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Yacine Mokhtari

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Thèse de doctorat de l'Université de Bourgogne Franche-Comté en  
co-tutelle avec l'Université des Sciences et de la Technologie  
Houari Boumediene

préparée à l'Université de Franche-Comté

par **Yacine MOKHTARI**

Pour obtenir le grade de  
Docteur en Mathématiques de l'Université Bourgogne Franche-Comté

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## Stabilisation et contrôle de quelques systèmes hyperboliques

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Thèse soutenue à Besançon le 08/07/2021, devant le jury composé de :

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## Résumé

Cette thèse est consacrée à l'étude de trois problèmes issus de la théorie du contrôle des EDP.

Dans un premier temps, on étudie le comportement asymptotique de la solution du système de von Kármán viscoélastique unidimensionnel avec retard. On montre que ce dernier est bien posé dans un espace fonctionnel convenable en utilisant la méthode de Faedo-Galerkin. Pour établir notre résultat de stabilité, on utilise la méthode de Lyapunov en construisant une fonctionnelle adéquate.

Dans un deuxième temps, on étudie le problème de la contrôlabilité et de la stabilisation par le bord pour l'équation des ondes unidimensionnelle dans un domaine non-cylindrique. Pour la contrôlabilité, on utilise la méthode des caractéristiques pour construire l'unique solution du problème. Nous sommes alors capables de donner l'expression explicite du contrôle pour lequel le système atteint le point d'équilibre après un certain temps. De plus, on montre que ce temps est optimal. Pour la stabilisation, nous fournissons une condition nécessaire et suffisante pour que l'énergie du système décroisse à un taux prédéterminé. De plus, l'influence de la géométrie du domaine et l'influence d'un amortisseur dépendant du temps sont clarifiées.

Dans un troisième temps, on étudie la contrôlabilité par le bord de deux équations d'ondes couplées par un couplage d'ordre un avec coefficients qui dépendent de l'espace et du temps. On fournit une condition nécessaire et suffisante pour la contrôlabilité exacte en haute fréquences dans le cas général et pour la continuation unique dans le cas cascade.

## Abstract

This thesis is devoted to the study of three problems related to the theory of control of PDE.

In a first time, we study the asymptotic behavior of the solution to the one-dimensional viscoelastic von Kármán system with delay. We prove that the latter system is well-posed in a suitable functional space by using the Faedo-Galerkin method. To establish our stability result, we employ the Lyapunov method by using a suitable candidate functional.

In a second time, we study the problem of boundary controllability and stabilization for the one-dimensional wave equation in non-cylindrical domains. For the controllability, one uses the characteristics method to build the unique solution. We will then be able to give the explicit expression of the controls for which the system reaches the equilibrium point after a certain time. Moreover, we show that this time is optimal. For the stabilization, we provide a necessary and sufficient condition that guarantees the energy decay at any desired rate. In addition, the influences of the domain geometry and time-dependent feedback are clarified.

In a third time, we study the boundary controllability of two coupled one-dimensional wave equations with first-order coupling terms with coefficients depending on space and time. We give a necessary and sufficient conditions for both exact controllability in high frequency in the general case and for the unique continuation in the cascade case.

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**Part I**

**Introduction**

# Chapter 1

## Aim of the thesis

This thesis is devoted to the study of some problems related to stability and controllability of one dimensional hyperbolic systems.

In Chapter 3, we study the asymptotic behavior of the von Kármán system in presence of viscoelastic damping mechanism and time delay. We use the Lyapunov method to prove stability for a large class of kernels under natural assumptions.

In Chapter 4, we study controllability and stability of the one dimensional wave equation in non-cylindrical domains (or domains with moving boundary). We investigate the controllability and the stability (with time varying feedback) properties under the effect of the moving boundary. By using the characteristics method, we will be able to construct the unique solution and to give a full answer to both questions.

In chapter 5, we study boundary controllability of two coupled wave equations with a first order coupling terms depending on space and time. We give a necessary and sufficient conditions that guarantee both controllability in high frequency for general coupling matrix and unique continuation in the cascade case.

In the current chapter, we will present briefly the main results in this work. Chapter 2 will be devoted to the presentation of some general notions and tools that will be used later.

All the new results presented in this manuscript are take from [64], [90], and [14].

### 1.1 Stabilization of the delayed viscoelastic one dimensional von Kármán system

The first problem consists in stabilizing the so called "the delayed viscoelastic von Kármán system" given by

$$\begin{cases} \rho h D \psi_{tt} + \psi_{xxxx} - [\psi_x(\eta_x + \frac{1}{2}\psi_x^2)]_x - g * \psi_{xxxx} = 0, & \text{in } (0, \infty) \times (0, L), \\ \rho h \eta_{tt} - [\eta_x + \frac{1}{2}(\psi_x)^2]_x + \alpha_1 \eta_t + \alpha_2 \eta_t(t - \tau) = 0, & \text{in } (0, \infty) \times (0, L), \\ \eta_t(t - \tau, x) = f_0(t - \tau, x), & \text{in } (0, \tau) \times (0, L), \\ \psi = \psi_x = \eta_x = 0, & \text{in } (0, \infty) \times \{0, L\}, \\ \psi(0, x) = \psi_0(x), \quad \psi_t(0, x) = \psi_1(x), & \text{in } (0, L), \\ \eta(0, x) = \eta_0(x), \quad \eta_t(0, x) = \eta_1(x), & \text{in } (0, L), \end{cases} \quad (1.1)$$

where  $D = (I - \frac{h^2}{12} \frac{\partial^2}{\partial x^2})$  and the interval  $(0, L)$  is the segment occupied by the beam. The unknowns  $\psi := \psi(t, x)$ , and  $\eta := \eta(t, x)$  represent, respectively, the vertical displacement, and the longitudinal displacement at time  $t$  of the cross section located at  $x$  units from the endpoint  $x = 0$ . The positive constants  $h$  and  $\rho$  represent respectively the thickness and the mass density per unit volume of the beam. In System (1.1),  $\alpha_1 \eta_t$  with  $\alpha_1 > 0$ , and  $g * \psi_{xxxx}$  represent the frictional and the viscoelastic dampings respectively. The time delay is given by  $\alpha_2 \eta_t(t - \tau)$ , with  $\alpha_2, \tau > 0$ . The notation  $(g * f)(\cdot)$  denotes the usual convolution product



$$(g * f)(t) = \int_0^t g(t-s)f(s)ds.$$

System (1.1) has been studied in [15] by F. D. Araruna, P. B. E. Silva and E. Zuazua which appeared as a singular limit of the nonlinear one dimensional Mindlin-Timoshenko system when the modulus of elasticity in shear tends to infinity. For the sake of stabilizing the resultant system, the authors used two damping mechanisms to the equation satisfied by the vertical displacement component  $\psi$  of the form  $\psi_t - \psi_{txx}$  and a damping mechanism  $\alpha_1 \eta_t$  to the equation satisfied by the longitudinal displacement. As a consequence, they obtained an exponential stability result.

Our goal is to study the same system by replacing the two dampings acting on the first equation by a viscoelastic one. Furthermore, it is interesting to add a delay term to the second component to investigate how it effects the asymptotic behavior of the solution since it is well known that it may play a role of anti-damping (see [101]). Note that the presence of the viscoelastic damping  $g * \psi_{xxxx}$  with the frictional one  $\alpha_1 \eta_t$  does not entail necessarily stronger dissipation unless the kernel  $g$  decays with a suitable rate, even for one single equation (See [52] for more details).

To state our result, we assume that:

**A<sub>1</sub>)** The kernel and the constants  $\alpha_1$  and  $\alpha_2$  are such that:

$$\begin{aligned} g &\in L^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+) \text{ with } g(0) > 0, \\ l &= 1 - \int_0^\infty g(s)ds > 0, \\ \alpha_1 &> \alpha_2. \end{aligned}$$

**A<sub>2</sub>)** There exists a  $C^1$ -function  $H : (0, \infty) \rightarrow (0, \infty)$  which is either *linear* or an *increasing and strictly convex*  $C^2$ -function on  $[0, r)$  ( $r \leq g(0)$ ) with  $H(0) = H'(0) = 0$ , such that

$$g'(t) \leq -\zeta(t)H(g(t)), \quad \forall t \geq 0,$$

where  $\zeta$  is a positive non-increasing differentiable function.

We let  $H_*^1(0, 1)$  be the space

$$H_*^1(0, 1) = \left\{ v \in H^1(0, 1), \int_0^L v(x)dx = 0 \right\}.$$

Let us begin by a well-posedness result:

**Theorem 1** *Assume that the initial datum satisfy*

$$\begin{aligned} (\psi_0, \psi_1) &\in [H_0^2(I) \cap H^3(0, L)] \times H_0^2(0, L), \\ (\eta_0, \eta_1) &\in [H^2(0, L) \cap H_*^1(0, L)] \times H_*^1(0, L), \\ (x, p) &\mapsto f_0(-\tau p, x) \in H^1(0, 1; H_*^1(0, L)), \end{aligned}$$

*with the compatibility condition  $f(\cdot, 0) = \eta_1$ . Moreover, assume that the Hypotheses (A<sub>1</sub>), (A<sub>2</sub>) hold. Then System (1.1) admits a unique weak solution*

$$\begin{aligned} (\psi, \psi_t, \psi_{tt}) &\in L_{loc}^\infty \left( 0, \infty; [H_0^2(0, L)]^2 \times H_0^1(0, L) \right), \\ (\eta, \eta_t, \eta_{tt}) &\in L_{loc}^\infty \left( -\tau, \infty; [H_*^1(0, L)]^2 \times L^2(0, L) \right). \end{aligned}$$

The previous result is proved in Subsection 3.2 by using the Faedo-Galerkin method. Next, define the energy functional  $\mathcal{E}(\cdot)$  associated with System (1.1)

$$\begin{aligned}\mathcal{E}(t) &= \frac{1}{2}\rho h \|\psi_t(t)\|^2 + \frac{\rho h^3}{12} \|\psi_{tx}(t)\|^2 + \frac{1}{2} \left(1 - \int_0^t g(s) ds\right) \|\psi_{xx}(t)\|^2 \\ &\quad + \rho h \|\eta_t(t)\|^2 + \left\| \eta_x(t) + \frac{1}{2}(\psi_x)^2(t) \right\|^2 \\ &\quad + \int_0^t g(t-s) \|\psi_{xx}(t) - \psi_{xx}(s)\|^2 ds + \int_{t-\tau}^t \|\eta_t(s)\|^2 ds,\end{aligned}$$

The main stability result for System (1.1) is the following:

**Theorem 2** *Assume that  $(\mathbf{A}_1)$ ,  $(\mathbf{A}_2)$  hold. Then for any solution to (1.1), there exists two positive constants  $k_1 \leq 1$  and  $k_2$  such that the energy functional  $\mathcal{E}(\cdot)$  satisfies*

$$\mathcal{E}(t) \leq k_2 H_1^{-1} \left( k_1 \int_{g^{-1}(r)}^t \zeta(s) ds \right), \quad \forall t \geq 0. \quad (1.2)$$

where

$$H_1(t) = \int_t^r \frac{1}{sH'(s)} ds,$$

and,  $H_1$  is a decreasing convex function on  $(0, r]$ , with  $\lim_{t \rightarrow 0} H_1(t) = +\infty$ .

The above result is proved using the Lyapunov method. It consists of building a suitable functional equivalent to the energy  $\mathcal{E}$ . For more details, see Section 3.3.

Here, we make several observations:

- By (1.2), if  $\int_0^\infty \zeta(s) ds = +\infty$  the solution decays to zero when  $t \rightarrow \infty$ .
- The decay rate of the energy  $\mathcal{E}$  depends on  $g, \zeta$  and the function  $H$ . We can achieve any desired decay rate by a suitable choice of these functions (for some examples, see [96]). Moreover, the decay rate of the solution to System (1.1) is optimal in the sense that it is consistent with the decay rate of the kernel. Indeed, by assumption  $(\mathbf{A}_2)$  we have:

$$g'(t) \leq -\zeta(t)H(g(t)), \quad \forall t \geq 0,$$

therefore,

$$\int_{g^{-1}(r)}^t \frac{-g'(s)}{H(g(s))} ds = \int_{g(t)}^r \frac{s}{H(s)} ds \geq \int_{g^{-1}(r)}^t \zeta(s) ds.$$

Define  $H_0(t) = \int_t^r \frac{s}{H(s)} ds$ , which is decreasing and convex on  $(0, r]$  with  $H_0(s) \xrightarrow{s \rightarrow 0} +\infty$  and

$$H_0(g(t)) \geq \int_{g^{-1}(r)}^t \zeta(s) ds, \quad t \geq g^{-1}(r),$$

which leads to

$$g(t) \leq H_0^{-1} \left( \int_{g^{-1}(r)}^t \zeta(s) ds \right), \quad t \geq g^{-1}(r).$$

Also, by the properties of  $H_0, H_1$  and  $H$ , the following estimate holds

$$H_1(t) = \int_t^r \frac{1}{sH'(s)} ds \leq \int_t^r \frac{1}{H(s)} ds = H_0(t) \Rightarrow H_1^{-1}(t) \leq H_0^{-1}(t), \quad t \geq r. \quad (1.3)$$

This shows that the decay rate of  $\mathcal{E}$  cannot be weaker than the decay rate of the kernel  $g$ .

- Assume now that the equality  $g'(t) = -\zeta(t)H(g(t))$  holds. In addition, assume that  $-\zeta'/\zeta$  is a non-increasing  $C^1$  function for  $t$  large. Then there exist two positive constants  $k_1, k_2$  such that:

$$\mathcal{E}(t) \leq k_2 g(k_1 t). \quad (1.4)$$

To see this, we define the function  $\bar{H}(t) = \zeta(g^{-1}(t))H(t)$ . By computing the derivative of  $\bar{H}$  we obtain

$$\begin{aligned} \bar{H}'(t) &= \frac{\zeta'(g^{-1}(t))}{g'(g^{-1}(t))}H(t) + \zeta(g^{-1}(t))H'(t) = \frac{\zeta'(g^{-1}(t))}{-\zeta(g^{-1}(t))H(t)}H(t) + \zeta(g^{-1}(t))H'(t), \\ &= -\frac{\zeta'}{\zeta}(g^{-1}(t)) + \zeta(g^{-1}(t))H'(t). \end{aligned}$$

From the properties of  $g, \xi$ , and  $H$ , one can easily see that  $\bar{H}(t)$  is increasing and strictly convex on  $(0, r]$  and  $g'(t) = -\bar{H}(g(t))$ . But, by (1.3) we have:

$$\begin{aligned} g(t) &= \bar{H}_0^{-1}\left(\int_{g^{-1}(r)}^t ds\right) = \bar{H}_0^{-1}(t - g^{-1}(r)) \\ &\Rightarrow \bar{H}_0^{-1}(t) = g(t + g^{-1}(r)), \\ &\Rightarrow \bar{H}_1^{-1}(t) \leq g(t + g^{-1}(r)). \end{aligned}$$

Therefore, by (1.2) we obtain:

$$\begin{aligned} \mathcal{E}(t) &\leq k_2 \bar{H}_1^{-1}(t) \left(k_1 \int_{g^{-1}(r)}^t 1 ds\right) = k_2 \bar{H}_1^{-1}(k_1 t - k_1 g^{-1}(r)), \\ &\leq k_2 g(k_1 t + [1 - k_1]g^{-1}(r)) \leq k_2 g(k_1 t), \end{aligned}$$

where we have used the fact that  $[1 - k_1] \geq 0$  and  $g$  is decreasing. The equality case is interesting because it may sometimes be used even if  $H$  is not explicitly known. To illustrate this, if  $g$  is given as a decreasing function satisfying  $(\mathbf{A}_1)$  and  $H$  is implicitly defined by  $H(t) = -g'(g^{-1}(t))$ , which means  $g'(t) = -H(g(t))$ . Then if  $-g''/g'$  is positive and non-increasing, it follows that  $H(\cdot)$  satisfies hypothesis  $(\mathbf{A}_2)$  which leads to the following estimate

$$\mathcal{E}(t) \leq k_2 g(k_1 t).$$

For some illustrations, we refer to [96].

## 1.2 Boundary controllability and boundary time-varying feedback stabilization of the $1 - D$ wave equation in non-cylindrical domains

The second part of this thesis is devoted to the study of boundary controllability and boundary stabilization of the one-dimensional wave equation in non-cylindrical domains. Consider the systems

$$\begin{cases} y_{tt}(t, x) = y_{xx}(t, x), & \text{in } Q, \\ y(t, \alpha(t)) = \frac{1}{2}u(t), \quad y(t, \beta(t)) = 0, & \text{in } (0, \infty), \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & \text{in } (0, 1), \end{cases} \quad (1.5)$$

and

$$\begin{cases} y_{tt}(t, x) = y_{xx}(t, x), & \text{in } Q, \\ y_t(t, \alpha(t)) = f(t)y_x(t, \alpha(t)), \quad y(t, \beta(t)) = 0, & \text{in } (0, \infty), \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & \text{in } (0, 1). \end{cases} \quad (1.6)$$

where the set  $Q$  is defined by

$$Q = \{(t, x) \in \mathbb{R}^2, x \in (\alpha(t), \beta(t)), \alpha(t) < \beta(t), t \in (0, \infty)\},$$

with  $\alpha(0) = 0$  and  $\beta(0) = 1$ .

Define spaces family  $\left[ H_{(\beta(t))}^1(\alpha(t), \beta(t)) \right]_{t \geq 0}$  by:

$$H_{(\beta(t))}^1(\alpha(t), \beta(t)) = \{h \in H^1(\alpha(t), \beta(t)), h(\beta(t)) = 0, t \geq 0\}.$$

In the sequel, we use the notation  $z^\pm$  to denote the quantities

$$z^\pm(t) = t \pm z(t) \tag{1.7}$$

for any real function  $z$  defined on  $(0, \infty)$ .

Finally, assume that the boundary functions satisfy

$$\alpha(t) < \beta(t), \forall t > 0, \alpha, \beta \in C^1(0, \infty), \max\left(\|\alpha'\|_{L^\infty(\mathbb{R}_+)}, \|\beta'\|_{L^\infty(\mathbb{R}_+)}\right) < 1. \tag{1.8}$$

Let us start by the controllability part.

### 1.2.1 Controllability result

The first result that will be presented in this subsection is an existence result:

**Theorem 3** *For any  $(y_0, y_1, u) \in H_{(1)}^1(0, 1) \times L^2(0, 1) \times H_{loc}^1(0, \infty)$ , there exists a unique solution to System (1.5) satisfying*

$$y \in C\left([0, t]; H_{(\beta(t))}^1(\alpha(t), \beta(t))\right) \cap C^1\left([0, t]; L^2(\alpha(t), \beta(t))\right), t \geq 0.$$

Now, we come to the controllability result of System (1.5).

**Theorem 4** *Let  $(y_0, y_1) \in H_{(1)}^1(0, 1) \times L^2(0, 1)$ . Assume that the boundary functions  $\alpha$  and  $\beta$  satisfy (1.8). System (1.5) is exactly controllable at time  $T > 0$  if, and only if,  $T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)$ . Furthermore, if  $T = T^*$ , there exists a unique control  $u \in H^1(0, T^*)$  steering the solution  $(y, y_t)$  to System (1.5) to the equilibrium point  $(0, 0)$  given by*

$$u(t) = \begin{cases} \int_0^t y_1(\alpha^+(s)) ds + y_0(\alpha^+(t)), & \text{if } t \in [0, (\alpha^+)^{-1}(1)], \\ \begin{aligned} & y_0\left(-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(t)\right) \\ & + \int_0^{(\alpha^+)^{-1}(1)} y_1(\alpha^+(s)) ds \\ & - \int_{(\alpha^+)^{-1}(1)}^t y_1\left(-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(s)\right) ds, \end{aligned} & \text{if } t \in [(\alpha^+)^{-1}(1), T^*]. \end{cases} \tag{1.9}$$

Controllability of the wave equation in non-cylindrical domains has been investigated first by C. Bardos and G. Chen in [20]. They proved exact controllability with a distributed control acting on the whole domain under some geometrical conditions. Since then, many results have been established especially in the one-dimensional case by using several methods: domain transformation [38, 89, 9], multipliers technique [39, 123], non-harmonic analysis [60, 119], d'Alembert's formula [58], and Carleman estimate for the multidimensional wave equation [120]. In all these works an observability inequality has been proved. The technique used in our work is the characteristics method as it has been proposed in [117] in more general context. For more details, see Section 4.

### 1.2.2 Stability result

As previously, we start by an existence result:

**Theorem 5** *For any  $(y_0, y_1, f) \in H^1_{(1)}(0, 1) \times L^2(0, 1) \times C([0, \infty))$ . There exists a unique solution to System (1.6) satisfying the regularity*

$$y \in C\left([0, t]; H^1_{(\beta(t))}(\alpha(t), \beta(t))\right) \cap C^1\left([0, t]; L^2(\alpha(t), \beta(t))\right), \quad t \geq 0.$$

The proof of Theorem 3 and Theorem 5 will be a consequence of the construction of the explicit solution that will be done in Section 4.2.

Define the energy functional  $E$  associated with System (1.6) by

$$\begin{aligned} E(t) & : = \|(y, y_t)\|_{H^1_{(\beta(t))}(\alpha(t), \beta(t)) \times L^2(\alpha(t), \beta(t))}^2 \\ & = \int_{\alpha(t)}^{\beta(t)} (y_t^2(t, x) + y_x^2(t, x)) dx, \quad t \geq 0. \end{aligned} \tag{1.10}$$

A simple computation shows that  $E$  satisfies:

$$\frac{dE}{dt}(t) = \beta'(t)y_x^2(t, \beta(t)) - (\alpha'(t) + \alpha'(t)f(t) + 2f(t))y_x^2(t, \alpha(t)), \quad t \geq 0.$$

Observe that the sign of  $\frac{dE}{dt}(t)$  depends the boundary functions which are not supposed of constant sign. The situation is more simple if  $Q$  is cylindrical ( $\alpha \equiv 0, \beta \equiv 1$ ) and we obtain in this case:

$$\frac{d}{dt}E(t) = -2f(t)y_x^2(t, 0), \quad t \geq 0.$$

It suffices to choose  $f$  positive for large times. Though, even with such a choice, determining the optimal decay rate of the energy remains tricky. To overcome this problem, we have studied the solution to System (1.6) along the characteristic lines which provides an explicit formula of the solution.

To set the stability result, we need to fix some notations. Introduce the function  $\phi := \phi(\alpha, \beta)$  defined by

$$\phi := \alpha^- \circ (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}.$$

By assumption (1.8) and notation (1.7), the function  $\phi : [-1, \infty) \rightarrow [\alpha^- \circ (\alpha^+)^{-1}(1), \infty)$  is well defined and increasing function as composition of increasing functions. Let  $(\psi_n)_{n \geq 0}$  be a sequence of functions defined by:

$$\begin{aligned} \psi_n & : [0, \phi(0)) \rightarrow [0, \infty) \\ \tau & \mapsto \psi_n(\tau) = \prod_{i=0}^n \left| F\left((\alpha^-)^{-1} \circ \phi^{[i]}(\tau)\right) \right|, \end{aligned}$$

where  $F = \frac{1-f}{1+f}$ . The notation  $\phi^{[n]}$  refers to

$$\phi^{[n]} = \underbrace{\phi \circ \dots \circ \phi}_{n \text{ times}}$$

with the convention  $\phi^{[0]} = I$ .

**Theorem 6 (General decay)** Let  $(y_0, y_1) \in H_{(1)}^1(0, 1) \times L^2(0, 1)$ . Assume that the boundary functions  $\alpha$  and  $\beta$  satisfy (1.8). In addition, assume that

$$\phi(\tau) < \dots < \phi^{[n]}(\tau) < \phi^{[n+1]}(\tau) \xrightarrow[n \rightarrow \infty]{} \infty, \quad \forall \tau \in [0, \phi(0)).$$

Then:

$$E(t) \xrightarrow[t \rightarrow \infty]{} 0 \Leftrightarrow \left( \psi_n(\tau) \xrightarrow[n \rightarrow \infty]{} 0, \quad \forall \tau \in [0, \phi(0)) \right).$$

Further, if there exists  $g \in C(\mathbb{R}, (0, \infty))$  and positive constant  $C > 0$  such that

$$\psi_n(\tau) \underset{n \rightarrow \infty}{\sim} Cg\left(\phi^{[n]}(\tau)\right), \quad \forall \tau \in [0, \phi(0)),$$

then  $E$  decays like  $g$ , i.e.

$$E(t) \leq Cg(t)E(0), \quad t \geq 0. \tag{1.11}$$

The following result gives a characterization of exponential stabilization of System (1.6):

**Theorem 7 (Exponential decay)** Under the assumptions of Theorem 6, the energy  $E$  decays exponentially to zero with growth bound  $\omega < 0$ , i.e. there exists  $M \geq 1$  such that

$$E(t) \leq Me^{t\omega}E(0), \quad \forall t \geq 0,$$

if, and only if:

$$\sup_{\tau \in [0, \phi(0))} \lim_{n \rightarrow \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = \omega.$$

Another result for System (1.6) in connection with the notion of finite time stability is the following:

**Theorem 8 (Extinction in finite time)** Under the assumptions of Theorem 6, the energy  $E$  vanishes in finite time  $T$  if, and only if,  $f \equiv 1$  and  $T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)$ , i.e.

$$E(T) \equiv 0, \quad \forall T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0).$$

One of the advantages with working with non-autonomous damping is that we are able to stabilize System (1.6) with any decay rate by a suitable choice of  $f$ . One of the main features of our result is that we can do the converse: the function  $f$  can be chosen, starting from a desired decay rate, using the formula

$$\frac{g(\alpha^-(t)) - g(\phi \circ \alpha^-(t))}{g(\alpha^-(t)) + g(\phi \circ \alpha^-(t))} = f_g(t).$$

In particular, if  $Q$  is cylindrical, we obtain

$$\frac{g(t) - g(t+2)}{g(t) + g(t+2)} = f_g(t).$$

When moreover  $f$  is constant, the stability problem (1.6) has been studied in [116] on cylindrical domains by using a spectral approach. It has been shown that the eigenvectors of the underlying operator form a Riesz basis in the space  $H_{(1)}^1(0, 1) \times L^2(0, 1)$  with eigenvalues having  $\frac{1}{2} \ln \left| \frac{1-f}{1+f} \right|$  as real part. Therefore, exponential decay occurs with growth bound  $\frac{1}{2} \ln \left| \frac{1-f}{1+f} \right|$ . If the feedback function  $f$  depends on time, to our knowledge the only existing result is due to P. Martinez and J. Vancostenoble for  $\alpha \equiv 0$ ,  $\beta \equiv 1$  in [86, Section 3.2]. They assumed that there exists a non-increasing function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying

$$\sigma(t) \leq f(t) \leq \frac{1}{\sigma(t)}, \quad t \geq 0, \quad \text{with} \quad \int_0^\infty \sigma(s) ds = +\infty, \tag{1.12}$$

and proved the following estimate:

$$E(t) \leq E(0)e^{1-\int_0^t \sigma(s)ds}, \quad t \geq 0. \quad (1.13)$$

The energy estimate (1.13) is not optimal. For instance, it can be checked that with the choice  $f(t) = \frac{t}{2+t}$ , assumption (1.12) is satisfied for  $\sigma(t) = f(t)$  which yields an exponential decay by (1.13). However, with this choice of  $f$  we can prove by Theorem 7 that the energy decays to zero faster than any exponential function (see the fourth items of Example 54 for the computation).

A recent result has been obtained in non-cylindrical domain with one fixed end-point by K. Ammari et al. in [8] where the boundary function is assumed 1-periodic. An exponential decay result has been obtained for a particular class of feedback functions  $f$ .

**Novelties of the work** The novelties of this work are:

- The control function that steers the solution to System (1.5) to the equilibrium state is given explicitly by (1.9). We can also find the exact formula of  $u$  at any time and for any target state in  $H^1_{(1)}(0,1) \times L^2(0,1)$  (see the proof of Theorem 4 (Proof 4.3.1)).
- The effect of time-varying feedbacks on the behavior of the solution is clarified and a necessary and sufficient condition is provided for the stability of System (1.6).
- Any decay rate can be achieved on the contrary to the autonomous case where only exponential decay rate is achievable. Furthermore, the decay rate is optimal.
- We can do the converse for System (1.6), the feedback function  $f$  can be chosen based on the desired decay rate.
- The influence of the moving boundary on the asymptotic behavior of the solution to System (1.6) is clarified (the boundary nature can increase or decrease the decay rate).
- Since our approach is based on the construction of the exact solutions, both of controllability and stability results still hold in  $W^{1,p}_{(\beta(t))}(\alpha(t),\beta(t)) \times L^p(\alpha(t),\beta(t))$ ,  $1 \leq p < \infty$ , or in the space of continuous function.

### 1.3 Boundary controllability of two coupled wave equations with space-time first order coupling in $1 - D$

The third part of this thesis is devoted to the study of boundary controllability of the following system

$$\begin{cases} y_{tt} = y_{xx} + M((ay)_t + (by)_x), & \text{in } Q_T := (0, T) \times (0, 1), \\ y(t, 0) = Bu(t), \quad y(t, 1) = 0, & \text{in } (0, T), \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & \text{in } (0, 1), \end{cases} \quad (1.14)$$

where  $y = (y_1, y_2)^t$  is a vector function and

$$M = (m_{ij})_{1 \leq i, j \leq 2} \in \mathcal{L}(\mathbb{R}^2), \quad B = (b_1, b_2)^t \in \mathbb{R}^2, \quad a, b \in C^1(\overline{Q_T}; \mathbb{R}).$$

It is well-known (see for instance [130] and the references therein) that System (1.14) is well-posed in a suitable functional framework. More precisely, we have:

**Proposition 9** *Let  $T > 0$ . Suppose that:*

$$(y_0, y_1, u) \in L^2(0, 1)^2 \times H^{-1}(0, 1)^2 \times L^2(0, T).$$

*Then there exists a unique weak solution  $y$  to System (1.14) such that*

$$(y, y_t) \in C\left([0, T], L^2(0, 1)^2 \times H^{-1}(0, 1)^2\right).$$

*Moreover, there exists a constant  $C = C(T, a, b) > 0$  such that:*

$$\|y\|_{C([0, T], L^2(0, 1)^2 \times H^{-1}(0, 1)^2)} \leq C \left( \|Bu\|_{L^2(0, T)^2} + \|(y_0, y_1)\|_{L^2(0, 1)^2 \times H^{-1}(0, 1)^2} \right).$$

The adjoint system associated with (1.14) writes:

$$\begin{cases} \varphi_{tt} = \varphi_{xx} + M^*(a_T \varphi_t - b_T \varphi_x), & \text{in } (0, T) \times (0, 1), \\ \varphi|_{x=0, 1} = 0, & \text{in } (0, T), \\ (\varphi, \varphi_t)|_{t=0} = (\varphi_0, \varphi_1) \in H_0^1(0, 1)^2 \times L^2(0, 1)^2, & \text{in } (0, 1), \end{cases} \quad (1.15)$$

where

$$a_T(t, x) := a(T - t, x), \quad b_T(t, x) := b(T - t, x), \quad (t, x) \in \overline{Q_T}.$$

It is well-known that controllability of System (1.14) is linked to the observability of its adjoint system (1.15) (see for instance [36, 129, 126]). Namely,

- System (1.14) is exactly controllable at time  $T > 0$  if, and only if, there exists  $C = C_T > 0$  such that for any  $(\varphi_0, \varphi_1) \in H_0^1(0, 1)^2 \times L^2(0, 1)^2$ , the associated solution  $\varphi$  to (1.15) satisfies the *observability inequality*:

$$\|(\varphi_0, \varphi_1)\|_{H_0^1(0, 1)^2 \times L^2(0, 1)^2}^2 \leq C \int_0^T |B^* \varphi_x(t, 0)|^2 dt. \quad (1.16)$$

In this case, the adjoint system is said exactly observable.

- System (1.14) is weakly exactly controllable at time  $T > 0$  if, and only if, there exist  $C = C_T > 0$  and a compact operator  $N : H_0^1(0, 1)^2 \times L^2(0, 1)^2 \rightarrow L^2(0, T)$  such that for any  $(\varphi_0, \varphi_1) \in H_0^1(0, 1)^2 \times L^2(0, 1)^2$ , the associated solution  $\varphi$  to (1.15) satisfies the *weak observability inequality*:

$$\|(\varphi_0, \varphi_1)\|_{H_0^1(0, 1)^2 \times L^2(0, 1)^2}^2 \leq C \int_0^T |B^* \varphi_x(t, 0)|^2 dt + C \|N(\varphi_0, \varphi_1)\|_{L^2(0, T)}^2. \quad (1.17)$$

In this case, the adjoint system is said weakly observable.

- System (1.14) is approximately controllable at time  $T > 0$  if, and only if, for any  $(\varphi_0, \varphi_1) \in H_0^1(0, 1)^2 \times L^2(0, 1)^2$ , the associated solution  $\varphi$  to (1.15) satisfies the *unique continuation property*:

$$(B^* \varphi_x(t, 0) = 0, t \in (0, T)) \Rightarrow \varphi \equiv 0 \text{ in } Q_T. \quad (1.18)$$

Actually, weak observability inequality (1.17) means that the space of the target states is finite co-dimensional. In addition, if the unique continuation property (1.18) holds, the space of the target states can be extended to the whole energy space  $H_0^1(0, 1)^2 \times L^2(0, 1)^2$  (see [125]).

Introduce the function  $\phi : [2, \infty) \rightarrow \mathbb{R}$  defined by

$$\phi(t) = \frac{1}{2} \int_{t-2}^{t-1} (a_T + b_T)(\tau, \tau - (t-2)) d\tau + \frac{1}{2} \int_{t-1}^t (a_T - b_T)(\tau, t - \tau) d\tau,$$

and denote by  $\lambda_1, \lambda_2$  the eigenvalues of  $M^*$  if it is diagonalizable and by  $\mu$  the multiple eigenvalue of  $M^*$  if it is not. Also, we denote by  $\chi_{(a, b)}$  the characteristic function of the interval  $(a, b)$ , namely, if  $a < b$ ,  $\chi_{(a, b)}(x) = 1$  when  $x \in (a, b)$  and 0 otherwise.



### 1.3.1 Weak observability

We start by a negative controllability result:

**Theorem 10** *If  $T < 4$ , the weak observability inequality (1.17) doesn't hold. More precisely, there is an infinite dimensional space of unreachable target states.*

Now, we present a positive controllability results.

**Theorem 11** *Let  $n \geq 2$  be an integer.*

- *If  $2n \leq T < 2n + 1$ . Then the weak observability (1.17) holds if, and only if the following three conditions are satisfied:*

1.  $\text{rank}[B \mid MB] = 2$ .

2. *For any  $x \in [0, 1]$ , there exists  $1 \leq k \leq n - 1$  such that:*

$$\begin{cases} \phi(2k + 2 - x) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(2k + 2 - x) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

3. *For any  $x \in [0, T - 2n)$  and  $x^* \in [T - 2n, 1)$ , there exist  $1 \leq k \leq n$  and  $1 \leq k^* \leq n - 1$  respectively such that:*

$$\begin{cases} \phi(x + 2k) \neq 0, \phi(x^* + 2k^*) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(x + 2k), \phi(x^* + 2k^*) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

- *If  $2n + 1 \leq T < 2n + 2$ . Then the weak observability (1.17) holds if, and only if the following three conditions are satisfied:*

1.  $\text{rank}[B \mid MB] = 2$ .

2. *For any  $x \in [2n + 2 - T, 1)$  and  $x^* \in [0, 2n + 2 - T)$ , there exist  $1 \leq k \leq n$  and  $1 \leq k^* \leq n - 1$  respectively such that:*

$$\begin{cases} \phi(2k + 2 - x) \neq 0, \phi(2k^* + 2 - x^*) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(2k + 2 - x), \phi(2k^* + 2 - x^*) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

3. *For any  $x \in [0, 1]$ , there exists  $1 \leq k \leq n$  such that:*

$$\begin{cases} \phi(x + 2k) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(x + 2k) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

### 1.3.2 Unique continuation

#### The constant case

Let us assume first that  $M$  is diagonalizable.

**Proposition 12** Assume that  $M^*$  has 2 distinct eigenvalues  $\mu_1, \mu_2$ . Then, the unique continuation property (1.18) holds true at time  $T \geq 4$  if, and only if

$$a^2 (\mu_1 - \mu_2) (\mu_2 \xi_{n_1}^2 - \mu_1 \xi_{n_2}^2) \neq (\xi_{n_1}^2 - \xi_{n_2}^2)^2, \quad \forall n_1, n_2 \in \mathbb{Z},$$

where  $\xi_{n_i} = \frac{1}{4}b^2\mu_i^2 + (n_i\pi)^2$ ,  $i = 1, 2$ .

If  $M$  is not diagonalizable we have the following result:

**Proposition 13** Assume that  $M^*$  is not diagonalizable and let  $\mu$  be its eigenvalue. Then, the unique continuation property (1.18) holds true at time  $T \geq 4$  if, and only if

$$\frac{1}{2}b^2\mu + a \neq 0.$$

### The cascade case

Assume now that  $M$  and  $B$  are given by

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Define the kernels  $K_n^{i,j}(\cdot, \cdot)$ ,  $1 \leq i, j \leq 2$ ,  $n \geq 1$ , by

$$K_n^{22}(s, x) = \frac{1}{4} \begin{cases} (a_T + b_T) \left( \frac{4n+x+s}{2}, \frac{x-s}{2} \right) + (a_T - b_T) \left( \frac{4n-2+x+s}{2}, \frac{2+s-x}{2} \right), & \text{if } 0 \leq s \leq x, \\ (a_T + b_T) \left( \frac{4n-2+s+x}{2}, \frac{x-s+2}{2} \right) + (a_T - b_T) \left( \frac{4n-4+s+x}{2}, \frac{s-x}{2} \right), & \text{if } x \leq s \leq 1, \end{cases}$$

$$K_n^{11}(s, x) = \frac{1}{4} \begin{cases} (a_T + b_T) \left( \frac{4n+2-x-s}{2}, \frac{2-x+s}{2} \right) + (a_T - b_T) \left( \frac{4n-x-s}{2}, \frac{x-s}{2} \right), & \text{if } 0 \leq s \leq x, \\ (a_T + b_T) \left( \frac{4n+4-x-s}{2}, \frac{s-x}{2} \right) + (a_T - b_T) \left( \frac{4n+2-x-s}{2}, \frac{2+x-s}{2} \right) & \text{if } x \leq s \leq 1, \end{cases}$$

$$K_n^{21}(s, x) = -\frac{1}{4} (a_T + b_T) \left( \frac{4n+x-s}{2}, \frac{x+s}{2} \right) - \frac{1}{4} (a_T - b_T) \left( \frac{4n+x-2-s}{2}, \frac{2-s-x}{2} \right), \quad (s, x) \in (0, 1)^2,$$

$$K_n^{12}(s, x) = -\frac{1}{4} (a_T + b_T) \left( \frac{4n+2-x+s}{2}, \frac{2-x-s}{2} \right) - \frac{1}{4} (a_T - b_T) \left( \frac{s+4n-x}{2}, \frac{s+x}{2} \right), \quad (s, x) \in (0, 1)^2,$$

and let  $\mathbb{K}_{n,k,l}^i(\cdot, \cdot)$ ,  $1 \leq i \leq 2$ ,  $n \geq 2$ ,  $k, l \geq 1$ , be given by

$$\mathbb{K}_{n,k,l}^1(s, x) = \begin{cases} \mathbf{K}_{k,l}(s, x), & 1 \leq k \leq n-1, 1 \leq l \leq n, & \text{if } x \in [0, T-2n), \\ \mathbf{K}_{k,l}(s, x), & 1 \leq k \leq n-1, 1 \leq l \leq n-1, & \text{if } x \in [T-2n, 1], \end{cases}$$

$$\mathbb{K}_{n,k,l}^2(s, x) = \begin{cases} \mathbf{K}_{k,l}(s, x), & 1 \leq k \leq n, 1 \leq l \leq n, & \text{if } x \in [0, 2n+2-T), \\ \mathbf{K}_{k,l}(s, x), & 1 \leq k \leq n+1, 1 \leq l \leq n, & \text{if } x \in [2n+2-T, 1], \end{cases}$$

where

$$A_{k,l}(x) = \begin{pmatrix} \phi(2k+2-x) & 0 \\ 0 & \phi(2l+x) \end{pmatrix}, \quad x \in [0, 1], \quad k, l \geq 1,$$

$$\mathbf{K}_{k,l}(s, x) = \begin{pmatrix} K_k^{11}(s, x) & K_k^{12}(s, x) \\ K_l^{21}(s, x) & K_l^{22}(s, x) \end{pmatrix}, \quad (s, x) \in [0, 1]^2, \quad k, l \geq 1,$$

associated with the third kind Fredholm integral equations

$$A_{k,l}(x) \begin{pmatrix} p_0^-(x) \\ q_0^-(x) \end{pmatrix} = \int_0^1 \mathbb{K}_{n,k,l}^1(s,x) \begin{pmatrix} p_0^-(s) \\ q_0^-(s) \end{pmatrix} ds, \quad (1.19)$$

$$A_{k,l}(x) \begin{pmatrix} p_0^-(x) \\ q_0^-(x) \end{pmatrix} = \int_0^1 \mathbb{K}_{n,k,l}^2(s,x) \begin{pmatrix} p_0^-(s) \\ q_0^-(s) \end{pmatrix} ds. \quad (1.20)$$

**Theorem 14** *Let  $n \geq 2$  be an integer.*

- *If  $2n \leq T < 2n + 1$  : Then, the unique continuation property (1.18) holds true at time  $T$  if, and only if, there exist  $k, l \geq 1$  such that the unique solution  $(p_0^-, q_0^-)$  to Equation (1.19) is the null one.*
- *If  $2n + 1 \leq T < 2n + 2$  : Then, the unique continuation property (1.18) holds true at time  $T$  if, and only if, there exist  $k, l \geq 1$  such that the unique solution  $(p_0^-, q_0^-)$  to Equation (1.20) is the null one.*

Observe that the weak observability (1.17) holds (see Theorem 11) if, and only if the matrix  $A_{k,l}(\cdot)$  is invertible on  $[0, 1]$  for some  $k, l \geq 1$ . However, if we make the latter assumption, we obtain a characterization of the unique continuation problem (1.18).

Let us assume that:

- If  $2n \leq T < 2n + 1$  :

1. There exist  $1 \leq k \leq n - 1, 1 \leq l \leq n$ , such that

$$\phi(2k + 2 - x) \neq 0, \phi(2l + x) \neq 0, \forall x \in [0, T - 2n]. \quad (1.21)$$

2. There exist  $1 \leq k \leq n - 1, 1 \leq l \leq n - 1$ , such that

$$\phi(2k + 2 - x) \neq 0, \phi(2l + x) \neq 0, \forall x \in [T - 2n, 1]. \quad (1.22)$$

- If  $2n + 1 \leq T < 2n + 2$  :

1. There exist  $1 \leq k \leq n - 1, 1 \leq l \leq n$ , such that

$$\phi(2k + 2 - x) \neq 0, \phi(2l + x) \neq 0, \forall x \in [2n + 2 - T, 0). \quad (1.23)$$

2. There exist  $1 \leq k \leq n, 1 \leq l \leq n$ , such that

$$\phi(2k + 2 - x) \neq 0, \phi(2l + x) \neq 0, \forall x \in [2n + 2 - T, 1]. \quad (1.24)$$

Under assumptions (1.21) and (1.22) (resp. (1.23) and (1.24)), Equations (1.19) and (1.20) write

$$\begin{pmatrix} p_0^-(x) \\ q_0^-(x) \end{pmatrix} = \int_0^1 A_{k,l}^{-1}(x) \mathbb{K}_{n,k,l}^i(s,x) \begin{pmatrix} p_0^-(s) \\ q_0^-(s) \end{pmatrix} ds := \mathcal{K}_{k,l}^i \begin{pmatrix} p_0^- \\ q_0^- \end{pmatrix} (x), \quad i = 1, 2,$$

which are a second kind Fredholm integral equations. The following corollary is a straightforward consequence of the Fredholm alternative:

**Corollary 15** *Let  $n \geq 2$  be an integer.*

- *If  $2n \leq T < 2n + 1$  : Assume that (1.21) and (1.22) hold for some  $k, l \geq 1$ . The unique continuation property for (1.18) holds at time  $T$  if, and only if  $1 \notin \sigma \left( \mathcal{K}_{k,l}^1 \right)$ .*
- *If  $2n + 1 \leq T < 2n + 2$  : Assume that (1.23) and (1.24) hold for some  $k, l \geq 1$ . The unique continuation property for (1.18) holds at time  $T$  if, and only if  $1 \notin \sigma \left( \mathcal{K}_{k,l}^2 \right)$ .*

It is clear that solving such a system of a third kind of Fredholm integral equations (1.19) or (1.20) is not an easy task. However, we made use of the particular form of the kernels  $\mathbb{K}_{n,k,l}^i(\cdot, \cdot)$ ,  $i = 1, 2$ , to provide a class of coupling functions  $a$  and  $b$  for which unique continuations holds. See Subsection 5.4.3 for more details.

## Preceding results

Controllability of non-scalar hyperbolic systems has been intensively investigated in the recent pas years by many authors. The aim of these researches was to understand how a system of several wave equations (with less number of control than the states) behaves from a control point of view. So, the question that should be asked for this kind of systems: what are the optimal conditions on the coupling operators (or matrices), and the domains of both coupling functions and the control should satisfy so that such a system is exactly approximately or controllable ?.

Give an answer to this question was the subject of many works. Controllability of two weakly-coupled wave equations has been studied in [1, 2, 3]. In particular, in [3], the geometric condition appeared for both supports of the coupling function and the control. Both distributed and boundary controllability were considered. The authors considered the following control system

$$\begin{cases} u_{tt} - \Delta u + au + pv = bf, & \text{in } (0, T) \times \Omega, \\ v_{tt} - \Delta v + av + pu = 0, & \text{in } (0, T) \times \Omega, \\ u = b_{\partial} f, v = 0, & \text{on } (0, T) \times \partial\Omega, \\ (u, u_t, v, v_t)|_{x=0} = (u_0, u_1, v_0, v_1), & \text{in } \Omega, \end{cases} \quad (1.25)$$

where  $\Omega$  is regular open in  $\mathbb{R}^n$ , the functions  $a, p, b, b_{\partial}$  are real and smooth, and  $f$  is the control. Clearly, System (1.25) is distributed (resp. boundary) control problem when  $b \neq 0$  and  $b_{\partial} = 0$  (resp.  $b_{\partial} \neq 0$  and  $b = 0$ ). The following theorem is proved:

**Theorem 16 (F. Alabau-Boussouira, M. Léautaud, 2013)** *Assume that:*

- *The operator  $-\Delta + a$  is coercive on  $H^2(\Omega) \cap H_0^1(\Omega)$  for  $L^2$ -norm.*
- *$p \geq 0$  on  $\Omega$  and  $\{p > 0\}$  satisfies the (GCC).*
- *$b \geq 0$  on  $\Omega$  (resp.  $b_{\partial} \geq 0$  on  $\partial\Omega$ ) and  $\{b > 0\}$  (resp.  $\{b_{\partial} > 0\}$ ) satisfies the (GCC) (resp. (GCC $_{\partial}$ )).*
- *$\|p\|_{L^\infty(\Omega)}$  is small enough,*

*Then there exists  $T$  (resp.  $T^*$ ) such that distributed controllability (resp. boundary controllability) of System (1.25) holds in the space*

$$H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times H^2(\Omega) \cap H_0^1(\Omega),$$

*(resp.  $L^2(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ ) at time  $T$  (resp.  $T^*$ ).*

Note that exact controllability cannot be established in the natural energy space since the coupling operator will be then a compact perturbation for the uncontrolled state.

In [46], it has been got rid of the smallness condition of the coupling function  $p$ . In addition, a sharp estimate to the control time is provided for cascade system type on connected compact manifold  $\mathcal{M}$  without boundary (periodic boundary condition in one dimension). More precisely, the authors studied the system

$$\begin{cases} u_{tt} - \Delta u + pv = bf, & \text{in } (0, T) \times \mathcal{M}, \\ v_{tt} - \Delta v = 0, & \text{in } (0, T) \times \mathcal{M}, \\ (u, u_t, v, v_t)|_{t=0} = (u_0, u_1, v_0, v_1) & \text{in } \mathcal{M}, \end{cases} \quad (1.26)$$

where  $p$  and  $b$  are smooth real functions and  $f$  is the control. The following result is proved:

**Theorem 17 (B. Dehman, J. Le Rousseau and M. Léautaud)** *Assume that  $p \geq 0$  on  $\Omega$  and both sets  $\{b \neq 0\}$  and  $\{p \neq 0\}$  satisfy the (GCC). Then, System (1.26) is exactly controllable in the space*

$$H^1(\mathcal{M}) \times L^2(\mathcal{M}) \times H^2(\mathcal{M}) \cap H^1(\mathcal{M}),$$

*in sufficiently large time  $T$ .*

Likewise, it has been shown that more regularity is needed so that exact controllability holds for System (1.26) with different speeds of propagation. In the same paper [46], the authors could provide an answer to exact controllability question with space-time coupling coefficient with zero order coupling term in high frequency. This last result does not require a sign assumption on the coupling coefficient  $p$  since the latter condition is only required to cover the invisible target states in low frequency. We should note that approximate controllability remains an open question when  $p$  depends on both space and time or if it changes sign on  $\Omega$ .

The first controllability result without sign assumption on the coupling function is proved in [24] for boundary controllability problem of two coupled wave equations through velocity. More precisely, the system

$$\begin{cases} y_{tt} = y_{xx} + A(x)y_t, & \text{in } (0, 1) \times (0, T), \\ y(t, 0) = Cu(t), \quad y(t, 1) = 0, & \text{in } (0, T), \\ (y, y_t)|_{t=0} = (y_0, y_1), & \text{in } (0, 1), \end{cases} \quad (1.27)$$

has been studied where

$$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in W^{1,\infty}(0, 1; M_{2 \times 2}(\mathbb{R})), \quad C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The authors proved the following result:

**Theorem 18 (A. Bennour, F. Ammar Khodja and D. Teniou, 2017)** *System (1.27) is exactly controllable in  $[L^2(0, 1) \times H^{-1}(0, 1)]^2$  at time  $T$  if, and only if*

- $T \geq 4$ .

- 

$$\int_0^1 a(s) \sin^2(n\pi s) ds \neq 0, \quad \forall n \geq 0. \quad (1.28)$$

- 

$$\int_0^1 a(s) ds \neq 0. \quad (1.29)$$

Note that  $T = 4$  is the natural time for boundary controllability of two coupled wave equations since the time required for one equation is 2. Assumption (1.28) is to guarantee approximate controllability (in low frequency) while assumption (1.29) concerns the high frequency part. Note that assumptions  $T \geq 4$  and (1.29) are necessary, otherwise, the space of the target states will be infinite co-dimensional.

Note that by a suitable change of variables (we set  $\partial_t y_2 = \tilde{y}_2$ ), we deduce that the above result entails that System (1.27) is exactly controllable with zero order coupling in the space

$$L^2(0, 1) \times H^{-1}(0, 1) \times H^{-1}(0, 1) \times H^{-2}(0, 1).$$

In [81], a necessary and sufficient condition is given for distributed exact controllability of  $N$  multi-dimensional zero-order coupled wave equations with constant coupling matrix. The authors studied the following control system

$$\begin{cases} y_{tt} = \Delta y + Ay + 1_\omega Bu, & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{in } (0, T) \times \partial\Omega, \\ (y, y_t)|_{t=0} = (y_0, y_1) & \text{in } \Omega, \end{cases} \quad (1.30)$$

where  $\omega \subset \Omega$  with  $\Omega$  is an open of  $\mathbb{R}^n$  with smooth boundary and  $A \in M_{N \times N}(\mathbb{R})$ . Of course, the control region must satisfies the **(GCC)**, and like for parabolic systems (see [13]), the Kalman rank condition showed up in a natural way. The theorem is stated as follows:

**Theorem 19 (T. Liard, P. Lissy, 2017)** Assume that  $(\omega, T)$  satisfies the (GCC) at time  $T > 0$ . System (1.30) is exactly controllable in  $\left[ H_{(0)}^{2N}(\Omega) \times H_{(0)}^{2N-1}(\Omega) \right]^N$ ,  $N \geq 1$ , at time  $T$  if, and only if

$$\text{rank}[B \mid AB \mid \cdots \mid A^{N-1}B] = N,$$

where

$$H_{(0)}^{2N}(\Omega) = \{v \in H^{2N}(\Omega), v = \Delta v = \cdots = \Delta^{N-1}v = 0 \text{ on } \partial\Omega\}, N \geq 1.$$

The last result is just an application of the main theorem announced in [81] which deals with more general systems.

The above result has been extended for space variable coupling matrix  $A = A(x)$  of cascade form by M. Duprez and G. Olive In [48], i.e.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,N} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,N} \\ 0 & a_{3,2} & a_{3,3} & \cdots & a_{3,N} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{N,N-1} & a_{N,N} \end{pmatrix}. \quad (1.31)$$

The authors showed that exact controllability holds in  $\left[ H_{(0)}^{2N}(\Omega) \times H_{(0)}^{2N-1}(\Omega) \right]^N$ ,  $N \geq 1$  if

$$\text{supp } \omega \cap a_{i,i-1} \neq \emptyset, i \in \{2, \dots, N\}, A \in C^{2(N-3)}(\omega, M_{N \times N}(\mathbb{R})).$$

Observe that in the last two results, the time of controllability of the whole system is the time required for the controllability of a single wave equation. This is due to the non-empty intersection of the coupling and the control domains. We refer to [46] for a sharp estimate of the control time when this intersection is empty.

Concerning boundary controllability of zero order  $N$ -coupled wave equations, exact controllability has been studied in [18] with one control force. More precisely, the following system

$$\begin{cases} y_{tt} = y_{xx} + Ay, & \text{in } (0, T) \times (0, 1), \\ y(t, 0) = Cu(t), y(t, 1) = 0, & \text{in } (0, T), \\ (y, y_t)|_{t=0} = (y_0, y_1), & \text{in } (0, 1), \end{cases} \quad (1.32)$$

has been studied and the authors proved the following result:

**Theorem 20 (S. A. Avdonin, J. Park, L. de Teresa, 2020)** Let  $A \in M_{N \times N}(\mathbb{R})$  and  $C = (0, \dots, 0, 1)$  is a vector in  $\mathbb{R}^N$  and  $u$  is a scalar control function. Let  $(\lambda_i)_{1 \leq i \leq N}$  be the eigenvalues of the matrix  $A$  (they are not necessarily distinct), then System (1.32) is exactly controllable in  $L^2(0, 1; \mathbb{R}^N) \times H^{-1}(0, 1; \mathbb{R}^N)$  at time  $T$  if and only if

- $T \geq 2\pi n$ .

