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Lukas Anzeletti

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Lukas Anzeletti. Stochastic differential equations with singular drift and (fractional) Brownian noise. Probability [math.PR]. Université Paris-Saclay, 2023. English. NNT : 2023UPASM025 . tel-04576910

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Stochastic differential equations with singular drift and (fractional) Brownian noise

*Équations différentielles stochastiques avec une dérive singulière et un
bruit brownien (fractionnaire)*

Thèse de doctorat de l'université Paris-Saclay

École doctorale n° 574 École doctorale de Mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques appliquées
Graduate School : Mathématiques. Référent : CentraleSupélec

Thèse préparée au laboratoire MICS (Université Paris-Saclay, CentraleSupélec), sous la direction d'**Alexandre RICHARD**, Maître de conférences, CentraleSupélec, Chercheur à la Fédération de Mathématiques de CentraleSupélec et au laboratoire MICS et d'**Etienne TANRÉ**, Chargé de recherche à l'Inria Sophia-Antipolis, Université Côte d'Azur.

Thèse soutenue à Paris-Saclay, le 26 Octobre 2023, par

Lukas ANZELETTI

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Titre : Équations différentielles stochastiques avec une dérive singulière et un bruit brownien

Mots clés : Analyse stochastique, EDS singulières, Brownien fractionnaire, temps local

Résumé : On étudie l'existence et l'unicité des solutions d'équations différentielles stochastiques avec une dérive singulière ou même distributionnelle et un bruit brownien (fractionnaire) additif. Dans le cas brownien fractionnaire, l'existence et l'unicité sont démontrées pour un ensemble de dérivées plus grand que ce qui était connu auparavant, en utilisant à la fois les intégrales de Young non linéaires et le lemme de la couturière stochastique (Stochastic Sewing Lemma) récemment découvert. De plus, pour tout temps positif, on montre que la loi de la solution a une densité et on étudie sa régularité.

Certaines des techniques utilisées sont de na-

ture plutôt déterministe. Cela conduit naturellement à la notion d'unicité "trajectoire par trajectoire" qui est conceptuellement plus forte que les notions habituelles d'unicité pour les équations différentielles stochastiques. Dans le cas brownien, l'unicité "trajectoire par trajectoire" est démontrée pour une classe de dérivées singulières, étendant ainsi la littérature précédente sur les dérivées bornées. En outre, des contre-exemples sont fournis, montrant que l'unicité "trajectoire par trajectoire" est effectivement plus forte que les notions habituelles d'unicité. En particulier, cela implique qu'il existe des équations différentielles stochastiques dont les solutions ne sont pas adaptées.

Title : Stochastic differential equations with singular drift and (fractional) Brownian noise

Keywords : Stochastic Analysis, singular SDEs, fractional Brownian Motion, local time

Abstract : Existence and uniqueness of solutions to stochastic differential equations with singular or even distributional drift and additive (fractional) Brownian noise are studied. In the fractional Brownian case existence and uniqueness is proven for a larger class of drifts compared to the previous literature, making use of both nonlinear Young integrals and the recently introduced Stochastic Sewing Lemma. Moreover, for any positive time, the law of the solution is shown to have a density and its regularity is investigated.

Some of the techniques involved are of a ra-

ther deterministic nature. This naturally leads to the notion of "path-by-path" uniqueness which is conceptually stronger than the usual notions of uniqueness to stochastic differential equations. In the Brownian case, path-by-path uniqueness is shown for a class of singular drifts extending the previous literature on bounded drifts. Additionally, counterexamples are provided, exhibiting that path-by-path uniqueness is indeed stronger than the usual notions of uniqueness. In particular, this implies that there are stochastic differential equations with solutions that are not adapted.

ÉQUATIONS DIFFÉRENTIELLES STOCHASTIQUES AVEC UNE
DÉRIVE SINGULIÈRE ET UN BRUIT BROWNIEN
(FRACTIONNAIRE)

Lukas ANZELETTI

Acknowledgments

First of all I want to express my gratitude to Laure Coutin and Nicolas Perkowski for carefully reviewing the thesis and to Paul Gassiat, Xue-Mei Li and Stéphane Menozzi for accepting to be part of the committee.

A huge thank you to my supervisors Alexandre and Etienne. I really enjoyed to discuss math with you in a laid back environment without being scared of saying something stupid (at least after the first few meetings). Also, I appreciated that you always had an open door and ear and gave me enough freedom to pursue my own research interests while still guiding and giving helpful advice (both regarding research and life - Etienne's capability of fitting some wise sentences in between a hodgepodge of dad jokes is quite unique).

Speaking of dad jokes, I want to take this opportunity to thank you Mama and Pezi for everything that you have done for me: the countless hours of driving me to practices and tournaments; always standing up for me; looking out that I made at least some minimal effort in important subjects in school while giving me the freedom to decide which ones are not that important after all (not paying attention in french was a bad call from my side). Thanks Dani for sharing almost all my passions in life and for being empathetic, kind-hearted while also being the funniest guy I know.

To Schurl and Giacomo, for all the hours of studying together and the countless hours of political discussions throughout our studies.

Thank you to my collaborators Chengcheng and Khoa, my desk-neighbors Theo and El Mehdi as well as to everyone who I had the pleasure of meeting at conferences over the last years. I genuinely enjoyed the mathematical discussions and the lunches/dinners/drinks/etc.

Also I would like to thank Fabienne and Brigitte whose support made it possible to navigate through the jungle of french bureaucracy; Pauline for the ongoing support, in particular during the planning of the CJC-MA 2023 and Ludovic for his creative intuitive way of doing math and the open wallet whenever there was a need for funding! Moreover, I would like to acknowledge the financial support of Labex Mathématique Hadamard.

A huge thanks to my friends in Austria. For visiting me, all the video-calls and welcoming me back home whenever I came for a visit.

Finally, I want to thank Elisa for the amazing three years in Paris!

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Preprints and publications

The thesis is based on the following preprints and publications as well as on ongoing work with my advisors Alexandre Richard and Etienne Tanré.

- [LA1] Lukas Anzeletti. Comparison of classical and path-by-path solutions to SDEs. *Preprint [arXiv:2204.07866](#)*, 2022, *submitted*.
- [LA2] Lukas Anzeletti. A note on weak existence for SDEs driven by fractional Brownian motion. *Preprint [arXiv:2303.17970](#)*, 2023, *accepted in *Mathématica Contemporânea**.
- [LA3] Lukas Anzeletti, Khoa Lê and Chengcheng Ling. Path-by-path uniqueness for stochastic differential equations under Krylov-Röckner condition. *Preprint [arXiv:2304.06802](#)*, 2023, *submitted*.
- [LA4] Lukas Anzeletti, Alexandre Richard and Etienne Tanré. Regularisation by fractional noise for one-dimensional differential equations with distributional drift. *Electron. J. Probab. 28: 1-49*, 2023.
- [LA5] Lukas Anzeletti, Alexandre Richard and Etienne Tanré. Properties of the density of the law of the solution for SDEs with distributional drift. 2023+, *in preparation*.

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I.1 Version française

Imaginons que l'on veuille modéliser le mouvement de particules dans un fluide. Cela dépend à la fois du mouvement intrinsèque (diffusion) des particules et d'une dérive due à la vitesse du fluide environnant. Ce qui semble être un problème d'EDP peut en réalité être directement lié à la théorie des probabilités, en particulier à la théorie des équations différentielles stochastiques (EDS). Comme le mouvement Brownien peut modéliser le mouvement intrinsèque aléatoire de la particule, la littérature classique sur les EDS permet d'étudier de tels phénomènes pour une dérive régulière. Cependant, il est clair que cela n'est pas suffisant pour couvrir tous les cas physiquement pertinents, car la vitesse du fluide peut être singulière en certains points de l'espace, par exemple en présence

de tourbillons dans les écoulements turbulents (pour plus de détails sur de telles diffusions avec une dérive singulière, voir par exemple [90]). Une question naturelle se pose alors :

Quelle irrégularité peut avoir la dérive, tout en maintenant le caractère bien posé
de l'EDS correspondante ? (Q)

La réponse habituelle (qui peut être assez surprenante à première vue!) est la suivante : si le bruit est très irrégulier (c'est-à-dire qu'il oscille beaucoup), le caractère bien posé peut être garanti même pour des dérivées très singulières - un phénomène souvent appelé "régularisation par le bruit" (voir e.g. [42, 48]). En particulier, le caractère bien posé est obtenu pour des équations qui seraient extrêmement mal posées sans la présence de bruit.

Grâce au théorème de Donsker, l'exemple le plus courant de bruit est le mouvement Brownien. Cependant, dans diverses situations, les accroissements futurs peuvent dépendre du comportement du passé. Le mouvement Brownien fractionnaire est un processus gaussien simple qui autorise des corrélations entre les différents accroissements temporels, et il est donc souvent utilisé pour modéliser de telles situations, dites avec "mémoire".

Tout au long de la thèse, l'objectif principal est de répondre à la question (Q), en considérant soit un bruit Brownien fractionnaire, soit un bruit Brownien.

I.1.1 Régularisation par le bruit - une perspective EDO

Cette section fournit une motivation et des explications informelles sur le phénomène de régularisation par le bruit. Pour des idées similaires et une compréhension plus approfondie, voir [42, 48].

Considérons une équation différentielle ordinaire (EDO) autonome de la forme

$$\dot{x}_t = b(x_t). \quad (\text{I.1.1})$$

Le célèbre théorème de Picard-Lindelöf affirme qu'il suffit que b soit lipschitzienne afin d'obtenir une solution unique à l'équation ci-dessus pour une condition initiale donnée $x_0 \in \mathbb{R}^d$. Pour des fonctions non lipschitziennes b , l'unicité peut être perdue, comme le montrent les deux exemples suivants.

Exemple 1 : En prenant $d = 1$, $b(\cdot) = 2\text{sgn}(\cdot)\sqrt{|\cdot|}$ et une condition initiale $x_0 = 0$, toute fonction de l'ensemble non dénombrable suivant

$$\{(t - t_0)^2 \mathbb{1}_{t \geq t_0} : t_0 \geq 0\}$$

résout le problème de Cauchy ci-dessus. Cependant, cette sorte de "division" en différentes solutions ne peut se produire qu'en 0 (en l'espace). Pour des conditions initiales $x_0 \neq 0$, l'équation est localement bien posée.

Exemple 2 : La situation peut être encore pire lorsque b est discontinue. Prenons $d = 1$ et

$$b(x) = \begin{cases} 0 & \text{pour } x \in \mathbb{Q}, \\ 1 & \text{pour } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Pour une condition initiale rationnelle x_0 , à la fois $x_t = x_0$ et $x_t = x_0 + t$ résolvent le problème de Cauchy correspondant. Pour une condition initiale irrationnelle x_0 , $x_t = x_0 + t$ est une solution. De plus, $x_t = x_0 + (q - x_0) \wedge t$ est une solution pour tout rationnel $q > x_0$. Nous constatons dans

ces deux exemples unidimensionnels que les points où b s'annule et n'est pas lipschitzienne posent problème. Soit $\gamma : [0, \infty) \rightarrow \mathbb{R}$ une fonction donnée fortement oscillante avec $\gamma_0 = 0$. Lorsque l'équation (I.1.1) est perturbée par γ , écrite sous forme d'équation intégrale, elle prend la forme

$$x_t = x_0 + \int_0^t b(x_s) ds + \gamma_t. \quad (\text{I.1.2})$$

On pourrait espérer que γ empêche l'apparition d'une "division" comme dans les exemples ci-dessus, car les oscillations du "bruit" γ pourraient empêcher que la solution reste sur les points critiques où b s'annule. Pour renforcer l'intuition de cet argument informel, on peut réécrire l'équation après avoir posé $\tilde{x} = x - \gamma$, par

$$\tilde{x}_t = x_0 + \int_0^t b(\tilde{x}_s + \gamma_s) ds. \quad (\text{I.1.3})$$

Si l'on sait *a priori* que \tilde{x} oscille moins vite que γ , alors sur un petit intervalle de temps $[u, v]$,

$$\int_u^v b(\tilde{x}_s + \gamma_s) ds \approx \int_u^v b(\tilde{x}_u + \gamma_s) ds. \quad (\text{I.1.4})$$

Donc, de manière informelle, pour de petits intervalles de temps, on peut considérer que \tilde{x} est constant. Ainsi, pour étudier (I.1.2) et l'effet de régularisation de γ , on peut examiner l'opérateur de moyennisation, défini par

$$T^\gamma b: (t, z) \mapsto \int_0^t b(z + \gamma_r) dr, \quad (\text{I.1.5})$$

avec l'espoir de retrouver la continuité lipschitzienne en espace de $T^\gamma b$, tout en conservant la régularité ε -Hölder en temps. Ainsi, pour des intervalles de temps suffisamment petits $[s, t]$, et pour deux solutions \tilde{x}, \tilde{y} ,

$$\begin{aligned} |\tilde{x}_t - \tilde{x}_s - (\tilde{y}_t - \tilde{y}_s)| &= \left| \int_s^t b(\tilde{x}_r + \gamma_r) dr - \int_s^t b(\tilde{y}_r + \gamma_r) dr \right| \\ &\approx \left| \int_s^t b(\tilde{x}_s + \gamma_r) dr - \int_s^t b(\tilde{y}_s + \gamma_r) dr \right| \\ &= \left| T^\gamma b(t, \tilde{x}_s) - T^\gamma b(s, \tilde{x}_s) - (T^\gamma b(t, \tilde{y}_s) - T^\gamma b(s, \tilde{y}_s)) \right| \\ &\lesssim (t - s)^\varepsilon |\tilde{x}_s - \tilde{y}_s|. \end{aligned}$$

Dans certaines situations, cela suffit pour garantir l'unicité en étudiant un espace qui inclut toutes les solutions (voir par exemple [34]) ou en faisant appel à l'existence d'un flot (voir [101] ou le Chapitre V). En général, cette condition permet de garantir l'existence d'une solution à (I.1.3) et donc à (I.1.2) via un schéma d'Euler (voir la Section II.3.2).

L'idée générale à garder en mémoire est que même si b présente des discontinuités ou des singularités, cela ne devrait pas avoir une grande influence sur la fonction $T^\gamma b$, car les oscillations rapides de γ empêchent $z + \gamma$ de passer beaucoup de temps autour de ces discontinuités/singularités. Cette idée est un thème récurrent tout au long de la thèse, permettant d'analyser l'effet de régularisation de γ et de définir une notion de solution à (I.1.2), y compris lorsque b est une distribution.

Il est important de noter que dans tous les arguments (heuristiques) ci-dessus, aucune technique probabiliste n'est requise, ce qui conduit naturellement à des notions d'existence et d'unicité des solutions de nature déterministe, même si γ est échantillonné à partir de la loi d'un processus stochastique (voir la Section I.1.4).

I.1.2 Régularisation par le bruit - une perspective EDS

Cette section offre une brève revue de littérature dans le cas où γ est échantillonné à partir de la loi d'un mouvement Brownien (fractionnaire). Classiquement, cela est fait dans le sens des EDS avec les concepts courants d'existence et d'unicité des solutions adaptées (faibles/fortes) et d'unicité parmi elles (soit en loi, soit trajectorielle). Cette section se concentre sur ces notions classiques. Pour des résultats plus récents fondés sur une approche trajectoire par trajectoire, nous renvoyons le lecteur à la Section I.2.3. Pour une comparaison de ces concepts, voir le Chapitre VI et la Section V.B.

Afin d'introduire l'EDS qui nous intéresse, rappelons d'abord la définition d'un mouvement Brownien fractionnaire.

Mouvement Brownien fractionnaire. Le mouvement Brownien fractionnaire (mBf) de paramètre de Hurst $H \in (0, 1)$ est le processus gaussien centré $(B_t)_{t \geq 0}$ de covariance

$$\mathbb{E}[B_s B_t] = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}), \quad \forall s, t \geq 0.$$

La définition s'étend immédiatement à d dimensions en considérant d mouvements Browniens fractionnaires indépendants de paramètre de Hurst H . Le cas $H = 1/2$ coïncide avec la définition d'un mouvement Brownien ; $H \in (1/2, 1)$ conduit à des accroissements positivement corrélés ainsi qu'à une mémoire longue et $H < 1/2$ conduit à des accroissements négativement corrélés et à des trajectoires très irrégulières. Ces corrélations entre les accroissements font du mouvement Brownien fractionnaire un exemple intéressant de processus stochastique pour décrire des phénomènes réels (par exemple en finance, voir [17, 31, 56]). De plus, en utilisant le théorème de continuité de Kolmogorov, il est facile de voir que pour tout $\alpha \in (0, H)$, il existe une version du processus qui est α -Hölder continue. Ce n'est pas le cas pour $\alpha = H$. En fait, pour presque chaque trajectoire, il n'y a pas de points de continuité H -Hölder, ce qui fait du mBf un exemple de processus (fortement) oscillant. Par conséquent, pour de petites valeurs de H , il devrait y avoir un fort effet de régularisation du mBf en raison de cette nature oscillante, de par les intuitions esquissées dans la Section I.1.1. Ainsi, tout au long de cette thèse, nous considérons $H \leq 1/2$, ce qui permet de prendre en compte des dérives distributionnelles. Un lien rigoureux entre le mBf et le mouvement Brownien via une formule de représentation intégrale est établi dans la Section II.A.2.

EDS étudiées. Considérons l'EDS

$$X_t = X_0 + \int_0^t b(s, X_s) ds + B_t, \quad t \in [0, T], \quad (\text{I.1.6})$$

où $T > 0$ est un temps déterministe fixé, $b: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ est mesurable et B est un mouvement Brownien fractionnaire d -dimensionnel. Dans la suite, une revue de littérature succincte sur (I.1.6) est présentée, en se concentrant spécifiquement sur les conditions sur b pour assurer le caractère bien posé de l'équation. Certains cas particuliers de (I.1.6) sont considérés (par exemple une dérive unidimensionnelle, autonome). Cependant, à ce stade, pour simplifier, nous choisissons de négliger cela.

Cas Brownien. Si B est un mouvement Brownien unidimensionnel et b est une fonction mesurable bornée, il est montré dans [116] qu'il existe une solution forte à (I.1.6) et que l'unicité trajectorielle est aussi vérifiée. Cela est réalisé par la transformation de Zvonkin ; une transformation déterministe

découlant de la solution d'une EDP bien choisie, afin de supprimer le terme de dérive irrégulière. Cette idée est réutilisée dans [110] pour étendre le résultat au cas multidimensionnel. Krylov et Röckner [68] ont étendu ces résultats, dans le cas d'un bruit additif, à des dérivées b non bornées satisfaisant la condition d'intégrabilité

$$\left(\int_0^T \left(\int_{\mathbb{R}^d} |b(r, x)|^p dx \right)^{\frac{q}{p}} dr \right)^{\frac{1}{q}} < \infty, \quad (p, q) \in (2, \infty)^2 \text{ avec } \frac{2}{q} + \frac{d}{p} < 1. \quad (\text{I.1.7})$$

L'existence de solutions faibles et l'unicité trajectorielle sont démontrées séparément ce qui donne également l'existence de solutions fortes par le principe de Yamada-Watanabe (voir [113]). L'existence de solutions faibles est prouvée de manière classique (voir par exemple [90, Chapitre 1, Section 4]) en supprimant le terme de dérive via une transformation de Girsanov. Cela est également possible pour un coefficient de diffusion non constant (voir [115]).

Cas du mouvement Brownien fractionnaire. Comme le mouvement Brownien fractionnaire n'est ni un processus de Markov ni une semimartingale pour $H \neq 1/2$, les techniques EDP mentionnées ci-dessus et le calcul stochastique classique ne sont pas disponibles, il faut donc trouver d'autres idées. Néanmoins, une transformation de Girsanov est toujours possible (voir [35]). Dans ce paragraphe, nous esquissons brièvement la chronologie détaillée des résultats, pour une présentation des résultats, voir la Section III.A.

Une première tentative en ce sens est due à Nualart et Ouknine [83, 84], qui montrent l'existence et l'unicité pour certaines dérivées non Lipschitziennes. Cela est fait en utilisant d'abord la transformation de Girsanov pour obtenir une existence et une unicité faible et l'observation que dans ce cas, ces deux propriétés sont suffisantes pour obtenir également une unicité trajectorielle. Dans un travail fondateur, Catellier et Gubinelli [24] prouvent le caractère bien-posé (au sens trajectoire par trajectoire, voir la Section I.2.3) pour des dérivées distributionnelles indépendantes du temps $b \in \mathcal{C}^\beta$ pour $\beta \in \mathbb{R}$ avec $\beta > 1 - 1/(2H)$. Lorsque $\beta < 0$, \mathcal{C}^β est un espace de Hölder généralisé contenant des distributions. Cela est réalisé par une étude approfondie de l'opérateur de moyennisation associé à chaque trajectoire du mouvement Brownien fractionnaire, en appliquant la transformation de Girsanov et en reformulant l'équation comme une équation intégrale de Young non linéaire (voir la Section II.3). Dans le Chapitre II, différents résultats sur l'existence et l'unicité sont présentés pour des classes de dérivées distributionnelles qui ne sont pas couvertes dans [24]. En particulier, l'existence d'une solution faible est prouvée pour $b \in \mathcal{C}^\beta$ avec $\beta > 1/2 - 1/(2H)$ et l'unicité trajectorielle est démontrée pour $\beta = 1 - 1/(2H)$ lorsque b est une mesure. En termes d'existence de solutions faibles, ces résultats sont affinés pour une dérive à valeurs dans les mesures dans le Chapitre III. Depuis mon travail, dans des cas spécifiques, c'est-à-dire lorsque b satisfait une certaine condition d'intégrabilité ou est une mesure finie, des résultats plus fins ont été obtenus dans [20], qui améliore davantage l'unicité et l'existence.

Dans [32], les auteurs adoptent une approche basée sur la formule de Clark-Ocone pour étudier les dérivées aléatoires dépendant du temps. Dans [49], les auteurs étendent le résultat principal de [24] aux dérivées distributionnelles dépendant du temps avec une singularité dans le temps, en récupérant le régime sous-critique complet en termes de régularité spatiale, pour des cas particuliers d'intégrabilité en temps.

À ce stade, mentionnons que tout au long de cette thèse, nous ne considérons pas de bruit multiplicatif. Cela simplifie les choses, car il n'y a pas d'ambiguïté dans la définition d'une intégrale par rapport au bruit. Sinon, il faudrait aborder cette question, par exemple en supposant une

régularité suffisante de la diffusion et du mouvement Brownien fractionnaire pour les interpréter comme une intégrale de Young ou une intégrale rugueuse. Ainsi, le bruit multiplicatif est délicat dans le sens où un bruit plus rugueux, d’une part aide à régulariser l’équation, d’autre part complique la définition du terme intégral. Nous mentionnons seulement une sélection de travaux : Dans [33], le caractère bien-posé des équations intégrales de Young et des équations intégrales rugueuses avec une dérive irrégulière est démontrée. Les auteurs dans [80] affaiblissent les hypothèses sur la régularité du terme de diffusion via une extension du lemme de la couturière stochastique, en considérant l’équation comme une équation intégrale de Young.

Une autre question intéressante qui n’est pas abordée dans cette thèse est celle de la limite de bruit nul : remplacer B_t dans (I.1.6) par εB_t et faire tendre ε vers zéro. Dans le cas où il n’y a pas d’unicité pour l’équation non perturbée, quelle solution récupérerons-nous à la limite ? Dans le cas Brownien, cette question a été résolue dans [10] pour une classe de dérives positives, cependant dans le cas du mouvement Brownien fractionnaire, cette question semble encore très ouverte. Pour une présentation plus détaillée de ce sujet, voir [42, Section 1.4].

Mouvement Brownien (fractionnaire) biaisé. De nombreuses situations physiques impliquent des phénomènes de diffusion dans des milieux hétérogènes (voir [74] pour des références décrivant ces phénomènes en physique, mais aussi en biologie, en finance, etc.). Le mouvement Brownien biaisé (mouvement Brownien “partiellement réfléchi”) est un objet probabiliste qui permet de décrire ce comportement.

Le mouvement Brownien biaisé a été introduit pour la première fois dans [66] avec l’idée suivante. On considère un mouvement Brownien et chaque fois qu’il atteint l’axe des abscisses, une variable aléatoire de Bernoulli indépendante est introduite. Avec une probabilité η , le mouvement Brownien est “poussé dans \mathbb{R}_+ ”, avec une probabilité $1 - \eta$, le mouvement Brownien est “poussé dans \mathbb{R}_- ”. Lorsque $\eta = 1$, on retrouve un mouvement Brownien réfléchi. Un tel processus peut également être construit via une approximation par des marches aléatoires biaisées (voir [74]). Notons qu’en dehors de 0, le processus ci-dessus se comporte comme un mouvement Brownien. Comme cela avait été suggéré dans [66], Walsh [111] a prouvé que le mouvement Brownien biaisé définit un processus de diffusion et a donné deux motivations principales pour l’étude de cet objet. Premièrement, il permet de modéliser le comportement d’une particule qui se déplace entre deux milieux différents. Deuxièmement, il fournit un exemple d’objet avec un temps local discontinu en espace dû au comportement en 0. Bien que le mouvement Brownien réfléchi partage cette propriété, ce n’est pas un exemple très satisfaisant car le codomaine naturel pour le mouvement Brownien réfléchi est $[0, \infty)$, contrairement au mouvement Brownien biaisé qui évalue dans tout \mathbb{R} . Pour plus de détails et diverses constructions du mouvement Brownien biaisé, voir [74]. De plus, il existe un lien clair entre le mouvement Brownien biaisé et le type d’équations qui nous intéressent, clarifié ci-dessous.

Pour le bruit Brownien et la dérive autonome $b = \delta_0$, il est connu qu’il existe une unique solution forte à l’équation (I.1.6) donnée par le mouvement Brownien réfléchi (voir e.g. [73]). Ainsi, le Dirac en zéro peut être interprété comme la force exacte nécessaire pour réfléchir la solution le long de l’axe des abscisses. De plus, on peut donner un sens à l’intégrale formelle $\int_0^\cdot \delta_0(X_r) dr$ en l’interprétant comme le temps local de X en 0. Cette idée a été exploitée dans [63] (avec une généralisation ultérieure dans [73]) à une dérive sous forme de temps local correspondant formellement à $b = a\delta_0$, pour $a \in [-1, 1]$. La solution à une telle équation est le mouvement Brownien biaisé introduit ci-dessus avec le paramètre $\eta = (a + 1)/2$. De plus, [63, 73] montrent qu’il n’existe pas de solution pour $a \in \mathbb{R}$ avec $|a| > 1$.

Ayant à l'esprit l'intuition selon laquelle un bruit plus rugueux devrait conduire à un effet de régularisation plus important, il est naturel de chercher à étendre le résultat ci-dessus au bruit Brownien fractionnaire avec $H < 1/2$. Cependant, à la fois [63] et [73] reposent sur la formule d'Itô et donc des idées similaires ne se transposent pas simplement dans le cas du mouvement Brownien fractionnaire. Néanmoins, pour $d = 1$ et $b = a\delta_0$ avec $a \in \mathbb{R}$, l'existence et l'unicité de cette équation ont été établies pour $H < 1/4$ par Catellier and Gubinelli [24] en utilisant des intégrales non linéaires de Young. Cependant, pour $d = 1$, un écart par rapport au cas du mouvement Brownien ($H = 1/2$), avec un caractère bien-posé pour $|a| \leq 1$ prouvée dans [73], subsiste. L'un des objectifs de notre travail était de combler en partie cet écart.

Chronologiquement, une première étape dans cette direction pour l'existence et l'unicité peut être trouvée dans [LA4] grâce à une approche basée sur les intégrales de Young non linéaires et une approche de couture stochastique (voir le Chapitre II). Finalement, en termes d'existence faible en dimension 1, cet écart a été complètement comblé dans [LA2] (voir le Chapitre III). Simultanément à [LA2], dans [20] les auteurs ont obtenu le même résultat en dimension 1. De plus, ils ont prouvé un résultat plus fort pour la dimension $d > 1$ et ont également partiellement comblé l'écart d'unicité, en étendant le résultat de [LA4] dans le cas particulier où b est une mesure finie. Néanmoins, combler complètement l'écart en termes d'unicité reste une question ouverte.

Il convient de mentionner à ce stade que la plupart des résultats ci-dessus sont des cas particuliers de résultats plus généraux concernant les dérivées sous forme de mesures ou même des distributions plus générales. Afin de rendre la présentation plus claire à ce stade, nous avons choisi de mettre en évidence les différents résultats en considérant $b = \delta_0$. Pour un résumé plus détaillé de la littérature pour (I.1.6) dans le cas du mouvement Brownien fractionnaire ainsi que des références à des travaux connexes, nous renvoyons le lecteur à la Section III.A.

I.1.3 À propos des solutions trajectoire par trajectoire et des solutions adaptées

Le caractère bien posé de (I.1.2) est actuellement un domaine de recherche très actif. À partir des travaux fondateurs de Davie [34] et Catellier–Gubinelli [24], il y a eu un intérêt croissant pour le problème, voir [23, 33, 49, 50, 62, 101]. Un résultat typique de ces articles est que lorsque γ est une trajectoire d'un processus stochastique irrégulier (comme le mouvement Brownien fractionnaire), alors pour presque chaque trajectoire de ce type, (I.1.2) a une solution unique, une propriété connue sous le terme d'*unicité trajectoire par trajectoire*. De manière générale, la régularité nécessaire pour le champ vectoriel est inversement proportionnelle à la rugosité du processus stochastique à partir duquel γ est échantillonné. Dans une autre direction, lorsque les coefficients sont suffisamment réguliers pour que la théorie des chemins rugueux de Lyons puisse être appliquée, l'unicité dans l'équation différentielle stochastique correspondante implique alors l'unicité trajectoire par trajectoire. Cela est dû au fait que la base stochastique n'est utilisée que pour construire un relèvement de chemin rugueux de γ et l'analyse des chemins rugueux de Lyons [79] est applicable pour toute trajectoire fixe de relèvement.

D'un point de vue pratique, l'unicité trajectoire par trajectoire est la plus pertinente dans les situations où seuls quelques échantillons du signal peuvent être observés. Récemment, ce type d'unicité a été largement étudié à la fois pour les EDS et les EDPS, voir les références dans le paragraphe précédent et [15, 21, 24, 42, 91, 92, 101]. D'un point de vue théorique, l'unicité trajectoire par trajectoire est plus forte que l'unicité trajectorielle plus classique (l'unicité trajectorielle permet

d'identifier deux solutions adaptées à (I.1.6) tandis que l'unicité trajectoire par trajectoire permet d'identifier toutes les solutions). En particulier, l'unicité trajectoire par trajectoire pour (I.1.6) avec une condition initiale $Y_0(\omega) = x$ ne garantit pas (conceptuellement) l'unicité pour l'EDO suivante dirigée par $B(\omega)$

$$Y_t = Y_0 + \int_0^t b(r, Y_r) dr + B_t(\omega), \quad t \in [0, T], \quad (\text{I.1.8})$$

pour un ω donné. La question de savoir si des exemples d'EDS existent pour lesquels les concepts d'unicité ci-dessus ne coïncident pas a été abordée dans [42, page 12] et [1, page 3]. De tels exemples d'EDS pilotées par le mouvement Brownien pour lesquelles l'unicité trajectoire est vérifiée mais pas l'unicité trajectoire par trajectoire, sont donnés dans [103, LA1] (pour plus de détails, voir le Chapitre VI). L'idée générale est que si les solutions sont autorisées à “prendre en compte le comportement futur” du mouvement Brownien, elles peuvent décider du “bon comportement” au moment présent. En faisant cela, les zones critiques au-delà desquelles une solution ne peut pas être prolongée peuvent être évitées, mais le caractère adapté est perdue. Par conséquent, cela n'est pas possible si on se limite aux solutions adaptées. Pour une discussion plus détaillée et une comparaison des différents concepts, nous renvoyons le lecteur au Chapitre V et au Chapitre VI. Nous rappelons tout de même rapidement les résultats de référence. Comme mentionné dans la Section I.2.2, l'unicité trajectoire pour les EDS avec dérive singulière b satisfaisant la condition de Krylov-Röckner (I.1.7) a été démontrée dans [68]. L'unicité trajectoire par trajectoire a été démontrée dans [34] pour les dérivées mesurables et bornées b . La démonstration dans [34] ne repose pas sur les résultats précédents concernant l'unicité trajectoire par trajectoire, mais établit plutôt l'unicité trajectoire par trajectoire de manière relativement autonome (et technique). Le même résultat est (à quelques améliorations mineures près) retrouvé par Shaposhnikov [101], mais avec une démonstration plus simple qui utilise l'existence d'un flot régulier (et donc aussi l'unicité trajectoire).

La contribution de ce travail concernant l'unicité trajectoire par trajectoire est double, tous les résultats ne concernant que le cas Brownien :

- (a) Construction de contre-exemples unidimensionnels pour lesquels les notions d'unicité trajectoire et d'unicité trajectoire par trajectoire ne coïncident pas (voir le Chapitre VI).
- (b) Réponse à la question ouverte de l'unicité trajectoire par trajectoire pour les EDS dirigées par un mouvement Brownien sous la condition de Krylov-Röckner (voir le Chapitre V).

Notons qu'en (b), il est clair que des contre-exemples comme ceux en (a) ne peuvent pas être construits pour des dérivées b satisfaisant la condition de Krylov-Röckner. Au lieu de cela, les contre-exemples mentionnés en (a) reposent sur des EDS bien connues avec une dérive encore plus singulière, comme l'EDS du processus de Bessel et l'EDS du pont de Bessel.

L'idée principale de la démonstration de (b) est similaire à celle de [101], c'est-à-dire comparer, pour toute réalisation du mouvement Brownien, toute solution de l'équation intégrale déterministe à la solution donnée par l'unique flot de solutions de l'EDS correspondante. Néanmoins, au niveau technique, des idées différentes sont nécessaires. En particulier, comme b n'est pas bornée, pour une solution Y de (I.1.8), $\int_0^t b(r, Y_r) dr$ n'est pas nécessairement lipschitzienne. À la place, nous travaillons dans l'espace des fonctions à variation finie. De plus, nous démontrons que pour presque toutes les trajectoires Browniennes, un flot de solutions existe. Enfin, nous utilisons l'inégalité de

John-Nirenberg quantitative récemment obtenue (voir [71]) pour assurer une régularité suffisante de l'opérateur d'approximation $T^B b$ introduit dans la Section I.1.1.

I.1.4 Comment définir et obtenir des solutions pour une dérive distributionnelle

Pour étudier (I.1.2) avec une dérive b distributionnelle, il n'est pas clair *a priori* comment interpréter l'intégrale du membre de droite de l'équation et donc comment définir une solution. Pour y parvenir deux idées peuvent être de

- (a) définir une solution x de manière à ce que l'intégrale puisse être identifiée comme la limite de la suite $\int_0^\cdot b^n(x_s) ds$ pour toute suite b^n convergeant vers b dans un espace de distributions choisi de manière appropriée.
- (b) faire directement appel aux oscillations de γ , via les intégrales de Young non linéaires.

La plupart du temps, nous considérons que γ est échantillonnée selon la loi d'un mouvement Brownien fractionnaire afin d'avoir à notre disposition des outils tels que le non-déterminisme local et l'existence d'un temps local (régulier). Cependant, notons que le schéma de construction d'une solution au sens de (b) est de nature déterministe. Dans la plupart des cas, (a) et (b) conduisent à une définition équivalente d'une solution (voir le théorème II.2.15). Néanmoins, la manière "naturelle" d'obtenir une solution correspondant à l'une ou l'autre des notions diffère. Nous présentons brièvement les idées générales. (a) : L'outil principal de cette approche est l'utilisation du lemme de la couturière stochastique, récemment introduit par Lê [70]. Tout comme le lemme de couturière habituel (déterministe), le lemme de la couturière stochastique permet d'approcher un processus à deux paramètres par un processus à un paramètre et vice versa. Cela permet de trouver des bornes *a priori* de la solution par une approximation similaire à celle de (I.1.4) (voir la Section I.1.5).

Ayant des estimations *a priori* en main, nous passons par un argument de tension-stabilité pour obtenir l'existence de solutions faibles. Encore une fois, par le lemme de la couturière stochastique, l'unicité trajectorielle peut être démontrée et finalement un argument de type Yamade-Watanabe conduit à l'existence d'une unique solution forte.

(b) : Soit $f \in \mathcal{C}_b(\mathbb{R})$ et rappelons l'opérateur de moyennisation T^γ associé au chemin γ défini par $T_s^\gamma f(z) := \int_0^s f(z + \gamma_r) dr$. Alors pour $t > 0$, une suite de partitions $(\{t_i^N\}_{0 \leq i \leq N})_{N \in \mathbb{N}}$ de $[0, t]$ et $\tilde{x} = x - \gamma$, nous avons formellement que

$$\begin{aligned} \int_0^t f(\tilde{x}_r + \gamma_r) dr &= \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} \int_{t_i^N}^{t_{i+1}^N} f(\tilde{x}_{t_i^N} + \gamma_r) dr \\ &= \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} \left(T_{t_{i+1}^N}^\gamma f(\tilde{x}_{t_i^N}) - T_{t_i^N}^\gamma f(\tilde{x}_{t_i^N}) \right) \text{ "=: " } \int_0^t T_{dr}^\gamma f(\tilde{x}_r). \end{aligned}$$

Pour l'instant, la dernière expression n'est qu'une définition formelle car la convergence de la somme de type Riemann n'est pas claire. Si la convergence est vérifiée, le dernier terme est une intégrale de Young non linéaire. De la régularité pour $T^\gamma f$ et \tilde{x} suffisent pour que cet objet soit bien défini, permettant une certaine compensation entre les deux. Pour γ oscillant suffisamment rapidement, il est possible d'étendre l'opérateur de moyennisation à des distributions f . Cela est

assez intuitif car l'opérateur de moyennisation peut être écrit comme une convolution avec le temps local de γ (voir la Section II.2.3). Ainsi, si γ a un temps local suffisamment régulier (ce qui est le cas pour les processus qui oscillent suffisamment rapidement), il compensera la faible régularité de f . Pour plus de détails, voir la Section II.4.1. Les intégrales de Young non linéaires dans des espaces de Hölder ont été utilisées dans [24, 50] pour étudier des équations comme (I.1.2). Pour une revue détaillée des intégrales de Young non linéaires et des équations intégrales de Young non linéaires, voir [47]. Dans certaines situations, il est utile de travailler dans des espaces de fonctions à p -variation finie plutôt que dans des espaces de Hölder. Par exemple, si \tilde{x} est connu pour être monotone et donc à variation finie, ce degré de régularité (en termes de variation) permet de diminuer les hypothèses de régularité sur $T^\gamma f$ afin que l'intégrale de Young non linéaire soit bien définie. Une application naturelle de cela se présente lorsque l'on considère (I.1.2) avec une dérive positive b , puisque nous savons *a priori* que toute solution de l'équation intégrale de Young correspondante sera croissante et donc à variation finie. La preuve de l'existence est ensuite effectuée par un schéma d'Euler, en recherchant une solution dans l'espace des fonctions à variation finie (voir la Section II.3.2).

Notons que l'approche qui consiste à définir l'opérateur de moyennisation comme une convolution avec un temps local est moins claire dans le cas d'une dérive dépendant du temps. Néanmoins, le lemme de la couturière stochastique fonctionne toujours bien. Cela a été exploré récemment dans [49].

Enfin, il convient de souligner que ces deux constructions coïncident pour des solutions régulières (voir le théorème II.2.15), et même dans des cas non triviaux où la dérive est singulière.

I.1.5 Comment appliquer le lemme de la couturière stochastique

Dans cette section, nous présentons un calcul simplifié pour illustrer comment définir et estimer la régularité de $\mathcal{A}_t := \int_0^t f(B_r)dr$ pour $f \in L^p(\mathbb{R})$ et $p \in [1, \infty)$. Dans les sections suivantes, des calculs similaires seront effectués pour des distributions f et pour $\int_0^t f(X_r)dr$, où X peut être décomposé en $X = B + \phi$ avec ϕ oscillant plus lentement que B .

Pour appliquer le lemme de la couturière stochastique, nous introduisons le processus à deux paramètres

$$A_{s,t} := \mathbb{E}^s \int_s^t f(B_r)dr,$$

où $\mathbb{E}^s[\cdot]$ est utilisé pour désigner $\mathbb{E}[\cdot | \mathcal{F}^s]$.

Afin d'estimer $A_{s,t}$, nous utilisons la propriété de non-déterminisme local du mouvement Brownien fractionnaire (dont on déduit que $\text{Var}(B_r - \mathbb{E}^s[B_r]) = C(r-s)^{2H}$) et une estimation du noyau de la chaleur :

$$\begin{aligned} \left| \mathbb{E}^s \int_s^t f(B_r)dr \right| &\leq \int_s^t |\mathbb{E}^s f(\mathbb{E}^s B_r + B_r - \mathbb{E}^s B_r)|dr \\ &= \int_s^t |g_{C(r-s)^{2H}} * f(\mathbb{E}^s B_r)|dr \\ &\leq \int_s^t \|g_{C(r-s)^{2H}} * f\|_{L^\infty(\mathbb{R})}dr \\ &\leq C\|f\|_{L^p(\mathbb{R})}(t-s)^{1-H/p}, \end{aligned}$$

où $g_t(x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$.

En utilisant le lemme de la couturière stochastique, on peut conclure que, pour tout $m \geq 2$, il existe une constante $C > 0$ telle que

$$\left\| \int_s^t f(B_r) dr \right\|_{L^m(\Omega)} \leq C \|f\|_{L^p(\mathbb{R})} (t-s)^{1-H/p}.$$

En utilisant la même idée, on peut estimer $\int_s^t f(B_r + \phi_s) dr$ pour un processus ϕ . Si ϕ oscille plus lentement que B , cela donnera également une estimation de $\int_s^t f(B_r + \phi_r) dr$ pour $|t-s|$ petit. Cela conduira à des estimations *a priori*, qui sont essentielles pour construire une solution à (I.1.6).

I.1.6 Densité de la solution et équations de McKean-Vlasov

Les idées des sections précédentes conduisent à des conditions sur la dérive b dans (I.1.6) afin d'obtenir l'existence et l'unicité d'une solution. Cependant, les schémas de preuve ne permettent pas d'en déduire directement des propriétés de la loi d'une telle solution. Pour le mouvement Brownien fractionnaire, des résultats sur la décroissance gaussienne de la densité sont disponibles pour une dérive qui est au moins une fonction mesurable (voir par exemple [16, 76]). De plus, des résultats sur la régularité de la loi de la solution pour une dérive dans un espace de Besov existent à la fois pour des dérivées avec une régularité positive [85] et des dérivées avec une régularité possiblement négative [51]. Dans ce dernier cas, la motivation de l'étude de la régularité de la densité est de donner des conditions sur la dérive afin d'obtenir le caractère bien posé d'une EDS avec une dérive dépendant de la loi de la solution. En particulier, nous pouvons espérer affaiblir les hypothèses sur \mathfrak{b} de manière à ce que l'équation de McKean-Vlasov suivante avec un bruit Brownien fractionnaire soit bien posée :

$$X_t = X_0 + \int_0^t \mathfrak{b}_s * \mu_s(X_s) ds + B_t, \quad \text{Loi}(X_t) = \mu_t. \quad (\text{I.1.9})$$

On peut s'attendre à ce que l'équation (I.1.9) soit bien posée pour \mathfrak{b} avec une régularité inférieure à celle que nous demandons pour b dans l'équation (I.1.6). L'idée est que $\mathfrak{b}_s * \mu_s$ sera plus régulier que \mathfrak{b}_s lui-même, si μ_s appartient à un espace de Besov avec une régularité positive. Dans la littérature actuelle, les conditions sur b assurant le caractère bien-posé de l'équation (I.1.6) et celles sur \mathfrak{b} assurant le caractère bien-posé de l'équation (I.1.9) sont identiques (voir [51]).

Soit b tel qu'une solution forte unique X de (I.1.6) existe. Alors nos contributions au Chapitre IV sur la densité de la loi de X peuvent être résumées comme suit :

- Pour tout $t > 0$, la loi de X_t est absolument continue par rapport à la mesure de Lebesgue et la densité $f_t(\cdot)$ se trouve dans un espace de Besov de régularité positive. De plus, une dérive plus régulière b donne directement plus de régularité à la densité.
- En tant que fonction sur $[0, T] \times \mathbb{R}$, f jouit d'une certaine intégrabilité en temps tout en étant considérée dans un espace de Besov de régularité positive en espace.
- Pour b satisfaisant des hypothèses de régularité légèrement plus fortes que les points ci-dessus, on peut démontrer à la fois par le lemme de la couturière stochastique et le calcul de Malliavin que pour tout $t > 0$, il existe des bornes supérieures gaussiennes pour la densité $f_t(\cdot)$.

De plus, dans la Section IV.4.3, nous esquissons comment construire une solution forte à (I.1.9) pour \mathbf{b} avec une régularité inférieure à celle que nous demandons pour b dans l'équation (I.1.6). Cela est basé sur les intuitions présentées précédemment dans cette section et les résultats ci-dessus sur les densités. La preuve rigoureuse est laissée au travail en cours [LA5].

I.1.7 Structure et résumé des principaux résultats

Dans le Chapitre II, l'existence et l'unicité de (I.1.6) sont démontrées pour une dérive à valeurs dans un espace de Besov. L'existence est démontrée via deux approches différentes. Premièrement, pour une dérive positive, en utilisant une approche plutôt déterministe qui réécrit l'équation comme une équation intégrale non linéaire de type Young et en utilise des propriétés fines du temps local du mouvement Brownien fractionnaire. Deuxièmement, pour une dérive générale, en utilisant le lemme de la couturière stochastique. Une adaptation récente du lemme de la couturière stochastique permet également d'obtenir l'unicité dans des cas qui n'ont pas été considérés dans Galeati et Gubinelli [50]. Tout le chapitre est basé sur un travail commun avec mes directeurs de thèse A. Richard et E. Tanré [LA4].

Dans le Chapitre III, des techniques similaires impliquant le lemme de la couturière stochastique comme dans le Chapitre II sont utilisées, mais uniquement pour l'existence de solutions faibles dans le cas d'une dérive positive. Le résultat principal de ce chapitre couvre le régime sous-critique complet en dimension 1. Le chapitre est basé sur [LA2]. Simultanément à [LA2], la pré-publication [20] est parue, comprenant un résultat sur l'existence faible pour le régime sous-critique complet dans n'importe quelle dimension. Cependant, le résultat principal du Chapitre III fournit une preuve simple. De plus, la Section III.A donne un examen détaillé de la littérature sur (I.1.6), en comparant notamment les conditions sur b pour obtenir l'existence et l'unicité des solutions.

Le Chapitre IV est basé sur un travail en cours avec mes directeurs de thèse. Par conséquent, nous nous attendons à ce que certains résultats de ce chapitre soient affinés. Cependant, à notre connaissance, les résultats sont nouveaux car les propriétés de la densité de la loi de la solution des EDS avec une dérive distributionnelle n'ont pas été approfondies jusqu'à présent. Le chapitre comprend des résultats sur les queues de type gaussien par le calcul de Malliavin et sur la régularité de la densité. De plus, on présente de potentielles applications de ces résultats à une EDS avec une dérive dépendante de la loi de la solution – l'équation de McKean-Vlasov. Encore une fois, tous ces résultats font intervenir des techniques de couture stochastique.

Dans le Chapitre V, l'unicité trajectorielle de (I.1.6) pour un bruit Brownien est démontrée sous la condition de Krylov-Röckner (I.1.7). En particulier, cela étend les résultats principaux de [34, 101] en permettant une plus grande classe de dérivées. Le schéma de preuve est similaire à [101], mais des résultats différents sont nécessaires d'un point de vue technique, notamment l'inégalité de John-Nirenberg récemment introduite [71]. Par ailleurs, la complétude forte est démontrée, c'est-à-dire que pour presque chaque trajectoire du mouvement Brownien, la famille de solutions avec différentes données initiales forme un semi-flot continu pour tous les temps positifs. Tout le chapitre est basé sur un travail commun avec Khoa Lê et Chengcheng Ling [LA3].

Dans le Chapitre VI, des dérivées unidimensionnelles b sont construites de telle manière que l'unicité trajectorielle (c'est-à-dire l'unicité classique parmi les solutions adaptées) de (I.1.6) soit satisfaite, mais l'unicité trajectoire par trajectoire ne l'est pas. De tels contre-exemples étaient connus auparavant uniquement en dimension $d > 1$. En particulier, nous construisons des solutions (trajectoire par trajectoire) d'EDS qui ne sont pas adaptées. Le chapitre est basé sur [LA1].

L'Annexe A.1 contient un bref résumé de la définition et de quelques propriétés utiles des espaces de Besov (similaire à [LA4, Annexe A]). Dans l'Annexe A.2 (basée sur [LA4, Annexe C]), différentes versions du lemme de la couturière stochastique sont rappelées.

I.2 English version

Imagine we want to model the movement of particles in a fluid. This is both determined by the intrinsic movement (diffusion) of the particles and by a drift due to the velocity of the surrounding fluid. What looks like a PDE-problem at first sight can actually be directly related to probability theory, in particular to the theory of Stochastic Differential Equations (SDEs). As the Brownian motion can model the random intrinsic movement of the particle, the classical literature on SDEs allows to study such phenomena for “nice” drift. However it is clear that this is not sufficient to cover all physically relevant cases since the speed of the fluid might be singular at some points in space, for example with the presence of vortices in turbulent flows (for more details on such diffusions with singular drift see for example the monograph of Portenko [90]). A natural question arises:

How “bad” can the drift be, while maintaining well-posedness of the corresponding SDE? (Q)

The usual answer (which might be quite surprising at first sight!) is the following: If the noise is very rough (i.e. oscillating a lot), well-posedness can be guaranteed even for very singular drifts – a phenomenon often referred to as “regularization by noise” (see also [42, 48]). In particular, well-posedness holds for equations that would be extremely ill-posed without the presence of noise.

Due to Donsker’s theorem, the most common example of noise is Brownian motion. However, in various situations, future increments might depend on the behavior of the past. Fractional Brownian motion is the simple Gaussian process that allows for correlations between different time-increments and is therefore often used to model such situations with “memory”.

Throughout the thesis the guiding aim is to answer Question (Q), considering either fractional Brownian noise or Brownian noise.

I.2.1 Regularization by noise - an ODE perspective

This section gives some motivation and hand-waving explanations for the regularization by noise phenomenon. For similar ideas and deeper insight, see [42, 48].

Let us consider an autonomous ODE of the form

$$\dot{x}_t = b(x_t). \tag{I.2.1}$$

The well-known Picard-Lindelöf theorem gives the sufficient condition of b being Lipschitz continuous, in order to get a unique solution to the above equation for a specified initial condition $x_0 \in \mathbb{R}^d$. For non-Lipschitz functions b , uniqueness can be lost as the following two examples show.

Example 1: Considering $d = 1$, $b(\cdot) = 2\text{sgn}(\cdot)\sqrt{|\cdot|}$ and initial condition $x_0 = 0$, any function in the uncountable set

$$\{(t - t_0)^2 \mathbb{1}_{t \geq t_0} : t_0 \geq 0\}$$

solves the above initial value problem. Nevertheless, this kind of “splitting” into different solutions can only occur at 0 (in space). For initial conditions $x_0 \neq 0$, the equation is locally well-posed.

Example 2: Even more can go wrong when considering b to be discontinuous. To see this, let $d = 1$ and

$$b(x) = \begin{cases} 0 & \text{for } x \in \mathbb{Q}, \\ 1 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

For a rational initial condition x_0 , both $x_t = x_0$ and $x_t = x_0 + t$ solve the corresponding initial value problem. For an irrational initial condition x_0 , $x_t = x_0 + t$ is a solution. Furthermore $x_t = x_0 + (q - x_0) \wedge t$ is a solution for any rational $q > x_0$.

We see in both of the above one-dimensional examples that the points on which b vanishes and is not smooth pose a problem. Let $\gamma : [0, \infty) \rightarrow \mathbb{R}$ be some given highly oscillating function with $\gamma_0 = 0$. When (I.2.1) is perturbed by γ , written as an integral equation, it takes the form

$$x_t = x_0 + \int_0^t b(x_s) ds + \gamma_t. \quad (\text{I.2.2})$$

One could hope that γ prevents the occurrence of “splitting” like in the examples above as the oscillations of the “noise” γ might prevent the solution from staying at the critical points on which b vanishes. To strengthen the intuition of this hand-waving argument, one can rewrite the equation after setting $\tilde{x} = x - \gamma$, by

$$\tilde{x}_t = x_0 + \int_0^t b(\tilde{x}_s + \gamma_s) ds. \quad (\text{I.2.3})$$

If it is known *a priori* that \tilde{x} oscillates slower than γ , then for a small time interval $[u, v]$,

$$\int_u^v b(\tilde{x}_s + \gamma_s) ds \approx \int_u^v b(\tilde{x}_u + \gamma_s) ds. \quad (\text{I.2.4})$$

So heuristically, for small time intervals, \tilde{x} can be considered to be constant. Hence, in order to study (I.2.2) and the regularization effect of γ , one can look at the so-called averaging operator

$$T^\gamma b: (t, z) \mapsto \int_0^t b(z + \gamma_r) dr, \quad (\text{I.2.5})$$

with the hope of recovering Lipschitz continuity in space of $T^\gamma b$, while jointly keeping ε -Hölder regularity in time, so that, for small enough time intervals $[s, t]$, and two solutions \tilde{x}, \tilde{y} ,

$$\begin{aligned} |\tilde{x}_t - \tilde{x}_s - (\tilde{y}_t - \tilde{y}_s)| &= \left| \int_s^t b(\tilde{x}_r + \gamma_r) dr - \int_s^t b(\tilde{y}_r + \gamma_r) dr \right| \\ &\approx \left| \int_s^t b(\tilde{x}_s + \gamma_r) dr - \int_s^t b(\tilde{y}_s + \gamma_r) dr \right| \\ &= \left| T^\gamma b(t, \tilde{x}_s) - T^\gamma b(s, \tilde{x}_s) - (T^\gamma b(t, \tilde{y}_s) - T^\gamma b(s, \tilde{y}_s)) \right| \\ &\lesssim (t - s)^\varepsilon |\tilde{x}_s - \tilde{y}_s|. \end{aligned}$$

In certain situations this is sufficient to obtain uniqueness by investigating a space that includes all solutions (see e.g. Davie [34]) or by making use of existence of a flow map (see Shaposhnikov [101] or Chapter V). In general however, this condition suffices to guarantee existence of a solution to (I.2.3) and therefore to (I.2.2) via an Euler scheme (see Section II.3.2).

The very rough idea to have in mind is that even if b has some discontinuities or singularities, this should not have a big influence on the mapping (I.2.5) as the fast oscillations of γ prevent $z + \gamma$ from spending a “long amount of time” in or around these discontinuities/singularities. This idea is a recurrent theme throughout the thesis, allowing to analyze the regularization effect of γ and to define a notion of a solution to (I.2.2) for merely distributional b .

Note that in all the (vague) arguments above no probability is required, eventually leading to a notion of existence and uniqueness of solutions that is of a deterministic flavor, even if γ is sampled from the law of a stochastic process (see Section I.2.4).

I.2.2 Regularization by noise - an SDE perspective

This section gives a brief overview of the literature in the case where γ from the previous section is sampled from the law of a (fractional) Brownian motion. Classically this is done in the sense of SDEs with the common concepts of existence and uniqueness of adapted (weak/strong) solutions and uniqueness among them (either in law or pathwise). This section focuses on these classical notions. For more recent results based on a path-by-path approach, we refer the reader to Section I.2.3. For a comparison of these concepts see Chapter VI and Section V.B.

In order to introduce the SDE we are interested in, we first recall the definition of a fractional Brownian motion.

Fractional Brownian motion. The fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $(B_t)_{t \geq 0}$ with covariance

$$\mathbb{E}[B_s B_t] = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}), \quad \forall s, t \geq 0.$$

The definition immediately extends to arbitrary dimension d , by considering d independent fractional Brownian motions with Hurst parameter H . The case $H = 1/2$ coincides with the definition of a Brownian motion; $H \in (1/2, 1)$ leads to positively correlated increments as well as long memory and $H < 1/2$ leads to negatively correlated increments and very rough sample paths. These correlations between increments make the fractional Brownian motion an interesting example of a stochastic process to describe real-world phenomena (for example in finance, see [17, 31, 56]). Furthermore, using Kolmogorov's continuity theorem, it is easy to see that for any $\alpha \in (0, H)$ there exists a version of the process which is α -Hölder continuous. However, this is not the case for $\alpha = H$. In fact, for almost every trajectory, there are no points of H -Hölder continuity, which makes the fBm an example of a (highly) oscillating process. Therefore for small values of H , there should be a large regularization effect of the fBm by the intuitions sketched in Section I.2.1. Hence, throughout the thesis we consider $H \leq 1/2$ which will allow to consider distributional drifts. For a clear link between fBm and Brownian motion via an integral representation formula, see Section II.A.2.

SDE of interest. Consider the SDE

$$X_t = X_0 + \int_0^t b(s, X_s) ds + B_t, \quad t \in [0, T], \quad (\text{I.2.6})$$

where $T > 0$ is a fixed deterministic time, $b: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable and B is a d -dimensional fractional Brownian motion. In the following an overview of the literature on (I.2.6) is presented, specifically focused on the conditions on b to ensure well-posedness. Sometimes particular cases of (I.2.6) are actually considered (e.g. one-dimensional, time-independent drift). However at this point, for simplicity, we choose to disregard this.

Brownian case. For B being a one-dimensional Brownian motion and b a measurable bounded function, it was shown by Zvonkin [116] that there exists a pathwise unique strong solution to (I.2.6). This is achieved by the so-called Zvonkin transformation; a deterministic transformation arising from the solution of a well-chosen PDE, in order to remove the irregular drift term. This idea is reused in [110] to extend the result to the multidimensional case. Both [116] and [110] allow for

multiplicative noise as well. Krylov and Röckner [68] extended these results in the case of additive noise to unbounded drifts b fulfilling the integrability condition

$$\left(\int_0^T \left(\int_{\mathbb{R}^d} |b(r, x)|^p dx \right)^{\frac{q}{p}} dr \right)^{\frac{1}{q}} < \infty, \quad (p, q) \in (2, \infty)^2 \text{ with } \frac{2}{q} + \frac{d}{p} < 1. \quad (\text{I.2.7})$$

This is done by separately showing existence of weak solutions and pathwise uniqueness, which additionally gives existence of strong solutions by the Yamada-Watanabe principle (see [113]). Existence of weak solutions is proven in a classical way (see e.g. [90, Chapter 1, Section 4]) by removing the drift term via a Girsanov transform. This is also possible for nonconstant diffusion coefficient (see [115]).

Fractional Brownian case. As fractional Brownian motion is neither a Markov process nor a semimartingale for $H \neq 1/2$, PDE techniques like the one mentioned above and classical stochastic calculus are not available and one has to come up with other ideas. Nevertheless a Girsanov transform is still available (see [35]). In this paragraph we briefly sketch the chronology of results; for a presentation of the state of the art results, see Section III.A.

A first attempt in this direction is due to Nualart and Ouknine [83, 84], who prove existence and uniqueness for some non-Lipschitz drifts. This is done by using the Girsanov transform in order to obtain weak existence and uniqueness in law and the observation that in this case these two properties are sufficient to also obtain pathwise uniqueness. In a seminal work by Catellier and Gubinelli [24] the authors prove well-posedness (in a path-by-path sense, see Section I.2.3) for time-independent distributional drifts $b \in \mathcal{C}^\beta$ for $\beta \in \mathbb{R}$ with $\beta > 1 - 1/(2H)$, where \mathcal{C}^β for $\beta < 0$ is a generalized Hölder space containing distributions. This is done by a careful study of the averaging operator associated to each path of the fractional Brownian motion, applying the Girsanov transform and reformulating the equation as a nonlinear Young integral equation (see Section II.3). In Chapter II different results on both existence and uniqueness are presented for classes of distributional drifts that are not covered in [24]. In particular, existence of a weak solution is proven for $b \in \mathcal{C}^\beta$ with $\beta > 1/2 - 1/(2H)$ and pathwise uniqueness is shown for $\beta = 1 - 1/(2H)$ for b being a measure. In terms of existence of weak solutions, these results are refined for measure-valued drift in Chapter III. In specific cases, i.e. for b fulfilling a certain integrability condition or being a finite measure, the results of Chapter II and Chapter III are refined in the recent work [20], which is further improving in terms of uniqueness and existence.

In [32] the authors take an approach via the Clark-Ocone formula to investigate random time-dependent drifts. In [49] the authors extend the main result in [24] to time-dependent distributional drifts with a singularity in time, recovering the full subcritical regime in terms of spatial regularity, for particular cases of time-integrability.

At this point, let us mention that throughout this thesis we do not consider multiplicative noise. This simplifies matters, as there is no ambiguity in the definition of an integral. Otherwise, one has to address this issue, for example by assuming enough regularity of the diffusion and the fractional Brownian motion to be interpreted as a Young integral or a rough integral. Hence multiplicative noise is delicate in the sense that on one hand rougher noise is helping to regularize the equation, whereas on the other hand rougher noise complicates the definition of the integral term. We only mention a selection of works: In [33] well-posedness for Young integral equations and rough integral equations with irregular drift is shown. The authors in [80] decrease the assumptions on the regularity of the diffusion term via an extension of the stochastic sewing lemma, considering the equation as a Young integral equation.

Another interesting question that is not addressed in this thesis is the one of the so-called zero-noise limit: replace B_t in (I.2.6) by εB_t and let ε go to zero. In case that there is no uniqueness to the unperturbed equation, which solution will we recover in the limit? In the Brownian case, this question was answered in [10] for a class of positive drifts, however in the fractional Brownian case this question still seems to be very much open. For a more detailed presentation of this topic see [42, Section 1.4].

Skew (fractional) Brownian motion. Many physical situations involve diffusion phenomena in heterogeneous media (see [74] for references describing these phenomena in physics, but also in biology, finance, etc.). The so-called skew Brownian motion (“partially reflected” Brownian motion) is a probabilistic object that allows to describe this behaviour.

Skew Brownian motion was first introduced in [66] with the following simple idea: One considers a Brownian motion and whenever it hits the x -axis, an independent Bernoulli random variable is introduced. With probability η the Brownian motion is “pushed into \mathbb{R}_+ ”, with probability $1 - \eta$ the Brownian motion is “pushed into \mathbb{R}_- ”. When setting $\eta = 1$ one recovers a reflected Brownian motion. Such a process can also be constructed via an approximation with skewed random walks (see [74]). Note that outside of 0 the above process behaves like a Brownian motion. Even though already suggested in [66], Walsh [111] proved that the skew Brownian motion defines a diffusion process and gave two main motivations for the study of this object. First, it allows to model a particle’s behavior when moving in between two different media. Secondly, it gives an example of an object with a discontinuous local time in space due to the behavior at 0. Even though reflected Brownian motion shares this property considered as a mapping to the codomain \mathbb{R} , it is a rather unsatisfying example as the natural codomain for the reflected Brownian motion is $[0, \infty)$. For more details and various constructions of the skew Brownian motion see [74]. Moreover, there is a clear link between skew Brownian motion and the kind of equations we are interested in, which is described below.

For Brownian noise and the time-independent drift $b = \delta_0$, it is known that there exists a unique strong solution to (I.2.6) which is given by the reflected Brownian motion (see e.g. [73]). Hence the Dirac in zero can be identified as the precise force that is needed to reflect the solution along the x -axis. One can make sense of the formal integral $\int_0^\cdot \delta_0(X_r) dr$ by interpreting it as the local time of X at 0. This idea was exploited in [63] (with further generalization in [73]) to a local time drift formally corresponding to $b = a\delta_0$, for $a \in [-1, 1]$. The solution to such an equation is the skew Brownian motion introduced above with parameter $\eta = (a + 1)/2$. Moreover, [63, 73] show that no solution exists for $a \in \mathbb{R}$ with $|a| > 1$.

Having the intuition in mind that rougher noise should lead to a higher effect of regularization, it is natural to try to extend the above result to fractional Brownian noise with $H < 1/2$. Note however that both [63] and [73] rely on Itô’s formula and therefore similar ideas cannot directly be used in the fractional Brownian case. Nevertheless, for $d = 1$ and $b = a\delta_0$ with $a \in \mathbb{R}$, the well-posedness of this equation was established for $H < 1/4$ by Catellier and Gubinelli [24] using nonlinear Young integrals. Yet, for $d = 1$, a gap to the Brownian case ($H = 1/2$), with well posedness for $|a| \leq 1$ proven in [73], remains. One of the aims of our work was to close this gap.

Chronologically, a first step in this direction for existence and uniqueness can be found in [LA4] by both a nonlinear Young integral approach and a stochastic sewing approach (see Chapter II). Eventually, in terms of weak existence in dimension one this gap was fully closed in [LA2] (see Chapter III). Simultaneously to [LA2], in [20] the authors recovered the same result in dimension 1. Additionally, they prove a stronger result for dimension $d > 1$ and they also partially close the gap

of uniqueness, extending the results from [LA4] (in the particular case where b is a finite measure). Nevertheless fully closing the gap in terms of uniqueness remains an open question.

