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Jiaxin Qiao

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THÈSE DE DOCTORAT
DE L'UNIVERSITÉ PSL

Préparée à École normale supérieure - PSL

**Sur la rotation de Wick de la fonction à quatre points dans
les théories de champ conformes**
**On the Wick Rotation of the Four-point Function in
Conformal Field Theories**

Soutenue par
Jiaxin Qiao
Le 14/06/2022

École doctorale n°564
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gies - Théorie**

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Abstract

Conformal field theories (CFTs) in Euclidean signature satisfy well-accepted rules, such as conformal invariance and the convergent Euclidean operator product expansion (OPE). Nowadays, it is common to assume that CFT correlators exist and have various properties in the Lorentzian signature. Some of these properties may represent extra assumptions, and it is an open question if they hold for familiar statistical-physics CFTs such as the critical 3d Ising model. In this thesis, we clarify that at the level of four-point functions, the Euclidean CFT axioms imply the standard quantum field theory axioms such as Osterwalder-Schrader axioms (in Euclidean) and Wightman axioms (in Lorentzian).

In part I, we study the Euclidean CFT four-point functions in the cross-ratio space. We show that the four-point functions in conformal field theory are defined as tempered distributions on the boundary of the region of convergence of the conformal block expansion. The conformal block expansion converges in the sense of distributions on this boundary, i.e. it can be integrated term by term against appropriate test functions. This result can be interpreted as giving a new class of functionals that commute with the summation of the conformal blocks when applied to the crossing equation, and we comment on the relation of our construction to other types of functionals. Our language is useful in all considerations involving the boundary of the region of convergence, e.g. for deriving the dispersion relations. We establish our results by elementary methods, relying only on crossing symmetry and the standard convergence properties of the conformal block expansion.

In part II, we consider CFT four-point functions of scalar primary operators in Lorentzian signature. We derive a minimal set of their properties solely from the Euclidean unitary CFT axioms. We establish all Wightman axioms (temperedness, Poincaré invariance, unitarity, spectral property, local commutativity, clustering), Lorentzian conformal invariance, and distributional convergence of the s-channel Lorentzian OPE. This is done constructively by analytically continuing the 4-point functions using the s-channel OPE expansion in the radial cross-ratios $\rho, \bar{\rho}$. We prove a crucial fact that $|\rho|, |\bar{\rho}| < 1$ inside the forward tube, and set bounds on how fast $|\rho|, |\bar{\rho}|$ may tend to 1 when the four-point configuration approaches the Minkowski space.

We also provide a guide to the axiomatic QFT literature for the modern CFT audience. We review the Wightman and Osterwalder-Schrader (OS) axioms for Lorentzian and Euclidean QFTs, and the celebrated OS theorem connecting them. We also review a classic result of Mack about the distributional OPE convergence. Some of the classic arguments turn out useful in our setup. Others fall short of our needs due to Lorentzian assumptions (Mack) or unverifiable Euclidean assumptions (OS theorem).

In part III, we study the OPE convergence properties of the CFT scalar four-point correlation functions in the sense of functions. We establish the criteria on OPE convergence in s-, t- and u-channels. We give a complete classification of four-point configurations in the Lorentzian signature. We show that all configurations in each class have the same OPE convergence properties in s-, t- and u-channels. We give tables including the information of OPE convergence for all classes. By taking the union of all Lorentzian configurations where at least one OPE channel converges, we get a minimal domain of analyticity of Lorentzian CFT four-point functions.

In part IV, we give a preview on two possible generalizations: one is the CFT four-point functions of operators with general $SO(d)$ spins, the other is the CFT four-point functions in the Minkowski cylinder.

Résumé

Les théories des champs conformes (*CFTs*) en signature euclidienne satisfont plusieurs règles bien acceptées, telles que l'invariance conforme et la convergence de l'expansion du produit d'opérateurs (*OPE*) en signature euclidienne. De nos jours, il est courant de supposer l'existence des fonctions de corrélation d'une *CFT* et d'assumer diverses propriétés en signature lorentzienne. Certaines de ces propriétés peuvent représenter des hypothèses supplémentaires, et leur validité reste incertaine dans les *CFT* de physique statistique familières telles que le modèle d'Ising critique en trois dimensions. Dans cette thèse, nous clarifions qu'au niveau des fonctions de corrélation à quatre points, les axiomes *CFT* euclidiens impliquent les axiomes standards de la théorie quantique des champs tels que les axiomes d'Osterwalder-Schrader (en signature euclidienne) et les axiomes de Wightman (en signature lorentzienne).

Dans la partie I, nous étudions les fonctions à quatre points de *CFT* euclidienne dans l'espace des birapports. Nous montrons que les fonctions à quatre points dans les théories des champs conformes sont définies comme des distributions tempérées sur la frontière de la région de convergence de l'expansion en blocs conformes. L'expansion en blocs conformes converge dans le sens des distributions sur cette frontière, c'est-à-dire qu'elle peut être intégrée terme à terme contre des fonctions tests appropriées. Ce résultat peut être interprété comme donnant une nouvelle classe de fonctionnelles qui commutent avec la sommation des blocs conformes lorsqu'elle est appliquée à l'équation de croisement, et nous commentons la relation de notre construction avec d'autres types de fonctionnelles. Notre langage est utile pour toutes questions impliquant la frontière de la région de convergence de l'*OPE*, par exemple, pour déduire les relations de dispersion. Nous établissons nos résultats par des méthodes élémentaires, en nous appuyant uniquement sur la symétrie de croisement et les propriétés de convergence standards de l'expansion en blocs conformes.

Dans la partie II, nous considérons les fonctions à quatre points des opérateurs primaires scalaires dans une *CFT* en signature lorentzienne. Nous dérivons un ensemble minimal de leurs propriétés uniquement à partir des axiomes *CFT* unitaires euclidiens. Nous établissons tous les axiomes de Wightman (caractère tempéré, invariance de Poincaré, unitarité, propriété spectrale, commutativité locale, *clustering*), l'invariance conforme lorentzienne et la convergence distributionnelle de l'*OPE* lorentzien dans le *canal s*. Ceci est fait de manière constructive en continuant analytiquement les fonctions à 4 points en utilisant l'expansion *OPE* du canal *s* dans les birapports radiaux $\rho, \bar{\rho}$. Nous prouvons un fait crucial que $|\rho|, |\bar{\rho}| < 1$ à l'intérieur du tube avant, et fixons des limites sur la vitesse à laquelle $|\rho|, |\bar{\rho}|$ tendent vers 1 lorsque la configuration à quatre points se rapproche de l'espace de Minkowski.

Nous fournissons également un guide de la littérature axiomatique QFT pour le public de *CFT*. Nous passons en revue les axiomes de Wightman et Osterwalder-Schrader (OS) pour les QFT lorentziennes et euclidiennes, ainsi que le célèbre théorème d'OS qui les relie. Nous passons également en revue un résultat classique de Mack sur la convergence *OPE* distributionnelle. Certains des arguments classiques s'avèrent utiles dans notre configuration. D'autres ne répondent pas à nos besoins en raison d'hypothèses lorentziennes (Mack) ou d'hypothèses euclidiennes invérifiables (théorème d'OS).

Dans la partie III, nous étudions les propriétés de convergence au sens des fonctions de l'*OPE* des fonctions de corrélation scalaires à quatre points dans une *CFT*. Nous établissons les critères de convergence de l'*OPE* dans les canaux *s*, *t* et *u*. Nous donnons une classification complète des configurations à quatre points possibles dans la signature lorentzienne. Nous montrons que toutes les configurations de chaque classe ont les mêmes propriétés de convergence de l'*OPE* dans les canaux *s*, *t* et *u*. Nous donnons des tableaux incluant les informations de convergence de l'*OPE* pour toutes les classes. En prenant l'union de toutes les configurations lorentziennes où au moins un canal de l'*OPE* converge, nous obtenons un domaine minimal d'analyticité des fonctions à quatre points lorentziennes *CFT*.

Dans la partie IV, nous donnons un aperçu de deux généralisations possibles : l'une, les fonctions à quatre points des opérateurs avec des spins généraux $SO(d)$; l'autre, les fonctions à quatre points dans le cylindre de Minkowski.

Foreword

This thesis is based on the following materials.

Part **I** is based on:

P. Kravchuk, J. Qiao and S. Rychkov, “Distributions in CFT. Part I. Cross-ratio space,” [JHEP **05**, 137 \(2020\)](#), [arXiv:2001.08778 \[hep-th\]](#).

Part **II** is based on:

P. Kravchuk, J. Qiao and S. Rychkov, “Distributions in CFT. Part II. Minkowski space,” [JHEP **08**, 094 \(2021\)](#), [arXiv:2104.02090 \[hep-th\]](#).

Part **III** is based on

J. Qiao, “Classification of Convergent OPE Channels for Lorentzian CFT Four-Point Functions,” [arXiv:2005.09105 \[hep-th\]](#).

Part **IV** is based on some upcoming works:

P. Kravchuk, J. Qiao and S. Rychkov, “Distributions in CFT III. Spinning Fields in Minkowski space,” work in progress.

P. Kravchuk, J. Qiao and S. Rychkov, “Distributions in CFT IV. Lorentzian Cylinder,” work in progress.

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have no way of naming them all. They are the ones who made my entire Ph.D. less tedious and more enriching, and our interactions have become some of the best memories of my life.

Introduction

The study of conformal field theories (CFTs) is essential for understanding strongly-coupled physics such as thermodynamic phase transition, confinement, quantum gravity, etc. In the framework of quantum field theories (QFTs), CFTs are the fixed points of the renormalization group (RG) flows. They are relatively easier to study than general QFTs because of the additional conformal symmetry and the algebraic structure.

In the past, there were fruitful results and applications in two-dimensional CFTs [4, 5]. Due to the infinite-dimensional conformal algebra, namely the Virasoro algebra, one can make many predictions solely from studying the representations of the Virasoro algebra. With the extra self-consistency conditions such as crossing symmetry and modular invariance, the two-dimensional CFTs are even more constrained. In higher-dimensional CFTs, there are much fewer constraints than in $d = 2$, mainly because the conformal group there is finite-dimensional. For this reason, few predictions were made in higher-dimensional CFTs.

Things have changed since a decade ago, thanks to the revival of bootstrap philosophy [6]. The conformal bootstrap approach is based on several Euclidean CFT assumptions (which we call the *Euclidean CFT axioms*), and it analyzes the constraints of self-consistency conditions on the CFT correlation functions. This approach makes precise numerical predictions of experimentally measurable quantities, such as the critical exponents of the 3d Ising model [7–10], $O(N)$ model [11–14], and other critical systems (see the review [15]).

While the basic CFT assumptions are made in Euclidean, many attempts have been made to study the bootstrap equations in the Lorentzian CFT. Due to the causal structure and richer singularities, many constraints are packaged into more visible forms in the Lorentzian signature. These studies include (but are not limited to) the conformal collider physics [16], the light-cone bootstrap [17, 18], the causality constraints and the averaged null energy condition (ANEC) [3, 19–21], and the Lorentzian inversion formula [22, 23]. It would be interesting to know whether these Lorentzian constraints are available for Euclidean CFTs such as the critical Ising model.

For general QFTs, the standard set of Lorentzian constraints are the Wightman axioms [24]:

- (W0) Temperedness.
- (W1) Poincaré invariance.
- (W2) Unitarity.
- (W3) Microcausality.
- (W4) Cluster property.
- (W5) Spectral condition.

In the early days, these axioms and some extra CFT assumptions (weak conformal invariance, asymptotic operator product expansion) were considered to be the defining properties of Lorentzian CFTs [25, 26]. However, it is not known yet whether these Lorentzian axioms follow from Euclidean CFT axioms. In particular, it was not known whether a generic Euclidean CFT correlator becomes a tempered distribution (i.e., satisfying (W0)) in the Lorentzian signature. Although not fully realized in this thesis, our dream goal is to derive Wightman axioms from Euclidean CFT axioms.

Osterwalder-Schrader reconstruction theorem.

One motivation of this thesis project comes from the celebrated *Osterwalder-Schrader reconstruction theorem* [1, 2]. The theorem describes the relation between Euclidean and Lorentzian QFT axioms. The Euclidean version of Wightman axioms, namely *Osterwalder-Schrader axioms*, is listed as follows

- (OS0) Euclidean temperedness.
- (OS1) Euclidean invariance.
- (OS2) Reflection positivity.
- (OS3) Permutation symmetry.
- (OS4) Cluster property.

Axioms (OS1) - (OS4) are the natural Euclidean analog of Wightman axioms (W1)-(W4). The two main differences between OS and Wightman axioms are the temperedness properties and the spectral condition. In Wightman axioms, temperedness means that correlators are tempered distributions, i.e., continuous linear functionals acting on Schwartz test functions. The Fourier transform of a tempered distribution is still a tempered distribution, so we can consider the momentum-space correlators and formulate the spectral condition (W5). While in OS axioms, the Euclidean temperedness means that correlators are continuous linear functionals acting on a more restricted function space, where the test functions vanish rapidly at coincident points. For this reason, it is not clear a priori whether the Fourier transform of a Euclidean correlator exists in a proper distribution space where we can formulate the spectral condition.

The OS theorem says that under (OS0) - (OS4) plus some extra assumptions, one can Wick rotate the Euclidean correlation functions to the Lorentzian signature and get the Lorentzian correlators that satisfy Wightman axioms. We will describe the exact procedure of Wick rotation in chapter 9.

We expect that the OS axioms are satisfied in Euclidean CFTs since they are also Euclidean QFTs. There are two versions of the extra assumptions: one assumes a stronger form of Euclidean temperedness; another assumes the growth of n -point function as a function of n , which is called the *linear growth condition*. In practice, both of them are rather difficult to verify. We can check these technical conditions only when there is good control of all the correlators in a QFT, e.g., Gaussian free field, 2d critical Ising model, etc. We summarize the current status in figure 0.0.1.

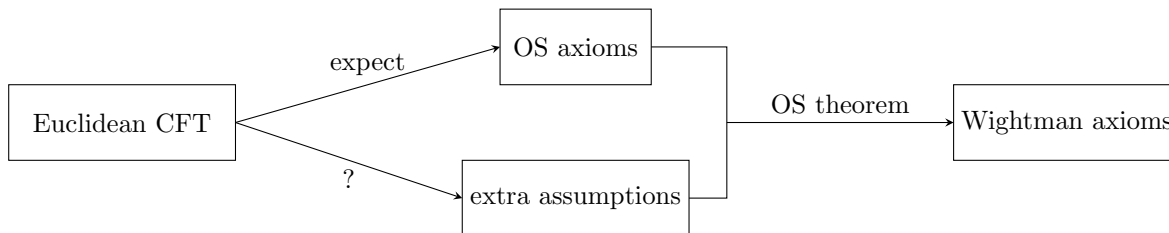


Figure 0.0.1: The current status of the (partial) relationship between Euclidean CFT, OS axioms, and Wightman axioms.

In the proof of OS theorem, the main use for introducing extra conditions is to prove power-law bounds on the analytically-continued Euclidean CFT correlators. These bounds imply that correlators are tempered distributions ((W0)) in the Lorentzian signature. Proving (W0) is the most technical part of the theorem: once this step is done, the other Wightman axioms can be obtained from OS axioms almost for free. For this reason, in this thesis we will discuss a lot on tempered distributions.

In CFT, the two- and three-point functions are kinematically fixed by conformal invariance. So it is not hard to prove Wightman axioms at the level of two- and three-point functions. As a modest step, we analyze the four-point functions, which is the first non-trivial CFT correlator. In this thesis, we will derive Wightman axioms for CFT four-point functions, starting from Euclidean CFT axioms.

Distributional properties in cross-ratio space.

As a warm-up exercise, we study the conformally invariant part of the CFT four-point function, denoted as $g(\rho, \bar{\rho})$. Our setup is the series expansion in radial cross-ratios $\rho, \bar{\rho}$ (in 1d there is only one independent cross-ratio variable):

$$\begin{aligned} g(\rho) &= \sum_h a_h \rho^h & (d = 1), \\ g(\rho, \bar{\rho}) &= \sum_{h, \bar{h}} a_{h, \bar{h}} \rho^h \bar{\rho}^{\bar{h}} & (d \geq 2). \end{aligned} \tag{0.0.1}$$

It is well known that in the unitarity CFT, this expansion converges in the sense of functions inside the unit disk $|\rho|, |\bar{\rho}| < 1$ for the radial variable. This setup is more accessible than the position space because we do not have to study the complicated relation between cross-ratios and position-space coordinates. At this point, it is not necessary to introduce the whole Euclidean CFT axioms.

We would like to show that the above expansion is convergent in the sense of tempered distributions on the boundary of the unit disk: $|\rho| = |\bar{\rho}| = 1$. To prove this, we introduce Vladimirov's theorem [27], which tells us that if we have a function $f(z)$ that is holomorphic in the upper half plane, then the limit $\lim_{y \rightarrow 0^+} f(x + iy)$ exists in the sense of distributions if the function does not blow up faster than power laws $(1 + |x|)^\alpha y^{-\beta}$. The same conclusion holds for functions of several complex variables, if we let the imaginary parts of all variables go to zero simultaneously.

Using Vladimirov's theorem and the fact that $g(\rho)$ has a power-law bound $(1 - |\rho|)^{-\alpha}$, we show that the limit $\lim_{r \rightarrow 1^-} g(re^{i\theta})$ exists in the sense of tempered distributions in the variable θ . The same is true for $g(\rho, \bar{\rho})$, using the two-variable version of Vladimirov's theorem.

Furthermore, we will show that the above expansion converges in the sense of distributions on the boundary, which means that we can integrate (0.0.1) term by term with any smooth test function $\varphi(\theta)$:

$$\begin{aligned} \lim_{r \rightarrow 1^-} \int d\theta g(re^{i\theta}) \varphi(\theta) &= \sum_h a_h \int d\theta e^{ih\theta} \varphi(\theta) & (d = 1), \\ \lim_{r, \bar{r} \rightarrow 1^-} \int d\theta d\bar{\theta} g\left(re^{i\theta}, \bar{r}e^{i\bar{\theta}}\right) \varphi(\theta, \bar{\theta}) &= \sum_{h, \bar{h}} a_{h, \bar{h}} \int d\theta d\bar{\theta} e^{ih\theta} e^{i\bar{h}\bar{\theta}} \varphi(\theta, \bar{\theta}) & (d \geq 2). \end{aligned} \tag{0.0.2}$$

Our results provide a new point of view about the *bootstrap functionals* that are used in the conformal bootstrap. These functionals satisfy the swapping property [28] when they are applied to the crossing equation. We will show that under proper coordinates, many bootstrap functionals can be interpreted as integrals against test functions, and the swapping property means distributional convergence, like eq. (0.0.2).

Osterwalder-Schrader axioms from Euclidean CFT.

We will postulate the well-accepted Euclidean CFT rules that were used in numerical conformal bootstrap and call them *Euclidean CFT axioms* (see section 7.3 for details):

- (ECFT0) Real analyticity.
- (ECFT1) Conformal invariance.
- (ECFT2) Reflection positivity of two-point functions.
- (ECFT3) Hermiticity of two-point and three-point functions.
- (ECFT4) Permutation symmetry.
- (ECFT5) Convergence of operator product expansion (OPE).

Since OS axioms are the properties of Euclidean QFT correlators, it may not be surprising that they follow from Euclidean CFT axioms. In chapter 8, we will derive OS axioms from Euclidean CFT axioms at the level of two-, three- and four-point functions. Since in the Euclidean CFT axioms, we only postulate a weak form of OPE convergence, extra assumptions are required for deriving OS axioms of higher-point functions. In appendix B.2, we will discuss how CFT axioms with a stronger form of OPE convergence imply OS axioms of higher-point functions.

Wightman axioms from Euclidean CFT four-point functions.

As mentioned above, the OS reconstruction theorem is not applicable here because we cannot check whether the extra assumption in the OS theorem is satisfied.

In chapter 9, we will introduce the basic strategy for deriving Wightman axioms from the Euclidean CFT axioms, following the spirit of OS reconstruction theorem. The most crucial step is to show that the analytically continued Euclidean correlators $G_n^E(x_1, \dots, x_n)$ become tempered distributions in the Lorentzian signature. For this step, we rely on Vladimirov's theorem, which requires two properties of G_n^E

1. G_n^E has analytic continuation to a regime of complex coordinates, which is called the *forward tube*.
2. When approaching the Lorentzian regime from the forward tube (which roughly means $x_k = (\epsilon_k + it_k, \mathbf{x}_k + i\mathbf{y}_k)$ with $\epsilon_k, \mathbf{y}_k \rightarrow 0$), the correlator should satisfy some power-law bound.

We first consider the cases of two- and three-point functions in chapter 10. One can easily prove the analyticity and power-law bound since they are fixed (up to constant factors) by conformal invariance. Therefore, the analytically continued Euclidean two- and three-point functions become tempered distributions in the Lorentzian signature.

In chapter 11, we consider the scalar CFT four-point functions (which we will simply call four-point functions). The four-point function can be factorized into a scaling prefactor and a conformally invariant factor. The scaling prefactor is the product of two-point functions, so its analyticity and power-law bound are easy to verify. The conformally invariant factor, $g(\rho, \bar{\rho})$, was already mentioned above. It has a series expansion (0.0.1) in $\rho, \bar{\rho}$, convergent when $|\rho|, |\bar{\rho}| < 1$. We will prove two key facts on the variables $\rho, \bar{\rho}$. The first fact is that $|\rho|, |\bar{\rho}| < 1$ when the four-point configuration is in the forward tube. The second fact is that when approaching the Lorentzian regime (which is on the boundary of the forward tube), the inverse distances between $\rho, \bar{\rho}$ and the unit circle (i.e., $(1 - |\rho|)^{-1}, (1 - |\bar{\rho}|)^{-1}$) are bounded by some power laws. Using these geometrical facts and the series expansion in $\rho, \bar{\rho}$, we prove the analyticity and power-law bound of the four-point function in the forward tube.

Then we apply Vladimirov's theorem to show that the four-point function is a tempered distribution in the Lorentzian regime. Once temperedness is proved, the other Wightman axioms (Lorentzian conformal invariance, clustering, microcausality) follow from Euclidean assumptions by some standard argument in the OS paper [1].

Since the Lorentzian four-point function is a tempered distribution instead of a genuine function, it is interesting to understand how its regularity depends on the scaling dimensions of the operators. In chapter 12, we will show that the radial cross-ratios $\rho, \bar{\rho}$ satisfy a Cauchy-Schwarz type inequality. Using this inequality, we derive an optimal power-law bound of $(1 - r)^{-1}$, where $r = \max\{|\rho|, |\bar{\rho}|\}$.¹ The optimal bound of $\rho, \bar{\rho}$ implies an optimal bound of the four-point function, which is saturated in the generalized free theory.

In chapter 13, we will discuss the OPE convergence properties, similarly to the discussion in part I. We show that the s-channel OPE converges in the sense of distributions in Minkowski space. We estimate the OPE convergence rate in the case when the test functions are compactly supported. In Euclidean space, it is well-known that the OPE converges exponentially fast when the scaling-dimension cutoff goes to infinity. However, in Minkowski space, the error term of the truncated OPE decays in a power law in the scaling-dimension cutoff. We also rephrase the OPE convergence in the forward tube, from the point of view of the two-operator states $|\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$. We show that in the forward tube region,² $|\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$ is an Hilbert-space-valued analytic function, and its OPE converges in the sense of the CFT Hilbert space.

Domain of analyticity of the CFT four-point functions.

In Minkowski space, one can ask: in which regions the CFT four-point function is analytic, not just a distribution. In the framework of Wightman QFTs, the correlators are known to be analytic functions when the operators are pairwise space-like separated [29]. However, the correlators involving time-like separated operators are not thoroughly studied. Since there are more constraints in CFT, coming from conformal invariance and OPE convergence, we expect that the CFT correlators are analytic functions in bigger domains than general QFT correlators. In part III, we study this problem at the level of four-point functions. Since our Lorentzian four-point function is

¹By optimal we mean that the power indices are optimal.

²Here we mean that the three-point configuration $(0, x_1, x_2)$ is in the forward tube.

constructed by the OPE of analytically-continued Euclidean four-point functions, our analysis will be based on the convergence properties of OPE in the Lorentzian regime.

In chapter 18, we will establish some criteria for OPE convergence in s-, t- and u-channels. In the previous part, we have already shown that $|\rho|, |\bar{\rho}| < 1$ for four-point configurations in the forward tube, which implies the s-channel OPE convergence. Since the Lorentzian configurations lie on the boundary of the forward tube, by continuity $|\rho|, |\bar{\rho}| < 1$ is equivalent to $|\rho|, |\bar{\rho}| \neq 1$ there. Thus the s-channel OPE at a Lorentzian configuration is convergent if $|\rho|, |\bar{\rho}| \neq 1$, or equivalently $z, \bar{z} \notin [1, +\infty)$ in terms of z, \bar{z} variables. This is the criterion of s-channel convergence.

There are two other ways of expanding the CFT four-point functions: t-channel and u-channel expansions. The three expansions satisfy the *crossing symmetry*, meaning that they agree in their common domain of convergence in the Euclidean signature. By uniqueness of analytic continuation, they also agree in the complex regime. In the Lorentzian signature, it is possible to have some configuration where the four-point function has convergent expansion in only one OPE channel. Therefore, it is essential to know the OPE convergence properties in all channels. We establish criteria for OPE convergence in the t- and u-channels, based on $|\rho_t|, |\bar{\rho}_t| < 1$ and $|\rho_u|, |\bar{\rho}_u| < 1$ for radial variables in t- and u-channels. A big difference between the s-channel and other channels is that deciding the s-channel convergence only relies on the values of the cross-ratio variables at the Lorentzian configuration. In contrast, for the t- and u-channel convergence, one must consider the behaviors of cross-ratio variables along the analytic continuation path.

We will use the above-established criteria to check OPE convergence for all possible Lorentzian four-point configurations. Chapter 19 gives a classification of Lorentzian four-point configuration, which reduces the OPE check from the whole Lorentzian configuration space to finitely many cases. The classification is basically according to the causal ordering and the range of the cross-ratio variables z, \bar{z} . Then we check whether each configuration class satisfies the OPE convergence criteria. As there are still 81 cases to check after reduction, we show the results in terms of some tables in appendix C.2.

Four-point functions of spinning operators

In chapter 22, we will discuss the generalization to the four-point functions of operators with $SO(d)$ -spins (which exclude the fermionic operators). For the spinning case, the main extra difficulty is that the tensor structures appear. In $d = 2$, the rotation group, which is $SO(2)$, is abelian. As a result, there is only one tensor structure for each spinning four-point function. Then the treatment of the spinning four-point functions in $d = 2$ is quite similar to the scalar case. However, in $d \geq 3$, a four-point function may contain several tensor structures. It is hard to show that each tensor structure has analytic continuation to the forward tube. Another point of view on this difficulty is that in $d \geq 3$, a four-point configuration has a $SO(d-2)$ or $SO(d-1)$ little group, which can be nontrivial. The little group ambiguity will cause non-analyticity of the rotation matrices in the formulation of conformal invariance.

To bypass the above difficulty, we will take another strategy for the analytic continuation of the spinning four-point function. In the Euclidean signature, the four-point function has a conformal partial wave expansion, which is convergent by the OPE convergence assumption in the Euclidean CFT axioms. We would like to show directly that each conformal partial wave, denoted as $G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}$, has analytic continuation to the forward tube, and satisfies a Cauchy-Schwarz type inequality. Using these two properties, we will show that the conformal partial wave expansion is convergent in the forward tube, which performs the analytic continuation of the four-point function. The remaining derivation of the Wightman axioms is similar to the scalar case.

We will show the analyticity and the Cauchy-Schwarz type inequality of the conformal partial wave $G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}$, for generic internal scaling dimension $\Delta_{\mathcal{O}}$. There exists a discrete set of exceptional $\Delta_{\mathcal{O}}$'s, where our proof does not apply, but we hope that the same results still hold by the existence of $G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}$ in the Euclidean signature and its continuity in $\Delta_{\mathcal{O}}$ in the generic case.

Scalar four-point functions in the Minkowski cylinder

In chapter 23, we will discuss the generalization to the scalar four-point functions in the Minkowski cylinder $\mathbb{R} \times S^{d-1}$, which is another important container of Lorentzian CFTs. In the framework of AdS_{d+1}/CFT_d -correspondence, it is well-known that the Minkowski cylinder is the boundary of the Lorentzian AdS_{d+1} [30].

The CFT correlators in the Minkowski cylinder are defined by analytically continuing the CFT correlators in the Euclidean flat space, using the cylinder coordinates. This Wick rotation procedure has been studied in the early days by Lüscher and Mack [25]. Their basic assumptions are the Wightman axioms and the so called *weak*

conformal invariance, which basically means conformal invariance in the Euclidean signature. Using the same argument as Glaser, Osterwalder and Schrader [2, 31], they showed that the CFT correlation function has analytic continuation to the cylinder forward tube, where the real parts of the cylinder temporal variables are ordered: $\text{Re}(\tau_1) > \text{Re}(\tau_2) > \dots > \text{Re}(\tau_n)$. The Minkowski cylinder points correspond to $\text{Re}(\tau) = 0$. However, without extra assumptions, they did not manage to show that the limit exists when all the cylinder temporal variables become imaginary. Therefore, deriving Wightman axioms for CFT correlators in the Minkowski cylinder is still an open problem.

We would like to show that in the Minkowski cylinder, the CFT four-point functions are tempered distributions in the cylinder temporal variables and continuous functions in the spherical variables. Our strategy is quite similar to the case of the scalar four-point function in Minkowski flat space. We will show that $|\rho|, |\bar{\rho}| < 1$ for four-point configurations in the forward tube. We will also derive power-law bounds of $(1 - |\rho|)^{-1}, (1 - |\bar{\rho}|)^{-1}$ in the forward tube. Then the remaining arguments are the same as the case of Minkowski flat space.

We will briefly demonstrate the Wightman axioms in the Minkowski cylinder. There are three main differences in the Minkowski cylinder. First, the global conformal invariance holds in the cylinder. In contrast, conformal invariance only holds infinitesimally in the Minkowski flat space. Second, there is no cluster property in the cylinder, because the spatial directions are compactified. Last, the spectral condition in the cylinder case holds only for the Fourier transform of temporal variables.

The Minkowski cylinder is an ∞ -covering of the (compactified) Minkowski flat space. Since people usually identify the Minkowski space to the Poincaré patch of the Minkowski cylinder [30], it is natural to ask whether the CFT correlators in the Poincaré patch are the same as in the Minkowski space, up to the scaling prefactors. We claim that the answer is yes, for general CFT n -point functions. The key observation for the proof is that each configuration in the Poincaré patch is connected to the Euclidean cylinder by a path in the cylinder forward tube, which is mapped to a path in the flat-space forward tube. Then the uniqueness of analytic continuation tells us that the correlators on two sides must match. This result is true not only for four-point functions but also for any n -point functions.

The CFT four-point functions have richer singularities in the Minkowski cylinder than in the Minkowski space. One may expect a more complicated classification of the four-point cylinder configurations for the OPE convergence properties in the sense of functions. However, we will show that in the cylinder case, the classification of the four-point cylinder configurations can be easily reduced to the case when all the points are in the Poincaré patch, with the help of the action of a \mathbb{Z} -group, which is the center of the cylinder conformal group.

Part I

Distributions in CFT I: cross-ratio space

Chapter 1

Introduction

Historically, distributions played a big role in axiomatic approaches to quantum field theory (QFT), via Wightman axioms [24] or Osterwalder-Schrader axioms [1, 2]. In particular, the language of tempered distributions allows clean treatment of correlation functions singularities at $x^2 = 0$ in a UV-complete QFT, where x^2 may be Euclidean or Lorentzian distance.

In recent years, a new axiomatic approach—the conformal bootstrap—has emerged in the study of conformal field theories (CFTs) in dimension $d \geq 2$, i.e. quantum field theories invariant under the action of conformal group (see review [15]). This approach is both rigorous and calculable. On the numerical side, it has allowed precise determinations of many experimentally measurable quantities, such as the critical exponents of the 3d Ising [7–11], $O(N)$ [11–14] and other critical points. On the analytic side, it also led to many insights into the structure of operator spectrum of general CFTs, in particular concerning how operators organize themselves in infinite families (Regge trajectories) [17, 18, 22]. Numerical bootstrap studies typically take place deep in the Euclidean region, staying away from the contact term singularities of correlation functions at short distances. In this regime, the rules of the game are well-understood and comprise the Euclidean bootstrap axioms.

On the other hand, analytical bootstrap studies often boldly go into the Lorentzian space, probe light-cone or other types of singularities. In this regime the most common set of assumptions for correlation functions are the Wightman axioms [24], but it has never been shown how these assumptions follow from the well-understood Euclidean bootstrap axioms. To achieve this is the goal of this series of papers. The uniting theme of this work will be tempered distributions, hence the title.

In this first paper of the series (part I) we will study convergence of the conformal block decomposition. As is well known, it converges in the sense of functions inside the unit disk $|\rho|, |\bar{\rho}| < 1$ for the radial variable. We will show that it converges in the sense of distributions also on the boundary of this unit disk. This is done using Vladimirov’s theorem [27]—a key result in the theory of functions of several complex variables that we will carefully introduce.

Vladimirov’s theorem provides the answer to the following question: if we have a function $g(\rho)$ that is holomorphic in the open unit disc $|\rho| < 1$, what can we say about its values for $|\rho| = 1$? If $g(\rho)$ were bounded, then the limit $\lim_{r \rightarrow 1} g(re^{i\theta})$ would be guaranteed to exist for almost every θ and give rise to a bounded function $g(e^{i\theta})$. However, the functions of cross-ratios that we encounter in conformal field theory are not bounded and instead can blow up near the boundary. Crucially though, it is easy to show (as we do in this part of the thesis) that they blow up only as power laws $(1 - r)^{-K}$. In this case, Vladimirov’s theorem guarantees that the limit $\lim_{r \rightarrow 1} g(re^{i\theta})$ exists in the space of distributions in the variable θ . We will explain that this conclusion holds both for the correlation function itself as well as for the individual terms in the conformal block expansion, which will allow us to prove convergence of conformal block expansion in the space of distributions. A simple yet illustrative example of distributional convergence is the sum

$$\sum_{n=-\infty}^{+\infty} e^{in\theta} = 2\pi\delta(\theta), \tag{1.0.1}$$

where $\theta \in (-\pi, \pi]$ is the coordinate on the unit circle. This sum doesn’t converge in the usual sense because every

term is of absolute value 1, but it does converge after being smeared with a smooth test function $f(\theta)$. We will study a more realistic toy example in section 3.1.

Our results can be interpreted as introducing a new class of functionals which satisfy the swapping property [28] when applied to the crossing equation. This point of view might be helpful for readers with interest in analytic functional bootstrap [32–39]. Specifically, we show that integration (appropriately defined) of the crossing equation with a test function over the boundary of the crossing region¹ can be exchanged with the sum over conformal blocks. We prove this result for infinitely smooth test functions, and argue that it likely can be strengthened to enlarge the class of test functions sufficiently so that our new class of functionals will include all functionals currently known to satisfy the swapping property.

In our second paper [40] (part II), CFT Wightman four-point functions in Lorentzian space will be shown to be tempered distributions, thus establishing Wightman axioms. In the third paper [41] (chapter 23), we will study analytic continuations of CFT correlation functions to the Lorentzian cylinder (also known as the boundary of the AdS space). Our goal is to establish everything from Euclidean bootstrap axioms, without any extra assumptions. When the time comes, we will explain that the existing classic results in the literature, like the Osterwalder-Schrader theorem [1, 2] or the construction of Lüscher and Mack [25], all require additional assumptions. So our conclusions cannot be recovered from the classic papers. Fortunately, we found a different way of reasoning which recovers all the results commonly assumed to be true, for the most important in applications case of four-point functions.² The good news is that our alternative arguments are really easy, and the main idea can really be summarized in one sentence: “Look for a powerlaw bound.” This should be contrasted with the classic papers which are quite intricate.

This part of the thesis is organized as follows. In chapter 2 we discuss the motivation for our work from the point of view of computing Euclidean and Lorentzian correlation functions. In chapter 3 we consider the simplified case of one cross-ratio, starting with a toy example of power series. We also use this simplified setting to discuss possible applications of our results to analytic functional bootstrap (section 3.6) and to proper definition of discontinuities (section 3.7). In chapter 4 we consider the case of two cross-ratios in scalar correlators in general number of dimensions. We comment on applications and generalization to spinning correlators. In section 4.6 we discuss an application in the context of a single-variable dispersion relation recently proposed by Bissi, Dey and Hansen [42]. We conclude in chapter 5.

¹By crossing region we mean the region in cross-ratio space where both s - and t -channel conformal block expansions converge. In the standard z -cross ratio it is given by \mathbb{C} minus the cuts along $[1, +\infty)$ and $(-\infty, 0]$.

²It’s an interesting open problem how to extend our arguments to higher point functions.

Chapter 2

Conformal block expansion

In this chapter we will state our basic problem, and the main idea how to solve it. Let us consider the conformal block expansion of a four-point function of identical scalar operators (we will consider more general four-point functions later)

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{(x_{12}^2)^{\Delta_\phi}(x_{34}^2)^{\Delta_\phi}} g(u, v), \quad (2.0.1)$$

where, as usual

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (2.0.2)$$

We will mostly be working with the radial coordinates $\rho, \bar{\rho}$ [43, 44] defined as

$$\rho = \frac{z}{(1 + \sqrt{1 - z})^2}, \quad \bar{\rho} = \frac{\bar{z}}{(1 + \sqrt{1 - \bar{z}})^2}, \quad (2.0.3)$$

where z, \bar{z} are determined by

$$z\bar{z} = u, \quad (1 - z)(1 - \bar{z}) = v. \quad (2.0.4)$$

We will abuse the notation a bit by writing $g(u, v), g(z, \bar{z})$, or $g(\rho, \bar{\rho})$ depending on which set of cross-ratios we want to use.

The function $g(\rho, \bar{\rho})$ can be expanded in conformal blocks in $\phi(x_1) \times \phi(x_2)$ OPE channel as follows,

$$g(\rho, \bar{\rho}) = \sum_{\Delta, J} p_{\Delta, J} g_{\Delta, J}(\rho, \bar{\rho}), \quad (2.0.5)$$

where $p_{\Delta, J} \geq 0$ are the OPE coefficients squared, and $g_{\Delta, J}(\rho, \bar{\rho})$ are the conformal blocks. This expansion is known to be absolutely convergent in the region $|\rho| < 1, |\bar{\rho}| < 1$, which we will denote by \mathcal{C} in what follows.

We will only use the global conformal invariance $SO(d+1, 1)$. Under these assumptions, the region \mathcal{C} is the largest region of convergence of the conformal block decomposition of a general CFT four-point function (we are not aware of any results to the contrary). In 2d CFT, using Virasoro, the region of convergence can be extended further in terms of Al. Zamolodchikov's uniformizing q variable, being given by $|q|, |\bar{q}| < 1$ which is a strictly larger region than \mathcal{C} [44, 45]. So our results should be best possible in $d > 2$ but not in $d = 2$.

Above we focused on the 12 OPE channel (s -channel) but the same discussion can be made for the t -channel 23 and u -channel 13, whose convergence is characterized by the conditions $|\rho_t|, |\bar{\rho}_t| < 1$ and $|\rho_u|, |\bar{\rho}_u| < 1$.

Let us briefly describe what the region \mathcal{C} corresponds to in the physical space of x_i . In Euclidean signature, this region includes all configurations when the four points x_i do not lie on a circle, which is the generic case. If x_i do lie on a circle, the cross-ratios belong to \mathcal{C} if x_1 and x_2 are next to each other on the circle. If the points instead fall on the circle in the ordering x_1, x_3, x_2, x_4 (read in some direction), then we find $|\rho| = |\bar{\rho}| = 1$. Therefore, only a measure zero set of Euclidean configurations does not belong to \mathcal{C} and is instead on its boundary $\partial\mathcal{C}$. For these configurations the s -channel expansion does not converge. However, it does converge for t - and u -channels, and so

the value of the Euclidean four-point function can be determined from the OPE for any configuration of the four points.

Our basic problem is to make sense of the four-point function (2.0.1) in Lorentzian signature. In order to talk about a Lorentzian four-point function, we need to specify which operator ordering we are interested in. We will only consider here the Wightman functions, i.e. we fix operator ordering:

$$W(x_1, x_2, x_3, x_4) = \langle 0 | \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) | 0 \rangle. \quad (2.0.6)$$

However, we wish to consider all possible time and causal ordering of the points x_i .¹ Once we have fixed the operator ordering, the Lorentzian four-point function can be obtained from the Euclidean one by an appropriate analytic continuation. While in Euclidean we always have $\bar{\rho} = \rho^*$, this property is generally lost after the analytic continuation. Furthermore, there are open regions in the Lorentzian configuration space of x_i where $|\rho|$ and/or $|\bar{\rho}|$ end up ≥ 1 after the analytic continuation. Then the corresponding conformal block expansion (2.0.5) diverges and thus cannot be used to determine the correlator.

One such well known case is the Regge regime [46–48], when x_1, x_4 and x_2, x_3 pairs are timelike separated, while all other intervals are spacelike (see Fig. 2.0.1). One may be tempted to use the 13 OPE for this Lorentzian correlator,

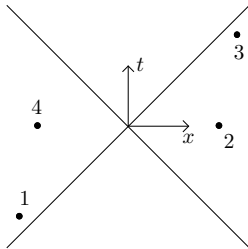


Figure 2.0.1: Regge kinematics.

because this channel is the most symmetric with respect to the origin, and also because one may be interested in the limit $x_{13}^2 \rightarrow 0$. However, although this channel converges when points x_1 and x_3 stay close to the origin, it starts diverging when they cross the lightcones of x_2 and x_4 and move into the Regge regime, because the corresponding ρ_u variable become larger than 1.² In this particular case, one can switch to the 23 OPE for which $|\rho_t|, |\bar{\rho}_t| < 1$ is less than 1, and so this channel converges. However, this is not always possible: there exist kinematic configurations when no channel converges (appendix A.1).

In this series of works we will propose a different way to solve this problem, and recover the Wightman function in all kinematic configurations. In our construction the key role will be played by the 12 OPE-channel. We call it the “vacuum channel”, because it involves the two leftmost operators in the Wightman ordering (2.0.6), i.e. the ones acting on the vacuum. While the vacuum channel OPE does not always converge, it *almost converges* for all possible configurations. What this means is that $|\rho| \leq 1$ and $|\bar{\rho}| \leq 1$ for all values of x_i . This crucial fact will be shown in [40]. It is only true for the vacuum OPE channel, but would not be true for the 23 or 13 channels, for which sometimes $|\rho|$ and/or $|\bar{\rho}|$ will be strictly greater than 1. In particular, as we show in appendix A.1, there exist configurations for which both 23 and 13 channels diverge with $|\rho|, |\bar{\rho}| > 1$, while for 12 channel $|\rho| = |\bar{\rho}| = 1$.

In other words, all possible Lorentzian configurations belong to the closure $\bar{\mathcal{C}}$. One can ask how large are the regions in configuration space of x_i which belong in $\partial\mathcal{C}$ but not in \mathcal{C} . In Euclidean, we have seen that these configurations were measure zero, but in Lorentzian this is no longer true: extended regions with non-empty interior have $|\rho| = 1$, $|\bar{\rho}| = 1$. So, a fraction of configurations are in $\partial\mathcal{C}$ and not in \mathcal{C} .

If the conformal block expansion converged in $\bar{\mathcal{C}}$ and not \mathcal{C} , we would be able to use it to compute any Lorentzian correlator in any configuration of the points x_i . Of course, this is not the case, and the conformal block expansion

¹Other often considered Lorentzian correlators (retarded, advanced, time-ordered) can be obtained by multiplying Wightman functions with appropriate factors enforcing the needed ordering. The Wightman functions being distributions, and the time-ordering factors being singular, this procedure introduces extra singularities and requires care in a rigorous treatment.

²Conformal Regge theory [46–48] provides a way to resum the expansion (2.0.5) in the limit $\bar{\rho}_u \rightarrow 0$, $\rho_u \rightarrow \infty$ with $\rho_u \bar{\rho}_u$ fixed. We will not consider such resummations in this part of the thesis since they rely on analytically-continued OPE data that we have little control over.

converges in the usual sense only in \mathcal{C} . However, our goal in this part of the thesis will be to extend the notion of convergence so that it will become valid in $\bar{\mathcal{C}}$. Specifically, we will show that the expansion (2.0.5) converges in the sense (to be clarified below) of distributions on the boundary $\partial\mathcal{C}$ in the cross-ratio space. In the forthcoming work [40, 41] we will extend this result to convergence in the sense of distributions in the physical space of x_i , either in Minkowski space, or on the Lorentzian cylinder.

One may be wondering what is special about the vacuum channel compared to other OPE channels. Intuitively, the distinguishing feature of vacuum channel is that we can understand it as inserting a complete set of states in the Wightman four-point function. Since Wightman four-point functions are distributions, we cannot generally expect this sum to make sense in terms of functions, but only in terms of distributions. Mack [26] understood the vacuum channel OPE expansion in distributional sense in position space. Mack’s reasoning is rather nontrivial, and it crucially relies on assuming from the start that Wightman axioms hold in Lorentzian signature—an assumption that we are here not willing to accept. Although our results in cross-ratio space are inspired by Mack’s considerations in position space, they do not follow from his results, since we rely on a different and simpler set of assumptions, natural from the modern bootstrap perspective. Also, we are only using rather elementary methods. Position space will be discussed in [40].

Chapter 3

One-dimensional case

First, let us simplify the problem by considering the one-dimensional case where there is a single cross-ratio. The conformal block expansion takes the form

$$g(\rho) = \sum_{\Delta} p_{\Delta} g_{\Delta}(\rho), \quad (3.0.1)$$

where the conformal blocks are given by¹

$$g_{\Delta}(\rho) = (4\rho)^{\Delta} {}_2F_1(1/2, \Delta; \Delta + 1/2; \rho^2). \quad (3.0.2)$$

Furthermore, the sum is over $\Delta \geq 0$ and $p_{\Delta} \geq 0$. This expansion converges, in the usual sense, in the interior of the unit disk $|\rho| < 1$, and our goal is to understand whether it can be made convergent, in some generalized sense, on the boundary $|\rho| = 1$.

If we look at (3.0.1) more closely, we will notice that the conformal blocks (3.0.2) are not single-valued in the unit disk of ρ . Therefore, we are really interested in the behavior of this sum on the universal cover of the unit disk branched at 0, which can be conveniently parametrized by writing

$$\rho = e^{i\tau}. \quad (3.0.3)$$

The expansion (3.0.1) is then absolutely convergent in the upper-half plane of τ , and we are interested in its convergence for real τ .

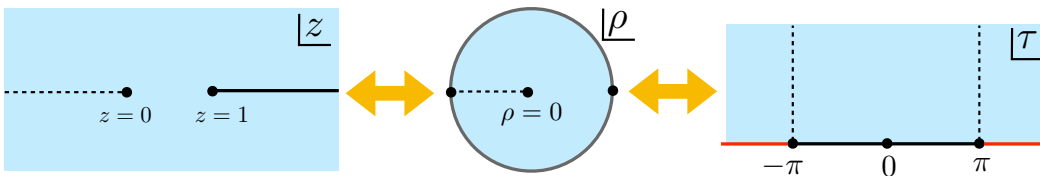


Figure 3.0.1: Transformation from the z cut plane to the ρ disk to the τ upper-half plane, see the text.

In Fig. 3.0.1 we show the transformation from the z cut plane to the ρ disk to the τ upper-half plane. The two sides of the cut $z \in [1, +\infty)$ are mapped on the boundary of the unit disk $|\rho| = 1$, and then to the black part ($\tau \in [-\pi, \pi]$) of the upper-half plane boundary. The rest of the τ boundary (marked in red) can be accessed in the ρ variable by first going through the cut $\rho \in [-1, 0]$ (dashed) and then approaching $|\rho| = 1$.

On the black interval $\tau \in [-\pi, \pi]$ (except at $\tau = 0$) the four-point function is actually analytic, as can be shown using the t -channel expansion. On the rest of the boundary (red part), the t -channel expansion does not converge and provides no information. Below we will show, using the s -channel, that the four-point function is a tempered distribution on the whole boundary. We will also show that the s -channel conformal block expansion converges in the sense of distributions. When using the s -channel, we have to use distributional convergence even on the black part of the boundary, although the function itself is analytic there as explained above.

¹This equation follows from the more familiar one in the z coordinate $g_{\Delta}(z) = z^{\Delta} {}_2F_1(\Delta, \Delta; 2\Delta; z)$ by a hypergeometric identity [44].

3.1 A toy problem

In order to gain some intuition, it is useful to consider the following toy problem. Let us study the power series

$$\frac{1}{1-\rho} = \sum_{n=0}^{\infty} \rho^n. \quad (3.1.1)$$

It has the similar feature that it converges absolutely for $|\rho| < 1$ and that the resulting function has a power-like singularity at $\rho = 1$, much like the physical four-point functions do.

In terms of τ variable we find the sum

$$\sum_{n=0}^{\infty} e^{in\tau}, \quad (3.1.2)$$

which clearly does not converge for any real τ . We claim that it does converge as a tempered distribution.² For example, let us compute its real part using the standard formulas of Fourier analysis

$$\operatorname{Re} \sum_{n=0}^{\infty} e^{in\tau} = \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{in\tau} + \frac{1}{2} = \frac{1}{2} + \pi \sum_{k=-\infty}^{\infty} \delta(\tau - 2\pi k). \quad (3.1.3)$$

It is a bit harder to compute the imaginary part, but we can run the following simple argument for the full sum (3.1.2). Let $f(\tau)$ be a Schwartz test function, i.e. a smooth³ function which, together with its derivatives, decays at infinity faster than any power. In order to show that (3.1.2) converges as a tempered distribution, we need to show, by definition, that the partial sums

$$\int d\tau f(\tau) \sum_{n=0}^N e^{in\tau} = \sum_{n=0}^N \int d\tau f(\tau) e^{in\tau} = \sum_{n=0}^N \tilde{f}(n) \quad (3.1.4)$$

converge to a finite limit as $N \rightarrow \infty$. Here, $\tilde{f}(n)$ is the Fourier transform of f . Since $f(\tau)$ is a Schwartz test function, so is $\tilde{f}(n)$ (where n is understood as a real parameter) and thus $\tilde{f}(n)$ decays faster than any power of n as $n \rightarrow \infty$. This implies that the partial sums (3.1.4) indeed converge. Strictly speaking, we also need to show that the limit is continuous with respect to f in an appropriate topology. We will delay this question until later. Here the important message is that even though (3.1.2) does not converge in the usual sense, it starts to converge after being smeared with a nice test function.

So far we have learned two things. First, the sum (3.1.2) converges in distributional sense for real τ . Second, the value of this sum is a genuine distribution, since we computed its real part and it is a sum of δ -functions. Now, we also know that in the upper-half plane of τ the sum converges to

$$g(\tau) = \frac{1}{1-\rho} = \frac{1}{1-e^{i\tau}}. \quad (3.1.5)$$

This suggests that on the real line $g(\tau)$ should have a limit that is the tempered distribution computed by (3.1.2). So we can conjecture that, for real τ ,

$$\sum_{n=0}^{\infty} e^{in\tau} = \lim_{\epsilon \rightarrow +0} g(\tau + i\epsilon) \equiv \lim_{\epsilon \rightarrow +0} \frac{1}{1-e^{-\epsilon+i\tau}}, \quad (3.1.6)$$

where everything is understood in the sense of tempered distributions.

How can we guarantee that the limit in the right-hand side exists? In the sense of functions, it clearly exists for $\tau \neq 2\pi k$ and is given by $g(\tau)$. However, $g(\tau)$ for real τ is not obviously a distribution, since it involves non-integrable singularities near $\tau = 2\pi k$ that we need to regulate. Specifically, we need to prove that for any Schwartz function $f(\tau)$ the limit

$$\lim_{\epsilon \rightarrow +0} \int d\tau g(\tau + i\epsilon) f(\tau) \quad (3.1.7)$$

²A tempered distribution is a distribution that can be paired with Schwartz test functions (see below).

³In this part of the thesis “smooth” means C^∞ and the two terms are used interchangeably.

exists and depends continuously on f in an appropriate topology. Notice that if $f(\tau)$ were a holomorphic function, for example $f(\tau) = e^{-\tau^2}$, then the existence of the limit would be simple to show.⁴ However, the class of holomorphic test functions is too restricted for many purposes. It is more customary to develop the theory of distributions using compactly supported C^∞ test functions, or the even larger class of Schwartz test functions.⁵ For a general Schwartz $f(\tau)$, existence of the limit (3.1.7) requires an argument which will be explained in the next section.

We would like to emphasize that the existence of the limits (3.1.6), (3.1.7) is not just some abstract nonsense, but a very concrete prediction. Integrating both parts of (3.1.6) against an arbitrary Schwartz test function $f(\tau)$, we obtain:

$$\lim_{\epsilon \rightarrow +0} \int d\tau f(\tau) \frac{1}{1 - e^{-\epsilon + i\tau}} = \sum_{n=0}^{\infty} \tilde{f}(n). \tag{3.1.8}$$

Let us test this prediction. We pick a function $f(\tau)$ given by $\exp(-1/(1 - \tau^2))$ for $\tau \in (-1, 1)$, extended by zero outside this interval. It is a compactly supported C^∞ function (in particular Schwartz, but not analytic). We evaluate both sides of the previous equation numerically for $0 < \epsilon < 1$, and check the limit (see Fig. 3.1.1).

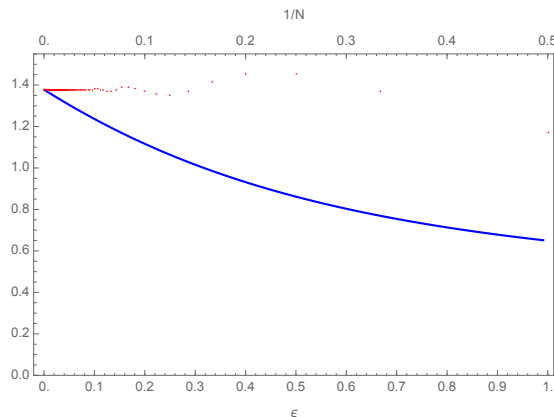


Figure 3.1.1: A numerical check of the existence of the limit (3.1.8), for $f(\tau)$ given in the text. The curve is the integral under the limit sign, and the red dots are the partials sums of Fourier coefficients in the r.h.s. of (3.1.8) up to $n = N$.

3.2 Vladimirov's theorem

Fortunately, there is a general result that immediately establishes that (3.1.6) is valid, i.e. that both left and right hand sides converge as tempered distributions and are indeed equal. Before stating this result, let us first clean up some formal definitions.

For a smooth function $f(x)$ define the semi-norms

$$\|f\|_{m,n} = \sup_{x \in \mathbb{R}} |(1 + |x|^m) \partial_x^n f(x)|. \tag{3.2.1}$$

The Schwartz space $\mathcal{S}(\mathbb{R})$ consists of smooth functions f for which $\|f\|_{m,n}$ is finite for all non-negative integer m and n . This is a vector space which is given a topology where a sequence f_k is said to converge to g if $h_k = f_k - g$ converges to 0. In turn, h_k converges to 0 iff for all m, n the sequence $\|h_k\|_{m,n}$ converges to 0.

⁴We will just give an idea. Expand $f(\tau)$ in Taylor series around $f(\tau + i\epsilon)$ as $f(\tau) = f(\tau + i\epsilon) + (-i\epsilon)f'(\tau + i\epsilon) + \dots + O(\epsilon^m)$. The terms involving $f^{(k)}(\tau + i\epsilon)$ are easy to analyze: the integrals don't depend on ϵ at all because by analyticity we can shift the contour. So only the first of these terms survives. The error term goes to zero provided that $\epsilon^m g(\tau + i\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. This will hold for $m > M$ if g satisfies the slow-growth condition (3.2.2) below. This shows that one could equivalently define the pairing between g and holomorphic f by shifting the integration contour for both, as $\int d\tau g(\tau + i\epsilon) f(\tau + i\epsilon)$. This is independent of ϵ and there is no limit to talk about.

⁵This is not just for the reasons of generality. Compactly supported test functions are needed if one wants to define a very basic notion of support of the distribution. This notion allows as to make statements such as “distributions $f(x)$ and $g(x)$ agree for $x \in [0, 1]$ but disagree outside of this interval.” Then the class of Schwartz test functions, being invariant under the Fourier transform, plays an important role in all questions involving the Fourier transform of distributions.

The space $\mathcal{S}'(\mathbb{R})$ of tempered distributions is defined as the space of continuous linear functionals on $\mathcal{S}(\mathbb{R})$. We say that a linear functional α is continuous if $\alpha(h_k) \rightarrow 0$ for any sequence $h_k \in \mathcal{S}(\mathbb{R})$ for which $h_k \rightarrow 0$. We say that a sequence of tempered distributions α_k converges to a tempered distribution β if for any $f \in \mathcal{S}(\mathbb{R})$ we have $\alpha_k(f) \rightarrow \beta(f)$.

Now, let $a > 0$ and $g(\tau)$ be a function holomorphic in the strip $0 < \text{Im } \tau < a$. Suppose there exist $N, M \in \mathbb{Z}_{\geq 0}$ and $C > 0$ such that in the strip

$$|g(x + iy)| \leq C(1 + |x|^N)y^{-M} \quad (3.2.2)$$

for all $x \in \mathbb{R}$ and $y \in (0, a)$. We then say that g satisfies a slow-growth condition near \mathbb{R} . What this means is that for any y the function $g(x + iy)$ is bounded by a polynomial of fixed degree, and the overall size of this polynomial grows at most as a fixed powerlaw when $y \rightarrow 0$. Note that thanks to this condition for any y , $0 < y < a$, the function $g_y(x) \equiv g(x + iy)$ is a tempered distribution in $\mathcal{S}'(\mathbb{R})$. We can ask whether the limit $\lim_{y \rightarrow +0} g_y$ exists in $\mathcal{S}'(\mathbb{R})$. If it does, we say that boundary value of g on \mathbb{R} exists in $\mathcal{S}'(\mathbb{R})$ and denote it by $\text{bv } g$,

$$\text{bv } g \equiv \lim_{y \rightarrow +0} g_y. \quad (3.2.3)$$

We can now state the theorem

Theorem 3.2.1. *Let $g(\tau)$ be a function holomorphic for $0 < \text{Im } \tau < a$ for some $a > 0$, satisfying the slow-growth condition near \mathbb{R} as defined above. Then the boundary value $\text{bv } g$ of g on \mathbb{R} exists in $\mathcal{S}'(\mathbb{R})$. Furthermore, if a sequence of functions g_n , holomorphic in the same region, satisfies the slow-growth condition with the same constants C, M, N for all n (uniform slow-growth condition), and converges pointwise to g for $0 < \text{Im } \tau < a$, then g satisfies the same slow-growth condition and*

$$\lim_{n \rightarrow \infty} \text{bv } g_n = \text{bv } g \quad \text{in } \mathcal{S}'(\mathbb{R}). \quad (3.2.4)$$

Such results are rather standard in the theory of distributions (an early mathematics reference is [49]). In mathematical physics they are very useful in the study of QFT Wightman functions. The standard reference is the book of Vladimirov [27] (section 26), and we will therefore refer to such results as ‘‘Vladimirov’s theorems’’. A self-contained proof of Theorem 3.2.1 will be given below. A more general Vladimirov’s theorem will be stated and used in [40].

Let us see how this result applies to our toy problem. We have

$$g_n(\tau) = \sum_{k=0}^n e^{ik\tau}, \quad g(\tau) = \frac{1}{1 - e^{i\tau}}. \quad (3.2.5)$$

Let us check the slow growth condition for g_n on $0 < \text{Im } \tau < 1$:

$$|g_n(x + iy)| \leq \sum_{k=0}^n |e^{ikx - ky}| \leq \sum_{k=0}^{\infty} e^{-ky} = \frac{1}{1 - e^{-y}} \leq Cy^{-1} \quad (3.2.6)$$

for some $C > 0$. So we see that the slow growth condition is satisfied with $N = 0, M = 1$. The same condition is then true for $g(\tau)$, as is easy to check. Then theorem 3.2.1 immediately implies our conjecture (3.1.6).

3.3 Proof of Vladimirov’s theorem 3.2.1

We first prove that $\text{bv } g$ exists and is a tempered distribution. So we pick a Schwartz test function $f(x)$ and study the integral

$$L(y) := \int dx g(x + iy)f(x). \quad (3.3.1)$$

We need to show that this has a limit as $y \rightarrow +0$. This looks a bit magic: estimating naively by absolute value one would conclude that the integral may blow up as y^{-M} . It won’t blow up only because of cancellations, not

captured by the naive estimate. In other words, when an analytic function tends somewhere to infinity, it will tend to minus infinity nearby, so that the integral will remain finite. For intuition, recall the Sochocki formula:

$$\lim_{y \rightarrow +0} \frac{1}{x + iy} = \text{PV} \frac{1}{x} - i\pi\delta(x) \quad (3.3.2)$$

Principal value PV represents a kind of cancellations whose existence we need to exhibit in general.

Going back to (3.3.1),⁶ the first key idea is that we can estimate not just L but any its derivative. By the Cauchy-Riemann equations, y -derivatives of $L(y)$ can be transformed into x derivatives acting on g which then can be integrated by parts to act on f :

$$L^{(j)}(y) = i^j \int dx g^{(j)}(x + iy) f(x) = (-i)^j \int dx g(x + iy) f^{(j)}(x). \quad (3.3.3)$$

Using then the slow-growth condition (3.2.2) we get an estimate of *any* y -derivative $L^{(j)}(y)$ by y^{-M} times a constant:

$$|L^{(j)}(y)| \leq C y^{-M}. \quad (3.3.4)$$

The constant here is proportional to the semi-norm $\|f\|_{N+2,j}$, see (3.2.1); order $N+2$ is needed to make the integral convergent, while derivative order j appears because of integrating by parts.

This is still growing as $y \rightarrow 0$. Here comes the second key idea: since we have this bound on any derivative, we can strengthen it recursively using the Newton-Leibnitz formula:

$$L^{(j-1)}(y) = - \int_y^{y_0} dy L^{(j)}(y) + L^{(j-1)}(y_0). \quad (3.3.5)$$

Here y_0 can be any fixed number in the strip of analyticity, e.g. $y_0 = a/2$ will do.

Every time we use this, we obtain a bound on $L^{(j-1)}$ of the same type as in (3.3.4) but with the order of singularity in y reduced by 1 w.r.t. $L^{(j)}$. Let us do this repeatedly, starting from $j = M + 2$.⁷ Then doing this M times we will prove that $L''(y)$ has an at most $\log(y)$ singularity, and doing this once more we prove that $L'(y)$ has no singularity at all, i.e. it is bounded by a constant, call it C_1 .

Now we can finally prove that $L(y)$ has a limit. From the $j = 1$ case of (3.3.5) we can write

$$(\text{bv } g)(f) = \lim_{y \rightarrow +0} L(y) = - \int_0^{y_0} dy L'(y) + L(y_0). \quad (3.3.6)$$

The limit exists, since by $|L'(y)| \leq C_1$ the integral in the r.h.s. converges absolutely at the lower limit of integration. Thus $\text{bv } g$ exists as a linear functional on $\mathcal{S}(\mathbb{R})$. All constants in the above argument are bounded by some semi-norms of f . This proves that $\text{bv } g$ is a *continuous* linear functional on $\mathcal{S}(\mathbb{R})$, i.e. a tempered distribution.