- 

$$\text{rank}[C \mid AC \mid \cdots \mid A^{N-1}C] = N. \quad (1.33)$$

- $\lambda_i - \lambda_j \neq (\pi k)^2 - (\pi l)^2$ ,  $i \neq j$ ,  $k \neq l$ ,  $1 \leq i, j \leq N$ ,  $k, l \in \mathbb{Z}$ . (this last condition can be removed if  $A$  has one eigenvalue of algebraic multiplicity  $N$ ).

Note that the last assumption guarantees only approximate controllability, it just means that there is no multiple eigenvalues for the operator  $\partial_{xx} + A$ , while the first and the second are to ensure controllability in high frequency.

Since hyperbolic control is stronger than the parabolic one by the transmutation method, the above results entails controllability for parabolic system. We refer the reader to [13] for more details.

The most challenging problem in this context is when the coupling matrix depends on both space and time. An interesting result in this direction is stated in [41]. The authors studied the following system

$$\begin{cases} u_{tt} - \Delta u + Lv = Bf, & \text{in } (0, T) \times \mathcal{M}, \\ (u, u_t, v, v_t)|_{t=0} = (u_0, u_1, v_0, v_1) & \text{in } \mathcal{M}, \end{cases} \quad (1.34)$$

where  $\mathcal{M}$  is compact manifold without boundary (periodic boundary conditions in the one dimensional case) with  $u = (u_1, \dots, u_N)$ ,  $f = (f_1, \dots, f_N)$  are the vector state and the control respectively and

$$\begin{aligned} L &= A_1 \partial_t + A_0, \quad B = B_0 \partial_t + B_1, \\ A_i &\in \mathcal{C}^\infty(\mathbb{R}; S_{phg}^i(T^*\mathcal{M}; M_{N \times N})), \quad i = 0, 1, \\ B_i &\in \mathcal{C}^\infty(\mathbb{R}; S_{phg}^i(T^*\mathcal{M}; M_{m \times N})), \quad i = 0, 1. \end{aligned}$$

We refer to [61, Definition 18.1.5] for more of details about the symbols space  $S_{phg}^i(\mathcal{M})$ .

The authors proved that the controllability in high frequency is equivalent to the controllability of a finite dimensional system along the Hamiltonian flow.

**Theorem 21 (Y. Cui, C. Laurent, Z. Wang, 2020)** *System (1.34) is weakly controllable in  $[L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})]$  with control  $u \in L^2(0, T; [L^2(\mathcal{M})]^N)$  if, and only if for any  $\rho_0 \in S^*\mathcal{M}$  the following finite dimensional system*

$$\begin{cases} X'(t) = \frac{1}{2}a(t, \varphi_t(\rho_0))X(t) + \frac{1}{2}d(t, \varphi_t(\rho_0))u(t) \\ X(0) = X_0 \in \mathbb{C}^N, \quad u \in L^2(0, T; \mathbb{C}^N), \end{cases}$$

is controllable in  $[0, T]$  where:

- $(X_1(t), X_2(t), \dots, X_N(t)) \in \mathbb{C}^N$  is the state variable.
- $S^*\mathcal{M}$  is the co-sphere bundle of  $\mathcal{M}$ .
- $\varphi_t(\rho_0)$  is the Hamiltonian flow of  $|\xi|_x$  initiated at  $\rho_0$  defined by the formula

$$\varphi_t(\rho_0) = (x(t), \xi(t)), \quad \varphi_t(\rho_0) = \rho_0.$$

- $a =: a_0 - \frac{a_1}{|\xi|_x}$ ,  $a_i \in \mathcal{C}^\infty(\mathbb{R}; S_{phg}^i(T^*\mathcal{M}; M_{N \times N}))$ ,  $i = 0, 1$ , is the homogeneous principal symbol of coupling matrix  $A_i$ ,  $i = 1, 2$ .
- $d =: b_0 - \frac{b_1}{|\xi|_x}$ ,  $b_i \in \mathcal{C}^\infty(\mathbb{R}; S_{phg}^i(T^*\mathcal{M}; M_{N \times m}))$ ,  $i = 0, 1$ , is the homogeneous principal symbol of the control operator matrices  $B_i$ ,  $i = 1, 2$ .
- $T^*\mathcal{M}$  is the co-tangent bundle.

Note that controllability of System (1.34) amounts to control a parametrized family of ODEs. The same idea appeared in [5] for internal controllability of  $1 - D$  hyperbolic systems.

As pointed out before, to extend the controllability result to the whole energy space we have to ensure that the unique continuation holds. This is done in [41, Appendix A.2] for time-independent constant sign coupling functions with non-empty intersection of both coupling and control supports.

## Novelties of the work

- Providing a necessary and sufficient condition for boundary exact controllability for system (1.14) in high frequency with Dirichlet boundary condition on the contrary of the previous works where only periodic boundary conditions are considered.
- Providing a necessary and sufficient condition for the unique continuation in the cascade case for space-time coupling function without any sign or size assumptions.

## 1.4 Perspectives and open questions

Let us discuss briefly some possible variations and generalization of the obtained results in this work.

### The von Kàrmàn system:

- It will be interesting to study the asymptotic behavior of System (1.1) with less damping mechanisms (for instance by using only the viscoelastic one). Like for Timoshenko system [10], it is quite natural to expect that exponential decay occurs at least for the linearized system of (1.1) when the two wave speeds are equal.
- Another interesting question is to study the controllability of System (1.1) and prove local controllability result by using the linearization method. (See [36] or [131]).

### The wave equation:

- Only boundary control is considered in our work. It is our hope that the tools developed to deal with System (1.6) may help in dealing with the distributed control case

$$\begin{cases} y_{tt}(t, x) = y_{xx}(t, x) + \chi_{\omega_T} h(t, x), & \text{in } Q_T, \\ y(t, \alpha(t)) = y(t, \beta(t)) = 0, & \text{in } (0, T), \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & \text{in } (0, 1), \end{cases} \quad (1.35)$$

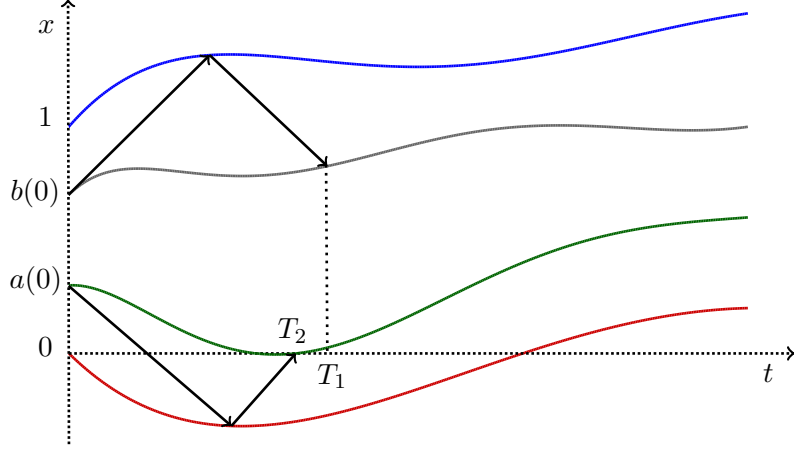
where  $\omega_T$  is a moving subset of  $Q_T := (0, T) \times (0, 1)$  defined by

$$\omega_T = \{(t, x) \in Q_T, x \in (a(t), b(t))\},$$

for some smooth real functions  $\alpha(t) < a(t) < b(t) < \beta(t)$ ,  $t \in (0, T)$ , and  $(y_0, y_1, h) \in H_0^1(0, 1) \times L^2(0, 1) \times L^2(\omega_T)$ . Actually, we can determine the minimal time for which the time-dependent geometric control condition introduced by J. Le Rousseau et al. in [80, Definition 1.6] is satisfied. The latter condition states that every generalized bicharacteristic must meet the moving control region at some time  $T$ . It is easy to verify this condition in the one dimensional settings. Indeed, under assumption (1.8) with  $a, b \in C^1(0, T)$  and  $\|a'\|_{L^\infty(0, T)}, \|b'\|_{L^\infty(0, T)} < 1$ , we find that all the characteristics with positive slope or negative slope emerging from the point  $(0, x)$ , for any  $x \in (0, 1)$  meet  $\omega_T$  if, and only if  $T > T^*$  where  $T^*$  is given by

$$T^* = \max \{T_1, T_2\} = \max \left\{ b^+ \circ \beta^+ \circ (\beta^-)^{-1} \circ b(0), a^- \circ \alpha^- \circ (\alpha^+)^{-1} \circ a(0) \right\}.$$





In particular, if  $\alpha \equiv 0$  and  $\beta \equiv 1$ , the time  $T^*$  is given by  $T^* = 2 \max \{a, 1 - b\}$  which is exactly the time given by E. Zuazua in [131]. Distributed controllability of System (1.35) has been studied by C. Castro et al. [27] in cylindrical domains with moving control support, i.e.  $\alpha \equiv 0$  and  $\beta \equiv 1$ . The authors proved that exact controllability holds if the moving control support  $\omega_T$  satisfies the geometric control condition without the restriction  $\|a'\|_{L^\infty(0,T)}, \|b'\|_{L^\infty(0,T)} < 1$ . It is worth mentioning that problem (1.35) has been recently studied by L. Cui in [40] with very particular boundary functions and moving control support. The critical time of control seems far from being optimal.

- Distributed stabilization of the  $1-D$  damped wave equation on non-cylindrical domains with damping coefficient depending on both space and time remains an open problem. More precisely, consider the following problem:

$$\begin{cases} y_{tt}(t, x) = y_{xx}(t, x) + a(t, x)y_t, & \text{in } Q_\infty, \\ y(t, \alpha(t)) = y(t, \beta(t)) = 0, & \text{in } (0, \infty), \\ y(0, x) = y_0(x), y_t(0, x) = y_1(x), & \text{in } (0, 1). \end{cases} \quad (1.36)$$

We ask the same question as for System (1.6): How do the solution to System (1.36) behaves for large times? What are the optimal assumptions should the damping function  $a$  satisfy to ensure stability?. It is also quite interesting to study pointwise stabilization of System (1.36) ( $a := \delta_{c(t)}$  with  $c(t)$  is some smooth function such that  $\alpha(t) < c(t) < \beta(t)$ ,  $t > 0$ ). This last question has been solved by K. Ammari et al. for periodic boundary functions [8].

Note that for  $\alpha \equiv 0$  and  $\beta \equiv 1$ , the stabilization problem (1.36) has been studied by J. Rauch and M. Taylor in [115] on a manifold without boundary in the usual energy space modulo compact operator. Concerning the periodic case, exponential stability has been recently obtained for System (1.36) by J. Le Rousseau et al. in [80].

It would be interesting to extend the result in [115] to the whole energy space for a general function  $a(t, x)$  at least for the  $1 - D$  case and to figure out the behavior of the solution to System (1.36). Note that the spectral approach or the observability inequality are useless in this case.

### Boundary controllability of two coupled wave equations

- We have only studied unique continuation for System (1.14) in the cascade case. It would be very interesting to extend this result to a general coupling matrix  $M$ .

- In Theorem 21, the manifold  $\mathcal{M}$  is supposed to be without boundary. A sufficient and a necessary condition for exact controllability is given in high frequency. A more challenging issue would be to proving the same result on Riemannian manifold with boundary for both distributed and boundary controllability for space-time coupling matrices.
- Boundary controllability of  $N$ -coupled multi-dimensional wave equations remains an open questions even for constant coupling matrix. It seems that more assumptions in addition to the Kalman rank condition have to be added to guarantee exact controllability since it is the case for  $n = 1$  (see Theorem 20).

## Chapter 2

# Concepts from control and stability theory

### 2.1 Stability of infinite dimensional systems

In this section, we recall some basic notions from stability theory of PDEs. In particular, the Lyapunov method. Henceforth,  $C$  will denote a positive constant that might change from a line to another.

Consider the linear autonomous Cauchy problem

$$\begin{cases} y'(t) = \mathcal{A}y(t), & t \in (0, \infty), \\ y(0) = y_0, \end{cases} \quad (2.1)$$

where  $\mathcal{A} : D(\mathcal{A}) \subset H \rightarrow H$  and  $H$  is a Hilbert space. Assume that  $\mathcal{A}$  generates a  $C_0$ -semigroup  $(\mathcal{S}(t))_{t \geq 0} \in \mathcal{L}(H)$  (see for instance [112]). Then, the unique solution to the Cauchy problem is given by  $y(t) = \mathcal{S}(t)y_0$ ,  $t \geq 0$ . The semigroup  $(\mathcal{S}(t))_{t \geq 0}$  is called:

- Strongly (asymptotically) stable if for any  $y_0 \in H$  :

$$\|\mathcal{S}(t)y_0\|_H \xrightarrow{t \rightarrow +\infty} 0.$$

- Exponentially stable (uniformly stable) if there exist two constants  $\omega > 0$  and  $M \geq 1$  such that for any  $y_0 \in H$  :

$$\|\mathcal{S}(t)y_0\|_H \leq Me^{-\omega t} \|y_0\|_H, \quad \forall t > 0. \quad (2.2)$$

- Stable with decay rate  $g(t) \xrightarrow{t \rightarrow +\infty} 0$  if for any  $y_0 \in D(\mathcal{A})$  :

$$\|\mathcal{S}(t)y_0\|_H \leq Cg(t) \|y_0\|_{D(\mathcal{A})}, \quad t > 0, \quad (2.3)$$

where  $\|y_0\|_{D(\mathcal{A})} = \|y_0\|_H + \|\mathcal{A}y_0\|_H$ .

- Stable in finite time  $T > 0$  if for any  $y_0 \in H$  :  $\mathcal{S}(T+t)y_0 = 0$ ,  $\forall t \geq 0$ .

Note that the norm  $\|\cdot\|_{D(\mathcal{A})}$  in (2.3) cannot be replaced by  $\|\cdot\|_H$ . Otherwise, there will exist  $t_0 > 0$  such that  $\|\mathcal{S}(t_0)\|_{\mathcal{L}(H)} < 1$  which yields (2.2).

We also define the growth bound (or the type) of the semigroup  $(\mathcal{S}(t))_{t \geq 0}$  by

$$\omega_0 = \inf \left\{ \gamma \in \mathbb{R} : \exists M \geq 1 \text{ such that } \|\mathcal{S}(t)\|_{\mathcal{L}(H)} \leq Me^{\gamma t} \right\}.$$

It is known that  $\omega_0(\mathcal{S})$  characterized by

$$\omega_0 = \max\{s(\mathcal{A}), \omega_{\text{ess}}(\mathcal{S})\},$$

where  $s(\mathcal{A})$  and  $\omega_{\text{ess}}(\mathcal{S})$  are respectively the spectral bound and the essential growth bound defined by

$$s(\mathcal{A}) = \sup \{ \Re(\lambda), \lambda \in \sigma(\mathcal{A}) \}, \quad \omega_{\text{ess}}(\mathcal{S}) = \inf_{t>0} \frac{\|\mathcal{S}(t)\|_{\text{ess}}}{t},$$

and  $\|\mathcal{D}\|_{\text{ess}}$  is the quotient norm on the Calkin algebra which is defined for any  $\mathcal{D} \in \mathcal{L}(H)$  by

$$\|\mathcal{D}\|_{\text{ess}} = \inf \left\{ \|\mathcal{D} - \mathcal{K}\|_{\mathcal{L}(H)}, \mathcal{K} \text{ is compact on } H \right\}.$$

For more details, we refer to [99] or [100].

There exists various approaches and characterizations of stability of infinite dimensional systems. For instance, time domain criteria [44], frequency domain criteria [57, 110], spectral method (for instance for  $C_0$ -semigroups generated by Riesz spectral operators [42, Theorem 2.3.5]), Lyapunov or multiplier method [65]. For more details, we refer the reader for instance to [129, 113, 84, 100, 99].

One of the main features of the Lyapunov method is that it can be also adapted for nonlinear problem for suitable nonlinearities (see [65, Chapter 9] and the references therein). Next, we will pay some attention to the Lyapunov method since it will be used to prove a stability result for the nonlinear von Kármán system. See Section 3.

### 2.1.1 Lyapunov method

The Lyapunov method consists in constructing (or finding) a functional  $\mathcal{V} \in C^1(H, \mathbb{R})$  equivalent to the  $H$ -norm of the solution and tending to zero with some decay rate along the trajectory  $y$ . More rigorous definition is given in the following:

**Definition 22** *We call a Lyapunov functional for System (2.1) a functional  $\mathcal{V} \in C^1(H, \mathbb{R}_+)$  satisfying:*

- *There exist two positive constants  $C_1$  and  $C_2$  such that for any solution to (2.1)*

$$C_1 \|\mathcal{S}(t)y_0\|_H \leq \mathcal{V}(y(t)) \leq C_2 \|\mathcal{S}(t)y_0\|_H, \quad t \geq 0.$$

- *The functional  $\mathcal{V}$  satisfies along the trajectory  $y$*

$$\frac{d}{dt} \mathcal{V}(y(t)) \leq -\omega \mathcal{V}(y(t)), \quad t > 0, \tag{2.4}$$

for some  $\omega > 0$ .

Observe that a simple integration of inequality (2.4) yields

$$\mathcal{V}(y(t)) \leq C e^{-\omega t} \mathcal{V}(y_0), \quad t > 0,$$

which entails exponential decay, i.e.

$$\|\mathcal{S}(t)y_0\|_H \leq C e^{-\omega t} \|y_0\|_H, \quad t > 0.$$

A classical application of this method is the damped wave equation. Let  $\Omega$  be an open set in  $\mathbb{R}^n$  and consider the system

$$\begin{cases} y_{tt} = \Delta y - a(x)y_t, & \text{in } (0, \infty) \times \Omega, \\ y = 0, & \text{in } (0, \infty) \times \partial\Omega, \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & \text{in } \Omega. \end{cases} \tag{2.5}$$

where  $a(x) > C > 0$  is a smooth real function. It can be shown by using the semigroups theory (in particular the Lummer-Philips lemma [112, Theorem 4.3]) that for any initial data  $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ , there exists a unique solution to System (2.5)

$$(y, y_t) \in C([0, \infty[; H_0^1(\Omega) \times L^2(\Omega)).$$

It is not difficult to see that the solution to System (2.5) satisfies the energy estimate

$$\frac{1}{2} \frac{d}{dt} E(t) = - \int_{\Omega} a(x) |y_t(t, x)|^2 dx, \quad (2.6)$$

where

$$E(t) = \|\mathcal{S}(t)(y_0, y_1)\|_{H_0^1(\Omega) \times L^2(\Omega)}^2 = \int_{\Omega} |y_t(t, x)|^2 dx + \int_{\Omega} |\nabla y(t, x)|^2 dx.$$

Clearly, by (2.6) the function  $t \mapsto E(t)$  is non-increasing. Our goal is to show that it decreases exponentially to zero. To do this, consider the candidate Lyapunov functional

$$\mathcal{V}(y, y_t)(t) = E(t) + \varepsilon \int_{\Omega} y_t(t, x) y(t, x) dx, \quad (2.7)$$

where  $\varepsilon$  is a positive constant. First we show that the functional  $\mathcal{V}$  is equivalent to  $E$ . Indeed, by using Young's and Poincaré's inequalities we get

$$\begin{aligned} |\mathcal{V}(y, y_t)(t) - E(t)| &= \left| \varepsilon \int_{\Omega} y_t(t, x) y(t, x) dx \right| \\ &\leq \frac{\varepsilon}{2} \int_{\Omega} |y_t(t, x)|^2 dx + \varepsilon C \int_{\Omega} |\nabla y(t, x)|^2 dx \\ &\leq \varepsilon C E(t). \end{aligned}$$

To conclude, it suffices to take  $\varepsilon < 1/C$ .

Now, differentiating  $\mathcal{V}$  with respect to time and using (2.6) then integrating by parts and using Young's and Poincaré's inequalities yields

$$\begin{aligned} \frac{d}{dt} \mathcal{V}(y, y_t)(t) &= - \int_{\Omega} a(x) |y_t(t, x)|^2 dx + \varepsilon \int_{\Omega} y_t^2(t, x) dx \\ &\quad - \varepsilon \int_{\Omega} |\nabla y(t, x)|^2 dx - \varepsilon \int_{\Omega} a(x) y_t(t, x) y(t, x) dx \\ &\leq \left( -a(x) + \frac{\|a\|_{\infty}^2 \varepsilon}{8\delta} + \varepsilon \right) \int_{\Omega} |y_t(t, x)|^2 dx + (-\varepsilon + C\delta) \int_{\Omega} |\nabla y(t, x)|^2 dx. \end{aligned}$$

Taking  $\varepsilon$  small enough and  $\delta < \varepsilon/C$  yields

$$\frac{d}{dt} \mathcal{V}(y, y_t)(t) = -\omega E(t) \leq -C\omega \mathcal{V}(y, y_t),$$

which proves the result. Note that the damping function  $a$  is acting on the whole domain. When  $\text{supp } a$  is a proper subset of  $\Omega$  ( $\text{supp } a \subsetneq \Omega$ ) more work is needed.

An optimal result in this direction has been gotten by J. Rauch and Taylor in [114] on manifold without boundary. The same result has been generalized later by C. Bardos, G. Lebeau and J. Rauch in [21] on any connected compact manifolds under the so called geometric control condition. It has been shown by G. Lebeau in [78] that the energy decays in a logarithmic way in absence of this last condition.

Likewise, the wave equation can be stabilized by a damping acting on a part of the boundary under geometric constraints. Consider the system

$$\begin{cases} y_{tt} - \Delta y = 0, & \text{in } (0, \infty) \times \Omega, \\ \partial_n y + a(x)y_t = 0, & \text{on } (0, \infty) \times \Gamma_1, \\ y = 0, & \text{on } (0, \infty) \times \Gamma_2, \\ y(0) = y_0, y_t(0) = y_1, & \text{in } \Omega, \end{cases} \quad (2.8)$$

where  $\Gamma_1 \cup \Gamma_2 = \partial\Omega$  with  $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$  and  $\partial_n$  denotes the normal derivative. The authors in [21] proved exponential stability under the geometric control condition using microlocal analysis techniques. Exponential decay has been also proved under stronger geometric condition (it is known as the multiplier geometric condition) by G. Chen in [32] by using the multipliers technique. Since then, many successive works have been done in this direction, see for instance [33, 65, 66, 71, 72, 77] and the references therein. It is worth to mention that finite time stability might occur for certain class of boundary feedback as it has been shown in [67].

If the operator  $\mathcal{A}$  depends on time, i.e.  $\mathcal{A} = \mathcal{A}(t)$ , the role of the semigroup  $(\mathcal{S}(t))_{t \geq 0}$  is played by the evolution family  $(\mathcal{S}(t, s))_{t \geq s \geq 0}$ . The study of the asymptotic behavior of such systems is different from the autonomous case. For a survey we refer the reader to [118].

## 2.1.2 Stability of the 1 – D wave equation and the characteristics method

### Boundary stability

Let us focus now on the stability of the one-dimensional wave equation. Consider the system

$$\begin{cases} y_{tt} = y_{xx}, & \text{in } (0, \infty) \times (0, 1), \\ y_x(t, 0) = ay_t(t, 1), y(t, 1) = 0. & \text{in } (0, \infty), \\ y(0, x) = y_0(x), y_t(0, x) = y_1(x), & \text{in } (0, 1). \end{cases} \quad (2.9)$$

where  $a$  is a positive constant. The stability result of the above system is well known since the eighties. A fine spectral analysis has been done in [116] to prove that the corresponding eigenvectors form a Riesz basis in  $H_{(1)}^1(0, 1) \times L^2(0, 1)$  where

$$H_{(1)}^1(0, 1) = \{v \in H^1(0, 1), v(1) = 0\}.$$

As a consequence, the solution to System (2.9) is exponentially stable if and only if the spectrum is located in the left half-plane, namely  $\omega_0 = \Re(\lambda_n) = \frac{1}{2} \ln \left| \frac{1-a}{1+a} \right| < 0$ . Now, we will give an alternative proof by using the characteristics method.

We begin by transforming (2.9) to a first order hyperbolic system by introducing the transformation

$$p = y_t - y_x, \quad q = y_t + y_x, \quad (2.10)$$

which takes  $(y, y_t)$  to  $(p, q)$ . A simple computation shows that  $(p, q)$  satisfies the system

$$\begin{cases} p_t + q_x = 0, & \text{in } (0, \infty) \times (0, 1), \\ q_t - p_x = 0, & \text{in } (0, \infty) \times (0, 1), \\ (p - \bar{a}q)|_{x=0} = (p + q)|_{x=1} = 0, & \text{in } (0, \infty), \\ p|_{t=0} = \bar{p}, \quad q|_{t=0} = \bar{q}, & \text{in } (0, 1), \end{cases} \quad (2.11)$$

where  $\bar{a} = \frac{1-a}{1+a}$ . Conversely, it can be checked that if  $(p, q)$  is solution to (2.11) then it satisfies (2.9). In fact, the transformation (2.10) defines an isomorphism from  $H_{(1)}^1(0, 1) \times L^2(0, 1)$  to  $L^2(0, 1) \times L^2(0, 1)$ . Moreover, the following equality holds:

$$\|p\|_{L^2(0,1)}^2 + \|q\|_{L^2(0,1)}^2 = 2 \|y_t\|_{L^2(0,1)}^2 + 2 \|y_x\|_{L^2(0,1)}^2 = 2 \|(y, y_t)\|_{H_{(1)}^1(0,1) \times L^2(0,1)}^2.$$

Observe that the first and the second equation in (2.11) are not coupled and satisfy

$$\frac{d}{dt}p(t, t+c) = \frac{d}{dt}p(q, -t+c) = 0, \quad t > 0, \quad c \in \mathbb{R},$$

which allows to find an explicit expression of the unique solution to this system.

First, assume that the initial data  $\bar{p}, \bar{q}$  are smooth (for instance  $C_0^1(0, 1)$ ). We can prove by using the reflection of the characteristic lines on the boundary  $x = 0, 1$  to show that the exact solution to System (2.11) is given for any  $n \geq 0$  by the formulas

$$p(t, x) = \begin{cases} \bar{a}^n \bar{p}(x - t + 2n), & \text{if } t - x \in [2n - 1, 2n), \\ -\bar{a}^{n+1} \bar{q}(t - x - 2n), & \text{if } t - x \in [2n, 2n + 1), \end{cases}$$

$$q(t, x) = \begin{cases} \bar{a}^n \bar{q}(x + t - 2n), & \text{if } t + x \in [2n, 2n + 1), \\ -\bar{a}^n \bar{p}(2n + 2 - x - t), & \text{if } t + x \in [2n + 1, 2n + 2), \end{cases}$$

Now, we estimate the solution  $(p, q)$  in the  $L^2$ -norm. We have two cases:

- If  $t \in [2n, 2n + 1)$ , we have for any  $(\bar{p}, \bar{q}) \in C_0^1(0, 1) \times C_0^1(0, 1)$  :

$$\begin{aligned} & \|p(t)\|_{L^2(0,1)}^2 + \|q(t)\|_{L^2(0,1)}^2 \\ &= \bar{a}^{2n+2} \int_0^{t-2n} \bar{q}^2(t-x-2n) dx + \bar{a}^{2n} \int_{t-2n}^1 \bar{p}^2(x-t+2n) dx \\ & \quad + \bar{a}^{2n} \int_0^{2n+1-t} \bar{q}^2(t+x-2n) dx + \bar{a}^{2n} \int_{2n+1-t}^1 \bar{p}^2(2n+2-x-t) dx \\ &= \bar{a}^{2n+2} \int_0^{t-2n} \bar{q}^2(x) dx + \bar{a}^{2n} \int_0^{1-t+2n} \bar{p}^2(x) dx \\ & \quad + \bar{a}^{2n} \int_{t-2n}^1 \bar{q}^2(x) dx + \bar{a}^{2n} \int_{2n+1-t}^1 \bar{p}^2(x) dx \\ &\leq C \bar{a}^{|t|} \left( \|\bar{p}\|_{L^2(0,1)}^2 + \|\bar{q}\|_{L^2(0,1)}^2 \right) \end{aligned}$$

- Similarly, if  $t \in [2n + 1, 2n + 2)$  we obtain for any initial states  $(\bar{p}, \bar{q}) \in C_0^1(0, 1) \times C_0^1(0, 1)$  :

$$\|p(t)\|_{L^2(0,1)}^2 + \|q(t)\|_{L^2(0,1)}^2 \leq C \bar{a}^{|t|} \left( \|\bar{p}\|_{L^2(0,1)}^2 + \|\bar{q}\|_{L^2(0,1)}^2 \right).$$

Therefore, we get for any  $(\bar{p}, \bar{q}) \in C_0^1(0, 1) \times C_0^1(0, 1)$  :

$$\begin{aligned} \|(\mathcal{S}(t))\|_{L^2(0,1) \times L^2(0,1)} &\leq C \bar{a}^{\frac{t}{2}} \|(\bar{p}, \bar{q})\|_{L^2(0,1) \times L^2(0,1)} \\ &= C e^{t \frac{\ln|\bar{a}|}{2}} \|(\bar{p}, \bar{q})\|_{L^2(0,1) \times L^2(0,1)}. \end{aligned} \tag{2.12}$$

At this level, since  $C_0^1(0, 1)$  is dense in  $L^2(0, 1)$ , a standard density argument allows to extend the estimate (2.12) to the energy space  $L^2(0, 1) \times L^2(0, 1)$ . Indeed, the solution  $(p, q)$  can be expressed as

$$(p, q)^t(t) = \mathcal{S}(t) (\bar{p}, \bar{q})^t, \quad t \geq 0,$$

where  $(\mathcal{S}(t))_{t \geq 0}$  is a  $C_0$ -semigroup (which is completely determined in our case). By (2.12), the estimate

$$\|\mathcal{S}(t) (\bar{p}, \bar{q})\|_{L^2(0,1) \times L^2(0,1)} \leq C e^{t \frac{\ln|\bar{a}|}{2}} \|(\bar{p}, \bar{q})\|_{L^2(0,1) \times L^2(0,1)}, \quad t \geq 0,$$

holds for any  $(\bar{p}, \bar{q}) \in C_0^1(0, 1) \times C_0^1(0, 1)$  and therefore for any  $(\bar{p}, \bar{q}) \in L^2(0, 1) \times L^2(0, 1)$ .

The above approach will be used to study the stability of the solution to System (2.9) with  $a := a(t)$  ( $a$  depends on time) and moving boundary (see Chapter 4). It can be also extended to more general wave operators but without lower order terms, for instance  $\partial_{tt} - \rho(x)\partial_{xx}$  (resp.  $\partial_{tt} - \partial_x \rho(x)\partial_x$ ) for a smooth positive function  $\rho$ . Then wave equation be transformed to a first order hyperbolic system with the transformation:

$$p = y_t - \sqrt{\rho(x)}y_x, \quad q = y_t + \sqrt{\rho(x)}y_x,$$

(resp.  $p = y_t - \left(\sqrt{\rho(x)}y\right)_x$ ,  $q = y_t + \left(\sqrt{\rho(x)}y\right)_x$ ). In this case the characteristics are not anymore lines but solutions to the ODEs:

$$(x^\pm)'(t) = \pm\sqrt{\rho(x^\pm(t))}, \quad x^\pm(t_0) = x_0.$$

As before, a stability result can be proved by constructing the exact solution. For instance, we can recover the result proved by S. Cox and E. E. Zuazua in [37].

### Internal stability

Observe that System (2.11) is not coupled in the interior on the domain which allowed to find the solution to a such system. The above approach can be adapted for coupled cascade systems as well.

One of the disadvantages of the characteristics method is that it becomes complex and difficult to use for genuinely coupled systems. To see that, we consider the internal damped wave equation

$$\begin{cases} y_{tt} = y_{xx} - 2ay_t, & \text{in } (0, \infty) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & \text{in } (0, \infty), \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & \text{in } (0, 1), \end{cases} \quad (2.13)$$

where  $a$  is a positive constant. If we try to follow the same steps as before, after using the transformation (2.10) we obtain

$$\begin{cases} p_t + p_x - ap - aq = 0, & \text{in } (0, \infty) \times (0, 1), \\ q_t - q_x - ap - aq = 0, & \text{in } (0, \infty) \times (0, 1), \\ (p + q)|_{x=0} = (p + q)|_{x=1} = 0, & \text{in } (0, \infty), \\ p|_{t=0} = \bar{p}, \quad q|_{t=0} = \bar{q}, & \text{in } (0, 1). \end{cases} \quad (2.14)$$

At first sight, the above system does not seem solvable because of the coupling. Nevertheless, we can get a high frequency result (or a stability result up to finite dimensional space of initial states) by considering only the diagonal part of System (2.14) which becomes decoupled and thus the computations are easier. More precisely, let  $(\mathcal{S}(t))_{t \geq 0}$  be the semigroup solution to System (2.14) on  $X = L^2(0, 1) \times L^2(0, 1)$  and let  $(\mathcal{S}_{\text{diag}}(t))_{t \geq 0}$  the semigroup solution to the diagonal system

$$\begin{cases} p_t + q_x - ap = 0, & \text{in } (0, \infty) \times (0, 1), \\ q_t - q_x - aq = 0, & \text{in } (0, \infty) \times (0, 1), \\ (p + q)|_{x=0} = (p + q)|_{x=1} = 0, & \text{in } (0, \infty), \\ p|_{t=0} = \bar{p}, \quad q|_{t=0} = \bar{q}, & \text{in } (0, 1), \end{cases} \quad (2.15)$$

on the same energy space. It has been shown by AF Neves et al. in [111] that the difference  $\mathcal{K}(t) := \mathcal{S}(t) - \mathcal{S}_{\text{diag}}(t)$ ,  $t \geq 0$ , is compact on  $X$ , therefore, they share the same essential spectrum (since it is invariant under compact perturbations), hence exponential stability holds on  $X$  up to finite dimensional space of initial states since the set  $\sigma(\mathcal{A}) \cap \{\lambda \in \mathbb{C}, \Re(\lambda) > \omega_{\text{ess}}\}$  consists of at most countably many isolated eigenvalues (see for instance [99, Corollary 2.11], [100, Chapter 3.6]).



In our case ( $a$  is a positive constant),  $\omega_{\text{ess}}(\mathcal{S})$  can be easily determined. After solving the diagonal System (2.15) by using the characteristics method we get for any  $n \geq 0$ :

$$\begin{aligned} p(t, x) &= \begin{cases} e^{-at}\bar{p}(x-t+2n), & \text{if } t-x \in [2n-1, 2n), \\ -e^{-at}\bar{q}(t-x-2n), & \text{if } t-x \in [2n, 2n+1), \end{cases} \\ q(t, x) &= \begin{cases} e^{-at}\bar{q}(x+t-2n), & \text{if } t+x \in [2n, 2n+1), \\ -e^{-at}\bar{p}(2n+2-x-t), & \text{if } t+x \in [2n+1, 2n+2). \end{cases} \end{aligned} \quad (2.16)$$

At this level, it can be seen that we have  $\omega_{\text{ess}}(\mathcal{S}) = -a < 0$ . The aim now is to prove that  $S(\mathcal{A}) < 0$  which amounts to solve the following eigenvalue problem

$$y''(x) - a\lambda y(x) - \lambda^2 y(x) = 0, \quad x \in (0, 1),$$

subject to

$$y(0) = y(1) = 0.$$

It is not difficult to prove that

$$S(\mathcal{A}) = \sup \left\{ \Re(-a \pm \sqrt{a^2 - n^2\pi^2}), \quad n \geq 0 \right\} < 0,$$

which entails  $\omega_0(\mathcal{S}) = S(\mathcal{A})$ .

## 2.2 Control of infinite dimensional systems

In this section, we recall some general notions about controllability of infinite dimensional systems. For more details we refer the reader to [36] or [126].

Throughout this thesis, we will focus on the study of the linear control system

$$\begin{cases} y'(t) = \mathcal{A}(t)y(t) + \mathcal{B}(t)u(t), & t \in (0, T), \\ y(0) = y_0 \in H, \end{cases} \quad (2.17)$$

where  $T$  is a positive real number. For the time being, assume that  $(\mathcal{A}(t))_{0 \leq t \leq T}, (\mathcal{B}(t))_{0 \leq t \leq T}$  are families of linear operators (bounded or not) respectively defined on  $D(\mathcal{A}(t)) \subset H$  and  $L^2(0, T; U)$  with values in  $H$  where  $H$  and  $U$  are some functional spaces. In (2.17) the function  $u$  is the control function. Suppose that (2.17) is well-posed in the Hadamard sense. The exact controllability issue is the following: Given a time  $T > 0$ , and initial state  $y_0 \in H$  and a desired state  $y_1 \in H$ , does there exist a control  $u \in L^2(0, T; U)$  such that  $y(T, u) = y_1$ ? Or at least, can we force the state  $y(T, u)$  to be close to the target state  $y_1$  as approximate controllability, which is referred as the approximate controllability issue? The answer to these questions depends genuinely on the nature of System (2.17). We distinguish two main classes of control systems, on whether  $H$  is finite or infinite dimensional space.

First, let us take a look on the finite dimensional case. Take  $H = \mathbb{R}^n, U = \mathbb{R}^m$ , for any  $n, m \geq 1$ . It is well known that by Cauchy-Lipschitz theorem, System (2.17) is well-posed under the assumptions

$$\mathcal{A} \in L^\infty((T_0, T_1), M_{n \times n}(\mathbb{R})), \quad \mathcal{B} \in L^\infty((T_0, T_1), M_{n \times m}(\mathbb{R})).$$

If  $\mathcal{A}$  and  $\mathcal{B}$  do not depend on time, the system writes

$$\begin{cases} y'(t) = \mathcal{A}y(t) + \mathcal{B}u(t), & t \in (0, T), \\ y(0) = y_0 \in \mathbb{R}^n, \end{cases} \quad (2.18)$$

and the Kalman rank condition provides a full answer to the controllability issue:

**Theorem 23** ([63]) *Let  $\mathcal{A} \in M_{n \times n}(\mathbb{R})$ , and  $\mathcal{B} \in M_{n \times m}(\mathbb{R})$ . System (2.18) is controllable at any time  $T > 0$  if, and only if*

$$\text{rank} [\mathcal{B} \mid \mathcal{A}\mathcal{B} \mid \mathcal{A}^2\mathcal{B} \mid \cdots \mid \mathcal{A}^{n-1}\mathcal{B}] = n. \quad (2.19)$$

If the matrices  $\mathcal{A}$  and  $\mathcal{B}$  are time dependent, there exists an answer due to A. Chang in [31] and L. Silverman and H. Meadows in [124]. For more details, we refer the reader to [36, Theorem 1.18].

Observe that in the previous situation, the control  $u$  does not depend necessarily on the state  $y$  (the output has no effect on the input): this type of control is called open-loop.

When the control depends on the state, i.e.  $u := u(t, y(t))$  it is called a closed-loop control or a feedback law. These feedbacks are built in such a way to make the solution reach an equilibrium state (in general  $y = 0$ ) in finite or infinite time.

Unlike the finite dimensional case, control of systems might require a minimal time of control. A typical example is the one-dimensional transport equation.

**Example 24** *Consider the following boundary control system*

$$\begin{cases} y_t(t, x) + y_x(t, x) = 0, \\ y(0, x) = y_0(x), \quad y(t, 0) = u(t), \end{cases} \quad (t, x) \in (0, T) \times (0, 1), \quad (2.20)$$

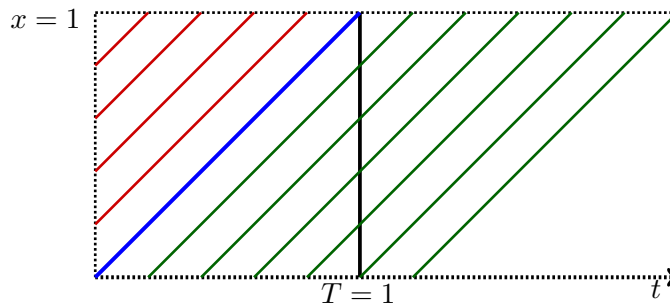
where  $T > 0$  and  $u$  is the control. By using the characteristics method for smooth initial state and control, we can obtain an explicit formula of the solution to (2.20) which is given by

$$y(t, x; y_0, u) = \begin{cases} y_0(x - t), & \text{if } x - t \in (0, 1), \\ u(t - x), & \text{if } t \geq x. \end{cases}$$

One can see easily that if  $T < 1$  System (2.20) is not controllable. Indeed, the solution at this time is written as

$$y(T, x; y_0, u) = \begin{cases} y_0(x - T), & \text{if } x \in (T, 1), \\ u(T - x), & \text{if } x \in (0, T), \end{cases}$$

which confirms the claim. If  $T \geq 1$ , it is enough to choose the control  $u$  such that  $u(T - x) = y_1(x)$ ,  $x \in (0, 1)$ , for any target state  $y_1$ .



Also, in the infinite dimensional settings it may happen that in addition to the minimal time of control, some geometrical constraints effect the controllability properties as it is shown in the following example:

**Example 25** *Consider the transport equation but this time with internal localized control*

$$\begin{cases} y_t(t, x) - y_x(t, x) = \chi_{(a,b)}(x)u(t, x), \\ y(0, x) = y_0(x), \quad y(t, 0) = 0, \end{cases} \quad (t, x) \in (0, T) \times (0, 1), \quad (2.21)$$

where  $\chi_{(a,b)}(x)$  equals to 1 if  $x \in (a,b)$  and 0 otherwise and  $u$  is the control function. We assume that  $y_0$  and  $u$  are smooth enough. Exactly as before, by using the characteristics method we can find an explicit formula to the solution of (2.21) which is given at time  $T \geq 0$  by

$$y(T, x; y_0, u) = \begin{cases} y_0(x - T) + \int_{x-T}^x \chi_{(a,b)}(s)u(s + T - x, s)ds, & \text{if } x - T \in (0, 1), \\ \int_0^x \chi_{(a,b)}(s)u(s + T - x, s)ds, & \text{if } T - x \geq 0. \end{cases} \quad (2.22)$$

The above formula shows that System (2.21) is not controllable if  $a > 0$ . Indeed, let  $y_1$  be a smooth target state. From the second formula in (2.22) we can see that if  $x \in (0, a)$  we get  $y_1(x) = 0, \forall T > 0$ , which is a constraint on the target state. Also, if  $T \leq 1 - b$  we obtain from the first formula in (2.22)  $y_1(x) = y_0(x - T), x \in (T, 1)$  which is again a constraint on both the initial and the target states. Consequently, System (2.21) is not exactly controllable at any  $T > 0$  if  $a > 0$  or if  $T \leq 1 - b$ . Note that even we have an explicit formula of the solution, proving controllability is not straightforward. If we look to the map  $\mathcal{K} : u \mapsto y(T, x; 0, u)$  as a map from  $L^2(0, T)$  in  $L^2(0, 1)$ , then a standard functional analysis result allows to study the surjectivity of  $\mathcal{K}$  by proving the following inequality for its adjoint (see for instance [26, Theorem 2.20.])

$$\exists C > 0 : \|\mathcal{K}^* x\|_{L^2(0,T)} \geq C \|x\|_{L^2(0,1)}, \quad \forall x \in L^2(0, 1). \quad (2.23)$$

A simple proof of this inequality will be given later on after developing the duality concept (see for instance [117]). It will be presented to deal with controllability of more general systems. Actually, it will be seen that inequality (2.23) holds if, and only if  $a = 0$  and  $T > 1 - b$ .

## 2.2.1 General theory

In this subsection we present the general theory of controllability of infinite dimensional systems in an appropriate functional framework. We will define different types of controllability and approaches to handle this type of problems. For the sake of the presentation, we will restrict ourselves to the hilbertian frame. All the given results can be extended to any reflexive Banach space.

Let  $H, U, V$  three separable Hilbert spaces. In all what follows, we assume that  $V$  is a dense subspace in  $H$  and assume that the embedding  $V \hookrightarrow H$  is continuous. As a consequence, we have

$$V \subset H \subset V',$$

where  $V'$  is the dual space of  $V$  with respect to the pivot space  $H$ . Consider the system

$$\begin{cases} y'(t) = \mathcal{A}(t)y(t) + \mathcal{B}(t)u(t), \\ y(0) = y_0 \in H, \end{cases} \quad (2.24)$$

where  $u \in L^2(0, T; U)$  is the control. We assume that the families of linear operators  $(\mathcal{A}(t))_{t \in [0, T]}$  and  $(\mathcal{B}(t))_{t \in [0, T]}$  satisfy

$$\mathcal{A}(\cdot) : L^2(0, T; V) \longrightarrow L^2(0, T; V'), \quad (2.25)$$

$$\mathcal{B}(\cdot) \in \mathcal{L}(L^2(0, T; U), L^2(0, T; V')). \quad (2.26)$$

In addition, we assume that:

- The function  $t \mapsto \langle \mathcal{A}(t)\varphi, \psi \rangle_{V', V}$  is measurable for all  $\varphi, \psi \in V$ .
- There exists a positive constant  $M$  such that for all  $t \in [0, T]$  and for all  $\varphi, \psi \in V$

$$\left| \langle \mathcal{A}(t)\varphi, \psi \rangle_{V', V} \right| \leq M \|\varphi\|_V \|\psi\|_V. \quad (2.27)$$

- The bilinear form  $(\varphi, \psi) \mapsto \langle \mathcal{A}(t)\varphi, \psi \rangle_{V',V}$  satisfies the coercivity property on  $V$ , namely, there exist  $\lambda \in \mathbb{R}$  and  $\alpha > 0$  such that

$$- \langle \mathcal{A}(t)\varphi, \varphi \rangle_{V',V} + \lambda \|\varphi\|_H^2 \geq \alpha \|\varphi\|_V^2, \text{ for all } \varphi \in V, t \in [0, T]. \quad (2.28)$$

The following theorem is valid in the parabolic context:

**Theorem 26** ([83, p.102]) *Under the assumption (2.25)-(2.28), there exists a unique solution to the problem (2.24) in the space*

$$W(0, T; V, V') = \{y \in L^2(0, T, V), y_t \in L^2(0, T, V')\} \xrightarrow{\text{continuous}} C([0, T], H).$$

Furthermore, there exist  $C, C' > 0$  such that the following estimation holds

$$\|y\|_{W(0, T; V, V')} \leq C e^{(T+1)C'} \left( \|y_0\|_H + \|\mathcal{B}u\|_{L^2(0, T, V')} \right).$$

The proof of this theorem is based on the Faedo-Galerkin method. We will use this approach to prove an existence theorem for the nonlinear von Kàrmàn system (see Section 3.2). If  $\mathcal{A}(\cdot)$  is independent of time, semigroups theory can be used to deal with the well-posedness of System (2.24). We refer to the same book for the hyperbolic type.

Now, let us move to the controllability of System (2.24).

**Definition 27** *We say that System (2.24) is :*

- Exactly controllable in  $H$  at time  $T > 0$  if for any initial state  $y_0 \in H$  and for any target state  $y_1 \in H$ , there exists a control  $u \in L^2(0, T; U)$  such that

$$y(T, y_0, u) = y_1.$$

- Null controllable in  $H$  at time  $T > 0$  if for any initial state  $y_0 \in H$ , there exists a control  $u \in L^2(0, T; U)$  such that

$$y(T, y_0, u) = 0_H.$$

- Approximately controllable in  $H$  at time  $T > 0$  if for any  $\varepsilon > 0$ , initial state  $y_0 \in H$  and target state  $y_1 \in H$ , there exists a control  $u \in L^2(0, T; U)$  such that

$$\|y(T, y_0, u) - y_1\|_H < \varepsilon.$$

Clearly, null controllability is a particular case of exact controllability, and if a system is exactly controllable then it is approximately controllable. The converse in general is not true and there is no relationship between null controllability and approximate controllability. We should note that the three types of controllability presented above are equivalent in the finite dimensional case.

In the sequel, we denote by  $y(t, y_0, u)$  the solution to System (2.24). Since the latter system is linear, the solution can be decomposed as a sum of homogeneous solution ( $u = 0$ ) and the solution with zero initial state ( $y_0 = 0$ ). i.e.

$$y(t, y_0, u) = S_t y_0 + L_t u, \quad t \in (0, T), \quad (2.29)$$

where the operators  $S_t$  and  $L_t$  are defined by

$$\begin{aligned} S_t : H &\longrightarrow H & L_t : L^2(0, T; U) &\longrightarrow H \\ & & \text{and} & \\ y_0 &\mapsto y(t, y_0, 0), & u &\mapsto y(t, 0, u), \end{aligned} \quad (2.30)$$

Let us go back to the controllability problem. Henceforth, we assume that the solution to problem (2.24) exists. Observe that studying the controllability of System (2.24) at time  $T$  amounts to study the range of the operator  $L_T$ . The following lemma will be used to provide a useful characterization of  $R(L_T)$ , the range of  $L_T$  (see for instance [129, Theorem 2.2, p. 208]).

**Lemma 28** *Let  $X_1, X_2$  and  $X_3$  be three Hilbert spaces and let  $L_1 \in \mathcal{L}(X_1, X_3)$  and  $L_2 \in \mathcal{L}(X_2, X_3)$ . The following equivalence holds*

$$R(L_1) \subset R(L_2) \Leftrightarrow \exists C > 0, \|L_1^*x\|_{X_1} \leq C \|L_2^*x\|_{X_2}, \forall x \in X_3.$$

Let  $T > 0$  and  $y_1 \in H$  be the desired time and the target state respectively. By (2.29), we have at time  $T$

$$y_1 - S_T y_0 = L_T u,$$

thus, exact controllability is equivalent to  $H \subset R(L_T)$  and the null controllability is equivalent to  $S_T H \subset R(L_T u)$ .

**Characterization of exact controllability:**

In order to get a characterization of exact controllability, we apply Lemma 28 for

$$\begin{aligned} X_1 &= H, & X_2 &= H, & X_3 &= L^2(0, T; U), \\ L_1 &= I_d, & L_2 &= L_T, \end{aligned}$$

we find

$$H \subset R(L_T u) \Leftrightarrow \exists C > 0, \|L_T^* v\|_{L^2(0, T; U)} \geq C \|v\|_H, \forall v \in H. \quad (2.31)$$

**Characterization of null controllability:**

We apply Lemma 28 with

$$\begin{aligned} X_1 &= H, & X_2 &= H, & X_3 &= L^2(0, T; U), \\ L_1 &= S_T, & L_2 &= L_T, \end{aligned}$$

we find

$$S_T H \subset R(L_T u) \Leftrightarrow \exists C > 0, \|L_T^* v\|_{L^2(0, T; U)} \geq C \|S_T^* v\|_H, \forall v \in H. \quad (2.32)$$

**Characterization of approximate controllability:**

Note that from the definition of approximate controllability, System (2.24) is approximately controllable if, and only if  $\overline{R(L_T)}^H = H$  ( $R(L_T u)$  is dense in  $H$ ), which means by [26, Corollary 2.18] that

$$\overline{R(L_T)}^H = H \Leftrightarrow \ker(L_T^*) = \{0_H\},$$

namely,

$$L_T^* v = 0 \text{ in } (0, T) \implies v = 0. \quad (2.33)$$

Inequalities (2.31) and (2.32) are known as observability inequalities, while relation (2.33) is known as the unique continuation property.

**Remark 29** *We should note that if the family of operators  $(\mathcal{A}(t))_{t \in [0, T]}$  generates an evolution family  $(\mathcal{S}(t, s))_{0 \leq s \leq t \leq T}$ , on  $H$  (see [112, Chapter 5] for more details), the operator  $S_t$  is the evolution family  $(\mathcal{S}(t, s))_{0 \leq s \leq t \leq T}$ , and by the Duhamel's formula, the operator  $L_t$  will be defined by*

$$L_t u := \int_0^t \mathcal{S}(t, s) \mathcal{B}(s) u(s) ds, \quad t \in (0, T).$$

Therefore, the operator  $L_t^*$  takes the form

$$\begin{aligned} L_t^* &: H \longrightarrow L^2(0, T; U) \\ L_t^* x &= B^*(\cdot) \mathcal{S}^*(t, \cdot) x, \end{aligned} \quad (2.34)$$

where  $(\mathcal{S}^*(t, s))_{0 \leq s \leq t \leq T}$ , is the adjoint evolution family solution to the backward system

$$\begin{aligned}\frac{d}{ds}\mathcal{S}^*(t, s) &= -\mathcal{A}^*(s)\mathcal{S}^*(t, s), \\ \mathcal{S}^*(t, t) &= I_d.\end{aligned}$$

In particular, if  $\mathcal{A}(t) = \mathcal{A}$ , then  $\mathcal{S}^*(t, s) = \mathcal{S}^*(t - s)$  is a  $C_0$ -semigroup.

As it is pointed out in Example 25, even though we have an explicit formula of the control operator  $L_T$  it is not straightforward to provide an answer to the exact controllability question. Let us try to deal with System (2.21) by using the observability inequality defined in (2.31). In the setting of Example 25, we have:

$$\begin{aligned}H &:= L^2(0, 1), \quad U := L^2(a, b), \quad \mathcal{B} = \chi_{(a, b)} \in \mathcal{L}(L^2(0, 1)), \\ \mathcal{A} &:= \partial_x, \quad D(\mathcal{A}) = \{v \in H^1(0, 1), v(0) = 0\}.\end{aligned}$$

So, the exact controllability issue amounts to prove that there exists a positive constant  $C_{T, a, b} > 0$  such that for any  $v \in H$ :

$$\|L_T^*v\|_{L^2(0, T; U)} = \int_0^1 \int_0^T |\mathcal{B}^*\mathcal{S}^*(T - s)v(x)|^2 ds dx \geq C_{T, a, b} \int_0^1 v^2(x) dx, \quad (2.35)$$

where  $\mathcal{B}^* = \mathcal{B}$  and  $z(s, x) = \mathcal{S}^*(T - s)v(x)$  is the solution to the homogeneous transport equation

$$\begin{cases} z_s + z_x = 0, & \text{if } (s, x) \in (0, T) \times (0, 1), \\ z(s, 1) = 0, \quad z(T) = v \in H, & \text{if } s \in (0, T). \end{cases} \quad (2.36)$$

To simplify the computation, we make the variable substitution  $t = T - s$ , then System (2.36) becomes

$$\begin{cases} z_t - z_x = 0, & \text{if } (t, x) \in (0, T) \times (0, 1), \\ z|_{x=0} = 0, \quad z|_{t=0} = v \in H, & \text{if } t \in (0, T). \end{cases} \quad (2.37)$$

By using the characteristics method, we find that the exact solution to System (2.37) is given by

$$z(t, x) = \begin{cases} v(t + x), & \text{if } t + x \in (0, 1), \\ 0, & \text{if } t + x \geq 1. \end{cases}$$

Therefore, inequality (2.35) turns to

$$\int \int_{\Xi} \chi_{(a, b)}(x) v^2(x + t) dt dx \geq C_{T, a, b} \int_0^1 v^2(x) dx, \quad (2.38)$$

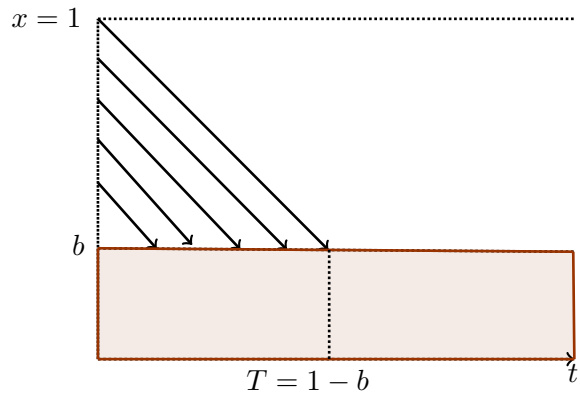
where  $\Xi = \{(t, x) \in (0, T) \times (0, 1), t + x \in (0, 1)\}$ .