It should be mentioned at this point that most of the results above are special cases of more general results considering measure-valued drifts or more general distributional drifts. In order to keep the presentation cleaner at this point, we chose to highlight the different results by considering $b = \delta_0$. For a more detailed summary of the literature for (I.2.6) in the fractional Brownian motion case as well as references to related works, we refer the reader to Section III.A.

I.2.3 About path-by-path solutions and adapted solutions

As mentioned in the previous section, well-posedness for (I.2.2) has been a very active field of investigation recently. Starting from the seminal works of Davie [34] and Catellier–Gubinelli [24], there has been an increasing interest in the problem, see [23, 33, 49, 50, 62, 101]. A typical result of these articles is that when γ is a sample path of an irregular stochastic process (such as fractional Brownian motion), then for almost every such sample path, (I.2.2) has a unique solution, a property known by the terminology *path-by-path uniqueness*. Generally speaking the necessary regularity for the vector field is inversely proportional to the roughness of the stochastic process from which γ is sampled. In a different direction, when the coefficients are sufficiently regular so that Lyons’ rough path theory can be applied, then uniqueness of the corresponding rough differential equation implies path-by-path uniqueness. This is because the stochastic basis is only used to construct a rough path lift of the driving signal and Lyons’ rough path analysis [79] is applicable for any fixed trajectory of the lift.

From a practical perspective, path-by-path uniqueness is most relevant in situations where only a handful of samples of the signal can be observed. Recently this kind of uniqueness has been heavily investigated both for SDEs and SPDEs, see references in the previous paragraph and [15, 21, 24, 42, 91, 92, 101]. Theoretically, “path-by-path uniqueness” is stronger than “pathwise uniqueness” (pathwise uniqueness allows to identify two adapted solutions to (I.2.6) whereas path-by-path uniqueness allows to identify “all” solutions). In particular pathwise uniqueness for (I.2.6) (conceptually) does not provide uniqueness for the $B(\omega)$ -driven ODE

$$Y_t = Y_0 + \int_0^t b(r, Y_r) dr + B_t(\omega), \quad t \in [0, T], \quad (\text{I.2.8})$$

for a given ω . The question whether examples of SDEs actually exist for which the above concepts of uniqueness do not coincide was raised in [42, page 12] and [1, page 3]. Such examples of SDEs driven by Brownian motion for which pathwise uniqueness holds but not path-by-path uniqueness are given in [103, LA1] (for details see Chapter VI). The rough idea is that if solutions are allowed to “look into the future behaviour” of the Brownian path, they can decide for the “right behaviour” at the current moment. By doing this, critical areas after which a solution cannot be extended, can be avoided, however adaptedness is lost. Hence this is not possible if restricting to adapted solutions. For further discussion and a comparison of the different concepts we refer the reader to Chapter V and Chapter VI. Nevertheless we quickly recall the benchmark results here. As mentioned in Section I.2.2 pathwise uniqueness for SDEs with singular drift b fulfilling the Krylov–Röckner condition (I.2.7) was shown in [68]. Path-by-path uniqueness was shown in [34] for measurable and bounded drifts b . The proof in [34] does not rely on the previous results on pathwise uniqueness, but instead establishes path-by-path uniqueness in a rather self-contained (technical) way. The same result is

(up to minor improvements) recovered by Shaposhnikov [101], but with a simpler proof that makes use of the existence of a regular flow (and therefore also of pathwise uniqueness).

The contribution of this work regarding path-by-path uniqueness is twofold, with all results considering the Brownian case:

- (a) Construction of one-dimensional counterexamples as mentioned above for which the notions of pathwise and path-by-path uniqueness do not coincide (see Chapter VI).
- (b) Answering the open question of path-by-path uniqueness for SDEs driven by a Brownian motion under Krylov-Röckner condition (see Chapter V).

Note that by giving a positive answer to (b), it is clear that counterexamples like the ones in (a) cannot be constructed for drifts b fulfilling the Krylov-Röckner condition. Instead, the counterexamples mentioned in (a) rely on well-known SDEs with even more singular drift, like Bessel SDE and Bessel bridge SDE.

The main idea of the proof of (b) is similar to [101], i.e. comparing, for any realization of the Brownian motion, any solution of the now deterministic integral equation to the solution given by the unique solution-flow of the corresponding SDE. Nevertheless on the technical level different ideas are needed. In particular, as b is not bounded, for a solution Y to (I.2.8), $\int_0^\cdot b(r, Y_r) dr$ is not necessarily Lipschitz continuous. Instead we work in the space of functions with finite 1-variation. Also, we prove that for a.e. Brownian path a solution-flow exists. Finally, we use the recently obtained quantitative John-Nirenberg inequality (see [71]) to ensure sufficient regularity of the averaging operator $T^B b$ introduced in Section I.2.1.

I.2.4 How to define and obtain solutions for distributional drift

In case of considering (I.2.2) for distributional drift b , it is *a priori* not clear how to interpret the integral on the right hand side of the equation and therefore how to define a solution. Two ideas are

- (a) to define a solution x in a way such that the integral can be identified with the limit of the sequence $\int_0^\cdot b^n(x_s) ds$ for any sequence b^n converging to b in a properly chosen space of distributions.
- (b) to directly make use of the oscillations of γ , via so-called nonlinear Young integrals.

Throughout most of the thesis we consider γ to be sampled according to the law of a fractional Brownian motion in order to have tools available such as local nondeterminism and existence of a (regular) local time. However, note that the scheme of constructing a solution in the sense of (b) is of a deterministic nature. In most cases, (a) and (b) lead to an equivalent definition of a solution (see Theorem II.2.15). Nevertheless, the “natural” way to obtain a solution corresponding to either of the notions differs. We briefly present the rough ideas and leave the rigorous arguments to later sections.

(a): The main tool for this approach is the use of the stochastic sewing lemma, recently introduced by Lê [70]. Similar to the usual (deterministic) sewing lemma, the stochastic sewing lemma allows to approximate a two-parameter process by a one-parameter process and vice versa. This allows to find *a priori* bounds of the solution by a similar approximation as in (I.2.4) (see Section I.2.5).

Having *a priori* estimates at hand, we go through a tightness-stability argument to obtain existence of weak solutions. Again via stochastic sewing, pathwise uniqueness can be shown and eventually a Yamade-Watanabe kind of argument leads to existence of a unique strong solution.

(b): Let $f \in \mathcal{C}_b(\mathbb{R})$ and recall the averaging operator T^γ associated to the path γ defined by $T_s^\gamma f(z) := \int_0^s f(z + \gamma_r) dr$. Then for $t > 0$, a sequence of partitions $(\{t_i^N\}_{0 \leq i \leq N})_{N \in \mathbb{N}}$ of $[0, t]$ and $\tilde{x} = x - \gamma$, we formally have that

$$\begin{aligned} \int_0^t f(\tilde{x}_r + \gamma_r) dr &= \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} \int_{t_i^N}^{t_{i+1}^N} f(\tilde{x}_{t_i^N} + \gamma_r) dr \\ &= \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} \left(T_{t_{i+1}^N}^\gamma f(\tilde{x}_{t_i^N}) - T_{t_i^N}^\gamma f(\tilde{x}_{t_i^N}) \right) \text{ “=:” } \int_0^t T_{dr}^\gamma f(\tilde{x}_r). \end{aligned}$$

For now the last expression is only a formal definition as convergence of the Riemann-type sum is not clear. If convergence holds, the last term is a so-called nonlinear Young integral. A sufficient condition for this object to be well-defined requires a certain regularity of $T^\gamma f(\cdot)$ and \tilde{x} , allowing for some compensation between the two. For γ oscillating sufficiently fast, it is possible to extend the averaging operator to distributional f . This is rather intuitive as the averaging operator can be written as a convolution with the local time of γ (see Section II.2.3). Hence, if γ has a smooth enough local time (which is the case for processes that oscillate sufficiently fast), it will compensate the low regularity of f . For further details see Section II.4.1. Nonlinear Young integrals with regularity measured in Hölder spaces have been used in [24, 50] to investigate equations like (I.2.2). For a detailed review of nonlinear Young integrals and nonlinear Young integral equations, see [47]. In some situations it is useful to work in p -variation spaces instead of Hölder spaces, for example if \tilde{x} is known to be nondecreasing and therefore of finite 1-variation as this amount of smoothness (in terms of variation) allows to lower the regularity assumptions on $T^\gamma f$ in order for the nonlinear Young integral to be well-defined. An application thereof naturally arises when considering (I.2.2) with nonnegative drift b , since we know *a priori* that any solution to the corresponding nonlinear Young integral equation will be nondecreasing and therefore of finite 1-variation. The proof of existence is then carried out by an Euler scheme, looking for a solution in the space of functions of finite 1-variation (see Section II.3.2).

Note that the approach of defining the averaging operator as a convolution with a local time is less clear in the case of time-dependent drift. Nevertheless stochastic sewing still works well. This has been explored recently in [49].

Finally, it should be pointed out that both of the above constructions of solutions coincide for solutions that are sufficiently regular (see Theorem II.2.15), even in nontrivial cases where the drift is singular.

I.2.5 How to use stochastic sewing

In this section we make a toy calculation on how to define and estimate the regularity of $\mathcal{A}_t := \int_0^t f(B_r) dr$ for $f \in L^p(\mathbb{R})$ and $p \in [1, \infty)$. In later sections, such calculations will be made for distributional f and $\int_0^t f(X_r) dr$, where X can be decomposed into $X = B + \phi$ for ϕ oscillating slower than B .

In order to apply stochastic sewing, we introduce the two-parameter process

$$A_{s,t} := \mathbb{E}^s \int_s^t f(B_r) dr,$$

where $\mathbb{E}^s[\cdot]$ is used to denote $\mathbb{E}[\cdot|\mathcal{F}^s]$. We expect A to approximate small increments of the one-parameter process \mathcal{A} . Under certain conditions on the increments of A (which are weaker than the ones of the usual sewing lemma), stochastic sewing allows to deduce estimates on the distance between $\mathcal{A}_t - \mathcal{A}_s$ and $A_{s,t}$, and therefore transfer estimates of the two-parameter process A to increments of \mathcal{A} . The former can be estimated, using the local nondeterminism property of fractional Brownian motion (which implies $\text{Var}(B_r - \mathbb{E}^s[B_r]) = C(r-s)^{2H}$) and a heat kernel estimate, in the following way:

$$\begin{aligned} \left| \mathbb{E}^s \int_s^t f(B_r) dr \right| &\leq \int_s^t |\mathbb{E}^s f(\mathbb{E}^s B_r + B_r - \mathbb{E}^s B_r)| dr \\ &= \int_s^t |g_{C(r-s)^{2H}} * f(\mathbb{E}^s B_r)| dr \\ &\leq \int_s^t \|g_{C(r-s)^{2H}} * f\|_{L^\infty(\mathbb{R})} dr \\ &\leq C \|f\|_{L^p(\mathbb{R})} (t-s)^{1-H/p}, \end{aligned}$$

where $g_t(x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$. Then using stochastic sewing one can conclude that, for any $m \geq 2$, there exists $C > 0$ such that

$$\left\| \int_s^t f(B_r) dr \right\|_{L^m(\Omega)} \leq C \|f\|_{L^p(\mathbb{R})} (t-s)^{1-H/p}.$$

By the same idea as above one can estimate $\int_s^t f(B_r + \phi_s) dr$ for some process ϕ . If ϕ is oscillating slower than B , this will also give an estimate of $\int_s^t f(B_r + \phi_r) dr$ for $|t-s|$ small. This will lead to *a priori* estimates, which are crucial to construct a solution to (I.2.6).

I.2.6 Density of the solution and McKean-Vlasov equations

The ideas of the previous sections lead to conditions on the drift b in (I.2.6) in order to get existence and uniqueness of a solution. However, the schemes of proof do not allow to directly infer properties of the law of such a solution. For fractional Brownian noise, results on Gaussian-like tails of the density are available for a drift that is at least a measurable function (see e.g. [16, 76]). Moreover, results on the regularity of the law of the solution for Besov-valued drift exist both for drifts with positive regularity [85] and general Besov-valued drift [51]. In the latter, the motivation to study the regularity of the density is to give conditions on the drift in order to get well-posedness to an SDE with distribution-dependent drift. In particular, we can hope to loosen the assumptions on \mathfrak{b} such that the following McKean-Vlasov Equation with fractional Brownian noise is well-posed:

$$X_t = X_0 + \int_0^t \mathfrak{b}_s * \mu_s(X_s) ds + B_t, \quad \text{Law}(X_t) = \mu_t. \quad (\text{I.2.9})$$

Equation (I.2.9) can be expected to be well-posed for \mathfrak{b} with lower regularity than what we ask for b in Equation (I.2.6). The idea is that $\mathfrak{b}_s * \mu_s$ will be more regular than \mathfrak{b}_s itself, if μ_s lives in a Besov space with positive regularity. In the current literature however, the conditions on b for well-posedness for Equation (I.2.6) are the same as the conditions on \mathfrak{b} for well-posedness to Equation (I.2.9) (see [51]).

Let b be such that a unique strong solution X to (I.2.6) exists. Then our contributions in Chapter IV on the density of the law of X can be summarized as follows:

- For any $t > 0$, the law of X_t is absolutely continuous w.r.t. Lebesgue measure and the density $f_t(\cdot)$ lies in a Besov space of positive regularity. Moreover, a smoother drift b directly gives more smoothness of the density.
- As a function on $[0, T] \times \mathbb{R}$, f enjoys a certain time integrability while still considering it to be in a Besov space of positive regularity in space.
- For b fulfilling slightly stronger regularity assumptions than in the points above, it can be proven by using both stochastic sewing and Malliavin calculus that for any $t > 0$, there exist Gaussian upper bounds for the density $f_t(\cdot)$.

Additionally, in Section IV.4.3, we sketch how to construct a strong solution to (I.2.9) for \mathfrak{b} of lower regularity than what we ask for b in Equation (I.2.6). This is based on the intuitions presented earlier in this section and the above results on densities. The rigorous proof is left to the ongoing work [LA5].

1.2.7 Structure and overview of the main results

In Chapter II, existence and uniqueness to (I.2.6) is proven for Besov space-valued drift b . Existence is shown via two different approaches. First, for a nonnegative drift, via a rather deterministic approach rewriting the equation as a nonlinear Young integral equation and using fine properties of the local time of fractional Brownian motion. Secondly, for a general drift, by making use of the stochastic sewing lemma. A recent adaptation of the stochastic sewing lemma also leads to uniqueness in cases that were not considered in Galeati and Gubinelli [50]. The entire chapter is based on joint work with my advisors A. Richard and E. Tanré [LA4].

In Chapter III, similar techniques involving stochastic sewing as in Chapter II are used, however purely focused on existence of weak solutions in the case of nonnegative drift. The main result therein is covering the full subcritical regime in dimension 1. The chapter is based on [LA2]. Simultaneously to [LA2], the preprint [20] appeared, including a result on weak existence for the full subcritical regime in any dimension. However, the main result in Chapter III provides a simple proof. Moreover, Section III.A gives a detailed review of the literature on (I.2.6), in particular comparing the conditions on b to obtain existence and uniqueness of solutions.

Chapter IV is based on ongoing work with my advisors. Hence, we expect that some results therein will be refined. However, to the best of our knowledge the results are still new as properties of the density of the law of the solution to SDEs with distributional drift have not been investigated in depth so far. The chapter includes results on Gaussian-like tails via Malliavin calculus and on regularity of the density. Also, a brief outlook is given to explain how these results could be applied to an SDE with distribution-dependent drift – the McKean-Vlasov equation. Again, all of these results involve stochastic sewing techniques.

In Chapter V, path-by-path uniqueness to (I.2.6) for Brownian noise is shown under the Krylov-Röckner condition (I.2.7). In particular, this is extending the main results in [34, 101] by allowing for a larger class of drifts. The scheme of proof is similar to [101], however on a technical level different results are needed, including the recently introduced quantitative John-Nirenberg inequality [71]. As a byproduct, strong completeness is proven, i.e. for almost every Brownian trajectory the family of solutions with different initial data forms a continuous semiflow for all nonnegative times. The whole chapter is based on joint work with Khoa Lê and Chengcheng Ling [LA3].

In Chapter VI, one-dimensional drifts b are constructed such that pathwise uniqueness (i.e. classical uniqueness among adapted solutions) to (I.2.6) holds but path-by-path uniqueness does not. Such counterexamples were previously known only in dimension $d > 1$. In particular we construct (path-by-path) solutions to SDEs that are not adapted. The chapter is based on [LA1].

Appendix A.1 contains a brief summary on the definition and some useful properties of Besov spaces (similar to [LA4, Appendix A]). In Appendix A.2 (based on [LA4, Appendix C]) different versions of the stochastic sewing lemma are recalled.

I.2.8 Notations and definitions.

Various notations. Throughout the thesis, we use the following notations and conventions:

- Constants C might vary from line to line.
- We write $H \lesssim G$ if $H \leq CG$ for some universal positive constant C .
- For $p \in [1, \infty]$, $p' \in [1, \infty]$ is such that $1/p + 1/p' = 1$.
- For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, let $|x| = \sum_{i=1}^d |x_i|$.
- For topological spaces X, Y we denote the set of continuous functions from X to Y by $\mathcal{C}_X(Y)$.
- For a topological space X , we denote the Borel sets of X by $\mathcal{B}(X)$.
- For a Banach space E , the ball of radius $R > 0$ is denoted by $D_R := \{x \in E : \|x\| \leq R\}$.
- Let $s < t$ be two real numbers and $\Pi = (s = t_0 < t_1 < \dots < t_n = t)$ be a partition of $[s, t]$. We denote $|\Pi| = \sup_{i=1, \dots, n} (t_i - t_{i-1})$ the mesh of Π .
- For $s, t \in \mathbb{R}$ with $s \leq t$, we denote $\Delta_{[s, t]} := \{(u, v) : s \leq u \leq v \leq t\}$.
- For any function f defined on $[s, t]$, we denote $f_{u, v} := f_v - f_u$ for $(u, v) \in \Delta_{[s, t]}$.
- For any function g defined on $\Delta_{[s, t]}$ and $s \leq r \leq u \leq v \leq t$, we denote $\delta g_{r, u, v} := g_{r, v} - g_{r, u} - g_{u, v}$.
- For a probability space Ω and $p \in [1, \infty]$, the norm on $L^p(\Omega)$ is denoted by $\|\cdot\|_{L^p}$.
- We denote by $(B_t)_{t \geq 0}$ a fractional Brownian motion with Hurst parameter $H \leq 1/2$.
- The filtration $(\mathcal{F}_t)_{t \geq 0}$ is denoted by \mathbb{F} .
- The filtration generated by a process $(Z_t)_{t \geq 0}$ is denoted by \mathbb{F}^Z .
- All filtrations are assumed to fulfill the usual conditions.
- Let \mathbb{F} be a filtration. We call $(W_t)_{t \geq 0}$ a \mathbb{F} -Brownian motion if $(W_t)_{t \geq 0}$ is a Brownian motion adapted to \mathbb{F} and for $0 \leq s \leq t$, $W_t - W_s$ is independent of \mathcal{F}_s . For such a filtration, the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_s]$ is denoted by $\mathbb{E}^s[\cdot]$.
- For a σ -algebra \mathcal{F} , a filtration \mathbb{F} and a stopping time τ we define the stopping time σ -algebra by $\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$.
- For $0 \leq s \leq t$, define $\mathbb{L}_p^q([s, t])$ to be the space of all measurable functions $f : [s, t] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\|f\|_{\mathbb{L}_p^q([s, t])} := \left(\int_s^t \left(\int_{\mathbb{R}^d} |f(r, x)|^p dx \right)^{\frac{q}{p}} dr \right)^{\frac{1}{q}} < \infty$$

for $p, q \in [1, \infty)$ and with obvious modification when p or q is infinity. We abbreviate $\mathbb{L}_p^q = \mathbb{L}_p^q(\mathbb{R})$. By abuse of notation we also denote by $\mathbb{L}_p^q([s, t])$ the space of \mathbb{R}^d -valued functions such that the expression above is finite.

Gaussian semigroup. For any $t > 0$ and $x \in \mathbb{R}^d$, let $g_t(x) := \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}}$. For a tempered distribution ϕ , let

$$G_t \phi(x) := (g_t * \phi)(x). \tag{I.2.10}$$

The occupation time formula. Let $T > 0$, $w : [0, T] \rightarrow \mathbb{R}$ be a measurable function and let λ denote the Lebesgue measure on \mathbb{R} . For $A \in \mathcal{B}([0, T])$, let μ_A be the occupation measure defined by $\mu_A(\Lambda) := \lambda(\{t \in A : w_t \in \Lambda\})$ for $\Lambda \in \mathcal{B}(\mathbb{R})$. If $\mu_{[0, T]} \ll \lambda$, then there exists a measurable map $\ell : \mathcal{B}([0, T]) \times \mathbb{R} \rightarrow \mathbb{R}_+$ such that for $A \in \mathcal{B}([0, T])$, $\mu_A(dx) = \ell(A, x)\lambda(dx)$. For any bounded

measurable function f , the occupation time formula reads (see [57] for more details)

$$\int_A f(w_r) dr = \int_{\mathbb{R}} f(x) \ell(A, x) dx. \quad (\text{I.2.11})$$

We define a local time $L : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ by $L_t(x) := \ell([0, t], x)$. By (I.2.11), it comes that for any bounded measurable f and $t \in [0, T]$,

$$\int_0^t f(w_r) dr = \int_{\mathbb{R}} f(x) L_t(x) dx. \quad (\text{I.2.12})$$

Note that if $w : [0, T] \rightarrow K \subset \mathbb{R}$ for some compact K , then $L_t(\cdot)$ vanishes on K^c for all $t \in [0, T]$.

Finite variations spaces. Let $p \in [1, \infty)$ and $(F, \|\cdot\|_F)$ be a Banach space. Define the p -variation seminorm of a function $x : [s, t] \rightarrow F$ as

$$[x]_{\mathcal{C}_{[s,t]}^{p\text{-var}}(F)} := \sup_{\{t_i\}} \left(\sum_{i=0}^{N-1} \|x_{t_i, t_{i+1}}\|_F^p \right)^{\frac{1}{p}},$$

where the supremum runs over all partitions $\{t_i\}_{i=0}^N$, $N \in \mathbb{N}$, of $[s, t]$. We denote by $\mathcal{C}_{[s,t]}^{p\text{-var}}(F)$ the set of continuous functions with finite p -variation.

If F is just a Fréchet space, we say that a function mapping from $[s, t]$ to F has finite p -variation if its p -variation is finite with respect to any continuous seminorm and we also use the notation $\mathcal{C}_{[s,t]}^{p\text{-var}}(F)$. If $F = \mathbb{R}^d$, we use the alleviate notations $\mathcal{C}_{[s,t]}^{p\text{-var}}$ and $[x]_{\mathcal{C}_{[s,t]}^{p\text{-var}}}$.

Control functions. A continuous function $\varkappa : \Delta_{[s,t]} \rightarrow [0, \infty)$ is a control function if, for $s \leq r \leq u \leq v \leq t$,

$$\varkappa(r, u) + \varkappa(u, v) \leq \varkappa(r, v), \quad (\text{I.2.13})$$

and $\varkappa(r, r) = 0$ for all $r \in [s, t]$. A typical example of a control function is $[x]_{\mathcal{C}_{[s,t]}^{p\text{-var}}(F)}^p$ (see [45, Prop. 5.8]).

We call a measurable function $\lambda : \Delta_{[s,t]} \times \Omega \rightarrow \mathbb{R}_+$ a random control if there exists a set Ω' of full measure such that for $\omega \in \Omega'$, $\lambda(\cdot, \omega)$ is a control.

Besov and Hölder spaces.

Definition I.2.1. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we denote the nonhomogeneous Besov space with these parameters by $\mathcal{B}_{p,q}^s$. For a precise definition, see Appendix A.1.

Besides, for a bounded smooth domain $\mathcal{I} \subset \mathbb{R}^d$, we denote by $\mathcal{B}_{p,q}^s(\mathcal{I})$ the space of all distributions u on \mathcal{I} for which there exists $v \in \mathcal{B}_{p,q}^s$ such that $u = v|_{\mathcal{I}}$ (see [109, Def. 1, p.192]).

If $q = \infty$, we write \mathcal{B}_p^s instead of $\mathcal{B}_{p,\infty}^s$. We have the following important embeddings between Besov spaces.

Remark I.2.2. Let $s \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$. From [11, Prop. 2.71], the space \mathcal{B}_{p_1,q_1}^s continuously embeds into $\mathcal{B}_{p_2,q_2}^{s-d(p_1^{-1}-p_2^{-1})}$, which we write as $\mathcal{B}_{p_1,q_1}^s \hookrightarrow \mathcal{B}_{p_2,q_2}^{s-d(p_1^{-1}-p_2^{-1})}$.

Remark I.2.3. Let \mathcal{I} be a bounded smooth domain. Let $p_1, p_2, q_1, q_2 \in [1, \infty]$ and $-\infty < s_2 < s_1 < \infty$. If $s_1 - d/p_1 > s_2 - d/p_2$, then from [109, Th. p.196] we have $\mathcal{B}_{p_1, q_1}^{s_1}(\mathcal{I}) \hookrightarrow \mathcal{B}_{p_2, q_2}^{s_2}(\mathcal{I})$.

For $s \in \mathbb{R}_+ \setminus \mathbb{N}$ and $p = q = \infty$, Besov spaces coincide with Hölder spaces (see [11, p.99]). We now give a definition of Hölder spaces in space domain for $s \in (0, 1]$.

Definition I.2.4. Let E, F be Banach spaces, $U \subset E$ and $\beta \in (0, 1]$.

- We denote the supremum norm of $f \in \mathcal{C}_U(F)$ by $\|f\|_{\mathcal{C}_U(F)} = \sup_{x \in U} \|f(x)\|_F$. When U and F are clear from the context, we might also denote $\|f\|_\infty = \|f\|_{\mathcal{C}_U(F)}$.
- The Hölder space $\mathcal{C}_U^\beta(F)$ is the collection of all $f \in \mathcal{C}_U(F)$ such that $\|f\|_{\mathcal{C}_U^\beta(F)}$ is finite, where

$$\|f\|_{\mathcal{C}_U^\beta(F)} := [f]_{\mathcal{C}_U^\beta(F)} + \|f\|_{\mathcal{C}_U(F)} \quad \text{with} \quad [f]_{\mathcal{C}_U^\beta(F)} := \sup_{x \neq y \in U} \frac{\|f(x) - f(y)\|_F}{\|x - y\|_E^\beta}.$$

If $U = E$, we alleviate the notations and write $\|f\|_{\mathcal{C}^\beta(F)}$ and $[f]_{\mathcal{C}^\beta(F)}$. Similarly, if $F = \mathbb{R}$ or if F is clear from the context, we write $\|f\|_{\mathcal{C}_U^\beta}$ and $[f]_{\mathcal{C}_U^\beta}$. Finally, if $U = F = \mathbb{R}$ we just write \mathcal{C}^β .

- The space $\mathcal{C}_{E,loc}^\beta(F)$ of locally Hölder continuous functions is the collection of all $f \in \mathcal{C}_E(F)$ such that $\|f\|_{\mathcal{C}_{D_R}^\beta(F)}$ is finite for all $R > 0$.

Remark I.2.5. In some results, we assume that the drift is a (nonnegative) measure in some \mathcal{B}_p^β . This is equivalent to the assumption that b is a nonnegative distribution in \mathcal{B}_p^β . Indeed, thanks to [108, Exercise 22.5], any nonnegative distribution is given by a Radon measure (i.e. a locally finite, complete measure fulfilling regularity conditions). Throughout the thesis we only deal with such measures and just write measure instead of specifying the Radon property every time.

Existence and uniqueness for one-dimensional equations with distributional drift and fractional noise



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II.1 Introduction

In this chapter, which is entirely based on [LA4], we are interested in the well-posedness of the one-dimensional equation

$$dX_t = b(X_t)dt + dB_t, \tag{II.1.1}$$

when b is a distribution in some Besov space and $(B_t)_{t \geq 0}$ is a fractional Brownian motion of Hurst parameter H . We will explain in which sense this equation can be solved when b is a genuine distribution. This equation encompasses at least two classes of equations which have frequently been studied in the literature. Even though a part of the following literature was already mentioned in Section I.2.2, we give a more detailed review here.

First, when B is the standard Brownian motion, let us mention the early work of Veretenikov [110] for bounded measurable drifts, then the more general $L^p - L^q$ criterion of Krylov and Röckner [68] for which the authors proved strong existence and uniqueness (both works allowing for time inhomogeneous drifts in dimension $d \geq 1$). Flandoli, Russo and Wolf [43] developed a weak well-posedness theory while Bass and Chen [13] proved existence and uniqueness of strong solutions with the drift being the distributional derivative of a γ -Hölder function for $\gamma > 1/2$. Then Davie [34] provided conditions for path-by-path existence and uniqueness, which is a stronger form of uniqueness, for time inhomogeneous bounded measurable drift. Using rough path methods, Delarue and Diel [36] proved weak existence and uniqueness in dimension 1 when the drift is the distributional derivative of a γ -Hölder function for $\gamma > 1/3$. In higher dimension, Flandoli, Issoglio and Russo [44] identified a class of SDEs with distributional drifts in Bessel spaces such that there exists a solution that is unique in law. In addition, when the drift is random, another well-posedness result is given by Duboscq and Réveillac [38]. We also point out the work [63], with extensions in [73] on an SDE involving the local time at 0 of the solution, which formally corresponds to a drift $b = a\delta_0$, for some $a \in [-1, 1]$ and δ_0 being the Dirac distribution. This setting corresponds to the so-called skew Brownian motion, see [74] for more details and various constructions.

This leads to a second class of interesting problems, namely solving Equation (II.1.1) when b is a distribution and B is a fractional Brownian motion with sufficiently small Hurst parameter H . A first attempt in this direction seems to be due to Nualart and Ouknine [83], who proved existence and uniqueness for some non-Lipschitz drifts. When $b = a\delta_0$ with $a \in \mathbb{R}$, the well-posedness of this equation was established for $H < 1/4$ by Catellier and Gubinelli [24] (who also consider multidimensional drifts in negative Hölder spaces) and independently for $H < 1/6$ in [2, 9] with extensions to dimension $d \geq 1$ in the three papers [2, 9, 24]. The solution is generally referred to as skew fractional Brownian motion. We observe a gap between the one-dimensional Brownian case ($H = 1/2$), with well posedness for $|a| \leq 1$ proven in [73], and the aforementioned result for fractional Brownian motion with H smaller than $1/4$. The intent of this chapter is to partially close this gap. Note also that the case $a = 1$ corresponds to reflection above 0 in the Brownian case. The well-posedness of reflected equations was established even for multiplicative rough noises in case X is one-dimensional [37, 96], while uniqueness might fail as soon as the dimension is greater than 2 (see [54]).

Finally, let us mention that regularization by noise was also investigated for other types of noise, for instance α -stable noises [27], regular noises [58] and other classes of rough processes [61, 62]. Recently, the regularization phenomenon was studied for SDEs with multiplicative noise (whether fractional Brownian motion [33], or more general rough paths [23]).

In this chapter, the drift b is in some Besov space $\mathcal{B}_{p,\infty}^\beta$ (denoted \mathcal{B}_p^β hereafter). The solutions we consider are stochastic processes of the form

$$X_t = X_0 + K_t + B_t, \quad (\text{II.1.2})$$

where K_t is the limit in probability of $\int_0^t b^n(X_s) ds$ for any sequence (b^n) of smooth approximations of b (in line with [8, 13]). Roughly, when such a solution exists and X and B are adapted to the filtration of the underlying probability space, we call it a weak solution. When X is adapted to the natural filtration of B , it is called a strong solution. We refer to Definition II.2.4 for more details.

Our first main result, Theorem II.2.5, gives conditions on β , p and H that ensure the existence of a weak solution to (II.1.1) when b is a measure. In particular when $b = a\delta_0$, for $a \in \mathbb{R}$, we obtain the existence of weak solutions to (II.1.1) for any $H < \sqrt{2} - 1$. In the standard Brownian case, Theorem II.2.5 provides weak solutions when the drift is in $\mathcal{B}_1^{\frac{1}{4}+}$. This space contains functions which, to the best of our knowledge, are not covered by the existing literature (see Remark II.2.7).

To prove Theorem II.2.5, we consider another approach to study Equation (II.1.1) which is via nonlinear Young integrals as introduced in [24], extending Young's theory of integration [114]. Consider the more general equation

$$dX_t = b(X_t) dt + dZ_t, \quad (\text{II.1.3})$$

where Z is a continuous stochastic process. The idea is to define path-by-path solutions to (II.1.3), that is, to solve this equation for a fixed realization of the noise $(Z_t(\omega))_t$. In order to do this one rewrites the equation as a random ODE: $dY_t = b(Y_t + Z_t) dt$, with $Y_t = X_t - Z_t$ and studies the regularity of the averaging operator $T_t^Z b : y \mapsto \int_0^t b(y + Z_s) ds$. In some interesting cases, $T_t^Z b$ is more regular than b itself, which permits to have solutions of the form $Y_t = Y_0 + \int_0^t T_{ds}^Z b(Y_s)$, where the integral is a so-called nonlinear Young integral. We refer to [47] for a review of nonlinear Young integrals in the Hölder setting and also to the recent work [50].

In Theorem II.2.15 we give conditions such that solutions w.r.t. the probabilistic approach via approximation of the drift and w.r.t. the approach via nonlinear Young integral theory are equivalent (i.e. weak solutions coincide). Therefore, in order to show existence of a weak solution to Equation (II.1.1), it is sufficient to construct a solution to the corresponding nonlinear Young integral equation that is additionally adapted to a small enough filtration.

To do so, we construct nonlinear Young integrals in the p -variation setting (instead of Hölder). In our one-dimensional setting this allows us to exploit the nonnegativity of b to get existence of solutions to (II.1.3) under some milder conditions on the regularity of b . To obtain the regularity of the averaging operator, we proceed similarly to Harang and Perkowski [62] by rewriting it as a convolution between b and the local time of Z . We are then able to deduce the Hölder regularity of the averaging operator from the Besov regularity of the local time of Z (see Lemma II.4.4). These two ingredients permit to construct path-by-path solutions by convergence of the Euler scheme associated to the equation, see Theorem II.2.14.

Alternatively, if the local time of Z has some probabilistic properties, as in the case $Z = B$, we are able to show that the averaging operator has a certain tightness property, see Lemma II.4.11. This

permits to prove that the sequence of (random) Euler schemes which approximate the nonlinear Young solution is compact in the space of continuous adapted stochastic processes, from which adaptedness of X is deduced.

Our second main result, Theorem II.2.10, states pathwise uniqueness of weak solutions in a class of processes which have some Hölder regularity. The main condition is that b is in \mathcal{B}_p^β with β and p satisfying $\beta - \frac{1}{p} \geq 1 - \frac{1}{2H}$. As with the Yamada-Watanabe Theorem, weak existence and pathwise uniqueness also lead here, under the same conditions on β , p and H , to the existence of a strong solution. Moreover, when b is a measure, any strong solution is proven to have sufficient Hölder regularity, thus ensuring pathwise uniqueness. For instance, we get strong existence and pathwise uniqueness to (II.1.1) when $b = a\delta_0$ for $a \in \mathbb{R}$ and any $H \leq 1/4$. This extends the previously known condition $H < 1/4$ from [24, 50] to $H = 1/4$.

To prove this theorem, we follow the recent approach of Athreya et al. [8]. In this paper, the authors proved the strong well-posedness of the one-dimensional stochastic heat equation with Besov drift by a tightness-stability argument. The main regularity estimates are obtained *via* the recent and powerful stochastic sewing Lemma introduced by Lê [70]. To adapt this argument to our setting, we control the Hölder norm of the conditional expectation of $x \mapsto \int_s^t f(x + B_r) dr$ in terms of the Besov norm of f (see Lemma II.7.1). One difficulty that arises here is the non-Markovian nature of B , which we could compensate by using a slightly adapted version of the local nondeterminism property of the fractional Brownian motion.

In the development of the proof of Theorem II.2.10, we obtain that weak solutions to (II.1.1) are limits of strong solutions to (II.1.1) with b replaced by a smooth bounded drift b^n , where the sequence (b^n) converges to b in Besov norm. This result is detailed in Theorem II.2.8 and can be of independent interest in view of numerical applications.

Lastly, let us mention that all techniques and results involving stochastic sewing in this chapter can be generalized to d dimensions in a straightforward way (see [59]).

Structure of the chapter. In Section II.2, the main definitions and results are stated as well as the organization of the proofs. In Section II.3, we develop the construction of nonlinear Young integrals in p -variation (Theorem II.3.1) and use it to find solutions to nonlinear Young integral equations with nonnegative (or nonpositive) drifts, see Theorem II.3.4. Then in Section II.4, we prove successively Theorem II.2.15 (relation between different notions of solution) and Theorem II.2.14 (existence of path-by-path solutions) which then permit to conclude with the proof of Theorem II.2.5 about the existence of weak solutions. The regularity of weak solutions is studied in Section II.5. The uniqueness part of Theorem II.2.10 is proven in Section II.6. The tightness-stability argument which leads to the existence of strong solutions is in Section II.7.

Other relevant sections are: Section II.A.2 including the local nondeterminism property of the fBm and the proof of the important regularity estimates of the conditional expectation of the fBm (Lemma II.5.1); Section II.B as it includes several Hölder bounds on the integrals of fBm; Appendix A.1 recalling some useful results on Besov spaces and Appendix A.2 containing different versions of the stochastic sewing lemma.

II.2 Main results

II.2.1 Definitions of solution

We define here weak and strong solutions to (II.1.1).

Definition II.2.1. Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$. We say that $(f_n)_{n \in \mathbb{N}}$ converges to f in $\mathcal{B}_p^{\beta-}$ as $n \rightarrow \infty$ if $\sup_{n \in \mathbb{N}} \|f_n\|_{\mathcal{B}_p^\beta} < \infty$ and

$$\forall \beta' < \beta, \quad \lim_{n \rightarrow \infty} \|f_n - f\|_{\mathcal{B}_p^{\beta'}} = 0.$$

Remark II.2.2. For any $f \in \mathcal{B}_p^\beta$, there exists a sequence $(f_n)_n$ of bounded smooth functions converging to f in the sense of Definition II.2.1: e.g. $f_n := G_{\frac{1}{n}} f$, where G is the Gaussian semigroup introduced in (I.2.10). This can be seen to hold true by applying Lemmas A.1.7 and A.1.9.

We recall here a link between fractional Brownian motion and Brownian motion. For each $H \in (0, \frac{1}{2})$, there exist operators \mathcal{A} and $\bar{\mathcal{A}}$, where both can be given in terms of fractional integrals and derivatives (see (II.A.1) and [87, Th. 11]), such that

$$\text{if } B \text{ is a fractional Brownian motion, } W = \mathcal{A}(B) \text{ is a Brownian motion,} \quad (\text{II.2.1})$$

$$\text{if } W \text{ is a Brownian motion, } B = \bar{\mathcal{A}}(W) \text{ is a fractional Brownian motion.} \quad (\text{II.2.2})$$

Besides, B and W generate the same filtration.

We give here the definition of an \mathbb{F} -fractional Brownian motion, for a given filtration \mathbb{F} .

Definition II.2.3. Let \mathbb{F} be a filtration. We say that B is an \mathbb{F} -fractional Brownian motion if $W = \mathcal{A}(B)$ is an \mathbb{F} -Brownian motion.

Definition II.2.4. Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\beta$, $T > 0$ and $X_0 \in \mathbb{R}$.

- *Weak solution:* We call a couple $((X_t)_{t \in [0, T]}, (B_t)_{t \in [0, T]})$ defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a weak solution to (II.1.1) on $[0, T]$, with initial condition X_0 , if
 - B is a \mathbb{F} -fBm;
 - X is adapted to \mathbb{F} ;
 - there exists a process $(K_t)_{t \in [0, T]}$ such that, a.s.,

$$X_t = X_0 + K_t + B_t \text{ for all } t \in [0, T]; \quad (\text{II.2.3})$$

- for every sequence $(b^n)_{n \in \mathbb{N}}$ of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$, we have that

$$\sup_{t \in [0, T]} \left| \int_0^t b^n(X_r) dr - K_t \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0. \quad (\text{II.2.4})$$

If the couple is clear from the context, we simply say that $(X_t)_{t \in [0, T]}$ is a weak solution.

- *Pathwise uniqueness:* As in the classical literature on SDEs, we say that pathwise uniqueness holds if for any two solutions (X, B) and (Y, B) defined on the same filtered probability space with the same fBm B and same initial condition $X_0 \in \mathbb{R}$, X and Y are indistinguishable.
- *Strong solution:* A weak solution (X, B) such that X is \mathbb{F}^B -adapted is called a strong solution.

II.2.2 Existence and uniqueness results

Theorem II.2.5. *Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$ and $H \in (0, \frac{1}{2}]$. Let $b \in \mathcal{B}_p^\beta$ be a measure. Assume that one of the following conditions holds:*

- (i) $H \geq \frac{1}{3}$ and $\beta > 1 + \frac{H}{2} - \frac{1}{2H}$;
- (ii) $H < \frac{1}{3}$ and $\beta > 2H - 1$;
- (iii) $p \in [2, \infty]$ and $\beta > -\frac{1}{2H} + 1$.

Then,

- (a) *there exists a weak solution X to (II.1.1) such that the convergence in (II.2.4) holds a.s.*
- (b) *Additionally, $X - B \in \mathcal{C}_{[0,T]}^\kappa(L^m)$ for any $\kappa \in (0, 1 + H(\beta - \frac{1}{p}) \wedge 0] \setminus \{1\}$ and $m \geq 2$.*

Corollary II.2.6. *For any finite measure b , there exists a weak solution to (II.1.1) for $H < \sqrt{2} - 1$. If $b = a\delta_0$, for some $a \in \mathbb{R}$, we call it an a -skew fractional Brownian motion.*

Remark II.2.7. In the Brownian motion case ($H = 1/2$), we obtain existence of a weak solution in new cases. For instance, consider, for $\varepsilon > 0$,

$$b(x) = |x|^{-3/4+\varepsilon} a(x),$$

where a is a smooth, compactly supported, nonnegative function equal to 1 on $[-1, 1]$. We have that $b \in \mathcal{B}_1^{\frac{1}{4}+}$ (see [11, Prop. 2.21] for similar computations), and the space $\mathcal{B}_1^{\frac{1}{4}+}$ is covered by Assumption (i) of Theorem II.2.5. Since b is neither in $\bigcup_{p \geq 2} L^p(\mathbb{R})$ nor in $\mathcal{C}^{-\frac{2}{3}+}$, we cannot directly apply results from [68] or [36].

The following theorem gives, as Theorem II.2.5 does, conditions on the drift and the Hurst parameter such that (II.1.1) has a weak solution. Note that this time, there is no nonnegativity assumption on b . Even in the case of considering b to be a measure, none of the two theorems is stronger than the other. However, if b is a finite measure, Theorem II.2.5 allows for a wider range of Hurst parameters to get existence of weak solutions to (II.1.1). Moreover, even though Theorem II.2.5(iii) is fully covered by (II.2.5), it still adds value as it gives a.s. convergence in (II.2.4).

Theorem II.2.8. *Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\beta$ and $X_0 \in \mathbb{R}$. Let $(b^n)_{n \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$. Let X^n be the unique strong solution to (II.1.1) with drift b^n . Assume*

$$\beta - \frac{1}{p} > -\frac{1}{2H} + \frac{1}{2}. \quad (\text{II.2.5})$$

Then, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(X^{n_k})_{k \in \mathbb{N}}$ converges in law w.r.t. $\|\cdot\|_\infty$ to a process X which is a weak solution to (II.1.1) with drift b . Furthermore, $X - B \in \mathcal{C}_{[0,T]}^\kappa(L^m)$ for any $\kappa \in (0, 1 + H(\beta - \frac{1}{p}) \wedge 0] \setminus \{1\}$ and $m \geq 2$.

Remark II.2.9. Without loss of generality, the previous theorem can directly be formulated for Hölder spaces by fixing $p = \infty$, using the embedding from Remark I.2.2. However, this is not the case for Theorem II.2.10 below. Hence we keep working in general Besov spaces for a better comparison of the results.

Under slightly stronger assumptions than (II.2.5), the following theorem states strong existence and pathwise uniqueness. In particular, it implies that under this stronger condition, convergence in probability of the approximation scheme in Theorem II.2.8 holds without passing to a subsequence.

Theorem II.2.10. *Let $H < 1/2$, $\beta \in \mathbb{R}$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\beta$ and $X_0 \in \mathbb{R}$. Assume*

$$\beta > -\frac{1}{2H} + 1 \quad \text{and} \quad \beta - \frac{1}{p} \geq -\frac{1}{2H} + 1. \quad (\text{II.2.6})$$

Then,

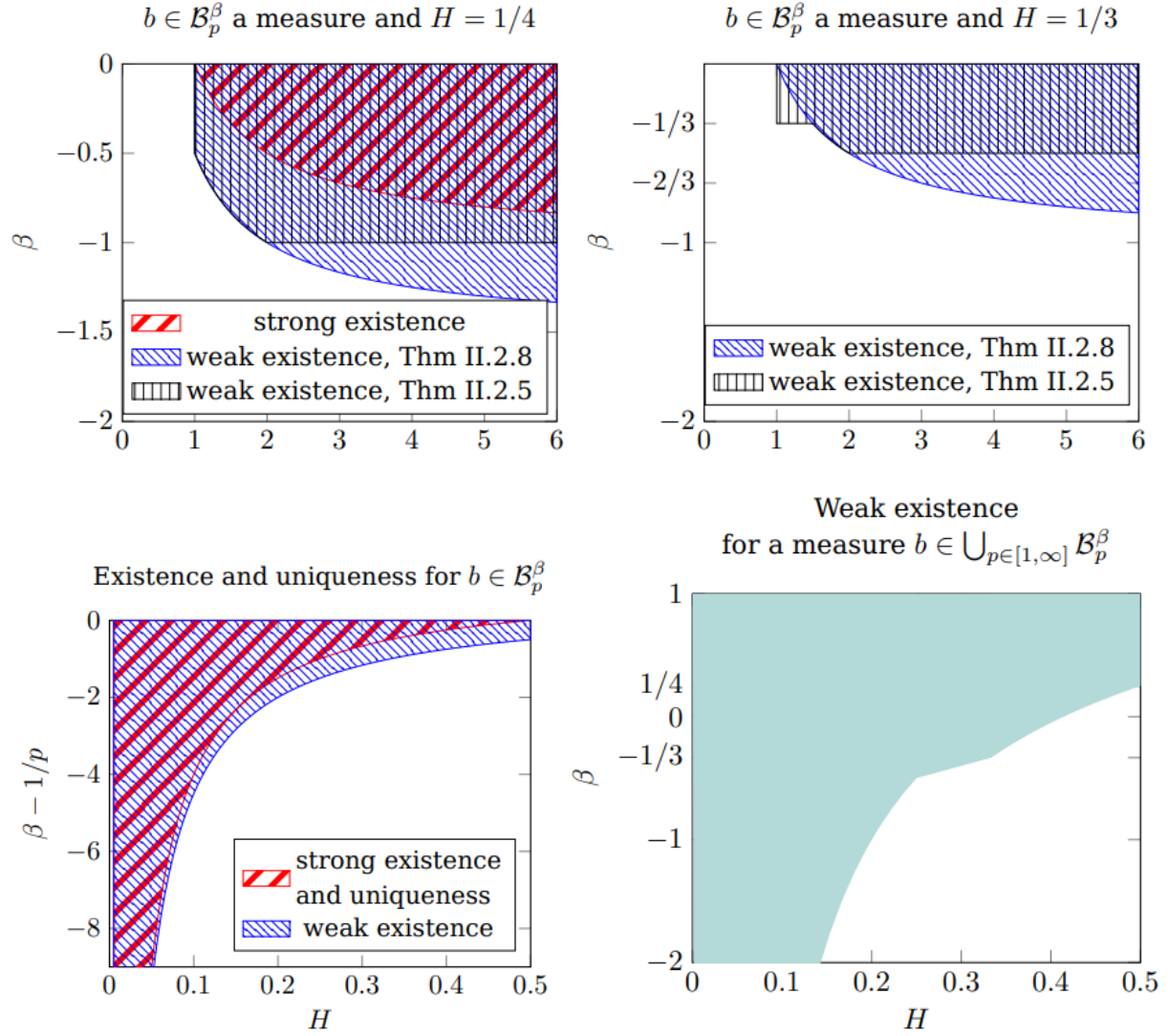
- (a) *there exists a strong solution X to (II.1.1) such that $X - B \in \mathcal{C}_{[0,T]}^{\frac{1}{2}+H}(L^m)$ for any $m \geq 2$;*
- (b) *pathwise uniqueness holds in the class of all solutions X such that $X - B \in \mathcal{C}_{[0,T]}^{\frac{1}{2}+H}(L^2)$;*
- (c) *for any sequence $(b^n)_{n \in \mathbb{N}}$ of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$, the corresponding sequence of strong solutions $(X^n)_{n \in \mathbb{N}}$ to (II.1.1) with drift b^n converges in probability to the unique strong solution for which $X - B \in \mathcal{C}_{[0,T]}^{\frac{1}{2}+H}(L^2)$. In particular X is independent of the chosen sequence of approximations.*
- (d) *if b is a measure, there exists a pathwise unique strong solution to (II.1.1).*

Remark II.2.11. • For b being a finite measure (hence in \mathcal{B}_1^0), Theorem II.2.10 gives existence of a unique strong solution to (II.1.1) for $H \leq 1/4$ and Theorem II.2.8 gives existence of a weak solution to (II.1.1) for $H < 1/3$.

Such a finite measure b is also in \mathcal{B}_∞^{-1} . In this space, the existence of a unique strong solution was shown in [51] for $H < 1/4$ (elaborating on the path-by-path result of [24]). Hence in this case, Theorem II.2.10 extends this result to $H = 1/4$.

- In the Brownian motion case, Theorem II.2.8 gives existence of weak solutions for $b \in \mathcal{B}_p^\beta$ when $\beta - 1/p > -1/2$. In this regime strong existence and pathwise uniqueness are already known by [13].
- Note that $H = 1/2$ is excluded from Theorem II.2.10. Strong existence and uniqueness result are already known under our assumptions (see [68]).

The following diagrams display for which Besov-valued distributions b we have well-posedness for Equation (II.1.1). The black-hatched area and the turquoise area correspond to the result obtained in Theorem II.2.5. The graphics visualize that the weak solution constructed in Theorem II.2.5 is a solution that, in some cases, does not arise from the weak solution constructed in Theorem II.2.8.


 Figure 2: Existence (and uniqueness) for variable H

II.2.3 Reformulation as a nonlinear Young equation

Let $w \in \mathcal{C}_{[0, T]}$. Rewriting Equation (II.1.3) with $Z \equiv w$ and $\tilde{X}_t = X_t - w_t$, we formally obtain

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t b(\tilde{X}_r + w_r) dr. \quad (\text{II.2.7})$$

For a bounded measurable function b , we define the averaging operator T^w by

$$T_t^w b(x) := \int_0^t b(x + w_r) dr, \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}. \quad (\text{II.2.8})$$

Assuming that w has a local time L and using the occupation time formula (I.2.12), it comes that

$$T_{s,t}^w b(x) = \int_s^t b(x + w_r) dr = \int_{\mathbb{R}} b(x + z) L_{s,t}(z) dz = b * \check{L}_{s,t}(x), \quad \forall x \in \mathbb{R}, \quad (\text{II.2.9})$$

where $\check{L}_{s,t}(x) = L_{s,t}(-x)$. This operator and its connection to the local time was already considered in [24]. In view of the expression of $T^w b$ as a convolution, one can expect that for w with a sufficiently regular local time, the definition of $T^w b$ will extend to b with lower regularity (we will consider suitable Besov spaces, see Section II.4). This idea was exploited in [62], in the case of noises with infinitely differentiable local times.

Using (II.2.7), we get that for b continuous and bounded, $\tilde{X} \in \mathcal{C}_{[0,T]}$ and a sequence of partitions $\{t_i^n\}_{i=1}^{N_n}$ of $[0, t]$ with mesh size converging to 0,

$$\begin{aligned} \int_0^t b(\tilde{X}_r + w_r) dr &= \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n-1} \int_{t_i^n}^{t_{i+1}^n} b(\tilde{X}_{t_i^n} + w_r) dr \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{N_n-1} T_{t_i^n, t_{i+1}^n}^w b(\tilde{X}_{t_i^n}) \\ &= \int_0^t T_{dr}^w b(\tilde{X}_r), \end{aligned} \quad (\text{II.2.10})$$

where the final equality is only formal at this point. We give a rigorous definition of this integral in Section II.3 and call it a nonlinear Young integral. This also suggests to rewrite Equation (II.1.3) as a nonlinear Young integral equation. Thus we give another definition of a solution to (II.1.3) and (II.1.1). Combining the theory of nonlinear Young integrals from Section II.3 and the extension of the averaging operator applied to distributions (see Section II.4) will give sense to the following definition.

Definition II.2.12. Let Z be a stochastic process. We call $X : \Omega \rightarrow \mathcal{C}_{[0,T]}$ a path-by-path solution to (II.1.3) if there exists a null-set \mathcal{N} , $\eta \in (0, 1]$ and $p, q \geq 1$ such that $1/p + \eta/q > 1$ and for any $\omega \notin \mathcal{N}$, $T^{Z(\omega)} b \in \mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}^\eta)$, $X(\omega) - Z(\omega) \in \mathcal{C}_{[0,T]}^{q\text{-var}}$ and

$$X_t(\omega) = X_0 + \int_0^t T_{dr}^{Z(\omega)} b(X_r(\omega) - Z_r(\omega)) + Z_t(\omega), \quad \text{for all } t \in [0, T]. \quad (\text{II.2.11})$$

The assumption involving $\eta \in (0, 1]$ and $p, q \geq 1$ is simply the (sufficient) condition formulated in Theorem II.3.1 that ensures the existence of the nonlinear Young integral. Note that in the above definition no measurability or adaptedness of X is required.

Theorem II.2.14 gives conditions on the local time of the process Z such that there exists a path-by-path solution to (II.1.3). These conditions will be needed multiple times throughout the chapter, hence we give them here.

Assumption II.2.13. Let $\gamma, \eta \in (0, 1), \beta \in \mathbb{R}$ and $p \in [1, \infty]$. Let $b \in \mathcal{B}_p^\beta$ and $w \in \mathcal{C}_{[0,T]}$ with local time L .

(I) There exists $\tilde{p} \in [1, \infty]$ with $1/p + 1/\tilde{p} > 1$ and $L \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^{-\beta + \eta + 1/p + 1/\tilde{p} - 1})$;

(II) There exists $\tilde{p} \in [1, \infty]$ with $1/p + 1/\tilde{p} \leq 1$ and $L \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^{-\beta + \eta})$.

Theorem II.2.14. *Let $\gamma, \eta \in (0, 1)$ with $\gamma + \eta > 1$. Let $b \in \mathcal{B}_p^\beta$ be a measure with $\beta \in \mathbb{R}$, $p \in [1, \infty]$. Let $Z : \Omega \rightarrow \mathcal{C}_{[0, T]}$ be a continuous stochastic process with a local time $L^Z : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$. Assume that there exists a null-set \mathcal{N} such that for any $\omega \notin \mathcal{N}$, $L^Z(\omega)$ satisfies (I) or (II). Then there exists a path-by-path solution to Equation (II.1.3).*

Recall that Theorem II.2.14 does not imply existence of a measurable/adapted solution. However in the case of fBm, using properties of its local time, additionally adaptedness of the path-by-path solution can be proven (see Theorem II.2.5).

The following theorem provides a comparison between solutions constructed by approximation with a smooth drift and solutions in the nonlinear Young sense. More precisely, it shows that being a solution to (II.1.3) (i.e. for a noise Z) in the sense of Definition II.2.12 implies being a solution in the sense of Definition II.2.4. Under some regularity restrictions, the reverse implication holds as well. We rephrase (II.2.3) and (II.2.4) (without specifying the mode of convergence yet) for a noise Z instead of B :

$$X_t = X_0 + K_t + Z_t \text{ for all } t \in [0, T]; \quad (\text{II.2.12})$$

$$\sup_{t \in [0, T]} \left| \int_0^t b^n(X_r) dr - K_t \right| \xrightarrow{n \rightarrow \infty} 0 \quad (\text{II.2.13})$$

for every sequence $(b^n)_{n \in \mathbb{N}}$ of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$.

Theorem II.2.15. *Let $\gamma, \eta \in (0, 1)$ and $q \geq 1$ with $\gamma + \eta/q > 1$, $X_0 \in \mathbb{R}$, $p \in [1, \infty]$, $\beta \in \mathbb{R}$, $b \in \mathcal{B}_p^\beta$ and $X : \Omega \rightarrow \mathcal{C}_{[0, T]}$. Let $Z : \Omega \rightarrow \mathcal{C}_{[0, T]}$ be a continuous stochastic process and $L^Z : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ its local time. Assume that there exists a null-set \mathcal{N} such that for any $\omega \notin \mathcal{N}$, $L^Z(\omega)$ satisfies (I) or (II).*

- (a) *Assume that X is a path-by-path solution to (II.1.3) and that $X - Z \in \mathcal{C}_{[0, T]}^{q\text{-var}}$. Then for any sequence of smooth bounded functions $(b^n)_{n \in \mathbb{N}}$ that converges to b in $\mathcal{B}_p^{\beta-}$, $X(\omega) - Z(\omega) - X_0 = K(\omega) = \int_0^\cdot T_{dr}^{Z(\omega)} b(X_r(\omega) - Z_r(\omega))$ is the uniform limit of $\int_0^\cdot b^n(X_r(\omega)) dr$ for all $\omega \notin \mathcal{N}$ (i.e. (II.2.13) holds on the set of full measure \mathcal{N}^c).*
- (b) *Assume that there exists a process $K : \Omega \rightarrow \mathcal{C}_{[0, T]}$ such that (II.2.12) and (II.2.13) hold, where the convergence in (II.2.13) is in probability. Assume further that a.s., $X(\omega) - Z(\omega) \in \mathcal{C}_{[0, T]}^{q\text{-var}}$. Then X is a path-by-path solution to (II.1.3).*

Remark II.2.16. In particular, statement (a) implies that if X is an \mathbb{F} -adapted path-by-path solution, it is a weak solution. Moreover, for the weak solution constructed in Theorem II.2.5, the convergence in (II.2.4) holds on a set of full measure instead of convergence in probability.

II.2.4 Organization of the proofs

In Section II.3, we construct nonlinear Young integrals in p -variation via a classical sewing argument. In particular, we establish existence of solutions to a nonlinear Young integral equation with monotone drift under milder regularity constraints than in the non-monotone case.

Then we rigorously rewrite Equation (II.1.1) as a nonlinear Young integral equation. The existence of a solution to the more general Equation (II.1.3), when Z has a sufficiently regular local time, is stated in Theorem II.2.14 and proven in Section II.4.2. We then establish that a

solution to (II.1.1) in this nonlinear Young sense is also a solution in the sense of Definition II.2.4 (see Theorem II.2.15, which is proven in Section II.4.2). For instance, when applied to the case of a fractional Brownian noise (after investigating the regularity properties of its local time), these theorems imply the existence of path-by-path solutions. However, in Theorem II.2.5(a) one wants to prove the existence of solutions that are adapted to the filtration \mathbb{F} of the underlying filtered probability space. Theorems II.2.14 and II.2.15 cannot be applied directly and a modified approach is developed using the tightness of the averaging operator of the fBm (Lemma II.4.11) and the continuity of the operator \mathcal{A} transforming a fBm to a Bm, see (II.2.1) and Lemma II.A.1. These arguments are given in Section II.4.4 and lead to the proof of Theorem II.2.5(a).

In Section II.5, we use some new regularity estimates on conditional expectations of the fBm (Lemma II.5.1) and the stochastic sewing Lemma with random control (see Lemma A.2.2) to establish that any weak solution X satisfies $X - B \in \mathcal{C}_{[0,T]}^\kappa(L^m)$ for any $\kappa \in (0, 1 + H(\beta - \frac{1}{p}) \wedge 0] \setminus \{1\}$ and $m \geq 2$ when b is a measure in \mathcal{B}_p^β . This proves Theorem II.2.5(b) and Theorem II.2.10(d).

In Section II.6, in order to establish pathwise uniqueness of weak solutions to (II.1.1) (see Proposition II.6.1), we adapt an approach developed recently for the stochastic heat equation with distributional drift, see [8]. This requires several regularity estimates on solutions which are derived from the crucial regularity Lemma II.B.1 and the stochastic sewing Lemma with critical exponent (Theorem 4.5 in [8]). The proof of Lemma II.B.1 relies on the stochastic sewing Lemma and the aforementioned regularity estimates on conditional expectations of the fBm (Lemma II.5.1).

Theorem II.2.8 is proven in Section II.7 by an approximation of the drift with smooth bounded functions. The corresponding sequence of strong solutions will be shown to be tight and furthermore reveal a stability property, such that we can identify the limit as a solution to (II.1.1), where continuity of the operator linking fBm to Brownian motion (see Lemma II.A.1) is needed to prove adaptedness. This works thanks to *a priori* regularity estimates of solutions, see Lemma II.7.2 and Lemma II.7.3.

Strong existence will follow by a Yamada-Watanabe type argument, using Gyöngy-Krylov's lemma ([60, Lem. 1.1]) and requires the uniqueness result Proposition II.6.1. The proofs of Theorems II.2.8 and II.2.10 are then completed in Section II.7.3.

II.3 Nonlinear Young integrals and nonlinear Young equations in $\mathcal{C}^{p\text{-var}}$

II.3.1 Construction of nonlinear Young integrals and properties

Throughout this subsection, E and F denote arbitrary Banach spaces. Theorem II.3.1 provides conditions for the existence of a nonlinear Young integral in terms of p -variations (rather than Hölder continuity as in [24, Th. 2.4] and [47]).

Theorem II.3.1. *Let $\eta \in (0, 1]$ and $p, q \in [1, \infty)$ such that $\theta := 1/p + \eta/q > 1$. Let $A \in \mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}_{E,loc}^\eta(F))$ and $x \in \mathcal{C}_{[0,T]}^{q\text{-var}}(E)$. Then for $(s, t) \in \Delta_{[0,T]}$ and any sequence of partitions $(\Pi_n)_n$ of $[s, t]$ with $\lim_n |\Pi_n| = 0$, the sum $\sum_{\Pi_n} A_{t_i, t_{i+1}}(x_{t_i})$ converges. Besides, the limit is independent of the sequence of partitions $(\Pi_n)_n$. We denote it by $\int_s^t A_{dr}(x_r)$ and call it the nonlinear Young integral with respect to A and x .*

In addition, there exists $C(\theta) > 0$ such that for $(s, t) \in \Delta_{[0, T]}$ and $R > 0$ with $\|x\|_\infty \leq R$, one has

$$\left\| \int_s^t A_{dr}(x_r) - A_{s,t}(x_s) \right\|_F \leq C(\theta) [A]_{\mathcal{C}_{[s,t]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} [x]_{\mathcal{C}_{[s,t]}^{q\text{-var}}}^\eta. \quad (\text{II.3.1})$$

Proof. We apply the sewing lemma [46, Theorem 2.2 and Remark 2.3] formulated with controls. Let $0 \leq u \leq v \leq w \leq T$. Then, for $R > 0$ such that $\|x\|_\infty \leq R$,

$$\begin{aligned} |A_{u,w}(x_u) - A_{u,v}(x_u) - A_{v,w}(x_v)| &= |A_{v,w}(x_u) - A_{v,w}(x_v)| \\ &\leq [A_{v,w}]_{\mathcal{C}_{D_R}^\eta} [x]_{\mathcal{C}_{[u,w]}^{q\text{-var}}}^\eta \\ &\leq [A]_{\mathcal{C}_{[u,w]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} [x]_{\mathcal{C}_{[u,w]}^{q\text{-var}}}^\eta, \end{aligned}$$

which gives the result as the last expression in the above defines a control raised to the power θ by [45, Exercise 1.9]. \square

As a corollary we obtain the following result.

Corollary II.3.2. *Let $T > 0$. Let $p, q \in [1, \infty)$ and $\eta \in (0, 1]$ with $\theta := 1/p + \eta/q > 1$. Let $x \in \mathcal{C}_{[0, T]}^{q\text{-var}}$ and $A \in \mathcal{C}_{[0, T]}^{p\text{-var}}(\mathcal{C}^\eta)$. If a sequence A^n converges to A in $\mathcal{C}_{[0, T]}^{p\text{-var}}(\mathcal{C}^\eta)$, then*

$$\sup_{t \in [0, T]} \left| \int_0^t (A^n - A)_{dr}(x_r) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Proof. By (II.3.1), we have

$$\begin{aligned} \left| \int_0^t (A^n - A)_{dr}(x_r) \right| &\leq \left| \int_0^t (A^n - A)_{dr}(x_r) - (A^n - A)_{0,t}(x_0) \right| + |(A^n - A)_{0,t}(x_0)| \\ &\leq C(\theta) [A^n - A]_{\mathcal{C}_{[0, T]}^{p\text{-var}}(\mathcal{C}^\eta)} [x]_{\mathcal{C}_{[0, T]}^{q\text{-var}}}^\eta + [A^n - A]_{\mathcal{C}_{[0, T]}^{p\text{-var}}(\mathcal{C}^\eta)}. \end{aligned}$$

Taking the supremum over all $t \in [0, T]$ and letting n go to ∞ gives the result. \square

The next lemma establishes the sensitivity in the “ x ” variable of the nonlinear Young integral. See [47, Th. 2.7] for its counterpart in the Hölder setting.