Now let us prove the second part of the theorem, about convergence. Replacing g_n by $g_n - g$, it's enough to consider the case $g = 0$. We pick an arbitrary Schwartz function f and consider

$$(\text{bv } g_n)(f) = \lim_{y \rightarrow +0} L_n(y). \quad (3.3.7)$$

Here $L_n(y)$ is defined by the integral (3.3.1) with g replaced by g_n . The existence of the limit for each n is guaranteed by the above argument. As a byproduct of the argument, we have also seen that $|L'_n(y)| \leq C_1$ uniformly in n and y , where C_1 is bounded by some semi-norm of f .

Furthermore, we claim that $L_n(y)$ tends to zero as $n \rightarrow \infty$ for any fixed $y \in (0, a)$. Indeed the integrand in (3.3.1) satisfies two conditions: (a) it tends to zero as $n \rightarrow \infty$ because $g_n(x + iy)$ goes pointwise to zero; (b) it is bounded in absolute value by an integrable function which does not depend on n :

$$|g_n(x + iy)f(x)| \leq \|f\|_{N+2,0} \frac{|g_n(x + iy)|}{1 + |x|^{N+2}} \leq C \|f\|_{N+2,0} \frac{1 + |x|^N}{y^M(1 + |x|^{N+2})}, \quad (3.3.8)$$

⁶We follow the proof in [24], Theorem 2-10.

⁷Exercise: once you understand the proof below, show that $j = M + 1$ will do as well. Hint: the key requirement is that $L'(y)$ end up bounded by some integrable function.

where we bounded f by its semi-norm, and then used the slow-growth condition (3.2.2). So the claim follows by Lebesgue's dominated convergence theorem.

Finally we wish to prove that $(\text{bv } g_n)(f)$ tends to zero as $n \rightarrow \infty$, as this is what is meant by $\text{bv } g_n \rightarrow 0$ in $\mathcal{S}'(\mathbb{R})$. From definition (3.3.7), we can bound this quantity as:

$$|(\text{bv } g_n)(f)| \leq \sup_{y \in (0, \epsilon)} |L_n(y)| \leq |L_n(\epsilon)| + C_1 \epsilon, \quad (3.3.9)$$

where in the second inequality we used $|L'_n(y)| \leq C_1$. We proved above that $L_n(\epsilon)$ goes to zero for any ϵ . So by picking first ϵ small enough, and then n large enough, the sum of the two terms in the r.h.s. is arbitrarily small. This implies that $\limsup_{n \rightarrow \infty} |(\text{bv } g_n)(f)|$ is arbitrarily small. Thus it is zero.

The attentive reader may notice that the last steps of the proof are not constructive, i.e. they do not provide a bound on how fast $(\text{bv } g_n)(f)$ tends to zero. This is because the used assumption, that g_n converges to zero pointwise, is very general. It allows to conclude, via dominated convergence, that $L_n(y)$ tends to zero pointwise as $n \rightarrow \infty$, but it does not tell us how fast this limit is reached. If more detailed information about the rate of the limit $g_n \rightarrow 0$ is available, as it usually is in practical applications, then a simple modification of the above argument makes the conclusion $\text{bv } g_n \rightarrow 0$ in $\mathcal{S}'(\mathbb{R})$ constructive.

3.4 Distributional convergence of conformal block expansion

Let us now turn back to the 1-dimensional conformal block expansion (3.0.1). We would like to claim that it converges as a tempered distribution for real τ (recall $\rho = e^{i\tau}$). To prove this, we will use Vladimirov's theorem 3.2.1, for which we need to establish a uniform slow-growth condition on the partial sums in the left-hand side of (3.0.1).

As a first step, let us derive a slow-growth condition for the four-point function $g(\rho)$ itself. First, note that for $|\rho| < 1$ we have

$$g(\rho) = \sum_{\Delta} \tilde{p}_{\Delta} \rho^{\Delta}. \quad (3.4.1)$$

with some positive coefficients \tilde{p}_{Δ} . This follows from radial quantization in an appropriate conformal frame [44]. Equivalently, we can expand the conformal blocks (3.0.2) in the right-hand side of (3.0.1) in powers of ρ and use the fact that these expansions have positive coefficients. In particular, the sum (3.4.1) can be turned back into the sum (3.0.1) by appropriately grouping the terms. Now, we can write

$$|g(\rho)| = \left| \sum_{\Delta} \tilde{p}_{\Delta} \rho^{\Delta} \right| \leq \sum_{\Delta} \tilde{p}_{\Delta} |\rho|^{\Delta} = g(|\rho|), \quad (3.4.2)$$

so it suffices to bound $g(\rho)$ for real $\rho \in (0, 1)$. This maps to $z \in (0, 1)$, and in terms of z variable we know that $g(z)$ satisfies the crossing equation

$$z^{-2\Delta_{\phi}} g(z) = (1-z)^{-2\Delta_{\phi}} g(1-z). \quad (3.4.3)$$

When $z \rightarrow 1$, we have $g(1-z) = O(1)$, which implies for $z \in (0, 1)$ the bound

$$|g(z)| \leq C(1-z)^{-2\Delta_{\phi}} \quad (3.4.4)$$

for some $C > 0$. Using the fact that $1-z \sim (1-\rho)^2/4$ as $z \rightarrow 1$, we find

$$|g(\rho)| \leq g(|\rho|) \leq C'(1-|\rho|)^{-4\Delta_{\phi}} \quad (3.4.5)$$

for some $C' > 0$. In terms of $\tau = x + iy$ this implies a powerlaw bound

$$|g(\tau)| \leq C'' y^{-4\Delta_{\phi}}, \quad (3.4.6)$$

near $y = 0$ for a $C'' > 0$, which is the required slow-growth condition. Therefore, by theorem 3.2.1, $\text{bv } g$ exists and is a tempered distribution.

An easy modification establishes the slow-growth condition for the partial sums in (3.0.1). Let I be any (possibly infinite) subset of the terms in (3.4.1) and write

$$\left| \sum_{\Delta \in I} \tilde{p}_\Delta \rho^\Delta \right| \leq \sum_{\Delta \in I} \tilde{p}_\Delta |\rho|^\Delta \leq \sum_{\Delta} \tilde{p}_\Delta |\rho|^\Delta = g(|\rho|) \leq C'' y^{-4\Delta_\phi}. \quad (3.4.7)$$

Taking $I = I_{\Delta_*} = \{\Delta | \Delta < \Delta_*\}$ we get a uniform slow-growth condition for partial sums of (3.4.1). Similarly, by allowing $I = I_n$ to contain the terms corresponding to the first n conformal blocks in (3.0.1) we get a uniform slow-growth condition on partial sums of (3.0.1). Therefore, by theorem 3.2.1, we conclude that the expansion (3.0.1) converges for the boundary values,

$$\text{bv } g = \sum_{\Delta} p_\Delta \text{bv } g_\Delta \quad \text{in } \mathcal{S}'(\mathbb{R}). \quad (3.4.8)$$

Let us unpack this equation a bit. Notice that in the case at hand, $\text{bv } g_\Delta$ is an ordinary locally integrable function which is the easiest kind of distribution. This is because the conformal blocks (3.0.2) only have a logarithmic singularity at $\rho = 1$. Written in full, this equation says that for any Schwartz function $f(\tau)$

$$\lim_{\epsilon \rightarrow +0} \int d\tau g(\rho = e^{-\epsilon + i\tau}) f(\tau) = \sum_{\Delta} p_\Delta \int d\tau (\text{bv } g_\Delta)(\rho = e^{i\tau}) f(\tau), \quad (3.4.9)$$

in the sense that the $\epsilon \rightarrow +0$ limit in the l.h.s. exists (it defines $(\text{bv } g)(f)$), the series in the r.h.s. made of ordinary integrals converges, and that the two sides independently defined in this way are equal.

3.5 Convergence for other normalizations and on other boundaries

We have proven that the conformal block expansion (3.0.1) converges as a distribution on the boundary $\partial\mathcal{C}$ of the normal function-like domain of convergence \mathcal{C} . We motivated this question in chapter 2 from the point of view of computing the Wightman functions. However, in other applications the domain \mathcal{C} may not be the most natural one to consider. For example, one of the main objects of study in CFT is the crossing equation

$$z^{-2\Delta_\phi} g(z) = (1-z)^{-2\Delta_\phi} g(1-z), \quad (3.5.1)$$

where both left- and right-hand side are expanded in conformal blocks. The two expansions are conventionally referred to as the s - and t -channel expansions. It is then natural to consider the domain $\mathcal{C}^{st} = \mathcal{C}^s \cap \mathcal{C}^t$ in which both expansions converge in the sense of functions, as well as distributional convergence on its boundary $\partial\mathcal{C}^{st}$. Additionally, the function $g(z)$ is multiplied by a factor $z^{-2\Delta_\phi}$ in the above equation, so we should also ask whether such modifications alter our result.

It is easy enough to address the latter question. Indeed, if a function $q(\rho)$ satisfies a slow-growth condition near $|\rho| = 1$, so does the function $q(\rho)g(\rho)$ and the partial sums of conformal block expansion (3.0.1) multiplied by $q(\rho)$. So we can state the straightforward corollary to theorem 3.2.1:

Corollary 3.5.1. *If function $q(\rho)$ is holomorphic in the branched unit ρ -disc and satisfies a slow-growth condition near $\tau \in \mathbb{R}$ (recall $\rho = e^{i\tau}$), then we have*

$$\text{bv}(q \cdot g) = \sum_{\Delta} p_\Delta \text{bv}(q \cdot g_\Delta) \quad \text{in } \mathcal{S}'(\mathbb{R}). \quad (3.5.2)$$

In the example (3.5.1) we have $q(\rho) = z^{-2\Delta_\phi}$ and it satisfies the assumptions of this theorem as can be seen from the identity $z = \frac{4\rho}{(1+\rho)^2}$.

In order to address the questions related to restricting the domain \mathcal{C} to smaller domains such as \mathcal{C}^{st} , we can prove the following theorem (see Fig. 3.5.1).

Theorem 3.5.2. *Let \mathbb{D} be the open unit disk parametrized by w and let $\varphi : w \mapsto \varphi(w)$ be a holomorphic map which maps \mathbb{D} one-to-one onto a domain S inside the cut unit disk of the ρ variable, $S \subset \mathbb{D} \setminus (-1, 0]$. Replacing $\rho = \varphi(w)$ in the conformal block expansion (3.0.1), we pull it back to $w \in \mathbb{D}$. Then this pulled-back conformal block expansion in w variable converges on the boundary $|w| = 1$ in the sense of distributions (i.e. when integrated against an arbitrary smooth function on the circle). Furthermore, the same conclusion holds for (3.5.2) with $q(\rho) = z^{-2\Delta_\phi}$.*

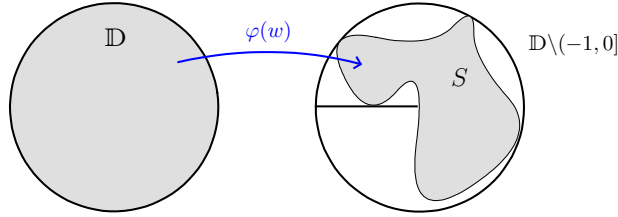


Figure 3.5.1: The setting of theorem 3.5.2. We give one particular example of a possible region S . In practical applications discussed below S will be either all of $\mathbb{D} \setminus (-1, 0]$ or an upper or lower half.

The proof will be based on a simple

Lemma 3.5.3. *For any one-to-one holomorphic function φ from \mathbb{D} onto $S \subset \mathbb{D} \setminus (-1, 0]$ there are lower bounds*

$$\begin{aligned} 1 - |\varphi(w)| &\geq C(1 - |w|), \\ |\varphi(w)| &\geq C'(1 - |w|)^2, \end{aligned} \tag{3.5.3}$$

with some $C, C' > 0$, and for any $w \in \mathbb{D}$. In other words, the first bound says that $|\varphi(w)|$ cannot approach 1 near the boundary faster than linearly in w . Similarly, $|\varphi(w)|$ cannot approach 0 near the boundary faster than quadratically in w .

To see why this is intuitively reasonable, consider some model situations. For the first bound, suppose that $\varphi(w)$ has the leading behavior $\varphi_0 + \text{const.}(w - w_0)^\alpha$, $|\varphi_0| = 1$, near some boundary point $|w_0| = 1$. This asymptotics is consistent with (3.5.3) as long as $\alpha \leq 1$. The latter condition is implied by the assumption that $\varphi : \mathbb{D} \rightarrow \mathbb{D}$: the argument of $w - w_0$ is multiplied by α , and for $\alpha > 1$ some points will end up outside of the unit circle. A similar check works also for the second bound.

It should be noted that in practical applications the domain S will typically be either the whole of $\mathbb{D} \setminus (-1, 0]$ or its upper or lower half. In these cases the functions $\varphi(w)$ will be explicitly known, and bounds (3.5.3) can be verified by an explicit computation. For completeness, a rigorous general proof of this lemma is given in appendix A.2.

By the first inequality of the lemma, we have the bound $(1 - |\rho|)^{-4\Delta_\phi} \leq C'(1 - |w|)^{-4\Delta_\phi}$ for some $C' > 0$. So the conformal block expansion pulled back to the unit disk $w \in \mathbb{D}$ satisfies the same bounds throughout the disk as the ρ -expansion bounds (3.4.5)-(3.4.7). Recall in particular that $g(\rho)$ is bounded near $\rho = 0$ so whatever happens if the boundary of S touches $\rho = 0$, as in figure 3.5.1, is not important for this part of the argument. Therefore, the first claim of the theorem follows by the same arguments as in section 3.4. There is even one simplification: since the circle is compact, temperedness of distributions having to do with behavior of infinity is of no importance in the case at hand. The space of test functions are C^∞ functions on the unit circle.

The second claim does not follow immediately because $z^{-2\Delta_\phi}$ blows up near $\rho = 0$. However, thanks to the second bound in (3.5.3), this does not spoil the slow-growth conditions near $|w| = 1$. This finishes the proof of the theorem.

Note that we can replace the unit ρ -disk by unit $\rho^{1/n}$ -disk for some n if we wish to allow the domain parametrized by w to go under the cut. Similarly, the same result can be proven for a wider class of functions $q(\rho)$ than just $z^{-2\Delta_\phi}$. We won't need these generalizations in this part of the thesis.

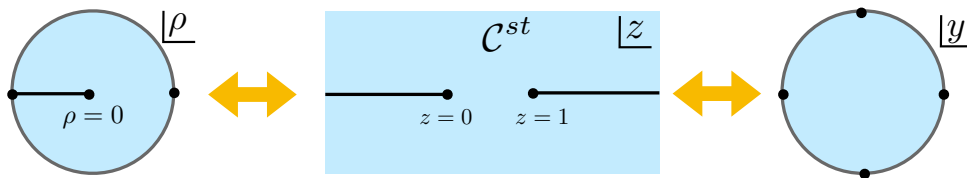


Figure 3.5.2: The crossing region \mathcal{C}^{st} and its parametrization using the ρ -coordinate and the Zhukovsky y -coordinate.

3.6 Analytic functionals

For the first application of theorem 3.5.2, consider the common region of convergence \mathcal{C}^{st} of the two OPE channels for the crossing equation given in z -coordinate by the cut plane

$$\mathcal{C}^{st} = \mathbb{C} \setminus ((-\infty, 0] \cup [1, \infty)), \quad (3.6.1)$$

see figure 3.5.2. In ρ -variable for either channel it becomes precisely the cut unit disk $\mathbb{D} \setminus (-1, 0]$. Following [32], it is convenient to parametrize domain \mathcal{C}^{st} via the Zhukovsky map⁸

$$z(y) = \frac{(1+y)^2}{2(1+y^2)}. \quad (3.6.2)$$

This is a holomorphic one-to-one mapping of the unit disk \mathbb{D} onto \mathcal{C}^{st} . Using the function $\varphi(y) = \rho(z(y))$, the s -channel conformal block expansion is pulled back to the unit disk of the Zhukovsky variable. Since the region \mathcal{C}^{st} is symmetric under $z \rightarrow 1-z$, the same statement is true for the t -channel block expansion (the crossing $z \rightarrow 1-z$ corresponds to $y \rightarrow -y$).

We will now apply theorem 3.5.2 with $S = \mathbb{D} \setminus (-1, 0]$. The first conclusion is that the four-point function (both with and without the factor $z^{-2\Delta_\phi}$) is a distribution on the boundary of the unit y -disk. This statement is only interesting near the points $y = \pm 1$, $y = \pm i$ where the four-point function is singular: on the rest of the boundary it is analytic, as can be shown using the s - and t -channel expansions.

The second conclusion is that both s - and t -channel conformal block expansions converge as a distribution on $|y| = 1$. This statement is interesting, because in the usual sense each channel converges only on one half of the boundary (the left half for the s -channel and the right half for the t -channel).

Distributional convergence has an interesting consequence for the study of the crossing equation using the method of linear functionals [6] and in particular for constructing a wide class of functionals satisfying the swapping property of [28]. We write the crossing in the usual sum rule form

$$\sum p_\Delta F_\Delta(z) = 0, \quad F_\Delta(z) = z^{-2\Delta_\phi} g_\Delta(z) - (1-z)^{-2\Delta_\phi} g_\Delta(1-z). \quad (3.6.3)$$

Denote by $F_\Delta(y)$ the same functions pulled back to the unit Zhukovsky disk. They are analytic in the interior and have boundary values (bv F_Δ) at $|y| = 1$. By theorem 3.5.2 we know that (3.6.3) converges on the $|y| = 1$ boundary to zero in the sense of distributions. This means that we can integrate it term by term with a smooth function $f(\theta)$:

$$\sum p_\Delta \int_0^{2\pi} d\theta (\text{bv } F_\Delta)(y = e^{i\theta}) f(\theta) = 0. \quad (3.6.4)$$

The 1d conformal blocks having only logarithmic singularities, the nature of their boundary values is determined by the singularity of prefactors $z^{-2\Delta_\phi}$ and $(1-z)^{-2\Delta_\phi}$. Thus they are ordinary locally integrable functions for $2\Delta_\phi < 1$, and distributions otherwise.

⁸The original Zhukovsky (Joukowski) map $\zeta = y + 1/y$ maps the unit circle onto the interval $(-2, 2)$. We have $z = 1/2 + 1/\zeta$ so that the unit circle is mapped onto the two cuts $(-\infty, 0] \cup [1, \infty)$. The Zhukovsky map is famous in aerodynamics: applying it to offcentric circles one can parametrize airfoil shapes and compute the lift force analytically by conformal invariance of incompressible 2d flows.

Now, let us fix an infinitely smooth $f(\theta)$ on the boundary of the unit y -disk. The support of this function may include points in both halves of the circle, including the points where the four-point function is singular. Consider a linear functional α_f defined by the formula

$$g(y) \mapsto \alpha_f[g] \equiv \int_0^{2\pi} d\theta (\text{bv } g)(y = e^{i\theta}) f(\theta). \quad (3.6.5)$$

We can write (3.6.4) equivalently as

$$\sum p_\Delta \alpha_f[F_\Delta] = 0. \quad (3.6.6)$$

This means, in the terminology of [28], that the functional (3.6.5) satisfies swapping property.

Note that many simple functionals can be rewritten in the form (3.6.5). For example, the derivative evaluation functional α_{n,y_0}

$$g_{n,y_0}(y) \mapsto \alpha_{n,y_0}[g] \equiv g^{(n)}(y_0) \quad (3.6.7)$$

for integer $n \geq 0$ and $|y_0| < 1$ can be written using Cauchy theorem as⁹

$$\alpha_{n,y_0}[g] = \frac{n!}{2\pi i} \int_0^{2\pi} d\theta \frac{ie^{i\theta}}{(e^{i\theta} - y_0)^{n+1}} (\text{bv } g)(y = e^{i\theta}). \quad (3.6.8)$$

This coincides with $\alpha_{f_{n,y_0}}$ with f_{n,y_0} given by

$$f_{n,y_0}(\theta) = \frac{n!}{2\pi i} \frac{ie^{i\theta}}{(e^{i\theta} - y_0)^{n+1}}. \quad (3.6.9)$$

A type of functionals commonly used in analytic functional conformal bootstrap [32–39] can be described as

$$g(y) \mapsto \alpha_{h,\Gamma}[g] \equiv \int_\Gamma dy h(y) g(y), \quad (3.6.10)$$

where $h(y)$ is some holomorphic function and Γ is a contour in \mathbb{D} which is allowed to have end points on the boundary $|y| = 1$. Conditions on $h(y)$ that guarantee the swapping property for $\alpha_{h,\Gamma}$ were studied in [28]. We can try to identify $\alpha_{h,\Gamma}$ with $\alpha_{f_{h,\Gamma}}$ where

$$f_{h,\Gamma}(\theta) = \int_\Gamma dy h(y) f_{0,y}(\theta), \quad (3.6.11)$$

with $f_{n,y}$ defined in (3.6.9). Unfortunately, if Γ ends or starts on $|y| = 1$ then for generic $h(y)$ the function $f_{h,\Gamma}(\theta)$ will not be smooth (and so will not be a test function) and thus we have not proven that $\alpha_{f_{h,\Gamma}}$ is well defined and satisfies the swapping property for this class of functionals. In other words, so far the class of functionals (3.6.5) is too small to accommodate the modern results in analytic functional bootstrap.

However, the swapping conditions of [28] require $h(y)$ to decay sufficiently quickly (as some power-law) near the end points of Γ that are on $|y| = 1$. In this case $f_{h,\Gamma}(\theta)$ is still generically not infinitely smooth, but it will have some finite number of derivatives, i.e. we will have $f_{h,\Gamma}(\theta) \in C^k(S^1)$ for some $k > 0$. In particular, under the swapping conditions on $h(y)$ derived in [28] k is proportional to Δ_ϕ . On the other hand, by examining the proof of Vladimirov's theorem 3.2.1 given in section 3.3, we can see that we only use a finite number of semi-norms of the test functions, corresponding to derivatives of order related to the power M in the slow-growth condition (3.2.2), which in turn is related to the dimension Δ_ϕ . This implies that in order for the functionals (3.6.5) to be well-defined and satisfy the swapping property, we only need f to have k' derivatives with k' proportional to Δ_ϕ .

We thus see that if the functional (3.6.10) satisfies the swapping conditions derived in [28], then the function (3.6.11) has $k \propto \Delta_\phi$ derivatives. Similarly, we concluded that our results can be strengthened so that the functional (3.6.5)

⁹In Cauchy theorem we integrate over the contour at $|y| = 1 - \epsilon$, where the function is analytic. As $\epsilon \rightarrow 0$, the Cauchy kernel tends to $\frac{1}{(e^{i\theta} - y_0)^{n+1}}$ in the C^∞ topology of test functions on the circle, while $g(y)$ tends to $\text{bv } g$ in the sense of distributions by theorem 3.5.2. This justifies pushing the contour all the way to the boundary $|y| = 1$.

is well defined and satisfies the swapping property if f has $k' \propto \Delta_\phi$ derivatives. This suggests that it is possible to define a space $\mathcal{B}_{\Delta_\phi}$ of functions on S^1 with the following properties. First, we would like α_f to be well defined and satisfy the swapping property for all $f \in \mathcal{B}_{\Delta_\phi}$. Furthermore, all functionals used in analytic functional bootstrap should be representable by α_f with $f \in \mathcal{B}_{\Delta_\phi}$, i.e. we want $f_{h,\Gamma} \in \mathcal{B}_{\Delta_\phi}$ for all h and Γ which satisfy the swapping conditions of [28].

As alluded to above, the first approximation to the space $\mathcal{B}_{\Delta_\phi}$ is $C^k(S^1)$ with appropriately chosen k . However, this seems too coarse, since k is a discrete parameter, while Δ_ϕ is continuous. Moreover, not all the points y with $|y| = 1$ are equal—there are special points $y = \pm 1, \pm i$, where the correlator might have a singularity that needs to be controlled, but at all other points we know from crossing that the correlator is smooth (but this does not imply that the conformal block expansion converges there pointwise). It would be interesting to find the appropriate definition for $\mathcal{B}_{\Delta_\phi}$ since it would provide a uniform description of all functionals suitable for analyzing the crossing equation. We leave these questions for future work.

3.7 Dispersion relation in cross-ratio space and the discontinuity

For a second application, we consider the upper half-plane in z variable. This region is a subset of \mathcal{C}^{st} and thus we can again use theorem 3.5.2 (this time with S being the upper half of \mathbb{D}) to conclude that both s - and t - conformal block expansions converge as distributions on the boundary of unit disk in the variable $w = \frac{z-i}{z+i}$. This boundary minus one point is smoothly mapped to the real line in z -plane, and so both s - and t -channels also converge as distributions on the real line \mathbb{R} in z -plane when approached from above. By repeating the same arguments for the lower half-plane mapped to the unit disk via $\tilde{w} = \frac{z+i}{z-i}$, we find that both channels converge as distributions on the real line in z -plane when approached from below.

Let us now see how this kind of arguments can be used to write rigorous dispersion relations and give a proper definition of discontinuity (including the point at infinity). Let z_0 be a point in the upper half-plane, and C and \tilde{C} be contours in the upper and lower half-planes, with C surrounding z_0 . Then we have

$$\begin{aligned} g(z_0) &= \frac{1}{2\pi i} \oint_C \frac{dz}{z - z_0} g(z), \\ 0 &= \frac{1}{2\pi i} \oint_{\tilde{C}} \frac{dz}{z - z_0} g(z). \end{aligned} \quad (3.7.1)$$

Intuitively, to derive the dispersion relation we push C and \tilde{C} to the real axis and infinity, and take the difference of the two equations, which gives a dispersion relation

$$g(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dx}{x - z_0} \text{Disc } g(x) + \text{contribution at infinity}, \quad (3.7.2)$$

where $\text{Disc } g(x)$ is the difference in two limits of $g(z)$. Contribution at infinity cannot be generally computed in this approach, unless one has some information about the asymptotics of $g(z)$ as $z \rightarrow \infty$.

Let us now turn this reasoning into a rigorous dispersion relation, including the contribution at infinity. First of all we pull Eqs. (3.7.1) to the unit discs of w and \tilde{w} variable, which gives:

$$\begin{aligned} g(w_0) &= \frac{1}{2\pi i} (w_0 - 1) \oint_C \frac{dw}{w - w_0} \frac{g_+(w)}{w - 1}, \\ 0 &= \frac{1}{2\pi i} (w_0 - 1) \oint_{\tilde{C}} \frac{d\tilde{w}/\tilde{w}^2}{\tilde{w}^{-1} - w_0} \frac{g_-(\tilde{w})}{\tilde{w}^{-1} - 1}. \end{aligned} \quad (3.7.3)$$

where we denoted by $g_+(w)$, $g_-(\tilde{w})$ the function $g(z)$ from the upper/lower half-plane pulled to the corresponding unit disk. Then we push the contours C , \tilde{C} to $|w| = 1$, $|\tilde{w}| = 1$ and get:

$$\begin{aligned} g(w_0) &= \frac{1}{2\pi i} (w_0 - 1) \oint_{|w|=1} \frac{dw}{w - w_0} \text{bv} \frac{g_+(w)}{w - 1}, \\ 0 &= \frac{1}{2\pi i} (w_0 - 1) \oint_{|\tilde{w}|=1} \frac{d\tilde{w}/\tilde{w}^2}{\tilde{w}^{-1} - w_0} \text{bv} \frac{g_-(\tilde{w})}{\tilde{w}^{-1} - 1}. \end{aligned} \quad (3.7.4)$$

Notice that we have to include the singular factors $1/(w-1)$ and $1/(\tilde{w}-1)$, arising due to the transformation of the measure dz , under the “bv” sign. Since these factors are power-like, the resulting limiting boundary values exist as distributions also in presence of these factors. Finally we take the difference of the two equations and we get:

$$g(w_0) = \frac{1}{2\pi} (w_0 - 1) \int_0^{2\pi} \frac{d\theta e^{i\theta}}{e^{i\theta} - w_0} D(\theta), \quad (3.7.5)$$

$$D(\theta) = \text{bv} \frac{g_+(w)}{w-1} \Big|_{w=e^{i\theta}} - \text{bv} \frac{g_-(\tilde{w})}{\tilde{w}^{-1}-1} \Big|_{\tilde{w}=e^{-i\theta}}. \quad (3.7.6)$$

Here $D(\theta)$ is a distribution on the unit circle, which plays the role of a rigorously defined discontinuity, including the point $z = \infty$ mapped to $\theta = 0$. For points away from $\theta = 0$ and $\theta = 2\pi$ we can pull the factors $1/(w-1)$ and $1/(\tilde{w}-1)$ from under bv and $D(\theta)$ becomes just

$$D(\theta) = \frac{1}{e^{i\theta} - 1} \text{Disc } g(x = -\cot \frac{\theta}{2}), \quad \theta \neq 0, 2\pi. \quad (3.7.7)$$

Here $\text{Disc } g(x = -\cot \frac{\theta}{2})$ is defined as $\text{bv} g_+(w)|_{w=e^{i\theta}} - \text{bv} g_-(\tilde{w})|_{\tilde{w}=e^{-i\theta}}$, which is equivalent to taking the boundary values in z -space from above and below the real axis, which is simply the intuitive definition of discontinuity. Using this value of $D(\theta)$ in (3.7.5) and changing back to x variable, we recover (3.7.2). So we see that (3.7.5) is indeed an analogue of (3.7.2). However, using $D(\theta)$ allows us to rigorously include the contribution at $x = \infty$.

An intuitive way to think about this construction is that it defines $\text{Disc } g(x)$ as a distribution on a class of test functions $\mathcal{S}_0(\mathbb{R})$ larger than $\mathcal{S}(\mathbb{R})$. The space $\mathcal{S}_0(\mathbb{R})$ consists of smooth functions $f(x)$ such that $f(1/x')$ is smooth and vanishing at $x' = 0$. Pairing with $\text{Disc } g(x)$ is defined by the formula

$$\int dx f(x) \text{Disc } g(x) \equiv -2 \int d\theta e^{i\theta} \tilde{f}(\theta) D(\theta), \quad (3.7.8)$$

where

$$\tilde{f}(\theta) \equiv \frac{1}{e^{i\theta} - 1} f(-\cot \frac{\theta}{2}) \quad (3.7.9)$$

is a smooth function on the circle parametrized by θ .¹⁰ With this definition we can write the dispersion relation (3.7.5) as

$$g(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dx}{x - z_0} \text{Disc } g(x), \quad (3.7.10)$$

since the Cauchy kernel $\frac{1}{x-z_0}$ belongs to our new class of test functions. In this language our results imply that both Disc of the four-point function and Disc of partial sums of the conformal block expansion are distributions in $\mathcal{S}'_0(\mathbb{R})$, and the partial sums converge to the four-point function in this space (i.e. discontinuity can be computed term-by-term).

Let us consider an example. First take $g(z) = \log z$. This is not a good four-point function since it does not satisfy crossing, but it will allow us to clarify the notion of the discontinuity as a distribution and how it can be concretely computed. Going to the ρ variable we easily see that the slow-growth condition is satisfied. For finite $x < 0$ we have $\text{Disc } g(x) = 2\pi i$. This is a distribution in $\mathcal{S}'(\mathbb{R})$, but not obviously in $\mathcal{S}'_0(\mathbb{R})$. To extend it to $\mathcal{S}'_0(\mathbb{R})$ let us write

$$\log z = - \lim_{\alpha \rightarrow +0} \partial_\alpha z^{-\alpha}. \quad (3.7.11)$$

The point here is that $z^{-\alpha}$ also satisfies a power-law bound and for $\alpha > 0$ the discontinuity

$$\text{Disc } x^{-\alpha} = -2i \sin \pi \alpha |x|^{-\alpha} \quad (3.7.12)$$

is in $\mathcal{S}'_0(\mathbb{R})$. We can then obtain $\text{Disc } g(x)$ by taking derivative and limit $\alpha \rightarrow +0$.¹¹ Pairing $\text{Disc } g(x)$ with functions that vanish as $1/x^2$ or faster we get integrals that converge in the usual sense. So we only need to use the limiting

¹⁰This equation established isomorphism between $\mathcal{S}_0(\mathbb{R})$ and $C^\infty(S^1)$ in the sense that $f \in \mathcal{S}_0(\mathbb{R})$ if and only if $\tilde{f} \in C^\infty(S^1)$.

¹¹Justification for this comes from the limit part of the statement of theorem 3.2.1 and (for the derivative) from arguments as in appendix A.3.

construction to define the pairing with $1/x$. We have:

$$\begin{aligned} \int_{-\infty}^{-1} dx \frac{1}{x} \text{Disc } g(x) &= - \lim_{\alpha \rightarrow +0} \partial_{\alpha} \int_{-\infty}^{-1} dx \frac{1}{x} (-2i \sin \pi \alpha |x|^{-\alpha}) \\ &= -2i \lim_{\alpha \rightarrow +0} \partial_{\alpha} \frac{\sin \pi \alpha}{\alpha} = 0, \end{aligned} \quad (3.7.13)$$

where the choice of the integral's upper limit -1 is just for convenience since it leads to a simple answer (zero). We can therefore define the distribution $\text{Disc } g(x)$ by

$$\int dx f(x) \text{Disc } g(x) = \int_{-\infty}^{-1} dx (f(x) - f_1 x^{-1}) 2\pi i + \int_{-1}^0 dx f(x) 2\pi i. \quad (3.7.14)$$

where $f_1 \equiv \lim_{x \rightarrow \infty} x f(x)$. The dispersion relation (3.7.10) then becomes

$$\log z_0 = \int_{-\infty}^{-1} dx \left(\frac{1}{x - z_0} - \frac{1}{x} \right) + \int_{-1}^0 dx \frac{1}{x - z_0}. \quad (3.7.15)$$

This is easy to verify.

Another example, which we will find useful in section 4.6, is $\text{Disc } 1$. Naively, this discontinuity must be zero. This is indeed correct, except at $x = \infty$. Indeed, analogously to the above, we have

$$1 = \lim_{\alpha \rightarrow +0} z^{-\alpha}, \quad (3.7.16)$$

so

$$\begin{aligned} \int_{-\infty}^{-1} dx \frac{1}{x} (\text{Disc } 1)(x) &= \lim_{\alpha \rightarrow +0} \int_{-\infty}^{-1} dx \frac{1}{x} (-2i \sin \pi \alpha |x|^{-\alpha}) \\ &= 2i \lim_{\alpha \rightarrow +0} \frac{\sin \pi \alpha}{\alpha} = 2\pi i, \end{aligned} \quad (3.7.17)$$

and thus

$$\int dx f(x) (\text{Disc } 1)(x) = 2\pi i f_1, \quad (3.7.18)$$

where as before $f_1 = \lim_{x \rightarrow \infty} x f(x)$.

Chapter 4

Scalar four-point functions in higher dimensions

We will now generalize our results to general scalar four-point functions in any number of dimensions d . This generalization is mostly technical, and all the conceptual points were already explained in chapter 3. Our strategy is therefore very similar: first we will introduce analogues of the expansions (3.0.1) and (3.4.1), and then use these expansions to prove bounds on the correlation function and partial sums of the conformal block expansion. Finally, we will apply a higher-dimensional version of Vladimirov's theorem 3.2.1 to conclude that the conformal block expansion converges in the sense of distributions on the boundary of the region $|\rho|, |\bar{\rho}| < 1$.

4.1 Conformal block expansion

We consider a correlation function of four not necessarily identical scalar operators ϕ_i with scaling dimensions Δ_i ,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4) \rangle = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}} g_{1234}(\rho, \bar{\rho}), \quad (4.1.1)$$

which is a simple generalization of (2.0.1). The subscript 1234 on g_{1234} indicates that it relates to the four-point function of ϕ_1, \dots, ϕ_4 . The function $g_{1234}(\rho, \bar{\rho})$ has a conformal block expansion of the form

$$g_{1234}(\rho, \bar{\rho}) = \sum_{\mathcal{O}} \lambda_{12\bar{\mathcal{O}}} \lambda_{43\mathcal{O}} g_{\Delta, J}(\rho, \bar{\rho}), \quad (4.1.2)$$

where we sum over primaries \mathcal{O} in $\phi_1 \times \phi_2$ OPE, λ 's are the three-point coefficients, Δ, J are the spin and dimension of \mathcal{O} , and $g_{\Delta, J}(\rho, \bar{\rho})$ are the conformal blocks. The conformal blocks also depend implicitly on $\Delta_{12} = \Delta_1 - \Delta_2$ and $\Delta_{34} = \Delta_3 - \Delta_4$.

We would like to show that the function $g_{1234}(\rho, \bar{\rho})$ satisfies a powerlaw bound as ρ and $\bar{\rho}$ approach the boundaries of their respective unit disks. We would also like to show that partial sums of the conformal block expansion (4.1.2) satisfy a uniform powerlaw bound. We will prove this by relating $g_{1234}(\rho, \bar{\rho})$ to the four-point function where operators are inserted symmetrically with respect to the origin [43].

Let us focus on configurations when all points x_i lie in the 2-plane P defined by $x^\mu = 0$ for $\mu > 2$. It is convenient to introduce complex coordinates y, \bar{y} in this plane

$$y = x^1 + ix^2, \quad \bar{y} = x^1 - ix^2. \quad (4.1.3)$$

Notice that in Euclidean configurations (i.e. when x^μ are real) we have $\bar{y} = y^*$. Using the notation $\phi_i(y, \bar{y})$ for operator insertions in P parametrized by y, \bar{y} , we consider for $\bar{\rho} = \rho^*$ a symmetrically-inserted four-point function

$$\tilde{g}_{1234}(\rho, \bar{\rho}) = (\rho\bar{\rho})^{\frac{\Delta_1+\Delta_2}{2}} \langle \phi_1(-\rho, -\bar{\rho})\phi_2(\rho, \bar{\rho})\phi_3(1, 1)\phi_4(-1, -1) \rangle, \quad (4.1.4)$$

where the factor $(\rho\bar{\rho})^{\frac{\Delta_1+\Delta_2}{2}}$ is inserted for further convenience (basically to make Eq. (4.1.6) look maximally nice). For operators inserted as shown, the meaning of ρ in (4.1.4) and (4.1.1) is the same, justifying the notation. Evaluating also the prefactor in (4.1.1), we find the following relation between \tilde{g}_{1234} and g_{1234} :¹

$$\tilde{g}_{1234}(\rho, \bar{\rho}) = 2^{-\Delta_1-\Delta_2-\Delta_3-\Delta_4} \left(\frac{(1+\rho)(1+\bar{\rho})}{(1-\rho)(1-\bar{\rho})} \right)^{\frac{1}{2}(\Delta_{12}-\Delta_{34})} g_{1234}(\rho, \bar{\rho}). \quad (4.1.5)$$

For $\bar{\rho} = \rho^*$ (4.1.4) is a Euclidean configuration, radial quantization of which [3, 18, 43, 44] gives the following absolutely convergent expansion for $|\rho| = |\bar{\rho}| < 1$

$$\tilde{g}_{1234}(\rho, \bar{\rho}) = \sum_{\psi} \tilde{\lambda}_{12\psi} \tilde{\lambda}_{43\bar{\psi}} \rho^h \bar{\rho}^{\bar{h}}, \quad (4.1.6)$$

where we sum over eigenstates ψ of dilatations and planar rotations in radial quantization, and h, \bar{h} are appropriate combinations of the corresponding eigenvalues. Since it converges absolutely for $|\rho| = |\bar{\rho}| < 1$ when $\bar{\rho} = \rho^*$, it also does so for independent ρ and $\bar{\rho}$ when $|\rho|, |\bar{\rho}| < 1$. Furthermore, the conformal block expansion (4.1.2) can be understood as a reorganization of expansion (4.1.6) by grouping ψ into conformal families.

4.2 Bounds on $g(\rho, \bar{\rho})$ and partial sums of the conformal block expansion

Consider the following analogues of (4.1.4), (4.1.6) where two pairs of operators are hermitean conjugates of each other:

$$\tilde{g}_{12\bar{2}\bar{1}}(\rho, \bar{\rho}) = \sum_{\psi} \tilde{\lambda}_{12\psi} \tilde{\lambda}_{\bar{1}\bar{2}\bar{\psi}} \rho^h \bar{\rho}^{\bar{h}} = \sum_{\psi} |\tilde{\lambda}_{12\psi}|^2 \rho^h \bar{\rho}^{\bar{h}}, \quad (4.2.1)$$

$$\tilde{g}_{\bar{4}\bar{3}\bar{3}\bar{4}}(\rho, \bar{\rho}) = \sum_{\psi} \tilde{\lambda}_{\bar{4}\bar{3}\bar{\psi}} \tilde{\lambda}_{43\bar{\psi}} \rho^h \bar{\rho}^{\bar{h}} = \sum_{\psi} |\tilde{\lambda}_{43\bar{\psi}}|^2 \rho^h \bar{\rho}^{\bar{h}}, \quad (4.2.2)$$

where we use $\bar{1}$, etc., to denote three-point coefficients of hermitian conjugates ϕ_1^\dagger , etc.. As shown, because of $\tilde{\lambda}_{\bar{1}\bar{2}\bar{\psi}} = (\tilde{\lambda}_{12\psi})^*$ and $\tilde{\lambda}_{\bar{4}\bar{3}\bar{\psi}} = (\tilde{\lambda}_{43\bar{\psi}})^*$, these two expansions have non-negative real coefficients. Furthermore, estimating by absolute value and applying Cauchy-Schwarz, we can bound (4.1.6) in terms of (4.2.1), (4.2.2):

$$|\tilde{g}_{1234}(\rho, \bar{\rho})| \leq \sum_{\psi} |\tilde{\lambda}_{12\psi}| |\tilde{\lambda}_{43\bar{\psi}}| r^{h+\bar{h}} \leq [\tilde{g}_{12\bar{2}\bar{1}}(r, r) \tilde{g}_{\bar{4}\bar{3}\bar{3}\bar{4}}(r, r)]^{1/2} \quad (4.2.3)$$

where $r = \max(|\rho|, |\bar{\rho}|)$.² Note that the same bound holds if we replace the sum over ψ by a sum over a subset of all allowed ψ 's. This, similarly to the argument in section 3.4, implies that the partial sums of expansions (4.1.2) and (4.1.6) satisfy the same bound (4.2.3) (with \tilde{g}_{1234} related to g_{1234} via (4.1.5) where needed).

To proceed we need a bound on $\tilde{g}_{12\bar{2}\bar{1}}(r, r)$ and $\tilde{g}_{\bar{4}\bar{3}\bar{3}\bar{4}}(r, r)$. This bound is easy to obtain from the corresponding definition (4.1.4). In the limit $r \rightarrow 1$ two pairs of hermitean conjugate operators approach each other. Using OPE between the approaching pairs, we get a leading asymptotics for the correlator.³ This implies a bound of the same functional form as the leading asymptotics times a constant. The resulting bounds have the form:

$$\tilde{g}_{12\bar{2}\bar{1}}(r, r) \leq C(1-r)^{-2\Delta_1-2\Delta_2}, \quad (4.2.4)$$

$$\tilde{g}_{\bar{4}\bar{3}\bar{3}\bar{4}}(r, r) \leq C(1-r)^{-2\Delta_3-2\Delta_4}, \quad (4.2.5)$$

¹Both \tilde{g}_{1234} and g_{1234} depend on $\rho, \bar{\rho}$ and both can pretend to be called the conformally invariant part of the general four-point function. One could switch from one convention to the other by changing the prefactor in (4.1.1). We will still express our final results in terms of g_{1234} , since Eq. (4.1.1) is the most standard convention.

²We also have a more nuanced bound by $[\tilde{g}_{12\bar{2}\bar{1}}(|\rho|, |\bar{\rho}|) \tilde{g}_{\bar{4}\bar{3}\bar{3}\bar{4}}(|\rho|, |\bar{\rho}|)]^{1/2}$ but we won't need it.

³This can be equivalently formulated via crossing symmetry in z space and then transforming to the ρ space, as in section 3.4.

with some $C > 0$. Notice that there is no blowup as $r \rightarrow 0$ since it's overcome by the prefactor in (4.1.4). Combining these with (4.2.3) we find

$$|\tilde{g}_{1234}(\rho, \bar{\rho})| \leq C(1-r)^{-\Delta_1 - \Delta_2 - \Delta_3 - \Delta_4}, \quad (4.2.6)$$

and finally via (4.1.5) we get a bound for g_{1234}

$$|g_{1234}(\rho, \bar{\rho})| \leq C'(1-r)^{-\Delta_1 - \Delta_2 - \Delta_3 - \Delta_4 - |\Delta_{12} - \Delta_{34}|}, \quad r = \max(|\rho|, |\bar{\rho}|), \quad (4.2.7)$$

for some $C' > 0$. Again, the same bound with the same C' holds for the partial sums of expansions (4.1.2) and (4.1.6).

We repeat the logic of this argument. The key idea is to use OPE in the cross channel to infer the leading singularity of the correlator and then to argue that a similar bound holds throughout the range $|\rho|, |\bar{\rho}| < 1$. This does not work directly for g_{1234} , but only for 4pt functions with non-negative $\rho, \bar{\rho}$ expansion coefficients, such as $g_{12\bar{2}\bar{1}}$ and $g_{\bar{4}\bar{3}34}$. So we run the argument for those, and recover the general case by Cauchy-Schwarz.

4.3 Vladimirov's theorem

Now that we have the bound (4.2.7) we would like to use a higher-dimensional version of Vladimirov's theorem 3.2.1 to argue for the distributional convergence of conformal block expansion (4.1.2).

Theorem 4.3.1. *Consider $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^d$ with coordinates w_k on \mathbb{C}^n and $u_k = x_k + iy_k$ on \mathbb{C}^d . Let U be an open subset of \mathbb{C}^n and let $M = U \times \mathbb{R}^d$ be the manifold defined by $w \in U, y_k = 0, k = 1 \dots d$. Let V be a convex open cone in \mathbb{R}^d with vertex at $y = 0$ that doesn't contain $y = 0$. Let \mathcal{W} be the subset of \mathbb{C}^N for which $y \in V, |y_k| < a$ for some $a > 0$, and $w \in U$. Let $g(w, u)$ be a function holomorphic in \mathcal{W} that satisfies the slow-growth condition near M ⁴*

$$|g(w, u)| \leq C \left(1 + \sum_k x_k^2\right)^L \left(\sum_k y_k^2\right)^{-K}. \quad (4.3.1)$$

Finally, let v be a vector in V . Then for fixed w the boundary value

$$(\text{bv } g)(w, x) = \lim_{\epsilon \rightarrow +0} g(w, x + iv\epsilon) \quad (4.3.2)$$

exists in $\mathcal{S}'(\mathbb{R}^d)$ and is independent of the choice of v . Furthermore, this boundary value depends on w holomorphically, which means that for any $f \in \mathcal{S}(\mathbb{R}^d)$ the function $h(w)$ defined by⁵

$$h(w) \equiv \int d^d x f(x) (\text{bv } g)(w, x) \quad (4.3.3)$$

is holomorphic for $w \in U$. Furthermore, suppose that sequence of functions g_n holomorphic in \mathcal{W} converges to g in \mathcal{W} pointwise and satisfies the slow-growth condition near M uniformly in n . Then for all $w \in U$

$$(\text{bv } g_n)(w, x) \rightarrow (\text{bv } g)(w, x) \quad \text{in } \mathcal{S}'(\mathbb{R}^d). \quad (4.3.4)$$

The proof of this theorem is very similar to the proof of theorem 3.2.1 given in section 3.3, and we summarize it in appendix A.3. For more general results in this direction see, for example, [27] and [50].

Let us now apply theorem 4.3.1 to the conformal block expansion (4.1.2). As a first step, we introduce the coordinates τ and $\bar{\tau}$ via

$$\rho = e^{i\tau}, \quad \bar{\rho} = e^{i\bar{\tau}}. \quad (4.3.5)$$

⁴More precisely, we'd like to have this condition satisfied uniformly on compact subsets $w \in \mathcal{K} \subset U$ with C, L, K allowed to depend on \mathcal{K} .

⁵Here the integral of course just means the pairing of the distribution with the test function.

Note that in Euclidean configurations we have $\bar{\tau} = -\tau^*$. The function $g_{1234}(\tau, \bar{\tau})$ as well as the partial sums of (4.1.2) are holomorphic functions in the region

$$\mathcal{W}_0 = \{(\tau, \bar{\tau}) | \text{Im } \tau, \text{Im } \bar{\tau} > 0\}, \quad (4.3.6)$$

which is the universal cover of the product of open unit discs of ρ and $\bar{\rho}$. Furthermore, the expansion (4.1.2) converges absolutely in \mathcal{W}_0 . We can apply theorem 4.3.1 in two essentially different ways.

Firstly, we can take $\text{Im } \bar{\tau}$ to zero while keeping τ fixed. This corresponds to $n = d = 1$ case of theorem 4.3.1, in which \mathbb{C}^n is parametrized by τ and \mathbb{C}^d by $\bar{\tau}$. The open set U is then given by $\text{Im } \tau > 0$ and the cone V is given by $y_1 = \text{Im } \bar{\tau} > 0$. The set \mathcal{W} is then

$$\mathcal{W} = \{(\tau, \bar{\tau}) | \text{Im } \tau > 0, a > \text{Im } \bar{\tau} > 0\}, \quad (4.3.7)$$

for some $a > 0$, say $a = 1$. The slow-growth condition for $g_{1234}(\tau, \bar{\tau})$ and the partial sums of (4.1.2) follows from (4.2.7). In this way, for each τ we get a distribution

$$(\text{bv } g_{1234})(\tau, \text{Re } \bar{\tau}) = \sum_{\mathcal{O}} \lambda_{12\bar{\mathcal{O}}} \lambda_{43\mathcal{O}} (\text{bv } g_{\Delta, J})(\tau, \text{Re } \bar{\tau}) \quad \text{in } \mathcal{S}'(\mathbb{R}) \quad (4.3.8)$$

that is holomorphic in τ . Similarly, we can send $\text{Im } \tau$ to 0 while keeping τ fixed to get

$$(\text{bv } g_{1234})(\text{Re } \tau, \bar{\tau}) = \sum_{\mathcal{O}} \lambda_{12\bar{\mathcal{O}}} \lambda_{43\mathcal{O}} (\text{bv } g_{\Delta, J})(\text{Re } \tau, \bar{\tau}) \quad \text{in } \mathcal{S}'(\mathbb{R}), \quad (4.3.9)$$

holomorphic in $\bar{\tau}$.

Secondly, we can take the simultaneous limit $\text{Im } \tau, \text{Im } \bar{\tau} \rightarrow 0$. This corresponds to $n = 0$ and $d = 2$ in theorem 4.3.1. A small subtlety is that with \mathcal{W} as in (4.3.7) the slow-growth condition doesn't follow from (4.2.7), since r in (4.2.7) can approach 1 even if only one of $\text{Im } \tau, \text{Im } \bar{\tau}$ is small. To fix this, choose any $\alpha < 1$ and define

$$V = \{(\text{Im } \tau, \text{Im } \bar{\tau}) | \text{Im } \tau, \text{Im } \bar{\tau} > 0, \alpha^{-1} < \text{Im } \tau / \text{Im } \bar{\tau} < \alpha\}. \quad (4.3.10)$$

The corresponding \mathcal{W} has form as in figure 4.3.1. In this new \mathcal{W} we have $1 - r > C[(\text{Im } \tau)^2 + (\text{Im } \bar{\tau})^2]^{1/2}$ for some $C > 0$ and the slow-growth condition follows from (4.2.7). We therefore conclude the existence of the boundary values and the distributional convergence of the boundary value series:

$$(\text{bv } g_{1234})(\text{Re } \tau, \text{Re } \bar{\tau}) = \sum_{\mathcal{O}} \lambda_{12\bar{\mathcal{O}}} \lambda_{43\mathcal{O}} (\text{bv } g_{\Delta, J})(\text{Re } \tau, \text{Re } \bar{\tau}) \quad \text{in } \mathcal{S}'(\mathbb{R}^2). \quad (4.3.11)$$

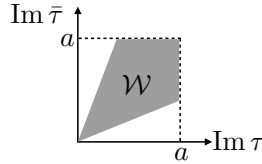


Figure 4.3.1: The region \mathcal{W} relevant for the second application of Vladimirov's theorem.

4.4 Analytic functionals

Similarly to the one-dimensional case, we can consider various generalizations. In particular, we have the obvious generalizations of corollary 3.5.1 and theorem 3.5.2.

Corollary 4.4.1. *If function $q(\rho, \bar{\rho})$ is holomorphic in the branched unit $\rho, \bar{\rho}$ -polydisc and satisfies the appropriate slow-growth conditions near $\tau, \bar{\tau} \in \mathbb{R}$ (recall $\rho = e^{i\tau}, \bar{\rho} = e^{i\bar{\tau}}$), then we have*

$$(\text{bv } q \cdot g_{1234})(\tau, \text{Re } \bar{\tau}) = \sum_{\mathcal{O}} \lambda_{12\bar{\mathcal{O}}} \lambda_{43\mathcal{O}} (\text{bv } q \cdot g_{\Delta, J})(\tau, \text{Re } \bar{\tau}) \quad \text{in } \mathcal{S}'(\mathbb{R}), \quad (4.4.1)$$

$$(\text{bv } q \cdot g_{1234})(\text{Re } \tau, \bar{\tau}) = \sum_{\mathcal{O}} \lambda_{12\bar{\mathcal{O}}} \lambda_{43\mathcal{O}} (\text{bv } q \cdot g_{\Delta, J})(\text{Re } \tau, \bar{\tau}) \quad \text{in } \mathcal{S}'(\mathbb{R}), \quad (4.4.2)$$

$$(\text{bv } q \cdot g_{1234})(\text{Re } \tau, \text{Re } \bar{\tau}) = \sum_{\mathcal{O}} \lambda_{12\bar{\mathcal{O}}} \lambda_{43\mathcal{O}} (\text{bv } q \cdot g_{\Delta, J})(\text{Re } \tau, \text{Re } \bar{\tau}) \quad \text{in } \mathcal{S}'(\mathbb{R}^2). \quad (4.4.3)$$

Theorem 4.4.2. *Let \mathbb{D} be the open unit disk parametrized by w and let $\varphi : w \mapsto \varphi(w)$ be a holomorphic map which maps \mathbb{D} one-to-one onto a domain S inside the cut unit disk of the ρ variable, $S \subset \mathbb{D} \setminus (-1, 0]$. Let $\bar{\varphi}$ be a map of the same kind with S replaced by $\bar{S} \subset \mathbb{D} \setminus (-1, 0]$. Replacing $\rho = \phi(w), \bar{\rho} = \bar{\phi}(\bar{w})$ in the conformal block expansion (4.1.2), we pull it back to $w, \bar{w} \in \mathbb{D} \times \mathbb{D}$. Then this pulled-back conformal block expansion in w, \bar{w} variables converges on the boundaries $|w| = 1, |\bar{w}| = 1$, or $|w| = |\bar{w}| = 1$ in the sense of distributions. Furthermore, the same conclusion holds if expansion (4.1.2) is multiplied by $q(\rho, \bar{\rho}) = (z\bar{z})^{-\frac{\Delta_1 + \Delta_2}{2}}$.*

For example, the discussion of analytic bootstrap functionals in section 3.6 can be extended to the two-variable case as follows. In Zhukovsky variables y, \bar{y} the crossing domain \mathcal{C}^{st} is given by $\mathbb{D} \times \mathbb{D}$. The boundary $\partial(\mathbb{D} \times \mathbb{D})$ is topologically a 3-sphere S^3 . This S^3 is a disjoint union

$$S^3 = (\mathbb{D} \times S^1) \sqcup (S^1 \times \mathbb{D}) \sqcup \mathbb{T}^2, \quad (4.4.4)$$

where the first solid torus $\mathbb{D} \times S^1$ corresponds to $|y| = 1$ and $|\bar{y}| < 1$, the second solid torus corresponds to $|\bar{y}| = 1$ and $|y| < 1$, while the torus $\mathbb{T}^2 = S^1 \times S^1$ corresponds to $|y| = |\bar{y}| = 1$. We have shown that the conformal block expansion in either s - or t -channel converges in the sense of distributions on each of these boundary components.

Let us focus on the component $\mathbb{T}^2 = S^1 \times S^1$. Our results imply that the functionals α_f of the form

$$g(y, \bar{y}) \mapsto \alpha_f[g] \equiv \int_0^{2\pi} \int_0^{2\pi} d\theta d\bar{\theta} f(\theta, \bar{\theta}) g(y = e^{i\theta}, \bar{y} = e^{i\bar{\theta}}) \quad (4.4.5)$$

where $f(\theta, \bar{\theta})$ is a smooth function, satisfy the swapping property. As in section 3.6, by taking f to be the Cauchy kernel

$$f_{m, n; y_0, \bar{y}_0}(\theta, \bar{\theta}) = \frac{m!n!}{(2\pi)^2} \frac{e^{i\theta}}{(e^{i\theta} - y_0)^{m+1}} \frac{e^{i\bar{\theta}}}{(e^{i\bar{\theta}} - \bar{y}_0)^{n+1}}, \quad (4.4.6)$$

we can reproduce the evaluation functionals $\alpha_{m, n; y_0, \bar{y}_0}$

$$g(y, \bar{y}) \mapsto \alpha_{m, n; y_0, \bar{y}_0}[g] \equiv \partial_y^m \partial_{\bar{y}}^n g(y_0, \bar{y}_0). \quad (4.4.7)$$

We can again ask about the space of functions f for which the functional α_f satisfies the swapping property and try to see if this space is large enough to incorporate the functionals that are useful in analytic conformal bootstrap. Just as in section 3.6, we leave these questions for future work.

4.5 Spinning operators

Another natural generalization available in higher dimensions is to operators with spin. In cross-ratio space this question is somewhat non-canonical due to the freedom of choosing the tensor structures for spin indices, which is similar to the freedom of selecting the prefactor in (4.1.1). Nevertheless, it is clear that for reasonable choices of the basis of tensor structures, the four-point functions of spinning operators should satisfy similar power-law bounds in cross-ratio space. For example, one could use equation (4.1.4) with ϕ_i replaced by plane-rotation eigencomponents of some spinning operators \mathcal{O}_i , and the arguments of sections 4.1 and 4.2 would still go through. This would correspond to using the ‘‘conformal frame’’ basis of four-point structures [51], which is related to all reasonable

choices of tensor structures by matrices which themselves satisfy power-law bounds.⁶ While it would be a good exercise to explicitly repeat our arguments in the case of spinning correlators, we do not do it in this part of the thesis for the sake of space.

4.6 Single-variable dispersion relation for the four-point function in $d \geq 2$

This section generalizes section 3.7 to the case of two cross-ratios z, \bar{z} . Ref. [42] presented a single-variable dispersion relation recovering the four-point function in terms of its discontinuities. We will state their story in our language, clarifying some issues. Consider the four-point function satisfying the crossing equation

$$F(z, \bar{z}) = F(1 - z, 1 - \bar{z}) = (z\bar{z})^{-\Delta_\phi} F(1/z, 1/\bar{z}), \quad (4.6.1)$$

where $F(z, \bar{z}) = (z\bar{z})^{-\Delta_\phi} g(z, \bar{z})$ and the third equation corresponds to the u -channel. This channel representation does not exist for a general 1d four-point function considered in section 3.5.

Ref. [42] considers a dispersion relation for the function $F(z, \bar{z})$ using the discontinuity w.r.t. z and keeping \bar{z} fixed. In our language this dispersion relation would be written in the form

$$F(z, \bar{z}) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dx}{z' - z} \text{Disc}_{z'} F(z', \bar{z}) \quad (4.6.2)$$

where the discontinuity has to be understood in a distributional sense, including the contribution at infinity, as discussed in section 3.7.

Then the question arises how to compute the discontinuity. There are three cases: $-\infty < z' \leq 0$, $1 \leq z' < +\infty$, and $z' = \infty$. In the first case we can use the s -channel conformal block decomposition, which converges in the sense of distributions (in fact in ordinary sense for $z' < 0$). The discontinuity at $z' \geq 1$ is reduced to the one at $z' \leq 0$ via the first crossing equation in (4.6.1).⁷

One can try to fix the contribution at infinity using the u -channel conformal block expansion, which determines the behavior of the correlator at $z' = \infty$. Let us assume that

$$F(z', \bar{z}) = 1 + O((z')^{-\tau/2}). \quad (4.6.3)$$

Ref. [42] argued this by appealing to the second crossing relation in (4.6.1), expanding $F(1/z, 1/\bar{z})$ in conformal blocks, keeping only the unit operator and dropping all the other operators which seem to be naively suppressed by $(1/z)^\tau$ where $\tau = \min(\Delta - \ell)$ is the minimal twist, assumed positive. This reasoning includes a subtlety, see below. But assuming (4.6.3) we can argue that, in the language of section 3.7,

$$\text{Disc}_z F(z, \bar{z}) = (\text{Disc } 1)(z) + \text{Disc}'_z F(z, \bar{z}), \quad (4.6.4)$$

where $\text{Disc } 1$ was computed in section 3.7 and $\text{Disc}'_z F(z, \bar{z})$ is a distribution that is represented near $z = \infty$ by an ordinary function. In other words, Disc' is the discontinuity “without the contribution at ∞ .”⁸

⁶There is a subtlety for $z = \bar{z}$, in which case the transition matrices to/from conformal frame basis become singular. These singularities are canceled by special conditions satisfied by four-point functions in conformal frame basis near this locus (see appendix A of [51] and appendix D of [52]).

⁷It is also possible to compute the discontinuity at $z \geq 1$ by summing the s -channel conformal block expansion since by our results it converges on this cut in the sense of distributions. Ref. [42] mentions this result in footnote 1, attributing it to Mack [26]. This is not correct: Mack’s paper studies distributional convergence of OPE expansion in position space, not in the cross-ratio space as needed here. In due fairness, footnote 1 is not central for [42], being only used in section 4.2.2.

We note in passing that Mack [26] relied on validity of Wightman axioms and rather non-trivial representation theory. It is only in [40, 41] that we will show, for the first time, how some of Mack’s assumptions follow from more mundane Euclidean CFT rules. In comparison, our arguments here are very elementary and rely only on the well-established properties of the conformal block expansion.

⁸Note, however, that we can only unambiguously define such discontinuity because of (4.6.3). For example, this is not possible for $\log z$ example from section 3.7.

Using this decomposition of $\text{Disc}_z F(z, \bar{z})$, one obtains from (4.6.2) a dispersion relation in the form given by [42]

$$F(z, \bar{z}) = 1 + \left(\frac{1}{2\pi i} \int_{-\infty}^0 \frac{dz'}{z' - z} \text{Disc}'_{z'} F(z', \bar{z}) + (z, \bar{z} \rightarrow 1 - z, 1 - \bar{z}) \right), \quad (4.6.5)$$

where, as mentioned above, the discontinuity Disc' does not include the contribution at infinity that is instead explicitly included as “1+”, and we used crossing symmetry to account for discontinuity on the cut $[1, +\infty)$.

Note that independently of the assumption (4.6.3), our results imply that $\text{Disc}_z F(z, \bar{z})$ can be computed term-by-term in conformal block expansion (including the contribution at infinity), and then used in (4.6.2), although it is not guaranteed that the decomposition (4.6.5) exists in that case.