By letting  $x + t = s$  in (2.38), the left hand side will be

$$\int \int_{\Xi} \chi_{(a, b)}(x) v^2(x + t) dt dx = \int_0^1 \left( \int_0^{\min\{s, T\}} \chi_{(a, b)}(s - t) dt \right) v^2(s) ds.$$

Hence, the function  $s \mapsto \int_0^{\min\{s, T\}} \chi_{(a, b)}(s - t) dt$  must be positive in  $(0, 1)$  which amounts to say that the family of lines  $t \mapsto s - t$ ,  $t \in (0, T)$  must enter the control region  $(a, b)$  for any  $s \in (0, 1)$  which is possible if and only if  $a = 0$  or  $T > 1 - b$ .

This condition is known as the geometric control condition, it states that every characteristic line corresponding to the adjoint system (2.36) must meet the control region at time before  $T$ . (See [21] for a precise definition in more general context).



## Part II

# Main results



## Chapter 3

# General decay of the solution to a nonlinear viscoelastic modified von Kármán system with delay

This Chapter is taken from [64].

We are concerned with the following nonlinear modified von Kármán system with time delay and a memory term,

$$\begin{cases} \rho h D \psi_{tt} + \psi_{xxxx} - [\psi_x(\eta_x + \frac{1}{2}\psi_x^2)]_x - g * \psi_{xxxx} = 0, & \text{in } (0, \infty) \times (0, L), \\ \rho h \eta_{tt} - [\eta_x + \frac{1}{2}(\psi_x)^2]_x + \alpha_1 \eta_t + \alpha_2 \eta_t(t - \tau) = 0, & \text{in } (0, \infty) \times (0, L), \\ \eta_t(t - \tau, x) = f_0(t - \tau, x), & \text{in } (0, \tau) \times (0, L), \end{cases} \quad (3.1)$$

where  $D = (I - \frac{h^2}{12} \frac{\partial^2}{\partial x^2})$  and the interval  $(0, L)$  is the segment occupied by the beam. The unknowns  $\psi = \psi(t, x)$ , and  $\eta = \eta(t, x)$  represent, respectively, the vertical displacement, and the longitudinal displacement at time  $t$  of the cross section located at  $x$  units from the endpoint  $x = 0$ .

In (3.1), subscripts mean partial derivatives and  $h > 0$  is a parameter related to the rotational inertia of the beam.

When  $\alpha_1 = \alpha_2 = 0$ , this system describes approximately the planar motion of a uniform prismatic beam of length  $L$  with memory term. Here,  $h$  and  $\rho$  are two positive constants represent respectively the thickness and the mass density per unit volume of the beam. In System (3.1),  $\alpha_1 \eta_t$  represents a frictional damping. The time delay is given by  $\alpha_2 \eta_t(t - \tau)$ , where  $\alpha_1, \alpha_2, \tau$  are positive constants.

In (3.1),  $(g * f)(\cdot)$  is defined by

$$(g * f)(t) = \int_0^t g(t - s) f(s) ds.$$

This integral term or the viscoelastic damping term that appears in the equations describes the relationship between the stress and the history of the strain in the beam, according to Boltzmann theory. The function  $g$  represents the kernel of the memory term or the relaxation function.

To the System (3.1) we add the boundary conditions

$$\psi = \psi_x = \eta_x = 0, \text{ in } (0, \infty) \times \{0, L\}, \quad (3.2)$$

and the initial conditions

$$(\psi(0, \cdot), \psi_t(0, \cdot), \eta(0, \cdot), \eta_t(0, \cdot)) = (\psi_0(\cdot), \psi_1(\cdot), \eta_0(\cdot), \eta_1(\cdot), \eta_t(0, \cdot)) \text{ in } (0, L) \quad (3.3)$$

The main purpose about problems (3.1)-(3.3) is to deal with the well-posedness and asymptotic behavior of solutions. Before stating and proving our results, let us recall some other results related to our work.

Several authors have studied the Mindlin-Timoshenko system of equations. This Model is a widely used and fairly complete mathematical model for describing the transverse vibrations of beams. It is a more accurate model than the Euler-Bernoulli one, since it also takes into account transverse shear effects. The Mindlin-Timoshenko system is used, for example, to model aircraft wings (see, e.g., [47]).

For a beam of length  $L > 0$ , this one-dimensional nonlinear system reads as

$$\begin{aligned} \frac{h^3}{12}\phi_{tt} - \phi_{xx} + k[\phi + \psi_x] &= 0, \quad \text{in } (0, \infty) \times (0, L), \\ \rho h\psi_{tt} - k[\phi + \psi_x]_x + \left[ \psi_x \left( \eta_x + \frac{1}{2}\psi_x^2 \right) \right]_x &= 0, \quad \text{in } (0, \infty) \times (0, L), \\ \rho h\eta_{tt} - \left[ \eta_x + \frac{1}{2}(\psi_x)^2 \right]_x &= 0, \quad \text{in } (0, \infty) \times (0, L). \end{aligned} \quad (3.4)$$

Here, the unknown  $\phi = \phi(x, t)$  represent the angle of rotation. The parameter  $k$  is the so called modulus of elasticity in shear. It is given by the expression  $k = \widehat{k}Eh/2(1 + \epsilon)$ , where  $\widehat{k}$  is a shear correction coefficient,  $E$  is the Young's modulus and  $\epsilon$  is the Poisson's ratio,  $0 < \epsilon < 1/2$ .

For Mindlin-Timoshenko system, there is a large literature, addressing problems of existence, uniqueness and asymptotic behavior in time when some damping effects are considered, as well as some other important properties (see [35, 68] and references therein).

When one assumes the linear filament of the beam to remain orthogonal to the deformed middle surface, the transverse shear effects are neglected, and one obtains, from the Mindlin-Timoshenko system of equations, the following von Kármán system (see [70]).

$$\begin{cases} \rho h D\psi_{tt} + \psi_{xxxx} - [\psi_x(\eta_x + \frac{1}{2}\psi_x^2)]_x = 0, & \text{in } (0, \infty) \times (0, L), \\ \rho h \eta_{tt} - [\eta_x + \frac{1}{2}(\psi_x)^2]_x = 0, & \text{in } (0, \infty) \times (0, L). \end{cases} \quad (3.5)$$

There is also an extensive literature about System (3.5) (see [35, 68, 51, 69, 91, 104, 106, 107, 108, 109] and references therein).

Lagnese and Leugering [69] considered a one-dimensional version of the von Kármán system describing the planar motion of a uniform prismatic beam of length  $L$ . More precisely, in [69] the following system was considered:

$$\begin{cases} \psi_{tt} + \psi_{xxxx} - h\psi_{xxtt} - [\psi_x(\eta_x + \frac{1}{2}\psi_x^2)]_x = 0, & \text{in } (0, \infty) \times (0, L), \\ \eta_{tt} - [\eta_x + \frac{1}{2}(\psi_x)^2]_x = 0, & \text{in } (0, \infty) \times (0, L). \end{cases} \quad (3.6)$$

In [69], suitable dissipative boundary conditions at  $x = 0$ ,  $x = L$  and initial conditions at  $t = 0$  were given and the stabilization problem was studied.

In [15], Araruna et al. have showed how the so called von Kármán model (3.6) can be obtained as a singular limit of a modified Mindlin-Timoshenko System (3.4) when the modulus of elasticity in shear  $k$  tends to infinity, provided a regularizing term through a fourth order dispersive operator is added. Introducing damping mechanisms, the authors also show that the energy of solutions for this modified Mindlin-Timoshenko system decays exponentially, uniformly with respect to the parameter  $k$ . As  $k \rightarrow \infty$ , the authors obtain the damped von Kármán model with associated energy exponentially decaying to zero as well.

**Remark 30** *Since  $k$  is inversely proportional to the shear angle, we note that neglecting the shear effects of the beam is formally equivalent to considering the modulus  $k$  tending to infinity in the Mindlin-Timoshenko system.*

The subject of stability of von Kármán system has received a lot of attention in the last years. It is important to mention that the authors in [34, 51, 68] proved uniform decay rates for the von Kármán

system with frictional dissipative effects in the boundary. The stability for a von Kármán system with memory and boundary memory conditions was treated in [51, 92, 103, 79]. They proved the exponential or polynomial decay rate when the relaxation function decay is at the same rate. As for the works about general decay for viscoelastic system, we refer the reader to [30, 105] and references therein.

Delay effects are very important because most natural phenomena are in many cases very complicated and do not depend only on the current state but also on the past history of the system. The presence of delay can be a source of instability. In recent years, the stabilization of PDEs with delay effects has draw attention for many authors and becomes an active area of research (see [62, 30, 45, 101, 102, 121, 122, 128, 127]).

In [22], Benaissa et al. studied a system of viscoelastic wave equations with a linear frictional damping term and a delay

$$\left\{ \begin{array}{ll} u_{tt}(x, t) - \Delta u(x, t) + \int_0^t g(t-s)\Delta u(x, s)ds & \text{in } \mathbb{R}_+ \times \Omega, \\ +\alpha_1 h_1(u_t(x, t)) + \alpha_2 h_2(u_t(x, t-\tau)) = 0, & \\ \\ u(x, t) = 0, & \text{on } \mathbb{R}_+ \times \Gamma, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & \text{in } \Omega, \\ u_t(t-\tau, x) = f_0(t-\tau, x), & \text{in } (0, \tau) \times \Omega, \end{array} \right. \quad (3.7)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ , with a smooth boundary  $\partial\Omega = \Gamma$ ,  $g$  is a positive non-increasing function defined on  $\mathbb{R}^n$ ,  $h_1$  and  $h_2$  are two functions,  $\tau > 0$  is a time delay,  $\alpha_1$  and  $\alpha_2$  are positive real numbers and the initial datum  $(u_0, u_1, f_0)$  belong to a suitable function space.

In the case  $g \equiv 0$ , problem (3.7) has been studied by many authors (see [101, 128, 23]) and in the case  $g \neq 0$ , Cavalcanti et al. [29] studied (3.7) for  $h_2 \equiv 0$  and with a linear localized frictional damping  $a(x)u_t$ . This work was later improved by Berrimi and Messaoudi [25] by introducing a different functional which allowed them to weaken the conditions on  $g$ .

For a wider class of relaxation functions, Messaoudi [87, 88] considered

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = b|u|^\gamma u, \quad (3.8)$$

for  $\gamma > 0$  and  $b = 0$  or  $b = 1$ , and the relaxation function satisfies

$$g'(t) \leq -\zeta(t)g(t), \quad (3.9)$$

where  $\zeta$  is a differentiable non-increasing positive function. He established a more general decay result, from which the usual exponential and polynomial decay results are only special cases. Such a condition was then employed in a series of papers, see for instance [54, 94, 95, 105].

Recently, Mustafa and Messaoudi [98] studied the problem (3.8) with  $b = 0$  for the relaxation functions satisfying

$$g'(t) \leq -H(g(t)), \quad (3.10)$$

where  $H$  is a non-negative function, with  $H(0) = H'(0) = 0$  and  $H$  is increasing and strictly convex on  $]0, k[$  for some  $k_0 > 0$ . The authors showed a general relation between the decay rate for the energy and that of the relaxation function  $g$  without imposing restrictive assumptions on the behavior of  $g$  at infinity. On the other hand, a condition of the form (3.10) where  $H$  is a convex function satisfying some smoothness properties, was introduced by Alabau-Boussouira and Cannarsa [7] and used then by several authors with different approaches. We refer to [76] where not only general but also optimal result was established by Lasiecka and Wang.

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (3.1) for linear damping, time delay terms and finite memory. We would like

to see the influence of frictional and viscoelastic dampings on the rate of decay of solutions in the presence of linear delay term.

Our aim is to investigate (3.1) for relaxation functions  $g$  of more general type than the ones in (3.9) and (3.10). We consider the condition

$$g'(t) \leq -\zeta(t)H(g(t)), \quad (3.11)$$

where  $H$  is increasing and convex without any additional constraints on  $H$  and the coefficients, and establish energy decay results that address both the optimality and generality. The energy decay rates are optimal in the sense that they decay qualitatively the same as the viscoelastic kernels  $g$  do.

To obtain global solutions of problem (3.1)-(3.3), we use the Galerkin approximation scheme (see Lions [82]) together with the energy estimate method. To prove decay estimates, we use a perturbed energy method and some properties of convex functions. In order to accomplish this goal, we shall pursue a strategy based on an adaptation of non linear differential inequalities technique developed in [93, 96, 97]. Arguments of convexity were introduced and developed by many authors [28, 43, 74, 75, 85, 50, 6].

We observe that our problem is set in a context where:

- a The memory damping is defined only on the equation for the vertical displacement.
- b The presence of a frictional damping and a time delay on the equation for the longitudinal displacement.
- c Energy decay estimates under a nonlinear tension.

Our work is organized as follows. In the next section, we prepare some materials needed in the proof of our result, like some lemmas (Poincaré's and Young's inequalities) and some useful notations. We introduce the different functionals by which we modify the classical energy to get an equivalent useful one. In Section 3.2, we state and prove the well-posedness of the problem. Finally, in Section 3.3, we will prove our main results concerning the decay of the energy associated to the solutions to the problem.

### 3.1 Statement of the results

As in [101], we introduce the new variable

$$z(t, x, p) = \eta_t(t - p\tau, x),$$

which satisfies

$$\begin{cases} \tau z_t(t, x, p) + z_p(t, x, p) = 0, & \text{in } (0, \infty) \times (0, L) \times (0, 1), \\ z_x(t, x, p) = 0, & \text{in } (0, \infty) \times \{0, L\} \times (0, 1), \\ z_0(x, p) = z(0, x, p) = f_0(x, -p\tau), & \text{in } (0, L) \times (0, 1). \end{cases}$$

Therefore, problem (3.1)-(3.3) is equivalent to

$$\begin{cases} \rho h D\psi_{tt} + \psi_{xxxx} - \left[ \psi_x(\eta_x + \frac{1}{2}\psi_x^2) \right]_x - g * \psi_{xxxx} = 0, & \text{in } (0, \infty) \times (0, L) \\ \rho h \eta_{tt} - \left( \eta_x + \frac{1}{2}\psi_x^2 \right)_x + \alpha_1 \eta_t + \alpha_2 z(1) = 0, & \text{in } (0, \infty) \times (0, L) \\ \tau z_t + z_p = 0, & \text{in } (0, \infty) \times (0, L) \times (0, 1), \end{cases} \quad (3.12)$$

with boundary conditions

$$\begin{cases} \psi = \psi_x = \eta_x = 0, & \text{in } (0, \infty) \times \{0, L\}, \\ z_x = 0, & \text{in } (0, \infty) \times \{0, L\} \times (0, 1), \end{cases} \quad (3.13)$$

and initial conditions

$$\begin{cases} (\psi(0, x), \psi_t(0, x), \eta(0, x), \eta_t(0, x)) = (\psi_0(x), \psi_1(x), \eta_0(x), \eta_1(x)), & \text{in } (0, L), \\ z(0, x, p) = z_0(x, p) = f_0(x, -p\tau), & \text{in } (0, L) \times (0, 1). \end{cases} \quad (3.14)$$

### Assumptions

To state and prove our result, we use the following assumptions:

**A<sub>1</sub>)** The kernel is such that  $g \in L^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+)$  with  $g(0) > 0$  and

$$l = 1 - \int_0^\infty g(s)ds > 0, \quad (3.15)$$

**A<sub>2</sub>)** There exists a  $C^1$  function  $H : (0, \infty) \rightarrow (0, \infty)$  which is linear or increasing and strictly convex  $C^2$ -function on  $[0, r)$  ( $r \leq g(0)$ ) with  $H(0) = H'(0) = 0$ , such that

$$g'(t) \leq -\zeta(t)H(g(t)), \quad \forall t \geq 0, \quad (3.16)$$

where  $\zeta$  is a positive non-increasing differentiable function.

Now, we prepare some notations and hypotheses which will be needed in the proof of our result. Let  $L^2(0, L)$  be the usual Hilbert space with the inner product  $\langle \cdot, \cdot \rangle$  and the inner product induced norm  $\|\cdot\|$ . Throughout this paper, we define

$$H_0^2(0, L) = \{v \in H^2(\Omega), v(0) = v(L) = v_x(0) = v_x(L) = 0\},$$

equipped with the norm  $\|w\|_{H_0^2(0,L)} = \|w_{xx}\|$ , and the space

$$H_*^1(0, L) = \left\{v \in H^1(0, L), \int_0^L v(x)dx = 0\right\},$$

equipped with the norm  $\|w\|_{H_*^1(0,L)} = \|w_x\|$ .

$C$  and  $c$  denote some general positive constants, which may be different in different estimates.

The following inequality will be used repeatedly in the sequel.

**Lemma 31** *We have the following Young's inequality*

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2, \quad a, b \in \mathbb{R}, \quad \delta > 0. \quad (3.17)$$

Also, we define the Hilbert space  $L^2((0, L) \times (0, 1))$  which is endowed with the inner product

$$\langle\langle z, \tilde{z} \rangle\rangle = \int_0^L \int_0^1 z(x, p)\tilde{z}(x, p)dpdx.$$

Note that the norms

$$\|z\|^2 = \int_0^L \int_0^1 z^2(x, p)dpdx,$$

and

$$\|z\|_*^2 = \int_0^L \int_0^1 e^{-2\tau p} z^2(x, p)dpdx,$$

are equivalent in  $L^2((0, L) \times (0, 1))$ . In the sequel, we use the notation  $\|\cdot\|$  to denote both of norms. We shall use the second one in the study of the asymptotic behavior since it is more flexible.

**Lemma 32** *Assume that  $(\psi, \psi_t, \eta, \eta_t, z)$  is a strong solution of the problem (3.12)-(3.14). Then we have*

$$\begin{aligned} \xi \frac{d}{dt} \int_0^L \int_0^1 z^2(t, x, p)dpdx &= -\frac{\xi}{\tau} \int_0^L \int_0^1 \frac{\partial}{\partial p} z^2(t, x, p)dpdx \\ &= \frac{\xi}{\tau} \int_0^L z^2(t, x, 0) - z^2(t, x, 1)dx. \end{aligned} \quad (3.18)$$

for any  $\xi \in \mathbb{R}$ .

**Proof.** We multiply the third equation in (3.12) by  $\xi z$  and integrate the result over  $(0, L) \times (0, 1)$  with respect to  $p$  and  $x$ , respectively, to get the desired result. ■

Throughout this paper, we denote by  $\circ$  and  $\diamond$  the binary operators defined by

$$(g \circ u)(t) = \int_0^t g(t-s) \|u(t) - u(s)\|^2 ds,$$

and

$$(g \diamond u)(t) = \int_0^t g(t-s) (u(s) - u(t)) ds, \quad (3.19)$$

where  $u \in C([0, T]; L^2(0, L))$ .

We define the energy associated with the solution of System (3.12)-(3.14) by

$$\begin{aligned} \mathcal{E}(t) = & \frac{1}{2} \left\{ \rho h \|\psi_t(t)\|^2 + \frac{\rho h^3}{12} \|\psi_{tx}(t)\|^2 + \frac{1}{2} \left( 1 - \int_0^t g(s) ds \right) \|\psi_{xx}(t)\|^2 \right. \\ & \left. + \rho h \|\eta_t(t)\|^2 + \left\| \eta_x(t) + \frac{1}{2}(\psi_x)^2(t) \right\|^2 + (g \circ \psi_{xx})(t) + \xi \|z(t)\|^2 \right\}, \end{aligned} \quad (3.20)$$

where  $\xi$  is a positive constant such that

$$\alpha_2 \tau < \xi < (2\alpha_1 - \alpha_2)\tau, \quad (3.21)$$

with  $\alpha_1$  and  $\alpha_1$  satisfying

$$\alpha_2 < \alpha_1. \quad (3.22)$$

The next lemma gives an identity for the convolution product.

**Lemma 33** ([10, Lemma 2.1]) *For real functions  $g, \varphi \in C^1(\mathbb{R}^+)$ , we have*

$$\begin{aligned} 2(g * \varphi)(t) \varphi_t(t) = & -g(t) |\varphi(t)|^2 + (g' \circ \varphi)(t) \\ & - \frac{d}{dt} \left( g \circ \varphi - \left( \int_0^t g(s) ds \right) |\varphi(t)|^2 \right). \end{aligned}$$

**Lemma 34** *Assume that  $(\psi, \psi_t, \eta, \eta_t, z)$  is a strong solution of the problem (3.12)-(3.14). Then the derivative of  $\mathcal{E}(\cdot)$  satisfies*

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} = & \left( \frac{\xi}{2\tau} - \alpha_1 \right) \|\eta_t(t)\|^2 - \alpha_2 \int_0^L \eta_t(t, x) z(t, x, 1) dx - \frac{\xi}{2\tau} \|z(t, 1)\|^2 \\ & - \frac{1}{2} g(t) \|\psi_{xx}(t)\|^2 + \frac{1}{2} (g' \circ \psi_{xx})(t). \end{aligned} \quad (3.23)$$

Moreover, for all  $t \geq 0$ , we have

$$\begin{aligned} \frac{d\mathcal{E}(t)}{dt} \leq & \left( \frac{\xi}{2\tau} + \frac{\alpha_2}{2} - \alpha_1 \right) \|\eta_t(t)\|^2 + \left( \frac{\alpha_2}{2} - \frac{\xi}{2\tau} \right) \|z(t, 1)\|^2 \\ & + \frac{1}{2} (g' \circ \psi_{xx})(t) \\ \leq & 0. \end{aligned} \quad (3.24)$$

**Proof.** Multiplying the first equation in (3.12) by  $\psi_t$ , the second and the fourth equations by  $\eta_t$  and  $\xi z$ , respectively, taking into account (3.18), (3.21) and the boundary conditions (3.13), we obtain the identity (3.23) after integration over  $(0, L)$ . Making use of Young's inequality, then (3.24) follows from (3.21). ■

**Lemma 35 (Jensen inequality)** *Let  $F$  be a convex function on  $[a, b]$ ,  $f : \Omega \rightarrow [a, b]$  and  $h$  are integrable functions on  $\Omega$ ,  $h(x) \geq 0$ , and  $\int_{\Omega} h(x) dx = k > 0$ , then Jensen's inequality states that*

$$F \left[ \frac{1}{k} \int_{\Omega} f(x) h(x) dx \right] \leq \frac{1}{k} \int_{\Omega} F[f(x)] h(x) dx. \quad (3.25)$$

## 3.2 Global well-posedness

In this section we show the existence and the regularity of solutions of the one dimensional viscoelastic von Kàrmàn System (3.12)-(3.14).

The existence and uniqueness result of problem (3.12)-(3.14) is stated as follows.

**Theorem 36 (Well-posedness)** *Assume that the initial datum satisfy*

$$\begin{aligned}(\psi_0, \psi_1) &\in [H_0^2(I) \cap H^3(0, L)] \times H_0^2(0, L), \\(\eta_0, \eta_1) &\in [H^2(0, L) \cap H_*^1(0, L)] \times H_*^1(0, L), \\f_0 &\in H^1(0, 1; H_*^1(0, L)),\end{aligned}$$

with the compatibility condition  $f(\cdot, 0) = \eta_1$ .

Moreover, assume that the Hypotheses  $(\mathbf{A}_1)$ ,  $(\mathbf{A}_2)$  hold. Then problem (3.12)-(3.14) admits a unique weak solution

$$\begin{aligned}(\psi, \psi_t, \psi_{tt}) &\in L_{loc}^\infty\left(0, \infty; [H_0^2(0, L)]^2 \times H_0^1(0, L)\right), \\(\eta, \eta_t, \eta_{tt}) &\in L_{loc}^\infty\left(-\tau, \infty; [H_*^1(0, L)]^2 \times L^2(0, L)\right).\end{aligned}$$

**Proof.** Let  $T > 0$  be fixed and denote by  $\mathcal{V}^m$  and  $\mathcal{W}^m$  and  $\mathcal{Z}^m$  be the spaces generated by  $\{e_i\}_{1 \leq i \leq m}$ ,  $\{\sigma_i\}_{1 \leq i \leq m}$  and  $\{\phi_i\}_{1 \leq i \leq m}$ , where the families  $\{e_k\}_{k \geq 1}$ ,  $\{\sigma_k\}_{k \geq 1}$  and  $\{\phi_k\}_{k \geq 1}$  are orthogonal basis for the spaces  $H_0^2(0, L)$ ,  $H_*^1(0, L)$  and  $H_*^1(0, L; H^1(0, 1))$  respectively and satisfying

$$\|e_k\| = \|\sigma_k\| = \|\phi_k\| = 1, \text{ for any } k \geq 1.$$

Our starting point is to construct the Galerkin approximation  $(\psi^m, \eta^m, z^m)$  of the solution

$$\psi^m(t, x) = \sum_{i=1}^m u_i^m(t) e_i(x), \quad \eta^m(t, x) = \sum_{i=1}^m v_i^m(t) \sigma_i(x), \quad z^m(t, x, p) = \sum_{i=1}^m z_i^m(t) \phi_i(x, p),$$

where  $u_i^m, v_i^m$ , and  $w_i^m$ ,  $i = 1, 2, \dots, m$ , are determined by the following ordinary integro-differential equations:

$$\begin{cases} \rho h \langle \psi_{tt}^m, e_i \rangle + \frac{\rho h^3}{12} \langle \psi_{ttx}^m, e_{ix} \rangle + \langle \psi_{xx}^m, e_{ixx} \rangle + \left\langle \psi_x^m \left( \eta_x^m + \frac{1}{2} (\psi_x^m)^2 \right), e_{ix} \right\rangle - \langle g * \psi_{xx}^m, e_{ixx} \rangle = 0 \\ \rho h \langle \eta_{tt}^m, \sigma_i \rangle + \left\langle \eta_x^m + \frac{1}{2} (\psi_x^m)^2, \sigma_{ix} \right\rangle + \alpha_1 \langle \eta_t^m, \sigma_i \rangle + \alpha_2 \langle z(1), \sigma_i \rangle = 0, \\ \tau \langle \langle z_t^m, \phi_i \rangle \rangle + \langle \langle z_p^m, \phi_i \rangle \rangle = 0, \end{cases} \quad (3.26)$$

with initial conditions

$$\begin{aligned}\psi^m(0, x) &= \sum_{i=1}^m u_i^m(0) e_i(x), & \psi_t^m(0, x) &= \sum_{i=1}^m (u_i^m)'(0) e_i(x), \\ \eta^m(0, x) &= \sum_{i=1}^m v_i^m(0) \sigma_i(x), & \eta_t^m(0, x) &= \sum_{i=1}^m (v_i^m)'(0) \sigma_i(x), \\ z^m(0, x, p) &= \sum_{i=1}^m z_i^m(0) \phi_i(x, p).\end{aligned} \quad (3.27)$$

The above sequences are chosen so that

$$\begin{aligned}\psi^m(0, x) &\rightarrow \psi_0(0, x), & \text{strongly in } & H^3(0, L) \cap H_0^2(0, L), \\ \psi_t^m(0, x) &\rightarrow \psi_1(0, x), & \text{strongly in } & H_0^2(0, L), \\ \eta^m(0, x) &\rightarrow \eta_0(0, x), & \text{strongly in } & H^2(0, L) \cap H_*^1(0, L), \\ \eta_t^m(0, x) &\rightarrow \eta_1(0, x), & \text{strongly in } & H_*^1(0, L), \\ z^m(0, x, p) &\rightarrow z_0(0, x, p), & \text{strongly in } & H_*^1(0, L; H^1(0, 1)).\end{aligned} \quad (3.28)$$

By virtue of the theory of ordinary differential equations, the System (3.26)-(3.27) has a unique local solution which is extended to a maximal interval  $[0, T_m[$  (with  $0 < T_m \leq \infty$ ). In the next step, we obtain a priori estimates for the solution such that it can be extended beyond  $[0, T_m[$  to obtain a single solution defined for all  $t > 0$ .

In order to use a standard compactness argument for the limiting procedure, it suffices to derive some a priori estimates for  $u_i^m(t)$ ,  $v_i^m(t)$ .

**The first estimate.** At standard calculations, using (3.26)-(3.27), similar to those used to derive (3.24), yields

$$\frac{d}{dt} \mathcal{E}_m(t) \leq 0, \quad \forall t \geq 0, \quad (3.29)$$

where

$$\begin{aligned} \mathcal{E}_m(t) = & \frac{1}{2} \left\{ \rho h \|\psi_t^m(t)\|^2 + \frac{\rho h^3}{12} \|\psi_{xt}^m(t)\|^2 + \left(1 - \int_0^t g(s) ds\right) \|\psi_{xx}^m(t)\|^2 + \rho h \|\eta_t^m(t)\|^2 \right. \\ & \left. + \left\| \eta_x^m(t) + \frac{1}{2}(\psi_x^m)^2(t) \right\|^2 + g \circ \psi_{xx}^m(t) + \xi \|z^m(t)\|^2 \right\}. \end{aligned} \quad (3.30)$$

Integrating (3.29) over  $(0, t)$  yields

$$\mathcal{E}_m(t) \leq \mathcal{E}_m(0) \leq C, \quad \forall t > 0. \quad (3.31)$$

for some positive constant  $C$  independent of  $m \in \mathbb{N}$ .

Also, to get an a priori estimate for  $\eta$ , we make use of the embedding  $H^1(0, L) \hookrightarrow L^\infty(0, L)$  and the boundedness of  $\mathcal{E}_m(\cdot)$  to obtain

$$\begin{aligned} \|\eta_x^m(t)\|^2 & \leq \left\| \eta_x^m(t) + \frac{1}{2}(\psi_x^m)^2(t) \right\|^2 + \frac{1}{4} \|(\psi_x^m)^2(t)\|^2 \\ & \leq \left\| \eta_x^m(t) + \frac{1}{2}(\psi_x^m)^2(t) \right\|^2 + \frac{1}{4} \|(\psi_x^m)^2(t)\|_{L^\infty(0,L)} \|(\psi_x^m)^2(t)\|^2 \\ & \leq \left\| \eta_x^m(t) + \frac{1}{2}(\psi_x^m)^2(t) \right\|^2 + \frac{L^2}{4} \|(\psi_x^m)^2(t)\|_{L^\infty(0,L)}^2 \|(\psi_{xx}^m)^2(t)\|^2 \\ & \leq \left\| \eta_x^m(t) + \frac{1}{2}(\psi_x^m)^2(t) \right\|^2 + c \|(\psi_{xx}^m)^2(t)\|^2 \|(\psi_{xx}^m)^2(t)\|^2 \\ & \leq c (\mathcal{E}_m(0) + \mathcal{E}_m^2(0)) \leq C. \end{aligned} \quad (3.32)$$

where  $C$  is a positive constant independent of  $m \in \mathbb{N}$ .

These estimates imply that the solution  $(\psi^m, \eta^m, z^m)$  of System (3.26)-(3.27) exists globally in  $[0, T[$ .

Estimates (3.31)-(3.33) yield

$$\begin{array}{ll} \psi^m & \text{is bounded in } L^\infty(0, T, H_0^2(0, L)), \\ \psi_t^m & \text{is bounded in } L^\infty(0, T, H_0^1(0, L)), \\ \eta^m & \text{is bounded in } L^\infty(0, T, H_*^1(0, L)), \\ \eta_t^m & \text{is bounded in } L^\infty(0, T, L^2(0, L)), \\ \eta_x^m + \frac{1}{2}(\psi_x^m)^2 & \text{is bounded in } L^\infty(0, T, L^2(0, L)), \\ z^m & \text{is bounded in } L^\infty(0, T, L^2((0, L) \times (0, 1))), \end{array} \quad (3.33)$$

for any  $T > 0$ .

**The second estimate.** We have to estimate  $\psi_{tt}^m(0)$ ,  $\psi_{ttx}^m(0)$  and  $\eta_{tt}^m(0)$  in  $L^2$  norm.



Considering  $t = 0$  in the first equation of (3.26), then multiplying it by  $(u_i^m)''(0)$  and summing up over  $i$  from 1 to  $m$ , it follows that

$$\begin{aligned} 0 &= \rho h \|\psi_{tt}^m(0)\|^2 + \frac{\rho h^3}{12} \|\psi_{ttx}^m(0)\|^2 + \langle \psi_{0xx}^m, \psi_{ttxx}^m(0) \rangle \\ &\quad + \left\langle \psi_{0x}^m \left( \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2 \right), \psi_{ttx}^m(0) \right\rangle. \end{aligned} \quad (3.34)$$

Integrating by parts and using Young's inequality, we get

$$\begin{aligned} \langle \psi_{0xx}^m, \psi_{ttxx}^m(0) \rangle &= - \langle \psi_{0xxx}^m, \psi_{ttx}^m(0) \rangle, \\ &\leq C_\delta \|\psi_{0xxx}^m\|^2 + \delta \|\psi_{ttx}^m(0)\|^2. \end{aligned} \quad (3.35)$$

Similarly

$$\left\langle \psi_{0x}^m \left( \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2 \right), \psi_{ttx}^m(0) \right\rangle \leq \delta \|\psi_{ttx}^m(0)\|^2 + C_\delta \left\| \psi_{0x}^m \left( \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2 \right) \right\|^2. \quad (3.36)$$

Using the embedding  $H^1(0, L) \hookrightarrow L^\infty(0, L)$ , we estimate the second term of the right hand side of (3.36) as follows

$$\begin{aligned} \left\| \psi_{0x}^m \left( \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2 \right) \right\|^2 &\leq \|\psi_{0x}^m\|_{L^\infty(0, L)}^2 \left\| \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2 \right\|^2 \\ &\leq c \|\psi_{0xx}^m\|^2 \left\| \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2 \right\|^2 \\ &\leq c \mathcal{E}_m^2(0) \\ &\leq C. \end{aligned} \quad (3.37)$$

After choosing a suitable  $\delta$ , we infer from (3.27)-(3.28) and (3.34) that there exists a positive constant  $C$  independent of  $m$  such that

$$\|\psi_{tt}^m(0)\|^2 + \|\psi_{ttx}^m(0)\|^2 \leq C. \quad (3.38)$$

Next, multiplying the second equation of (3.26) by  $(v_i^m)''(0)$ , choosing  $t = 0$  and summing up over  $i$  from 1 to  $m$ , we get

$$\rho h \|\eta_{tt}^m(0)\|^2 + \left\langle \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2, \eta_{ttx}^m(0) \right\rangle + \alpha_1 \langle \eta_1^m, \eta_{tt}^m(0) \rangle + \alpha_2 \langle z_0^m(1), \eta_{tt}^m(0) \rangle = 0. \quad (3.39)$$

Applying Young's and Poincaré's inequalities, using the embedding  $H^1(0, L) \hookrightarrow L^\infty(0, L)$  and the fact that  $\mathcal{E}_m(\cdot)$  is non-increasing, we conclude that

$$\begin{aligned} \left\langle \eta_{0x}^m + \frac{1}{2} (\psi_{0x}^m)^2, \eta_{ttx}^m(0) \right\rangle &= - \langle \eta_{0xx}^m + \psi_{0x}^m \psi_{0xx}^m, \eta_{tt}^m(0) \rangle \\ &\leq \delta \|\eta_{tt}^m(0)\|^2 + C_\delta \|\eta_{0xx}^m + \psi_{0x}^m \psi_{0xx}^m\|^2 \\ &\leq \delta \|\eta_{tt}^m(0)\|^2 + C_\delta \|\eta_{0xx}^m\|^2 + C_\delta \|\psi_{0x}^m\|_{L^\infty(I)} \|\psi_{0xx}^m\|^2 \\ &\leq \delta \|\eta_{tt}^m(0)\|^2 + c \mathcal{E}_m^2(0) \\ &\leq \delta \|\eta_{tt}^m(0)\|^2 + C. \end{aligned} \quad (3.40)$$

Then we use Young's inequality to obtain, for any  $\delta > 0$ ,

$$\langle \eta_{1t}^m, \eta_{tt}^m(0) \rangle \leq \delta \|\eta_{tt}^m(0)\|^2 + C_\delta \|\eta_1^m\|^2, \quad (3.41)$$

and

$$\langle z_0^m(1), \eta_{tt}^m(0) \rangle \leq \delta \|\eta_{tt}^m(0)\|^2 + C_\delta \|z_0^m(1)\|^2. \quad (3.42)$$

Hence, from (3.27)-(3.28) and with a suitable choice of  $\delta$ , there exists a positive constant  $C$  independent of  $m$  such that

$$\|\eta_{tt}^m(0)\| \leq C. \quad (3.43)$$

Next, differentiating the first equation of (3.26) with respect to  $t$ , and multiplying the result by  $(u_i^m)''(t)$ , adding from  $i = 1$  to  $m$ , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \rho h \|\psi_{tt}^m(t)\|^2 + \frac{\rho h^3}{12} \|\psi_{ttx}^m(t)\|^2 \right\} + g(0) \|\psi_{txx}^m(t)\|^2 \\ & + \left\langle \frac{d}{dt} \left( \psi_x^m(\eta_x^m + \frac{1}{2}(\psi_x^m)^2)(t) \right), \psi_{ttx}^m(t) \right\rangle \\ & = \frac{d}{dt} \langle g' * \psi_{xx}^m(t), \psi_{ttx}^m(t) \rangle - \langle g'' * \psi_{xx}^m(t), \psi_{ttx}^m(t) \rangle - g'(0) \langle \psi_{xx}^m(t), \psi_{ttx}^m(t) \rangle \\ & + g(0) \frac{d}{dt} \langle \psi_{xx}^m(t), \psi_{ttx}^m(t) \rangle. \end{aligned} \quad (3.44)$$

We have

$$\begin{aligned} \left\langle \frac{d}{dt} \left( \psi_x^m(\eta_x^m + \frac{1}{2}(\psi_x^m)^2)(t) \right), \psi_{ttx}^m(t) \right\rangle & = \left\langle \psi_{tx}^m(\eta_x^m + \frac{1}{2}(\psi_x^m)^2)(t), \psi_{ttx}^m(t) \right\rangle \\ & + \langle \psi_x^m(\eta_{tx}^m + \psi_{xt}^m \psi_x^m)(t), \psi_{ttx}^m(t) \rangle. \end{aligned} \quad (3.45)$$

Making use of Young's inequality, the embedding  $H^1(0, L) \hookrightarrow L^\infty(0, L)$  and (3.33), we conclude that

$$\begin{aligned} \left\langle \psi_{tx}^m(\eta_x^m + \frac{1}{2}(\psi_x^m)^2)(t), \psi_{ttx}^m(t) \right\rangle & \leq C_\delta \left\| \psi_{tx}^m(\eta_x^m + \frac{1}{2}(\psi_x^m)^2)(t) \right\|^2 + \delta \|\psi_{ttx}^m(t)\|^2, \\ & \leq C + \delta \|\psi_{ttx}^m(t)\|^2. \end{aligned} \quad (3.46)$$

Making use of the embedding  $H^1(0, L) \hookrightarrow L^\infty(0, L)$ , we get

$$\begin{aligned} \langle \psi_x^m(\eta_{tx}^m + \psi_{xt}^m \psi_x^m)(t), \psi_{ttx}^m(t) \rangle & = \langle \psi_x^m(t) \eta_{tx}^m(t), \psi_{ttx}^m(t) \rangle + \langle \psi_{xt}^m(t) (\psi_x^m(t))^2, \psi_{ttx}^m(t) \rangle \\ & \leq \delta \|\psi_x^m(t)\|_{L^\infty(0, L)}^2 \|\eta_{tx}^m(t)\|^2 + C_\delta \|\psi_{ttx}^m(t)\|^2 \\ & \quad + \delta \|\psi_{xt}^m(t)\|_{L^\infty(0, L)} \left\| (\psi_x^m)^2(t) \right\|^2 \\ & \leq L^2 \delta \|\psi_{xx}^m(t)\|^2 \|\eta_{tx}^m(t)\|^2 + C_\delta \|\psi_{ttx}^m(t)\|^2 \\ & \quad + \delta c \mathcal{E}_m(0) \|\psi_{xx}^m(t)\|^2 \|\psi_{xxt}^m(t)\|^2 \\ & \leq \delta C \|\psi_{ttx}^m(t)\|^2 + C \|\eta_{tx}^m(t)\|^2 + C \|\psi_{ttx}^m(t)\|^2. \end{aligned} \quad (3.47)$$

Since  $g'$  and  $g''$  are continuous functions on  $[0, T]$  then  $m_1 = \sup_{t \in [0, T]} |g'|$  and  $m_2 = \sup_{t \in [0, T]} |g''|$  exist for all  $T < \infty$ . Using Cauchy-Shwarz and Young's inequalities produce the estimates

$$\langle g' * \psi_{xx}^m, \psi_{ttx}^m \rangle \leq m_1^2 C_\delta \int_0^t \|\psi_{xx}^m(s)\|^2 ds + \delta \|\psi_{ttx}^m\|^2, \quad (3.48)$$

and

$$\langle g'' * \psi_{xx}^m, \psi_{ttx}^m \rangle \leq m_2^2 C_\delta \int_0^t \|\psi_{xx}^m(s)\|^2 ds + \delta \|\psi_{ttx}^m\|^2. \quad (3.49)$$

Employing Young's inequality, combining (3.45)-(3.49), then integrating (3.44) over  $(0, t)$ , we obtain

$$\begin{aligned}
& \rho h \|\psi_{tt}^m(t)\|^2 + \frac{\rho h^3}{12} \|\psi_{ttx}^m(t)\|^2 + (g(0) - C\delta) \|\psi_{txx}^m(t)\| \\
\leq & C_\delta \int_0^t \|\psi_{txx}^m(s)\|^2 + C_\delta \int_0^t \|\psi_{ttx}^m(s)\|^2 ds \\
& C_\delta \int_0^t \|\eta_{tx}^m(s)\|^2 ds + C_\delta \int_0^t \int_0^s \|\psi_{xx}^m(\chi)\|^2 d\chi ds + \rho h \|\psi_{tt}^m(0)\|^2 \\
& + \frac{\rho h^3}{12} \|\psi_{ttx}^m(0)\|^2 + C,
\end{aligned} \tag{3.50}$$

where  $C$  is a positive constant independent of  $m$  but depends on  $T$  and the initial data. The term  $\int_0^t \int_0^s \|\psi_{xx}^m(\chi)\|^2 d\chi ds$  can be estimated as follows

$$\int_0^t \int_0^s \|\psi_{xx}^m(\chi)\|^2 d\chi ds \leq c\mathcal{E}(0) \int_0^t \int_0^s d\chi ds \leq \frac{CT^2}{2}. \tag{3.51}$$

Next, Multiplying the second equation of (3.26) by  $(v_i^m)''$  and summing up over  $i$  from 1 to  $m$ , we arrive at

$$\frac{1}{2} \frac{d}{dt} \left\{ \rho h \|\eta_{tt}^m(t)\|^2 + \|\eta_{xt}^m(t)\|^2 \right\} + \langle \psi_{xt}^m(t) \psi_x^m(t), \eta_{ttx}^m(t) \rangle + \alpha_1 \|\eta_{tt}^m(t)\|^2 + \alpha_2 \langle z_t(t, 1), \eta_{tt}^m(t) \rangle = 0. \tag{3.52}$$

Now, applying Young's and Poincaré's inequalities, we get

$$\begin{aligned}
\langle \psi_{xt}^m(t) \psi_x^m(t), \eta_{ttx}^m(t) \rangle &= -\langle \psi_{xxt}^m(t) \psi_x^m(t), \eta_{tt}^m(t) \rangle - \langle \psi_{xt}^m(t) \psi_{xx}^m(t), \eta_{tt}^m(t) \rangle \\
&\leq 2\delta \|\eta_{tt}^m(t)\|^2 + C_\delta \|\psi_x^m(t)\|_{L^\infty(I)} \|\psi_{xxt}^m(t)\|^2 \\
&\leq 2\delta \|\eta_{tt}^m(t)\|^2 + C_\delta \|\psi_{xxt}^m(t)\|^2 \|\psi_{xx}^m(t)\|^2 \\
&\leq 2\delta \|\eta_{tt}^m(t)\|^2 + C_\delta \|\psi_{xxt}^m(t)\|^2.
\end{aligned} \tag{3.53}$$

A differentiation with respect to  $t$  of the third equation of (3.26) implies

$$\langle \langle z_{tt}^m, \phi_i \rangle \rangle + \frac{1}{\tau} \langle \langle z_{pt}^m, \phi_i \rangle \rangle = 0. \tag{3.54}$$

Multiplying (3.54) by  $(z_i^m)'(t)$ , integrating by parts and adding from  $i = 1$  to  $m$ , we obtain

$$\tau \frac{d}{dt} \|\|z_t^m(t)\|\|^2 + \|z_t^m(t, 1)\|^2 - \|z_t^m(t, 0)\|^2 = 0. \tag{3.55}$$

Taking the sum of (3.52) and (3.55) and integrating over  $(0, t)$ , we obtain

$$\begin{aligned}
& \frac{\rho h}{2} \|\eta_{tt}^m(t)\|^2 + \frac{1}{2} \|\eta_{xt}^m(t)\|^2 + \tau \|\|z_t^m(t)\|\|^2 + \alpha_1 \int_0^t \|\eta_{tt}^m(s)\|^2 ds + \frac{1}{2} \int_0^t \|z_t^m(s, 1)\|^2 ds \\
\leq & \frac{1}{2} \|\eta_{tt}^m(0)\|^2 + \frac{1}{2} \|\eta_{1x}^m\|^2 + \tau \int_0^1 \|z_t^m(0, p)\|^2 dp + C_\delta \int_0^t \|\eta_{tt}^m(s)\|^2 ds \\
& + C_\delta \int_0^t \|\psi_{xxt}^m(s)\|^2 ds + c\delta \int_0^t \|z_t(s, 1)\|^2 ds.
\end{aligned} \tag{3.56}$$

Combining (3.50) and (3.56) with a suitable choice of  $\delta$ , then using Gronwall's lemma, we arrive at

$$\|\psi_{tt}^m(t)\|^2 + \|\psi_{ttx}^m(t)\|^2 + \|\psi_{txx}^m(t)\| + \|\eta_{tt}^m(t)\|^2 + \|\eta_{xt}^m(t)\|^2 + \|\|z_t^m(t)\|\|^2 \leq C, \tag{3.57}$$

where  $C$  is independent of  $m \in \mathbb{N}$ .

Replacing  $\phi_i$  by  $-\phi_{ixx}$  in the third equation of (3.26), multiplying the resulting equation by  $z_i^m(t)$ , summing over  $i$  from 1 to  $m$ , leads to

$$\tau \frac{d}{dt} \|z_x^m(t)\|^2 + \|z_x^m(t, 1)\|^2 - \|z_x^m(t, 0)\|^2 = 0.$$

A integration over  $(0, t)$  yields

$$\tau \|z_x^m(t)\|^2 + \int_0^t \|z_x^m(s, 1)\|^2 ds = \int_0^t \|\eta_{xt}^m(s)\|^2 ds + \tau \int_0^1 \|f_{0x}^m(p)\|^2 dp \leq C, \quad (3.58)$$

where  $C$  independent of  $m$ .

From (3.31), (3.33), (3.57) and (3.58), we infer that

$$\begin{aligned} \psi^m & \text{ is bounded in } L^\infty(0, T, H_0^2(0, L)), \\ \psi_t^m & \text{ is bounded in } L^\infty(0, T, H_0^2(0, L)), \\ \eta^m & \text{ is bounded in } L^\infty(0, T, H_*^1(0, L)), \\ \eta_t^m & \text{ is bounded in } L^\infty(0, T, H_*^1(0, L)), \\ z^m & \text{ is bounded in } L^\infty(0, T; H_*^1(0, L; L^2((0, 1))). \end{aligned} \quad (3.59)$$

Now, using (3.33) and (3.59), we deduce that

$$\psi^m \rightarrow \psi \text{ weak-star in } W^{1,\infty}(0, T; H_0^2(0, L)) \cap W^{2,\infty}(0, T; H_0^1(0, L)), \quad (3.60)$$

$$\eta^m \rightarrow \eta \text{ weak-star in } W^{1,\infty}(0, T; H_*^1(0, L)) \cap W^{2,\infty}(0, T; L^2(0, L)), \quad (3.61)$$

$$\eta_x^m + \frac{1}{2}(\psi_x^m)^2 \rightarrow f \text{ weak-star in } L^\infty(0, T; L^2(0, L)), \quad (3.62)$$

$$z^m \rightarrow z \text{ weak-star in } L^\infty(0, T; H_*^1((0, L); L^2(0, L)) \cap W^{1,\infty}(0, T; L^2((0, L) \times (0, 1))), \quad (3.63)$$

as  $m \rightarrow \infty$ , for a suitable function  $f \in L^\infty(0, T; L^2(0, L))$ .

According to (3.59),  $\psi^m$  is uniformly bounded in  $W^{1,\infty}(0, T; H_0^2(I))$ , in particular, it belongs to  $W^{1,2}(0, T; H_0^2(I))$ . In this way, we can extract a subsequence  $\psi^m$  such that

$$\psi^m \rightarrow \psi \text{ strongly in } L^2(0, T; H_0^1(0, L)) \text{ as } m \rightarrow \infty, \quad (3.64)$$

Therefore,

$$(\psi_x^m)^2 \rightarrow \psi_x^2 \text{ almost everywhere in } (0, \infty) \times (0, L) \text{ as } m \rightarrow \infty, \quad (3.65)$$

Similarly, by using the compact embedding  $H^1(0, L) \hookrightarrow L^2(0, L)$ , we infer that

$$\eta^m \rightarrow \eta \text{ strongly in } L^2((0, \infty) \times (0, L)), \quad (3.66)$$

which implies

$$\eta^m \rightarrow \eta \text{ almost everywhere in } (0, \infty) \times (0, L) \text{ as } m \rightarrow \infty.$$

Combining (3.60)-(3.63) and (3.65), it follows that  $f = \eta_x + \frac{1}{2}\psi_x^2$ , and

$$\psi_x^m(\eta_x^m + \frac{1}{2}(\psi_x^m)^2) \rightarrow \psi_x(\eta_x + \frac{1}{2}\psi_x^2) \text{ weakly in } L^2((0, \infty) \times (0, L)).$$

Similarly, we obtain

$$z^m \rightarrow z \text{ weakly in } L^2(0, T; L^2((0, L) \times (0, 1))), \text{ as } m \rightarrow \infty \quad (3.67)$$

The convergence (3.64)-(3.67) allows us to pass to the limit in (3.26)-(3.27). Thus, the problem (3.12)-(3.14) admits a global weak solution  $(\psi, \eta, z)$ .

Uniqueness can be proved by the straightforward methods and Gronwall's inequality.  $\blacksquare$

### 3.3 General decay

In this section we consider a wider class of kernel functions, and we establish a general decay result, which contains the usual exponential and polynomial decay rates as special cases. The main result of general decay is the following.

**Theorem 37** *Assume that  $(\mathbf{A}_1)$ ,  $(\mathbf{A}_2)$  holds. Then for any solution of (3.12)-(3.14), there exist two positive constants  $k_1 \leq 1$  and  $k_2$  such that the energy functional satisfies*

$$\mathcal{E}(t) \leq k_2 H_1^{-1} \left( k_1 \int_{g^{-1}(r)}^t \xi(s) ds \right), \quad \forall t \geq 0, \quad (3.68)$$

where

$$H_1(t) = \int_t^r \frac{1}{sH'(s)} ds,$$

and,  $H_1$  is decreasing and convex function on  $(0, r]$ , with  $\lim_{t \rightarrow 0} H_1(t) = +\infty$  when  $t \rightarrow 0$ .

To prove Theorem 37, we first proceed to prepare a series of useful lemmas.

**Lemma 38** *The following inequalities hold*

$$(g \diamond \psi)^2(t) \leq \int_0^t g(t-s) (\psi(t) - \psi(s))^2 ds, \quad (3.69)$$

$$(g' \diamond \psi)^2(t) \leq -c \int_0^t g'(t-s) (\psi(t) - \psi(s))^2 ds, \quad (3.70)$$

where  $g \diamond \psi$  is given by (3.19).

**Proof.** For inequality (3.69), we have

$$\begin{aligned} (g \diamond \psi)^2(t) &= \left( \int_0^t g(t-s) (\psi(t) - \psi(s)) ds \right)^2 \\ &= \left( \int_0^t \sqrt{g(t-s)} \sqrt{g(t-s)} (\psi(t) - \psi(s)) ds \right)^2. \end{aligned}$$

Cauchy-Schwarz inequality leads to

$$\begin{aligned} (g \diamond \psi)^2(t) &\leq \int_0^t g(s) ds \int_0^t g(t-s) (\psi(t) - \psi(s))^2 ds \\ &\leq \int_0^t g(t-s) (\psi(t) - \psi(s))^2 ds. \end{aligned}$$

Similarly, we prove (3.70) by replacing  $\sqrt{g(t-s)}$  by  $\sqrt{-g'(t-s)}$ . ■

Let  $\mathcal{F}$  be the functional defined by

$$\mathcal{F}(t) = \frac{N_1}{2} \mathcal{I}(t) + N_1 \mathcal{J}(t) + N_2 \mathcal{K}(t) + \mathcal{L}(t), \quad \forall t \geq 0, \quad (3.71)$$

where

$$\mathcal{I}(t) = \langle \rho h D \psi_t(t), \psi(t) \rangle, \quad (3.72)$$

$$\mathcal{J}(t) = \langle \rho h \eta_t(t), \eta(t) \rangle, \quad (3.73)$$

$$\mathcal{K}(t) = -\langle \rho h D \psi_t(t), g \diamond \psi(t) \rangle, \quad (3.74)$$

$$\mathcal{L}(t) = \int_0^L \int_0^1 e^{-2\tau p} z^2(t, x, p) dp dx, \quad (3.75)$$

and  $N_1$  and  $N_2$  are positive constants that will be chosen later.