Lemma II.3.3. *Let $\eta \in (0, 1]$ and $p, q \in [1, \infty)$ with $\theta := 1/p + \eta/q > 1$. Let $A \in \mathcal{C}_{[0, T]}^{p\text{-var}}(\mathcal{C}_{E, \text{loc}}^\eta(F))$, $x, y \in \mathcal{C}_{[0, T]}^{q\text{-var}}(E)$. We denote $R = \text{Max}(\|x\|_\infty, \|y\|_\infty)$. Then, for $\delta \in (q(1 - \frac{1}{p}), \eta)$ and $(s, t) \in \Delta_{[0, T]}$, one has*

$$\begin{aligned} \left\| \int_s^t A_{dr}(x_r) - \int_s^t A_{dr}(y_r) \right\|_F &\leq C(\theta) [A]_{\mathcal{C}_{[s,t]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} \left([x]_{\mathcal{C}_{[s,t]}^{q\text{-var}}} + [y]_{\mathcal{C}_{[s,t]}^{q\text{-var}}} \right)^\delta \|x - y\|_\infty^{\eta - \delta} \\ &\quad + \|A_{s,t}\|_{\mathcal{C}_{D_R}^\eta} \|x - y\|_\infty^\eta. \end{aligned} \quad (\text{II.3.2})$$

Proof. For $(u, w) \in \Delta_{[0, T]}$, let

$$\Gamma_{u, w} := (A_{u, w}(x_u) - A_{u, w}(y_u)).$$

Again, we aim to apply the sewing lemma. For any $z \in \mathcal{C}_{[0, T]}^{q\text{-var}}(E)$, let $\varkappa_z(u, w)^\theta := [A]_{\mathcal{C}_{[u, w]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} [z]_{\mathcal{C}_{[u, w]}^{q\text{-var}}}^\eta$. Furthermore, let

$$\hat{\varkappa}(u, w)^{\delta/q+p^{-1}} := (\varkappa_x(u, w) + \varkappa_y(u, w))^{\frac{\delta}{\eta}\theta} \left([A]_{\mathcal{C}_{[u, w]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} \right)^{1-\frac{\delta}{\eta}}.$$

By [45, Exercise 1.9] and by definition of a control, we get that \varkappa_x , \varkappa_y , $\varkappa_x + \varkappa_y$ and $\hat{\varkappa}$ are control functions.

Note that, for $v \in [u, w]$,

$$\begin{aligned} \|\Gamma_{u, w} - \Gamma_{u, v} - \Gamma_{v, w}\|_F &\leq \|A_{v, w}(x_v) - A_{v, w}(x_u)\|_F + \|A_{v, w}(y_v) - A_{v, w}(y_u)\|_F \\ &\leq [A]_{\mathcal{C}_{[u, w]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} ([x]_{\mathcal{C}_{[u, v]}^{q\text{-var}}}^\eta + [y]_{\mathcal{C}_{[u, v]}^{q\text{-var}}}^\eta) \\ &\leq (\varkappa_x(u, w) + \varkappa_y(u, w))^\theta. \end{aligned}$$

Furthermore, we also have that

$$\begin{aligned} \|\Gamma_{u, w} - \Gamma_{u, v} - \Gamma_{v, w}\|_F &\leq \|A_{v, w}(x_v) - A_{v, w}(y_v)\|_F + \|A_{v, w}(x_u) - A_{v, w}(y_u)\|_F \\ &\leq 2[A]_{\mathcal{C}_{[u, w]}^{p\text{-var}}(\mathcal{C}_{D_R}^\eta)} \|x - y\|_\infty^\eta. \end{aligned}$$

It clearly holds true that $a \wedge b \leq a^\xi b^{1-\xi}$ for $a, b \geq 0$ and $\xi \in [0, 1]$. Hence, for $\xi = \delta/\eta \in (0, 1)$, it comes

$$\|\Gamma_{u, w} - \Gamma_{u, v} - \Gamma_{v, w}\|_F \leq C \hat{\varkappa}(u, w)^{\frac{\delta}{q}+p^{-1}} \|x - y\|_\infty^{\eta(1-\frac{\delta}{\eta})}.$$

Applying the sewing lemma and the inequality

$$\|\Gamma_{u, w}\|_F \leq \|A_{u, w}\|_{\mathcal{C}_{D_R}^\eta} \|x - y\|_\infty^\eta$$

gives (II.3.2). □

II.3.2 Solving nonlinear Young equations

We are now ready to state a result of existence of solutions to nonlinear Young integral equations with a positive drift. From now on we will work with real-valued functions and vector fields. The proof resembles the one in the Hölder setting, see [47, Th. 3.2]. However, there is one crucial step where the p -variation permits to take into account the nonnegative drift and which then allows to assume milder regularity conditions than in the Hölder setting.

Theorem II.3.4. *Let $\eta \in (0, 1]$ and $p \geq 1$ with $1/p + \eta > 1$. Let $A \in \mathcal{C}_{[0, T]}^{p\text{-var}}(\mathcal{C}^\eta)$ with $A_{s, t}(y) \geq 0$ for all $y \in \mathbb{R}$ and all $(s, t) \in \Delta_{[0, T]}$. Then for any $x_0 \in \mathbb{R}$, there exists a solution $x \in \mathcal{C}_{[0, T]}^{1\text{-var}}$ to the nonlinear Young equation*

$$x_t = x_0 + \int_0^t A_{dr}(x_r), \quad \forall t \in [0, T]. \quad (\text{II.3.3})$$

Proof. Without loss of generality, let $T = 1$. Let $\theta := 1/p + \eta$. For $n \in \mathbb{N}$ and $0 \leq k \leq n$, let $t_k^n := k/n$, $\bar{x}_0^n := x_0$ and define recursively

$$\bar{x}_{k+1}^n = \bar{x}_k^n + A_{t_k^n, t_{k+1}^n}(\bar{x}_k^n).$$

We embed $(\bar{x}_k^n)_{k=0}^n$ into $\mathcal{C}_{[0,1]}$ by setting

$$x_t^n = x_0 + \sum_{0 \leq k \leq \lfloor nt \rfloor} A_{t_k^n, t_{k+1}^n}(\bar{x}_k^n),$$

which we write as follows

$$x_t^n = x_0 + \int_0^t A_{dr}(x_r^n) + \sum_{0 \leq k \leq \lfloor nt \rfloor} \left(\int_{t_k^n}^{t \wedge t_{k+1}^n} A_{dr}(x_{t_k^n}^n) - A_{dr}(x_r^n) \right). \quad (\text{II.3.4})$$

Denote the sum in (II.3.4) by ψ_t^n . Let us introduce the control function

$$\varkappa^n(s, t) := \left([A]_{\mathcal{C}_{[s,t]}^{p\text{-var}}(\mathcal{C}^\eta)} [x^n]_{\mathcal{C}_{[s,t]}^{1\text{-var}}}^\eta \right)^{\frac{1}{\theta}}.$$

Then using (II.3.1) and the superadditivity property (I.2.13) of a control,

$$\begin{aligned} |\psi_t^n| &= \left| \sum_{0 \leq k \leq \lfloor nt \rfloor} \left(\int_{t_k^n}^{t \wedge t_{k+1}^n} A_{dr}(x_{t_k^n}^n) - A_{dr}(x_r^n) \right) \right| \\ &\leq C(\theta) \sum_{0 \leq k \leq \lfloor nt \rfloor} \varkappa^n(t_k^n, t \wedge t_{k+1}^n)^\theta \\ &\leq C(\theta) \varkappa^n(0, t) \max_{0 \leq k \leq \lfloor nt \rfloor} \{ \varkappa^n(t_k^n, t \wedge t_{k+1}^n) \}^{\theta-1}. \end{aligned} \quad (\text{II.3.5})$$

Let $0 \leq s \leq u \leq T = 1$. We note $i = \lfloor ns \rfloor$ and $i + j = \lfloor nu \rfloor$. We assume that $j > 0$ as the case $j = 0$ is simpler. Then

$$\begin{aligned} |\psi_{s,u}^n| &\leq \left| \int_s^{t_{i+1}^n} A_{dr}(x_{t_i^n}^n) - A_{dr}(x_r^n) \right| + \sum_{k=1}^j \left| \int_{t_{i+k}^n}^{u \wedge t_{i+k+1}^n} A_{dr}(x_{t_{i+k}^n}^n) - A_{dr}(x_r^n) \right| \\ &\leq \left| \int_s^{t_{i+1}^n} A_{dr}(x_s^n) - A_{dr}(x_r^n) \right| + |A_{s, t_{i+1}^n}(x_s^n) - A_{s, t_{i+1}^n}(x_{t_i^n}^n)| \\ &\quad + \sum_{k=1}^j \left| \int_{t_{i+k}^n}^{u \wedge t_{i+k+1}^n} A_{dr}(x_{t_{i+k}^n}^n) - A_{dr}(x_r^n) \right|, \end{aligned}$$

so by the estimate (II.3.1) it comes

$$|\psi_{s,u}^n| \leq C(\theta) \left(\varkappa^n(s, u)^\theta + [A]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{p\text{-var}}(\mathcal{C}^\eta)} [x^n]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{1\text{-var}}}^\eta \right). \quad (\text{II.3.6})$$

We now look for a bound on the 1-variation norm of x^n . Due to the non-negativity assumption on A , x^n is non-decreasing. Hence

$$[x^n]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{1\text{-var}}} = |x_{t_i^n}^n - x_{t_{i+1}^n}^n| = |A_{t_i^n, t_{i+1}^n}(x_{t_i^n}^n)| \leq [A]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{p\text{-var}}(\mathcal{C}^\eta)}.$$

Then in view of (II.3.4), (II.3.6) and applying (II.3.1) to $\int_s^u A_{dr}(x_r^n)$, we get

$$[x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} \leq C(\theta) \left((|A_{s,u}(x_s^n)| + \varkappa^n(s, u)^\theta) + \left(\varkappa^n(s, u)^\theta + [A]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{p\text{-var}}}(\mathcal{C}^\eta)^{1+\eta} \right) \right).$$

Let now $\varepsilon > 0$ and choose n such that $[A]_{\mathcal{C}_{[t_i^n, t_{i+1}^n]}^{p\text{-var}}}(\mathcal{C}^\eta) < \varepsilon^{1/(1+\eta)}$ for all $i = 0, \dots, n-1$. Then one obtains

$$[x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} \leq 2C(\theta) \left([A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}}(\mathcal{C}^\eta) + [A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}}(\mathcal{C}^\eta) [x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}}^\eta + \varepsilon \right).$$

Using that $a^\eta \leq 1 + a$ for $a \geq 0$ and $\eta \in [0, 1]$, it follows that

$$[x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} \leq 2C(\theta) \left([A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}}(\mathcal{C}^\eta) (1 + [x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}}) + \varepsilon \right).$$

Hence, for $s \leq u$ such that $2C(\theta)[A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}}(\mathcal{C}^\eta) < 1$, this leads to

$$[x^n]_{\mathcal{C}_{[s,u]}^{1\text{-var}}} \leq 2C(\theta) \frac{[A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}}(\mathcal{C}^\eta) + \varepsilon}{1 - 2C(\theta)[A]_{\mathcal{C}_{[s,u]}^{p\text{-var}}}(\mathcal{C}^\eta)}, \quad (\text{II.3.7})$$

which gives both uniform boundedness and equicontinuity. In view of Equation (II.3.5), the uniform boundedness of $[x^n]_{\mathcal{C}_{[0,t]}^{1\text{-var}}}$ combined with the continuity of $[A]_{\mathcal{C}_{[0,t]}^{p\text{-var}}}(\mathcal{C}^\eta)$ give that ψ^n converges to 0 uniformly on $[0, 1]$. By the Arzelà-Ascoli Theorem, we deduce that x^n converges uniformly along a subsequence to some non-decreasing $x \in \mathcal{C}_{[0,1]}$. Without loss of generality, we still denote by x^n this subsequence. Then using the uniform boundedness of $[x^n]_{\mathcal{C}_{[0,1]}^{1\text{-var}}}$ and the uniform convergence of x^n to x , one gets from Lemma II.3.3 that for any t ,

$$\int_0^t A_{dr}(x_r^n) \xrightarrow{n \rightarrow \infty} \int_0^t A_{dr}(x_r).$$

Hence, passing to the limit in (II.3.4), we finally obtain that x solves (II.3.3). \square

Remark II.3.5. Theorem II.3.4 extends to mappings A which are only locally Hölder in space, giving existence possibly up to a blow-up time. Its proof can be done using typical localization arguments.

Since we are interested in differential equations perturbed by noise, it is natural to look for an extension of Theorem II.3.4 in case A is random, and to look for a measurable solution. Therefore we conclude this section with an extension of Theorem II.3.4 for random A . We omit the proof as it is similar to the first part of the proof of Theorem II.2.5 which is presented in Section II.4.4 (i.e. showing tightness and then using Skorokhod's representation Theorem to pass to an almost sure limit).

Corollary II.3.6. *Let $\eta \in (0, 1]$ and $p \geq 1$ with $1/p + \eta > 1$. Let $A : \Omega \rightarrow \mathcal{C}_{[0,T]}^{p\text{-var}}(\mathcal{C}^\eta)$ be a random variable such that almost surely, $A_{s,t}(y) \geq 0$ for all $y \in \mathbb{R}$ and $(s, t) \in \Delta_{[0,T]}$. Furthermore, assume that for any $\lambda > 0$,*

$$\lim_{\delta \rightarrow 0} \mathbb{P}(\Omega_{\delta, \lambda}) = 1, \quad \text{where } \Omega_{\delta, \lambda} := \{\omega : \sup_{|t-s| < \delta} [A]_{\mathcal{C}_{[s,t]}^{p\text{-var}}}(\mathcal{C}^\eta) < \lambda\}. \quad (\text{II.3.8})$$

Then for any $Y_0 \in \mathbb{R}$, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{E}}, \tilde{\mathbb{P}})$, a measurable map \tilde{A} which satisfies $\text{Law}(\tilde{A}) = \text{Law}(A)$ and a measurable map $Y : \tilde{\Omega} \rightarrow C_{[0,T]}^{1\text{-var}}$ such that almost surely,

$$Y_t = Y_0 + \int_0^t \tilde{A}_{dr}(Y_r), \quad \forall t \in [0, T].$$

II.4 Existence of weak solutions with nonnegative drift via nonlinear Young integrals

We prove Theorem II.2.5, Corollary II.2.6, Theorem II.2.14 and Theorem II.2.15 in this section. First, we extend the averaging operator defined in (II.2.8) to distributions b in Besov spaces. Obtaining some Hölder regularity properties for this object allows to prove Theorem II.2.15, which states roughly that solutions to (II.1.1) in the sense of Definition II.2.4 are equivalent to solutions in the sense of Definition II.2.12. Hence, we then only consider solutions in the sense of Definition II.2.12.

The remaining subsections are dedicated to the proof of Theorem II.2.5, using some fine results on the joint regularity in time and space of the local time of fractional Brownian motion and the results on nonlinear Young equations from Section II.3.

II.4.1 Definition and properties of the averaging operator

In this section we give the definition of the averaging operator $T^w b$ for distributional b and $w \in C_{[0,T]}$, extending the construction of Section II.2.3. Note that this was already done in a very general setup in [50, Section 3.1]. We take a less general approach by directly mollifying b , which is in line with the definition in [50] by Lemma 3.9 therein.

Definition II.4.1. Let $w \in C_{[0,T]}$. Let $\beta \in \mathbb{R}$ and $p \in [1, \infty]$, $b \in \mathcal{B}_p^\beta$. The averaging operator is defined by

$$T_t^w b(x) := \lim_{n \rightarrow \infty} T_t^w b^n(x),$$

if the limit exists for any sequence (b^n) of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ and is independent of the choice of the sequence.

Lemma II.4.2. Consider $\beta, \tilde{\beta} \in \mathbb{R}$ with $\beta - \tilde{\beta} \in (0, 1]$, $\gamma \in (0, 1]$, $p \in [1, \infty]$, $b \in \mathcal{B}_p^\beta$ and $w \in C_{[0,T]}$. Assume that w has a local time $L \in C_{[0,T]}^\gamma(\mathcal{B}_{p'}^{-\tilde{\beta}})$. For the above choice of b and w , $T^w b$ is well-defined in $C^\gamma(C^\rho)$ for any $\rho \in (0, \beta - \tilde{\beta})$.

Remark II.4.3. Recall from [11, Prop. 2.76] that for $\eta \in \mathbb{R}$, $p, p', q, q' \in [1, \infty]$ with $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, there is a continuous bilinear functional $\langle \cdot, \cdot \rangle : \mathcal{B}_{p,q}^\eta \times \mathcal{B}_{p',q'}^{-\eta} \rightarrow \mathbb{R}$ extending the L^2 inner product.

Proof of Lemma II.4.2. Let $\varepsilon = \beta - \tilde{\beta}$. After an embedding of Besov spaces (see Remark I.2.2), we know that $b \in \mathcal{B}_p^{\beta-\varepsilon/2}$ and $L \in C_{[0,T]}^\gamma(\mathcal{B}_{p',1}^{-\tilde{\beta}-\varepsilon/2})$. Let $(b^n)_{n \in \mathbb{N}}$ be any sequence of smooth bounded

functions converging to b in $\mathcal{B}_p^{\beta-}$. By the convolutional representation (II.2.9) and Remark II.4.3, we have that

$$T_t^w b^n(x) = \langle b^n, L_t(\cdot - x) \rangle.$$

Using the continuity of the bilinear functional $\langle \cdot, \cdot \rangle$, we get, for $n, m \in \mathbb{N}$,

$$\|T^w b^n(\cdot) - T^w b^m(\cdot)\|_\infty \leq C \|L\|_{\mathcal{C}^\gamma(\mathcal{B}_{p',1}^{-\tilde{\beta}-\varepsilon/2})} \|b^n - b^m\|_{\mathcal{B}_p^{\beta-\varepsilon/2}}.$$

Therefore $T^w b^n$ forms a Cauchy sequence and is uniformly convergent. Hence, $T^w b$ is well defined and is easily seen to be independent of the approximating sequence.

Let $\rho \in (0, \beta - \tilde{\beta})$. In order to show that $T^w b$ actually has the required regularity, we have to check that

$$\begin{aligned} \|T^w b\|_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^\rho)} &= \sup_{s \neq t, x \neq y} \frac{|T_{s,t}^w b(y) - T_{s,t}^w b(x)|}{|t-s|^\gamma |x-y|^\rho} + \sup_{s \neq t} \sup_x \frac{|T_{s,t}^w b(x)|}{|t-s|^\gamma} \\ &\quad + \sup_t \sup_{x \neq y} \frac{|T_t^w b(y) - T_t^w b(x)|}{|x-y|^\rho} + \sup_t \sup_x |T_t^w b(x)| \end{aligned} \quad (\text{II.4.1})$$

is finite. Fix $s \neq t$ and $x \neq y$. For any $n \in \mathbb{N}$,

$$|T_{s,t}^w b(x) - T_{s,t}^w b(y)| \leq 2 \|T_{s,t}^w b(\cdot) - T_{s,t}^w b^n(\cdot)\|_\infty + |T_{s,t}^w b^n(x) - T_{s,t}^w b^n(y)|.$$

Choosing $n = n(s, t, x, y)$ large enough, we have

$$\|T_{s,t}^w b(\cdot) - T_{s,t}^w b^n(\cdot)\|_\infty \leq |t-s|^\gamma |x-y|^{\beta-\tilde{\beta}}.$$

Moreover, using continuity of the bilinear form, an embedding of Besov spaces and Lemma A.1.8(b), we have, for $\bar{\varepsilon} = \beta - \tilde{\beta} - \rho > 0$,

$$\begin{aligned} |T_{s,t}^w b^n(x) - T_{s,t}^w b^n(y)| &\leq C \|b^n\|_{\mathcal{B}_p^{\beta-\bar{\varepsilon}/2}} \|L_{s,t}(\cdot - x) - L_{s,t}(\cdot - y)\|_{\mathcal{B}_{p'}^{-\beta+\bar{\varepsilon}}} \\ &\leq C |t-s|^\gamma |x-y|^\rho \|b\|_{\mathcal{B}_p^\beta} \|L\|_{\mathcal{C}^\gamma(\mathcal{B}_{p'}^{\tilde{\beta}})}. \end{aligned}$$

As the other terms in (II.4.1) can be controlled similarly, the result follows. \square

Lemma II.4.4. *Let $p, \tilde{p} \in [1, \infty]$, $\beta \in \mathbb{R}$, $\eta, \gamma \in (0, 1)$ and $b \in \mathcal{B}_p^\beta$. Assume that the local time L of w satisfies Assumption II.2.13 (I) or (II). Then, for any $\varepsilon \in (0, \eta)$, $T^w b \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})$. Moreover, for any bounded open interval \mathcal{I} that contains the (compact) support of L , there exists a constant $C_{\mathcal{I}}$ such that:*

- If (I) holds, then $[T^w b]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})} \leq C_{\mathcal{I}} \|b\|_{\mathcal{B}_p^\beta} [L]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^{-\beta+\eta+1/p+1/\tilde{p}-1})}$ and $\|T^w b\|_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})} \leq C_{\mathcal{I}} \|b\|_{\mathcal{B}_p^\beta} \|L\|_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^{-\beta+\eta+1/p+1/\tilde{p}-1})}$;
- If (II) holds, then $[T^w b]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})} \leq C_{\mathcal{I}} \|b\|_{\mathcal{B}_p^\beta} [L]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^{-\beta+\eta})}$ and $\|T^w b\|_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})} \leq C_{\mathcal{I}} \|b\|_{\mathcal{B}_p^\beta} \|L\|_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^{-\beta+\eta})}$.

Remark II.4.5. In particular Lemma II.4.4 shows the following: Assume (I) or (II) holds for $\gamma, \eta \in (0, 1)$ with $\gamma + \eta > 1$. Then $T^w b \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\tilde{\eta}})$ for $\tilde{\eta} \in (0, \eta)$ with $\gamma + \tilde{\eta} > 1$.

Proof. The proof follows along the same lines as Lemma II.4.2, making use of several Besov embeddings, including $\mathcal{B}_{\tilde{p}}^{-\beta+\eta+1/p+1/\tilde{p}-1}(\mathcal{I}) \hookrightarrow \mathcal{B}_{p',1}^{-\beta+\eta-\varepsilon}(\mathcal{I})$ (see Remark I.2.3). \square

II.4.2 Path-by-path solutions: Existence and comparison of solutions

In this section we prove Theorem II.2.14 on the existence of path-by-path solutions to Equation (II.1.3) and Theorem II.2.15 on the comparison of solutions.

Proof of Theorem II.2.14. The idea of the proof is to identify a set of full measure on which $T^Z b$ is sufficiently regular and has nonnegative increments. Then, for ω in this set, we can apply the (deterministic) theory of nonlinear Young integral equations developed in Section II.3. In particular, for any such ω , we use Theorem II.3.4 to pick a solution. As the proof of Theorem II.3.4 is non-constructive (it relies on Arzelà-Ascoli's Theorem), the axiom of choice is needed to pick a solution simultaneously for all such ω . For ω outside of this full-measure set, we can define the solution to be identically 0. Note that this construction is indeed done in a path-by-path sense so that the solution solves (II.1.3) on a set of full measure. However, *a priori* there is no reason why the constructed solution should be adapted.

Let $0 \leq s < t$. Provided that

$$T^Z b \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\tilde{\eta}}) \text{ for } \gamma, \tilde{\eta} \in (0, 1) \text{ with } \gamma + \tilde{\eta} > 1 \text{ and} \quad (\text{II.4.2})$$

$$T_{s,t}^Z b(x) \geq 0 \text{ for all } x \in \mathbb{R} \quad (\text{II.4.3})$$

hold a.s., using $\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\tilde{\eta}}) \subset \mathcal{C}_{[0,T]}^{1/\gamma\text{-var}}(\mathcal{C}^{\tilde{\eta}})$, Theorem II.3.4 will give a solution.

Under the assumptions of Theorem II.2.14, Lemma II.4.4 ensures that (II.4.2) holds. To see that (II.4.3) holds, let $\varepsilon \in (0, \eta)$ and $\delta := \eta - \varepsilon$. Both under Assumption II.2.13 (I) or (II), we have that $L_{s,t}^Z \in \mathcal{B}_{p',1}^{-\beta+\delta}$ and $L_{s,t}^Z \in \mathcal{B}_{p'}^{-\beta+2\delta/3}$ by the Besov embeddings in Remark I.2.2 and Remark I.2.3. By Remark II.2.2, the sequence of nonnegative function $\phi_n^x := G_{1/n} L_{s,t}^Z(\cdot - x)$ converges to $L_{s,t}^Z(\cdot - x)$ in $\mathcal{B}_{p'}^{(-\beta+2\delta/3)-}$. Hence

$$\lim_n \|L_{s,t}^Z(\cdot - x) - \phi_n^x\|_{\mathcal{B}_{p',1}^{-\beta}} \leq C \lim_n \|L_{s,t}^Z(\cdot - x) - \phi_n^x\|_{\mathcal{B}_{p'}^{-\beta+\delta/3}} = 0. \quad (\text{II.4.4})$$

Then we get, for any sequence of nonnegative smooth bounded functions b^m converging to b in $\mathcal{B}_p^{\beta-}$,

$$\begin{aligned} T_{s,t}^Z b(x) &= \lim_{m \rightarrow \infty} T_{s,t}^Z b^m(x) \\ &= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \langle b^m, \phi_n^x \rangle, \end{aligned}$$

where the first line holds true by definition and the second line by the continuity of $\langle \cdot, \cdot \rangle$ and (II.4.4). The inequality (II.4.3) now follows from the fact that $\langle b^m, \phi_n^x \rangle \geq 0$, for any n, m and x . \square

Proof of Theorem II.2.15. (a): Let $\varepsilon \in (0, \eta)$ such that $\gamma + (\eta - \varepsilon)/q > 1$. As Assumption II.2.13 (I) or (II) is fulfilled, we know by Lemma II.4.4 that $T^Z b \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})$ on \mathcal{N}^c (which we do not mention in the rest of the proof, although all equalities and convergences happen on this set). Then, by the definition of the nonlinear Young integral of $T^Z b^n$ in Theorem II.3.1, we get that Equality (II.2.10) does hold for b^n :

$$\int_0^t b^n(X_r) dr = \int_0^t T_{dr}^Z b^n(X_r - Z_r). \quad (\text{II.4.5})$$

By Lemma II.4.4, we get that $T^Z b^n$ converges to $T^Z b$ in $C_{[0,T]}^\gamma(\mathcal{C}^{\eta-\varepsilon})$. Hence, by Corollary II.3.2, we obtain that

$$\sup_{t \in [0,T]} \left| \int_0^t b^n(X_r) dr - \int_0^t T_{dr}^Z b(X_r - Z_r) \right| \xrightarrow{n \rightarrow \infty} 0.$$

Therefore the convergence in the statement holds for $K_t := \int_0^t T_{dr}^Z b(X_r - Z_r)$.

(b): By (II.2.13), we know that for some subsequence $(n_k)_{k \in \mathbb{N}}$,

$$\sup_{t \in [0,T]} \left| \int_0^t b^{n_k}(X_r) dr - K_t \right| \longrightarrow 0 \text{ a.s.}$$

We have again that (II.4.5) holds true and we deduce that

$$\sup_{t \in [0,T]} \left| \int_0^t T_{dr}^Z b^{n_k}(X_r - Z_r) - K_t \right| \longrightarrow 0 \text{ a.s.}$$

By Lemma II.4.4 and Corollary II.3.2, we get that almost surely, $K_t = \int_0^t T_{dr}^Z b(X_r - Z_r)$ for all $t \in [0, T]$. Equation (II.2.11) now follows from (II.2.12). \square

II.4.3 Joint regularity of the local time of the fractional Brownian motion

In the rest of this section, L denotes the local time of a one-dimensional fBm B and T^B denotes the averaging operator associated to it, as constructed in Section II.4.1.

First, we recall Theorem 3.1 from [64]. This result is stated in [64] for a compact hypercube with side length equal to one. By dilatation, the result also holds for an arbitrary large hyperrectangle.

Theorem II.4.6 (Th. 3.1 in [64]). *Consider the rectangle $\mathcal{R} = [m_x, M_x] \times [m_y, M_y]$ for some $m_x < M_x$ and $m_y < M_y$. Let $Y : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous stochastic process. Suppose that for $\gamma, \beta > 0$, $\alpha > 1$ and all $(x_1, y_1), (x_2, y_2) \in \mathcal{R}$,*

$$\mathbb{E} [|Y(x_2, y_2) - Y(x_2, y_1) - Y(x_1, y_2) + Y(x_1, y_1)|^\alpha] \leq K |x_2 - x_1|^{1+\gamma} |y_2 - y_1|^{1+\beta}.$$

Then for every $\varepsilon_1, \varepsilon_2$ with $0 < \varepsilon_1 \alpha < \gamma$ and $0 < \varepsilon_2 \alpha < \beta$, there exists a random variable ρ with $\mathbb{E}[\rho^\alpha] \leq K$ and a constant $C = C(\mathcal{R}) > 0$ such that almost surely,

$$|Y(x_2, y_2) - Y(x_2, y_1) - Y(x_1, y_2) + Y(x_1, y_1)| \leq C \rho |x_2 - x_1|^{\gamma/\alpha - \varepsilon_1} |y_2 - y_1|^{\beta/\alpha - \varepsilon_2},$$

for all $(x_1, y_1), (x_2, y_2) \in \mathcal{R}$.

Remark II.4.7. Let $T > 0$ and $n \in \mathbb{N}$. By Lemma 8.12 in [112], for any $n \geq 1$ there exists a constant $C > 0$ such that for any $(x, y) \in \mathbb{R}^2$, $(s, t) \in \Delta_{[0,T]}$ and $0 < \bar{\beta} < (1/(2H) - 1/2) \wedge 1$, the local time L of a one-dimensional fBm fulfills

$$\mathbb{E} [|L_{s,t}(y) - L_{s,t}(x)|^n] \leq C |t - s|^{n(1-H(1+\bar{\beta}))} |x - y|^{n\bar{\beta}}. \quad (\text{II.4.6})$$

Lemma II.4.8. *Let $0 < \bar{\beta} < (1/(2H) - 1/2) \wedge 1$ and $0 < \gamma < 1 - H(1 + \bar{\beta})$. Then, almost surely, $L \in C_{[0,T]}^\gamma(\mathcal{C}^{\bar{\beta}})$. Additionally for any $n \in \mathbb{N}$ there exists a random variable ρ with $\mathbb{E}[\rho^n] < \infty$ such that for any $M > 0$, $(s, t) \in \Delta_{[0,T]}$ and x, y with $|x|, |y| \leq M$, there exists a constant $C_M > 0$ such that*

$$|L_{s,t}(x) - L_{s,t}(y)| \leq C_M \rho |t - s|^\gamma |x - y|^{\bar{\beta}}. \quad (\text{II.4.7})$$

Proof. Note that by [57, Th. 26.1] we can assume L to be jointly continuous in (t, x) . Then choosing n large enough in (II.4.6) and by Theorem II.4.6, there exists a random variable ρ with finite n -th moment such that, a.s., for $(s, t) \in \Delta_{[0, T]}$ and x, y with $|x|, |y| \leq M$,

$$|L_{s,t}(x) - L_{s,t}(y)| \leq C_M \rho |t - s|^\gamma |x - y|^{\bar{\beta}}.$$

As L is a.s. compactly supported, after exhausting $[0, T] \times \mathbb{R}$ with an increasing sequence of compacts $[-M_i, M_i]$, it follows that $L \in \mathcal{C}_{[0, T]}^\gamma(\mathcal{C}^{\bar{\beta}})$ a.s. \square

Note that Lemma II.4.8 does not give differentiability in space for the local time of a fBm, even in case of a small Hurst parameter. It is also possible to get time-space Sobolev regularity of L by following the methodology of [62], and in particular to obtain differentiability for small enough H . This is used in proving Theorem II.2.5 under Assumption (iii).

Remark II.4.9. With the same parameters as in the previous lemma, if $b \in \mathcal{B}_p^\beta$ for $\beta \in (-\bar{\beta}, 1 - \bar{\beta})$ and $p \in [1, \infty]$, then Lemma II.4.4 and Lemma A.1.10 imply that $T^{Bb} \in \mathcal{C}_{[0, T]}^\gamma(\mathcal{C}^{\bar{\beta} + \beta - \varepsilon})$ almost surely, for any $\varepsilon \in (0, \bar{\beta} + \beta)$.

II.4.4 Existence of a weak solution

We first show in Proposition II.4.10 that the assumptions of Theorem II.2.5 imply that either Assumption (I) or Assumption (II) holds. In particular, we get existence of a path-by-path solution by Theorem II.2.14. Then we observe, by ‘‘randomizing’’ the Euler scheme, that it is actually possible to construct a weak solution (i.e. adapted solution).

Proposition II.4.10. *Let β, p, b be as in Theorem II.2.5.*

- (a) *Assume that (i) or (ii) from Theorem II.2.5 holds. Then, a.s., L satisfies Assumption II.2.13 (II) for $\gamma, \eta \in (0, 1)$ with $\gamma + \eta > 1$, $0 < -\beta + \eta < 1$ and $\tilde{p} = \infty$.*
- (b) *Assume that (iii) from Theorem II.2.5 holds. Then, a.s., L satisfies Assumption II.2.13 (II) for $\gamma, \eta \in (0, 1)$ with $\gamma + \eta > 1$, $\eta < (\frac{1}{2H} + \beta - 1/2)$ and $\tilde{p} = 2$.*

Hence, by Theorem II.2.14, whenever one of the conditions (i)-(iii) from Theorem II.2.5 holds, there exists a path-by-path solution to (II.1.3).

Proof of Proposition II.4.10. It is used multiple times throughout the proof that we can also consider b to be in $\mathcal{B}_p^{\tilde{\beta}}$ for any $\tilde{\beta} < \beta$ by an embedding.

(a): Assume that (i) holds. W.l.o.g. assume that $\beta \in (1 + \frac{H}{2} - \frac{1}{2H}, \frac{3}{2} - \frac{1}{2H})$. In view of Lemma II.4.8 and the assumption $H \geq \frac{1}{3}$, $L \in \mathcal{C}_{[0, T]}^\gamma(\mathcal{C}^{-\beta + \eta})$ for $\eta \in \mathbb{R}$ such that $0 < -\beta + \eta < \frac{1}{2H} - \frac{1}{2}$ and $0 < \gamma < 1 - H(1 - \beta + \eta)$. Thus Assumption (II) is fulfilled for $\tilde{p} = \infty$. Choose $\eta = \beta + \frac{1}{2H} - \frac{1}{2} - \varepsilon$ and $\gamma = 1 - H(\frac{1}{2} - \varepsilon + \frac{1}{2H}) - \varepsilon$. Then for small enough ε one gets $\gamma \in (0, 1)$, $\eta \in (0 \vee \beta, 1)$ and $\gamma + \eta > 1$.

Assume that (ii) holds. W.l.o.g. assume that $\beta \in (2H - 1, 0)$. In view of Lemma II.4.8 and the assumption $H < \frac{1}{3}$, we have that $L \in \mathcal{C}_{[0, T]}^\gamma(\mathcal{C}^{-\beta + \eta})$ for $\eta \in \mathbb{R}$ such that $0 < -\beta + \eta < 1$ and $0 < \gamma < 1 - H(1 - \beta + \eta)$. Thus Assumption (II) is fulfilled for $\tilde{p} = \infty$. Choose $\eta = \beta + 1 - \varepsilon$ and $\gamma = 1 - H(2 - \varepsilon) - \varepsilon$. Then again, one gets $\gamma \in (0, 1)$, $\eta \in (0 \vee \beta, 1)$ and $\gamma + \eta > 1$ for small enough ε .

(b): Assume that (iii) holds. By Theorem 3.1 in [62], we know that $L \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{2,2}^\lambda)$ almost surely for $\lambda < \frac{1}{2H} - 1/2$ and $0 < \gamma < 1 - H(\lambda + 1/2)$ ¹. Hence, after a Besov space embedding (see Remark I.2.3), Assumption (II) is fulfilled for $\tilde{p} = 2$, $0 < \eta < (\frac{1}{2H} - 1/2 + \beta) \wedge 1$ and $0 < \gamma < (1 - H(\eta - \beta + 1/2)) \wedge 1$. The assumption $\beta > 1 - \frac{1}{2H}$ ensures that we can choose η and γ such that $\eta + \gamma > 1$. \square

To construct weak solutions, we will proceed with an approximation by an Euler scheme, similarly to the proof of Theorem II.3.4. Although this time, we need the following lemma, which gives tightness and is a crucial step in showing adaptedness by an argument using Skorokhod's representation Theorem.

Lemma II.4.11. *Assume that one of the assumptions (i)-(iii) in Theorem II.2.5 holds. Then there exist $\gamma, \tilde{\eta} \in (0, 1)$ with $\gamma + \tilde{\eta} > 1$ such that for any $\lambda > 0$,*

$$\lim_{\delta \rightarrow 0} \mathbb{P}(\Omega_{\delta, \lambda}) = 1, \text{ where } \Omega_{\delta, \lambda} := \left\{ \omega : \sup_{|t-s| < \delta} [T^B b]_{\mathcal{C}_{[s,t]}^{1/\gamma\text{-var}}(\mathcal{C}^{\tilde{\eta}})} < \lambda \right\}.$$

Proof. In view of Proposition II.4.10, L satisfies either Assumption II.2.13 (I) or (II) for some $\gamma, \eta \in (0, 1)$ such that $\gamma + \eta > 1$. Hence by Lemma II.4.4, there exists $\tilde{\eta} < \eta$ such that $\tilde{\eta} + \gamma > 1$ and $T^B b \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\tilde{\eta}})$.

For $M > 0$, define $L^M := \mathbb{1}_{\Omega_M} L$, where $\Omega_M = \{\omega : \sup_{t \in [0,T]} |B_t| \leq M\}$. Then,

$$\mathbb{P}(\Omega_{\delta, \lambda}^c) \leq \mathbb{P}\left(\left\{ \omega : \delta^\gamma \sup_{|t-s| < \delta} [T^B b]_{\mathcal{C}_{[s,t]}^\gamma(\mathcal{C}^{\tilde{\eta}})} \geq \lambda \right\} \cap \Omega_M\right) + \mathbb{P}(\Omega_M^c), \quad (\text{II.4.8})$$

as $[T^B b]_{\mathcal{C}_{[s,t]}^{1/\gamma\text{-var}}(\mathcal{C}^{\tilde{\eta}})} \leq |t-s|^\gamma [T^B b]_{\mathcal{C}_{[s,t]}^\gamma(\mathcal{C}^{\tilde{\eta}})}$.

We will distinguish two cases, depending on whether (i) or (ii) is satisfied (first case), or (iii) is satisfied (second case).

First case. Assume that (i) or (ii) in Theorem II.2.5 holds true. Then by Proposition II.4.10(a), $L \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{-\beta+\eta})$ with $0 < -\beta + \eta < 1$ and by Lemma II.4.8 $\mathbb{E}[[L^M]_{\mathcal{C}_{[s,t]}^\gamma(\mathcal{C}^{-\beta+\eta})}^m]$ is finite for any $m \geq 1$. Moreover, by Lemma II.4.4, we have on Ω_M that $[T^B b]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\tilde{\eta}})} \leq C_M [L]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{-\beta+\eta})}$, where C_M depends on M (but not on the realization ω). Hence, we can bound the right hand side of (II.4.8) from above by

$$\begin{aligned} \forall m \geq 1, \quad & \mathbb{P}\left(\left\{ \omega : \delta^\gamma \sup_{|t-s| < \delta} C_M [L^M]_{\mathcal{C}_{[s,t]}^\gamma(\mathcal{C}^{-\beta+\eta})} \geq \lambda \right\}\right) + \mathbb{P}(\Omega_M^c) \\ & \leq C_M^m \lambda^{-m} \delta^{m\gamma} \mathbb{E}[[L^M]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{-\beta+\eta})}^m] + \mathbb{P}(\Omega_M^c). \\ & \leq C(m, M) \lambda^{-m} \delta^{m\gamma} + \mathbb{P}(\Omega_M^c). \end{aligned} \quad (\text{II.4.9})$$

Let $\varepsilon > 0$. By Fernique's theorem, $\|B\|_\infty$ has exponential moments and we can choose M such that $\mathbb{P}(\Omega_M^c) < \varepsilon/2$. Then δ can be chosen such that the other term in (II.4.9) can also be controlled by $\varepsilon/2$.

¹Actually in [62] they use a Bessel space instead of $\mathcal{B}_{2,2}^\lambda$, but by Proposition 2.(iii) on page 47 and the Theorem on page 88 in [109], this is equivalent.

Second case. Now assume that (iii) in Theorem II.2.5 holds. By Proposition II.4.10(b), L satisfies Assumption II.2.13 (II) with $\gamma < (1 - H(-\beta + \eta + 1/2)) \wedge 1$, $\tilde{p} = 2$ and $\eta < (\frac{1}{2H} - 1/2 + \beta) \wedge 1$. In view of Lemma II.4.4 and an embedding of Besov spaces (see Remark I.2.2), we have that on Ω_M ,

$$[T^B b]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{C}^{\tilde{\eta}})} \leq C_M [L]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{2,2}^{-\beta+\eta})} \quad (\text{II.4.10})$$

for some constant $C_M > 0$ that depends on M but not on the realization ω . Next, we use the chain of inequalities on page 12 in [62] setting λ' therein equal to $-\beta + \eta + 1/2 + \varepsilon$. To check the condition $\lambda' < 1/(2H)$ appearing in [62], note that $-\beta + \eta + 1/2 + \varepsilon < 1/(2H)$ for ε small enough. Hence, we get for any $m \geq 2$ and some $\tilde{\varepsilon} > 0$,

$$\mathbb{E}[\|L_{u,v}^M\|_{\mathcal{B}_{2,2}^{-\beta+\eta}}^m] \leq C |u - v|^{m(1-H(-\beta+\eta+1/2+\varepsilon))} \leq C |u - v|^{m(\gamma+\tilde{\varepsilon})}.$$

An application of Kolmogorov's continuity theorem for Banach-valued stochastic processes (see [105, Th. 4.3.2]) gives that $\mathbb{E}[[L^M]_{\mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{2,2}^{-\beta+\eta})}^m]$ is finite. Using (II.4.10), we deduce as in the first case (see (II.4.9)) that

$$\mathbb{P}(\Omega_{\delta,\lambda}^c) \leq C(m, M) \lambda^{-m} \delta^{m\gamma} + \mathbb{P}(\Omega_M^c),$$

for a constant $C(m, M)$ only depending on m and M . We can make the right-hand side arbitrarily small by choosing M large enough and δ small enough. \square

Proof of Theorem II.2.5. Proof of (a): By Proposition II.4.10, Assumption II.2.13 (I) or (II) is fulfilled and thus there are some $\gamma \in (0, 1)$ and $\tilde{p} \in [1, \infty]$ such that $L \in \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^\xi)$ a.s., for some ξ , with its precise value given in Assumption II.2.13 (I) or (II). Besides it follows by Remark II.4.5 that $T^B b \in \mathcal{C}^\gamma(\mathcal{C}^{\tilde{\eta}})$ for some $\tilde{\eta} \in (0, 1)$ such that $\gamma + \tilde{\eta} > 1$.

W.l.o.g. let $T = 1$. First, we use the Euler scheme as in the proof of Theorem II.3.4 in order to construct a measurable solution. Let X^n be the random counterpart of x^n in the proof of Theorem II.3.4 for $A = T^B b$, $p = 1/\gamma$ and $\theta := \gamma + \tilde{\eta}$. Let $\nu > 0$. The computations done in the proof of Theorem II.3.4 hold for almost every $\omega \in \Omega$ until equation (II.3.7). Choose $\lambda \in (0, 1)$ such that $2\lambda C(\theta) < 1$ for $C(\theta)$ as in (II.3.7). Let δ such that $\mathbb{P}(\Omega_{\delta,\lambda}) \geq 1 - \nu$, which is possible by Lemma II.4.11. Then one can choose $N = N(\delta)$ large enough so that for any $n \geq N$ and any $\omega \in \Omega_{\delta,\lambda}$,

$$\begin{aligned} [X^n]_{\mathcal{C}_{[s,t]}^{1-\text{var}}} &\leq 2C(\theta) \frac{[T^B b]_{\mathcal{C}_{[s,t]}^{1/\gamma-\text{var}}(\mathcal{C}^{\tilde{\eta}})} + \lambda}{1 - 2C(\theta)[T^B b]_{\mathcal{C}_{[s,t]}^{1/\gamma-\text{var}}(\mathcal{C}^{\tilde{\eta}})}} \\ &\leq 2C(\theta) \frac{2\lambda}{1 - 2C(\theta)\lambda}, \quad \forall (s, t) \in \Delta_{[0,1]} \text{ with } |t - s| < \delta. \end{aligned}$$

It follows that we can choose M sufficiently large so that for any $n \geq N$,

$$\mathbb{P}(\|X^n\|_\infty > M) \leq \mathbb{P}(\Omega_{\delta,\lambda} \cap \{\|X^n\|_\infty > M\}) + \mathbb{P}(\Omega_{\delta,\lambda}^c) < \nu.$$

Therefore the sequence $(\text{Law}(X^n))_{n \in \mathbb{N}}$ is tight in the space of continuous functions. Hence, along some subsequence that we do not relabel, (X^n, B, L) converges in law in $\mathcal{C}_{[0,T]} \times \mathcal{C}_{[0,T]} \times \mathcal{C}_{[0,T]}^\gamma(\mathcal{B}_{\tilde{p}}^\xi)$ to

some $(X, \tilde{B}, \tilde{L})$. By Skorokhod's representation Theorem, there exists a sequence $(Y^n, B^n, L^{B^n})_{n \in \mathbb{N}}$ with $\text{Law}(Y^n, B^n, L^{B^n}) = \text{Law}(X^n, B, L)$ for all $n \in \mathbb{N}$, such that $(Y^n, B^n, L^{B^n})_{n \in \mathbb{N}}$ converges a.s. to some (Y, \hat{B}, \hat{L}) . To get that \hat{L} is the local time of \hat{B} , observe that for any bounded measurable function f ,

$$\begin{aligned} 1 &= \mathbb{P}\left(\left\{\omega : \forall t \in [0, T], \int_0^t f(B_r) dr = \int_{\mathbb{R}} f(x) L_t(x) dx\right\}\right) \\ &= \hat{\mathbb{P}}\left(\left\{\omega : \forall t \in [0, T], \int_0^t f(\hat{B}_r) dr = \int_{\mathbb{R}} f(x) \hat{L}_t(x) dx\right\}\right). \end{aligned} \quad (\text{II.4.11})$$

As the local time of a fBm is characterised by the occupation time formula, we deduce that \hat{L} is the local time of \hat{B} . By (II.3.5), we have that for $T^B b(x) = \langle b, L^B(\cdot - x) \rangle$,

$$\left| X_t^n - X_0 - \int_0^t T_{dr}^B b(X_r^n) \right| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0.$$

Hence for $T^{B^n} b(x) = \langle b, L^{B^n}(\cdot - x) \rangle$,

$$\left| Y_t^n - X_0 - \int_0^t T_{dr}^{B^n} b(Y_r^n) \right| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

and therefore a.s. along a subsequence, which we do not relabel. Hence, using that $\hat{L} \in \mathcal{C}_{[0, T]}^\gamma(\mathcal{B}_{\hat{p}}^\xi)$, we get by Remark II.4.5 and Lemma II.3.3 that for $T^{\hat{B}} b(x) = \langle b, \hat{L}(\cdot - x) \rangle$,

$$Y_t = X_0 + \int_0^t T_{dr}^{\hat{B}} b(Y_r), \text{ for all } t \in [0, T] \text{ a.s.}$$

In order for Y to be a weak solution, it remains to show that Y is adapted to a filtration $\hat{\mathbb{F}}$ such that \hat{B} is an $\hat{\mathbb{F}}$ -fBm. First note that by construction X^n is adapted to \mathbb{F}^B . Hence, Y^n is \mathbb{F}^{B^n} measurable as

$$\text{Law}(Y^n, B^n, L^{B^n}) = \text{Law}(X^n, B, L).$$

Therefore B^n is an $\hat{\mathbb{F}}_t^n$ -fBm for $\hat{\mathcal{F}}_t^n := \sigma(Y_s^n, B_s^n, s \in [0, t])$. By definition this implies that, for $s < t$, $W_t^n - W_s^n = \mathcal{A}(B^n)_t - \mathcal{A}(B^n)_s$ is independent of $\hat{\mathcal{F}}_s^n$. After passing to the limit and using Lemma II.A.1 we infer that $W_t - W_s = \mathcal{A}(\hat{B})_t - \mathcal{A}(\hat{B})_s$ is independent of $\hat{\mathcal{F}}_s := \sigma(Y_r, \hat{B}_r, r \in [0, s])$. Hence, $\hat{B} = \bar{\mathcal{A}}(W)$ is an $\hat{\mathbb{F}}$ -fBm and therefore (Y, \hat{B}) is a weak solution as Y is clearly $\hat{\mathbb{F}}$ -adapted.

Proof of (b): this proof will be presented at the end of Section II.5 as it follows from the stochastic sewing arguments developed in the next section. \square

Proof of Corollary II.2.6. Any finite measure lies in \mathcal{B}_1^0 (see Remark A.1.4). Hence, if b is a measure, we can choose $\beta = 0$ in Theorem II.2.5. For $H < 1/3$ condition (i) therein is clearly fulfilled and condition (ii) is fulfilled if $H^2 + 2H - 1 < 0$, giving existence of a weak solution for $H < \sqrt{2} - 1$. \square

II.5 Regularity of weak solutions

We first state Lemma II.5.1 which establishes various regularity estimates on the conditional expectation of fractional Brownian motion. It is an extension to the fBm of [8, Lemma C.4] (which was

for standard Brownian motion). It is used several times in the remainder of this chapter and its proof is postponed to Section II.A.2. In particular, the proof of Lemma II.5.1(d) relies on a variant of local nondeterminism of the fBm, see Lemma II.A.2. Note that Lemma II.5.1(a) was already stated and proven in Proposition 3.6(iii) of [18].

Lemma II.5.1. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and B be an \mathbb{F} -fBm. Let $\gamma < 0$ and $p \in [1, \infty]$. Let $d \in \mathbb{N}$, $(t_1, t_2) \in \Delta_{[0, T]}$ and $f : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a bounded measurable function and Ξ be an \mathcal{F}_{t_1} -measurable \mathbb{R}^d -valued random variable. Assume that $\|f(\cdot, \Xi)\|_{\mathcal{C}^1} < \infty$ almost surely. Then there exists a constant $C > 0$ such that*

$$(a) \quad \mathbb{E}^{t_1}[f(B_{t_2}, \Xi)] = \int_{\mathbb{R}} g_{\sigma_{t_1, t_2}^2}(x) f(\mathbb{E}^{t_1}[B_{t_2}] - x, \Xi) dx, \text{ alternatively also written } \mathbb{E}^{t_1}[f(B_{t_2}, \Xi)] = G_{\sigma_{t_1, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}], \Xi), \text{ where } g \text{ is the Gaussian density and } G \text{ is the Gaussian semigroup introduced in (I.2.10) and } \sigma_{t_1, t_2}^2 := \text{Var}(B_{t_2} - \mathbb{E}^{t_1}[B_{t_2}]);$$

$$(b) \quad |\mathbb{E}^{t_1}[f(B_{t_2}, \Xi)]| \leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} (t_2 - t_1)^{H(\gamma-1/p)};$$

$$(c) \quad \|f(B_{t_2}, \Xi) - \mathbb{E}^{t_1}[f(B_{t_2}, \Xi)]\|_{L^1} \leq C \| \|f(\cdot, \Xi)\|_{\mathcal{C}^1} \|_{L^2} (t_2 - t_1)^H.$$

Furthermore, for $n \in [1, p]$, there exists a constant $C > 0$ such that for any \tilde{t} in the interval (t_1, t_2) ,

$$(d) \quad \|\mathbb{E}^{\tilde{t}}[f(B_{t_2}, \Xi)]\|_{L^n} \leq C \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (t_2 - \tilde{t})^{H\gamma} (\tilde{t} - t_1)^{-\frac{1}{2p}} (t_2 - t_1)^{\frac{1-2H}{2p}}.$$

Remark II.5.2. In this section we assume that $b \in \mathcal{B}_p^\beta$ for $p \in [1, \infty]$ and $\beta \in \mathbb{R}$ with $\beta - 1/p > -1/(2H)$. Note that this condition allows negative values of β for any $H < \frac{1}{2}$. In the proofs of this section, we consider the case $\beta < 0$ and $p \geq m$ for some $m \geq 2$. Indeed, it is always possible to come back to these cases in the following way: If $\beta \geq 0$, $p \in [1, \infty]$ and $m \geq 2$, there exist $\tilde{\beta} < 0$, $\tilde{p} \geq m$ fulfilling $\tilde{\beta} - 1/\tilde{p} > -1/(2H)$ such that $\mathcal{B}_p^\beta \hookrightarrow \mathcal{B}_{\tilde{p}}^{\tilde{\beta}}$. This can be seen using the embeddings $\mathcal{B}_p^\beta \hookrightarrow \mathcal{B}_{\tilde{p}}^{\beta - (\frac{1}{p} - \frac{1}{\tilde{p}})}$ (see Remark I.2.2) and $\mathcal{B}_p^s \hookrightarrow \mathcal{B}_{\tilde{p}}^{\tilde{s}}$ for $s > \tilde{s}$.

The following proposition ensures smoothness of $X - B$ for any solution X of (II.1.1).

Proposition II.5.3. *Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$ with $0 > \beta - 1/p > -1/(2H)$. Suppose that $b \in \mathcal{B}_p^\beta$ is a measure. Then every weak solution X to (II.1.1) fulfills $X - B \in \mathcal{C}_{[0, T]}^{1+H(\beta-1/p)}(L^m)$, for any $m \geq 2$.*

Thus if X is a solution to (II.1.1) with b a finite measure, we deduce that $X - B \in \mathcal{C}_{[0, T]}^{1-H}(L^m)$, for any $m \geq 2$. It will imply Theorem II.2.5(b) and is the main step in the proof of Theorem II.2.10(d).

Proof of Proposition II.5.3. Let X be a weak solution to (II.1.1) and $m \geq 2$. W.l.o.g. we assume that $X_0 = 0$. We choose a sequence $(b^k)_{k \in \mathbb{N}}$ of smooth nonnegative bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ with $\|b^k\|_{\mathcal{B}_p^\beta} \leq \|b\|_{\mathcal{B}_p^\beta}$ for all k . For $k \in \mathbb{N}$ and $t \in [0, T]$, let

$$K_t^k := \int_0^t b^k(X_r) dr = \int_0^t b^k(B_r + K_r) dr.$$

By Definition II.2.4 of a weak solution, we know that K^k converges in probability to K with respect to $\|\cdot\|_\infty$ on $[0, T]$. Hence it also converges almost surely, up to passing to a subsequence

(without loss of generality, we do not relabel K^k and assume it converges a.s.). By nonnegativity of b^k , K^k is monotone for all k , and therefore K is monotone as well. For any $(s, t) \in \Delta_{[0, T]}$, let

$$A_{s,t}^k := \int_s^t b^k(B_r + K_s) dr.$$

We now want to apply the stochastic sewing Lemma with random controls (see Lemma A.2.2) for $K^k = \mathcal{A}^k$. In order to see that all conditions of this theorem are fulfilled (i.e. (A.2.2) and (A.2.4)), we will show the following for a constant C that is independent of k and for any $u \in (s, t)$,

$$(i) \quad \|A_{s,t}^k\|_{L^m} \leq C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)};$$

$$(ii) \quad |\mathbb{E}^u[\delta A_{s,u,t}^k]| \leq C \|b\|_{\mathcal{B}_p^\beta} |K_u - K_s| (t-u)^{H(\beta-1/p-1)+1};$$

(iii) Identifying K^k with the limit of $\sum_{i=0}^{N_n-1} A_{t_i^n, t_{i+1}^n}^k$ along any sequence of partitions $\Pi_n = \{t_i^n\}_{i=0}^{N_n}$ of $[0, t]$ with mesh converging to 0.

Notice that (i) gives (A.2.2) for $n = m$ and that $1 + H(\beta - 1/p) > 1/2$. Furthermore, (ii) gives condition (A.2.4) for $\alpha_1 = H(\beta - 1/p - 1) + 1 > 0$, $\beta_1 = 1$ and $\lambda(s, t) := K_t - K_s$, which is a random control function by monotonicity of K .

Assume, for the moment, that (i)-(ii)-(iii) hold true. Then by Lemma A.2.2, there exists a process D^k such that

$$|K_t^k - K_s^k - A_{s,t}^k| \leq C \|b\|_{\mathcal{B}_p^\beta} (K_t - K_s) (t-s)^{H(\beta-1/p-1)+1} + D_{s,t}^k, \quad (\text{II.5.1})$$

with $\|D_{s,t}^k\|_{L^m} \leq C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)}$. Hence, by (i),

$$\|K_t^k - K_s^k\|_{L^m} \leq C \|b\|_{\mathcal{B}_p^\beta} \|K_t - K_s\|_{L^m} (t-s)^{H(\beta-1/p-1)+1} + C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)}.$$

Hence, after letting k go to ∞ in the previous inequality, we obtain that for $(s, t) \in \Delta_{[0, T]}$ such that $C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{H(\beta-1/p-1)+1} < 1/2$,

$$\|K_t - K_s\|_{L^m} \leq C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)}.$$

After covering $[0, T]$ with a finite number of small enough intervals, we obtain the result.

Let us now verify (i)-(ii)-(iii).

Proof of (i): By Lemma II.B.1 applied to $\Xi = K_s$, $f(z, x) = b^k(z + x)$ and $n = m$, we have

$$\|A_{s,t}^k\|_{L^m} \leq C \|b^k(\cdot + K_s)\|_{\mathcal{B}_p^\beta} \|L^m\| (t-s)^{1+H(\beta-1/p)}.$$

Using Lemma A.1.8(a), we thus get

$$\begin{aligned} \|A_{s,t}^k\|_{L^m} &\leq C \|b^k\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)} \\ &\leq C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)}. \end{aligned} \quad (\text{II.5.2})$$

Proof of (ii): By Lemma II.5.1(a) applied to $\Xi = (K_s, K_u)$ and $f(z, (x_1, x_2)) = b^k(z + x_1) - b^k(z + x_2)$, we get

$$|\mathbb{E}^u[\delta A_{s,u,t}^k]| = \left| \int_u^t \mathbb{E}^u[b^k(B_r + K_s) - b^k(B_r + K_u)]dr \right| \leq \int_u^t \|G_{\sigma_{u,r}^2} b^k\|_{\mathcal{C}^1} |K_u - K_s| dr,$$

where we recall $\sigma_{u,r}^2 = \text{Var}(B_r - \mathbb{E}^u[B_r])$. From (II.A.4), we have that $\sigma_{u,r}^2 = C(u-r)^{2H}$. Apply now Lemma A.1.7(d), which is possible as $\beta - 1/p < 0$, and then use that $\beta - 1/p > -1/2H$ to ensure integrability. This gives

$$\begin{aligned} |\mathbb{E}^u[\delta A_{s,u,t}^k]| &\leq C \int_u^t |r-u|^{H(\beta-1/p-1)} \|b^k\|_{\mathcal{B}_p^\beta} |K_u - K_s| dr \\ &\leq C \|b\|_{\mathcal{B}_p^\beta} |K_u - K_s| (t-u)^{H(\beta-1/p-1)+1}. \end{aligned}$$

Proof of (iii): For a sequence $\Pi_n = \{t_i^n\}_{i=0}^{N_n}$ of partitions of $[0, t]$ with mesh size going to 0, we have

$$\begin{aligned} |K_t^k - \sum_{i=0}^{N_n-1} A_{t_i^n, t_{i+1}^n}^k| &\leq \sum_i \int_{t_i^n}^{t_{i+1}^n} |b^k(B_r + K_r) - b^k(B_r + K_{t_i^n})| dr \\ &\leq \sum_i \int_{t_i^n}^{t_{i+1}^n} \|b^k\|_{\mathcal{C}^1} |K_r - K_{t_i^n}| dr \\ &\leq \sum_i \|b^k\|_{\mathcal{C}^1} (t_{i+1}^n - t_i^n) |K_{t_{i+1}^n} - K_{t_i^n}| \\ &\leq \|b^k\|_{\mathcal{C}^1} |\Pi_n| (K_t - K_0) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0. \end{aligned}$$

□

We conclude this section with the proof of Theorem II.2.5(b).

Proof of Theorem II.2.5(b). It is a direct consequence of Proposition II.5.3. More precisely, in order to apply Proposition II.5.3, we check that $\beta - 1/p > -1/(2H)$ holds in all cases (i)-(iii). In case (i) holds, the assumption $\beta - 1 > -1/(2H) + H/2$ and $p \geq 1$ yields $\beta - 1/p > -1/(2H)$. In case (ii) holds, then $\beta - 1/p > 2H - 2$ and $2H - 2 > -1/(2H)$ for $H < 1/3$. Finally, in case (iii) the result follows from $p \geq 1$. □

II.6 Uniqueness

In this section we state and prove Proposition II.6.1, which gives pathwise uniqueness to (II.1.1) among all weak solutions X fulfilling $X - B \in \mathcal{C}_{[0,T]}^{1/2+H}(L^2)$. This is a crucial step towards the proof of the uniqueness part in Theorem II.2.10. Note that we do not assume anymore that b is a measure. The scheme of proof of Proposition II.6.1, which is briefly described in Section II.2.4, is inspired by the proof of Proposition 2.1 in [34] and closely follows the steps of the proof of Proposition 3.6 in [8]. More precisely, we establish that if two solutions X and Y are such that $X - B$ and $Y - B$ are $(\frac{1}{2} + H)$ -Hölder continuous, then their difference Z satisfies inequality (II.6.9), and since any continuous function which satisfies such an inequality must be 0, the uniqueness follows. The most

technical part is to establish (II.6.9). This relies on variations of the stochastic sewing Lemma and results based thereon in Sections II.6.1, II.6.2.

Even when it is not explicitly written, we assume in the whole section that $H < 1/2$.

In the rest of the chapter we need the following definition. Let $0 \leq s \leq t$. Let $\psi : [s, t] \times \Omega \rightarrow \mathbb{R}$ be a stochastic process. For $\alpha \in (0, 1]$ and $m, n \in [1, \infty]$, define

$$[\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,n})} := \sup_{(u,v) \in \Delta_{[s,t]}} \frac{\|\mathbb{E}^u[|\psi_v - \psi_u|^m]^{\frac{1}{m}}\|_{L^n}}{(v-u)^\alpha}. \quad (\text{II.6.1})$$

where the conditional expectation is taken w.r.t. the filtration the space is equipped with. By the tower property and Jensen's inequality for conditional expectation, we know that, for $1 \leq m \leq n \leq \infty$,

$$[\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^m)} = [\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,m})} \leq [\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,n})} \leq [\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^n)}. \quad (\text{II.6.2})$$

Proposition II.6.1. *Let $H < 1/2$. Let $\beta \in \mathbb{R}$, $p \in [1, \infty]$ such that (II.2.6) holds. Let $b \in \mathcal{B}_p^\beta$, and $(X_t)_{t \in [0, T]}$ and $(Y_t)_{t \in [0, T]}$ be two weak solutions to (II.1.1) in the sense of Definition II.2.4 with the same initial condition X_0 , both being defined on the same probability space and adapted to the same filtration \mathbb{F} . Suppose that $(X - B)$ and $(Y - B)$ are in $\mathcal{C}_{[0, T]}^{1/2+H}(L^2)$. Further assume that*

$$[X - B]_{\mathcal{C}_{[0, T]}^{1/2+H}(L^{2, \infty})} < \infty \quad a.s. \quad (\text{II.6.3})$$

Then X and Y are indistinguishable.

Remark II.6.2. In the above proposition, note that condition (II.2.6) allows negative values of β for any $H < \frac{1}{2}$. In the proofs in this section, we will thus only consider the case $\beta < 0$ and $p \geq 2$. Indeed, it is always possible to come back to these cases in the following way: If $\beta \geq 0$ and $p \in [1, \infty]$, there exist $\tilde{\beta} < 0$, $\tilde{p} \geq 2$ fulfilling (II.2.6) such that $\mathcal{B}_p^\beta \hookrightarrow \mathcal{B}_{\tilde{p}}^{\tilde{\beta}}$. This can be seen using the embeddings $\mathcal{B}_p^\beta \hookrightarrow \mathcal{B}_{\tilde{p}}^{\beta - (\frac{1}{p} - \frac{1}{\tilde{p}})}$ (see Remark I.2.2) and $\mathcal{B}_p^s \hookrightarrow \mathcal{B}_{\tilde{p}}^s$ for $s > \tilde{s}$.