Let us now discuss the subtlety in the asymptotics (4.6.3). Upon a closer look, this asymptotic is only justified provided that z and \bar{z} belong to the different halfplanes of the region \mathcal{C}^{st} , i.e. if $\text{Im } z$ and $\text{Im } \bar{z}$ have opposite sign. This is because the u -channel conformal block expansion stops converging when z crosses the cut $(0, 1)$ and moves into the same half-plane as \bar{z} . Thus, if \bar{z} is fixed, asymptotics (4.6.3) is rigorously true only on one of the two arcs at infinity z . The asymptotics on the second arc is somewhat similar to the Regge limit asymptotics, in the sense that $1/z$ goes through the s -channel cut and then is sent to zero (while, unlike in the Regge limit, \bar{z} stays fixed).

There are two ways around this difficulty. One way is to take $\bar{z} \in (0, 1)$ real. Then, by our results, the u -channel OPE expansion converges in the sense of distributions on both arcs. In this case the asymptotics (4.6.3) is true provided that the error term is understood in the sense of distributions, and it goes to zero as $z \rightarrow \infty$. Since a zero distribution is a zero function, we recover the dispersion relation (4.6.5).

The second way around the difficulty is to apply the dispersion relation in perturbation theory around a mean field theory, which was in fact the main focus of [42]. In their case the zeroth order term satisfies the asymptotics (4.6.3) by inspection, while perturbative corrections have an even better behavior. The use of dispersion relation in such a limited context is justified.

Chapter 5

Conclusions

In this part of the thesis we studied the properties of the conformal block expansion on the boundary of its region of convergence. We showed that both the correlation functions and conformal blocks can be interpreted as distributions on this boundary, and that the conformal block expansion converges in the space of distributions. We have proven these results in one- and higher-dimensional cases for correlators of scalar operators, but the extension to general spinning four-point functions is straightforward.

An important feature of our analysis is that we did not rely on anything but the modern Euclidean bootstrap axioms. Specifically, we essentially only used the reality properties of OPE coefficients and the usual convergence properties of the conformal block expansion. There is a growing consensus that the Euclidean bootstrap axioms provide a good conceptual and practical definition for CFTs. Their conceptual appeal is due to them being rooted in cutting-and-gluing properties of Euclidean path integrals, which is a natural expected consequence of locality. The practical utility of these axioms has been demonstrated by the numerical conformal bootstrap studies, which have yielded extremely precise values of critical exponents and other parameters in various strongly-coupled CFTs such as the 3d Ising CFT and the $O(2)$ model (see [11, 14] for the most precise determination to date). These values are in agreement with a plethora of other completely independent methods (most notably Monte Carlo simulations and the ϵ -expansion).

Our results are important for understanding the nature of conformal correlation functions in Lorentzian signature. Indeed, as we show in appendix A.1, the best one can guarantee in general configurations in Lorentzian signature is that the conformal cross-ratios are on the boundary of the region of convergence for one of the OPE channels. It is thus important to understand the value of CFT four-point functions on this boundary. We have shown that the conformal block expansion converges there in distributional sense, which gives a practical way for computing correlation functions. For example, we can now imagine collecting numerical OPE data for 3d Ising CFT as in [53] and using it to compute pairings of the boundary value of $\langle\sigma\sigma\sigma\sigma\rangle$ four-point function with various test functions.

One important byproduct of our results, which we discuss in section 3.6, is a hint at a uniform description of the space of functionals with which we can probe the crossing equation. Starting with numerical conformal bootstrap [6], it has become standard to disprove the existence (under certain spectral assumptions) of solutions to the crossing equation by exhibiting functionals that separate the left-hand side of the crossing equation from the right-hand side. In numerical bootstrap (see [15] for review) these functionals are finite combinations of evaluation functionals $\alpha_{n,y}$ (3.6.7), while in more recent analytical functional bootstrap [32–39] the appropriate functionals are given by contour integrals $\alpha_{h,\Gamma}$ (3.6.10). Having a uniform description of a sufficiently large class $\mathcal{B}_{\Delta_\phi}$ of functionals (that in particular would include $\alpha_{n,y}$ and $\alpha_{h,\Gamma}$) would allow us to formulate and hopefully answer some interesting conceptual questions. For example,

- is it true that for any spectral assumption for which there is no solution to crossing equation there exists a functional in $\mathcal{B}_{\Delta_\phi}$ that disproves the existence of a solution?
- Is it true that when the spectral assumption is not “extremal,” this functional can be taken as a finite linear combination of evaluation functionals? (In other words, is numerical conformal bootstrap complete?)
- When the spectral assumption is extremal, is it true that there exists a unique extremal functional?

Most practitioners would probably guess that the answer to these three questions should be “yes”, “yes” and “generically yes”. To put this intuition on firm footing we need first of all understand better the space $\mathcal{B}_{\Delta_\phi}$ and the appropriate topology on this space. Answering these questions will be important for advancing our analytical understanding of conformal bootstrap.

Part II

Distributions in CFT II: Minkowski space

Chapter 6

Introduction

Quantum Field Theory (QFT) can be studied via constructive or axiomatic approaches. The former use microscopic formulations, while the latter rely on axioms. There are many constructive approaches, e.g. using Hamiltonian, path integral, lattice, etc. There are also many axiomatic approaches, corresponding to various sets of axioms (Wightman [24], Osterwalder-Schrader [1, 2], Haag-Kastler [54, 55], etc.). Historically, axiomatic approaches played an important role in clarifying general QFT properties, but they did not have a tremendous success in making predictions about concrete theories in $d > 2$ dimensions.¹ This started to change recently, with the revival of the bootstrap philosophy [6]. Our focus here will be on conformal field theories (CFTs) in dimension $d \geq 2$, i.e. QFTs invariant under the action of conformal group, which are nowadays studied via the *conformal bootstrap*. This axiomatic approach led to precise determinations of many experimentally measurable quantities, such as the critical exponents of the 3d Ising [7–11], $O(N)$ [11–14, 57] and other critical points (see review [15]).²

The numerical conformal bootstrap relies on the “Euclidean CFT axioms”,³ which specify properties of correlation functions in any unitary CFT in \mathbb{R}^d via a set of simple and commonly accepted rules, such as the unitarity bounds on primary operator dimensions, conformally invariant and convergent Operator Product Expansion (OPE), and reality constraints on OPE coefficients.

On the other hand, correlation functions in a general unitary QFT (and in particular in a CFT) should satisfy Osterwalder-Schrader (OS) and Wightman axioms. It is then interesting and important to inquire what is the relation of Euclidean CFT axioms to these other sets of axioms.⁴ To carry out this analysis will be the main goal of this thesis. Our main conclusion will be that the Euclidean CFT axioms imply OS axioms and Wightman axioms for 2, 3 and 4-point functions. In this thesis we only consider bosonic operators.

The relation of Euclidean CFT and OS axioms is perhaps not so surprising since they both deal with the Euclidean correlation functions. It is more interesting that we are able to construct Minkowski n -point functions (for $n = 2, 3, 4$), and show that they satisfy Wightman axioms, such as temperedness, spectral condition, and unitarity. Temperedness (being a tempered distribution) is a crucial property of Minkowski correlation functions: it shows that in a certain averaged sense they are meaningful everywhere including the light-cone and double light-cone singularities. One might be tempted to think that in CFT this question is relatively trivial due to OPE. However, as discussed in part I and part III, already for 4-point functions there exist causal configurations of points in Minkowski space, away from the null cones, for which no OPE channel is convergent in the conventional sense. We briefly discuss one such example in the conclusions (chapter 15).

A theorem of Osterwalder and Schrader [1, 2] says that, under some extra assumption, OS axioms imply Wightman axioms. Unfortunately this extra assumption, the so called “linear growth condition”, which involves the Euclidean n -point functions with arbitrarily high n , appears impossible to verify from the Euclidean CFT axioms (see chapter 14). For this reason we cannot appeal to the OS theorem, and we will give an independent derivation of Wightman

¹In $d = 2$ significant progress has been achieved axiomatically for massive integrable models using the S-matrix bootstrap [56] as well as for rational CFTs [4].

²There is also an ongoing revival of the S-matrix bootstrap applicable to nonintegrable massive QFTs in $d \geq 2$ [58–72].

³The term “Euclidean bootstrap axioms” is also sometimes used.

⁴Clarifying the relation to the Haag-Kastler axioms appears more challenging as those axioms do not deal with correlation functions but with operator algebras.

axioms for CFT correlators.

The study of distributional properties of CFT correlators started in part I. There, we considered expansions of the CFT 4-point function $g(\rho, \bar{\rho})$ in terms of conformally invariant cross-ratios $\rho, \bar{\rho}$. While such expansions converge in the usual sense for $|\rho|, |\bar{\rho}| < 1$, in part I we showed that they also converge for $|\rho|, |\bar{\rho}| = 1$ in the sense of distributions. As explained in part I, results of this sort follow naturally from the theory of functions of several complex variables (namely Vladimirov’s theorem), given some a priori information about the growth of the analytically continued correlator. That key insight of part I, “*Look for a powerlaw bound!*”, will be transported here from the cross-ratio to the position space.

The readers interested in our main technical result—analytic continuation of a scalar Euclidean CFT 4-point function to the forward tube and showing that the Minkowski 4-point function is a tempered distribution—may follow the **fast track**: start with the executive summary in Sec. 6.1, proceed to Secs. 9 and 10 (skipping 9.1 and 10.1), then continue with Secs. 11.1-11.5 (optionally including chapter 12) and finish with the discussion in chapter 15. This is only about 20-25 pages.

On the other hand, we made an effort to make the exposition self-contained and to review main ideas and results of the axiomatic quantum field literature, directly or tangentially related to our discussion. This explains the great total length of our work. The reader will find here:

- A review of classic QFT axioms: Wightman (Sec. 7.1), OS (Sec. 7.2). A review of main implications among these axioms: how OS reflection positivity robustly implies Wightman positivity (Sec. 11.7). A review of the Osterwalder-Schrader theorem about how OS axioms imply Wightman axioms under the additional assumptions of the linear growth condition (chapter 14).
- A formulation of ‘Euclidean CFT axioms’ for unitary CFT in Euclidean space \mathbb{R}^d (Sec. 7.3). We consider bosonic fields in arbitrary tensor representations. Our axioms encode in a consistent and non-redundant manner the main properties used in the numerical conformal bootstrap calculations.⁵ They are applicable to any globally conformally invariant theory in $d \geq 2$. We do not include the axioms involving the local stress tensor and the conserved currents. In particular our axioms would be too weak (but valid) when applied to local 2d CFTs, as they know nothing about the Virasoro algebra.⁶ See Remark 7.3.1 for a comparison between our axioms and the CFT rules gathered in the conformal bootstrap review [15].
- A derivation of OS axioms from Euclidean CFT axioms for 4-point function (chapter 8). A notable result is a rigorous proof that the state produced by two operators in lower Euclidean half-space belongs to the CFT Hilbert space generated by single-point operator insertions. The higher-point case is discussed in App. B.2, where we need a somewhat stronger form of the OPE axiom than in Sec. 7.3.
- A derivation of Wightman axioms from Euclidean CFT axioms for scalar 4-point functions (chapter 11). As mentioned above, this is the main technical result of this part of the thesis. The key observation is that the analytic continuation from Euclidean to Minkowski can be done in a way which keeps the s-channel $\rho, \bar{\rho}$ less than 1 in absolute value along the continuation contour. When we take the Minkowski limit $|\rho|, |\bar{\rho}|$ stay less than 1 for some causal orderings and approaches 1 for others (see part III for a classification) but even if $|\rho|, |\bar{\rho}| \rightarrow 1$ they approach this limit sufficiently slowly (“powerlaw bound”) which guarantees that the Minkowski 4-point function is a tempered distribution everywhere. E.g. using this argument we can show for the first time that the CFT 4-point function is a tempered distribution on the double light-cone singularity.
- We include also a derivation of other expected properties of Minkowski 4-point functions, such as conformal invariance, unitarity, clustering, and local commutativity (Secs. 11.6-11.9). The reader may find it curious how some of the steps do not use conformal invariance at all but follow simply from analyticity and/or OS positivity.
- chapter 12 proves a curious geometric “Cauchy-Schwarz” inequality for $\rho, \bar{\rho}$ variables which provides an alternative way of understanding why $|\rho|, |\bar{\rho}| < 1$ in the forward tube. It bounds $\rho, \bar{\rho}$ for a generic configuration

⁵See also [75] for a recent informal exposition of Euclidean CFT axioms (incomplete as it omits tensor fields) for mathematical physics audience. Ref. [76] attempted the axiomatization of Euclidean CFT in $d > 2$ dimensions similar to Segal’s axioms in $d = 2$ [77]. It is not immediately obvious if the axioms of Ref. [76] are equivalent to ours, or how to connect them to practical CFT calculations.

⁶Recall that while in $d = 2$ assuming the existence of a local stress tensor immediately implies Virasoro symmetry, no such dramatic statements are currently known in $d > 2$.

by $\rho, \bar{\rho}$ for reflection-symmetric configurations. It would be interesting to find an elementary proof of this inequality (our proof uses some facts about conformal blocks).

- chapter 13 shows that the (s-channel) conformal block expansion of 4-point Wightman functions converges in the sense of distributions for all configurations of points in Minkowski space. It is also shown that the OPE for the state-valued distributions $|\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$ with $x_1, x_2 \in \mathbb{R}^{d-1,1}$ converges in the sense of distributions. We discuss the relationship of these results to the classic work of Mack [26] and prove estimates for the convergence rates of these expansions.
- chapter 14 contains a review of the papers [1, 2] by Osterwalder and Schrader. In particular, we discuss the gap in the arguments of [1] which precludes the derivation of Wightman axioms from the OS axioms of [1], and explain in detail how this gap is filled in [2] with the addition of new axioms.
- App. B.1 is a guide to the modern Lorentzian CFT literature: conformal collider bounds, light-cone bootstrap, causality constraints, the Lorentzian OPE inversion formula, light-ray operators, etc. Our results help put some of these considerations on a firmer footing. We indicate the most critical remaining steps, which still wait to be rigorously derived from the Euclidean CFT axioms.

We conclude in Sec. 15. Some additional technical details are given in Apps. B.2-B.4.

6.1 Executive summary of results for CFT experts

This part of the thesis is rather lengthy as a result of our attempt to make it self-contained. In this section we give a brief summary of the main technical results, aimed at the more expert readers who may not wish to read the expository parts of this part. Note, however, that here we omit many secondary results, some of which are mentioned above.

The basic question we address in this part is the question of the distributional properties of Wightman 4-point functions in CFTs. As is well-known, Wightman n -point distributions are recovered from the boundary values of functions holomorphic in the forward tube \mathcal{T}_n . For an n -point function

$$\langle 0 | \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | 0 \rangle \quad (6.1.1)$$

the forward tube is defined as the set of $x_i \in \mathbb{C}^{1,d-1}$ subject to

$$\text{Im } x_1 \prec \text{Im } x_2 \prec \cdots \prec \text{Im } x_n, \quad (6.1.2)$$

where \prec denotes the causal ordering in $\mathbb{R}^{1,d-1}$. Analyticity in \mathcal{T}_n and existence of the boundary value as $\text{Im } x_i \rightarrow 0$ is usually derived from Wightman or OS axioms (with extra assumptions in the latter case). In this part of the thesis we want to understand this question from the point of view of CFT axioms.

With the cases $n = 2, 3$ being relatively trivial in a CFT, our main observation is that a particular OPE channel for 4-point functions converges everywhere in the interior of \mathcal{T}_4 . Specifically, we take the OPE $\mathcal{O}(x_1) \times \mathcal{O}(x_2)$ in the Wightman function

$$\langle 0 | \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) | 0 \rangle. \quad (6.1.3)$$

This OPE is expected to converge, at least distributionally for real x_i , from the results of Mack [26]. However, his work assumes Wightman axioms from the beginning, and our goal here is to clarify the implications of Euclidean CFT axioms, which only assume convergence of the Euclidean OPE.

To see that this OPE channel converges, we show in Lemma 11.3.1 that for any configuration of x_i in \mathcal{T}_4 the radial cross-ratios ρ and $\bar{\rho}$ for this OPE belong to the open unit disk,

$$|\rho|, |\bar{\rho}| < 1. \quad (6.1.4)$$

This implies convergence of the conformal block expansion in $\mathcal{O}(x_1) \times \mathcal{O}(x_2)$ channel in the interior of \mathcal{T}_4 . A technical way to see this is to note that the expansion in descendants

$$g(\rho, \bar{\rho}) = \sum_{h, \bar{h} > 0} p_{h, \bar{h}} \rho^h \bar{\rho}^{\bar{h}}, \quad (6.1.5)$$

where g is the conformally-invariant part of the 4-point function and $p_{h,\bar{h}} > 0$, can be bounded term-by-term by

$$|p_{h,\bar{h}}\rho^h\bar{\rho}^{\bar{h}}| \leq p_{h,\bar{h}}r^hr^{\bar{h}}, \quad (6.1.6)$$

where $r = \max(|\rho|, |\bar{\rho}|) < 1$. The right-hand side of this inequality is a term in the expansion of $g(r, r)$, a Euclidean configuration in which the OPE is known to converge, so (6.1.5) is dominated by a convergent series. Therefore, (6.1.5) is convergent for $r < 1$, and moreover uniformly so on compact subsets, since each term $p_{h,\bar{h}}r^hr^{\bar{h}}$ is monotonic. We can then conclude that the sum $g(\rho, \bar{\rho})$ is a holomorphic function.

This reasoning also gives us the inequality

$$|g(\rho, \bar{\rho})| \leq g(r, r). \quad (6.1.7)$$

So, we find that the correlator can be recovered inside of \mathcal{T}_4 from the $\mathcal{O}(x_1) \times \mathcal{O}(x_2)$ OPE, is analytic there, and is bounded by a Euclidean configuration.

In Sec. 11.5.1 we establish a stronger form of Lemma 11.3.1, schematically,

$$1 - r(c) \geq \text{dist}(c, \partial\mathcal{T}_4)^a \quad (6.1.8)$$

for some $a > 0$, where $c \in \mathcal{T}_4$ is a configuration of 4 points in \mathcal{T}_4 . (The more precise form also bounds $1 - r(c)$ as c goes to infinity.) This immediately implies a powerlaw bound on $g(r, r)$ near the boundary of \mathcal{T}_4 . Indeed, near $r \rightarrow 1$ the correlator is dominated by the identity in the crossed channel, and so

$$g(r(c), r(c)) \leq C(1 - r(c))^{-4\Delta_\varphi}, \quad (6.1.9)$$

and thus

$$|g(c)| \leq C \text{dist}(c, \partial\mathcal{T}_4)^b \quad (6.1.10)$$

for some real b . This allows us to use Vladimirov's Theorem 9.0.1, which implies that the boundary limit (as x_i approach real Minkowski values) of (6.1.3) exists in the space of tempered distribution. (We establish a more refined bound for $x_i \rightarrow \infty$ to claim temperedness.)

The above bounds hold just as well for the truncated conformal block expansion as for the full correlator. A variant of Theorem 9.0.1 then allows us to conclude that the conformal block expansion, while converging in the sense of functions in the interior of \mathcal{T}_4 , converges in the space of tempered distributions on the Minkowski boundary.

We extend the above results to correlators of non-identical scalars by replacing the term-by-term bound (6.1.6) with a standard Cauchy-Schwartz argument, bounding the correlator in terms of a product of two reflection-symmetric Euclidean correlators. While it is intuitively obvious that similar arguments should also work for operators with spin, we found that the extension to spinning operators, due to the complexity of tensor structures, requires enough additional work to warrant a separate paper [78].

Finally, in chapter 12 we prove Theorem 12.0.2, which gives an optimal bound of the form (6.1.8). Specifically, it is

$$r(c)^2 \leq r(c_{12})r(c_{34}), \quad (6.1.11)$$

together with a bound for the right-hand side. Here, if $c = (x_1, x_2, x_3, x_4)$ (where x_i are real in Minkowski space), then $c_{12} \equiv (x_1, x_2, x_2^*, x_1^*)$ and $c_{34} \equiv (x_4^*, x_3^*, x_3, x_4)$. The bound for the right-hand side is easier to obtain because the configurations c_{12} and c_{34} are reflection symmetric. This is done in Sec. 12.2. The bound (6.1.11) looks like a Cauchy-Schwartz-type inequality, and is indeed derived from the Cauchy-Schwartz inequality for unitary conformal blocks (12.3.6). The latter is true because of the unitarity of conformal representations corresponding to these blocks. In the limit $\Delta + \ell \rightarrow \infty$, $\Delta - \ell$ fixed, conformal blocks are dominated by $r^{(\Delta+\ell)/2}$, which reduces the conformal block Cauchy-Schwartz inequality to (6.1.11).

Chapter 7

Axioms

7.1 Wightman axioms

In this section we will state the properties of Wightman correlation functions in a unitary QFT, to which we will refer here as “Wightman axioms.” These axioms appear as “properties of the vacuum expectation values” in [24], Sec. 3-3, and as (W1)-(W5) in [79], Theorem II.1. Refs. [24, 79] give in addition another set of axioms (called Gårding-Wightman axioms in [79]) saying that fields are operator-valued distributions in the Hilbert space on which the Lorentz group acts, etc. This other set of axioms will not be used in this thesis. In any case, the Wightman reconstruction theorem [24] says that the two sets of axioms are equivalent.

A unitary QFT in Minkowski space studies n -point correlators

$$\langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle, \quad (7.1.1)$$

(Wightman functions) of local operators $\varphi_i(x)$, $x \in \mathbb{R}^{1,d-1}$. For simplicity in this part of the thesis we will only consider bosonic operators, although more generally one should allow fermionic operators and spinor representations. Wightman functions are translation and $\text{SO}(1, d-1)$ invariant. We will choose a basis of local operators \mathcal{O}_i transforming in irreducible $\text{SO}(1, d-1)$ representations ρ_i . Then, Wightman functions remain invariant when

$$\mathcal{O}_i^\alpha(x) \rightarrow \rho_i(g)^\alpha_\beta \mathcal{O}_i^\beta(g^{-1}x), \quad (7.1.2)$$

where $g \in \text{SO}(1, d-1)$, and $\rho_i(g)$ are finite-dimensional matrices of the representation ρ_i ($\alpha, \beta = 1 \dots \dim \rho_i$). Let \mathcal{C} be the complex vector space whose elements are arbitrary components of \mathcal{O}_i 's, and their *finite* linear combinations with constant complex coefficients. Operators φ_i in (7.1.1) can be arbitrary elements of \mathcal{C} , and Wightman function (7.1.1) is multi-linear in φ_i . Note that in this and the next section derivatives of local operators (of any order) are counted as independent operators, while in the CFT Sec. 7.3 we will start making distinction between primaries and their derivatives.

Wightman functions (7.1.1) are required to be tempered distributions, i.e. can be paired with Schwartz class test functions $f(x_1, \dots, x_n)$. For this reason they are sometimes referred to as “Wightman distributions”. Note that the test functions $f(x_1, \dots, x_n)$ with which Wightman functions are paired do not have to vanish at coincident points (unlike for the Schwinger functions discussion in the next section). This means that, in a distributional sense, Wightman functions have meaning for all configurations, including coincident points and light-cone singularities. Translation and Lorentz invariance of Wightman functions are also understood not pointwise but in the sense of distributions (i.e. that the pairing should remain invariant if the test function is transformed in the dual way).¹

We will not consider here other Minkowski correlators, such as retarded, advanced, or time ordered, which are obtained from Wightman functions multiplying by theta-functions of time coordinate differences, and whose distributional properties require a separate discussion.

¹Although Wightman functions can be shown to be real-analytic at some totally spacelike-separated configurations (Jost points), in general they may be singular even away from light cones (in particular when there are timelike separations).

Limiting to the bosonic case as we are, **local commutativity** (also called microcausality) morally says that operators commute at spacelike separation. Wightman axioms impose this as a constraint on Wightman functions:

$$\langle \varphi_1(x_1) \dots \varphi_p(x_p) \varphi_{p+1}(x_{p+1}) \dots \varphi_n(x_n) \rangle = \langle \varphi_1(x_1) \dots \varphi_{p+1}(x_{p+1}) \varphi_p(x_p) \dots \varphi_n(x_n) \rangle \quad (7.1.3)$$

whenever $x_p - x_{p+1}$ is spacelike: $(x_p - x_{p+1})^2 > 0$. Since we are talking about distributions, this constraint means that (7.1.3) holds when paired with any test function whose support is contained in $(x_p - x_{p+1})^2 > 0$.

Clustering says that correlators should factorize if two groups of points are far separated in a spacelike direction:

$$\langle \varphi_1(x_1) \dots \varphi_p(x_p) \varphi_{p+1}(x_{p+1} + \lambda a) \dots \varphi_n(x_n + \lambda a) \rangle \rightarrow \langle \varphi_1(x_1) \dots \varphi_p(x_p) \rangle \langle \varphi_{p+1}(x_{p+1}) \dots \varphi_n(x_n) \rangle \quad (7.1.4)$$

as $\lambda \rightarrow \infty$ for any spacelike vector a , limit understood in the sense of distributions.

We next discuss the spectral condition. By translation invariance we can write

$$\langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle = W(\xi_1, \dots, \xi_{n-1}), \quad \xi_k = x_k - x_{k+1}, \quad (7.1.5)$$

where W is a tempered distribution in one less variable. Consider its Fourier transform:

$$\widehat{W}(q_1, \dots, q_{n-1}) = \int W(\xi_1, \dots, \xi_{n-1}) e^{i \sum_{k=1}^{n-1} q_k \cdot \xi_k} d\xi_1 \dots d\xi_n, \quad (7.1.6)$$

where $q_k = (E_k, \mathbf{q}_k)$, $\xi_k = (t_k, \boldsymbol{\xi}_k)$, $q_k \cdot \xi_k = -E_k t_k + \mathbf{q}_k \cdot \boldsymbol{\xi}_k$. Since W is a tempered distribution, the Fourier transform \widehat{W} is well defined and is also a tempered distribution. The **spectral condition** then says that \widehat{W} must be supported in the product of closed forward light cones, i.e. in the region

$$E_k \geq \mathbf{q}_k, \quad k = 1, 2, \dots, n-1. \quad (7.1.7)$$

For the two remaining conditions we need to discuss conjugation. Physically, each operator φ should have a conjugate φ^\dagger . In the discussed framework we cannot define φ^\dagger as an adjoint of an operator acting on a Hilbert space, since we do not have a Hilbert space. Instead, we will simply assume that there is a rule which associates φ^\dagger to φ , and impose the expected relations at the level of correlation functions (Eq. (7.1.10) below). This rule, conjugation map $\dagger : \mathcal{C} \rightarrow \mathcal{C}$, associates to each independent component \mathcal{O}_i^α ($\alpha = 1 \dots \dim \rho_i$) of the above-mentioned basis of \mathcal{C} a conjugate operator $(\mathcal{O}_i^\alpha)^\dagger$. This map is required to be an involution, i.e. $\dagger \dagger = 1$. Furthermore, it is extended to the whole of \mathcal{C} by anti-linearity, i.e. $(c_1 \varphi_1 + c_2 \varphi_2)^\dagger = c_1^* \varphi_1^\dagger + c_2^* \varphi_2^\dagger$.²

Let us group operators $(\mathcal{O}_i^\alpha)^\dagger$ in a multiplet which we denote by \mathcal{O}_i^\dagger , i.e. $(\mathcal{O}_i^\dagger)^\alpha = (\mathcal{O}_i^\alpha)^\dagger$. We will see below that $(\mathcal{O}_i^\dagger)^\alpha$ transform under $g \in \text{SO}(1, d-1)$ with matrices complex-conjugate to those of \mathcal{O}_i^α :

$$\mathcal{O}_i^\alpha \rightarrow \rho_i(g)^\alpha_\beta \mathcal{O}_i^\beta \quad \Rightarrow \quad (\mathcal{O}_i^\dagger)^\alpha \rightarrow \overline{\rho_i(g)^\alpha_\beta} (\mathcal{O}_i^\dagger)^\beta. \quad (7.1.8)$$

In other words, \mathcal{O}_i^\dagger transforms in the conjugate representation $\overline{\rho_i}$.

Since we are considering only bosonic operators, the relevant representations ρ_i are tensors $T^{\mu_1 \dots \mu_l}$, on which $g \in \text{SO}(1, d-1)$ act as:

$$T^{\mu_1 \dots \mu_l} \rightarrow (\rho_i(g) T)^{\mu_1 \dots \mu_l} = g^{\mu_1}_{\nu_1} \dots g^{\mu_l}_{\nu_l} T^{\nu_1 \dots \nu_l}. \quad (7.1.9)$$

Depending on ρ_i , these tensors have some fixed rank and mixed symmetry properties. In addition, in even d , for tensors with $d/2$ antisymmetric indices, (anti-)chirality³ constraints must be imposed. All tensor representations of $\text{SO}(1, d-1)$ are real (i.e. matrices $\rho_i(g)^\alpha_\beta$ in (7.1.8) can be chosen real), except for (anti-)chiral representations in $d = 0 \bmod 4$ which are complex-conjugate to each other. For operators in real representations we can choose a basis such that $\mathcal{O}_i = \mathcal{O}_i^\dagger$.

After this intermezzo we are ready to formulate hermiticity and positivity conditions. **Hermiticity** says that complex conjugate correlators equal correlators of conjugated operators in inverted order:

$$\overline{\langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle} = \langle \varphi_n^\dagger(x_n) \dots \varphi_1^\dagger(x_1) \rangle. \quad (7.1.10)$$

²The \dagger operation is denoted by $*$ in [24].

³Chiral and anti-chiral representations are sometimes also called “self-dual” and “anti-self-dual”. We use “chiral” and “anti-chiral” to avoid the clash with “dual representation” in mathematician’s sense.

This would be true of course if φ 's were operators acting on a Hilbert space, with φ^\dagger 's their adjoints. In the present framework without Hilbert space it is imposed as an axiom. This axiom implies in particular (7.1.8), i.e. that \mathcal{O}_i^\dagger transforms in the conjugate irrep $\bar{\rho}_i$.

The last Wightman axiom, **positivity**, is most conveniently written down using the language of states. One considers basic ket states $|\psi(f, \varphi_1, \dots, \varphi_n)\rangle$, associated with n local operators $\varphi_1, \dots, \varphi_n \in V$ and a complex Schwartz test function of n variables f . One defines the inner product on basic ket states by

$$\begin{aligned} \langle \psi(g, \chi_1, \dots, \chi_m) | \psi(f, \varphi_1, \dots, \varphi_n) \rangle &:= \int dx dy \overline{g(x_1, \dots, x_m)} f(y_1, \dots, y_n) \\ &\times \langle \chi_m^\dagger(x_m) \dots \chi_1^\dagger(x_1) \varphi_1(y_1) \dots \varphi_n(y_n) \rangle. \end{aligned} \quad (7.1.11)$$

The vector space of ket states \mathcal{H}_0 consists of *finite* linear combinations $|\Psi\rangle$ of basic ket states, with the inner product extended to it by (anti)linearity. **Positivity** then says that the so defined inner product is positive semidefinite:

$$\langle \Psi | \Psi \rangle \geq 0 \quad \forall |\Psi\rangle \in \mathcal{H}_0. \quad (7.1.12)$$

Remark 7.1.1. A comment is in order concerning the meaning of these states. They may be seen as just a convenient notation, since Eq. (7.1.12) can be rewritten without ever using the word “state” (see [24], Eq. (3-35)). But they are more than that: the vector space of states \mathcal{H}_0 is “almost” the Hilbert space \mathcal{H} of our QFT. The only difference between \mathcal{H}_0 and \mathcal{H} is that \mathcal{H}_0 is not complete and may contain some states of zero norm. However, since \mathcal{H}_0 has a positive semidefinite inner product, as expressed by Eq. (7.1.12), we can obtain from it a Hilbert space \mathcal{H} via a standard procedure of completion and modding out by states of zero norm. This is the first step of the Wightman reconstruction theorem [24], and the resulting Hilbert space \mathcal{H} turns out to be (possibly a superselection sector of) *the* Hilbert space of the QFT, on which fields can then be realized as operator valued distributions.

Remark 7.1.2. Although we included hermiticity as a separate axiom because of its suggestive form, it can be derived from positivity, considering the states of the form $|\Psi\rangle = |\psi(f_0, 1)\rangle + |\psi(f, \varphi_1, \dots, \varphi_n)\rangle$ where $f_0 \in \mathbb{C}$ and 1 is the unit operator.

Remark 7.1.3. Another interesting positivity property of Wightman functions is called Rindler Reflection positivity, or Wedge Reflection positivity [80]. A restricted version of this property (with wedge-ordered points) can be derived from Wightman axioms, while a stronger version (no wedge-ordering) follows from Tomita-Takesaki theory which relies on Haag-Kastler axioms [80]. In CFT context this property has been discussed, e.g., in [20]. We will not discuss these properties in this part of the thesis. However, it would be interesting to see whether the stronger form of Rindler positivity (including distributional information) can be derived from CFT axioms without the appeal to Tomita-Takesaki theory (the weaker version following from our results on Wightman axioms and [80]). We believe this can be done, and it could be a nice exercise for someone wishing to master our techniques.

7.2 Osterwalder-Schrader axioms

We next describe a version of the Osterwalder and Schrader axioms [1, 2] of Euclidean unitary QFT (see the end of the section about the relation to the original OS axioms). The setup is similar to Wightman axioms with $\text{SO}(d)$ replacing $\text{SO}(1, d-1)$. We consider a basis of local bosonic operators \mathcal{O}_i transforming⁴ in $\text{SO}(d)$ irreps ρ_i , counting derivatives as independent operators. Finite linear combinations of their components span a complex vector space \mathcal{C} of local operators. The axioms specify properties of translation and $\text{SO}(d)$ invariant n -point correlators (often called Schwinger functions)

$$\langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle, \quad \varphi_i \in \mathcal{C}, \quad x_i \in \mathbb{R}^d. \quad (7.2.1)$$

These correlators are defined away from coincident points (i.e. whenever $x_i \neq x_j$ for each i, j). We will assume that⁵

$$\text{correlators are real-analytic,} \quad (7.2.2)$$

⁴In the sense of Eq. (7.1.2) where now $g \in \text{SO}(d)$.

⁵Recall that a C^∞ function of m real variables is called real-analytic in a domain $D \subset \mathbb{R}^m$ if it has a convergent Taylor series expansion in a small ball around every point of this domain. Equivalently, such a function has an analytic extension to a small open neighborhood of this domain inside \mathbb{C}^m .

and grow not faster than some power when some points approach each other or go to infinity, i.e.

$$|\langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle| \leq C \left(1 + \max_{i \neq j} \left(\frac{1}{|x_i - x_j|}, |x_i - x_j| \right) \right)^p \quad (7.2.3)$$

with some correlator-dependent positive constants C, p . Unlike Wightman axioms, OS axioms do not bother what happens *precisely* at coincident points (not even in the sense of distributions).

As we are limiting to the bosonic case, correlators remain invariant when operators are permuted:⁶

$$\langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle = \langle \varphi_{\pi(1)}(x_{\pi(1)}) \dots \varphi_{\pi(n)}(x_{\pi(n)}) \rangle. \quad (7.2.4)$$

To formulate the Euclidean version of hermiticity and positivity, we will need some simple facts about $\text{SO}(d)$ representations. Abstractly, for any irrep ρ acting $T^\alpha \rightarrow \rho(g)^\alpha_\beta T^\beta$, the conjugate representation $\bar{\rho}$ acts with complex conjugate matrices $\overline{\rho(g)^\alpha_\beta}$. Since $\text{SO}(d)$ is compact, we have $\bar{\rho} \simeq \rho^*$, the dual representation. The $\text{SO}(d)$ irreps ρ are again tensors $T^{\mu_1 \dots \mu_i}$ like in (7.1.9), of in general mixed symmetry, and with (anti)-chirality constraints if having $d/2$ antisymmetric indices in even d . All of them are real, except for chiral representations in $d = 2 \bmod 4$ which are complex-conjugate to the anti-chiral ones.⁷

We will also need the reflected representation ρ^R with matrices $\rho^R(g) = \rho(g^R)$, where $g \rightarrow g^R = \Theta g \Theta$, $\Theta = \text{diag}(-1, 1, \dots, 1)$, is an automorphism of $\text{SO}(d)$. For tensor representations, we can consider the map

$$T^{\mu_1 \dots \mu_i} \rightarrow \Theta_{\nu_1}^{\mu_1} \dots \Theta_{\nu_i}^{\mu_i} T^{\nu_1 \dots \nu_i}, \quad (7.2.5)$$

which preserves rank and mixed symmetry properties. It also maps chiral to anti-chiral tensors in any even d . Whenever the representation space is preserved, this map serves as an intertwiner between ρ^R and ρ . This means that $\rho^R \simeq \rho$ for all tensor representations without chirality constraints, while this operation interchanges chiral and antichiral irreps in any even d .⁸

Applying both conjugation and reflection we get the conjugate reflected representation $\bar{\rho}^R$ (isomorphic to dual reflected). From the above it follows that $\bar{\rho}^R \simeq \rho$ for all $\text{SO}(d)$ irreps, except for (anti)-chiral tensors in $d = 0 \bmod 4$ which are interchanged.

Just as for Wightman axioms, we will need a conjugation operation $\dagger : \mathcal{C} \rightarrow \mathcal{C}$ on the vector space of local operators, which is involutive, anti-linear, and associates to each independent component \mathcal{O}_i^α ($\alpha = 1 \dots \dim \rho_i$) a conjugate operator $(\mathcal{O}_i^\dagger)^\alpha := (\mathcal{O}_i^\alpha)^\dagger$. Then the **hermiticity** axiom takes the form⁹

$$\overline{\langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle} = \langle \varphi_n^\dagger(x_n^\theta) \dots \varphi_1^\dagger(x_1^\theta) \rangle, \quad (7.2.6)$$

similar to the Minkowski counterpart (7.1.10) but with an important difference that the operators in the r.h.s. are put at reflected positions

$$x^\theta := \Theta x. \quad (7.2.7)$$

This change has a consequence that \mathcal{O}_i^\dagger transforms in the conjugate *reflected* representation $\bar{\rho}_i^R$, explaining why we introduced this concept in the first place.¹⁰ For self-conjugate-reflected representations we may choose a basis such that

$$(\mathcal{O}_i^\dagger)^\mu = \Theta_{(\nu)}^{(\mu)} \mathcal{O}_i^\nu, \quad (7.2.8)$$

where $\Theta_{(\nu)}^{(\mu)} := \Theta_{\nu_1}^{\mu_1} \dots \Theta_{\nu_i}^{\mu_i}$ is the intertwiner (7.2.5).

To write positivity, basic ket states $|\psi(f, \varphi_1, \dots, \varphi_n)\rangle$ are associated with n local operators $\varphi_1, \dots, \varphi_n \in \mathcal{C}$ and a complex compactly supported Schwartz test function of n variables $f(x_1, \dots, x_n)$ which vanishes unless all points

⁶In particular one can sort all operators so that the Euclidean time coordinates are ordered $x_1^0 \geq x_2^0 \geq \dots \geq x_n^0$, and Euclidean correlator for any other ordering can be obtained by trivially permuting field labels.

⁷This well-known shift from $d = 0 \bmod 4$ for $\text{SO}(1, d-1)$ is induced by raising the indices of the ε -tensor. E.g. $\varepsilon^{01} \varepsilon_{10} = -1$ for $\text{SO}(2)$, while it is 1 for $\text{SO}(1, 1)$.

⁸In odd d , Θ is a product of -1 and an $\text{SO}(d)$ matrix, so that $g \rightarrow g^R$ is an inner automorphism. This provides another argument why $\rho^R \simeq \rho$ for all irreducible $\text{SO}(d)$ representations in odd d .

⁹Although we write the operators in the r.h.s. in the inverted order like in (7.1.10), permutation invariance renders this detail unimportant for the OS axioms.

¹⁰Indeed we have $\langle (\mathcal{O}_i^\dagger)^\alpha(x) \dots \rangle = \overline{\langle \mathcal{O}_i^\alpha(x^\theta) \dots \rangle} = \overline{\rho(g)^\alpha_\beta \langle \mathcal{O}_i^\beta(g^{-1}x^\theta) \dots \rangle} = \overline{\rho(g)^\alpha_\beta} \langle (\mathcal{O}_i^\dagger)^\beta((g^R)^{-1}x) \dots \rangle$.

are in the lower half space and have time variables ordered: $0 > x_1^0 > x_2^0 > \dots > x_n^0$. These support requirements were absent in the Wightman case. The inner product on the basic ket states is defined by

$$\begin{aligned} \langle \psi(g, \chi_1, \dots, \chi_m) | \psi(f, \varphi_1, \dots, \varphi_n) \rangle &:= \int dx dy \overline{g(y_1^\theta, \dots, y_m^\theta)} f(x_1, \dots, x_n) \\ &\times \langle \chi_m^\dagger(y_m) \dots \chi_1^\dagger(y_1) \varphi_1(x_1) \dots \varphi_n(x_n) \rangle, \end{aligned} \quad (7.2.9)$$

and is extended by (anti)linearity to the vector space $\mathcal{H}_0^{\text{OS}}$ of *finite* linear combinations $|\Psi\rangle$ of basic ket states. In this notation, **positivity** takes the same form as (7.1.12), i.e. that the so defined inner product must be positive semidefinite:

$$\langle \Psi | \Psi \rangle \geq 0 \quad \forall |\Psi\rangle \in \mathcal{H}_0^{\text{OS}}. \quad (7.2.10)$$

This is referred to as ‘‘OS reflection positivity’’ because of the reflected g arguments in (7.2.9), differently from the Wightman case. Because of this reflection and the above test function support requirements, all operators in (7.2.9) sit at separated positions. This is one reason why the OS axioms involve ordinary functions, without worrying about coincident points. In contrast, Wightman positivity integrates operator insertions over coincident points and makes sense only for distributions.

Just as in the Wightman case (Remark 7.1.1), we can complete the vector space $\mathcal{H}_0^{\text{OS}}$, mod out by states of zero norm, and obtain a Hilbert space \mathcal{H}^{OS} of the Euclidean theory.

Although we included hermiticity as an independent axiom, it can be derived from positivity, just as in Remark 7.1.2 in the Wightman case.

One simple consequence of OS reflection positivity is pointwise positivity of $2n$ -point functions at reflection invariant configurations of points:

$$\langle \varphi_n^\dagger(x_n^\theta) \dots \varphi_1^\dagger(x_1^\theta) \varphi_1(x_1) \dots \varphi_n(x_n) \rangle \geq 0 \quad (7.2.11)$$

for any x_1, \dots, x_n in the lower half space.¹¹ This follows from (7.2.10) by taking $|\Psi\rangle = |\psi(f, \varphi_1, \dots, \varphi_n)\rangle$ and localizing f near one configuration of points. In general, imposing (7.2.11) for all φ 's and x 's would be clearly weaker than full OS reflection positivity. E.g. (7.2.10), but not (7.2.11), can be used to bound 3-point functions in terms of 2- and 4-point functions, or non-reflection-invariant 4-point functions by reflection-invariant ones. However for CFTs we will see below that OS reflection positivity can be reduced to a form of (7.2.11) for 2-point functions plus a form of (7.2.6) for 3-point functions.

Finally, the OS **clustering** asserts that

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \int dx dy \overline{g(y_1^\theta, \dots, y_m^\theta)} f(x_1, \dots, x_n) \langle \chi_m^\dagger(y_m) \dots \chi_1^\dagger(y_1) \varphi_1(x_1 + \lambda a) \dots \varphi_n(x_n + \lambda a) \rangle \\ = \int dx dy \overline{g(y_1^\theta, \dots, y_m^\theta)} f(x_1, \dots, x_n) \langle \chi_m^\dagger(y_m) \dots \chi_1^\dagger(y_1) \rangle \langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle \end{aligned} \quad (7.2.12)$$

for any Schwartz test functions $f(x_1, \dots, x_n)$ and $g(y_1, \dots, y_m)$ supported for $0 > x_1^0 > x_2^0 > \dots > x_n^0$ and $0 > y_1^0 > y_2^0 > \dots > y_m^0$, for any local fields $\varphi_1, \dots, \varphi_n$ and χ_1, \dots, χ_m , and for any $a \in \mathbb{R}^d$ which is parallel to the x^0 plane ($a^0 = 0$). The latter requirement is somewhat analogous to having the Wightman cluster property (7.1.4) to be satisfied only for spacelike a .¹²

Note that the Minkowski operators can be mapped to Euclidean operators. In particular any $\text{SO}(1, d-1)$ irrep can be mapped to an $\text{SO}(d)$ irrep. This map of irreps originates from the map between the two Lie algebras which have the same complexification. It can then be shown that a pair of conjugate $\text{SO}(1, d-1)$ irreps is mapped to a pair of $\text{SO}(d)$ irreps which are conjugate-reflected to each other. This gives another rationale for the appearance of reflected irreps in the OS axioms.

Remark 7.2.1. The stated version of OS axioms includes the assumption of real analyticity (7.2.2) and the bound (7.2.3). These assumptions are natural from physics perspective; they also easily follow from Wightman axioms. The original OS axioms did not include (7.2.2) nor (7.2.3), but included instead a differently stated assumption:

$$\text{correlators are distributions on } {}^0\mathcal{S}, \quad (7.2.13)$$

¹¹For tensor operator in self-conjugate-reflected representations, choosing the real basis (7.2.8), this becomes $\langle \dots \Theta_{(\nu)}^{(\mu)} \mathcal{O}^{(\nu)}(x^\theta) \mathcal{O}^{(\mu)}(x) \dots \rangle \geq 0$ (no sum on μ), i.e. tensor indices are also reflected.

¹²This is axiom E4 in [1]. Ref. [1] also mentions a stronger axiom E4', but we will be content here with checking the easier axiom E4.

where ${}^0\mathcal{S}$ is the space of Schwartz test function vanishing at coincident points with all their derivatives.

We would like to discuss here the relation between (7.2.2)+(7.2.3) and (7.2.13). In one direction this is easy: clearly (7.2.3) implies (7.2.13). In the other direction it can be shown that (7.2.13) and other OS axioms (in particular OS positivity and rotation invariance) imply real analyticity (7.2.2). This is a result of [1, 2] and [31]. It is also possible to derive (7.2.3) from (7.2.13) and other OS axioms [2]. These issues will be reviewed further in chapter 14.

7.3 Euclidean CFT axioms

Wightman and OS axioms stated in the previous two sections are standard. We took care to present them for general operator representations and in general d . We will now present axioms for Euclidean unitary CFT. Just as OS axioms, these concern correlators in Euclidean signature, but there is an extra assumption of conformal invariance. Another feature of the CFT axioms is that assumptions are imposed on simple building blocks (2- and 3-point functions) from which more complicated correlators can be constructed. Properties of these complicated correlators then follow. The point of this part is how one can recover OS axioms and (after Wick rotation) Wightman axioms in this setup.

A Euclidean unitary CFT in \mathbb{R}^d ($d \geq 2$) deals with local *primary* operators $\mathcal{O}_i(x)$ and with their n -point correlation functions $\langle \mathcal{O}_{i_1}(x_1) \dots \mathcal{O}_{i_n}(x_n) \rangle$. Correlators are real-analytic functions defined away from coincident points, which are permutation-invariant as in (7.2.4). Each primary is characterized by its scaling dimension Δ_i and is an $\text{SO}(d)$ tensor transforming in an irreducible representation ρ_i .¹³ The scaling dimensions are real and nonnegative, with the unit operator having dimension zero. The set of scaling dimensions (“spectrum”) is assumed to be discrete, by which we mean that there are finitely many Δ_i ’s in any finite interval $[a, b] \subset \mathbb{R}$.¹⁴

The set of all local operators of a CFT consists of primaries $\mathcal{O}_i(x)$ and their space-time derivatives $\partial_{\mu_1} \dots \partial_{\mu_n} \mathcal{O}_i(x)$, often referred to as descendants. The correlation functions of the descendant operators are simply the derivatives of the correlation functions of primary operators. They are well-defined since the correlators of primaries are assumed to be real-analytic.

Parameters Δ_i and ρ_i determine transformation properties of $\mathcal{O}_i(x)$ under the conformal group $\text{SO}(d+1, 1)$, and correlators remain invariant under these standard transformations which we will not write down. These constraints determine the functional form of 1,2,3-point functions. In particular, the unit operator is the only one with a nonzero 1-point function. See, e.g., [15] for a review of these facts.

An important fact that follows from the conformal invariance of correlation functions is that one is allowed to insert an operator at spatial infinity. This is defined as

$$\langle \mathcal{O}_i(\infty) \dots \rangle \equiv \lim_{L \rightarrow +\infty} L^{2\Delta_i} \langle \mathcal{O}_i(L\hat{e}_0) \dots \rangle. \quad (7.3.1)$$

To see that this limit exists one can use a conformal map that takes ∞ to a finite point and moves no other operators to infinity. After applying this map the limit (7.3.1) turns into a limit in which all points approach finite values. We conclude that (7.3.1) exists, and is then of course independent of the concrete conformal map that we chose. In the definition (7.3.1) we have chosen a particular direction (\hat{e}_0) for the limit. Using conformal symmetry it is easy to show that (7.3.1) is independent of this direction, up to a rotation on the indices of \mathcal{O}_i . In what follows we will always allow for Euclidean CFT correlators to have one of the operators to be at ∞ .

For every primary \mathcal{O}_i there is a unique conjugate primary \mathcal{O}_i^\dagger (where \dagger is involutive) such that the 2-point function $\langle \mathcal{O}_i^\dagger \mathcal{O}_i \rangle$ does not vanish. The \mathcal{O}_i and \mathcal{O}_i^\dagger have equal scaling dimensions, and transform in the conjugate-reflected irreps. Recall that in Sec. 7.2 we saw that most $\text{SO}(d)$ irreps are self-conjugate-reflected, $\rho_i \simeq \bar{\rho}_i^R$, the only exception being (anti-)chiral tensors in $d = 0 \bmod 4$ which are exchanged by this operation. For operators in self-conjugate-

¹³Operators can also be grouped into multiplets of the global symmetry group G which a CFT might have, but we will not discuss global symmetry here. For simplicity we will only consider bosonic operators. More generally one should allow fermionic operators and spinor representations.

¹⁴There exist 2d unitary CFTs, such as the Liouville theory, with a continuous spectrum of scaling dimensions. In this case axioms need to be modified. All known unitary CFTs in $d \geq 3$ have a discrete spectrum.

reflected irreps we may choose operator basis such that Eq. (7.2.8) holds, which we copy here:

$$(\mathcal{O}_i^\dagger)^{(\mu)} = \Theta_{(\nu)}^{(\mu)} \mathcal{O}_i^{(\nu)}. \quad (7.3.2)$$

The functional form of the $\langle \mathcal{O}_i^\dagger(x) \mathcal{O}_i(y) \rangle$ 2-point function is fixed by conformal symmetry:

$$\langle (\mathcal{O}_i^\dagger)^{(\mu)}(x) \mathcal{O}_i^{(\nu)}(y) \rangle = \mathcal{N}_i I^{(\mu),(\nu)}(x-y), \quad (7.3.3)$$

where $(\mu), (\nu)$ are collections of tensor indices (of equal length), $I^{(\mu),(\nu)}(x-y)$ is a tensor function depending only on Δ_i, ρ_i , and \mathcal{N}_i is a constant whose phase is determined by the positivity condition discussed below. The remaining freedom to rescale \mathcal{N}_i by a positive real number is fixed in some arbitrary unimportant way, e.g. so that some component of the 2-point function is one at unit separation.

Positivity is imposed in Euclidean CFT axioms only on 2-point functions. We write it again using the language of states. Basic ket states are $|\partial^{(\beta)} \mathcal{O}_i^{(\nu)}\rangle$ where $\mathcal{O}_i^{(\nu)}$ is a primary component and $\partial^{(\beta)}$ an arbitrary derivative. The inner product is defined as

$$\langle \partial^{(\alpha)} \mathcal{O}_i^{(\mu)} | \partial^{(\beta)} \mathcal{O}_i^{(\nu)} \rangle = \Theta_{(\alpha')}^{(\alpha)} \langle \partial^{(\alpha')} (\mathcal{O}_i^\dagger)^{(\mu)}(x_N) \partial^{(\beta)} \mathcal{O}_i^{(\nu)}(x_S) \rangle, \quad (7.3.4)$$

i.e. as the value of the shown 2-point function inserting the operators at $x_S = (-1, 0, \dots, 0)$ and $x_N = (1, 0, \dots, 0) = (x_S)^\theta$ (where N,S stands for north, south). For ket states with $i \neq j$ the inner product vanishes since the 2-point function is zero. This inner product is extended by (anti)linearity to the vector space $\mathcal{H}_0^{\text{CFT}}$ of finite complex linear combinations of basic ket states. In this language, Euclidean CFT **positivity** reads exactly as the Wightman and OS positivity: $\langle \Psi | \Psi \rangle \geq 0$ for all states of this restricted form. More prosaically, this can also be stated that the infinite matrices $M_i^{(\alpha)(\mu),(\beta)(\nu)}$ built out of 2-point functions in the r.h.s. of (7.3.4) are all positive semidefinite when restricted to finite subspaces.

CFT positivity can be analyzed primary by primary, and it depends only on the primary 2-point function, Eq. (7.3.3) which determines the full matrix $M_i^{(\alpha)(\mu),(\beta)(\nu)}$. Clearly, only one phase of the normalization constant \mathcal{N}_i in Eq. (7.3.3) can give rise to a positive definite matrix, so that phase is uniquely fixed. Once the phase of \mathcal{N}_i is fixed, positivity for a given primary depends only on its Δ, ρ . It then can be shown that CFT positivity holds if and only if every Δ , in addition to being real and non-negative, lies above a certain minimal ρ -dependent value (“unitarity bound”):

$$\Delta \geq \Delta_{\min}(\rho). \quad (7.3.5)$$

These unitarity bounds are documented in the literature, e.g. we have $\Delta_{\min} = d/2 - 1$ for scalars, and $d + \ell - 2$ for spin- ℓ , $\ell \geq 1$. For arbitrary $\text{SO}(d)$ representations see [81].¹⁵

For future uses, we wish to define the CFT Hilbert space \mathcal{H}^{CFT} via completion of $\mathcal{H}_0^{\text{CFT}}$, after modding out by zero norm states (for operators saturating the unitarity bounds, some descendants have zero norm). This can be done abstractly, or explicitly using a basis as we now describe. Throwing out zero-norm descendants, the remaining states can be organized choosing an orthonormal basis. We may choose such a basis independently among descendants of each primary, and then combine all these bases, e.g. in the order of non-decreasing scaling dimensions. The elements of \mathcal{H}^{CFT} are then formal linear combinations $\sum_n c_n |n\rangle$, where $|n\rangle$ are orthonormal basis elements, and c_n is an arbitrary complex ℓ_2 sequence. The norm on \mathcal{H}^{CFT} is the ℓ_2 norm of the sequence c_n . Restricting to sequences c_n which have only a finite number of nonzero elements, we get elements of $\mathcal{H}_0^{\text{CFT}}$ (modulo the zero-norm states).

Let us continue with the axioms. CFT **hermiticity** condition is imposed only on the 2-point and 3-point functions, namely:

$$\langle (\mathcal{O}_i^\dagger)^{(\mu)}(x_1) \mathcal{O}_i^{(\nu)}(x_2) \rangle = \overline{\langle \mathcal{O}_i^{(\mu)}(x_1^\theta) (\mathcal{O}_i^\dagger)^{(\nu)}(x_2^\theta) \rangle}, \quad (7.3.6)$$

¹⁵We chose to express CFT positivity inserting operators at the points $(\pm 1, 0, \dots, 0)$ which corresponds to the N-S quantization (see [82]) and will facilitate the comparison with the Osterwalder-Schrader reflection positivity. Equivalently, one could go via a conformal transformation to the more familiar radial quantization corresponding to inserting the operators at 0 and ∞ . CFT positivity is then equivalent to radial quantization states having positive norm on every level, which is how the unitarity bounds are usually worked out in Euclidean CFTs [81]. In mathematical language, this latter condition corresponds to having a positive-definite Shapovalov form on the parabolic Verma module. Recent work [83, 84] explained how the determinant formulas by Jantzen [85] provide a rigorous justification of the Euclidean unitarity bounds (both in the necessary and sufficient directions).

which is also a consequence CFT positivity and in particular fixed the phase of \mathcal{N}_i up to a sign,¹⁶ and

$$\langle \mathcal{O}_i^{(\mu)}(x_1) \mathcal{O}_j^{(\nu)}(x_2) \mathcal{O}_k^{(\lambda)}(x_3) \rangle = \overline{\langle (\mathcal{O}_i^\dagger)^{(\mu)}(x_1^\theta) (\mathcal{O}_j^\dagger)^{(\nu)}(x_2^\theta) (\mathcal{O}_k^\dagger)^{(\lambda)}(x_3^\theta) \rangle}. \quad (7.3.7)$$

for any 3 primaries $\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k$.¹⁷ Similarly to the 2-point function case, this condition can be simplified using the conformally invariant tensor structures, with an important difference that the normalization of operators has already been fixed. Conformal invariance constrains the 3-point functions to take the form

$$\langle \mathcal{O}_i^{(\mu)}(x_1) \mathcal{O}_j^{(\nu)}(x_2) \mathcal{O}_k^{(\lambda)}(x_3) \rangle = \sum_{a=1}^{N_{ijk}} f_{ijk}^a \langle \mathcal{O}_i^{(\mu)}(x_1) \mathcal{O}_j^{(\nu)}(x_2) \mathcal{O}_k^{(\lambda)}(x_3) \rangle_a, \quad (7.3.8)$$

where $\langle \mathcal{O}_i^{(\mu)}(x_1) \mathcal{O}_j^{(\nu)}(x_2) \mathcal{O}_k^{(\lambda)}(x_3) \rangle_a$ span the finite-dimensional space (of dimension N_{ijk}) of solutions of conformal invariance constraints on the 3-point functions of the operators with given Δ_s, ρ_s ($s = i, j, k$). On the other hand the coefficients $f_{ijk}^a \in \mathbb{C}$ are not fixed by conformal symmetry (no sum on i, j, k in the r.h.s. of (7.3.8)). We often refer to f_{ijk}^a as the ‘‘OPE coefficients.’’ It is always possible to choose the basis structures $\langle \mathcal{O}_i^{(\mu)}(x_1) \mathcal{O}_j^{(\nu)}(x_2) \mathcal{O}_k^{(\lambda)}(x_3) \rangle_a$ to satisfy the hermiticity constraint (7.3.7) individually, in which case the OPE coefficients must satisfy

$$(f_{ijk}^a)^* = f_{i\bar{j}k}^a, \quad (7.3.9)$$

where the barred indices refer to the conjugate operators $\mathcal{O}_i^\dagger, \mathcal{O}_j^\dagger, \mathcal{O}_k^\dagger$. In particular, when all three operators are self-conjugate-reflected i.e. satisfy (7.3.2), the OPE coefficients f_{ijk}^a must be real.

Finally, unitary CFTs enjoy a **convergent operator product expansion** (OPE). This means that any correlation function¹⁸ satisfies

$$\langle \mathcal{O}_i^{(\mu)}(x_1) \mathcal{O}_j^{(\nu)}(x_2) \mathcal{O}_m^{(\rho)}(x_3) \cdots \rangle = \sum_k \sum_{a=1}^{N_{ijk}} f_{ijk}^a C_{a,(\lambda)}^{(\mu)(\nu)}(x_1, x_2, x_0, \partial_0) \langle (\mathcal{O}_k^\dagger)^{(\lambda)}(x_0) \mathcal{O}_m^{(\rho)}(x_3) \cdots \rangle, \quad (7.3.10)$$

where the first sum runs over all primary operators \mathcal{O}_k in the theory, and $C_{a,(\lambda)}^{(\mu)(\nu)}(x_1, x_2, x_0, \partial_0)$ is a formal sum of the form

$$C_{a,(\lambda)}^{(\mu)(\nu)}(x_1, x_2, x_0, \partial_0) = \sum_\alpha C_{a,(\lambda),\alpha}^{(\mu)(\nu)}(x_1, x_2, x_0) (\partial/\partial x_0)^\alpha. \quad (7.3.11)$$

This differential operator is determined by conformal symmetry¹⁹ and depends only on Δ_s, ρ_s ($s = i, j, k$). Here $(\partial/\partial x)^\alpha = (\partial/\partial x^0)^{\alpha_0} \cdots (\partial/\partial x^{d-1})^{\alpha_{d-1}}$ with $\alpha = (\alpha_0, \dots, \alpha_{d-1}) \in (\mathbb{Z}_{\geq 0})^d$ a multiindex. Convergence of OPE means that the sum (7.3.10), with C expanded as in (7.3.11), converges whenever

$$x_1, x_2 \in B(x_0, R), \quad R = \min(|x_3 - x_0|, |x_4 - x_0|, \dots) \quad (7.3.12)$$

where $B(x_0, R)$ is an open ball centered at x_0 and of radius R . In other words, OPE converges whenever x_1 and x_2 are the two closest operator insertions to x_0 (in Euclidean distance). Convergence should be understood carefully as follows. For each \mathcal{O}_k^\dagger in the r.h.s. of (7.3.10), and for each $n \in \mathbb{Z}_+$, we perform finite summation over a, λ , and all multiindices α with $|\alpha| = n$. We are left then with the doubly infinite sum

$$\sum_k \sum_{n=0}^{\infty} g_{k,n}(\{x_i\}). \quad (7.3.13)$$

This doubly infinite sum has to converge absolutely for every x_1, x_2 as in (7.3.12).²⁰

¹⁶For the special case $x_1 = x_N, x_2 = x_S$, Eq. (7.3.6) is nothing but hermiticity of the matrix $M_i^{(\mu),(\nu)}$, a consequence of positive-semidefiniteness. The general case reduces to the special one mapping x_1, x_2 , to x_N, x_S by a conformal transformation (both sides of (7.3.6) have the same conformal transformation properties).

¹⁷Note that since \dagger is involutive, this covers the case when an operator and its conjugate are interchanged between the two sides of this equation.

¹⁸Importantly, we allow here for one of the operators to be inserted at spatial infinity.

¹⁹There is an ambiguity when \mathcal{O}_k is in a short conformal representation (in unitary theories this happens only if \mathcal{O}_k is a conserved current or a free field). This subtlety will not play any role in this part.

²⁰The requirement of absolute convergence can be somewhat relaxed, see Sec. 8.1.

That the same coefficients f_{ijk}^a appear in the OPE (7.3.10) and the 3-point function (7.3.8) follows immediately by using the OPE inside the latter 3-point function.

Local Euclidean CFTs contain the conserved stress tensor operator $T_{\mu\nu}$ of dimension d , and in case of continuous global symmetry, conserved global symmetry currents J_μ of dimensions $d - 1$. We will not discuss here additional axioms involving 3-point functions and OPE coefficients of these operators, related to their conservation and Ward identities, see e.g. [86, 87].