Let  $\lambda > 0$  and define  $\mathcal{Y}(\cdot)$  by

$$\mathcal{Y}(t) = \lambda \mathcal{E}(t) + \mathcal{F}(t), \quad \forall t \geq 0, \quad (3.76)$$

where  $\mathcal{E}(\cdot)$  is defined in (3.20).

The following proposition gives the equivalence between  $\mathcal{E}(\cdot)$  and the functional  $\mathcal{Y}(\cdot)$ .

**Proposition 39** *Assume that  $(\mathbf{A}_1)$  holds, then there exist two positive constants  $\delta_1, \delta_2$  such that*

$$\delta_1 \mathcal{E}(t) \leq \mathcal{Y}(t) \leq \delta_2 \mathcal{E}(t), \quad \forall t \geq 0. \quad (3.77)$$

**Proof.** To compare  $\mathcal{Y}(\cdot)$  with  $\mathcal{E}(\cdot)$ , we have to estimate the terms of the right hand side of (3.71) and to show that

$$|\mathcal{F}(t)| \leq c^* \mathcal{E}(t), \quad \forall t \geq 0,$$

for some  $c^* > 0$ .

From (3.72), (3.73), (3.74) and (3.75), we obtain

• **Estimate for  $\mathcal{I}(\cdot)$**

$$\mathcal{I}(t) = \rho h \langle D\psi_t(t), \psi(t) \rangle = \rho h \left\langle \psi_t(t) - \frac{h^2}{12} \psi_{txx}(t), \psi(t) \right\rangle.$$

Integration by parts and applying Poincaré's and young's inequalities yield

$$\begin{aligned} \mathcal{I}(t) &= \rho h \langle \psi_t(t), \psi(t) \rangle + \frac{\rho h^3}{24} \|\psi_{tx}(t)\|^2 \\ &\leq \frac{\rho h}{2} \|\psi_t(t)\|^2 + \frac{\rho h L^4}{2} \|\psi_{xx}(t)\|^2 + \frac{\rho h^3}{12} \|\psi_{xt}(t)\|^2 \\ &\leq c_1 \mathcal{E}(t). \end{aligned} \quad (3.78)$$

• **Estimate for  $\mathcal{J}(\cdot)$**

Using Young's inequality, we obtain

$$\mathcal{J}(t) = \rho h \langle \eta_t(t), \eta(t) \rangle \leq \frac{\rho h}{2} \|\eta_t(t)\|^2 + \frac{\rho h}{2} \|\eta(t)\|^2.$$

Applying Poincaré's inequality, one gets

$$\|\eta(t)\|^2 \leq L^2 \|\eta_x(t)\|^2 \leq L^2 \left\| \eta_x(t) + \frac{1}{2} \psi_x^2(t) \right\|^2 + \frac{L^2}{4} \|\psi_x^2(t)\|^2.$$

Applying Poincaré's inequality and the embedding  $H^1(0, L) \hookrightarrow L^\infty(0, L)$  and then using the fact that  $\mathcal{E}(\cdot)$  is decreasing, yields

$$\begin{aligned} \|\eta(t)\|^2 &\leq L^2 \left\| \eta_x(t) + \frac{1}{2} \psi_x^2(t) \right\|^2 + \frac{L^4}{4} \|\psi_x(t)\|_{L^\infty(0, L)}^2 \|\psi_{xx}(t)\|^2 \\ &\leq L^2 \left\| \eta_x(t) + \frac{1}{2} \psi_x^2(t) \right\|^2 + c \|\psi_{xx}(t)\|^2 \|\psi_{xx}(t)\|^2 \\ &\leq L^2 \left\| \eta_x(t) + \frac{1}{2} \psi_x^2(t) \right\|^2 + c \mathcal{E}(0) \|\psi_{xx}(t)\|^2 \\ &\leq c_2 \mathcal{E}(t). \end{aligned} \quad (3.79)$$

• **Estimate for  $\mathcal{K}(\cdot)$**

Using (3.69), applying Young's and then Poincaré inequalities, one gets

$$\begin{aligned}
\mathcal{K}(t) &= -\langle \rho h D \psi_t(t), g \diamond \psi(t) \rangle = -\rho h \langle \psi_t(t), g \diamond \psi(t) \rangle + \frac{\rho h^3}{12} \langle \psi_{tx}(t), g \diamond \psi(t) \rangle \\
&\leq \frac{\rho h}{2} \|\psi_t(t)\|^2 + \frac{\rho h^3}{24} \|\psi_{tx}(t)\|^2 + \left( \frac{\rho h L^4}{2} + \frac{\rho h^3 L^2}{24} \right) g \circ \psi_{xx}(t) \\
&\leq c_3 \mathcal{E}(t).
\end{aligned} \tag{3.80}$$

• **Estimate for  $\mathcal{L}(\cdot)$**

Since  $\mathcal{L}(\cdot)$  defines a norm in  $L^2(0, L; L^2(0, 1))$  which is equivalent to the one induced by  $L^2(0, L; L^2(0, 1))$ , then we have

$$\mathcal{L}(t) \leq \int_0^L \int_0^1 e^{-2\tau p} z^2 dp dx \leq \int_0^L \int_0^1 z^2 dp dx \leq \mathcal{E}(t). \tag{3.81}$$

According to (3.78), (3.79), (3.80) and (3.81), we have

$$|\mathcal{F}(t)| \leq c^* \mathcal{E}(t), \quad \forall t \geq 0,$$

for

$$c^* = \max\{c_1, c_2, c_3, 1\}. \tag{3.82}$$

Therefore, we obtain

$$|\mathcal{Y}(t) - \lambda \mathcal{E}(t)| \leq c^* \mathcal{E}(t), \quad \forall t \geq 0,$$

that is

$$(\lambda - c^*) \mathcal{E}(t) \leq \mathcal{Y}(t) \leq (\lambda + c^*) \mathcal{E}(t), \quad \forall t \geq 0.$$

So, we can choose  $\lambda$  large enough such that  $\delta_1 = \lambda - c^* > 0$ ,  $\delta_1 = \lambda + c^* > 0$ . Then (3.77) holds true which completes the proof. ■

In order to proof the main theorem, we need some additional lemmas.

**Lemma 40** *Suppose that  $(\psi, \psi_t, \eta, \eta_t, z)$  is the solution of (3.12)-(3.14). Then the derivative of the functional  $\mathcal{I}(\cdot)$  satisfies*

$$\begin{aligned}
\frac{d}{dt} \mathcal{I}(t) &\leq -(l - \varepsilon) \|\psi_{xx}(t)\|^2 + \rho h \|\psi_t(t)\|^2 + \frac{\rho h^3}{12} \|\psi_{tx}(t)\|^2 \\
&\quad - \left\langle \psi_x(t) \left( \eta_x + \frac{1}{2} \psi_x^2 \right) (t), \psi_x(t) \right\rangle + C_\varepsilon g \circ \psi_{xx}(t),
\end{aligned} \tag{3.83}$$

where  $\varepsilon$  is an arbitrary positive constant.

**Proof.** Using (3.12) and (3.69), we have

$$\begin{aligned}
\frac{d}{dt} \mathcal{I}(t) &= \langle \rho h D \psi_{tt}(t), \psi(t) \rangle + \langle \rho h D \psi_t(t), \psi_t(t) \rangle \\
&= I_1(t) + I_2(t) + I_3(t) + \left\langle \rho h \left( I - \frac{h^2}{12} \partial_x^2 \right) \psi_t(t), \psi_t(t) \right\rangle,
\end{aligned} \tag{3.84}$$

where

$$\begin{aligned}
I_1(t) &= -\langle \psi_{xxxx}(t), \psi(t) \rangle, \\
I_2(t) &= \left\langle \psi(t), \int_0^t g(t-s) \psi_{xxxx}(s) ds \right\rangle, \\
I_3(t) &= \left\langle \left[ \psi_x \left( \eta_x + \frac{1}{2} \psi_x^2 \right) \right]_x, \psi(t) \right\rangle.
\end{aligned}$$

Integrating  $I_1(\cdot)$  and  $I_2(\cdot)$  by parts twice, and using the boundary conditions, we obtain

$$I_1(t) = -\|\psi_{xx}(t)\|_2^2, \quad (3.85)$$

$$I_2(t) = \left\langle \psi_{xx}(t), \int_0^t g(t-s)\psi_{xx}(s)ds \right\rangle. \quad (3.86)$$

A integration by parts in  $I_3(\cdot)$ , leads to

$$I_3(t) = -\left\langle \psi_x(t) \left( \eta_x + \frac{1}{2}\psi_x^2 \right) (t), \psi_x(t) \right\rangle. \quad (3.87)$$

Substituting (3.85)–(3.87) in (3.84), we get

$$\begin{aligned} & \frac{d}{dt}\mathcal{I}(t) \\ &= -\|\psi_{xx}(t)\|^2 + \left\langle \psi_{xx}(t), \int_0^t g(t-s)\psi_{xx}(s)ds \right\rangle - \left\langle \psi_x(t) \left( \eta_x + \frac{1}{2}\psi_x^2 \right) (t), \psi_x(t) \right\rangle \\ & \quad + \rho h \|\psi_t(t)\|^2 + \frac{\rho h^3}{12} \|\psi_{tx}(t)\|^2. \end{aligned} \quad (3.88)$$

But

$$\left\langle \psi_{xx}(t), \int_0^t g(t-s)\psi_{xx}(s)ds \right\rangle = \left\langle \psi_{xx}(t), \int_0^t g(t-s)(\psi_{xx}(s) - \psi_{xx}(t)) ds \right\rangle + \int_0^t g(s)ds \|\psi_{xx}(t)\|^2. \quad (3.89)$$

Substituting (3.89) in (3.88), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{I}(t) &= -(1 - \int_0^t g(s)ds) \|\psi_{xx}\|^2 + \left\langle \psi_{xx}(t), \int_0^t g(t-s)(\psi_{xx}(s) - \psi_{xx}(t)) ds \right\rangle \\ & \quad - \left\langle \left[ \psi_x \left( \eta_x + \frac{1}{2}\psi_x^2 \right) \right], \psi_x(t) \right\rangle \\ & \quad + \rho h \|\psi_t\|^2 + \frac{\rho h^3}{12} \|\psi_{tx}\|^2. \end{aligned} \quad (3.90)$$

Making use of Young's and Cauchy-Schwarz inequalities for the second term in the right-hand side of (3.90), we get, for any  $\varepsilon > 0$ ,

$$\left\langle \psi_{xx}(t), \int_0^t g(t-s)(\psi_{xx}(s) - \psi_{xx}(t)) ds \right\rangle \leq \varepsilon \|\psi_{xx}(t)\|^2 + C_\varepsilon g \circ \psi_{xx}(t). \quad (3.91)$$

From (3.90)–(3.91) and (3.15), we infer that

$$\begin{aligned} \frac{d}{dt}\mathcal{I}(t) &\leq -(l - \varepsilon) \|\psi_{xx}(t)\|^2 - \left\langle \psi_x(t) \left( \eta_x + \frac{1}{2}\psi_x^2 \right) (t), \psi_x(t) \right\rangle + \rho h \|\psi_t(t)\|^2 \\ & \quad + \frac{\rho h^3}{12} \|\psi_{tx}(t)\|^2 + C_\varepsilon g \circ \psi_{xx}(t), \end{aligned}$$

which proves Lemma 40. ■



**Lemma 41** Assume that  $(\psi, \psi_t, \eta, \eta_t, z)$  is the solution of (3.12)-(3.14). Then the derivative of the functional  $\mathcal{J}(\cdot)$  satisfies

$$\begin{aligned} \frac{d}{dt}\mathcal{J}(t) &\leq \rho h \|\eta_t(t)\|^2 - \left\langle \eta_x(t) + \frac{1}{2}\psi_x^2(t), \eta_x(t) \right\rangle + \varepsilon c \left\| \eta_x(t) + \frac{1}{2}\psi_x^2(t) \right\|^2 + \varepsilon c \|\psi_{xx}(t)\|^2 \\ &\quad + C_\varepsilon \|\eta_t(t)\|^2 + C_\varepsilon \|z(t, 1)\|^2. \end{aligned} \quad (3.92)$$

where  $\varepsilon$  is an arbitrary positive constant.

**Proof.** A differentiation of  $\mathcal{J}(\cdot)$  yields

$$\frac{d}{dt}\mathcal{J}(t) \leq \rho h \|\eta_t(t)\|^2 - \left\langle \eta_x(t) + \frac{1}{2}\psi_x^2(t), \eta_x(t) \right\rangle - \alpha_1 \langle \eta_t(t), \eta(t) \rangle - \alpha_2 \langle z(t, 1), \eta(t) \rangle.$$

Applying Young's inequality, we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{J}(t) &\leq \rho h \|\eta_t(t)\|^2 - \left\langle \eta_x(t) + \frac{1}{2}\psi_x^2(t), \eta_x(t) \right\rangle + (\alpha_1 + \alpha_2) \varepsilon \|\eta(t)\|^2 \\ &\quad + C_\varepsilon \|\eta_t(t)\|^2 + C_\varepsilon \|z(t, 1)\|^2. \end{aligned} \quad (3.93)$$

Keeping in mind (3.79), the proof follows. ■

**Lemma 42** Suppose that  $(\psi, \psi_t, \eta, \eta_t, z)$  is the solution of (3.12)-(3.14). Then the derivative of the functional  $\mathcal{K}(\cdot)$  satisfies

$$\begin{aligned} \frac{d}{dt}\mathcal{K}(t) &\leq -\rho h \left( g_0 - \frac{h^2}{12}\varepsilon \right) \|\psi_t(t)\|^2 - (g_0 - \varepsilon) \frac{\rho h^3}{12} \|\psi_{tx}(t)\|^2 \\ &\quad + (c\varepsilon^3 + 2\varepsilon + \rho h\varepsilon) \|\psi_{xx}(t)\|^2 + c\varepsilon \left\| \eta_x(t) + \frac{1}{2}\psi_x^2(t) \right\|^2 \\ &\quad - C_\varepsilon g' \circ \psi_{xx}(t) + C_\varepsilon g \circ \psi_{xx}(t), \end{aligned} \quad (3.94)$$

where  $\varepsilon$  is an arbitrary positive constant and

$$g_0 := \int_0^{t_0} g(s)ds \leq \int_0^t g(s)ds, \quad \forall t \geq t_0.$$

**Proof.** Differentiating  $\mathcal{K}(\cdot)$  and using (3.12), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{K}(t) &= - \left\langle \rho h D\psi_{tt}(t), \int_0^t g(t-s)\psi(s)ds \right\rangle \\ &\quad - \left\langle \rho h D\psi_t(t), g' \diamond \psi(t) + \int_0^t g(s)ds\psi_t(t) \right\rangle \\ &= - \int_0^t g(s)ds \left( \rho h \|\psi_t(t)\|^2 + \frac{\rho h^3}{12} \|\psi_{tx}(t)\|^2 \right) + \langle \psi_{xx}(t), g \diamond \psi_{xx}(t) \rangle \\ &\quad - \left\langle \psi_x(t)(\eta_x + \frac{1}{2}\psi_x^2)(t), g \circ \psi_x(t) \right\rangle \\ &\quad + \left\langle \int_0^t g(t-s)\psi_{xx}(s)ds, g \diamond \psi_{xx}(t) \right\rangle \\ &\quad - \rho h \langle \psi_t(t), g' \diamond \psi(t) \rangle + \frac{\rho h^3}{12} \langle \psi_t(t), g' \diamond \psi_{xx}(t) \rangle \\ &: = -g_0 \left( \rho h \|\psi_t(t)\|^2 + \frac{\rho h^3}{12} \|\psi_{tx}(t)\|^2 \right) + J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t). \end{aligned}$$

Next, we shall analyze the terms  $J_i(\cdot)$ ,  $i = 1, \dots, 5$ .

*Estimate for  $J_1(\cdot)$*

Applying Young's inequality and (3.69), we get

$$J_1(t) = \langle \psi_{xx}(t), g \diamond \psi_{xx}(t) \rangle \leq \varepsilon \|\psi_{xx}(t)\|^2 + C_\varepsilon g \diamond \psi_{xx}(t), \quad (3.95)$$

where  $\varepsilon$  is an arbitrary positive constant.

*Estimate for  $J_2(\cdot)$*

We follow the previous steps, with applying Poincaré's inequality, we get

$$\begin{aligned} J_2(t) &= - \left\langle \psi_x(t) \left( \eta_x + \frac{1}{2} \psi_x^2 \right) (t), g \diamond \psi_x(t) \right\rangle \\ &\leq \varepsilon^2 \left\| \psi_x(t) \left( \eta_x + \frac{1}{2} \psi_x^2 \right) (t) \right\|^2 + C_\varepsilon g \diamond \psi_{xx}(t). \end{aligned} \quad (3.96)$$

On the other hand, we know that  $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ , so

$$\begin{aligned} \left\| \psi_x(t) \left( \eta_x + \frac{1}{2} \psi_x^2 \right) (t) \right\|^2 &\leq \|\psi_x(t)\|_{L^\infty(I)}^2 \left\| \eta_x(t) + \frac{1}{2} \psi_x^2(t) \right\|^2 \\ &\leq c \|\psi_{xx}(t)\|^2 \left\| \eta_x(t) + \frac{1}{2} \psi_x^2(t) \right\|^2 \\ &\leq \varepsilon c \|\psi_{xx}(t)\|^2 + \frac{c}{\varepsilon} \left\| \eta_x(t) + \frac{1}{2} \psi_x^2(t) \right\|^2. \end{aligned}$$

This implies that

$$J_2(t) \leq c\varepsilon^3 \|\psi_{xx}\|^2 + \frac{c\varepsilon}{4} \left\| \eta_x + \frac{1}{2} \psi_x^2 \right\|^2 + C_\varepsilon g \diamond \psi_{xx}.$$

*Estimate for  $J_3(\cdot)$*

We use (3.69) to get

$$\begin{aligned} J_3(t) &= \left\langle \int_0^t g(t-s) \psi_{xx}(s) ds, g \diamond \psi_{xx}(t) \right\rangle \\ &= - \left\| \int_0^t g(t-s) (\psi_{xx}(t) - \psi_{xx}(s)) ds \right\|^2 \\ &\quad + \int_0^t g(s) ds \langle \psi_{xx}(t), g \diamond \psi_{xx}(t) \rangle \\ &\leq \varepsilon \|\psi_{xx}(t)\|^2 + C_\varepsilon g \diamond \psi_{xx}(t). \end{aligned} \quad (3.97)$$

*Estimate for  $J_4(\cdot)$*

Applying Young's, Poincaré's inequalities and using (3.70), we conclude that

$$J_4(t) = -\rho h \langle \psi_t(t), g' \diamond \psi(t) \rangle \leq \rho h \varepsilon \|\psi_{xx}(t)\|^2 - C_\varepsilon g' \diamond \psi_{xx}(t). \quad (3.98)$$

*Estimate for  $J_5(\cdot)$*

Finally, for  $J_5(\cdot)$ , invoking (3.70), we obtain

$$J_5(t) = \frac{\rho h^3}{12} \langle \psi_t(t), g' \diamond \psi_{xx}(t) \rangle \leq \frac{\rho h^3}{12} \varepsilon \|\psi_t(t)\|^2 - C_\varepsilon g' \diamond \psi_{xx}(t). \quad (3.99)$$

Combining (3.95)-(3.99), we arrive at the proof of (42). ■

**Lemma 43** Suppose that  $(\psi, \psi_t, \eta, \eta_t, z)$  is the solution of (3.12)-(3.14). Then the time derivative of the functional  $\mathcal{L}(\cdot)$  satisfies

$$\frac{d}{dt}\mathcal{L}(t) \leq -2\mathcal{L}(t) + \frac{\alpha_1 e^{-2\tau}}{\tau} \|z(t, 1)\|^2 + \frac{\alpha_2}{\tau} \|\eta_t(t)\|^2. \quad (3.100)$$

**Proof.** Keeping in mind that  $z_t(t, x, p) = -\frac{1}{\tau} z_p(t, x, p)$ , we infer

$$\begin{aligned} \frac{d}{dt}\mathcal{L}(t) &= -\frac{2}{\tau} \int_0^L \int_0^1 e^{-2\tau p} z_p(t, x, p) z(t, x, p) dp dx = -\frac{1}{\tau} \int_0^L \int_0^1 e^{-2\tau p} (z^2(t, x, p))_p dp dx \\ &= -2\mathcal{L}(t) + \frac{\alpha_1 e^{-2\tau}}{\tau} \|z(t, 1)\|^2 + \frac{\alpha_2}{\tau} \|\eta_t(t)\|^2. \end{aligned}$$

■

**Proposition 44** Assume that  $(\mathbf{A}_1)$  and  $(\mathbf{A}_2)$  hold, then there exists two positive constants  $\beta_1, \beta_2$  such that

$$\frac{d}{dt}\mathcal{Y}(t) \leq -\beta_1 \mathcal{E}(t) + \beta_2 g \circ \psi_{xx}, \quad \forall t \geq 0. \quad (3.101)$$

**Proof.** By using (3.71), (3.76) and combining (3.83)-(3.100), we get

$$\begin{aligned} &\frac{d}{dt}\mathcal{Y}(t) \\ \leq &\left\{ -\frac{N_1}{2} (l - \varepsilon) + N_1 c \varepsilon + N_2 (c \varepsilon^3 + 2\varepsilon + \rho h \varepsilon) \right\} \|\psi_{xx}(t)\|^2 \\ &+ \rho h \left\{ -N_2 \left( g_0 - \frac{h^2}{12} \varepsilon \right) + \frac{N_1}{2} \right\} \|\psi_t(t)\|^2 + \left\{ N_1 C_\varepsilon + \lambda \left( \frac{\xi}{2\tau} + \frac{\alpha_2}{2} - \alpha_1 \right) \right\} \|\eta_t(t)\|^2 \\ &+ \frac{\rho h^3}{12} \left\{ \frac{N_1}{2} - N_2 (g_0 - \varepsilon) \right\} \|\psi_{tx}(t)\|^2 + \{ N_1 (\varepsilon c - 1) + c N_2 \varepsilon \} \left\| \eta_x(t) + \frac{1}{2} \psi_x^2(t) \right\|^2 \\ &- \left\{ N_2 C_\varepsilon - \frac{\lambda}{2} \right\} g' \circ \psi_{xx}(t) + C_\varepsilon (N_1 + N_2) g \circ \psi_{xx}(t) - 2\mathcal{M}(t) \\ &+ \left\{ N_2 C_\varepsilon - \frac{e^{-2\tau} \alpha_1}{\tau} + \lambda \left( \frac{\alpha_2}{2} - \frac{\xi}{2\tau} \right) \right\} \|z(t, 1)\|^2. \end{aligned} \quad (3.102)$$

We want to impose suitable conditions on the different parameters so that the coefficients on the right hand side of (3.102) are all negative. That is to obtain the following inequalities

$$N_1 c \varepsilon + N_2 (c \varepsilon^3 + 2\varepsilon + \rho h \varepsilon) < \frac{N_1}{2} (l - \varepsilon), \quad (3.103)$$

$$N_1 < 2N_2 \left( g_0 - \frac{h^2}{12} \varepsilon \right), \quad (3.104)$$

$$N_1 < 2N_2 (g_0 - \varepsilon), \quad (3.105)$$

$$c N_2 \varepsilon < N_1 (1 - \varepsilon c), \quad (3.106)$$

$$N_2 C_\varepsilon < \frac{\lambda}{2}, \quad (3.107)$$

$$N_2 C_\varepsilon < \frac{e^{-2\tau} \alpha_1}{\tau} - \lambda \left( \frac{\alpha_2}{2} - \frac{\xi}{2\tau} \right), \quad (3.108)$$

$$N_1 C_\varepsilon < -\lambda \left( \frac{\xi}{2\tau} + \frac{\alpha_2}{2} - \alpha_1 \right). \quad (3.109)$$

First, we pick  $\varepsilon$  such that

$$\varepsilon < \min \left\{ l, \frac{12g_0}{h^2}, g_0, \frac{1}{c} \right\},$$

so that the above system makes sense. Now, observe that for  $\varepsilon \ll 1$ , we get from inequality (3.103)

$$N_1 c \varepsilon + N_2 (c \varepsilon^3 + 2\varepsilon + \rho h \varepsilon) \leq \varepsilon (N_1 c + N_2 (c + 2 + \rho h)),$$

which means that it is satisfied for all  $\varepsilon$  such that

$$\varepsilon < \frac{N_1 l}{(N_1 c + N_2 (c + 2 + \rho h))}, \quad (3.110)$$

for any  $N_1, N_2 > 0$ . Similarly, inequality (3.106) is verified for all  $\varepsilon$  satisfies

$$\varepsilon < \frac{N_1}{(c N_2 + N_1 c)}, \quad (3.111)$$

for any  $N_1, N_2 > 0$ . To satisfy (3.103) and (3.106), it is enough to pick  $N_1$  and  $N_2$  such that

$$N_1 < \min \left\{ 2N_2 \left( g_0 - \frac{h^2}{12} \varepsilon \right), 2N_2 (g_0 - \varepsilon) \right\},$$

with  $\varepsilon$  is smaller than in (3.110) and (3.111) if needed. Concerning (3.107)-(3.109), we pick

$$\lambda > \max \left\{ \frac{\alpha_2}{a_1 \tau}, -\frac{N_2 C_\varepsilon}{\left(\frac{\alpha_2}{2} - \frac{\xi}{2\tau}\right)}, -\frac{N_1 C_\varepsilon}{\left(\frac{\xi}{2\tau} + \frac{\alpha_2}{2} - \alpha_1\right)}, c^* \right\},$$

where  $c^*$  is defined in (3.82). (Recall that  $\frac{\alpha_2}{2} - \frac{\xi}{2\tau}$  and  $\frac{\xi}{2\tau} + \frac{\alpha_2}{2} - \alpha_1$  are negative by assumption (3.21)). ■

We consider the following two cases.

**Case I.  $H(t)$  is linear:**

By Multiplying (3.101) by  $\zeta(\cdot)$  and using (3.24), we get

$$\begin{aligned} \frac{d}{dt} \mathcal{Y}(t) \zeta(t) &\leq -\beta_1 \zeta(t) \mathcal{E}(t) + \beta_2 \zeta(t) g \circ \psi_{xx} \\ &\leq -\beta_1 \zeta(t) \mathcal{E}(t) + \beta_2 g \zeta \circ \psi_{xx} \\ &\leq -\beta_1 \zeta(t) \mathcal{E}(t) - \beta_2 a g' \circ \psi_{xx} \\ &\leq -\beta_1 \zeta(t) \mathcal{E}(t) - c \mathcal{E}'(t), \end{aligned}$$

which gives, since  $\zeta(\cdot)$  is non-increasing,

$$\frac{d}{dt} (\mathcal{Y}(t) \zeta(t) + c \mathcal{E}(t)) \leq -\beta_1 \zeta(t) \mathcal{E}(t), \quad \forall t \geq t_1.$$

Hence, using the fact that  $\mathcal{Y}(\cdot) \zeta(\cdot) + c \mathcal{E}(\cdot)$  is equivalent to  $\mathcal{E}(\cdot)$ , it is easy to see that

$$\frac{d}{dt} (\mathcal{Y}(t) \zeta(t) + c \mathcal{E}(t)) \leq -\beta_1 \zeta(t) (\mathcal{Y}(t) \zeta(t) + c \mathcal{E}(t)), \quad \forall t \geq t_1,$$

for some  $\beta_1 > 0$ . Then

$$(\mathcal{Y}(t) \zeta(t) + c \mathcal{E}(t)) \leq \gamma_2 e^{-\beta_1 \int_{t_1}^t \zeta(s) ds}, \quad \forall t \geq t_1,$$

from which we deduce

$$\mathcal{E}(t) \leq \gamma_2 e^{-\beta_1 \int_{t_1}^t \zeta(s) ds}, \quad \forall t \geq t_1,$$

for some  $\gamma_2 > 0$ . Furthermore, using the continuity and boundedness of  $\mathcal{E}(t)$  in  $[0, t_1]$ , we get

$$\mathcal{E}(t) \leq C e^{-\beta_1 \int_0^t \zeta(s) ds}, \quad \forall t \geq 0.$$

**Case II.**  $H(t)$  is **nonlinear**:

Next, with  $f(t) = \int_t^\infty g(s) ds$ , we use the functional

$$\mathcal{M}(t) = \int_0^t f(t-s) \|\psi_{xx}(s)\|^2 ds, \quad \forall t \geq 0.$$

**Lemma 45** Assume that  $(\mathbf{A}_1)$  and  $(\mathbf{A}_2)$  hold. The functional  $\mathcal{M}(\cdot)$  satisfies, for any  $\varepsilon > 0$ , the estimate

$$\frac{d}{dt} \mathcal{M}(t) \leq (2\varepsilon - 1) g \circ \psi_{xx}(t) + (f(t) + C_\varepsilon) \|\psi_{xx}(t)\|^2, \quad \forall t \geq 0. \quad (3.112)$$

**Proof.** By Young's inequality and the fact  $f'(t) = -g(t)$ , we see that

$$\begin{aligned} \frac{d}{dt} \mathcal{M}(t) &= f(0) \|\psi_{xx}(t)\|^2 - \int_0^t g(t-s) \|\psi_{xx}(s)\|^2 ds \\ &= -g \circ \psi_{xx}(t) - 2 \langle \psi_{xx}(t), g \diamond \psi_{xx}(t) \rangle + f(t) \|\psi_{xx}(t)\|^2. \end{aligned} \quad (3.113)$$

But

$$-2 \langle \psi_{xx}(t), g \diamond \psi_{xx}(t) \rangle \leq C_\varepsilon \|\psi_{xx}\|^2 + 2\varepsilon g \circ \psi_{xx}(t). \quad (3.114)$$

Combining (3.113) and (3.114), we obtain (3.112). ■

Let us introduce the functional

$$\tilde{\mathcal{Y}}(t) = \mathcal{Y}(t) + \kappa \mathcal{M}(t), \quad \forall t \geq 0,$$

where  $\kappa$  is a positive constant. Then we have  $\tilde{\mathcal{Y}}(\cdot) \sim \mathcal{E}(\cdot)$ . Therefore, it is always possible to pick  $N_1$  (in 3.102) and  $\kappa$  large enough to get

$$\frac{d}{dt} \tilde{\mathcal{Y}}(t) \leq -C \mathcal{E}(t), \quad \forall t \geq 0.$$

Integrating over  $(t_0, \infty)$ , we get

$$C \int_{t_0}^\infty \mathcal{E}(s) ds \leq \tilde{\mathcal{Y}}(t_0) < \infty. \quad (3.115)$$

Next, let us define the functional  $\mathcal{P}(\cdot)$

$$\mathcal{P}(t) = q \int_{t_0}^t \|\psi_{xx}(s) - \psi_{xx}(t-s)\|^2 ds, \quad \forall t \geq t_0,$$

where  $q > 0$ . Thanks to (3.115), we can always choose  $q$  such that

$$\mathcal{P}(t) < 1, \quad \forall t \geq t_0. \quad (3.116)$$

Next we define

$$\mathcal{P}_g(t) = - \int_{t_0}^t g'(s) \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds, \quad \forall t \geq t_0.$$

Observe that

$$\mathcal{P}_g(t) \leq -C \mathcal{E}'(t), \quad \forall t \geq 0,$$

for some positive constant  $C$ . Since  $H$  is strictly convex on  $(0, r]$  and  $H(0) = 0$  we have

$$H(\theta x) \leq \theta H(x), \quad (\theta, x) \in [0, 1] \times (0, r]. \quad (3.117)$$

Using  $(\mathbf{A}_2)$ , we get

$$\begin{aligned}
\mathcal{P}_g(t) &= \frac{1}{q\mathcal{P}(t)} \int_{t_0}^t \mathcal{P}(t) (-g'(s)) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds \\
&\geq \frac{1}{q\mathcal{P}(t)} \int_{t_0}^t \mathcal{P}(t) \zeta(s) H(g(s)) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds \\
&\geq \frac{\zeta(t)}{q\mathcal{P}(t)} \int_{t_0}^t H(\mathcal{P}(t)g(s)) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds.
\end{aligned}$$

Keeping in mind (3.116) and applying inequality (3.117) for  $\theta := \mathcal{P}(\cdot)$  and  $x = g(\cdot)$ , yields

$$\mathcal{P}_g(t) \geq \frac{\zeta(t)}{q\mathcal{P}(t)} \int_{t_0}^t H(\mathcal{P}(t)g(s)) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds. \quad (3.118)$$

Applying Jensen's inequality (3.25) we obtain

$$\begin{aligned}
\mathcal{P}_g(t) &\geq \frac{\zeta(t)}{q\mathcal{P}(t)} \int_{t_0}^t H(\mathcal{P}(t)g(s)) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds \\
&\geq \frac{\zeta(t)}{q} H\left(\frac{1}{\mathcal{P}(t)} \int_{t_0}^t \mathcal{P}(t)g(s) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds\right) \\
&= \frac{\zeta(t)}{q} H\left(\int_{t_0}^t g(s) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds\right) \\
&= \frac{\zeta(t)}{q} \bar{H}\left(\int_{t_0}^t g(s) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds\right),
\end{aligned}$$

where  $\bar{H}$  is an extension of  $H$  such that  $\bar{H}$  is increasing and strictly convex  $C^2$  function on  $(0, \infty)$  and this leads to

$$\int_{t_0}^t g(s) q \|\psi_{xx}(t) - \psi_{xx}(t-s)\|^2 ds \leq \frac{1}{q} \bar{H}^{-1}\left(\frac{q\mathcal{P}_g(t)}{\zeta(t)}\right), \quad \forall t \geq t_0.$$

So (3.101) becomes

$$\tilde{\mathcal{Y}}(t) \leq -\beta_1 \mathcal{E}(t) + \beta_2 \frac{1}{q} \bar{H}^{-1}\left(\frac{q\mathcal{L}_g(t)}{\zeta(t)}\right), \quad \forall t \geq t_0. \quad (3.119)$$

Let  $\epsilon_0 < r$ , using the fact that  $\mathcal{E}(\cdot)$  is non-increasing and  $\bar{H}' > 0, \bar{H}'' > 0$ , we observe that the functional  $\mathcal{N}(\cdot)$  defined by

$$\mathcal{N}(t) := \bar{H}'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) \mathcal{Y}(t) + \mathcal{E}(t), \quad \forall t \geq 0,$$

is equivalent to  $\mathcal{E}(\cdot)$ . Using (3.119), we find that  $\mathcal{N}(\cdot)$  satisfies

$$\begin{aligned}
\frac{d}{dt} \mathcal{N}(t) &= \epsilon_0 \frac{\mathcal{E}'(t)}{\mathcal{E}(0)} \bar{H}''\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) \mathcal{Y}(t) + \bar{H}'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) \mathcal{Y}'(t) + \mathcal{E}'(t) \\
&\leq \epsilon_0 \frac{\mathcal{E}'(t)}{\mathcal{E}(0)} \bar{H}''\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) \mathcal{Y}(t) \\
&\quad + \bar{H}'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) \left[-\beta_1 \mathcal{E}(t) + \beta_2 \frac{1}{q} \bar{H}^{-1}\left(\frac{q\mathcal{P}_g(t)}{\zeta(t)}\right)\right] + \mathcal{E}'(t) \\
&\leq -\beta_1 \mathcal{E}(t) \bar{H}'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) \\
&\quad + \frac{\beta_2}{q} \bar{H}^{-1}\left(\frac{q\mathcal{P}_g(t)}{\zeta(t)}\right) \bar{H}'\left(\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right) + \mathcal{E}'(t).
\end{aligned} \quad (3.120)$$

Let us denote by  $G^*$  the conjugate function of the convex function  $G$  defined by  $G^*(s) = \sup_{t \in \mathbb{R}^+} (st - G(t))$ , then

$$st \leq G^*(s) + G(t), \quad (3.121)$$

and, thanks to the arguments given in [16, 28, 43, 73, 74]

$$\overline{G}^*(s) = s(\overline{G}')^{-1}(s) - \overline{G} \left[ (\overline{H}')^{-1}(s) \right], \quad \forall s \geq 0.$$

This and the definition of  $H$  give

$$\overline{H}^*(s) = s(\overline{H}')^{-1}(s) - \overline{H} \left[ (\overline{H}')^{-1}(s) \right], \quad \forall s \geq 0. \quad (3.122)$$

Taking  $s := \frac{C_2}{q} \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right)$  and  $t := \overline{H}^{-1} \left( \frac{q\mathcal{P}_g(t)}{\zeta(t)} \right)$  in (3.121), then making use of (3.120), (3.121) and (3.122), we arrive at

$$\begin{aligned} & \frac{d}{dt} \mathcal{N}(t) \\ & \leq -\beta_1 \mathcal{E}(t) \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + \overline{H} \left[ \overline{H}^{-1} \left( \frac{q\mathcal{P}_g(t)}{\zeta(t)} \right) \right] + \overline{H}^* \left[ \frac{\beta_2}{q} \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \right] + \mathcal{E}'(t) \\ & \leq -\beta_1 \mathcal{E}(t) \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + \frac{q\mathcal{P}_g(t)}{\zeta(t)} + \overline{H}^* \left[ \frac{\beta_2}{q} \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) \right] + \mathcal{E}'(t) \\ & \leq -\beta_1 \mathcal{E}(t) \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + \frac{q\mathcal{P}_g(t)}{\zeta(t)} + \frac{\beta_2 \epsilon_0}{q} \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + \mathcal{E}'(t). \end{aligned} \quad (3.123)$$

Next, multiplying (3.123) by  $\zeta(t)$  and using the fact that  $\epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} < r$ ,  $\overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) = H' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right)$ , we get

$$\begin{aligned} \zeta(t) \frac{d}{dt} \mathcal{N}(t) & \leq -\beta_1 \mathcal{E}(t) \zeta(t) \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + q\mathcal{P}_g(t) \\ & \quad + \frac{\beta_2 \epsilon_0}{q} \zeta(t) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + \zeta(t) \mathcal{E}'(t) \\ & \leq -\beta_1 \mathcal{E}(t) \zeta(t) \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) + \frac{\beta_2 \epsilon_0}{q} \zeta(t) \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) - c\mathcal{E}'(t). \end{aligned}$$

Now, let us define the functional  $\tilde{\mathcal{N}}(\cdot)$  by

$$\tilde{\mathcal{N}}(t) = \mathcal{N}(t) \zeta(t) + \mathcal{E}(t), \quad \forall t \geq 0.$$

It is not difficult to see that there exist two positive constants  $\rho_1$  and  $\rho_2$  for which we have

$$\rho_1 \tilde{\mathcal{N}}(t) \leq \mathcal{E}(t) \leq \rho_2 \tilde{\mathcal{N}}(t), \quad \forall t \geq 0. \quad (3.124)$$

Consequently, with an appropriate choice of  $\epsilon_0$ , there exists a positive constant  $k$  such that

$$\frac{d}{dt} \tilde{\mathcal{N}}(t) \leq -k \zeta(t) \overline{H}' \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right) = -k \zeta(t) H_2 \left( \epsilon_0 \frac{\mathcal{E}(t)}{\mathcal{E}(0)} \right), \quad \forall t \geq t_0 \quad (3.125)$$

where  $H_2(s) = sH'(\epsilon_0 s)$ .

Since  $H'_2(s) = H'(\epsilon_0 s) + \epsilon_0 s H''(\epsilon_0 s)$ , we use the strict convexity of  $H$  on  $[0, r)$ , we observe that  $H_2 > 0$ ,  $H'_2 > 0$  on  $(0, r]$ .

Defining now

$$\mathcal{R}(t) = \frac{\delta_1 \tilde{\mathcal{N}}(t)}{\mathcal{E}(0)}, \quad \forall t \geq 0.$$

thanks to (3.124) and (3.125) we have  $\mathcal{E}(\cdot) \sim \mathcal{R}(\cdot)$  and for a positive constant  $\tilde{k}$

$$\frac{d}{dt} \mathcal{R}(t) \leq -\tilde{k} \zeta(t) H_2(\mathcal{R}(t)), \quad \forall t \geq t_0.$$

Then, integrating over  $(t_0, t)$  yields

$$\int_{t_0}^t \frac{\mathcal{R}'(s)}{H_2(\mathcal{R}(s))} ds \leq - \int_{t_0}^t \tilde{k} \zeta(s) ds,$$

and this leads to

$$\int_{\epsilon_0 \mathcal{R}(t)}^{\epsilon_0 \mathcal{R}(t_0)} \frac{\mathcal{R}'(s)}{H'(\mathcal{R}(s))} ds \geq \tilde{k} \int_{t_0}^t \zeta(s) ds,$$

which gives us

$$\mathcal{R}(t) \leq \frac{1}{\epsilon_0} H_1^{-1} \left( \tilde{k} \int_{t_0}^t \zeta(s) ds \right), \quad \forall t \geq t_0,$$

where  $H_1(t) = \int_t^r \frac{ds}{s H'(s)}$ . This completes the proof.



## Chapter 4

# Boundary controllability and boundary time-varying feedback stabilization of the $1 - D$ wave equation in non-cylindrical domains

This Chapter is taken from [90].

We are interested in the boundary controllability and stabilization of the one dimensional wave equation in non-cylindrical domains. More precisely, let  $\alpha$  and  $\beta$  be two real functions defined on  $\mathbb{R}_+$  and  $Q$  be the set

$$Q = \{(t, x) \in \mathbb{R}^2, x \in (\alpha(t), \beta(t)), \alpha(t) < \beta(t), t \in (0, \infty)\},$$

with  $\alpha(0) = 0$  and  $\beta(0) = 1$ . We consider the following two systems

$$\begin{cases} y_{tt}(t, x) = y_{xx}(t, x), & \text{in } Q, \\ y(t, \alpha(t)) = \frac{1}{2}u(t), y(t, \beta(t)) = 0, & \text{in } (0, \infty), \\ y(0, x) = y_0(x), y_t(0, x) = y_1(x), & \text{in } (0, 1), \end{cases} \quad (4.1)$$

and

$$\begin{cases} y_{tt}(t, x) = y_{xx}(t, x), & \text{in } Q, \\ y_t(t, \alpha(t)) = f(t)y_x(t, \alpha(t)), y(t, \beta(t)) = 0, & \text{in } (0, \infty), \\ y(0, x) = y_0(x), y_t(0, x) = y_1(x), & \text{in } (0, 1). \end{cases} \quad (4.2)$$

The functions  $u \in H_{\text{loc}}^1(0, \infty)$ , and  $f \in C([0, \infty))$  in (4.1) and (4.2) represent the control force and the feedback function respectively.

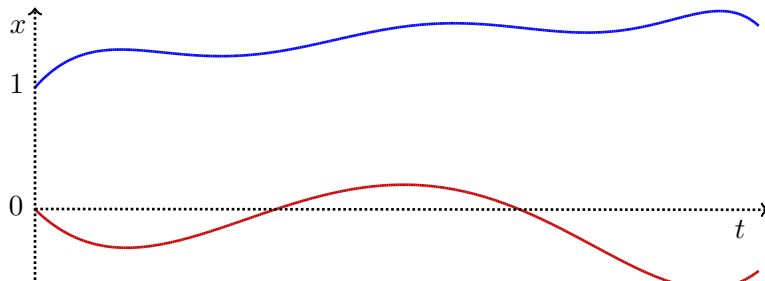


Figure 4.1: The curve  $(t, \alpha(t))_{t \geq 0}$  in red and  $(t, \beta(t))_{t \geq 0}$  in blue.

Controllability of System (4.1) has been extensively studied in the recent past years; most of the papers dealt with the case of one moving endpoint with boundary conditions of the form

$$y(t, 0) = 0, \quad y(t, kt + 1) = u(t), \quad k \in (0, 1), \quad t \in (0, \infty).$$

In [38], it has been shown that, with these boundary conditions, exact controllability holds for all times  $T > \frac{e^{\frac{2k(k+1)}{1-k}} - 1}{2}$ . The same authors came back in [39] and improved the latter result to  $T > \frac{e^{\frac{2k(k+1)}{(1-k)^3}} - 1}{2}$ . Later, in [123], the controllability time has been improved to be  $T > \frac{2}{1-k}$ . In these papers, only a sufficient condition is provided for the exact controllability.

Concerning the two moving endpoints case, the boundary functions considered in [119] are of the form

$$\alpha(t) = -kt, \quad \beta(t) = rt + 1, \quad t \in (0, \infty), \quad k, r \in [0, 1] \text{ with } r + k > 0.$$

It has been shown that exact controllability holds if, and only if  $T \geq \frac{2}{(1-k)(1-r)}$ .

Another kind of boundary functions has been considered in [9]. An observability inequality has been established for the dual of System (4.1) with  $\beta \equiv 1$  for sufficiently large time under the assumption that the boundary function  $\alpha$  must be periodic and satisfies  $\|\alpha'\|_{L^\infty(0, \infty)} < 1$ . More general boundary functions are considered in [60] with boundary conditions

$$y(t, 0) = 0, \quad y(t, s(t)) = u(t), \quad t \in (0, \infty),$$

where  $s : [0, \infty) \rightarrow (0, \infty)$  is assumed to be a  $C^1$  function satisfying  $\|s'\|_{L^\infty(0, \infty)} < 1$ . Furthermore, it has been assumed that  $s$  must be in some admissible class of curves (see [60] for more details). Under these assumptions, the authors proved that exact controllability holds if, and only if  $T \geq s^+ \circ (s^-)^{-1}(0)$ , where  $s^\pm(t) = t \pm s(t)$ . Also, they provided a controllability result when the control is located on the non-moving part of the boundary. By considering the boundary conditions

$$y(t, 0) = u(t), \quad y(t, s(t)) = 0, \quad t \in (0, \infty),$$

they proved that exact controllability holds if, and only if  $T \geq (s^-)^{-1}(1)$ . The same result has been proved in [58] by using a different approach. In all the cited works, the proofs rely on the multipliers technique, domain transform, the non-harmonic Fourier analysis or the d'Alembert solution of the wave equation.

Recently, in [120], a new Carleman estimate has been established for the wave equation in non-cylindrical domains in more general settings. As a consequence, it has been shown for a boundary conditions as in (4.1) where  $\alpha(t) < \beta(t)$ ,  $t \in (0, \infty)$ , are smooth functions satisfying  $\|\alpha'\|_{L^\infty(0, \infty)}, \|\beta'\|_{L^\infty(0, \infty)} < 1$ , that System (4.1) is exactly controllable at time  $T$  if  $T > T^*$  and not exactly controllable if  $T < T^*$  where  $T^*$  is the required time by the geometric control condition, in other words, it is the time where a characteristic line with slope one emanating from the point  $(0, 0)$  hits the curve  $(t, \beta(t))_{t \geq 0}$  and reflected to intersect the curve  $(t, \alpha(t))_{t \geq 0}$  in the point  $(T^*, \alpha(T^*))$ . Actually, this time can be computed explicitly in terms of the boundary functions, that is  $T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)$  where the functions  $\alpha^\pm, \beta^\pm$  are defined by  $\alpha^\pm(t) = t \pm \alpha(t)$ ,  $\beta^\pm(t) = t \pm \beta(t)$ . However, the result doesn't cover the critical case  $T = T^*$ .

As for the boundary stability of System (4.2) with non-autonomous damping, to the best of our knowledge, the only existing result in the literature is in [8] where the authors dealt with the same system but with only one moving endpoint, i.e.

$$y(t, 0) = 0, \quad y_t(t, a(t)) + f(t)y_x(t, a(t)) = 0, \quad t \in (0, \infty), \quad (4.3)$$

where  $a$  is a strictly positive 1-periodic function with  $\|a'\|_{L^\infty(0, \infty)} < 1$  and  $f$  is the feedback function. The authors proved exponential stability of System (4.2) for a particular class of feedbacks  $f$ . The proof relies on transforming problem (4.2) which is posed on non-cylindrical domain into a problem posed on cylindrical one, then making use of some known results of boundary stability of the  $1-D$  wave equation. If the damping

function  $f$  is constant and the boundary function  $a$  is not periodic with derivative  $\|a'\|_{L^\infty(0,\infty)} < 1$ , it has been shown in [59] for  $f = 1$  that the solution vanishes at time  $T$  for any  $T \geq a^+ \circ (a^-)^{-1}(0)$ .

In this paper, we will improve all the previous results either for the boundary control or the boundary stability of the  $1 - D$  wave equation by using the characteristics method. We shall build the unique exact solution to both Systems (4.1) and (4.2) in an appropriate energy space. To do so, we proceed by transforming both of systems to a first order hyperbolic system by introducing the Riemann invariants

$$\begin{cases} p = y_t - y_x, \\ q = y_t + y_x. \end{cases} \quad (4.4)$$

An elementary computation shows that System (4.1) transforms into

$$\begin{cases} p_t + p_x = 0, & \text{in } Q, \\ q_t - q_x = 0, & \text{in } Q, \\ (p + q)(t, \alpha(t)) = u'(t), \quad (p + q)(t, \beta(t)) = 0, & \text{in } (0, \infty), \\ p(0, x) = \tilde{p}(x), \quad q(0, x) = \tilde{q}(x). & \text{in } (0, 1). \end{cases} \quad (4.5)$$

In the same way, System (4.2) becomes

$$\begin{cases} p_t + p_x = 0, & \text{in } Q, \\ q_t - q_x = 0, & \text{in } Q, \\ (p + F(t)q)(t, \alpha(t)) = 0, \quad (p + q)(t, \beta(t)) = 0, & \text{in } (0, \infty), \\ p(0, x) = \tilde{p}(x), \quad q(0, x) = \tilde{q}(x), & \text{in } (0, 1), \end{cases} \quad (4.6)$$

where  $F(t) = \frac{1-f(t)}{1+f(t)}$  with  $1 + f(t) \neq 0, \forall t \geq 0$ .

Henceforth, we use the following notations:

- the spaces family  $[L^2(\alpha(t), \beta(t))]_{t \geq 0}$  will be denoted by  $L^2(\alpha(t), \beta(t))$ .
- The spaces family  $[H^1_{(\beta(t))}(\alpha(t), \beta(t))]_{t \geq 0}$  will be denoted by  $H^1_{(\beta(t))}(\alpha(t), \beta(t))$  where  $H^1_{(\beta(t))}(\alpha(t), \beta(t)) = \{h \in H^1(\alpha(t), \beta(t)), h(\beta(t)) = 0, t \geq 0\}$ .
- For any function  $z$ , the functions  $z^\pm$  will represent the quantities  $z^\pm(t) = t \pm z(t)$ .
- $C$  denotes a generic positive constant which might be different from line to line.

The Riemann coordinates introduced in (4.4) guarantee the equivalence of the transformed Systems (4.5),(4.6), with the original Systems (4.1),(4.2) up to an additive constant. All the results for the transformed systems will be proved in  $[L^2(\alpha(t), \beta(t))]^2$ , then the results for the original ones can be deduced by inverting the transformation.

Since our approach consists in constructing the unique exact solutions to Systems (4.5) and (4.6), instead of studying each system separately, we consider the following system

$$\begin{cases} p_t + p_x = 0, & \text{in } Q, \\ q_t - q_x = 0, & \text{in } Q, \\ (p + F(t)q)(t, \alpha(t)) = v(t), \quad (p + q)(t, \beta(t)) = 0, & \text{in } (0, \infty), \\ p(0, x) = \tilde{p}(x), \quad q(0, x) = \tilde{q}(x). & \text{in } (0, 1), \end{cases} \quad (4.7)$$

where  $v \in L^2_{\text{loc}}(0, \infty)$  stands for  $u'$ . Note that if  $F \equiv 1$  then System (4.7) turns to be (4.5), and if  $v \equiv 0$ , System (4.7) turns to be (4.6). Observe that the solutions to the first and the second equations of (4.7) satisfy

$$\frac{d}{dt}p(t, c+t) = \frac{d}{dt}q(t, c-t) = 0, \quad t \geq 0, c \in \mathbb{R}, c \pm t \in (0, 1). \quad (4.8)$$

Hence,  $p$  (resp.  $q$ ) is constant along the characteristic lines  $x - t = c$  (resp.  $x + t = c$ ). The idea is to use the boundary conditions

$$(p + F(t)q)(t, \alpha(t)) = v(t), \quad (p + q)(t, \beta(t)) = 0, \quad t > 0, \quad (4.9)$$

and the reflection of the characteristic lines  $x \pm t = c$ ,  $c \in \mathbb{R}$ , on the boundary curves  $(t, \alpha(t))_{t \geq 0}$  and  $(t, \beta(t))_{t \geq 0}$  to find the unique solution to system (4.7). Along this work, we assume that the boundary functions satisfy

$$\alpha(t) < \beta(t), \quad \forall t > 0, \quad \alpha, \beta \in C^1(0, \infty), \quad \max\left(\|\alpha'\|_{L^\infty(\mathbb{R}_+)}, \|\beta'\|_{L^\infty(\mathbb{R}_+)}\right) < 1. \quad (4.10)$$

The size assumption in (4.10) guarantees that the characteristic lines  $x = t + c$  (resp.  $x = c - t$ ) meet the curve  $(t, \alpha(t))_{t \geq 0}$  (resp.  $(t, \beta(t))_{t \geq 0}$ ) in finite time; also, they serve to ensure that the characteristic lines  $x \pm t = c$  are not gliding on the boundary curves or are not out of  $Q$ . In fact, assumption (4.10) is necessary for the existence of solutions.

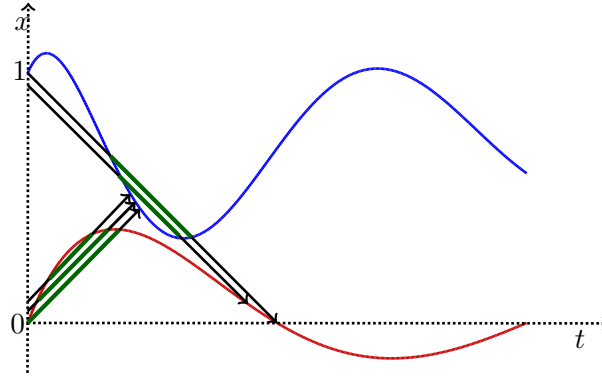


Figure 4.2: An example of a boundary curves  $(t, \alpha(t))_{t \geq 0}$  and  $(t, \beta(t))_{t \geq 0}$  that do not satisfy assumption (4.10). The values of the solution are not defined on the green part of the characteristic lines lying under or above these curves.

A straightforward consequence of assumption (4.10) is that the functions  $\alpha^\pm : [0, \infty) \rightarrow [0, \infty)$  and  $\beta^\pm : [0, \infty) \rightarrow [\pm 1, \infty)$  are invertible. In the sequel, we use the standard notations to denote their inverses by  $(\alpha^\pm)^{-1}$  and  $(\beta^\pm)^{-1}$ .