II.6.1 Uniqueness: Proof of Proposition II.6.1

For the proof of Proposition II.6.1, which is detailed at the end of this subsection, we will use Lemma II.6.3, Lemma II.6.4 and Lemma II.6.5. From now on, assume w.l.o.g. that $X_0 = 0$. Let $K := X - B$ and $\tilde{K} := Y - B$. Let $Z := X - Y = K - \tilde{K}$.

Lemma II.6.3. *Let the assumptions of Proposition II.6.1 hold. Let $(b^n)_{n \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ with $\sup_n \|b^n\|_{\mathcal{B}_p^\beta} \leq \|b\|_{\mathcal{B}_p^\beta}$. Then for any $(s, t) \in \Delta_{[0, T]}$, there exist random variables $\tilde{T}_{s,t}^K$ and $\tilde{T}_{s,t}^{\tilde{K}}$ such that*

$$\int_s^t b^n(B_r + K_s) dr \xrightarrow[n \rightarrow \infty]{L^2} \tilde{T}_{s,t}^K \quad \text{and} \quad \int_s^t b^n(B_r + \tilde{K}_s) dr \xrightarrow[n \rightarrow \infty]{L^2} \tilde{T}_{s,t}^{\tilde{K}}. \quad (\text{II.6.4})$$

Moreover, there exists $C > 0$ such that, for any $(s, t) \in \Delta_{[0, T]}$, we have that

$$\|\tilde{T}_{s,t}^K - \tilde{T}_{s,t}^{\tilde{K}}\|_{L^2} \leq C \|b\|_{\mathcal{B}_p^\beta} \|Z_s\|_{L^2} (t-s)^{1/2} \quad \text{and} \quad (\text{II.6.5})$$

$$\|\tilde{T}_{s,t}^K - \tilde{T}_{s,t}^{\tilde{K}}\|_{L^2} \leq C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1/2+H}. \quad (\text{II.6.6})$$

The proof of this lemma is moved to Section II.6.2.

For $(s, t) \in \Delta_{[0, T]}$, let

$$R_{s,t} := (K_t - K_s - \tilde{T}_{s,t}^K) - (\tilde{K}_t - \tilde{K}_s - \tilde{T}_{s,t}^{\tilde{K}}).$$

It is now necessary to estimate the regularity of this remainder term. The proof of the following lemma, which heavily relies on stochastic sewing, is moved to Section II.6.2.

Lemma II.6.4. *Let the assumptions of Proposition II.6.1 hold. There exists $\delta > 0$ such that for any $\alpha \in (1/2, 1)$, there exists $C > 0$ for which the following holds: for all $(u, v) \in \Delta_{[0, T]}$, we have*

$$\begin{aligned} \|Z_v - Z_u\|_{L^2} &\leq C \|R\|_{C_{[u,v]}^\alpha(L^2)} (v - u)^{1/2+\delta} \\ &\quad + C \|Z\|_{C_{[u,v]}(L^2)} |\log \|Z\|_{C_{[u,v]}(L^2)}| (v - u) + C \|Z\|_{C_{[u,v]}(L^2)} (v - u)^{1/2}, \end{aligned} \quad (\text{II.6.7})$$

$$\begin{aligned} \|R_{u,v}\|_{L^2} &\leq C \left(\|Z\|_{C_{[u,v]}(L^2)} + \|R\|_{C_{[u,v]}^\alpha(L^2)} \right) (v - u)^{1/2+\delta} \\ &\quad + C \|Z\|_{C_{[u,v]}(L^2)} |\log \|Z\|_{C_{[u,v]}(L^2)}| (v - u). \end{aligned} \quad (\text{II.6.8})$$

One can now deduce a regularity estimate on Z alone.

Lemma II.6.5. *Let the assumptions of Proposition II.6.1 hold. Let for $\tilde{T} \in (0, T]$ such that $\|Z\|_{C_{[0, \tilde{T}]}(L^2)} \leq 1/e$, there exist $C > 0$ and $l > 0$ such that for any $(s, t) \in \Delta_{[0, \tilde{T}]}$ with $t - s < l$, we have*

$$\|Z_t - Z_s\|_{L^2} \leq C \|Z\|_{C_{[s,t]}(L^2)} (t - s)^{1/2} + C \|Z\|_{C_{[s,t]}(L^2)} |\log \|Z\|_{C_{[s,t]}(L^2)}| (t - s). \quad (\text{II.6.9})$$

Proof. Consider δ as in Lemma II.6.4 and assume w.l.o.g. that $\delta < 2H$. By assumption on X and Y and by (II.6.6) we know that $\|R\|_{C_{[0, T]}^{1/2+H}(L^2)} < \infty$. Let $(s, t) \in \Delta_{[0, \tilde{T}]}$. After dividing both sides of Equation (II.6.8) by $(v - u)^{1/2+\delta/2}$ and taking the supremum over all $(u, v) \in \Delta_{[s,t]}$, we obtain for $\alpha = 1/2 + \delta/2$ that

$$\begin{aligned} \|R\|_{C_{[s,t]}^\alpha(L^2)} &\leq C \left(\|Z\|_{C_{[s,t]}(L^2)} + \|R\|_{C_{[s,t]}^\alpha(L^2)} \right) (t - s)^{\delta/2} \\ &\quad + C \|Z\|_{C_{[s,t]}(L^2)} |\log \|Z\|_{C_{[s,t]}(L^2)}| (t - s)^{1/2-\delta/2}. \end{aligned} \quad (\text{II.6.10})$$

In the above we used that

$$\sup_{(u,v) \in \Delta_{[s,t]}} \|Z\|_{C_{[u,v]}(L^2)} |\log \|Z\|_{C_{[u,v]}(L^2)}| \leq \|Z\|_{C_{[s,t]}(L^2)} |\log \|Z\|_{C_{[s,t]}(L^2)}|$$

as $f(x) = x \log(1/x)$ is increasing on $[0, 1/e]$ and $\|Z\|_{C_{[s,t]}(L^2)} \leq 1/e$ by assumption. Using $\delta < 2H$, we have that $\|R\|_{C_{[0, T]}^\alpha(L^2)} \leq C \|R\|_{C_{[0, T]}^{1/2+H}(L^2)} < \infty$. Let $l \in (0, \tilde{T})$ be such that $Cl^{\delta/2} < 1/2$ for C as in (II.6.10). For $t - s < l$, we have

$$\begin{aligned} \frac{1}{2} \|R\|_{C_{[s,t]}^\alpha(L^2)} &\leq C \|Z\|_{C_{[s,t]}(L^2)} (t - s)^{\delta/2} \\ &\quad + C \|Z\|_{C_{[s,t]}(L^2)} |\log \|Z\|_{C_{[s,t]}(L^2)}| (t - s)^{1/2-\delta/2}. \end{aligned}$$

Plugging this into (II.6.7) for $u = s$ and $v = t$ finishes the proof. \square

Proof of Proposition II.6.1. Let $\tilde{T} := \sup\{t \in [0, T] : \sup_{0 \leq s \leq t} \|Z_s\|_{L^2} \leq 1/e\}$. We show in the next paragraph that Z is indistinguishable from 0 on $[0, \tilde{T}]$. Note that this implies that $\tilde{T} = T$: Indeed, by definition of \tilde{T} and continuity of $Z : [0, T] \rightarrow L^2$, $\|Z_{\tilde{T}}\|_{L^2} = 1/e$ if $\tilde{T} < T$. This would contradict Z being indistinguishable from 0 on $[0, \tilde{T}]$. Hence, we get that Z is actually indistinguishable from 0 on $[0, T]$.

Assume that $\|Z_t\|_{L^2}$ is not identically 0 on $[0, \tilde{T}]$. Choose $k_0 \in \mathbb{N}$ such that $2^{-k_0} < \sup_{t \in [0, \tilde{T}]} \|Z_t\|_{L^2}$. For $k \geq k_0$ let $t_k := \inf\{t : \|Z_t\|_{L^2} \geq 2^{-k}\}$. We have that $\|Z_t\|_{L^2} < 2^{-k}$ for $t < t_k$ and $\|Z_{t_k}\|_{L^2} = 2^{-k}$ as $t \mapsto \|Z_t\|_{L^2}$ is continuous by (II.6.9). For l as in Lemma II.6.5, choose $a \in (0, l)$ such that $Ca^{1/2} < 1/4$ for C as in (II.6.9). As the sequence t_k is strictly decreasing we can choose M such that $t_k - t_{k+1} < a$ for $k \geq M$. Therefore by (II.6.9) we have, for $k \geq M$, that

$$2^{-(k+1)} \leq \|Z_{t_k} - Z_{t_{k+1}}\|_{L^2} \leq 2^{-(k+2)} + C 2^{-k} k \log(2) (t_k - t_{k+1})$$

and therefore

$$\frac{1}{4C \log(2)k} \leq t_k - t_{k+1}, \tag{II.6.11}$$

which leads to a contradiction as the sequence t_k is strictly decreasing and the left hand side of (II.6.11) is diverging when summing over $k \geq M$. Hence, $\|Z_t\|_{L^2}$ is identically 0 on $[0, \tilde{T}]$. Now using that $Z \in \mathcal{C}_{[0, T]}^{1/2+H}(L^2)$, we get that $\mathbb{E}\|Z_t - Z_s\|^2 \leq \|Z\|_{\mathcal{C}_{[0, T]}^{1/2+H}(L^2)}^2 |t - s|^{1+2H}$ for any $s, t \in [0, \tilde{T}]$.

Hence by Kolmogorov's continuity theorem, Z is a.s. continuous and is therefore indistinguishable from 0. \square

II.6.2 Intermediate regularity results: Proofs of Lemma II.6.3 and Lemma II.6.4

The proofs of Lemma II.6.3 and Lemma II.6.4 rely on several results about the regularity of $\int_s^t f(B_r, \Xi) dr$ in terms of $t - s$ and of the Besov norm of the random mapping $f(\cdot, \Xi)$, for Ξ an \mathcal{F}_s -measurable random variable. These regularity results are stated and proven in Section II.B, and are derived from two main ingredients: the stochastic sewing Lemma (Lemma A.2.1) and Lemma II.5.1.

We now turn to the proofs of Lemmas II.6.3 and II.6.4.

Proof of Lemma II.6.3. Recall that we assumed $\beta < 0$ at the beginning of this section (Remark II.6.2). Let $\beta' \in (-\frac{1}{2H}, \beta)$ and $(s, t) \in \Delta_{[0, T]}$. Let $k, l \in \mathbb{N}$. Applying the crucial regularity Lemma II.B.1 for $\Xi = K_s$ and $f(z, x) = b^k(z + x) - b^l(z + x)$, we get that

$$\begin{aligned} \left\| \int_s^t b^k(B_r + K_s) dr - \int_s^t b^l(B_r + K_s) dr \right\|_{L^2} &\leq C \left\| \|b^k(\cdot + K_s) - b^l(\cdot + K_s)\|_{\mathcal{B}_p^{\beta'}} \right\|_{L^2} \\ &\leq C \|b^k - b^l\|_{\mathcal{B}_p^{\beta'}}, \end{aligned}$$

using Lemma A.1.8(a) in the last inequality. Hence, $(\int_s^t b^k(B_r + K_s) dr)_{k \in \mathbb{N}}$ is a Cauchy sequence in L^2 , so it converges to some $\tilde{T}_{s,t}^K$. The same holds for the sequence with K being replaced by \tilde{K} .

Using Fatou's Lemma and Corollary II.B.2, for $\lambda \in [0, 1]$ we obtain

$$\begin{aligned} \|\tilde{T}_{s,t}^K - \tilde{T}_{s,t}^{\tilde{K}}\|_{L^2} &\leq \liminf_n \left\| \int_s^t b^n(B_r + K_s) dr - \int_s^t b^n(B_r + \tilde{K}_s) dr \right\|_{L^2} \\ &\leq C \sup_n \|b^n\|_{\mathcal{B}_p^\beta} \|K_s - \tilde{K}_s\|_{L^2}^\lambda (t-s)^{1+H(\beta-\lambda-1/p)} \\ &\leq C \|b\|_{\mathcal{B}_p^\beta} \|K_s - \tilde{K}_s\|_{L^2}^\lambda (t-s)^{1/2+H-\lambda H}, \end{aligned}$$

using that $\beta - 1/p \geq -1/(2H) + 1$. Setting $\lambda = 1$ gives (II.6.5) and setting $\lambda = 0$ gives (II.6.6). \square

Proof of Lemma II.6.4. Let $(b^k)_{k \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ with $\sup_k \|b^k\|_{\mathcal{B}_p^\beta} \leq \|b\|_{\mathcal{B}_p^\beta}$. Fix $(u, v) \in \Delta_{[0, T]}$. For $(s, t) \in \Delta_{[u, v]}$, let

$$\begin{aligned} A_{s,t}^k &:= \int_s^t \left(b^k(B_r + K_s) - b^k(B_r + \tilde{K}_s) \right) dr, \\ \mathcal{A}_t^k &:= \int_u^t \left(b^k(B_r + K_r) - b^k(B_r + \tilde{K}_r) \right) dr. \end{aligned} \quad (\text{II.6.12})$$

Then, for $\theta \in (s, t)$,

$$\delta A_{s,\theta,t}^k := \int_\theta^t \left(b^k(B_r + K_s) - b^k(B_r + \tilde{K}_s) - b^k(B_r + K_\theta) + b^k(B_r + \tilde{K}_\theta) \right) dr. \quad (\text{II.6.13})$$

We now verify the conditions of the stochastic sewing Lemma with critical exponents, as stated in [8, Th. 4.5]. To show that the conditions of this theorem hold (i.e. (A.2.1) and (A.2.2) with $\alpha_2 = \beta_2 = 0$ and $m = n = 2$ in the current paper, and (4.11) from [8]), we verify that there exists $C > 0$ independent of s, t, u, v and θ such that for any $k \in \mathbb{N}$,

- (i) $\|\mathbb{E}^s \delta A_{s,\theta,t}^k\|_{L^2} \leq C(t-s)^{1+H}$;
- (ii) $\|\delta A_{s,\theta,t}^k\|_{L^2} \leq C \left(\|Z\|_{\mathcal{C}_{[u,v]}(L^2)} + \|R\|_{\mathcal{C}_{[u,v]}^\alpha(L^2)} \right) (t-s)^{1/2+\delta}$;
- (iii) $\|\mathbb{E}^s \delta A_{s,\theta,t}^k\|_{L^2} \leq C \|Z\|_{\mathcal{C}_{[u,v]}(L^2)} (t-s) + C \|R\|_{\mathcal{C}_{[u,v]}^\alpha(L^2)} (t-s)^{1/2+\alpha}$;
- (iv) Identifying \mathcal{A}^k , as given in (II.6.12), with the limit of $\sum_{i=0}^{N_n-1} A_{t_i^n, t_{i+1}^n}^k$ along any sequence of partitions $\Pi_n = \{t_i^n\}_{i=0}^{N_n}$ of $[u, t]$ with mesh converging to 0.

Assume for the moment that the above properties hold. Then applying Theorem 4.5 in [8], we get that for any $(s, t) \in \Delta_{[u, v]}$,

$$\begin{aligned} &\|\mathcal{A}_t^k - \mathcal{A}_s^k - A_{s,t}^k\|_{L^2} \\ &\leq C \|Z\|_{\mathcal{C}_{[u,v]}(L^2)} \left(1 + \left| \log \frac{T^H}{\|Z\|_{\mathcal{C}_{[u,v]}(L^2)}} \right| \right) |t-s| \\ &\quad + C \left(\|Z\|_{\mathcal{C}_{[u,v]}(L^2)} + \|R\|_{\mathcal{C}_{[u,v]}^\alpha(L^2)} \right) (t-s)^{1/2+\delta} + C \|R\|_{\mathcal{C}_{[u,v]}^\alpha(L^2)} (t-s)^{1/2+\alpha} \\ &\leq C \left(\|Z\|_{\mathcal{C}_{[u,v]}(L^2)} + \|R\|_{\mathcal{C}_{[u,v]}^\alpha(L^2)} \right) (t-s)^{1/2+\delta} \\ &\quad + C \|Z\|_{\mathcal{C}_{[u,v]}(L^2)} \left| \log \|Z\|_{\mathcal{C}_{[u,v]}(L^2)} \right| (t-s). \end{aligned} \quad (\text{II.6.14})$$

By Definition II.2.4 and Lemma II.6.3, we have the following convergence in probability:

$$\lim_{k \rightarrow \infty} \mathcal{A}_t^k - \mathcal{A}_s^k - A_{s,t}^k = Z_t - Z_s - (\tilde{T}_{s,t}^K - \tilde{T}_{s,t}^{\tilde{K}}) = R_{s,t}.$$

Hence and letting k go to ∞ in (II.6.14) and using Fatou's Lemma, we get (II.6.8) by choosing $(s, t) = (u, v)$. Putting together (II.6.8) and (II.6.5), we get (II.6.7).

Proof of (i): Recall that we consider times $0 \leq u \leq s \leq \theta \leq t \leq v \leq T$. Let $f_1 : \mathbb{R} \times \mathbb{R}^4 \rightarrow \mathbb{R}$ and $f_2 : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} f_1(z, (x_1, x_2, x_3, x_4)) &= b^k(z + x_1) - b^k(z + x_2) + b^k(z + x_3) - b^k(z + x_4), \\ f_2(z, (x_1, x_2)) &= b^k(z + x_1) - b^k(z + x_2), \end{aligned}$$

and consider the \mathcal{F}_θ -measurable random vectors Ξ_1 and Ξ_2 given by:

$$\begin{aligned} \Xi_1 &= (K_s, \tilde{K}_s, K_\theta + \tilde{K}_s - K_s, K_\theta) \\ \Xi_2 &= (\tilde{K}_\theta, K_\theta + \tilde{K}_s - K_s). \end{aligned}$$

We can rewrite the integrand on the right hand side of (II.6.13) as $f_1(B_r, \Xi_1) + f_2(B_r, \Xi_2)$. Hence using Lemma II.5.1(b), we get that for $\lambda \in [0, 1]$,

$$\begin{aligned} |\mathbb{E}^s \delta A_{s,\theta,t}^k| &= |\mathbb{E}^s \mathbb{E}^\theta \delta A_{s,\theta,t}^k| \\ &\leq C \int_\theta^t \left\{ (r - \theta)^{H(\beta-1-\lambda-1/p)} \mathbb{E}^s \|f_1(\cdot, \Xi_1)\|_{\mathcal{B}_p^{\beta-1-\lambda}} \right. \\ &\quad \left. + (r - \theta)^{H(\beta-1-1/p)} \mathbb{E}^s \|f_2(\cdot, \Xi_2)\|_{\mathcal{B}_p^{\beta-1}} \right\} dr. \end{aligned} \quad (\text{II.6.15})$$

Apply Lemma A.1.8(c) first, then Jensen's inequality for conditional expectation to get that

$$\begin{aligned} \mathbb{E}^s \|f_1(\cdot, \Xi_1)\|_{\mathcal{B}_p^{\beta-1-\lambda}} &\leq C \|b^k\|_{\mathcal{B}_p^\beta} |K_s - \tilde{K}_s|^\lambda \mathbb{E}^s |K_\theta - K_s| \\ &\leq C \|b\|_{\mathcal{B}_p^\beta} |K_s - \tilde{K}_s|^\lambda (\mathbb{E}^s [|K_\theta - K_s|^2])^{1/2} \\ &\leq C \|b\|_{\mathcal{B}_p^\beta} |Z_s|^\lambda [K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)} (\theta - s)^{1/2+H}. \end{aligned}$$

Hence,

$$\|\mathbb{E}^s \|f_1(\cdot, \Xi_1)\|_{\mathcal{B}_p^{\beta-1-\lambda}}\|_{L^2} \leq C \|b\|_{\mathcal{B}_p^\beta} \|Z\|_{\mathcal{C}_{[u,v]}^\lambda(L^2)} [K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)} (t - s)^{1/2+H}. \quad (\text{II.6.16})$$

By Lemma A.1.8(b), we have that $\mathbb{E}^s \|f_2(\cdot, \Xi_2)\|_{\mathcal{B}_p^{\beta-1}} \leq C \|b^k\|_{\mathcal{B}_p^\beta} \mathbb{E}^s |Z_\theta - Z_s|$. Therefore using the contraction property of conditional expectation and (II.6.2), we get that

$$\|\mathbb{E}^s \|f_2(\cdot, \Xi_2)\|_{\mathcal{B}_p^{\beta-1}}\|_{L^2} \leq C \|b\|_{\mathcal{B}_p^\beta} \|Z_\theta - Z_s\|_{L^2} \quad (\text{II.6.17})$$

$$\leq C \|b\|_{\mathcal{B}_p^\beta} \left([K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)} + [\tilde{K}]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2)} \right) (t - s)^{1/2+H}. \quad (\text{II.6.18})$$

After putting $\lambda = 0$ in (II.6.15) and using inequalities (II.6.16) and (II.6.18), we get

$$\begin{aligned} |\mathbb{E}^s \delta A_{s,\theta,t}^k| &\leq C \|b\|_{\mathcal{B}_p^\beta} \left([K]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2, \infty)} + [\tilde{K}]_{\mathcal{C}_{[0,T]}^{1/2+H}(L^2)} \right) (t - s)^{1/2+H} \int_\theta^t (r - \theta)^{H(\beta-1-1/p)} dr. \end{aligned}$$

Now using that $\beta - 1/p \geq -1/(2H) + 1$, we obtain that for some C depending only on T , $\|b\|_{\mathcal{B}_p^\beta}$, $[K]_{C_{[0,T]}^{1/2+H}(L^2,\infty)}$ and $[\tilde{K}]_{C_{[0,T]}^{1/2+H}(L^2)}$,

$$\|\mathbb{E}^s \delta A_{s,\theta,t}^k\|_{L^2} \leq C(t-s)^{1+H}.$$

Proof of (ii): We use Corollary II.B.3 with $f = b^k$, $\gamma = \beta$, $\varepsilon > 0$ small enough to ensure that $0 > \beta > -1/(2H) + 1 + \varepsilon$, $\lambda = \lambda_1 = 1$, $\lambda_2 = \varepsilon$, $\kappa_1 = K_s$, $\kappa_2 = \tilde{K}_s$, $\kappa_3 = K_\theta$ and $\kappa_4 = \tilde{K}_\theta$.

Hence, we get

$$\begin{aligned} \|\delta A_{s,\theta,t}^k\|_{L^2} &\leq C\|b\|_{\mathcal{B}_p^\beta} \|\mathbb{E}^s [|K_\theta - K_s|^2]^{1/2}\|_{L^\infty}^\varepsilon \|K_s - \tilde{K}_s\|_{L^2} (t-\theta)^{1+H(\beta-1/p-1-\varepsilon)} \\ &\quad + C\|b\|_{\mathcal{B}_p^\beta} \|Z_\theta - Z_s\|_{L^2} (t-\theta)^{1+H(\beta-1-1/p)}. \end{aligned} \quad (\text{II.6.19})$$

We have $\|Z_\theta - Z_s\|_{L^2} \leq \|R_{s,\theta}\|_{L^2} + \|\tilde{T}_{s,\theta}^K - \tilde{T}_{s,\theta}^{\tilde{K}}\|_{L^2}$. Thus by Lemma II.6.3,

$$\|Z_\theta - Z_s\|_{L^2} \leq \|R\|_{C_{[u,v]}^\alpha(L^2)} (\theta-s)^\alpha + C\|b\|_{\mathcal{B}_p^\beta} \|Z_s\|_{L^2} (\theta-s)^{1/2}. \quad (\text{II.6.20})$$

It follows directly from the definitions that

$$\|\mathbb{E}^s [|K_\theta - K_s|^2]^{1/2}\|_{L^\infty} \leq C(\theta-s)^{1/2+H} [K]_{C_{[0,T]}^{1/2+H}(L^2,\infty)}$$

and that $\|K_s - \tilde{K}_s\|_{L^2} \leq \|Z\|_{C_{[u,v]}(L^2)}$. Plugging the last three inequalities into (II.6.19) we get that

$$\begin{aligned} \|\delta A_{s,\theta,t}^k\|_{L^2} &\leq C\|b\|_{\mathcal{B}_p^\beta} [K]_{C_{[0,T]}^{1/2+H}(L^2,\infty)}^\varepsilon \|Z\|_{C_{[u,v]}(L^2)} (t-s)^{1-H+H(\beta-1/p)+\varepsilon/2} \\ &\quad + C\|b\|_{\mathcal{B}_p^\beta} \|R\|_{C_{[u,v]}^\alpha(L^2)} (t-s)^{\alpha+1+H(\beta-1-1/p)} \\ &\quad + C\|b\|_{\mathcal{B}_p^\beta}^2 \|Z\|_{C_{[u,v]}(L^2)} (t-s)^{3/2-H+H(\beta-1/p)}. \end{aligned}$$

After using that $\beta - 1/p \geq -1/(2H) + 1$, we obtain that for some C depending only on T , $\|b\|_{\mathcal{B}_p^\beta}$ and $[K]_{C_{[0,T]}^{1/2+H}(L^2,\infty)}$ and some $\delta > 0$,

$$\|\delta A_{s,\theta,t}^k\|_{L^2} \leq C \left(\|Z\|_{C_{[u,v]}(L^2)} + \|R\|_{C_{[u,v]}^\alpha(L^2)} \right) (t-s)^{1/2+\delta}.$$

Proof of (iii): We use (II.6.15) with $\lambda = 1$, (II.6.16), (II.6.17) and (II.6.20) to obtain that

$$\|\mathbb{E}^s \delta A_{s,\theta,t}^k\|_{L^2} \leq C\|Z\|_{C_{[u,v]}(L^2)} (t-s) + C\|R\|_{C_{[u,v]}^\alpha(L^2)} (t-s)^{1/2+\alpha}.$$

That (iv) holds can be shown by similar arguments as for (iii) in the proof of Proposition II.5.3. \square

II.7 Existence of weak and strong solutions

In this section we prove Theorem II.2.8 and Theorem II.2.10. As in Section II.6, recall that b is not assumed to be a measure.

In the proofs, we assume that $p \in [m, \infty]$ and $\beta < 0$ for some $m \geq 2$. As explained in Remarks II.5.2 and II.6.2, it is not a restriction as we can always reduce to this case under our assumptions.

The scheme of proof is the following: we use a sequence of smooth approximations of the drift $(b^n)_{n \in \mathbb{N}}$ that converges in $\mathcal{B}_p^{\beta^-}$ to b . We prove that the sequence $(X^n)_{n \in \mathbb{N}}$ of solutions associated to b^n is tight and that each limit point is a weak solution to (II.1.1). To prove the existence of a strong solution (which is in some sense a Yamada-Watanabe result), we rely on a classical argument of Krylov [60], a new result on the continuity of fractional operators (Lemma II.A.1) and the aforementioned construction of weak solutions.

On the technical level, we follow the approach of [8] which is based on applications of the stochastic sewing Lemma. However there are noticeable differences due to the non-Markovian character of the fBm. In particular we state immediately below:

- Lemma II.7.1, which is a crucial estimate on the regularity of the integrals of the fBm, relying on fine properties of the fBm (notably Lemma II.5.1) and stochastic sewing, and which will be used in Section II.7.1 to establish *a priori* estimates on the solutions;
- Lemma II.A.1, which will be used in Section II.7.3 to prove that solutions are adapted.

Lemma II.7.1. *Let $(\psi_t)_{t \in [0, T]}$ be a stochastic process adapted to \mathbb{F} . Let $m \in [2, \infty)$, $n \in [m, \infty]$, $p \in [n, \infty]$ and $\gamma < 0$ such that $(\gamma - 1/p)H > -1/2$. Let $\alpha \in (0, 1)$ such that $H(\gamma - 1/p - 1) + \alpha > 0$. There exists a constant $C > 0$ such that for any $f \in \mathcal{C}_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\gamma$ and any $(s, t) \in \Delta_{[0, T]}$ we have*

$$\begin{aligned} \left\| \mathbb{E}^s \left[\left| \int_s^t f(B_r + \psi_r) dr \right|^m \right]^{1/m} \right\|_{L^n} &\leq C \|f\|_{\mathcal{B}_p^\gamma} (t-s)^{1+H(\gamma-1/p)} \\ &\quad + C \|f\|_{\mathcal{B}_p^\gamma} [\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,n})} (t-s)^{1+H(\gamma-1-1/p)+\alpha}, \end{aligned} \tag{II.7.1}$$

where we recall that the seminorm $[\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,n})}$ is defined in (II.6.1).

A similar regularization result was proposed in [18, Lemma 4.7] for functions of positive regularity. There, the deterministic sewing lemma is used instead of the stochastic sewing Lemma, hence the conditional expectation does not appear. Note that due to an embedding argument, it would be no restriction here to work in Hölder spaces, yet we directly work in Besov spaces to stay consistent.

The proof of Lemma II.7.1 is postponed to Section II.B.

II.7.1 A priori estimates

Before starting the main proofs of existence of solutions, we need *a priori* estimates on solutions. These estimates are established for drifts in $\mathcal{C}_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\beta$ and therefore the solution is unique and strong. Note that the following lemma looks similar to Proposition II.5.3. However we work here within the assumptions of Theorem II.2.8, that is with a drift that may not be a measure and $\beta - 1/p > 1/2 - 1/(2H)$ (recall that Proposition II.5.3 allows the milder condition $\beta - 1/p > -1/(2H)$).

Lemma II.7.2. *Let $m \in [2, \infty)$. There exists $C > 0$, such that, for any $b \in \mathcal{C}_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\beta$ with $\beta - 1/p > 1/2 - 1/(2H)$, one has*

$$[X - B]_{\mathcal{C}_{[0,T]}^{1+H(\beta-1/p)}(L^{m,\infty})} \leq C \|b\|_{\mathcal{B}_p^\beta} \left(1 + \|b\|_{\mathcal{B}_p^\beta}^{\frac{H(1/p-\beta)}{1+H(\beta-1/p)-H}} \right) \leq C \left(1 + \|b\|_{\mathcal{B}_p^\beta}^2 \right), \quad (\text{II.7.2})$$

where X is the strong solution to (II.1.1) with drift b , and we recall that the seminorm $[\cdot]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,n})}$ was defined in (II.6.1).

Another difference with Proposition II.5.3 is that we obtain here an estimate on the seminorm $[\cdot]_{\mathcal{C}_{[0,T]}^{1+H(\beta-1/p)}(L^{m,\infty})}$, which is stronger (see (II.6.2)) than the seminorm $[\cdot]_{\mathcal{C}_{[0,T]}^{1+H(\beta-1/p)}(L^m)}$ used in Proposition II.5.3.

Proof. Note that it is sufficient to show the first inequality in (II.7.2) as the second one follows immediately from $1 + H(\beta - 1/p) - H > H(1/p - \beta) > 0$. Without loss of generality, we assume that $X_0 = 0$ and denote $K = X - B$. Then $[K]_{\mathcal{C}_{[0,T]}^\alpha(L^{m,\infty})}$ is finite for any $\alpha \in (0, 1]$ as

$$|K_t - K_s| = \left| \int_s^t b(B_r + K_r) dr \right| \leq \|b\|_\infty |t - s|.$$

We will apply Lemma II.7.1 with $n = \infty$, $\alpha = 1 + H(\beta - 1/p)$ and considering $b \in \mathcal{B}_\infty^{\beta-1/p}$ after an embedding. Remark that $\alpha > \alpha - H > 1/2 - H/2 > 0$ and since we assume $\beta < 0$ in the whole section, $\alpha \in (0, 1)$. In addition, $H(\beta - 1/p) > H/2 - 1/2 > -1/2$ and $H(\beta - 1/p - 1) + \alpha > 0$. So, the assumptions of Lemma II.7.1 are fulfilled. Then we get

$$\begin{aligned} & \|\mathbb{E}^s [|K_t - K_s|^m]^{1/m}\|_{L^\infty} \\ & \leq C \|b\|_{\mathcal{B}_\infty^{\beta-1/p}} \left((t-s)^{1+H(\beta-1/p)} + [K]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,\infty})} (t-s)^{1+H(\beta-1/p)+\alpha-H} \right) \\ & = C \|b\|_{\mathcal{B}_p^\beta} (t-s)^{1+H(\beta-1/p)} \left(1 + [K]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,\infty})} (t-s)^{\alpha-H} \right). \end{aligned} \quad (\text{II.7.3})$$

Choose $l = (4C\|b\|_{\mathcal{B}_p^\beta})^{1/(H-\alpha)}$ so that $C\|b\|_{\mathcal{B}_p^\beta} l^{\alpha-H} < 1/2$. Let $u \in [0, T]$. After dividing both sides in (II.7.3) by $(t-s)^{1+H(\beta-1/p)}$ and taking the supremum over $(s, t) \in \Delta_{[u, (u+l) \wedge T]}$ we get

$$[K]_{\mathcal{C}_{[u, (u+l) \wedge T]}^{1+H(\beta-1/p)}(L^{m,\infty})} \leq \left(C\|b\|_{\mathcal{B}_p^\beta} + 1/2 [K]_{\mathcal{C}_{[u, (u+l) \wedge T]}^\alpha(L^{m,\infty})} \right)$$

and therefore

$$[K]_{\mathcal{C}_{[u, (u+l) \wedge T]}^{1+H(\beta-1/p)}(L^{m,\infty})} \leq 2C\|b\|_{\mathcal{B}_p^\beta}. \quad (\text{II.7.4})$$

If $l > T$, then (II.7.2) follows immediately from (II.7.4). Hence, assume $l \leq T$. In order to obtain (II.7.2) we will iteratively apply inequality (II.7.4). Let $(s, t) \in \Delta_{[0, T]}$ be arbitrary. Let $N = \lceil T/l \rceil$ and define the sequence $(s_k)_{k=0}^N$ by $s_k = s + k(t-s)/N$. Using the triangle inequality, that $\mathcal{F}_s \subset \mathcal{F}_{s_{k-1}}$

for $k \geq 1$, the contraction property of conditional expectation and (II.7.4) we get

$$\begin{aligned}
 \|\mathbb{E}^s[|K_t - K_s|^m]^{1/m}\|_{L^\infty} &\leq \sum_{k=1}^N \|\mathbb{E}^s[|K_{s_k} - K_{s_{k-1}}|^m]^{1/m}\|_{L^\infty} \\
 &\leq \sum_{k=1}^N \|\mathbb{E}^{s_{k-1}}[|K_{s_k} - K_{s_{k-1}}|^m]^{1/m}\|_{L^\infty} \\
 &\leq C \|b\|_{\mathcal{B}_p^\beta} \sum_{k=1}^N (s_k - s_{k-1})^{1+H(\beta-1/p)} \\
 &\leq C \|b\|_{\mathcal{B}_p^\beta} \sum_{k=1}^N \left(\frac{t-s}{N}\right)^{1+H(\beta-1/p)} \\
 &\leq C \|b\|_{\mathcal{B}_p^\beta} N^{H(1/p-\beta)} (t-s)^{1+H(\beta-1/p)}.
 \end{aligned}$$

Using $N \leq 1 + \frac{T}{t} \leq 2\frac{T}{t} \leq C \|b\|_{\mathcal{B}_p^\beta}^{-\frac{1}{H-\alpha}}$, we get

$$\|b\|_{\mathcal{B}_p^\beta} N^{H(1/p-\beta)} \leq C \|b\|_{\mathcal{B}_p^\beta} \|b\|_{\mathcal{B}_p^\beta}^{-\frac{H(1/p-\beta)}{H-\alpha}}$$

and therefore (II.7.2). \square

Lemma II.7.3. *Let $b, h \in C_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\beta$ where $\beta - 1/p > 1/2 - 1/(2H)$. Let X be the strong solution to (II.1.1) with drift b . Let $\delta \in (0, 1 + H(\beta - 1/p))$. Then there exists a constant $C > 0$ which is independent of X_0 , b and h , and a nonnegative random variable Z which satisfies $\mathbb{E}[Z] \leq C \|h\|_{\mathcal{B}_p^\beta} (1 + \|b\|_{\mathcal{B}_p^\beta}^2)$ such that*

$$\left| \int_s^t h(X_r) dr \right| \leq Z |t - s|^\delta. \quad (\text{II.7.5})$$

Proof. Let $\Lambda_t^h := \int_0^t h(X_r) dr$. We apply Lemma II.7.1 to establish an upper bound for $\|\Lambda_t^h - \Lambda_s^h\|_{L^m}$, with the parameters $n = m$, $\gamma = \beta (< 0)$ and $\alpha = 1 + H(\beta - 1/p)$. Hence, we get

$$\|\Lambda_t^h - \Lambda_s^h\|_{L^m} \leq C \|h\|_{\mathcal{B}_p^\beta} (1 + [X - B]_{\mathcal{C}_{[0,T]}^\alpha(L^m)})(t - s)^{1+H(\beta-1/p)}. \quad (\text{II.7.6})$$

By (II.6.2) and Lemma II.7.2 we know that

$$[X - B]_{\mathcal{C}_{[0,T]}^\alpha(L^m)} \leq [X - B]_{\mathcal{C}_{[0,T]}^\alpha(L^{m,\infty})} \leq C (1 + \|b\|_{\mathcal{B}_p^\beta}^2).$$

Since for any m there exists a constant C such that (II.7.6) holds, the result follows from Kolmogorov's continuity criterion. \square

II.7.2 Tightness and stability

In Proposition II.7.4 we show that if $(b^n)_{n \in \mathbb{N}}$ approximates b , the solution X^n to (II.1.1) with drift b^n converges weakly (up to taking a subsequence). Proposition II.7.6 states that the limit in probability of such a sequence $(X^n)_{n \in \mathbb{N}}$ is a solution to the original equation with drift b .

Proposition II.7.4. *Let $(b^n)_{n \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ where $b \in \mathcal{B}_p^\beta$ for $\beta - 1/p > 1/2 - 1/(2H)$. For $n \in \mathbb{N}$, let X^n be the strong solution to (II.1.1) with initial condition X_0 and drift b^n . Then there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(X^{n_k}, B)_{k \in \mathbb{N}}$ converges weakly in the space $[\mathcal{C}_{[0,T]}]^2$.*

Proof. Let $K_t^n := \int_0^t b^n(X_r^n) dr$. For $M > 0$, let

$$A_M := \{f \in \mathcal{C}_{[0,T]} : f(0) = 0, |f(t) - f(s)| \leq M(t-s)^{1+H(\beta-1/p)}, \forall (s, t) \in \Delta_{[0,T]}\}.$$

By Arzelà-Ascoli's Theorem, A_M is compact in $\mathcal{C}_{[0,T]}$. Applying Lemma II.7.2 and Markov's inequality, we get

$$\begin{aligned} \mathbb{P}(K^n \notin A_M) &\leq \mathbb{P}(\exists (s, t) \in \Delta_{[0,T]} : |K_{s,t}^n| > M(t-s)^{1+H(\beta-1/p)}) \\ &\leq C(1 + \sup_{n \in \mathbb{N}} \|b^n\|_{\mathcal{B}_p^\beta}^2) M^{-1}. \end{aligned}$$

Hence, the sequence $(K^n)_{n \in \mathbb{N}}$ is tight in $\mathcal{C}_{[0,T]}$. So $(K^n, B)_{n \in \mathbb{N}}$ is tight in $[\mathcal{C}_{[0,T]}]^2$. Thus by Prokhorov's Theorem, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(K^{n_k}, B)_{k \in \mathbb{N}}$ converges weakly in the space $[\mathcal{C}_{[0,T]}]^2$, and so does $(X^{n_k}, B)_{k \in \mathbb{N}}$. \square

Remark II.7.5. The previous proposition can be generalised to any pair of sequences of approximations $(X^{1,n}, X^{2,n})_{n \in \mathbb{N}}$ of strong solution to (II.1.1) with two sequences $(b_1^n)_{n \in \mathbb{N}}$ and $(b_2^n)_{n \in \mathbb{N}}$ of smooth approximations of b .

Proposition II.7.6. *Let $(\tilde{b}^k)_{k \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$ where $b \in \mathcal{B}_p^\beta$ for $\beta - 1/p > 1/2 - 1/(2H)$. Let \tilde{B}^k have the same law as B . We consider \tilde{X}^k the strong solution to (II.1.1) for $B = \tilde{B}^k$, initial condition X_0 and drift \tilde{b}^k . We assume that there exist stochastic processes $\tilde{X}, \tilde{B} : [0, T] \rightarrow \mathbb{R}$ such that $(\tilde{X}^k, \tilde{B}^k)_{k \in \mathbb{N}}$ converges to (\tilde{X}, \tilde{B}) on $[\mathcal{C}_{[0,T]}]^2$ in probability. Then \tilde{X} fulfills (II.2.3) and (II.2.4) from Definition II.2.4 and for any $m \in [2, \infty)$, there exists $C > 0$ such that*

$$[\tilde{X} - \tilde{B}]_{\mathcal{C}_{[0,T]}^{1+H(\beta-1/p)}(L^{m,\infty})} \leq C(1 + \sup_{k \in \mathbb{N}} \|\tilde{b}^k\|_{\mathcal{B}_p^\beta}^2) < \infty. \quad (\text{II.7.7})$$

Proof. We again assume here w.l.o.g. that $X_0 = 0$ and let $\tilde{K} := \tilde{X} - \tilde{B}$, so that (II.2.3) is automatically verified. Let now $(b^n)_{n \in \mathbb{N}}$ be any sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$. In order to verify that \tilde{K} and \tilde{X} fulfill (II.2.4) from Definition II.2.4, we have to show that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r) dr - \tilde{K}_t \right| = 0 \text{ in probability.} \quad (\text{II.7.8})$$

By the triangle inequality we have that for $k, n \in \mathbb{N}$ and $t \in [0, T]$,

$$\begin{aligned} \left| \int_0^t b^n(\tilde{X}_r) dr - \tilde{K}_t \right| &\leq \left| \int_0^t b^n(\tilde{X}_r) dr - \int_0^t b^n(\tilde{X}_r^k) dr \right| + \left| \int_0^t b^n(\tilde{X}_r^k) dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr \right| \\ &\quad + \left| \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr - \tilde{K}_t \right| =: A_1 + A_2 + A_3. \end{aligned} \quad (\text{II.7.9})$$

Now we will show that all summands on the right hand side of (II.7.9) converge to 0 uniformly on $[0, T]$ in probability as $n \rightarrow \infty$, choosing $k = k(n)$ accordingly.

First we bound A_1 . Notice that

$$\begin{aligned} \left| \int_0^t b^n(\tilde{X}_r) dr - \int_0^t b^n(\tilde{X}_r^k) dr \right| &\leq \|b^n\|_{C^1} \int_0^t |\tilde{X}_r - \tilde{X}_r^k| dr \\ &\leq \|b^n\|_{C^1} T \sup_{t \in [0, T]} |\tilde{X}_t - \tilde{X}_t^k|. \end{aligned}$$

For any $\varepsilon > 0$, we can choose an increasing sequence $(k(n))_{n \in \mathbb{N}}$ such that

$$\mathbb{P} \left(\|b^n\|_{C^1} T \sup_{t \in [0, T]} |\tilde{X}_t - \tilde{X}_t^{k(n)}| > \varepsilon \right) < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Hence, we get that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r) dr - \int_0^t b^n(\tilde{X}_r^{k(n)}) dr \right| = 0 \text{ in probability.}$$

Now, let us bound A_2 . Let $\beta' < \beta$ with $\beta' - 1/p > -1/(2H) + 1/2$. By Lemma II.7.3 applied to \tilde{X}^k , $h = b^n - \tilde{b}^k$ and β' instead of β , we know that there exists a random variable $Z_{n,k}$ such that

$$\begin{aligned} \mathbb{E}[Z_{n,k}] &\leq C \|b^n - \tilde{b}^k\|_{\mathcal{B}_p^{\beta'}} (1 + \|\tilde{b}^k\|_{\mathcal{B}_p^{\beta'}}^2) \\ &\leq C (\|b^n - b\|_{\mathcal{B}_p^{\beta'}} + \|\tilde{b}^k - b\|_{\mathcal{B}_p^{\beta'}}) (1 + \sup_{m \in \mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_p^{\beta'}}^2), \end{aligned} \quad (\text{II.7.10})$$

for C independent of k, n and that we have

$$\sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r^k) dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr \right| \leq Z_{n,k} (1 + T).$$

Using Markov's inequality and (II.7.10) we obtain that

$$\begin{aligned} \mathbb{P} \left(\sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r^k) dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr \right| > \varepsilon \right) &\leq \varepsilon^{-1} \mathbb{E}[Z_{n,k}] (1 + T) \\ &\leq C \varepsilon^{-1} (1 + T) (\|b^n - b\|_{\mathcal{B}_p^{\beta'}} + \|\tilde{b}^k - b\|_{\mathcal{B}_p^{\beta'}}) (1 + \sup_{m \in \mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_p^{\beta'}}^2). \end{aligned}$$

Choosing $k = k(n)$ as before we get

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r^{k(n)}) dr - \int_0^t \tilde{b}^{k(n)}(\tilde{X}_r^{k(n)}) dr \right| = 0 \text{ in probability.}$$

To bound the last summand A_3 , recall that $\tilde{X}_t^k = \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr + \tilde{B}_t^k$. Hence, we get that

$$\sup_{t \in [0, T]} \left| \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr - \tilde{K}_t \right| \leq \sup_{t \in [0, T]} (|\tilde{X}_t^k - \tilde{X}_t| + |\tilde{B}_t^k - \tilde{B}_t|).$$

Since by assumption $(\tilde{X}^k, \tilde{B}^k)_{k \in \mathbb{N}}$ converges to (\tilde{X}, \tilde{B}) on $[\mathcal{C}_{[0,T]}]^2$ in probability, we get that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0,T]} \left| \int_0^t \tilde{b}^{k(n)}(\tilde{X}_r^{k(n)}) dr - \tilde{K}_t \right| = 0 \text{ in probability}$$

and therefore (II.7.8) holds true.

It remains to show that (II.7.7) holds true. By Lemma II.7.2, we know that there exists some $C > 0$ such that for any $(s, t) \in \Delta_{[0,T]}$,

$$\|\mathbb{E}^s[|(\tilde{X}_t^k - \tilde{B}_t^k) - (\tilde{X}_s^k - \tilde{B}_s^k)|^m]^{1/m}\|_{L^\infty} \leq C (1 + \sup_{m \in \mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_p^\beta}^2) (t - s)^{1+H(\beta-1/p)}. \quad (\text{II.7.11})$$

Recall that $\int_0^t \tilde{b}^k(\tilde{X}_r^k) dr$ converges to K_t on $\mathcal{C}_{[0,T]}$ in probability by assumption. Also, $\sup_{m \in \mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_p^\beta}$ is finite and therefore get (II.7.7) by applying Fatou's Lemma to (II.7.11). \square

II.7.3 Approximation by smooth drifts

Proof of Theorem II.2.8. Instead of constructing a single weak solution, we will construct a couple (Y^1, Y^2) . This does not change the nature of the proof, but it will be extremely useful in the proof of Theorem II.2.10 when we will seek to construct a strong solution, *via* a Gyöngy-Krylov argument.

Let $(b_1^n)_{n \in \mathbb{N}}, (b_2^n)_{n \in \mathbb{N}}$ be sequences of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$. By Proposition II.7.4 and Remark II.7.5, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ so that $(X^{1,n_k}, X^{2,n_k}, B)_{k \in \mathbb{N}}$ converges weakly in $[\mathcal{C}_{[0,T]}]^3$. Without loss of generality, we assume that $(X^{1,n}, X^{2,n}, B)_{n \in \mathbb{N}}$ converges weakly. By the Skorokhod representation Theorem, there exists a sequence of random variables $(Y^{1,n}, Y^{2,n}, \hat{B}^n)_{n \in \mathbb{N}}$ defined on a common probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, such that

$$\text{Law}(Y^{1,n}, Y^{2,n}, \hat{B}^n) = \text{Law}(X^{1,n}, X^{2,n}, B), \quad \forall n \in \mathbb{N}, \quad (\text{II.7.12})$$

and $(Y^{1,n}, Y^{2,n}, \hat{B}^n)$ converges a.s. to some (Y^1, Y^2, \hat{B}) in $[\mathcal{C}_{[0,T]}]^3$. As $X^{i,n}$ solves the SDE (II.1.1) with drift b_i^n , we know by (II.7.12) that $Y^{i,n}$ also solves (II.1.1) with drift b_i^n and \hat{B}^n instead of B . As $X^{i,n}$ is a strong solution, we have that $X^{i,n}$ is adapted to \mathbb{F}^B . Hence by (II.7.12), we know that $Y^{i,n}$ is adapted to $\mathbb{F}^{\hat{B}^n}$ as the conditional laws of $Y^{i,n}$ and $X^{i,n}$ agree and therefore they are strong solutions to (II.1.1) with \hat{B}^n instead of B .

By Proposition II.7.6, we know that Y^1 and Y^2 fulfill (II.2.3) and (II.2.4) from Definition II.2.4 with \hat{B} instead of B and they are both adapted with respect to the filtration $\hat{\mathbb{F}}$ defined by $\hat{\mathcal{F}}_t := \sigma(Y_s^1, Y_s^2, \hat{B}_s, s \in [0, t])$.

Following the same arguments as in the proof of Theorem II.2.5, after passing to the limit, we know that \hat{B} is an $\hat{\mathbb{F}}$ -fBm and Y^1 and Y^2 are weak solutions adapted to $\hat{\mathbb{F}}$.

Lastly, (II.7.7) gives that

$$[Y^i - \hat{B}]_{\mathcal{C}_{[0,T]}^{1+H(\beta-1/p)}(L^{m,\infty})} < \infty, \quad i = 1, 2, \quad (\text{II.7.13})$$

and thus $Y^1 - \hat{B}$ and $Y^2 - \hat{B}$ belong to $\mathcal{C}_{[0,T]}^{1+H(\beta-1/p)}(L^m)$. \square

II.7.4 Existence of strong solutions

In order to prove strong existence, we follow a Yamada-Watanabe argument: this is done here by combining the construction of weak solutions from the previous proof, the uniqueness result Proposition II.6.1 and Gyöngy-Krylov's result.

Lemma II.7.7. *Let $H < 1/2$, $p \in [1, \infty]$ and β such that $\beta > -1/(2H) + 1$ and $\beta - 1/p \geq -1/(2H) + 1$. Let $b \in \mathcal{B}_p^\beta$, and $(b^n)_{n \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_p^{\beta-}$. For $n \in \mathbb{N}$, consider X^n the strong solution to (II.1.1) with drift b^n and initial condition $X_0 \in \mathbb{R}$. Then there exists an \mathbb{F}^B -adapted process $X : [0, T] \times \Omega \rightarrow \mathbb{R}$ such that*

$$(i) \sup_{t \in [0, T]} |X_t^n - X_t| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0;$$

$$(ii) [X - B]_{\mathcal{C}_{[0, T]}^{1/2+H}(L^{2, \infty})} < \infty \text{ a.s.}$$

Proof. Let $(X^{\phi(n)})_{n \in \mathbb{N}}$ and $(X^{\psi(n)})_{n \in \mathbb{N}}$ be two arbitrary subsequences of $(X^n)_{n \in \mathbb{N}}$. From the previous proof, we know that there exists a filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \hat{P})$, an $\hat{\mathbb{F}}$ -fBm \hat{B} and a pair (Y^1, Y^2) of weak solutions to (II.1.1) adapted to $\hat{\mathbb{F}}$. By tightness (Proposition II.7.4 and Remark II.7.5), we also have that there exist subsequences $\tilde{\phi}(n)$ and $\tilde{\psi}(n)$ such that $(X^{\tilde{\phi}(n)}, X^{\tilde{\psi}(n)})$ converges weakly to (Y^1, Y^2) on $[\mathcal{C}_{[0, T]}]^2$ for $n \rightarrow \infty$.

Combining (II.7.13), (II.6.2) and that by assumption $1 + H(\beta - 1/p) \geq 1/2 + H$, we get that

$$[Y^i - \hat{B}]_{\mathcal{C}_{[0, T]}^{1/2+H}(L^2)} \leq [Y^i - \hat{B}]_{\mathcal{C}_{[0, T]}^{1/2+H}(L^{2, \infty})} < \infty, \quad i = 1, 2.$$

By Proposition II.6.1, this gives $Y^1 = Y^2$ a.s. Hence by Lemma 1.1 in [60], we get that there exists X such that X^n converges in probability to X on $\mathcal{C}_{[0, T]}$. Notice that X^n is adapted to \mathbb{F}^B , for any $n \in \mathbb{N}$, as they are strong solutions to (II.1.1). So X is adapted to the same filtration. Lastly, (ii) follows from (II.7.7). \square

Proof of Theorem II.2.10. (a) Let X be the process constructed in Lemma II.7.7. Proposition II.7.6 yields that X is a strong solution to (II.1.1) fulfilling (II.7.7) for any $m \geq 2$. Since $\beta - 1/p \geq 1 - 1/(2H)$, we get that $[X - B]_{\mathcal{C}_{[0, T]}^{1/2+H}(L^{m, \infty})} < \infty$ for any $m \geq 2$.

(b) Let (Y^1, B) and (Y^2, B) be weak solutions defined on the same probability space, with $Y^1 - B$ and $Y^2 - B$ in $\mathcal{C}_{[0, T]}^{1/2+H}(L^2)$. On this probability space, let X be a strong solution which satisfies (II.6.3) with the same fBm B . Since X is also a weak solution adapted to \mathbb{F} , it follows from Proposition II.6.1 that $X = Y^i$ for $i = 1, 2$.

(c) By Lemma II.7.7 the sequence of strong solutions corresponding to an approximation sequence of the drift b converges in probability to a strong solution X to (II.1.1) such that

$$[X - B]_{\mathcal{C}_{[0, T]}^{1/2+H}(L^{2, \infty})} < \infty. \tag{II.7.14}$$

The limit X is the same for any approximation sequence as uniqueness in the class of solutions fulfilling (II.7.14) holds by (b).

(d) Observe that if b is a measure, then we know by applying Proposition II.5.3 and using $\beta - 1/p \geq 1 - 1/(2H)$ that any solution X to (II.1.1) verifies $X - B \in \mathcal{C}_{[0, T]}^{1/2+H}(L^2)$. Hence by the previous uniqueness statement, X is the pathwise unique solution to (II.1.1). \square

II.A Properties of fBm

II.A.1 Operator linking Brownian motion and fBm

In Lemma II.A.1 we prove a continuity property of the operator \mathcal{A} introduced in (II.2.1). This is crucial in order to show adaptedness in the proof of II.2.5(a) and Theorem II.3.4. The explicit form of the operator \mathcal{A} is:

$$\mathcal{A} = \tilde{\Pi}^{H-1/2} \mathcal{I}^{1/2-H} \tilde{\Pi}^{1/2-H}, \quad (\text{II.A.1})$$

where the operators $\tilde{\Pi}^h$ and \mathcal{I}^h are defined as follows:

$$\begin{aligned} \forall h \in \mathbb{R}, \quad (\tilde{\Pi}^h f)(t) &:= t^h f(t) - h \int_0^t s^{h-1} f(s) ds, \\ \forall h > 0, \quad (\mathcal{I}^h f)(t) &:= \frac{1}{\Gamma(h)} \int_0^t (t-s)^{h-1} f(s) ds. \end{aligned}$$

Lemma II.A.1. *Let $H < 1/2$. The operator \mathcal{A} defined in (II.A.1) continuously maps the space $(\mathcal{C}_{[0,T]}, \|\cdot\|_\infty)$ to itself.*

Proof. The operator \mathcal{A} is linear, therefore it is sufficient to show that it is bounded. For $f \in \mathcal{C}_{[0,T]}$ and $t \in [0, T]$, we have by (II.A.1) that

$$\tilde{\Pi}^{H-1/2} \mathcal{I}^{1/2-H} \tilde{\Pi}^{1/2-H} f(t) = \sum_{i=1}^4 f_i(t),$$

where

$$\begin{aligned} f_1(t) &= t^{H-1/2} \frac{1}{\Gamma(1/2-H)} \int_0^t (t-y)^{-1/2-H} y^{1/2-H} f(y) dy \\ f_2(t) &= t^{H-1/2} (1/2-H) \frac{-1}{\Gamma(1/2-H)} \int_0^t (t-y)^{-1/2-H} \int_0^y x^{-1/2-H} f(x) dx dy \\ f_3(t) &= (1/2-H) \frac{1}{\Gamma(1/2-H)} \int_0^t s^{H-3/2} \int_0^s (s-y)^{-1/2-H} y^{1/2-H} f(y) dy ds \\ f_4(t) &= (1/2-H)^2 \frac{-1}{\Gamma(1/2-H)} \int_0^t s^{H-3/2} \int_0^s (s-y)^{-1/2-H} \int_0^y r^{-1/2-H} f(r) dr dy ds. \end{aligned}$$

We have, after doing a change of variables of the form $\xi = y/t$, uniformly on $[0, T]$, that

$$\begin{aligned} |f_1(t)| &\leq C t^{H-1/2} \|f\|_\infty \int_0^1 t^{-1/2-H} t^{1/2-H} t (1-z)^{-1/2-H} z^{1/2-H} dz \leq C T^{1/2-H} \|f\|_\infty; \\ |f_2(t)| &\leq C t^{H-1/2} \|f\|_\infty \int_0^t (t-y)^{-1/2-H} y^{1/2-H} dy \leq C T^{1/2-H} \|f\|_\infty; \\ |f_3(t)| &\leq C \|f\|_\infty \int_0^t s^{H-3/2} s^{1-2H} ds \leq C T^{1/2-H} \|f\|_\infty; \\ |f_4(t)| &\leq C \|f\|_\infty \int_0^t s^{H-3/2} \int_0^s (s-y)^{-1/2-H} y^{1/2-H} dy ds \leq C T^{1/2-H} \|f\|_\infty. \end{aligned}$$

Hence, the operator is bounded. It remains to prove that $\mathcal{A}(f)$ is a bounded continuous function. For f_1 , we perform the change of variables $z = y/t$ and it comes

$$f_1(t) = t^{1/2-H} \frac{1}{\Gamma(1/2-H)} \int_0^1 (1-z)^{-1/2-H} z^{1/2-H} f(tz) dz.$$

By the dominated convergence theorem, this is a continuous function. We proceed similarly for f_2 . As for $i = 3, 4$, we have

$$f_i(t) = \int_0^t g_i(s) ds$$

for functions g_i that are integrable on $[0, T]$, which implies the continuity of f_3 and f_4 . \square

II.A.2 Local nondeterminism of fBm and proof of Lemma II.5.1

The goal of this section is to prove Lemma II.5.1. As an intermediate step, we obtain first the following lemma, which is a local nondeterminism property for the fractional Brownian motion.

Lemma II.A.2. *Let $(B_t)_{t \geq 0}$ be an \mathbb{F} -fractional Brownian motion with $H \leq 1/2$. There exists $C > 0$ such that for any $0 \leq s < u < t$,*

$$\mathbb{E} [(\mathbb{E}^u[B_t] - \mathbb{E}^s[B_t])^2] \geq C(u-s)(t-s)^{2H-1}.$$

Note that in the above the conditional expectation is taken as usual with respect to \mathbb{F} and not with respect to the filtration generated by B .

Proof. The case $H = 1/2$ is trivial, so we assume $H < 1/2$. Recall that the process $W = \mathcal{A}(B)$, where \mathcal{A} is the operator given in (II.2.1), is an \mathbb{F} -Brownian motion. Moreover, by Theorem 11 in [87], it satisfies

$$B_t = \int_0^t K_H(t, r) dW_r, \tag{II.A.2}$$

where, for some $d_H > 0$,

$$K_H(t, r) = d_H \left[\left(\frac{t}{r} \right)^{H-1/2} (t-r)^{H-1/2} + (1/2-H)r^{1/2-H} \int_r^t z^{H-3/2} (z-r)^{H-1/2} dz \right].$$

Therefore

$$\begin{aligned} \mathbb{E} [(\mathbb{E}^u[B_t] - \mathbb{E}^s[B_t])^2] &= \mathbb{E} \left[\left(\mathbb{E}^u \left[\int_0^t K_H(t, r) dW_r \right] - \mathbb{E}^s \left[\int_0^t K_H(t, r) dW_r \right] \right)^2 \right] \\ &= \mathbb{E} \left[\left(\int_s^u K_H(t, r) dW_r \right)^2 \right] \\ &= \int_s^u K_H(t, r)^2 dr \\ &\geq (u-s) \min_{r \in [s, u]} K_H(t, r)^2. \end{aligned} \tag{II.A.3}$$

Notice that by the change of variables $\tilde{z} = z/r$,

$$\int_r^t z^{H-3/2}(z-r)^{H-1/2} dz = r^{2H-1} \beta_H(t/r),$$

for $\beta_H(\tau) = \int_1^\tau \tilde{z}^{H-3/2}(\tilde{z}-1)^{H-1/2} d\tilde{z}$. Hence,

$$K_H(t, r) = d_H \left[\left(\frac{t}{r} \right)^{H-1/2} (t-r)^{H-1/2} + (1/2-H)r^{H-1/2} \beta_H(t/r) \right].$$

Using that

$$\begin{aligned} \beta_H\left(\frac{t}{r}\right) &= \int_1^{t/r} z^{H-3/2}(z-1)^{H-1/2} dz \\ &\geq \int_1^{t/r} z^{2H-2} dz \\ &= \frac{1}{1-2H} \left(1 - \left(\frac{t}{r} \right)^{2H-1} \right) \end{aligned}$$

and subadditivity of $x \mapsto x^\alpha$ for $\alpha \in (0, 1)$ we get, for $0 < r < t$, that

$$\begin{aligned} \frac{1}{d_H} K_H(t, r) &\geq \left(\frac{t}{r} \right)^{H-1/2} (t-r)^{H-1/2} + \frac{1/2-H}{1-2H} r^{H-1/2} \left(1 - \left(\frac{t}{r} \right)^{2H-1} \right) \\ &\geq (t-r)^{H-1/2} \left(\left(\frac{r}{t} \right)^{1/2-H} + \frac{1}{2} (t-r)^{1/2-H} r^{H-1/2} \left(1 - \left(\frac{r}{t} \right)^{1-2H} \right) \right) \\ &\geq (t-r)^{H-1/2} \left(\left(\frac{r}{t} \right)^{1/2-H} + \frac{1}{2} (t^{1/2-H} - r^{1/2-H}) r^{H-1/2} \left(1 - \left(\frac{r}{t} \right)^{1-2H} \right) \right) \\ &= \frac{1}{2} (t-r)^{H-1/2} \left(\left(\frac{r}{t} \right)^{1/2-H} + \left(\frac{t}{r} \right)^{1/2-H} + \left(\frac{r}{t} \right)^{1-2H} - 1 \right) \\ &\geq \frac{1}{2} (t-r)^{H-1/2}, \end{aligned}$$

where the last line holds true as $x + x^{-1} \geq 2$ for $x > 0$. Plugging this into (II.A.3) we get that

$$\begin{aligned} \mathbb{E} [(\mathbb{E}^u[B_t] - \mathbb{E}^s[B_t])^2] &\geq (u-s) \min_{r \in [s, u]} K_H(t, r)^2 \\ &\geq \frac{d_H^2}{4} (u-s)(t-s)^{2H-1}. \end{aligned}$$

□

Proof of Lemma II.5.1. Proof of (a): Notice that $B_{t_2} - \mathbb{E}^{t_1}[B_{t_2}]$ is Gaussian with zero mean and variance σ_{t_1, t_2}^2 . Furthermore, it is independent of \mathcal{F}_{t_1} , which can be seen using the integral representation (see (II.A.2)). Hence, we have

$$\begin{aligned} \mathbb{E}^{t_1}[f(B_{t_2}, \Xi)] &= \mathbb{E}^{t_1}[f(B_{t_2} - \mathbb{E}^{t_1}[B_{t_2}] + \mathbb{E}^{t_1}[B_{t_2}], \Xi)] \\ &= G_{\sigma_{t_1, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}], \Xi). \end{aligned}$$

Proof of (b): By the local nondeterminism property of fBm (see Lemma 7.1 in [89]), there exists $C > 0$ such that

$$\sigma_{t_1, t_2}^2 = C(t_2 - t_1)^{2H}. \quad (\text{II.A.4})$$

To see that (b) holds true we use (a), Lemma A.1.7(a), the embedding $\mathcal{B}_p^\gamma \hookrightarrow \mathcal{B}_\infty^{\gamma-1/p}$ and (II.A.4) to get that

$$\|\mathbb{E}^{t_1}[f(B_{t_2}, \Xi)]\| \leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_\infty^{\gamma-1/p} \sigma_{t_1, t_2}^{\gamma-1/p}} \leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} (t_2 - t_1)^{H(\gamma-1/p)}.$$

Proof of (c): First notice that for fixed $x \in \mathbb{R}$, $t > 0$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ with $\|\phi\|_{\mathcal{C}^1} < \infty$,

$$|\phi(x) - G_t \phi(x)| \leq C \|\phi\|_{\mathcal{C}^1} \sqrt{t}. \quad (\text{II.A.5})$$

To see that (c) holds true note that due to (a), (II.A.5) and Cauchy-Schwarz's inequality we get

$$\begin{aligned} \mathbb{E}[|f(B_{t_2}, \Xi) - \mathbb{E}^{t_1}[f(B_{t_2}, \Xi)]|] &= \mathbb{E}[|f(B_{t_2}, \Xi) - G_{\sigma_{t_1, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}], \Xi)|] \\ &\leq \mathbb{E}[|f(B_{t_2}, \Xi) - f(\mathbb{E}^{t_1}[B_{t_2}], \Xi)|] + \mathbb{E}[|f(\mathbb{E}^{t_1}[B_{t_2}], \Xi) - G_{\sigma_{t_1, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}], \Xi)|] \\ &\leq \mathbb{E}[\|f(\cdot, \Xi)\|_{\mathcal{C}^1} |B_{t_2} - \mathbb{E}^{t_1}[B_{t_2}]|] + C \mathbb{E}[\|f(\cdot, \Xi)\|_{\mathcal{C}^1}] \sigma_{t_1, t_2} \\ &\leq C \| \|f(\cdot, \Xi)\|_{\mathcal{C}^1} \|_{L^2} (t_2 - t_1)^H, \end{aligned}$$

where the last line holds true by (II.A.4) and as

$$\|(B_{t_2} - \mathbb{E}^{t_1}[B_{t_2}])\|_{L^2} \leq \|B_{t_2} - B_{t_1}\|_{L^2} + \|B_{t_1} - \mathbb{E}^{t_1}[B_{t_2}]\|_{L^2} \leq 2(t_2 - t_1)^H$$

by Jensen's inequality for conditional expectations.