Remark 7.3.1. The just given Euclidean CFT axioms are more careful in what concerns reality constraints than the set of CFT rules gathered in the conformal bootstrap review [15]. They are also more economical: e.g. Ref. [15] assumed OS reflection positivity and clustering for n -point functions, which for us will be theorems to prove, not assumptions. Ref. [15] also included some constraints on the CFT data which emerge when considering CFT in Minkowski signature, most notably the Averaged Null Energy Condition (ANEC). In this part of the thesis we will establish all Wightman axioms for scalar Minkowski CFT 4-point functions from the Euclidean CFT axioms, but we will not discuss ANEC. A proof of ANEC [88] has been given using the Haag-Kastler axioms for general QFT. CFT arguments have also been given in [20, 89], but they rely on some assumptions which have not been rigorously proven from axioms. It would be interesting to fill these gaps and establish ANEC as a theorem from Euclidean CFT axioms.²¹

²¹ The argument in [20] uses an OPE asymptotic expansion on the second sheet, outside of the range of convergence of the OPE rigorously implied by the Euclidean CFT axioms (see App. B.1.1 where we review this method going back to [3]). In [89] ANEC is derived using manipulations with a generalization of the Lorentzian inversion formula of [22], of which some have not been rigorously justified. For example, the derivation starts with the Euclidean inversion formula, which is readily justified from Harmonic analysis only for external scaling dimensions on principal series and square-integrable correlators, none of which is generically the case in the required setup.

Chapter 8

Euclidean CFT \Rightarrow Osterwalder-Schrader

In this section we will discuss some simple consequences of CFT axioms and in particular will show that they imply OS axioms for 4-point functions (the case of higher-point functions is more subtle and is discussed in App. B.2). Here we will prove only the OS reflection-positivity and the cluster property. The ‘‘Euclidean temperedness’’ bound (7.2.3) will follow from our arguments in the following sections, where we establish power-law bounds on CFT correlation functions. The remaining OS axioms are a subset of the CFT axioms.

8.1 OS reflection positivity

In this section we prove OS positivity for compactly-supported test functions. The extension to Schwartz functions is easy once we establish (7.2.3) for CFT correlators, see Remark 9.0.2.

First let us slightly reformulate the OPE convergence property. Consider an n -point correlation function with operators inserted at $x_1 \dots x_n$. Let S be a hyperplane and x_0 be a point such that x_1, x_2, x_0 are on one side of S while all the other points $x_i, i > 2$, are on the other side. Using a conformal transformation we can map S to a sphere S' so that x_0 is mapped to the center of S' which we denote by x'_0 . Let x'_i denote the positions of all the other points x_i after this map. We can then use the OPE (7.3.10) for the correlation function evaluated at x'_i ,

$$\langle \mathcal{O}_i^{(\mu)}(x'_1) \mathcal{O}_j^{(\nu)}(x'_2) \mathcal{O}_m^{(\rho)}(x'_3) \dots \rangle = \sum_k \sum_{a=1}^{N_{ijk}} f_{ijk}^a C_{a,(\lambda)}^{(\mu)(\nu)}(x'_1, x'_2, x'_0, \partial_0) \langle (\mathcal{O}_k^\dagger)^{(\lambda)}(x'_0) \mathcal{O}_m^{(\rho)}(x'_3) \dots \rangle. \quad (8.1.1)$$

Transforming this expansion term-by-term to the original coordinates x_i we find the convergent expansion (with convergence understood in the same sense as in the previous section¹)

$$\langle \mathcal{O}_i^{(\mu)}(x_1) \mathcal{O}_j^{(\nu)}(x_2) \mathcal{O}_m^{(\rho)}(x_3) \dots \rangle = \sum_k \sum_{a=1}^{N_{ijk}} f_{ijk}^a \tilde{C}_{a,(\lambda)}^{(\mu)(\nu)}(x_1, x_2, x_0, \mathcal{D}_0) \langle (\mathcal{O}_k^\dagger)^{(\lambda)}(x_0) \mathcal{O}_m^{(\rho)}(x_3) \dots \rangle. \quad (8.1.2)$$

Here the differential operators $\mathcal{D}^{(\alpha)}$ are simply the derivatives $(\frac{\partial}{\partial x'})^{(\alpha)}$ expressed in the original coordinates x , and conjugated by the conformal transformation factor of \mathcal{O}_k^\dagger . The functions $\tilde{C}_{a,(\lambda)}^{(\mu)(\nu)}$ are obtained from $C_{a,(\lambda)}^{(\mu)(\nu)}$ by our conformal transformation. The important point is that truncation of $C_{a,(\lambda)}^{(\mu)(\nu)}$ in order of derivatives $\partial^{(\alpha)}$ corresponds to truncation of $\tilde{C}_{a,(\lambda)}^{(\mu)(\nu)}$ in order of operators $\mathcal{D}^{(\alpha)}$.

We now specialize to S being the $x^0 = 0$ plane, $x_0 = x_S = (-1, 0, \dots)$, and take S' to be the unit sphere with the center $x'_0 = 0$. Then the derivatives $(\frac{\partial}{\partial x'})^{(\alpha)} \mathcal{O}_k^{(\mu)}(x'_0)$ are eigenstates of the standard dilatation generator D with

¹Careful reading of the argument below shows that, in the 4-point case, the requirement of absolute convergence of (7.3.13) could be replaced by a weaker requirement that we can find any subsequence of partial sums of (7.3.13) which approximates the correlator pointwise.

eigenvalues $\Delta_k + |\alpha|$. Note that D has two fixed points: $x'_0 = 0$ and infinity. Applying our conformal map, we find that the derivatives $\mathcal{D}^{(\alpha)}\mathcal{O}_k^{(\mu)}(x_S)$ are in turn eigenstates, with the same eigenvalues, of the conformal generator $D' = (K^0 - P^0)/2$ that preserves x_S and $x_N = x_S^\theta$ (which is the image of infinity under our conformal map) and acts by dilatations near these two points. This, together with the conformal invariance and diagonality of 2-point functions, implies

$$\langle (\mathcal{D}^{(\alpha)})^\theta \mathcal{O}_j^{\dagger(\mu)}(x_N) \mathcal{D}^{(\beta)} \mathcal{O}_k^{(\nu)}(x_S) \rangle \propto \delta_{|\alpha|, |\beta|} \delta_{j,k}, \quad (8.1.3)$$

where $(\mathcal{D}^{(\alpha)})^\theta$ is obtained from $\mathcal{D}^{(\alpha)}$ by replacing $x \rightarrow x^\theta$.

The OPE (8.1.2) gives an expansion for ket states $|\Psi\rangle \in \mathcal{H}_0^{\text{OS}}$ created by two local operators in terms of ket states created by a single operator, which are elements of $\mathcal{H}_0^{\text{CFT}}$. We would like to have a dual version of this expansion for bra states $\langle\Psi|$. For this we need to understand how the OPE transforms under the conjugation. Note that the formal differential operators $\tilde{C}_{a,(\lambda)}^{(\mu)(\nu)}(x_1, x_2, x_0, \mathcal{D}_0)$ can be uniquely determined by the equation

$$\langle \mathcal{O}_1^{(\mu)}(x_1) \mathcal{O}_2^{(\nu)}(x_2) \mathcal{O}_3^{(\rho)}(x_3) \rangle_a = \tilde{C}_{a,(\lambda)}^{(\mu)(\nu)}(x_1, x_2, x_0, \mathcal{D}_0) \langle (\mathcal{O}_3^\dagger)^{(\lambda)}(x_0) \mathcal{O}_3^{(\rho)}(x_3) \rangle, \quad (8.1.4)$$

where it is understood that the points are arranged as above, so that the formal sum defined by $\tilde{C}_{a,(\lambda)}^{(\mu)(\nu)}(x_1, x_2, x_0, \mathcal{D}_0)$ actually converges. By applying complex conjugation on both sides and using the 2-point and 3-point hermiticity constraints (7.3.6) and (7.3.7) we find

$$\langle (\mathcal{O}_1^\dagger)^{(\mu)}(x_1^\theta) (\mathcal{O}_2^\dagger)^{(\nu)}(x_2^\theta) (\mathcal{O}_3^\dagger)^{(\rho)}(x_3^\theta) \rangle_a = [\tilde{C}_{a,(\lambda)}^{(\mu)(\nu)}(x_1, x_2, x_0, \mathcal{D}_0)]^* \langle \mathcal{O}_3^{(\lambda)}(x_0) (\mathcal{O}_3^\dagger)^{(\rho)}(x_3) \rangle, \quad (8.1.5)$$

which implies that

$$[\tilde{C}_{a,(\lambda)}^{(\mu)(\nu)}(x_1^\theta, x_2^\theta, x_0^\theta, \mathcal{D}_0^\theta)]^* = \tilde{C}_{a,(\lambda)}^{\approx(\mu)(\nu)}(x_1, x_2, x_0, \mathcal{D}_0), \quad (8.1.6)$$

where $\tilde{C}_{a,(\lambda)}^{\approx(\mu)(\nu)}(x_1, x_2, x_0, \mathcal{D}_0)$ is the formal sum that appears in the OPE for operators with conjugate-reflected quantum numbers.

We are now ready to prove OS positivity for 4-point functions. Let $|\Psi_0\rangle$ be an OS ket state involving at most two local operators, i.e.

$$|\Psi_0\rangle = \sum_{i,j,\alpha,\beta} \int dx_1 dx_2 f_{i,j,(\alpha)(\beta)}(x_1, x_2) \mathcal{O}_i^{(\alpha)}(x_1) \mathcal{O}_j^{(\beta)}(x_2), \quad (8.1.7)$$

where $f(x_1, x_2)$ is a compactly supported test function vanishing unless $0 > x_1^0 > x_2^0$. (Terms with one or no operators are realized by setting one or both operators to the identity.) Since by Euclidean CFT axioms the correlation functions are real-analytic, the integrals that appear in the expression for $\langle\Psi_0|\Psi_0\rangle$ can be approximated by finite Riemann sums, reflection-symmetric if necessary. This implies that, for any $\varepsilon > 0$, we can pass from $|\Psi_0\rangle$ to a ket state $|\Psi\rangle$ which is created by a *finite* linear combination of insertions of up to two local operators with $x_1^0, x_2^0 < 0$:

$$|\Psi\rangle = \sum_{i,j,x_1,x_2,\alpha,\beta} c_{i,j,(\alpha)(\beta)} \mathcal{O}_i^{(\alpha)}(x_1) \mathcal{O}_j^{(\beta)}(x_2), \quad (8.1.8)$$

and has the property that

$$|\langle\Psi_0|\Psi_0\rangle - \langle\Psi_0|\Psi\rangle| < \varepsilon, \quad |\langle\Psi_0|\Psi\rangle - \langle\Psi|\Psi\rangle| < \varepsilon, \quad (8.1.9)$$

so that as a result

$$|\langle\Psi_0|\Psi_0\rangle - \langle\Psi|\Psi\rangle| < 2\varepsilon \quad (8.1.10)$$

We are therefore reduced to proving the nonnegativity of $\langle\Psi|\Psi\rangle$.

Now, the OPE convergence axiom implies that, for any $\varepsilon > 0$, starting from $|\Psi\rangle$ and using the OPE (8.1.2) in the half-space $x^0 < 0$, we can construct a state $|\psi\rangle = |\psi_\Lambda\rangle \in \mathcal{H}_0^{\text{CFT}}$ created by a finite linear combination of local operators at x_S such that

$$|\langle\Psi|\psi\rangle - \langle\Psi|\Psi\rangle| < \varepsilon. \quad (8.1.11)$$

Here Λ is an OPE truncation cutoff which we need to increase appropriately as ε gets smaller. Namely, we will obtain $|\psi\rangle$ by keeping in the OPE all terms with $\Delta_k + |\alpha| < \Lambda$, where Δ_k is the dimension of a primary \mathcal{O}_k appearing in the OPE, and α is the order of the descendant $\mathcal{D}^{(\alpha)}\mathcal{O}_k$.

We can then repeat this procedure in the upper half-plane $x^0 > 0$ and construct a state $\langle \psi' | = \langle \psi_{\Lambda'} |$ of local operators inserted at x_N such that

$$|\langle \Psi | \psi \rangle - \langle \psi' | \psi \rangle| < \varepsilon. \quad (8.1.12)$$

Eq. (8.1.6) and the reality constraint (7.3.9) for the OPE coefficients imply that the state $\langle \psi' |$ differs from $\langle \psi |$ at most by where the OPE expansion was truncated. Furthermore, we can always assume that $\langle \psi' |$ contains at least all the terms that $\langle \psi |$ does (i.e. $\Lambda' \geq \Lambda$), since adding more OPE terms to $\langle \psi' |$ can only improve (8.1.12).

Eq. (8.1.3) then implies that $\langle \psi' | \psi \rangle = \langle \psi | \psi \rangle$ and therefore

$$|\langle \Psi | \psi \rangle - \langle \psi | \psi \rangle| < \varepsilon. \quad (8.1.13)$$

Combining this with (8.1.11) we conclude:

$$|\langle \Psi | \Psi \rangle - \langle \psi | \psi \rangle| < 2\varepsilon, \quad (8.1.14)$$

Since by the CFT positivity axiom $\langle \psi | \psi \rangle$ is non-negative, we conclude that $\langle \Psi | \Psi \rangle$ is also non-negative. This completes the proof of OS positivity for states created by up to two operator insertions.

8.2 Denseness and Hilbert space implications

Here we will describe some useful byproducts of the just given argument. Note that (8.1.11) and (8.1.13), in addition to (8.1.14), also implies

$$\|\Psi - \psi_\Lambda\| \equiv \langle \Psi - \psi_\Lambda | \Psi - \psi_\Lambda \rangle = \langle \Psi | \Psi \rangle - \langle \Psi | \psi_\Lambda \rangle + \langle \psi_\Lambda | \psi_\Lambda \rangle - \langle \psi_\Lambda | \Psi \rangle < 2\varepsilon. \quad (8.2.1)$$

This means that any $|\Psi\rangle$ can be approximated arbitrarily well by a $|\psi_\Lambda\rangle \in \mathcal{H}_0^{\text{CFT}}$. In other words, $\mathcal{H}_0^{\text{CFT}}$ is a dense subspace of the vector space of Ψ 's.

This fact has a simple but quite powerful consequence involving the CFT Hilbert space \mathcal{H}^{CFT} , defined in Sec. 7.3 as the completion of $\mathcal{H}_0^{\text{CFT}}$. Eq. (8.2.1) implies, using the triangle inequality $\|\psi_{\Lambda_1} - \psi_{\Lambda_2}\| \leq \|\Psi - \psi_{\Lambda_1}\| + \|\Psi - \psi_{\Lambda_2}\|$, that the states $|\psi_\Lambda\rangle$ corresponding to smaller and smaller ε form a Cauchy sequence. Therefore, these states have a limit in \mathcal{H}^{CFT} as $\Lambda \rightarrow \infty$, which we call $|\psi_\infty\rangle$. This ψ_∞ is nothing but the full, untruncated, OPE expansion of the state Ψ . We claim that the map mapping Ψ 's to the corresponding ψ_∞ 's is isometric, i.e. it preserves the inner products:

$$\langle \Phi | \Psi \rangle = \langle \varphi_\infty | \psi_\infty \rangle. \quad (8.2.2)$$

Here the inner product on the l.h.s. is the OS inner product, computed using CFT 4-point functions with operators inserted in the lower and upper half-spaces, while the inner product in the r.h.s. is the \mathcal{H}^{CFT} inner product, defined as the limit of $\mathcal{H}_0^{\text{CFT}}$ inner product removing the cutoff:

$$\langle \varphi_\infty | \psi_\infty \rangle := \lim_{\Lambda \rightarrow \infty} \langle \varphi_\Lambda | \psi_\Lambda \rangle. \quad (8.2.3)$$

The proof of (8.2.2) is straightforward. We write:

$$\langle \varphi_\Lambda | \psi_\Lambda \rangle = \langle \Phi + (\varphi_\Lambda - \Phi) | \Psi + (\psi_\Lambda - \Psi) \rangle = \langle \Phi | \Psi \rangle + \text{err}(\Lambda), \quad (8.2.4)$$

$$\text{err}(\Lambda) = \langle \varphi_\Lambda - \Phi | \Psi \rangle + \langle \Phi | \psi_\Lambda - \Psi \rangle + \langle \varphi_\Lambda - \Phi | \psi_\Lambda - \Psi \rangle. \quad (8.2.5)$$

By Eq. (8.2.1) we know that $\|\Psi - \psi_\Lambda\|$, $\|\Phi - \varphi_\Lambda\|$ go to zero as $\Lambda \rightarrow \infty$. Hence, $\text{err}(\Lambda) \rightarrow 0$ and (8.2.2) is proved.

Eqs. (8.2.2) and (8.2.3) mean that OPE converges in the sense of the CFT Hilbert space. This property is often used in the CFT literature (see Sec. 11.2). Note that CFT axioms in Sec. 7.3 only assume pointwise OPE convergence, which is a weaker statement. Curiously, by the given arguments, pointwise OPE convergence plus CFT positivity imply Hilbert space convergence, at least for the 4-point functions.

In the above argument we used the Hilbert space \mathcal{H}^{CFT} , the completion of $\mathcal{H}_0^{\text{CFT}}$. We may introduce a second Hilbert space as the completion of the space of Ψ 's, call it $\mathcal{H}^{(2)}$. This Hilbert space contains e.g. all Ψ_0 states (8.1.7). (Similarly to (8.2.1), Eqs. (8.1.9) and (8.1.10) imply that Ψ states are dense in the Ψ_0 states.) Although $\mathcal{H}^{(2)}$ may look like a ‘‘bigger’’ space than \mathcal{H}^{CFT} , actually it's not. Indeed, the map from Ψ to ψ_∞ extends to an isometric map from $\mathcal{H}^{(2)}$ to \mathcal{H}^{CFT} . In other words, Eq. (8.2.2) remains true for any $\Phi, \Psi \in \mathcal{H}^{(2)}$. Since we can view $\mathcal{H}_0^{\text{CFT}}$ as a subspace of the space of Ψ 's, we also have that this map is surjective. The Hilbert spaces $\mathcal{H}^{(2)}$ and \mathcal{H}^{CFT} are thus unitarily equivalent and may be identified.

8.3 OS clustering

Here we will derive the OS clustering (7.2.12) from CFT axioms. We will consider only $m + n = 4$, i.e. when the left-hand side of (7.2.12) can be written in terms of a 4-point function (this also covers $m + n < 4$ since we can choose some of φ 's or χ 's to be the trivial identity field). We can assume that all χ 's and φ 's in (7.2.12) are primary fields, since any derivatives can be integrated by parts.

First as a general remark, assuming OS positivity, clustering (for any m, n) only needs to be established point-wise, i.e.

$$\lim_{\lambda \rightarrow \infty} \langle \chi_m^\dagger(y_m) \dots \chi_1^\dagger(y_1) \varphi_1(x_1 + \lambda a) \dots \varphi_n(x_n + \lambda a) \rangle = \langle \chi_m^\dagger(y_m) \dots \chi_1^\dagger(y_1) \rangle \langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle. \quad (8.3.1)$$

This follows from the dominated convergence theorem. Indeed, OS positivity and translation invariance (recall that we only consider $a^0 = 0$) implies a uniform in λ bound²

$$\begin{aligned} & | \langle \chi_m^\dagger(y_m) \dots \chi_1^\dagger(y_1) \varphi_1(x_1 + \lambda a) \dots \varphi_n(x_n + \lambda a) \rangle | \leq \\ & \langle \chi_m^\dagger(y_m) \dots \chi_1^\dagger(y_1) \chi_1(y_1^\theta) \dots \chi_n(y_n^\theta) \rangle \times \langle \varphi_n^\dagger(x_n^\theta) \dots \varphi_1^\dagger(x_1^\theta) \varphi_1(x_1) \dots \varphi_n(x_n) \rangle. \end{aligned} \quad (8.3.2)$$

It then follows that the integrand in (7.2.12) is bounded by a λ -independent integrable function, and the dominated convergence theorem is applicable.

Going back to the 4-point function case which is our focus in this section, let us start with $m = n = 2$. Since we already proved OS positivity for states created by at most two operators (Sec. 8.1), in this case we can rely on the above observation and we only need to check the point-wise limit:

$$\lim_{\lambda \rightarrow \infty} \langle \chi_2^\dagger(y_2) \chi_1^\dagger(y_1) \varphi_1(x_1 + \lambda a) \varphi_2(x_2 + \lambda a) \rangle = \langle \chi_2^\dagger(y_2) \chi_1^\dagger(y_1) \rangle \langle \varphi_1(x_1) \varphi_2(x_2) \rangle. \quad (8.3.3)$$

To see this, we apply the OPE (8.1.2) to $\chi_2^\dagger(y_2) \chi_1^\dagger(y_1)$ in the left-hand side. The results of the previous section imply that this OPE can be interpreted as expanding the state in the Hilbert space \mathcal{H} created by $\chi_2^\dagger(y_2) \chi_1^\dagger(y_1)$ in terms of eigenstates of $(K^0 - P^0)/2$. This implies that the OPE converges uniformly in λ since the norm of $|\varphi_1(x_1 + \lambda a) \varphi_2(x_2 + \lambda a)\rangle$ is independent of λ due to translation invariance ($a^0 = 0$). We can thus use the OPE to approximate $\langle \chi_2^\dagger(y_2) \chi_1^\dagger(y_1) \varphi_1(x_1 + \lambda a) \varphi_2(x_2 + \lambda a) \rangle$ for any λ to within any $\varepsilon > 0$ by a finite sum of 3-point functions of the form $\langle (\mathcal{D}^{(\alpha)})^\theta \mathcal{O}_i^{(\nu)}(x_N) \varphi_1(x_1 + \lambda a) \varphi_2(x_2 + \lambda a) \rangle$ times some λ -independent coefficients. Of these, only the term corresponding to the identity operator, i.e. the one with $(\mathcal{D}^{(\alpha)})^\theta \mathcal{O}_i^{(\nu)} = 1$, does not decay with λ . It is easily verified that the contribution of this term is precisely equal to $\langle \chi_2^\dagger(y_2) \chi_1^\dagger(y_1) \rangle \langle \varphi_1(x_1) \varphi_2(x_2) \rangle$. This finishes the proof of clustering for $m = n = 2$.

In the remaining case $m = 3, n = 1$, we will consider the limit for the integral (since we have not yet proved OS positivity for states involving 3 operators). Note that $\langle \varphi_1(x) \rangle = 0$ unless $\varphi_1 \propto 1$, in which case the cluster property becomes trivial. This means that we only need to prove

$$\lim_{\lambda \rightarrow \infty} \int dx dy \overline{g(y_1^\theta, y_2^\theta, y_3^\theta)} f(x_1) \langle \chi_3^\dagger(y_3) \chi_2^\dagger(y_2) \chi_1^\dagger(y_1) \varphi_1(x_1 + \lambda a) \rangle = 0 \quad (8.3.4)$$

for $\varphi \neq 1$. This in turn is a very simple consequence of conformal invariance, and of the fact that $\Delta_\varphi > 0$ for all operators but the identity. We will be somewhat schematic. The main point is that the configuration (y_3, y_2, y_1, ∞) is nonsingular from the conformal kinematics point of view (for which the conformal compactification S_d of the Euclidean space \mathbb{R}^d is the appropriate arena). One way to see it is that the cross ratios are finite in this limit. Thinking in a pedestrian way, we can find a conformal transformation g_λ which will move points $(y_3, y_2, y_1, x_1 + \lambda a)$ to some points which have finite limits as $\lambda \rightarrow \infty$. Transforming the integral (8.3.4) to this coordinate system, the only singular behavior at large λ comes from the Weyl transformation factor as the operator φ_1 is moved from near infinity to a finite position. This factor implies that the integral (8.3.4) will go to zero as $\lambda^{-2\Delta_\varphi}$, proving clustering in this particular case. See Sec. 11.8.2 for additional details. More generally, the same argument will also work for arbitrary m as long as $n = 1$.

²This follows, similarly to (7.2.11), by applying OS positivity to the state $|\Psi\rangle = |\psi(F, \varphi_1 \dots \varphi_n)\rangle + e^{i\alpha} |\psi(G, \chi_1 \dots \chi_n)\rangle$ where F, G tend to delta functions localizing the operators at points $x_1 + \lambda a, \dots, x_n + \lambda a$ and $y_1^\theta, \dots, y_n^\theta$ respectively, and choosing the phase α appropriately.

Chapter 9

Euclidean CFT \Rightarrow Wightman: Basic strategy

We will now pass to the main task of this part of the thesis: given a Euclidean unitary CFT, recover Minkowski correlators and show that they satisfy Wightman axioms.

Let us first discuss this problem without assuming conformal invariance. Suppose we know correlators $G_n^E(x_1, \dots, x_n)$ of a scalar field in a Euclidean QFT which is translationally and rotationally invariant, but not necessarily conformally invariant. We are assuming, as discussed above, that the correlators G_n^E are defined and real-analytic (see footnote 5) for non-coincident Euclidean points ($x_k \in \mathbb{R}^d, x_i \neq x_j$).

We would like to recover correlators in Minkowski signature. We are only interested here in Wightman correlation functions, where the operator ordering is fixed while the Minkowski time coordinates vary independently. We will call them simply “Minkowski correlators”. Starting from this section we will focus on correlators of scalar primaries; correlators of fields in general $SO(d)$ representations will be considered in our future publication [78] (chapter 22).

To understand the equations below, it helps to keep in mind the basic heuristic. If we had a Hilbert space, field operators ϕ , and a Hamiltonian H , then the Minkowski correlators would be given by

$$G_n^M(x_1^M, \dots, x_n^M) = \langle 0 | \phi(0, \mathbf{x}_1) e^{-iH(t_1-t_2)} \phi(0, \mathbf{x}_2) e^{-iH(t_2-t_3)} \dots | 0 \rangle, \quad x_k^M = (t_k, \mathbf{x}_k). \quad (9.0.1)$$

while the Euclidean correlators by

$$G_n^E(x_1, \dots, x_n) = \langle 0 | \phi(0, \mathbf{x}_1) e^{-H(\epsilon_1-\epsilon_2)} \phi(0, \mathbf{x}_2) e^{-H(\epsilon_2-\epsilon_3)} \dots | 0 \rangle, \quad x_k = (\epsilon_k, \mathbf{x}_k), \quad \epsilon_k > \epsilon_{k+1}, \quad (9.0.2)$$

We stress that the r.h.s. of these two equations will never be used in this part of the thesis. We just use them to illustrate the intuitive property that G_n^M can be recovered from G_n^E by analytic continuation $\epsilon_k \rightarrow \epsilon_k + it_k$ and sending $\epsilon_k \rightarrow 0$ while respecting $\epsilon_k > \epsilon_{k+1}$. In other words, there is a holomorphic function G_n which reduces to G_n^M in one limit and to G_n^E in another. The precise domain of analyticity of G_n can be clarified from the Wightman axioms [24]. Their basic consequence is that Minkowski correlators can be analytically continued to the “forward tube” (see below), which contains the Euclidean space as a section. In this part of the thesis we will derive Wightman axioms, rather than assume them. In particular, we will carry out analytic continuation to the forward tube just from the properties of the Euclidean correlators.

Let us put these observations into a definition of what it means to recover G_n^M from G_n^E . We consider n -point configurations with complexified coordinates:

$$c = (x_1, \dots, x_n), \quad x_k = (x_k^0, \mathbf{x}_k) \in \mathbb{C}^d. \quad (9.0.3)$$

The “forward tube” \mathcal{T}_n is defined as the set of all such configurations for which the differences $y_k = x_k - x_{k+1} = (y_k^0, \mathbf{y}_k) \in \mathbb{C}^d$ satisfy the constraint:

$$\text{Re } y_k^0 > |\text{Im } \mathbf{y}_k|, \quad k = 1, \dots, n-1. \quad (9.0.4)$$

Equivalently, this means that vectors $\text{Im}(iy_k^0, \mathbf{y}_k)$ belong to the open forward light cone of $\mathbb{R}^{1,d-1}$, explaining the name “forward tube”.¹

Let \mathcal{D}_n be the subset of the forward tube consisting of the configurations with real spatial parts \mathbf{x}_k . Equivalently, we have:

$$\mathcal{D}_n = \{c \mid x_k^0 = \epsilon_k + it_k, \mathbf{x}_k \in \mathbb{R}^{d-1}, \epsilon_1 > \epsilon_2 > \dots > \epsilon_n\}. \quad (9.0.5)$$

Finally, we denote by \mathcal{D}_n^E the Euclidean part of \mathcal{D}_n obtained by setting all $t_k = 0$.

Minkowski correlators are then defined by the following two-step procedure:

Step 1. One finds an extension G_n^E from \mathcal{D}_n^E to a function $G_n(x_1, \dots, x_n)$ such that one of the two conditions is satisfied:

$$G_n \text{ is defined on } \mathcal{T}_n, \text{ and holomorphic in all variables } x_k^0, \mathbf{x}_k, \quad (9.0.6)$$

or

$$G_n \text{ is defined on } \mathcal{D}_n, \text{ is holomorphic in variables } x_k^0 \text{ and is real-analytic in } \mathbf{x}_k. \quad (9.0.7)$$

Real analyticity in \mathbf{x}_k means that G_n can be extended from \mathcal{D}_n to a holomorphic function defined on a neighborhood of \mathcal{D}_n which allows small imaginary parts for \mathbf{x}_k . This neighborhood can be arbitrarily small. Condition (9.0.7) is thus weaker than (9.0.6) and may be easier to check, although Theorem 9.0.1 below shows that the two conditions are equivalent under the “powerlaw bound” assumption.

Step 2. Minkowski correlators are defined as the limits of G_n from inside \mathcal{D}_n by sending $\epsilon_i \rightarrow 0$:²

$$G_n^M(x_1^M, \dots, x_n^M) = \lim_{\epsilon_i \rightarrow 0} G_n(x_1, \dots, x_n), \quad x_k^M = (t_k, \mathbf{x}_k), \quad k = 1 \dots n. \quad (9.0.8)$$

As mentioned several times, Minkowski correlators are expected to be tempered distributions, and therefore this limit has to be understood in the distributional sense. To show that the limit exists and has properties required by Wightman axioms, one relies on the following powerful theorem of several complex variables:

Theorem 9.0.1 (Vladimirov’s theorem). *Suppose that the function G_n is translation- and rotation-invariant, satisfies (9.0.7) and in addition satisfies everywhere on \mathcal{D}_n the following ‘powerlaw bound’ with some positive constants C_n, A_n, B_n :*

$$|G_n(x_1, \dots, x_n)| \leq C_n \left(1 + \max_k \frac{1}{\epsilon_k - \epsilon_{k+1}}\right)^{A_n} (1 + \max_i |x_i - x_{i+1}|)^{B_n}, \quad (9.0.9)$$

$$|x_i - x_j|^2 \equiv |\epsilon_i + it_i - \epsilon_j - it_j|^2 + |\mathbf{x}_i - \mathbf{x}_j|^2. \quad (9.0.10)$$

Then:

1. Limit (9.0.8) exists in the sense of tempered distributions. The limiting value G_n^M is a tempered distribution.³
2. The distribution G_n^M is Poincaré-invariant and satisfies the Wightman spectral condition. I.e. its Fourier transform $W(p_1, \dots, p_{n-1})$ with respect to the differences $x_k^M - x_{k+1}^M$ has support in the product of closed forward light cones, which is the region $E_k \geq |\mathbf{p}_k|$, $p_k = (E_k, \mathbf{p}_k)$.
3. The function G_n can be extended from \mathcal{D}_n to an holomorphic function on the whole forward tube \mathcal{T}_n . The limit (9.0.8) exist also from the forward tube, i.e. when $\text{Re } y_k^0 \rightarrow 0$, $|\text{Im } \mathbf{y}_k| \rightarrow 0$, satisfying (9.0.4).

See App. B.3 for the proof of Vladimirov’s theorem and a reminder of what the limit in the sense of distributions means. In the process of the proof, it will be established that the holomorphic function G_n on \mathcal{T}_n can be written as a “Fourier-Laplace” transform

$$G_n(x_1, \dots, x_n) = \int dp_1 \dots dp_{n-1} W(p_1, \dots, p_{n-1}) e^{\sum_{k=1}^{n-1} (-E_k(x_k^0 - x_{k+1}^0) + i\mathbf{p}_k \cdot (\mathbf{x}_k - \mathbf{x}_{k+1}))}, \quad (9.0.11)$$

¹The just given definition of the forward tube is adapted to the Euclidean coordinates. In Sec. 6.1, Eq. (6.1.2), we wrote the same definition in terms of Minkowski coordinates $x_k^M = (-ix_k^0, \mathbf{x}_k)$.

²We will see in Theorem 9.0.1 that this limit has to be taken along a fixed direction and is independent of direction. If the stronger condition (9.0.6) holds, the limit can in fact be taken along any direction in the forward null cone.

³This part of the theorem does not need translation- and rotation-invariance of G_n .

where W is a tempered distribution which is the Fourier transform of the tempered distribution G_n^M , mentioned in Part 2 of the theorem.

To use Theorem 9.0.1, one needs to verify the powerlaw bound (9.0.9). This strategy was first developed by Osterwalder and Schrader (OS) [1, 2].⁴ Their full list of assumptions included, in addition to reflection positivity and other OS axioms listed in Sec. 7.2, a less widely known *linear growth condition*, which roughly says that G_n^E (appropriately integrated) grows with n not faster than a power of $n!$ and the degree of its singularities grows not faster than linearly in n . The proof of the powerlaw bound was the most technical part of the OS construction, and it crucially relied on the linear growth condition. See App. 14 for the review.

In this part of the thesis we aim to define Minkowski correlators of a *conformal* field theory, given Euclidean correlators satisfying the CFT axioms of Sec. 7.3. As seen in Sec. 8.1, reflection positivity is robustly encoded in CFT via the positivity requirements for 2-point functions and reality constraints on the OPE coefficients. On the other hand, not much is known about how CFT n -point functions grow with n . In particular, we are unable to justify the OS growth condition in our setup, hence we cannot appeal to the OS theorem.

In this part of the thesis we will be able to circumvent this difficulty, by giving an alternative proof of the powerlaw bound for the most important in applications cases of 2, 3 and 4-point functions. Then, by Theorem 9.0.1, these correlators exist in Minkowski space and are tempered distributions. Our proof of the powerlaw bound uses only the Euclidean CFT axioms. In fact, the two- and 3-point function case is almost trivial, these correlators being fixed by conformal invariance. The 4-point function case is much deeper and is one of our main results. Remaining Wightman axioms not mentioned in Theorem 9.0.1 (positivity and clustering) will also be shown to hold.

Remark 9.0.1. In practice, to compute the Minkowski correlator one may connect a Minkowski configuration to a Euclidean configuration by a curve $c(s)$, $0 \leq s \leq 1$, where $c(0)$ is Euclidean, $c(1)$ Minkowskian, and $c(s)$, $0 < s < 1$, belong to the forward tube. In general, the curve should remain in the forward tube except for the endpoint $c(1)$. This means that we must have strict inequalities:

$$\operatorname{Re} x_1^0(s) > \operatorname{Re} x_2^0(s) > \cdots > \operatorname{Re} x_n^0(s) \quad (9.0.12)$$

everywhere along the analytic continuation contour, except for $s = 1$. See Fig. 9.0.1.

In the literature, one sometimes encounters a different prescription for computing the Minkowski correlators (see e.g. [3], Sec. 3.1), where one puts all points but one at Minkowski positions, and considers correlators as a holomorphic function of the complexified coordinate of the remaining point. One then imagines that Wightman functions are holomorphic functions branching at light-cone separation, and that one can access different operator orderings by going around branch points. We would like to warn the reader that this prescription has to be taken with a grain of salt. To our knowledge there is no general result that the only Wightman functions singularities are branch cuts on the light cones. This is known to be true only in some special cases, e.g. for CFT 2-point and 3-point functions, as well as for CFT 4-point functions in $d = 2$ [45]. While some analytic continuation beyond the forward tube can be done in a general QFT (to the so called permuted extended tube), it does not suffice to justify the analytic continuation prescription of [3] in a general QFT. In CFTs in higher dimensions, the prescription of Ref. [3] has some applicability, with the understanding that the correlator is analytic along the continuation contour but may stop being analytic at the endpoint (see App. B.1.1).

Remark 9.0.2. A powerlaw bound in the forward tube (9.0.9) of course implies in a powerlaw bound for the Euclidean 4pt function itself. Together with rotation invariance, this will imply the remaining OS axiom, the ‘‘Euclidean temperedness bound’’ (7.2.3). Indeed, by rotation invariance, we can choose the direction of the x^0 axis before applying the Euclidean powerlaw bound. Let us choose the x^0 direction so that, after ordering the operators according to $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$, we have $\epsilon_k - \epsilon_{k+1} \geq \alpha |x_k - x_{k+1}|$ for each k . Such a direction exists for a sufficiently small positive α , depending on d and the number of points but not on x_i .⁵ Applying the Euclidean case of the powerlaw bound (9.0.9) in this frame we obtain (7.2.3).

⁴OS used a slightly stronger version of Theorem 9.0.1 with real analyticity in \mathbf{x}_k replaced by the weaker assumption of continuity in these variables, but this difference is not essential.

⁵For each pair of points (x_i, x_j) we consider the set of direction \hat{e}_0 such that $|(x_i - x_j) \cdot \hat{e}_0| \leq |x_i - x_j| \sin \delta$. This gives a subset \mathcal{U}_δ of the sphere S^{d-1} with $\operatorname{Vol}(\mathcal{U}_\delta) \leq 2\delta \operatorname{Vol}(S^{d-2})$. If we choose $\delta_* = \frac{\operatorname{Vol}(S^{d-1})}{2n(n-1) \operatorname{Vol}(S^{d-2})}$, then the total volume excluded by considering all possible (x_i, x_j) pairs is less than $\operatorname{Vol}(S^{d-1})/2$. Therefore, we can find a direction \hat{e}_0 such that the opposite inequality $|(x_i - x_j) \cdot \hat{e}_0| \geq |x_i - x_j| \sin \delta_*$ holds for all pairs. Then, renumbering the points in the order of decreasing x_i^0 , we obtain $x_k^0 - x_{k+1}^0 \geq |x_k - x_{k+1}| \sin \delta_*$.

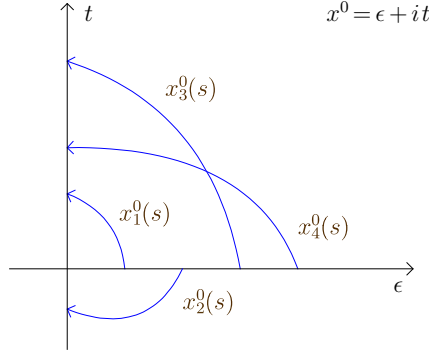


Figure 9.0.1: Inequalities (9.0.12) should be satisfied along the analytic continuation contour.

9.1 Recovering Minkowski averages from Euclidean averages

Minkowski correlators provided by Theorem 9.0.1, being tempered distributions, can be paired with a Schwartz test function F :

$$(G_n^M, F) = \int d^d x_1 \dots d^d x_n G_n^M(x_1 \dots x_n) F(x_1 \dots x_n). \quad (9.1.1)$$

Here we will discuss how these pairings can be computed given the Euclidean correlators (compare [1], Sec. 4.3). This discussion will be needed in Sec. 11.7 below and may be skipped on the first reading.

Eq. (9.1.1) can equivalently be expressed via the Fourier transform W of G_n^M with respect to $x_k - x_{k+1}$:

$$(G_n^M, F) = \int d^d p_1 \dots d^d p_{n-1} W(p_1, \dots, p_{n-1}) f(p_1 \dots p_{n-1}), \quad (9.1.2)$$

where $f(p_1 \dots p_{n-1}) = \widehat{F}(-p_1, p_1 - p_2, \dots, p_{n-2} - p_{n-1}, p_{n-1})$. Natural pairings for Euclidean correlators are

$$(G_n^E, \varphi) = \int d^d x_1 \dots d^d x_n G_n^E(x_1 \dots x_n) \Phi(x_1 \dots x_n). \quad (9.1.3)$$

where Φ is a C^∞ test function compactly supported in $x_1^0 > \dots > x_n^0$. We wish to discuss how pairings (9.1.1) or (9.1.2) can be found given (9.1.3).

By the Fourier-Laplace representation (9.0.11), we can write (9.1.3) as

$$(G_n^E, \varphi) = \int d^d p_1 \dots d^d p_{n-1} W(p_1, \dots, p_{n-1}) g(p_1, \dots, p_{n-1}), \quad (9.1.4)$$

where g is any Schwartz class function which agrees inside the forward light cones with $\tilde{\varphi}$, the Fourier-Laplace transform of $\varphi(y_1, \dots, y_{n-1}) = \int d^d x_n \Phi(x_n + \sum_{i=1}^{n-1} y_i, x_n + \sum_{i=2}^{n-1} y_i, \dots, x_n)$:

$$g \in \mathcal{S}, \quad g(p) = \tilde{\varphi}(p) \quad (p_k \in \overline{V}_+), \quad (9.1.5)$$

$$\tilde{\varphi}(p_1, \dots, p_{n-1}) = \int d^d y_1 \dots d^d y_{n-1} \varphi(y_1 \dots y_{n-1}) e^{\sum_{k=1}^{n-1} (-p_k^0 y^0 + i \mathbf{p}_k \cdot \mathbf{y}_k)}.$$

Note that we cannot just put $g = \tilde{\varphi}$ because $\tilde{\varphi}$ is by itself not a Schwartz function (it may grow exponentially in the negative p_k^0 directions, although it will decrease exponentially in the positive one, since φ is supported at $y_k^0 > 0$). On the other hand the values of g outside the light cones, where W is supported, are unimportant. We can for example take

$$g(p_1, \dots, p_{n-1}) = \chi(p_1^0) \dots \chi(p_{n-1}^0) \tilde{\varphi}(p_1, \dots, p_{n-1}), \quad (9.1.6)$$

where $\chi(s)$ is a C^∞ function which equals identically 1 for $s \geq 0$ and 0 for $s < -1$.

Suppose then that we find a sequence of C^∞ functions $\{\varphi_r\}_{r=1}^\infty$ compactly supported at $y_k^0 > 0$, the corresponding functions $g_r \in \mathcal{S}$ such that $g_r = \tilde{\varphi}_r$ inside the light cones, and in addition that $g_r \rightarrow f$ in the sense of the Schwartz space (i.e. that all Schwartz space seminorms of the difference go to zero), where f is the function in (9.1.2). Let us put $\Phi_r(x_1, \dots, x_n) = \varphi(x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n)\omega(x_n)$ where ω is any C_0^∞ function of integral one. Then we will have

$$(G_n^E, \Phi_r) = (W, g_r) \longrightarrow (W, f) = (G_n^M, F) \quad (r \rightarrow \infty), \quad (9.1.7)$$

and so we will solve the problem of computing Minkowski averages given Euclidean averages. The following lemma, loosely related to Lemma 8.2 in [1], shows that it is indeed possible to find such sequences φ_r and g_r for any Schwartz class f .

Lemma 9.1.1. *The set of functions $g \in \mathcal{S}(\mathbb{R}^{d(n-1)})$ which satisfy (9.1.5) for some φ a C^∞ test function compactly supported in $y_k^0 > 0$ is dense in the Schwartz space.*

We will give a formal proof; see App. B.4 for some intuition. We will consider the case $n = 2$ as $n > 2$ is no more complicated. For each φ we consider the set A_φ of Schwartz functions g which satisfy (9.1.5):

$$A_\varphi := \left\{ g \in \mathcal{S}(\mathbb{R}^d) \mid g|_{\overline{V_+}} = \tilde{\varphi} \right\}. \quad (9.1.8)$$

We know that A_φ is non-empty, e.g. we can take g from (9.1.6) (it is not hard to show that this is a Schwartz function). Our lemma says that $A \equiv$ “the union of A_φ over all φ ” is dense in $\mathcal{S}(\mathbb{R}^d)$. The proof will be by contradiction. Note that A is a linear subspace of $\mathcal{S}(\mathbb{R}^d)$. If $\overline{A} \neq \mathcal{S}(\mathbb{R}^d)$, then there exists a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^d)$ such that T vanishes on all test functions from A but does not vanish identically.⁶

So T in particular vanishes on $A_0 = \left\{ g \in \mathcal{S}(\mathbb{R}^d) \mid g|_{\overline{V_+}} = 0 \right\}$ (take $\varphi = 0$). This means that the support of the distribution T is contained inside $\overline{V_+}$. Consider the Fourier transform of T ,

$$\widehat{T}(x) := \int \frac{d^d p}{(2\pi)^d} T(p) e^{ip^0 x^0 - i\mathbf{p} \cdot \mathbf{x}}. \quad (9.1.9)$$

We can consider $\widehat{T}(x)$ for real x where it is a distribution. Since $\text{supp}(T) \subseteq \overline{V_+}$ it is also natural to consider $\widehat{T}(\xi + i\eta)$ where ξ, η are real and η is in the forward cone. We know that $\widehat{T}(x)$ is a holomorphic function for such $x = \xi + i\eta$. We also know that the distribution $\widehat{T}(x)$ for real x can be obtained as a limit of the holomorphic function $\widehat{T}(\xi + i\eta)$ as $\eta \rightarrow 0$.

Let us now come back to the assumption that $(T, g) = 0$ for any $g \in A_\varphi$. We will apply this to a function g of the form $g = X(p)\tilde{\varphi}$ where $X(p)$ is a C^∞ function identically 1 on the forward light cone and such that $X(p)e^{-p^0 x^0 + i\mathbf{p} \cdot \mathbf{x}}$ is in Schwartz class for any $x^0 > 0$. It is easy to see that such functions $X(p)$ exist. Writing (T, g) in full we get:

$$\begin{aligned} 0 = (T, g) &= \int dp T(p) X(p) \int dx e^{-p^0 x^0 + i\mathbf{p} \cdot \mathbf{x}} \varphi(x) \\ &= \int dx \varphi(x) \int dp T(p) X(p) e^{-p^0 x^0 + i\mathbf{p} \cdot \mathbf{x}} \\ &= \int dx \varphi(x) \widehat{T}(ix^0, \mathbf{x}). \end{aligned} \quad (9.1.10)$$

The swap of the order of integration between the first and the second line can be justified as follows. Since $T(p)$ is a tempered distribution, we can write it as a finite sum of derivatives of continuous functions of power growth: $T(p) = \sum_\alpha \partial_p^\alpha F_\alpha(p)$. Using distributional integration by parts, we can rewrite the first line of (9.1.10) as a sum of ordinary integrals, apply Fubini’s theorem to swap the integration order, and integrate by parts back to express the answer in terms of $\widehat{T}(p)$.

⁶The corresponding statement for normed spaces is standard, being a well-known consequence of the Hahn-Banach theorem (see e.g. [90], Corollary 1.8). For the Schwartz space, we can first find a Schwartz norm $|\cdot|_n$, such that \overline{A} is not everywhere dense with respect to this norm, and then apply the standard statement with the norm $|\cdot|_n$. This gives a linear functional T on $\mathcal{S}(\mathbb{R}^d)$ continuous with respect to $|\cdot|_n$, hence $T \in \mathcal{S}'(\mathbb{R}^d)$.

Because $\varphi(x)$ has compact support in the region $x^0 > 0$, the argument of \widehat{T} in the last line of (9.1.10) is of the form $\xi + i\eta$ with $\eta = (x^0, \mathbf{0})$ in the forward light cone, where we know \widehat{T} is analytic. So, from the fact that the last line of (9.1.10) vanishes for any φ we conclude that

$$\widehat{T}(\xi + i\eta) = 0, \quad \xi = (0, \mathbf{x}), \quad \eta = (ix^0, \mathbf{0}), \quad x^0 > 0. \quad (9.1.11)$$

The set of these points is a totally real submanifold, and so by analyticity we conclude that $\widehat{T}(\xi + i\eta)$ is identically zero for any $\xi \in \mathbb{R}^d$ and any η in the forward cone. Furthermore, as mentioned above, $\widehat{T}(x)$ for real x is a boundary value of $\widehat{T}(\xi + i\eta)$. Therefore, $\widehat{T} = 0$ in the sense of distributions. However we assumed above that T was not identically zero. The reached contradiction shows that A is dense in \mathcal{S} .

Chapter 10

Two- and 3-point functions

Let us see how the strategy from chapter 9 works for the CFT 2-point and 3-point functions. The Euclidean 2-point and 3-point correlators of scalar primaries are given by $[x_{ij}^2 = (x_i - x_j)^2]$

$$G_2^E(x_1, x_2) = \frac{1}{(x_{12}^2)^\Delta}, \quad (10.0.1)$$

$$G_3^E(x_1, x_2, x_3) = \frac{c_{123}}{(x_{12}^2)^{h_{123}}(x_{13}^2)^{h_{132}}(x_{23}^2)^{h_{23}}}, \quad h_{ijk} = (\Delta_i + \Delta_j - \Delta_k)/2. \quad (10.0.2)$$

In this case, the standard way to obtain the Wightman functions is to write these Euclidean correlators in terms of x_{ij}^2 with $i < j$ (as we did). Substituting the analytic continuation of x_{ij}^2 ,

$$x_{ij}^2 = (x_i - x_j)^\mu (x_i - x_j)_\mu = (x_i^0 - x_j^0)^2 + (\mathbf{x}_i - \mathbf{x}_j)^2, \quad x_i = (x_i^0, \mathbf{x}_i) \in \mathbb{C}^d. \quad (10.0.3)$$

into the Euclidean 2-point and 3-point functions expressions, we obtain their analytic continuations. Suppose further that $x_{ij}^2 \neq 0$, $i < j$, in the forward tube (this will be shown below). Then the functions

$$c \mapsto x_{ij}^2 \quad (i < j) \quad (10.0.4)$$

are holomorphic functions from the forward tube to $\widetilde{\mathbb{C} \setminus \{0\}}$, the universal covering of the complex plane minus the origin. On the other hand $z \mapsto z^h$ is holomorphic from this universal covering to \mathbb{C} . Composing these two holomorphic functions, we conclude that $(x_{ij}^2)^h$, $i < j$, are holomorphic on the forward tube. Hence this procedure analytically extends the Euclidean 2-point and 3-point functions to the whole forward tube \mathcal{T}_n ($n = 2, 3$).

We will now give a simple lemma which proves that indeed $x_{ij}^2 \neq 0$, $i < j$, in the forward tube. Actually the lemma says that a bit more is true, namely $x_{ij}^2 \in \mathbb{C} \setminus (-\infty, 0]$. This has the following practical consequence. In general, to compute the analytic continuation of $(F(c))^h$, where $F(c)$ is a nonzero holomorphic function, we need to know the phase of $F(c)$, i.e. to which sheet of the Riemann surface $\widetilde{\mathbb{C} \setminus \{0\}}$ it belongs. To compute the phase we need to connect c to a c_E by a curve and analytically continue along this curve, following the phase. However, this is unnecessary for $(x_{ij}^2)^h$. Indeed, by the lemma below x_{ij}^2 always belongs to the principal sheet. So there is no need to use a curve to compute the phase: it can be computed unambiguously just by plugging the coordinates into (10.0.3).

Lemma 10.0.1. *Let $y = (y^0, \mathbf{y}) \in \mathbb{C}^d$ satisfy $\text{Re } y^0 > |\text{Im } \mathbf{y}|$. Then*

$$y^2 \equiv (y^0)^2 + \mathbf{y}^2 \in \mathbb{C} \setminus (-\infty, 0]. \quad (10.0.5)$$

Proof. We will denote by Greek letters ξ, η , etc., vectors of Minkowski space $\mathbb{R}^{1,d-1}$ with the Minkowski inner product $\xi^2 = -(\xi^0)^2 + \boldsymbol{\xi}^2$.¹ Decomposing the vector (iy^0, \mathbf{y}) into its real and imaginary parts:

$$(iy^0, \mathbf{y}) = \xi + i\eta, \quad (10.0.6)$$

¹Everywhere in this part of the thesis we are using $-, +, \dots, +$ Minkowski signature.

condition $\text{Re } y^0 > |\text{Im } \mathbf{y}|$ means that $\eta^0 > 0$ and $-\eta^2 > 0$, i.e. η is in the open forward light cone, which we will denote by $\eta \succ 0$. In this notation, we have to prove that

$$(\xi + i\eta)^2 = \xi^2 - \eta^2 + 2i(\xi\eta) \notin \mathbb{C} \setminus (-\infty, 0]. \quad (10.0.7)$$

where by our conventions all inner products involving ξ, η are Minkowski. Suppose this is violated, i.e.

$$(\xi\eta) = 0, \quad \xi^2 - \eta^2 < 0, \quad (10.0.8)$$

for some ξ, η . Since η is timelike, $(\xi\eta) = 0$ implies that ξ is spacelike. But then $\xi^2 - \eta^2 = \xi^2 + (-\eta^2) > 0$. Thus the two conditions in (10.0.8) cannot both be true, and (10.0.7) is proved. \square

As the next step of implementing the strategy from chapter 9, we need to check that the constructed analytic continuations satisfy a powerlaw bound so that we can apply Theorem 9.0.1. Although we already constructed analytic continuation to the whole \mathcal{T}_n , we only need to check the bound on \mathcal{D}_n which is somewhat easier. The powerlaw bound follows from the following lemma.

Lemma 10.0.2. (a) Let $y = (\varepsilon + is, \mathbf{y})$, $\varepsilon, s \in \mathbb{R}$, $\mathbf{y} \in \mathbb{R}^{d-1}$. Then y^2 is bounded above and below in the absolute value, as follows:

$$\varepsilon^2 \leq |y^2| \leq |y|^2 \equiv |\varepsilon + is|^2 + \mathbf{y}^2; \quad (10.0.9)$$

(b) On \mathcal{D}_n ($n = 2, 3$), each $1/(x_{ij}^2)^h$ factor in (10.0.1), (10.0.2) ($h \in \mathbb{R}$) satisfies a powerlaw bound:

$$\left| \frac{1}{(x_{ij}^2)^h} \right| \leq \frac{|x_i - x_j|^B}{(\varepsilon_i - \varepsilon_j)^A} \quad (i > j), \quad (10.0.10)$$

where $A = 2h$, $B = 0$ for h positive and $A = 0$, $B = -2h$ for h negative.

Proof. (a) The upper bound is obvious. Let us show the lower bound by an explicit computation (see Lemma 11.5.1 below for an alternative proof). We have:

$$|y^2|^2 \equiv |(\varepsilon + is)^2 + \mathbf{y}^2|^2 = (\varepsilon^2 - s^2 + \mathbf{y}^2)^2 + 4\varepsilon^2 s^2, \quad (10.0.11)$$

Minimizing this in \mathbf{y} , we get

$$\min_{\mathbf{y}} |y^2|^2 = \begin{cases} (\varepsilon^2 - s^2)^2 + 4\varepsilon^2 s^2 = (\varepsilon^2 + s^2)^2, & |s| \leq \varepsilon, \\ 4\varepsilon^2 s^2, & |s| \geq \varepsilon. \end{cases}$$

Minimizing this next in s , we find that the absolute minimum is located at $\mathbf{y} = 0$, $s = 0$, and is equal to ε^4 . Part (b) follows from (a). \square

Now that we have the powerlaw bound, we can apply Theorem 9.0.1. We conclude that the Minkowski 2-point and 3-point functions, defined as $\varepsilon_i \rightarrow 0$ limits of the analytically continued Euclidean correlators, exist, are Lorentz-invariant tempered distributions, and satisfy the spectral condition.²

10.1 Comparison with the $i\varepsilon$ -prescription

Here we will comment on the “ $i\varepsilon$ -prescription” often used in the literature to define Minkowski 2-point and 3-point correlators, and how it compares with our definition. We will focus on the 2-point case for definiteness (same remarks hold for the 3-point case).

The $i\varepsilon$ -prescription defines the Minkowski 2-point correlator $G_2^M(x_1^M, x_2^M)$ as

$$\frac{1}{(-(s - i\varepsilon)^2 + \mathbf{y}^2)^\Delta}, \quad (10.1.1)$$

²Since these are tempered distributions, their Fourier transforms are well defined. Explicit expressions for these Fourier transforms are known in many cases. See [91] for $\langle \mathcal{O}_{\Delta, l}(p) \mathcal{O}_{\Delta, l}(-p) \rangle$ and [92] for $\langle \mathcal{O}_{\Delta_1}(p_1) \mathcal{O}_{\Delta_2}(p_2) \mathcal{O}_{\Delta, l}(p_3) \rangle$.

with $s = t_1 - t_2$, $\mathbf{y} = \mathbf{x}_1 - \mathbf{x}_2$ and taking the $\varepsilon \rightarrow 0^+$ limit. The precise meaning of the limit is often left implicit in the physics literature. Away from the light cone the 2-point correlator is an ordinary function, the limit can be understood pointwise and it agrees with our definition. Clearly, on the light cone the limit must be understood in distributional sense, integrating against a test function $f(s, \mathbf{y})$. That is what we showed above: Vladimirov's theorem guarantees that the limit $\varepsilon \rightarrow 0$ exists as a tempered distribution and can be therefore integrated against any Schwartz test function. In physics literature, one instead often hears that such integrals should be defined by "shifting the integration contour". Note however that this alternative way of understanding the $\varepsilon \rightarrow 0$ limit would only work for analytic test functions. Let us discuss the consequences of this limitation.

It is helpful to recall that the theory of distributions commonly uses three classes of test functions, denoted $\mathcal{S}, \mathcal{K}, \mathcal{Z}$ [93]. Here \mathcal{S} is the space of Schwartz functions, \mathcal{K} (denoted sometimes by \mathcal{D}) is the space of compactly supported C^∞ functions, and \mathcal{Z} consists of entire holomorphic functions decreasing faster than any power in the real directions and bounded by some fixed exponential in the imaginary directions. Note that $\mathcal{K}, \mathcal{Z} \subset \mathcal{S}$. The corresponding distribution spaces thus satisfy the opposite inclusion: $\mathcal{S}' \subset \mathcal{K}', \mathcal{Z}'$. The elements of \mathcal{S}' are precisely the tempered distributions discussed above, \mathcal{K}' are distributions on the compactly supported test functions³, while \mathcal{Z}' is yet another distributional class.

Importantly, the Fourier transform \mathcal{F} leaves \mathcal{S} invariant. Since the Fourier transform is defined in the theory of distributions by duality, we also have $\mathcal{F}(\mathcal{S}') = \mathcal{S}'$: the Fourier transform of a tempered distribution is also a tempered distribution. On the other hand, one can show (see [93]) that $\mathcal{F}(\mathcal{K}) = \mathcal{Z}$. This is the rationale behind introducing the space \mathcal{Z} , and this also implies that \mathcal{F} maps \mathcal{K}' to \mathcal{Z}' and vice versa.

Coming back to (10.1.1), shifting the integration contour defines this distribution as an element of \mathcal{Z}' . The pairing with a test function $f \in \mathcal{Z}$ is thus defined by

$$\int_C dz \int d\mathbf{y} \frac{1}{(-z^2 + \mathbf{y}^2)^\Delta} f(z, \mathbf{y}) \quad (10.1.2)$$

with the contour C running parallel to the real axis in the lower half plane.⁴ By the previous paragraph, this is then sufficient to define the Fourier transform of the 2-point function as an element of \mathcal{K}' . By moving the contour far away from the real axis, one shows that the Fourier transform vanishes for negative energies, and by Lorentz invariance one concludes that the support must belong to the forward light cone. These arguments have parallels in the proof of Part 2 of Theorem 9.0.1 (see App. B.3.2).

Compared to this simple and almost elementary discussion, Theorem 9.0.1 proves a stronger statement that the 2-point distribution (10.1.1) can be extended to test functions of Schwartz class and, furthermore, to functions which have only a finite number of derivatives as expressed by Eq. (B.3.3). This can be seen as a finer characterization of the singularity structure at short distances. The Fourier transform is then also a tempered distribution, thus bounded by some power, which is a stronger statement than it being an element of \mathcal{K}' since those can grow arbitrarily fast at infinity.

Since the 2-point and 3-point correlators are known in closed form, one can in principle verify that their Fourier transform does not grow too fast at infinity by an explicit computation. This would provide an alternative proof of temperedness. Our point here is that Theorem 9.0.1 reaches this conclusion without any computations. For the 4-point correlators considered below, the Fourier transform cannot be evaluated easily, and Theorem 9.0.1 appears to be the only realistic way to show temperedness.

It is instructive to discuss why we insist so much on temperedness. In other words, why Wightman axioms require that the Minkowski n -point correlators must be tempered distributions, and not of some other class? There is a simple reason why temperedness is a natural requirement, while \mathcal{K}' or \mathcal{Z}' would not suffice. The point is that Wightman axioms include **both** commutativity at spacelike separation and the spectral condition (the Fourier transform supported in the forward tube). Both these conditions need compactly supported test functions: the former in position space, the latter in momentum space. The space \mathcal{S} is large enough to write both these conditions, while \mathcal{K} or \mathcal{Z} are inadequate as we would lose one of them.⁵

³They are briefly mentioned in the proof of Theorem 9.0.1, App. B.3.2, after Eq. (B.3.13).

⁴We can somewhat relax the condition $f \in \mathcal{Z}$. At the very least, f must be holomorphic in the lower half-plane close to the real axis and decrease sufficiently fast at infinity for the integral to be convergent.

⁵For completeness it should be noted that one can reduce \mathcal{S} a bit and still be able to formulate both these axioms, as for Jaffe fields [94], which may have stronger-than-powerlaw singularities at short distances. For CFTs and for any theory which asymptotes to a CFT at short distances, there is no reasons to consider such modifications, and \mathcal{S} remains the natural choice.

Finally, sometimes by the $i\varepsilon$ -prescription one means the following simplified form of (10.1.1):

$$\frac{1}{(-s^2 + i0^+ \operatorname{sign}(s) + \mathbf{y}^2)^\Delta}, \quad (10.1.3)$$

which agrees with (10.1.1) away from the light cone. By Vladimirov's theorem, this defines a distribution for $s > 0$ (including the light cone) and another distribution for $s < 0$, but it is not an adequate starting point for defining the distribution around $(s, \mathbf{y}) = (0, 0)$.

Chapter 11

Scalar 4-point function

This section is the heart of this part of the thesis. In it we will show how to define Minkowski 4-point functions starting from Euclidean 4-point functions of a unitary CFT. We will follow the strategy of chapter 9 and in particular will rely on Theorem 9.0.1. To avoid inessential details, we will focus on the case of four identical scalars. Non-identical scalars can be treated by the same argument (see Sec. 11.10). Additional complications arise for spinning operators; this case is postponed to a future publication [78].

So, we consider the Euclidean CFT 4-point function of four identical scalar Hermitian primaries, which by conformal invariance can be written as:

$$G_4^E(c_E) \equiv \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{O}}}} g(c_E). \quad (11.0.1)$$

Here $c_E = (x_1, x_2, x_3, x_4)$ denotes an ordered configuration of four Euclidean non-coincident points ($x_k \in \mathbb{R}^d$, $x_i \neq x_j$), $\Delta_{\mathcal{O}}$ is the scaling dimension of \mathcal{O} , and $g(c_E)$ is a real function which depends only on the conformal class of c_E . It can be written as a function of two conformally invariant cross-ratios u, v :

$$g(c_E) = g(u, v), \quad u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (11.0.2)$$

Our plan is as follows. After a discussion of the basic issues involved in the analytic continuation of the 4-point function (Sec. 11.1), we will introduce a representation in terms of the radial coordinates $\rho, \bar{\rho}$ (Sec. 11.2), and use it to construct the analytic continuation to the whole forward tube \mathcal{T}_4 (Sec. 11.3). This construction works because $\rho, \bar{\rho}$ remain strictly inside the unit disc everywhere in the forward tube (Lemma 11.3.1 and Eq. (11.3.5)), a fundamental fact proved in Sec. 11.4. We then briefly review the well-established powerlaw bound on $g(\rho, \bar{\rho})$ with respect to $\rho, \bar{\rho}$, and prove a powerlaw bound on $|\rho(c)|, |\bar{\rho}(c)|$ with respect to $c \in \mathcal{T}_4$. Combining these powerlaw bounds together, we will get a powerlaw bound on the analytically continued 4-point function $G_4(c)$, which by Theorem 9.0.1 implies (as c approaches the Minkowski region) the existence of the boundary limit of $G_4(c)$ as a tempered distribution (Sec. 11.5). After establishing temperedness, we will derive the Minkowski conformal invariance (Sec. 11.6), Wightman positivity (Sec. 11.7), Wightman clustering (Sec. 11.8) and local commutativity (Sec. 11.9). Some of them do not rely on conformal properties: for these we will use the standard arguments given by Osterwalder and Schrader [1]. In Sec. 11.10, we will generalize the above results to non-identical scalars by using Cauchy-Schwarz arguments.