## 4.1 Main results

We start by giving the well-posedness result for System (4.7).

**Theorem 46** *Let  $(\tilde{p}, \tilde{q}, v, F) \in [L^2(0, 1)]^2 \times L^2_{\text{loc}}(0, \infty) \times C([0, \infty))$ . Assume that the boundary functions  $\alpha$  and  $\beta$  satisfy (4.10). Then, there exists a unique solution to System (4.7) satisfying*

$$(p, q) \in C\left([0, t]; [L^2(\alpha(t), \beta(t))]^2\right), \quad t \geq 0. \quad (4.11)$$

The proof of this theorem is a straightforward consequence of the explicit construction of the unique solution that will be done in Section 4.2.

**Remark 47** By inverting the transformation given in (4.4), we obtain

$$y_t = \frac{p+q}{2}, \quad y_x = \frac{q-p}{2},$$

hence, for any  $(y_0, y_1, u, f) \in H_{(1)}^1(0, 1) \times L^2(0, 1) \times H_{loc}^1(0, \infty) \times C([0, \infty))$ , the solutions to Systems (4.1) and (4.2) satisfy the regularity

$$y \in C\left([0, t]; H_{(\beta(t))}^1(\alpha(t), \beta(t))\right) \cap C^1\left([0, t]; L^2(\alpha(t), \beta(t))\right), \quad t \geq 0.$$

#### 4.1.1 Controllability result

**Definition 48** System (4.1) is said to be exactly controllable at time  $T > 0$  if for any initial state  $(y_0, y_1) \in H_{(1)}^1(0, 1) \times L^2(0, 1)$  and for any target state  $(h, k) \in H_{(\beta(T))}^1(\alpha(T), \beta(T)) \times L^2(\alpha(T), \beta(T))$ , there exists a control  $u \in H_{loc}^1(0, \infty)$  such that  $(y(T), y_t(T)) = (h, k)$ .

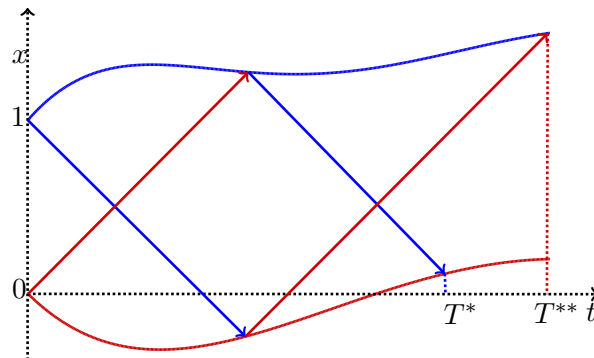
The following result shows that the minimal time  $T^*$  where exact controllability is possible depends on the movement of the boundaries and can be represented explicitly in terms of the functions  $\alpha^\pm$  and  $\beta^\pm$ . Moreover, also the unique exact control for  $T^*$  can be represented explicitly using these functions.

**Theorem 49** Let  $(y_0, y_1) \in H_{(1)}^1(0, 1) \times L^2(0, 1)$ . Assume that the boundary functions  $\alpha$  and  $\beta$  satisfy (4.10). System (4.1) is exactly controllable at time  $T > 0$  if, and only if  $T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)$ . Further, if  $T = T^*$ , there exists a unique control  $u \in H^1(0, T^*)$  steering the solution  $(y, y_t)$  to System (4.1) to the equilibrium point  $(0, 0)$  given by

$$u(t) = \begin{cases} \int_0^t y_1(\alpha^+(s)) ds + y_0(\alpha^+(t)), & \text{if } t \in [0, (\alpha^+)^{-1}(1)], \\ y_0(-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(t)) \\ + \int_0^{(\alpha^+)^{-1}(1)} y_1(\alpha^+(s)) ds \\ - \int_{(\alpha^+)^{-1}(1)}^t y_1(-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(s)) ds, & \text{if } t \in [(\alpha^+)^{-1}(1), T^*], \end{cases} \quad (4.12)$$

**Remark 50** The controllability result still makes sense even if the boundary curves  $(t, \alpha(t))_{t \geq 0}$  and  $(t, \beta(t))_{t \geq 0}$  are allowed to intersect in time larger than  $T^*$ .

**Remark 51** Let us consider the particular case  $\alpha(t) = kt$ ,  $\beta(t) = rt + 1$ ,  $k, r \in (-1, 1)$ , with  $\frac{2(k-r)}{(1-r)(1+k)} < \frac{1}{2}$  (The last assumption guarantees that the boundary curves do not intersect before  $T^*$ ). In this case, it can be checked that  $T^*$  is given by  $T^* = \frac{2}{(1-r)(k+1)}$  which is the same time found in [119]. In particular, if  $\alpha \equiv 0$  and  $\beta \equiv 1$ , we obtain the classical result  $T^* = 2$ .



**Remark 52** *The minimal time  $T^*$  is precisely the necessary time for the main characteristic line issued from the point  $(0, 0)$  to touch again the curve  $(t, \alpha(t))_{t \geq 0}$  in the point  $(T^*, \alpha(T^*))$  after having been reflected from the curve  $(t, \beta(t))_{t \geq 0}$ . More precisely, the characteristic line  $x = t$  hits the curve  $(t, \beta(t))_{t \geq 0}$  in the point  $\left((\beta^-)^{-1}(0), \beta\left((\beta^-)^{-1}(0)\right)\right)$ . The reflected characteristic line passing through the last point, i.e.  $x = -t + \beta^+ \circ (\beta^-)^{-1}(0)$  hits the curve  $(t, \alpha(t))_{t \geq 0}$  in the point  $(T^*, \alpha(T^*))$ . If the control  $u$  is located on the curve  $(t, \beta(t))_{t \geq 0}$  instead of  $(t, \alpha(t))_{t \geq 0}$ , then  $T^{**}$  is the analogous time for the main characteristic line issued from the point  $(0, 1)$  with negative slope. In this case  $T^{**} = (\beta^-)^{-1} \circ \alpha^- \circ (\alpha^+)^{-1}(1)$ .*

### 4.1.2 Stability result

For the sake of lighting notations, we introduce the function  $\phi := \phi(\alpha, \beta)$  defined by

$$\phi := \alpha^- \circ (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}. \quad (4.13)$$

By assumption (4.10), the function  $\phi : [-1, \infty) \rightarrow [\alpha^- \circ (\alpha^+)^{-1}(1), \infty)$  is well defined and increasing function as composition of increasing functions, and hence invertible with inverse

$$\phi^{-1} := \beta^- \circ (\beta^+)^{-1} \circ \alpha^+ \circ (\alpha^-)^{-1}.$$

Let  $(\psi_n(\cdot))_{n \geq 0}$  be a sequence of functions such that

$$\begin{aligned} \psi_n(\cdot) &: [0, \phi(0)) \rightarrow [0, \infty) \\ \tau &\mapsto \psi_n(\tau) = \prod_{i=0}^n \left| F \left( (\alpha^-)^{-1} \circ \phi^{[i]}(\tau) \right) \right|. \end{aligned} \quad (4.14)$$

The notation  $\phi^{[n]}$  refers to

$$\phi^{[n]} = \underbrace{\phi \circ \dots \circ \phi}_{n \text{ times}},$$

with the convention  $\phi^{[0]} = I$ . The following result shows that the asymptotic behavior of the solution to System (4.2) relies heavily on the behavior of the sequence of functions  $(\psi_n(\tau))_{n \geq 0}$  defined in (4.14) when  $n \rightarrow \infty$ .

**Theorem 53** *Let  $(y_0, y_1) \in H_{(1)}^1(0, 1) \times L^2(0, 1)$ . Assume that the boundary functions  $\alpha$  and  $\beta$  satisfy (4.10). In addition, assume that*

$$\phi(\tau) < \dots < \phi^{[n]}(\tau) < \phi^{[n+1]}(\tau) \xrightarrow[n \rightarrow \infty]{} \infty, \quad \forall \tau \in [0, \phi(0)), \quad (4.15)$$

then,

$$\|(y(t), y_t(t))\|_{H_{(\beta(t))}^1(\alpha(t), \beta(t)) \times L^2(\alpha(t), \beta(t))} \xrightarrow[t \rightarrow \infty]{} 0,$$

if, and only if

$$\psi_n(\tau) \xrightarrow[n \rightarrow \infty]{} 0, \quad \forall \tau \in [0, \phi(0)). \quad (4.16)$$

If there exists  $g \in C(\mathbb{R}, (0, \infty))$  such that

$$\psi_n(\tau) \underset{n \rightarrow \infty}{\sim} Cg\left(\phi^{[n]}(\tau)\right), \quad \forall \tau \in [0, \phi(0)), \quad (4.17)$$

for some positive constant  $C > 0$ , then the solution to System (4.2) decays like  $g$ , i.e.

$$\|(y(t), y_t(t))\|_{H_{(\beta(t))}^1(\alpha(t), \beta(t)) \times L^2(\alpha(t), \beta(t))} \leq Cg(t) \|(y_0, y_1)\|_{H_{(1)}^1(0, 1) \times L^2(0, 1)}. \quad (4.18)$$

In particular, there exists  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that

$$\|(y(t), y_t(t))\|_{H^1_{(\beta(t))}(\alpha(t), \beta(t)) \times L^2(\alpha(t), \beta(t))} \leq M e^{t\omega} \|(y_0, y_1)\|_{H^1_{(1)}(0,1) \times L^2(0,1)}, \quad \forall t \geq 0,$$

if, and only if

$$\sup_{\tau \in [0, \phi(0)]} \lim_{n \rightarrow \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = \omega < \infty. \quad (4.19)$$

If  $f \equiv 1$ , the solution to System (4.2) vanishes in finite time  $T$  if, and only if  $T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)$ , i.e.

$$y(T) \equiv y_t(T) \equiv 0, \quad \forall T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0).$$

Let us illustrate the previous theorem by some examples.

**Example 54 (Cylindrical domain)** If  $Q$  is cylindrical domain, i.e.  $\alpha \equiv 0$  and  $\beta \equiv 1$ , the function  $\phi$  defined in (4.13) is given by  $\phi(\tau) = \tau + 2$ , then,  $\phi^{[n]}(\tau) = \tau + 2n$ . Therefore, the sequence of functions  $(\psi_n(\cdot))_{n \geq 0}$  defined in (4.14) takes the form

$$\begin{aligned} \psi_n &: [0, 2) \rightarrow [0, \infty) \\ \tau &\mapsto \psi_n(\tau) = \prod_{i=0}^n |F(\tau + 2i)|. \end{aligned} \quad (4.20)$$

In this case, the assumptions of Theorem 53 can be checked easily. Note that since the feedback law in System (4.2) is non-autonomous ( $f$  is time dependent), we can achieve any decay rate we want (even faster than exponential) with a suitable choice of  $f$ . Below, we illustrate this fact by several examples:

- *Exponential decay:*

Let  $f(t) = \frac{2 - \sin(\pi t)}{2 + \sin(\pi t)}$ , therefore,  $F(t) = \frac{\sin(\pi t)}{2}$ , thus,

$$\psi_n(\tau) = \prod_{i=0}^n |F(\tau + 2i)| = \left[ \frac{\sin(\pi \tau)}{2} \right]^{n+1}.$$

By (4.19), we have

$$\begin{aligned} \sup_{\tau \in (0,1) \cup (1,2)} \lim_{n \rightarrow \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} &= \sup_{\tau \in (0,1) \cup (1,2)} \lim_{n \rightarrow \infty} \frac{(n+1) \ln \left| \frac{\sin(\pi \tau)}{2} \right|}{\tau + 2n} \\ &= \sup_{\tau \in (0,1) \cup (1,2)} \frac{1}{2} \ln \left| \frac{\sin(\pi \tau)}{2} \right| = -\frac{\ln 2}{2}, \end{aligned}$$

therefore, exponential decay occurs with growth bound  $\omega = -\frac{\ln 2}{2}$ .

- *Polynomial decay:*

Let  $f(t) = \frac{(t+1)^{-s} - (t+3)^{-s}}{(t+1)^{-s} + (t+3)^{-s}}$ ,  $s > 0$ , then  $F(t) = \left( \frac{t+3}{t+1} \right)^{-s}$ , consequently, the sequence of functions  $(\psi_n(\cdot))_{n \geq 0}$  defined in (4.20) takes the form

$$\psi_n(\tau) = \prod_{i=0}^n |F(\tau + 2i)| = \prod_{i=0}^n \left| \left( \frac{\tau + 2i + 3}{\tau + 2i + 1} \right)^{-s} \right| = \left( \frac{\tau + 2n + 3}{\tau + 1} \right)^{-s}.$$

Set  $g(t) = (t+1)^{-s}$ ,  $s > 0$ . A simple computation shows that

$$\lim_{n \rightarrow \infty} \frac{\psi_n(\tau)}{g(\phi^{[n]})} = \lim_{n \rightarrow \infty} \frac{\psi_n(\tau)}{g(\tau + 2n)} = \frac{1}{\tau + 1}, \quad \tau \in [0, 2),$$

thus, by (4.18), the solution to System (4.2) decays like  $(t+1)^{-s}$ ,  $s > 0$ .

- *Logarithmic decay:*

Let  $f(t) = \frac{\log^{-s}(t+1) - \log^{-s}(t+3)}{\log^{-s}(t+1) + \log^{-s}(t+3)}$ ,  $s > 0$ , then  $F(t) = \left(\frac{\log(t+3)}{\log(t+1)}\right)^{-s}$ , consequently, we obtain

$$\psi_n(\tau) = \prod_{i=0}^n |F(\tau + 2i)| = \prod_{i=0}^n \left| \left( \frac{\log(\tau + 2i + 3)}{\log(\tau + 2i + 1)} \right)^{-s} \right| = \left| \left( \frac{\log(\tau + 2n + 3)}{\log(\tau + 1)} \right)^{-s} \right|.$$

By letting  $g(t) = \log^{-s}(t + 1)$ ,  $s > 0$ , we get

$$\lim_{n \rightarrow \infty} \frac{\psi_n(\tau)}{g(\phi^{[n]})} = \log^s(\tau + 1) \lim_{n \rightarrow \infty} \left| \left( \frac{\log(\tau + 2n + 3)}{\log(\tau + 2n + 1)} \right)^{-s} \right| = \log^s(\tau + 1), \quad \tau \in [0, 2),$$

hence, (4.18) is satisfied with  $g(t) = \log^{-s}(t + 1)$ ,  $s > 0$ .

- *Super-stability:*

Let  $f(t) = \frac{t}{2+t}$ , therefore,  $F(t) = \frac{1}{t+1}$ , consequently, we obtain

$$\psi_n(\tau) = \prod_{i=0}^n \frac{1}{\tau + 2i + 1} = \frac{1}{(\tau + 1) 2^n n!} \prod_{i=1}^n \left( \frac{\tau + 1}{2i} + 1 \right)^{-1}.$$

A simple computation shows that

$$\log \prod_{i=1}^n \left( \frac{\tau + 1}{2i} + 1 \right)^{-1} \underset{n \rightarrow \infty}{\sim} C(\tau) \log n^{-\frac{\tau+1}{2}},$$

where  $C(\tau)$  is a positive constant depending on  $\tau$ . So, we get

$$\psi_n(\tau) \underset{n \rightarrow \infty}{\sim} \frac{C(\tau)}{(\tau + 1) n^{\frac{\tau+1}{2}} 2^n n!}, \quad (4.21)$$

which by (4.16) implies that the solution to System (4.2) decays to zero. To check whether the stability is exponential it suffices to use (4.19) to compute the growth bound  $\omega$ . By using (4.21) we obtain

$$\lim_{n \rightarrow \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = - \lim_{n \rightarrow \infty} \frac{\ln n^{\frac{\tau+1}{2}} + n \ln 2 + \ln n!}{2n + \tau} = -\infty = \omega.$$

Therefore, the decay rate for this choice of  $f$  is faster than any exponential function. Actually, this phenomena is called *super-stability*. For more of details, we refer the reader to [19].

**Example 55 (Non cylindrical domain)** Things are more delicate in the non-cylindrical case. Consider a boundary functions of the form  $\alpha(t) = rt$ ,  $\beta(t) = kt + 1$ ,  $r, k \in (-1, 1)$ . To guarantee that  $\alpha(t) \neq \beta(t)$ ,  $\forall t \geq 0$ , we assume that  $k \geq r$ . The function  $\phi$  defined in (4.13) will be given by

$$\phi(\tau) = \frac{(1+k)(1-r)}{(1-k)(1+r)} \tau + \frac{2(1-r)}{(1-k)(1+r)} = a\tau + b,$$

therefore, we obtain

$$\phi^{[n]}(\tau) = \begin{cases} a^n \left( \tau - \frac{b}{1-a} \right) + \frac{b}{1-a}, & \text{if } r < k, \\ \tau + \frac{2n}{1+r}, & \text{if } r = k. \end{cases} \quad (4.22)$$



Consequently,

$$(\alpha^-)^{-1} \circ \phi^{[n]}(\tau) = \begin{cases} a^n \left( \frac{\tau}{1-r} - \frac{b}{(1-a)(1-r)} \right) + \frac{b}{(1-a)(1-r)}, & \text{if } r < k, \\ \frac{\tau}{1-r} + \frac{2n}{(1+r)(1-r)}, & \text{if } r = k. \end{cases}$$

For simplicity, let us take  $f$  as in the previous example,  $f(t) = \frac{t}{2+t}$  which implies that  $F(t) = \frac{1}{t+1}$ . So, we have:

- If  $r < k$ :

From (4.22), we can check that (4.15) is satisfied if, and only if  $a > 1$ . To verify (4.16), it is enough to estimate its asymptotic behavior for a large  $n$ . So, we have

$$\begin{aligned} \psi_n(\tau) &= \prod_{i=0}^n \frac{1}{\left| a^i \left( \frac{\tau}{1-r} - \frac{b}{(1-a)(1-r)} \right) + \frac{b}{(1-a)(1-r)} + 1 \right|} = \prod_{i=0}^n \frac{1}{|a^i s(\tau) + z|} \\ &= \frac{1}{a^{\frac{n(n+1)}{2}} s^{n+1}(\tau)} \prod_{i=0}^n \left| 1 + \frac{z}{a^i s(\tau)} \right|^{-1}. \end{aligned}$$

Since  $a > 1$ , the series  $\sum_{i=0}^{\infty} \ln \left( 1 + \frac{z}{a^i s(\tau)} \right)$  converges, we obtain,

$$\psi_n(\tau) \underset{n \rightarrow \infty}{\sim} C(r, k, \tau) a^{-\frac{n(n+1)}{2}} s^{-n-1}(\tau), \quad \forall \tau \in [0, b), \quad (4.23)$$

where  $C(r, k, \tau)$  is a positive constant depending on  $r, k$  and  $\tau$ . In view of (4.23), if  $a > 1$ , the solution to System (4.2) decays to zero. On the contrary of the cylindrical domain case, even with this choice of the feedback function  $f$ , (4.19) is not satisfied, and hence, exponential stability cannot occur. Indeed,

$$\lim_{n \rightarrow \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = - \lim_{n \rightarrow \infty} \frac{n^2}{2a^n} \ln |a| = 0, \quad \forall \tau \in [0, b).$$

Nevertheless, we still be able to get an idea about the decay rate. From (4.23), we observe that the term that really matters is  $a^{-\frac{n^2}{2}}$ , so, for  $g(t) = e^{-\frac{1}{2} \log_a^2(t)}$ , we obtain

$$a^{-\frac{n^2}{2}} \underset{n \rightarrow \infty}{\sim} Cg(a^n s(\tau) + z), \quad \forall \tau \in [0, b).$$

Note that we did not lose too much since  $g$  decays to zero faster than any polynomial function. This loss can be justified by the fact that the characteristic lines will need a larger time to reflect on the two boundary lines when  $t$  becomes larger.

- If  $k = r$ :

In this case, the lines  $x = rt$  and  $x = kt + 1$  are parallel, therefore, the needed time from the characteristics to reflect on the two endpoints is the same, so we might expect super-stability with this choice of  $f$ . Let us first check that whether the solution to System (4.2) decays exponentially or not. By (4.21), the sequence of functions  $(\psi_n(\cdot))_{n \geq 0}$  behaves like

$$\begin{aligned} \psi_n(\tau) &= \prod_{i=0}^n \left| \frac{1}{\frac{\tau}{1-r} + 1 + \frac{2i}{(1+r)(1-r)}} \right| \\ &= \frac{(1+r)^n (1-r)^n}{2^n n! \left( \frac{\tau}{1-r} + 1 \right)} \prod_{i=1}^n \left| \frac{1}{\frac{(1+r)(1-r)}{2i} \left( \frac{\tau}{1-r} + 1 \right) + 1} \right|. \\ &\underset{n \rightarrow \infty}{\sim} C(r, \tau) \frac{(1+r)^n (1-r)^{n+1}}{2^n n! (\tau + 1 - r) n^{\frac{(1+r)(\tau+1-r)}{2}}}, \end{aligned}$$

where  $C(r, \tau)$  is a positive constant depending on  $r$  and  $\tau$ . By using (4.19), we get

$$\lim_{n \rightarrow \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = \lim_{n \rightarrow \infty} -\frac{\ln n!}{2n} = -\infty = \omega,$$

therefore, the solution to System (4.2) is super-stable.

**Example 56 (Constant feedback)** Consider the case when  $f$  is a constant such that  $f \neq 1$  with keeping  $\alpha$  and  $\beta$  as in the previous example. A simple computation yields

$$\psi_n = F^{n+1} = \left| \frac{f-1}{f+1} \right|^{n+1}.$$

Therefore, by using the formula (4.19), we arrive at:

- If  $r < k$ :

We can check that the decay is not exponential. Indeed,

$$\lim_{n \rightarrow \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = \lim_{n \rightarrow \infty} \frac{(n+1) \ln \left| \frac{f-1}{f+1} \right|}{a^n} = 0, \quad \forall \tau \in [0, b).$$

Nonetheless, by (4.18), we can determine the decay rate for a particular values of  $f$ . Let  $g(t) = t^{-s}$ . It is easy to check that if  $a^{-s} = \frac{f-1}{f+1}$  for some  $s > 0$  then

$$\lim_{n \rightarrow \infty} \frac{\psi_n(\tau)}{g(\phi^{[n]}(\tau))} = \lim_{n \rightarrow \infty} \frac{\left| \frac{f-1}{f+1} \right|^{n+1}}{(a^n s(\tau) + z)^{-s}} = C(\tau, r, k), \quad \forall \tau \in [0, b),$$

where  $C(r, k, \tau)$  is a positive constant depending on  $r, k$  and  $\tau$ . Hence, the solution decays like  $t^{-s}$ ,  $s > 0$ .

- If  $r = k$ :

In this case, we have

$$\lim_{n \rightarrow \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = \lim_{n \rightarrow \infty} \frac{(n+1) \ln \left| \frac{f-1}{f+1} \right|}{\frac{2n}{(1+r)(1-r)}} = \frac{(1+r)(1-r)}{2} \ln \left| \frac{f-1}{f+1} \right| = \omega,$$

hence, exponential decay occurs with growth bound  $-\omega$ . In particular, if  $Q$  is a cylindrical domain ( $r = 0$ ), the solution to System (4.2) is exponentially stable if, and only if

$$\frac{1}{2} \ln \left| \frac{f-1}{f+1} \right| = \omega < 0,$$

which is a known result from [116].

**Remark 57** We have seen in the previous examples that the decay rate is determined in a crucial way by the boundary functions and the damping function. Actually, we can do the converse for System (4.2). Namely, by setting

$$F(t) = \frac{g(\phi \circ \alpha^-(t))}{g(\alpha^-(t))}, \quad \forall t \geq 0,$$

with  $g(t) \neq 0$ , for all  $t \geq 0$ , we obtain

$$\psi_n(\tau) = \prod_{i=0}^n \left| F \left( (\alpha^-)^{-1} \circ \phi^{[i]}(\tau) \right) \right| = \prod_{i=0}^n \left| \frac{g(\phi^{[i+1]}(\tau))}{g(\phi^{[i]}(\tau))} \right| = \left| \frac{g(\phi^{[n+1]}(\tau))}{g(\phi^{[0]}(\tau))} \right|.$$

In this case, (3.57) is automatically satisfied, and since  $F = \frac{1-f}{1+f}$ , we obtain

$$\frac{g(\alpha^-(t)) - g(\phi \circ \alpha^-(t))}{g(\alpha^-(t)) + g(\phi \circ \alpha^-(t))} = f_g(t), \quad \forall t \geq 0. \quad (4.24)$$

The last expression provides an explicit relation between the decay rate and the feedback function  $f$ . This means that  $f$  can be determined based on the desired decay rate. Formula (4.24) has been used to construct  $f$  in the second and the third points in example (54).

**Remark 58** Examples 55 and 56 illustrate the big influence of the boundary nature on the decay rate of the solution to System (4.2).

## 4.2 Construction of the exact solution

The aim now is to find the solution  $(p, q)$  to System (4.7) in all  $Q$ . To this end, let us start by splitting  $Q$  into an infinite number of parts. Namely,

$$Q = \cup_{n \geq 0} \Sigma_n^p = \cup_{n \geq 0} \Sigma_n^q, \quad \Sigma_i^p \cap \Sigma_j^p, \Sigma_i^q \cap \Sigma_j^q = \emptyset, \quad i \neq j,$$

where  $\Sigma_n^p, \Sigma_n^q$  are given for  $n = 0, 1$ , by

$$\Sigma_0^p = \{(t, x) \in Q, t \in [0, x]\}, \quad (4.25)$$

$$\Sigma_1^p = \{(t, x) \in Q, t - x \in [0, \alpha^- \circ (\alpha^+)^{-1}(1)]\}, \quad (4.26)$$

$$\Sigma_0^q = \{(t, x) \in Q, t \in [0, 1 - x]\}, \quad (4.27)$$

$$\Sigma_1^q = \{(t, x) \in Q, t + x \in [1, \beta^+ \circ (\beta^-)^{-1}(0)]\}, \quad (4.28)$$

and for all  $n \geq 1$

$$\Sigma_{2n}^p = \{(t, x) \in Q, t - x \in [\phi^{[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1}(1), \phi^{[n]}(0)]\}, \quad (4.29)$$

$$\Sigma_{2n+1}^p = \{(t, x) \in Q, t - x \in [\phi^{[n]}(0), \phi^{[n]} \circ \alpha^- \circ (\alpha^+)^{-1}(1)]\}, \quad (4.30)$$

$$\Sigma_{2n}^q = \{(t, x) \in Q, t + x \in [\xi^{[n-1]} \circ \beta^+ \circ (\beta^-)^{-1}(0), \xi^{[n]}(1)]\}, \quad (4.31)$$

$$\Sigma_{2n+1}^q = \{(t, x) \in Q, t + x \in [\xi^{[n]}(1), \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0)]\}, \quad (4.32)$$

where  $\xi$  is defined by

$$\xi := \beta^+ \circ (\beta^-)^{-1} \circ \alpha^- \circ (\alpha^+)^{-1}. \quad (4.33)$$

The construction of these regions relies on the reflection of the principal characteristic lines with positive and negative slopes emerging from the points  $(0, 0)$  and  $(0, 1)$  and reflected along the boundary curves. More precisely, the lines  $x = t$  and  $x = -t + 1$  emerging respectively from  $(0, 0)$  and  $(0, 1)$  meet the curves  $(t, \beta(t))_{t \geq 0}$  and  $(t, \alpha(t))_{t \geq 0}$  in the points  $\left( (\beta^-(0))^{-1}, \beta((\beta^-(0))^{-1}) \right)$  and  $\left( (\alpha^+)^{-1}(1), \alpha((\alpha^+(1))^{-1}) \right)$  respectively. The regions  $\Sigma_0^p$  and  $\Sigma_0^q$  are those located between  $t = 0$  and these lines. We can do similarly to

construct the regions  $\Sigma_n^p, \Sigma_n^q$ ,  $n \geq 1$ , given above. In the sequel, we denote by  $p_n$  and  $q_n$  the restriction of  $p$  and  $q$  solutions to System (4.7) on  $\Sigma_n^p$  and  $\Sigma_n^q$ ,  $n \geq 0$ .

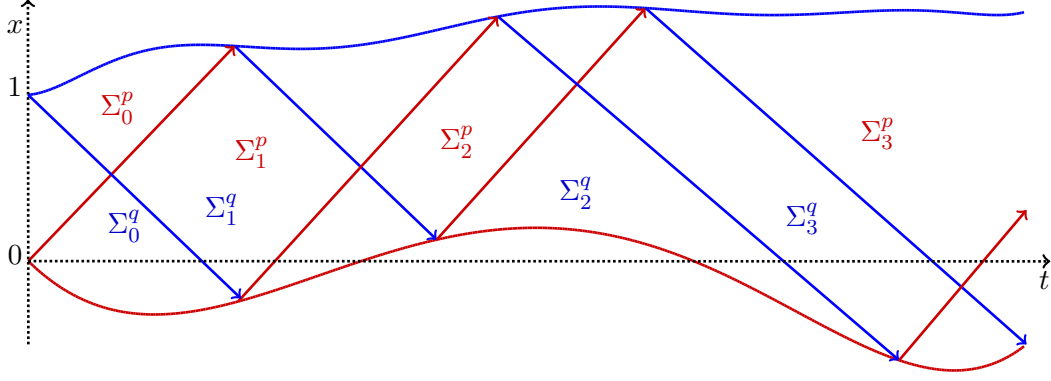


Figure 4.3: The regions  $\Sigma_i^p$  are those between the red lines and  $\Sigma_i^q$  are those between the blue lines.

**Remark 59** In particular, if  $\alpha \equiv 0$  and  $\beta \equiv 1$ , the regions  $\Sigma_n^p, \Sigma_n^q$ ,  $n \geq 0$ , are simply given by

$$\Sigma_n^p = \{(t, x) \in \mathbb{R}_+ \times [0, 1], t - x \in [n - 1, n)\},$$

$$\Sigma_n^q = \{(t, x) \in \mathbb{R}_+ \times [0, 1], x + t \in [n, n + 1)\}.$$

During the construction below, we use the standard density argument by assuming first that the initial states are sufficiently regular then passing to the limit. So, the constructed solutions must be understood in the weak sense. Let us start by finding  $p_0$  and  $q_0$  :

**Lemma 60** Let  $(\tilde{p}, \tilde{q}) \in [L^2(0, 1)]^2$ . The solution  $(p_0, q_0)$  to System (4.7) is given by

$$p_0(t, x) = \tilde{p}(x - t), \quad q_0(t, x) = \tilde{q}(x + t). \quad (4.34)$$

**Proof.** The proof readily follows from (4.8). ■

Now, let us find the solution in the regions  $\Sigma_1^p, \Sigma_1^q$  :

**Lemma 61** Let  $(\tilde{p}, \tilde{q}) \in [L^2(0, 1)]^2$ . The solution  $(p_1, q_1)$  to System (4.7) is given by

$$p_1(t, x) = v \left( (\alpha^-)^{-1}(t - x) \right) - F \left( (\alpha^-)^{-1}(t - x) \right) \tilde{q} \left( \alpha^+ \circ (\alpha^-)^{-1}(t - x) \right), \quad (4.35)$$

$$q_1(t, x) = -\tilde{p} \left( -\beta^- \circ (\beta^+)^{-1}(x + t) \right). \quad (4.36)$$

**Proof.** By using (4.34), we have at the boundary

$$p_0(\tau, \beta(\tau)) = \tilde{p}(-\beta^-(\tau)), \quad \tau \in \left[0, (\beta^-)^{-1}(0)\right),$$

$$q_0(\chi, \alpha(\chi)) = \tilde{q}(\alpha^+(\chi)), \quad \chi \in \left[0, (\alpha^+)^{-1}(1)\right).$$

By using the boundary conditions given in (4.9), we get

$$\begin{aligned} p_1(\tau, \alpha(\tau)) &= v(\tau) - F(\tau)q_0(\tau, \alpha(\tau)) \\ &= v(\tau) - F(\tau)\tilde{q}(\alpha^+(\tau)), \quad \tau \in \left[0, (\alpha^+)^{-1}(1)\right), \end{aligned}$$

$$q_1(\chi, \beta(\chi)) = -p_0(\chi, \beta(\chi)) = -\tilde{p}(-\beta^-(\chi)), \quad \chi \in \left[0, (\beta^-)^{-1}(0)\right).$$

Consider the latter values as initial states on both regions  $\Sigma_1^p, \Sigma_1^q$  and use (4.8), we write

$$p_1(t, c-t) = p_1(\tau, \tau-s) \quad , \quad q_1(\chi, c+\chi) = q_1(\chi, c+\chi).$$

By using the fact that  $p$  and  $q$  are constant along the characteristic lines  $x = t - \alpha^-(\tau)$  and  $x = -t + \beta^+(\chi)$  respectively, we obtain

$$p_1(t, t - \alpha^-(\tau)) = p_1(\tau, \alpha(\tau)) = v(\tau) - F(\tau)\tilde{q}(\alpha^+(\tau)), \quad (4.37)$$

and

$$q_1(t, -t + \beta^+(\chi)) = q_1^-(\chi, \beta(\chi)) = -\tilde{p}(-\beta^-(\chi)). \quad (4.38)$$

Now, letting  $(\alpha^-)^{-1}(t-x) = \tau$  in (4.37) and  $\chi = (\beta^+)^{-1}(x+t)$  in (4.38) yields the desired result. ■

**Remark 62** Note that  $\alpha^+ \circ (\alpha^-)^{-1}(t-x)$ ,  $(t, x) \in \Sigma_1^p$  and  $-\beta^- \circ (\beta^+)^{-1}(x+t)$ ,  $(t, x) \in \Sigma_1^q$  belong to  $(0, 1)$  and the above expressions make perfectly sense. To clarify more things, let  $(t, x) \in \Sigma_1^p$  and let  $\tilde{x}(s) = s - t + x$  the line passing through the point  $(t, x)$ . By moving backwards, this line meets the curve  $(s, \alpha(s))_{s \geq 0}$  at the point  $\left((\alpha^-)^{-1}(t-x), \alpha(\alpha^-)^{-1}(t-x)\right)$  where  $(\alpha^-)^{-1}(t-x) \in \left[0, (\alpha^-)^{-1}(1)\right)$ . We use again the reflection of the characteristic line with negative slope passing through the latter point. i.e.  $\tilde{x}(s) = -s + \alpha^+ \circ (\alpha^-)^{-1}(t-x)$  lying in  $\Sigma_0^p$ , for  $s = 0$ , we obtain  $\tilde{x}(0) = \alpha^+ \circ (\alpha^-)^{-1}(t-x) \in (0, 1)$ . We can do similarly for  $-\beta^- \circ (\beta^+)^{-1}(x+t)$ ,  $(t, x) \in \Sigma_1^q$ .

**Lemma 63** Let  $(\tilde{p}, \tilde{q}) \in [L^2(0, 1)]^2$ . The solution  $(p_2, q_2)$  to System (4.7) is given by

$$\begin{aligned} p_2(t, x) &= v\left((\alpha^-)^{-1}(t-x)\right) \\ &+ F\left((\alpha^-)^{-1}(t-x)\right)\tilde{p}(-\phi^{-1}(t-x)), \end{aligned} \quad (4.39)$$

$$\begin{aligned} q_2(t, x) &= -v\left((\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1}(x+t)\right) \\ &+ F\left((\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1}(x+t)\right)\tilde{q}(\xi^{-1}(x+t)), \end{aligned} \quad (4.40)$$

where  $\phi$  and  $\xi$  are defined in (4.13) and (4.33).

**Proof.** From (4.35) and (4.36), we have at the boundary

$$\begin{aligned} p_1(\tau, \beta(\tau)) &= v\left((\alpha^-)^{-1} \circ \beta^-(\tau)\right) \\ &- F\left((\alpha^-)^{-1} \circ \beta^-(\tau)\right)\tilde{q}\left(\alpha^+ \circ (\alpha^-)^{-1} \circ \beta^-(\tau)\right), \\ \tau &\in \left[(\beta^-)^{-1}(0), (\beta^-)^{-1} \circ \alpha^- \circ (\alpha^+)^{-1}(1)\right) \end{aligned} \quad (4.41)$$

and

$$\begin{aligned} q_1(\chi, \alpha(\chi)) &= -\tilde{p}\left(-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(\chi)\right), \\ \chi &\in \left[(\alpha^+)^{-1}(1), (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)\right]. \end{aligned} \quad (4.42)$$

In order to find  $p_2$  and  $q_2$ , we use the boundary conditions (4.9) and the values of  $p_1$  and  $q_1$  at the boundary given in (4.41) and (4.42) as initial states. Namely, for any  $\tau$  and  $\chi$  such that

$$\tau \in \left[(\beta^-)^{-1}(0), (\beta^-)^{-1} \circ \alpha^- \circ (\alpha^+)^{-1}(1)\right]$$

and

$$\chi \in \left[(\alpha^+)^{-1}(1), (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)\right],$$

we have along the lines  $x = t - \alpha^-(\tau)$  and  $x = -t + \beta^+(\chi)$  respectively

$$p_2(t, t - \alpha^-(\tau)) = p_2(\tau, \alpha(\tau)) = v(\tau) - F(\tau)q_1(\tau, \alpha(\tau)), \quad (4.43)$$

$$q_2(t, \beta^+(\chi) - t) = q_2(\chi, \beta(\chi)) = -p_1^-(\chi, \beta(\chi)). \quad (4.44)$$

Plugging (4.41) and (4.42) in (4.43) and (4.44), we get

$$p_2(t, t - \alpha^-(\tau)) = v(\tau) + F(\tau)\tilde{p}\left(-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(\tau)\right),$$

and

$$\begin{aligned} q_2(t, \beta^+(\chi) - t) &= -v\left((\alpha^-)^{-1} \circ \beta^-(\chi)\right) \\ &+ F\left((\alpha^-)^{-1} \circ \beta^-(\chi)\right)\tilde{q}\left(\alpha^+ \circ (\alpha^-)^{-1} \circ \beta^-(\chi)\right). \end{aligned}$$

The proof follows immediately for  $\tau = (\alpha^-)^{-1}(t - x)$  and  $(\beta^+)^{-1}(x + t) = \chi$ . ■

**Remark 64** *In the same spirit of Remark 62, the expressions (4.39) and (4.40) make perfectly sense. We can use the same reasoning to show that*

$$\begin{aligned} -\beta^- \circ (\beta^+)^{-1} \circ \alpha^+ \circ (\alpha^-)^{-1}(t - x) &\in (0, 1), \quad \forall (t, x) \in \Sigma_2^p, \\ \alpha^+ \circ (\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1}(x + t) &\in (0, 1), \quad \forall (t, x) \in \Sigma_2^q. \end{aligned}$$

More generally, we have:

**Lemma 65** *Let  $(\tilde{p}, \tilde{q}) \in [L^2(0, 1)]^2$ . The solutions  $p_{2n+1}, p_{2n+2}, q_{2n+1}, q_{2n+2}$ ,  $n \geq 1$ , to System (4.7) are given by*

$$\begin{aligned} &p_{2n+1}(t, x) \\ &= \sum_{k=0}^n v\left((\alpha^-)^{-1} \circ (\phi^{-1})^{[k]}(t - x)\right) \prod_{i=0}^{k-1} F\left((\alpha^-)^{-1} \circ (\phi^{-1})^{[i]}(t - x)\right) \\ &\quad - \tilde{q}\left((\xi^{-1})^{[n]} \circ \alpha^+ \circ (\alpha^-)^{-1}(t - x)\right) \prod_{k=0}^n F\left((\alpha^-)^{-1} \circ (\phi^{-1})^{[k]}(t - x)\right), \end{aligned} \quad (4.45)$$

$$\begin{aligned}
& p_{2n+2}(t, x) \tag{4.46} \\
= & \sum_{k=0}^n v \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]}(t-x) \right) \prod_{i=0}^{k-1} F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]}(t-x) \right) \\
& + \tilde{p} \left( -(\phi^{-1})^{[n+1]} \circ (t-x) \right) \prod_{k=0}^n F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]}(t-x) \right),
\end{aligned}$$

$$\begin{aligned}
& q_{2n+1}(t, x) \tag{4.47} \\
= & - \sum_{k=0}^{n-1} v \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^- \circ (\beta^+)^{-1}(x+t) \right) \times \\
& \prod_{i=0}^{k-1} F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]} \circ \beta^- \circ (\beta^+)^{-1}(x+t) \right) \\
& - \tilde{p} \left( -(\phi^{-1})^{[n]} \circ \beta^- \circ (\beta^+)^{-1}(x+t) \right) \times \\
& \prod_{k=0}^{n-1} F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^- \circ (\beta^+)^{-1}(x+t) \right),
\end{aligned}$$

$$\begin{aligned}
& q_{2n+2}(t, x) \tag{4.48} \\
= & - \sum_{k=0}^n v \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^- \circ (\beta^+)^{-1}(x+t) \right) \times \\
& \prod_{i=0}^{k-1} F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]} \circ \beta^- \circ (\beta^+)^{-1}(x+t) \right) \\
& + \tilde{q} \left( (\xi^{-1})^{[n+1]}(x+t) \right) \times \\
& \prod_{k=0}^n F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^- \circ (\beta^+)^{-1}(x+t) \right),
\end{aligned}$$

with the convention  $\prod_{k=0}^{-1} = 1$ . The functions  $\phi$  and  $\xi$  are defined in (4.13) and (4.33).

**Proof.** The above expressions can be proved by induction. Let us start by proving (4.48). At the boundary  $x = \beta(t)$ , (4.45) becomes

$$\begin{aligned}
& p_{2n+1}(t, \beta(t)) \\
= & \sum_{k=0}^n v \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^-(t) \right) \prod_{i=0}^{k-1} F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]} \circ \beta^-(t) \right) \\
& - \tilde{q} \left( (\xi^{-1})^{[n]} \circ \alpha^+ \circ (\alpha^-)^{-1} \circ \beta^-(t) \right) \prod_{k=0}^n F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^-(t) \right).
\end{aligned}$$

Now, we use the boundary condition given in (4.9), i.e.

$$\begin{aligned}
q_{2n+2}(\chi, \beta(\chi)) &= -p_{2n+1}(\chi, \beta(\chi)), \\
\chi &\in \left[ (\beta^{-1})^{-1} \circ \phi^{[n]}(0), (\beta^{-1})^{-1} \circ \phi^{[n]} \circ \alpha^- \circ (\alpha^+)^{-1}(1) \right],
\end{aligned}$$

we find

$$\begin{aligned}
& q_{2n+2}(\chi, \beta(\chi)) \tag{4.49} \\
= & -\sum_{k=0}^n v \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^-(\chi) \right) \prod_{i=0}^{k-1} F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]} \circ \beta^-(t) \right) \\
& + \tilde{q} \left( (\xi^{-1})^{[n]} \circ \alpha^+ \circ (\alpha^-)^{-1} \circ \beta^-(\chi) \right) \prod_{k=0}^n F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^-(\chi) \right).
\end{aligned}$$

Since  $q$  is constant along the characteristic lines of the form  $x = c - t$ , in particular, on the line  $x = \beta^+(\chi) - t$ , we have

$$q_{2n+2}(t, \beta^+(\chi) - t) = q_{2n+2}(\chi, \beta(\chi)).$$

Finally, by letting  $\chi = (\beta^+)^{-1}(x + t)$  in (4.49), we obtain the formula in (4.48). Let us do similarly for  $p_{2n+2}$ . By taking (4.47) for  $x = \alpha(t)$ , we obtain

$$\begin{aligned}
& q_{2n+1}(t, \alpha(t)) \tag{4.50} \\
= & -\sum_{k=0}^{n-1} v \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^- \circ (\beta^+)^{-1} \circ \alpha^+(t) \right) \times \\
& \prod_{i=0}^{k-1} F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]} \circ \beta^- \circ (\beta^+)^{-1} \circ \alpha^+(t) \right) \\
& - \tilde{p} \left( -(\phi^{-1})^{[n]} \circ \beta^- \circ (\beta^+)^{-1} \circ \alpha^+(t) \right) \times \\
& \prod_{k=0}^{n-1} F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k]} \circ \beta^- \circ (\beta^+)^{-1} \circ \alpha^+(t) \right).
\end{aligned}$$

Using the boundary condition

$$\begin{aligned}
p_{2n+2}(\tau, \alpha(\tau)) &= v(\tau) - F(\tau)q_{2n+1}(\tau, \alpha(\tau)), \\
\tau &\in \left[ (\alpha^+)^{-1} \circ \xi^{[n]}(1), (\alpha^+)^{-1} \circ \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0) \right],
\end{aligned}$$

and the fact that  $q$  is constant along the characteristic lines  $x = c - t$ , in particular, on the line  $x = t - \alpha^-(\tau)$ , we obtain

$$p_{2n+2}(\tau, t - \alpha^-(\tau)) = v(\tau) - F(\tau)q_{2n+1}(\tau, \alpha(\tau)). \tag{4.51}$$

By letting  $\tau = (\alpha^-)^{-1}(t - x)$  in (4.50) and plugging the result in (4.51) then using the definition of  $\phi$  given in (4.13), we get

$$\begin{aligned}
& p_{2n+2}(t, x) \\
= & v((\alpha^-)^{-1}(t - x)) + \sum_{k=0}^{n-1} v \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k+1]}(t - x) \right) \times \\
& \prod_{i=0}^{k-1} F \left( (\alpha^-)^{-1}(t - x) \right) F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i+1]}(t - x) \right) \\
& + F \left( (\alpha^-)^{-1}(t - x) \right) \tilde{p} \left( -(\phi^{-1})^{[n+1]}(t - x) \right) \times \\
& \prod_{k=0}^{n-1} F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[k+1]}(t - x) \right).
\end{aligned}$$

After some manipulation we obtain the formula in (4.46). ■





**Proof.** It suffices to prove it for  $T = T^*$ . Let  $(h, k) \in L^2(\alpha(T^*), \beta(T^*))$  be a target state and let  $T^{**} = (\beta^-)^{-1} \circ \alpha^- \circ (\alpha^+)^{-1}(1)$ . We have three possible configurations:

Case 1:  $T^{**} = T^*$

In this case, we have  $p(T^*) = p_2(T^*)$  and  $q(T^*) = q_2(T^*)$ , then by making use of (4.39) and (4.40) we obtain

$$\begin{aligned} h(x) &= p_2(T^*, x) = v \left( (\alpha^-)^{-1}(T^* - x) \right) + \tilde{p}(-\phi^{-1}(T^* - x)), \quad x \in (\alpha(T^*), \beta(T^*)) \\ k(x) &= -v \left( (\alpha^-)^{-1} \circ \beta^- \circ (\beta^+)^{-1}(x + T^*) \right) \\ &\quad + \tilde{q}(\xi^{-1}(x + T^*)), \quad x \in (\alpha(T^*), \beta(T^*)). \end{aligned}$$

Therefore, the control  $v$  is given by

$$v(t) = \begin{cases} h(T^* - \alpha^-(t)) - \tilde{p}(-\phi^{-1} \circ \alpha^-(t)), & \text{if } t \in \left( (\alpha^-)^{-1} \circ \beta^-(T^*), T^* \right), \\ \tilde{q}(\alpha^+(t)) \\ -k \left( \beta^+ \circ (\beta^-)^{-1} \circ \alpha^-(t) - T^* \right), & \text{if } t \in \left( 0, (\alpha^-)^{-1} \circ \beta^-(T^*) \right). \end{cases}$$

Case 2:  $T^{**} < T^*$

In this case,  $p(T^*)$  and  $q(T^*)$  are defined by

$$p(T^*, x) = \begin{cases} p_1(T^*, x), & \text{if } x \in \left( T^* - \alpha^- \circ (\alpha^+)^{-1}(1), \beta(T^*) \right), \\ p_2(T^*, x), & \text{if } x \in \left( \alpha(T^*), T^* - \alpha^- \circ (\alpha^+)^{-1}(1) \right), \end{cases}$$

and  $q(T^*) = q_2(T^*)$ . Thus, by making use of (4.35), (4.39) and (4.40), then making some variable substitutions, we arrive at

$$v(t) = \begin{cases} h_1(T^* - \alpha^-(t)) + \tilde{q}(\alpha^+(t)), & \text{if } t \in \left( (\alpha^-)^{-1} \circ \beta^-(T^*), (\alpha^+)^{-1}(1) \right), \\ h_2(T^* - \alpha^-(t)) - \tilde{p}(-\phi^{-1} \circ \alpha^-(t)), & \text{if } t \in \left( (\alpha^+)^{-1}(1), T^* \right), \\ \tilde{q}(\alpha^+(t)) \\ -k \left( \beta^+ \circ (\beta^-)^{-1} \circ \alpha^-(t) - T^* \right), & \text{if } t \in \left( 0, (\alpha^-)^{-1} \circ \beta^-(T^*) \right), \end{cases}$$

where  $h_1$  and  $h_2$  are the restrictions of the target state  $h$  on the regions  $\Sigma_1^p$  and  $\Sigma_2^p$  respectively.

Case 3:  $T^{**} > T^*$

In this case, we have  $p(T^*) = p_2(T^*)$ , and  $q(T^*)$  is defined by

$$q(T^*, x) = \begin{cases} q_2(T^*, x), & \text{if } x \in (\alpha(T^*), \xi(1) - T^*), \\ q_3(T^*, x), & \text{if } x \in (\xi(1) - T^*, \beta(T^*)). \end{cases}$$

By using (4.39), (4.40) and (4.47) for  $n = 1$  and  $t = T^*$ , then making some variable substitutions, we obtain

$$v(t) = \begin{cases} h(T^* - \alpha^-(t)) - \tilde{p}(-\phi^{-1} \circ \alpha^-(t)), & \text{if } t \in \left( (\alpha^-)^{-1} \circ \beta^-(T^*), T^* \right), \\ \tilde{q}(\alpha^+(t)) \\ -k_2(\beta^+ \circ (\beta^-)^{-1} \circ \alpha^-(t) - T^*), & \text{if } t \in \left( 0, (\alpha^+)^{-1}(1) \right), \\ -\tilde{p} \left( -\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(t) \right) \\ -k_3(\beta^+ \circ (\beta^-)^{-1} \circ \alpha^-(t) - T^*), & \text{if } t \in \left( (\alpha^+)^{-1}(1), (\alpha^-)^{-1} \circ \beta^-(T^*) \right), \end{cases}$$

where  $k_2$  and  $k_3$  are the restrictions of the target state  $k$  on the regions  $\Sigma_2^q$  and  $\Sigma_3^q$  respectively. The above expressions are well defined and the control  $v$  is uniquely determined on  $[0, T^*)$ . In particular, from (4.39) and (4.40), we can see that the control

$$v(t) = \begin{cases} \tilde{q}(\alpha^+(t)), & \text{if } t \in [0, (\alpha^+)^{-1}(1)], \\ -\tilde{p}(-\beta^- \circ (\beta^+)^{-1} \circ \alpha^+(t)), & \text{if } t \in [(\alpha^+)^{-1}(1), T^*), \\ 0, & \text{if } t \geq T^*, \end{cases}$$

makes  $p_2$  and  $q_2$  vanish, then by the boundary conditions given in (4.9) all the solutions  $p_n, q_n, n \geq 2$ , will be zero. To get an explicit formula of the control  $u$ , it suffices to inverse the transformation defined in (4.4), then using the compatibility condition  $y_0(0) = u(0)$  to obtain (4.12). ■

**Remark 70** *Since we have an explicit formula of the solution for all  $t \geq 0$ , we can prove that exact controllability holds at any time  $T > T^*$  with loss of uniqueness of the control.*

### 4.3.2 Proof of the stability theorem

In this subsection, we let  $v \equiv 0$ . We start by proving the sufficient part.

At time  $t \geq 0$ , the components  $p$  and  $q$  might involve at most three values of the restrictive solutions  $p_n$  and  $q_n$  respectively on the contrary of the cylindrical case where  $p$  and  $q$  might involve at most two values (see Figure 4.2), (if  $p$  or  $q$  are defined on four regions, we obtain  $\alpha(t) > \beta(t)$ ). Let us deal with the worst case that might occur. We have for the component  $p$ :

Case 1:  $t \in [(\alpha^-)^{-1} \circ \phi^{[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1}(1), (\alpha^-)^{-1} \circ \phi^{[n]}(0)]$ .