Proof of (d): In order to prove (d) we first state the following inequality, for some $C > 0$: for any $t_1 < \tilde{t} < t_2$,

$$\hat{\sigma}_{t_1, \tilde{t}, t_2}^2 := \text{Var}(\mathbb{E}^{\tilde{t}}[B_{t_2}] - \mathbb{E}^{t_1}[B_{t_2}]) \geq C(\tilde{t} - t_1)(t_2 - t_1)^{-1+2H}. \quad (\text{II.A.6})$$

The above inequality holds true by Lemma II.A.2. Notice that $\mathbb{E}^{\tilde{t}}[B_{t_2}] - \mathbb{E}^{t_1}[B_{t_2}]$ is Gaussian with mean zero and variance $\hat{\sigma}_{t_1, \tilde{t}, t_2}^2$ and it is independent of \mathcal{F}_{t_1} . Using this, (a) and Hölder's inequality for $q = p/n \geq 1$ and $q' = q/(q-1)$, we get

$$\begin{aligned} \mathbb{E}^{t_1} \left[|\mathbb{E}^{\tilde{t}}[f(B_{t_2}, \Xi)]|^n \right] &= \mathbb{E}^{t_1} [|G_{\sigma_{\tilde{t}, t_2}^2} f(\mathbb{E}^{\tilde{t}}[B_{t_2}], \Xi)|^n] \\ &= \int g_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2}(z) |G_{\sigma_{\tilde{t}, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}] + z, \Xi)|^n dz \\ &\leq \|g_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2}\|_{L^{q'}(\mathbb{R})} \left\| \left(G_{\sigma_{\tilde{t}, t_2}^2} f(\mathbb{E}^{t_1}[B_{t_2}] + \cdot, \Xi) \right)^n \right\|_{L^q(\mathbb{R})} \\ &= \|G_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2} \delta_0\|_{L^{q'}(\mathbb{R})} \|G_{\sigma_{\tilde{t}, t_2}^2} f(\cdot, \Xi)\|_{L^p(\mathbb{R})}^n. \end{aligned}$$

Using Lemma A.1.7(a), (II.A.4), (II.A.6) and that $\|\delta_0\|_{\mathcal{B}_x^{-1+1/x}} < \infty$ for $x \geq 1$, we get

$$\begin{aligned} \|G_{\hat{\sigma}_{t_1, \tilde{t}, t_2}^2} \delta_0\|_{L^{q'}(\mathbb{R})} \|G_{\sigma_{\tilde{t}, t_2}^2} f(\cdot, \Xi)\|_{L^p(\mathbb{R})}^n &\leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma}^n \|\delta_0\|_{\mathcal{B}_q^{-n/p} \sigma_{\tilde{t}, t_2}^{\gamma n} \hat{\sigma}_{t_1, \tilde{t}, t_2}^{-n/p}} \\ &\leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma}^n \|\delta_0\|_{\mathcal{B}_q^{-n/p}} (t_2 - \tilde{t})^{H\gamma n} (\tilde{t} - t_1)^{-n/(2p)} (t_2 - t_1)^{n(1-2H)/(2p)}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}^{t_1} \left[\left| \mathbb{E}^{\tilde{t}} [f(B_{t_2}, \Xi)] \right|^n \right] \\ \leq C \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma}^n \|\delta_0\|_{\mathcal{B}_q^{-n/p}} (t_2 - \tilde{t})^{H\gamma n} (\tilde{t} - t_1)^{-n/(2p)} (t_2 - t_1)^{n(1-2H)/(2p)}. \end{aligned} \quad (\text{II.A.7})$$

After taking expectations in (II.A.7) and raising both sides to the power $1/n$, we obtain (d). \square

II.B Regularising properties of the fBm

In this section we state and prove some results that are crucial in the current chapter, Chapter III and Chapter IV. The statements and their proofs, that heavily rely on stochastic sewing, are extensions to fBm of Lemma 6.1, its corollaries and Lemma 5.2 in [8].

Lemma II.B.1. *Let $\gamma \in (-1/(2H), 0)$, $m \in [2, \infty)$, $n \in [m, \infty]$, $p \in [n, \infty]$ and $e \in \mathbb{N}$. Then there exists a constant $C > 0$ such that for any $0 \leq S \leq T$, any \mathcal{F}_S -measurable random variable Ξ in \mathbb{R}^e and any bounded measurable function $f : \mathbb{R} \times \mathbb{R}^e \rightarrow \mathbb{R}$ fulfilling*

$$(i) \quad \mathbb{E} \left[\|f(\cdot, \Xi)\|_{\mathcal{C}^1}^2 \right] < \infty;$$

$$(ii) \quad \mathbb{E} \left[\|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma}^n \right] < \infty,$$

we have for any $t \in [S, T]$ that

$$\left\| \mathbb{E}^S \left[\left| \int_S^t f(B_r, \Xi) dr \right|^{m\gamma} \right]^{1/m} \right\|_{L^n} \leq C \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (t - S)^{1+H(\gamma-1/p)}. \quad (\text{II.B.1})$$

Proof. In order to show (II.B.1), we will apply Lemma A.2.1. For $S \leq s \leq t \leq T$, let

$$A_t := \int_S^t f(B_r, \Xi) dr \quad \text{and} \quad A_{s,t} := \mathbb{E}^s \left[\int_s^t f(B_r, \Xi) dr \right].$$

Notice that we have $\mathbb{E}^s[\delta A_{s,u,t}] = 0$, so (A.2.1) trivially holds.

In order to establish inequality (A.2.2), we show that

$$\|\delta A_{s,u,t}\|_{L^n} \leq \Gamma_2(u - S)^{-\alpha_2} (t - s)^{1/2+\varepsilon_2} \quad (\text{II.B.2})$$

holds true for some $\alpha_2 \in [0, 1/2)$ and $\varepsilon_2 > 0$, which is sufficient by the tower property and conditional Jensen's inequality. For $u = (s+t)/2$ we have by Minkowski's integral inequality, Jensen's inequality

for conditional expectation and Lemma II.5.1(d) that

$$\begin{aligned}
 \|\delta A_{s,u,t}\|_{L^n} &\leq \left\| \mathbb{E}^s \left[\int_u^t f(B_r, \Xi) dr \right] \right\|_{L^n} + \left\| \mathbb{E}^u \left[\int_u^t f(B_r, \Xi) dr \right] \right\|_{L^n} \\
 &\leq \int_u^t (\|\mathbb{E}^s f(B_r, \Xi)\|_{L^n} + \|\mathbb{E}^u f(B_r, \Xi)\|_{L^n}) dr \\
 &\leq 2 \int_u^t \|\mathbb{E}^u f(B_r, \Xi)\|_{L^n} dr \\
 &\leq C \int_u^t \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (r-u)^{H\gamma} (u-S)^{-\frac{1}{2p}} (r-S)^{\frac{1-2H}{2p}} dr \\
 &\leq C \int_u^t \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (r-u)^{H\gamma} (u-S)^{-\frac{H}{p}} dr \\
 &\leq C \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (t-u)^{1+H\gamma} (u-S)^{-\frac{H}{p}},
 \end{aligned}$$

where the penultimate inequality holds true as $r-S \leq 2(u-S)$. Hence, we have (II.B.2) for $\varepsilon_2 = 1/2 + H\gamma > 0$ and $\alpha_2 = H/p < 1/2$.

Let $t \in [S, T]$. Let $(\Pi_k)_{k \in \mathbb{N}}$ be a sequence of partitions of $[S, t]$ with mesh size converging to zero. For each k , denote $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$. By Lemma II.5.1(c) we have that

$$\begin{aligned}
 \|\mathcal{A}_t - \sum_i A_{t_i^k, t_{i+1}^k}\|_{L^1} &\leq \sum_i \int_{t_i^k}^{t_{i+1}^k} \|f(B_r, \Xi) - \mathbb{E}^{t_i^k} f(B_r, \Xi)\|_{L^1} dr \\
 &\leq C \| \|f(\cdot, \Xi)\|_{C^1} \|_{L^2} (t-S) |\Pi_k|^H \longrightarrow 0.
 \end{aligned}$$

Hence (A.2.3) holds true.

Applying Lemma A.2.1, we get

$$\begin{aligned}
 \|\mathbb{E}^S[|\mathcal{A}_t - \mathcal{A}_S|^m]^{1/m}\|_{L^n} &\leq \|A_{S,t}\|_{L^n} + C \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} \left(\int_S^t (r-S)^{-\frac{2H}{p}} dr \right)^{\frac{1}{2}} (t-S)^{\frac{1}{2}+H\gamma} \\
 &\leq \|A_{S,t}\|_{L^n} + C \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (t-S)^{1+H(\gamma-1/p)}.
 \end{aligned}$$

Applying Minkowski's integral inequality and Lemma II.5.1(b), we get that

$$\begin{aligned}
 \|A_{S,t}\|_{L^n} &= \left\| \mathbb{E}^S \int_S^t f(B_r, \Xi) dr \right\|_{L^n} \\
 &\leq \int_S^t \|\mathbb{E}^S f(B_r, \Xi)\|_{L^n} dr \\
 &\leq C \int_S^t \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (r-S)^{H(\gamma-1/p)} dr \\
 &\leq C \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (t-S)^{1+H(\gamma-1/p)}.
 \end{aligned}$$

Hence

$$\|\mathbb{E}^S[|\mathcal{A}_t - \mathcal{A}_S|^m]^{1/m}\|_{L^n} \leq C \| \|f(\cdot, \Xi)\|_{\mathcal{B}_p^\gamma} \|_{L^n} (t-S)^{1+H(\gamma-1/p)}.$$

□

Corollary II.B.2. *Let $\gamma \in (-1/(2H), 0)$, $m \in [2, \infty)$ and $p \in [m, \infty]$. Let $\lambda \in [0, 1]$ and assume that $\gamma > -1/(2H) + \lambda$. There exists $C > 0$ such that for any $f \in C_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\gamma$, any $0 \leq s \leq t \leq T$ and any \mathcal{F}_s -measurable random variables $\kappa_1, \kappa_2 \in L^m$, one has*

$$\begin{aligned} & \left\| \int_s^t (f(B_r + \kappa_1) - f(B_r + \kappa_2)) dr \right\|_{L^m} \\ & \leq C \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2\|_{L^m}^\lambda (t - s)^{1+H(\gamma-\lambda-1/p)}. \end{aligned} \quad (\text{II.B.3})$$

Proof. We aim to apply Lemma II.B.1 for the function $(z, (x_1, x_2)) \mapsto f(z + x_1) - f(z + x_2)$ with $m = n$ and $\Xi = (\kappa_1, \kappa_2)$. By Lemma A.1.8(b) and Jensen's inequality, we have that

$$\begin{aligned} & \|f(\cdot + \kappa_1) - f(\cdot + \kappa_2)\|_{\mathcal{B}_p^{\gamma-\lambda}} \|L^m\| \leq C \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2\|_{L^m}^\lambda \|L^m\| \\ & \leq C \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2\|_{L^m}^\lambda. \end{aligned}$$

Moreover

$$\|f(\cdot + \kappa_1) - f(\cdot + \kappa_2)\|_{C^1} \leq 2\|f\|_{C^1} < \infty.$$

Therefore all assumptions of Lemma II.B.1 are fulfilled and the result follows. \square

Corollary II.B.3. *Let $\gamma \in (-1/(2H), 0)$, $m \in [2, \infty)$ and $p \in [m, \infty]$. Let $\lambda, \lambda_1, \lambda_2 \in (0, 1]$ and assume that $\gamma > -1/(2H) + \lambda$ and $\gamma > -1/(2H) + \lambda_1 + \lambda_2$. There exists a constant $C > 0$ such that for any $f \in C_b^\infty(\mathbb{R}) \cap \mathcal{B}_p^\gamma$, any $0 \leq s \leq u \leq t \leq T$, any \mathcal{F}_s -measurable random variables $\kappa_1, \kappa_2 \in L^m$ and any \mathcal{F}_u -measurable random variables $\kappa_3, \kappa_4 \in L^m$, we have that*

$$\begin{aligned} & \left\| \int_u^t (f(B_r + \kappa_1) - f(B_r + \kappa_2) - f(B_r + \kappa_3) + f(B_r + \kappa_4)) dr \right\|_{L^m} \\ & \leq C \|f\|_{\mathcal{B}_p^\gamma} \|\mathbb{E}^s[|\kappa_1 - \kappa_3|^m]^{1/m}\|_{L^\infty}^{\lambda_2} \|\kappa_1 - \kappa_2\|_{L^m}^{\lambda_1} (t - u)^{1+H(\gamma-\lambda_1-\lambda_2-1/p)} \\ & \quad + C \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2 - \kappa_3 + \kappa_4\|_{L^m}^\lambda (t - u)^{1+H(\gamma-\lambda-1/p)}. \end{aligned} \quad (\text{II.B.4})$$

Proof. Let $h : \mathbb{R} \times \mathbb{R}^4 \rightarrow \mathbb{R}$ be defined by

$$h : (z, (x_1, x_2, x_3, x_4)) \mapsto f(z + x_1) - f(z + x_2) - f(z + x_3) + f(z + x_3 + x_2 - x_1).$$

and $g : \mathbb{R} \times \mathbb{R}^4 \rightarrow \mathbb{R}$ by

$$g : (z, (x_1, x_2, x_3, x_4)) \mapsto f(z + x_4) - f(z + x_3 + x_2 - x_1).$$

Let $\Xi = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$. Hence the integrand on the left hand side of (II.B.4) is $h(B_r, \Xi) + g(B_r, \Xi)$. The proof will consist in applying Lemma II.B.1 to the integral of h , and Corollary II.B.2 to the integral of g . By Lemma A.1.8(c), we have

$$\|h(\cdot, \Xi)\|_{\mathcal{B}_p^{\gamma-\lambda_1-\lambda_2}} \leq \|f\|_{\mathcal{B}_p^\gamma} \|\kappa_1 - \kappa_2\|_{L^m}^{\lambda_1} \|\kappa_1 - \kappa_3\|_{L^m}^{\lambda_2}. \quad (\text{II.B.5})$$

Using that $\kappa_1 - \kappa_2$ is \mathcal{F}_s -measurable, by the tower property, Jensen's inequality for conditional expectation and (II.B.5), we get that

$$\begin{aligned} & \mathbb{E} \|h(\cdot, \Xi)\|_{\mathcal{B}_p^{\gamma-\lambda_1-\lambda_2}}^m \leq \|f\|_{\mathcal{B}_p^\gamma}^m \mathbb{E} \left[\|\kappa_1 - \kappa_2\|_{L^m}^{m\lambda_1} \mathbb{E}^s[|\kappa_1 - \kappa_3|^{m\lambda_2}] \right] \\ & \leq \|f\|_{\mathcal{B}_p^\gamma}^m \mathbb{E} \left[\|\kappa_1 - \kappa_2\|_{L^m}^{m\lambda_1} \right] \left\| \mathbb{E}^s[|\kappa_1 - \kappa_3|^{m\lambda_2}] \right\|_{L^\infty} \\ & \leq \|f\|_{\mathcal{B}_p^\gamma}^m \|\kappa_1 - \kappa_2\|_{L^m}^{m\lambda_1} \|\mathbb{E}^s[|\kappa_1 - \kappa_3|^m]^{1/m}\|_{L^\infty}^{m\lambda_2}. \end{aligned}$$

Furthermore

$$\|h(\cdot, \Xi)\|_{\mathcal{C}^1} \leq 4\|f\|_{\mathcal{C}^1}.$$

Hence, we get the result by applying Lemma II.B.1 for $S = u$ to the integral of h and Corollary II.B.2 to the integral of g . \square

As another consequence of the stochastic sewing Lemma, we finally prove Lemma II.7.1.

Proof of Lemma II.7.1. We assume that $[\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,n})} < \infty$, otherwise (II.7.1) trivially holds. For $(\tilde{s}, \tilde{t}) \in \Delta_{[s,t]}$, let

$$A_{\tilde{s}, \tilde{t}} := \int_{\tilde{s}}^{\tilde{t}} f(B_r + \psi_{\tilde{s}}) dr \text{ and } \mathcal{A}_{\tilde{t}} := \int_s^{\tilde{t}} f(B_r + \psi_r) dr. \quad (\text{II.B.6})$$

In the following, we check the necessary conditions in order to apply Lemma A.2.1. In order to show that (A.2.1) and (A.2.2) hold true with $\alpha_2 = \beta_2 = 0$, $\varepsilon_1 = H(\gamma - 1/p - 1) + \alpha > 0$ and $\varepsilon_2 = 1/2 + H(\gamma - 1/p) > 0$, we show that there exists a constant $C > 0$ independent of s, t, \tilde{s} and \tilde{t} such that

$$(i) \quad \|\mathbb{E}^{\tilde{s}} \delta A_{\tilde{s}, u, \tilde{t}}\|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} [\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,n})} (\tilde{t} - \tilde{s})^{1+H(\gamma-1-1/p)+\alpha}.$$

$$(ii) \quad \|\mathbb{E}^s [|\delta A_{\tilde{s}, u, \tilde{t}}|^m]^{1/m}\|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} (\tilde{t} - \tilde{s})^{1+H(\gamma-1/p)}.$$

(iii) If (i) and (ii) are verified, then (A.2.3) gives the convergence in probability of $\sum_{i=0}^{N_n-1} A_{t_i^k, t_{i+1}^k}$ along any sequence of partitions $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ of $[s, \tilde{t}]$ with mesh converging to 0. We will show that the limit is the process \mathcal{A} given in (II.B.6).

Assume for now that (i), (ii) and (iii) hold. Applying Lemma A.2.1, we obtain that

$$\begin{aligned} \left\| \mathbb{E}^s \left[\left| \int_{\tilde{s}}^{\tilde{t}} f(B_r + \psi_r) dr \right|^m \right]^{1/m} \right\|_{L^n} &\leq C \|f\|_{\mathcal{B}_p^\gamma} (\tilde{t} - \tilde{s})^{1+H(\gamma-1/p)} \\ &\quad + C \|f\|_{\mathcal{B}_p^\gamma} [\psi]_{\mathcal{C}_{[s,t]}^\alpha(L^{m,n})} (\tilde{t} - \tilde{s})^{1+H(\gamma-1-1/p)+\alpha} \\ &\quad + \|\mathbb{E}^s [A_{\tilde{s}, \tilde{t}}^m]^{1/m}\|_{L^n}. \end{aligned}$$

In (II.B.8) we will see that $\|\mathbb{E}^s [A_{\tilde{s}, \tilde{t}}^m]^{1/m}\|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} (\tilde{t} - \tilde{s})^{1+H(\gamma-1/p)}$. Then, choosing $(\tilde{s}, \tilde{t}) = (s, t)$ we get (II.7.1).

We now check that the conditions (i), (ii) and (iii) actually hold.

Proof of (i): For $s \leq \tilde{s} \leq u \leq \tilde{t} \leq t$, we have

$$\delta A_{\tilde{s}, u, \tilde{t}} = \int_u^{\tilde{t}} f(B_r + \psi_{\tilde{s}}) - f(B_r + \psi_u) dr.$$

Hence, by the tower property of conditional expectation and Fubini's Theorem, we get

$$|\mathbb{E}^{\tilde{s}} \delta A_{\tilde{s}, u, \tilde{t}}| = \left| \mathbb{E}^{\tilde{s}} \int_u^{\tilde{t}} \mathbb{E}^u [f(B_r + \psi_{\tilde{s}}) - f(B_r + \psi_u)] dr \right|.$$

Now using Lemma II.5.1(b) with the \mathcal{F}_u -measurable variable $\Xi = (\psi_{\tilde{s}}, \psi_u)$, Lemma A.1.8(b) for $\alpha = 1$ and again Fubini's Lemma, we obtain that

$$\begin{aligned} & \left| \mathbb{E}^{\tilde{s}} \int_u^{\tilde{t}} \mathbb{E}^u [f(B_r + \psi_{\tilde{s}}) - f(B_r + \psi_u)] dr \right| \\ & \leq \mathbb{E}^{\tilde{s}} \int_u^{\tilde{t}} \|f(\cdot + \psi_{\tilde{s}}) - f(\cdot + \psi_u)\|_{\mathcal{B}_p^{\gamma-1}}(r-u)^{H(\gamma-1-1/p)} dr \\ & \leq C \|f\|_{\mathcal{B}_p^\gamma} \int_u^{\tilde{t}} \mathbb{E}^{\tilde{s}} [|\psi_u - \psi_{\tilde{s}}|](r-u)^{H(\gamma-1-1/p)} dr. \end{aligned}$$

Hence we get

$$\|\mathbb{E}^{\tilde{s}} \delta A_{\tilde{s}, u, \tilde{t}}\|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} \int_u^{\tilde{t}} \|\mathbb{E}^{\tilde{s}} |\psi_u - \psi_{\tilde{s}}|\|_{L^n}(r-u)^{H(\gamma-1-1/p)} dr. \quad (\text{II.B.7})$$

By the conditional Jensen's inequality, we have

$$\|\mathbb{E}^{\tilde{s}} [|\psi_u - \psi_{\tilde{s}}|]\|_{L^n} \leq [\psi]_{C_{[s, \tilde{t}]}^\alpha(L^{m, n})}(u - \tilde{s})^\alpha.$$

Combining this with Equation (II.B.7), we get

$$\|\mathbb{E}^{\tilde{s}} \delta A_{\tilde{s}, u, \tilde{t}}\|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} [\psi]_{C_{[s, \tilde{t}]}^\alpha(L^{m, n})}(\tilde{t} - \tilde{s})^{1+H(\gamma-1-1/p)+\alpha}.$$

Proof of (ii): Note that by Jensen's inequality for conditional expectation, tower property and Lemma II.B.1 we have that

$$\begin{aligned} \|\mathbb{E}^s [|A_{\tilde{s}, \tilde{t}}|^m]^{1/m}\|_{L^n} &= \left(\mathbb{E} [(\mathbb{E}^s \mathbb{E}^{\tilde{s}} |A_{\tilde{s}, \tilde{t}}|^m)^{n/m}] \right)^{1/n} \leq \left(\mathbb{E} \mathbb{E}^s [\mathbb{E}^{\tilde{s}} [|A_{\tilde{s}, \tilde{t}}|^m]^{n/m}] \right)^{1/n} \\ &= \left(\mathbb{E} [\mathbb{E}^{\tilde{s}} [|A_{\tilde{s}, \tilde{t}}|^m]^{n/m}] \right)^{1/n} \\ &\leq C \|f\|_{\mathcal{B}_p^\gamma} (\tilde{t} - \tilde{s})^{1+H(\gamma-1/p)}. \end{aligned} \quad (\text{II.B.8})$$

After similarly controlling $\|\mathbb{E}^s [|A_{\tilde{s}, u}|^m]^{1/m}\|_{L^n}$ and $\|\mathbb{E}^s [|A_{u, \tilde{t}}|^m]^{1/m}\|_{L^n}$, we get

$$\|\mathbb{E}^s [|\delta A_{\tilde{s}, u, \tilde{t}}|^m]^{1/m}\|_{L^n} \leq C \|f\|_{\mathcal{B}_p^\gamma} (\tilde{t} - \tilde{s})^{1+H(\gamma-1/p)}.$$

The proof of (iii) can be done by similar arguments as for (iii) in the proof of Proposition II.5.3. \square

Existence of weak solutions with nonnegative drift via stochastic sewing



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III.1 Main result and proof

This chapter is based on [LA2]. In the previous chapter, we proved in Theorem II.2.8 the existence of a weak solution to (II.1.1) using the stochastic sewing lemma. In this chapter, we will use a refined approach, however working with the d -dimensional equation

$$dX_t = b(X_t)dt + dB_t, \tag{III.1.1}$$

where each component of b is assumed to be measure-valued.

The overall scheme of proof, via a tightness-stability argument is similar to Section II.7. However the nonnegativity of the drift allows to make use of the stochastic sewing lemma with random controls (see Lemma A.2.2). Throughout this chapter we work under the following assumption:

Assumption III.1.1. *Consider (b, β) such that b is a measure with*

$$b \in \mathcal{B}_\infty^\beta \text{ for } \beta \in \mathbb{R} \text{ with } \beta > -\frac{1}{2H}. \tag{III.1.2}$$

Our main result reads as follows:

Theorem III.1.2. *Let (b, β) satisfy Assumption III.1.1. Then there exists a weak solution to Equation (III.1.1) in the sense of Definition II.2.4.*

Note that, even though Assumption III.1.1 refines the conditions in Theorem II.2.5(i)-(iii), Theorem III.1.2 still gives a slightly different statement as Theorem II.2.5 since it gives a.s. convergence in (II.2.4).

Combining the fact that finite measures are in \mathcal{B}_1^0 (see Remark A.1.4) and Theorem III.1.2, we get the following:

Theorem III.1.3. *Let b be a finite \mathbb{R}^d -valued measure and $H < 1/(2d)$. Then there exists a weak solution to (III.1.1).*

The remainder of the section is dedicated to proving Theorem III.1.2. This will be done by regularizing the drift, considering the sequence of strong solutions to the corresponding approximated equations and proceeding via a tightness-stability argument. In order to do so we state two *a priori* estimates in Lemma III.1.4, quantifying the regularization effect of any solution. First, recall Lemma II.B.1 since it captures the regularization effect of the fBm. The lemma immediately transfers to the d -dimensional setting (for a proof see [59, Lemma 3.4]).

Intuitively the regularization effect of any solution X will be similar to the one of a fBm, since it is expected to have a similar oscillatory behaviour. This is because $X - B$ is expected to be more regular than B . For another perspective on this, note that any solution X has a jointly continuous local time by [20, Theorem 2.16], where the authors make use of the fact that $X - B$ is of finite variation.

Lemma III.1.4. (a) *Let $\beta < 0$. Let $m \in \mathbb{N}$. Then there exists a constant $C > 0$ such that for any b such that (b, β) satisfies Assumption III.1.1 and $(s, t) \in \Delta_{[0, T]}$, any weak solution X to (III.1.1) fulfills*

$$\|X_{s,t} - B_{s,t}\|_{L^m} \leq C \|b\|_{\mathcal{B}_\infty^\beta} \left(1 + \|b\|_{\mathcal{B}_\infty^\beta}^{\frac{-H\beta}{1+H(\beta-1)}}\right) (t-s)^{1+\beta H}, \quad (\text{III.1.3})$$

where by assumption $\frac{-H\beta}{1+H(\beta-1)} \in (0, \infty)$.

(b) *Let $\beta < 0$ and $m \in \mathbb{N}$. Let $\delta \in (0, 1 + H\beta)$. Then there exists a constant $C > 0$ and there exists a nonnegative random variable Z such that for $\phi, h \in \mathcal{C}_b^\infty(\mathbb{R}^d, \mathbb{R}^d) \cap \mathcal{B}_\infty^\beta$ such that (ϕ, β) and (h, β) satisfy Assumption III.1.1,*

$$\left| \int_s^t h(X_r) dr \right| \leq Z |t-s|^\delta, \quad (\text{III.1.4})$$

where X is the unique strong solution to (III.1.1) with drift ϕ . Moreover,

$$\|Z\|_{L^m} \leq C \|h\|_{\mathcal{B}_\infty^\beta} \left(1 + \|\phi\|_{\mathcal{B}_\infty^\beta}^{1 + \frac{-H\beta}{1+H(\beta-1)}}\right). \quad (\text{III.1.5})$$

Proof of Lemma III.1.4. (a): The proof is almost identical to Proposition II.5.3. Therein we use nonnegativity of the drift in order to obtain an increasing process for $d = 1$. The same can be done here for dimension $d \geq 1$ obtaining a process that is increasing in each component and therefore we omit most details (for a similar argument see the proof of (b)).

The proof of Proposition II.5.3 gives existence of a constant $\tilde{C} > 0$ such that for $(s, t) \in \Delta_{[0, T]}$ fulfilling $\tilde{C} \|b\|_{\mathcal{B}_\infty^\beta} (t-s)^{1+H(\beta-1)} < 1/2$ and $K = X - B$,

$$\|K_t - K_s\|_{L^m} \leq \tilde{C} \|b\|_{\mathcal{B}_\infty^\beta} (t-s)^{1+H\beta}. \quad (\text{III.1.6})$$

Choose $l = (3\tilde{C} \|b\|_{\mathcal{B}_\infty^\beta})^{\frac{1}{H(1-\beta)-1}}$ so that $\tilde{C} \|b\|_{\mathcal{B}_\infty^\beta} l^{1+H(\beta-1)} < 1/2$. Let $u \in [0, T]$. In particular, by (III.1.6),

$$[K]_{\mathcal{C}_{[u, (u+l) \wedge T]}^{1+H\beta}(L^{m, \infty})} \leq \tilde{C} \|b\|_{\mathcal{B}_\infty^\beta}. \quad (\text{III.1.7})$$

If $l > T$, then (III.1.3) follows from (III.1.7). Hence, we assume $l \leq T$. To obtain (III.1.3), we will iteratively apply inequality (III.1.7). Let $(s, t) \in \Delta_{[0, T]}$ be arbitrary. Let $N = \lceil T/l \rceil$ and let the sequence $(s_k)_{k=0}^N$ be defined by $s_k = s + k(t-s)/N$. By triangle inequality and (III.1.7), we get

$$\begin{aligned} \|K_t - K_s\|_{L^m} &\leq \sum_{k=1}^N \|K_{s_k} - K_{s_{k-1}}\|_{L^m} \\ &\leq \tilde{C} \|b\|_{\mathcal{B}_\infty^\beta} \sum_{k=1}^N (s_k - s_{k-1})^{1+H\beta} \\ &\leq \tilde{C} \|b\|_{\mathcal{B}_\infty^\beta} \sum_{k=1}^N \left(\frac{t-s}{N}\right)^{1+H\beta} \\ &\leq \tilde{C} \|b\|_{\mathcal{B}_\infty^\beta} N^{-H\beta} (t-s)^{1+H\beta}. \end{aligned}$$

Using $N \leq 1 + \frac{T}{l} \leq 2\frac{T}{l} \leq C \|b\|_{\mathcal{B}_\infty^\beta}^{\frac{1}{1+H(\beta-1)}}$, it follows that

$$\|b\|_{\mathcal{B}_\infty^\beta} N^{-H\beta} \leq C \|b\|_{\mathcal{B}_\infty^\beta} \|b\|_{\mathcal{B}_\infty^\beta}^{\frac{-H\beta}{1+H(\beta-1)}}$$

and therefore (III.1.3).

(b): By nonnegativity of ϕ , $K = X - B$ is monotone in each component. In particular $(v, w) \mapsto |K_w - K_v|$ defines a random control as the choice of norm on \mathbb{R}^d gives superadditivity (see list of notations). For $(s, t) \in \Delta_{[0, T]}$ let

$$A_{s,t} := \int_s^t h(B_r + K_s) dr.$$

We apply Lemma A.2.2 for $K^h = \mathcal{A}$ defined by $K^h = \int_0^\cdot h(X_r) dr$. In order to see that all conditions are fulfilled, we prove the following for $u \in [s, t]$ and some constant $C > 0$ independent of s, t and u :

- (i) $\|A_{s,t}\|_{L^m} \leq C \|h\|_{\mathcal{B}_\infty^\beta} (t-s)^{1+H\beta}$;
- (ii) $|\mathbb{E}^u[\delta A_{s,u,t}]| \leq C \|h\|_{\mathcal{B}_\infty^\beta} |K_u - K_s| (t-u)^{H(\beta-1)+1}$;
- (iii) $\sum_{i=0}^{N_n-1} A_{t_i^n, t_{i+1}^n} \xrightarrow{a.s.} K_t^h$ along any sequence of partitions $\Pi_n = \{t_i^n\}_{i=0}^{N_n}$ of $[0, t]$ with mesh converging to 0.

Notice that $1 + H\beta > 1/2$ and hence (i) gives (A.2.2). Furthermore, (ii) gives (A.2.1) for $\alpha_1 = H(\beta-1) + 1 > 1/2 - H \geq 0$, $\beta_1 = 1$ and $\lambda(s, t) := |K_t - K_s|$.

Assume, for the moment, that (i)-(ii)-(iii) hold true. Then by Lemma A.2.2, there exists a process D such that

$$|K_t^h - K_s^h - A_{s,t}| \leq C \|h\|_{\mathcal{B}_\infty^\beta} |K_t - K_s| (t-s)^{H(\beta-1)+1} + D_{s,t}, \quad (\text{III.1.8})$$

with $\|D_{s,t}\|_{L^m} \leq C\|h\|_{\mathcal{B}_\infty^\beta} (t-s)^{1+H\beta}$. Hence, by (i) and Lemma III.1.4(a) and as $H(\beta-1)+1 > 0$

$$\begin{aligned} \|K_t^h - K_s^h\|_{L^m} &\leq C\|h\|_{\mathcal{B}_\infty^\beta} \|K_t - K_s\|_{L^m} (t-s)^{H(\beta-1)+1} + C\|h\|_{\mathcal{B}_\infty^\beta} (t-s)^{1+H\beta} \\ &\leq C\|h\|_{\mathcal{B}_\infty^\beta} (1 + \|\phi\|_{\mathcal{B}_\infty^\beta}^{1+\frac{-H\beta}{1+H(\beta-1)}}) (t-s)^{1+H\beta}. \end{aligned}$$

The result now follows from Kolmogorov's continuity theorem after choosing m large enough, which is no restriction as (III.1.5) then also holds true for any smaller choice of m .

Let us now verify (i)-(ii)-(iii).

Proof of (i): By the d -dimensional version of Lemma II.B.1 applied to $\Xi = K_s$ and $f(z, x) = h(z+x)$, we have

$$\|A_{s,t}\|_{L^m} \leq C\| \|h(\cdot + K_s)\|_{\mathcal{B}_\infty^\beta} \|_{L^m} (t-s)^{1+H\beta}.$$

Using that $\|h(\cdot + K_s)\|_{\mathcal{B}_\infty^\beta} = \|h\|_{\mathcal{B}_\infty^\beta}$ (see [8, Lemma A.2] for $d = 1$, which easily generalizes to $d > 1$), we thus get

$$\|A_{s,t}\|_{L^m} \leq C\|h\|_{\mathcal{B}_\infty^\beta} (t-s)^{1+H\beta}. \quad (\text{III.1.9})$$

Proof of (ii): From the local nondeterminism property of fractional Brownian motion (see [89, Lemma 7.1]), we have that

$$\sigma_{s,t}^2 \geq C(t-s)^{2H}, \quad (\text{III.1.10})$$

where

$$\sigma_{s,t}^2 I_d = \text{Var}(B_t - \mathbb{E}^s[B_t]).$$

Then by [59, Lemma 3.3] applied to $\Xi = (K_s, K_u)$ and $f(z, (x_1, x_2)) = h(z+x_1) - h(z+x_2)$, we obtain

$$\begin{aligned} |\mathbb{E}^u[\delta A_{s,u,t}]| &= \left| \int_u^t \mathbb{E}^u[h(B_r + K_s) - h(B_r + K_u)] dr \right| \\ &= \left| \int_u^t G_{\sigma_{u,r}^2} h(\mathbb{E}^u[B_r] + K_s) - G_{\sigma_{u,r}^2} h(\mathbb{E}^u[B_r] + K_u) dr \right| \\ &\leq \int_u^t \|G_{\sigma_{u,r}^2} h\|_{\mathcal{C}^1} |K_u - K_s| dr. \end{aligned}$$

We now use [8, Lemma A.3 (iv)] which again easily generalizes to $d > 1$. Note that it can be used as we can assume w.l.o.g. that $\beta < 0$. Hence

$$\|G_{\sigma_{u,r}^2} h\|_{\mathcal{C}^1} \leq C\|h\|_{\mathcal{B}_\infty^\beta} (\sigma_{u,r}^2)^{(\beta-1)/2}.$$

The above, (III.1.10) and using that $H(\beta-1) > -1$ to ensure integrability gives

$$\begin{aligned} |\mathbb{E}^u[\delta A_{s,u,t}]| &\leq C \int_u^t |r-u|^{H(\beta-1)} \|h\|_{\mathcal{B}_\infty^\beta} |K_u - K_s| dr \\ &\leq C\|h\|_{\mathcal{B}_\infty^\beta} |K_u - K_s| (t-u)^{H(\beta-1)+1}. \end{aligned}$$

Proof of (iii): For a sequence $\Pi_n = \{t_i^n\}_{i=0}^{N_n}$ of partitions of $[0, t]$ with mesh size going to 0, we have

$$\begin{aligned}
 |K_t^h - \sum_{i=0}^{N_n-1} A_{t_i^n, t_{i+1}^n}| &\leq \sum_i \int_{t_i^n}^{t_{i+1}^n} |h(B_r + K_r) - h(B_r + K_{t_i^n})| dr \\
 &\leq \sum_i \int_{t_i^n}^{t_{i+1}^n} \|h\|_{C^1} |K_r - K_{t_i^n}| dr \\
 &\leq \sum_i \|h\|_{C^1} (t_{i+1}^n - t_i^n) |K_{t_{i+1}^n} - K_{t_i^n}| \\
 &\leq \|h\|_{C^1} |\Pi_n| |K_t - K_0| \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}
 \end{aligned}$$

□

Below we state the two propositions that ensure tightness and stability of the approximation scheme. The proofs are similar to the ones of Proposition II.7.4 and Proposition II.7.6. The only differences are allowing for $d \geq 1$ instead of $d = 1$ and that Assumption III.1.1 is weaker than the corresponding assumption therein. The latter is due to the crucial Lemma III.1.4(b) being stated under Assumption III.1.1. For the reader's convenience we prove both statements.

Proposition III.1.5 (Tightness). *Assume (b, β) fulfills Assumption III.1.1. Let $(b^n)_{n \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_\infty^{\beta-}$. For $n \in \mathbb{N}$, let X^n be the strong solution to (III.1.1) with initial condition x_0 and drift b^n . Then there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(X^{n_k}, B)_{k \in \mathbb{N}}$ converges weakly in the space $[\mathcal{C}_{[0, T]}]^2$.*

Proposition III.1.6 (Stability). *Assume (b, β) fulfills Assumption III.1.1. Let $(\tilde{b}^k)_{k \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_\infty^{\beta-}$. Let \tilde{B}^k have the same law as B and assume all \tilde{B}^k are defined on the same filtered probability space. We consider \tilde{X}^k to be the unique strong solution to (III.1.1) for $B = \tilde{B}^k$, initial condition x_0 and drift \tilde{b}^k . We assume that there exist stochastic processes $\tilde{X}, \tilde{B} : [0, T] \rightarrow \mathbb{R}^d$ such that $(\tilde{X}^k, \tilde{B}^k)_{k \in \mathbb{N}}$ converges to (\tilde{X}, \tilde{B}) on $[\mathcal{C}_{[0, T]}]^2$ in probability. Then \tilde{X} fulfills (II.2.3) and (II.2.4) from Definition II.2.4.*

Proof of Proposition III.1.5. Assume w.l.o.g. $\beta < 0$. Let $K_t^n := \int_0^t b^n(X_r^n) dr$. For $M > 0$, let

$$A_M := \{f \in \mathcal{C}_{[0, T]} : f(0) = 0, |f(t) - f(s)| \leq M(t - s)^{1+H\beta}, \forall (s, t) \in \Delta_{[0, T]}\}.$$

By Arzelà-Ascoli's Theorem, A_M is compact in $\mathcal{C}_{[0, T]}$. Applying Lemma III.1.4(a) and Markov's inequality we get

$$\begin{aligned}
 \mathbb{P}(K^n \notin A_M) &\leq \mathbb{P}(\exists (s, t) \in \Delta_{[0, T]} : |K_{s, t}^n| > M(t - s)^{1+H\beta}) \\
 &\leq C \sup_{n \in \mathbb{N}} (1 + \|b^n\|_{\mathcal{B}_\infty^\beta}^{1 + \frac{-H\beta}{1+H(\beta-1)}}) M^{-1}.
 \end{aligned}$$

Hence, the sequence $(K^n)_{n \in \mathbb{N}}$ is tight in $\mathcal{C}_{[0, T]}$. So $(K^n, B)_{n \in \mathbb{N}}$ is tight in $[\mathcal{C}_{[0, T]}]^2$. Thus by Prokhorov's Theorem, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(K^{n_k}, B)_{k \in \mathbb{N}}$ converges weakly in the space $[\mathcal{C}_{[0, T]}]^2$, and so does $(X^{n_k}, B)_{k \in \mathbb{N}}$. □

Proof of Proposition III.1.6. Assume w.l.o.g. that $\beta < 0$ and $X_0 = 0$. Let $\tilde{K} := \tilde{X} - \tilde{B}$, so that (II.2.3) is verified. Let now $(b^n)_{n \in \mathbb{N}}$ be any sequence of smooth bounded functions converging to b in $\mathcal{B}_\infty^{\beta-}$. In order to verify that \tilde{K} and \tilde{X} fulfill (II.2.4) from Definition II.2.4, we have to show that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r) dr - \tilde{K}_t \right| = 0 \text{ in probability.} \quad (\text{III.1.11})$$

By the triangle inequality we have that for $k, n \in \mathbb{N}$ and $t \in [0, T]$,

$$\begin{aligned} \left| \int_0^t b^n(\tilde{X}_r) dr - \tilde{K}_t \right| &\leq \left| \int_0^t b^n(\tilde{X}_r) dr - \int_0^t b^n(\tilde{X}_r^k) dr \right| + \left| \int_0^t b^n(\tilde{X}_r^k) dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr \right| \\ &\quad + \left| \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr - \tilde{K}_t \right| =: A_1 + A_2 + A_3. \end{aligned} \quad (\text{III.1.12})$$

We show that all summands on the right hand side of (III.1.12) converge to 0 uniformly on $[0, T]$ in probability as $n \rightarrow \infty$, choosing $k = k(n)$ accordingly.

A₁: Notice that

$$\begin{aligned} \left| \int_0^t b^n(\tilde{X}_r) dr - \int_0^t b^n(\tilde{X}_r^k) dr \right| &\leq \|b^n\|_{C^1} \int_0^t |\tilde{X}_r - \tilde{X}_r^k| dr \\ &\leq \|b^n\|_{C^1} T \sup_{t \in [0, T]} |\tilde{X}_t - \tilde{X}_t^k|. \end{aligned}$$

The result follows as for any $\varepsilon > 0$, by assumption, we can choose a sequence $(k(n))_{n \in \mathbb{N}}$ such that

$$\mathbb{P} \left(\|b^n\|_{C^1} T \sup_{t \in [0, T]} |\tilde{X}_t - \tilde{X}_t^{k(n)}| > \varepsilon \right) < \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

A₂: Let $\beta' \in (-1/(2H), \beta)$. By Lemma III.1.4(b) applied to \tilde{X}^k , $h = b^n - \tilde{b}^k$ and β' instead of β , we know that there exists a random variable $Z_{n,k}$ such that

$$\begin{aligned} \mathbb{E}[Z_{n,k}] &\leq C \|b^n - \tilde{b}^k\|_{\mathcal{B}_\infty^{\beta'}} \left(1 + \|\tilde{b}^k\|_{\mathcal{B}_\infty^{\beta'}}^{1 + \frac{-H\beta'}{1+H(\beta'-1)}} \right) \\ &\leq C (\|b^n - b\|_{\mathcal{B}_\infty^{\beta'}} + \|\tilde{b}^k - b\|_{\mathcal{B}_\infty^{\beta'}}) \left(1 + \sup_{m \in \mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_\infty^{\beta'}}^{1 + \frac{-H\beta'}{1+H(\beta'-1)}} \right), \end{aligned} \quad (\text{III.1.13})$$

for C independent of k, n and that we have

$$\sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r^k) dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr \right| \leq Z_{n,k} (1 + T).$$

Using Markov's inequality and (III.1.13) we obtain that

$$\begin{aligned} \mathbb{P} \left(\sup_{t \in [0, T]} \left| \int_0^t b^n(\tilde{X}_r^k) dr - \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr \right| > \varepsilon \right) &\leq \varepsilon^{-1} \mathbb{E}[Z_{n,k}] (1 + T) \\ &\leq C \varepsilon^{-1} (1 + T) (\|b^n - b\|_{\mathcal{B}_\infty^{\beta'}} + \|\tilde{b}^k - b\|_{\mathcal{B}_\infty^{\beta'}}) \left(1 + \sup_{m \in \mathbb{N}} \|\tilde{b}^m\|_{\mathcal{B}_\infty^{\beta'}}^{1 + \frac{-H\beta'}{1+H(\beta'-1)}} \right). \end{aligned}$$

Choosing $k = k(n)$ as before gives the convergence.

A₃: Recall that $\tilde{X}_t^k = \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr + \tilde{B}_t^k$. Hence, we get that

$$\sup_{t \in [0, T]} \left| \int_0^t \tilde{b}^k(\tilde{X}_r^k) dr - \tilde{K}_t \right| \leq \sup_{t \in [0, T]} (|\tilde{X}_t^k - \tilde{X}_t| + |\tilde{B}_t^k - \tilde{B}_t|).$$

Since by assumption $(\tilde{X}^k, \tilde{B}^k)_{k \in \mathbb{N}}$ converges to (\tilde{X}, \tilde{B}) on $[\mathcal{C}_{[0, T]}]^2$ in probability, we get the result. \square

Proof of Theorem III.1.2. Let $(b^n)_{n \in \mathbb{N}}$ be a sequence of smooth bounded functions converging to b in $\mathcal{B}_\infty^{\beta-}$. By Proposition III.1.5, there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $(X^{n_k}, B)_{k \in \mathbb{N}}$ converges weakly in $[\mathcal{C}_{[0, T]}]^2$. W.l.o.g., we assume that $(X^n, B)_{n \in \mathbb{N}}$ converges weakly. By the Skorokhod representation Theorem, there exists a sequence of random variables $(Y^n, \hat{B}^n)_{n \in \mathbb{N}}$ defined on a common probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, such that

$$\text{Law}(Y^n, \hat{B}^n) = \text{Law}(X^n, B), \quad \forall n \in \mathbb{N}, \quad (\text{III.1.14})$$

and (Y^n, \hat{B}^n) converges a.s. to some (Y, \hat{B}) in $[\mathcal{C}_{[0, T]}]^2$. As X^n solves the SDE (III.1.1) with drift b^n , we know by (III.1.14) that Y^n also solves (III.1.1) with drift b^n and \hat{B}^n instead of B . Since X^n is a strong solution, we have that X^n is adapted to \mathbb{F}^B . Hence by (III.1.14), we know that Y^n is adapted to $\mathbb{F}^{\hat{B}^n}$ as the conditional laws of Y^n and X^n agree and therefore Y^n is a strong solution to (III.1.1) with \hat{B}^n instead of B .

By Proposition III.1.6, we know that Y fulfills (II.2.3) and (II.2.4) from Definition II.2.4 with \hat{B} instead of B . Clearly Y is adapted to the filtration $\hat{\mathbb{F}}$ defined by $\hat{\mathcal{F}}_t := \sigma(Y_s, \hat{B}_s, s \in [0, t])$. By (II.2.2) and (II.2.1), we have

$$\hat{B}^n = \bar{\mathcal{A}}(W^n) \text{ a.s. and} \quad (\text{III.1.15})$$

$$W^n = \mathcal{A}(\hat{B}^n) \text{ a.s.}, \quad (\text{III.1.16})$$

where W^n is a sequence of Brownian motions with $\mathbb{F}^{\hat{B}^n} = \mathbb{F}^{W^n}$. By definition of a Brownian motion with respect to a filtration, for $(s, t) \in \Delta_{[0, T]}$, $W_t^n - W_s^n$ is independent of $\mathcal{F}_s^{W^n} = \mathcal{F}_s^{\hat{B}^n} = \sigma(Y_r^n, \hat{B}_r^n, r \in [0, s])$. By (III.1.15), (III.1.16), Lemma II.A.1 and the a.s.-convergence of \hat{B}^n , we know that W^n converges a.s. uniformly on $[0, T]$ to a Bm W such that $\hat{B} = \bar{\mathcal{A}}(W)$. Moreover \hat{B} and W generate the same filtration. Hence, we deduce that $W_t - W_s$ is independent of $\hat{\mathcal{F}}_s$ and so W is an $\hat{\mathbb{F}}$ -Bm. Therefore, \hat{B} is a $\hat{\mathbb{F}}$ -fBm and Y is adapted to $\hat{\mathbb{F}}$. Hence Y is a weak solution. \square

III.A Summary of the literature

The following Theorem gives an overview of the developments for Equation (II.1.1) with $H \neq 1/2$. In (a) as well as partly in (b) and (d) a nonlinear Young integral approach was followed. To ensure readability, some results do not represent the full scope of the actual results proven. Below we partly also allow for time-dependent drift b .

Theorem III.A.1. (a) [24, Theorem 1.9] combined with [51, Theorem 3.13]: Let $b \in \mathcal{B}_\infty^\beta$ for $\beta > 1 - 1/(2H)$. Then there exists a strong solution to (II.1.1) and path-by-path uniqueness holds (i.e. uniqueness to the integral equation for almost every realization of the noise, giving a stronger notion of uniqueness than the classical notion of pathwise uniqueness)

- (b) *Theorem II.2.8, Theorem II.2.10 and Corollary II.2.6: Let $d = 1$. For $H < \sqrt{2} - 1$, there exists a weak solution for a finite measure b . Additionally there exists a weak solution for $b \in \mathcal{B}_\infty^\beta$ for $\beta > 1/2 - 1/(2H)$. Pathwise uniqueness and strong existence is shown for $b \in \mathcal{B}_p^\beta$ for $\beta \in \mathbb{R}$ and $p \in [1, \infty]$ with*

$$\beta > 1 - \frac{1}{2H} \text{ and } \beta - \frac{1}{p} \geq 1 - \frac{1}{2H}.$$

- (c) *[59, Theorem 2.3]: Generalization of the latter two results in (b) to d dimensions; i.e. there exists a weak solution for $b \in \mathcal{B}_\infty^\beta$ with $\beta > 1/2 - 1/(2H)$ and pathwise uniqueness and strong existence holds for $b \in \mathcal{B}_p^\beta$ with*

$$\beta > 1 - \frac{1}{2H} \text{ and } \beta - d/p \geq 1 - \frac{1}{2H}.$$

- (d) *[49, Theorem 1.4]: Strong existence, path-by-path uniqueness, Malliavin differentiability and existence of a flow for time-dependent drift $b \in L^q([0, T], \mathcal{B}_\infty^\beta)$ for $q \in (1, 2]$ and $\beta > 1 - 1/(Hq')$. Additionally weak existence for $b \in L^q([0, T], \mathcal{B}_\infty^\beta)$ for $q \in (2, \infty]$ and*

$$\beta > \left(1 - \frac{1}{Hq'}\right) \vee \left(\frac{1}{2} - \frac{1}{2H}\right)$$

is shown.

- (e) *[20, Theorem 2.11 and Theorem 2.14]: For a finite measure b : Existence of a weak solution for $H < 1/(1 + d)$ for any $d \in \mathbb{N}$; pathwise uniqueness and existence of a strong solution for $H < (\sqrt{13} - 3)/2$ and $d = 1$.*

- (f) *Theorem III.1.2: Let $d \in \mathbb{N}$. Existence of a weak solution for nonnegative $b \in \mathbb{B}_\infty^\beta$ with $\beta > -1/(2H)$. In particular, existence of a weak solution for a finite measure b and $H < 1/(2d)$.*

Remark III.A.2. In particular, (f) can be seen as an extension of the existence result in (b) and as a slightly weaker result than (e) for a finite measure b .

Finally let us mention that also similar equations were investigated, such as the case of local time drift [2, 9], distribution-dependent drift [51], multiplicative fractional noise [33], Lévy noise [67], “infinitely regularizing” noises [62], regular noise [58], generic noises [50] and equations with reflection or additional perturbation [55].

Properties of the density of a solution

IV

Contents

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IV.1 Introduction

We are interested in the properties of the law of SDEs driven by an \mathbb{R}^d -valued fractional Brownian motion with a distributional, time-dependent drift:

$$X_t = X_0 + \int_0^t b(s, X_s) ds + B_t, \quad t \in [0, T]. \quad (\text{IV.1.1})$$

We consider drifts $b \in L^q([0, T], \mathcal{B}_\infty^\beta)$ with parameters satisfying:

$$H \in (0, \frac{1}{2}), \quad q \in (1, 2], \quad 1 - \frac{1}{Hq'} < \beta < 0. \quad (\text{A})$$

Recall that, for time-independent drift b , the above condition on the spatial regularity of b coincides with the one required for well-posedness of (IV.1.1), given in Chapter II. However, even for $b \in L^2([0, T], \mathcal{B}_p^\beta)$, (A) is not asking for more spatial regularity. The choice of condition (A) is due to the recent work of Galeati and Gerencsér [49] which is extending the well-posedness results in Chapter II to time-dependent drifts.

Since we obtain both Gaussian tails of the density as well as regularity properties, we briefly summarize the related literature in the fractional Brownian case.

Gaussian tails. Let us first mention the following related literature for smooth drifts. Besalú et al. [16] obtained Gaussian bounds for additive one-dimensional drifts with $H \in (0, 1)$ as well as for multiplicative noise if a Doss-Sussman-type transformation is possible. Also, results in higher dimensions for $H > 1/2$ are given. For $H > 1/4$, Baudoin et al. [14] showed existence of a strictly positive density together with Gaussian-like bounds, considering the case of multiplicative noise $H > 1/4$, interpreting the equation as a rough integral equation.

Li et al. [76] obtained Gaussian upper and lower bounds for the density of the law of (IV.1.1) under mild conditions on the drift b , while considering additive noise. Namely, the condition for $H > 1/2$ is that $b \in \mathcal{C}^\beta(\mathbb{R}^d)$ with $\beta > 1 - 1/(2H)$. This seems optimal as it coincides with the assumption that ensures strong existence and path-by-path uniqueness, see [24]. For $H < 1/2$, they ask that b is merely a measurable function with linear growth. The first goal of this chapter is to obtain Gaussian upper bounds under relaxed assumptions on b when $H < 1/2$. In that case, we can hope to have Gaussian bounds under the assumption that $b \in \mathcal{B}_\infty^\beta(\mathbb{R}^d)$ with $\beta > 1 - 1/(2H)$. We will not reach such a large class of drifts, but we will obtain Gaussian upper bounds for $\beta > 2 - 1/(2H)$, which is more general than what has been obtained in [76], at least for $H < 1/4$. We believe the condition $\beta > 2 - 1/(2H)$ is not optimal and the approach we chose with Malliavin calculus might be responsible for such non-optimality. This gap might be filled using a mix of the density representation from [76] and the stochastic sewing techniques with singularities that we introduce in Section IV.3.2. We plan to investigate this in future research.

Regularity of the density. Olivera and Tudor [85] proved, for any $t > 0$, existence of a density of the solution to (IV.1.1) in a Besov space of positive regularity if the drift b is Hölder continuous. In [51] distributional drifts were considered and regularity of the density was proven, while keeping track of its integrability as $t \rightarrow 0$.

Our main contribution in this direction is generalizing the method of [85] to distributional drifts and improving the regularity of the density observed in [51].

McKean-Vlasov Equations. Due to their connection with interacting particle systems, SDEs with Brownian noise and distribution-dependent drift have been thoroughly investigated after being introduced by McKean [81], see for example [53, 107]. These kinds of equations were also recently considered in a rough path-setting in [12, 22, 30]. The case of distributional drift for fractional Brownian noise was first considered in [51]. However, following the intuitions built in Section I.2.6, for equations with convolutional-type drift the results in [51] can be expected to be improved (for such a result in the Brownian setting, see [26]). We sketch an argument for existence of a strong solution to such an equation in Section IV.4.3, making use of our results on the regularity of densities mentioned in the previous paragraph. The rigorous proof is left to the ongoing work [LA5].

IV.1.1 Notations and definitions

We introduce some notations and definitions, in particular to clarify the interpretation of (IV.1.1) for distributional drift.

Notations.

- For $q \in [1, \infty]$, let $L^q([0, T], \mathcal{C}_b^\infty) := \bigcap_{k \in \mathbb{N}} L^q([0, T], \mathcal{C}_b^k)$.
- We denote by $\Psi(\cdot)$ a nondecreasing function defined from \mathbb{R}_+ to \mathbb{R}_+ , which can change from line to line.

Definition IV.1.1. Let $\beta \in \mathbb{R}$ and $p, q \in [1, \infty]$. We say that $(f^n)_{n \in \mathbb{N}}$ converges to f in $L^q([0, T], \mathcal{B}_\infty^{\beta-})$ as $n \rightarrow \infty$ if $\sup_{n \in \mathbb{N}} \|f^n\|_{L^q_{[0, T]} \mathcal{B}_\infty^\beta} < \infty$ and

$$\forall \beta' < \beta, \quad \lim_{n \rightarrow \infty} \|f^n - f\|_{L^q_{[0, T]} \mathcal{B}_\infty^{\beta'}} = 0. \quad (\text{IV.1.2})$$

Lemma IV.1.2. Let $\beta \in \mathbb{R}$, $q \in [1, \infty]$. For any $f \in L^q([0, T], \mathcal{B}_\infty^\beta)$, there exists a sequence $(f^n)_{n \in \mathbb{N}}$, with $f^n \in L^q([0, T], \mathcal{C}_b^\infty)$, such that $(f^n)_{n \in \mathbb{N}}$ converges to f in the sense of Definition IV.1.1.

Proof. We assume $q < \infty$ as $q = \infty$ can be proven by similar (easier) arguments. Let $f_t^n(\cdot) = \mathbb{1}_{\{t \geq \frac{1}{n}\}} G_{\frac{1}{n}} f_t(\cdot)$. That $f^n \in L^q([0, T], \mathcal{C}_b^\infty)$ holds by standard heat kernels estimates, and that $\sup_n \|f^n\|_{L^q_{[0, T]} \mathcal{B}_\infty^\beta} < \infty$ follows directly from Lemma A.1.7(c). Now we check that (IV.1.2) holds true. Let $\beta' < \beta$. Using dominated convergence and Lemma A.1.7(b), we obtain that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|f^n - f\|_{L^q_{[0, T]} \mathcal{B}_\infty^{\beta'}}^q &= \lim_{n \rightarrow \infty} \int_0^T \|f_t^n - f_t\|_{\mathcal{B}_\infty^{\beta'}}^q dt \\ &= \lim_{n \rightarrow \infty} \left(\int_0^{\frac{1}{n}} \|f_t\|_{\mathcal{B}_\infty^{\beta'}}^q dt + \int_{\frac{1}{n}}^T \|G_{\frac{1}{n}} f_t - f_t\|_{\mathcal{B}_\infty^{\beta'}}^q dt \right) \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

□

Control functions. We recall two properties that will often be used throughout this chapter. For $q \in [1, \infty)$, E a Banach space and $\phi \in L^q([0, T], E)$, the mapping $(s, t) \mapsto \|\phi\|_{L^q_{[s, t]} E}^q$ is a control function. If $\varkappa_1, \dots, \varkappa_n$ are control functions and a_1, \dots, a_n are such that $\sum_{i=1}^n a_i > 1$, then $\prod_{i=1}^n \varkappa_i^{a_i}$ is also a control function (see e.g. [45, Exercise 1.9]).

Definition of a solution. In this chapter we interpret (IV.1.1) as a nonlinear Young integral equation. Throughout the chapter, Assumption (A) always holds. Then, by [49, Section 5], there is strong existence and pathwise uniqueness to Equation (IV.1.1) interpreted as a nonlinear Young integral equation. In particular for any approximation sequence b^n of the drift b in the sense of Definition IV.1.1, the corresponding sequence of solutions X^n converges weakly to the unique strong solution X to (IV.1.1) (see [49, Theorem 3.2]).

IV.2 Main results

IV.2.1 Existence of a density and Gaussian bound

We start with a result in dimension one that requires one more degree of regularity than for the mere well-posedness of (IV.1.1) that comes from Assumption (A). As mentioned in the introduction, we do not claim that this requirement is optimal. In the next results, we will also have existence of a density but precisely under Assumption (A).

Theorem IV.2.1. *Let $H < 1/2$, $\beta > 2 - 1/(2H)$ and $b \in L^\infty([0, T], \mathcal{B}_\infty^\beta(\mathbb{R}))$. Then the pathwise unique strong solution to Equation (IV.1.1) with drift b has a density $p(t, \cdot)$ for any $t \in (0, T]$ and there exists $C, \tilde{C} > 0$ such that, for any $t \in (0, T]$,*

$$p(t, x) \leq Ct^{-H} \exp\{-\tilde{C}t^{-2H}(x - x_0)^2\}, \quad \forall x \in \mathbb{R}. \quad (\text{IV.2.1})$$

Moreover, $p(t, \cdot)$ is α -Hölder continuous for any $\alpha < 1/2$.

Section IV.3 is dedicated to the proof of this result.

In [76], a similar upper bound is obtained, as well as a lower bound. Their result holds in any dimension but is restricted to functions with linear growth, while here we can deal with distributions when H is small enough. Based on the decomposition used in the proof, we could also expect a lower bound, but this is left for future research.

IV.2.2 Results on existence and regularity of the density

The first result below establishes existence and regularity of a density for fixed time $t > 0$. Then, in Theorem IV.2.4, the behaviour of the density as $t \rightarrow 0$ is investigated by proving certain integrability in time. In particular, Theorem IV.2.4 can be seen as an extension of [51, Proposition 3.20]. Note that, in contrast to Theorem IV.2.2, keeping track of the singularity around $t = 0$, does not allow to deduce a higher amount of smoothness for regular drift b compared to distributional drift b . The proofs can be found in Section IV.4.

Theorem IV.2.2. *Let $b \in L^q([0, T], \mathcal{B}_\infty^\beta(\mathbb{R}^d))$ with q and β satisfying Assumption (A). Let X be the pathwise unique strong solution to (IV.1.1) with drift b . Then, for fixed $t > 0$ and*

$$\eta < \beta - 1 + \frac{1}{Hq'},$$

the law of X_t admits a density w.r.t Lebesgue measure that is in $\mathcal{B}_1^\eta(\mathbb{R}^d)$.

Remark IV.2.3. This result is consistent with the literature on drifts in Besov spaces with positive regularity, see [85, Theorem 1] which gives exactly the same result when $\beta > 0$ and $q = \infty$.

We now consider the integrability in time of the density, which comes with a lower regularity in space.

Theorem IV.2.4. *Let $b \in L^q([0, T], \mathcal{B}_\infty^\beta(\mathbb{R}^d))$ with q and β satisfying Assumption (A). Let X be the pathwise unique strong solution to (IV.1.1) with drift b . Let $\lambda \in (0, 1]$ and let $\tilde{q} \in (1, \infty]$, $\tilde{p} \in [1, \infty]$, $\tilde{\beta} \in \mathbb{R}$ which satisfy the condition*

$$\max \left\{ \lambda \left(1 - \frac{1}{H} - \beta \right), \lambda - \frac{1}{\tilde{q}'H}, \frac{-1}{2H} \right\} < \tilde{\beta} - \frac{d}{\tilde{p}} < 0. \quad (\text{IV.2.2})$$

Then there exist constants $C, \eta > 0$ such that for any $(s, t) \in \Delta_{[0, T]}$,

$$\|\mathcal{L}(X.)\|_{L_{[s, t]}^{\tilde{q}'} \mathcal{B}_{\tilde{p}'}^{-\tilde{\beta}}} \leq C \left((t-s)^{\frac{1}{\tilde{q}'} + H(\tilde{\beta} - \frac{d}{\tilde{p}})} + (1 + \|b\|_{L_{[0, T]}^q \mathcal{B}_\infty^\beta}^\eta) \|b\|_{L_{[s, t]}^q \mathcal{B}_\infty^\beta}^\lambda (t-s)^{\frac{\lambda}{\tilde{q}'} + \frac{1}{\tilde{q}'} + H(\beta + \tilde{\beta} - \frac{d}{\tilde{p}} - \lambda)} \right),$$

where $\mathcal{L}(X.)$ denotes the law of $X.$. In particular, if $\tilde{q}' \geq \frac{1}{1+H\beta}$, one can choose $\lambda = \frac{1}{\tilde{q}'(1+H\beta)} \in (0, 1]$ to minimize left-hand side in (IV.2.2). Then one gets the condition

$$\max \left\{ \frac{1}{\tilde{q}'} \left(\frac{1}{1+H\beta} - \frac{1}{H} \right), \frac{-1}{2H} \right\} < \tilde{\beta} - \frac{d}{\tilde{p}} < 0. \quad (\text{IV.2.3})$$

Then under condition (IV.2.3) and if $\tilde{q}' \geq \frac{1}{1+H\beta}$,

$$\mathcal{L}(X.) \in L^{\tilde{q}'}([0, T], \mathcal{B}_{\tilde{p}'}^{-\tilde{\beta}}).$$

Similarly for $\tilde{q}' = 1$, one can choose $\lambda = \frac{1}{2H\beta + 2 - 2H}$ and obtain that

$$\mathcal{L}(X.) \in L^1([0, T], \mathcal{B}_1^{1/(2H) - \varepsilon}).$$

Remark IV.2.5. Assume that (A) holds, then the following holds true for any $\varepsilon > 0$:

- choosing $\tilde{q} = \infty$ and $\tilde{p} = 1$, we have that $\mathcal{L}(X.) \in L_{[0, T]}^1 \mathcal{B}_\infty^{1/(2H) - d - \varepsilon}$;
- choosing $\tilde{q} = \infty$ and $\tilde{p} = \infty$, we have that $\mathcal{L}(X.) \in L_{[0, T]}^1 \mathcal{B}_1^{1/(2H) - \varepsilon}$;
- choosing $\tilde{q} = 2$ and $\tilde{p} = 1$ gives $\mathcal{L}(X.) \in L_{[0, T]}^2 \mathcal{B}_\infty^{1/(2H) - 1/(2(1+H\beta)) - d - \varepsilon}$.
- choosing $\tilde{q} = 2$ and $\tilde{p} = \infty$ gives $\mathcal{L}(X.) \in L_{[0, T]}^2 \mathcal{B}_1^{1/(2H) - 1/(2(1+H\beta)) - \varepsilon}$.