11.1 Informal introduction to basic issues

Here we wish to outline a few basic difficulties which must be dealt with when analytically continuing the 4-point function. We will be using u, v coordinates as an example, although we will see below that other coordinates will be more suitable for our task. Readers uninterested in philosophical discussions may skip directly to Sec. 11.2.

Given any point c of the forward tube, we can connect it to a Euclidean point c_E by a curve, and analytically continue the 4-point function along the curve (see Fig. 11.1.1, left). The forward tube being simply connected, the

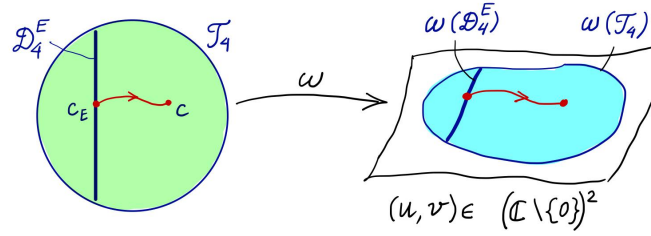


Figure 11.1.1: Illustration of the discussion in Sec. 11.1.

analytic continuation (if it exists) does not depend on the curve. Furthermore, let us take into account that our conformal 4-point function effectively only depends on two variables u, v . Applying Lemma 10.0.1, we see that u, v are both nonzero holomorphic functions on the forward tube. Consider the map:

$$\omega : c \rightarrow (u, v). \quad (11.1.1)$$

Since the forward tube is simply connected, we can consider this map as acting from \mathcal{T}_4 to $(\widetilde{\mathbb{C} \setminus \{0\}})^2$, where tilde denotes the universal cover. Denote by $\omega(\mathcal{T}_4)$ and $\omega(\mathcal{D}_4^E)$ the images of the forward tube and of its Euclidean part under this map (see Fig. 11.1.1, right).

Now suppose that we found an analytic continuation of $g(u, v)$ from $\omega(\mathcal{D}_4^E)$, where it is initially defined, to the whole of $\omega(\mathcal{T}_4)$. Then we could immediately write down the analytic continuation of the 4-point function to the forward tube as follows:¹

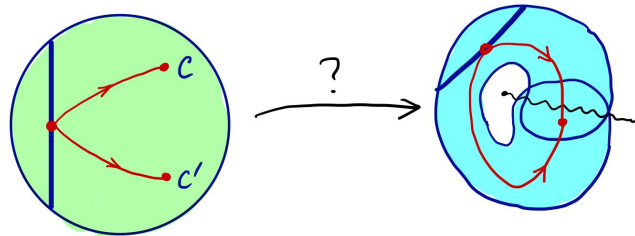
$$G_4(c) = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\sigma}} g(u(c), v(c)). \quad (11.1.2)$$

This formula defines the analytic continuation to the forward tube as a composition of two holomorphic functions:

$$\mathcal{T}_4 \xrightarrow{\omega} \omega(\mathcal{T}_4) \xrightarrow{g} \mathbb{C}. \quad (11.1.3)$$

We would like to use this strategy, but its direct implementation is hindered by a couple of difficulties:

- We don't know much about the shape or even topology of $\omega(\mathcal{T}_4)$. E.g. we don't know if this set is simply connected. The continuous image of a simply connected set, such as the forward tube, does not have to be simply connected (Fig. 11.1.2). If $\omega(\mathcal{T}_4)$ is not simply connected, there is no guarantee that $g(u, v)$ will be single-valued on it. And if $g(u, v)$ has branch cuts, then a simple formula like (11.1.2) using only the endpoint values $(u(c), v(c))$ will not work; we will need to know in addition “from which side of the cut” we got to this point along the analytic continuation contour (see Fig. 11.1.2).
- To be sure, we don't know if the above difficulty is actually realized. Perhaps the set $\omega(\mathcal{T}_4)$ is, after all, simply connected, and $g(u, v)$ has a single-valued analytic continuation to it. Even if this is the case, how can we construct this extension starting from $g(u, v)$ in the Euclidean region?

Figure 11.1.2: Illustration of a potential difficulty if the set $\omega(\mathcal{T}_4)$ were not simply connected (see Sec. 11.1).

¹The prefactor analytically continues just as the 2-point and 3-point functions in chapter 10.

In this part of the thesis we will circumvent these difficulties rather than attacking them head-on. In the Euclidean region, one often uses different variables to parametrize the cross-ratios u, v , such as the Dolan-Osborn variables z, \bar{z} , or the radial variables $\rho, \bar{\rho}$. As one can imagine, a smart choice of Euclidean variables can greatly simplify the analytic continuation. We will see that the radial variables are ideally suited for this task, allowing a natural resolution of the above-mentioned difficulties.

11.2 Euclidean 4-point function in radial coordinates

We first recall the well-known Dolan-Osborn variables z, \bar{z} [95, 96], which are two complex variables related to u, v by

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}), \quad \text{or} \quad (11.2.1)$$

$$z, \bar{z} = \frac{1}{2} \left(1 + u - v \pm \sqrt{(1+u-v)^2 - 4u} \right). \quad (11.2.2)$$

Since in the Euclidean case we only consider non-coincident points, we have $u, v \neq 0$, and hence $z, \bar{z} \neq 0, 1$. It is possible to fix a Euclidean conformal frame by setting the four points to positions

$$x_1 = 0, \quad x_2 = a\hat{e}_0 + b\hat{e}_1, \quad x_3 = \hat{e}_0, \quad x_4 = \infty\hat{e}_0, \quad (11.2.3)$$

where \hat{e}_μ is the standard orthonormal basis of \mathbb{R}^d . Using this frame, we obtain $z, \bar{z} = a \pm ib$. This shows that in the Euclidean, the variables z, \bar{z} are complex-conjugate ($\bar{z} = z^*$).

Euclidean configurations with real $z = \bar{z}$ correspond to four points lying on a circle, which maps in the frame (11.2.3) to four points on a line. The three possibilities $z < 0$, $z \in (0, 1)$, $z \in (1, +\infty)$ are then realized for different cycling orderings.

The radial variables $\rho, \bar{\rho} \in \mathbb{C}$ [43, 44] are defined in terms of the Dolan-Osborn variables by the formula:

$$\rho = f(z), \quad \bar{\rho} = f(\bar{z}), \quad f(w) := \frac{w}{(1 + \sqrt{1-w})^2}. \quad (11.2.4)$$

The function $f(w)$ in this definition² is the uniformization map for the complex plane minus the cut $(1, +\infty)$, i.e. it is a one-to-one map of $\mathbb{C} \setminus [1, +\infty)$ onto the unit disk. Eq. (11.2.4) thus associates with any Euclidean configuration a pair of complex conjugate $\rho, \bar{\rho}$ ($\bar{\rho} = \rho^*$) belonging to the unit disk: $|\rho| \leq 1$. Moreover we have $|\rho| < 1$ except for the Euclidean configurations with $z = \bar{z} \in (1, +\infty)$. As explained above, this happens when four points lie on a circle in the cyclic order 1324. For such exceptional configurations one may define $\rho, \bar{\rho}$ by continuity so that $|\rho| = 1$, $\bar{\rho} = \rho^*$.

The meaning of the coordinate ρ is clarified by mapping the 4-point configuration to a conformal frame (compare (11.2.3))

$$x_1 = -\alpha\hat{e}_0 - \beta\hat{e}_1, \quad x_2 = \alpha\hat{e}_0 + \beta\hat{e}_1, \quad x_3 = \hat{e}_0, \quad x_4 = -\hat{e}_0, \quad (11.2.5)$$

Using this frame, we obtain $\rho, \bar{\rho} = \alpha \pm i\beta$.

There is a small difference between $d = 2$ and $d \geq 3$ dimensions. In $d \geq 3$, conformal frames (11.2.3) and (11.2.5) are unique only up to a sign of b and β (flipped rotating by π in the 12 plane), which implies that pairs (z, \bar{z}) and $(\rho, \bar{\rho})$ are defined only up to permutation. On the other hand in $d = 2$ flipping the sign of b or β is a parity transformation, which is not in the identity component of the conformal group. Hence the conformal frames are unique and z, \bar{z} as well as $\rho, \bar{\rho}$ are individually meaningful.

In a unitary Euclidean CFT, the 4-point function admits a power-series expansion in the ρ coordinate, absolutely convergent whenever $|\rho| < 1$ [43]. Specifically, *the function $g(c_E)$ appearing in the 4-point function (11.0.1) of four identical scalar Hermitean primaries has a series expansion of the form*

$$g(c_E) = \sum_{\delta, m} p_{\delta, m} r^\delta e^{im\theta}, \quad (11.2.6)$$

²The definition assumes the standard branch of the square root function.

where the sum runs over a discrete set of pairs (δ, m) with $\delta \geq 0$, $m \in 2\mathbb{Z}$, and the variables $r, \theta \in \mathbb{R}$ are the modulus and the phase of $\rho(c_E) = re^{i\theta}$. The sum is absolutely convergent when $r = |\rho(c_E)| < 1$. In addition, we know that $|m| \leq \delta$ and $p_{\delta, m} \geq 0$ for all terms in (11.2.6). Finally, when $d \geq 3$ we have $p_{\delta, -m} = p_{\delta, m}$, so that the r.h.s. of (11.2.6) is uniquely defined in spite of $\rho(c_E)$ being defined only up to complex conjugation.

The readers familiar with this fact may skip to Sec. 11.3 where we will use it to perform analytic continuation. In the rest of this section we recall how it follows from the CFT axioms [18, 43].

We consider the 4-point function in the conformal frame configuration (11.2.5) and write it as the inner product of two states created by the operators outside and inside a unit sphere S centered at the origin:

$$\langle \mathcal{O}(1, 0, \mathbf{0}) \mathcal{O}(-1, 0, \mathbf{0}) | \mathcal{O}(\alpha, \beta, \mathbf{0}) \mathcal{O}(-\alpha, -\beta, \mathbf{0}) \rangle \quad (11.2.7)$$

We can find a conformal transformation which maps the sphere S to $x^0 = 0$ plane, its center 0 to x_S and the infinity to x_N . This is the setup in which we developed the CFT Hilbert space picture in Sec. 8.2. Applying the inverse transformation, we are allowed to use the Hilbert space language in the frame (11.2.7), which is the familiar setting of radial quantization. We decompose the radial quantization Hilbert space, produced by local operators inserted at the origin, in orthonormalized eigenstates $|\delta, m\rangle$ of the dilatation D and the planar rotation M_{01} . The ket state is expanded in this basis as

$$|\mathcal{O}(\alpha, \beta, \mathbf{0}) \mathcal{O}(-\alpha, -\beta, \mathbf{0})\rangle = \sum_{\delta, m} c_{\delta, m} r^{\delta - 2\Delta_\varphi} e^{im\theta} |\delta, m\rangle. \quad (11.2.8)$$

The dependence of the expansion coefficients in this formula on r and θ is fixed by knowing how the state in the l.h.s. transforms under rotations and dilatations. The transformation $\theta \rightarrow \theta + \pi$ swaps the two operators leaving the state invariant for the considered case of identical operators. Hence the state in the r.h.s. also must remain invariant, proving that m must be even.

Setting $r = 1, \theta = 0$ in (11.2.8), we get

$$|\mathcal{O}(1, 0, \mathbf{0}) \mathcal{O}(-1, 0, \mathbf{0})\rangle = \sum_{\delta, m} c_{\delta, m} |\delta, m\rangle. \quad (11.2.9)$$

In the considered frame the OS reflection is the inversion with respect to the sphere S : $x^\mu \rightarrow x^\mu/x^2$. In particular, this leaves x_3 and x_4 invariant. Applying this transformation to (11.2.9), we get

$$\langle \mathcal{O}(1, 0, \mathbf{0}) \mathcal{O}(-1, 0, \mathbf{0}) | = \sum_{\delta, m} c_{\delta, m}^* \langle \delta, m |. \quad (11.2.10)$$

Taking the inner product of (11.2.8) and (11.2.10), we get

$$\langle \mathcal{O}(1, 0, \mathbf{0}) \mathcal{O}(-1, 0, \mathbf{0}) | \mathcal{O}(\alpha, \beta, \mathbf{0}) \mathcal{O}(-\alpha, -\beta, \mathbf{0}) \rangle = \sum_{\delta, m} |c_{\delta, m}|^2 r^{\delta - 2\Delta_\varphi} e^{im\theta}. \quad (11.2.11)$$

Comparing this with Eq. (11.0.1), and using that $x_{12}^2 = 4r^2$, $x_{34}^2 = 4$ in the considered conformal frame, we obtain (11.2.6) with $p_{\delta, m} = 16^{\Delta_\varphi} |c_{\delta, m}|^2 \geq 0$.

In the above argument we chose for simplicity the sphere of radius 1, but any sphere of radius $r < r_0 < 1$ would work equally well and give rise to the same expression. Absolute convergence for $r < 1$ follows, because both the bra and the ket states are normalizable for such r_0 (while for $r_0 = 1$ as above the bra state $\langle \varphi(x_3) \varphi(x_4) |$ is not normalizable).

The restriction $|m| \leq \delta$ follows from the 2d unitarity bounds. The 2d unitarity bound applies, as any d -dimensional CFT restricted to a plane can be seen as a unitary 2d CFT. For 2d primaries of spin J and dimension Δ , the 2d unitarity bound says $|J| \leq \Delta$. The descendants at level $n \in \mathbb{Z}_{\geq 0}$ have $\delta = \Delta + n$, $|m - J| \leq n$, hence $|m| \leq \delta$ follows.

Finally, let us prove that $p_{\delta, m} = p_{\delta, -m}$ in $d \geq 3$. We consider Eq. (11.2.9) and perform a π rotation in the 12 plane. In the r.h.s. $|\delta, m\rangle \rightarrow |\delta, -m\rangle$ because $M_{01} \rightarrow -M_{01}$ under such a rotation. On the other hand the l.h.s. does not change. This implies that we must have $c_{\delta, m} = c_{\delta, -m}$, and hence $p_{\delta, m} = p_{\delta, -m}$. (In $d = 2$, these properties also hold under the additional assumption of parity invariance.)

11.3 Analytic continuation

In this section we will construct the analytic continuation of the Euclidean 4-point function (11.0.1) to the forward tube \mathcal{T}_4 (recall the forward tube definition (9.0.4)). Analytic continuation to $\mathcal{D}_4 \subset \mathcal{T}_4$ has already been given in [74], Sec. 3.4,³ and we will use a somewhat streamlined version of that construction. We will analytically continue to the full forward tube \mathcal{T}_4 , since this does not lead to additional complications.

The analytic continuation will be given by the formula

$$G_4(c) = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{D}}}} g(c), \quad c \in \mathcal{T}_4. \quad (11.3.1)$$

Here the prefactor trivially analytically continues to \mathcal{T}_4 similarly to the 2-point and 3-point functions discussed in chapter 10. We will construct $g(c)$, analytic continuation of $g(c_E)$, starting from Eq. (11.2.6).

First we have to define the variables $z(c)$, $\bar{z}(c)$ on the forward tube, which is naturally done as follows. Given a configuration $c \in \mathcal{T}_4$, we evaluate $u = u(c)$, $v = v(c)$ via (11.0.2). By Lemma 10.0.1, $u(c)$ and $v(c)$ are nonzero holomorphic functions on the forward tube. We then define $z(c)$, $\bar{z}(c)$ via (11.2.2):

$$z(c), \bar{z}(c) = \frac{1}{2} \left(1 + u(c) - v(c) \pm \sqrt{[1 + u(c) - v(c)]^2 - 4u(c)} \right). \quad (11.3.2)$$

Unlike for Euclidean configurations, for a general configuration $c \in \mathcal{T}_4$ these are two complex numbers unrelated by conjugation. Since $u(c)$ and $v(c)$ are nonzero, Eq. (11.2.1) implies $z(c), \bar{z}(c) \in \mathbb{C} \setminus \{0, 1\}$.

Since Eq. (11.3.2) only defines $z(c), \bar{z}(c)$ up to permutation, we view it as a map from the forward tube to $\mathbb{C}^2/\mathbb{Z}_2$, the set of unordered pairs of complex numbers. This map is continuous, and is analytic everywhere except on $\Gamma \subset \mathcal{T}_4$ where the expression under the square root vanishes:

$$\Gamma = \{c \in \mathcal{T}_4 : [1 + u(c) - v(c)]^2 - 4u(c) = 0\}. \quad (11.3.3)$$

Actually, it turns out that in $d = 2$ one can resolve the ambiguity inherent in Eq. (11.3.2) and define $z(c)$, $\bar{z}(c)$ as individually globally holomorphic functions on \mathcal{T}_4 . We will bring up this fact below when we need it. Ref. [74], App. A, showed that such an improvement is impossible in $d \geq 3$.

The following result is fundamental for our construction. The proof is elementary but a bit tricky and is postponed to Sec. 11.4.

Lemma 11.3.1. *For any $c \in \mathcal{T}_4$ we have $z(c), \bar{z}(c) \notin [1, +\infty)$.*

We next define $\rho(c)$, $\bar{\rho}(c)$ on \mathcal{T}_4 , via

$$\rho(c) = f(z(c)), \quad \bar{\rho}(c) = f(\bar{z}(c)), \quad (11.3.4)$$

where f is the same function as in (11.2.4), mapping $\mathbb{C} \setminus [1, +\infty)$ onto the unit disk. By Lemma 11.3.1, we then have

$$0 < |\rho(c)|, |\bar{\rho}(c)| < 1 \quad \text{for any } c \in \mathcal{T}_4.^4 \quad (11.3.5)$$

Moreover, $\rho(c)$ and $\bar{\rho}(c)$ are locally holomorphic away from Γ . Because of this, and since Eq. (11.2.6) for the 4-point function converges in the Euclidean for any $|\rho| < 1$, we may hope to use Eq. (11.2.6) to analytically continue $g(c)$ to the whole forward tube. We will now carry out this strategy. Note that some extra care is needed, because $\rho(c)$ and $\bar{\rho}(c)$ are, just as $z(c)$ and $\bar{z}(c)$, not globally holomorphic and are defined only up to permutation (except in $d = 2$, see below), and because (11.2.6) contains in general non-integer powers.

To begin with, we rewrite Eq. (11.2.6) equivalently as

$$g(c_E) = \sum_{\delta, 0 \leq m \leq \delta} (\rho \bar{\rho})^{\delta/2 - m/2} (p_{\delta, m} \rho^m + p_{\delta, -m} \bar{\rho}^m), \quad (11.3.6)$$

³Since the analytic continuation result is similar, we do not present that section of [74] in this thesis.

⁴Note that the converse is not true: the region in which $0 < |\rho|, |\bar{\rho}| < 1$ is larger than the forward tube. For example, it includes the extended forward tube (see Sec. 11.9.1).

Various pieces of this formula need to be analytically continued to the forward tube. Consider first

$$R(c) = \rho(c)\bar{\rho}(c), \quad (11.3.7)$$

which is a candidate for the analytic continuation of $\rho\bar{\rho}$ from the Euclidean region. We can view it as a composition of two functions: $c \mapsto (\rho(c), \bar{\rho}(c))$ which is a continuous function from \mathcal{T}_4 to $\mathbb{C}^2/\mathbb{Z}_2$ analytic away from Γ , followed by $(\rho, \bar{\rho}) \mapsto \rho\bar{\rho}$ which is a continuous holomorphic function from $\mathbb{C}^2/\mathbb{Z}_2$ to \mathbb{C} . Hence $R(c)$ is a continuous function on the forward tube, analytic everywhere except perhaps on Γ . However, manifold Γ has complex codimension one, and by an analogue of Riemann's theorem about removable singularities we conclude that $R(c)$ is in fact analytic also on Γ , and thus on the whole \mathcal{T}_4 .⁵

In addition, $R(c)$ is nonzero in the forward tube. Thus we can lift $R(c)$ to a holomorphic function $\tilde{R}(c)$ from the forward tube to the universal cover $\widetilde{\mathbb{C}\setminus\{0\}}$. Composing this function with $z^h : \widetilde{\mathbb{C}\setminus\{0\}} \rightarrow \mathbb{C}$, we obtain an analytic continuation of $(\rho\bar{\rho})^h$ for any $h \in \mathbb{R}$, which we denote by $R_h(c)$. This discussion mirrors the one around Eq. (10.0.4) in chapter 10. However, unlike x_{ij}^2 in that discussion, it is not true that $\tilde{R}(c)$ always belongs to the principal sheet of $\widetilde{\mathbb{C}\setminus\{0\}}$. So, in general, to compute the phase of the analytically continued function, one should follow the phase of $\rho\bar{\rho}$ along a curve joining c_E to c .

Following a curve is perfectly fine as a theoretical device. For practical computations of the phase, one may wish to use instead the following trick which avoids having to look at the curve. (The reader happy to follow the curve may skip the trick and go directly to Eq. (11.3.10).) Consider the identity:

$$\rho\bar{\rho} = \frac{1}{16}u(1+\rho)^2(1+\bar{\rho})^2 = \frac{1}{16} \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} Y^2, \quad Y = (1+\rho)(1+\bar{\rho}), \quad (11.3.8)$$

which follows by using $z = \frac{4\rho}{(1+\rho)^2}$, the inverse of the relation (11.2.4) between ρ and z , as well as $u = z\bar{z}$ and the expression for u . The function $Y(c) = (1+\rho(c))(1+\bar{\rho}(c))$ is holomorphic on \mathcal{T}_4 by the same ‘‘analyticity on $\mathcal{T}_4 \setminus \Gamma$ plus Riemann's theorem’’ argument as used above for $\rho\bar{\rho}$. In addition, and this is the key point, because $|\rho(c)|, |\bar{\rho}(c)| < 1$, we know that $Y(c) \in \mathbb{C} \setminus (-\infty, 0]$. The upshot of the trick is that Eq. (11.3.8) expresses $\rho\bar{\rho}$ as a product of factors which all remain on the principal sheet of z^h upon the analytic continuation. Hence we can compute the analytic continuation of $(\rho\bar{\rho})^h$ by

$$R_h(c) = \frac{1}{16^h} \frac{(x_{12}^2)^h (x_{34}^2)^h}{(x_{13}^2)^h (x_{24}^2)^h} Y(c)^{2h} \quad (h \in \mathbb{R}), \quad (11.3.9)$$

This determines the phase of $R_h(c)$ unambiguously without having to look at the curve joining c_E to c .

Next, we consider for an integer m a function

$$\Phi_m(c) = \rho(c)^m + \bar{\rho}(c)^m. \quad (11.3.10)$$

Just as $\rho\bar{\rho}$ and Y , it is continuous on \mathcal{T}_4 and holomorphic on $\mathcal{T}_4 \setminus \Gamma$, and thus holomorphic on the whole \mathcal{T}_4 .

We can now define the analytic continuation of (11.3.6). Consider first $d \geq 3$, when $p_{\delta, -m} = p_{\delta, m}$. In this case the analytic continuation is given by the formula

$$g(c) = \sum_{m, \delta, 0 \leq m \leq \delta} p_{\delta, m} R_{\delta/2 - m/2}(c) \Phi_m(c). \quad (11.3.11)$$

This series consists of holomorphic functions, and it reduces to (11.3.6) in the Euclidean region. Furthermore, every term in the series can be bounded in absolute value by:

$$\begin{aligned} |p_{\delta, m} R^{\delta/2 - m/2}(c) \Phi_m(c)| &\leq p_{\delta, m} |\rho(c)\bar{\rho}(c)|^{\delta/2 - m/2} (|\rho(c)|^m + |\bar{\rho}(c)|^m) \\ &\leq p_{\delta, m} r^{\delta - m} (r^m + r^m), \end{aligned} \quad (11.3.12)$$

⁵The precise argument is as follows. Let us keep all complex coordinates fixed and vary just one, say x_1^0 . There are two cases: either (11.3.3) is identically zero as a function of x_1^0 , or it is a nonzero polynomial of x_1^0 . In the first case $R(c)$ is trivially holomorphic in x_1^0 . In the second case (11.3.3) vanishes at most for a few isolated values of x_1^0 . We can then apply 1d Riemann's theorem to say that $R(c)$ is also analytic at those isolated points. By these arguments, we conclude that $R(c)$ is holomorphic in each variable separately. Finally, a continuous function of several complex variables holomorphic in each variable separately is jointly holomorphic [97].

where $r = r(c) = \max(|\rho(c)|, |\bar{\rho}(c)|)$, which is < 1 by Eq. (11.3.5). Here we used $p_{\delta, m} \geq 0$ in the first line, and $\delta - m \geq 0$ in the second line. The terms in the r.h.s. of (11.3.12) comprise a positive convergent series whose sum is the Euclidean 4-point function (11.3.6) evaluated at $\rho = \bar{\rho} = r(c)$. This proves that (11.3.11) converges uniformly on compact subsets of \mathcal{T}_4 , and hence defines a holomorphic function in \mathcal{T}_4 .

It remains to consider $d = 2$. As anticipated above, in this case the functions $z(c), \bar{z}(c)$ are individually globally holomorphic on \mathcal{T}_4 . This can be seen introducing coordinates (see [74], Sec. 3.5)

$$z_k = x_k^0 + ix_k^1, \quad \bar{z}_k = x_k^0 - ix_k^1, \quad k = 1, 2, 3, 4. \quad (11.3.13)$$

Then the explicit formulas for $z(c), \bar{z}(c)$ are given by:

$$z(c) = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \quad \bar{z}(c) = \frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_3 - \bar{z}_4)}{(\bar{z}_1 - \bar{z}_3)(\bar{z}_2 - \bar{z}_4)}. \quad (11.3.14)$$

The functions $\rho(c), \bar{\rho}(c)$ defined by (11.3.4) are also individually globally holomorphic on \mathcal{T}_4 . As a consequence, the functions $\rho(c)^m$ and $\bar{\rho}(c)^m$ are individually holomorphic in $d = 2$, and not just their sum (11.3.10). We can therefore define the analytic continuation of $g(c)$ by the formula (compare (11.3.11)):

$$g(c) = \sum_{m, \delta, 0 \leq m \leq \delta} R_{\delta/2 - m/2}(c) [p_{\delta, m} \rho(c)^m + p_{\delta, -m} \bar{\rho}(c)^m]. \quad (11.3.15)$$

This formula would be appropriate for non-parity invariant 2d CFTs which may have $p_{\delta, m} \neq p_{\delta, -m}$. Analyticity follows from the uniform convergence on compact subsets, by the same argument as for $d \geq 3$.

Finally, we wish to explain how the above construction may be translated into the language of Sec. 11.1, to see how the issues raised there are resolved. This is instructive but not strictly speaking necessary, so we will be schematic. In the 2d case, when $\rho(c), \bar{\rho}(c)$ are individually defined, the translation is in terms of the map

$$\Omega : c \mapsto (\rho(c), \bar{\rho}(c)) \in (\mathbb{D} \setminus \{0\})^2, \quad (11.3.16)$$

where \mathbb{D} is the open unit disk, and we lifted each of the maps $\rho(c), \bar{\rho}(c)$ to the universal cover of $\mathbb{D} \setminus \{0\}$. This map is the present analogue of ω in (11.1.1). The function $g(\rho, \bar{\rho})$ extends analytically to the whole $(\mathbb{D} \setminus \{0\})^2$, which makes it unnecessary to understand the precise shape of $\Omega(\mathcal{T}_4)$.

For $d \geq 3$, $\rho(c), \bar{\rho}(c)$ are defined only up to permutation. Translation can then be done in terms of their symmetric combinations $\rho\bar{\rho}, \rho + \bar{\rho}$. Any symmetric polynomial in $\rho, \bar{\rho}$, such as the r.h.s. of (11.3.10), can be expressed as a polynomial in these coordinates. Let then X be the image of $(\mathbb{D} \setminus \{0\})^2$ under the map $(\rho, \bar{\rho}) \mapsto (\rho\bar{\rho}, \rho + \bar{\rho})$. The following map is holomorphic on \mathcal{T}_4 :

$$\Omega : c \mapsto (\rho(c)\bar{\rho}(c), \rho(c) + \bar{\rho}(c)) \in \tilde{X}, \quad (11.3.17)$$

where we lifted to the universal cover. The above argument can be interpreted as showing that the function $g(\rho, \bar{\rho})$ extends analytically to the whole \tilde{X} . Understanding the precise shape of $\Omega(\mathcal{T}_4)$ is once again unnecessary.

11.4 Proof of $z, \bar{z} \notin [1, +\infty)$

Here we will prove Lemma 11.3.1 which played such a fundamental role in the previous section. Just as for Lemma 10.0.1, it will be helpful to use the Minkowski metric. Thus we pass from Euclidean complex coordinates $x_k \in \mathbb{C}^d$ to Minkowski complex coordinates $\zeta_k = (ix_k^0, \mathbf{x}_k) \in \mathbb{C}^{1, d-1}$. Definitions of u, v are then rewritten equivalently as

$$u = \frac{\zeta_{12}^2 \zeta_{34}^2}{\zeta_{13}^2 \zeta_{24}^2}, \quad v = \frac{\zeta_{23}^2 \zeta_{14}^2}{\zeta_{13}^2 \zeta_{24}^2}, \quad (11.4.1)$$

where $\zeta_{ij} = \zeta_i - \zeta_j$ and $\zeta^2 = -(\zeta^0)^2 + \zeta^2$. We denote

$$\hat{\zeta}_k = \xi_k + i\eta_k, \quad \xi_k, \eta_k \in \mathbb{R}^{1, d-1}. \quad (11.4.2)$$

We will thus use Minkowski norm for ξ 's, η 's and their differences. The forward tube condition on x_k is rewritten as $\eta_k - \eta_{k+1} \succ 0$ which is the notation for

$$\eta_k^0 - \eta_{k+1}^0 > 0 \quad \text{and} \quad -(\eta_k - \eta_{k+1})^2 > 0. \quad (11.4.3)$$

We will need the following lemma which is related to Lemma 10.0.1 (see the proof at the end of the section).

Lemma 11.4.1. *Let $\zeta = \xi + i\eta$ and $\eta^2 < 0$. Then*

(a) $\zeta^2 \neq 0$;

(b) Define $\zeta' = \xi' + i\eta'$ by

$$\zeta' = \zeta/\zeta^2, \quad (11.4.4)$$

which is finite by Part (a). Then η' belongs to the same causal part of the light cone (future or past) as η . I.e. $\eta \succ 0 \Rightarrow \eta' \succ 0$. Analogously, $\eta \prec 0 \Rightarrow \eta' \prec 0$.

Let us start the proof of Lemma 11.3.1. The z, \bar{z} are defined from u, v via (11.2.1). It is not hard to see from the first line of (11.2.1) that z, \bar{z} are precisely the two solutions of the quadratic equation

$$z^2 - (1 + u - v)z + u = 0. \quad (11.4.5)$$

We thus have to show that, assuming (11.4.3), this equation has no solutions which are real and belong to the interval $[1, +\infty)$.

Without loss of generality, we can assume that $\zeta_3 = 0$.⁶ Then we have $\eta_1, \eta_2 \succ 0$ while $\eta_4 \prec 0$. Then we apply Lemma 11.4.1 and map the configuration $(\zeta_1, \zeta_2, 0, \zeta_4)$ to the configuration $(\zeta'_1, \zeta'_2, \infty, \zeta'_4)$ with $\eta'_1, \eta'_2 \succ 0$ while $\eta'_4 \prec 0$. These relations imply $\eta'_{14} \succ 0, \eta'_{24} \succ 0$ which will be used below.

Since u, v are invariant under the inversion, we have (this can be checked by a direct computation)

$$u = \frac{(\zeta'_{12})^2}{(\zeta'_{24})^2}, \quad v = \frac{(\zeta'_{14})^2}{(\zeta'_{24})^2}, \quad (11.4.6)$$

and Eq. (11.4.5) reduces to

$$(\zeta'_{24})^2 z^2 - [(\zeta'_{24})^2 + (\zeta'_{12})^2 - (\zeta'_{14})^2]z + (\zeta'_{12})^2 = 0. \quad (11.4.7)$$

Using that $\zeta'_{12} = \zeta'_{14} - \zeta'_{24}$, this equation can be written equivalently as

$$(\zeta'_{14} + (z - 1)\zeta'_{24})^2 = 0. \quad (11.4.8)$$

Now let us suppose that $z \in [1, +\infty)$. Then

$$\text{Im}[\zeta'_{14} + (z - 1)\zeta'_{24}] = \eta'_{14} + (z - 1)\eta'_{24} \succ 0. \quad (11.4.9)$$

Then Eq. (11.4.8) is in contradiction with Lemma 10.0.1. Lemma 11.3.1 is demonstrated.

Proof of Lemma 11.4.1. This was shown in [89], footnote 74, and we reproduce the argument here for completeness. Part (a) is a partial case of Lemma 10.0.1 (for $\eta \prec 0$ we should apply it to the complex conjugate vector $\zeta^* = \xi - i\eta$). Let us show Part (b). To show that $\eta \succ 0 \Rightarrow \eta' \succ 0$, we write

$$\zeta' = \frac{\xi + i\eta}{\xi^2 - \eta^2 + 2i(\xi, \eta)} = \frac{(\xi + i\eta)(\xi^2 - \eta^2 - 2i(\xi, \eta))}{(\xi^2 - \eta^2)^2 + 4(\xi, \eta)^2}. \quad (11.4.10)$$

So, up to a positive factor, η' is given by

$$(\xi^2 - \eta^2)\eta - 2(\xi, \eta)\xi. \quad (11.4.11)$$

For $\xi = 0$ this is given by $(-\eta^2)\eta \succ 0$. More generally, this squares to

$$(\xi^2 - \eta^2)^2 \eta^2 + 4(\xi, \eta)^2 \xi^2 - 4(\xi, \eta)^2 (\xi^2 - \eta^2) = \eta^2 ((\xi^2 - \eta^2)^2 + 4(\xi, \eta)^2) < 0. \quad (11.4.12)$$

Therefore, for all ξ , we have that η' is timelike. Since we have shown that $\eta' \succ 0$ for $\xi = 0$, by continuity it follows that $\eta' \succ 0$ for all ξ .

Finally, the implication $\eta \prec 0 \Rightarrow \eta' \prec 0$ follows by complex conjugation.

⁶It is important to move ζ_3 (or ζ_2) to zero rather than ζ_1 or ζ_4 , because only then, after applying the inversion, one gets causal information not only on η'_k 's but also on some of their differences.

11.5 4-point function powerlaw bound

We wish to show next that the analytically continued 4-point function satisfies a powerlaw bound, so that we can apply Theorem 9.0.1. The prefactor in Eq. (11.3.1) satisfies a powerlaw bound by Lemma 10.0.2. Furthermore, Eq. (11.3.12) implies that the analytic continuation $g(c)$ constructed in Sec. 11.3 is bounded by a Euclidean 4-point function, namely:

$$|g(c)| \leq g_E(c_*), \quad (11.5.1)$$

where c_* is any Euclidean 4-point function configuration having $\rho(c_*) = \bar{\rho}(c_*) = r = r(c) = \max(|\rho(c)|, |\bar{\rho}(c)|)$. We choose the conformal frame (11.2.5):

$$c_* : \quad x_1 = -r\hat{e}_0, \quad x_2 = r\hat{e}_0, \quad x_3 = \hat{e}_0, \quad x_4 = -\hat{e}_0. \quad (11.5.2)$$

Using the convergent OPE in the $x_2 \rightarrow x_3, x_1 \rightarrow x_4$ channel, we have the asymptotics

$$G_4^E(c_*) \sim \frac{1}{(1-r)^{4\Delta_\varphi}} \quad (r \rightarrow 1). \quad (11.5.3)$$

The function $g_E(c_*)$ satisfies the same asymptotics up to a constant, being related to $G_4^E(c_*)$ via Eq. (11.0.1) by a factor which is non-singular in the $r \rightarrow 1$ limit. Since $g_E(c_*)$ is a positive monotonically increasing function for $0 \leq r < 1$ (see Eq. (11.2.6)), we conclude that it has a bound

$$g_E(c_*) \leq \frac{\text{const.}}{(1-r(c))^{4\Delta_\varphi}}, \quad (11.5.4)$$

and $|g(c)|$ by (11.5.1) satisfies the same bound.

The upshot of this discussion is that we will have a powerlaw bound on $G_4(c)$ if we manage to get a powerlaw bound on $\frac{1}{1-r(c)}$. We will next state and prove such a bound.

Before launching into the technical discussion, let us discuss intuitively why a result like this is expected to be true. We know (Lemma 11.3.1) that $|\rho(c)|, |\bar{\rho}(c)| < 1$ and now we wish to prove that $|\rho(c)|, |\bar{\rho}(c)|$ do not approach 1 too quickly as c goes to the Minkowski boundary of the forward tube. This may remind the reader of the Schwarz-Pick lemma, which says that if $f(w)$ is a function from a unit disk to itself and $f(0) = 0$, then $|f(w)| \leq |w|$, hence providing a bound on how fast $|f(w)|$ can approach 1 as $|w| \rightarrow 1$. In the 2d case, when $\rho(c)$ and $\bar{\rho}(c)$ are individually defined holomorphic functions in the forward tube, it is indeed possible to use the Schwarz-Pick lemma to prove a powerlaw bound on $\max(|\rho(c)|, |\bar{\rho}(c)|)$ [98]. It should be possible to generalize the Schwarz-Pick argument to any d , although we have not worked it out in full details.⁷ The proof below will be different and more direct: it will simply mimic the proof of Lemma 11.3.1, replacing all “ > 0 ” inequalities by “ $\geq \varepsilon$ ” with an explicit positive ε .

11.5.1 A powerlaw bound on $\frac{1}{1-r(c)}$

Let us introduce some notation. We will measure the size of a complex vector $\zeta \in \mathbb{C}^{1,d-1}$ by $|\zeta|$,

$$|\zeta|^2 = |\zeta^0|^2 + |\zeta^1|^2 + \dots + |\zeta^{d-1}|^2. \quad (11.5.5)$$

Clearly $|(\zeta_1, \zeta_2)| \leq |\zeta_1||\zeta_2|$. We also define for $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbb{R}^{1,d-1}$, and $\eta^2 < 0$ (i.e. timelike)

$$S(\zeta) = \max\left(\frac{1}{\sqrt{-\eta^2}}, |\zeta|\right). \quad (11.5.6)$$

Thus $S(\zeta)$ is large either if some component of ζ (real or imaginary) is large or if η approaches the light cone. Note that $S(\zeta) \geq 1$ for any ζ . We will never need $S(\xi + i\eta)$ for spacelike η .

Finally we consider an analogous function on \mathcal{T}_4 :

$$S(c) = \max_{i < j} S(\zeta_{ij}), \quad (11.5.7)$$

⁷For any d , the Schwarz-Pick lemma allows a natural generalization to holomorphic functions in the forward tube [99].

which becomes large if any of $S(\zeta_{ij})$ become large. We claim that there is the following bound (recall $r(c) = \max(|\rho(c)|, |\bar{\rho}(c)|)$)

$$\frac{1}{1-r(c)} \leq 720S(c)^{12} \quad (c \in \mathcal{T}_4). \quad (11.5.8)$$

This bound will be shown for any c in the forward tube, which is the natural setting. When we specify to $c \in \mathcal{D}_4 \subset \mathcal{T}_4$ [see Eq. (9.0.5)], we have

$$S(c) = \max_{i < j} \max \left\{ \frac{1}{|\epsilon_i - \epsilon_j|}, |x_i - x_j| \right\}. \quad (11.5.9)$$

Eq. (11.5.8) then becomes a powerlaw bound for $\frac{1}{1-r(c)}$ on \mathcal{D}_4 of the form (9.0.9), precisely as needed for applying Theorem 9.0.1.

The proof of the bound (11.5.8) will build upon the proof of $z, \bar{z} \notin [1, +\infty)$ given in Sec. 11.4. There we showed that z solves Eq. (11.4.8), which however is inconsistent for $z \in [1, +\infty)$ and c in the forward tube. Here we will use the same Eq. (11.4.8), but make the rest of the argument quantitative, by showing that if c stays away from the boundary or infinity of the forward tube, so that $S(c)$ is bounded, then both $z(c)$ and $\bar{z}(c)$ must stay a finite distance away from $[1, +\infty)$, as measured by an upper bound on $\frac{1}{1-r(c)}$ expressed by Eq. (11.5.8). The proof is straightforward but somewhat technical and we split it into a series of lemmas.

Lemma 11.5.1. *Let $\zeta = \xi + i\eta$, $\eta^2 < 0$. Then for any ξ*

$$|\zeta^2| \geq (-\eta^2). \quad (11.5.10)$$

Proof. This is a generalization of Lemma 10.0.2(a) and could be proven analogously. We give a slightly different proof for a change. We have

$$|\zeta^2|^2 = (\xi^2 - \eta^2)^2 + 4(\xi, \eta)^2 = (\xi^2)^2 + (\eta^2)^2 + 2[2(\xi, \eta)^2 - \xi^2\eta^2]. \quad (11.5.11)$$

The lemma now follows from the inequality:

$$2(\xi, \eta)^2 - \xi^2\eta^2 \geq 0. \quad (11.5.12)$$

Eq. (11.5.12) is obvious for $\xi^2 \geq 0$, so let us consider $\xi^2 < 0$. By Lorentz invariance and homogeneity it's enough to consider $\xi = (\pm 1, 0, \dots, 0)$ in which case the l.h.s. of (11.5.12) becomes $(\eta^0)^2 + \boldsymbol{\eta}^2 \geq 0$. \square

Then we have the following strengthening of Lemma 11.4.1(b):

Lemma 11.5.2. *Let $\zeta = \xi + i\eta$, $\eta^2 < 0$, and $\zeta' = \zeta/\zeta^2$. Then*

$$S(\zeta') \leq [S(\zeta)]^3. \quad (11.5.13)$$

Proof. We have

$$|\zeta'| = \frac{|\zeta|}{|\zeta^2|} \leq [\text{by Lemma 11.5.1}] \frac{|\zeta|}{-\eta^2} \leq S(\zeta)^3. \quad (11.5.14)$$

We also have (see the proof of Lemma 11.4.1, in particular Eq. (11.4.12)) that $\eta'^2 < 0$ and

$$\frac{1}{-\eta'^2} = \frac{|\zeta^2|^2}{-\eta^2} \leq [\text{by Lemma 11.5.1}] S(\zeta)^6. \quad \square$$

Lemma 11.5.3. *Let $\zeta_i \in \mathbb{C}^{1, d-1}$, $\eta_i \succ 0$ ($i = 1, 2$). Then*

$$S(\zeta_1 + \zeta_2) \leq S(\zeta_1) + S(\zeta_2). \quad (11.5.15)$$

Proof. We have $|\zeta_1 + \zeta_2| \leq |\zeta_1| + |\zeta_2|$ and $-(\eta_1 + \eta_2)^2 \geq -\eta_1^2 - \eta_2^2$ (since $\eta_1 \cdot \eta_2 < 0$). \square

Lemma 11.5.4. *Let $\Upsilon_i = \Phi_i + i\Psi_i \in \mathbb{C}^{1,d-1}$, $\Phi_i, \Psi_i \in \mathbb{R}^{1,d-1}$, $\Psi_i \succ 0$ ($i=1,2$), and z solves the equation*

$$(\Upsilon_1 + (z-1)\Upsilon_2)^2 = 0. \quad (11.5.16)$$

Then

$$1 - |\rho(z)| \geq \delta_0 := \frac{1}{45S^4}, \quad S = \max(S(\Upsilon_1), S(\Upsilon_2)). \quad (11.5.17)$$

Proof. Note that $z = 4\rho/(1+\rho)^2$, and so Eq. (11.5.16) can be rewritten as

$$((\rho+1)^2\Upsilon_1 - (\rho-1)^2\Upsilon_2)^2 = 0. \quad (11.5.18)$$

For $\rho = e^{i\alpha}$, multiplying this equation by $e^{-2i\alpha}$, it becomes

$$(\Upsilon)^2 = 0, \quad \Upsilon \equiv \left(2\cos\frac{\alpha}{2}\right)^2 \Upsilon_1 + \left(2\sin\frac{\alpha}{2}\right)^2 \Upsilon_2, \quad (11.5.19)$$

which contradicts Lemma 11.4.1(a), since $\text{Im } \Upsilon \succ 0$. So ρ cannot lie precisely on the unit circle (as we already knew). It should then not be surprising that it also cannot get too close to the unit circle, which is what (11.5.17) says. This can be shown by a straightforward although somewhat technical generalization of the above argument.

Denoting $\rho = re^{i\alpha} = e^{i\alpha} - \delta e^{i\alpha}$, $\delta = 1 - r > 0$, and multiplying (11.5.18) by $e^{-2i\alpha}$, it becomes

$$\left(\left(2\cos\frac{\alpha}{2} - \delta e^{i\alpha/2}\right)^2 \Upsilon_1 + \left(2\sin\frac{\alpha}{2} + i\delta e^{i\alpha/2}\right)^2 \Upsilon_2 \right)^2 = 0, \quad (11.5.20)$$

or

$$(\Upsilon + \Upsilon')^2 = 0 \quad (11.5.21)$$

with

$$\Upsilon = 4\cos^2\frac{\alpha}{2}\Upsilon_1 + 4\sin^2\frac{\alpha}{2}\Upsilon_2, \quad (11.5.22)$$

$$\Upsilon' = \kappa_1\Upsilon_1 + \kappa_2\Upsilon_2, \quad (11.5.23)$$

$$\kappa_1 = -4\cos\frac{\alpha}{2}\delta e^{i\alpha/2} + \delta^2 e^{i\alpha}, \quad \kappa_2 = 4i\sin\frac{\alpha}{2}\delta e^{i\alpha/2} - \delta^2 e^{i\alpha}. \quad (11.5.24)$$

So for δ small, Υ' is a small correction to Υ . We write $\text{Im}(\Upsilon + \Upsilon') = \Psi + \Psi'$, where

$$\Psi = \text{Im } \Upsilon = 4\cos^2\frac{\alpha}{2}\Psi_1 + 4\sin^2\frac{\alpha}{2}\Psi_2, \quad \Psi' = \text{Im } \Upsilon'.$$

We know that $\Psi \succ 0$. In addition we also have a lower bound on $-\Psi^2$:

$$-\Psi^2 \geq 16\cos^4\frac{\alpha}{2}(-\Psi_1^2) + 16\sin^4\frac{\alpha}{2}(-\Psi_2^2) \geq \frac{1}{S^2} \times 16 \min\left\{\cos^4\frac{\alpha}{2}, \sin^4\frac{\alpha}{2}\right\} = \frac{4}{S^2}. \quad (11.5.25)$$

We will now show that $-(\Psi + \Psi')^2$ remains strictly positive if $\delta < \delta_0$. This will imply, by Lemma 11.4.1(a), that Eq. (11.5.21) cannot hold, and hence we must have $\delta \geq \delta_0$, i.e. Eq. (11.5.17), proving the lemma.

To implement this natural strategy, we will need only crude estimates of the size of various terms. Note that $\delta_0 < 1$ since $S \geq 1$, so in particular we have $\delta^2 \leq \delta$. Using this we have the bounds $|\kappa_i| \leq 5\delta$, and hence an upper bound

$$|\Psi'| \leq |\Upsilon'| \leq 10\delta S. \quad (11.5.26)$$

We also have an upper bound $|\Psi| \leq 4S$. Using these, (11.5.25), and $\delta^2 \leq \delta$, we have:

$$\begin{aligned} -(\Psi + \Psi')^2 &= -\Psi^2 - 2(\Psi, \Psi') - (\Psi')^2 \geq \frac{4}{S^2} - 2|\Psi||\Psi'| - |\Psi'|^2 \\ &\geq \frac{4}{S^2} - 80\delta S^2 - 100\delta^2 S^2 \\ &\geq \frac{4}{S^2} - 180\delta S^2 = \frac{4(1 - \delta/\delta_0)}{S^2}, \end{aligned} \quad (11.5.27)$$

which is strictly positive for $\delta < \delta_0$. As explained above this proves the lemma. \square

Finally we can prove (11.5.8). We repeat the proof of Lemma 11.3.1 given in Sec. 11.4. As there, we reduce to configuration having $\zeta_3 = 0$ and obtain that z (as well as \bar{z}) is a solution of Eq. (11.4.8), which has the form (11.5.16) with

$$\Upsilon_1 = \zeta'_{14} = \zeta'_1 - \zeta'_4, \quad \Upsilon_2 = \zeta'_{24} = \zeta'_2 - \zeta'_4, \quad \zeta'_i = \zeta_i / \zeta^2 \quad (i = 1, 2, 4). \quad (11.5.28)$$

Let us write $\Upsilon_i = \Phi_i + i\Psi_i \in \mathbb{C}^{1,d-1}$, $\Phi_i, \Psi_i \in \mathbb{R}^{1,d-1}$. As was already pointed out in Sec. 11.4, we have $\Psi_i > 0$ ($i = 1, 2$). Furthermore, by Lemma 11.5.2 we know that $S(\zeta'_i) \leq S(c)^3$, and then applying Lemma 11.5.3 that $S(\Upsilon_i) \leq 2S(c)^3$. Thus Lemma 11.5.4 implies (11.5.8) (note that $720 = 45 \times 16$).

Remark 11.5.1. The bound (11.5.8) is not optimal. We will prove a better bound in chapter 12, by a different argument.

Let us recap. In Sec. 11.3 we have analytically continued the Euclidean 4-point function to the forward tube, and here we showed that this analytic continuation satisfies a powerlaw bound. Then by Theorem 9.0.1, the Minkowski 4-point function defined as the limit (9.0.8) exists, is a Lorentz-invariant tempered distribution, and satisfies Wightman spectral condition. In the remainder of this section we will show that this distribution is also conformally invariant (Sec. 11.6), that it satisfies the remaining Wightman axioms (positivity in Sec. 11.7, clustering in Sec. 11.8, and local commutativity in Sec. 11.9). Later in Sec. 13 we will also show that it can be computed by a convergent (in the sense of distributions) OPE.

Now that we know that the Minkowski 4-point function is a distribution everywhere, one may inquire about the regularity of this distribution. E.g. for some configurations the 4-point function is actually real-analytic (see part III). We will come back to this question in the conclusion section.

11.6 Conformal invariance

Conformal invariance of Euclidean 4-point function (11.0.1) can be described as invariance under finite conformal transformations $x \rightarrow x' = f(x)$,

$$\Omega_1 \Omega_2 \Omega_3 \Omega_4 G_4^E(x'_1, x'_2, x'_3, x'_4) = G_4^E(x_1, x_2, x_3, x_4), \quad (11.6.1)$$

where $\Omega_i = J(x_i)^{\Delta_{\mathcal{O}}}$ and $J(x) = \det(\partial f^\mu / \partial x^\nu)^{1/d}$ is the local scale factor. Alternatively, and equivalently, this can be expressed as invariance under infinitesimal conformal transformations, a conformal Ward identity, which says that the Euclidean correlator is annihilated by a sum of differential operators, one per point:

$$\sum_{i=1}^4 \mathcal{D}(x_i, \partial_{x_i}) G_4^E(x_1, x_2, x_3, x_4) = 0. \quad (11.6.2)$$

There is a differential operator per conformal group generator (∂^μ for P_μ , $x^\mu \partial^\nu - x^\nu \partial^\mu$ for $M_{\mu\nu}$, $x \cdot \partial + \Delta_{\mathcal{O}}$ for D , $x^2 \partial^\mu - 2x^\mu (x \cdot \partial) - 2x^\mu \Delta_{\mathcal{O}}$ for K_μ).

Since all these differential operators have polynomial coefficients, Ward identities (11.6.2) continue to hold in the forward tube for the function $G(x_1, x_2, x_3, x_4)$. Taking the limit to the Minkowski boundary, we obtain that the Minkowski 4-point function satisfies infinitesimal Minkowski conformal invariance expressed by the Ward identities.

The possibility to take the limit is guaranteed by the standard result that distributional limits commute with derivatives. Indeed, suppose that we have, in the sense of distributions, $\lim_{\varepsilon \rightarrow 0} f_\varepsilon = g$. This means that for any test function φ , we have $\lim_{\varepsilon \rightarrow 0} (f_\varepsilon, \varphi) = (g, \varphi)$. But then for any derivative ∂ ,

$$(\partial g, \varphi) = -(g, \partial \varphi) = -\lim_{\varepsilon \rightarrow 0} (f_\varepsilon, \partial \varphi) = \lim_{\varepsilon \rightarrow 0} (\partial f_\varepsilon, \varphi), \quad (11.6.3)$$

which implies that $\lim_{\varepsilon \rightarrow 0} \partial f_\varepsilon = \partial g$. A similar argument shows that the limit commutes with multiplication of distributions by polynomials. All this is analogous to how we prove Lorentz invariance of the Minkowski correlator in App. B.3.

So we have shown that the Minkowski 4-point function satisfies Lorentzian conformal Ward identities. This means that

$$\sum_{i=1}^4 (\mathcal{D}_i G_4^M, \varphi) = 0, \quad (11.6.4)$$

where \mathcal{D}_i are the analytic continuations of the Euclidean differential operators to Minkowski space, and the pairing with the Schwartz test functions is defined by integration by parts. Note that the conformal Ward identities in Minkowski space hold also at coincident points (i.e. the test function φ does not have to be zero at coincident points).

Now let us discuss invariance of Minkowski 4-point function under *finite* Lorentzian conformal transformations. Since G_4^M is a distribution, the appropriate form of writing is to transform the test function:

$$(G_4^M, \varphi) = (G_4^M, \varphi^f), \tag{11.6.5}$$

where $\varphi^f(x_1, \dots, x_4) = \varphi(f^{-1}(x_1), \dots, f^{-1}(x_4)) \prod_{i=1}^4 J(f^{-1}(x_i))^{\Delta_{\sigma}-d}$. However we have to be careful. This invariance is true not for every test function φ but only for an f -dependent subset of test functions.

Let f_t be a smooth family of Lorentzian conformal transformations connecting f to the identity: $f_0 = \text{id}$, $f_1 = f$. Suppose that

$$\varphi^{f_t} \text{ is a Schwartz function for any } f_t \text{ in the family.} \tag{11.6.6}$$

Then we can integrate infinitesimal conformal invariance and prove that (11.6.5) is true. For translations, Lorentz transformations and dilatations, Eq. (11.6.6) is clearly satisfied and Eq. (11.6.5) holds for any φ . However, for general conformal transformations, (11.6.6) may not necessarily be true. The problems will appear if f is singular on the support of φ , as φ^f may then not be a Schwartz function. As a concrete example, consider the Lorentzian special conformal transformation:

$$f(x) = \frac{x^\mu + x^2 b^\mu}{1 + 2x \cdot b + x^2 b^2}. \tag{11.6.7}$$

The corresponding scale factor is $J(x) = \frac{1}{1+2x \cdot b + x^2 b^2}$. Take for definiteness spacelike $b = \beta \hat{e}_1$, where $\beta > 0$ and \hat{e}_1 is the unit vector in the x^1 direction. The transformation (11.6.7) is then singular for $x^0 = \pm|x^1 + \beta^{-1}|$, where the scale factor blows up, i.e. on the light cone whose vertex is at $x = -\beta^{-1}\hat{e}_1$.⁸ Scaling β to zero we can connect the transformation (11.6.7) to the identity. Under this scaling the light cone of singularities moves away to infinity along the negative x^1 direction. Requirement (11.6.6), and hence finite invariance (11.6.5), will hold if the light cone of singularities, while moving away, does not touch the support of φ (see Fig. 11.6.1 for the 2d case).

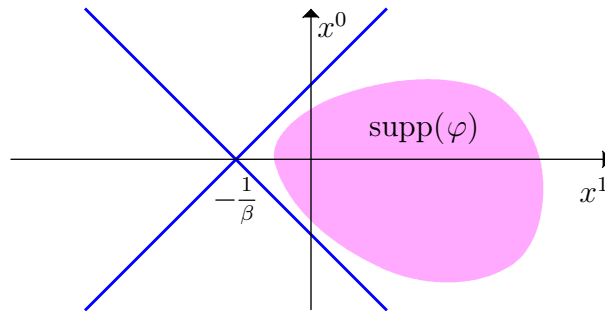


Figure 11.6.1: In the 2d case, the special conformal transformation (11.6.7) is singular on the blue light cone $x^0 = \pm|x^1 + \beta^{-1}|$. Suppose φ is supported as shown on the right of the light cone. As $\beta \rightarrow 0$, the light cones moves towards the left infinity and does not touch $\text{supp}(\varphi)$. Therefore, such a φ satisfies the condition for the invariance under a finite special conformal transformation (11.6.7).

Note that such a support requirement still leaves possibility for both spacelike and timelike separations among the points x_i in the support of φ . For $x_i \in \text{supp} \varphi$, the points $f(x_i)$ will have the same causal structure as the points x_i , i.e. $(f(x_i) - f(x_j))^2$ will have the same sign as $(x_i - x_j)^2$. This follows from the fact that $J(x) > 0$ on $\text{supp}(\varphi)$, as guaranteed by being able to continuously connect to the identity without singularities.

In the early CFT days, it was considered puzzling that Lorentzian special conformal transformations may change the causal structure of a point configuration. As we see here, the puzzle can be avoided by either limiting to infinitesimal conformal invariance, or by restricting the class of test functions so that the causal structure is preserved. A third way to deal with the puzzle is to consider the Lorentzian conformal transformations acting on the Lorentzian cylinder as opposed to the Minkowski space [25]. We will revisit the Lorentzian cylinder in our future publication [41].

⁸Recall that we are using $-, + \dots +$ Minkowski signature.

Remark 11.6.1. We would like to contrast the Minkowski conformal Ward identities (11.6.4) with conformal Ward identities valid for Euclidean correlators. Euclidean correlators are real-analytic away from coincident points and naturally satisfy conformal Ward identities for such configurations. Although in this part of the thesis we don't need it, in some questions it might be useful to extend Euclidean correlators, in the sense of distributions, also to coincident points. One may ask if such an extension can be done in a way so that the resulting distributional correlators satisfy conformal Ward identities analogous to (11.6.4). In general the answer is no, already for 2-point functions. Namely 2-point functions of primaries of dimension Δ such that $2\Delta - d \in \mathbb{Z}$ will in general not allow even a scale invariant extension at coincident points, let alone conformally invariant one. E.g. this feature will always be present for the stress tensor 2-point function.

11.6.1 Conformal invariance in terms of cross ratios

So as we have just seen, Minkowski correlator G_4^M is conformally invariant. If it were a function, conformal invariance would imply that we could write it as the usual prefactor times a function of the cross-ratios. Since it is a distribution, one might hope that it can be written as the prefactor times a *distribution* of the cross-ratios. We will now develop this point of view and show that it indeed works, at least locally.⁹

Our goal will be to make sense of the formula:

$$G_4(c) = \frac{g(z(c), \bar{z}(c))}{(x_{12}^2 x_{34}^2)^{\Delta_\sigma}}, \quad (11.6.8)$$

where $g(z, \bar{z})$ will be in general a distribution in two variables, and $g(z(c), \bar{z}(c))$ its pullback to the space \mathbb{R}^{4d} of Minkowski 4-point configurations c . This equation will be understood in the sense of integrating both parts with a test function. We will only consider compactly supported C^∞ test functions $\varphi(c)$, with the additional requirement that all $c \in \text{supp } \varphi$ have the same causal ordering. In particular, this implies that $\text{supp } \varphi$ contains no c 's with lightlike separated pairs. The causal ordering of a configuration $c = (x_1, x_2, x_3, x_4)$ is encoded by the directed graph with vertices 1, 2, 3, 4 and edges $i \rightarrow j$ if x_j belongs to the open future light cone of x_i (no edge if two points are spacelike).

Since u, v are real in Minkowski space, z, \bar{z} are either both real (excluding 0, 1) or complex conjugate. Part III divided all causal orderings into 4 classes according to possible values of (z, \bar{z}) :

- Class S: $(z, \bar{z}) \in (0, 1) \times (-\infty, 0)$ or the other way around
- Class T: $(z, \bar{z}) \in (0, 1) \times (1, +\infty)$ or the other way around
- Class U: $(z, \bar{z}) \in (-\infty, 0) \times (1, +\infty)$ or the other way around
- Class E causal orderings which contain configurations realizing the remaining possibilities:
 - E_{su} : $(z, \bar{z}) \in (-\infty, 0) \times (-\infty, 0)$
 - E_{st} : $(z, \bar{z}) \in (0, 1) \times (0, 1)$
 - E_{tu} : $(z, \bar{z}) \in (1, +\infty) \times (1, +\infty)$
 - E_{stu} : z, \bar{z} are complex-conjugate and not real

Some class E causal orderings realize only one of the four subclasses, while others contain configurations in each subclass. In the latter case different subclasses are connected along configurations with $z = \bar{z}$ (see Fig. 19.1.1).

To simplify the discussion, we will assume that $\text{supp } \varphi$ does not include any configurations with $z = \bar{z}$. In particular, this implies that all configurations from $\text{supp } \varphi$ fall into a single class S, T, U or a single subclass E_{su} , E_{st} , E_{tu} , E_{stu} . Below we will comment how one can add the $z = \bar{z}$ configurations.

⁹One does not expect a very nice global statement. Indeed, the cross-ratio space is morally the moduli space of four points on Minkowski cylinder \mathcal{M} . (We ignore the fact that we have not constructed the distributions on Minkowski cylinder yet in favor of having an actual action of the conformal group.) This is a quotient space $(\mathcal{M}^4)/G$ where G is the universal cover of Lorentzian conformal group. This quotient space is rather singular, which has to do with different configurations in \mathcal{M}^4 having different stability subgroups (light-cones, $z = \bar{z}$). The quotient space $(\mathcal{M}^4)/G$ is not only not smooth, it is not even Hausdorff. So away from some regular regions of $(\mathcal{M}^4)/G$ one shouldn't expect a simple statement of the form (11.6.8), unless one finds a smoother model of this moduli space.

If $\text{supp } \varphi$ falls into class S, E_{su} , E_{st} , E_{stu} , we will have $|\rho|, |\bar{\rho}| < 1$. These cases do not require special treatment, since the correlator is a function, and Eq. (11.6.8) is true in the ordinary sense of functions.

If $\text{supp } \varphi$ falls into class T or U, we will have $|\rho| < 1, |\bar{\rho}| = 1$ or the other way around. Then $g(z, \bar{z})$ will be a function in z and a distribution in \bar{z} .¹⁰ This case can be treated analogously, and simpler, than the $|\rho|, |\bar{\rho}| = 1$ case discussed below.

Finally, if $\text{supp } \varphi$ falls into class E_{tu} , we will have $|\rho|, |\bar{\rho}| = 1$. Then $g(z, \bar{z})$ will generally be a distribution in two variables. This is the case we will focus on. E.g. it is realized for the causal ordering $1 \rightarrow 3 \rightarrow 2 \rightarrow 4$.