In this case,  $p$  might expressed in function of  $p_{2n-1}, p_{2n}, p_{2n+1}$ ,

$$p(t, x) = \begin{cases} p_{2n-1}(t, x), & \text{if } x \in [t - \phi^{[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1}(1), \beta(t)], \\ p_{2n}(t, x), & \text{if } x \in [t - \phi^{[n]}(0), t - \phi^{[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1}(1)], \\ p_{2n+1}(t, x), & \text{if } x \in [\alpha(t), t - \phi^{[n]}(0)]. \end{cases} \quad (4.52)$$

By definition of the regions  $\Sigma_n^p, n \geq 0$ , given in (4.25)-(4.32), we have for  $k = 1, 2, 3$

$$\|p(t)\|_{L^2(\alpha(t), \beta(t))}^2 = \sum_{k=1}^3 \int_{(t,x) \in \Sigma_{2n+k-2}^p} |p_{2n+k-2}(t, x)|^2 dx, \quad (4.53)$$

which amounts to estimate the right hand side of (4.53). By using the exact solution formulas given in (4.45) and (4.46), we obtain for  $k = 1, 2, 3$

$$\begin{aligned} & \sqrt{\sum_{k=1}^3 \int_{(t,x) \in \Sigma_{2n+k-2}^p} |p_{2n+k-2}(t, x)|^2 dx} \\ & \leq \|(\tilde{p}, \tilde{q})\|_{[L^2(0,1)]^2} \sum_{k=1}^3 \sup_{x, (t,x) \in \Sigma_{2n+k-2}^p} \prod_{i=0}^{n-1 + \lceil \frac{k-1}{2} \rceil} \left| F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]}(t-x) \right) \right|. \end{aligned}$$

By definition of the regions  $\Sigma_n^p, n \geq 0$  given in (4.25)-(4.32), we have

$$\begin{aligned} (t, x) \in \Sigma_{2n}^p & \Leftrightarrow t - x \in [\phi^{[n-1]} \circ \alpha^- \circ (\alpha^+)^{-1}(1), \phi^{[n]}(0)], \\ (t, x) \in \Sigma_{2n+1}^p & \Leftrightarrow t - x \in [\phi^{[n]}(0), \phi^{[n]} \circ \alpha^- \circ (\alpha^+)^{-1}(1)], \end{aligned}$$

therefore, there exist a sequences  $\tau_1^n(t, x) \in \left[ \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0) \right)$  and  $\tau_2^n(t, x) \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)$  such that

$$\begin{aligned} (t, x) \in \Sigma_{2n}^p &\Leftrightarrow t - x = \phi^{[n-1]}(\tau_1^n(t, x)), \\ (t, x) \in \Sigma_{2n+1}^p &\Leftrightarrow t - x = \phi^{[n]}(\tau_2^n(t, x)). \end{aligned} \quad (4.54)$$

Observe that when  $(t, x)$  runs  $\Sigma_{2n}^p$  (resp.  $\Sigma_{2n+1}^p$ ), the bounded sequence  $\tau_1^n(t, x)$  (resp.  $\tau_2^n(t, x)$ ) rises  $\left[ \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0) \right)$  (resp.  $\left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)$ ). These sequences will play the role of two parameters  $\tau_1 \in \left[ \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0) \right)$  and  $\tau_2 \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)$ . With these notations, we have

$$\begin{aligned} &\sum_{k=1}^3 \sup_{x, (t, x) \in \Sigma_{2n+k-2}^p} \prod_{i=0}^{n+k-2} \left| F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]}(t - x) \right) \right| \\ &\leq \sup_{\tau_2 \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)} \prod_{i=0}^{n-1} \left| F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]} \circ \phi^{[n-1]}(\tau_2) \right) \right| \\ &\quad + \sup_{\tau_1 \in \left[ \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0) \right)} \prod_{i=0}^{n-1} \left| F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]} \circ \phi^{[n-1]}(\tau_1) \right) \right| \\ &\quad + \sup_{\tau_2 \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)} \prod_{i=0}^n \left| F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]} \circ \phi^{[n]}(\tau_2) \right) \right| \\ &= \sup_{\tau \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)} \psi_{n-1}(\tau) + \sup_{\tau \in \left[ \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0) \right)} \psi_{n-1}(\tau) \\ &\quad + \sup_{\tau \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)} \psi_n(\tau). \end{aligned}$$

So,

$$\begin{aligned} \|p(t)\|_{L^2(\alpha(t), \beta(t))} &\leq \|(\tilde{p}, \tilde{q})\|_{[L^2(0,1)]^2} \sup_{\tau \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)} \psi_{n-1}(\tau) \\ &\quad + \|(\tilde{p}, \tilde{q})\|_{[L^2(0,1)]^2} \sup_{\tau \in \left[ \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0) \right)} \psi_{n-1}(\tau) \\ &\quad + \|(\tilde{p}, \tilde{q})\|_{[L^2(0,1)]^2} \sup_{\tau \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)} \psi_n(\tau). \end{aligned} \quad (4.55)$$

Case 2:  $t \in \left[ (\alpha^-)^{-1} \circ \phi^{[n]}(0), (\alpha^-)^{-1} \circ \phi^{[n]} \circ \alpha^- \circ (\alpha^+)^{-1}(1) \right)$ .

In this case,  $p$  might be expressed in function of  $p_{2n}$ ,  $p_{2n+1}$ ,  $p_{2n+2}$

$$p(t, x) = \begin{cases} p_{2n}(t, x), & \text{if } x \in [t - \phi^{[n]}(0), \beta(t)], \\ p_{2n+1}(t, x), & \text{if } x \in [t - \phi^{[n]} \circ \alpha^- \circ (\alpha^+)^{-1}(1), t - \phi^{[n]}(0)], \\ p_{2n+2}(t, x), & \text{if } x \in [\alpha(t), t - \phi^{[n]} \circ \alpha^- \circ (\alpha^+)^{-1}(1)]. \end{cases} \quad (4.56)$$

In the same way, we obtain the estimate

$$\begin{aligned} \|p(t)\|_{L^2(\alpha(t), \beta(t))}^2 &\leq \|(\tilde{p}, \tilde{q})\|_{[L^2(0,1)]^2}^2 \sup_{\tau \in \left[ \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0) \right)} \psi_{n-1}(\tau) \\ &\quad + \|(\tilde{p}, \tilde{q})\|_{[L^2(0,1)]^2}^2 \sup_{\tau \in \left[ 0, \alpha^- \circ (\alpha^+)^{-1}(1) \right)} \psi_n(\tau) \\ &\quad + \|(\tilde{p}, \tilde{q})\|_{[L^2(0,1)]^2}^2 \sup_{\tau \in \left[ \alpha^- \circ (\alpha^+)^{-1}(1), \phi(0) \right)} \psi_n(\tau). \end{aligned} \quad (4.57)$$

Analogously, we have for the component  $q$ :

Case 1:  $t \in \left[ (\beta^+)^{-1} \circ \xi^{[n-1]} \circ \beta^+ \circ (\beta^-)^{-1} (0), (\beta^+)^{-1} \circ \xi^{[n]}(1) \right)$ .

The expression of  $q$  might involve the expressions of  $q_{2n-1}, q_{2n}, q_{2n+1}$

$$q(t, x) = \begin{cases} q_{2n-1}(t, x), & \text{if } x \in \left[ \alpha(t), \xi^{[n-1]} \circ \beta^+ \circ (\beta^-)^{-1} (0) - t \right), \\ q_{2n}(t, x), & \text{if } x \in \left[ \xi^{[n-1]} \circ \beta^+ \circ (\beta^-)^{-1} (0) - t, \xi^{[n]}(1) - t \right), \\ q_{2n+1}(t, x), & \text{if } x \in \left[ \xi^{[n]}(1) - t, \beta(t) \right). \end{cases} \quad (4.58)$$

So, we have

$$\begin{aligned} \|q(t)\|_{L^2(\alpha(t), \beta(t))} &= \sqrt{\sum_{k=1}^3 \int_{(t,x) \in \Sigma_{2n+k-2}^q} |q_{2n+k-2}(t, x)|^2 dx} \\ &\leq \|(\tilde{p}, \tilde{q})\|_{[L^2(0,1)]^2} \times \\ &\quad \sum_{k=1}^3 \sup_{x, (t,x) \in \Sigma_{2n+k-2}^q} \prod_{i=0}^{n-2+\lfloor \frac{k-1}{2} \rfloor} \left| F \left( (\alpha^-)^{-1} \circ (\phi^-)^{[k]} \circ \beta^- \circ (\beta^+)^{-1} (x+t) \right) \right|. \end{aligned} \quad (4.59)$$

By definition of the regions  $\Sigma_n^q$ ,  $n \geq 0$  given in (4.25)-(4.32), we have

$$(t, x) \in \Sigma_{2n}^q \Leftrightarrow t + x \in \left[ \xi^{[n-1]} \circ \beta^+ \circ (\beta^-)^{-1} (0), \xi^{[n]}(1) \right), \quad (4.60)$$

$$(t, x) \in \Sigma_{2n+1}^q \Leftrightarrow t + x \in \left[ \xi^{[n]}(1), \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1} (0) \right), \quad (4.61)$$

and since  $\xi$  is defined as

$$\xi = \beta^+ \circ (\beta^-)^{-1} \circ \phi \circ \beta^- \circ (\beta^+)^{-1},$$

(4.60) and (4.61) turns to

$$\begin{aligned} (t, x) &\in \Sigma_{2n}^q \Leftrightarrow \\ t + x &\in \left[ \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n-1]}(0), \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]} \circ \beta^- \circ (\beta^+)^{-1} (1) \right), \end{aligned} \quad (4.62)$$

$$\begin{aligned} (t, x) &\in \Sigma_{2n+1}^q \Leftrightarrow \\ t + x &\in \left[ \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]} \circ \beta^- \circ (\beta^+)^{-1} (1), \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]}(0) \right), \end{aligned} \quad (4.63)$$

therefore, there exist  $\chi_1 := \chi^n(t, x) \in \left[ 0, \phi \circ \beta^- \circ (\beta^+)^{-1} (1) \right)$  and  $\chi_2 := \chi_2^n(t, x) \in \left[ \phi \circ \beta^- \circ (\beta^+)^{-1} (1), \phi(0) \right)$  such that

$$\begin{aligned} (t, x) \in \Sigma_{2n}^q &\Leftrightarrow t + x = \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n-1]}(\chi_1), \\ (t, x) \in \Sigma_{2n+1}^q &\Leftrightarrow t + x = \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]}(\chi_2). \end{aligned}$$

Thus, by combining (4.59), (4.62) and (4.63), we obtain

$$\begin{aligned}
& \sum_{k=1}^3 \sup_{x, (t,x) \in \Sigma_{2n+k-2}^q} \prod_{i=0}^{n-2+\lceil \frac{k-1}{2} \rceil} \left| F \left( (\alpha^-)^{-1} \circ (\phi^-)^{[k]} \circ \beta^- \circ (\beta^+)^{-1} (x+t) \right) \right| \\
\leq & \sup_{\chi_2 \in [\phi \circ \beta^- \circ (\beta^+)^{-1}(1), \phi(0)]} \prod_{i=0}^{n-2} \left| F \left( (\alpha^-)^{-1} \circ (\phi^-)^{[i]} \circ \beta^- \circ (\beta^+)^{-1} \right. \right. \\
& \left. \left. \circ \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n-1]} (\chi_2) \right) \right| \\
+ & \sup_{\chi_1 \in [0, \phi \circ \beta^- \circ (\beta^+)^{-1}(1)]} \prod_{i=0}^{n-2} \left| F \left( (\alpha^-)^{-1} \circ (\phi^-)^{[i]} \circ \beta^- \circ (\beta^+)^{-1} \right. \right. \\
& \left. \left. \circ \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n-1]} (\chi_1) \right) \right| \\
+ & \sup_{\chi_2 \in [\phi \circ \beta^- \circ (\beta^+)^{-1}(1), \phi(0)]} \prod_{i=0}^{n-1} \left| F \left( (\alpha^-)^{-1} \circ (\phi^-)^{[i]} \circ \beta^- \circ (\beta^+)^{-1} \right. \right. \\
& \left. \left. \circ \beta^+ \circ (\beta^-)^{-1} \circ \phi^{[n]} (\chi_2) \right) \right| \\
= & \sup_{\chi_2 \in [\phi \circ \beta^- \circ (\beta^+)^{-1}(1), \phi(0)]} \prod_{i=0}^{n-2} \left| F \left( (\alpha^-)^{-1} \circ \phi^{[n-i-1]} (\chi_2) \right) \right| \\
= & \sup_{\chi_1 \in [0, \phi \circ \beta^- \circ (\beta^+)^{-1}(1)]} \prod_{i=0}^{n-2} \left| F \left( (\alpha^-)^{-1} \circ \phi^{[n-i-1]} (\chi_1) \right) \right| \\
= & \sup_{\chi_2 \in [\phi \circ \beta^- \circ (\beta^+)^{-1}(1), \phi(0)]} \prod_{i=0}^{n-1} \left| F \left( (\alpha^-)^{-1} \circ \phi^{[n-i]} (\chi_2) \right) \right|.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
\|q(t)\|_{L^2(\alpha(t), \beta(t))}^2 & \leq C \|\tilde{p}, \tilde{q}\|_{L^2(0,1)}^2 \sup_{\chi \in [\phi \circ \beta^- \circ (\beta^+)^{-1}(1), \phi(0)]} \psi_{n-1}(\chi) \\
& + C \|\tilde{p}, \tilde{q}\|_{L^2(0,1)}^2 \sup_{\chi \in [0, \phi \circ \beta^- \circ (\beta^+)^{-1}(1)]} \psi_{n-1}(\chi) \\
& + C \|\tilde{p}, \tilde{q}\|_{L^2(0,1)}^2 \sup_{\chi \in [\phi \circ \beta^- \circ (\beta^+)^{-1}(1), \phi(0)]} \psi_n(\chi).
\end{aligned} \tag{4.64}$$

Case 2:  $t \in [(\beta^+)^{-1} \circ \xi^{[n]}(1), (\beta^+)^{-1} \circ \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0)]$ .

As previously,  $q$  might involve the values of  $q_{2n}, q_{2n+1}, q_{2n+2}$

$$q(t, x) = \begin{cases} q_{2n}(t, x), & \text{if } x \in [\alpha(t), \xi^{[n]}(1) - t], \\ q_{2n+1}(t, x), & \text{if } x \in [\xi^{[n]}(1) - t, \xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0) - t], \\ q_{2n+2}(t, x), & \text{if } x \in [\xi^{[n]} \circ \beta^+ \circ (\beta^-)^{-1}(0) - t, \beta(t)]. \end{cases} \tag{4.65}$$

In the same way, the following estimate holds

$$\begin{aligned}
\|q(t)\|_{L^2(\alpha(t), \beta(t))} & \leq C \|\tilde{p}, \tilde{q}\|_{[L^2(0,1)]^2} \sup_{\chi \in [0, \phi \circ \beta^- \circ (\beta^+)^{-1}(1)]} \psi_{n-1}(\chi) \\
& + C \|\tilde{p}, \tilde{q}\|_{[L^2(0,1)]^2} \sup_{\chi \in [\phi \circ \beta^- \circ (\beta^+)^{-1}(1), \phi(0)]} \psi_{n-1}(\chi) \\
& + C \|\tilde{p}, \tilde{q}\|_{[L^2(0,1)]^2} \sup_{\chi \in [0, \phi \circ \beta^- \circ (\beta^+)^{-1}(1)]} \psi_n(\chi).
\end{aligned} \tag{4.66}$$

From (4.55), (4.57), (4.64) and (4.66), we deduce that

$$\sup_{\tau \in [0, \phi(0)]} \psi_n(\tau) \xrightarrow{n \rightarrow \infty} 0 \implies \|(p, q)(t)\|_{[L^2(\alpha(t), \beta(t))]^2} \xrightarrow{t \rightarrow \infty} 0, \tag{4.67}$$

which finishes the proof of the first statement of Theorem 53. The proof of the second and the third statements are just a consequences of (4.67). By definition of the regions  $\Sigma_n^p, \Sigma_n^q, n \geq 0$ , given in (4.25)-(4.32), we can see that letting  $t \rightarrow \infty$  is the same as  $\phi^{[n]}(\tau) \rightarrow \infty, \forall \tau \in [0, \phi(0))$ , so, if there exists a positive function  $g$  such that

$$Cg\left(\phi^{[n]}(\tau)\right) \underset{n \rightarrow \infty}{\sim} \psi_n(\tau), \quad \forall \tau \in [0, \phi(0)),$$

then obviously (4.18) holds. In particular, exponential stability follows immediately from

$$\sup_{\tau \in [0, \phi(0))} \psi_n(\tau) = \sup_{\tau \in [0, \phi(0))} \exp \left[ \phi^{[n]}(\tau) \left( \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} \right) \right].$$

The proof of the necessary part is straightforward. Without loss of generality, assume that the limit of  $\inf_{\tau \in [0, \phi(0))} \psi_n(\tau)$  is not zero. From (4.45), (4.46) and (4.54), we obtain

$$\begin{aligned} & \sqrt{\int_{(t,x) \in \Sigma_{2n+1}^p} |p_{2n+1}(t,x)|^2 dx + \int_{(t,x) \in \Sigma_{2n+2}^p} |p_{2n+2}(t,x)|^2 dx} \\ \geq & C \|\tilde{q}\|_{L^2(0,1)} \inf_{x, (t,x) \in \Sigma_{2n+1}^p} \prod_{i=0}^n \left| F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]}(t-x) \right) \right| \\ & + C \|\tilde{p}\|_{L^2(0,1)} \inf_{x, (t,x) \in \Sigma_{2n+2}^p} \prod_{i=0}^n \left| F \left( (\alpha^-)^{-1} \circ (\phi^{-1})^{[i]}(t-x) \right) \right| \\ \geq & C \left( \|\tilde{q}\|_{L^2(0,1)} + \|\tilde{p}\|_{L^2(0,1)} \right) \times \\ & \left[ \inf_{\tau \in [0, \alpha^- \circ (\alpha^+)^{-1}(1))} \psi_n(\tau) + \inf_{\tau \in [\alpha^- \circ (\alpha^+)^{-1}(1), \phi(0))} \psi_n(\tau) \right], \end{aligned}$$

therefore, if (4.16) is not satisfied then clearly stability cannot occur.

Let us prove the second claim of Theorem 53. If  $f \equiv 1$  then  $F \equiv 0$ . In this case, we infer from the exact formula of solutions given in (4.35), (4.36) and (4.40) that we have  $p_1 \equiv 0$  while  $q_1 \neq 0$ , and since  $q$  is constant along the characteristic lines,  $q$  is identically zero from the time that  $q_2$  will be zero, that is  $t \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)$  which is the same time for boundary controllability of System (4.1).

## Chapter 5

# Boundary controllability of two coupled wave equations with space-time first order coupling in $1 - D$

This Chapter is taken from [14].

### 5.1 Introduction

We are interested in the boundary controllability of the following system of two strongly coupled  $1 - D$  wave equations

$$\begin{cases} y_{tt} = y_{xx} + M((ay)_t + (by)_x), & \text{in } Q_T := (0, T) \times (0, 1), \\ y(t, 0) = Bu(t), \quad y(t, 1) = 0, & \text{in } (0, T), \\ y(0, x) = y_0(x), \quad y_t(0, x) = y_1(x), & \text{in } (0, 1), \end{cases} \quad (5.1)$$

where  $y = (y_1, y_2)^t$  is a vector function and

$$M = (m_{ij})_{1 \leq i, j \leq 2} \in \mathcal{L}(\mathbb{R}^2), \quad B = (b_1, b_2)^t \in \mathbb{R}^2, \quad a, b \in C^1(\overline{Q_T}; \mathbb{R}), \quad (5.2)$$

and  $u$  a scalar control function acting at  $x = 0$ .

This work is motivated by some previous papers. One of them is the result of Zhang [130]: a single wave equation in any space dimension with lower order terms is proved to be exactly controllable by a control acting on part of the boundary under a suitable geometric condition and independently from the lower terms. The author extended earlier Carleman inequalities proved by Fursikov-Imanuvilov [53] for the wave equation without these lower order terms.

The same issue arises for systems of  $n$  ( $\geq 2$ ) coupled wave equations with boundary or distributed controls. In [1], Alabau-Boussouira studied the controllability of 2-coupled wave equations with zero order coupling operator with constant coefficients. Later, this result has been generalized by Alabau-Boussouira and Léautaud in [2], [3], for coupling coefficients depending on the space variable under the geometric control condition introduced in [21]. In these works, one of the coupling coefficients is supposed to be small.

Dehman, Le Rousseau and Léautaud [46] studied distributed controllability of 2-coupled wave equations on a Riemannian manifold without boundary (periodic boundary conditions in the  $1 - D$  case) with a particular zero order coupling operator of cascade type. They proved that exact controllability holds provided that the Geometric Control Condition is satisfied. Further, they gave a characterization of the minimal time of control. An abstract result on the exact controllability of cascade systems is due to



Alabau-Boussouira [4] (abstract setting with application to various coupled second order PDEs). In all these cited works, it has been assumed that the coupling functions are of constant sign.

A boundary controllability result has been established without the sign or the smallness conditions by Bennour et al. [24] for 2-coupled wave equations in  $1 - D$  with cascade type coupling through velocity.

Concerning the constant case, Avdonin and De Tereza gave a complete answer for the exact boundary controllability issue for 2-coupled wave equations by zero order operator in  $1 - D$ . The same authors came back in [18] and generalized their result to  $n (\geq 2)$  coupled wave equations in  $1 - D$  but always with constant coupling coefficients under a Kalman rank condition. The same condition appears for distributed controllability of  $n (\geq 2)$ -coupled multidimensional wave equations with zero order coupling matrix with constant coefficients. It has been proved by Liard and Lissy in [81] that it is necessary and sufficient for the exact controllability in more regular energy space. An extension of this result can be found in Duprez and Olive [48] for cascade systems with zero order coupling operator whose coefficients depend on the space variable. However, boundary controllability has not been treated yet.

Recently, in [41], Cui et al. studied distributed controllability of  $n (\geq 2)$ -coupled wave equations with zero and first order coupling operator whose coefficients depend on both space and time variables on a compact Riemannian manifold without boundary (periodic boundary conditions in the 1-D case). It has been shown that the exact controllability issue can be reduced to the controllability of a finite dimensional system along the associated Hamiltonian flow. The authors also gave a unique continuation results in the *autonomous* case under classical support and sign assumptions. The same idea appears in [5] by Alabau-Boussouira et al. where distributed controllability of  $1 - D$  first order system with periodic boundary conditions is considered. The authors proved that exact controllability is reduced to the controllability of parametrized non-autonomous finite dimensional system.

In light of all of the cited works, we can see that the main issue that has to be solved is to figure out the optimal assumptions the coupling coefficients (or operators) of such systems must satisfy so that exact or approximate controllability hold.

In this article, we give a necessary and sufficient condition for the boundary exact controllability of System (5.1) in high frequency for a general matrix  $M$ . We shall also propose a criterion for the unique continuation property in the cascade case. Actually, we will prove that the unique continuation property is equivalent to solving a system a 2-coupled Fredholm integral equations of the third kind. We apply this criterion to nontrivial examples.

This paper is organized as follows: after some preliminaries fixing some notations, well-posedness and equivalence with a first order symmetric hyperbolic system gathered in Section 5.2, we present the main results of exact controllability of System (5.1) in high frequency (weak observability) in Section 5.3. Section 5.4 is devoted to the unique continuation issue for System (5.1). Appendix 5.5 contains the proof of some technical lemmas used in the previous sections.

## 5.2 Preliminaries

In this section, we recall some results about well-posedness which can be proved exactly as in the scalar case. For the proof of these results in the scalar case, we refer for instance to [130] and the references therein.

**Proposition 71** *Let  $T > 0$ . Under the assumption (5.2), suppose that:*

$$(y_0, y_1, u) \in L^2(0, 1)^2 \times H^{-1}(0, 1)^2 \times L^2(0, T).$$

*Then there exists a unique weak solution  $y$  to System (5.1) such that*

$$(y, y_t) \in C\left([0, T], L^2(0, 1)^2 \times H^{-1}(0, 1)^2\right).$$

Moreover, there exists a constant  $C = C(T, a, b) > 0$  such that:

$$\|y\|_{C([0,T], L^2(0,1)^2 \times H^{-1}(0,1)^2)} \leq C \left( \|Bu\|_{L^2(0,T)^2} + \|(y_0, y_1)\|_{L^2(0,1)^2 \times H^{-1}(0,1)^2} \right).$$

The adjoint problem associated with (5.1) writes:

$$\begin{cases} \varphi_{tt} = \varphi_{xx} - M^*(a\varphi_t + b\varphi_x), & \text{in } (0, T) \times (0, 1), \\ \varphi|_{x=0,1} = 0, & \text{in } (0, T), \\ (\varphi, \varphi_t)|_{t=T} = (\varphi_0, \varphi_1), & \text{in } (0, 1). \end{cases} \quad (5.3)$$

**Proposition 72** *Let  $T > 0$ . Under the assumption (5.2), suppose that:*

$$(\varphi_0, \varphi_1) \in H_0^1(0, 1)^2 \times L^2(0, 1)^2.$$

*Then there exists a unique weak solution  $\varphi$  to System (5.3) such that*

$$(\varphi, \varphi_t) \in C\left([0, T], H_0^1(0, 1)^2 \times L^2(0, 1)^2\right).$$

*Moreover,  $\varphi_{x|_{x=0,1}} \in L^2(0, T)^2$  and there exists a constant  $C = C(T, a, b) > 0$  such that:*

$$\|(\varphi, \varphi_t)\|_{C([0,T], H_0^1(0,1)^2 \times L^2(0,1)^2)} + \|\varphi_{x|_{x=0,1}}\|_{L^2(0,T)^2} \leq C \|(\varphi_0, \varphi_1)\|_{H_0^1(0,1)^2 \times L^2(0,1)^2}.$$

We are interested in the controllability issue for System (5.1). Recall that:

- System (5.1) is said to be

1. *exactly controllable at time  $T > 0$  if for any*

$$(y_0, y_1), (\tau_0, \tau_1) \in L^2(0, 1)^2 \times H^{-1}(0, 1)^2,$$

*there exists  $u \in L^2(0, T)$  such that the associated solution  $y$  to (5.1) satisfies*

$$(y, y_t)|_{t=T} = (\tau_0, \tau_1), \text{ in } (0, 1).$$

2. *approximately controllable at time  $T > 0$  if for any*

$$(y_0, y_1), (\tau_0, \tau_1) \in L^2(0, 1)^2 \times H^{-1}(0, 1)^2,$$

*and any  $\varepsilon > 0$ , there exists  $u \in L^2(0, T)$  such that the associated solution  $y$  to (5.1) satisfies:*

$$\left\| (y, y_t)|_{t=T} - (\tau_0, \tau_1) \right\|_{L^2(0,1)^2 \times H^{-1}(0,1)^2} < \varepsilon.$$

These controllability concepts are known to be connected with the observability properties of the adjoint system (5.3) (see [129, Part 4, Chapter 2]). Namely:

- System (5.1) is exactly controllable at time  $T > 0$  if, and only if, there exists  $C = C_T > 0$  such that for any  $(\varphi_0, \varphi_1) \in H_0^1(0, 1)^2 \times L^2(0, 1)^2$ , the associated solution  $\varphi$  to (5.3) satisfies the *observability inequality*:

$$\|(\varphi_0, \varphi_1)\|_{H_0^1(0,1)^2 \times L^2(0,1)^2}^2 \leq C \int_0^T |B^* \varphi_x(t, 0)|^2 dt. \quad (5.4)$$

In this case, the adjoint system is said exactly observable.

- System (5.1) is approximately controllable at time  $T > 0$  if, and only if, for any  $(\varphi_0, \varphi_1) \in H_0^1(0, 1)^2 \times L^2(0, 1)^2$ , the associated solution  $\varphi$  to (5.3) satisfies the *unique continuation property*:

$$(B^* \varphi_x(t, 0) = 0, t \in (0, T)) \Rightarrow \varphi \equiv 0 \text{ in } Q_T. \quad (5.5)$$

To study the observability inequality (5.4) for solutions to (5.3), we transform this system into a hyperbolic system of order one. Introduce the Riemann invariants:

$$p = \varphi_t - \varphi_x, \quad q = \varphi_t + \varphi_x, \text{ in } Q_T. \quad (5.6)$$

We will have, under the assumptions of Proposition 72:

$$p|_{t=T} = \varphi_1 - \frac{d\varphi_0}{dx} = p_0 \in L^2(0, 1)^2, \quad q|_{t=T} = \varphi_1 + \frac{d\varphi_0}{dx} = q_0 \in L^2(0, 1)^2.$$

Thus,  $(p_0, q_0) \in L^2(0, 1)^2 \times L^2(0, 1)^2$ . Moreover, since  $\varphi_0 \in H_0^1(0, 1)$  and  $q_0 - p_0 = 2\varphi_x$ , we must have:

$$\int_0^1 (q_0 - p_0) = 0. \quad (5.7)$$

It is readily seen that System (5.3) writes:

$$\begin{cases} p_t + p_x + M^*(\alpha_1 p + \alpha_2 q) = 0, & \text{in } Q_T, \\ q_t - q_x + M^*(\alpha_1 p + \alpha_2 q) = 0, & \text{in } Q_T, \\ (p + q)|_{x=0,1} = 0_{\mathbb{R}^2}, & \text{in } (0, T), \\ (p, q)|_{t=T} = (p_T, q_T), & \text{in } (0, 1), \end{cases} \quad (5.8)$$

where:

$$\alpha_1 = \frac{a-b}{2}, \quad \alpha_2 = \frac{a+b}{2}, \quad (5.9)$$

( $a, b$  and  $M$  being defined in (5.2)). From which it appears in particular that

$$\alpha_1, \alpha_2 \in C^1(\overline{Q_T}; \mathbb{R}). \quad (5.10)$$

Thanks to Proposition 72, we have

$$\begin{aligned} (p, q) &\in C\left([0, T], L^2(0, 1)^2 \times L^2(0, 1)^2\right), \\ (q - p)|_{x=0,1} &\in L^2(0, T)^2, \\ \|(p, q)\|_{C([0, T], L^2(0, 1)^2 \times L^2(0, 1)^2)} + \|(q - p)|_{x=0,1}\|_{L^2(0, T)^2} \\ &\leq C \|(p_T, q_T)\|_{L^2(0, 1)^2 \times L^2(0, 1)^2}. \end{aligned}$$

Conversely, if  $(p, q)$  is a solution to (5.8) associated with  $(p_0, q_0)$  satisfying (5.7), then there exists  $\varphi \in H^1(Q_T)$  such that:

$$\begin{pmatrix} \varphi_t \\ \varphi_x \end{pmatrix} = \begin{pmatrix} \frac{q+p}{2} \\ \frac{q-p}{2} \end{pmatrix},$$

since in  $Q_T$ , from System (5.8), the scalar curl of  $\begin{pmatrix} q+p \\ q-p \end{pmatrix}$  is:

$$(q+p)_x - (q-p)_t = (p_x + p_t) - (q_t - q_x) \equiv 0.$$

Moreover, taking into account the definition of  $\alpha_1$  and  $\alpha_2$  in (5.9), it is straightforward that:

$$\varphi_{tt} - \varphi_{xx} = -M^*(a\varphi_t + b\varphi_x), \text{ in } Q_T.$$

We note moreover that

$$\varphi_x = \frac{q-p}{2} \Rightarrow \varphi(t, x) = \int_0^x \frac{q-p}{2}(\xi) d\xi + C.$$

From (5.8), it appears that

$$(q-p)_t = (q+p)_x \Rightarrow \left( \int_0^1 (q-p) \right)_t = 0 \Rightarrow \int_0^1 (q-p) = 0,$$

the last equality coming from (5.7) and the continuity in time of  $(p, q)$ . It follows that:

$$\varphi|_{x=0,1} = 0, \text{ in } (0, T).$$

To summarize, let us introduce the space:

$$H = \left\{ (f, g) \in L^2(0, 1)^2 \times L^2(0, 1)^2, \int_0^1 (f - g) = 0 \right\}. \quad (5.11)$$

This is clearly a closed subspace of  $L^2(0, 1)^2 \times L^2(0, 1)^2$  and thus a Hilbert space with the usual norm (and scalar product) of  $L^2(0, 1)^2 \times L^2(0, 1)^2$ . In view of the previous considerations, we have:

**Proposition 73** *Let  $T > 0$  and  $H$  defined in (5.11).*

1. *For any  $(p_T, q_T) \in H$ , there exists a unique weak solution  $(p, q)$  to (5.8) such that  $(p, q) \in C([0, T], H)$ . Moreover  $(p - q)|_{x=0,1} \in L^2(0, T)^2$  and there exists a constant  $C = C(T, \alpha_1, \alpha_2) > 0$  such that:*

$$\|(p, q)\|_{C([0, T], H)} + \left\| (p - q)|_{x=0,1} \right\|_{L^2(0, T)^2} \leq C \|(p_T, q_T)\|_H.$$

2. *The observability inequality (5.4) is equivalent to*

$$\|(p, q)_{t=0}\|_H^2 \leq C_T \int_0^T |B^*p(t, 0)|^2 dt. \quad (5.12)$$

We will need in an essential way the block diagonal system associated with System (5.8):

$$\begin{cases} p_t + p_x + M^*\alpha_1 p = 0, & \text{in } Q_T, \\ q_t - q_x + M^*\alpha_2 q = 0, & \text{in } Q_T, \\ (p + q)|_{x=0,1} = 0_{\mathbb{R}^2}, & \text{in } (0, T), \\ (p, q)|_{t=T} = (p_T, q_T), & \text{in } (0, 1). \end{cases} \quad (5.13)$$

System (5.8) is a perturbation of System (5.13) by the multiplication operator defined on  $H$  by:

$$\mathcal{P} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0_{2 \times 2} & \alpha_2 I_{2 \times 2} \\ \alpha_1 I_{2 \times 2} & 0_{2 \times 2} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \quad (5.14)$$

The plan now is the following:

1. In a first step (Section 5.3), we will give necessary and sufficient condition for the solution to the diagonal system (5.13) to satisfy the observability inequality (5.12).

2. In a second step and in the same section (Subsection 5.3.5), starting from a compactness result due to [111], we will prove that if the solutions of (5.13) satisfy (5.12), then up to a finite dimensional subspace of initial data in  $H$ , the same is true for solutions to System (5.8). More precisely, we will prove that there exists a compact operator  $N : H \rightarrow L^2(0, T)$ , such that the following *weak observability* inequality

$$\|(p, q)_{t=0}\|_H^2 \leq C_T \int_0^T |B^* p(t, 0)|^2 dt + \|N(p_T, q_T)\|_{L^2(0, T)}, \quad (5.15)$$

holds. Actually, we will see that  $N := p(t, 0) - p_d(t, 0)$  where  $p$  and  $p_d$  are the solutions of Systems (5.8) and (5.13) respectively.

3. The last step (Section 5.4) will provide a necessary and sufficient condition for the unique continuation property to be satisfied for some particular matrix  $M$  and functions  $\alpha_1$  and  $\alpha_2$ . Nontrivial examples will be developed at the end the section.

### 5.3 Weak observability

In the block diagonal system (5.13), the change of variables  $t \Leftarrow T - t$  (we keep the same notations) leads to a system of the form

$$\begin{cases} p_t + p_x - M^* \eta_1 p = 0, & \text{in } Q_T, \\ q_t - q_x - M^* \eta_2 q = 0, & \text{in } Q_T, \\ (p + q)|_{x=0,1} = 0_{\mathbb{R}^2}, & \text{in } (0, T), \\ (p, q)|_{t=0} = (p_0, q_0), & \text{in } (0, 1), \end{cases} \quad (5.16)$$

where

$$\eta_1(t, x) = \alpha_2(T - t, x), \quad \eta_2(t, x) = \alpha_1(T - t, x), \quad (t, x) \in Q_T. \quad (5.17)$$

Note that the observed component does not change since  $(p + q)|_{x=0} = 0_{\mathbb{R}^2}$ .

In this subsection, for any numbers  $s, T$  such that  $0 < s < T$ , we compute the explicit solution  $Z = (p, q)$  to the system

$$\begin{cases} p_t + p_x - M^* \eta_1 p = 0, & \text{in } Q_T, \\ q_t - q_x - M^* \eta_2 q = 0, & \text{in } Q_T, \\ (p + q)|_{x=0,1} = 0_{\mathbb{R}^2}, & \text{in } (0, T), \\ (p, q)|_{t=s} = (p_s, q_s), & \text{in } (0, 1). \end{cases} \quad (5.18)$$

For given real numbers  $s, t$  such that  $0 \leq s < t$ , the function  $Z(t, x) = Z(t, x; s, Z_s) = (p, q)(t, x; s, Z_s)$  for  $t \in (s, T)$  and  $x \in (0, 1)$  will denote the solution to (5.18) with its dependence on the starting time  $s \geq 0$  and the initial data  $Z_s = (p_s, q_s) \in H$ .

When  $s = 0$ , we simply write  $Z(t, x) = Z(t, x; Z_0)$  but unless necessary, *all along this section*,  $Z = (p, q)$  will denote the solution to (5.16).

The following assumption is fixed and is assumed in all the results of this section:

$$\eta_i \in C^1(\overline{Q_T}, \mathbb{R}), \quad i = 1, 2.$$

It is simply derived from the assumption on  $a, b$  in (5.2).

Notice that the exact observability property of System (5.16) amounts to the observability inequality:

$$\exists C_T > 0, \quad \|(p_0, q_0)\|_H^2 \leq C_T \int_0^T |B^* p(t, 0)|^2 dt, \quad \forall (p_0, q_0) \in H.$$

To express more compactly the formulas for the solutions to (5.18), we introduce the function  $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by

$$\phi(t, s) = \begin{cases} 0, & \text{if } t \leq s, \\ \int_{\max\{s, t-2\}}^{\max\{s, t-1\}} \eta_1(\tau, \tau - (t-2)) d\tau + \int_{\max\{s, t-1\}}^t \eta_2(\tau, t - \tau) d\tau, & \text{if } t > s, \end{cases} \quad (5.19)$$

and the sequence of functions:

$$f_n(t, s) = \sum_{k=0}^n \phi(t - 2k, s), \quad n \geq 0. \quad (5.20)$$

When  $s = 0$ , we simply write:

$$\phi(t, 0) = \phi(t), \quad f_n(t, 0) = f_n(t), \quad t \in \mathbb{R}, \quad n \geq 0.$$

At this level, it is useful to clarify the geometric meaning of the function  $\phi$ . Actually the characteristic curves associated with the hyperbolic systems (5.13), (5.16) are the lines

$$x + t = c^{te}, \quad x - t = c^{te}.$$

Introduce the vector field  $\mathbf{F} = (\frac{\eta_1 + \eta_2}{2}, \frac{\eta_1 - \eta_2}{2})$  and let  $\gamma_j$  ( $j = 1, 2$ ) the two directions  $\gamma_1 = (1, 1)$  and  $\gamma_2 = (1, -1)$  of the characteristic lines. For the canonical scalar product in  $\mathbb{R}^2$ , one has  $\mathbf{F} \cdot \gamma_j = \eta_j$ , ( $j = 1, 2$ ) and for  $t > 0$ ,  $\phi(t)$  is then the line integral of the vector field  $\mathbf{F}$  along the line  $\Gamma_t$  defined by the function:

$$\gamma_t(\tau) = \begin{cases} (\tau, \tau - (t-2)), & \text{if } \max\{0, t-2\} \leq \tau \leq \max\{0, t-1\}, \\ (\tau, t - \tau), & \text{if } \max\{0, t-1\} \leq \tau \leq t. \end{cases} \quad (5.21)$$

As a consequence, for  $t > 0$  and  $n \geq 0$ , the function  $f_n(t)$  is the line integral of the vector field  $\mathbf{F}$  along the lines  $\cup_{1 \leq k \leq n+1} \Gamma_{t-2k}$  with the convention that if  $t - 2k < 0$ ,  $\Gamma_{t-2k} = \emptyset$  (see the figure below for the representation of these lines).

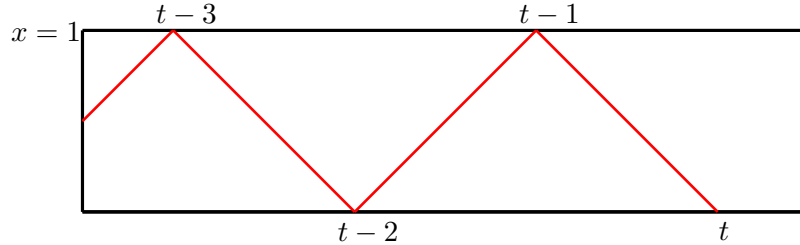


Figure 5.1:  $\cup_{1 \leq k \leq 2} \Gamma_{t-2k}$  is represented by the union of the reflected red lines on the boundary.

### 5.3.1 Main results

As pointed out above, observability inequality for System (5.8) will hold modulo compact operator. A classical functional analysis result shows that the space of invisible target is finite codimension and it might be reduced to zero if approximate controllability (or the unique continuation property) holds (See Section 5.4).

Henceforth, we denote by  $\chi_{(a,b)}$  the characteristic function of the interval  $(a, b)$ , namely, if  $a < b$ ,  $\chi_{(a,b)}(x) = 1$  when  $x \in (a, b)$  and 0 otherwise. We start by a negative controllability result:

**Theorem 74** *If  $T < 4$ , the weak observability inequality (5.15) doesn't hold. More precisely, there is an infinite dimensional space of unreachable target states.*

Now, we present a positive controllability results.

Denote by  $\lambda_1, \lambda_2$  the eigenvalues of  $M^*$  if it is diagonalizable and by  $\mu$  the multiple eigenvalue of  $M^*$  if it is not. We have the following controllability result:

**Theorem 75** *Let  $n \geq 2$  be an integer.*

- *If  $2n \leq T < 2n + 1$ . Then System (5.8) is weakly observable (see (5.15)) if, and only if the following three conditions are satisfied:*

1.  $\text{rank}[B \mid MB] = 2$ .
2. *For any  $x \in [0, 1]$ , there exists  $1 \leq k \leq n - 1$  such that:*

$$\begin{cases} \phi(2k + 2 - x) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(2k + 2 - x) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

3. *For any  $x \in [0, T - 2n)$  and  $x^* \in [T - 2n, 1)$ , there exist  $1 \leq k \leq n$  and  $1 \leq k^* \leq n - 1$  respectively such that:*

$$\begin{cases} \phi(x + 2k) \neq 0, \phi(x^* + 2k^*) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(x + 2k), \phi(x^* + 2k^*) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

- *If  $2n + 1 \leq T < 2n + 2$ . Then System (5.8) is weakly observable (see (5.15)) if, and only if the following three conditions are satisfied:*

1.  $\text{rank}[B \mid MB] = 2$ .
2. *For any  $x \in [2n + 2 - T, 1)$  and  $x^* \in [0, 2n + 2 - T)$ , there exist  $1 \leq k \leq n$  and  $1 \leq k^* \leq n - 1$  respectively such that:*

$$\begin{cases} \phi(2k + 2 - x) \neq 0, \phi(2k^* + 2 - x^*) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(2k + 2 - x), \phi(2k^* + 2 - x^*) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

3. *For any  $x \in [0, 1]$ , there exists  $1 \leq k \leq n$  such that:*

$$\begin{cases} \phi(x + 2k) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(x + 2k) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

Let us make several observations:

**Remark 76** *To illustrate geometrically the assertions of the above Theorem, we recall that from each point  $(0, x)$  (with  $x \in [0, 1]$ ) come two characteristics which stop at some point of the line  $t = T$ . If for example  $2n \leq T < 2n + 1$  with  $n \geq 2$ , these characteristics touch the observability boundary  $[0, T] \times \{0\}$  at least two times at points of the form  $(2k - x, 0)$  for one of them and of the form  $(2l + x, 0)$  for the other. The conditions on  $\phi$  means that there exist at least two consecutive points of the form  $(2k - x, 0)$  and two consecutive points of the form  $(2l + x, 0)$  such that the line integrals of the vector field  $\mathbf{F} = \left(\frac{\eta_1 + \eta_2}{2}, \frac{\eta_1 - \eta_2}{2}\right)$ , namely  $\int_{\Gamma_{(2\ell+x, 0)}^{(2(\ell+1)+x, 0)}} \mathbf{F} \cdot \gamma$  and  $\int_{\Gamma_{(2k-x, 0)}^{(2(k+1)-x, 0)}} \mathbf{F} \cdot \gamma$ , are not, depending on the coupling matrix nature, zero or are not in some discrete set.*

**Remark 77** Let us first recall that for any  $t \geq 2$ , the function  $\phi$  defined in (5.19) is given by

$$\phi(t) = \int_{t-2}^{t-1} \eta_1(\tau, \tau - (t-2)) d\tau + \int_{t-1}^t \eta_2(\tau, t - \tau) d\tau.$$

Then, by (5.17) and (5.10) we get

$$\phi(t) = \frac{1}{2} \int_0^1 (a+b)(T - \tau + t - 2, \tau) d\tau + \frac{1}{2} \int_0^1 (a-b)(T - t + \tau, \tau) d\tau.$$

Observe that in the autonomous case ( $a$  and  $b$  are time independent) the above formulas becomes

$$\phi(t) = \int_0^1 a(s) ds. \quad (5.22)$$

Hence, the coupling with first order derivative in space doesn't have any influence on the controllability of System (5.16) in high frequency unless  $b$  depends on time. More precisely, if we let  $a = 0$  and  $b = b(x)$ , then the weak observability inequality (5.15) doesn't hold in any time and for any  $b$  since  $\phi$  will be zero. The situation is not the same for parabolic systems. In [49], boundary controllability of a cascade system of two parabolic equations in  $1 - D$  has been studied with coupling acting on first order component. It has been shown that the underlying system is exactly controllable if the coupling function satisfies a moment assumption for the low frequency part and an average assumption like (5.22) for the high frequency. This shows that differences between hyperbolic and parabolic systems are not limited to the geometric control condition introduced in [21] or the minimal time of control.

**Remark 78** All the results stated above can be generalized to  $n \times n$  matrix  $M$  with a slight modifications. The minimal time of control becomes  $2n$  rather than 4 and the assumption  $\text{rank}[B \mid MB] = 2$  should be replaced by

$$\text{rank}[B \mid MB \mid M^2B \mid \dots \mid M^{n-1}B] = n.$$

### 5.3.2 Construction of the solution to the diagonal system

Given  $(t, x) \in Q_T$ , the value of  $p(t, x)$  and  $q(t, x)$  is determined either by  $(p_s, q_s)$  or by their values at  $x = 0$  or  $x = 1$ . More precisely, we have by the characteristics method:

$$p(t, x) = \begin{cases} \exp\left(M^* \int_{t-x}^t \eta_1(\tau, \tau - (t-x)) d\tau\right) p(t-x, 0), & \text{if } t-x > s, \\ \exp\left(M^* \int_s^t \eta_1(\tau, \tau - (t-x)) d\tau\right) p_s(x-t+s), & \text{if } t-x < s, \end{cases} \quad (5.23)$$

and

$$q(t, x) = \begin{cases} \exp\left(M^* \int_{x+t-1}^t \eta_2(\tau, t+x-\tau) d\tau\right) q(t+x-1, 1), & \text{if } t+x-1 > s, \\ \exp\left(M^* \int_s^t \eta_2(\tau, t+x-\tau) d\tau\right) q_s(x+t-s), & \text{if } t+x-1 < s. \end{cases} \quad (5.24)$$

Thus, computing  $p(t, x)$  and  $q(t, x)$  amounts to evaluate  $p(t, 0)$  and  $q(t, 1)$  (respectively) as functions of the initial data  $(p_0, q_0)$ , keeping in mind the boundary conditions. The following lemma can be proved by induction:

**Lemma 79** Let  $n \geq 0$  be an integer and  $Z_s = (p_s, q_s) \in H$ . Then if  $Z = (p, q)$  is the solution to System (5.18), one has:

$$p(t, 0) = -e^{f_n(t,s)M^*} q_s(t-s-2n), \text{ if } 2n \leq t-s < 2n+1, \quad (5.25)$$



$$p(t, 0) = e^{f_n(t,s)M^*} p_s(2n+2-t+s), \text{ if } 2n+1 \leq t-s < 2n+2. \quad (5.26)$$

As a consequence:

$$q(t, 1) = e^{M^* \left( \int_s^t \eta_1(\tau, \tau-(t-1)) d\tau \right)} p_s(t-s-2n), \text{ if } 0 \leq t-s < 1, \quad (5.27)$$

$$q(t, 1) = -e^{M^* \left( \int_s^t \eta_1(\tau, \tau-(t-1)) d\tau \right)} p(t-1, 0), \text{ if } t-s > 1. \quad (5.28)$$

**Proof.** We give the proof for  $s = 0$ , and a simple change of variable  $t \Rightarrow t - s$  leads to the formulas of the lemma.

Assume  $n = 0$  in (5.25)-(5.26). For  $0 < t < 1$ , the characteristics method, the boundary conditions and (5.19)-(5.20) give:

$$\begin{aligned} q(t, 0) &= e^{M^* \int_0^t \eta_2(\tau, t-\tau) d\tau} q_0(t) \\ &= e^{f_0(t)M^*} q_0(t). \end{aligned}$$

For  $1 \leq t < 2$ , as previously:

$$\begin{aligned} q(t, 0) &= e^{M^* \int_{t-1}^t \eta_2(\tau, t-\tau) d\tau} q(t-1, 1) \\ &= -e^{M^* \int_{t-1}^t \eta_2(\tau, t-\tau) d\tau} p(t-1, 1) \\ &= -e^{M^* \phi(t)} p_0(2-t) \\ &= -e^{f_0(t)M} p_0(2-t). \end{aligned}$$

Thus (5.25)-(5.26) are verified for  $n = 0$ .

Given  $n \geq 0$ , let us assume (5.25) and (5.26). Let  $2n+2 \leq t < 2n+3$ . Then, by the same computations using the characteristics method:

$$q(t, 0) = -e^{\phi(t)M^*} p(t-2, 0). \quad (5.29)$$

Since  $2n \leq t-2 < 2n+1$ , formula (5.25) applies and gives

$$p(t-2, 0) = e^{f_n(t-2)M^*} q_0(t-2(n+1)). \quad (5.30)$$

Now, from (5.20)

$$\begin{aligned} f_n(t-2) &= \sum_{k=0}^n \phi(t-2-2k) \\ &= \sum_{k=1}^{n+1} \phi(t-2k) \\ &= f_{n+1}(t) - \phi(t). \end{aligned} \quad (5.31)$$

Thus, inserting (5.30)-(5.31) in (5.29) leads to:

$$2n+2 \leq t < 2n+3 \Rightarrow q(t, 0) = e^{f_{n+1}(t)M^*} q_0(t-2(n+1)),$$

and (5.25) is proved with  $n$  replaced by  $n+1$ .

The proof by induction of (5.26) can be performed in the same way. ■

### Remark 80

1. System (5.18) defines an evolution family  $(U_{\text{diag}}(t, s))_{0 \leq s \leq t}$  on  $H$  (see [111], [117] for instance) which is explicitly computed by mean of formulas (5.23)-(5.24)-(5.25)-(5.26):

$$U_{\text{diag}}(t, s)Z_0 = (p, q)(t, \cdot; s, Z_0), \quad 0 \leq s \leq t, \quad Z_0 \in H. \quad (5.32)$$

2. From (5.25)-(5.26), the following formula follows: if  $Z = (p, q)$  is a solution to System (5.16) associated with an initial data  $Z_s = (p_s, q_s)$ , then

$$B^*p(t, 0; s, Z_s) = \begin{cases} -B^*e^{f_n(t,s)M^*}q_s(t-s-2n), & \text{if } 2n \leq t-s < 2n+1, \\ B^*e^{f_n(t,s)M^*}p_s(2n+2-t+s), & \text{if } 2n+1 \leq t-s < 2n+2. \end{cases} \quad (5.33)$$

for  $n \geq 0$ . Formula (5.33) will be used in the next subsection in the study of the observability issue for System (5.16).

### 5.3.3 Some technical results on multiplication operators

In order to make clear the proof of our exact observability results, we will need some preliminary results on multiplication operators defined from  $L^2(0, 1)^s$  in  $L^2(0, 1)^n$  for some positive integers  $s, n$ .

Let  $M = (m_{ij})_{1 \leq i, j \leq n}$  a  $n \times n$  matrix whose entries satisfies  $m_{ij} \in C([0, 1], \mathbb{R})$ . The multiplication operator  $\mathbb{M} : L^2(0, 1)^s \rightarrow L^2(0, 1)^n$  associated with  $M$  is defined by:

$$(\mathbb{M}h)(x) = M(x)h(x), \quad x \in (0, 1), \quad h \in L^2(0, 1)^n.$$

Clearly  $\mathbb{M}$  is a bounded operator. When  $s = n$ , the following characterization of the invertibility of  $\mathbb{M}$  is derived from [55, Proposition 2.2]:

**Proposition 81** *The operator  $\mathbb{M} : L^2(0, 1)^s \rightarrow L^2(0, 1)^s$  is invertible if, and only if:*

$$\inf_{x \in [0, 1]} |\det M(x)| > 0.$$

We are now interested by the case  $s < n$ .

**Proposition 82** *Let  $s < n$  and  $\mathbb{M} : L^2(0, 1)^s \rightarrow L^2(0, 1)^n$ . The following properties are equivalent:*

1. *There exists a constant  $C > 0$  such that*

$$\|h\|_{L^2(0, 1)^s} \leq C \|\mathbb{M}h\|_{L^2(0, 1)^n}, \quad \forall h \in L^2(0, 1)^s.$$

2. *For all  $x \in [0, 1]$ , there exists a  $s \times s$  matrix  $M_{\text{ext}}$ , extracted from  $M$ , such that*

$$\det M_{\text{ext}}(x) \neq 0.$$

**Proof.** For the proof, see Appendix 5.5. ■

### 5.3.4 Observability results

Let us start with the following Lemma:

**Lemma 83** *Let  $T > 0$  and  $(p, q)$  be the solution to (5.16) associated with  $Z_0 = (p_0, q_0) \in H$ . Then:*

• *If  $2n \leq T < 2n + 1$  for some  $n \geq 0$ , one has:*

$$\int_0^T |B^*p(t, 0)|^2 dt = \int_0^T \left| B^*e^{f_0(x)M^*}q_0(x) \right|^2 dx, \quad \text{if } n = 0, \quad (5.34)$$

and for any  $n \geq 1$

$$\begin{aligned} \int_0^T |B^* p(t, 0)|^2 dt &= \sum_{k=0}^{n-1} \int_0^1 \left| B^* e^{f_k(2k+2-x)M^*} p_0(x) \right|^2 dx \\ &+ \sum_{k=0}^{n-1} \int_0^1 \left| B^* e^{f_k(x+2k)M^*} q_0(x) \right|^2 dx \\ &+ \int_0^{T-2n} \left| B^* e^{f_n(x+2n)M^*} q_0(x) \right|^2 dx. \end{aligned} \quad (5.35)$$

- If  $2n + 1 \leq T < 2n + 2$  for some  $n \geq 0$ , then for  $n = 0$

$$\int_0^T |B^* p(t, 0)|^2 dt = \int_0^{T-1} \left| B^* e^{f_0(2-x)M^*} p_0(x) \right|^2 dx + \int_0^1 \left| B^* e^{f_0(x)M^*} q_0(x) \right|^2 dx, \quad (5.36)$$

and for any  $n \geq 1$

$$\begin{aligned} \int_0^T |B^* p(t, 0)|^2 dt &= \sum_{k=0}^n \int_0^1 \left| e^{f_k(x+2k)M^*} q_0(x) \right|^2 dx \\ &+ \sum_{k=0}^{n-1} \int_0^1 \left| e^{f_k(2k+2-x)M^*} p_0(x) \right|^2 dx \\ &+ \int_{2n+2-T}^1 \left| e^{f_n(2n+2-x)M^*} p_0(x) \right|^2 dx. \end{aligned} \quad (5.37)$$

**Proof.** Let  $n \geq 0$  be an integer and suppose that  $2n \leq T < 2n + 1$ . Then, if  $n = 0$ , we have from (5.25):

$$\int_0^T |B^* p(t, 0)|^2 dt = \int_0^T \left| B^* e^{f_0(t)M^*} q_0(t) \right|^2 dt.$$

If  $n \geq 1$ :

$$\begin{aligned} \int_0^T |B^* p(t, 0)|^2 dt &= \sum_{k=0}^{n-1} \left( \int_{2k}^{2k+1} + \int_{2k+1}^{2k+2} \right) |B^* p(t, 0)|^2 dt + \int_{2n}^T |B^* p(t, 0)|^2 dt \\ &: = \sum_{k=1}^n (I_k + J_k) + \int_{2n}^T |B^* p(t, 0)|^2 dt. \end{aligned} \quad (5.38)$$

Using (5.33) leads to:

$$\begin{aligned} I_k &= \int_{2k}^{2k+1} |B^* p(t, 0)|^2 dt \\ &= \int_{T-2k}^{T-2k+1} \left| B^* e^{f_k(t)M^*} q_0(t-2k) \right|^2 dt \\ &= \int_0^1 \left| B^* e^{f_k(x+2k)M^*} q_0(x) \right|^2 dx. \end{aligned}$$

For the second integral, in the same way:

$$\begin{aligned} J_k &= \int_{2k+1}^{2k+2} |B^* p(t, 0)|^2 dt \\ &= \int_{2k+1}^{2k+2} \left| B^* e^{f_k(t)M^*} p_0(2k+2-t) \right|^2 dt \\ &= \int_0^1 \left| B^* e^{f_k(2k+2-x)M^*} p_0(x) \right|^2 dx. \end{aligned}$$

And last:

$$\begin{aligned} \int_{2n}^T |B^* p(t, 0)|^2 dt &= \int_{2n}^T \left| B^* e^{f_n(t)M^*} q_0(t-2n) \right|^2 dt \\ &= \int_0^{T-2n} \left| B^* e^{f_n(x+2n)M^*} q_0(x) \right|^2 dx. \end{aligned}$$

Inserting the last formula in (5.38), we get (5.35).