IV.3 Gaussian tails of the density of the solution

IV.3.1 General decomposition of the density

For any $H \in (0, 1)$, the fractional Brownian motion has the following Wiener integral representation:

$$B_t = \int_0^t K_H(t, s) dW_s, \quad t \geq 0, \quad (\text{IV.3.1})$$

where the kernel K_H is given by

$$\begin{cases} \forall s \geq t > 0, & K_H(t, s) = 0, \\ \forall t > s > 0, & K_H(t, s) = c_H \left\{ \left(\frac{t(t-s)}{s} \right)^{H-\frac{1}{2}} - (H - \frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right\}, \end{cases}$$

and

$$c_H = \left(\frac{2H \Gamma(3/2 - H)}{\Gamma(H + \frac{1}{2}) \Gamma(2 - 2H)} \right)^{\frac{1}{2}}. \quad (\text{IV.3.2})$$

By a simple change of variable the following inequality holds true for $H < 1/2$:

$$|K_H(t, s)| \leq C \left((t-s)^{-(\frac{1}{2}-H)} + s^{-(\frac{1}{2}-H)} \right). \quad (\text{IV.3.3})$$

Based on the representation (IV.3.1), it was proven in [83] that a Girsanov transform holds for the fBm. This transform involves the operator K_H^{-1} , which is the inverse of the operator with kernel K_H defined on $L^2([0, T])$ (see [83] for a definition using fractional operators). For $H < 1/2$ and any absolutely continuous function h , K_H^{-1} reads

$$K_H^{-1}h(s) = \frac{s^{H-\frac{1}{2}}}{\Gamma(\frac{1}{2}-H)} \int_0^s (s-u)^{-(H+\frac{1}{2})} u^{\frac{1}{2}-H} h'(u) du, \quad s \geq 0. \quad (\text{IV.3.4})$$

Let $T > 0$. Denote by $(\Omega, \mathcal{H}, \mathbb{P})$ the classical Wiener space, that is $\Omega = \mathcal{C}([0, T], \mathbb{R})$, $\mathcal{H} = H^1([0, T])$ the Sobolev space of square integrable functions with square integrable derivatives, and \mathbb{P} the standard Gaussian measure on Ω . On this space, the Malliavin derivative D of a smooth random variable $F = f(W_{t_1}, \dots, W_{t_n})$, for some smooth function f and times $0 \leq t_1 \leq \dots \leq t_n \leq T$, is the process $(D_s F)_{s \in [0, T]}$ defined by

$$D_s F = \sum_{k=1}^n \partial_k f(W_{t_1}, \dots, W_{t_n}) \mathbb{1}_{[0, t_k]}(s).$$

For $p \geq 1$, the domain of D in $L^p(\Omega)$ is the Malliavin-Sobolev space $\mathbb{D}^{1,p}$. For more details we refer e.g. to [82]. If B has the representation (IV.3.1), then one has

$$D_s B_t = K_H(t, s), \quad s, t \in [0, T].$$

We first consider the solution X to Equation (IV.1.1) with a smooth drift $b \in L^\infty([0, T], \mathcal{C}_b^\infty(\mathbb{R}))$. We will apply the Girsanov transform to X as established in [83, Theorem 2]: there exists a new probability measure $\tilde{\mathbb{P}}$ under which $X_t - X_0$ is an \mathbb{F}^B -fractional Brownian motion. This measure $\tilde{\mathbb{P}}$ is characterised by its Radon-Nikodym derivative ξ_t with respect to \mathbb{P} , i.e. $\xi_t = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t}$ given by

$$\xi_t = \exp \left(\int_0^t \mathcal{M}_s d\tilde{W}_s - \frac{1}{2} \int_0^t \mathcal{M}_s^2 ds \right), \quad (\text{IV.3.5})$$

where

$$\tilde{W}_t = W_t + \int_0^t K_H^{-1} \left(\int_0^\cdot b(r, x_0 + B_r) dr \right) (s) ds \quad \text{and} \quad \tilde{B}_t = \int_0^t K_H(t, s) d\tilde{W}_s$$

are respectively a standard Brownian motion and a fractional Brownian motion under $\tilde{\mathbb{P}}$, and

$$\mathcal{M}_t = K_H^{-1} \left(\int_0^\cdot b(v, x_0 + \tilde{B}_v) dv \right) (t). \quad (\text{IV.3.6})$$

In addition, the Malliavin derivatives of ξ_t and \mathcal{M}_t will be needed: in view of the previous equations, they read

$$D_s \xi_t = \xi_t \left(\mathcal{M}_s \mathbb{1}_{[0, t]}(s) + \int_0^t D_s \mathcal{M}_u d\tilde{W}_u - \int_0^t \mathcal{M}_u D_s \mathcal{M}_u du \right) \quad (\text{IV.3.7})$$

and

$$D_s \mathcal{M}_t = K_H^{-1} \left(\int_0^\cdot K_H(v, s) b'(v, x_0 + \tilde{B}_v) dv \right) (t). \quad (\text{IV.3.8})$$

In Eq. (43) of [16], a formula for the density $p(t, x)$ was given:

$$p(t, x) = p_1(t, x) - p_2(t, x), \quad (\text{IV.3.9})$$

where for $x \geq x_0$,

$$\begin{aligned} p_1(t, x) &= \tilde{\mathbb{E}} \left[\mathbb{1}_{\{x_0 + \tilde{B}_t \geq x\}} \frac{\xi_t \tilde{B}_t}{t^{2H}} \right], \\ p_2(t, x) &= \tilde{\mathbb{E}} \left[\mathbb{1}_{\{x_0 + \tilde{B}_t \geq x\}} \frac{\langle D \cdot \xi_t, K_H(t, \cdot) \rangle_{L^2[0, t]}}{t^{2H}} \right]. \end{aligned} \quad (\text{IV.3.10})$$

The symmetric case $x \leq x_0$ is obtained by replacing $\mathbb{1}_{\{x_0 + \tilde{B}_t \geq x\}}$ by $\mathbb{1}_{\{x_0 + \tilde{B}_t \leq x\}}$ in the above expressions.

IV.3.2 Stochastic sewing with a singularity

Before proving the main result of this section, Theorem IV.2.1, we state and prove a couple of regularising properties of the fractional Brownian motion derived from the stochastic sewing lemma.

Lemma IV.3.1. *Let $r \in [0, T)$, $m \in [2, \infty)$, $n \in [m, \infty]$ and $\gamma \in (1 - 1/(2H), 0)$. Then there exists $C > 0$ such that for any measurable function g defined on $[r, T]$ with $|g(\cdot)| \lesssim g_0 (T - \cdot)^{-(1/2+H)}$ for some $g_0 > 0$, any $f \in L^\infty([0, T], \mathcal{B}_\infty^\gamma) \cap L^\infty([0, T], \mathcal{C}_b^\infty(\mathbb{R}^d))$ and any $(s, t) \in \Delta_{[r, T]}$,*

$$\left\| \mathbb{E}^s \left[\left| \int_s^t g(v) f(v, x + B_v) dv \right|^m \right]^{1/m} \right\|_{L^n} \leq C g_0 \|f\|_{L_{[s, t]}^\infty \mathcal{B}_\infty^\gamma} (t - s)^{\frac{1}{2} + H(\gamma - 1)}. \quad (\text{IV.3.11})$$

Lemma IV.3.2. *Let $r \in [0, T)$, $\kappa \in (0, 1/2)$, $m \in [2, \infty)$, $n \in [m, \infty]$ and $\gamma \in (-1/(2H), 0)$. Then there exists $C > 0$ such that for any measurable function g defined on $[r, T]$ with $|g(\cdot)| \lesssim g_0 (\cdot - r)^{-\kappa}$ for some $g_0 > 0$, any $f \in L^\infty([0, T], \mathcal{B}_\infty^\gamma) \cap L^\infty([0, T], \mathcal{C}_b^\infty(\mathbb{R}^d))$ and any $(s, t) \in \Delta_{[r, T]}$,*

$$\left\| \mathbb{E}^s \left[\left| \int_s^t g(v) f(v, x + B_v) dv \right|^m \right]^{1/m} \right\|_{L^n} \leq C g_0 \|f\|_{L_{[s, t]}^\infty \mathcal{B}_\infty^\gamma} (t - s)^{1 + H\gamma - \kappa}. \quad (\text{IV.3.12})$$

We only prove Lemma IV.3.1 as Lemma IV.3.2 follows along similar arguments.

Proof. Let $\delta > 0$. We will show (IV.3.11) for $(s, t) \in \Delta_{[r, T - \delta]}$ with a constant C on the right-hand side of (IV.3.11) that does not depend on δ . Letting δ go to zero then gives the statement for $(s, t) \in [r, T)$. Then by continuity the statement also holds for $(s, t) \in [r, T]$.

We aim to apply Theorem A.2.1 on the interval $[r, T - \delta]$. Let

$$\begin{aligned} A_{s, t} &= \mathbb{E}^s \left[\int_s^t g(v) f(v, x + B_v) dv \right] \\ \mathcal{A}_t &= \int_r^t g(v) f(v, x + B_v) dv. \end{aligned}$$

Clearly (A.2.1) is fulfilled as $\mathbb{E}^s[\delta A_{s,u,t}] = 0$ for $u \in [s, t]$.

To verify (A.2.2), after applying Minkowski's inequality, Lemma II.5.1(b) and Hölder's inequality, we obtain that for any $\varepsilon \in (0, 1/2)$,

$$\begin{aligned}
\|A_{s,t}\|_{L^n} &\leq \left\| \mathbb{E}^s \int_s^t g(v) f(v, x + B_v) dv \right\|_{L^n} \\
&\leq \int_s^t |g(v)| \|\mathbb{E}^s f(v, x + B_v)\|_{L^n} dv \\
&\leq C \int_s^t (T-v)^{-(\frac{1}{2}+H)} \|f(v, \cdot)\|_{\mathcal{B}_\infty^\gamma} (v-s)^{H\gamma} dv \\
&\leq C \|f\|_{L_{[s,t]}^\infty \mathcal{B}_\infty^\gamma} (T-t)^{-\frac{1}{2}+\varepsilon} \int_s^t (t-v)^{-(H+\varepsilon)} (v-s)^{H\gamma} dv \\
&\leq C \|f\|_{L_{[s,t]}^\infty \mathcal{B}_\infty^\gamma} (T-\delta-t)^{-\frac{1}{2}+\varepsilon} (t-s)^{1+H(\gamma-1)-\varepsilon}.
\end{aligned}$$

Hence, after choosing ε small enough, we have (A.2.2) for $\alpha_2 = 0$, $\gamma_2 = 1/2 - \varepsilon$ and $\varepsilon_2 = 1/2 + H(\gamma - 1) - \varepsilon > 0$.

Lastly, we verify that (A.2.3) holds. Let $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ be a sequence of partitions of $[r, t]$ with mesh size going to zero. Then, Lemma II.5.1(c) and Hölder's inequality,

$$\begin{aligned}
\left\| \mathcal{A}_t - \sum_{i=0}^{N_k} A_{t_i^k, t_{i+1}^k} \right\|_{L^1} &\leq \sum_{i=0}^{N_k} \int_{t_i^k}^{t_{i+1}^k} (T-r)^{-(H+\frac{1}{2})} \|f(v, x + B_v) - \mathbb{E}^{t_i^k} f(v, x + B_v)\|_{L^1} dv \\
&\leq C \delta^{-(H+\frac{1}{2})} \sum_{i=0}^{N_k} \int_{t_i^k}^{t_{i+1}^k} \|f(v, \cdot)\|_{\mathcal{C}^1} (v-t_i)^H dv \\
&\leq C \delta^{-(H+\frac{1}{2})} (t_{i+1}^k - t_i^k)^{1+H} \\
&\leq C \delta^{-(H+\frac{1}{2})} (t-r) |\Pi_k|^H \xrightarrow[k \rightarrow \infty]{} 0.
\end{aligned}$$

Therefore, by Theorem A.2.1, we have that

$$\begin{aligned}
\|\mathbb{E}^s[|\mathcal{A}_t - \mathcal{A}_s|^m]^{1/m}\|_{L^n} &\leq \|A_{s,t}\|_{L^n} + C \|f\|_{L_{[s,t]}^\infty \mathcal{B}_\infty^\gamma} (t-s)^{\frac{1}{2}+H(\gamma-1)} \\
&\leq C \|f\|_{L_{[s,t]}^\infty \mathcal{B}_\infty^\gamma} (t-s)^{\frac{1}{2}+H(\gamma-1)}.
\end{aligned}$$

□

IV.3.3 Proof of the Gaussian upper bound on the density

We now aim to prove Theorem IV.2.1. The proof relies on the decomposition of the density given in Equations (IV.3.9)-(IV.3.10) and carefully bounding the moments of the terms appearing therein. The latter is done by the quantification of the regularization effect of fBm for integrals with an additional time singularity established in Section IV.3.2 and by borrowing a result on exponential moments of such integrals without the additional time singularity (see [51, Proposition 3.8]). This is first done for smooth bounded drifts. As all bounds will be given in terms of the Besov norm of the drift, it is possible to pass to the limit eventually.

Lemma IV.3.3. *Let $\beta > 2 - 1/(2H)$. There exists $\tilde{C} > 0$ and a nondecreasing function $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, for $\beta \in L^\infty([0, T], \mathcal{B}_\infty^\beta(\mathbb{R})) \cap L^\infty([0, T], \mathcal{C}_b^\infty(\mathbb{R}))$ and for $t \in (0, T]$, the unique strong solution to Equation (IV.1.1) with drift β has a density $p(t, \cdot)$ fulfilling*

$$p(t, x) \leq \Psi(\|\beta\|_{L_{[0,t]}^\infty \mathcal{B}_\infty^\beta}) t^{-H} \exp\{-\tilde{C}t^{-2H}(x - x_0)^2\}, \quad \forall x \in \mathbb{R}.$$

Furthermore, $p(t, \cdot)$ is α -Hölder continuous for any $\alpha < 1/2$.

In order to prove Lemma IV.3.3 and ultimately Theorem IV.2.1 we first state and prove the following lemma.

Lemma IV.3.4. *Let $\beta > 1 - 1/(2H)$ and $\beta \in L^\infty([0, T], \mathcal{B}_\infty^\beta(\mathbb{R})) \cap L^\infty([0, T], \mathcal{C}_b^\infty(\mathbb{R}))$. Then for any $N \in \mathbb{N}$ and $t \in [0, T]$, there exists a non-decreasing function $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$\tilde{\mathbb{E}}[\xi_t^N] = \tilde{\mathbb{E}}\left[\exp\left\{N \int_0^t \mathcal{M}_s d\tilde{W}_s - \frac{N}{2} \int_0^t \mathcal{M}_s^2 ds\right\}\right] \leq \Psi(\|\beta\|_{L_{[0,t]}^\infty \mathcal{B}_\infty^\beta}).$$

Proof. Assume that $\beta < 0$. This is no loss of generality after applying a Besov space embedding. Let $t \in [0, T]$. By the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} & \tilde{\mathbb{E}}\left[\exp\left\{N \int_0^t \mathcal{M}_s d\tilde{W}_s - \frac{N}{2} \int_0^t \mathcal{M}_s^2 ds\right\}\right] \\ & \leq \tilde{\mathbb{E}}\left[\exp\left\{2N \int_0^t \mathcal{M}_s d\tilde{W}_s - N^2 \int_0^t \mathcal{M}_s^2 ds\right\}\right]^{\frac{1}{2}} \tilde{\mathbb{E}}\left[\exp\left\{(N-1)N \int_0^t \mathcal{M}_s^2 ds\right\}\right]^{\frac{1}{2}}. \end{aligned} \quad (\text{IV.3.13})$$

We show that the second factor in the last line above is finite. Hence, Novikov's condition is fulfilled so the first factor is equal to 1. We know from [50, Lemma 4.17] that we can bound the L^2 norm of a process of the form $K_H^{-1}Y$ (such as \mathcal{M}) by a Hölder norm of Y . In this context, for $1 + H\beta > \bar{\beta} > 1/2 + H$, this reads precisely as follows:

$$\int_0^t \mathcal{M}_s^2 ds \leq \left[\int_0^t \beta(v, x_0 + \tilde{B}_v) dv\right]_{\mathcal{C}_{[0,t]}^{\bar{\beta}}}^2.$$

Note that the right-hand side in the above can be chosen to be the seminorm since the function vanishes at 0. Furthermore, by [51, Proposition 3.8] (using that $\beta < 0$), the right-hand side has exponential moments which can be bounded by the quantity $\Psi(\|\beta\|_{L_{[0,t]}^\infty \mathcal{B}_\infty^\beta})$. \square

Proof of Lemma IV.3.3. Throughout the proof we assume without loss of generality that $\beta - 1 < 0$ in order to be able to apply the stochastic sewing results from the previous section. We will bound successively p_1 and p_2 from (IV.3.10). In this proof, we adopt the notation $\|\cdot\|_{\tilde{L}^p}$ to denote the $L^p(\Omega, \tilde{\mathbb{P}})$ norm. By Hölder's inequality, Lemma IV.3.4 and using that for $x \geq x_0$ there is $\mathbb{P}(x_0 + \tilde{B}_t \geq x) \leq \exp\{-\tilde{C}t^{-2H}(x - x_0)^2\}$, we obtain

$$\begin{aligned} p_1(t, x) & \leq \|\mathbb{1}_{\{x_0 + \tilde{B}_t \geq x\}}\|_{\tilde{L}^4} \|\xi_t\|_{\tilde{L}^4} \left\| \frac{\tilde{B}_t}{t^{2H}} \right\|_{\tilde{L}^2} \\ & \leq \Psi(\|\beta\|_{L_{[0,t]}^\infty \mathcal{B}_\infty^\beta}) t^{-H} \exp\{-\tilde{C}t^{-2H}(x - x_0)^2\}. \end{aligned}$$

Now to bound p_2 , let $m \in (1, 2)$. By Hölder's inequality, we have

$$\begin{aligned} p_2(t, x) &\leq Ct^{-2H} \left\| \mathbb{1}_{\{x_0 + \tilde{B}_t \geq x\}} \right\|_{\tilde{L}^{m'}} \left\| \langle D.\xi_t, K_H(t, \cdot) \rangle_{L^2_{[0,t]}} \right\|_{\tilde{L}^m} \\ &\leq Ct^{-2H} \exp\{-\tilde{C}t^{-2H}(x - x_0)^2\} \left\| \langle D.\xi_t, K_H(t, \cdot) \rangle_{L^2_{[0,t]}} \right\|_{\tilde{L}^m}. \end{aligned}$$

It remains to control the last factor in the above expression by Ct^H . By the Cauchy-Schwarz inequality,

$$\begin{aligned} \left\| \langle D.\xi_t, K_H(t, \cdot) \rangle_{L^2_{[0,t]}} \right\|_{\tilde{L}^m} &\leq \left\| \|D.\xi_t\|_{L^2_{[0,t]}} \right\|_{\tilde{L}^m} \|K_H(t, \cdot)\|_{L^2_{[0,t]}} \\ &= t^H \left\| \|D.\xi_t\|_{L^2_{[0,t]}} \right\|_{\tilde{L}^m}, \end{aligned}$$

using (IV.3.1) to get that

$$\|K_H(t, \cdot)\|_{L^2_{[0,t]}}^2 = \mathbb{E}[B_t^2] = t^{2H}.$$

Choose $n > 1$ such that $1/n + 1/2 = 1/m$. From Equation (IV.3.7) and Hölder's inequality, there is

$$\begin{aligned} \left\| \|D.\xi_t\|_{L^2_{[0,t]}} \right\|_{\tilde{L}^m} &\leq \|\xi_t\|_{\tilde{L}^n} \left(\left\| \|\mathcal{M}.\mathbb{1}_{[0,t]}(\cdot)\|_{L^2_{[0,t]}} \right\|_{\tilde{L}^2} \right. \\ &\quad \left. + \left\| \left\| \int_0^t D.\mathcal{M}_u d\tilde{W}_u \right\|_{L^2_{[0,t]}} \right\|_{\tilde{L}^2} + \left\| \left\| \int_0^t \mathcal{M}_u D.\mathcal{M}_u du \right\|_{L^2_{[0,t]}} \right\|_{\tilde{L}^2} \right) \\ &=: \|\xi_t\|_{\tilde{L}^n} (S_1 + S_2 + S_3). \end{aligned} \tag{IV.3.14}$$

Applying Lemma IV.3.4, we have that $\|\xi_t\|_{\tilde{L}^n} \leq \Psi(\|\beta\|_{L^\infty_{[0,t]} \mathcal{B}^\beta_\infty})$.

We will bound S_1 , S_2 and S_3 separately. First, from (IV.3.4), Lemma IV.3.1 and a Besov space embedding,

$$\begin{aligned} S_1 &\leq \left\| \left\| K_H^{-1} \left(\int_0^\cdot \beta(s, x_0 + \tilde{B}_s) ds \right) (\cdot) \right\|_{L^2_{[0,t]}} \right\|_{\tilde{L}^2} \\ &\leq C \left(\int_0^t \left\| \int_0^u u^{H-\frac{1}{2}} s^{\frac{1}{2}-H} (u-s)^{-(\frac{1}{2}+H)} \beta(s, x_0 + \tilde{B}_s) ds \right\|_{\tilde{L}^2}^2 du \right)^{\frac{1}{2}} \\ &\leq Ct^{1+H(\beta-2)} \|\beta\|_{L^\infty_{[0,t]} \mathcal{B}^{\beta-1}_\infty} \\ &\leq Ct^{1+H(\beta-2)} \|\beta\|_{L^\infty_{[0,t]} \mathcal{B}^\beta_\infty}. \end{aligned}$$

As for S_2 , we get from Minkowski's integral inequality and Itô's isometry that

$$\begin{aligned} S_2 &= \tilde{\mathbb{E}} \left[\left(\int_0^t \left(\int_0^t D_r \mathcal{M}_u d\tilde{W}_u \right)^2 dr \right) \right]^{\frac{1}{2}} \\ &\leq \left(\int_0^t \tilde{\mathbb{E}} \left(\int_0^t D_r \mathcal{M}_u d\tilde{W}_u \right)^2 dr \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^t \int_0^t \tilde{\mathbb{E}}(D_r \mathcal{M}_u)^2 dudr \right)^{\frac{1}{2}} \leq \left(\int_0^t \int_0^u \tilde{\mathbb{E}}(D_r \mathcal{M}_u)^2 drdu \right)^{\frac{1}{2}}. \end{aligned}$$

While bounding S_2 , we will prove that for any $p \in [2, \infty)$ and any $r \leq u$, there is

$$\|D_r \mathcal{M}_u\|_{\tilde{L}^p} \leq C \|\beta'\|_{L_{[0,u]}^\infty \mathcal{B}_\infty^{\beta-1}} \left((u-r)^{H-\frac{1}{2}} + r^{-(\frac{1}{2}-H)} \right) (u-r)^{\frac{1}{2}+H(\beta-2)}. \quad (\text{IV.3.15})$$

We do the proof for $p = 2$, but it works the same for any p . From (IV.3.8), we get

$$\begin{aligned} S_2^2 &\leq \int_0^t \int_0^u \tilde{\mathbb{E}} \left[\left(K_H^{-1} \left(\int_0^\cdot K_H(v, r) \beta'(v, x_0 + \tilde{B}_v) dv \right) (u) \right)^2 \right] dr du \\ &= \int_0^t \int_0^u \tilde{\mathbb{E}} \left[\left(\int_r^u u^{H-\frac{1}{2}} v^{\frac{1}{2}-H} (u-v)^{-(H+\frac{1}{2})} K_H(v, r) \beta'(v, x_0 + \tilde{B}_v) dv \right)^2 \right] dr du. \end{aligned} \quad (\text{IV.3.16})$$

Recall from (IV.3.3) that $K_H(v, r) = K_{H,1}(v, r) + K_{H,2}(v, r)$ with $|K_{H,1}(v, r)| \lesssim (v-r)^{-(1/2-H)}$ and $|K_{H,2}(v, r)| \lesssim r^{-(1/2-H)}$. Decompose the expectation in (IV.3.16) as follows:

$$\begin{aligned} &\left\| \int_r^u u^{H-\frac{1}{2}} v^{\frac{1}{2}-H} (u-v)^{-(H+\frac{1}{2})} K_H(v, r) \beta'(v, x_0 + \tilde{B}_v) dv \right\|_{\tilde{L}^2} \\ &\leq \left\| \int_{\frac{u+r}{2}}^u u^{H-\frac{1}{2}} v^{\frac{1}{2}-H} (u-v)^{-(H+\frac{1}{2})} K_{H,1}(v, r) \beta'(v, x_0 + \tilde{B}_v) dv \right\|_{\tilde{L}^2} \\ &\quad + \left\| \int_r^{\frac{u+r}{2}} u^{H-\frac{1}{2}} v^{\frac{1}{2}-H} (u-v)^{-(H+\frac{1}{2})} K_{H,1}(v, r) \beta'(v, x_0 + \tilde{B}_v) dv \right\|_{\tilde{L}^2} \\ &\quad + \left\| \int_r^u u^{H-\frac{1}{2}} v^{\frac{1}{2}-H} (u-v)^{-(H+\frac{1}{2})} K_{H,2}(v, r) \beta'(v, x_0 + \tilde{B}_v) dv \right\|_{\tilde{L}^2} \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

For I_1 we apply Lemma IV.3.1 with $s \equiv (u+r)/2$, $t \equiv T \equiv u$ and $g(v) = (v/u)^{1/2-H} (u-v)^{-(H+1/2)} K_{H,1}(v, r)$. Since $g(v) \lesssim (u-v)^{-(H+1/2)} (u-r)^{-(1/2-H)}$ when $v \in ((u+r)/2, u)$, we get for $g_0 = (u-r)^{-(1/2-H)}$ that

$$I_1 \leq C \|\beta'\|_{L_{[0,u]}^\infty \mathcal{B}_\infty^{\beta-1}} (u-r)^{H(\beta-1)}.$$

For I_2 we apply Lemma IV.3.2 with $s \equiv r$, $t \equiv (u+r)/2$, $\kappa = 1/2 - H$ and the same g . Since $g(v) \lesssim (u-r)^{-(H+1/2)} (v-r)^{-(1/2-H)}$ when $v \in (r, (u+r)/2)$, we get for $g_0 = (u-r)^{-(H+1/2)}$ that

$$I_2 \leq C \|\beta'\|_{L_{[0,u]}^\infty \mathcal{B}_\infty^{\beta-1}} (u-r)^{H(\beta-1)}.$$

As for I_3 , we have similarly by applying Lemma IV.3.1 that

$$I_3 \leq C \|\beta'\|_{L_{[0,u]}^\infty \mathcal{B}_\infty^{\beta-1}} r^{-(\frac{1}{2}-H)} (u-r)^{\frac{1}{2}+H(\beta-2)}.$$

Hence

$$\begin{aligned} &\left\| \int_r^u u^{H-\frac{1}{2}} v^{\frac{1}{2}-H} (u-v)^{-(H+\frac{1}{2})} K_H(v, r) \beta'(v, x_0 + \tilde{B}_v) dv \right\|_{\tilde{L}^2} \\ &\leq C \|\beta'\|_{L_{[0,u]}^\infty \mathcal{B}_\infty^{\beta-1}} \left((u-r)^{H-\frac{1}{2}} + r^{-(\frac{1}{2}-H)} \right) (u-r)^{\frac{1}{2}+H(\beta-2)}, \end{aligned}$$

which proves (IV.3.15). Plugging this back into (IV.3.16), we obtain by Lemma A.1.6 that

$$\begin{aligned} S_2 &\leq Ct^{1+H(\beta-1)} \|\beta'\|_{L^\infty_{[0,t]} \mathcal{B}_\infty^{\beta-1}} \\ &\leq Ct^{1+H(\beta-1)} \|\beta\|_{L^\infty_{[0,t]} \mathcal{B}_\infty^\beta}. \end{aligned}$$

Now for S_3 , we proceed with an argument similar to what was done for S_1 and S_2 . First, recall (IV.3.14) and (IV.3.6), and apply the Cauchy-Schwarz inequality to get that

$$\begin{aligned} S_3 &\leq \left\| \left\| \left(\int_0^t \mathcal{M}_u^2 du \right)^{\frac{1}{2}} \left(\int_0^t (D \cdot \mathcal{M}_u)^2 du \right)^{\frac{1}{2}} \right\|_{L^2_{[0,t]}} \right\|_{\tilde{L}^2} \\ &\leq \left\| \left\| K_H^{-1} \left(\int_0^\cdot \beta(v, x_0 + \tilde{B}_v) dv \right) (\cdot) \right\|_{L^2_{[0,t]}} \right\|_{\tilde{L}^2} \left\| \left(\int_0^t (D \cdot \mathcal{M}_u)^2 du \right)^{\frac{1}{2}} \right\|_{L^2_{[0,t]}} \right\|_{\tilde{L}^2}. \end{aligned}$$

Applying the Cauchy-Schwarz inequality again and Minkowski's integral inequality, it comes

$$S_3 \leq C \left\| \left\| K_H^{-1} \left(\int_0^\cdot \beta(v, x_0 + \tilde{B}_v) dv \right) (\cdot) \right\|_{\tilde{L}^4} \right\|_{L^2_{[0,t]}} \left\| \left(\int_0^t \int_0^u (D_r \mathcal{M}_u)^2 dr du \right)^{\frac{1}{2}} \right\|_{\tilde{L}^4}.$$

By (IV.3.4) and Lemma IV.3.1 (as in bounding S_1),

$$\left\| \left\| K_H^{-1} \left(\int_0^\cdot \beta(v, x_0 + \tilde{B}_v) dv \right) (\cdot) \right\|_{\tilde{L}^4} \right\|_{L^2_{[0,t]}} \leq Ct^{1+H(\beta-2)} \|\beta\|_{L^\infty_{[0,t]} \mathcal{B}_\infty^\beta}.$$

By Minkowski's integral inequality and (IV.3.15), we have

$$\begin{aligned} \left\| \left(\int_0^t \int_0^u (D_r \mathcal{M}_u)^2 dr du \right)^{\frac{1}{2}} \right\|_{\tilde{L}^4} &\leq \left(\int_0^t \int_0^u \left(\tilde{\mathbb{E}}(D_r \mathcal{M}_u)^4 \right)^{\frac{1}{2}} dr du \right)^{\frac{1}{2}} \\ &\leq C \|\beta'\|_{L^\infty \mathcal{B}_\infty^{\beta-1}} \left(\int_0^t \int_0^u (u-r)^{2H(\beta-1)} + r^{-(1-2H)} (u-r)^{1+2H(\beta-2)} dr du \right)^{\frac{1}{2}} \\ &\leq C \|\beta'\|_{L^\infty_{[0,t]} \mathcal{B}_\infty^{\beta-1}} t^{1+H(\beta-1)}. \end{aligned}$$

Hence

$$S_3 \leq C \|\beta'\|_{L^\infty_{[0,t]} \mathcal{B}_\infty^{\beta-1}}^2 t^{2+2H(\beta-1)} \leq C \|\beta\|_{L^\infty_{[0,t]} \mathcal{B}_\infty^\beta}^2 t^{2+2H(\beta-1)}$$

where we used Lemma A.1.6 in the final inequality. \square

Proof of Theorem IV.2.1. Let $(b^n)_{n \in \mathbb{N}}$ be a sequence of functions in $L^\infty([0, T], \mathcal{C}_b^\infty)$ converging to b in $L^\infty([0, T], \mathcal{B}_\infty^{\beta-})$. By stability (see [49, Theorem 3.2]), we know that the strong solution X^n to (IV.1.1) with drift b^n converges in law to the unique strong solution X to (IV.1.1) with drift b . We know from Lemma IV.3.3 that (IV.2.1) holds for the density $p^n(t, \cdot)$ corresponding to X^n , with constants that can be chosen independently of n .

We aim to apply the Arzelà-Ascoli Theorem, which will give that along some subsequence, $(p^n)_{n \in \mathbb{N}}$ converges uniformly on compacts to some function p . We know by Lemma IV.3.3 that $(p^n)_{n \in \mathbb{N}}$ is uniformly bounded. Next we check that $(p^n)_{n \in \mathbb{N}}$ is uniformly globally Hölder continuous and therefore uniformly equicontinuous. W.l.o.g. let $x_0 < \tilde{x} < x$ (the case of arbitrary x, \tilde{x} follows by similar arguments). Let $m \in (1, 2)$. Then for all $n \in \mathbb{N}$ we have by the same arguments as in the proof of Lemma IV.3.3 that

$$\begin{aligned}
 |p^n(t, x) - p^n(t, \tilde{x})| &\leq |p_1^n(t, x) - p_1^n(t, \tilde{x})| + |p_2^n(t, x) - p_2^n(t, \tilde{x})| \\
 &\leq \left| \tilde{\mathbb{E}} \left[\mathbb{1}_{\{x_0 + \tilde{B}_t \in [\tilde{x}, x]\}} \frac{\xi_t \tilde{B}_t}{t^{2H}} \right] \right| + \left| \tilde{\mathbb{E}} \left[\mathbb{1}_{\{x_0 + \tilde{B}_t \in [\tilde{x}, x]\}} \frac{\langle D \cdot \xi_t, K_H(t, \cdot) \rangle_{L^2[0, t]}}{t^{2H}} \right] \right| \\
 &\leq C \Psi(\|b^n\|_{L^\infty \mathcal{B}_\infty^\beta}) t^{-H} \tilde{\mathbb{P}}(\tilde{B}_t \in [\tilde{x}, x])^{1/m'} \\
 &\leq C \Psi(\|b\|_{L^\infty \mathcal{B}_\infty^\beta}) t^{-H-H/m'} |x - \tilde{x}|^{1/m'}.
 \end{aligned} \tag{IV.3.17}$$

Hence by Arzelà-Ascoli, we have uniform convergence on compacts along some subsequence of $(p^n)_{n \in \mathbb{N}}$. For notational simplicity we do not relabel the subsequence. In particular $(p^n)_{n \in \mathbb{N}}$ converges pointwise to some p , which fulfills the Gaussian bound (IV.2.1). Hence we can apply the dominated convergence theorem and obtain, for any $f \in \mathcal{C}_b(\mathbb{R})$, that

$$\int_{\mathbb{R}} p^n(t, x) f(x) dx \xrightarrow{n \rightarrow \infty} \int_{\mathbb{R}} p(t, x) f(x) dx.$$

By the weak convergence of $(X_t^n)_{n \in \mathbb{N}}$ to X_t , we can identify $p(t, \cdot)$ as the density of X_t . As $p(t, \cdot)$ is the pointwise limit of a sequence of functions fulfilling (IV.2.1), with a uniform constant on the right-hand side, $p(t, \cdot)$ fulfills (IV.2.1) as well. Also it is α -Hölder continuous for any $\alpha < 1/2$ by (IV.3.17) after choosing $m \in (1, 2)$ close to 2. \square

IV.4 Existence of density for the solution of the SDE and its Besov regularity

In this section we prove that under Assumption (A), any strong solution X to (IV.1.1) has a density which lies in a Besov space with positive regularity (Theorem IV.2.2) and track its integrability in time (Theorem IV.2.4).

IV.4.1 Existence of density and its regularity at fixed time $t > 0$

The proof of Theorem IV.2.2 mainly relies on Romito's Lemma that we recall below in Lemma IV.4.2 (for a proof see [98, Lemma A.1]). It gives a condition on a random variable that allows to deduce existence of a density with a certain regularity. To formulate Lemma IV.4.2, we need the following definition.

Definition IV.4.1. Let $h \in \mathbb{R}^d$ and $f: \mathbb{R}^d \rightarrow \mathbb{R}$. We define $\Delta_h^1 f: \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$(\Delta_h^1 f)(\cdot) := f(\cdot + h) - f(\cdot).$$

Iteratively we define $\Delta_h^n f: \mathbb{R}^d \rightarrow \mathbb{R}$ for $n \in \mathbb{N}^*$ by

$$(\Delta_h^n f)(\cdot) := (\Delta_h^1(\Delta_h^{n-1} f))(\cdot) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} f(\cdot + jh).$$

Lemma IV.4.2 (Romito's Lemma [98]). *Let Z be an \mathbb{R}^d -valued random variable. Assume there exist $s, \delta, C > 0$ and an integer $m \geq 1$ with $\delta < s < m$ such that, for every $\phi \in \mathcal{C}^\delta(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$ with $|h| < 1$,*

$$\left| \mathbb{E}[\Delta_h^m \phi(Z)] \right| \leq C|h|^s \|\phi\|_{\mathcal{C}^\delta}.$$

Then Z has a density f w.r.t. Lebesgue measure and there exists a constant $\tilde{C} > 0$ independent of Z such that $\|f\|_{\mathcal{B}_1^{s-\delta}} \leq \tilde{C}(1+C)$.

We aim to apply Lemma IV.4.2 to X_t , but to get a nice upper bound, we introduce an approximation scheme Y^ε as follows. Let $t \in (0, T]$. For $\varepsilon > 0$, consider the following process defined on $[0, t]$:

$$Y_s^\varepsilon := \begin{cases} X_s & \text{for } s \leq t - \varepsilon, \\ X_{(t-\varepsilon)\vee 0} + B_s - B_{(t-\varepsilon)\vee 0} & \text{for } s \geq (t - \varepsilon) \vee 0. \end{cases}$$

In order to apply Lemma IV.4.2, we look at the following quantity:

$$\mathbb{E}[\Delta_h^m \phi(X_t)] = \mathbb{E}[\Delta_h^m \phi(Y_t^\varepsilon)] + (\mathbb{E}[\Delta_h^m \phi(X_t)] - \mathbb{E}[\Delta_h^m \phi(Y_t^\varepsilon)]) =: E_1 + E_2. \quad (\text{IV.4.1})$$

The idea is to choose ε depending on h in an optimal way, so that $E_1 + E_2$ is minimized. The scheme of the proof of the following lemma is similar to [85, Proposition 1], but we will additionally make use of regularising properties of the fBm, due to stochastic sewing.

Lemma IV.4.3. *Let $\beta \in L^q([0, T], \mathcal{B}_\infty^\beta(\mathbb{R})) \cap L^q([0, T], \mathcal{C}_b^\infty(\mathbb{R}))$ with parameters fulfilling (A). Let X be the strong solution to (IV.1.1) with drift β . Let $\delta \in (0, 1]$, $m \in \mathbb{N}^*$, $h \in \mathbb{R}^d$ with $|h| < 1$ and $\phi \in \mathcal{C}^\delta(\mathbb{R}^d)$. Then, we have for any $\varepsilon > 0$ that*

$$\left| \mathbb{E}[\Delta_h^m \phi(X_t)] \right| \leq C\Psi(\|\beta\|_{L^q_{[0,T]} \mathcal{B}_\infty^\beta}) \|\phi\|_\infty \left(\frac{|h|}{(t \wedge \varepsilon)^H} \right)^m + \|\phi\|_{\mathcal{C}^\delta} (t \wedge \varepsilon)^{\delta(\frac{1}{q} + H\beta)}.$$

where we recall that $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ denotes some non-decreasing function.

Proof. We use (IV.4.1) and bound E_1 and E_2 separately.

Bounding E_1 . Note that, by (IV.3.1),

$$\begin{aligned} Y_t^\varepsilon &= X_{(t-\varepsilon)\vee 0} + B_t - B_{(t-\varepsilon)\vee 0} \\ &= X_{(t-\varepsilon)\vee 0} + \int_0^{(t-\varepsilon)\vee 0} (K_H(t, s) - K_H((t-\varepsilon)\vee 0, s)) dW_s + \int_{(t-\varepsilon)\vee 0}^t K_H(t, s) dW_s \\ &=: Z_t^\varepsilon + I_t^\varepsilon, \end{aligned}$$

where $Z_t^\varepsilon = X_{(t-\varepsilon)\vee 0} + \int_0^{(t-\varepsilon)\vee 0} (K_H(t, s) - K_H((t-\varepsilon)\vee 0, s)) dW_s$ is $\mathcal{F}_{(t-\varepsilon)\vee 0}$ -measurable and I_t^ε is Gaussian and independent of $\mathcal{F}_{(t-\varepsilon)\vee 0}$. By the tower property we have that

$$\begin{aligned} E_1 &= \mathbb{E}[\Delta_h^m \phi(Z_t^\varepsilon + I_t^\varepsilon)] \\ &= \mathbb{E}[\mathbb{E}^{(t-\varepsilon)\vee 0}[\Delta_h^m \phi(Z_t^\varepsilon + I_t^\varepsilon)]] \\ &= \mathbb{E} \left[\int_{\mathbb{R}^d} \Delta_h^m \phi(Z_t^\varepsilon + x) g_{\sigma_{(t-\varepsilon)\vee 0, t}^2}(x) dx \right] \\ &= \mathbb{E} \left[\int_{\mathbb{R}^d} \phi(Z_t^\varepsilon + x) \Delta_{-h}^m g_{\sigma_{(t-\varepsilon)\vee 0, t}^2}(x) dx \right], \end{aligned}$$

where g is the Gaussian density with variance $\sigma_{(t-\varepsilon)\vee 0, t}^2 = \text{Var}(I_t^\varepsilon)$. Hence

$$|E_1| \leq \|\phi\|_\infty \|\Delta_{-h}^m g_{\sigma_{(t-\varepsilon)\vee 0, t}^2}\|_{L^1(\mathbb{R}^d)}.$$

As in [98] (see two lines after Equation (2.7)),

$$\|\Delta_{-h}^m g_{\sigma_{(t-\varepsilon)\vee 0, t}^2}\|_{L^1(\mathbb{R}^d)} \leq C \left(\frac{|h|}{\sqrt{\text{Var}(I_t^\varepsilon)}} \right)^m.$$

Now observe that $\text{Var}(I_t^\varepsilon) = \mathbb{E} |B_t - \mathbb{E}^{(t-\varepsilon)\vee 0}[B_t]|^2$ and by the local nondeterminism property of the fBm (see [89, Lemma 7.1]), there exists $C > 0$ such that $\text{Var}(I_t^\varepsilon) = C(t - (t - \varepsilon) \vee 0)^{2H} = C(t \wedge \varepsilon)^{2H}$. Hence

$$\|\Delta_{-h}^m g_{\sigma_{(t-\varepsilon)\vee 0, t}^2}\|_{L^1(\mathbb{R}^d)} \leq C \left(\frac{|h|}{(t \wedge \varepsilon)^H} \right)^m.$$

Bounding E_2 . By the Hölder regularity of ϕ , we have for $n \geq 2$,

$$|E_2| \leq C \|\phi\|_{\mathcal{C}^\delta} \mathbb{E} \left[|X_t - Y_t^\varepsilon|^\delta \right] \leq C \|\phi\|_{\mathcal{C}^\delta} \mathbb{E} \left[\left| \int_{0 \vee (t-\varepsilon)}^t \beta(u, X_u) du \right|^{n\delta/n} \right].$$

Now apply [49, Lemma 2.4] (which is possible since β is smooth in the space variable) to get

$$|E_2| \leq C \Psi(\|\beta\|_{L_{[0, T]}^q \mathcal{B}_\infty^\beta}) \|\phi\|_{\mathcal{C}^\delta} (t \wedge \varepsilon)^{\delta(1/q' + H\beta)}.$$

□

We now proceed with the proof of Theorem IV.2.2.

Proof of Theorem IV.2.2. Let $b \in L^q([0, T], \mathcal{C}^\beta)$. Let $(b^n)_{n \in \mathbb{N}}$ be a sequence converging to b in $L^q([0, T], \mathcal{C}^{\beta-})$ such that $b^n \in L^q([0, T], \mathcal{C}^\infty)$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, let X^n be the strong solution to (IV.1.1) with drift b^n . By the stability estimate [49, Theorem 3.2] we know that X^n converges weakly to the unique strong solution to (IV.1.1). Denote $\zeta = 1/q' + H\beta$. Let h be such that $|h| < 1$, then by Lemma IV.4.3, we obtain

$$\begin{aligned} |h|^{-s} |\mathbb{E}[\Delta_h^m \phi(X_t)]| &= |h|^{-s} \lim_{n \rightarrow \infty} |\mathbb{E}[\Delta_h^m \phi(X_t^n)]| \\ &\leq C(1 + \Psi(\|b\|_{L_{[0, T]}^q \mathcal{C}^\beta})) \|\phi\|_{\mathcal{C}^\delta} \left(|h|^{-s} (t \wedge \varepsilon)^{\delta\zeta} + \frac{|h|^{m-s}}{(t \wedge \varepsilon)^{Hm}} \right) \\ &\leq C(1 + \Psi(\|b\|_{L_{[0, T]}^q \mathcal{C}^\beta})) \|\phi\|_{\mathcal{C}^\delta} \left(|h|^{-s} \varepsilon^{\delta\zeta} + \frac{|h|^{m-s}}{(t \wedge 1)^{Hm}} + \frac{|h|^{m-s}}{\varepsilon^{Hm}} \right). \end{aligned}$$

Choosing $\varepsilon = |h|^{\frac{m}{\delta\zeta + mH}}$, we get

$$|h|^{-s} |\mathbb{E}[\Delta_h^m \phi(X_t^n)]| \leq C \|\phi\|_{\mathcal{C}^\delta} \left(|h|^{\frac{m\delta\zeta}{\delta\zeta + mH} - s} + \frac{|h|^{m-s}}{(t \wedge 1)^{Hm}} + |h|^{m-s - \frac{m^2 H}{\delta\zeta + mH}} \right).$$

To apply Lemma IV.4.2, we want the exponents in the right-hand side of the previous inequality to be nonnegative, hence it suffices to impose $s < \frac{m\delta\zeta}{\delta\zeta + mH}$. As we want to maximize $s - \delta$ to obtain the best regularity for the law of X_t , this comes to finding the supremum of the function g defined by

$$g(m, \delta) = \frac{m\delta\zeta}{\delta\zeta + mH} - \delta.$$

Letting $m \rightarrow \infty$ in the above and choosing $\delta = 1$ yields that we can have $s - \delta$ positive and close to $\beta - 1 + 1/(Hq')$. \square

IV.4.2 Joint time integrability and space regularity of the density

In this section, we prove Theorem IV.2.4 via a duality argument. We first prove the following lemma via stochastic sewing formulated with controls (see Lemma A.2.3).

Lemma IV.4.4. *Assume (A) holds. Let $\lambda \in (0, 1]$ and let $\tilde{q} \in (1, \infty]$, $\tilde{p} \in [1, \infty]$, $\tilde{\beta} \in \mathbb{R}$ which satisfy condition (IV.2.2). Then there exist constants $C, \eta > 0$ such that for the unique strong solution X to (IV.1.1), any $f \in L^{\tilde{q}}([0, T], \mathcal{B}_{\tilde{p}}^{\tilde{\beta}}) \cap L^{\tilde{q}}([0, T], \mathcal{C}_b^\infty(\mathbb{R}^d))$ and any $(s, t) \in \Delta_{[0, T]}$,*

$$\begin{aligned} & \left\| \int_s^t f(r, X_r) dr \right\|_{L^m} \\ & \leq C \|f\|_{L_{[s, t]}^{\tilde{q}} \mathcal{B}_{\tilde{p}}^{\tilde{\beta}}} \left((t-s)^{\frac{1}{\tilde{q}'} + H(\tilde{\beta} - \frac{d}{\tilde{p}})} + (1 + \|b\|_{L_{[0, T]}^q \mathcal{B}_\infty^\beta}^\eta) \|b\|_{L_{[s, t]}^q \mathcal{B}_\infty^\beta}^\lambda (t-s)^{\frac{\lambda}{\tilde{q}'} + \frac{1}{\tilde{q}'} + H(\beta + \tilde{\beta} - \frac{d}{\tilde{p}} - \lambda)} \right). \end{aligned}$$

Proof. Let $K = X - B$. For $(u, v) \in \Delta_{[0, T]}$, let

$$A_{u, v} := \int_u^v f(r, K_u + B_r) dr.$$

In order to verify the conditions from Lemma A.2.3, we will show that:

(a)

$$\|A_{u, v}\|_{L^m} \lesssim \|f\|_{L_{[u, v]}^{\tilde{q}} \mathcal{B}_{\tilde{p}}^{\tilde{\beta}}} (v-u)^{\frac{1}{\tilde{q}'} + H(\tilde{\beta} - \frac{d}{\tilde{p}})}.$$

(b) For some $\eta > 0$ and all $\xi \in [u, v]$,

$$\|\mathbb{E}^u \delta A_{u, \xi, v}\|_{L^m} \lesssim (1 + \|b\|_{L_{[0, T]}^q \mathcal{B}_\infty^\beta}^\eta) \|b\|_{L_{[u, v]}^q \mathcal{B}_\infty^\beta}^\lambda \|f\|_{L_{[u, v]}^{\tilde{q}} \mathcal{B}_{\tilde{p}}^{\tilde{\beta}}} (v-u)^{\frac{1}{\tilde{q}'} + H(\tilde{\beta} - \frac{d}{\tilde{p}} - \lambda)} (v-u)^{\lambda(1/\tilde{q}' + H\beta)}.$$

(c) for any $v \in [0, T]$, the convergence in probability of $\sum_{t_i^n \in \Pi^n} A_{t_i^n, t_{i+1}^n}$ to $\int_0^v f(r, X_r) dr$ along any sequence of partitions $|\Pi^n|$ of $[0, v]$ with mesh going to 0.

In the above, (a) implies that condition (A.2.7) is fulfilled: Indeed, if $\tilde{q} \leq 2$, $\|f\|_{L_{[u, v]}^{\tilde{q}} \mathcal{B}_{\tilde{p}}^{\tilde{\beta}}}$ is a control function to the power $1/\tilde{q}$ which is greater than $1/2$, and $1/\tilde{q}' + H(\tilde{\beta} - d/\tilde{p}) > 0$ (using

$\tilde{\beta} - d/\tilde{p} > -1/(H\tilde{q}')$. Now if $\tilde{q} > 2$, we have that $\|f\|_{L_{[u,v]}^{\tilde{q}}\mathcal{B}_{\tilde{p}}^{\tilde{\beta}}}(v-u)^{1/2-1/\tilde{q}}$ is a control function to the power 1/2, and there remains $(v-u)^{1/\tilde{q}-1/2+1/\tilde{q}'+H(\tilde{\beta}-d/\tilde{p})}$, with $1/\tilde{q}-1/2+1/\tilde{q}'+H(\tilde{\beta}-d/\tilde{p}) = 1/2+H(\tilde{\beta}-d/\tilde{p}) > 0$ (using $\tilde{\beta}-d/\tilde{p} > -1/2H$).

As for (b), it gives condition (A.2.8): Indeed, first note that by (IV.2.2) and (A) the power of $(v-u)$ is strictly positive. Also, given the properties of control functions recalled at the beginning of this chapter, we have that the right-hand side of the equation in (b) is a control function if

$$1 + \lambda + H\left(\tilde{\beta} - \frac{d}{\tilde{p}} - \lambda\right) + \lambda H\beta > 1,$$

which is fulfilled as by assumption

$$\tilde{\beta} - \frac{d}{\tilde{p}} > \lambda\left(1 - \frac{1}{H} - \beta\right). \quad (\text{IV.4.2})$$

Hence given that (a), (b) and (c) are satisfied, the result follows after applying Lemma A.2.3.

Proof of (a). Lemma IV.A.1 for $S = u$ directly gives (a) (note that the constant therein is independent of S).

Proof of (b). Denoting $\tilde{f}(r, \cdot) = f(r, K_u + \cdot) - f(r, K_\xi + \cdot)$, Lemma II.5.1(b) gives for any $\lambda \in (0, 1]$ that

$$\begin{aligned} \|\mathbb{E}^u \delta A_{u,\xi,v}\|_{L^m} &\leq \int_\xi^v \left\| \mathbb{E}^u \left[\mathbb{E}^\xi [f(r, K_u + B_r) - f(r, K_\xi + B_r)] \right] \right\|_{L^m} dr \\ &\lesssim \int_\xi^v \|\mathbb{E}^u \|\tilde{f}\|_{\mathcal{B}_{\tilde{p}}^{\tilde{\beta}-\lambda}}\|_{L^m} (r-\xi)^{H(\tilde{\beta}-\frac{d}{\tilde{p}}-\lambda)} dr. \end{aligned}$$

Hence, we obtain using Lemma A.1.8(b) that

$$\|\mathbb{E}^u \delta A_{u,\xi,v}\|_{L^m} \lesssim \|\mathbb{E}^u |K_\xi - K_u|^\lambda\|_{L^m} \int_\xi^v (r-\xi)^{H(\tilde{\beta}-d/\tilde{p}-\lambda)} \|f(r, \cdot)\|_{\mathcal{B}_{\tilde{p}}^{\tilde{\beta}}} dr.$$

To apply Hölder's inequality in the last upper bound, we need $\tilde{q}'H(\tilde{\beta}-d/\tilde{p}-\lambda) > -1$, which is implied by our assumption (IV.2.2). Hence by Hölder's inequality,

$$\|\mathbb{E}^u \delta A_{u,\xi,v}\|_{L^m} \lesssim \|f\|_{L_{[u,v]}^{\tilde{q}}\mathcal{B}_{\tilde{p}}^{\tilde{\beta}}}(v-u)^{\frac{1}{\tilde{q}'}+H(\tilde{\beta}-\frac{d}{\tilde{p}}-\lambda)} \|\mathbb{E}^u |K_\xi - K_u|^\lambda\|_{L^m}. \quad (\text{IV.4.3})$$

Now by [49, Lemma 2.4], there exist constants $C, \eta > 0$ such that for all $(s, t) \in [0, T]$,

$$\|K_{s,t}\|_{L^m} \leq C(1 + \|b\|_{L_{[0,T]}^q\mathcal{B}_\infty^\beta}^\eta) \|b\|_{L_{[s,t]}^q\mathcal{B}_\infty^\beta} (t-s)^{1/q'+H\beta}. \quad (\text{IV.4.4})$$

Combining this with Jensen's inequality and (IV.4.3), we get

$$\|\mathbb{E}^u \delta A_{u,\xi,v}\|_{L^m} \lesssim (1 + \|b\|_{L_{[0,T]}^q\mathcal{B}_\infty^\beta}^\eta) \|b\|_{L_{[u,v]}^q\mathcal{B}_\infty^\beta}^\lambda \|f\|_{L_{[u,v]}^{\tilde{q}}\mathcal{B}_{\tilde{p}}^{\tilde{\beta}}}(v-u)^{\frac{1}{\tilde{q}'}+H(\tilde{\beta}-\frac{d}{\tilde{p}}-\lambda)} (v-u)^{\lambda(1/q'+H\beta)}.$$

Proof of (c). Using Lemma II.5.1(c) and (IV.4.4), we get that there exists $\delta > 0$ and a control $\tilde{\varkappa}$ such that for any sequence of partitions $\Pi^n = \{t_i^n\}_i$ of $[0, v]$ with mesh going to zero,

$$\begin{aligned}
\left\| \int_0^v f(r, X_r) dr - \sum_i A_{t_i^n, t_{i+1}^n} \right\|_{L^m} &\leq \sum_i \int_{t_i^n}^{t_{i+1}^n} \|f(r, K_r + B_r) - f(r, K_{t_i^n} + B_r)\|_{L^1} dr \\
&\leq \sum_i \int_{t_i^n}^{t_{i+1}^n} \|f(r, \cdot)\|_{C^1} \|K_r - K_{t_i^n}\|_{L^m} dr \\
&\leq \sum_i \|f\|_{L^1_{[t_i^n, t_{i+1}^n]} C^1} \|K_{t_{i+1}^n} - K_{t_i^n}\|_{L^m} \\
&\leq \sum_i \|f\|_{L^1_{[t_i^n, t_{i+1}^n]} C^1} (t_{i+1}^n - t_i^n)^{1/q' + H\beta} \\
&\leq |\Pi^n|^\delta \tilde{\varkappa}(0, v) \xrightarrow[n \rightarrow \infty]{} 0.
\end{aligned}$$

□

Proof of Theorem IV.2.4. Let $r \in (1, \infty)$ and $(u, v) \in \Delta_{[0, T]}$. Denote by $\mathcal{S}_{u, v}^{(r)}$ the set of functions $f \in L^{\tilde{q}}([u, v], C_b^\infty)$ such that $\|f\|_{L^{\tilde{q}}_{[u, v]} \mathcal{B}_{\tilde{p}, r}^\beta} \leq 1$. Then applying a Besov embedding (see Remark A.1.5) first and then Lemma IV.4.4, we obtain

$$\begin{aligned}
\sup_{f \in \mathcal{S}_{u, v}^{(r)}} \left| \int_u^v \langle f_s, \mathcal{L}(X_s) \rangle ds \right| &\leq C \sup_{f \in \mathcal{S}_{u, v}^{(\infty)}} \left| \int_u^v \langle f_s, \mathcal{L}(X_s) \rangle ds \right| \\
&\leq C \sup_{f \in \mathcal{S}_{u, v}^{(\infty)}} \left\| \int_u^v f(s, X_s) ds \right\|_{L^m} \\
&\leq C \left((v - u)^{\frac{1}{\tilde{q}'} + H(\tilde{\beta} - \frac{d}{\tilde{p}})} + (1 + \|b\|_{L^q_{[0, T]} \mathcal{B}_\infty^\beta}^\eta) \|b\|_{L^q_{[u, v]} \mathcal{B}_\infty^\beta}^\lambda (v - u)^{\frac{\lambda}{\tilde{q}'} + \frac{1}{\tilde{q}'} + H(\beta + \tilde{\beta} - \frac{d}{\tilde{p}} - \lambda)} \right).
\end{aligned}$$

As Schwartz functions are dense in $\mathcal{B}_{\tilde{p}, r}^{\tilde{\beta}}$, it is sufficient to take the supremum over $f \in L^{\tilde{q}}([u, v], C_b^\infty)$, to conclude that $\mathcal{L}(X.) \in L^{\tilde{q}}([u, v], \mathcal{B}_{\tilde{p}, r}^{\tilde{\beta}})^*$. By [25, Corollary 2] we know that $(L^{\tilde{q}}([u, v], \mathcal{B}_{\tilde{p}, r}^{\tilde{\beta}}))^* = L^{\tilde{q}'}([u, v], \mathcal{B}_{\tilde{p}', r'}^{-\tilde{\beta}})$ as the latter is separable. Hence we get that

$$\|\mathcal{L}(X.)\|_{L^{\tilde{q}'}_{[u, v]} \mathcal{B}_{\tilde{p}', r'}^{-\tilde{\beta}}} \leq C \left((v - u)^{\frac{1}{\tilde{q}'} + H(\tilde{\beta} - \frac{d}{\tilde{p}})} + (1 + \|b\|_{L^q_{[0, T]} \mathcal{B}_\infty^\beta}^\eta) \|b\|_{L^q_{[u, v]} \mathcal{B}_\infty^\beta}^\lambda (v - u)^{\frac{\lambda}{\tilde{q}'} + \frac{1}{\tilde{q}'} + H(\beta + \tilde{\beta} - \frac{d}{\tilde{p}} - \lambda)} \right).$$

Using the embedding from Remark A.1.5, this gives the same upper bound for $\|\mathcal{L}(X.)\|_{L^{\tilde{q}'}_{[u, v]} \mathcal{B}_{\tilde{p}'}^{-\tilde{\beta}}}$.

Finally, under the assumption that $\tilde{q}' \geq \frac{1}{1 + H\beta}$, we minimize the left-hand side of (IV.2.2) over all $\lambda \in (0, 1]$ and get the condition (IV.2.3) by choosing

$$\lambda = \frac{1}{\tilde{q}'(1 + H\beta)} \in (0, 1].$$

□

IV.4.3 Some words on the McKean-Vlasov Equation

Following the intuitions built in Section I.2.6 an important motivation to study the regularity of Equation (IV.1.1) is to improve on existence and uniqueness results to the McKean-Vlasov equation, for any initial distribution X_0 ,

$$\begin{cases} X_t = X_0 + \int_0^t \mathbf{b} * \mu_s(X_s) ds + B_t \\ \mu_t = \mathcal{L}(X_t), t > 0. \end{cases} \quad (\text{IV.4.5})$$

The benchmark result (for $H < 1/2$) is that well-posedness holds for $\mathbf{b} \in \mathcal{B}_\infty^\beta$ with $\beta > 1 - 1/(2H)$ (see [51], [49, Theorem 7.4]). In the following, we sketch an argument how Theorem IV.2.4 could be used to get existence of a strong solution to (IV.4.5) for distributions $\mathbf{b} \in \mathcal{B}_\infty^\beta$ with

$$\beta > 2 - 1/H \quad (\text{IV.4.6})$$

and therefore with less regularity than the current state of the art theory allows for. The argument given below is done for a small time interval, however by an iteration argument this can be extended to any interval $[0, T]$. Also, via a bootstrap argument, condition (IV.4.6) can be improved. We leave the detailed arguments to the work in progress [LA5]. In particular, the arguments actually work for $\mathbf{b} \in \mathcal{B}_p^\beta$ for p arbitrarily large but finite, instead of \mathcal{B}_∞^β , however we also disregard this subtlety here.

Let $(\mathbf{b}_n)_{n \in \mathbb{N}}$ be a sequence of smooth functions that converges to \mathbf{b} in $\mathcal{B}_\infty^{\beta-}$. By the standard theory of McKean-Vlasov equations, the equation

$$\begin{cases} X_t^n = X_0 + \int_0^t \mathbf{b}_n * \mu_s^n(X_s) ds + B_t \\ \mu_t^n = \mathcal{L}(X_t^n), t > 0, \end{cases}$$

has a pathwise unique, strong solution for any $n \in \mathbb{N}$. Denote $b^n = \mathbf{b}_n * \mu^n$. Since for any $t \in [0, T]$, μ^n is a probability measure and $\|\cdot\|_{\mathcal{B}_1^0}$ is uniformly bounded among all probability measures, we have $\mu^n \in L^\infty([0, T], \mathcal{B}_1^0)$. Hence using that \mathbf{b}_n is smooth, we get by a convolution inequality and a Besov embedding that $b^n \in L^\infty([0, T], \mathcal{B}_\infty^k)$, for any $k > 0$. Thus b^n satisfies Assumption (A).

Let $t > 0$ be small. Applying Theorem IV.2.4 (apart from the subtlety of the additional power of $\|b\|_{L^q \mathcal{B}_\infty^\beta}$) with $\alpha < \frac{1}{2H} - 1$, $\tilde{\beta} = -\alpha$, $\tilde{p} = \infty$, $\tilde{q} = 2$ and $\lambda = 1$, and some $\varepsilon_1, \varepsilon_2 > 0$, we get

$$\|\mu^n\|_{L^2_{[0,t]} \mathcal{B}_{1,1}^\alpha} \leq C(t^{\varepsilon_1} + t^{\varepsilon_2} \|\mathbf{b}_n * \mu^n\|_{L^2_{[0,t]} \mathcal{B}_\infty^{\beta+\alpha}}) \quad (\text{IV.4.7})$$

$$\leq C(t^{\varepsilon_1} + t^{\varepsilon_2} \|\mathbf{b}_n\|_{\mathcal{B}_\infty^\beta} \|\mu^n\|_{L^2_{[0,t]} \mathcal{B}_{1,1}^\alpha}) \quad (\text{IV.4.8})$$

where we used Young's inequality for Besov spaces in the second step (see [69, Theorem 2.2]). Hence, after choosing t small enough we can uniformly bound $\|\mu^n\|_{L^2_{[0,t]} \mathcal{B}_{1,1}^\alpha}$.