Let us define the distribution $g(z, \bar{z})$ for $z, \bar{z} \in (1, +\infty)$. We first define the distribution $g(\rho, \bar{\rho})$ with $|\rho|, |\bar{\rho}| = 1$. This is done using the series in the r.h.s. of Eq. (11.3.6), which we now consider as a function of two independent variables $\rho, \bar{\rho}$. To be precise we consider the series:

$$g(\rho, \bar{\rho}) = \sum_{\delta, 0 \leq m \leq \delta} e^{i\Phi(\delta/2 - m/2)} (\rho\bar{\rho})^{\delta/2 - m/2} (p_{\delta, m} \rho^m + p_{\delta, -m} \bar{\rho}^m), \quad (11.6.9)$$

which we view as a holomorphic function on $(\mathbb{D} \setminus (-1, 0])^2$. Here $e^{i\Phi}$, $\Phi \in \{0, \pm 2\pi, \pm 4\pi\}$, is the phase acquired by $\rho(c)\bar{\rho}(c)$ upon analytic continuation from Euclidean space (as discussed in Sec. 11.3 this phase is the same as for $u(c)$). This phase is constant for each causal ordering and it may be determined by following a path from c_E to c for any particular c . Alternatively, the phase can also be determined from (11.3.8). E.g. the causal ordering $1 \rightarrow 3 \rightarrow 2 \rightarrow 4$ has $\Phi = 0$.¹¹

It's easy to see that function (11.6.9) satisfies a powerlaw bound as $|\rho|, |\bar{\rho}| \rightarrow 1$. This is a baby version of the problems studied in this part of the thesis, which was considered in part I. The limit of $g(re^{i\theta_1}, re^{i\theta_2})$ as $r \rightarrow 1$ defines a tempered distribution on the boundary of the domain of analyticity, parametrized by the two angles θ_1, θ_2 . We can write it as $g(\rho, \bar{\rho})$, with $\rho, \bar{\rho} \in S^1$.¹²

In fact we are interested only in a part of this distribution, because $\rho, \bar{\rho} \neq \pm 1$ for each fixed causal ordering. The points $-1, 1$ divide the circle into two open arcs, and within $\text{supp } \varphi$, ρ and $\bar{\rho}$ will each live in one or the other arc. Each arc is mapped smoothly and one-to-one to $(1, +\infty)$ by the $\rho \mapsto z$ map. Thus we obtain the distribution $g(z, \bar{z})$ defined for $z, \bar{z} > 1$. Although in general $z(c), \bar{z}(c)$ are defined only up to permutation, let us define them in the case at hand, with real $z \neq \bar{z}$, so that $\bar{z}(c) > z(c)$.

Now let us go back to making sense of (11.6.8). Suppose first $g(z(c), \bar{z}(c))$ were a function. Integrating (11.6.8) against a test function we have:

$$\int d^4c G_4(c) \varphi(c) = \int d^4c g(z(c), \bar{z}(c)) \tilde{\varphi}(c), \quad \tilde{\varphi}(c) = \frac{\varphi(c)}{(x_{12}^2 x_{34}^2)^{\Delta_\sigma}}. \quad (11.6.10)$$

Note that $\tilde{\varphi}(c)$ is still C^∞ since we are away from light cones. We would like to continue by expressing the r.h.s. of the previous equation as an integral of $g(z, \bar{z})$ against a two-dimensional test function:

$$\begin{aligned} \int d^4c g(z(c), \bar{z}(c)) \tilde{\varphi}(c) &= \int dz d\bar{z} g(z, \bar{z}) \psi(z, \bar{z}), \\ \psi(x_1, x_2) &= \int d^4c \delta(x_1 - z(c)) \delta(x_2 - \bar{z}(c)) \tilde{\varphi}(c). \end{aligned} \quad (11.6.11)$$

We would like to know if $\psi(x_1, x_2)$ is a smooth function. By our assumptions, $\tilde{\varphi}(c)$ is supported away from $\bar{z}(c) = z(c)$. In this region the map $c \rightarrow (z(c), \bar{z}(c))$ is a submersion, which means that the Jacobian has maximal rank (i.e. 2). Alternatively, this means that the form $dz \wedge d\bar{z}$ does not vanish anywhere away from $z = \bar{z}$. Showing this is a matter of an easy computation.¹³

¹⁰For some (but not all) of these causal orderings, it can be shown using another OPE channel that $g(z, \bar{z})$ is actually a function of both variables. See part III.

¹¹We have that $x_{12}^2, x_{34}^2, x_{13}^2, x_{24}^2$ all acquire phase $-\pi$, hence $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ acquires phase 0.

¹²In part I we also discussed a more general distribution defined on the product of universal covers of two circles. Here Eq. (11.6.9) with fixed α will be sufficient for our purposes.

¹³Start by noting that, away from $z = \bar{z}$, we have $dz \wedge d\bar{z} \propto du \wedge dv$ with a nonvanishing prefactor. We need to understand where ∇u can become proportional to ∇v . Using the embedding space formalism [100] we write $u = \frac{(X_1 X_2)(X_3 X_4)}{(X_1 X_3)(X_2 X_4)}$, $v = \frac{(X_1 X_2)(X_3 X_4)}{(X_1 X_3)(X_2 X_4)}$ where X_i

Using the fact that $c \rightarrow (z(c), \bar{z}(c))$ is an submersion, it's easy to show that $\psi(x_1, x_2)$ is smooth for $\tilde{\varphi}(c)$ supported away from $\bar{z}(c) = z(c)$ (see Chapter III.1 of [93] for such arguments). To summarize, for every smooth function $\varphi(c)$ compactly supported away from $\bar{z}(c) = z(c)$ and from the light cones, we constructed a smooth function $\psi(z, \bar{z})$ compactly supported in $1 < z < \bar{z}$ such that

$$\int d^{4d}c G_4(c) \varphi(c) = \int dz d\bar{z} g(z, \bar{z}) \psi(z, \bar{z}) \quad (11.6.12)$$

holds in case $g(z, \bar{z})$ is a function. We now claim that this equation continues to hold, with the same ψ , in case $g(z, \bar{z})$ is a distribution. The point is that we can find a sequence of functions $g_n(z, \bar{z})$ which tend to $g(z, \bar{z})$ in the sense of distributions, so that the corresponding $\frac{g_n(z(c), \bar{z}(c))}{(x_1^2 x_2^2 x_3^2 x_4^2)^{\Delta_{\mathcal{O}}}}$ tend to $G_4(c)$ in the sense of distributions on \mathbb{R}^{4d} . Since both φ and ψ are smooth, we are allowed to pass to the limit on both sides of the equation, proving the claim. The functions $g_n(z, \bar{z})$ are given e.g. by the partial sums of the series (11.6.9), transformed from the ρ to the z coordinates.

Let us now discuss how configurations where $z = \bar{z}$ can be included into this discussion. The basic difficulty is that the map $c \mapsto (z, \bar{z})$ fails to be a submersion near such configurations. So in general the function $\psi(z, \bar{z})$ will not be smooth. Consider e.g. the causal ordering $1 \rightarrow 3 \rightarrow 2 \rightarrow 4$. In this case it's possible to show (we omit the proof) that the function $\psi(z, \bar{z})$ behaves like

$$|z - \bar{z}|^{d-2} \text{ times a smooth function near } z = \bar{z}, \quad (11.6.13)$$

which in general is not smooth unless d is even.

We need to be able to make sense of the r.h.s. of (11.6.12) for such non-fully-smooth test functions. This is possible due to the following observation. Above we explained, following the arguments first presented in part I, that $g(\rho, \bar{\rho})$ is a distribution for $|\rho|, |\bar{\rho}| = 1$. But in fact it's a bit better than that (the fact not mentioned in part I): it is a distribution in ρ for each fixed value of $\bar{\rho}/\rho = e^{i\alpha}$! Indeed if we substitute $\bar{\rho} = e^{i\alpha}\rho$ with a fixed α into (11.6.9), we get a holomorphic function in the unit disk of ρ , which satisfies a powerlaw bound, hence its boundary value is a distribution. This can be generalized to holomorphic maps $\bar{\rho} = f(\rho)$ which maps the unit disk into itself (or at least a portion of the unit disk near $\rho = \rho_0$ into the unit disk). Translating to z, \bar{z} , this implies in particular that $g(z, z+t)$ is a distribution for any fixed t . In fact, the map $\bar{z} = z+t$ corresponds to a map $\bar{\rho} = f_t(\rho)$ to which the previous argument is applicable. So $g(z, \bar{z})$ is by no means the most general distribution in two variables, as it allows the restriction to the submanifold $\bar{z} = z+t$ for any t . E.g. $\delta(z - \bar{z})$ is not allowed by this property, while $\delta(z + \bar{z})$ is allowed. Following this logic a bit more carefully, it can be shown (we omit the proof) that $g(z, \bar{z})$ can be paired with test functions $\psi(z, \bar{z})$ which, when expressed in terms of $s = z + \bar{z}$, $t = \bar{z} - z$, have the following property: $\psi(s, t)$ is C^∞ with respect to s for any fixed t , with bounds on derivatives in the s direction which are integrable in the t direction. Eq. (11.6.13) is compatible with this requirement.

A further complications arises near the $z = \bar{z} > 1$ locus for the causal orderings which include configurations in both E_{tu} and E_{stu} subclasses. In this case the function $\psi(z, \bar{z})$ defined in (11.6.11) will consist of two functions $\psi_1(z, \bar{z})$ and $\psi_2(z, \bar{z})$: one defined for real z, \bar{z} , another for complex-conjugate z, \bar{z} . The two functions ψ_i will be glued along the $z = \bar{z} > 1$ line. The resulting glued function will not in generally be smooth on the $z = \bar{z} > 1$ line (while it will be smooth away from it). However the directions orthogonal to the line turn out analogous to the t direction in the previous paragraph, i.e. the test function is actually not required to be smooth in these directions for the pairing to be defined. This allows to make sense of the formula (11.6.12) also in this case. We omit the details.

11.6.2 Fixing points

We would like to put the results of the previous section in the context of a general question of ‘‘fixing points’’ in a distribution. E.g. we know that the Minkowski 4-point function is a translationally invariant distribution. Using translation invariance we can always fix one of the 4 points to a given position, e.g. zero, and consider it as a distribution with respect to the remaining 3 positions. One could ask if one can do better than that, i.e. to fix n

are null cone $d+2$ dimensional vectors. For any X_i, X_j, X_k the direction $R_{i,jk} = X_j(X_i X_k) - X_k(X_i, X_j)$ is tangent to the null cone at X_i . Imposing $R_{i,jk} \cdot \nabla_{X_i}(u - \alpha v) = 0$ for all unequal i, j, k where α is a constant, one gets a set of simple algebraic constraints on u, v . These constraints can be easily solved to show that $\alpha = \frac{2u}{-1+u+v}$ while $(1+u-v)^2 - 4u = 0$. The latter is precisely the constraint characterizing $z = \bar{z}$.

points to given positions and consider the 4-point function as a distribution with respect to the remaining $4 - n$ positions. Where the 4-point function is real-analytic we can of course consider all four points as fixed.

Now, results of Sec. 11.6.1 show that, if one excludes lightlike separations limiting to configurations having some fixed causal ordering, one can fix a conformal frame, i.e. fix three points to some fixed positions, and the fourth point to a position characterized by two conformal cross ratios, and consider the distribution as a distribution in only two variables (cross ratios). It is not clear if results of Sec. 11.6.1 can be generalized to cover lightlike separations.

In some cases it is possible to argue that one can fix more than one point without using conformal invariance. E.g. we may always fix a consecutive pair of points, i.e. (x_k, x_{k+1}) , where $k = 1, 2$ or 3 , to spacelike-separated positions in Minkowski space, while allowing the remaining two points to approach Minkowski limit from the forward tube. The proof of Lemma 11.3.1 can be slightly modified to show that $|\rho|, |\bar{\rho}| < 1$ for such configurations (see Sec. 11.9.1 below). Moreover, a powerlaw bound also holds, by a slight modification of the argument after Eq. (11.5.28).¹⁴ Then our arguments show that the Minkowski 4-point function is a distribution with respect to the two unfixed coordinates, which depends analytically on the fixed coordinates. In this case the unfixed coordinates may have any causal orderings and also lightlike separation.

One interesting case is that of the double light cone (DLC) singularity, i.e. the region close to $x_1 = 0, x_3 = \hat{e}_1, x_4 = \infty$, while x_2 on the light cones of x_1, x_3 . Our results are the first ones which establish the existence of the Wightman 4-point function in a neighborhood of DLC. However, there is a difference between restricting to one causal ordering near DLC or studying an open neighborhood of DLC which includes several causal orderings (see Fig. 15.0.1 in Conclusions). In the former case we can use directly the results of Sec. 11.6.1 and represents the 4-point function as a distribution in two variables z, \bar{z} . In the latter case we can fix, by the above argument, two successive spacelike points x_3 and x_4 . We are left with a distribution depending on x_1, x_2 , i.e. $2 \times d$ coordinates. This distribution still satisfies conformal invariance Ward identities w.r.t. infinitesimal conformal transformations preserving x_2 . It would be interesting to understand how this constrains the distribution at the DLC.

Although it is not directly related, we would also like to mention here the classic result of Borchers [101] which says that it is enough to smear Wightman functions $G_M(x_1, \dots, x_n = 0)$ with respect to the time variables only, i.e. integrating with respect to $h_1(x_1^0) \dots h_{n-1}(x_{n-1}^0)$ where $h_i \in \mathcal{S}(\mathbb{R})$, after which they become C^∞ functions in the remaining spatial variables \mathbf{x}_i . This result is valid in any QFT satisfying Wightman axioms. It holds because smearing in time, which acts as an energy cutoff, is effectively also a momentum cutoff because $|\mathbf{p}| \leq E$.

11.7 Wightman positivity

Recall that in Sec. 8.1 we showed that CFT axioms imply OS reflection positivity for 4-point functions. That discussion gives us access to OS states $|\mathcal{O}(x)\mathcal{O}(y)\rangle$ with $0 > x^0 > y^0$, with finite norm, and inner products measured by the Euclidean 4-point function. We know that these states belong to the CFT Hilbert space, i.e. can be arbitrarily well approximated in norm by states produced by inserting finite linear combinations of CFT local operators at one point in the half-space $x^0 < 0$, e.g. the south pole $x_S = (-1, \mathbf{0})$.

Now that we analytically continued the 4-point function, we can consider other states involving operators at complexified coordinates. We wish to prove that those states belong to the CFT Hilbert space and have a positive definite inner product. This can be shown by a robust argument, going back to Osterwalder and Schrader [1], Sec. 4.3. The argument uses only OS positivity and the Fourier-Laplace representation, but not directly the CFT axioms.

We will consider two new kinds of states. First, states generated by a pair of Minkowski operators smeared with respect to an arbitrary Schwartz test function:

$$|\Psi_M(F)\rangle = \int dx dy F(x, y) |\mathcal{O}(ix^0, \mathbf{x})\mathcal{O}(iy^0, \mathbf{y})\rangle, \quad (11.7.1)$$

¹⁴Since $S(c) = \infty$ in these cases, we cannot rely on (11.5.8). Instead we directly show powerlaw bounds on $S(\Upsilon_1), S(\Upsilon_2)$ defined in Eq. (11.5.28). Then the powerlaw bound on $|\rho|, |\bar{\rho}|$ holds by Lemma 11.5.4. For $k = 1$, by fixing $\zeta_3 = 0$, and using Lemmas 11.5.2 and 11.5.3, we have $S(\Upsilon_i) \leq S(\zeta'_i) + S(-\zeta'_i) \leq S(x_{i3})^3 + S(x_{i4})^3$ ($i = 1, 2$). This is the desired powerlaw bound with respect to x_3 and x_4 . For $k = 2$, $S(\Upsilon_1)$ is bounded as for $k = 1$, while for $S(\Upsilon_2)$ we argue as follows. Since x_2 and x_3 are spacelike separated, after fixing $\zeta_3 = 0$, ζ_2 is a spacelike Minkowski point, hence so is ζ'_2 , i.e. $\text{Im}(\zeta'_4 + \zeta'_2) = \text{Im}(\zeta'_4)$. Then by Lemma 11.5.2, $S(\Upsilon_2) \leq S(\zeta'_4) + |\zeta'_2| \leq S(\zeta_4)^3 + |\zeta'_2|$, which is the needed bound. Case $k = 3$ follows by similar arguments or by mapping it to $k = 1$ via $(x_1, x_2, x_3, x_4) \rightarrow (x'_1 = x_4, x'_2 = x_3, x'_3 = x_2, x'_4 = x_1)$ which maps ρ and $\bar{\rho}$ are to their complex conjugates.

and second, states generated by a pair of Euclidean operators at complexified time positions:

$$|\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle, \quad x_i = (\epsilon_i + it_i, \mathbf{x}_i), \quad 0 > \epsilon_3 > \epsilon_4. \quad (11.7.2)$$

The inner products of states (11.7.1) are given by integrals of the Minkowski 4-point function

$$\langle \Psi_M(F_1) | \Psi_M(F_2) \rangle = \int dx G_4^M(x_1, x_2, x_3, x_4) \overline{F_1(x_2, x_1)} F_2(x_3, x_4), \quad (11.7.3)$$

while the natural inner product on the states (11.7.2) is:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2) | \mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = G_4(x_2^\theta, x_1^\theta, x_3, x_4), \quad (11.7.4)$$

where the OS reflection operation extends to points with complex time coordinates by:

$$x = (\epsilon + it, \mathbf{x}) \mapsto x^\theta = (-\epsilon + it, \mathbf{x}). \quad (11.7.5)$$

The states (11.7.1) also have a natural inner product $\langle \mathcal{O}(x_1)\mathcal{O}(x_2) | \Psi(F) \rangle$ with the OS states.¹⁵

We wish to show that all these new inner products are positive definite and, moreover, that the new states can be approximated in norm by the smeared OS 2-operator states at Euclidean positions. Note that the positive definiteness of (11.7.3) is precisely Wightman positivity for the 4-point case.

11.7.1 Wightman states

Let us start with (11.7.3). Rewriting the inner product in terms of $W(p_1, p_2, p_3)$, the (distributional) Fourier transform of G_4^M with respect to $y_k = x_k - x_{k+1}$, we obtain

$$\langle \Psi_M(F_1) | \Psi_M(F_2) \rangle = \int dp W(p_1, p_2, p_3) [\widehat{F_1}(p_2 - p_1, p_1)]^* \widehat{F_2}(p_2 - p_3, p_3). \quad (11.7.6)$$

We will also need the inner products of the (smeared) OS states

$$|\Psi(H)\rangle = \int dx dy H(x, y) |\mathcal{O}(x)\mathcal{O}(y)\rangle \quad (11.7.7)$$

where H is any C^∞ function compactly supported at $0 > x^0 > y^0$. Their inner products are given by

$$\langle \Psi(H_1) | \Psi(H_2) \rangle = \int dx G_4^E(x_1, x_2, x_3, x_4) \overline{H_1(x_2^\theta, x_1^\theta)} H_2(x_3, x_4). \quad (11.7.8)$$

This can be expressed using the Fourier-Laplace representation (9.0.11). We obtain

$$\langle \Psi(H_1) | \Psi(H_2) \rangle = \int dp W(p_1, p_2, p_3) \overline{g(H_1)(p_2, p_1)} g(H_2)(p_2, p_3), \quad (11.7.9)$$

where $g(H)(p, q)$ is a Schwartz class function related to $H(x, y)$ as follows. First we form the function $h(y_1, y_2) = H(-y_1, -y_1 - y_2)$ which has support at $y_1^0, y_2^0 > 0$. Next we consider \tilde{h} , the Fourier-Laplace transform of $h(y_1, y_2)$:

$$\tilde{h}(p_1, p_2) = \int dy_1 dy_2 e^{-p_1^0 y_1^0 + i\mathbf{p}_1 \cdot \mathbf{y}_1 - p_2^0 y_2^0 + i\mathbf{p}_2 \cdot \mathbf{y}_2} h(y_1, y_2). \quad (11.7.10)$$

Finally, $g(H)$ is an arbitrary Schwartz class function which coincides with \tilde{h} inside the forward light cones. We also have an analogous formula for the inner product between states of two types:

$$\langle \Psi(H) | \Psi_M(F) \rangle = \int dp W(p_1, p_2, p_3) \overline{g(H)(p_2, p_1)} \widehat{F_2}(p_2 - p_3, p_3). \quad (11.7.11)$$

¹⁵We start from the analytically continued Euclidean 4-point function $G_4(x_1, x_2, x_3, x_4)$ and take the limit where x_1, x_2 are kept at fixed Euclidean positions, while x_3, x_4 approach the Minkowski space. By Theorem 9.0.1, the limit is a distribution in x_3, x_4 , and the inner product is its pairing with the test function F .

At this point we recall Lemma 9.1.1 from Sec. 9.1. That lemma implies that Schwartz functions of the form $g(H)$ are dense in the Schwartz space. In particular, for any Schwartz F , we can find a sequence of functions $\{H_r\}_{r=1}^{\infty}$ such that $g(H_r)(p_2, p_3) \rightarrow \widehat{F}(p_2 - p_3, p_3)$ in the Schwartz space. Then it follows from (11.7.6), (11.7.9), (11.7.11) that

$$\begin{aligned}\langle \Psi(H_r) | \Psi(H_r) \rangle &\rightarrow \langle \Psi_M(F) | \Psi_M(F) \rangle, \\ \langle \Psi(H_r) | \Psi_M(F) \rangle &\rightarrow \langle \Psi_M(F) | \Psi_M(F) \rangle.\end{aligned}\tag{11.7.12}$$

From the first equation we conclude that $\langle \Psi_M(F) | \Psi_M(F) \rangle \geq 0$, proving Wightman positivity. The two equations taken together imply that

$$\langle \Psi(H_r) - \Psi_M(F) | \Psi(H_r) - \Psi_M(F) \rangle \rightarrow 0,\tag{11.7.13}$$

i.e. OS states can approximate Wightman states in norm.

11.7.2 OS states for complexified times

Let us discuss next the states (11.7.2) obtained by putting operators at complexified time positions. In these states we don't take the limit to Minkowski space, so they are defined without smearing. Using the Fourier-Laplace representation, their inner product (11.7.4) is expressed as

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) | \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle = G_4(x_2^\theta, x_1^\theta, x_3, x_4) = \int dp W(p_1, p_2, p_3) \overline{f_{x_1, x_2}(p_2, p_1)} f_{x_3, x_4}(p_2, p_3),\tag{11.7.14}$$

where $f_{x,y}(p, q)$, where $0 > \text{Re}(x^0) > \text{Re}(y^0)$, is any Schwartz function which agrees with

$$e^{p^0 x^0 - i\mathbf{p} \cdot \mathbf{x} - q^0 (x^0 - y^0) + i\mathbf{q} \cdot (\mathbf{x} - \mathbf{y})}.\tag{11.7.15}$$

for p, q in the forward light cone (where this function is exponentially decreasing) and extends it somehow outside the light cones (it does not matter how because W has support in the forward light cones).

Since $f_{x,y}$ is a Schwartz function, it can be approximated by Schwartz functions of the form $g(H)$. This implies that non-smearred complexified OS states can be approximated in norm by Euclidean OS states smeared with compactly supported test functions. In particular, the inner product (11.7.4) is positive definite, providing an extension of pointwise OS positivity to complexified times:

$$G_4(y^\theta, x^\theta, x, y) \geq 0, \quad (0 > \text{Re } x^0 > \text{Re } y^0).\tag{11.7.16}$$

As usual, positive-definite inner product implies a Cauchy-Schwarz inequality for the complexified times:

$$|G_4(x_1, x_2, x_3, x_4)|^2 \leq G_4(x_1, x_2, x_2^\theta, x_1^\theta) G_4(x_4^\theta, x_3^\theta, x_3, x_4),\tag{11.7.17}$$

valid for $\text{Re } x_1^0 > \text{Re } x_2^0 > 0 > \text{Re } x_3^0 > \text{Re } x_4^0$. The analogues of these properties for conformal blocks will be useful in chapter 12.

Remark 11.7.1. We can extend the reflection operation further for points with complexified both time and space coordinates, as

$$x = (\varepsilon + it, \mathbf{x} + i\mathbf{y}) \mapsto x^\theta = (-\varepsilon + it, \mathbf{x} - i\mathbf{y}).\tag{11.7.18}$$

With this definition, we can show by the same arguments as above that $G_4(y^\theta, x^\theta, x, y) \geq 0$ (pointwise OS positivity) remains true for $0 \succ (\text{Re } x^0, \text{Im } \mathbf{x}) \succ (\text{Re } y^0, \text{Im } \mathbf{y})$ where $\eta_1 \succ \eta_2$ means $\eta_1 - \eta_2 \succ 0$ (i.e. in the forward light cone). We can then show that the states $|\mathcal{O}(x)\mathcal{O}(y)\rangle$ make sense for such x, y and can be approximated in norm by integrated Euclidean OS states.

11.8 Wightman clustering

11.8.1 2 + 2 split

In this section we will derive clustering (7.1.4) for Wightman 4-point functions (see [1], Sec. 4.4). As in Sec. 8.3 for the OS case, we will consider 2+2 and 3+1 splits separately. The property we need to prove in the 2+2 case can be written conveniently in the language of Wightman states $|\Psi_M(F)\rangle$, at our disposal by the discussion in Sec. 11.7:

$$\langle \Psi_M(F_1) | \Psi_M(U_{\lambda a} F_2) \rangle \rightarrow \langle \Psi_M(F_1) | \Omega \rangle \langle \Omega | \Psi_M(F_2) \rangle \quad (11.8.1)$$

as $\lambda \rightarrow \infty$ for any spacelike vector a and any Schwartz test functions F_1, F_2 , where $U_{\lambda a}$ is translation: $(U_{\lambda a} F_2)(x, y) = F_2(x - \lambda a, y - \lambda a)$, and Ω is the vacuum state corresponding to inserting the unit operator. By Lorentz invariance it's enough to prove this for $a = (0, \mathbf{a})$, purely spatial vector. In Sec. 8.3 we showed the OS clustering, which we can also write using the integrated OS states (11.7.7), as

$$\langle \Psi(H_1) | \Psi(U_{\lambda a} H_2) \rangle \rightarrow \langle \Psi(H_1) | \Omega \rangle \langle \Omega | \Psi(H_2) \rangle. \quad (11.8.2)$$

As explained in Sec. 11.7, we can find states $|\Psi(H_1)\rangle$ and $|\Psi(H_2)\rangle$ which approximate $|\Psi_M(F_1)\rangle$ and $|\Psi_M(F_2)\rangle$ in norm within any $\varepsilon > 0$. Moreover it's obvious from that construction that the norm is invariant under shifts in purely spatial direction (i.e. the operator $U_{\lambda a}$ is unitary). Hence we have $\|\Psi(U_{\lambda a} H_2) - \Psi_M(U_{\lambda a} F_2)\| = \|\Psi(H_2) - \Psi_M(F_2)\| \leq \varepsilon$ for any λ . By these properties, (11.8.2) implies (11.8.1).¹⁶

11.8.2 3 + 1 split

Let us first restate the Euclidean 3+1 clustering argument from Sec. 8.3 in a somewhat more explicit form, and specializing to scalars. So let $\varphi(x_1), \chi(x_2, x_3, x_4)$ be two smooth functions with compact support¹⁷

$$\text{supp}(\varphi) \subset \{x_1^0 > 0\}, \quad \text{supp}(\chi) \subset \{0 > x_2^0 > x_3^0 > x_4^0\}. \quad (11.8.3)$$

We would like to show

$$\lim_{\lambda \rightarrow \infty} (G, \varphi_\lambda \otimes \chi) = 0, \quad \varphi_\lambda := \varphi(\cdot - \lambda \hat{e}_1) \quad (11.8.4)$$

where $G = G_4^E$ is the Euclidean 4-point function of four identical scalars, and \hat{e}_1 is the x^1 unit vector. The main idea is that we can find a conformal transformation which moves the point at infinity as well as all the other points to some finite positions. The suppression of the integral then comes from the Jacobian of this transformation. Consider a special conformal transformation $f(x) = \frac{x^\mu + x^2 b^\mu}{1 + 2x \cdot b + x^2 b^2} = \mathcal{J} \circ T_b \circ \mathcal{J}$, where \mathcal{J} is inversion and T_b is a translation by $b = \hat{e}_1$. We have $f(-\hat{e}_1) = \infty$, while f is non-singular on $\text{supp}(\varphi_\lambda)$ and $\text{supp}(\chi)$. We also have $f(\infty) = \hat{e}_1$. By conformal invariance we have (compare (11.6.5)) $(G, \Phi) = (G, \Phi^f)$ where $\Phi^f(x_1, \dots, x_4) = \Phi(f^{-1}(x_1), \dots, f^{-1}(x_4)) \prod_{i=1}^4 J(f^{-1}(x_i))^{\Delta_{\mathcal{O}} - d}$, where $J(x) = \frac{1}{1 + 2x \cdot b + x^2 b^2}$. We apply this equation with $\Phi = \varphi_\lambda \otimes \chi$. The function χ is mapped by this transformation to some smooth function. Suppression of the integral in the limit $\lambda \rightarrow \infty$ will come from the transformation of φ_λ , which is mapped to

$$\varphi_\lambda^f(x_1) := \varphi(f^{-1}(x_1) - \lambda \hat{e}_1) J(f^{-1}(x_1))^{\Delta_{\mathcal{O}} - d}. \quad (11.8.5)$$

Namely we have

$$|(G, \varphi_\lambda^f \otimes \chi^f)| \leq C(\lambda) I, \quad I = \int dx_1 |\varphi_\lambda^f(x_1)|, \quad C(\lambda) = \sup_{x_1 \in \text{supp} \varphi_\lambda^f} |(G(x_1, \cdot), \chi^f)|. \quad (11.8.6)$$

The function φ_λ^f is nonzero for $f^{-1}(x_1) \in \text{supp}(\varphi) + \lambda \hat{e}_1$, which is a point near infinity for λ large. We conclude that φ_λ^f is supported in a small neighborhood, order $1/\lambda$, of $f(\infty) = \hat{e}_1$. Since G is real-analytic at nonzero point

¹⁶Indeed we have $|\langle \Psi_M(F_1) | \Psi_M(U_{\lambda a} F_2) \rangle - \langle \Psi(H_1) | \Psi(U_{\lambda a} H_2) \rangle| \leq C\varepsilon$ with some C independent of λ . Now passing to the limit $\lambda \rightarrow \infty$ and using (11.8.2) we obtain $\limsup_{\lambda \rightarrow \infty} \langle \Psi_M(F_1) | \Psi_M(U_{\lambda a} F_2) \rangle \leq \langle \Psi_M(F_1) | \Omega \rangle \langle \Omega | \Psi_M(F_2) \rangle + C'\varepsilon$, and an analogous lower bound on $\liminf_{\lambda \rightarrow \infty}$. Since $\varepsilon > 0$ is arbitrary we obtain (11.8.1).

¹⁷For simplicity, in this section we prove clustering for compactly-supported, as opposed to Schwartz, test functions. We expect that it should be possible to find a proof for Schwartz test functions as well. In any case, the most natural proof would use positivity and the OPE similarly to 2+2 split, provided positivity for higher-point functions is proven (which we don't do in this part of the thesis).

separation, this implies that $C(\lambda)$ is bounded by some constant for $\lambda \geq \lambda_0$. To compute I , we do the change of variables $x_1 = f(y)$:

$$I = \int dy |\varphi(y - \lambda \hat{e}_1)| J(y)^{\Delta_\sigma} \sim \frac{\text{const}}{\lambda^{2\Delta_\sigma}}. \quad (11.8.7)$$

This finishes the proof of Euclidean 3+1 clustering, Eq. (11.8.4).

Let us proceed next to show Wightman 3+1 clustering. We will show the same equation as (11.8.4), namely

$$\lim_{\lambda \rightarrow +\infty} (G, \varphi_\lambda \otimes \chi) = 0, \quad (11.8.8)$$

where now $G = G_4^M$ is the Minkowski 4-point function, which is a tempered distribution, and $\varphi(x_1)$ and $\chi(x_2, x_3, x_4)$ are arbitrary compactly supported test functions (i.e. no support requirements analogous to (11.8.3)).¹⁸ The proof will be based on the same idea of moving the point at infinity to a finite position, paying attention to G now being distribution, and to the requirement (11.6.6) on invariance under finite Minkowski conformal transformations.

We will use the same transformation $f(x) = \frac{x^\mu + x^2 b^\mu}{1 + 2x \cdot b + x^2 b^2}$, $b = \hat{e}_1$. By translation invariance, we may assume that $\text{supp}(\chi)$ lies at larger x^1 values than of the singularity light cone $x^0 = \pm |\mathbf{x} + \hat{e}_1|$ of this transformation (see Sec. 11.6). For sufficiently large λ , $\text{supp}(\varphi_\lambda)$ will also satisfy this condition. As we scale b to zero to connect f to the identity, the singularity light cone moves away to infinity along the negative x^1 direction, without touching $\text{supp}(\varphi_\lambda)$ nor $\text{supp}(\chi)$, see Fig. 11.8.1. Hence requirement (11.6.6) is satisfied and we may apply invariance (11.6.5), which says $(G, \varphi_\lambda \otimes \chi) = (G, \varphi_\lambda^f \otimes \chi^f)$.

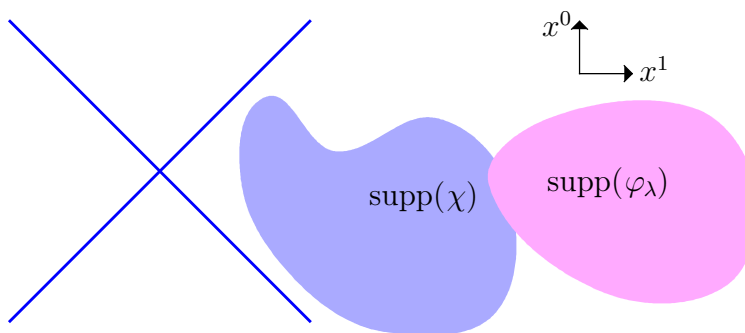


Figure 11.8.1: Location of supports of φ_λ and χ with respect to the singularity light cone of f .

Now, using translation invariance of the 4-point function, $G(x_1, x_2, x_3, x_4) = \tilde{G}(x_2 - x_1, x_3 - x_1, x_4 - x_1)$ we may write

$$(G, \varphi_\lambda^f \otimes \chi^f) = \int dx_1 \varphi_\lambda^f(x_1) F(x_1), \quad (11.8.9)$$

where $F(x_1) = (\tilde{G}, T_{x_1} \cdot \chi^f)$ and T_a is a translation. \tilde{G} is a distribution, but since translation is a continuous operation in the space of test functions, we know that $F(x_1)$ is a continuous function of x_1 . When λ goes to $+\infty$, the support of φ_λ^f shrinks to the point \hat{e}_1 .¹⁹ Hence for $\lambda \geq \lambda_0$ we can bound $|F|$ on $\text{supp}(\varphi_\lambda^f)$ by a constant, and estimate (11.8.9) in absolute value by $\text{const} \times \int dx_1 |\varphi_\lambda^f(x_1)|$. This remaining integral is computed via the change of variables as the Euclidean one, and goes to zero as $\lambda^{-2\Delta_\sigma}$, completing the proof.

11.9 Local commutativity

Let us show that the constructed Minkowski correlators satisfy local commutativity. This follows by a robust argument which uses only Lorentz invariance, analyticity in the forward tube, existence of the boundary distribution, and real analyticity of the Euclidean correlators away from coincident points (OS [1], Sec. 4.5). Here for completeness

¹⁸The method described below cannot be straightforwardly generalized to the case of Schwartz test functions.

¹⁹It is important for the argument that, as one can easily check, $\text{supp}(\varphi_\lambda^f)$ shrinks to a compact set (in fact, a point) and not, say, spreads out along some light cone.

we will provide this argument for n -point functions which is its natural setting. In Sec. 11.9.1 below we will make some remarks specific to CFT 4-point functions.

So, we start from the Euclidean correlator $G^E(x_1, \dots, x_n)$ at $x_1^0 > x_2^0 > \dots > x_n^0$ and its analytic continuation $G(x_1, \dots, x_n)$ to the forward tube \mathcal{T}_n which is the set of points $x_k \in \mathbb{C}^d$ such that their differences $y_k = x_k - x_{k+1}$ satisfy $\text{Re } y_k^0 > |\text{Im } \mathbf{y}_k|$ or equivalently $\eta_k \succ 0$ in terms of $\zeta_k = (iy_k^0, \mathbf{y}_k) = \xi_k + i\eta_k$, $\xi_k, \eta_k \in \mathbb{R}^{1, d-1}$. We will write G instead of G_n . We know by Theorem 9.0.1 that this analytic continuation is invariant under Lorentz transformations $\zeta_k \rightarrow \Lambda \zeta_k$ where $\Lambda \in L_+^\uparrow$, the identity component of the real Lorentz group. Since G is translationally invariant, it depends only on ζ_k , and we will abuse of notation by sometimes writing $G(\zeta_1, \dots, \zeta_{n-1})$ and $(\zeta_1, \dots, \zeta_{n-1}) \in \mathcal{T}_n$ instead of $G(x_1, \dots, x_n)$ and $(x_1, \dots, x_n) \in \mathcal{T}_n$.

Step 1. We will extend domain of analyticity of G using the complex Lorentz group $L(\mathbb{C})$, defined as the set of complex matrices A preserving the Minkowski metric, i.e. $A^T g A = g$ where $g = \text{diag}(-1, 1 \dots 1)$. We will only need the component of $L(\mathbb{C})$ connected to the identity, denoted $L_+(\mathbb{C})$. For any $\Lambda \in L_+(\mathbb{C})$ consider the equation

$$G(\zeta_1, \dots, \zeta_{n-1}) = G(\Lambda^{-1}\zeta_1, \dots, \Lambda^{-1}\zeta_{n-1}). \quad (11.9.1)$$

The two sides of this equation coincide for real $\Lambda \in L_+^\uparrow$ (by Lorentz invariance of G_n), and hence by analyticity in the components of Λ also for complex $\Lambda \in L_+(\mathbb{C})$, at least for Λ close to 1. In other words, Eq. (11.9.1) is just an identity if $\Lambda \approx 1$ and the arguments of G_n on both sides are in the forward tube. But a general $\Lambda \in L_+(\mathbb{C})$ does not preserve the forward tube. For such Λ , Eq. (11.9.1) extends analytically G from the forward tube to the set

$$\mathcal{T}'_n = \bigcup_{\Lambda \in L_+(\mathbb{C})} \Lambda \cdot \mathcal{T}_n, \quad (11.9.2)$$

called the extended tube. The Bargmann-Hall-Wightman theorem shows that no further topological obstructions arise in this analytic continuation; see [102], p.78 for details. Call this extension \tilde{G} .

Step 2. Let us consider $\tilde{G}(x_1, \dots, x_n)$ for

$$\epsilon_1 > \dots > \epsilon_{k-1} > 0 > \epsilon_{k+2} > \dots > \epsilon_n, \quad (11.9.3)$$

while assuming that $\epsilon_k, \epsilon_{k+1}$ are near zero and much smaller than other ϵ_i 's, and $|t_k - t_{k+1}| < |\mathbf{x}_k - \mathbf{x}_{k+1}|$, $\mathbf{x}_k, \mathbf{x}_{k+1}$ real, so that $x_k - x_{k+1}$ approaches a spacelike separation. For $\epsilon_k > \epsilon_{k+1}$ this configuration is in the forward tube, so we know \tilde{G} is analytic there and agrees with $G(x_1, \dots, x_n)$. Let us show that the configurations with $\epsilon_k < \epsilon_{k+1}$ are in the extended tube. We may set $x_{k+1} = 0$ for this argument, so that

$$\zeta_k = (t_k + i\epsilon_k, \mathbf{x}_k). \quad (11.9.4)$$

We may assume without loss of generality that $\mathbf{x}_k = (x_k^1, 0, \dots, 0)$, $x_k^1 > |t_k|$. Then acting on ζ_k with the complexified Lorentz transformation

$$\Lambda_\theta = \begin{pmatrix} \cosh(i\theta) & \sinh(i\theta) \\ \sinh(i\theta) & \cosh(i\theta) \end{pmatrix} \in L_+(\mathbb{C}), \quad (11.9.5)$$

with small θ we get, using $\Lambda_\theta \approx \begin{pmatrix} 1 & i\theta \\ i\theta & 1 \end{pmatrix}$, $\zeta'_k = \Lambda_\theta \zeta_k \approx (t_k, x_k^1) + i(\theta x_k^1 + \epsilon_k, \theta t_k)$, and thus $\eta'_k \approx (\theta x_k^1 + \epsilon_k, \theta t_k)$. If ϵ_k is negative but very small, we can achieve $\eta'_k \succ 0$ by choosing an appropriate small θ . We need θ small so that all the other ζ'_i remain in the forward light cone, and this will work because we assume that ϵ_k is very much smaller than all the other ϵ_i 's.

The bottom line is that the extended tube contains an open set of configurations as above, with $|t_k - t_{k+1}| < |\mathbf{x}_k - \mathbf{x}_{k+1}|$ and $\epsilon_k, \epsilon_{k+1}$ small, with $\epsilon_k - \epsilon_{k+1}$ of any sign. Let us call this set $\mathcal{Q}_{n,k}$. By restricting this set a bit, we may assume that $\mathcal{Q}_{n,k}$ is invariant under permutations of x_k and x_{k+1} . By Step 1 we know that function \tilde{G} is holomorphic in the extended tube and hence also in $\mathcal{Q}_{n,k}$. In particular, it is analytic if we set $\epsilon_k = \epsilon_{k+1} = 0$. This already has an interesting consequence: Minkowski correlator is analytic with respect to a pair of spacelike-separated points (while it remains a distribution with respect to all the other points). Projection of $\mathcal{Q}_{n,k}$ to the plane $(\epsilon_k - \epsilon_{k+1}, t_k - t_{k+1})$ is shown schematically in Fig. 11.9.1.

The set $\mathcal{Q}_{n,k}$ contains real configurations (horizontal axis in Fig. 11.9.1, setting other $t_i \rightarrow 0$ as well). Restriction of \tilde{G} to the real part of $\mathcal{Q}_{n,k}$ agrees with the Euclidean correlator G^E . (They agree for $\epsilon_k > \epsilon_{k+1}$ by construction and

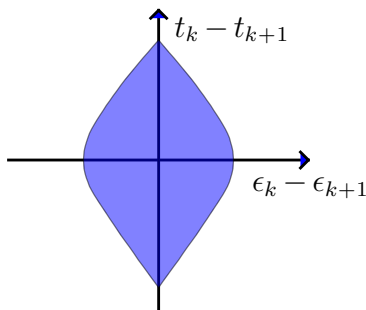


Figure 11.9.1: Projection of the set $\mathcal{Q}_{n,k}$, where the function \tilde{G} is holomorphic, to the plane $(\epsilon_k - \epsilon_{k+1}, t_k - t_{k+1})$. The vertical extent of this region is determined by the condition $|t_k - t_{k+1}| < |\mathbf{x}_k - \mathbf{x}_{k-1}|$. The horizontal extent is determined, among other things, by the condition that $\epsilon_k, \epsilon_{k+1}$ have to be much smaller than all the other ϵ_i 's.

for $\epsilon_k < \epsilon_{k+1}$ by the uniqueness of analytic continuation. Recall that the Euclidean correlator G^E is real analytic everywhere away from coincident points, i.e. for $\epsilon_k - \epsilon_{k+1}$ of any sign as long as $\mathbf{x}_k \neq \mathbf{x}_{k-1}$.) One consequence of this fact is that \tilde{G} restricted to the real part $\mathcal{Q}_{n,k}$ is permutation invariant w.r.t. $x_k \leftrightarrow x_{k+1}$:

$$\tilde{G}(\dots x_k, x_{k+1} \dots) = \tilde{G}(\dots x_{k+1}, x_k \dots), \quad (11.9.6)$$

because the Euclidean correlator has this property. Finally, since $\mathcal{Q}_{n,k}$ is connected to the real configurations (see Fig. 11.9.1), we conclude that permutation invariance (11.9.6) holds everywhere in $\mathcal{Q}_{n,k}$.²⁰

We now see the meaning of \tilde{G} for configurations with $\epsilon_k < \epsilon_{k+1}$. Via permutation invariance (11.9.6), such configurations are mapped to the forward tube and hence can be evaluated as G for the permuted configurations.

Step 3. We are now ready to show local commutativity. We have to prove that boundary value limits of two holomorphic functions agree:

$$\lim_{\epsilon_i \rightarrow 0} G(\dots x_k, x_{k+1} \dots) = \lim_{\epsilon_i \rightarrow 0} G(\dots x_{k+1}, x_k \dots), \quad (11.9.7)$$

when approaching a Minkowski configuration in which $x_k - x_{k+1}$ is spacelike. Note that, by the original definition, the two limits are from different forward tubes: the first one must respect the condition $\epsilon_k > \epsilon_{k+1}$, while the second $\epsilon_{k+1} > \epsilon_k$. By Theorem 9.0.1, Part 3, we can take the limits $\epsilon_i \rightarrow 0$ in any order, so let us send $\epsilon_k, \epsilon_{k+1} \rightarrow 0$ first, while keeping other ϵ_i fixed for the moment. For very small $\epsilon_k, \epsilon_{k+1}$, the configurations on both sides will be in $\mathcal{Q}_{n,k}$ where both sides are restrictions of the function \tilde{G} analytic around $\epsilon_k, \epsilon_{k+1} = 0$ and satisfying permutation invariance (11.9.6). It follows that the two sides of (11.9.7) agree in the limit $\epsilon_k, \epsilon_{k+1} \rightarrow 0$. Sending the remaining $\epsilon_i \rightarrow 0$ we recover the local commutativity.

11.9.1 Local commutativity for CFT 4-point functions

In this part of the thesis we analytically continued the CFT 4-point function $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle$ to the forward tube using $\rho, \bar{\rho}$ coordinates. We would like to indicate here that this provides an alternative path to understanding local commutativity. We have shown previously that $0 < |\rho|, |\bar{\rho}| < 1$ in the forward tube. Since the extended tube is obtained from the forward tube by complexified Lorentz transformations and $\rho, \bar{\rho}$ are invariant under such transformations, it follows that $0 < |\rho|, |\bar{\rho}| < 1$ also in the extended tube. Below we will show this explicitly for the configurations used in the proof of local commutativity. We consider separately $k = 1$ and $k = 2$ ($k = 3$ being analogous to $k = 1$).

$k = 1$: Here x_1, x_2 approach spacelike-separated Minkowski points. We know that $|\rho|, |\bar{\rho}| < 1$ in \mathcal{D}_4 , $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$. Extended tube analyticity suggests that this must remain true also for $\epsilon_1 = \epsilon_2 > \epsilon_3 > \epsilon_4$. Indeed, this follows from critical rereading of the proof of Lemma 11.3.1 (Sec. 11.4, Eq. (11.4.5) and below). (That proof does not use

²⁰In fact, \tilde{G} can be extended to a single-valued holomorphic function on the ‘‘permuted extended tube’’ $\bigcup_{\pi \in S_n} \pi \mathcal{T}'_n$, and satisfied permutation invariance (11.9.6) on this large set. See [102], App. II, [103] and [104], Sec. 9.D. However for our purposes analyticity and permutation invariance on $\mathcal{Q}_{n,k}$ will suffice.

the condition $\eta_1 \succ \eta_2$ but only $\eta_1, \eta_2 \succ 0$.²¹ It is also important for analyticity that $\rho, \bar{\rho}$ not vanish. In the forward tube $\rho, \bar{\rho}$ do not vanish because $x_{ij}^2 \neq 0, i < j$ (Lemma 10.0.1). When $\epsilon_1 = \epsilon_2$ we have $x_{12}^2 > 0$ (spacelike separation), hence also nonzero. These observations show that the CFT 4-point function can be analytically extended, using the $\rho, \bar{\rho}$ expansion, to a neighborhood of points with $\epsilon_1 = \epsilon_2 > 0 > \epsilon_3 > \epsilon_4, x_{12}^2 > 0$, in agreement with the general QFT arguments given above.

Let us now permute the first two points: $(\epsilon_1 + it_1, \mathbf{x}_1) \leftrightarrow (\epsilon_2 + it_2, \mathbf{x}_2)$. In the Euclidean region, this transformation maps $\rho \rightarrow -\rho, \bar{\rho} \rightarrow -\bar{\rho}$ and leaves the 4-point function of identical scalars invariant because the expansion (11.2.6) contains only even m . The same transformation remains true for complexified times for spacelike separation. Taking the limit $\epsilon_1, \epsilon_2 \rightarrow 0$, we recover local commutativity very explicitly.

$k = 2$: Now we are interested in the limit $\epsilon_2 \rightarrow \epsilon_3$ from inside $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$.²² As for $k = 1$, critical rereading of the proof of Lemma 11.3.1 shows that $|\rho|, |\bar{\rho}|$ remain less than 1. (We put in that proof $\zeta_2 = \xi_2 + i\eta_2, \xi_2 = (t_2, \mathbf{x}_2)$ spacelike, and $\eta_2 = (\epsilon_2, \mathbf{0}), \epsilon_2 > 0$. The proof does not use the condition $\eta_2' \succ 0$ but only $\eta_{24}' \succ 0$. The latter condition remains true for $\epsilon_2 \rightarrow \epsilon_3 = 0$, as ζ_2' goes to a finite real vector.) Hence, the CFT 4-point function can be analytically extended, using the $\rho, \bar{\rho}$ expansion, to a neighborhood of points with $\epsilon_1 > \epsilon_2 = \epsilon_3 > \epsilon_4, x_{23}^2 < 0$.

To finish the proof of local commutativity, we fall back on the general argument, appealing to the permutation invariance of the (real-analytic) CFT 4-point function under $x_2 \leftrightarrow x_3$. (Unlike for $k = 1$, the s-channel OPE expansion (11.2.6) cannot be used to make this step more explicit, as it does not manifestly have this invariance.)

11.10 Generalization to non-identical scalars

In the previous subsections we proved that the 4-point function of identical scalars has analytic continuation to the forward tube \mathcal{T}_4 , and its boundary value in the Minkowski region is a tempered distribution. Then Minkowski conformal invariance, Wightman positivity, Wightman clustering and local commutativity follow from their Euclidean analogues.

In this section we will indicate how to generalize analytic continuation and temperedness to 4-point functions of non-identical scalars. The proof of the other properties is the same as in the case of identical scalars.

We consider the 4-point function of scalar primary operators \mathcal{O}_i with scaling dimensions Δ_i ,

$$\begin{aligned} G_{1234}^E(c_E) &:= \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle \\ &= \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \left(\frac{x_{24}^2}{x_{14}^2} \right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2} \right)^{\frac{\Delta_3-\Delta_4}{2}} g_{1234}(c_E), \end{aligned} \quad (11.10.1)$$

which reduces to (11.0.1) when Δ_i 's are identical. The analytic continuation of the prefactor to the forward tube \mathcal{T}_4 is straightforward. The function $g_{1234}(c_E)$ only depends on the conformal equivalence class of c_E , i.e. $g_{1234}(c_E) = g_{1234}(\rho(c_E), \bar{\rho}(c_E))$. By the similar argument to that in Sec. 11.2, the function $g_{1234}(c_E)$ has the following series expansion

$$g_{1234}(c_E) = \left[\frac{(1-\rho)(1-\bar{\rho})}{(1+\rho)(1+\bar{\rho})} \right]^{\frac{\Delta_1-\Delta_2-\Delta_3+\Delta_4}{2}} \sum_{\delta, m} a_{12}(\delta, m) a_{43}(\delta, m)^* r^\delta e^{im\theta}, \quad \rho(c_E) = r e^{im\theta}, \quad (11.10.2)$$

where the sum runs over a discrete set of pairs (δ, m) with $\delta \geq 0, m \in \mathbb{Z}$ (not necessarily even for non-identical scalars), $|m| \leq \delta$. Analogously to the case of identical scalars, the sum is absolutely convergent when $|\rho(c_E)| < 1$

²¹An alternative argument is as follows. In chapter 12 we will show the Cauchy-Schwarz inequality for $\rho, \bar{\rho}$, Theorem 12.0.2, which bounds $\rho, \bar{\rho}$ for any configuration in the forward tube with $\epsilon_1 > \epsilon_2 > 0 > \epsilon_3 > \epsilon_4$ in terms of $\rho, \bar{\rho}$ of “reflection-symmetric” configurations having $\epsilon_3 = -\epsilon_2, \epsilon_4 = -\epsilon_1$. The proof of Lemma 12.0.1, Eq. (12.1.4) shows that $\rho, \bar{\rho}$ remain less than 1 for the latter configurations in the limit $\epsilon_1 \rightarrow \epsilon_2$.

²²The discussion on the local commutativity of this type can also be found in the study of causality in a shockwave background (see Sec. 5 of [3]). In [3], the 2-point function in a shockwave background is defined by $\langle \mathcal{O}(x)\mathcal{O}(y) \rangle_\Psi := \frac{\langle \Psi(i\delta)\mathcal{O}(x)\mathcal{O}(y)\Psi(-i\delta) \rangle}{\langle \Psi(i\delta)\Psi(-i\delta) \rangle}$, where x, y are Minkowski points and “ $i\delta$ ” means the Euclidean point $(\delta, 0, \dots, 0)$. In our language it corresponds to the 4-point function $\langle \Psi(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\Psi(x_4) \rangle$ with $\epsilon_1 = -\epsilon_4 = \delta > 0$ and $\epsilon_2 = \epsilon_3 = 0$. We know that the 4-point function is regular analytic at such configurations. So the commutator $[\mathcal{O}(x), \mathcal{O}(y)]$ vanishes in the shockwave background when x and y are spacelike separated.

(see below). Also, when $d \geq 3$ we have $p_{\delta,-m} = p_{\delta,m}$, where $p_{\delta,m} = a_{12}(\delta, m)a_{\overline{43}}(\delta, m)^*$. Analogously to Sec. 11.3, the analytic continuation of $g_{1234}(c)$ in $d \geq 3$ will be given by the formula (compare (11.3.11))

$$g_{1234}(c) = \left(\frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \right)^{\frac{\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4}{4}} \sum_{m, \delta, 0 \leq m \leq \delta} p_{\delta,m} R_{\delta/2 - m/2}(c) \Phi_m(c). \quad (11.10.3)$$

In $d = 2$, $p_{\delta,m} \neq p_{\delta,-m}$ but the functions $\rho(c)^m$ and $\bar{\rho}(c)^m$ are individually holomorphic. In this case the analytic continuation of $g_{1234}(c)$ is given by the formula (compare (11.3.15)):

$$g_{1234}(c) = \left(\frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \right)^{\frac{\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4}{2}} \sum_{m, \delta, 0 \leq m \leq \delta} R_{\delta/2 - m/2}(c) [p_{\delta,m} \rho(c)^m + p_{\delta,-m} \bar{\rho}(c)^m]. \quad (11.10.4)$$

We would like to show that

(a) when $r = \max\{|\rho|, |\bar{\rho}|\} < 1$, the series

$$\tilde{g}_{1234}(\rho, \bar{\rho}) = \sum_{\delta, m} a_{12}(\delta, m) a_{\overline{43}}(\delta, m)^* \rho^{(\delta+m)/2} \bar{\rho}^{(\delta-m)/2} \quad (11.10.5)$$

is absolutely convergent;

(b) the remainder $\tilde{g}_{1234}(\rho, \bar{\rho}; \delta_*) := \sum_{\delta \geq \delta_*, m} a_{12}(\delta, m) a_{\overline{43}}(\delta, m)^* \rho^{(\delta+m)/2} \bar{\rho}^{(\delta-m)/2}$ has a powerlaw bound, uniform in δ_* :

$$|\tilde{g}_{1234}(\rho, \bar{\rho}; \delta_*)| \leq C(1-r)^{-\Delta_1 - \Delta_2 - \Delta_3 - \Delta_4}. \quad (11.10.6)$$

This is done as follows (compare Sec. 4.2). Consider the 4-point functions $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_2^\dagger \mathcal{O}_1^\dagger \rangle$, $\langle \mathcal{O}_4^\dagger \mathcal{O}_3^\dagger \mathcal{O}_3 \mathcal{O}_4 \rangle$, and let $\tilde{g}_{12\overline{21}}$, $\tilde{g}_{\overline{43}34}$ be the analogues of (11.10.5):

$$\begin{aligned} \tilde{g}_{12\overline{21}}(\rho, \bar{\rho}) &= \sum_{\delta, m} |a_{12}(\delta, m)|^2 \rho^{(\delta+m)/2} \bar{\rho}^{(\delta-m)/2}, \\ \tilde{g}_{\overline{43}34}(\rho, \bar{\rho}) &= \sum_{\delta, m} |a_{\overline{43}}(\delta, m)|^2 \rho^{(\delta+m)/2} \bar{\rho}^{(\delta-m)/2}. \end{aligned} \quad (11.10.7)$$

Noticing that $|m| \leq \delta$, we estimate (11.10.5) by absolute value and apply Cauchy-Schwarz inequality:

$$|\tilde{g}_{1234}(\rho, \bar{\rho}; \delta_*)| \leq \sum_{\delta, m} |a_{12}(\delta, m)| |a_{\overline{43}}(\delta, m)| r^\delta \leq [\tilde{g}_{12\overline{21}}(r, r) \tilde{g}_{\overline{43}34}(r, r)]^{1/2}. \quad (11.10.8)$$

The functions $\tilde{g}_{12\overline{21}}(r, r)$ and $\tilde{g}_{\overline{43}34}(r, r)$ correspond to the 4-point functions at the Euclidean configurations with $\rho = \bar{\rho} = r < 1$, hence their series expansions (11.10.7) are convergent by the Euclidean OPE axiom. Therefore, (11.10.5) is absolutely convergent when $|\rho|, |\bar{\rho}| < 1$. This finishes the proof of part (a).

Using the t-channel OPE, we can show that for $0 \leq r < 1$,

$$\begin{aligned} \tilde{g}_{12\overline{21}}(r, r) &\leq C(1-r)^{-2\Delta_1 - 2\Delta_2}, \\ \tilde{g}_{\overline{43}34}(r, r) &\leq C(1-r)^{-2\Delta_3 - 2\Delta_4}, \end{aligned} \quad (11.10.9)$$

with some $C > 0$. Combining (11.10.9) with (11.10.8) we get (11.10.6). This finishes the proof of part (b).

Chapter 12

Optimal powerlaw bound from Cauchy-Schwarz $\rho, \bar{\rho}$ inequality

In Sec. 11.5 we provided a powerlaw bound for the 4-point function, based on the inequality (11.5.8) for $\max(|\rho(c)|, |\bar{\rho}(c)|)$. That did the job of allowing us to apply Theorem 9.0.1 and prove that the Minkowski 4-point function is a distribution, but the actual bound (11.5.8) is not optimal. It is interesting to get a better bound on $|\rho(c)|, |\bar{\rho}(c)|$, because this will translate into a better powerlaw bound for the 4-point function, allowing us to get a better idea about the regularity of the Minkowski 4-point function as a distribution, i.e. how many derivatives test functions must have. In the proof of Theorem 9.0.1, parameters A_n and B_n of the powerlaw bound enter into Eq. (B.3.6) which provides an upper bound on the regularity.

In this section we will provide such an optimal bound on $|\rho(c)|, |\bar{\rho}(c)|$. The main idea of the bound and of its proof is inspired by Sec. 11.7.2. Let us denote by $\mathcal{D}_4^{(0)}$ the subset of configurations $c \in \mathcal{D}_4$ satisfying the condition $\text{Re } x_1^0 > \text{Re } x_2^0 > 0 > \text{Re } x_3^0 > \text{Re } x_4^0$. We showed that the 4-point functions for complexified times satisfy the Cauchy-Schwarz inequality (11.7.17) for $c = (x_1, x_2, x_3, x_4) \in \mathcal{D}_4^{(0)}$. For a general configuration $c \in \mathcal{D}_4^{(0)}$ we define two configurations

$$c_{12} = (x_1, x_2, x_2^\theta, x_1^\theta), \quad c_{34} = (x_4^\theta, x_3^\theta, x_3, x_4), \quad (12.0.1)$$

where θ is the operation in (11.7.5) which generalizes the OS reflection to complexified times. We will call such configurations, for obvious reasons, reflection-symmetric. It is clear that both $c_{12}, c_{34} \in \mathcal{D}_4^{(0)}$. Eq. (11.7.17) can now be written as

$$|G_4(c)|^2 \leq G_4(c_{12})G_4(c_{34}) \quad (c \in \mathcal{D}_4^{(0)}). \quad (12.0.2)$$

Since we know that G_4 can be written as a convergent power series in $\rho, \bar{\rho}$, Eq. (12.0.2) suggests that there should be a corresponding bound for the $\rho, \bar{\rho}$ coordinates. This is indeed the case, as we have the following couple of results:

Lemma 12.0.1. *Any reflection-symmetric configuration $c \in \mathcal{D}_4^{(0)}$ has $\rho(c), \bar{\rho}(c) \in (0, 1)$.*

Theorem 12.0.2 (Cauchy-Schwarz inequality for $\rho, \bar{\rho}$). *For any configuration $c \in \mathcal{D}_4^{(0)}$ we have the inequality:*

$$\max\{|\rho(c)|, |\bar{\rho}(c)|\}^2 \leq \max\{\rho(c_{12}), \bar{\rho}(c_{12})\} \times \max\{\rho(c_{34}), \bar{\rho}(c_{34})\}. \quad (12.0.3)$$

We will next prove Lemma 12.0.1. We will then show how, combined with Theorem 12.0.2, this implies an optimal bound on $\rho, \bar{\rho}$. Finally we will present a proof of Theorem 12.0.2, which is surprisingly subtle.

12.1 Proof of Lemma 12.0.1

To prove the lemma, consider a reflection-symmetric configuration c as in (12.0.1) with:

$$\begin{aligned} x_1 &= (\epsilon_1 + it_1, \mathbf{x}_1), & x_2 &= (\epsilon_2 + it_2, \mathbf{x}_2), & \epsilon_1 &> \epsilon_2 > 0, \\ x_3 &= x_2^\theta = (-\epsilon_2 + it_2, \mathbf{x}_2), & x_4 &= x_1^\theta = (-\epsilon_1 + it_1, \mathbf{x}_1). \end{aligned} \quad (12.1.1)$$

We will compute $z(c)$, $\bar{z}(c)$ explicitly. We can use translations in the \mathbf{x} direction, as well as spatial rotations to simplify these computations. All these transformations do not change the conformal class of configuration, hence preserve u, v and z, \bar{z} . They also commute with time reflection, and so map reflection-symmetric configurations to reflection-symmetric ones. By using this freedom, we get an equivalent configuration c' with the same z, \bar{z} :

$$x'_1 = (\epsilon_1 + it_1, \mathbf{0}), \quad x'_2 = (\epsilon_2 + it_2, |\mathbf{x}_2 - \mathbf{x}_1|, 0, \dots, 0), \quad x'_3 = (x'_2)^\theta, \quad x'_4 = (x'_1)^\theta. \quad (12.1.2)$$

This is an effectively two-dimensional configuration. The z, \bar{z} variables of a two-dimensional 4-point configuration $x_k = (x_k^0, x_k^1)$ are given by Eq. (11.3.14), which we copy here

$$z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}, \quad \bar{z} = \frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_3 - \bar{z}_4)}{(\bar{z}_1 - \bar{z}_3)(\bar{z}_2 - \bar{z}_4)}, \quad z_k = x_k^0 + ix_k^1, \quad \bar{z}_k = x_k^0 - ix_k^1. \quad (12.1.3)$$

Applying this to the configuration c' , we get z, \bar{z} for c' (which are the same as for c). It's easy to see that $z_3 - z_4 = (z_1 - z_2)^*$, $z_1 - z_3 = (z_2 - z_4)^*$ as a consequence of reflection symmetry, and similarly for \bar{z} 's. So we get $z(c), \bar{z}(c)$ both real and positive. Explicit expressions come out to be

$$\begin{aligned} z(c) &= \frac{(\epsilon_1 - \epsilon_2)^2 + (t_1 - t_2 - |\mathbf{x}_1 - \mathbf{x}_2|)^2}{(\epsilon_1 + \epsilon_2)^2 + (t_1 - t_2 + |\mathbf{x}_1 - \mathbf{x}_2|)^2}, \\ \bar{z}(c) &= \frac{(\epsilon_1 - \epsilon_2)^2 + (t_1 - t_2 + |\mathbf{x}_1 - \mathbf{x}_2|)^2}{(\epsilon_1 + \epsilon_2)^2 + (t_1 - t_2 - |\mathbf{x}_1 - \mathbf{x}_2|)^2}. \end{aligned} \quad (12.1.4)$$

In particular we see that $0 < z(c), \bar{z}(c) < 1$. The function $f(\zeta)$ in the definition of ρ variables maps the interval $(0, 1)$ to itself. Hence also $0 < \rho(c), \bar{\rho}(c) < 1$, and the lemma and proved.