If  $2n+1 \leq T < 2n+2$ , exactly as in the previous computations, if  $n=0$ , we get:

$$\begin{aligned} \int_0^T |B^* p(t, 0)|^2 dt &= \int_0^1 |B^* p(t, 0)|^2 dt + \int_1^T |B^* p(t, 0)|^2 dt \\ &= \int_0^1 \left| B^* e^{f_0(x)M^*} q_0(x) \right|^2 dx + \int_0^{T-1} \left| B^* e^{f_0(2-x)M^*} p_0(x) \right|^2 dx, \end{aligned}$$

and if  $n \geq 1$ :

$$\begin{aligned} \int_0^T |B^* p(t, 0)|^2 dt &= \left( \sum_{k=0}^n \int_{2k}^{2k+1} + \sum_{k=0}^{n-1} \int_{2k+1}^{2k+2} \right) |B^* p(t, 0)|^2 dt + \int_{2n+1}^T |B^* p(t, 0)|^2 dt \\ &= \sum_{k=0}^n \int_0^1 \left| e^{f_k(x+2k)M^*} q_0(x) \right|^2 dx + \sum_{k=0}^{n-1} \int_0^1 \left| e^{f_k(2k+2-x)M^*} p_0(x) \right|^2 dx \\ &\quad + \int_{2n+2-T}^1 \left| e^{f_n(2n+2-x)M^*} p_0(x) \right|^2 dx, \end{aligned}$$

which is exactly (5.37). This ends the proof of the lemma. ■

As an immediate consequence, we have:

**Corollary 84** *Let  $n \geq 1$ .*

- *If  $2n \leq T < 2n+1$ , a necessary and sufficient condition for exact observability of System (5.16) is that:*

$$\exists C_T > 0 : \int_0^1 |p_0(x)|^2 dx \leq C_T \sum_{k=0}^{n-1} \int_0^1 \left| B^* e^{f_k(2k+2-x)M^*} p_0(x) \right|^2 dx, \quad \forall p_0 \in L^2(0, 1)^2, \quad (5.39)$$

and

$$\begin{aligned} &\sum_{k=0}^{n-1} \int_0^1 \left| B^* e^{f_k(x+2k)M^*} q_0(x) \right|^2 dx + C_T \int_0^{T-2n} \left| B^* e^{f_n(x+2n)M^*} q_0(x) \right|^2 dx \\ &\leq C_T \int_0^1 |q_0(x)|^2 dx, \quad \forall q_0 \in L^2(0, 1)^2. \end{aligned} \quad (5.40)$$

- *If  $2n+1 \leq T < 2n+2$ , a necessary and sufficient condition for exact observability of System (5.16) is:*

$$\begin{aligned} &\sum_{k=0}^{n-1} \int_0^1 \left| B^* e^{f_k(2k+2-x)M^*} p_0(x) \right|^2 dx + \int_{2n+2-T}^1 \left| B^* e^{f_n(2n+2-x)M^*} p_0(x) \right|^2 dx \\ &\geq C_T \int_0^1 |p_0(x)|^2 dx, \quad \forall p_0 \in L^2(0, 1)^2, \end{aligned} \quad (5.41)$$

and

$$\int_0^1 |q_0(x)|^2 dx \leq C_T \sum_{k=0}^n \int_0^1 \left| B^* e^{f_k(x+2k)M^*} q_0(x) \right|^2 dx, \quad \forall q_0 \in L^2(0, 1)^2. \quad (5.42)$$

**Remark 85** Let  $n \geq 2$ . For  $2n \leq T < 2n + 1$ , introduce the matrices:

$$P_{2n}(x, T) = \begin{bmatrix} B^* e^{f_0(2-x)M^*} \\ B^* e^{f_0(4-x)M^*} \\ \vdots \\ B^* e^{f_{n-1}(2n-x)M^*} \end{bmatrix}; \quad Q_{2n}(x, T) = \begin{bmatrix} B^* e^{f_0(x)M^*} \\ \vdots \\ B^* e^{f_{n-1}(x+2(n-1))M^*} \\ \chi_{(0, T-2n)}(x) B^* e^{f_n(x+2n)M^*} \end{bmatrix}, \quad (5.43)$$

and their associated multiplication operators  $\mathbb{P}_{2n} : L^2(0, 1)^2 \rightarrow L^2(0, 1)^n$  and  $\mathbb{Q}_{2n} : L^2(0, 1)^2 \rightarrow L^2(0, 1)^{n+1}$ . With these notations, (5.39) and (5.40) respectively write:

$$\exists C_T > 0, \quad \int_0^1 |p_0(x)|^2 dx \leq C_T \|\mathbb{P}_{2n} p_0\|_{L^2(0,1)^n}^2, \quad \forall p_0 \in L^2(0, 1)^2, \quad (5.44)$$

$$\exists C_T > 0, \quad \int_0^1 |q_0(x)|^2 dx \leq C_T \|\mathbb{Q}_{2n} q_0\|_{L^2(0,1)^{n+1}}^2, \quad \forall q_0 \in L^2(0, 1)^2. \quad (5.45)$$

For  $2n + 1 \leq T < 2n + 2$ , introduce the matrices:

$$P_{2n+1}(x, T) = \begin{bmatrix} B^* e^{f_0(2-x)M^*} \\ \vdots \\ B^* e^{f_{n-1}(2n-x)M^*} \\ \chi_{(1, 2n+2-T)}(x) B^* e^{f_n(2n+2-x)M^*} \end{bmatrix}; \quad Q_{2n+1}(x, T) = \begin{bmatrix} B^* e^{f_0(x)M^*} \\ \vdots \\ B^* e^{f_{n-1}(x+2(n-1))M^*} \\ B^* e^{f_n(x+2n)M^*} \end{bmatrix}, \quad (5.46)$$

and their associated multiplication operators  $\mathbb{P}_{2n+1} : L^2(0, 1)^2 \rightarrow L^2(0, 1)^{n+1}$  and  $\mathbb{Q}_{2n+1} : L^2(0, 1)^2 \rightarrow L^2(0, 1)^{n+1}$ . With these notations, (5.41) and (5.42) respectively write:

$$\exists C_T > 0, \quad \int_0^1 |p_0(x)|^2 dx \leq C_T \|\mathbb{P}_{2n+1} p_0\|_{L^2(0,1)^{n+1}}^2, \quad \forall p_0 \in L^2(0, 1)^2, \quad (5.47)$$

$$\exists C_T > 0, \quad \int_0^1 |q_0(x)|^2 dx \leq C_T \|\mathbb{Q}_{2n+1} p_0\|_{L^2(0,1)^{n+1}}^2, \quad \forall q_0 \in L^2(0, 1)^2. \quad (5.48)$$

We are ready to state our first (negative) results on the controllability of System (5.16).

**Lemma 86** For  $T < 4$ , there exists an infinite dimensional subspace of initial data  $(p_0, q_0) \in H$  for which the exact observability inequalities (5.39)-(5.42) are not satisfied by the associated solution  $(p, q)$  to System (5.16).

**Proof.** If  $0 \leq T < 1$ , from (5.34), one has:

$$\int_0^T |B^* p(t, 0)|^2 dt = \int_0^T \left| B^* e^{f_0(x)M^*} q_0(x) \right|^2 dx,$$

and clearly the observability inequality (5.4) does not hold for all  $(p_0, q_0) \in H \times V_T$  where

$$V_T = \left\{ q_0 \in H : q_0 \cdot e^{f_0 M} B = 0 \text{ in } (0, T) \right\}.$$

If  $1 \leq T < 2$ , from (5.36):

$$\int_0^T |B^* p(t, 0)|^2 dt = \int_0^{T-1} \left| B^* e^{f_0(2-x)M^*} p_0(x) \right|^2 dx + \int_0^1 \left| B^* e^{f_0(x)M^*} q_0(x) \right|^2 dx.$$

The observability inequality (5.4) does not hold for all nontrivial  $(p_0, q_0) \in U_T \times H$  (for instance) where

$$U_T = \left\{ p_0 \in H : p_0 \cdot e^{f_0(2-\cdot)M} B = 0 \text{ in } (0, T-1) \right\}.$$

If  $2 \leq T < 3$ , from (5.35) follows the equality:

$$\begin{aligned} \int_0^T |B^* p(t, 0)|^2 dt &= \int_0^1 |B^* e^{f_0(2-x)M^*} p_0(x)|^2 dx \\ &+ \int_0^1 |B^* e^{f_0(x)M^*} q_0(x)|^2 dx + \\ &+ \int_0^{T-2} |B^* e^{f_1(x+2)M^*} q_0(x)|^2 dx, \end{aligned}$$

and again (5.4) does not hold for all  $(p_0, q_0) \in U_{T=1} \times H$ .

Last, for  $3 \leq T < 4$ , from (5.37):

$$\begin{aligned} \int_0^T |B^* p(t, 0)|^2 dt &= \sum_{k=0}^{n-1} \int_0^1 |B^* e^{f_0(2-x)M^*} p_0(x)|^2 dx \\ &+ \int_0^{T-3} |B^* e^{f_1(4-x)M^*} p_0(x)|^2 dx \\ &+ \sum_{k=0}^1 \int_0^1 |B^* e^{f_k(x+2k)M^*} q_0(x)|^2 dx, \end{aligned}$$

and (5.4) does not hold for all  $(p_0, q_0) \in W \times H$  where

$$W = U_{T=1} \cap \{p_0 \in H : \text{supp}(p_0) \subset (T-3, 1)\}.$$

All the introduced subspaces of non-observable initial data are actually infinite dimensional.

In these spaces can be found initial data  $(p_0, q_0)$  for which approximate observability does not hold too: if  $0 < T < 4$ , there exists  $(p_0, q_0) \in H \times H$  such that  $\|(p_0, q_0)\|_{H \times H} = 1$  and for which the associated solution  $(p, q)$  satisfies:

$$B^* p(t, 0) = 0, \quad t \in (0, T).$$

■

It has to be pointed out that the previous negative observability result does not depend of the choice of  $\eta_1, \eta_2$  and  $M$ .

Before going one in the analysis, let us give a necessary condition for the exact observability to hold:

**Lemma 87** *Let  $T > 0$ . A necessary condition for the exact observability of System (5.16) is*

$$\text{rank}[B \mid MB] = 2. \tag{5.49}$$

**Proof.** If (5.49) does not hold, there exists  $\lambda \in \mathbb{R}$  such that  $MB = \lambda B$  (in other words,  $B$  is an eigenvector to  $M$ ). It follows that for any  $r \in \mathbb{R}$ ,  $e^{rM}B = e^{r\lambda}B$  ( $\Leftrightarrow B^*e^{rM^*} = e^{r\lambda}B^*$ ). Thus, in this case, (5.35) writes:

$$\begin{aligned} \int_0^T |B^* p(t, 0)|^2 dt &= \int_0^1 \left( \sum_{k=0}^{n-1} e^{2f_k(2k+2-x)\lambda} \right) |B^* p_0(x)|^2 dx \\ &+ \int_0^1 \left( \sum_{k=0}^{n-1} e^{2f_k(x+2k)\lambda} \right) |B^* q_0(x)|^2 dx \\ &+ \int_0^{T-2n} e^{2f_n(x+2n)\lambda} |B^* q_0(x)|^2 dx. \end{aligned}$$

Clearly, the exact observability property will not hold for initial data of the form  $p_0 = \alpha B^\perp$  and  $q_0 = \beta B^\perp$  where  $\alpha, \beta \in L^2(0, 1)$  and  $B^\perp$  denotes any orthogonal vector to  $B$ . The same conclusion is achieved starting from (5.37). ■

The purpose in the sequel is to give answers for the exact controllability when  $T \geq 4$ . We begin by the limit case  $T = 4$ . As a first step, we have the following necessary and sufficient condition for observability:

**Proposition 88** *For  $T = 4$ , System (5.16) is exactly observable if, and only if:*

$$\inf_{t \in [2, 4]} \left| \det \left[ B^* \mid B^* e^{\phi(t)M^*} \right] \right| > 0. \quad (5.50)$$

**Proof.** Let  $(p_0, q_0) \in H$  and  $(p, q)$  the associated solution to System (5.16). For  $n = 2$ , the matrices  $P_4$  and  $Q_4$  defined in (5.43) write:

$$P_4(x) = \begin{bmatrix} B^* e^{f_0(2-x)M^*} \\ B^* e^{f_1(4-x)M^*} \end{bmatrix}, \quad Q_4(x) = \begin{bmatrix} B^* e^{f_0(x)M^*} \\ B^* e^{f_1(x+2)M^*} \end{bmatrix}.$$

From Remark 85, System (5.16) is exactly observable if, and only if, the conditions (5.44)-(5.45) are satisfied with  $n = 2$  and  $T = 4$ . From Proposition 81, (5.44)-(5.45) are equivalent to

$$\inf_{x \in [0, 1]} |\det P_4(x)| > 0 \quad \text{and} \quad \inf_{x \in [0, 1]} |\det Q_4(x)| > 0.$$

But since the multiplication operator on  $L^2(0, 1)^2$  whose matrix is  $e^{f_0(2-x)M^*}$  (resp.  $e^{f_0(x)M^*}$ ) ( $x \in (0, 1)$ ) is invertible, the two last conditions are equivalent to the following:

$$\begin{aligned} \inf_{x \in [0, 1]} |\det (P_4(x) e^{-f_0(2-x)M^*})| &> 0 \\ \text{and} \\ \inf_{x \in [0, 1]} |\det (Q_4(x) e^{-f_0(x)M^*})| &> 0. \end{aligned}$$

Now:

$$P_4(x) e^{-f_0(2-x)M^*} = \begin{bmatrix} B^* \mid B^* e^{(f_1(4-x) - f_0(2-x))M^*} \\ B^* \mid B^* e^{\phi(4-x)M^*} \end{bmatrix},$$

and

$$\begin{aligned} Q_4(x) e^{-f_0(x)M^*} &= \begin{bmatrix} B^* \mid B^* e^{(f_1(x+2) - f_0(x))M^*} \\ B^* \mid B^* e^{\phi(x+2)M^*} \end{bmatrix} \\ &= \begin{bmatrix} B^* \mid B^* e^{\phi(x+2)M^*} \end{bmatrix}. \end{aligned}$$

This leads to the desired inequalities (5.50) after noting that  $3 \leq 4-x \leq 4$  and  $2 \leq x+2 \leq 3$  for  $0 \leq x \leq 1$ . ■

Denote by  $\lambda_1, \lambda_2$  the eigenvalues of  $M^*$  if it is diagonalizable and by  $\mu$  the multiple eigenvalue if it is not. The next lemma will provide an equivalent condition to (5.50).

**Lemma 89** *Let  $r \in \mathbb{R}$ . Then  $\det [B^* \mid B^* e^{rM^*}] \neq 0$  if, and only if:*

$$\text{rank } [B \mid MB] = 2 \quad \text{and} \quad \begin{cases} r \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M^*) = \{\mu\}, \\ r \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases} \quad (5.51)$$

**Proof.** Let  $P$  a  $2 \times 2$  invertible matrix and set  $\tilde{B} = P^{-1}B$ . Then

$$\begin{aligned} [B \mid e^{rM}B] &= [P\tilde{B} \mid e^{rM}P\tilde{B}] \\ &= P [ \tilde{B} \mid P^{-1}e^{rM}P\tilde{B} ] \\ &= P [ \tilde{B} \mid e^{rP^{-1}MP}\tilde{B} ]. \end{aligned}$$

But

$$\det [B^* \mid B^* e^{rM^*}] \neq 0 \Leftrightarrow \det P \times \det [ \tilde{B} \mid e^{rP^{-1}MP}\tilde{B} ] \neq 0.$$

If  $M$  is diagonalizable in  $\mathbb{R}$  then it admits a basis  $\{V_1, V_2\}$  of real eigenvectors associated with the real eigenvalues  $\{\lambda_1, \lambda_2\}$ . If  $P = [V_1 \mid V_2]$  is the eigenvectors matrix, we get

$$e^{rP^{-1}MP}\tilde{B} = \begin{pmatrix} e^{\lambda_1 r} & 0 \\ 0 & e^{\lambda_2 r} \end{pmatrix} \tilde{B}.$$

So that, if  $\tilde{B} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$  (so that  $B = \beta_1 V_1 + \beta_2 V_2$ ), then:

$$\det [ \tilde{B} \mid e^{rP^{-1}MP}\tilde{B} ] = \beta_1 \beta_2 (e^{\lambda_1 r} - e^{\lambda_2 r}).$$

Thus, in this case:

$$\det [B^* \mid B^* e^{rM^*}] \neq 0 \Leftrightarrow \beta_1 \beta_2 (e^{\lambda_1 r} - e^{\lambda_2 r}) \neq 0 \Leftrightarrow \begin{cases} \beta_1 \beta_2 \neq 0, \\ \text{and} \\ e^{\lambda_1 r} - e^{\lambda_2 r} \neq 0. \end{cases}$$

The condition  $\beta_1 \beta_2 \neq 0$  expresses that  $B$  is not an eigenvector for  $M$  and this is equivalent to  $\text{rank} [B \mid MB] = 2$ . For the second condition, one has:

$$e^{\lambda_1 r} - e^{\lambda_2 r} \neq 0 \Leftrightarrow \begin{cases} r \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R}, \\ r \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

If  $M$  is not diagonalizable, then there exists a  $2 \times 2$  invertible matrix  $P$  such that

$$M = P \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix} P^{-1},$$

then

$$e^{rP^{-1}MP}\tilde{B} = \begin{pmatrix} e^{\mu r} & r e^{\mu r} \\ 0 & e^{\mu r} \end{pmatrix} \tilde{B}$$

and in this case:

$$\det [ \tilde{B} \mid e^{rP^{-1}MP}\tilde{B} ] = -\beta_2^2 r e^{\mu r}.$$

The proof follows immediately. ( $\beta_2 \neq 0$  says that  $B$  is not an eigenvector to  $M$ ). ■

An immediate consequence of Proposition 88 and Lemma 89 is the following

**Corollary 90** For  $T = 4$ , System (5.16) is exactly observable if, and only if, for any  $t \in [2, 4]$ :

$$\text{rank} [B \mid MB] = 2 \text{ and } \begin{cases} \phi(t) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M^*) = \{\mu\}, \\ \phi(t) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$



Now, we deal with the case  $T \geq 4$ :

**Proposition 91** *Let  $n \geq 2$  be an integer and  $2n \leq T < 2n + 1$ . Then System (5.16) is exactly observable if, and only if the following three conditions are satisfied:*

1.  $\text{rank}[B \mid MB] = 2$ .
2. For any  $x \in [0, 1]$ , there exists  $2 \leq k \leq n$  such that:

$$\begin{cases} \phi(2k - x) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(2k - x) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

3. For any  $x \in [0, T - 2n)$  and  $x^* \in [T - 2n, 1)$ , there exist  $1 \leq k \leq n$  and  $1 \leq k^* \leq n - 1$  such that:

$$\begin{cases} \phi(x + 2k) \neq 0, \phi(x^* + 2k^*) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(x + 2k), \phi(x^* + 2k^*) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

**Proof.** From Remark 85, System (5.16) is exactly observable if, and only if the conditions (5.44)-(5.45) are satisfied where we recall that for  $x \in [0, 1]$ :

$$P_{2n}(x, T) = \begin{bmatrix} B^* e^{f_0(2-x)M^*} \\ B^* e^{f_0(4-x)M^*} \\ \vdots \\ B^* e^{f_{n-1}(2n-x)M^*} \end{bmatrix}; \quad Q_{2n}(x, T) = \begin{bmatrix} B^* e^{f_0(x)M^*} \\ \vdots \\ B^* e^{f_{n-1}(x+2(n-1))M^*} \\ \chi_{(0, T-2n)}(x) B^* e^{f_n(x+2n)M^*} \end{bmatrix}.$$

From Proposition 82 with  $s = 2$  and  $n$  given in the lemma, (5.44)-(5.45) amount to say that for any  $x \in [0, 1]$ , there exist  $2 \times 2$  matrices  $P_{2n}^{\text{ext}}$  and  $Q_{2n}^{\text{ext}}$  respectively extracted from  $P_{2n}$  and  $Q_{2n}$  such that

$$\det P_{2n}^{\text{ext}}(x, T) \neq 0 \text{ and } \det Q_{2n}^{\text{ext}}(x, T) \neq 0.$$

Fix  $x \in [0, 1]$  and let us first deal with  $P_{2n}(x, T)$ . We are going to prove that  $P_{2n}$  satisfies the required property if, and only if, the following matrix satisfies it too:

$$\tilde{P}_{2n}(x, T) = \begin{bmatrix} B^* \\ B^* e^{\phi(4-x)M^*} \\ \vdots \\ B^* e^{\phi(2n-x)M^*} \end{bmatrix}$$

The proof of this last point is based on the identity:

$$f_k(2(k+1) - x) - f_{k-1}(2k - x) = \phi(2(k+1) - x), \quad k \geq 1, \quad x \in [0, 1], \quad (5.52)$$

which is easily derived from the definitions of the function  $\phi$  and the sequence  $(f_n)$  in (5.19) and (5.20).

Assume first that there exists  $x_0 \in [0, 1]$  such that for any  $2 \times 2$  matrices  $\tilde{P}_{2n}^{\text{ext}}$  extracted from  $\tilde{P}_{2n}$  one has:

$$\det \tilde{P}_{2n}^{\text{ext}}(x_0, T) = 0.$$

It follows that

$$\det [B^* \mid B^* e^{\phi(4-x_0)M^*}] = 0 \Leftrightarrow \begin{cases} \phi(4 - x_0) = 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(4 - x_0) \in \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases},$$

by the equivalence given by Lemma 89. By induction, we get that if for  $k \geq 1$ ,  $\phi(2k - x_0) = 0$  then

$$\det \left[ B^* e^{\phi(2k-x_0)M^*} \mid B^* e^{\phi(2(k+1)-x_0)M^*} \right] = \det \left[ B^* \mid B^* e^{\phi(2(k+1)-x_0)M^*} \right] = 0,$$

$$\Updownarrow$$

$$\begin{cases} \phi(2(k+1) - x_0) = 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(2(k+1) - x_0) \in \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

In view of (5.52), it readily follows that in this case:

$$P_{2n}(x_0, T) = \begin{bmatrix} B^* \\ \vdots \\ B^* \end{bmatrix} e^{\phi(2-x_0)}$$

and thus for this  $x_0$ , any  $2 \times 2$  matrices  $P_{2n}^{\text{ext}}$  extracted from  $P_{2n}$  satisfies:  $\det P_{2n}^{\text{ext}}(x_0, T) = 0$ .

Conversely, if we assume there exists  $x_0 \in [0, 1]$  such that for any  $2 \times 2$  matrices  $P_{2n}^{\text{ext}}$  extracted from  $P_{2n}$  one has:

$$\det P_{2n}^{\text{ext}}(x_0, T) = 0,$$

then again using (5.52), it is easily deduced that for any  $1 \leq k \leq n$ :

$$\begin{cases} \phi(2k - x_0) = 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(2k - x_0) \in \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

The same considerations hold for  $Q_{2n}$  and this proves the proposition. ■

**Proposition 92** *Let  $n \geq 2$  be an integer and  $2n + 1 \leq T < 2n + 2$ . System (5.16) is exactly observable if, and only if the following three conditions are satisfied :*

1.  $\text{rank}[B \mid MB] = 2$ .

2. For any  $x \in [2n + 2 - T, 1)$  and  $x^* \in [0, 2n + 2 - T)$ , there exist  $2 \leq k \leq n + 1$  and  $2 \leq k^* \leq n$  respectively such that:

$$\begin{cases} \phi(2k - x), \phi(2k^* - x^*) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(2k - x), \phi(2k^* - x^*) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

3. For any  $x \in [0, 1]$ , there exists  $1 \leq k \leq n$  such that:

$$\begin{cases} \phi(x + 2k) \neq 0, & \text{if } \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \sigma(M) = \{\mu\}, \\ \phi(x + 2k) \notin \frac{\pi}{\Im(\lambda_1)}\mathbb{Z}, & \text{if } \lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

**Proof.** Exactly as previously, it suffices to develop the same arguments for the matrices

$$P_{2n+1}(x, T) = \begin{bmatrix} B^* e^{f_0(2-x)M^*} \\ \vdots \\ B^* e^{f_{n-1}(2n-x)M^*} \\ \chi_{(2n+2-T,1)}(x) B^* e^{f_n(2n+2-x)M^*} \end{bmatrix}; \quad Q_{2n+1}(x, T) = \begin{bmatrix} B^* e^{f_0(x)M^*} \\ \vdots \\ B^* e^{f_{n-1}(x+2(n-1))M^*} \\ B^* e^{f_n(x+2n)M^*} \end{bmatrix}.$$

■

The proof of Theorems 75 is a straightforward consequences of Lemma 89 and Propositions 91 and 92.

### 5.3.5 Compactness

In section 5.3, we have, after the change of variable  $t \rightleftharpoons T - t$ , considered the System (5.16) that we recall here:

$$\begin{cases} p_t + p_x - M^* \eta_1 p = 0, & \text{in } Q_T, \\ q_t - q_x - M^* \eta_2 q = 0, & \text{in } Q_T, \\ (p + q)|_{x=0,1} = 0_{\mathbb{R}^2}, & \text{in } (0, T), \\ (p, q)|_{t=0} = (p_0, q_0), & \text{in } (0, 1). \end{cases} \quad (5.53)$$

The associated whole system obtained by the same change of variable from (5.8) is:

$$\begin{cases} p_t + p_x - M^* \eta_1 p - M^* \eta_2 q = 0, & \text{in } Q_T, \\ q_t - q_x - M^* \eta_1 p - M^* \eta_2 q = 0, & \text{in } Q_T, \\ (p + q)|_{x=0,1} = 0_{\mathbb{R}^2}, & \text{in } (0, T), \\ (p, q)|_{t=0} = (p_0, q_0), & \text{in } (0, 1). \end{cases} \quad (5.54)$$

In the sequel, we denote by  $Z = Z(t, \cdot, s; p_0, q_0) = (p, q)(t, \cdot, s; p_0, q_0)$  the solution to (5.53) and by  $Z_d = Z_d(t, \cdot, s; p_0, q_0) = (p_d, q_d)(t, \cdot, s; p_0, q_0)$  the solution to the diagonal system (5.54).

This section is devoted to the proof of the compactness of the following operator:

$$D_T : \begin{array}{ccc} H & \rightarrow & L^2(0, T) \\ (p_0, q_0)^t & \mapsto & B^*(p - p_d)|_{x=0}. \end{array}$$

In fact, we have:

**Proposition 93** *Let  $T > 0$ . Then the operator  $D_T$  is compact.*

The proof of Theorem 93 will need some preliminaries. Recall that the solution to System (5.53) can be expressed in terms of the evolution family  $(U_d(t, s))_{s \leq t}$  as

$$U_d(t, s)Z_0 = (p_d, q_d)(t, \cdot, s; Z_0), \quad 0 \leq s \leq t, \quad Z_0 \in H.$$

Therefore, there exist two operators  $(S_d^\pm(t, s))_{s \leq t} \in H \rightarrow L^2(0, 1)^2$  such that

$$p_d(t, \cdot, s; Z_0) = S_d^-(t, s)Z_0(\cdot), \quad q_d(t, \cdot, s; Z_0) = S_d^+(t, s)Z_0(\cdot), \quad 0 \leq s \leq t, \quad Z_0 \in H.$$

Since System (5.54) is a bounded perturbation of System (5.53), then by [112, Chapter 5, Theorem 2.3], there exists a unique evolution family associated with System (5.54) defined by

$$U(t, s)Z_0 = (p, q)(t, \cdot, s; Z_0), \quad 0 \leq s \leq t, \quad Z_0 \in H.$$

Similarly, there exist two operators  $(S^\pm(t, s))_{s \leq t} \in H \rightarrow L^2(0, 1)^2$  such that

$$p(t, \cdot, s; Z_0) = S^-(t, s)Z_0, \quad q(t, \cdot, s; Z_0) = S^+(t, s)Z_0, \quad 0 \leq s \leq t, \quad Z_0 \in H.$$

With these new notations, the operator  $D_T$  takes the form

$$D_T Z_0 = \mathcal{C} (S^-(t, s) - S_d^-(t, s)) Z_0, \quad 0 \leq s \leq t, \quad Z_0 \in H,$$

where  $\mathcal{C}$  is the operator

$$\mathcal{C} : \begin{array}{ccc} C(0, T; L^2(0, 1)^2) & \rightarrow & L^2(0, T) \\ v & \mapsto & B^*v|_{x=0}. \end{array}$$

Since  $(S(t, s))_{s \leq t}$  is a perturbation of  $(S_d(t, s))_{s \leq t}$  by a multiplication operators with multiplier  $\mathcal{P}_T(s) = \mathcal{P}_T(T - s)$ , it is clear that the two evolutions families are linked by the Duhamel formula:

$$U(t, 0)Z_0 = U_d(t, 0)Z_0 + \int_0^t U_d(t, s)\mathcal{P}_T(s)U(s, 0)Z_0 ds, \quad Z_0 \in H.$$

Therefore,

$$\begin{aligned} (U(t, 0) - U_d(t, 0))Z_0 &= \int_0^t U_d(t, s)\mathcal{P}_T(s)(U(s, 0) - U_d(s, 0))Z_0 ds \\ &\quad + \int_0^t U_d(t, s)\mathcal{P}_T(s)U_d(s, 0)Z_0 ds. \end{aligned}$$

Consequently,

$$\begin{aligned} (S^-(t, s) - S_d^-(t, s))Z_0 &= \int_0^t S_d^-(t, s)\mathcal{P}_T(s)(U(s, 0) - U_d(s, 0))Z_0 ds \\ &\quad + \int_0^t S_d^-(t, s)\mathcal{P}_T(s)U_d(s, 0)Z_0 ds \\ &= \Psi_1(t, \cdot, Z_0) + \Psi_2(t, \cdot, Z_0). \end{aligned}$$

Thus,

$$D_T(p_0, q_0)^t = \mathcal{C}\Psi_1(t, \cdot, Z_0) + \mathcal{C}\Psi_2(t, \cdot, Z_0), \quad Z_0 \in H.$$

So, proving that  $D_T$  is compact amounts to prove the compactness of the operators  $\mathcal{C}\Psi_i(t, \cdot, Z_0)$ ,  $i = 1, 2$ .

Since  $(U_d(t, 0))_{0 \leq t \leq T}$  is completely known, the compactness will be just a consequence of the explicit formula of the operator  $\mathcal{C}\Psi_2(t, \cdot, Z_0)$ . To deal with  $\mathcal{C}\Psi_1(t, \cdot, Z_0)$  we use the following lemma inspired from ([48]) which has been used also in ([56]) in the same context to deal with more general autonomous hyperbolic systems.

**Lemma 94** *For any  $f \in C([0, T], H)$ , there exists  $C_T > 0$  such that*

$$\int_0^T \left| \mathcal{C} \int_0^t S_d^-(t, s)f(s)ds \right|^2 dt \leq C_T \|f\|_{L^2(0, T; H)}^2. \quad (5.55)$$

**Proof.** For the time being, let  $f = (f_1, f_2) \in C([0, T], C_0(0, 1)^2 \times C_0(0, 1)^2)$ . By using the characteristics method, we have by (5.25) and (5.26) for any  $n \geq 0$ :

$$(S_d^-(t, s)f(s))(x) = \begin{cases} -R_n(t, x; s)f_2(s, t - x - s - 2n), & \text{if } t - x - s \in [2n, 2n + 1), \\ R_n(t, x; s)f_1(s, 2n + 2 - t + x + s), & \text{if } t - x - s \in [2n + 1, 2n + 2), \end{cases} \quad (5.56)$$

where

$$R_n(t, x; s) = e^{f_n(t-x, s)M^* + \int_{t-x}^t \eta_1(\tau, \tau - t + x)d\tau M^*}, \quad n \geq 0.$$

In particular, if  $t - x \in (0, 1)$ :

$$(S_d^-(t, s)f(s))(x) = \begin{cases} -e^{\int_{t-x}^t \eta_1(\tau, \tau - t + x)d\tau + \int_s^{t-x} \eta_2(\tau, t - x - \tau)d\tau} f_2(s, t - x - s), & \text{if } s \in [0, t - x), \\ e^{\int_s^t \eta_1(\tau, \tau - t + x)d\tau} f_1(s, s - t + x), & \text{if } s \in [t - x, t). \end{cases}$$

Consider first the case  $t - x \in [0, 1)$ . Since  $f$  is a continuous function, the trace operator makes sense and thus we obtain

$$\begin{aligned} & C \int_0^t S_d^-(t, s) f(s) ds \\ &= B^* \left( \int_0^t S_d^-(t, s) f(s) ds \right) (0) \\ &= -B^* e^{\int_s^t \eta_2(\tau, t-\tau) d\tau} f_2(s, t-s) ds. \end{aligned}$$

Therefore

$$\left| B^* \left( \int_0^t S_d^-(t, s) f(s) ds \right) (0) \right|^2 \leq C \int_0^t |f_2(t-s, s)|^2 ds.$$

By integrating over  $(0, T)$  the above inequality we obtain for any  $T \leq 1$  :

$$\begin{aligned} \int_0^T \left| C \int_0^t S_d^-(t, s) f(s) ds \right|^2 dt &\leq C \int_0^T \int_0^t |f_2(t-s, s)|^2 ds dt \\ &\leq C \int_0^T \int_0^t |f_2(t-s, s)|^2 + |f_1(t-s, s)|^2 ds dt \\ &\leq C \int_0^T \int_0^1 |f_2(t, s)|^2 + |f_1(t, s)|^2 ds dt \\ &= C \|f\|_{L^2(0, T; H)}^2. \end{aligned} \tag{5.57}$$

Now, we deal with the case  $T \geq 1$ . Let  $t - x \in [2n - 1, 2n)$ ,  $n \geq 1$ . We write

$$\begin{aligned} & \left( \int_0^t S_d^-(t, s) f(s) ds \right) (x) \\ &= \left( \int_0^{t-x-2n+1} + \sum_{k=1}^{n-1} \int_{t-x-2k}^{t-x-2k+1} + \sum_{k=0}^{n-1} \int_{t-x-2k-1}^{t-x-2k} + \int_{t-x}^t \right) (S_d^-(t, s) f(s) ds) (x) \\ &= \int_{t-x}^t e^{\int_s^t \eta_1(\tau, t-x-\tau) d\tau} M^* f_1(s, s-t+x) ds \\ & \quad + \sum_{k=1}^{n-1} \int_{t-x-2k}^{t-x-2k+1} R_{k-1}(t, x; s) f_1(s, 2k-t+x+s) ds \\ & \quad - \sum_{k=0}^{n-1} \int_{t-x-2k-1}^{t-x-2k} R_k(t, x; s) f_2(s, t-x-s-2k) ds \\ & \quad + \int_0^{t-x-2n+1} R_{n-1}(t, x; s) f_1(s, 2n-t+x+s) ds. \end{aligned}$$

A simple variable substitution yields

$$\begin{aligned}
& \left( \int_0^t S_d^-(t, s) f(s) ds \right) (x) \\
= & \int_0^x e^{\int_{s+t-x}^t \eta_1(\tau, t-x-\tau) d\tau M^*} f_1(s+t-x, s) ds \\
& + \sum_{k=1}^{n-1} \int_0^1 R_{k-1}(t, x; s+t-x-2k) f_1(s+t-x-2k, s) ds \\
& - \sum_{k=0}^{n-1} \int_0^1 R_k(t, x; t-x-s-2k) f_2(t-x-s-2k, s) ds \\
& + \int_{2n-t+x}^1 R_{n-1}(t, x; s+t-x-2n) f_1(s+t-x-2n, s) ds.
\end{aligned}$$

Again, since  $f$  is a continuous function, the trace operator makes sense and we obtain:

$$\begin{aligned}
& \mathcal{C} \left( \int_0^t S_d^-(t, s) f(s) ds \right) \\
= & B^* \left( \int_0^t S_d^-(t, s) f(s) ds \right) (0) \\
= & \sum_{k=1}^{n-1} \int_0^1 B^* R_{k-1}(t, 0; s+t-2k) f_1(s+t-2k, s) ds \\
& - \sum_{k=0}^{n-1} \int_0^1 B^* R_k(t, 0; t-s-2k) f_2(t-s-2k, s) ds \\
& - \int_{2n-t}^1 B^* R_{n-1}(t, 0; s+t-2n) f_1(s+t-2n, s) ds.
\end{aligned}$$

Since  $\eta_1, \eta_2$  are bounded, we get, using Cauchy-Schwarz inequality:

$$\begin{aligned}
\left| \mathcal{C} \int_0^t S_d^-(t, s) f(s) ds \right|^2 & \leq C_{n,t} \sum_{k=1}^{n-1} \int_0^1 |f_1(s+t-2k, s)|^2 ds \\
& + C_{n,t} \sum_{k=0}^{n-1} \int_0^1 |f_2(t-s-2k, s)|^2 ds \\
& + C_{n,t} \int_{2n-t}^1 |f_1(s+t-2n, s)|^2 ds,
\end{aligned} \tag{5.58}$$

where  $C_{n,t}$  is a positive constant depending on  $t$  and  $n$ . Taking the integral of (5.58) over  $(0, T)$  for  $T \in [2n-1, 2n)$ ,  $n \geq 1$ , and using the fact

$$\begin{aligned}
\int_0^T \int_0^1 |f_1(s+t-2k, s)|^2 ds dt & \leq \int_0^T \int_0^1 |f_1(t, s)|^2 ds dt, \quad \forall k \in \{1, \dots, n-1\}, \\
\int_0^T \int_0^1 |f_2(t-s-2k, s)|^2 ds dt & \leq \int_0^T \int_0^1 |f_2(t, s)|^2 ds dt, \quad \forall k \in \{0, \dots, n-1\},
\end{aligned}$$

yields

$$\int_0^T \left| \int_0^t S_d^-(t, s) f(s) ds \right|^2 dt \leq n \|f\|_{L^2(0, T; H)}^2. \tag{5.59}$$

Similarly, we obtain the following estimate for  $T \in (2n, 2n + 1)$ ,  $n \geq 1$

$$\int_0^T \left| \int_0^t S_d^-(t, s) f(s) ds \right|^2 dt \leq (n + 1) \|f\|_{L^2(0, T; H)}^2, \quad (5.60)$$

which ends the proof. The estimates (5.57), (5.59) and (5.60) can be extended for any  $f \in C(0, T; H)$  by using a standard density argument. ■

**Proposition 95** *The operator  $Z_0 \mapsto \mathcal{C}\Psi_1(t, \cdot, Z_0)$  acting from  $H$  to  $L^2(0, T)$  is compact.*

**Proof.** The proof of Proposition 95 is a direct consequence of the result proved in [111] (and an extension of this result in [11] to the case where some wave speeds are equal) which asserts that the difference of the evolution operators defined by systems (5.54) and (5.53) are compact) and Lemma 94. More precisely, by letting

$$f(s) = \mathcal{P}_T(s) (U(s, 0) - U_d(s, 0)) Z_0, \quad Z_0 \in H, \quad s \leq t,$$

in (5.55), we obtain

$$\begin{aligned} \|\mathcal{C}\Psi_1(t, \cdot, Z_0)\|_{L^2(0, T)} &= \int_0^T \mathcal{C} \left| \int_0^t S_d^-(t, s) \mathcal{P}_T(s) (U(s, 0) - U_d(s, 0)) Z_0 ds \right|^2 dt \\ &\leq C_T \| (U(\cdot, 0) - U_d(\cdot, 0)) Z_0 \|_{L^2(0, T; H)}^2, \end{aligned}$$

which is a compact operator by the result in [11]. It remains to deal with the operator  $\mathcal{C}\Psi_2(t, \cdot, Z_0)$ . ■

**Proposition 96** *The operator  $Z_0 \mapsto \mathcal{C}\Psi_2(t, \cdot, Z_0)$  acting from  $H$  to  $L^2(0, T)$  is compact.*

**Proof.** The proof is purely constructive. First, we find the solution  $q(t, \cdot, s, Z_0) = S^+(t, s)Z_0$ .

Let  $(p_0, q_0) \in C_0(0, 1)^2 \times C_0(0, 1)^2$ . By using the characteristics method, we compute (5.56) for  $x = 1$  by (5.27 and (5.28) we obtain:

$$(S_d^+(t, s)Z_0)(x) = \begin{cases} N(t, x; s) f_2(s, x + t - s - 2n), & \text{if } x + t - s \in [2n, 2n + 1), \\ -N(t, x; s) f_1(s, 2n + 2 - x - t + s), & \text{if } x + t - s \in [2n + 1, 2n + 2), \end{cases} \quad (5.61)$$

where

$$N_n(t, x; s) = R_{n-1}(x + t - 1, 1; s) + e^{\int_{x+t-1}^t \eta_2(\tau, -\tau+t+x) d\tau}, \quad n \geq 0.$$

The aim now is to compute  $\Psi_2(t, \cdot, Z_0)$  explicitly. We recall that

$$\Psi_2(t, \cdot, Z_0) = \int_0^t S_d^-(t, s) \mathcal{P}_T(s) U_d(s, 0) Z_0 ds, \quad t \leq T,$$

where  $U_d(s, 0)Z_0 = (S_d^-(s, 0)Z_0, S_d^+(s, 0)Z_0)^t$ ,  $0 \leq s \leq t$ , and  $(S_d^\pm(t, s))_{s \leq t \leq T}$  are given in (5.56) and (5.61).

Applying  $\mathcal{P}_T(\cdot)$  yields

$$\mathcal{P}_T(s) U_d(s, 0) = (\eta_2(s) S_d^+(s, 0) Z_0, \eta_1(s) S_d^-(s, 0) Z_0)^t, \quad s \leq t.$$

Therefore,

$$\mathcal{C}\Psi_2(t, \cdot, Z_0) = \mathcal{C} \int_0^t S_d^-(t, s) (\eta_2(s) S_d^+(s, 0) Z_0, \eta_1(s) S_d^-(s, 0) Z_0)^t Z_0 ds, \quad t \leq T. \quad (5.62)$$

Let us start by computing the integrand in (5.62). We have for any  $n \geq 0$

$$\begin{aligned}
& S_d^-(t, s) (\eta_2(s) S_d^+(s, 0) Z_0, \eta_1(s) S_d^-(s, 0) Z_0)^t(x) \\
= & \begin{cases} M_1(t, x; s) (S_d^-(s, 0) Z_0) (t - x - s - 2n), & \text{if } t - x - s \in [2n, 2n + 1), \\ M_2(t, x; s) (S_d^+(s, 0) Z_0) (2n + 2 - t + x + s), & \text{if } t - x - s \in [2n + 1, 2n + 2), \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
M_n^1(t, x; s) &= -R_n(t, x; s) \eta_1(s, t - x - s - 2n), \quad t - x - s \in [2n, 2n + 1), \quad n \geq 0, \\
M_n^2(t, x; s) &= R_n(t, x; s) \eta_2(s, 2n + 2 - t + x + s), \quad t - x - s \in [2n + 1, 2n + 2), \quad n \geq 0.
\end{aligned}$$

Now, by using (5.56) and (5.61) for  $s = 0$  we get for any  $n, k \geq 0$

$$\begin{aligned}
& (S_d^-(\tau, 0) Z_0) (t - x - \tau - 2n) \\
= & \begin{cases} -R_k(\tau, t - x - \tau - 2n; 0) \times \\ q_0(2\tau - t + x + 2(n - k)), & \text{if } \tau \in \left(\frac{2(k-n)+t-x}{2}, \frac{2(k-n)+t-x+1}{2}\right) \cap (0, t), \\ R_k(\tau, t - x - \tau - 2n; 0) \times \\ p_0(2(k - n + 1) - 2\tau + t - x), & \text{if } \tau \in \left(\frac{2(k-n)+t-x+1}{2}, \frac{2(k-n)+t-x+2}{2}\right) \cap (0, t), \end{cases} \tag{5.63}
\end{aligned}$$

and

$$\begin{aligned}
& S^+(t, 0) Z_0 (2n + 2 - t + x + \tau) \\
= & \begin{cases} N_k(\tau, 2n + 2 - t + x + \tau; 0) \times \\ q_0(2(n - k + 1) - t + x + 2\tau), & \text{if } \tau \in \left(\frac{2(k-n-1)-t-x}{2}, \frac{2(k-n)-t-x-1}{2}\right) \cap (0, t), \\ -N_k(\tau, 2n + 2 - t + x + \tau; 0) \times \\ p_0(2(k - n) + t - x - 2\tau), & \text{if } \tau \in \left(\frac{2(k-n)-t-x-1}{2}, \frac{2(k-n)-t-x}{2}\right) \cap (0, t), \end{cases} \tag{5.64}
\end{aligned}$$

Therefore, we obtain for any  $k, n \geq 0$

$$\begin{aligned}
\Psi_2(t, \cdot, Z_0) &= - \sum_{k, n \geq 0} \int_{\frac{2(k-n)+t-x}{2}}^{\frac{2(k-n)+t-x+1}{2}} \mathbf{1}(\tau)_{(0,t)} P_{k,n}^1(t, x; \tau) q_0(2\tau - t + x + 2(n - k)) d\tau \\
&+ \sum_{k, n \geq 0} \int_{\frac{2(k-n)+t-x+1}{2}}^{\frac{2(k-n)+t-x+2}{2}} \mathbf{1}(\tau)_{(0,t)} P_{k,n}^1(t, x; \tau) p_0(2(k - n + 1) - 2\tau + t - x) d\tau \\
&+ \sum_{k, n \geq 0} \int_{\frac{2(k-n)-t-x-1}{2}}^{\frac{2(k-n)-t-x}{2}} \mathbf{1}(\tau)_{(0,t)} P_{k,n}^2(t, x; \tau) q_0(2(n - k + 1) - t + x + 2\tau) d\tau \\
&- \sum_{k, n \geq 0} \int_{\frac{2(k-n)-t-x}{2}}^{\frac{2(k-n)-t-x-1}{2}} \mathbf{1}(\tau)_{(0,t)} P_{k,n}^2(t, x; \tau) p_0(2(k - n) + t - x - 2\tau) d\tau,
\end{aligned}$$

where

$$\begin{aligned}
P_{k,n}^1(t, x; \tau) &= M_n^1(t, x; \tau) R_k(s, t - x - s - 2n; 0), \\
P_{k,n}^1(t, x; \tau) &= M_n^2(t, x; \tau) N_k(s, 2n + 2 - t + x + \tau; 0).
\end{aligned}$$



Consequently

$$\begin{aligned}
& \mathcal{C}\Psi_2(t, \cdot, Z_0) \\
= & -B^* \sum_{k,n \geq 0} \int_{\frac{2(k-n)+t}{2}}^{\frac{2(k-n)+t+1}{2}} \mathbf{1}(\tau)_{(0,t)} P_{k,n}^1(t, 0; \tau) q_0(2\tau - t + 2(n - k)) d\tau \\
& + B^* \sum_{k,n \geq 0} \int_{\frac{2(k-n)+t+1}{2}}^{\frac{2(k-n)+t+2}{2}} \mathbf{1}(\tau)_{(0,t)} P_{k,n}^1(t, 0; \tau) p_0(2(k - n + 1) - 2\tau + t) d\tau \\
& + B^* \sum_{k,n \geq 0} \int_{\frac{2(k-n)-t}{2}}^{\frac{2(k-n)-t-1}{2}} \mathbf{1}(\tau)_{(0,t)} P_{k,n}^2(t, 0; \tau) q_0(2(n - k + 1) - t + 2\tau) d\tau \\
& - B^* \sum_{k,n \geq 0} \int_{\frac{2(k-n)-t}{2}}^{\frac{2(k-n)-t-1}{2}} \mathbf{1}(\tau)_{(0,t)} P_{k,n}^2(t, 0; \tau) p_0(2(k - n) + t - 2\tau) d\tau.
\end{aligned}$$

The proof follows by [111, Lemma 4] which allows to conclude that  $\mathcal{C}\Psi_2(t, \cdot, Z_0)$  is a compact operator from  $H$  to  $L^2(0, T)$  since it is a finite sum of such operators. ■

## 5.4 Unique continuation

In this section, we deal with two particular cases for which we prove the *unique continuation* property holds true for System (5.1). The first one corresponds to constant coupling parameters  $a, b$ , and the second for matrices  $M$  in a cascade form with coupling functions  $a, b$  depending both on space and time.

### 5.4.1 The constant case

Here, we assume that  $a, b \in \mathbb{R}$ . Recall that the adjoint system of System (5.1) is given by

$$\begin{cases} \varphi_{tt} = \varphi_{xx} - M^*(a\varphi_t + b\varphi_x), & \text{in } (0, T) \times (0, 1), \\ \varphi|_{x=0,1} = 0, & \text{in } (0, T), \\ (\varphi, \varphi_t)|_{t=T} = (\varphi_0, \varphi_1), & \text{in } (0, 1). \end{cases} \quad (5.65)$$

The correspond *unique continuation property* reads:

$$B^* \varphi_x(t, 0) = 0, \quad \forall t \in (0, T) \Rightarrow (\varphi_0, \varphi_1) = (0, 0), \quad \forall (\varphi_0, \varphi_1) \in H_0^1(0, 1)^2 \times L^2(0, 1)^2. \quad (5.66)$$

For the sake of simplicity, we set  $e^{\frac{bx}{2}} M^* \tilde{\varphi}(x) = \varphi(x)$ . Then  $\tilde{\varphi}$  is the solution of the following system

$$\begin{cases} \tilde{\varphi}_{tt} = \tilde{\varphi}_{xx} - \frac{1}{4} b^2 (M^*)^2 \tilde{\varphi} - a M^* \tilde{\varphi}_t, & \text{in } (0, T) \times (0, 1), \\ \tilde{\varphi}|_{x=0,1} = 0, & \text{in } (0, T), \\ (\tilde{\varphi}, \tilde{\varphi}_t)|_{t=T} = (\tilde{\varphi}_0, \tilde{\varphi}_1), & \text{in } (0, 1). \end{cases}$$

By using a standard spectral decomposition of the solution to System (5.65) (See for instance [17, 24] in the hyperbolic context or [49, 12] in the parabolic one), we can see that proving that (5.66) holds in a time  $T \geq 4$  amounts to proving that all the eigenvalues of the corresponding spectral problem

$$\begin{cases} \lambda^2 \tilde{\varphi}(x) = \tilde{\varphi}''(x) - \left( \frac{1}{4} b^2 (M^*)^2 + a M^* \lambda \right) \tilde{\varphi}(x), & x \in (0, 1), \\ \tilde{\varphi}(0) = \tilde{\varphi}(1) = 0, & \tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2), \end{cases} \quad (5.67)$$

are simple. First, let us assume first that  $M^*$  is diagonalizable.

**Proposition 97** Assume that  $M^*$  has 2 distinct eigenvalues  $\mu_1, \mu_2$ . Then, all the eigenvalues of the Sturm-Liouville problem (5.67) are simple if, and only if

$$a^2 (\mu_1 - \mu_2) (\mu_2 \xi_{n_1}^2 - \mu_1 \xi_{n_2}^2) \neq (\xi_{n_1}^2 - \xi_{n_2}^2)^2, \quad \forall n_1, n_2 \in \mathbb{Z}, \quad (5.68)$$

where  $\xi_{n_i} = \frac{1}{4}b^2\mu_i^2 + (n_i\pi)^2$ ,  $i = 1, 2$ .

**Proof.** Since  $M^*$  is diagonalizable, there exists a diagonal matrix  $D = \text{diag}(\mu_1, \mu_2)$  and  $2 \times 2$  invertible matrix  $P$  such that  $M^* = PDP^{-1}$ . Letting  $z = P^{-1}\tilde{\varphi}$  with  $z = (z_1, z_2)$  in (5.67) yields the following Sturm-Liouville problem

$$\begin{cases} z_i''(x) = (\lambda^2 + a\lambda\mu_i + \frac{1}{4}b^2\mu_i^2) z_i(x), & x \in (0, 1), \\ z_i(0) = z_i(1) = 0, & i = 1, 2. \end{cases}$$

In order to prove that the eigenvalues of the above problem are simple we have to check that the following polynomial equations

$$\lambda^2 + \lambda a\mu_i + \frac{1}{4}b^2\mu_i^2 + (n_i\pi)^2 = 0, \quad n_i \in \mathbb{Z}, \quad i = 1, 2,$$

don't have a common roots which is equivalent to check that the following Sylvester matrix is invertible

$$S_{k,n} = \begin{pmatrix} 1 & a\mu_1 & \frac{1}{4}b^2\mu_1^2 + (n_1\pi)^2 & 0 \\ 0 & 1 & a\mu_1 & \frac{1}{4}b^2\mu_1^2 + (n_1\pi)^2 \\ 1 & a\mu_2 & \frac{1}{4}b^2\mu_2^2 + (n_2\pi)^2 & 0 \\ 0 & 1 & a\mu_2 & \frac{1}{4}b^2\mu_2^2 + (n_2\pi)^2 \end{pmatrix}, \quad n_1, n_2 \in \mathbb{Z},$$

which is the case if, and only if (5.68) is satisfied. ■

**Remark 98** In particular, if  $\mu_1 = -\mu_2 = \mu$ , assumption (5.68) becomes

$$-2a^2\mu^2 (\xi_{n_1}^2 + \xi_{n_2}^2) \neq (\xi_{n_1}^2 - \xi_{n_2}^2)^2, \quad \forall n_1, n_2 \in \mathbb{Z},$$

which might occur only if, and only if  $\mu \in i\mathbb{R}$ .

Now, we consider the case where  $M^*$  is not diagonalizable.