Again by Young's inequality for Besov spaces, there is

$$\|b^n\|_{L^2_{[0,t]} \mathcal{B}_\infty^{\beta+\alpha}} \lesssim \|\mathbf{b}_n\|_{\mathcal{B}_\infty^\beta} \|\mu^n\|_{L^2_{[0,t]} \mathcal{B}_{1,1}^\alpha} \lesssim \sup_{n \in \mathbb{N}} \|\mathbf{b}_n\|_{\mathcal{B}_\infty^\beta} < \infty. \quad (\text{IV.4.9})$$

Hence,

$$\sup_{n \in \mathbb{N}} \|b^n\|_{L^2_{[0,t]} \mathcal{B}_\infty^{\beta+\alpha}} < \infty. \quad (\text{IV.4.10})$$

Observe that the conditions on α and β from (IV.4.6) imply that α can be chosen large enough such that $\beta + \alpha > 1 - 1/(2H)$, hence the drifts b^n satisfy “uniformly in n ” Assumption (A). Now apply Lemma IV.4.4 to X^n , $f = b^n$, $\tilde{\beta} = \beta + \alpha$, $\tilde{p} = +\infty$, $\lambda = 1$ and $\tilde{q} = 2$. It follows, using (IV.4.10), that there exists $\eta > 0$ such that for any $n \in \mathbb{N}$ and any $(u, v) \in \Delta_{[0,t]}$,

$$\left\| \int_u^v b^n(r, X_r^n) dr \right\|_{L^m} \leq C \sup_{n \in \mathbb{N}} \|b^n\|_{L^2_{[0,t]} \mathcal{B}_\infty^{\beta+\alpha}} (1 + \|b^n\|_{L^2_{[0,t]} \mathcal{B}_\infty^{\beta+\alpha}}) (v - u)^\eta < \infty$$

Thus by Kolmogorov’s tightness criterion, $(X^n)_{n \in \mathbb{N}}$ is tight in $\mathcal{C}_{[0,t]}$.

We now aim at the identification of the limit points of $(X^n)_{n \in \mathbb{N}}$ and $(\mu^n)_{n \in \mathbb{N}}$. Up to taking a subsequence, we get that $(\mu^n)_{n \in \mathbb{N}}$ converges weakly in the space of probability measures on $\mathcal{C}_{[0,t]}$ to some probability measure μ^∞ . It follows from [51, Corollary A.5] (up to minor modification of the space) that for any $u \in [0, t]$, $\mathbf{b} * \mu_u^n$ converges to $\mathbf{b} * \mu_u^\infty$ in \mathcal{B}_∞^β . Then, after uniformly bounding $\|\mathbf{b} * \mu^n - \mathbf{b} * \mu^\infty\|_{L^2_{[0,t]} \mathcal{B}_\infty^{\beta+\alpha}}$ by similar arguments as in (IV.4.10), we get after an interpolation of Besov spaces that, for any $\varepsilon > 0$,

$$\|\mathbf{b} * \mu^n - \mathbf{b} * \mu^\infty\|_{L^2_{[0,t]} \mathcal{B}_\infty^{\beta+\alpha-\varepsilon}} \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand, we also have by (IV.4.7) and Remark A.1.5 that

$$\begin{aligned} \|\mathbf{b}_n * \mu^n - \mathbf{b} * \mu^n\|_{L^2_{[0,t]} \mathcal{B}_\infty^{\beta+\alpha-\varepsilon}} &\lesssim \|\mathbf{b}_n - \mathbf{b}\|_{\mathcal{B}_\infty^{\beta-\varepsilon}} \|\mu^n\|_{L^2_{[0,t]} \mathcal{B}_{1,1}^\alpha} \\ &\lesssim C \|\mathbf{b}_n - \mathbf{b}\|_{\mathcal{B}_\infty^{\beta-\varepsilon}} \xrightarrow{n \rightarrow \infty} 0. \end{aligned} \quad (\text{IV.4.11})$$

Then it follows that

$$\begin{aligned} &\|\mathbf{b}_n * \mu^n - \mathbf{b} * \mu^\infty\|_{L^2_{[0,t]} \mathcal{B}_\infty^{\beta+\alpha-\varepsilon}} \\ &\leq \|\mathbf{b}_n * \mu^n - \mathbf{b} * \mu^n\|_{L^2_{[0,t]} \mathcal{B}_\infty^{\beta+\alpha-\varepsilon}} + \|\mathbf{b} * \mu^n - \mathbf{b} * \mu^\infty\|_{L^2_{[0,T]} \mathcal{B}_\infty^{\beta+\alpha-\varepsilon}} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

In particular $b^\infty = \mathbf{b} * \mu^\infty \in L^2([0, t], \mathcal{B}_\infty^{\beta+\alpha-\varepsilon})$ for any $\varepsilon > 0$. Therefore it fulfills condition (A) and we know from [49] that there exists a pathwise unique strong solution X with drift b^∞ . Then by the stability estimate [49, Theorem 3.2], putting the same noise on the same probability space, X^n converges to X in $L^m(\Omega; \mathcal{C}_{[0,t]})$. Since μ^n converges weakly to μ^∞ , we get in particular that μ^∞ must coincide with the law of X . Hence X is a strong solution to (IV.4.5).

IV.A Another regularising property of fBm

Below we state and prove a lemma similar to Lemma II.B.1, however allowing for time dependence. The proof is based on stochastic sewing phrased with control functions, i.e. Lemma A.2.3.

Lemma IV.A.1. *Let $q \in (1, \infty]$, $m \in [2, \infty)$, $p \in [1, \infty]$ and $\gamma - d/p \in (\frac{-1}{2H} \vee \frac{-1}{qH}, 0)$. Then there exists a constant $C > 0$ such that for any deterministic times $0 < S < T$, any f_Ξ of the form $f_\Xi(t, \cdot) = \tilde{f}(t, \cdot + \Xi)$, for $\tilde{f} \in L^q([0, T], \mathcal{B}_p^\gamma) \cap L^q([0, T], \mathcal{C}^1)$ and an \mathcal{F}_S -measurable random variable Ξ ,*

$$\left\| \int_S^T f_\Xi(r, B_r) dr \right\|_{L^m} \leq C \|\tilde{f}\|_{L^q_{[S,T]} \mathcal{B}_p^\gamma} (T - S)^{1/q' + H(\gamma - d/p)}. \quad (\text{IV.A.1})$$

Proof. We aim to apply Lemma A.2.3. For $S \leq s \leq t \leq T$, define

$$\mathcal{A}_t := \int_S^t f_{\Xi}(r, B_r) dr \quad \text{and} \quad A_{s,t} := \mathbb{E}^s \left[\int_s^t f_{\Xi}(r, B_r) dr \right].$$

As $\mathbb{E}^s[\delta A_{s,u,t}] = 0$, condition (A.2.8) trivially holds.

Using Minkowski's integral inequality and Lemma II.5.1(b), we get that

$$\begin{aligned} \|A_{s,t}\|_{L^m} &= \left\| \mathbb{E}^s \int_S^t f_{\Xi}(r, B_r) dr \right\|_{L^m} \\ &\leq \int_s^t \|\mathbb{E}^s f(B_r, \Xi)\|_{L^m} dr \\ &\leq C \int_s^t \|\tilde{f}(r, \cdot)\|_{\mathcal{B}_p^\gamma} (r-s)^{H(\gamma-d/p)} dr \\ &\leq C \|\tilde{f}(r, \cdot)\|_{L_{[s,t]}^q \mathcal{B}_p^\gamma} (t-s)^{1/q'+H(\gamma-d/p)}. \end{aligned}$$

Note that the power of $(t-s)$ is positive as $\gamma - d/p > -1/(q'H)$. Also the above is a control raised to a power greater than $1/2$ as $\gamma - d/p > -1/(2H)$. Therefore condition (A.2.7) is fulfilled.

Let $t \in [S, T]$. Let $(\Pi_k)_{k \in \mathbb{N}}$ be a sequence of partitions of $[S, t]$ with mesh size converging to zero. For each k , denote $\Pi_k = \{t_i^k\}_{i=1}^{N_k}$. By Lemma II.5.1(c), we obtain

$$\begin{aligned} \|\mathcal{A}_t - \sum_i A_{t_i^k, t_{i+1}^k}\|_{L^1} &\leq \sum_i \int_{t_i^k}^{t_{i+1}^k} \|f_{\Xi}(r, B_r) - \mathbb{E}^{t_i^k} f_{\Xi}(r, B_r)\|_{L^1} dr \\ &\leq C \|\tilde{f}\|_{L_{[S,t]}^1 \mathcal{C}^1} |\Pi_k|^H \rightarrow 0. \end{aligned}$$

Now the result follows by applying Lemma A.2.3. □

Path-by-path uniqueness under Krylov-Röckner condition



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The rest of the thesis solely focuses on the case of Brownian noise. Hence, in order to avoid confusion, we hereon denote the noise by W instead of B . The equation under consideration is similar to the ones of the previous sections, however drifts are singular functions and no longer distributions. In this chapter, that is based on [LA3], path-by-path uniqueness is established as well as a fine property of the flow generated by such equations.

V.1 Introduction

For a measurable time-dependent vector field $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $d \geq 1$, and a measurable driving signal $\gamma : [0, T] \rightarrow \mathbb{R}^d$ with $\gamma_0 = 0$, we consider the deterministic ordinary integral equation (ODE)

$$y_t = y_0 + \int_0^t b(r, y_r) dr + \gamma_t, \quad t \in [0, T]. \tag{V.1.1}$$

Here, $T > 0$ is a finite time horizon which is fixed but arbitrary. Because b is only measurable, a solution $(y_t)_{t \in [0, T]}$, if it exists, should be measurable and satisfy the integrability condition

$$\int_0^T |b(r, y_r)| dr < \infty, \tag{V.1.2}$$

which ensures that the integral in (V.1.1) is well-defined in Lebesgue sense. For such solution, it is immediate that $y - \gamma$ is continuous.

When γ is sampled from a stochastic process W on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$, equation (V.1.1) transforms into the stochastic differential equation (SDE)

$$Y_t = Y_0 + \int_0^t b(s, Y_s) ds + W_t, \quad t \in [0, T]. \quad (\text{V.1.3})$$

A solution $Y : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ to (V.1.3) is a stochastic process such that for a.s. ω , $(Y_t(\omega))_{t \in [0, T]}$ is a solution to (V.1.1) with $W(\omega)$ in place of γ .

We recall that it is now well-understood that the addition of fast oscillating driving signals γ , W can make both (V.1.1) and (V.1.3) well-posed for very irregular vector fields b with references given in previous chapters. However, let us mention again the following benchmark result by Krylov and Röckner [68], establishing existence of a pathwise unique adapted solution to (V.1.3), for b fulfilling

$$\left(\int_0^T \left(\int_{\mathbb{R}^d} |b(r, x)|^p dx \right)^{\frac{q}{p}} dr \right)^{\frac{1}{q}} < \infty, \quad (p, q) \in \mathcal{J} := \left\{ (p, q) \in (2, \infty)^2 : \frac{2}{q} + \frac{d}{p} < 1 \right\}. \quad (\text{V.1.4})$$

This condition is also known as the *sub-critical Ladyzhenskaya-Prodi-Serrin* (LPS) condition in fluid dynamics, see more in [15, 40].

A typical result of recent articles on regularization by noise is that when γ is a sample path of an irregular stochastic process (such as fractional Brownian motion), then for almost every such sample path, ODE (V.1.1) has a unique solution, i.e. *path-by-path uniqueness* holds. As mentioned in Section I.2.3, having pathwise uniqueness for (V.1.3) does not provide uniqueness for the $W(\omega)$ -driven ODE

$$Y_t = Y_0 + \int_0^t b(r, Y_r) dr + W_t(\omega), \quad t \in [0, T], \quad (\text{V.1.5})$$

for a given ω . In fact, examples of SDEs driven by Brownian motion which have pathwise uniqueness but not path-by-path uniqueness are given in [103, LA1]. We point to Chapter VI and Section V.B for further discussions on the related concepts.

The main goal of the current chapter is the following result, which will be stated more precisely later in Theorems V.2.2, V.2.4 and V.2.5.

Theorem. *Assuming that W is a standard Brownian motion, then path-by-path uniqueness, path-by-path stability and strong completeness for (V.1.3) hold under the Krylov-Röckner condition (V.1.4).*

By path-by-path stability, we mean that for a sequence of functions (b^n) converging sufficiently fast to b in a suitable topology, there exists an event of full probability which depends only on (b^n) and T such that for every ω in such event, for every $x \in \mathbb{R}^d$, the solution Y^n to the ODE

$$Y_t^n = x + \int_0^t b^n(r, Y_r^n) dr + W_t(\omega), \quad t \in [0, T]$$

converges uniformly to the unique solution to (V.1.5) with initial condition $Y_0 = x$. This is stronger than standard stability results because the event of full probability measure only depends on (b^n) . Usually, for each n , we have to choose Y^n to be the pathwise unique strong solution and the set of full measure on which convergence holds depends on the choice of Y^n (respectively the set of ω for which it fulfills the equation). Additionally, with this procedure the event of full measure would also depend on the choice of the initial condition.

Strong completeness refers to the property that for almost all trajectories of the Brownian motion, the equation can be solved globally regardless of the initial data. Strong completeness is a rather demanding property and may fail even for SDEs with bounded smooth (but with unbounded derivatives) coefficients, with examples given by Li and Scheutzow in [77]. Strong completeness for SDEs driven by Brownian motion with standard regularity conditions are considered in [75, 100]. Alternatively, when the SDE under consideration can be lifted to a rough differential equation which has unique global solution, then it is strongly complete, see [97]. The SDE (V.1.3) under condition (V.1.4) falls outside of the standard Itô framework and Lyons' rough path theory. Hence, our result provides the first example of strongly complete singular SDEs driven by Brownian motion.

To the best of our knowledge, path-by-path stability is a new concept, and path-by-path uniqueness under (V.1.4) has not been established in the literature although there were some open discussions and attempts on the problem in [41, 101]. [34, 101] shows path-by-path uniqueness for SDEs with bounded drifts. For equations driven by general fractional noise with distributional drifts, this kind of result is shown in [24, 49, 62], by reformulating the original equation into the framework of nonlinear Young differential equations. In this framework, proofs are considerably simpler, however, the assumed regularity conditions on the drifts in the aforementioned works exclude the Krylov–Röckner condition (V.1.4). Although the solutions of nonlinear Young differential equations are apparently different from solutions of ordinary differential equations, it follows from our analysis that solutions of (V.1.5) can also be interpreted in nonlinear Young sense, see Remark V.4.5.

Our approach is closer to that of [34] taking into account the insights from [24, 49, 101] and the quantitative John–Nirenberg inequality recently obtained in [71]. Davie [34] delivers all arguments from first principles, which unfortunately increases the level of technicality. His proof consists of two main parts; one is the regularizing estimates for averaging operators along Brownian motion ([34, The basic estimate, Proposition 2.1]) and the other is the uniqueness argument. Catellier and Gubinelli in [24] reformulate and extend Davie's approach in the framework of nonlinear Young integration, leading to path-by-path uniqueness for a large class of fractional SDEs. Although more general, results in [24] do not reproduce Davie's result due to several technical reasons. Shaposhnikov in [101] gives a simpler justification for Davie's uniqueness arguments, calling out the importance of the flow generated by (V.1.3). Galeati and Gerencsér in [49] adopt this argument in combination with estimates for nonlinear Young integrals. It is known that [101] contains several technical issues some of which were corrected in [102]. Herein, we discover and justify another one which was not addressed in [101, 102], see Remark V.5.3.

As for our strategy, we apply John–Nirenberg inequality to show Davie estimates under (V.1.4) (see Proposition V.3.1). The John–Nirenberg inequality effectively reduces estimating all positive moments to estimating only the second moment. This shortens existing arguments and improves known results from [34, 95]. Building upon Davie estimates, we use nonlinear Young integration to obtain a key regularizing estimate for averaging operators along Brownian motion (Lemma V.4.3). This allows us to show that equation (V.1.3) generates a random continuous semiflow and obtain *a priori* estimates for any solution to (V.1.5) (Propositions V.5.1 and V.5.2). Applying these *a priori* estimates and the regularity of the semiflow generated by (V.1.3), we are able to identify any solution with the semiflow, see proof of Theorem V.2.2 in Section V.5. Although using similar tools as [24, 49, 101], our proofs are logically different from the aforementioned articles. Additionally, our arguments are generic and are not strictly tied to Brownian motion, as long as the random semiflow generated by the SDE (V.1.3) and the averaging operators along the driving signal W are sufficiently regular. This leads to a new uniqueness criterion for ODEs of the form (V.1.1), which does not require any regularity on the vector field, but instead relies on the regularizing effect of

the driving signal γ . These results are summarized in Section V.A.

Structure of the chapter. Main results are stated in Section V.2. In Section V.3, we show Davie estimates under (V.1.4). In Section V.4, we show some path-by-path regularizing estimates, which are consequences of Davie estimates and the sewing lemma. The proof of the main results are presented in Section V.5. In Section V.A, we summarize a uniqueness criterion for (V.1.1) which is based on the regularization effect of the driving signal. In Section V.B, we give a relation between “path-by-path” uniqueness and adaptedness of solutions.

V.2 Main results

A *semiflow* generated by the ODE (V.1.1) over the period $[0, T]$ is a measurable map $(s, t, x) \mapsto \phi_t^{s,x}$ defined on $\Delta_{[0,T]} \times \mathbb{R}^d$ such that for every $0 \leq s \leq u \leq t \leq T$ and $x \in \mathbb{R}^d$,

$$\int_s^T |b(r, \phi_r^{s,x})| dr < \infty, \quad \phi_t^{s,x} = x + \int_s^t b(r, \phi_r^{s,x}) dr + \gamma_t - \gamma_s \quad \text{and} \quad \phi_t^{s,x} = \phi_t^u \phi_u^{s,x}.$$

We say that ϕ is *locally κ -Hölder continuous* if for every compact set $K \subset \mathbb{R}^d$, there exists a finite constant $C = C(\kappa, K, T)$ such that for every $x, y \in K$ and $(s, t) \in \Delta_{[0,T]}$,

$$|\phi_t^{s,x} - \phi_t^{s,y}| \leq C|x - y|^\kappa.$$

Proposition V.2.1 (Random Hölder semiflow). *Under (V.1.4), there exists an almost surely locally κ -Hölder continuous semiflow to (V.1.3) for every $\kappa \in (0, 1)$. In other words, there exists a jointly measurable map $(s, t, x, \omega) \mapsto X_t^{s,x}(\omega)$ defined on $\Delta_{[0,T]} \times \mathbb{R}^d \times \Omega$ such that for a.s. ω , $(s, t, x) \mapsto X_t^{s,x}(\omega)$ is a locally κ -Hölder continuous semiflow to (V.1.5) for every $\kappa \in (0, 1)$.*

The semiflow $(X_t^{s,x})$ is constructed by taking the continuous extension of the family of strong solutions over the dyadics. Hence, it inherits other properties and as such, it is adapted and is a flow of homeomorphisms. However, we will not use these properties. Proposition V.2.1 thus improves upon [39, Theorem 1.2] by showing joint continuity in all parameters and allowing the set of full measure on which the flow solves the equation to be uniform with respect to the initial data. Because b is not continuous, showing that the continuous extension X forms a family of solutions to (V.1.3) is actually a major obstacle. This is overcome thanks to the path-by-path regularizing estimates obtained by sewing techniques and Davie estimates, see Lemma V.4.3 and Corollary V.4.4. These arguments can be applied for other situations and it is expected that the conclusions of Proposition V.2.1 hold for SDEs with bounded measurable drifts, more details are discussed in Remark V.5.3. Furthermore, it follows from our arguments that a solution to (V.1.5) is also a nonlinear Young solution and a regularized solution, see Remark V.4.5 for further details.

Theorem V.2.2 (Path-by-path uniqueness). *There exists an event $\Omega_{b,T} \in \mathcal{F}$ which depends only on b, T and has full probability measure such that for every $\omega \in \Omega_{b,T}$, any solution Y to (V.1.5) is identical to $X^{0,Y_0}(\omega)$. As a consequence, path-by-path uniqueness for Equation (V.1.3) holds.*

Remark V.2.3. The dependence on T can easily be removed by considering the event $\cap_{T=1}^\infty \Omega_{b,T}$. The dependence on b seems necessary. When an initial datum is specified so that $Y_0 = \xi$ for some $\xi \in \mathbb{R}^d$, the set $\Omega_{b,T}$ is independent from ξ . In addition, if ξ is a random variable which is not necessary \mathcal{F}_0 -measurable, Theorem V.2.2 implies uniqueness among non-adapted solutions.

The event $\Omega_{b,T}$ essentially contains ω such that

- (i) the semiflow $X_t^{s,x}(\omega)$ is almost Lipschitz in x ,
- (ii) the averaging operators

$$(t, x) \mapsto \int_0^t b(r, W_r(\omega) + x) dr, \int_0^t |b|(r, W_r(\omega) + x) dr$$

are jointly locally Hölder continuous with exponent $(\alpha, 1 - \varepsilon)$ for some $\alpha > 0$ and some $\varepsilon > 0$ sufficiently small.

These properties are rather generic and available not only when W is a Brownian motion but also for other stochastic processes. The case of fractional Brownian motion is currently investigated in [19].

To state the result on path-by-path stability, we introduce some additional notations. For $\nu \geq 0$, $W^{-\nu,p}(\mathbb{R}^d)$ is the Bessel potential space $W^{-\nu,p}(\mathbb{R}^d) := \{f : \|f\|_{W^{-\nu,p}(\mathbb{R}^d)} := \|(I - \Delta)^{-\frac{\nu}{2}} f\|_{L^p(\mathbb{R}^d)} < \infty\}$. When $\nu = 0$, we have $W^{-\nu,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$. A sequence (z^n) in a metric space (\mathcal{Z}, ρ) is summable convergent to an element $z \in \mathcal{Z}$ if there is a constant $\eta > 0$ such that

$$\sum_{n=1}^{\infty} \rho(z^n, z)^\eta < \infty.$$

In such case, we write $\Sigma\text{-lim}_n z^n = z$ in \mathcal{Z} . It is evident that summable convergence implies convergence in the usual sense. Conversely, if $\lim_n z^n = z$, then for every $\eta > 0$, there exists a subsequence $(z^{\ell(n)})$ such that $\sum_{n=1}^{\infty} \rho(z^{\ell(n)}, z)^\eta < \infty$, in particular $\Sigma\text{-lim}_n z^{\ell(n)} = z$.

Theorem V.2.4 (Path-by-path stability). *Let $(p, q) \in \mathcal{J}$ and (b^n) be a sequence of measurable functions in \mathbb{L}_p^q . We assume that $\Sigma\text{-lim}_n b^n = b$ in $L^q([0, T], W^{-\nu,p}(\mathbb{R}^d))$ where ν satisfies either*

$$\nu \in [0, 1 - \frac{2}{q}), \quad \frac{d}{p} + \frac{2}{q} + \nu < 2; \tag{V.2.1}$$

or

$$\nu \in [1 - \frac{2}{q}, 1], \quad \frac{d}{p} + \frac{4}{q} < 3 - 2\nu, \quad \frac{d}{p} < \nu. \tag{V.2.2}$$

Then there exists an event $\Omega'_{(b^n),b,T}$, which depends on $(b^n), b, T$ and has full probability measure, such that for every $\omega \in \Omega'_{(b^n),b,T}$, for every $(s, x) \in [0, T] \times \mathbb{R}^d$, whenever Y^n are solutions to ODEs

$$Y_t^n = x + \int_s^t b^n(r, Y_r^n) dr + W_t(\omega) - W_s(\omega), \quad t \in [s, T],$$

then $\lim_n Y^n = X^{s,x}(\omega)$ uniformly on $[s, T]$.

We say that the SDE (V.1.3) is *strongly complete* if for a.s. ω , the ODE (V.1.5) generates a continuous semiflow over $[0, \infty)$. This means that there exists a measurable map

$$[0, \infty)_{\leq}^2 \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$$

$$(s, t, x, \omega) \mapsto X_t^{s,x}(\omega)$$

such that for a.s. ω the following hold:

- (i) $(s, t, x) \mapsto X_t^{s,x}(\omega)$ is continuous;
- (ii) for every $0 \leq s \leq u \leq t$ and $x \in \mathbb{R}^d$

$$\int_s^t |b(r, X_r^{s,x}(\omega))| dr < \infty, \quad X_t^{s,x}(\omega) = x + \int_s^t b(r, X_r^{s,r}(\omega)) dr + W_t(\omega) - W_s(\omega)$$

and

$$X_t^{s,x}(\omega) = X_t^{u, X_u^{s,x}(\omega)}(\omega).$$

Our definition given above is different from the literatures' in that (ii) demands that the event on which the semiflow solves the equation is independent from initial data. Heuristically, strong completeness requires that the random semiflow is defined globally over all nonnegative times $[0, \infty)$ and continuously in (s, t, x) over $[0, \infty)_{\leq}^2 \times \mathbb{R}^d$. That the semiflow is generated by the equation means that for almost every ω and every initial data (s, x) , the map $t \mapsto X_t^{s,x}(\omega)$ is a global solution which does not blow up in finite time. The latter property can be thought as “path-by-path non-explosion”. SDEs which have global pathwise solutions without explosion may fail to have path-by-path non-explosion with examples given in [77].

Our next result provides a general principle which asserts that if over any finite period of time, an SDE generates a random continuous semiflow and has path-by-path uniqueness, then the SDE is strongly complete. Although this fact has been observed in [97] in the context of rough differential equations, singular SDEs form another distinct class of equations. It is therefore necessary to draw the connections.

Theorem V.2.5 (Strong completeness). *Suppose that for every $T > 0$, (V.1.3) generates a random continuous semiflow over $[0, T]$ and that path-by-path uniqueness holds over $[0, T]$. Then (V.1.3) is strongly complete.*

Proof. Let $(X_t^{s,x})$ and $(\bar{X}_t^{s,x})$ be random continuous semiflows respectively over $[0, T]$ and $[0, \bar{T}]$ for some $T < \bar{T}$. Then by path-by-path uniqueness $\bar{X}|_{\Delta_{[0,T] \times \mathbb{R}^d}} = X$. This implies the existence (and uniqueness) of a random continuous semiflow over all nonnegative time. \square

The proofs of Proposition V.2.1 and Theorems V.2.2 and V.2.4 are presented in Section V.5.

V.3 Davie estimates

We show a variant of *Davie estimates* in [34, Propositions 2.1, 2.2].

Proposition V.3.1. *Let f be a Schwartz function in \mathbb{L}_p^q for some $(p, q) \in \mathcal{J}$. Then there is a constant $C = C(d, p, q, T)$ such that for every $(s, t) \in \Delta_{[0,T]}$ and $m \geq 1$,*

$$\left\| \int_s^t \nabla f(r, W_r) dr \right\|_{L^m} \leq C \Gamma \left(m \left(\frac{1}{2} + \frac{d}{2p} \right) + 1 \right) \|f\|_{\mathbb{L}_p^q([s,t])} (t-s)^{\frac{1}{2} - \frac{1}{q} - \frac{d}{2p}}, \quad (\text{V.3.1})$$

where $\Gamma(r) = \int_0^\infty u^{r-1} e^{-u} du$ is the Gamma function. Consequently, for every bounded continuous function f in $\mathbb{L}_p^q([0, T])$, every $x, y \in \mathbb{R}^d$, every $(s, t) \in \Delta_{[0, T]}$ and $m \geq 1$, we have

$$\begin{aligned} & \left\| \int_s^t [f(r, W_r + x) - f(r, W_r + y)] dr \right\|_{L^m} \\ & \leq C \Gamma \left(m \left(\frac{1}{2} + \frac{d}{2p} \right) + 1 \right) \|f\|_{\mathbb{L}_p^q([s, t])} (t-s)^{\frac{1}{2} - \frac{1}{q} - \frac{d}{2p}} |x-y|. \end{aligned} \quad (\text{V.3.2})$$

The precise growth constant $\Gamma \left(m \left(\frac{1}{2} + \frac{d}{2p} \right) + 1 \right)$ is not essential to our current purposes. Nevertheless, it improves upon previous known estimates from [41, 95]. To prove this result, we follow a recent approach from [71] based on the quantitative John–Nirenberg inequality. We prepare two auxiliary lemmas, putting

$$p_t(x) := \mathbb{1}_{(t>0)} (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}.$$

Lemma V.3.2. *The operator $f \mapsto D(f)$ defined by*

$$D(f)(t, x) := \int_{\mathbb{R} \times \mathbb{R}^d} \nabla p_s(y) \cdot \nabla f(t-s, x-y) d(s, y),$$

is bounded on \mathbb{L}_p^q for every $(p, q) \in (1, \infty)^2$.

Proof. The operator D is associated to the Fourier multiplier

$$\Phi(x_0, x) = c \frac{x^2}{x^2 + ix_0}, \quad x^2 = x_1^2 + \dots + x_d^2,$$

for some absolute constant c , where i is the imaginary unit. A result from [78, Corollary 1, pg. 234] asserts that D is a bounded operator on \mathbb{L}_p^q provided that Φ admits continuous (for $x_0, \dots, x_d \neq 0$) purely mixed derivatives of orders $k \leq d+1$ such that

$$\left| x_{j_1} \dots x_{j_k} \partial_{x_{j_1} \dots x_{j_k}}^k \Phi \right| \leq M$$

for all distinct j_1, \dots, j_k and for some finite constant M . Such condition can be verified directly. \square

Lemma V.3.3 (Quantitative John–Nirenberg inequality). *Let V be an adapted continuous process, \varkappa be a deterministic control and $\alpha \in (0, 1)$. Assume that*

$$\|\mathbb{E}^s |V_t - V_s|\|_{L^\infty} \leq \varkappa(s, t)^\alpha \quad \forall (s, t) \in \Delta_{[0, T]}.$$

Then, there exists a finite constant $C_\alpha > 0$ such that for every $(s, t) \in \Delta_{[0, T]}$ and every $m \geq 1$,

$$\left\| \sup_{u \in [s, t]} |V_u - V_s| \right\|_{L^m} \leq C_\alpha \Gamma(m(1-\alpha) + 1) \varkappa(s, t)^\alpha.$$

Proof. Without the growth constant $\Gamma(m(1-\alpha) + 1)$, this result is a direct consequence of [106, Exercice A.3.2]. To obtain the precise growth constant, we follow [71] closely. Continuity and the assumption imply that

$$\|\mathbb{E}^\sigma |V_\tau - V_\sigma|\|_{L^\infty} \leq \varkappa(s, t)^\alpha$$

for every $(s, t) \in \Delta_{[0, T]}$ and all stopping times σ, τ satisfying $s \leq \sigma \leq \tau \leq t$. The class of all processes with such property is denoted by $\text{VMO}^{(1/\alpha)\text{-Var}}$ in [71, Section 3]. We then apply [71, Corollary 3.5] to obtain the desired estimate. \square

Proof of Proposition V.3.1. In view of Lemma V.3.3, it suffices to estimate the conditional second moment. Define for each $(s, t) \in \Delta_{[0, T]}$,

$$V_t := \int_0^t \nabla f(r, W_r) dr, \quad \text{and} \quad I_{s,t} := \frac{1}{2} \mathbb{E}^s |V_t - V_s|^2. \quad (\text{V.3.3})$$

Using integration by parts, tower property of conditional expectation and Fubini's Theorem we have, almost surely,

$$\begin{aligned} I_{s,t} &= \mathbb{E}^s \int_s^t \int_{r_2}^t \mathbb{E}^{r_2} [\nabla f(r_1, W_{r_1})] \cdot \nabla f(r_2, W_{r_2}) dr_1 dr_2 \\ &= -\mathbb{E}^s \int_s^t \int_{r_2}^t \left[\int_{\mathbb{R}^d} f(r_1, y + W_{r_2}) \nabla p_{r_1-r_2}(y) dy \right] \cdot \nabla f(r_2, W_{r_2}) dr_1 dr_2 \\ &= -\int_s^t \int_{r_2}^t \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} f(r_1, y + z + W_s) \nabla p_{r_1-r_2}(y) dy \right] \cdot \nabla f(r_2, z + W_s) p_{r_2-s}(z) dz dr_1 dr_2 \\ &=: I_1 + I_2 \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_s^t \int_{r_2}^t \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} f(r_1, y + z + W_s) \nabla p_{r_1-r_2}(y) dy \right] \cdot f(r_2, z + W_s) \nabla p_{r_2-s}(z) dz dr_1 dr_2, \\ I_2 &= \int_s^t \int_{r_2}^t \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \nabla f(r_1, y + z + W_s) \cdot \nabla p_{r_1-r_2}(y) dy \right] f(r_2, z + W_s) p_{r_2-s}(z) dz dr_1 dr_2. \end{aligned}$$

To estimate I_1 , we apply Hölder's inequality (below p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$) to see that for any $v \in \mathbb{R}^d$,

$$\begin{aligned} &\int_s^t \int_{r_2}^t \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} f(r_1, y + z + v) \nabla p_{r_1-r_2}(y) dy \right] f(r_2, z + v) \nabla p_{r_2-s}(z) dz dr_1 dr_2 \\ &\leq \int_s^t \int_{r_2}^t \|f(r_1, z + v + \cdot)\|_{L^p(\mathbb{R}^d)} \|\nabla p_{r_1-r_2}\|_{L^{p'}(\mathbb{R}^d)} \|f(r_2, v + \cdot)\|_{L^p(\mathbb{R}^d)} \|\nabla p_{r_2-s}\|_{L^{p'}(\mathbb{R}^d)} dr_1 dr_2 \\ &\lesssim \|f\|_{\mathbb{L}_p^q([s,t])} \int_s^t \left(\int_{r_2}^t (r_1 - r_2)^{\left(-\frac{1}{2} - \frac{d}{2p}\right) \frac{q}{q-1}} dr_1 \right)^{1-\frac{1}{q}} \|f(r_2)\|_{L^p(\mathbb{R}^d)} (r_2 - s)^{-\frac{1}{2} - \frac{d}{2p}} dr_2 \\ &\lesssim \|f\|_{\mathbb{L}_p^q([s,t])} \int_s^t (t - r_2)^{\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}} \|f(r_2)\|_{L^p(\mathbb{R}^d)} (r_2 - s)^{-\frac{1}{2} - \frac{d}{2p}} dr_2 \\ &\lesssim \|f\|_{\mathbb{L}_p^q([s,t])} (t - s)^{\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}} \int_s^t \|f(r_2)\|_{L^p(\mathbb{R}^d)} (r_2 - s)^{-\frac{1}{2} - \frac{d}{2p}} dr_2 \\ &\lesssim \|f\|_{\mathbb{L}_p^q([s,t])} (t - s)^{\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}} \|f\|_{\mathbb{L}_p^q([s,t])} (t - s)^{\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}} \lesssim \|f\|_{\mathbb{L}_p^q([s,t])}^2 (t - s)^{1 - \frac{2}{q} - \frac{d}{p}}. \end{aligned}$$

This yields that, almost surely,

$$|I_1| \lesssim \|f\|_{\mathbb{L}_p^q([s,t])}^2 (t - s)^{1 - \frac{2}{q} - \frac{d}{p}}.$$

To estimate I_2 , we apply Hölder inequality and Lemma V.3.2 to see that

$$|I_2| \lesssim \|f\|_{\mathbb{L}_p^q([s,t])} \|\mathbb{1}_{[s,t]} f^{W_s} p^s\|_{\mathbb{L}_{p'}^{q'}}$$

where

$$f^{W_s}(r, z) = f(r, z + W_s), \quad p_r^s(z) = p_{r-s}(z),$$

and p', q' denote the Hölder conjugates of p, q respectively. Applying Hölder inequality again, we have

$$\|\mathbb{1}_{[s,t]} f^{W_s} p^s\|_{\mathbb{L}^{q'}} \leq \|\mathbb{1}_{[s,t]} p^s\|_{\mathbb{L}^{q/(q-2)}} \|f\|_{\mathbb{L}_p^q([s,t])}.$$

Using the fact that $(p, q) \in \mathcal{J}$ and some elementary calculations, we have

$$\|\mathbb{1}_{[s,t]} p^s\|_{\mathbb{L}_p^{q/(q-2)}} = c(t-s)^{1-\frac{d}{p}-\frac{2}{q}}$$

for some constant c depending on p, q . Combining with the previous estimates, we obtain that

$$|I_{s,t}| \lesssim \|f\|_{\mathbb{L}_p^q([s,t])}^2 (t-s)^{1-\frac{2}{q}-\frac{d}{p}} \quad a.s.$$

Consequently,

$$\|\mathbb{E}^s |V_t - V_s|\|_{L^\infty} \lesssim \|f\|_{\mathbb{L}_p^q([s,t])} (t-s)^{\frac{1}{2}-\frac{1}{q}-\frac{d}{2p}}.$$

It is evident that V is continuous. Applying Lemma V.3.3, we obtain (V.3.1).

Next we show (V.3.2). By approximations, we can assume that f has bounded continuous first derivatives. We observe that

$$\int_s^t f(r, W_r + x) - f(r, W_r + y) dr = (x - y) \cdot \int_0^1 \int_s^t \nabla f(r, W_r + \theta x + (1 - \theta)y) dr d\theta.$$

Then it follows from (V.3.1) that

$$\begin{aligned} & \left\| \int_s^t f(r, W_r + x) - f(r, W_r + y) dr \right\|_{L^m} \\ & \leq \sup_{\theta \in [0,1]} \left\| \int_s^t \nabla f(r, W_r + \theta x + (1 - \theta)y) dr \right\|_{L^m} |x - y| \\ & \leq |x - y| \sup_{z \in \mathbb{R}^d} \left\| \int_s^t \nabla f(r, W_r + z) dr \right\|_{L^m} \\ & \lesssim \Gamma \left(m \left(\frac{1}{2} + \frac{d}{2p} \right) + 1 \right) \|f\|_{\mathbb{L}_p^q([s,t])} (t-s)^{\frac{1}{2}-\frac{1}{q}-\frac{d}{2p}} |x - y|. \end{aligned}$$

The proof is completed. □

V.4 Path-by-path estimates

Let f be a measurable function on $\mathbb{R} \times \mathbb{R}^d$. If ω is such that $\int_0^T |f(r, W_r(\omega))| dr < \infty$ then the function $t \mapsto \int_0^t f(r, W_r(\omega)) dr$ is continuous. The following result extends this argument for the function $(t, x) \mapsto \int_0^t f(r, W_r(\omega) + x) dr$. The integrability condition $\int_0^T |f(r, W_r(\omega) + x)| dr < \infty$ is no longer sufficient and one has to replace it with the quantity $\Xi_{T,K}(f)(\omega)$ defined below.

Lemma V.4.1. *Let $K \subset \mathbb{R}^d$ be a compact set containing 0, $\alpha \in (0, \frac{1}{2} - \frac{d}{2p} - \frac{1}{q})$ for some $(p, q) \in \mathcal{J}$ and $\varepsilon \in (0, 1)$. There exists a function $\Xi_{T,K,\alpha,\varepsilon} = \Xi_{T,K} : \mathbb{L}_p^q \rightarrow L^1(\Omega)$ such that*

- (i) $\Xi_{T,K}$ satisfies the triangle inequality, i.e. $\Xi_{T,K}(f + g) \leq \Xi_{T,K}(f) + \Xi_{T,K}(g)$;
- (ii) for every $f \in \mathbb{L}_p^q$,

$$\mathbb{E}[\Xi_{T,K}(f)] \lesssim \|f\|_{\mathbb{L}_p^q([0,T])}; \quad (\text{V.4.1})$$

- (iii) for every bounded measurable function f in \mathbb{L}_p^q , there exists an event $\Omega_{f,T,K}$ of full measure such that for every $\omega \in \Omega_{f,T,K}$, every $(s, t) \in \Delta_{[0,T]}$ and every $x, y \in K$,

$$\left| \int_s^t [f(r, W_r(\omega) + x) - f(r, W_r(\omega) + y)] dr \right| \leq \Xi_{T,K}(f)(\omega) |x - y|^{1-\varepsilon} (t - s)^\alpha, \quad (\text{V.4.2})$$

$$\left| \int_s^t f(r, W_r(\omega) + x) dr \right| \leq \Xi_{T,K}(f)(\omega) (t - s)^\alpha. \quad (\text{V.4.3})$$

Proof. For each function $f \in \mathbb{L}_p^q$, we define

$$\begin{aligned} \Xi_{T,K}(f) = C_{T,K} & \left(\iint_{[0,T]^2} \left| \frac{\int_s^t f(r, W_r) dr}{|t - s|^{\gamma_1}} \right|^m ds dt \right. \\ & \left. + \iint_{[0,T]^2 \times K^2} \left| \frac{\int_s^t [f(r, W_r + x) - f(r, W_r + y)] dr}{|t - s|^{\gamma_1} |x - y|^{\gamma_2}} \right|^m ds dt dx dy \right)^{\frac{1}{m}} \end{aligned}$$

for some absolute constants $C_{T,K} > 0$, $m \geq 2$ sufficiently large and $\gamma_1 \in (\frac{2}{m}, \frac{1}{m} + (\frac{1}{2} - \frac{d}{2p} - \frac{1}{q}))$, $\gamma_2 \in (\frac{2}{m}, 1 + \frac{1}{m})$ fixed. The constants $C_{T,K}$ and m will be determined at a later step. It is obvious that (i) holds. Using (V.3.2) and the following estimate from [72, Lemma 4.5]

$$\mathbb{E} \left(\int_s^t f(r, W_r) dr \right)^m \lesssim \|f\|_{\mathbb{L}_p^q([s,t])}^m (t - s)^{m(1 - \frac{d}{p} - \frac{2}{q})}, \quad (\text{V.4.4})$$

we see that (ii) holds.

It remains to show that (iii) holds. Assume first that f is bounded continuous so that the map $(t, x) \mapsto \int_0^t f(r, W_r(\omega) + x) dr$ is continuous for a.s. ω . From (V.3.2) and (V.4.4), applying the multiparameter Garsia–Rodemich–Rumsey inequality ([64, Theorem 3.1]), there exist constants $C_{T,K}$ and m such that (V.4.2) and (V.4.3) hold.

Below, we remove the continuity restriction.

Step 1. We show (iii) for $f = \mathbb{1}_U$ where U is an open set of finite measure in $\mathbb{R} \times \mathbb{R}^d$. By Urysohn lemma, there exists a sequence of increasing continuous functions f^n converging pointwise to f . From (i) and (ii), we can choose a further subsequence, still denoted by (f^n) such that $\lim_n \Xi_{T,K}(f^n) = \Xi_{T,K}(f)$ a.s. We then apply (V.4.2) and (V.4.3) for f^n and take limit in n to obtain (iii).

Step 2. We show (iii) for a general bounded measurable f in \mathbb{L}_p^q . Let M be a constant such that $|f| \leq M$. By Lusin theorem, for any $n \in \mathbb{N}$, there exists a continuous function f^n and an open set U^n such that $|U^n| \leq 2^{-n}$ and $|f - f^n| \leq 2M \mathbb{1}_{U^n}$. This implies that $\lim_n f^n = f$ in \mathbb{L}_p^q . Using (ii), we can choose subsequences, still denoted by n such that a.s.

$$\lim_n \Xi_{T,K}(\mathbb{1}_{U^n}) = 0 \quad \text{and} \quad \lim_n \Xi_{T,K}(f^n) = \Xi_{T,K}(f). \quad (\text{V.4.5})$$

Define

$$A[f](\omega) = \left| \int_s^t [f(r, W_r(\omega) + x) - f(r, W_r(\omega) + y)] dr \right|.$$

Then

$$A[f] \leq A[f^n] + A[f - f^n]. \quad (\text{V.4.6})$$

Applying (V.4.2) for the continuous function f^n , we have a.s.

$$A[f^n] \leq \Xi_{T,K}(f^n) |x - y|^{1-\varepsilon} (t - s)^\alpha \quad \forall (s, t, x, y) \in \Delta_{[0,T]} \times K^2.$$

Applying (V.4.2) and (V.4.3) for $\mathbb{1}_{U^n}$, we have a.s.

$$\begin{aligned} A[f - f^n] &\leq 2M \int_s^t [\mathbb{1}_{U^n}(r, W_r + x) + \mathbb{1}_{U^n}(r, W_r + y)] dr \\ &\leq 4M \Xi_{T,K}(\mathbb{1}_{U^n})(t - s)^\alpha \quad \forall (s, t, x, y) \in \Delta_{[0,T]} \times K^2. \end{aligned}$$

Hence, from (V.4.6), taking limit in n and using (V.4.5), we see that (V.4.2) holds. The estimate (V.4.3) is obtained in a similar way. \square

Remark V.4.2. Via a truncation procedure as is done in the last step in the proof of Lemma V.4.3 below, one can remove the assumption of boundedness in Lemma V.4.1(iii). The current formulation is sufficient for our purpose.

The following result provides an alternative perspective to [34, Lemmas 3.3 and 3.4] and [101, Lemmas 3.3 and 3.4].

Lemma V.4.3. *Let $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a function in \mathbb{L}_p^q for some $(p, q) \in \mathcal{J}$, $K \subset \mathbb{R}^d$ be a compact set containing 0, and $\alpha \in (0, \frac{1}{2} - \frac{d}{2p} - \frac{1}{q})$. Then there exists an event $\Omega_{f,T,K}$ with full probability such that for every $\omega \in \Omega_{f,T,K}$, every $\varepsilon \in (0, \alpha)$, there is a deterministic constant $C = C(\varepsilon, \alpha)$ such that for every $(s, t) \in \Delta_{[0,T]}$, every function $\psi : [0, T] \rightarrow K$ of finite variation, we have*

$$\left| \int_s^t f(r, W_r(\omega) + \psi_r) dr - \int_s^t f(r, W_r(\omega) + \psi_s) dr \right| \leq C \Xi_{T,K}(f)(\omega) [\psi]_{\mathcal{C}_{[s,t]}^{1-\varepsilon}}^{1-\varepsilon} (t - s)^\alpha \quad (\text{V.4.7})$$

and

$$\left| \int_s^t f(r, W_r(\omega) + \psi_r) dr \right| \leq C \Xi_{T,K}(f)(\omega) (1 + [\psi]_{\mathcal{C}_{[s,t]}^{1-\varepsilon}}) (t - s)^\alpha. \quad (\text{V.4.8})$$

Proof. Since (V.4.8) is a consequence of (V.4.7) and (V.4.3), we only focus on (V.4.7).

Step 1. We first show the result assuming that f is a bounded continuous function. For $(s, t) \in \Delta_{[0,T]}$ define $A_{s,t} = \int_s^t f(r, W_r + \psi_s) dr$ so that, for $u \in [s, t]$,

$$\delta A_{s,u,t} = \int_u^t [f(r, W_r + \psi_s) - f(r, W_r + \psi_u)] dr.$$

Applying (V.4.2), we have

$$|\delta A_{s,u,t}| \leq \Xi_{T,K}(f) |\psi_u - \psi_s|^{1-\varepsilon} (t - s)^\alpha \leq \Xi_{T,K}(f) [\psi]_{\mathcal{C}_{[s,t]}^{1-\varepsilon}}^{1-\varepsilon} (t - s)^\alpha.$$

That $\varepsilon < \alpha$ ensures that $1 - \varepsilon + \alpha > 1$. Since f is continuous, we have for every $t \in [0, T]$,

$$\int_0^t f(r, W_r + \psi_r) dr = \lim_{|\pi| \downarrow 0} \sum_{[a,b] \in \pi} A_{a,b}$$

where π is any partition of $[0, t]$. We apply the sewing lemma [46, Theorem 2.2 and Remark 2.3] formulated with controls to obtain (V.4.7).

Step 2. To remove the continuity assumption and show the result for a bounded measurable function f , we use similar arguments as in Lemma V.4.1.

First, using Urysohn lemma, one can show that (V.4.7) holds for $f = \mathbb{1}_U$ for any open set U of finite measure. In particular, we have a.s.

$$\left| \int_s^t \mathbb{1}_U(r, W_r + \psi_r) dr \right| \lesssim \Xi_{T,K}(\mathbb{1}_U) (1 + [\psi]_{\mathcal{C}_{[s,t]}^{1-\varepsilon}}^{1-\varepsilon}) (t-s)^\alpha \quad (\text{V.4.9})$$

for every $(s, t) \in \Delta_{[0,T]}$ and every ψ .

Next, applying Lusin theorem, we can find for each integer $n \geq 1$ a continuous bounded function f^n and an open set U^n such that $|U^n| \leq 2^{-n}$ and $|f - f^n| \leq 2M \mathbb{1}_{U^n}$, where M is a constant such that $|f| \leq M$. This implies that $\lim_n f^n = f$ in \mathbb{L}_p^q . For each $s \leq t$, we put

$$J[f] = \int_s^t f(r, W_r + \psi_r) dr - \int_s^t f(r, W_r + \psi_s) dr$$

so that $J[f] = J[f^n] + J[f - f^n]$. By the previous step,

$$|J[f^n]| \leq C \Xi_{T,K}(f^n) [\psi]_{\mathcal{C}_{[s,t]}^{1-\varepsilon}}^{1-\varepsilon} (t-s)^\alpha.$$

Using (V.4.3) and (V.4.9),

$$\begin{aligned} |J[f - f^n]| &\lesssim M \left| \int_s^t \mathbb{1}_{U^n}(r, W_r + \psi_r) dr \right| + \sup_{x \in K} \left| \int_s^t \mathbb{1}_{U^n}(r, W_r + x) dr \right| \\ &\lesssim MT^\alpha (1 + [\psi]_{\mathcal{C}_{[s,t]}^{1-\varepsilon}}^{1-\varepsilon}) \Xi_{T,K}(\mathbb{1}_{U^n}). \end{aligned}$$

By Lemma V.4.1(i)(ii), we can further choose a subsequence, still denoted by n , such that

$$\lim_n \Xi_{T,K}(f^n) = \Xi_{T,K}(f) \quad \text{and} \quad \lim_n \Xi_{T,K}(\mathbb{1}_{U^n}) = 0 \quad \text{a.s.}$$

We emphasize that the null events only depend on f, T, K and are independent from s, t, ψ . Then, by passing through the limit in n , we obtain that

$$|J[f]| \leq C \Xi_{T,K}(f) [\psi]_{\mathcal{C}_{[s,t]}^{1-\varepsilon}}^{1-\varepsilon} (t-s)^\alpha.$$

Step 3. Consider now the case $f \in \mathbb{L}_p^q$. For each integer $M \geq 1$, define $f^M = f \mathbb{1}_{(|f| \leq M)}$. Note that (f^M) (respectively $(|f^M|)$) is a sequence of bounded functions converging to f (respectively $|f|$) in \mathbb{L}_p^q . We can choose a sequence (M_n) such that $(\Xi_{T,K}(f^{M_n}), \Xi_{T,K}(|f^{M_n}|))$ converges to

$(\Xi_{T,K}(f), \Xi_{T,K}(|f|))$ a.s. Using this, (V.4.8) (with $|f^{M_n}|$) and monotone convergence we then have a.s. ω

$$\begin{aligned} \int_s^t |f(r, W_r(\omega) + \psi_r)| dr &= \lim_{n \rightarrow \infty} \int_s^t |f^{M_n}(r, W_r(\omega) + \psi_r)| dr \\ &\leq C \Xi_{T,K}(|f|)(\omega) (1 + [\psi]_{\mathcal{C}_{[s,t]}^{1-\varepsilon}})(t-s)^\alpha < \infty. \end{aligned}$$

From the above estimate and (V.4.3), applying the Lebesgue dominated convergence theorem, we have $\lim_n J[f - f^{M_n}] = 0$. As in the previous step, we have

$$\begin{aligned} |J[f]| &\leq |J[f^{M_n}]| + |J[f - f^{M_n}]| \\ &\lesssim \Xi_{T,K}(f^{M_n}) [\psi]_{\mathcal{C}_{[s,t]}^{1-\varepsilon}}(t-s)^\alpha + |J[f - f^{M_n}]|. \end{aligned}$$

We take the limit in n to obtain (V.4.7).

This concludes the proof. \square

Corollary V.4.4. *Let f be a function in \mathbb{L}_p^q for some $(p, q) \in \mathcal{J}$. Let (\mathcal{Z}, ρ) be a metric space and $(t, z) \mapsto \psi_t^z$ be a bounded function from $[0, T] \times \mathcal{Z}$ to \mathbb{R}^d such that*

(i) *for each $z \in \mathcal{Z}$, $t \mapsto \psi_t^z$ has finite variation and $\sup_{z \in \mathcal{Z}} [\psi^z]_{\mathcal{C}_{[0,T]}^{1-\text{var}}} < \infty$;*

(ii) *the family $\{z \mapsto \psi_t^z\}_{t \in [0, T]}$ is uniformly equicontinuous, i.e.*

$$\lim_{h \downarrow 0} \sup_{t \in [0, T]} \sup_{\rho(z, \bar{z}) \leq h} |\psi_t^z - \psi_t^{\bar{z}}| = 0.$$

Then there exist an event $\Omega_{f,T}$ of full measure and a sequence of bounded continuous functions (f^n) which are independent from ψ and (\mathcal{Z}, ρ) , such that $\lim_n f^n = f$ in \mathbb{L}_p^q and for every $\omega \in \Omega_{f,T}$,

$$\lim_n \int_0^t f^n(r, W_r(\omega) + \psi_r^z) dr = \int_0^t f(r, W_r(\omega) + \psi_r^z) dr \text{ uniformly in } (t, z) \in [0, T] \times \mathcal{Z}.$$

Consequently, the map

$$(t, z) \mapsto \int_0^t f(r, W_r(\omega) + \psi_r^z) dr$$

is continuous on $[0, T] \times \mathcal{Z}$.

Proof. Let (K^i) be an increasing sequence of compacts such that $\bigcup_i K^i = \mathbb{R}^d$. Let Ω_{f,T,K^i} be as in Lemma V.4.3. In view of (V.4.1), we can choose a sequence of bounded continuous functions (f^n) such that $\lim_n \Xi_{T,K^i}(f^n - f) = 0$ a.s. for all $i \in \mathbb{N}$. Because ψ is bounded, there exists $j \in \mathbb{N}$ such that $\psi_t^z \in K^j$ for all $(t, z) \in [0, T] \times \mathcal{Z}$. Applying (V.4.8) and (i), we have for any $\omega \in \Omega_{f,T} := \bigcap_i \Omega_{f,T,K^i}$,

$$\lim_n \int_0^t f^n(r, W_r(\omega) + \psi_r^z) dr = \int_0^t f(r, W_r(\omega) + \psi_r^z) dr \text{ uniformly in } (t, z) \in [0, T] \times \mathcal{Z}.$$

Due to (ii), the function $(t, z) \mapsto \int_0^t f^n(r, W_r(\omega) + \psi_r^z) dr$ is continuous for each n . This shows the claim. \square

Remark V.4.5. We record some consequential observations which may be useful for other purposes.

(i) From Lemma V.4.3, it follows that for a.s. ω , a solution of (V.1.5) is also a solution in the framework of nonlinear Young integrals. More precisely, for a.s. ω , if Y is a solution to (V.1.5), then there is a control \varkappa and a number $\beta > 1$ such that for $\psi = Y - W$,

$$|\psi_t - \psi_s - \int_s^t b(r, W_r(\omega) + \psi_s) dr| \leq \varkappa(s, t)^\beta \quad \forall (s, t) \in \Delta_{[0, T]}.$$

Via the sewing lemma, this means that ψ is a solution to the nonlinear Young integral equation ([65])

$$\psi_t = \psi_s + \int_s^t b^W(dr, \psi_r) \quad \forall (s, t) \in \Delta_{[0, T]},$$

where $\int_s^t b^W(dr, \psi_r)$ is the nonlinear Young integral defined as the limit of the Riemann sums

$$\sum_{[u, v]} \int_u^v b(r, W_r + \psi_u) dr.$$

While nonlinear Young integral equations have been a central theme in previous works [24, 49, 50, 62, LA4], it was not known if (V.1.5) under (V.1.4) can be formulated in this framework.

(ii) From Corollary V.4.4, it follows that for a.s. ω , a solution of (V.1.5) is also a *regularized solution* in the following sense: For a.s. ω , if Y is a solution to (V.1.5) then there exists a sequence of bounded continuous function (b^n) and a continuous function $V : [0, T] \rightarrow \mathbb{R}^d$ such that $V_t = \lim_n \int_0^t b^n(r, Y_r) dr$ uniformly on $[0, T]$ and $Y_t = Y_0 + V_t + W_t(\omega)$ for all $t \in [0, T]$. Regularized solutions of stochastic differential equations appear in [8, 13, 20, LA4].

V.5 Proof of main results

For a given ω and a solution Y to (V.1.5), we obtain in Proposition V.5.1 *a priori* estimates on the variations of $Y - W$ on arbitrary intervals. This allows us to show the existence of random Hölder continuous semiflow $(X_t^{s,x})_{s,t,x}$, i.e. Proposition V.2.1. We then show in Proposition V.5.2 another set of *a priori* estimates on $Y_t - X_t^{s,Y_s}(\omega)$ for any given $(s, t) \in \Delta_{[0, T]}$. Having these properties at our disposal, we proceed to prove path-by-path uniqueness and stability, i.e. Theorems V.2.2 and V.2.4.

Proposition V.5.1. *Let $K \subset \mathbb{R}^d$ be a compact set containing 0, $(u, v) \in \Delta_{[0, T]}$ and $\alpha \in (0, \frac{1}{2} - \frac{d}{2p} - \frac{1}{q})$. Let ω be such that (V.4.7) and (V.4.8) hold with $f = |b|$ and any function ψ of finite variation taking values in K . Let $Y : [u, v] \rightarrow \mathbb{R}^d$ be a solution to (V.1.5) on $[u, v]$, i.e. $\int_u^v |b(r, Y_r)| dr < \infty$ and*

$$Y_t = Y_s + \int_s^t b(r, Y_r) dr + W_t(\omega) - W_s(\omega) \quad \forall (s, t) \in \Delta_{[u, v]}.$$

Assume that $Y_t - W_t(\omega) \in K$ for all $t \in [u, v]$. Then there exists a finite constant $C = C(\varepsilon, \alpha)$ such that

$$\int_u^v |b(r, Y_r)| dr \leq C \Xi_{T, K}(|b|)(\omega) (v - u)^\alpha + C (\Xi_{T, K}(|b|)(\omega) (v - u)^\alpha)^{\frac{1}{\varepsilon}}. \quad (\text{V.5.1})$$

Proof. We omit the dependence on ω in this proof. Define $\psi = Y - W$ and note that

$$[\psi]_{\mathcal{C}_{[u,v]}^{1-\text{var}}} \leq \int_u^v |b|(r, Y_r) dr.$$

Applying (V.4.8), we have

$$\int_u^v |b|(r, Y_r) dr \leq C \Xi_{T,K}(|b|)(v-u)^\alpha + C \Xi_{T,K}(|b|) \left(\int_u^v |b|(r, Y_r) |dr \right)^{1-\varepsilon} (v-u)^\alpha.$$

Applying Young's inequality, we have for every $\varepsilon' > 0$

$$\Xi_{T,K}(|b|) \left| \int_u^v |b|(r, Y_r) dr \right|^{1-\varepsilon} (v-u)^\alpha \leq \varepsilon' \int_u^v |b|(r, Y_r) dr + C_{\varepsilon'} (\Xi_{T,K}(|b|)(v-u)^\alpha)^{\frac{1}{\varepsilon}}.$$

We choose ε' sufficiently small to get (V.5.1). \square

Proof of Proposition V.2.1. For each s, x , let $(\tilde{X}_t^{s,x})_{t \in [s,T]}$ be the unique strong solution to (V.1.3) started from x at time s . By definition of a solution, for every ω in a set of full measure that depends on s and x , we have $\int_s^T |b(r, \tilde{X}_r^{s,x}(\omega))| dr < \infty$ and

$$\tilde{X}_t^{s,x}(\omega) = x + \int_s^t b(r, \tilde{X}_r^{s,x}(\omega)) dr + W_t(\omega) - W_s(\omega) \quad \forall t \in [s, T]. \quad (\text{V.5.2})$$

Note that the construction above is also possible for random initial condition x (see [115, Theorem 1.1]), however, if not mentioned otherwise, throughout the rest of the proof x is deterministic.

Let $m \geq 2$ be a fixed number. By [52, Theorem 1.2 (1.2)]

$$\|\tilde{X}_t^{s,x} - \tilde{X}_t^{s,y}\|_{L^m} \lesssim \|x - y\|_{L^m} \quad (\text{V.5.3})$$

for any initial random points x, y which are \mathcal{F}_s -measurable. Applying [72, Lemma 4.5], we get

$$\|\tilde{X}_t^{s,x} - \tilde{X}_{t'}^{s,x}\|_{L^m} \lesssim \left\| \int_{t'}^t b(r, \tilde{X}_r^{s,x}) dr \right\|_{L^m} \lesssim |t - t'|^\alpha$$

for some $\alpha \in (\frac{1}{2}, 1)$. Using (V.5.3) and pathwise uniqueness, we have for $s' < s$,

$$\|\tilde{X}_t^{s,x} - \tilde{X}_t^{s',x}\|_{L^m} = \|\tilde{X}_t^{s,x} - \tilde{X}_t^{s, \tilde{X}_s^{s',x}}\|_{L^m} \lesssim \|x - \tilde{X}_s^{s',x}\|_{L^m} \lesssim |s - s'|^{\frac{1}{2}}.$$

It follows, that

$$\|\tilde{X}_t^{s,x} - \tilde{X}_{t'}^{s',y}\|_{L^m} \lesssim |x - y| + |t - t'|^\alpha + |s - s'|^{\frac{1}{2}}. \quad (\text{V.5.4})$$

Since m can be arbitrarily large, applying Kolmogorov continuity criterion ([94, Theorem (2.1) Chapter I]), we see that for a.s. ω , the map $(s, t, x) \mapsto \tilde{X}_t^{s,x}(\omega)$ is locally Hölder continuous on

$$\{(s, t, x) \in \Delta_{[0,T]} \times \mathbb{R}^d : s, t, x \text{ are dyadic}\}$$

with exponents $(\alpha', \beta, \kappa) \in (0, \alpha) \times (0, 1/2) \times (0, 1)$. Because for each s, x , $(\tilde{X}_t^{s,x}(\omega))_{t \in [s,T]}$ is continuous, the map $(s, t, x) \mapsto \tilde{X}_t^{s,x}(\omega)$ is locally Hölder continuous on

$$G := \{(s, t, x) \in \Delta_{[0,T]} \times \mathbb{R}^d : s, x \text{ are dyadic}\}$$

with the same exponents.

Let Ω' be the event of full measure on which (V.5.2) holds whenever s, x are dyadic, and \tilde{X} is locally Hölder continuous on G . For each $\omega \in \Omega'$, let $X(\omega)$ be the (unique) continuous extension of $\tilde{X}(\omega)|_G$ to $\Delta_{[0,T]} \times \mathbb{R}^d$. We show that $(X_t^{s,x})_{s,t,x}$ is the desired semiflow. Hölder regularity is clear and hence, we focus on showing the semiflow properties. Since b is not continuous, it is non-trivial that for almost every ω and for each (s, x) , the map $t \mapsto X_t^{s,x}(\omega)$ satisfies the equation (V.1.5). The main difficulty is to show that the continuous extension of the map

$$(s, t, x) \mapsto \int_s^t b(r, \tilde{X}_r^{s,x}(\omega)) dr = \int_s^t b(r, X_r^{s,x}(\omega)) dr$$

which is defined on G , is identical to the map

$$(s, t, x) \mapsto \int_s^t b(r, X_r^{s,x}(\omega)) dr,$$

which is defined on $\Delta_{[0,T]} \times \mathbb{R}^d$.

Let $H \subset \mathbb{R}^d$ be a compact set and put $\mathcal{Z} = [0, T] \times H$. Let $(K_n)_{n \in \mathbb{N}}$ be sequence of compacts exhausting \mathbb{R}^d . Let $\tilde{\Omega}_{|b|,T} := \Omega' \cap \bigcap_{n \in \mathbb{N}} \Omega_{|b|,T,K_n}$, for $\Omega_{|b|,T,K_n}$ as in Lemma V.4.3, i.e. (V.4.7) and (V.4.8) hold for $f = |b|$ and any function ψ of finite variation with values in K_n . We verify the conditions (i) and (ii) of Corollary V.4.4 for $\omega \in \tilde{\Omega}_{|b|,T}$. For each $(t, s, x) \in [0, T] \times \mathcal{Z}$, define $\psi_t^{s,x}(\omega) := X_{\min(s,t)}^{s,x}(\omega) - W_{\min(s,t)}(\omega)$, and if s, x are dyadic, define $\tilde{\psi}_t^{s,x}(\omega) := \tilde{X}_{\min(s,t)}^{s,x}(\omega) - W_{\min(s,t)}(\omega)$. Note that $\psi(\omega)$ is the continuous extension of $\tilde{\psi}(\omega)$. By continuity, there exists $N \in \mathbb{N}$ such that $\psi^{s,x}(\omega)$ takes values in K_N for all $(s, x) \in \mathcal{Z}$. As (V.4.8) holds, we can apply Proposition V.5.1 for the compact K_N and we can find a constant $C_T(\omega)$ such that whenever s, x are dyadic,

$$[\psi^{s,x}(\omega)]_{\mathcal{C}_{[0,T]}^{1\text{-var}}} = [\tilde{\psi}^{s,x}(\omega)]_{\mathcal{C}_{[0,T]}^{1\text{-var}}} \leq \int_s^T |b|(r, \tilde{X}_r^{s,x}(\omega)) dr \leq C_T(\omega).$$

This means that for any partition π of $[0, T]$ and (s, x) dyadic in \mathcal{Z}

$$\sum_{[u,v] \in \pi} |\psi_v^{s,x}(\omega) - \psi_u^{s,x}(\omega)| \leq C_T(\omega).$$

By continuity, the above estimate also holds for every $(s, x) \in \mathcal{Z}$. This implies that

$$\sup_{(s,x) \in \mathcal{Z}} [\psi^{s,x}(\omega)]_{\mathcal{C}_{[0,T]}^{1\text{-var}}} \leq C_T(\omega),$$

verifying (i). Uniform equicontinuity condition (ii) is satisfied because $X(\omega)$ is locally Hölder continuous on $\Delta_{[0,T]} \times \mathbb{R}^d$. Let $\Omega_{b,T}$ be the event in Corollary V.4.4. Let $\omega \in \tilde{\Omega}_{|b|,T} \cap \Omega_{b,T}$. Applying Corollary V.4.4, we see that the map

$$(t, s, x) \mapsto \int_0^t b(r, W_r(\omega) + \psi_r^{s,x}(\omega)) dr = \int_0^t b(r, X_r^{s,x}(\omega)) dr$$

is continuous on $[0, T] \times \mathcal{Z}$.