12.2 Optimal bound for $\rho, \bar{\rho}$

We wish to derive a powerlaw bound on $\frac{1}{1-r}$, $r = \max(|\rho|, |\bar{\rho}|)$, since by the arguments in Sec. 11.5 this implies a powerlaw bound for the 4-point function. Our aim here is to improve on (11.5.8), (11.5.9).

Consider first a configuration $c \in \mathcal{D}_4^{(0)}$. For such a configuration, by Theorem 12.0.2, we have

$$r(c) \leq \sqrt{r(c_{12})r(c_{34})} \leq \max(r(c_{12}), r(c_{34})), \quad (12.2.1)$$

and hence

$$\frac{1}{1-r(c)} \leq \max\left(\frac{1}{1-r(c_{12})}, \frac{1}{1-r(c_{34})}\right). \quad (12.2.2)$$

We are thus reduced to study $r(c)$ for reflection-symmetric configurations, like in (12.1.1). By definition (11.2.4) of ρ variables, we have

$$\frac{1}{1-\rho} = \frac{1 + \sqrt{1-z}}{2\sqrt{1-z}} \leq \frac{1}{\sqrt{1-z}}, \quad z \in [0, 1), \quad (12.2.3)$$

so it suffices to study $1/(1-z)$ and $1/(1-\bar{z})$. Using z, \bar{z} for reflection-symmetric configurations computed in Eqs. (12.1.4) we have

$$\frac{1}{1-z(c_{12})} = \frac{(\epsilon_1 + \epsilon_2)^2 + (t_1 - t_2 - |\mathbf{x}_1 - \mathbf{x}_2|)^2}{4\epsilon_1\epsilon_2}, \quad (12.2.4)$$

and an analogous relation for $\frac{1}{1-\bar{z}(c_{12})}$. From these equations, using $\epsilon_2 < \epsilon_1$, and estimating $\epsilon_1 - \epsilon_2, |t_1 - t_2|, |\mathbf{x}_1 - \mathbf{x}_2|$ from above by $|x_1 - x_2|$ (see (9.0.10)), we easily get

$$\frac{1}{1-z(c_{12})}, \frac{1}{1-\bar{z}(c_{12})} \leq \left(1 + \frac{1}{\epsilon_2}\right)^2 (1 + |x_1 - x_2|)^2, \quad (12.2.5)$$

Putting together this relation, an analogous relation for $z(c_{34}), \bar{z}(c_{34})$, Eqs. (12.2.2) and (12.2.3), we get

$$\frac{1}{1-r(c)} \leq \max\left\{\left(1 + \frac{1}{\epsilon_2}\right)(1 + |x_1 - x_2|), \left(1 + \frac{1}{|\epsilon_3|}\right)(1 + |x_3 - x_4|)\right\} \quad (c \in \mathcal{D}_4^{(0)}). \quad (12.2.6)$$

This was for $c \in \mathcal{D}_4^{(0)}$. For a general configuration $c \in \mathcal{D}_4$, we will shift the coordinates by a translation in time direction (which of course does not change $\rho, \bar{\rho}$), arranging so that the shifted configurations c' has $\epsilon_2 > 0 > \epsilon_3$, i.e. $c' \in \mathcal{D}_4^{(0)}$. Specifically we will choose

$$\epsilon_2(c') = \frac{1}{2}(\epsilon_2(c) - \epsilon_3(c)), \quad \epsilon_3(c') = -\frac{1}{2}(\epsilon_2(c) - \epsilon_3(c)). \quad (12.2.7)$$

Then, using (12.2.6) for c' , we obtain a bound on $\frac{1}{1-r(c)}$ which for example can be expressed as

$$\frac{1}{1-r(c)} \leq 2 \left(1 + \frac{1}{\epsilon_2 - \epsilon_3} \right) (1 + \max\{|x_1 - x_2|, |x_3 - x_4|\}) \quad (c \in \mathcal{D}_4). \quad (12.2.8)$$

This is a powerlaw bound of the type we were looking for. By considering reflection-symmetric configurations, it's easy to see that the exponents in this bound cannot be improved. Eq. (12.2.8) is much stronger than our previous suboptimal bound (11.5.8); in fact it implies a bound of the same form as (11.5.8) with the power exponent 12 replaced by 2.

12.3 Proof of Theorem 12.0.2

Although (12.0.3) looks like a simple-enough geometric inequality, we do not know an elementary proof of this fact. We essentially guessed this inequality, checked it numerically, and then looked for a proof. Our guess started in the Euclidean region, where $\bar{\rho} = \rho^*$, and (12.0.3) takes the form

$$|\rho(c)|^2 \leq \rho(c_{12})\rho(c_{34}) \quad (c \in \mathcal{D}_4^{(0)} \text{ Euclidean}). \quad (12.3.1)$$

Even in this case we don't know an elementary proof. We guessed that this must hold, because otherwise it was hard to imagine that the 4-point function itself would satisfy a Cauchy-Schwarz inequality. Indeed (12.3.1) implies the Euclidean version of (11.7.17). We then guessed (12.0.3) as a generalization of (12.3.1) for complexified times.

Our proof of (12.0.3) reverses this logic, by deriving it from (11.7.17). There exist many explicit CFT 4-point functions, e.g. mean field theories (MFT). One could imagine that by considering (11.7.17) for a family of such 4-point functions, and passing to some limit (e.g. of scaling dimension of the mean field going to infinity), one could recover (12.0.3). We haven't managed to make this work using MFTs, but a closely related strategy does work. Namely we will apply this sort of argument not to the full 4-point function, but to a single conformal block, since the latter also satisfy (11.7.17) (as we will explain).

Now that we explained the main idea, let us supply the details. By applying a translation, we may set $\mathbf{x}_1 = 0$. The remaining spacial component vectors $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ span at most three-dimensional subspace of \mathbb{R}^{d-1} . This shows that it is enough to prove the inequality (12.0.3) in the case $d = 4$. We assume that the readers are familiar with the conformal blocks, which encode contributions of a primary into a 4-point function. In the considered case of 4 identical Hermitean scalar, the relevant OPE has the form (simplifying the general case considered in Sec. 8.1)

$$\varphi(x_1)\varphi(x_2) = f_{\varphi\varphi\mathcal{O}}C_{(\lambda)}(x_1, x_2, x_0, \mathcal{D}_0)\mathcal{O}^{(\lambda)}(x_0) \quad (12.3.2)$$

where $\mathcal{O}^{(\lambda)}$ is a dimension Δ , spin ℓ symmetric traceless primary. The conformal block then can be computed by

$$g_{\Delta, \ell}(c) = C_{(\lambda)}(x_1, x_2, x_0, \mathcal{D}_0)C_{(\mu)}(x_3, x_4, x_0, \mathcal{D}_0^\theta)\langle \mathcal{O}^{(\lambda)}(x_0)\mathcal{O}^{\dagger(\mu)}(x_0^\theta) \rangle. \quad (12.3.3)$$

The 4d Euclidean conformal blocks are known explicitly [95, 96]:

$$g_{\Delta, \ell}(c) = \frac{z\bar{z}}{z - \bar{z}}[k_h(z)k_{\bar{h}-1}(\bar{z}) - k_h(\bar{z})k_{\bar{h}-1}(z)], \quad (12.3.4)$$

where $h, \bar{h} = (\Delta \pm \ell)/2$, and $k_\beta(z) = z^\beta {}_2F_1(\beta, \beta, 2\beta; z)$. (We only cite the result for equal external dimensions.) We will assume that the exchanged operator satisfies the 4d unitarity bound $\Delta \geq \ell + 2$. As Eq. (12.3.4) shows, Euclidean conformal blocks are real-analytic functions whenever $|z| < 1$. We can also use this formula to analytically continue them to the forward tube. We wish to show that this analytic continuation satisfies some properties. This

is best shown not from the explicit formula, but by adapting the robust 4-point function arguments from chapter 11. Indeed, conformal blocks allow an expansion of the same form as (11.2.6), with non-negative coefficients which are fixed by conformal invariance. This can be shown by arguments similar to those in Sec. 11.2. The existence of the representation (12.3.3) guarantees Hilbert space unitarity. Then, by the arguments of Sec. 11.3, conformal blocks admit an analytic extension to the forward tube (which is of course the same as the one following from the explicit formula (12.3.4)). The point of the current construction is that it shows that the analytic extension satisfies an inequality analogous to (11.5.1):

$$|g_{\Delta, \ell}(c)| \leq g_{\Delta, \ell}(c_*) \quad (12.3.5)$$

Then, by the arguments in Sec. 11.5, conformal blocks satisfy the powerlaw bound in the forward tube. (As is easy to see from (12.3.4), 4d Euclidean conformal blocks grow as $1/(1-z)$ as $z \rightarrow 1^-$ along the real axis, which replaces Eq. (11.5.3).) Finally, by the arguments analogous to Sec. 11.7.2 we conclude that the analytically continued conformal blocks satisfy Cauchy-Schwarz inequality:

$$|g_{\Delta, l}(c)|^2 \leq g_{\Delta, l}(c_{12})g_{\Delta, l}(c_{34}) \quad \text{for any } c \in \mathcal{D}_4^{(0)}. \quad (12.3.6)$$

(Euclidean reflection positivity of conformal blocks follows from the representation (12.3.3), which we assume to be valid in the Euclidean region.)

In the rest of the argument we will only need two facts, the Cauchy-Schwarz inequality (12.3.6) and the explicit Dolan-Osborn formula (12.3.4). We will apply (12.3.6) to the blocks of spin $\ell \geq 1$ at the unitarity bound, i.e. with $\bar{h} = 1, h = \ell + 1$. The Cauchy-Schwarz inequality for $\rho, \bar{\rho}$ will follow by extracting the asymptotics in the limit $h \rightarrow +\infty$. The asymptotic behavior of k_h is given by the following lemma:

Lemma 12.3.1. *For any fixed $z \in \mathbb{C} \setminus [1, +\infty)$, the function $k_h(z)$ has the following asymptotic behavior in terms of the ρ variable defined in (11.2.4):*

$$k_h(z) = (4\rho)^h \left[\frac{1}{\sqrt{1-\rho^2}} + o(1) \right], \quad h \rightarrow +\infty. \quad (12.3.7)$$

Proof. We have the following identity for $k_h(z)$ [44]:

$$k_h(z) = (4\rho)_2^h F_1(1/2, h; h + 1/2; \rho^2). \quad (12.3.8)$$

The region $z \notin [1, +\infty)$ corresponds to $|\rho| < 1$, where the hypergeometric function ${}_2F_1$ has the power series representation

$${}_2F_1(1/2, h; h + 1/2; \rho^2) = \sum_{n=0}^{\infty} \frac{(1/2)_n (h)_n}{n! (h + 1/2)_n} \rho^{2n}. \quad (12.3.9)$$

When $h \rightarrow +\infty$, each coefficient of the series increases monotonically, and tends to the coefficients of the convergent in the disk $|\rho| < 1$ series

$$\sum_{n=0}^{\infty} \frac{(1/2)_n}{n!} \rho^{2n} = \frac{1}{\sqrt{1-\rho^2}}.$$

This implies the statement of the lemma. \square

Consider now inequality (12.3.6) for the blocks with $\bar{h} = 1, h = \ell + 1$. Since $k_0 \equiv 1$, it reads:

$$|w \cdot [k_h(z) - k_h(\bar{z})]|^2 \leq w_{12}w_{34} \cdot [k_h(z_{12}) - k_h(\bar{z}_{12})][k_h(z_{34}) - k_h(\bar{z}_{34})], \quad (12.3.10)$$

where we denoted $w = \frac{z\bar{z}}{z-\bar{z}}$, and similarly w_{12}, w_{34} . Let us assume that the configuration $c \in \mathcal{D}_4^{(0)}$ is such that

$$|\rho| \neq |\bar{\rho}|, \quad \rho_{12} \neq \bar{\rho}_{12}, \quad \rho_{34} \neq \bar{\rho}_{34}. \quad (12.3.11)$$

Then, using Lemma 12.3.1, for large h inequality (12.3.10) becomes:

$$(A + o(1)) \max\{|\rho|, |\bar{\rho}|\}^{2h} \leq (B + o(1)) \max\{\rho_{12}, \bar{\rho}_{12}\}^h \max\{\rho_{34}, \bar{\rho}_{34}\}^h, \quad (12.3.12)$$

where A, B are some positive h -independent quantities. Now, taking the limit $h \rightarrow +\infty$, we obtain inequality (12.0.3).

It's easy to see that configurations which violate the condition (12.3.11) are non-generic. They can be approached by configurations which do satisfy (12.3.11). Therefore, by continuity inequality (12.0.3) is valid also for such exceptional configurations.

Chapter 13

OPE convergence in the forward tube and in Minkowski space

We have several OPE convergence statements scattered throughout this part of the thesis. The Euclidean CFT axioms assume convergence of the OPE series for $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ whenever the two points x_1, x_2 are closer to the OPE center than any other point. Then we established OPE convergence in the Hilbert space sense (Sec. 8.2) in the Euclidean region for states generated by two operators in the half-space. Then in Sec. 11.2 we used Hilbert space language to derive the power series representation (11.2.6) for the 4-point function, whose convergence is thus morally equivalent to OPE convergence (for the 4-point functions). We then used this power series representation to analytically continue the 4-point function to the forward tube, and then define the Minkowski 4-point function as a boundary value in the sense of distributions. Finally, in Sec. 11.7 we showed, by arguments not using conformal invariance, that the OS states $|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle$ can be extended to the forward tube and (when integrated against test functions) to the Minkowski region, and that the so obtained states can be arbitrarily well approximated by (integrated) OS states. Therefore, OPE convergence holds for these states, as for the OS states.

In this section we will give a more explicit discussion of the OPE convergence for the Minkowski 4-point function and for the 2-operator states in the forward tube and Minkowski space. We will also explain how our approach and results compare to the classic paper by Mack [26].

13.1 Convergence of conformal block decomposition for 4-point functions

Let us consider the 4-point function of identical scalars (11.0.1):

$$G(x_1, x_2, x_3, x_4) \equiv G(c) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \frac{g(\rho, \bar{\rho})}{(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{O}}}}. \quad (13.1.1)$$

The discussion below can be easily extended to non-identical scalars using the same ideas as in Sec. 11.10.

We know that in the Euclidean region the function $g(\rho, \bar{\rho})$ has a convergent conformal block decomposition

$$g(\rho, \bar{\rho}) = \sum_{\Delta, l} C_{\Delta, l}^2 g_{\Delta, l}(\rho, \bar{\rho}). \quad (13.1.2)$$

As in Sec. 12.3, we will assume that the reader is familiar with conformal blocks. The main point is that the conformal block decomposition is obtained by separating the series (11.2.6) into parts corresponding to the conformal multiplets of primary operators $\mathcal{O}_{\Delta, l}$ occurring in the $\mathcal{O} \times \mathcal{O}$ OPE with coefficients $C_{\Delta, l}$. Conformal blocks in the Euclidean region by themselves have power series expansions like (11.2.6) with positive coefficients (fixed by conformal symmetry). As in Sec. 11.3, we can use this expansion to analytically continue conformal blocks to the forward tube. By an analogue of the bound (11.3.12) we know that conformal block expansion remains convergent

everywhere in the forward tube, since $|\rho|, |\bar{\rho}| < 1$ there. Since individual conformal blocks are smaller than the 4-point function in the Euclidean region, by the arguments in Sec. 11.5 we know that they satisfy a powerlaw bound, and hence they become tempered distributions when going to the Minkowski region.^{1,2,3}

By the arguments like in Sec. 11.5, $g(\rho, \bar{\rho})$, the partial sums of the conformal block decomposition $g(\rho, \bar{\rho}; \Delta_*) = \sum_{\Delta \leq \Delta_*, l} C_{\Delta, l}^2 g_{\Delta, l}(\rho, \bar{\rho})$, and the corresponding remainders satisfy in the forward tube a uniform bound:

$$|g(\rho, \bar{\rho}; \Delta_*)|, |g(\rho, \bar{\rho}) - g(\rho, \bar{\rho}; \Delta_*)| \leq C(1-r)^{-4\Delta}, \quad r = \max\{|\rho|, |\bar{\rho}|\}. \quad (13.1.3)$$

Consider the 4-point partial sums including the prefactor $G(c; \Delta_*) = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{O}}}} g(\rho, \bar{\rho}; \Delta_*)$. By the powerlaw bound of $(1-r(c))^{-1}$, we have the powerlaw bounds

$$|G(c; \Delta_*)|, |G(c) - G(c; \Delta_*)| \leq C \left(1 + \max_k \frac{1}{\epsilon_k - \epsilon_{k+1}}\right)^A (1 + \max_i |x_i - x_{i+1}|)^B \quad (13.1.4)$$

for all $c \in \mathcal{D}_4$ and $\Delta_* \geq 0$. Consider the boundary value of $G(c; \Delta_*)$, call it $G^M(x_1, x_2, x_3, x_4; \Delta_*)$, where $x_i \in \mathbb{R}^{1, d-1}$; it is a tempered distribution by Theorem 9.0.1. The following theorem establishes distributional convergence of conformal block decomposition.

Theorem 13.1.1. *We have $G^M(x; \Delta_*) \rightarrow G^M(x)$ in the sense of tempered distributions.*

Proof. Denote $H(c; \Delta_*) = G(c) - G(c; \Delta_*)$. We have to show that, as Δ_* goes to infinity, the boundary value of $H(c; \Delta_*)$ converges to 0 in the sense of tempered distributions, i.e, for any Schwartz test function $f \in \mathcal{S}(\mathbb{R}^{4d})$

$$\lim_{\Delta_* \rightarrow \infty} \lim_{\lambda \rightarrow 0^+} \int H(\lambda\epsilon + it, \mathbf{x}; \Delta_*) f(t, \mathbf{x}) dt d\mathbf{x} = 0, \quad (\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4), \quad (13.1.5)$$

where we write for brevity t instead of t_1, t_2, t_3, t_4 etc. The proof is the same as in part I, theorem 3.2.1. We will retrace here the main steps for completeness and because we will need it to establish a stronger result below. Define $L_f(\lambda; \Delta_*) := \int H(\lambda\epsilon + it, \mathbf{x}; \Delta_*) f(x) dx$ with $x = (t, \mathbf{x})$. Since H is holomorphic in $\tau = \lambda\epsilon + it$ we have

$$\begin{aligned} L_f^{(n)}(\lambda; \Delta_*) &= \int \left(\left(\epsilon \cdot \frac{\partial}{i\partial t} \right)^n H(\lambda\epsilon + it, \mathbf{x}; \Delta_*) \right) f(x) dx \\ &= \int H(\lambda\epsilon + it, \mathbf{x}; \Delta_*) \left(\left(i\epsilon \cdot \frac{\partial}{\partial t} \right)^n f(x) \right) dx, \end{aligned} \quad (13.1.6)$$

which by the powerlaw bound (13.1.4) implies

$$L_f^{(n)}(\lambda; \Delta_*) \leq \frac{C_n}{\lambda^A} |f|_{p_n}, \quad \lambda \in (0, 1], \quad p_n = \max\{n, [B] + 4d + 1\}. \quad (13.1.7)$$

These bounds blow up in the $\lambda \rightarrow 0$ limit, but by using the Newton-Leibniz repeatedly one can get bounds which do not blow up:

$$L_f^{(n)}(\lambda; \Delta_*) \leq D_n |f|_{p_{n+[A]+1}}, \quad \lambda \in (0, 1]. \quad (13.1.8)$$

Using this for $n = 1$ one proves that the limit $L_f(0; \Delta_*) = \lim_{\lambda \rightarrow 0^+} L_f(\lambda; \Delta_*)$ exists and

$$|L_f(0; \Delta_*) - L_f(\lambda; \Delta_*)| \leq D_1 \lambda |f|_{\max\{[A]+2, [B]+4d+1\}}. \quad (13.1.9)$$

¹This argument shows that any conformal block which occurs in a reflection-positive CFT 4-point function satisfies a powerlaw bound. E.g. conformal blocks for $l \geq 0$, $\Delta \geq l + d - 2$ occur in a 4-point function $\langle \varphi_1 \varphi_2 \varphi_1 \varphi_2 \rangle$ where φ_1, φ_2 are two GFFs of appropriately chosen equal dimension. It should be also possible to show that conformal blocks satisfy a powerlaw bound without relying on a fiducial 4-point function. E.g. for $d = 4$ conformal blocks this follows from their explicit Dolan-Osborn expressions. For general d , powerlaw bound on the diagonal $z = \bar{z}$ can be shown using the differential equation found in [105] and extended to $z \neq \bar{z}$ by the usual arguments.

²It should be noted that away from light cones conformal blocks are better than distributions: they are real-analytic there (although this fact won't play a role for us). In even d this is obvious from their explicit expressions in terms of hypergeometric functions. For general d this follows from a first-order matrix ODE satisfied by a finite-length vector including the conformal block and its low-order derivatives. Such an ODE exists for a length-8 vector and can be built using the quadratic and quartic Casimir equations [106].

³Also ‘‘conformal partial waves’’ $\frac{g_{\Delta, l}(\rho, \bar{\rho})}{(x_{12}^2 x_{34}^2)^{\Delta_{\mathcal{O}}}}$ are tempered distributions in the Minkowski space. Therefore their Fourier transforms are well defined. Explicit expressions for these Fourier transforms were found recently in [107].

By Lebesgue's dominated convergence theorem, for any fixed λ in $(0, 1]$, $L_f(\lambda; \Delta_*)$ tends to zero as $\Delta_* \rightarrow +\infty$. Thus the previous bound implies

$$\begin{aligned} \overline{\lim}_{\Delta_* \rightarrow \infty} |L_f(0, \Delta_*)| &\leq \overline{\lim}_{\Delta_* \rightarrow \infty} |L_f(0, \Delta_*) - L_f(\lambda; \Delta_*)| + \overline{\lim}_{\Delta_* \rightarrow \infty} |L_f(\lambda; \Delta_*)| \\ &\leq D_1 \lambda |f|_{\max\{[A]+2, [B]+4d+1\}}. \end{aligned} \quad (13.1.10)$$

Since λ can be arbitrarily small, we get $\overline{\lim}_{\Delta_* \rightarrow \infty} |L_f(0, \Delta_*)| = 0$. This finishes the proof. \square

13.1.1 Convergence rate for compactly supported test functions

Because of the use of Lebesgue's theorem on dominated convergence, Theorem 13.1.1 does not give the rate of convergence. We will now give the rate in an important special case of compactly supported test functions. This provides an explicit example for the remark in the last paragraph of Sec. 3.3.

The idea is that not only $H(c, \Delta_*) \rightarrow 0$ pointwise but it does so exponentially fast. We will first derive the exponential convergence bound, upgrading the Euclidean argument from [43], to the forward tube, and then use it. Let $F(t)$ be the Laplace transform of a positive measure $\mu(E)$ on $E \geq 0$:

$$F(t) = \int_0^\infty \mu(E) e^{-Et} dE, \quad \mu(E) \geq 0. \quad (13.1.11)$$

We assume that the integral is convergent for $t > 0$ and $F(t) \sim t^{-\alpha}$ as $t \rightarrow 0^+$, and $\alpha > 0$. Then by the Hardy-Littlewood tauberian theorem we know that

$$M(E) = \int_0^E \mu(E') dE' \sim \frac{E^\alpha}{\Gamma(\alpha + 1)} \text{ as } E \rightarrow +\infty. \quad (13.1.12)$$

We can now estimate the remainder $F_{E_*}(t) := \int_{E_*}^\infty \mu(E) e^{-Et} dE$, via

$$F_{E_*}(t) = \int_{E_*}^\infty e^{-Et} dM(E) = -M(E_*) e^{-E_* t} + t \int_{E_*}^\infty M(E) e^{-Et} dE, \quad (13.1.13)$$

which gives $|F_{E_*}(t)| \leq C_1 e^{-E_* t} E_*^\alpha + C_2 e^{-E_* t} \left(\frac{1+E_* t}{t}\right)^\alpha$, and finally

$$|F_{E_*}(t)| \leq \text{const} \times e^{-E_* t} (t^{-1} + E_*)^\alpha. \quad (13.1.14)$$

for any $E_* \geq 1$ (say).

Now let's go back to the 4-point function of four identical scalars $G(c) = \langle \mathcal{O}(x_1) \mathcal{O}(x_2) \mathcal{O}(x_3) \mathcal{O}(x_4) \rangle$, $c \in \mathcal{D}_4$. The remainder $H(c; \Delta_*)$ can clearly be bounded by replacing $\rho, \bar{\rho}$ with $r = \max\{|\rho|, |\bar{\rho}|\}$:

$$|H(c; \Delta_*)| \leq \frac{1}{|x_{12}^2|^{\Delta_\mathcal{O}} |x_{34}^2|^{\Delta_\mathcal{O}}} [g(r, r) - g(r, r; \Delta_*)], \quad (13.1.15)$$

By setting $t = \log(1/r)$, $g(r, r)$ with its representation (11.2.6) is in the same form as (13.1.11) and by Eq. (11.5.3) we have that the corresponding $F(t) \sim t^{-\alpha}$ with $\alpha = 4\Delta_\mathcal{O}$. Bounding the remainder $g(r, r) - g(r, r; \Delta_*)$ by (13.1.14), and using (13.1.15), we get a bound on the remainder $|H(c; \Delta_*)|$ for any $\Delta_* \geq 1$ (say):

$$|H(c; \Delta_*)| \leq \frac{\text{const}}{|x_{12}^2|^{\Delta_\mathcal{O}} |x_{34}^2|^{\Delta_\mathcal{O}}} \times r(c)^{\Delta_*} \left(\frac{1}{\log(r(c)^{-1})} + \Delta_* \right)^{4\Delta_\mathcal{O}}. \quad (13.1.16)$$

Let us now convert this into an explicit estimate for the distributional convergence rate, improving on Theorem 13.1.1 for compactly supported test functions. Recall that we have an upper bound on $r(c) = \max\{|\rho(c)|, |\bar{\rho}(c)|\}$ ($c = (x_1, x_2, x_3, x_4)$), Eq. (12.2.8), which we copy here:

$$\frac{1}{1-r(c)} \leq 2 \left(1 + \frac{1}{\epsilon_2 - \epsilon_3} \right) (1 + \max\{|x_1 - x_2|, |x_3 - x_4|\}). \quad (13.1.17)$$

This bound tells us how much $r(c)$ is separated from 1. In turn, by (13.1.16) this translates into an explicit bound on $|H(c; \Delta_*)|$. Let us retrace the proof of Theorem 13.1.1, replacing Eq. (13.1.10) by

$$\begin{aligned} |L_f(0, \Delta_*)| &\leq |L_f(0, \Delta_*) - L_f(\lambda; \Delta_*)| + |L_f(\lambda; \Delta_*)| \\ &\leq D_1 \lambda |f|_{\max\{[A]+2, [B]+4d+1\}} + |L_f(\lambda; \Delta_*)|. \end{aligned} \quad (13.1.18)$$

We will now choose λ small, as a function of Δ_* , so that the second term in the r.h.s. is smaller than the first one. Let us choose and fix $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$. For $x_k^\lambda = (\lambda \epsilon_k + it_k, \mathbf{x}_k)$, $x_k^M = (it_k, \mathbf{x}_k)$, bound (13.1.17) gives

$$\frac{1}{1 - r(c^\lambda)} \leq \frac{C_\epsilon}{\lambda} (1 + \max\{|x_1^M - x_2^M|, |x_3^M - x_4^M|\}) \quad (0 < \lambda \leq 1), \quad (13.1.19)$$

where C_ϵ is a constant which depends only on ϵ_i but not on λ or x_k^M . If $f \in C_0^\infty(\mathbb{R}^{4d})$, a compactly supported test function, then (13.1.19) implies

$$\frac{1}{1 - r(c^\lambda)} \leq \frac{A_f}{\lambda} \quad (0 < \lambda \leq 1, (x_k^M) \in \text{supp}(f)), \quad (13.1.20)$$

where $A_f = 2C_\epsilon \sup_{(x_k^M) \in \text{supp}(f)} (1 + \max\{|x_1^M - x_2^M|, |x_3^M - x_4^M|\})$.

Now by (13.1.16), (13.1.20), and Lemma 10.0.2(b) we have a bound for a compactly supported test function f :

$$|L_f(\lambda; \Delta_*)| \leq \frac{\text{const}}{\lambda^{4\Delta_\circ}} e^{-\frac{\lambda}{A_f} \Delta_*} \left(\frac{A_f}{\lambda} + \Delta_* \right)^{4\Delta_\circ} \int |f| dx. \quad (13.1.21)$$

By this and (13.1.18) we have

$$|L_f(0; \Delta_*)| \leq A_1 \lambda + \frac{A_2}{\lambda^{4\Delta_\circ}} e^{-\frac{\lambda}{A_f} \Delta_*} \left(\frac{A_f}{\lambda} + \Delta_* \right)^{4\Delta_\circ}, \quad (13.1.22)$$

where all constants A_1, A_2, A_f are f -dependent. If we choose $\lambda = 1/\Delta_*^\gamma$ then for any $\gamma \in (0, 1)$ the first term dominates (the second term decays exponentially fast for large Δ_*). It is easy to see that the first term still dominates for $\lambda = \kappa \frac{\log \Delta_*}{\Delta_*}$ with sufficiently large κ . We therefore obtain the following promised strengthening of Theorem 13.1.1.

Theorem 13.1.2. *For any compactly supported C^∞ test function f , we have $(G^M(\Delta_*), f) - (G^M, f) \rightarrow 0$ as $O\left(\frac{\log \Delta_*}{\Delta_*}\right)$.*

It is somewhat surprising that conformal block decomposition converges so slowly in the Minkowski region, while it converges exponentially fast in the Euclidean region.

13.2 OPE convergence in the sense of \mathcal{H}^{CFT}

We will now rephrase the question of OPE convergence from the point of view of states generated by two operators (rather than 4-point functions which represent inner products of such states). We already discussed these questions to some extent in Sec. 8.2 in the Euclidean region, and Sec. 11.7 in the forward tube and in the Minkowski region. We will now update that discussion.

Recall that in Sec. 11.7 we defined states (see Eq. (11.7.2) and Remark 11.7.1)

$$\psi(x_1, x_2) = |\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle, \quad x_k = (\epsilon_k + it_k, \mathbf{x}_k + iy_k), \quad 0 \succ (\epsilon_1, \mathbf{y}_2) \succ (\epsilon_1, \mathbf{y}_2), \quad (13.2.1)$$

as elements of a vector space with inner products (11.7.4):

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2) | \mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = G_4(x_2^\theta, x_1^\theta, x_3, x_4), \quad (13.2.2)$$

where $x^\theta = (-\epsilon + it, \mathbf{x} - i\mathbf{y})$ for $x = (\epsilon + it, \mathbf{x} + i\mathbf{y})$. We then proved that this inner product was positive definite, and that these states could be approximated in norm by Euclidean states, and so belong to the same Hilbert space. The arguments of Sec. 11.7 did not use conformal symmetry.

So, by arguments of Sec. 11.7 we have a map $\psi(x_1, x_2)$ from x_1, x_2 as in (13.2.1) into \mathcal{H}^{CFT} . We claim that this map is holomorphic. To begin with, this map is continuous with respect to the \mathcal{H}^{CFT} norm, and in particular bounded on compact subsets. This follows easily from the continuity of G_4 . To show holomorphicity, we will use Morera's theorem and Osgood's lemma (which remain valid for Hilbert-space-valued functions of complex variables). Morera's theorem says that a locally continuous function of one complex variable is holomorphic if its integral over any small contour is zero. Let C be a small 1d contour in the region of $\xi = (x_1, x_2)$ where ψ is defined (we assume that one of the components of ξ goes around the contour while the others stay fix). We have

$$\left\| \int_C d\xi \psi(\xi) \right\|^2 = \int_{\xi' \in C} d\bar{\xi}' \int_{\xi \in C} d\xi G_4(\xi'^\theta, \xi) = 0, \quad (13.2.3)$$

the last integral being zero because G_4 is holomorphic in ξ . Hence $\int_C d\xi \psi(\xi) = 0$ and by Morera's theorem $\psi(x_1, x_2)$ is holomorphic in each component separately. Finally, by Osgood's lemma [97] $\psi(x_1, x_2)$ is holomorphic in all variables jointly.

Let us connect this discussion with the OPE. In the Euclidean region, OPE says

$$\begin{aligned} \psi(x_1, x_2) = |\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle &= \sum_{k,\lambda} f_{\mathcal{O}\mathcal{O}k} C_{k,\lambda}(x_1, x_2, x_S, \mathcal{D}) |(\mathcal{O}_k^\dagger)^{(\lambda)}(x_S)\rangle, \\ C_{k,\lambda}(x_1, x_2, x_S, \mathcal{D}) &= \sum_{\alpha} f_{\mathcal{O}\mathcal{O}k} C_{k,\lambda,\alpha}(x_1, x_2, x_S) \mathcal{D}^\alpha, \end{aligned} \quad (13.2.4)$$

where x_1, x_2 are two Euclidean points in the lower halfspace, $0 > x_1^0, x_2^0$, $x_S = (-1, 0, \dots)$ is the south pole, and $\mathcal{D} = \mathcal{D}_{x_S}$ is the image of $\partial/\partial x|_{x=0}$ under a conformal transformation which maps $0, \infty$ to x_S, x_N . We proved in Sec. 8.2, from the Euclidean CFT axioms, that the series in the r.h.s. converges in \mathcal{H}^{CFT} . As discussed in Sec. 8.2, convergence holds provided that the series is summed in a certain manner: for each Λ we define a partial sum $\psi_\Lambda(x_1, x_2)$ over all terms with $\Delta_k + |\alpha| < \Lambda$, and then tend $\Lambda \rightarrow \infty$. This procedure is needed because, although the states $\mathcal{D}^\alpha |(\mathcal{O}_k^\dagger)^{(\lambda)}(x_S)\rangle$ are orthogonal for different $|\alpha|$ (because they are eigenvectors of $\frac{K^0 - P^0}{2}$ with different eigenvalues), they are not orthonormal. This can be corrected as follows. For each k , let us orthonormalize the infinite multiplet of states $\mathcal{D}^\alpha |(\mathcal{O}_k^\dagger)^{(\lambda)}(x_S)\rangle$. Let $e_{k,n}$, $n \in \mathbb{Z}_{\geq 0}$, be the corresponding orthonormal basis of states (there is obviously a lot of arbitrariness in this basis). We then can write

$$\psi(x_1, x_2) = |\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \sum_k f_{\mathcal{O}\mathcal{O}k} \sum_n \tilde{C}_{k,n}(x_1, x_2) e_{k,n}, \quad (13.2.5)$$

where $\tilde{C}_{k,n}$'s are some finite linear combinations of $C_{k,\lambda,\alpha}(x_1, x_2, x_S)$. Since $e_{k,n}$'s with different k are also orthogonal, this equation is an expansion of the state $\psi(x_1, x_2)$ in an orthonormal basis. Convergence of the series is now equivalent to the finiteness of the norm of $\psi(x_1, x_2)$:

$$\|\psi(x_1, x_2)\|_{\mathcal{H}^{\text{CFT}}} = \sum_{k,n} |f_{\mathcal{O}\mathcal{O}k} \tilde{C}_{k,n}(x_1, x_2)|^2 < \infty. \quad (13.2.6)$$

Moreover, by definition, this norm is nothing but the 4-point function $\langle \mathcal{O}(x_2^\theta) \mathcal{O}(x_1^\theta) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle$.

Eqs. (13.2.4)-(13.2.6) were all in the Euclidean region, but we claim that they continue to make sense in the forward tube. The argument is as follows. We know by the arguments around (13.2.3) that $\psi(x_1, x_2)$ have analytic continuation to the region (13.2.1). The inner product $\langle e_{k,n} | \psi(x_1, x_2) \rangle$ is then the analytic continuation of $f_{\mathcal{O}\mathcal{O}k} \tilde{C}_{k,n}(x_1, x_2)$ from the Euclidean to the same region. (This inner product is a finite linear combination of x_N -derivatives of the 3-point function $\langle (\mathcal{O}_k)^{(\lambda)}(x_N) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle$, hence holomorphic in the forward tube.) We thus obtain the following fact:

Theorem 13.2.1. *Expansion (13.2.5), analytically continued from the Euclidean region to the forward tube term by term, converges in the sense of \mathcal{H}^{CFT} to the same states $\psi(x_1, x_2)$ in the region (13.2.1) that we defined in Sec. 11.7.*

For the subsequent discussion, we also define the states

$$\psi_k(x_1, x_2) = f_{\mathcal{O}\mathcal{O}k} \sum_n \tilde{C}_{k,n}(x_1, x_2) e_{k,n}. \quad (13.2.7)$$

The norms of these states is given by conformal blocks (up to prefactor $f_{\mathcal{O}\mathcal{O}k}^2/(x_{12}^2)^{4\Delta_{\mathcal{O}}}$). Just as the state $\psi(x_1, x_2)$, each state $\psi_k(x_1, x_2)$ is an \mathcal{H}^{CFT} -valued holomorphic function in the region (13.2.1), moreover in this region we have

$$\psi(x_1, x_2) = \sum_k \psi_k(x_1, x_2), \quad (13.2.8)$$

the series convergent in the sense of \mathcal{H}^{CFT} . The norm of the tail of this series is given by the function $H(c, \Delta_*)$ from the proof of Theorem 13.1.1:

$$\left\| \sum_{\Delta_k > \Delta_*} \psi_k(x_1, x_2) \right\| = H(c, \Delta_*), \quad c = (x_2^\theta, x_1^\theta, x_1, x_2). \quad (13.2.9)$$

Vladimirov's Theorem 9.0.1 remains true for Hilbert-space-valued holomorphic functions, whose norm satisfies a powerlaw bound in the forward tube. Applying such a version of Theorem 9.0.1, as well as arguments from the proof of Theorem 13.1.1 and from Sec. 13.1.1 (in particular using the bound (13.1.16)), it's easy to obtain the following result (we omit the proof).

Theorem 13.2.2. (a) *The boundary value $\text{bv}(\psi) = \lim_{\epsilon_i \rightarrow 0} \psi(x_1, x_2)$ exists as \mathcal{H}^{CFT} -valued tempered distributions, and similarly for each $\text{bv}(\psi_k)$: $\text{bv}(\psi), \text{bv}(\psi_k) \in \mathcal{S}'(\mathbb{R}^{2d}) \otimes \mathcal{H}^{\text{CFT}}$. Explicitly, the limit*

$$(\text{bv}(\psi), f) = \lim_{\epsilon \rightarrow 0} \int \psi(\epsilon + it, \mathbf{x}) f(t, \mathbf{x}) d^2t d^{2(d-1)}\mathbf{x} \quad (13.2.10)$$

exists as a vector in \mathcal{H}^{CFT} for any Schwartz function $f \in \mathcal{S}(\mathbb{R}^{2d})$, and is a continuous linear operator from $\mathcal{S}(\mathbb{R}^{2d})$ to \mathcal{H}^{CFT} , and analogously for $\text{bv}(\psi_k)$;

(b) *(Distributional OPE convergence in \mathcal{H}^{CFT}) For each Schwartz test function $f \in \mathcal{S}(\mathbb{R}^{2d})$, the series $\sum_k (\text{bv}(\psi_k), f)$ converges in \mathcal{H}^{CFT} norm to $(\text{bv}(\psi), f)$;*

(c) *(Convergence rate) For compactly supported test functions, the series in (b) summed over $\Delta_k \leq \Delta_*$ converges with rate $O\left(\sqrt{\frac{\log \Delta_*}{\Delta_*}}\right)$.*

Remark 13.2.1. The following more finegrained version of Theorem 13.2.2(b) also holds. Denote

$$E_{k,n}(x_1, x_2) = f_{\mathcal{O}\mathcal{O}k} \tilde{C}_{k,n}(x_1, x_2) e_{k,n}. \quad (13.2.11)$$

As explained above, $\tilde{C}_{k,n}(x_1, x_2)$ is a finite sum of terms like $(\mathcal{D}^\theta)^\alpha \langle (\mathcal{O}_k)^{(\lambda)}(x_N) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle$ (all having the same $|\alpha|$), so $\text{bv}(E_{k,n})$ exists by Vladimirov's theorem. Then *for each Schwartz test function $f \in \mathcal{S}(\mathbb{R}^{2d})$, the series $\sum_{k,n} (\text{bv}(E_{k,n}), f)$ converges in \mathcal{H}^{CFT} norm to $(\text{bv}(\psi), f)$.*

It would be interesting to prove a version Theorem 13.2.2(c), truncating the series $\sum_{k,n} (\text{bv}(E_{k,n}), f)$ to k, n such as the corresponding $\Delta_k + |\alpha| \leq \Delta_*$. To do so one would have to find an analogue of the bounds (13.2.9), (13.1.16) valid for such a truncation. This is not straightforward because the partial sums of the series (13.2.6) truncated to $\Delta_k + |\alpha| \leq \Delta_*$ do not correspond to a simple truncation of the $\rho, \bar{\rho}$ series of the full 4-point function (basically because the transformation which maps $x_1, x_2, x_1^\theta, x_2^\theta$ to their $\rho, \bar{\rho}$ conformal frame does not necessarily map x_S, x_N to $0, \infty$).

13.3 Comparison to Mack's work on OPE convergence

To assume OPE convergence as an axiom in Euclidean CFT, and to derive Minkowski physics from it, as we did in this part of the thesis, seems natural from the modern perspective. On the contrary, in the early days of CFTs it

was considered natural to assume standard Minkowski physics (such as Wightman axioms). The OPE convergence was not assumed at the time, but was something to be derived.

This was the underlying philosophy of the works by Lüscher and Mack [25] and of Mack [26, 108]. Written 45 years ago, these papers remain widely cited, but not everyone is familiar with what precisely has been derived there. Here we will present a short review for the benefit of the modern audience.

These works make two main assumptions. **First**, that we have a unitary quantum field theory in the Minkowski signature which satisfies Wightman axioms (in particular has a Hilbert space \mathcal{H} on which the Poincaré group acts unitarily and quantum fields are operator-valued distributions). Correlators then have the usual analyticity properties of Wightman functions, in particular they are real-analytic in the Euclidean. The **second** main assumption is that these Euclidean correlators are invariant under the action of the Euclidean conformal group.

Using these two assumptions, Lüscher and Mack [25] proved that the Hilbert space \mathcal{H} carries a unitary representation not just of the Poincaré but of the group G^* = universal cover of the Lorentzian conformal group $\text{SO}(d, 2)$.⁴ Mack [108] then classified all unitary positive energy representations of G^* . It should be mentioned that Refs. [25, 26, 108] only consider $d = 4$ spacetime dimensions. Many explicit group theoretic calculations are done only for this value of d . The results should however generalize to arbitrary d with appropriate modifications.

Continuing this program, Mack [26] studied distributional OPE convergence in Minkowski CFT. Since we also have results of this kind (Sec. 13.2), it will be particularly interesting to compare with Mack's discussion. So let us review his argument. Compared to [25, 108], Ref. [26] includes one extra assumption: that the OPE $\varphi_i(x_1)\varphi_j(x_2)$ of two fields acting on the Minkowski vacuum is valid in an asymptotic sense. Namely that for some dense set of states ψ we have⁵

$$\langle \psi | \varphi_i(x_1)\varphi_j(x_2) \rangle \sim \sum_k C_{ijk}(x_1, x_2) \langle \psi | \varphi_k(0) \rangle, \quad (13.3.1)$$

as an asymptotic expansion for rescaling $x_1, x_2 \rightarrow \lambda x_1, \lambda x_2$, where $C_{ijk}(x_1, x_2)$ are some ψ -independent homogeneous functions: $C_{ijk}(\lambda x_1, \lambda x_2) = \lambda^{\Delta_k - \Delta_i - \Delta_j} C_{ijk}(x_1, x_2)$.⁶ Asymptotic means that if we truncate the expansion at $\Delta_k = \Delta_*$ and take $\lambda \rightarrow 0$ limit for any fixed x_1, x_2 then the error is $o(\lambda^{\Delta_* - \Delta_i - \Delta_j})$. Note that there are both primaries and descendants among φ_k 's. The main result of [26] is to convert this asymptotic expansion to an expansion convergent in the Hilbert space sense.

The first step is to use a general result that any Hilbert space carrying a unitary representation of a semisimple Lie group can be decomposed as a direct integral of unitary irreducible representations [109]. Since, by the above-mentioned result of [25], \mathcal{H} carries a unitary representation of G^* , we thus have

$$\mathcal{H} = \int^{\oplus} d\mu_\chi d\tilde{\mu}_\nu \mathcal{H}^{\chi\nu}, \quad (13.3.2)$$

where χ labels different unitary irreps of G^* , $\chi = (\Delta_\chi, \rho_\chi)$ with Δ_χ the scaling dimension and ρ_χ a Lorentz group irrep, and ν labels different copies of the same irrep. Only positive energy irreps may occur, since all states of Wightman theory have positive energy. By definition, Eq. (13.3.2) identifies every vector $\psi \in \mathcal{H}$ with a Hilbert-space-valued function $(\chi, \nu) \mapsto \psi_{\chi\nu} \in \mathcal{H}^\chi$, some standard realization of the irrep χ . It is assumed that $\langle \psi | \psi \rangle < \infty$, inner products being given by the following integral:

$$\langle \psi | \psi' \rangle = \int d\mu_\chi d\tilde{\mu}_\nu \langle \psi_{\chi\nu} | \psi'_{\chi\nu} \rangle_{\mathcal{H}^\chi}, \quad (13.3.3)$$

Also G^* acts on ψ by acting on each $\psi_{\chi\nu}$. Furthermore, Ref. [108] realized \mathcal{H}^χ as a space of distributions $\psi(x)$ on Minkowski space⁷ taking values in the representation space of ρ_χ , with Fourier transform supported in the forward

⁴One also often quotes Lüscher and Mack [25] for proving that CFT correlation functions may be continued to the Minkowski cylinder $S^{d-1} \times \mathbb{R}$. This is a misquotation as they did not prove this, but posed it as a conjecture. What they did prove was that CFT correlation functions can be analytically continued to a domain of which $S^{d-1} \times \mathbb{R}$ is a real boundary. One still needs to establish a powerlaw bound to take the boundary limit and obtain a distribution. We plan to derive this fact in our future work [41], for 4-point functions, from the Euclidean CFT axioms, using the $\rho, \bar{\rho}$ expansion.

⁵Mack assumes $x_2 = -x_1$ but here for simplicity we will assume that this is valid for any x_1, x_2 .

⁶In fact C_{ijk} is a distribution so homogeneity should be understood in the sense of pairing with a rescaled test function.

⁷More properly $\psi(x)$ is a distribution on the Lorentzian cylinder on which the group G^* acts naturally, but due to a periodicity condition it may be reconstructed from its values on the Poincaré patch.

light cone, and for which the following inner product (defined initially on a dense subset of smooth ψ, ψ') is finite:

$$\langle \psi | \psi' \rangle_{\mathcal{H}^x} = \int dx dy \overline{\psi(x)} I^x(x-y) \psi'(y), \quad (13.3.4)$$

where I^x is an ‘‘intertwining kernel’’. Physically, I^x is the Minkowski CFT 2-point function of the primary in the ‘‘shadow irrep’’ of χ .

The above integration measure $d\mu_\chi d\tilde{\mu}_\nu$ depends on the theory; from the abstract arguments alone it may be continuous or discrete. Ref. [26] then proceeds to show that (a) this measure is actually discrete (a sum of delta-functions), so that the direct integral is a direct sum; (b) that the state $|\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle$ produced by two Minkowski primary operators acting on the Minkowski vacuum can be written as

$$|\mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \sum_{k,a} f_{ijk}^a \int dx B_{k,ij}^a(x, x_1, x_2) |\mathcal{O}_k(x)\rangle, \quad (13.3.5)$$

where $B_{ijk}^a(x, x_1, x_2)$, $a = 1 \dots N_{ijk}$, are some kinematically determined distributions, the convergence is in the Hilbert space sense after integrating with an arbitrary test functions $f(x_1, x_2)$, and the local primary operators \mathcal{O}_k occurring in this sum have quantum numbers in the discrete set where the integration measure $d\mu_\chi d\tilde{\mu}_\nu$ is supported.

To show how this comes about, let us focus on the case of scalar identical $\mathcal{O}_i = \mathcal{O}_j = \mathcal{O}$ for simplicity. In this case expansion (13.3.5) will end up being precisely our expansion (13.2.8) (although derived under very different assumptions), with $B(x, x_1, x_2)$ related to the OPE kernel in (13.2.4).

Applying (13.3.3) with $|\psi'\rangle = |\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle$ gives (Eq. (2.6) in [26]):

$$\langle \psi | \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = \int d\mu_\chi d\tilde{\mu}_\nu c_{\chi\nu} \int dx \overline{\psi_{\chi\nu}(x)} B_\chi(x, x_1, x_2), \quad (13.3.6)$$

where we denoted $\int dy I^x(x-y) \psi'_{\chi\nu}(y) = c_{\chi\nu} B_\chi(x, x_1, x_2)$ where $c_{\chi\nu}$ is a proportionality factor, and $B_\chi(x, x_1, x_2)$ is a kinematically determined distribution (it is basically the amputated Minkowski 3-point function $\langle \mathcal{O}_\chi(x)\mathcal{O}(x_1)\mathcal{O}(x_2) \rangle$). Mack then undertakes a meticulous study of $B_\chi(x, x_1, x_2)$ and of its Fourier transform with respect to the first argument $\widehat{B}_\chi(p, x_1, x_2)$. This actually takes most of his paper, and involves many explicit nontrivial calculations (e.g. it involves the first ever explicit characterization of the most general 3-point function of CFT primaries in arbitrary irreps). One of the main results is that $\widehat{B}_\chi(p, x_1, x_2)$ are entire functions of p :

$$\widehat{B}_\chi(p, x_1, x_2) = \sum_{|\alpha| \geq 0} b_\chi^\alpha(x_1, x_2) (-ip)_\alpha, \quad (13.3.7)$$

where $b_\chi^\alpha(x_1, x_2) = (x_{12}^2)^{-\Delta_\mathcal{O} + \Delta_\chi/2}$ times a polynomial in x_1, x_2 of degree $|\alpha|$, in particular $b_\chi^\alpha(\lambda x_1, \lambda x_2) = \lambda^{\Delta_\chi + |\alpha| - 2\Delta_\mathcal{O}} b_\chi^\alpha(x_1, x_2)$.

Let us now specialize to states ψ for which the function $\psi_{\chi\nu}(x)$ has Fourier transform of compact support (one can show that such states are dense in \mathcal{H}). Then the previous equations imply the following convergent expansion for the integrand in (13.3.6):

$$\int dx \overline{\psi_{\chi\nu}(x)} B_\chi(x, x_1, x_2) = \sum_\alpha b_\chi^\alpha(x_1, x_2) \overline{\partial^\alpha \psi_\chi(0)}, \quad (13.3.8)$$

Mack then claims (before Eq. (2.11')) that if, for each χ , this convergent expansion is truncated at $\Delta_\chi + |\alpha| = \Delta_*$, and inserted back into (13.3.6), this results in an asymptotic expansion for the l.h.s. of (13.3.6). I.e. for any Δ_* (Mack does not write this equation explicitly):

$$\langle \psi | \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = \left\{ \int d\mu_\chi d\tilde{\mu}_\nu c_{\chi\nu} \sum_{\Delta_\chi + |\alpha| \leq \Delta_*} b_\chi^\alpha(x_1, x_2) \overline{\partial^\alpha \psi_\chi(0)} \right\} + E(x_1, x_2; \Delta_*), \quad (13.3.9)$$

where the error term $E(\lambda x_1, \lambda x_2; \Delta_*) = O(\lambda^{\Delta_* - 2\Delta_\mathcal{O}})$ as $\lambda \rightarrow 0$ for any fixed x_1 . Unfortunately, Mack does not give any justification of this claim, which to us does not appear self-evident. The difficulty is that although for every χ, ν the truncated series has error $O(\lambda^{\Delta_* - 2\Delta_\mathcal{O}})$, the constant will certainly depend on χ, ν . How do we know that the error estimate survives after the integration in χ, ν ? It might be possible to close this omission in Mack's reasoning using normalizability of $|\psi\rangle$, but this needs extra arguments compared to what is given in his paper, and we have not investigated this in detail.⁸ Researchers relying on Mack's result should keep this caveat in mind.

⁸We also tried, but unfortunately we did not manage, to get feedback from Prof. Gerhard Mack concerning this matter.

Assuming that (13.3.9) is true, the argument is completed as follows. We now have two asymptotic expansions for the l.h.s. of $\langle \psi | \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle$, one coming from (13.3.9), and another from (13.3.1). The second one is discrete (by assumption), so the first one also must be discrete. This establishes that the measure $d\mu_\chi d\tilde{\mu}_\nu$ is discrete, a sum of delta functions, hence we can write (13.3.6) with the r.h.s. as a sum, not an integral:

$$\langle \psi | \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \sum_n c_{\chi_n} \int dx \overline{\psi_{\chi_n}(x)} B_{\chi_n}(x, x_1, x_2). \quad (13.3.10)$$

A more detailed comparison of this equation with (13.3.1) leads us to conclude that (a)

$$c_{\chi_n} \overline{\psi_{\chi_n}(x)} = f_n \langle \psi | \mathcal{O}_{\chi_n}(x) \rangle, \quad (13.3.11)$$

where \mathcal{O}_{χ_n} are primary operators related by rescaling to a subset of the local operators φ_k , we choose them unit-normalized (hence a coefficient f_n); (b) that all the other operators φ_k are the descendants \mathcal{O}_{χ_n} 's; and (c) that all coefficients $C_k(x_1, x_2)$ are basically the expansion coefficients $b_{\chi_n}^\alpha(x_1, x_2)$ in (13.3.7). From (13.3.10) and (13.3.11), we have

$$\langle \psi | \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \sum_n f_n \int dx \langle \psi | \mathcal{O}_{\chi_n}(x) \rangle B_{\chi_n}(x, x_1, x_2), \quad (13.3.12)$$

for a dense set of states ψ . Because of the orthogonality of different $|\mathcal{O}_{\chi_n}(x)\rangle$'s, this implies that

$$|\mathcal{O}(x_1) \mathcal{O}(x_2)\rangle = \sum_n f_n \int dx B_{\chi_n}(x, x_1, x_2) |\mathcal{O}_{\chi_n}(x)\rangle, \quad (13.3.13)$$

the sum convergent in the Hilbert space sense after integrating out with any test function $f(x_1, x_2)$. This is Eq. (13.3.5) in the considered case $\mathcal{O}_i = \mathcal{O}_j = \mathcal{O}$.

13.3.1 Relating Mack's kernel B to the Euclidean OPE kernel C

Now we would like to relate Mack's OPE kernel B to our OPE kernel $C_{a,(\chi)}^{(\mu)(\nu)}(x_1, x_2, x_0, \partial_0)$ defined by Eqs. (7.3.10) and (7.3.11). We only consider the OPE kernel for the scalar external operators for simplicity, i.e. $C_\chi(x_1, x_2, x_0, \partial_0)$, $\chi = (\Delta, \ell = 0)$; similar remarks apply in the general case. We first give the conclusion:

$$C_\chi(x_1, x_2, x_0, \partial_0) = \sum_\mu b_\chi^\mu(x_1 - x_0, x_2 - x_0) \partial_0^\mu, \quad (13.3.14)$$

where the coefficient functions b_χ^μ are the same as in (13.3.7). One could "derive" this by using (13.3.5) and formally manipulating the integral in the momentum space:⁹

$$\begin{aligned} |\mathcal{O}(x_1) \mathcal{O}(x_2)\rangle_\chi &= \int dx B_\chi(x, x_1, x_2) |\mathcal{O}_\chi(x)\rangle \\ &= \int dp \widehat{B}_\chi(p, x_1, x_2) |\widehat{\mathcal{O}}_\chi(p)\rangle = \sum_\mu b_\chi^\mu(x_1, x_2) |\partial^\mu \mathcal{O}_\chi(0)\rangle, \end{aligned} \quad (13.3.15)$$

which shows (13.3.14) in the case when $x_0 = 0$. The general x_0 case follows by translation invariance. This derivation is not rigorous for various reasons: (a) we did not clarify the meaning of $|\widehat{\mathcal{O}}_\chi(p)\rangle$; (b) why can we swap the order of summation and integration? (c) the above derivation is done in Minkowski region, how do we match the coefficients $b_\chi^\mu(x_1, x_2)$ with the Euclidean coefficients in (7.3.11)?

Below we will give a rigorous justification of (13.3.14), using only the two- and 3-point functions which are kinematically determined by conformal invariance. Recall that on the Euclidean side, the formal power series of C_χ (the scalar version of (7.3.11))

$$C_\chi(x_1, x_2, x_0, \partial_0) = \frac{1}{(x_{12}^2)^{\Delta_\mathcal{O} - \Delta_\chi/2}} \sum_\mu c_\chi^\mu(x_{10}, x_{20}) \partial_0^\mu \quad (13.3.16)$$

⁹Since here we are only interested in the OPE kernels B_χ and C_χ , we set $f_\chi = 1$ (the overall coefficient) for convenience.

is determined by the Euclidean two- and 3-point functions:

$$\langle \mathcal{O}_\chi^\dagger(y) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_E = \frac{1}{(x_{12}^2)^{\Delta_\sigma - \Delta_\chi/2}} \sum_\alpha c_{\chi, \alpha}(x_{10}, x_{20}) \langle \mathcal{O}_\chi^\dagger(y) \partial^\alpha \mathcal{O}_\chi(x_0) \rangle_E.$$

Here we already used translation invariance, which implies $C_{a, (\lambda), \alpha}^{(\mu)(\nu)}(x_1, x_2, x_0) = c_{a, (\lambda), \alpha}^{(\mu)(\nu)}(x_{10}, x_{20})$ on the r.h.s. of (7.3.11). One can match the coefficients $c_\chi^\mu(x_{10}, x_{20})$ with the Taylor expansion of $\langle \mathcal{O}_\chi^\dagger(y) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_E$ around $x_1 = x_2 = x_0$. In Euclidean one can always find a proper region for the matching: let y be sufficiently far from the (x_0, x_1, x_2) cluster, so that the Taylor expansions of $[(y - x_1)^2]^\#$ around $x_1 = x_0$ and $[(y - x_2)^2]^\#$ around $x_2 = x_0$ are convergent.

On the Minkowski side, the OPE kernel B_χ is kinematically determined by the equality

$$\widehat{G}_{\mathcal{O}_\chi \mathcal{O} \mathcal{O}}(p, x_1, x_2) = \widehat{B}_\chi(p, x_1, x_2) \widehat{G}_\chi(p), \quad (13.3.17)$$

where $\widehat{G}_{\mathcal{O}_\chi \mathcal{O} \mathcal{O}}(p, x_1, x_2) = \int dy \langle \mathcal{O}_\chi^\dagger(y) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_M e^{-ip \cdot y}$ and $\widehat{G}_\chi(p) = \int dy \langle \mathcal{O}_\chi^\dagger(y) \mathcal{O}_\chi(0) \rangle_M e^{-ip \cdot y}$ (see [26], Eq. (8.2)).¹⁰ All Fourier transforms here are in the sense of distributions. To get an equation valid in the sense of functions we pick a test function φ with compactly supported Fourier transform, and integrate (13.3.17) against $\widehat{\varphi}$, which gives:

$$\int \langle \mathcal{O}_\chi^\dagger(x) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_M \varphi(x) dx = \int dp \widehat{\varphi}(p) \widehat{B}_\chi(p, x_1, x_2) \widehat{G}_\chi(p). \quad (13.3.18)$$

The variable x ranges over the Minkowski space, while we will pick x_1, x_2 complex, in the forward tube region

$$\text{Im}(x_1), \text{Im}(x_2) < 0. \quad (13.3.19)$$

Then the 3-point function $\langle \mathcal{O}_\chi^\dagger(x) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_M$ is nonsingular as a function of x and the l.h.s. of (13.3.18) is a finite number. To transform the r.h.s. of (13.3.18) we will use the fact that \widehat{B}_χ has the following form (a more detailed version than (13.3.7)):

$$\widehat{B}_\chi(p, x_1, x_2) = \frac{e^{-ip \cdot x_1}}{(x_{12}^2)^{\Delta_\sigma - \Delta_\chi/2}} E_\chi(x_{12} \cdot p, x_{12}^2 p^2), \quad (13.3.20)$$

where $E_\chi(z_1, z_2)$ is some entire function on \mathbb{C}^2 . Hence as long as $x_{12}^2 \neq 0$ (not necessarily real), $\widehat{B}_\chi(p, x_1, x_2)$ has the following convergent expansion:

$$\widehat{B}_\chi(p, x_1, x_2) = \frac{e^{-ip \cdot x_1}}{(x_{12}^2)^{\Delta_\sigma - \Delta_\chi/2}} \sum_\alpha a_\chi^\alpha(x_{21}) (-ip)_\alpha, \quad (13.3.21)$$

where $a_\chi^\alpha(x)$ is some $\text{SO}(1, d-1)$ -covariant, homogeneous, symmetric polynomial of degree $|\alpha|$. Plugging this into (13.3.18), using that the expansion (13.3.21) converges uniformly on the support of $\widehat{\varphi}$ (assumed compact), and the fact that \widehat{G}_χ is a tempered measure,¹¹ we obtain:

$$\int \langle \mathcal{O}_\chi^\dagger(x) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_M \varphi(x) dx = \frac{1}{(x_{12}^2)^{\Delta_\sigma - \Delta_\chi/2}} \sum_\alpha a_\chi^\alpha(x_{21}) \int \langle \mathcal{O}_\chi^\dagger(x) \partial_\alpha \mathcal{O}_\chi(x_1) \rangle_M \varphi(x) dx. \quad (13.3.22)$$

At this stage we have established that for any x_1, x_2 as in (13.3.19), and for any test φ with compact $\widehat{\varphi}$, the series in the r.h.s. converges to the l.h.s.