**Proposition 99** Assume that  $M^*$  is not diagonalizable and let  $\mu$  be its eigenvalue. Then, all the eigenvalues of the Sturm-Liouville problem (5.67) are simple if, and only if

$$\frac{1}{2}b^2\mu + a \neq 0. \quad (5.69)$$

**Proof.** In this case, we write  $M^*$  in the Jordan form: there exists a matrix  $J$  and a  $2 \times 2$  invertible matrix  $P$  such that  $M^* = PJP^{-1}$ , where

$$J = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}.$$

Letting  $z = P^{-1}\tilde{\varphi}$  in (5.67) yields the following coupled Sturm-Liouville problem

$$\begin{cases} z_1''(x) = (\lambda^2 + a\mu\lambda + \frac{1}{4}b^2\mu^2) z_1(x) + (\frac{1}{2}b^2\mu + a) z_2, \\ z_2''(x) = (\lambda^2 + a\mu\lambda + \frac{1}{4}b^2\mu^2) z_2(x), \\ y(0) = y(1) = 0, \\ z(0) = z(1) = 0. \end{cases}$$

By the same reasoning as in [12, 49, Proposition 2.1], it can be seen that the above system has non-trivial solution if, and only if  $\frac{1}{2}b^2\mu + a \neq 0$  with  $\lambda$  fulfills the following second order polynomial equation for some  $n \in \mathbb{Z}$ :

$$\lambda^2 + a\mu\lambda + \frac{1}{4}b^2\mu^2 + (n\pi)^2 = 0.$$

It is clear that the above equation has simple roots for any  $n \in \mathbb{Z}$ . This finishes the proof. ■

**Remark 100** As it is shown, if we let  $a = 0$ , the coupling parameter  $b$  affects controllability in low frequency. Actually, conditions (5.68) and (5.69) become respectively

$$b^2 \neq 2\pi^2 \frac{n_2^2 - n_1^2}{\mu_1^2 - \mu_2^2}, \quad \forall n_1, n_2 \in \mathbb{Z} \quad \text{and } b \neq 0.$$

However,  $b$  doesn't affect controllability in high frequency unless it is time dependent. (See Remark 77).

### 5.4.2 Cascade coupling

In this subsection we prove the *unique continuation* property for a particular class of System (5.16). We assume in the sequel that the matrix  $M$  and the vector  $B$  have the following form:

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

With this in mind, and by decomposing the system by writing  $p = (p^-, p^+)$ ,  $q = (q^-, q^+)$ , the unique continuation problem of System (5.16) reads

$$\begin{cases} p_t^- + p_x^- = 0, & \text{in } Q_T, \\ q_t^- - q_x^- = 0, & \text{in } Q_T, \\ p_t^+ + p_x^+ - \eta_1 p^- - \eta_2 q^- = 0, & \text{in } Q_T, \\ q_t^+ - q_x^+ - \eta_1 p^- - \eta_2 q^- = 0, & \text{in } Q_T, \\ (p^- + q^-)|_{x=0,1} = (p^+ + q^+)|_{x=0,1} = 0, & \text{in } (0, T), \\ p_{x=0}^+ = 0, & \text{in } (0, T), \\ (p^-, p^+, q^-, q^+)|_{t=0} = (p_0^-, p_0^+, q_0^-, q_0^+), & \text{in } (0, 1), \end{cases} \implies (p_0^-, p_0^+, q_0^-, q_0^+) = 0_H, \quad (5.70)$$

for any  $(p_0^-, p_0^+, q_0^-, q_0^+) \in \tilde{H}$  where

$$\tilde{H} = \left\{ (p^-, p^+, q^-, q^+) \in H^1(0, 1)^4, \quad (p^\pm + q^\pm)|_{x=0,1} = 0 \right\}, \quad (5.71)$$

is the associated operator.

Observe that the first and the third equations of the above system are free. The idea is to solve these equations explicitly and then considering their solution as a second member for the second and the fourth equations. Let us start by solving the homogeneous part of System (5.70). i.e.

$$\begin{cases} p_t^- + p_x^- = 0, & \text{in } Q_T, \\ q_t^- - q_x^- = 0, & \text{in } Q_T, \\ (p^+ + q^+)|_{x=0,1} = 0, & \text{in } (0, T), \\ (p^-, q^-)|_{t=0} = (p_0^-, q_0^-), & \text{in } (0, 1). \end{cases} \quad (5.72)$$

**Lemma 101** The solution  $(p^-, q^-)$  to System (5.72) are given by

$$p^-(t, x) = \begin{cases} p_0^-(x - t + 2n), & \text{if } 2n - 1 \leq t - x \leq 2n, \\ -q_0^-(t - x - 2n), & \text{if } 2n \leq t - x \leq 2n + 1, \end{cases} \quad n \geq 0, \quad (5.73)$$

$$q^-(t, x) = \begin{cases} q_0^-(x + t - 2n), & \text{if } 2n \leq x + t \leq 2n + 1, \\ -p_0^-(2n + 2 - x - t), & \text{if } 2n + 1 \leq x + t \leq 2n + 2, \end{cases} \quad n \geq 0. \quad (5.74)$$

**Proof.** The proof follows immediately by using the characteristics method. ■

Now, we focus on the nonhomogeneous part of System (5.70) where boundary condition  $q^+_{|x=0} = 0$  has been removed, i.e.

$$\begin{cases} p_t^+ + p_x^+ = f, & \text{in } Q_T, \\ q_t^+ - q_x^+ = f, & \text{in } Q_T, \\ (p^+ + q^+)_{|x=1} = p^+_{|x=0} = 0, & \text{in } (0, T), \\ (p^+, q^+)_{|t=0} = (p_0^+, q_0^+). & \text{in } (0, 1). \end{cases} \quad (5.75)$$

Observe that the function  $f = \eta_1 p^- + \eta_2 q^-$  plays the role of a second member. The explicit solution to (5.75) is given in the following lemma:

**Lemma 102** *The solution  $(p^+, q^+)$  to System (5.75) are given by*

$$p^+(t, x) = \begin{cases} p_0^+(x-t) + \int_0^t f(\tau, \tau + x - t) d\tau, & \text{if } 0 \leq x - t \leq 1 \\ \int_{t-x}^t f(\tau, \tau - t + x) d\tau, & \text{if } t - x \geq 1, \end{cases}$$

and

$$q^+(t, x) = \begin{cases} q_0^+(x+t) + \int_0^t f(\tau, x+t-\tau) d\tau, & \text{if } 0 \leq x+t \leq 1, \\ \int_{x+t-1}^t f(\tau, x+t-\tau) d\tau - p_0^+(2-x-t) \\ - \int_0^{x+t-1} f(\tau, \tau+2-x-t) d\tau & \text{if } 1 \leq x+t \leq 2, \\ \int_{x+t-1}^t f(\tau, -\tau+x+t) d\tau \\ - \int_{x+t-2}^{x+t-1} f(\tau, \tau+2-t-x) d\tau, & \text{if } x+t \geq 2, \end{cases} \quad (5.76)$$

**Proof.** By using the characteristics method, it follows

$$p^+(t, x) = p_0^+(x-t) + \int_0^t f(\tau, \tau + x - t) d\tau, \text{ if } 0 \leq x - t \leq 1,$$

and

$$q^+(t, x) = q_0^+(x+t) + \int_0^t f(\tau, x+t-\tau) d\tau, \text{ if } 0 \leq x+t \leq 1.$$

Now, at  $x = 1$  we obtain

$$p^+(t, 1) = p_0^+(1-t) + \int_0^t f(\tau, \tau + 1 - t) d\tau, \text{ if } 0 \leq 1 - t \leq 1.$$

Using the boundary condition,  $q^+(s, 1) = -p^+(s, 1)$ ,  $s \geq 0$ , entails

$$q^+(s, 1) = -p_0^+(1-s) - \int_0^s f(\tau, \tau + 1 - s) d\tau, \text{ if } 0 \leq 1 - s \leq 1.$$

Solving the second equation of System (5.75) along the characteristic  $x(t) = -t + s + 1$  we get

$$\begin{aligned} q^+(t, -t + s + 1) &= q^+(s, 1) + \int_s^t f(\tau, s + 1 - \tau) d\tau \\ &= -p_0^+(1-s) - \int_0^s f(\tau, \tau + 1 - s) d\tau + \int_s^t f(\tau, s + 1 - \tau) d\tau. \end{aligned}$$

Letting  $x = -t + s + 1$  yields for any  $0 \leq 2 - x - t \leq 1$ :

$$q^+(t, x) = -p_0^+(2-x-t) - \int_0^{x+t-1} f(\tau, \tau + 2 - x - t) d\tau + \int_{x+t-1}^t f(\tau, x+t-\tau) d\tau.$$

Now, since  $p(s, 0) = 0$ ,  $s \geq 0$ , we have for any  $t - x \geq 1$

$$p^+(t, x) = \int_{t-x}^t f(\tau, \tau - t + x) d\tau. \quad (5.77)$$

By taking  $x = 1$  in (5.77), the using the boundary conditions  $q^+(s, 1) = -p^+(s, 1)$ ,  $s \geq 0$ , we get

$$q^+(t, x) = \int_{x+t-1}^t f(\tau, x + t - \tau) d\tau.$$

Similarly, we obtain for any  $x + t \geq 2$

$$q^+(t, x) = - \int_{x+t-2}^{x+t-1} f(\tau, \tau + 2 - t - x) d\tau + \int_{x+t-1}^t f(\tau, -\tau + x + t) d\tau,$$

which ends the proof. ■

To satisfy the remained boundary condition  $q|_{x=0} = 0$ , it suffices to replace  $x$  by zero in (5.76). This gives the following system of equations

$$0 = q_0^+(t) + \int_0^t f(\tau, t - \tau) d\tau, \text{ if } 0 \leq t \leq 1, \quad (5.78)$$

$$0 = -p_0^+(2 - t) - \int_0^{t-1} f(\tau, \tau + 2 - t) d\tau + \int_{t-1}^t f(\tau, t - \tau) d\tau = 0, \text{ if } 1 \leq t \leq 2, \quad (5.79)$$

$$0 = - \int_{t-2}^{t-1} f(\tau, \tau + 2 - t) d\tau + \int_{t-1}^t f(\tau, -\tau + t) d\tau = 0, \text{ if } t \geq 2. \quad (5.80)$$

As a consequence, we have:

**Proposition 103** *The unique continuation property (5.70) holds true if, and only if the solution  $(p_0, q_0)$  to System (5.78)-(5.80) is the null one.*

The strategy is the following: We solve Equation (5.80) which depends only on the initial states  $p_0^-, q_0^-$  since  $f$  does. Then, we prove that  $p_0^- \equiv q_0^- \equiv 0$  which entails that  $f \equiv 0$  since it depends linearly on  $p_0^-$  and  $q_0^-$ . This leads to  $p_0^+ \equiv q_0^+ \equiv 0$  by (5.78) and (5.79).

Let  $t \geq 2$ . Recall that  $f = \eta_1 p^- + \eta_2 q^-$ . Equation (5.80) becomes

$$\begin{aligned} 0 &= - \int_{t-2}^{t-1} (\eta_1 p^-)(\tau, \tau + 2 - t) d\tau + \int_{t-1}^t (\eta_2 q^-)(\tau, -\tau + t) d\tau \\ &\quad - \int_{t-2}^{t-1} (\eta_2 q^-)(\tau, \tau + 2 - t) d\tau + \int_{t-1}^t (\eta_1 p^-)(\tau, -\tau + t) d\tau \\ &: = I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned}$$

Since  $p^-$  (resp.  $q^-$ ) is defined on the characteristics of slope 1 (resp.  $-1$ ),  $I_1(\cdot)$  (resp.  $I_2(\cdot)$ ) can be easily computed. Indeed, by using the expressions of  $p^-$  and  $q^-$  given in (5.73) and (5.74) respectively, we obtain

$$\begin{aligned} I_1(t) &= - \int_{t-2}^{t-1} (\eta_1 p^-)(\tau, \tau + 2 - t) d\tau \\ &= - \left( \int_{t-2}^{t-1} \eta_1(\tau, \tau + 2 - t) d\tau \right) \times \\ &= \begin{cases} p_0^-(4 + 2n - t), & \text{if } 2n + 3 \leq t \leq 2n + 4, \\ -q_0^-(t - 2 - 2n), & \text{if } 2n + 2 \leq t \leq 2n + 3, \end{cases} \quad n \geq 0, \end{aligned} \quad (5.81)$$

and

$$\begin{aligned}
I_2(t) &= \int_{t-1}^t (\eta_1 q^-)(\tau, -\tau + t) d\tau \\
&= \left( \int_{t-1}^t \eta_1(\tau, -\tau + t) d\tau \right) \times \\
&= \begin{cases} q_0^-(t - 2n), & \text{if } 2n \leq t \leq 2n + 1, \\ -p_0^-(2n + 2 - t), & \text{if } 2n + 1 \leq t \leq 2n + 2, \end{cases} \quad n \geq 1.
\end{aligned} \tag{5.82}$$

It can be seen that  $I_1(\cdot)$  and  $I_2(\cdot)$  are the high frequency part of the solution. Now, we deal with the compact terms  $I_3(\cdot)$  and  $I_4(\cdot)$ .

By using the expression of  $q^-$  given in (5.74), we obtain for any  $n \geq 0$

$$\begin{aligned}
I_3(t) &= - \int_{t-2}^{t-1} (\eta_2 q^-)(\tau, \tau + 2 - t) d\tau \\
&\begin{cases} - \int_{t-2}^{t-1} \eta_2(\tau, \tau + 2 - t) q_0^-(2\tau + 2 - t - 2n) d\tau, & \text{if } \tau \in \left(\frac{2n-2+t}{2}, \frac{2n-1+t}{2}\right) \cap (0, t), \\ \int_{t-2}^{t-1} \eta_2(\tau, \tau + 2 - t) p_0^-(2n - 2\tau + t) d\tau, & \text{if } \tau \in \left(\frac{2n-1+t}{2}, \frac{2n+t}{2}\right) \cap (0, t). \end{cases}
\end{aligned} \tag{5.83}$$

- If  $t \in [2n - 1, 2n)$ ,  $n \geq 2$  : In this case, the interval  $(t - 2, t - 1)$  can be decomposed as

$$(t - 2, t - 1) = [t - 2, \frac{2n - 4 + t}{2}) \cup (\frac{2n - 4 + t}{2}, \frac{2n - 3 + t}{2}) \cup (\frac{2n - 3 + t}{2}, t - 1).$$

Therefore, by using (5.83) we get

$$\begin{aligned}
I_3(t) &= \int_{t-2}^{\frac{2n-4+t}{2}} \eta_2(\tau, \tau + 2 - t) p_0^-(2n - 4 + t - 2\tau) d\tau \\
&\quad - \int_{\frac{2n-4+t}{2}}^{\frac{2n-3+t}{2}} \eta_2(\tau, \tau + 2 - t) q_0^-(2\tau + 4 - t - 2n) d\tau \\
&\quad + \int_{\frac{2n-3+t}{2}}^{t-1} \eta_2(\tau, \tau + 2 - t) p_0^-(2n - 2 + t - 2\tau) d\tau.
\end{aligned}$$

And after a change of variables, we obtain

$$\begin{aligned}
I_3(t) &= \frac{1}{2} \int_0^{2n-t} \eta_2\left(\frac{2n - 4 + t - s}{2}, \frac{2n - t - s}{2}\right) p_0^-(s) ds \\
&\quad - \frac{1}{2} \int_0^1 \eta_2\left(\frac{s + t - 4 + 2n}{2}, \frac{s - t + 2n}{2}\right) q_0^-(s) ds \\
&\quad + \frac{1}{2} \int_{2n-t}^1 \eta_2\left(\frac{2n - 2 + t - s}{2}, \frac{2n + 2 - t - s}{2}\right) p_0^-(s) ds.
\end{aligned} \tag{5.84}$$

- If  $t \in [2n, 2n + 1)$ ,  $n \geq 1$  :

The interval  $[t - 2, t - 1)$  can be written as

$$(t - 2, t - 1) = (t - 2, \frac{2n - 3 + t}{2}) \cup (\frac{2n - 3 + t}{2}, \frac{2n - 2 + t}{2}) \cup (\frac{2n - 2 + t}{2}, t - 1).$$

Similarly, we obtain

$$\begin{aligned}
I_3(t) &= - \int_{t-2}^{\frac{2n-3+t}{2}} \eta_2(\tau, \tau+2-t) q_0^-(2\tau+4-t-2n) d\tau \\
&\quad + \int_{\frac{2n-3+t}{2}}^{\frac{2n-2+t}{2}} \eta_2(\tau, \tau+2-t) p_0^-(2n-2+t-2\tau) d\tau \\
&\quad - \int_{\frac{2n-2+t}{2}}^{t-1} \eta_2(\tau, \tau+2-t) q_0^-(2\tau+2-t-2n) d\tau.
\end{aligned}$$

And after a change of variables we get

$$\begin{aligned}
I_3(t) &= -\frac{1}{2} \int_{t-2n}^1 \eta_2\left(\frac{s+2n+t-4}{2}, \frac{s+2n-t}{2}\right) q_0^-(s) ds \\
&\quad + \frac{1}{2} \int_0^1 \eta_2\left(\frac{2n+t-2-s}{2}, \frac{2n-t+2-s}{2}\right) p_0^-(s) ds \\
&\quad - \frac{1}{2} \int_0^{t-2n} \eta_2\left(\frac{2n+t-2+s}{2}, \frac{2n-t+2+s}{2}\right) q_0^-(s) ds.
\end{aligned} \tag{5.85}$$

Now, we deal with  $I_4(\cdot)$ . In the same way, we have for any  $n \geq 0$  :

$$\begin{aligned}
I_4(t) &= \int_{t-1}^t (\eta_1 p^-)(\tau, -\tau+t) d\tau \\
&\quad \begin{cases} \int_{t-1}^t \eta_1(\tau, -\tau+t) p_0^-(-2\tau+t+2n) d\tau, & \text{if } \tau \in \left(\frac{2n-1+t}{2}, \frac{2n+t}{2}\right) \cap (0, t), \\ - \int_{t-1}^t \eta_1(\tau, -\tau+t) q_0^-(2\tau-t-2n) d\tau, & \text{if } \tau \in \left(\frac{2n+t}{2}, \frac{2n+t+1}{2}\right) \cap (0, t), \end{cases}
\end{aligned} \tag{5.86}$$

Consider the first case:

- If  $t \in [2n, 2n+1)$ ,  $n \geq 1$  : In this case, we write

$$(t-1, t) = \left(t-1, \frac{t+2n-1}{2}\right) \cup \left(\frac{t+2n-1}{2}, \frac{t+2n}{2}\right) \cup \left(\frac{t+2n}{2}, t\right),$$

$$\begin{aligned}
I_4(t) &= - \int_{t-1}^{\frac{t+2n-1}{2}} \eta_1(\tau, -\tau+t) q_0^-(2\tau-t-2n+2) d\tau \\
&\quad + \int_{\frac{t+2n-1}{2}}^{\frac{t+2n}{2}} \eta_1(\tau, -\tau+t) p_0^-(-2\tau+t+2n) d\tau \\
&\quad - \int_{\frac{t+2n}{2}}^t \eta_1(\tau, -\tau+t) q_0^-(2\tau-t-2n) d\tau,
\end{aligned}$$

which yields after a change of variables

$$\begin{aligned}
I_4(t) &= -\frac{1}{2} \int_{t-2n}^1 \eta_1\left(\frac{s+t+2n-2}{2}, \frac{t-2n-s+2}{2}\right) q_0^-(s) ds \\
&\quad + \frac{1}{2} \int_0^1 \eta_1\left(\frac{2n+t-s}{2}, \frac{t+s-2n}{2}\right) p_0^-(s) ds \\
&\quad - \frac{1}{2} \int_0^{t-2n} \eta_1\left(\frac{2n+t+s}{2}, \frac{t-2n-s}{2}\right) q_0^-(s) ds.
\end{aligned} \tag{5.87}$$

- If  $t \in [2n + 1, 2n + 2)$ ,  $n \geq 1$  : In the same way,

$$(t - 1, t) = (t - 1, \frac{t + 2n}{2}) \cup (\frac{t + 2n}{2}, \frac{t + 2n + 1}{2}) \cup (\frac{t + 2n + 1}{2}, t),$$

so, we obtain

$$\begin{aligned} I_4(t) &= \int_{t-1}^{\frac{t+2n}{2}} \eta_1(\tau, -\tau + t) p_0^-(-2\tau + t + 2n) d\tau \\ &\quad - \int_{\frac{t+2n}{2}}^{\frac{t+2n+1}{2}} \eta_1(\tau, -\tau + t) q_0^-(2\tau - t - 2n) d\tau \\ &\quad + \int_{\frac{t+2n+1}{2}}^t \eta_1(\tau, -\tau + t) p_0^-(-2\tau + t + 2n + 2) d\tau, \end{aligned}$$

and after a change of variables we obtain

$$\begin{aligned} I_4(t) &= \frac{1}{2} \int_0^{2n+2-t} \eta_1\left(\frac{2n+t-s}{2}, \frac{-2n+t+s}{2}\right) p_0^-(s) ds \\ &\quad - \frac{1}{2} \int_0^1 \eta_1\left(\frac{2n+t+s}{2}, \frac{t-2n-s}{2}\right) q_0^-(s) ds \\ &\quad + \frac{1}{2} \int_{2n+2-t}^1 \eta_1\left(\frac{2n+2+t-s}{2}, \frac{t+s-2n-2}{2}\right) p_0^-(s) ds. \end{aligned} \tag{5.88}$$

To summarize, the initial states  $p_0^-$ ,  $q_0^-$  are solutions of the following two equations:

- If  $t \in [2n, 2n + 1)$ ,  $n \geq 1$  :

By gathering (5.81), (5.82), (5.85), and (5.87) we get

$$\begin{aligned} 0 &= \left( \int_{t-2}^{t-1} \eta_1(\tau, \tau + 2 - t) d\tau + \int_{t-1}^t \eta_1(\tau, -\tau + t) d\tau \right) q_0^-(t - 2n) \\ &\quad - \frac{1}{2} \int_{t-2n}^1 \left[ \eta_1\left(\frac{s+t+2n-2}{2}, \frac{t-2n-s+2}{2}\right) + \eta_2\left(\frac{s+2n+t-4}{2}, \frac{s+2n-t}{2}\right) \right] q_0^-(s) ds \\ &\quad + \frac{1}{2} \int_0^1 \left[ \eta_1\left(\frac{2n+t-s}{2}, \frac{t+s-2n}{2}\right) + \eta_2\left(\frac{2n+t-2-s}{2}, \frac{2n-t+2-s}{2}\right) \right] p_0^-(s) ds \\ &\quad - \frac{1}{2} \int_0^{t-2n} \left[ \eta_1\left(\frac{2n+t+s}{2}, \frac{t-2n-s}{2}\right) + \eta_2\left(\frac{2n+t-2+s}{2}, \frac{2n-t+2+s}{2}\right) \right] q_0^-(s) ds. \end{aligned}$$

Letting  $x = t - 2n$  yields

$$\phi(x + 2n) q_0^-(x) - \int_0^1 (K_n^{21}(s, x) p_0^-(s) + K_n^{22}(s, x) q_0^-(s)) ds = 0, \quad x \in (0, 1), \tag{5.89}$$

where  $\phi$  is defined in (5.19) and the kernels  $K_n^{21}(\cdot, \cdot)$ ,  $K_n^{22}(\cdot, \cdot)$ , are given by

$$K_n^{21}(s, x) = -\frac{1}{2} \eta_1\left(\frac{4n+x-s}{2}, \frac{x+s}{2}\right) - \frac{1}{2} \eta_2\left(\frac{4n+x-2-s}{2}, \frac{2-s-x}{2}\right), \quad (s, x) \in (0, 1)^2, \tag{5.90}$$

$$K_n^{22}(s, x) = \frac{1}{2} \begin{cases} \eta_1\left(\frac{4n+x+s}{2}, \frac{x-s}{2}\right) + \eta_2\left(\frac{4n-2+x+s}{2}, \frac{2+s-x}{2}\right), & \text{if } 0 \leq s \leq x, \\ \eta_1\left(\frac{4n-2+s+x}{2}, \frac{x-s+2}{2}\right) + \eta_2\left(\frac{4n-4+s+x}{2}, \frac{s-x}{2}\right), & \text{if } x \leq s \leq 1. \end{cases} \tag{5.91}$$

Observe that when  $t$  varies in  $[2n, 2n + 1)$  the  $x$  varies in  $[0, T - 2n)$ .



- If  $t \in [2n + 1, 2n + 2)$ ,  $n \geq 1$  :

Gathering (5.81), (5.82), 5.84, and (5.88) yields

$$\begin{aligned}
0 &= - \left( \int_{t-2}^{t-1} \eta_1(\tau, \tau + 2 - t) d\tau + \int_{t-1}^t \eta_1(\tau, -\tau + t) d\tau \right) p_0^-(2n + 2 - t) \\
&+ \frac{1}{2} \int_0^{2n+2-t} \left[ \eta_1\left(\frac{2n+t-s}{2}, \frac{-2n+t+s}{2}\right) + \eta_2\left(\frac{2n-2+t-s}{2}, \frac{2n+2-t-s}{2}\right) \right] p_0^-(s) ds \\
&+ \frac{1}{2} \int_{2n+2-t}^1 \left[ \eta_1\left(\frac{2n+2+t-s}{2}, \frac{t+s-2n-2}{2}\right) + \eta_2\left(\frac{2n+t-s}{2}, \frac{2n+4-t-s}{2}\right) \right] p_0^-(s) ds \\
&- \frac{1}{2} \int_0^1 \left[ \eta_1\left(\frac{2n+t+s}{2}, \frac{t-2n-s}{2}\right) + \eta_2\left(\frac{s+t-2+2n}{2}, \frac{s-t+2n+2}{2}\right) \right] q_0^-(s) ds.
\end{aligned}$$

Letting  $2n + 2 - t = x$  yields

$$\phi(2n + 2 - x)p_0^-(x) - \int_0^1 [K_n^{11}(s, x)p_0^-(s) + K_n^{12}(s, x)q_0^-(s)] ds = 0, \quad x \in (0, 1), \quad n \geq 1, \quad (5.92)$$

where  $\phi$  is defined in (5.19) and the kernels  $K_n^{11}(\cdot, \cdot)$ ,  $K_n^{12}(\cdot, \cdot)$ , are given by

$$K_n^{11}(s, x) = \frac{1}{2} \begin{cases} \eta_1\left(\frac{4n+2-x-s}{2}, \frac{2-x+s}{2}\right) + \eta_2\left(\frac{4n-x-s}{2}, \frac{x-s}{2}\right), & \text{if } 0 \leq s \leq x, \\ \eta_1\left(\frac{4n+4-x-s}{2}, \frac{s-x}{2}\right) + \eta_2\left(\frac{4n+2-x-s}{2}, \frac{2+x-s}{2}\right) & \text{if } x \leq s \leq 1, \end{cases} \quad (5.93)$$

$$K_n^{12}(s, x) = -\frac{1}{2}\eta_1\left(\frac{4n+2-x+s}{2}, \frac{2-x-s}{2}\right) - \frac{1}{2}\eta_2\left(\frac{s+4n-x}{2}, \frac{s+x}{2}\right), \quad (s, x) \in (0, 1)^2. \quad (5.94)$$

Similarly, when  $t$  varies  $[2n + 1, 2n + 2)$  then  $x$  varies  $(2n + 2 - T, 1]$ .

Observe that equations (5.89) and (5.92) form a system of Fredholm Integral equations of third kind. Indeed, if  $t \in [2n, 2n + 1)$ , the interval  $[2, t)$  can be written as

$$[2, t) = (\cup_{l=1}^{n-1} [2l, 2l + 1)) \cup (\cup_{k=1}^{n-1} [2k + 1, 2k + 2)) \cup [2n, t),$$

In this case, we have to solve Equation (5.89) (resp. Equation (5.92)) in intervals of the form  $[2l, 2l + 1)$  (resp.  $[2k + 1, 2k + 2)$ ) for some  $k, l \geq 1$ . In the same way, for  $t \in [2n + 1, 2n + 2)$ , the interval  $[2, t)$  can be written as

$$[2, t) = (\cup_{l=1}^n [2l, 2l + 1)) \cup (\cup_{k=1}^{n-1} [2k + 1, 2k + 2)) \cup [2n + 1, T).$$

Similarly, we have to solve Equation (5.89) (resp. Equation (5.92)) in the intervals of the form  $[2l, 2l + 1)$  (resp.  $[2k + 1, 2k + 2)$ ) for some  $k, l \geq 1$ . More precisely, introduce the kernels  $\mathbb{K}_{n,k,l}^i(\cdot, \cdot)$ ,  $1 \leq i \leq 2$ ,  $n \geq 2$ ,  $k, l \geq 1$ , by

$$\mathbb{K}_{n,k,l}^1(s, x) = \begin{cases} \mathbf{K}_{k,l}(s, x), & 1 \leq k \leq n - 1, \quad 1 \leq l \leq n, & \text{if } x \in [0, T - 2n), \\ \mathbf{K}_{k,l}(s, x), & 1 \leq k \leq n - 1, \quad 1 \leq l \leq n - 1, & \text{if } x \in [T - 2n, 1], \end{cases}$$

$$\mathbb{K}_{n,k,l}^2(s, x) = \begin{cases} \mathbf{K}_{k,l}(s, x), & 1 \leq k \leq n, \quad 1 \leq l \leq n, & \text{if } x \in [0, 2n + 2 - T), \\ \mathbf{K}_{k,l}(s, x), & 1 \leq k \leq n + 1, \quad 1 \leq l \leq n, & \text{if } x \in [2n + 2 - T, 1], \end{cases}$$

where

$$A_{k,l}(x) = \begin{pmatrix} \phi(2k + 2 - x) & 0 \\ 0 & \phi(2l + x) \end{pmatrix}, \quad x \in [0, 1], \quad k, l \geq 1,$$

$$\mathbf{K}_{k,l}(s, x) = \begin{pmatrix} K_k^{11}(s, x) & K_k^{12}(s, x) \\ K_l^{21}(s, x) & K_l^{22}(s, x) \end{pmatrix}, \quad (s, x) \in [0, 1]^2, \quad k, l \geq 1,$$

associated with the third kind Fredholm integral equations

$$A_{k,l}(x) \begin{pmatrix} p_0^-(x) \\ q_0^-(x) \end{pmatrix} = \int_0^1 \mathbb{K}_{n,k,l}^1(s,x) \begin{pmatrix} p_0^-(s) \\ q_0^-(s) \end{pmatrix} ds, \quad (5.95)$$

$$A_{k,l}(x) \begin{pmatrix} p_0^-(x) \\ q_0^-(x) \end{pmatrix} = \int_0^1 \mathbb{K}_{n,k,l}^2(s,x) \begin{pmatrix} p_0^-(s) \\ q_0^-(s) \end{pmatrix} ds. \quad (5.96)$$

Now, we come to the main theorem of this section:

**Theorem 104** *Let  $n \geq 2$  be an integer.*

- *If  $2n \leq T < 2n + 1$  : Then the unique continuation property for (5.70) holds true at time  $T$  if, and only if, there exist  $k, l \geq 1$  such that the unique solution  $(p_0^-, q_0^-)$  to Equation (5.95) is the null one.*
- *If  $2n + 1 \leq T < 2n + 2$  : Then the unique continuation property for (5.70) holds true at time  $T$  if, and only if, there exist  $k, l \geq 1$  such that the unique solution  $(p_0^-, q_0^-)$  to Equation (5.96) is the null one.*

Now, let us assume that the weak observability holds, i.e

- If  $2n \leq T < 2n + 1$  :

1. There exist  $1 \leq k \leq n - 1, 1 \leq l \leq n$ , such that

$$\phi(2k + 2 - x) \neq 0, \phi(2l + x) \neq 0, \forall x \in [0, T - 2n]. \quad (5.97)$$

2. There exist  $1 \leq k \leq n - 1, 1 \leq l \leq n - 1$ , such that

$$\phi(2k + 2 - x) \neq 0, \phi(2l + x) \neq 0, \forall x \in [T - 2n, 1]. \quad (5.98)$$

- If  $2n + 1 \leq T < 2n + 2$  :

1. There exist  $1 \leq k \leq n - 1, 1 \leq l \leq n$ , such that

$$\phi(2k + 2 - x) \neq 0, \phi(2l + x) \neq 0, \forall x \in [2n + 2 - T, 0). \quad (5.99)$$

2. There exist  $1 \leq k \leq n, 1 \leq l \leq n$ , such that

$$\phi(2k + 2 - x) \neq 0, \phi(2l + x) \neq 0, \forall x \in [2n + 2 - T, 1]. \quad (5.100)$$

Under assumptions (5.97) and (5.98) (resp. (5.99) and (5.100)), Equations (5.95) and (5.96) write

$$\begin{pmatrix} p_0^-(x) \\ q_0^-(x) \end{pmatrix} = \int_0^1 A_{k,l}^{-1}(x) \mathbb{K}_{n,k,l}^i(s,x) \begin{pmatrix} p_0^-(s) \\ q_0^-(s) \end{pmatrix} ds := \mathcal{K}_{k,l}^i \begin{pmatrix} p_0^- \\ q_0^- \end{pmatrix} (x), \quad i = 1, 2, \quad (5.101)$$

which are a second kind Fredholm integral equations. The following corollary is a straightforward consequence of the above theorem:

**Corollary 105** *Let  $n \geq 2$  be an integer.*

- *If  $2n \leq T < 2n + 1$  : Assume that (5.97) and (5.98) hold for some  $k, l \geq 1$ . The unique continuation property for (5.70) holds at time  $T$  if, and only if  $1 \notin \sigma \left( \mathcal{K}_{k,l}^1 \right)$ .*

- If  $2n + 1 \leq T < 2n + 2$  : Assume that (5.99) and (5.100) hold for some  $k, l \geq 1$ . The unique continuation property (5.70) holds at time  $T$  if, and only if  $1 \notin \sigma(\mathcal{K}_{k,l}^2)$ .

**Proof.** By assumptions (4.19)-(5.100), the Equations (5.101) are Fredholm integral equations of second kind. By the Fredholm alternative, they possess a unique solution if, and only if  $1 \notin \sigma(\mathcal{K}_{k,l}^i)$ ,  $i = 1, 2$ , for some  $l, k \geq 1$ . ■

**Remark 106** Since the component of the kernel  $A_{k,l}^{-1}(\cdot)\mathbb{K}_{n,k,l}^i(\cdot, \cdot)$ ,  $i = 1, 2$ ,  $k, l \geq 1$ , are completely known, we can always prove that  $1 \notin \sigma(\mathcal{K}_{k,l}^i)$ ,  $i = 1, 2$ , by assuming that  $\|\mathcal{K}_{k,l}\|_{\mathcal{L}(L^2(0,1;M_{2 \times 2}(\mathbb{R})))} < 1$ . Notice that this is not necessarily a smallness assumption on the coupling coefficients since the kernel involves the matrix  $A_{k,l}^{-1}(\cdot)$ .

**Remark 107** It is not difficult to see that if  $\eta_1$  and  $\eta_2$  are time independent, the compact operator  $\mathcal{K}_{k,l}^i = \mathcal{K}$ , is symmetric. Therefore, its spectrum consists of real eigenvalues. However, giving a characterization of the spectrum needs more care.

**Remark 108** In [24], it has been shown that the unique continuation property for  $a(t, x) = a(x)$  and  $b = 0$  holds at time  $T > 0$  for System (5.70) if, and only if  $T \geq 4$ , and

$$\int_0^1 a(s) \sin^2(\pi n s) ds \neq 0, \quad \forall n \geq 1. \quad (5.102)$$

It is natural to expect that condition (5.102) is equivalent to the uniqueness of the solution to Equation (5.95) (or (5.96)) for  $\mathbb{K}_{n,k,l} = \mathbb{K}$ .

**Remark 109** With a simple change of variable (see [4, Remark 15]), we can deduce that all the results proved for cascade system with velocity coupling ( $a \neq 0$ ) hold true for zero order coupling (replace  $a\varphi_t$  by  $a\varphi$  in System (5.3)) in the space

$$H_0^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \times H^{-1}(0, 1).$$

### 5.4.3 Examples

Here, we will provide some illustrations of Theorem 104. For the sake of simplicity, we will use both coupling functions  $a$  and  $b$ , also, we will choose the time of control to be  $T = 2n$  for some  $n \geq 2$ .

**The case**  $\eta_1(t, x) = \alpha(t - x)$ ,  $\eta_2(t, x) = \beta(t + x)$

Set  $\eta_1(t, x) = \alpha(t - x)$  and  $\eta_2(t, x) = \beta(t + x)$  for some smooth functions  $\alpha$  and  $\beta$  in  $Q_T$ . After performing a simple computations, the components  $K_k^{1,i}(\cdot, \cdot)$ ,  $K_l^{2,i}(\cdot, \cdot)$ ,  $i = 1, 2$ ,  $k, l \geq 1$ , given in (5.90), (5.91), (5.93) and (5.94) and the function  $\phi$  defined in (5.19) take the form

$$\begin{aligned} 2K_k^{12}(s, x) &= -\alpha(2k + s) - \beta(2k + s), \quad s \in (0, 1), \\ 2K_l^{21}(s, x) &= -\alpha(2l - s) - \beta(2l - s), \quad s \in (0, 1), \\ 2K_k^{11}(s, x) &= \begin{cases} \alpha(2k - s) + \beta(2k - s), & \text{if } 0 \leq s \leq x, \\ \alpha(2k + 2 - s) + \beta(2k + 2 - s), & \text{if } x \leq s \leq 1, \end{cases} \\ 2K_l^{22}(s, x) &= \begin{cases} \alpha(2l + s) + \beta(2l + s), & \text{if } 0 \leq s \leq x, \\ \alpha(2l - 2 + s) + \beta(2l - 2 + s), & \text{if } x \leq s \leq 1. \end{cases} \end{aligned}$$

$$\begin{aligned}\phi(t) &= \int_{t-2}^{t-1} \alpha(t-2)d\tau + \int_{t-1}^t \beta(t)d\tau \\ &= \alpha(t-2) + \beta(t), \quad t \geq 2.\end{aligned}$$

Next, we define the functions  $S_j^i$ ,  $i = 1, 2$  on  $[0, 1]$  for any  $j \geq 1$  by

$$S_j^1(s) = (2\phi(2j+2-s))^{-1} \begin{pmatrix} -2\phi'(2j+2-s) - \alpha(2j+2-s) \\ +\beta(2j-s) - \beta(2j+2-s) + \alpha(2j-s) \end{pmatrix},$$

and

$$S_j^2(s) = (2\phi(2j+s))^{-1} \begin{pmatrix} -\phi'(2j+s) + \alpha(2j+s) + \beta(2j+s) \\ -\alpha(2j-2+s) - \beta(2j-2+s) \end{pmatrix}.$$

We have the following unique continuation result:

**Proposition 110** *Let  $n \geq 2$ . Assume that  $A_{k,l}^{-1}(\cdot)$  exists on  $[0, 1]$  for some  $1 \leq k, l \leq n-1$ . Then the unique continuation property (5.70) holds true in time  $T = 2n$  if*

$$\int_0^1 S_k^1(s)ds \neq \int_0^1 S_l^2(s)ds,$$

**Proof.** For  $T = 2n$ , the system of integral equations (5.95) writes:

$$\begin{aligned}2\phi(2k+2-x)p_0^-(x) &= \int_0^x (\alpha(2k-s) + \beta(2k-s))p_0^-(s)ds \\ &+ \int_x^1 (\alpha(2k+2-s) + \beta(2k+2-s))p_0^-(s)ds \\ &- \int_0^1 (\alpha(2k+s) + \beta(2k+s))q_0^-(s)ds,\end{aligned}\tag{5.103}$$

$$\begin{aligned}2\phi(2l+x)q_0^-(x) &= \int_0^x (\alpha(2l+s) + \beta(2l+s))q_0^-(s)ds \\ &+ \int_x^1 (\alpha(2l-2+s) + \beta(2l-2+s))q_0^-(s)ds \\ &- \int_0^1 (\alpha(2l-s) + \beta(2l-s))p_0^-(s)ds.\end{aligned}\tag{5.104}$$

Taking the derivative of (5.103) and (5.104) yields

$$2\phi(2k+2-x)(p_0^-(x))' = \begin{pmatrix} -2\phi'(k+2-x) - \alpha(2k+2-x) \\ +\beta(2k-x) - \beta(2k+2-x) + \alpha(2k-x) \end{pmatrix} p_0^-(x),\tag{5.105}$$

$$2\phi(2l+x)(q_0^-(x))' = \begin{pmatrix} -\phi'(2l+x) + \alpha(2l+x) + \beta(2l+x) \\ -\alpha(2l-2+x) - \beta(2l-2+x) \end{pmatrix} q_0^-(x).\tag{5.106}$$

Now, we divide by  $2\phi(2k+2-x)$  and  $2\phi(2l+x)$  the equations (5.105) and (5.106) and integrating the later two systems to get

$$p_0^-(x) = \exp\left(\int_0^x S_l^1(s)ds\right)p_0^-(0), \quad q_0^-(x) = \exp\left(\int_0^x S_k^2(s)ds\right)q_0^-(0),$$

where

$$S_k^1(s) = (2\phi(2k+2-s))^{-1} \begin{pmatrix} -2\phi'(k+2-s) - \alpha(2k+2-s) \\ +\beta(2k-s) - \beta(2k+2-s) - \alpha(2k-s) \end{pmatrix}$$

and

$$S_l^2(s) = (2\phi(2l+s))^{-1} \begin{pmatrix} -2\phi'(2l+s) + \alpha(2l+s) + \beta(2l+s) \\ -\alpha(2l-2+s) - \beta(2l-2+s) \end{pmatrix}.$$

Since  $p_0^-$  and  $q_0^-$  lie in  $\tilde{H}$ , they are linked by the boundary conditions  $(p_0^- + q_0^-)|_{x=0,1}$ , we get the system

$$\begin{cases} p_0^-(0) + q_0^-(0) = 0, \\ p_0^-(x) = \exp\left(\int_0^1 S_k^1(s)ds\right) p_0^-(0) + \exp\left(\int_0^1 S_l^2(s)ds\right) q_0^-(0) = 0, \end{cases}$$

which has a unique solution if, and only if

$$\int_0^1 S_k^1(s)ds \neq \int_0^1 S_l^2(s)ds.$$

■

**The case**  $\eta_1(t, x) = \alpha(t+x)$ ,  $\eta_2(t, x) = \beta(t-x)$

This time set  $\eta_1(t, x) = \alpha(t+x)$  and  $\eta_2(t, x) = \beta(t-x)$  for some smooth functions  $\alpha$  and  $\beta$  in  $Q_T$ . The components  $K_k^{1,i}(\cdot, \cdot)$ ,  $K_l^{2,i}(\cdot, \cdot)$ ,  $j = 1, 2$ ,  $k, l \geq 1$ , given in (5.90), (5.91), (5.93) and (5.94) and the function  $\phi$  defined in (5.19) take the form

$$\begin{aligned} 2K_k^{11}(x) &= -2K_k^{12}(x) = \alpha(2k+2-x) + \beta(2k-x), \quad x \in [0, 1], \\ 2K_l^{22}(x) &= -2K_l^{21}(x) = \alpha(2l+x) + \beta(2l-2+x), \quad x \in [0, 1], \end{aligned}$$

$$\begin{aligned} \phi(t) &= \int_{t-2}^{t-1} a(\tau, \tau - (t-2))d\tau + \int_{t-1}^t \beta(\tau, t-\tau)d\tau \\ &= \int_{t-2}^{t-1} \alpha(2\tau - (t-2))d\tau + \int_{t-1}^t \beta(2\tau - t)d\tau \\ &= \frac{1}{2} \int_{t-2}^t (\alpha(s) + \beta(s)) ds, \quad t \geq 2. \end{aligned}$$

At time  $T = 2n$ , the integral equation (5.101) turns to

$$(\mathcal{K}_{k,l}\Phi)(x) = A_{k,l}^{-1}(x)\mathbb{K}_{n,k,l}(x) \int_0^1 \Phi(s) ds, \quad \Phi \in L^2(0, 1)^2, \quad k, l \geq 1, \quad x \in [0, 1],$$

where

$$A_{k,l}^{-1}(x)\mathbb{K}_{n,k,l}(x) = \begin{pmatrix} \frac{K_k^{11}(x)}{\phi(2k+2-x)} & -\frac{K_k^{11}(x)}{\phi(2k+2-x)} \\ -\frac{K_l^{22}(x)}{\phi(2l+x)} & \frac{K_l^{22}(x)}{\phi(2l+x)} \end{pmatrix}, \quad 1 \leq k, l \leq n-1, \quad x \in [0, 1].$$

We have the following unique continuation result:

**Proposition 111** *Let  $n \geq 2$ . Assume that  $A_{k,l}^{-1}(\cdot)$  exists on  $[0, 1]$  for some  $1 \leq k, l \leq n-1$ . Then the unique continuation property (5.70) holds true in time  $T = 2n$  if, and only if*

$$1 \notin \sigma \left( \int_0^1 A_{k,l}^{-1}(x)\mathbb{K}_{n,k,l}(x)dx \right).$$

The proof of the above proposition is an immediate consequence of combining Corollary 105 and the following lemma:

**Lemma 112** Let  $J : L^2(0, 1)^n \rightarrow L^2(0, 1)^n$  be the compact operator

$$(Jf)(x) = M(x) \int_0^1 f(s) ds, \quad x \in (0, 1),$$

where  $M \in C([0, 1], M_{n \times n}(\mathbb{R}))$ . Then  $1 \in \sigma(J)$  if, and only if  $1 \in \sigma\left(\int_0^1 M\right)$ .

**Proof.** See Appendix 5.5.2. ■

**Remark 113** In the above two examples, we have used both  $a$  and  $b$  to simplify computations. Dealing with the integral equation (5.95) and (5.96) with only one of them seems to be not accessible unless they are in a very particular class of simple functions (for instance  $a(t, x) = \kappa_2 t + \kappa_1 x + \kappa_0$ ,  $\kappa_i \in \mathbb{R}$ ,  $i = 1, 2, 3$ ).

## 5.5 Appendix

### 5.5.1 Proof of Proposition 81

We start by proving that  $1 \Rightarrow 2$ .

Suppose there exists  $x_0 \in [0, 1]$  such for any  $s \times s$  matrix  $M_{\text{ext}}$ , extracted from  $M$ , we have  $\det M_{\text{ext}}(x_0) = 0$ . Denote by  $M_1, M_2, \dots, M_n$  the  $s$ -dimensional row vector functions of the matrix  $M$  so that:

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{bmatrix}.$$

Then from our assumption, the vector space spanned by the  $M_i(x_0)$  is at most of dimension  $s - 1$ . Thus there exists  $V \in \mathbb{R}^s$  such that

$$|V|_{\mathbb{R}^s} = 1 \text{ and } M_i(x_0) \cdot V = 0, \quad 1 \leq i \leq n.$$

Let  $u \in C_0^\infty(\mathbb{R})$  with  $\int_{\mathbb{R}} u^2(x) dx = 1$  and (for example)  $\text{supp}(u) = [-1, 1]$ . We are going to prove that the sequence:

$$h_j(x) = \sqrt{j} u(j(x - x_0)) V, \quad j \geq j_0, \quad x \in \mathbb{R},$$

is a singular sequence for the multiplication operator  $\mathbb{M}_{\text{ext}}$  on  $L^2(0, 1)^s$  whose matrix is some  $s \times s$  matrix  $M_{\text{ext}}$ , extracted from  $M$ . If  $x_0 \in (0, 1)$  (we leave to the reader to check that the proof works with  $x_0 = 0$  by choosing  $h_j(x) = \sqrt{j} u(jx - 1) V$  and with  $x_0 = 1$  by choosing  $h_j(x) = \sqrt{j} u(j(x - 1) + 1) V$ ), we see from the definition of  $u$  that for  $j_0$  sufficiently large

$$\text{supp}(h_j) = \left[ x_0 - \frac{1}{j}, x_0 + \frac{1}{j} \right] \subset [0, 1], \quad j \geq j_0.$$

Moreover, for all  $j \geq j_0$ .

$$\begin{aligned} \int_0^1 h_j^2(x) dx &= j \int_0^1 u^2(j(x - x_0)) |V|_{\mathbb{R}^s}^2 dx \\ &= j \int_{x_0 - \frac{1}{j}}^{x_0 + \frac{1}{j}} u^2(j(x - x_0)) dx \\ &= \int_{-1}^1 u^2(\xi) d\xi = 1. \end{aligned}$$

Since  $\text{supp}(h_j) \rightarrow \{x_0\}$  as  $j \rightarrow \infty$ ,  $(h_j)$  has no convergent subsequence in  $L^2(0,1)^n$ . On the other hand:

$$\begin{aligned} \|\mathbb{M}_{\text{ext}} h_j\|_{L^2(0,1)^s}^2 &= \int_0^1 |M_{\text{ext}}(x) h_j(x)|^2 dx \\ &= j \int_{x_0 - \frac{1}{j}}^{x_0 + \frac{1}{j}} u^2(j(x - x_0)) |M_{\text{ext}}(x) V|^2 dx \\ &= \int_{-1}^1 u^2(\xi) \left| M_{\text{ext}}\left(x_0 + \frac{1}{j}\xi\right) V \right|^2 d\xi. \end{aligned}$$

Thus, from Lebesgue's dominated convergence theorem, we get:

$$\lim_{j \rightarrow \infty} \|\mathbb{M}_{\text{ext}} h_j\|_{L^2(0,1)^s}^2 = 0.$$

Since, in particular, the choice of  $V$  and the continuity of  $M_{\text{ext}}$  give:

$$\lim_{j \rightarrow \infty} M_{\text{ext}}\left(x_0 + \frac{1}{j}\xi\right) V = M_{\text{ext}}(x_0) V = 0.$$

If  $n = ds + q$  with  $d \geq 1$  and  $0 \leq q \leq s - 1$ , we form the  $s \times s$  extracted matrices:

$$M_{\text{ext}}^1 = \begin{bmatrix} M_1 \\ \vdots \\ M_s \end{bmatrix}, \dots, M_{\text{ext}}^d = \begin{bmatrix} M_{(d-1)s+1} \\ \vdots \\ M_{ds} \end{bmatrix}, M_{\text{ext}}^{d+1} = \begin{bmatrix} M_{ds+1} \\ \vdots \\ M_n \\ M_1 \\ \vdots \\ M_{s-q} \end{bmatrix}.$$

We then have:

$$\begin{aligned} \|\mathbb{M} h_j\|_{L^2(0,1)^n}^2 &= \sum_{k=1}^n \|M_k \cdot h_j\|_{L^2(0,1)}^2 \\ &\leq \sum_{k=1}^{d+1} \|\mathbb{M}_{\text{ext}}^k h_j\|_{L^2(0,1)^s}^2. \end{aligned}$$

It readily follows that:

$$\lim_{j \rightarrow \infty} \|\mathbb{M} h_j\|_{L^2(0,1)^n}^2 = 0.$$

This proves that  $1 \Rightarrow 2$ .

To prove that  $2 \Rightarrow 1$ , we assume that for all  $x \in [0, 1]$ , there exists a  $s \times s$  matrix  $M_{\text{ext}}$ , extracted from  $M$ , such that

$$\det M_{\text{ext}}(x) \neq 0.$$

Each one of the functions  $|\det M_{\text{ext}}(x)|$  is uniformly continuous on  $[0, 1]$ :

$$\forall \varepsilon > 0, \exists \eta_{M_{\text{ext}}} > 0, |x - y| < \eta_{M_{\text{ext}}} \Rightarrow \left| |\det M_{\text{ext}}(x)| - |\det M_{\text{ext}}(y)| \right| < \varepsilon, \forall (x, y) \in [0, 1]^2.$$

In the sequel, we set  $\eta = \min \{\eta_{M_{\text{ext}}}, M_{\text{ext}} \text{ extracted } s \times s \text{ matrix}\}$ . Let  $0 \leq j \leq m - 1$  and  $\xi_j \in \left[\frac{j}{m}, \frac{j+1}{m}\right]$ .

There exists a  $s \times s$  matrix  $M_{\text{ext}}^j$

$$\left| \det M_{\text{ext}}^j(\xi_j) \right| := \delta_j > 0.$$

Choosing  $m$  such that  $\frac{1}{m} < \eta$ , we get with  $\varepsilon < \min_{0 \leq j \leq m-1} \delta_j = \delta$

$$\left| \det M_{\text{ext}}^j(x) \right| > \delta - \varepsilon, \quad \forall x \in \left[ \frac{j}{m}, \frac{j+1}{m} \right].$$

From Proposition 81, this ensures that for each  $0 \leq j \leq m-1$ ,  $\mathbb{M}_{\text{ext}}^j$  is invertible on  $L^2\left(\frac{j}{m}, \frac{j+1}{m}\right)^s$ . Now, we can write for any  $h \in L^2(0, 1)^s$ :

$$\begin{aligned} \|\mathbb{M}h\|_{L^2(0,1)^n}^2 &= \sum_{k=1}^n \|M_k \cdot h\|_{L^2(0,1)}^2 \\ &= \sum_{j=0}^{m-1} \sum_{k=1}^n \int_{\frac{j}{m}}^{\frac{j+1}{m}} |M_k(x) \cdot h(x)|^2 dx \\ &\geq \sum_{j=0}^{m-1} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \left| M_{\text{ext}}^j(x) h(x) \right|^2 dx \\ &\geq C \sum_{j=0}^{m-1} \int_{\frac{j}{m}}^{\frac{j+1}{m}} |h(x)|^2 dx \\ &\geq C \int_0^1 |h(x)|^2 dx. \end{aligned}$$

This ends the proof.

### 5.5.2 Proof of Lemma 112

Proof of  $\Rightarrow$ : If  $1 \in \sigma(J)$ , there will exist a non-zero  $f \in L^2(0, 1)^n$  such that

$$f(x) = M(x) \int_0^1 f(s) ds, \quad x \in (0, 1). \quad (5.107)$$

Integrating (5.107) over  $(0, 1)$  yields

$$\int_0^1 f(s) ds = \int_0^1 M(s) ds \int_0^1 f(s) ds.$$

This shows that  $\int_0^1 f$  is an eigenvector of the matrix  $\int_0^1 M$  associated with the eigenvalue 1.

Proof of  $\Leftarrow$ : If  $1 \in \sigma\left(\int_0^1 M\right)$ , then there exists an eigenvector of  $\int_0^1 M$  denoted by  $V \in \mathbb{R}^n$  such that

$$V = \left( \int_0^1 M(x) dx \right) V.$$

applying the matrix  $M(\cdot)$  yields

$$M(x)V = M(x) \left( \int_0^1 M(x)V dx \right), \quad x \in [0, 1].$$

This shows that the vector  $M(\cdot)V$  is an eigenvector of  $J$  associated with the eigenvalue 1.



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