It is now clear from (V.5.2) that for every $\omega \in \tilde{\Omega}_{|b|,T} \cap \Omega_{b,T}$, for every $(s, t, x) \in \Delta_{[0,T]} \times H$, we have

$$X_t^{s,x}(\omega) = x + \int_s^t b(r, X_r^{s,x}(\omega)) dr + W_t(\omega) - W_s(\omega).$$

By exhausting \mathbb{R}^d with a sequence of increasing compact sets (H^i) , we see that the above equation holds for every $(s, t, x) \in \Delta_{[0, T]} \times \mathbb{R}^d$. By pathwise uniqueness and using that existence of a strong solution also holds for random initial condition, we have for every $s \leq u \leq t$ and every $x \in \mathbb{R}^d$

$$X_t^{s,x} = X_t^{u, X_u^{s,x}} \quad \text{a.s.} \quad (\text{V.5.5})$$

Note that the exceptional null event (V.5.5) depends on s, u, t, x . However, because all processes in (V.5.5) are continuous, one can deduce that for a.s. ω , $X_t^{s,x} = X_t^{u, X_u^{s,x}}$ for any s, u, t, x , which means that $(X_t^{s,x}(\omega))_{s,t,x}$ is a semiflow. \square

The next result, together with Proposition V.5.1, provide *a priori* estimates for any solution.

Proposition V.5.2. *Let $K \subset \mathbb{R}^d$ be a compact set which contains 0. Let $\Omega_{b,T,K}$ be the event with full probability on which $(X_t^{s,x})$ is a locally κ -Hölder continuous semiflow for every $\kappa \in (0, 1)$ and (V.4.7) holds with $f \in \{b, |b|\}$ and any function ψ of finite variation taking values in K . Let $\omega \in \Omega_{b,T,K}$ and Y be a solution to (V.1.5). Suppose that $X_t^{s, Y_s}(\omega) - W_t(\omega)$ belongs to K for every $(s, t) \in \Delta_{[0, T]}$. Then, there exist*

(i) *a control \varkappa which depends only on $Y, \Xi_{T,K}(b)(\omega), \Xi_{T,K}(|b|)(\omega), d, p, q$;*

(ii) *a constant $\beta > 1$ which depends only on d, p, q*

such that

$$|Y_t - X_t^{s, Y_s}(\omega)| \leq \varkappa(s, t)^\beta \quad \forall (s, t) \in \Delta_{[0, T]}. \quad (\text{V.5.6})$$

Proof. We omit the dependence of ω in the argument below. Define $\psi = Y - W$ and $\xi^{s,x} = X_{\min(s, \cdot)}^{s,x} - W_{\min(s, \cdot)}$. The proof consists of two steps.

Step 1. We show that for $\varepsilon \in (0, \alpha)$, there exist a constant $C = C(\Xi_{T,K}(|b|), \varepsilon)$ such that

$$\left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right| \leq C \left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha + C(t-s)^{\frac{\alpha}{\varepsilon}} \quad \forall (s, t) \in \Delta_{[0, T]}. \quad (\text{V.5.7})$$

As $\psi_s = \xi_s^{s, Y_s}$, we obtain that

$$\begin{aligned} & \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \\ &= \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, W_r + \xi_r^{s, Y_s})] dr - \int_s^t [|b|(r, Y_r) - |b|(r, W_r + \psi_s)] dr. \end{aligned}$$

We note that ξ^{s, Y_s} and ψ are contained in K and have finite variation such that

$$[\xi^{s, Y_s}]_{\mathcal{C}_{[s, t]}^{1\text{-var}}} \leq \int_s^t |b|(r, X_r^{s, Y_s}) dr \quad \text{and} \quad [\psi]_{\mathcal{C}_{[s, t]}^{1\text{-var}}} \leq \int_s^t |b|(r, Y_r) dr.$$

Applying Lemma V.4.3, we have for every $\varepsilon \in (0, \alpha)$ and every $(s, t) \in \Delta_{[0, T]}$,

$$\begin{aligned} \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, W_r + \xi_s^{s, Y_s})] dr \right| &\lesssim \Xi_{T, K}(|b|) \left(\int_s^t |b|(r, X_r^{s, Y_s}) dr \right)^{1-\varepsilon} (t-s)^\alpha, \\ \left| \int_s^t [|b|(r, Y_r) - |b|(r, W_r + \psi_s)] dr \right| &\lesssim \Xi_{T, K}(|b|) \left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha. \end{aligned}$$

We also have

$$\left| \int_s^t |b|(r, X_r^{s, Y_s}) dr \right|^{1-\varepsilon} \leq \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right|^{1-\varepsilon} + \left| \int_s^t |b|(r, Y_r) dr \right|^{1-\varepsilon}. \quad (\text{V.5.8})$$

It follows that

$$\begin{aligned} \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right| &\lesssim \Xi_{T, K}(|b|) \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right|^{1-\varepsilon} (t-s)^\alpha \\ &\quad + \Xi_{T, K}(|b|) \left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha. \end{aligned}$$

Applying Young's inequality, we have for every $\varepsilon' > 0$

$$\begin{aligned} \Xi_{T, K}(|b|) \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right|^{1-\varepsilon} (t-s)^\alpha \\ \leq \varepsilon' \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right| + C_{\varepsilon'} (\Xi_{T, K}(|b|) (t-s)^\alpha)^{\frac{1}{\varepsilon}}. \end{aligned} \quad (\text{V.5.9})$$

Combining the previous two inequality and choosing ε' sufficiently small yields (V.5.7).

Step 2. We show (V.5.6). Let $(s, t) \in \Delta_{[0, T]}$ be fixed. Using the equations and the identity $\psi_s = \xi_s^{s, Y_s}$, we obtain that

$$\begin{aligned} |X_t^{s, Y_s} - Y_t| &= \left| \int_s^t [b(r, X_r^{s, Y_s}) - b(r, Y_r)] dr \right| \\ &\leq \left| \int_s^t [b(r, X_r^{s, Y_s}) - b(r, W_r + \xi_s^{s, Y_s})] dr \right| + \left| \int_s^t [b(r, Y_r) - b(r, W_r + \psi_s)] dr \right|. \end{aligned}$$

Applying Lemma V.4.3, we have for every $\varepsilon \in (0, \alpha)$,

$$\begin{aligned} \left| \int_s^t [b(r, X_r^{s, Y_s}) - b(r, W_r + \xi_s^{s, Y_s})] dr \right| &\lesssim \Xi_{T, K}(b) \left(\int_s^t |b|(r, X_r^{s, Y_s}) dr \right)^{1-\varepsilon} (t-s)^\alpha, \\ \left| \int_s^t [b(r, Y_r) - b(r, W_r + \psi_s)] dr \right| &\lesssim \Xi_{T, K}(b) \left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha. \end{aligned}$$

Combining with (V.5.8), we have

$$\begin{aligned} |X_t^{s, Y_s} - Y_t| &\lesssim \Xi_{T, K}(b) \left| \int_s^t [|b|(r, X_r^{s, Y_s}) - |b|(r, Y_r)] dr \right|^{1-\varepsilon} (t-s)^\alpha \\ &\quad + \Xi_{T, K}(b) \left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha. \end{aligned}$$

We apply (V.5.9) and (V.5.7) to get

$$|X_t^{s,Y_s} - Y_t| \leq C(\varepsilon, \Xi_{T,K}(|b|), \Xi_{T,K}(b)) \left(\left(\int_s^t |b|(r, Y_r) dr \right)^{1-\varepsilon} (t-s)^\alpha + (t-s)^{\alpha/\varepsilon} \right).$$

Choosing ε small enough such that $\alpha/\varepsilon > 1$ yields (V.5.6). \square

Proof of Theorem V.2.2. Let $K_n = \{x \in \mathbb{R}^d : |x| \leq n\}$ ($n \in \mathbb{N}$) and $\Omega_{b,T} = \bigcup_{k=1}^\infty \bigcap_{n=k}^\infty \Omega_{b,T,K_n}$ for Ω_{b,T,K_n} as in Proposition V.5.2 (i.e. the event with full probability on which $(X_t^{s,x})$ is a locally κ -Hölder continuous semiflow for every $\kappa \in (0, 1)$ and (V.4.7), with $f \in \{b, |b|\}$ and any function ψ of finite variation taking values in K_n , holds). Note that $\Omega_{b,T}$ has full probability. Let $\omega \in \bigcap_{n=k}^\infty \Omega_{b,T,K_n}$ for some k and let Y be a solution to (V.1.5). By continuity, we can choose $N \geq k$ such that K_N contains $X_t^{s,Y_s}(\omega) - W_t(\omega)$ for every $(s, t) \in \Delta_{[0,T]}$. Let \varkappa be the control and β be the constant found in Proposition V.5.2. We note that \varkappa depends on N . Let $\tau \in (0, T]$ be a fixed but arbitrary number and define

$$F(t) = X_\tau^{t,Y_t}, \quad t \in [0, \tau].$$

For $(s, t) \in \Delta_{[0,\tau]}$, we obtain by the semiflow property that

$$F(t) - F(s) = X_\tau^{t,Y_t} - X_\tau^{s,Y_s} = X_\tau^{t,Y_t} - X_\tau^{t,X_t^{s,Y_s}}.$$

Then, using Hölder continuity of the semiflow (Proposition V.2.1), we have

$$|F(t) - F(s)| \lesssim |Y_t - X_t^{s,Y_s}|^\kappa,$$

for some $\kappa \in (0, 1)$ which can be chosen to be arbitrarily close to 1. Applying (V.5.6), we have

$$|F(t) - F(s)| \lesssim \varkappa(s, t)^{\kappa\beta}.$$

Choosing κ so that $\kappa\beta > 1$, for any sequence of partitions $\Pi_n = \{t_i\}_{i=0}^{N_n}$ of $[0, \tau]$ with mesh converging to 0, we get that

$$|F(\tau) - F(0)| \leq \sum_{i=1}^{N_n} |F(t_i) - F(t_{i-1})| \lesssim \left| \sup_{i \leq N_n} \varkappa(t_{i-1}, t_i) \right|^{\kappa\beta-1} \varkappa(0, \tau) \xrightarrow{n \rightarrow \infty} 0.$$

Hence $F(\tau) = F(0)$. Since τ was arbitrarily chosen in $(0, T]$, this means that Y and X^{0,Y_0} are identical. \square

Proof of Theorem V.2.4. In the proof, we write $\|b^n - b\|$ for $\|b^n - b\|_{L^q_{[0,T]} W^{-\nu,p}(\mathbb{R}^d)}$, where p, q and ν satisfy either (V.2.1) or (V.2.2) as in the statement. Denote by $(X_t^{s,x,n})_{s,t,x}$ the Hölder continuous semiflow of solutions to

$$X_t = X_0 + \int_0^t b^n(r, X_r) dr + W_t, \quad t \in [0, T]. \tag{V.5.10}$$

In view of Theorem V.2.2, it suffices to show that there exists an event $\Omega'_{(b^n),b,T}$ such that for every $\omega \in \Omega'_{(b^n),b,T}$,

$$\lim_{n \rightarrow \infty} X^n(\omega) = X(\omega) \text{ uniformly over compact sets of } \Delta_{[0,T]} \times \mathbb{R}^d. \tag{V.5.11}$$

From (V.5.4) and [52, Theorem 1.2, Remark 2.13], putting $V_{s,t,x} = X_t^{s,x,n} - X_t^{s,x}$, we can find an $\alpha \in [\frac{1}{2}, 1)$ such that for every $m \geq 1$, $t, t' \in [0, T]$ and $x, x' \in \mathbb{R}^d$

$$\|V_{s,t,x} - V_{s',t',x'}\|_{L^m} \lesssim |s - s'|^{1/2} + |t - t'|^\alpha + |x - x'|,$$

and

$$\|V_{s,t,x}\|_{L^m} \lesssim \|b^n - b\|.$$

Hence,

$$\|V_{s,t,x} - V_{s',t',x'}\|_{L^m} \lesssim (|s - s'|^{1/2} + |t - t'|^\alpha + |x - x'|)^{1/2} \|b^n - b\|^{1/2}.$$

Applying Garsia–Rodemich–Rumsey inequality, for every compact set $K \subset \mathbb{R}^d$, every $m \geq 1$, we have

$$\left\| \sup_{(s,t,x) \in \Delta_{[0,T]} \times K} |V_{s,t,x}| \right\|_{L^m} \leq C_{m,T,K} \|b^n - b\|^{1/2} \quad (\text{V.5.12})$$

for some finite constant $C_{m,T,K}$.

Let (K^i) be an increasing sequence of compacts such that $\bigcup_{i \in \mathbb{N}} K^i = \mathbb{R}^d$. By definition of summable convergence, we can choose $m \geq 2$ so that

$$\sum_n \|b^n - b\|_{\frac{m}{4}} < \infty.$$

Using (V.5.12) and Markov's inequality, we have for every i

$$\mathbb{P} \left(\sup_{(s,t,y) \in \Delta_{[0,T]} \times K^i} |X_t^{s,y,n} - X_t^{s,y}| > \|b^n - b\|_{\frac{1}{4}} \right) \leq C_{m,T,K^i}^m \|b^n - b\|_{\frac{m}{4}}^m.$$

Applying Borel–Cantelli lemma, we get an event $\Omega_{(b^n),b,T}^i$ of full measure on which $X^n(\omega)$ converges to $X(\omega)$ uniformly on $\Delta_{[0,T]} \times K^i$. Putting $\Omega'_{(b^n),b,T} = \bigcap_i \Omega_{(b^n),b,T}^i$, it follows that (V.5.11) holds for every $\omega \in \Omega'_{(b^n),b,T}$. \square

Remark V.5.3. The fact that for almost every ω , for every s, x , $(X_t^{s,x}(\omega))_t$ is a solution to (V.1.5) is crucial for the proof of path-by-path uniqueness. When b is not continuous, this property becomes highly non-trivial. It was taken for granted without any justifications in [101, 102]. We achieved this property in Proposition V.2.1 by utilizing the regularizing estimates from Lemma V.4.3. This issue is irrelevant to [34] because of its different arguments. We take this chance to note that the exceptional null events in [34] depend on the initial condition. This dependence can be removed following our arguments herein. In fact, one just replaces Proposition V.3.1 by Davie's basic estimate, then the rest of the arguments follows with minimal adjustments. In particular, Proposition V.2.1 and Theorems V.2.2 and V.2.4 hold with $p = q = \infty$.

V.A ODE uniqueness

Most of the arguments in Section V.5 are independent from the probability space, and hence, independent from the probability law of the driving noise. This forms a uniqueness criterion for ODE (V.1.1) which does not require any regularity on the vector field, but instead relies on the regularizing effect of the driving signal γ with Lemma V.4.3 being a prototype. This regularizing effect is formalized by the following definition.

Definition V.A.1. Let $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a measurable function. We say that γ is $(1 - \varepsilon, \alpha)$ -regularizing for f if there exist a control η and a constant $\Xi_{T,K}$ for each compact set $K \subset \mathbb{R}^d$ such that

$$\left| \int_s^t [f(r, \gamma_r + \psi_r) - f(r, \gamma_r + \psi_s)] dr \right| \leq \Xi_{T,K} [\psi]_{C_{[s,t]}^{1-\varepsilon, \text{var}}} \eta(s, t)^\alpha$$

for every $(s, t) \in \Delta_{[0,T]}$ and every $\psi : [0, T] \rightarrow K$ of finite variation.

Theorem V.A.2. Let $K \subset \mathbb{R}^d$ be a compact set which contains 0. Suppose that γ is $(1 - \varepsilon, \alpha)$ -regularizing for $b, |b|$. Let y be a solution to (V.1.1). Suppose that $\phi_t^{s,y_s} - \gamma_t$ belongs to K for every $(s, t) \in \Delta_{[0,T]}$. Then, there exists a control \varkappa which depends only on $y, \Xi_{T,K}, \eta$ such that

$$|y_t - \phi_t^{s,y_s}| \leq \varkappa(s, t)^{\min(1-\varepsilon+\alpha, \frac{\alpha}{\varepsilon})} \quad \forall (s, t) \in \Delta_{[0,T]}. \tag{V.A.1}$$

Suppose furthermore that ϕ is locally κ -Hölder continuous for some $\kappa \in (0, 1]$ such that

$$\kappa \cdot \min(1 - \varepsilon + \alpha, \frac{\alpha}{\varepsilon}) > 1.$$

Then $(y_t)_{t \in [0,T]}$ is identical to $(\phi_t^{0,y_0})_{t \in [0,T]}$.

The proof of this result follows analogous arguments used in proving Proposition V.5.2 and Theorem V.2.2 and hence is omitted.

V.B Adapted solutions and path-by-path solutions

This section is devoted to proving Theorem V.B.1 (taken from [LA3]) that puts the intuition that path-by-path uniqueness is identifying all solutions that might possibly not be adapted into a rigorous result. In particular, if it is known *a priori* that all solutions are adapted, then pathwise uniqueness and path-by-path uniqueness are equivalent. To recall the different notions of existence and uniqueness, see Definition VI.1.1 and VI.1.1.

Theorem V.B.1. Consider

$$dX_t = b(t, X_t)dt + dW_t \tag{V.B.1}$$

for $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be measurable. Assume that

- (i) pathwise uniqueness to (V.B.1) holds,
- (ii) all path-by-path solutions to (V.B.1) are weak solutions.

Then path-by-path uniqueness holds.

Proof. Assume that path-by-path uniqueness does not hold. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a Brownian motion W is defined. Then there exists a set $A \subset \Omega$ of positive measure such that for $\omega \in A$, there exist multiple solutions to

$$X_t(\omega) = X_0 + \int_0^t b(s, X_s(\omega))ds + W_t(\omega),$$

with the same initial condition. Then we can define two path-by-path solutions X^1 and X^2 to (V.B.1) by letting them agree with the unique strong solution on A^c and letting $X^1(\omega) \neq X^2(\omega)$ on A , which is possible by the above and the axiom of choice. By assumption, both X^1 and X^2 are adapted w.r.t. filtrations (\mathcal{F}_t^1) and (\mathcal{F}_t^2) such that W is a Brownian motion w.r.t. these filtrations. Hence, W is also a Brownian motion with respect to $(\mathcal{F}_t^1 \cup \mathcal{F}_t^2)$. Therefore we constructed two solutions on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^1 \cup \mathcal{F}_t^2)_{t \in [0, T]}, \mathbb{P})$. By construction $X^1 \neq X^2$ on a set of positive measure. This contradicts pathwise uniqueness. \square

Pathwise and path-by-path uniqueness do not always agree VI

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In the current chapter we take a different viewpoint on the equation considered in Chapter V. Instead of showing path-by-path uniqueness for a certain class of singular drifts, we identify drifts that are even more singular such that uniqueness does no longer hold, however uniqueness among adapted solutions does. The material is based on [LA1].

VI.1 Introduction

We consider Stochastic Differential Equations (SDEs) of the type

$$X_t = X_0 + \int_0^t b(s, X_s) ds + W_t, \quad t \in [0, T], \tag{VI.1.1}$$

where $T > 0$ is a fixed deterministic time, $b: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable and W is a d -dimensional Brownian motion. As mentioned in previous sections, one can think of different kinds of solutions:

- (a) Weak and strong solutions to the SDE (VI.1.1).
- (b) Considering Equation (VI.1.1) as an ODE for each realization $(W_t(\omega))_{0 \leq t \leq T}$ of the Brownian path ("path-by-path" solutions).

The second of the above is possible as we consider an SDE with constant diffusion coefficient. Therefore we can evaluate the noise term therein for each Brownian path, which would generally not be possible with a more general diffusion term as this would lead to a stochastic integral.

In both [42, page 12] and [1, page 3] it was phrased as an open problem whether every solution of type (b) has to arise from an adapted solution. In a higher-dimensional setting this question was

answered via a counterexample in [103], making heavy use of the dimension $d \geq 2$. Hence, it is a natural question whether similar counterexamples can be constructed in a one-dimensional setting. We give a positive answer to this question in Section VI.3. The construction relies on patching together Bessel processes, Bessel bridges and a well-known SDE without a weak solution. It can easily be extended to any dimension $d \geq 1$.

Important definitions. Below we recall different types of solutions to (VI.1.1). In all cases, we assume that the following condition is, almost surely, satisfied (and do not repeat it for different types of solutions):

$$\int_0^T |b(s, X_s(\omega))| ds < \infty.$$

This is in line with the literature, see for example [94, Chapter 9]. Note that in the definitions below the initial condition X_0 might not be deterministic. Instead it prescribes the initial law of any solution to (VI.1.1).

Furthermore, $T > 0$ is a fixed deterministic time, i.e. we consider solutions defined on the whole prescribed time interval.

Definition VI.1.1 (Existence). (i) If there exists a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ equipped with a Brownian motion W and an \mathbb{F} -adapted process $(X_t)_{t \in [0, T]}$ such that (X, W) fulfills (VI.1.1) almost surely for $t \in [0, T]$, we say that (X, W) is a weak solution to (VI.1.1) on $[0, T]$. If the choice of W is clear from the context, we write that X is a weak solution.

(ii) We call X a strong solution if X is a weak solution and X is adapted to the filtration generated by W .

(iii) Let $X_0 \in \mathbb{R}^d$ be deterministic. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a Brownian motion W is defined. We call a mapping $X: \Omega \rightarrow \mathcal{C}_{[0, T]}$ a path-by-path solution to (VI.1.1) on $[0, T]$ if there exists a set $\tilde{\Omega} \subset \Omega$ of full measure such that, for $\omega \in \tilde{\Omega}$, $(X(\omega), W(\omega))$ fulfills equation (VI.1.1).

Definition VI.1.2 (Uniqueness). (i) We say that pathwise uniqueness for (VI.1.1) holds if for any two weak solutions $(X, W), (\tilde{X}, W)$ defined on the same filtered probability space with the same Brownian motion W and the same initial condition, X and \tilde{X} are indistinguishable.

(ii) Let $X_0 \in \mathbb{R}^d$ be deterministic. We say that path-by-path uniqueness for (VI.1.1) holds if for any probability space on which a Brownian motion W is defined, there exists a set of full measure $\tilde{\Omega}$, such that, for $\omega \in \tilde{\Omega}$, there exists a unique solution on $[0, T]$ to

$$X_t(\omega) = X_0 + \int_0^t b(s, X_s(\omega)) ds + W_t(\omega).$$

(iii) Let X be a nonnegative weak solution to (VI.1.1) such that all other nonnegative weak solutions defined on the same probability space with the same Brownian motion and the same initial condition are indistinguishable from X . Then we call X the pathwise unique nonnegative weak solution. In a symmetric manner we define a pathwise unique nonpositive weak solution.

In particular the definitions are given such that solutions have to be defined on the whole time interval $[0, T]$. The following implications follow directly from the definitions:

$$\boxed{\text{strong existence}} \implies \boxed{\text{weak existence}} \implies \boxed{\text{path-by-path existence}}$$

$$\boxed{\text{path-by-path uniqueness}} \implies \boxed{\text{pathwise uniqueness}}$$

Remark VI.1.3. It is also possible to equivalently rephrase the definition of path-by-path uniqueness in the following way (see [41]): For $\gamma \in \mathcal{C}_{[0, T]}$ and $X_0 \in \mathbb{R}^d$ deterministic, let $S(\gamma, X_0)$ be the set of solutions $X \in \mathcal{C}_{[0, T]}$ to the deterministic equation

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \gamma_t, \quad t \in [0, T].$$

Then, path-by-path uniqueness to Equation (VI.1.1) holds if there exists a null set $\mathcal{N} \subset \mathcal{C}_{[0, T]}$ with respect to Wiener measure such that $|S(\gamma, X_0)| = 1$ for $\gamma \in \mathcal{C}_{[0, T]} \setminus \mathcal{N}$. Note that no measurability condition on S is required here.

Comparison with previous results. In [103], for dimension $d \geq 2$, drifts b are constructed such that

- there exist multiple path-by-path solutions, but there exists no weak solution;
- there exists a pathwise unique weak solution to (VI.1.1), but path-by-path uniqueness is lost.

The main contribution of this chapter is that in Section VI.3, we construct drifts b such that the two situations described above happen for Equation (VI.1.1) with $d \geq 1$.

From Chapter V, we know that path-by-path uniqueness to (VI.1.1) holds for $b \in \mathbb{L}_p^q([0, T])$ and $p \in [2, \infty], q \in (2, \infty]$ with

$$\frac{d}{p} + \frac{2}{q} < 1. \tag{VI.1.2}$$

As existence of a unique strong solution in this regime holds by [68], counterexamples like the ones above can only be constructed with $b \notin \mathbb{L}_p^q([0, T])$ for any p, q fulfilling the previous conditions.

VI.2 Preparatory results and Bessel bridges

Lemma VI.2.1. *If there exists a weak solution on $[0, T]$ to (VI.1.1) with initial condition X_0 fulfilling $\mathbb{P}(X_0 = 0) > 0$, then there also exists a weak solution on $[0, T]$ with initial condition fulfilling $\mathbb{P}(X_0 = 0) = 1$.*

Proof. Let (X, W) , defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, be a weak solution to (VI.1.1) with X_0 fulfilling $\mathbb{P}(X_0 = 0) > 0$. Let $A := \{X_0 = 0\}$. Define (\tilde{X}, \tilde{W}) and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ in the following way: $\tilde{\Omega} = A$; $\tilde{\mathcal{F}} = \{E \cap A : E \in \mathcal{F}\}$; $\tilde{\mathcal{F}}_t = \{E \cap A : E \in \mathcal{F}_t\}$; for $C \in \tilde{\mathcal{F}}$, $\tilde{\mathbb{P}}(C) = \frac{\mathbb{P}(C)}{\mathbb{P}(A)}$; for $\omega \in A$, $\tilde{X}(\omega) = X(\omega)$ and $\tilde{W}(\omega) = W(\omega)$. First, we check that \tilde{X} is adapted to $\tilde{\mathbb{F}}$. Let $U \subset \mathbb{R}$ be measurable. Then

$$\tilde{X}_t^{-1}(U) = X_t^{-1}(U) \cap A \in \tilde{\mathcal{F}}_t.$$

Next, we check that \tilde{W} is an $\tilde{\mathbb{F}}$ -Brownian motion. As X_0 is \mathcal{F}_0 -measurable and W is an \mathbb{F} -Brownian motion, we know that, for $0 \leq s \leq t$,

$$\tilde{\mathbb{P}}(\tilde{W}_t - \tilde{W}_s \in U) = \frac{\mathbb{P}(\{W_t - W_s \in U\} \cap A)}{\mathbb{P}(A)} = \mathbb{P}(W_t - W_s \in U)$$

and using this we get, for $V \in \tilde{\mathcal{F}}_s$,

$$\tilde{\mathbb{P}}\left(\{\tilde{W}_t - \tilde{W}_s \in U\} \cap V\right) = \frac{\mathbb{P}(\{W_t - W_s \in U\} \cap V)}{\mathbb{P}(A)} = \tilde{\mathbb{P}}(\tilde{W}_t - \tilde{W}_s \in U)\tilde{\mathbb{P}}(V).$$

Hence, \tilde{W} is an $\tilde{\mathbb{F}}$ -Brownian motion.

Clearly (\tilde{X}, \tilde{W}) fulfills equation (VI.1.1) a.s. \square

The following lemma and its proof are similar to [15, Example 7.4] and [29, Example 2.1].

Lemma VI.2.2. *Let $T > 0$. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be measurable. There does not exist a weak solution on $[0, T]$ to*

$$X_t = X_0 + \int_0^t b(X_s)ds + W_t, \quad (\text{VI.2.1})$$

with initial condition X_0 fulfilling $\mathbb{P}(X_0 \in (-1, 1)) > 0$ and

$$b(x) := -\mathbb{1}_{\{x \neq 0, |x| \leq 1\}} \frac{1}{2x} + \mathbb{1}_{\{|x| > 1\}} f(x).$$

Proof. Step 1: First we show, by contradiction, that there exists no weak solution to (VI.2.1) on $[0, T/2]$ with initial condition $X_0 = 0$. Assume there exists a weak solution (X, W) on $[0, T/2]$ with $X_0 = 0$. Let $\tau_1 := \inf\{s \geq 0 : |X_s| \geq 1\}$, so that, on $[0, \tau_1]$, f does not play a role. Then by Itô's Lemma, we get that, for $t \in [0, T/2]$,

$$\begin{aligned} X_{t \wedge \tau_1}^2 &= \int_0^{t \wedge \tau_1} -\mathbb{1}_{\{X_s \neq 0\}} ds + \int_0^{t \wedge \tau_1} 1 ds + \int_0^{t \wedge \tau_1} 2X_s dW_s \\ &= \int_0^{t \wedge \tau_1} \mathbb{1}_{\{X_s = 0\}} ds + \int_0^{t \wedge \tau_1} 2X_s dW_s. \end{aligned}$$

Define $X_t^{\tau_1} := X_{t \wedge \tau_1}$. Note that X^{τ_1} is a continuous semimartingale with quadratic variation $\langle X^{\tau_1} \rangle_t = t \wedge \tau_1$, so we can use [94, Chapter 6, Corollary (1.6)] to obtain

$$\int_0^{t \wedge \tau_1} \mathbb{1}_{\{X_s = 0\}} ds = \int_0^{t \wedge \tau_1} \mathbb{1}_{\{X_s = 0\}} d\langle X \rangle_s = \int_{\mathbb{R}} \mathbb{1}_{\{x=0\}} L_{t \wedge \tau_1}^x(X) dx = 0,$$

where $L(X)$ denotes the local time of X . Hence, $(X_t^{\tau_1})^2 = \int_0^{t \wedge \tau_1} 2X_s dW_s$ is a local martingale and as $|X^{\tau_1}| \leq 1$ it is also a martingale. Note that $X_0^{\tau_1} = 0$ and $(X^{\tau_1})^2 \geq 0$. This implies that X^{τ_1} must be identically 0, which contradicts $\langle X^{\tau_1} \rangle_t = t \wedge \tau_1$ as $\tau_1 > 0$ a.s.

Step 2:

Assume that there exists a weak solution (X, W) to (VI.2.1) on $[0, T]$ (defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$) with initial condition X_0 fulfilling $\mathbb{P}(X_0 \in (-1, 1)) > 0$. W.l.o.g. we assume that there exists $a \in [0, 1)$ such that $\mathbb{P}(X_0 \in [0, a]) > 0$.

Let $\tau_0 := \inf\{t \geq 0 : X_t \leq 0\}$ and

$$C := \{X_0 \in [0, a]\} \cap \left\{ \sup_{t \in [0, T/2]} W_t < 1 - a, W_{T/2} \leq -a \right\}.$$

Then $\{\tau_0 \leq T/2\} \supset C$. To see this let $\omega \in C$ and assume that $\inf_{t \in [0, T/2]} X_t(\omega) > 0$. Then by construction $\sup_{t \in [0, T/2]} X_t(\omega) \leq 1$, so again f does not play a role and therefore

$$X_{T/2}(\omega) = X_0(\omega) - \int_0^{T/2} \frac{1}{2X_s(\omega)} ds + W_{T/2}(\omega) \leq a + W_{T/2}(\omega) \leq 0,$$

which gives a contradiction. Hence $\mathbb{P}(\tau_0 \leq T/2) > 0$ as $C \subset \{\tau_0 \leq T/2\}$ and $\mathbb{P}(C) > 0$.

Let $\tilde{\tau}_0 := \tau_0 \wedge (T/2)$, $\tilde{W}_t := W_{\tilde{\tau}_0+t} - W_{\tilde{\tau}_0}$ and $\tilde{X}_t := X_{\tilde{\tau}_0+t}$. Then $(\tilde{X}_t, \tilde{W}_t)_{t \in [0, T/2]}$ is a weak solution to (VI.2.1) on $[0, T/2]$ with initial condition $X_{\tilde{\tau}_0}$ which fulfills $\mathbb{P}(X_{\tilde{\tau}_0} = 0) > 0$. This holds true because first, for $t \in [0, T/2]$,

$$\begin{aligned} \tilde{X}_t = X_{\tilde{\tau}_0+t} &= X_{\tilde{\tau}_0} + \int_{\tilde{\tau}_0}^{\tilde{\tau}_0+t} b(X_s) ds + W_{\tilde{\tau}_0+t} - W_{\tilde{\tau}_0} \\ &= X_{\tilde{\tau}_0} + \int_0^t b(\tilde{X}_s) ds + \tilde{W}_t, \end{aligned}$$

where we used that $\tilde{\tau}_0 + T/2 \leq T$. Second, \tilde{W} is an $\mathbb{F}^{\tilde{\tau}_0}$ -Brownian motion for $\mathcal{F}_t^{\tilde{\tau}_0} := \mathcal{F}_{\tilde{\tau}_0+t}$ by [93, page 23, Theorem 32] and \tilde{X} is clearly adapted to $\mathbb{F}^{\tilde{\tau}_0}$. Hence, we constructed a solution to (VI.2.1) on $[0, T/2]$ starting from 0 with positive probability. This gives a contradiction to Step 1 due to Lemma VI.2.1. □

In the following lemma we consider two SDEs and present some properties of solutions to these SDEs. Both have been investigated thoroughly in the literature (references are given in the proof of Lemma VI.2.3). The lemma is stated to give a clearer presentation of the counterexamples in Section VI.3.

Lemma VI.2.3. *Let $y > 0$. Consider the following equations on $[0, T]$:*

$$X_t = \int_0^t \mathbb{1}_{\{X_s > 0\}} \left(\frac{y - X_s}{T - s} + \frac{1}{X_s} \right) ds + \int_0^t \mathbb{1}_{\{X_s < 0\}} \left(\frac{-y - X_s}{T - s} + \frac{1}{X_s} \right) ds + W_t, \quad (\text{VI.2.2})$$

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \mathbb{1}_{\{\tilde{X}_s \neq 0\}} \frac{1}{\tilde{X}_s} ds + W_t, \quad \tilde{X}_0 \in \mathbb{R}. \quad (\text{VI.2.3})$$

(a) *Any weak solution to (VI.2.2) on $[0, T]$ satisfies $|X_T| = y$ a.s. and does not change its sign on the interval $(0, T]$ a.s. Moreover, there exists a nonnegative strong solution and a nonpositive strong solution that are unique in the class of nonnegative/nonpositive solutions. We will call these solutions nonnegative/nonpositive Bessel bridge.*

(b) *For $\tilde{X}_0 \geq 0$, there exists a pathwise unique nonnegative strong solution on $[0, T]$ to (VI.2.3). For $\tilde{X}_0 > 0$, pathwise uniqueness holds.*

Proof. (a): Weak existence follows by [88, page 274, Equation (29)]. Pathwise uniqueness and strong existence follow by the same arguments as in the proof of Proposition 1 in [103], which is stated for $y = 1$ and $T = 1$.

(b): Special case of Theorem 3.2 in [28]. See also [94, Chapter 11]. \square

Lemma VI.2.4. *Let X be the nonnegative Bessel bridge on $[0, T]$ with $X_0 = 0$ and $X_T = 1$. Then there exists $\varepsilon > 0$ small enough such that $\{\sup_{t \in [0, T]} X_t < 2\} \supset \{\sup_{s \in [0, T]} |W_s + s| < \varepsilon\}$ and therefore*

$$\mathbb{P}\left(\sup_{t \in [0, T]} X_t < 2\right) > 0.$$

Proof. Recall that X satisfies

$$X_t = \int_0^t \mathbb{1}_{\{X_s > 0\}} \left(\frac{1 - X_s}{T - s} + \frac{1}{X_s} \right) ds + W_t.$$

Let $\varepsilon \in (0, 1/2)$ and consider the set $A := \{\sup_{s \in [0, T]} |W_s + s| < \varepsilon\}$ fulfilling $\mathbb{P}(A) > 0$. Let $\omega \in A$. Assume that there exists $\tau_2 \in [0, T]$ such that $X_{\tau_2}(\omega) = 2$. Let $\rho_1 := \sup\{s \in [0, \tau_2] : X_s(\omega) = 1\}$. By continuity $X_{\rho_1}(\omega) = 1$. Then, for ω chosen above,

$$\begin{aligned} X_{\tau_2} &= 1 + \int_{\rho_1}^{\tau_2} \left(\frac{1 - X_s}{T - s} + \frac{1}{X_s} \right) ds + W_{\tau_2} - W_{\rho_1} \\ &\leq 1 + \int_{\rho_1}^{\tau_2} \frac{1}{X_s} ds + 2\varepsilon - (\tau_2 - \rho_1) \\ &\leq 1 + 2\varepsilon < 2, \end{aligned}$$

which gives a contradiction. Hence, $\sup_{t \in [0, T]} X_t(\omega) < 2$ for $\omega \in A$. \square

Remark VI.2.5. Let $X = 2 - Y$ where Y is the Bessel bridge with $Y_0 = 0$ and $Y_T = 1$, i.e. (Y, W) is the unique nonnegative weak solution to (VI.2.2) on $[0, T]$. In particular, for $\tilde{W} = -W$, (X, \tilde{W}) is a weak solution on $[0, T]$ to

$$X_t = 2 + \int_0^t \mathbb{1}_{\{X_s < 2\}} \left(\frac{1 - X_s}{T - s} - \frac{1}{2 - X_s} \right) ds + \tilde{W}_t$$

with $X_0 = 2$ and $X_T = 1$. Lemma VI.2.4 gives that

$$\mathbb{P}\left(\inf_{t \in [0, T]} X_t > 0\right) > 0.$$

VI.3 Construction of Counterexamples

VI.3.1 Path-by-path existence but no weak existence

In the following proposition we construct an SDE without weak solutions, but with multiple path-by-path solutions. The construction is done in a one-dimensional setting, but can easily be extended to multiple dimensions.

Proposition VI.3.1. *Let $T > 0$. Consider, on $[0, 3T]$, the SDE*

$$X_t = \int_0^t b(s, X_s) ds + W_t, \quad (\text{VI.3.1})$$

where

$$b(t, x) = \begin{cases} \mathbb{1}_{\{x>0\}} \left(\frac{1-x}{T-t} + \frac{1}{x} \right) + \mathbb{1}_{\{x<0\}} \left(\frac{-1-x}{T-t} + \frac{1}{x} \right) & \text{if } 0 \leq t < T, \\ 0 & \text{if } T \leq t < 2T, \\ \mathbb{1}_{\{x \neq 0, |x| \leq 1\}} \frac{-1}{2x} + \mathbb{1}_{\{x>1\}} \frac{1}{x-1} + \mathbb{1}_{\{x<-1\}} \frac{1}{x+1} & \text{if } 2T \leq t \leq 3T. \end{cases}$$

There does not exist a weak solution to (VI.3.1) on $[0, 3T]$, but there exist path-by-path solutions.

Idea of the proof.

The drift is constructed in a way such that for a solution X the following must hold: X_T is forced to be equal to 1 or -1 . On the time interval $[T, 2T]$ we let a Brownian motion evolve freely without any drift. On the time interval $[2T, 3T]$ the drift is constructed in a way such that there exists no adapted solution if $X_{2T} \in (-1, 1)$. However if $|X_{2T}| > 1$, then X can be extended to the interval $[0, 3T]$ with $|X_t| > 1$ for all $t \in [2T, 3T]$ while still being adapted. Hence, any adapted solution must avoid taking a value in $(-1, 1)$ at time $2T$. If $W_{2T} - W_T = X_{2T} - X_T \in (0, 2)$, this can only be achieved if one can choose $X_T = 1$. Similarly, if $W_{2T} - W_T = X_{2T} - X_T \in (-2, 0)$, one must be allowed to choose $X_T = -1$. This necessity of “looking into the future” prohibits the existence of weak solutions, but not the construction of path-by-path solutions.

Proof. No weak solution.

Assume there exists a weak solution X defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. By Lemma VI.2.3(a), $\mathbb{P}(|X_T| = 1) = 1$. Assume that $\mathbb{P}(X_T = 1) > 0$. Let $A_1 := \{X_T = 1\}$ and $A_2 := \{W_{2T} - W_T \in (-2, 0)\}$. Then we have $X_{2T}(\omega) \in (-1, 1)$ for $\omega \in A_1 \cap A_2$. As A_1 is \mathcal{F}_T -measurable and $W_{2T} - W_T$ is independent of \mathcal{F}_T , $\mathbb{P}(A_1 \cap A_2) > 0$. Hence, after a time shift, we would have a weak solution on $[0, T]$ to the SDE

$$X_t = X_0 + \int_0^t \left(-\mathbb{1}_{\{X_s \neq 0, |X_s| \leq 1\}} \frac{1}{2X_s} + \mathbb{1}_{\{X_s > 1\}} \frac{1}{X_s - 1} + \mathbb{1}_{\{X_s < -1\}} \frac{1}{X_s + 1} \right) ds + W_t,$$

with $\mathbb{P}(X_0 \in (-1, 1)) > 0$, which contradicts Lemma VI.2.2. Hence, $X_T = -1$ a.s. By the same arguments as above we must have $X_T(\omega) = 1$ for $\omega \in \tilde{A}_2$, where $\tilde{A}_2 := \{W_{2T} - W_T \in (0, 2)\}$, which gives a contradiction to $X_T = -1$ a.s. as $\mathbb{P}(\tilde{A}_2) > 0$.

Existence of a path-by-path solution.

Let $X^+, X^- : \Omega \rightarrow \mathcal{C}_{[0, T]}$ be the nonnegative and nonpositive Bessel bridge with terminal value 1, respectively -1 . Let $C_1 := \{W_{2T} - W_T > 0\}$ and $C_2 := \{W_{2T} - W_T < 0\}$. Define $X : \Omega \rightarrow \mathcal{C}_{[0, 2T]}$ such that, for $t \in [0, 2T]$,

$$X_t(\omega) := \begin{cases} X_{t \wedge T}^+(\omega) + W_{t \vee T}(\omega) - W_T(\omega) & \text{if } \omega \in C_1, \\ X_{t \wedge T}^-(\omega) + W_{t \vee T}(\omega) - W_T(\omega) & \text{if } \omega \in C_2. \end{cases}$$

Hence $|X_{2T}(\omega)| > 1$ for $\omega \in C_1 \cup C_2$. By Lemma VI.2.3(b) (after a shift of the space variable), we can uniquely extend X to $[0, 3T]$ fulfilling (VI.3.1) so that $|X_t| > 1$ for all $t \in [2T, 3T]$. As $C_1 \cup C_2$ has full measure, X is indeed a path-by-path solution.

Existence of other path-by-path solutions.

Other path-by-path solutions can be constructed. Indeed on the set $C_3 := \{W_{2T} - W_T > 2\}$ we can choose freely if X coincides with the nonnegative or nonpositive Bessel bridge on $[0, T]$ since, for $\omega \in C_3$, $X_{2T}(\omega) > 1$. Then, we can proceed the same way as before. \square

VI.3.2 Pathwise unique weak solution, no path-by-path uniqueness

The following proposition gives an example of a one-dimensional SDE with a pathwise unique weak solution, but for which path-by-path uniqueness does not hold. Again, the construction can easily be extended to multiple dimensions.

Proposition VI.3.2. *Let $T > 0$. Consider, on $[0, 4T]$, the SDE*

$$X_t = \int_0^t b(s, X_s) ds + W_t, \quad (\text{VI.3.2})$$

where

$$b(t, x) = \begin{cases} \mathbb{1}_{\{x>0\}} \left(\frac{2-x}{T-t} + \frac{1}{x} \right) + \mathbb{1}_{\{x<0\}} \left(\frac{-2-x}{T-t} + \frac{1}{x} \right) & \text{if } 0 \leq t < T, \\ \mathbb{1}_{\{x<0\}} \frac{1}{x} + \mathbb{1}_{\{x>2\}} \left(\frac{3-x}{2T-t} + \frac{1}{x-2} \right) + \mathbb{1}_{\{0<x<2\}} \left(\frac{1-x}{2T-t} + \frac{1}{x-2} \right) & \text{if } T \leq t < 2T, \\ \mathbb{1}_{\{x<0\}} \frac{1}{x} & \text{if } 2T \leq t < 3T, \\ \mathbb{1}_{\{x<0\}} \frac{1}{x} - \mathbb{1}_{\{x \neq 2, |x-2| \leq 1\}} \frac{1}{2(x-2)} + \mathbb{1}_{\{x>3\}} \frac{1}{x-3} + \mathbb{1}_{\{0<x<1\}} \frac{1}{x-1} & \text{if } 3T \leq t \leq 4T. \end{cases}$$

There exists a pathwise unique weak solution to (VI.3.2) on $[0, 4T]$, but path-by-path uniqueness does not hold.

Idea of the proof. On the time interval $[0, T]$ the drift is the one of an SDE solved by a Bessel bridge. Hence, for $t \in [0, T]$ there exists a pathwise unique nonnegative weak solution and a pathwise unique nonpositive weak solution. On $[T, 4T] \times \mathbb{R}_-$ the drift is constructed such that the nonpositive Bessel bridge on $[0, T]$ can be extended in a unique way to a weak solution on the time interval $[0, 4T]$. On $[T, 4T] \times \mathbb{R}_+$ the drift is constructed so that we can extend the nonnegative Bessel bridge on the time interval $[0, T]$ to path-by-path solutions on $[0, 4T]$, but not to a weak solution. Hereby solutions are allowed to enter the negative half plane (which is no problem as the drift there ensures the existence of nonpositive solution), but with positive probability a situation as in Proposition VI.3.1 occurs; i.e. weak solutions X are forced to fulfill $X_{3T} \in (1, 3)$ with positive probability and on the time interval $[3T, 4T]$ the drift is constructed such that these solutions cannot be extended to $[0, 4T]$.

Proof. Existence of a pathwise unique nonpositive weak solution.

Note that there exists a pathwise unique nonpositive weak solution \tilde{X} on $[0, T]$ to (VI.3.2) with $\tilde{X}_T = -2$ by Lemma VI.2.3(a). We can extend this solution in a pathwise unique way to $[0, 4T]$ by Lemma VI.2.3(b).

No other weak solution.

Assume that there exists another weak solution X to (VI.3.2). Then we must have $\mathbb{P}(X_T = 2) > 0$. Let

$$\begin{aligned} A_1 &:= \{X_T = 2\}, \quad A_2 := \left\{ \inf_{t \in [T, 2T]} X_t > 0, X_{2T} = 1 \right\}, \\ A_3 &:= \left\{ \inf_{t \in [2T, 3T]} (W_t - W_{2T}) > -1, W_{3T} - W_{2T} \in (0, 2) \right\}. \end{aligned}$$

First assume that $\mathbb{P}(X_{2T} = 1 \mid X_T = 2) = 1$. Then by assumption and by Lemma VI.2.4 and Remark VI.2.5, for $\varepsilon > 0$ small enough,

$$\begin{aligned} \mathbb{P}(A_1 \cap A_2) &\geq \mathbb{P}(A_1 \cap \left\{ \sup_{t \in [T, 2T]} |W_t - W_T - (t - T)| < \varepsilon \right\}) \\ &= \mathbb{P}(A_1) \mathbb{P}\left(\sup_{t \in [T, 2T]} |W_t - W_T - (t - T)| < \varepsilon \right) > 0. \end{aligned}$$

Hence $\mathbb{P}(\bigcap_{i=1}^3 A_i) > 0$ as A_3 is independent of $A_1 \cap A_2$. Note that $X_{3T}(\omega) \in (1, 3)$ for $\omega \in \bigcap_{i=1}^3 A_i$. This gives a contradiction to Lemma VI.2.2 as after a space shift we would have a weak solution to Equation (VI.2.1) with initial condition X_0 fulfilling $\mathbb{P}(X_0 \in (-1, 1)) > 0$.

Assume now that $\mathbb{P}(X_{2T} = 1 \mid X_T = 2) < 1$ and therefore $\mathbb{P}(X_{2T} = 3 \mid X_T = 2) > 0$. Let $\tilde{A}_1 := A_1$, $\tilde{A}_2 := \{X_{2T} = 3\}$ and

$$\tilde{A}_3 := \left\{ \inf_{t \in [2T, 3T]} (W_t - W_{2T}) > -3, W_{3T} - W_{2T} \in (-2, 0) \right\}.$$

Then by assumption $\mathbb{P}(\bigcap_{i=1}^3 \tilde{A}_i) > 0$ and $X_{3T}(\omega) \in (1, 3)$ for $\omega \in \bigcap_{i=1}^3 \tilde{A}_i$ and again this leads to a contradiction to Lemma VI.2.2.

Construction of another path-by-path solution.

By separately considering the sets $C_1 := \{W_{3T} - W_{2T} > 0\}$ and $C_2 := \{W_{3T} - W_{2T} < 0\}$ we construct an additional path-by-path solution on the set $C_1 \cup C_2$ of full measure. Let $X: C_1 \rightarrow \mathcal{C}_{[0, 2T]}$ coincide on $[0, T]$ with the Bessel bridge such that $X_T = 2$ and on $[T, 2T]$ with the spatially shifted Bessel bridge (similar to Remark VI.2.5) such that $X_{2T} = 3$. Let $\tau := \inf\{t \geq 2T : W_t - W_{2T} = -3\}$. Then, by Lemma VI.2.3(b), for $\omega \in C_1 \cap \{\tau > 3T\}$ we can extend X to the time interval $[0, 4T]$ fulfilling equation (VI.3.2). Now consider $\omega \in C_1 \cap \{\tau \leq 3T\}$. We can clearly extend X to the time interval $[0, \tau]$ by adding the Brownian increment $W_t - W_{2T}$. Then we have that $X_\tau(\omega) = 0$. Hence, for $\omega \in C_1 \cap \{\tau \leq 3T\}$ and $t \in [\tau, 4T]$, we can choose X to be the pathwise unique nonpositive solution on $[\tau, 4T]$ to

$$X_t = X_\tau + \int_\tau^t \frac{1}{X_s} ds + W_t - W_\tau,$$

and therefore we can construct $X: C_1 \rightarrow \mathcal{C}_{[0, 4T]}$ fulfilling equation (VI.3.2).

Let

$$\tilde{b}(t, x) := b(t, x) - \mathbb{1}_{\{t \geq T, x < 0\}} \frac{1}{x} + \mathbb{1}_{\{t \geq 3T, x \leq 0\}} \frac{1}{x - 1}.$$

For $\omega \in C_2$ and $t \in [0, 4T]$, let \tilde{X} fulfill

$$\tilde{X}_t(\omega) = \int_0^t \tilde{b}(s, \tilde{X}_s(\omega)) ds + W_t(\omega)$$

such that \tilde{X} coincides with the Bessel bridge on $[0, T]$ and the spatially shifted Bessel bridge (see Remark VI.2.5) on $[T, 2T]$ so that $\tilde{X}_T = 2$ and $\tilde{X}_{2T} = 1$. This is possible by the same arguments as in the proof of Proposition VI.3.1. Let $\tau_0 := \inf\{t > T : \tilde{X}_t = 0\} \wedge 4T$. On $[0, \tau_0]$ let $\hat{X} := \tilde{X}$. Then \hat{X} is a solution to (VI.3.2) on $[0, \tau_0]$ as, for $x > 0$, $b(t, x) = \tilde{b}(t, x)$. On $[\tau_0, 4T]$, we choose \hat{X} to be the pathwise unique nonpositive solution on $[\tau_0, 4T]$ to

$$X_t = X_{\tau_0} + \int_{\tau_0}^t \frac{1}{X_s} ds + W_t - W_{\tau_0}.$$

As $\mathbb{P}(C_1 \cup C_2) = 1$, we can construct a path-by-path solution $Y: \Omega \rightarrow \mathcal{C}_{[0, 4T]}$ by setting

$$Y(\omega) = \begin{cases} X(\omega) & \text{if } \omega \in C_1, \\ \hat{X}(\omega) & \text{if } \omega \in C_2. \end{cases}$$

□

A.1 Elementary results on Besov spaces

Definition A.1.1 (Partition of unity). Let $\chi, \rho \in C^\infty(\mathbb{R}^d, \mathbb{R})$ be radial functions and for $j \geq 0$, $\rho_j(x) = \rho(2^{-j}x)$. We assume that χ is supported on a ball around 0 and ρ is supported on an annulus. Moreover, we have

$$\chi + \sum_{j \geq 0} \rho_j \equiv 1, \tag{A.1.1}$$

$$\text{supp}(\chi) \cap \text{supp}(\rho_j) = \emptyset, \quad \forall j \geq 1, \tag{A.1.2}$$

$$\text{supp}(\rho_j) \cap \text{supp}(\rho_i) = \emptyset, \quad \text{if } |i - j| \geq 2. \tag{A.1.3}$$

Then we call the pair (χ, ρ) a partition of unity.

Existence of a partition of unity is proven in [11, Prop. 2.10]. Let such a partition be fixed.

Definition A.1.2 (Littlewood-Paley blocks). Let f be an \mathbb{R}^d -valued tempered distribution. We define its j -th Littlewood-Paley block by

$$\Delta_j f = \begin{cases} \mathcal{F}^{-1}(\rho_j \mathcal{F}(f)) & \text{for } j \geq 0, \\ \mathcal{F}^{-1}(\chi \mathcal{F}(f)) & \text{for } j = -1, \\ 0 & \text{for } j \leq -2, \end{cases}$$

where \mathcal{F} and \mathcal{F}^{-1} denote the Fourier transform and its inverse.

Definition A.1.3. For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, let the nonhomogeneous Besov space $\mathcal{B}_{p,q}^s$ be the space of \mathbb{R}^d -valued tempered distributions u such that

$$\|u\|_{\mathcal{B}_{p,q}^s} := \left(\sum_{j \in \mathbb{Z}} \left(2^{js} \|\Delta_j u\|_{L^p(\mathbb{R}^d)} \right)^q \right)^{\frac{1}{q}} < \infty.$$

Remark A.1.4. Note that any finite measure lies in \mathcal{B}_1^0 by similar computations as in [11, Proposition 2.39]. Therefore, after an embedding of Besov spaces (see Remark I.2.2), it lies in \mathcal{B}_∞^{-d} as well.

Remark A.1.5. For $s \in \mathbb{R}$, $p, q \in [1, \infty]$ and any $\varepsilon > 0$,

$$\mathcal{B}_p^{s+\varepsilon} \hookrightarrow \mathcal{B}_{p,q}^s \hookrightarrow \mathcal{B}_p^s.$$

For a proof see [109, page 47, Proposition 2].

The following lemma comes directly from the definition of Besov spaces and Bernstein's inequality (see [11, Lemma 2.1]).

Lemma A.1.6. *Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]$. Let D denote the distributional derivative operator. Then, for $f \in \mathcal{B}_{p,q}^s$,*

$$\|Df\|_{\mathcal{B}_{p,q}^{s-1}} \leq \|f\|_{\mathcal{B}_{p,q}^s}.$$

Lemma A.1.7 and A.1.8, both taken from [8] (respectively [59] for $d > 1$), are used on a regular basis. Hence, we state them again below.

Lemma A.1.7. *Let $\gamma \in \mathbb{R}$ and $p \in [1, \infty]$ and the Gaussian semigroup G_t defined by (I.2.10). Then there exists $C > 0$ such that for any $f \in \mathcal{B}_p^\gamma$,*

- (a) $\|G_t f\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{\mathcal{B}_p^\gamma} t^{\gamma/2}$ for any $t > 0$, provided that $\gamma < 0$;
- (b) $\lim_{t \rightarrow 0} G_t f = f$ in $\mathcal{B}_p^{\tilde{\gamma}}$ for every $\tilde{\gamma} < \gamma$;
- (c) $\sup_{t > 0} \|G_t f\|_{\mathcal{B}_p^\gamma} \leq \|f\|_{\mathcal{B}_p^\gamma}$;
- (d) $\|G_t f\|_{\mathcal{C}^1} \leq C \|f\|_{\mathcal{B}_p^\gamma} t^{(\gamma-d/p-1)/2}$ for all $t > 0$, provided that $\gamma - d/p < 0$.

Lemma A.1.8. *Let f be an \mathbb{R}^d -valued tempered distribution, $\gamma \in \mathbb{R}$, $p \in [1, \infty]$. Then for any $a, a_1, a_2, a_3 \in \mathbb{R}$, $\alpha, \alpha_1, \alpha_2 \in [0, 1]$, there exists a constant $C > 0$ such that*

- (a) $\|f(a + \cdot)\|_{\mathcal{B}_p^\gamma} = \|f\|_{\mathcal{B}_p^\gamma}$;
- (b) $\|f(a_1 + \cdot) - f(a_2 + \cdot)\|_{\mathcal{B}_p^\gamma} \leq C |a_1 - a_2|^\alpha \|f\|_{\mathcal{B}_p^{\gamma+\alpha}}$;
- (c) $\|f(a_1 + \cdot) - f(a_2 + \cdot) - f(a_3 + \cdot) + f(a_3 + a_2 - a_1 + \cdot)\|_{\mathcal{B}_p^\gamma} \leq C |a_1 - a_2|^{\alpha_1} |a_1 - a_3|^{\alpha_2} \|f\|_{\mathcal{B}_p^{\gamma+\alpha_1+\alpha_2}}$.

Lemma A.1.9. *Let $\gamma \in \mathbb{R}$, $p \in [1, \infty]$ and $f \in \mathcal{B}_p^\gamma$ and $t > 0$, the function $G_t f$ defined by (I.2.10) is smooth and bounded.*

Proof. Smoothness comes from [104, Th 3.13]. We now prove boundedness: First, we consider the case $f \in \mathcal{B}_p^\gamma$ for $\gamma \in \mathbb{R}$ and $p \in [1, \infty]$ such that $\gamma - d/p < 0$. Using Lemma A.1.7(a) and the embedding $\mathcal{B}_p^\gamma \hookrightarrow \mathcal{B}_\infty^{\gamma-d/p}$, we have

$$\|G_t f\|_{L^\infty} \leq C_\gamma \|f\|_{\mathcal{B}_\infty^{\gamma-d/p}} t^{\frac{\gamma}{2} - \frac{d}{2p}} \leq C_\gamma \|f\|_{\mathcal{B}_p^\gamma} t^{\frac{\gamma}{2} - \frac{d}{2p}}.$$

So, $G_t f$ is bounded. Now for arbitrary $\gamma \in \mathbb{R}$ and $p \in [1, \infty]$ such that $f \in \mathcal{B}_p^\gamma$, one can choose $\gamma' < \gamma$ such that $\gamma' - d/p < 0$. We then have $f \in \mathcal{B}_p^{\gamma'}$ since $\mathcal{B}_p^\gamma \subset \mathcal{B}_p^{\gamma'}$. By the first part of the proof $G_t f$ is bounded. \square

The following lemma provides an embedding between Hölder and Besov spaces for functions with compact support. It is useful when we deal with local times.

Lemma A.1.10. *Let $M \subset \mathbb{R}^d$ be a compact set, $0 < \beta < \hat{\beta} < 1$ and $p, q \in [1, \infty]$. There exists constants $C, \tilde{C} > 0$ only depending on M such that for any $f \in \mathcal{C}^{\hat{\beta}}$ with support included in M , we have*

$$\|f\|_{\mathcal{B}_{p,q}^{\beta}} \leq C \|f\|_{\mathcal{B}_{\infty}^{\hat{\beta}}} \leq \tilde{C} \|f\|_{\mathcal{C}^{\hat{\beta}}}.$$

The first inequality follows by Remark I.2.3 and the second inequality by the Proposition on page 14 in [99].

A.2 Variations of the stochastic sewing lemma

In this section we recall recent extensions of the stochastic sewing Lemma (see [70, Th. 4.1 and Th. 4.7]) and [49, Lemma 2.5]. In all statements, let $0 < S < T$, let $m \in [2, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space and let $A : \Delta_{[S,T]} \rightarrow L^m$ such that $A_{s,t}$ is \mathcal{F}_t -measurable for $(s, t) \in \Delta_{[S,T]}$.

Lemma A.2.1 (Stochastic sewing Lemma). *Let $n \in [m, \infty]$. Assume that there exist constants $\Gamma_1, \Gamma_2 \geq 0$, $\varepsilon_1, \varepsilon_2 > 0$, $\alpha_2, \beta_2 \in [0, 1/2)$ with $\alpha_2 + \beta_2 < 1/2$ such that for every $(s, t) \in \Delta_{[S,T]}$ and $u := (s + t)/2$,*

$$\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^n} \leq \Gamma_1(t-s)^{1+\varepsilon_1}, \tag{A.2.1}$$

$$\|\mathbb{E}^S[|\delta A_{s,u,t}|^m]^{1/m}\|_{L^n} \leq \Gamma_2(u-S)^{-\alpha_2}(T-u)^{-\beta_2}(t-s)^{1/2+\varepsilon_2}. \tag{A.2.2}$$

Suppose there exists a process $(\mathcal{A}_t)_{t \in [S,T]}$ such that, for any $t \in [S, T]$ and any sequence of partitions $\Pi_k = \{t_i^k\}_{i=0}^{N_k}$ of $[S, t]$ with mesh size going to zero, we have

$$\mathcal{A}_t = \lim_{k \rightarrow \infty} \sum_{i=0}^{N_k} A_{t_i^k, t_{i+1}^k} \text{ in probability.} \tag{A.2.3}$$

Moreover, there exists a constant $C = C(\varepsilon_1, \varepsilon_2, \alpha_2, \beta_2, m)$ independent of S, T such that for every $(s, t) \in \Delta_{[S,T]}$ we have

$$\|\mathbb{E}^S[|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}|^m]^{1/m}\|_{L^n} \leq C\Gamma_2(t-s)^{1/2-\alpha_2-\beta_2+\varepsilon_2} + C\Gamma_1(t-s)^{1+\varepsilon_1}$$

and

$$\|\mathbb{E}^S[\mathcal{A}_t - \mathcal{A}_s - A_{s,t}]\|_{L^n} \leq C\Gamma_1(t-s)^{1+\varepsilon_1}.$$

Lemma A.2.2 (Stochastic sewing lemma with random controls). *Let λ be a random control. Assume that there exist constants $\Gamma_1, \alpha_1, \beta_1 \geq 0$ with $\alpha_1 + \beta_1 > 1$ such that*

$$|\mathbb{E}^u \delta A_{s,u,t}| \leq \Gamma_1 |t-s|^{\alpha_1} \lambda(s, t)^{\beta_1} \text{ a.s.} \tag{A.2.4}$$

for all $(s, t) \in \Delta_{[S, T]}$ and $u := (s + t)/2$. Additionally, assume that (A.2.2) holds for $m = n$ and $\alpha_2 = \beta_2 = 0$ and that there exists a process $(\mathcal{A}_t)_{t \in [S, T]}$ such that (A.2.3) holds. Then there is a map $B: \Delta_{[S, T]} \rightarrow L^m$ and a constant $C > 0$ such that for all $(s, t) \in \Delta_{[S, T]}$,

$$|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}| \leq C\Gamma_1 |t - s|^{\alpha_1} \lambda(s, t)^{\beta_1} + B_{s,t} \text{ a.s. and} \quad (\text{A.2.5})$$

$$\|B_{s,t}\|_{L^m} \leq C\Gamma_2 |t - s|^{1/2 + \varepsilon_2}. \quad (\text{A.2.6})$$

Lemma A.2.3 (Stochastic sewing lemma with controls). *Assume that there exist controls \varkappa_1, \varkappa_2 and constants $\varepsilon_1, \varepsilon_2 > 0$ such that, for $S \leq s \leq u \leq t \leq T$,*

$$\|A_{s,t}\|_{L^m} \leq \varkappa_1(s, t)^{1/2} (t - s)^{\varepsilon_1} \quad (\text{A.2.7})$$

$$\|\mathbb{E}^s[\delta A_{s,u,t}]\|_{L^m} \leq \varkappa_2(s, t) (t - s)^{\varepsilon_2}. \quad (\text{A.2.8})$$

Then there exists a process $(\mathcal{A}_t)_{t \in [S, T]}$ such that (A.2.3) holds and there exists a constant $C = C(m, \varepsilon_1, \varepsilon_2) > 0$ such that for all $(s, t) \in \Delta_{[S, T]}$,

$$\|\mathcal{A}_t - \mathcal{A}_s\|_{L^m} \leq C\varkappa_1(s, t)^{1/2} (t - s)^{\varepsilon_1} + C\varkappa_2(s, t) (t - s)^{\varepsilon_2}. \quad (\text{A.2.9})$$

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