Now the key point is that the r.h.s. of (13.3.22) is a convergent power series in x_{21} , while the l.h.s. can be expanded in such a convergent power series. Indeed we know the explicit form of $\langle \mathcal{O}_\chi^\dagger(x) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_M$:

$$\langle \mathcal{O}_\chi^\dagger(x) \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle_M = \frac{1}{(x_{12}^2)^{\Delta_\sigma - \Delta_\chi/2}} \frac{1}{[(x - x_1)^2 (x - x_2)^2]^{\Delta_\chi/2}}. \quad (13.3.23)$$

¹⁰In the unitary CFTs, $\widehat{G}_{\overline{n} \mathcal{O} \mathcal{O}}(p, x_1, x_2)$ and $\widehat{G}_{\chi_n}(p)$ vanish unless $p \in \overline{V}_+$, so the behavior of $\widehat{B}_{\chi_n}(p, x_1, x_2)$ outside the forward light cone is not important.

¹¹This is a consequence of the Bochner-Schwartz theorem: any positive tempered distribution is the Fourier transform of some positive tempered measure (see [110], Sec. 8.2).

For all x in the Minkowski space, the function $[(x - x_1)^2(x - x_2)^2]^{-\Delta_\chi/2}$ is holomorphic in x_1, x_2 as long as $\text{Im}(x_1), \text{Im}(x_2) \prec 0$. It's easy to show that this remains true after integration in φ . At this point we can match the expansions for the two sides of (13.3.22), and get

$$\sum_{|\alpha|=n} \int dx \varphi(x) \left\{ \frac{(x_{21})_\alpha}{\alpha!} \partial_{x_2}^\alpha [(x - x_1)^2(x - x_2)^2]^{-\Delta_\chi/2} \Big|_{x_2=x_1} - a_\chi^\alpha(x_{21}) \langle \mathcal{O}_\chi^\dagger(x) \partial_\alpha \mathcal{O}_\chi(x_1) \rangle_M \right\} = 0. \quad (13.3.24)$$

Up to this point it was crucial to keep the function φ in the game to keep convergence issues under control, but now we can get rid of it. Indeed $\sum_{|\alpha|=n}$ is a finite sum, also $\langle \mathcal{O}_\chi^\dagger(x) \partial_\alpha \mathcal{O}_\chi(x_1) \rangle_M$ is a holomorphic function in the forward tube \mathcal{T}_2 . For any Minkowski point x_0 , we choose a sequence of test functions φ_k of compact support $\widehat{\varphi}_k$ such that φ_k tends to $\delta(x - x_0)$. Passing to the limit, (13.3.24) implies the same equality for the integrand. I.e. for any fixed $n \in \mathbb{N}$, and any Minkowski x ,

$$\frac{(x_{21})_\alpha}{\alpha!} \partial_{x_2}^\alpha [(x - x_1)^2(x - x_2)^2]^{-\Delta_\chi/2} \Big|_{x_2=x_1} - a_\chi^\alpha(x_{21}) \langle \mathcal{O}_\chi^\dagger(x) \partial_\alpha \mathcal{O}_\chi(x_1) \rangle_M = 0. \quad (13.3.25)$$

Now as promised we are reduced to an equation which only involves 2-point and 3-point functions which are holomorphic. E.g. we can take $x = 0$ and x_1, x_2 in Euclidean. Then this equation is the same one as the equation which determines the Euclidean OPE kernel for $x_0 = x_1$, i.e. $C_\chi(x_1, x_2, x_1, \partial)$. For convenience in this discussion we use Minkowski coordinates for Euclidean correlators (i.e. we write the Euclidean correlators as $\langle\langle \mathcal{O}(-i\tau, \mathbf{x}) \dots \rangle_M$). Under this convention we have

$$C_\chi(x_1, x_2, x_1, \partial) = \frac{1}{(x_{12}^2)^{\Delta_\mathcal{O} - \Delta_\chi/2}} \sum_{(\mu)} a_\chi^\mu(x_{21}) \partial_\mu = \sum_{(\mu)} b_\chi^\alpha(0, x_{21}) \partial_\alpha. \quad (13.3.26)$$

This establishes (13.3.14) for $x_0 = x_1$. The general case reduces to this one by noticing that c_χ^α satisfies the relation:

$$c_\chi^\alpha(x_{10}, x_{20}) = \sum_{\beta \leq \alpha} \frac{1}{\beta!} c_\chi^{\alpha-\beta}(0, x_{21}) x_{10}^\beta, \quad (13.3.27)$$

where $\beta \leq \alpha$ means $\beta_i \leq \alpha_i$ for all i ; and b_χ satisfies the identical relation with $c_\chi \rightarrow b_\chi$. For c_χ^α this follows by translation invariance and analyticity of CFT two- and 3-point functions, and for b_χ^α from $\widehat{B}_\chi(p, x_{10}, x_{20}) = e^{-ip \cdot x_{10}} \widehat{B}_\chi(p, 0, x_{21})$.

Chapter 14

Review of Osterwalder-Schrader theorem

In this section we review the results of [1, 2] and, in particular, discuss the linear growth condition and why it was necessary for establishing Wightman axioms in [2].

In [1] Osterwalder and Schrader formulated an equivalence theorem which stated that a set of axioms for Euclidean correlation functions (a version of the Osterwalder-Schrader axioms described in Sec. 7.2) is equivalent to Wightman axioms for Euclidean correlation functions. Unfortunately, later a technical error was discovered in their proof, and in [2] Osterwalder and Schrader gave two new results.

The first result of [2] is a revised equivalence theorem, which shows that a stronger version of Euclidean axioms is in fact equivalent to Wightman axioms. The proof of this theorem is rather simple. However, as we will review, this is at the expense of the new version of Euclidean axioms being rather hard to verify.

The second result of [2] shows that the original OS axioms, plus a “linear growth condition,” imply Wightman axioms and a growth condition on Wightman distributions. A partial result in the reverse direction is also valid. It assumes a stronger growth condition on the Wightman distributions than follows from the direct result, and it yields a growth condition on Euclidean correlators which is weaker than the linear growth condition. Therefore, these latter results do not establish an equivalence of any two systems of axioms, but they do allow to establish Wightman axioms from OS axioms in some situations.

In what follows we will review the general structure of the arguments of [1, 2]. For our purposes it will suffice to ignore the space coordinates and focus only on the time arguments of the fields. We will not completely reproduce all arguments of [1, 2], and in some of the omitted steps the space arguments and Lorentz symmetry are important. We will also work with correlation functions involving a single hermitian scalar field ϕ , similarly to [1, 2]. In CFT applications we are interested in correlation functions of all local operators. It should be relatively straightforward to adapt the discussion of [1, 2] to this more general setup.

Our main goal is to construct an analytic continuation of the Euclidean correlation functions

$$G_n^E(t_1, \dots, t_n) \equiv \langle \phi(t_1) \dots \phi(t_n) \rangle \quad (14.0.1)$$

from real to complex t_k , and to establish that the Wightman functions recovered in the limit of pure imaginary t_k (real Lorentzian times) are tempered distributions. This is the most non-trivial part of the argument. Other Wightman axioms such as positivity, spectrum condition, etc., follow relatively easily and have been reviewed in chapter 11.

14.1 The argument of [1]

Physically, the analyticity of position-space correlation functions is due to positivity of energy. More concretely, the Euclidean evolution operator e^{-Ht} is well-defined and holomorphic in t for $\text{Re } t > 0$ due to the spectrum of H

being non-negative. The first step to establishing analyticity is thus to construct the operator H , and for this we first need to construct a Hilbert space on which it acts.

The Hilbert space \mathcal{H}^{OS} is constructed, as we discussed in Sec. 7.2, by considering the vector space $\mathcal{H}_0^{\text{OS}}$ of formal linear combinations of states¹

$$|\phi(t_1)\phi(t_2)\dots\phi(t_n)\rangle \quad (14.1.1)$$

with $0 > t_1 > t_2 > \dots > t_n$. A Hermitian inner product is introduced on $\mathcal{H}_0^{\text{OS}}$ by

$$\langle\phi(s_1)\phi(s_2)\dots\phi(s_m)|\phi(t_1)\phi(t_2)\dots\phi(t_n)\rangle \equiv G_n^E(-s_m, \dots, -s_1, t_1, \dots, t_n). \quad (14.1.2)$$

By OS reflection positivity-axiom this inner product is positive-semidefinite. The Hilbert space \mathcal{H}^{OS} is obtained from $\mathcal{H}_0^{\text{OS}}$ by modding out null states and completing the resulting quotient space with respect to the above inner product. We can naturally think of $|\phi(t_1)\phi(t_2)\dots\phi(t_n)\rangle$ as states in \mathcal{H}^{OS} .

Physically, to construct the Hamiltonian H , we first define it by its action on (14.1.1). Then we note that H has to be positive, otherwise the correlation functions would grow exponentially at large distances. Formally, one first defines for $t > 0$ an operator U_t on $\mathcal{H}_0^{\text{OS}}$ by

$$U_t|\phi(t_1)\phi(t_2)\dots\phi(t_n)\rangle \equiv |\phi(t_1 - t)\phi(t_2 - t)\dots\phi(t_n - t)\rangle. \quad (14.1.3)$$

The usual care must be taken to ensure that this defines an operator on \mathcal{H}^{OS} . For this one notes that for any $\Psi \in \mathcal{H}_0^{\text{OS}}$ we have $|\langle\Psi|U_t|\Psi\rangle| \leq P(t)$ for some polynomial $P(t)$ since the Euclidean correlation functions are assumed to be powerlaw-bounded when groups of points are separated to infinity. Then a simple estimate gives

$$|\langle\Psi|U_t|\Psi\rangle| \leq \|\Psi\| \|U_t\Psi\| = \|\Psi\| |\langle\Psi|U_{2t}|\Psi\rangle|^{1/2} \leq \dots \leq \|\Psi\|^{1+1/2+\dots+1/2^{n-1}} |\langle\Psi|U_{2^n t}|\Psi\rangle|^{1/2^n}. \quad (14.1.4)$$

Using $|\langle\Psi|U_t|\Psi\rangle| \leq P(t)$ we get in the limit $n \rightarrow \infty$

$$|\langle\Psi|U_t|\Psi\rangle| \leq \|\Psi\|^{1+1/2+\dots+1/2^{n-1}} (P(2^n t))^{1/2^n} \rightarrow \|\Psi\|^2. \quad (14.1.5)$$

This shows that U_t maps null states to null states and thus is defined on (a dense subset of) \mathcal{H}^{OS} . By the above, it is also a bounded operator, so it extends in a unique way to all of \mathcal{H}^{OS} . Furthermore, noting that it is symmetric, of norm at most 1, and we have the semigroup law $U_t U_s = U_{t+s}$, we find that $U_t = e^{-Ht}$ for a non-negative self-adjoint Hamiltonian H (see, e.g., [111] Sec. 141).

Since the domain in which we need to construct the analytic continuation of G_n^E is awkward to define in t_k variables, we introduce the difference variables $y_k \equiv t_k - t_{k+1}$. Due to translation invariance, $G_n^E(t_1 \dots t_n)$ can be rewritten as

$$G_n^E(t_1 \dots t_n) = S_{n-1}(y_1 \dots y_{n-1}) \quad (14.1.6)$$

for some functions S_n . Similarly, we will use the following notation for states in terms of y_k variables,

$$|\Psi_n(-t_1; y_1 \dots y_{n-1})\rangle \equiv |\phi(t_1)\phi(t_2)\dots\phi(t_n)\rangle. \quad (14.1.7)$$

Note that

$$\langle\Psi_m(t'; y'_1 \dots y'_{m-1})|\Psi_n(t; y_1 \dots y_{n-1})\rangle = S_{m+n-1}(y'_{m-1}, \dots, y'_1, t' + t, y_1, \dots, y_{n-1}). \quad (14.1.8)$$

In terms of $S_{n-1}(y_1 \dots y_{n-1})$, our goal is to construct an analytic continuation to $\text{Re } y_k > 0$ and show that the limit of all $\text{Re } y_k \rightarrow 0^+$ exists in the sense of tempered distributions.

With a positive H now constructed, we can define a holomorphic family of bounded operators $U_\tau = e^{-H\tau}$ for $\text{Re } \tau > 0$, which will be our main tool for analytically continuing the correlation functions S_n . In particular, we can now consider the matrix elements

$$\langle\Psi_m(t'; y'_1 \dots y'_{m-1})|U_\tau|\Psi_n(t; y_1 \dots y_{n-1})\rangle = S_{m+n-1}(y'_{m-1}, \dots, y'_1, t' + t + \tau, y_1, \dots, y_{n-1}), \quad (14.1.9)$$

which are analytic for $\text{Re } \tau > 0$. This establishes the desired analyticity of $S_{n-1}(y_1 \dots y_{n-1})$ in each variable y_k separately. In [1] they additionally establish some growth conditions on these individual holomorphic functions

¹In Sec. 7.2 the states are introduced as integrals of these quantities. This is also what is done in [1, 2], since they assume only that the Euclidean correlators are distributions. Here, for simplicity of discussion, we use the knowledge that correlators are functions and use states evaluated at points. The arguments easily generalize to distributions and smeared states, but become more technical.

which then imply that for fixed y_k, y'_k and $\text{Re } \tau > 0$ the above function can be represented as the Fourier-Laplace transform

$$S_{m+n-1}(y'_{m-1}, \dots, y'_1, \tau, y_1, \dots, y_{n-1}) = \int d\alpha e^{-\alpha\tau} \check{S}(\alpha) \tag{14.1.10}$$

for some tempered distribution $\check{S}(\alpha)$. In other words, S_{m+n-1} can be extended to a holomorphic function in the right-half plane in each variable separately, and each such holomorphic function can be represented as a Fourier-Laplace transform of a tempered distribution. The erroneous Lemma 8.8 of [1] states that under these conditions, the full function S_{m+n-1} is a simultaneous Fourier-Laplace transform in all its variables of a tempered distribution,

$$S_{m+n-1}(\tau_1 \dots \tau_{m+n-1}) = \int d\alpha e^{-\alpha_1\tau_1 - \dots - \alpha_{m+n-1}\tau_{m+n-1}} \check{S}_{m+n-1}(\alpha_1 \dots \alpha_{m+n-1}). \tag{14.1.11}$$

From this, the tempered Wightman distributions are obtained immediately by setting $\text{Re } \tau_k \rightarrow 0^+$ in which case the Fourier-Laplace transform above becomes a Fourier transform of a tempered distribution. Fourier transform of a tempered distribution is, of course, itself tempered.

14.2 The argument of [2]

14.2.1 Fixing the equivalence theorem

Unfortunately, Lemma 8.8 of [1] is wrong. As explained in [2], the function $S_2(y_1, y_2) = e^{-y_1 y_2}$ gives a simple counterexample. For fixed $y_2 > 0$, we find that $S_2(y_1, y_2)$ is holomorphic for $\text{Re } y_1 > 0$ and is there the Fourier-Laplace transform of the tempered distribution $\delta(\alpha - y_2)$. The same statements hold with y_1 and y_2 exchanged. However, $S_2(y_1, y_2)$ is not a Fourier-Laplace transform of a tempered distribution in both variables simultaneously. For if this were the case, the corresponding Wightman function $S_2(ix_1, ix_2) = e^{x_1 x_2}$ would be a tempered distribution, which it is not since it grows faster than any power in some directions.

The first result of [2] (see also the review in [79]) rescues Lemma 8.8 by making a stronger assumption about $S_n(y_1 \dots y_n)$ which they denote by $\check{E}0$. Concretely, let \mathbb{R}_+^n be the set of points (y_1, \dots, y_n) with $y_k > 0$. Let $\mathcal{S}(\mathbb{R}_+^n)$ be the subspace of the space of Schwartz functions, consisting of functions supported on \mathbb{R}_+^n with the induced topology. The functions $S_n(y_1 \dots y_n)$ can be viewed as distributions in the continuous dual space $\mathcal{S}'(\mathbb{R}_+^n)$ defined by, for $f \in \mathcal{S}(\mathbb{R}_+^n)$

$$S_n(f) \equiv \int dy_1 \dots dy_n S_n(y_1 \dots y_n) f(y_1 \dots y_n). \tag{14.2.1}$$

Note that smoothness of f together with its support properties ensures that $f(y_1 \dots y_n)$ vanishes with all derivatives whenever $y_k = y_j$ for $k \neq j$. The assumption that S_n has at most powerlaw singularities at coincident points and at infinity means that $S_n(f)$ is continuous in f in the topology of $\mathcal{S}(\mathbb{R}_+^n)$. The additional assumption $\check{E}0$ is that it is also continuous in f in a weaker topology. This weaker topology is defined by the usual Schwartz norms on $\overline{\mathbb{R}_+^n}$

$$|g|_{p,+} = \sup_{x \in \overline{\mathbb{R}_+^n}, |\alpha| \leq p} |(1+x^2)^{p/2} \partial^{(\alpha)} g(x)| \tag{14.2.2}$$

but applied not to f and instead to its Fourier-Laplace² transform \check{f}

$$\check{f}(q_1 \dots q_n) \equiv \int dy_1 \dots dy_n e^{-q_1 y_1 - \dots - q_n y_n} f(y_1 \dots y_n). \tag{14.2.3}$$

One establishes that $\check{f} = 0$ iff $f = 0$ (injectivity) and that the set of all images \check{f} is dense in an appropriate space of Schwartz functions (denseness). The proof of (14.1.11) then becomes straightforward: one first defines \check{S}_n by $\check{S}_n(\check{f}) = S_n(f)$. This definition makes sense due to the injectivity property. The assumption $\check{E}0$ ensures that \check{S}_n is continuous. The denseness property just mentioned then allows to extend \check{S}_n to an appropriate space of Schwartz functions by continuity, establishing temperedness of \check{S}_n and allowing one to define tempered Wightman distributions as Fourier transforms of \check{S}_n . It is similarly not difficult to show that Wightman axioms imply $\check{E}0$.

²As written, this is a Laplace transform. It is a Fourier transform in the spatial variables which we are omitting.

14.2.2 Wightman axioms from linear growth condition

As we can see, the axiom $\check{E}0$ is not very different from directly assuming temperedness of Wightman distributions, even though it is formulated for Euclidean correlators. It is also unclear how to verify this axiom in practice.³ For this reason, [2] introduced an alternative “linear growth condition” on the correlation functions S_n which is easier to verify and has been established in some models (see below), yet is also sufficient to establish temperedness of Wightman functions (though this condition is not known to follow from Wightman axioms). The construction of the analytic continuation of the functions S_n as well the proof of the temperedness of the resulting Wightman distributions is much more complicated than using $\check{E}0$. Therefore, our review of these arguments will be even more schematic than the above, and we will only try to illustrate the key ideas and explain why and how the linear growth condition is used. We are not aware of any previous attempt to review this part of [2].

First of all, let us state the linear growth condition of [2]. Note that the correlation functions G_n^E can be viewed as distributions in $({}^0\mathcal{S})'(\mathbb{R}^{d \cdot n})$, where ${}^0\mathcal{S}(\mathbb{R}^{d \cdot n})$ is the space of Schwartz functions of n arguments in \mathbb{R}^d which vanish with all derivatives at coincident points, by

$$G_n^E(f) = \int d^d x_1 \dots d^d x_n f(x_1, \dots, x_n) G_n^E(x_1, \dots, x_n). \quad (14.2.4)$$

Here we have temporarily restored the spatial coordinates. In fact, [1, 2] do not assume that G_n^E are functions, and only that they are distributions in $({}^0\mathcal{S})'(\mathbb{R}^{d \cdot n})$. It follows, however, from the OS axioms (without the linear growth condition) that G_n^E are real-analytic functions, as shown in [2, 31].

Note that the assumption that $G_n^E \in ({}^0\mathcal{S})'(\mathbb{R}^{d \cdot n})$ means G_n^E is sufficiently continuous as a linear functional or, equivalently, is sufficiently bounded. That is,

$$|G_n^E(f)| \leq \sigma_n |f|_{q_n} \quad (14.2.5)$$

for all $f \in {}^0\mathcal{S}(\mathbb{R}^{d \cdot n})$ and some $\sigma_n > 0$ and $q_n \in \mathbb{Z}_{\geq 0}$, where $|f|_p$ denotes the Schwartz norms on ${}^0\mathcal{S}(\mathbb{R}^{d \cdot n})$. The linear growth condition requires q_n to grow at most linearly, and σ_n at most as a power of a factorial. In other words, the linear growth condition is the statement that there exists a positive integer s and a sequence σ_n such that

$$|G_n^E(f)| \leq \sigma_n |f|_{n \cdot s} \quad (14.2.6)$$

for any n and $f \in {}^0\mathcal{S}(\mathbb{R}^{d \cdot n})$, and $\sigma_n \leq \alpha(n!)^\beta$ for some constants α, β .

The unusual feature of the linear growth condition is that this is a condition on all n -point correlation functions G_n^E . It has to hold for all n in order for the result of [2] to imply, say, even just the temperedness of 3-point Wightman distribution. In order to understand why this is required, below we will review the basic strategy behind the proof of [2]. There are two steps in the argument. In the first step, one establishes analyticity of $S_n(y_1, \dots, y_n)$ in the region $\text{Re } y_k > 0$. This does not require the linear growth condition [2, 31]. In the second step, which does use the linear growth condition, one proves a bound on S_n in this region, which allows the application of Vladimirov’s theorem and thus the construction of tempered Wightman distributions.

We conclude this section with additional comments about the linear growth condition. First of all, Appendix of [2] shows that the linear growth condition follows from requiring that $G_n^E \in \mathcal{S}'(\mathbb{R}^{dn})$ and imposing

$$|G_n^E(f_1 \otimes \dots \otimes f_n)| \leq \sigma_n \prod_{i=1}^n |f_i|_r, \quad (14.2.7)$$

for any n , where $f_i \in \mathcal{S}(\mathbb{R}^d)$, $|\cdot|_r$ is some fixed Schwartz space norm, and $\sigma_n \leq \alpha(n!)^\beta$. In other words, while in (14.2.6) the n -point function variables are smeared jointly, here each variable is smeared separately. Note that the total smearing $f_1 \otimes \dots \otimes f_n$ does not necessarily exclude coincident points, that’s why we need to assume $G_n^E \in \mathcal{S}'(\mathbb{R}^{dn})$ and not $G_n^E \in ({}^0\mathcal{S})'(\mathbb{R}^{d \cdot n})$ as above.

³We would also like to mention related work by Zinoviev [112]. Zinoviev replaces axiom $\check{E}0$ by an axiom E5 which imposes that certain limits exist which allow to compute the inverse Laplace transform of S_n . While E5 may look like a more constructive version of $\check{E}0$, in practice its verification appears just as hard as assuming outright that S_n is a Laplace transform (which is what $\check{E}0$ essentially does). We are grateful to David Brydges for an enlightening explanation of Zinoviev’s construction, and in particular for pointing out that it represents a generalization of Post’s Laplace transform inversion formula [113] to the case of distributions.

Although (14.2.7) is stronger than (14.2.6), it is easier to verify in particular models. E.g. it holds for any gaussian scalar field \mathcal{O} with a two point function G_2 having a powerlaw asymptotics in the UV.⁴ It has been also established in some non-gaussian models.⁵

More generally, bound (14.2.7) is natural for field theories realizable as random distributions.⁶ Imagine that there is a measure $d\mu$ in the space of distributions $\phi \in \mathcal{S}'(\mathbb{R}^d)$ such that for every test function $f \in \mathcal{S}(\mathbb{R}^d)$ the following expectation value is finite:

$$S(f) = \int d\mu e^{\phi(f)}. \quad (14.2.8)$$

Such measures make rigorous sense of the Feynman path integral. Eq. (14.2.8) is a rigorous version of generating functional, and differentiating with respect to f one defines correlation functions $\langle \phi(x_1) \dots \phi(x_n) \rangle$ which are in this framework automatically distributions in $\mathcal{S}'(\mathbb{R}^{dn})$. Bound (14.2.7) in this case can be reduced to an estimate on the growth of $S(f)$. The Osterwalder-Schrader and Wightman axioms then follow.

The field ϕ in (14.2.8) is naturally a ‘‘fundamental field’’ of some model, such as $P(\phi)_2$ [115] or $(\phi^4)_3$ (see [115], Sec. 23.1 for references). Sometimes this framework can be extended to generating functionals $\int d\mu e^{\phi'(f)}$ where ϕ' is a composite operator. E.g. $\phi' = : \phi^n :$, $n < \deg P$, in $P(\phi)_2$ is treated in [115]. See also [116] for the general problem to construct $: \phi^2 :$ as a random distribution given ϕ .

14.2.3 Analytic continuation

There are three tricks used together to construct the analytic continuation of S_n . The first trick was already used above: it is the observation that if the states $|\Psi_n(t; y_1, \dots, y_n)\rangle$ and $|\Psi_m(t'; y'_1, \dots, y'_m)\rangle$ are defined for some values of t, y_k and t', y'_k , then we can compute the matrix elements

$$\langle \Psi_m(t'; y'_1 \dots y'_{m-1}) | U_\tau | \Psi_n(t; y_1 \dots y_{n-1}) \rangle = S_{m+n-1}(\overline{y'_{m-1}}, \dots, \overline{y'_1}, t' + t + \tau, y_1, \dots, y_{n-1}) \quad (14.2.9)$$

with $\text{Re } \tau > 0$, thus potentially extending the domain of analyticity of S_{m+n-1} .

The second trick, intuitively, says that we can write

$$\langle \Psi_n(t; y_1 \dots y_{n-1}) | \Psi_n(t; y_1 \dots y_{n-1}) \rangle = S_{2n-1}(\overline{y_{n-1}}, \dots, \overline{y_1}, 2t, y_1, \dots, y_{n-1}), \quad (14.2.10)$$

and so the state $|\Psi_n(t; y_1 \dots y_{n-1})\rangle$, whose norm appears in the left-hand side, should be well-defined as long as the correlation function in the right-hand side is well-defined. That is, while we start with the states $|\Psi_n(t; y_1 \dots y_{n-1})\rangle$ defined for positive real y_k , we should be able to analytically continue them in y_k if we manage to analytically continue the correlators S_{2n-1} . Of course, this is not a proof that $|\Psi_n(t; y_1 \dots y_{n-1})\rangle$ is well-defined. We will give the proof below, after we get more information about the domain in which we wish to construct it.

The final trick is the idea of analytic completion for functions of several complex variables. Recall that for $n > 1$ not every domain in \mathbb{C}^n is the domain of holomorphy of some holomorphic function: there exist domains $\mathcal{D} \subset \mathbb{C}^n$ such that any f holomorphic in \mathcal{D} can be extended to a function holomorphic in a strictly larger domain $\mathcal{D}' \supset \mathcal{D}$. For our applications the relevant theorem is Bochner’s tube theorem, which states that any holomorphic function in a tube domain of the form $\mathcal{D} = \mathbb{R}^n + iX$, where X is a connected open subset of \mathbb{R}^n , can be extended to a holomorphic function on $\mathcal{D}' = \text{ch}(\mathcal{D}) = \mathbb{R}^n + i \text{ch}(X)$, where ch denotes the convex hull. Note that since \mathcal{D}' is a convex set, it is a domain of holomorphy⁷ and so f cannot be extended any further by analytic completion alone.

⁴Then $G_n^E(f_1 \otimes \dots \otimes f_n)$ is a sum of $(n-1)!!$ terms, products of Wick contractions $G_2(f_i \otimes f_j)$, which can be bounded by $A|f_i|_r|f_j|_r$ where r depends on the UV dimension of \mathcal{O} . We thus get (14.2.7) with $\sigma_n = (n-1)!!A^{n/2}$.

⁵See e.g. [114], Theorem 1.1.8, which establishes Eq. (14.2.7) for Schwinger functions of arbitrarily high normal-ordered powers $: \phi^n :$ of the fundamental field ϕ in weakly coupled $P(\phi)_2$ theories.

⁶See [115], Sec. 6. This book introduced axioms for random distributions, numbered OS0-OS5. This chosen name is a bit unfortunate because these axioms are quite different in spirit from the original Osterwalder-Schrader axioms described in Sec. 7.2, and appear much stronger. E.g. they make the recovery of Wightman axioms a relatively trivial task. We don’t know how to derive the axioms of [115] from the Euclidean CFT axioms.

⁷To see this, it suffices to show that for any point z on the boundary of \mathcal{D}' there exists a function holomorphic in \mathcal{D}' but singular at z . In general, such functions might not exist since the set of singularities of a holomorphic function cannot be arbitrary. However, it is easy to construct a function singular on any complex codimension-1 hyperplane in \mathbb{C}^n (take the reciprocal of an affine-linear function). For a convex \mathcal{D}' one can always find such a hyperplane passing through a given boundary point but staying away from the interior of \mathcal{D}' .

The requirement that X is open is a bit too restrictive and we'll need also a degenerate case of this theorem, as described below.

These three tricks are applied one by one infinitely many times in order to construct the full analytic continuation of S_n . Instead of setting up the procedure in its full glory, we will only follow the first steps to see how it works in principle. The full analysis is performed in [2].

First, it helps to introduce new variables w_i by

$$e^{w_i} = y_i. \quad (14.2.11)$$

Our domains of analyticity in terms of w_i will always be tubes of the form $(w_1 \dots w_n) \in \mathcal{D}(X) \equiv \mathbb{R}^n + iX$ for various $X \subset \mathbb{R}^n$, and so we'll often just describe X . For example, we start with S_n and Ψ_n defined for real positive y_i , which corresponds to the domain $\mathcal{D}(\{0\})$ in w_i .

Consider the 2-point function $S_1(y_1)$. We start with the domain $\mathcal{D}(\{0\}) = \mathbb{R}$ in w_1 , corresponding to real positive y_1 . Next, we apply the first trick. Specifically, we write

$$\langle \Psi_1(t') | U_\tau | \Psi_1(t) \rangle = S_1(t' + t + \tau), \quad (14.2.12)$$

and since we are free to choose $t > 0$ and $t' > 0$ arbitrarily small, while U_τ , as discussed above, is a well-defined bounded operator for $\text{Re } \tau > 0$, we obtain an analytic continuation of $S_1(y_1)$ to the right half-plane.

We are now done with the analytic continuation of S_1 , since our goal was to continue all y_k to the right-half plane. In terms of w_i , this corresponds to the domain $w_1 \in \mathcal{D}((-\frac{\pi}{2}, +\frac{\pi}{2}))$, i.e. a strip. For higher-point functions, in terms of w_i , we should stop when our domain of analyticity is $\mathcal{D}((-\frac{\pi}{2}, +\frac{\pi}{2}) \times \dots \times (-\frac{\pi}{2}, +\frac{\pi}{2}))$.

Consider now the 3-point function $S_2(y_1, y_2)$. We can again use the first trick and define it on $(w_1, w_2) \in \mathcal{D}(X_2)$, where $X_2 = \{0\} \times (-\frac{\pi}{2}, +\frac{\pi}{2}) \cup (-\frac{\pi}{2}, +\frac{\pi}{2}) \times \{0\}$ (see Fig. 14.2.1, left). In more detail, we write the following two equations for $S_2(y_1, y_2)$, representing it as an inner product in two ways, and inserting a U_τ ,

$$\langle \Psi_2(t'; y'_1) | U_\tau | \Psi_1(t) \rangle = S_2(y'_1, t' + t + \tau), \quad (14.2.13)$$

$$\langle \Psi_1(t') | U_\tau | \Psi_2(t, y_1) \rangle = S_2(t' + t + \tau, y_1), \quad (14.2.14)$$

where the left-hand sides are well-defined (at this point) for $t, t', y_1, y'_1 > 0$ and $\text{Re } \tau > 0$. We see that the first equation defines $S_2(y_1, y_2)$ for real $y_1 > 0$ and $\text{Re } y_2 > 0$ as a holomorphic function of y_2 . The second equation does the same, but with y_1 and y_2 exchanged. In terms of (w_1, w_2) this corresponds to the ‘‘analyticity domain’’ $\mathcal{D}(X_2)$ described above. We write ‘‘analyticity domain’’ in quotes because $\mathcal{D}(X_2)$ is not open (and has empty interior), and thus is not a domain. Correspondingly, we cannot say that S_2 is an holomorphic function of two variables on $\mathcal{D}(X_2)$. We will deal with this problem momentarily.

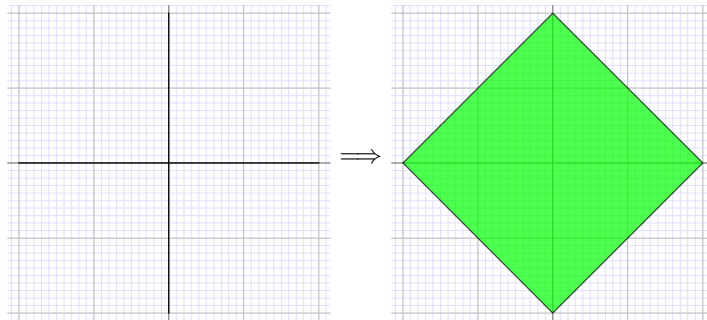


Figure 14.2.1: Left: set X_2 . Right: domain X'_2 which defines the envelope of holomorphy $\mathcal{D}(X'_2)$ of $\mathcal{D}(X_2)$.

To proceed with the analytic continuation of $S_2(y_1, y_2)$, we want to use the third trick, the tube theorem, to extend the analyticity domain from $\mathcal{D}(X_2)$ to $\mathcal{D}(X'_2)$, with $X'_2 \equiv \text{ch}(X_2)$ (Fig. 14.2.1, right).

The problem with this is that X_2 is not open, as mentioned above, so the tube theorem does not apply. Instead, for this step one has to use Malgrange-Zerner theorem [117], which allows X_2 to be a union of intervals, with $S_2(y_1, y_2)$ separately holomorphic in one variable on each of these intervals, as is the case in our setup. The conclusion is still that $S_2(y_1, y_2)$ can be analytically continued to $\mathcal{D}(X'_2)$.

Note that the domain $\mathcal{D}(X'_2)$ is not yet the full analyticity domain $\mathcal{D}\left(\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)\right)$ that we are aiming for. In particular, X'_2 is a proper subset of the square $\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$, see the right panel of Fig. 14.2.1. Importantly, it doesn't approach the corners $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$, which correspond to pure imaginary y_1, y_2 . Pure imaginary y_1, y_2 is, in turn, where we want to recover the Wightman distributions.

To extend the domain of analyticity of $S_2(y_1, y_2)$ even further, we need to first extend the domain of $\Psi_2(t, y_1)$, which can be done by the second trick above — via the equality

$$\langle \Psi_2(t, y_1) | \Psi_2(t, y_1) \rangle = S_3(\bar{y}_1, 2t, y_1). \quad (14.2.15)$$

Note that we are not interested in the analytic continuation in t here — it is automatic when we act on Ψ_2 with e^{-Ht} — so we can assume t is real. For S_3 we can run the same argument as we just did for S_2 and conclude that it is holomorphic in $\mathcal{D}(X'_3)$, where X'_3 is the convex hull of three intersecting intervals on coordinate axes (an octahedron). As discussed above, we expect that $\Psi_2(t, y_1)$ is defined whenever t and y_1 are such that the arguments of S_3 above are in its analyticity domain. This happens whenever

$$(\bar{w}_1, \log 2t, w_1) \in \mathcal{D}(X'_3),$$

which is equivalent to

$$(\text{Im } \bar{w}_1, \text{Im } \log 2t, \text{Im } w_1) \in X'_3. \quad (14.2.16)$$

Since we take t to be real and positive, we have $\text{Im } \log 2t = 0$ and so t is otherwise unconstrained. By construction of X'_3 and X'_2 , $(\text{Im } \bar{w}_1, 0, \text{Im } w_1) \in X'_3$ is equivalent to $(\text{Im } \bar{w}_1, \text{Im } w_1) \in X'_2$. Using $\text{Im } \bar{w}_1 = -\text{Im } w_1$, we conclude that w_1 is constrained by

$$(-\text{Im } w_1, \text{Im } w_1) \in X'_2. \quad (14.2.17)$$

This is equivalent to $|\text{Im } w_1| < \frac{\pi}{4}$, which is the same as $w_1 \in \mathcal{D}\left(\left(-\frac{\pi}{4}, +\frac{\pi}{4}\right)\right)$. To conclude, we expect $\Psi_2(t, y_1)$ to be defined and holomorphic in y_1 for $t > 0$ and $w_1 \in \mathcal{D}\left(\left(-\frac{\pi}{4}, +\frac{\pi}{4}\right)\right)$.

We can now apply the first trick to $S_2(y_1, y_2)$ again, writing it as inner product of Ψ_1 and Ψ_2 in the two ways (14.2.13) and (14.2.14). However, this time we can use $\Psi_2(t, y_1)$ in a wider domain of y_1 , as computed above, equivalent to $w_1 \in \mathcal{D}\left(\left(-\frac{\pi}{4}, +\frac{\pi}{4}\right)\right)$. From (14.2.13) we conclude that $S_2(y_1, y_2)$ is analytic for

$$(w_1, w_2) \in \mathcal{D}\left(\left(\left(-\frac{\pi}{4}, +\frac{\pi}{4}\right) \times \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)\right)\right), \quad (14.2.18)$$

where the domain of analyticity in w_1 comes from that of $\Psi_2(t, y_1)$, and in w_2 from $e^{-H\tau}$. Similarly, (14.2.14) now implies analyticity in the domain

$$(w_1, w_2) \in \mathcal{D}\left(\left(\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \times \left(-\frac{\pi}{4}, +\frac{\pi}{4}\right)\right)\right). \quad (14.2.19)$$

Combining the two together, we find that $S_2(y_1, y_2)$ is analytic for $(w_1, w_2) \in \mathcal{D}(X''_2)$, where

$$X''_2 \equiv \left(-\frac{\pi}{4}, +\frac{\pi}{4}\right) \times \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \times \left(-\frac{\pi}{4}, +\frac{\pi}{4}\right), \quad (14.2.20)$$

see the left panel of Fig. 14.2.2.

Using the tube theorem, we can now extend the analyticity domain from $\mathcal{D}(X''_2)$ further to $\mathcal{D}(X'''_2)$, where $X'''_2 \equiv \text{ch}(X''_2)$ is the convex hull of X''_2 shown in the right panel of Fig. 14.2.2.

We see that in order to analytically continue the 3-point function S_2 , it was useful to split it into an inner product of one-operator and two-operator states Ψ_1 and Ψ_2 , and use the information about the latter that is provided by its norm, the 4-point function S_3 . Still, we have not yet managed to analytically continue S_2 to the entire region of interest (we still have the corners missing in the right panel of Fig. 14.2.2). The only way to fix this is to extend the region of analyticity of S_3 . For that, we have to split it into a product of two states, and extend the region of analyticity of these states. It is useless to split it as a product of two Ψ_2 states, since their norm is computed by S_3 itself and we won't learn anything new in this way. Instead, we have to split it as a product of Ψ_1 and Ψ_3 . This will lead us to consider the norm of Ψ_3 , which is computed by the six-point function S_5 . Following this logic, eventually, we will be forced to consider S_n with arbitrarily high n just in order to construct the analytic continuation of S_2 .

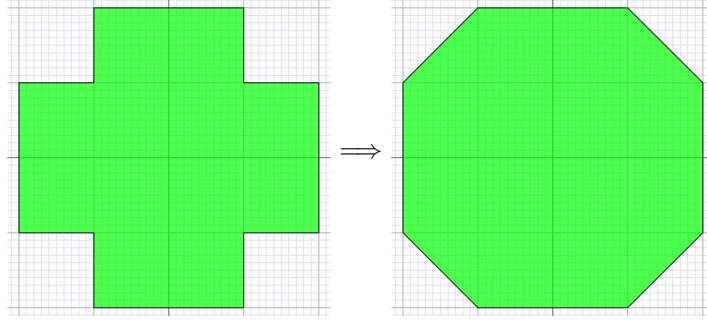


Figure 14.2.2: Left: set X_2'' . Right: domain X_2''' which defines the envelope of holomorphy $\mathcal{D}(X_2''')$ of $\mathcal{D}(X_2'')$.

Fortunately, it can be shown that this procedure converges to the desired domain $\mathcal{D}\left(\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \times \cdots \times \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)\right)$ for all S_n , see [2] for details.

To finish the discussion of the analytic continuation of S_n , let us justify the second trick, which constructs the states Ψ_n based on analyticity of their norm S_{2n-1} . Let C be the domain of analyticity of $S_{2n-1}(y_1 \dots y_{2n-1})$ known to us, expressed in terms of w_i , and let D be the domain of the arguments $t, w_1 \dots w_{n-1}$ of $\Psi_n(t; y_1 \dots y_{n-1})$ for which the arguments of S_{2n-1} in the right-hand side of

$$\langle \Psi_n(t; y_1 \dots y_{n-1}) | \Psi_n(t; y_1 \dots y_{n-1}) \rangle = S_{2n-1}(\overline{y_{n-1}}, \dots, \overline{y_1}, 2t, y_1, \dots, y_{n-1}), \quad (14.2.21)$$

belong to C . As is clear from the above discussion, C (expressed in terms of w_i) is always of the form $C = \mathcal{D}(X)$ for some X . We similarly have $D = \mathcal{D}(Y)$ for some Y . By the tube theorem (or Malgrange-Zerner theorem), we can assume that X (and thus also Y) is open, non-empty, and convex. Furthermore, it is easy to convince oneself that X , and thus Y , is invariant under reflections along any of the coordinate real axes (i.e. sending w_i to $\overline{w_i}$ for some i).

Suppose now that we have a point $(t; w_1^0 \dots w_{n-1}^0) \in D$. Then by definition of D we have

$$p \equiv (\overline{w_{n-1}^0} \dots \overline{w_1^0}, \log 2t, w_1^0 \dots w_{n-1}^0) \in C. \quad (14.2.22)$$

The above properties imply that there are $r_i > 0$ such that the polydisk

$$P = \{(w_1 \dots w_{2n-1}) \mid |w_i| < r_i\} + \text{Re } p \quad (14.2.23)$$

is contained in C , $P \subset C$, and moreover $p \in P$. Indeed, since $C = \mathcal{D}(X)$, this will be true if $\text{Im } P \subset X$ and $\text{Im } p \in \text{Im } P$.⁸ By construction, $\text{Im } P$ is a box with sides $2r_i$ centered at 0. On the other hand, the properties of X imply that together with any point x , X contains such a box with x being one of its vertices. We can then find an $\varepsilon > 0$ such that $(1 + \varepsilon)\text{Im } p \in X$, and take $\text{Im } P$ to be the box defined by the vertex $x = (1 + \varepsilon)\text{Im } p$. See Fig. 14.2.3 for an intuitive picture.

Writing temporarily the state Ψ_n as a function of w_k instead of y_k , we define it at w_k by the Taylor series

$$|\Psi_n(t; w_1 \dots w_{n-1})\rangle \equiv \sum_{\alpha} \frac{(w - \text{Re } w^0)^{\alpha}}{\alpha!} \partial^{\alpha} |\Psi_n(t; \text{Re } w_1^0 \dots \text{Re } w_{n-1}^0)\rangle \quad (14.2.24)$$

(α is a multiindex so $w^{\alpha} = w_1^{\alpha_1} w_2^{\alpha_2} \dots$ etc.). Note that the state in the right-hand side is well defined since the corresponding $y_k = e^{\text{Re } w_k^0} > 0$. To check whether this Taylor series converges, we look at its remainder

$$\sum_{|\alpha| > N} \frac{(w - \text{Re } w^0)^{\alpha}}{\alpha!} \partial^{\alpha} |\Psi_n(t; \text{Re } w_1^0 \dots \text{Re } w_{n-1}^0)\rangle, \quad (14.2.25)$$

whose norm squared is

$$\sum_{|\alpha| > N} \sum_{|\beta| > N} \frac{(w - \text{Re } w^0)^{\alpha}}{\alpha!} \frac{(\overline{w} - \text{Re } w^0)^{\beta}}{\beta!} \partial^{\alpha} \partial^{\beta} S_{2n-1}(\text{Re } w_{n-1}^0 \dots \text{Re } w_1^0, \log 2t, \text{Re } w_1^0 \dots \text{Re } w_{n-1}^0), \quad (14.2.26)$$

⁸The latter is because Im of the section of P by $\text{Re } x = \text{Re } p$ is $\text{Im } P$.

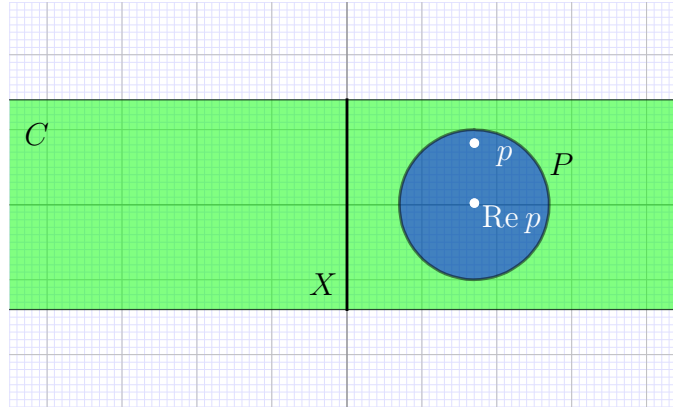


Figure 14.2.3: Schematic picture of the tube C and polydisk P .

where β -derivatives act on the first $n - 1$ arguments of S_{2n-1} , while α -derivatives act on the last $n - 1$ arguments. Here we also temporarily write S_{2n-1} as function of w_k . This norm is clearly just the tail of the Taylor series of S_{2n-1} expanded around the point $\text{Re } p$, and evaluated at $(\overline{w_{n-1}}, \dots, \overline{w_1}, \log 2t, w_1, \dots, w_{n-1})$. (We are not expanding in t .) Since S_{2n-1} is holomorphic in the polydisk P centered at $\text{Re } p$, this Taylor series converges in P and thus this remainder tends to 0 there.

Since $p = (\overline{w_{n-1}^0} \dots \overline{w_1^0}, \log 2t, w_1^0 \dots w_{n-1}^0) \in P$, the remainder tends to 0 at p , and thus (14.2.24) converges at $(t; w_1^0 \dots w_{n-1}^0)$. Furthermore, since P is open, it follows that (14.2.24) converges in some neighborhood of $(t; w_1^0 \dots w_{n-1}^0)$, defining $|\Psi_n(t; w_1 \dots w_{n-1})\rangle$ as a holomorphic \mathcal{H}^{OS} -valued function in that neighborhood. Since the choice of $(t; w_1^0 \dots w_{n-1}^0) \in D$ was arbitrary, we have defined $|\Psi_n(t; w_1 \dots w_{n-1})\rangle$ as a holomorphic function of w_i for all points in D .

14.2.4 Temperedness bound

Now that the correlation functions $S_n(y_1 \dots y_n)$ have been analytically continued from $y_k > 0$ to $\text{Re } y_k > 0$, we only need to establish a bound on their growth as $\text{Re } y_i \rightarrow 0$ in order to construct tempered Wightman distributions by an application of Vladimirov’s theorem. The logic proceeds by establishing a bound on $S_n(y_1 \dots y_n)$ for real y_k , and then repeating the analytic continuation described above, while keeping track of this bound. We will only sketch this rather technical argument in very general terms.

The final temperedness bound that we want to establish is

$$|S_n(y_1 \dots y_n)| \leq c_n \left(\left(1 + \sum_k |y_k| \right) \left(1 + \sum_k (\text{Re } y_k)^{-1} \right) \right)^{p_n}, \tag{14.2.27}$$

for some sequences c_n and p_n .⁹ We would like (14.2.27) to hold for all y_k , $\text{Re } y_k > 0$. For real positive y_k (i.e. in the Euclidean) this holds as a consequence of (7.2.3). As discussed in Remark 7.2.1, the original OS axioms did not include (7.2.3), so their first step was to derive (14.2.27) for $y_k > 0$ using (14.2.5).

In principle at fixed n , (14.2.27) looks reasonable given (14.2.5): both say, intuitively, that the correlation functions cannot be too singular at coincident points or grow too fast at infinity. However, (14.2.27) imposes this in a much more direct way. It turns out that in general one cannot derive direct bounds such as (14.2.27) from averaged statements such as (14.2.5), even if we know that S_n is real analytic.

Consider the real-analytic function $\sin(e^x)$, $x \in \mathbb{R}$. It is a bounded function, hence a tempered distribution. Thus its first derivative $e^x \cos(e^x)$ is also a tempered distribution. This is an example of a real-analytic tempered distribution which is not polynomially bounded. So some further assumptions are needed beyond real analyticity.¹⁰

⁹Here, for simplicity, we again ignore spatial arguments of the correlation functions, although they need to be taken care of at this step in order to establish “temperedness in spatial directions.” Furthermore, note that Osterwalder and Schrader establish additional bounds on c_k , etc., which are not important for the application of Vladimirov’s theorem.

¹⁰Incidentally, our example shows that the Corollary of Lemma 1 in [31] is wrong.

In our case, the functions $S_n(y_1 \dots y_n)$ are real-analytic and satisfy (14.2.5). In addition, they satisfy OS positivity. We already used OS positivity to show real-analyticity, and we will now have to invoke it again to prove (14.2.27) for $y_k > 0$. The full argument is rather technical; we will explain the main idea on the example of $S_1(y)$. Since we know that S_1 is holomorphic, in particular harmonic, by the mean value theorem for harmonic functions we can write it as a radially symmetric average

$$\begin{aligned} S_1(y) &= \int dx dt S_1(y + x + it) k_\rho(x, t) \\ &= \int_{|t|, |t'| < \rho} dt dt' T(t|g_\rho(\cdot, t + t'), g_\rho(\cdot, t')), \\ T(t|\varphi_1, \varphi_2) &:= \int dx dx' S_1(y + x + it) \varphi_1(x + x') \varphi_2(x'), \end{aligned} \quad (14.2.28)$$

where k_ρ is a C_0^∞ radial function supported in a ball of radius ρ and of integral 1, and we choose ρ sufficiently small so that all points under the integral sign are where S_2 is analytic. We also chose

$$k_\rho(x, t) = \int dx' dt' g_\rho(x + x', t + t') g_\rho(x', t'), \quad (14.2.29)$$

a convolution of another radial C_0^∞ function with itself (and hence a radial function). The point of this construction is that, for generic φ_1, φ_2 , $T(0|\varphi_1, \varphi_2)$ is an inner product $\langle \Psi_1 | \Psi_2 \rangle$ of two OS states:

$$\langle \Psi_1 | = \int dx \mathcal{O}(y/2 + x) \varphi_1(x), \quad | \Psi_2 \rangle = \int dx \mathcal{O}(-y/2 + x) \varphi_2(x). \quad (14.2.30)$$

The norm of these states, and hence their inner product, can be bounded using (14.2.5). Furthermore $T(t|\varphi_1, \varphi_2) = \langle \Psi_1 | e^{-iHt} | \Psi_2 \rangle$ satisfies the same bound. Using this bound for $\varphi_1 = g_\rho(\cdot, t + t')$, $\varphi_2 = g_\rho(\cdot, t')$, Eq. (14.2.28) gives a bound on $S_1(y)$. The same idea works for higher point functions. We first have to estimate the norm of some states using (14.2.5).¹¹ We then analytically continue separately in each time, and then use Malgrange-Zerner theorem to extend the bound on T to an open set. A single use of Malgrange-Zerner theorem suffices here, like in Fig. 14.2.1. We refer the reader to Sec. VI.1 of [2] for full details.

Once (14.2.27) is established for $y_k > 0$, one repeats the analytic continuation procedure that we described above, keeping track of the implications of (14.2.27). The analytic continuation used three tricks: (1) analytically continuing S_n by representing it in the form (14.2.9) (as $e^{-H\tau}$ inserted between two states), (2) expressing the norms of these states in terms of higher-point S_n as in (14.2.10), and (3) analytic completion.

The bound (14.2.27) propagates through the tricks (1) and (2) by the use of Cauchy-Schwartz inequality, as well as by using the fact that the norm of $e^{-H\tau}$ is bounded from above by 1 (i.e. Eq. (14.1.5)).

To propagate the bound through trick (3), the following simple idea is used. Suppose we have domains $\mathcal{D}' \supset \mathcal{D}$ such that any holomorphic function f on \mathcal{D} can be extended to a holomorphic function on \mathcal{D}' . Then we have the equality of images

$$f(\mathcal{D}') = f(\mathcal{D}), \quad (14.2.31)$$

and in particular

$$\sup_{z \in \mathcal{D}'} |f(z)| = \sup_{z \in \mathcal{D}} |f(z)|. \quad (14.2.32)$$

To see this, suppose $a \in \mathbb{C}$ is a value which f assumes in \mathcal{D}' but not in \mathcal{D} . Then the function $(f(z) - a)^{-1}$ is holomorphic in \mathcal{D} but has a singularity in \mathcal{D}' , which is a contradiction. This shows that if we have a bound on f in \mathcal{D} , it is also valid in \mathcal{D}' .

Finally, recall that in order to construct the analytic continuation of S_{n_0} for some fixed n_0 , we had to use S_n with arbitrarily high n in the process. This means that in order to establish the bound (14.2.27) on S_{n_0} for all $\text{Re } y_k$, we have to use (14.2.27) for $y_k > 0$ for S_n with arbitrarily high n . These bounds need to combine in a way that is strong enough to establish (14.2.27) for S_{n_0} . For this, it is important that c_n is of factorial growth and p_n grows at most linearly. This requires the same of the sequence σ_n and the index of the seminorm in (14.2.5), explaining the need for the linear growth condition.

¹¹Note that the linear growth condition is not needed at this point: Eq. (14.2.5) with some σ_n and q_n suffices to establish (14.2.27) with some c_n and p_n . The linear growth condition gives in addition c_n of factorial growth and p_n growing at most linearly. This turns out important later in the proof, see below.

Chapter 15

Conclusions

In this part of the thesis we studied the relationship between the modern Euclidean CFT axioms (which we formulated in Sec. 7.3) and the more traditional Osterwalder-Schrader and Wightman axioms. We showed that at least for $(n \leq 4)$ -point functions, both OS and Wightman axioms follow from the Euclidean CFT axioms. Our Euclidean CFT axioms are quite modest. In particular, beyond the minimal assumptions of regularity of correlators and the standard constraints of unitarity, we assumed only a very weak form of the convergent OPE.

Our derivation of Wightman axioms is of particular importance: it shows that the conformal Wightman 4-point functions are well-defined tempered distributions for arbitrary configurations of the 4 points, even when no OPE channel is convergent in the sense of functions. We have furthermore shown that these tempered distributions can always be computed by a conformal block expansion which is convergent in the sense of distributions, generalizing our previous results in part I, and giving a derivation of Mack's results [26] from Euclidean CFT axioms.

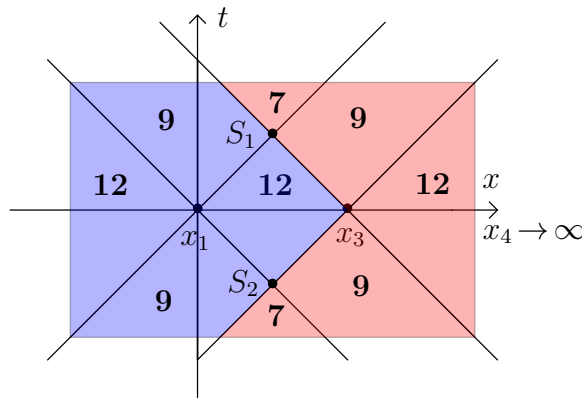


Figure 15.0.1: Minkowski configurations with $x_1 = 0, x_3 = \hat{e}_1, x_4 = \infty \hat{e}_1$ and $x_2 = t\hat{e}_0 + x\hat{e}_1$. Blue: configurations where $|\rho|, |\bar{\rho}| < 1$ and the 4-point functions is analytic. Red: configurations where $|\rho|$ and/or $|\bar{\rho}| = 1$. Boldface numbers $X = 7, 9, 12$ denote the causal type of the configuration according to part III (excluding lightlike separations). $S_{1,2}$ are double light-cone singularities.

For example, consider the configuration in Fig. 15.0.1, where the operators in a 4-point function are inserted at $x_1 = 0, x_3 = \hat{e}_1, x_4 = \infty \hat{e}_1$, while $x_2 = t\hat{e}_0 + x\hat{e}_1$ is allowed to move in a plane parametrized by (t, x) . The cross-ratios for this configuration are $z, \bar{z} = x \pm t$. It is then easy to see that for x_2 in the blue region of Fig. 15.0.1 $|\rho|, |\bar{\rho}| < 1$ and the s-channel OPE converges in the sense of functions. Our results imply that the s-channel OPE also converges in the red region where $|\rho|$ and/or $|\bar{\rho}| = 1$, but now the convergence is in the sense of distributions. In particular, the 4-point function is at least a distribution for all values of x_2 . Of course, in some regions of Fig. 15.0.1 this was obviously true – for example, in the red part of the regions **9, 12** (labeling according to the classification in part III), one can show that the 4-point function is real-analytic using the convergent t-channel OPE. One may hope to establish real-analyticity also in the region **7** using u-channel OPE. This would indeed be the case for the

ordering $\langle \mathcal{O}(x_2)\mathcal{O}(x_1)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle$. However, for the ordering $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle$ that we are discussing here, it turns out that no OPE channel converges in region **7** in the sense of functions.¹ Therefore, before our work it was not at all clear whether this correlator makes any sense in region **7** if we assume only the Euclidean CFT axioms.

While we have shown that the correlator is *at least* distributional in region **7**, we have not excluded the possibility of it being real-analytic there. For example, in 2 dimensions Virasoro symmetry implies that the 4-point function is analytic everywhere away from light-cone singularities [45]. This is perhaps too much to expect in higher dimensions, but one can still ask whether analyticity can be established in a larger domain. One approach is to ask for the envelope of holomorphy of the known domain of analyticity. Since the 4-point function is essentially only a function of two cross-ratios, this might be a tractable question [118]. We leave working out the full consequences of this idea for future work.²

In an upcoming paper [78], we will generalize our results to external operators with spins. In addition, there are many other fundamental open questions which we believe are important to understand. First of all, this part of the thesis is concerned with properties of CFT Wightman functions in Minkowski space. However, it is expected that Lorentzian CFTs should be naturally defined on Minkowski cylinder [25], which is the smallest physically-sensible space on which finite conformal transformations can act. Yet, it is not known whether CFT Wightman functions can be defined as tempered distributions on Minkowski cylinder (see note 4). Answering this question in the positive for CFT ($n \leq 4$)-point functions is the main goal of our forthcoming paper [41].

An important problem is to extend our results to ($n > 4$)-point functions. As we discuss in App. B.2, even deriving the OS axioms might require some strengthening of the OPE axiom. Another interesting possibility is to formulate Euclidean CFT axioms as OS axioms supplemented with a very weak form of the OPE (for example, asymptotic OPE in Euclidean space). This is perhaps less attractive, since it is desirable to formulate CFT axioms directly in terms of the CFT data (scaling dimensions and OPE coefficients). However, it will still be interesting to establish an equivalence between OS+(weak OPE axiom) and (possibly a stronger version of) our Euclidean CFT axioms, perhaps using arguments similar to those of [26]. Once OS axioms are established, it is likely that a strategy similar to that of the present paper can be pursued to establish Wightman axioms, using a comb-like OPE channel.

In this part of the thesis we only considered Wightman functions, but in practice one often needs time-order Minkowski correlators. Textbook definition of time-ordered correlators involves multiplying Wightman functions by θ -functions implementing time ordering. Since Wightman functions are in general distributions, this definition does not make rigorous sense at coincident points. As a matter of fact, time-ordered correlators have not been rigorously defined just from Wightman axioms alone (see e.g. [104], p.505) in a general QFT. In a general QFT setting, it is known that defining time-ordered Minkowski correlators is closely related to defining Euclidean correlators at coincident points [120]. In the future, it would be interesting to construct time-ordered CFT Minkowski correlators as distributions just from Euclidean CFT axioms.

A more ambitious goal is to understand the relationship of CFT axioms to Haag-Kastler axioms. This appears to be considerably harder since these axioms deal with operator algebras rather than local correlation functions, and some qualitatively new ideas seem to be required.

¹For a reader comfortable with cuts in z, \bar{z} plane the intuitive argument is simple: we have $z < 0$ (on what we'll call s-channel cut), $\bar{z} > 1$ (on t-channel cut). Furthermore according to the operator ordering, when z crosses 0 we need to make $\text{Im } t$ slightly negative (and thus $\text{Im } z$ slightly negative), because x_2 at this point crosses the null cone of x_1 , and when \bar{z} crosses 1 we need to make $\text{Im } t$ slightly positive (and thus $\text{Im } \bar{z}$ slightly negative) because it corresponds to x_2 crossing the null cone of x_3 . Thus both z and \bar{z} end on lower sides of their respective cuts, and so one of them must have crossed the u -channel cut at $(0, 1)$ when analytically continuing from a Euclidean configuration. We conclude that s- and t- channel OPEs are only distributionally convergent, while u -channel is badly divergent.

²Another approach could be via alternative representations of the 4-point function having an extended region of analyticity, e.g. [119].

Part III

Classification of Convergent OPE Channels for Lorentzian CFT Four-Point Functions

Chapter 16

Introduction

In this part of the thesis we study the convergence properties of operator product expansion (OPE) for Lorentzian four-point functions in conformal field theories (CFT).

Historically, analyticity of correlation functions is an important bridge connecting Lorentzian quantum field theories (QFT) and Euclidean QFTs. Starting from a Lorentzian correlator, we can get a Euclidean correlator by analytically continuing the time variables onto the imaginary axis [102]. Under certain conditions we can also do the reverse [1, 2, 31]. This procedure of analytic continuation, called Wick rotation, allows us to explore the Lorentzian nature of QFTs which may originate from statistical models in the Euclidean signature.

The Lorentzian correlators are not always genuine functions, instead they belong to a class of tempered distributions which are called Wightman distributions [24]. It is interesting to know at which Lorentzian configurations (x_1, \dots, x_n) the correlators $G_n(x_1, \dots, x_n)$ are indeed functions. The Wightman distributions are known to be analytic functions in some regions \mathcal{J}_n which are the sets of “Jost points” [121] (G_n are called Wightman functions in their domains of analyticity).¹ \mathcal{J}_n corresponds to some (not all) Lorentzian configurations with totally space-like separations. By using the microscopic causality constraints, one can extend G_n to a larger domain, including all configurations with totally space-like separations [29]. In Minkowski space $\mathbb{R}^{d-1,1}$,² two points can also have time-like or light-like separation. The Lorentzian correlators usually diverge at configurations with light-like separations, and these configurations are called light-cone singularities [122]. Except for some exactly solvable models, the Lorentzian correlators at configurations which contain time-like separations are not fully studied.

There are more constraints in CFTs. In general QFTs, the domains of Wightman functions are Poincaré invariant, while in CFTs this Poincaré invariant domain can be further extended by using conformal symmetry. Furthermore, in CFTs we have better control on correlators with the help of OPE [43]. A successful example is the four-point functions in 2d local unitary CFTs, where the conformal algebra is infinite dimensional [4]. In this case, by using Al. Zamolodchikov’s uniformizing variables q, \bar{q} [123], one can show that the four-point function is regular analytic at all possible Lorentzian configurations aside from light-cone singularities [45]. We are going to study a similar problem in $d \geq 3$, for which the conformal group is finite dimensional and the radial coordinates $\rho, \bar{\rho}$ [44] are used in our analysis.³ In addition, in 2d there exists non-local unitary CFTs, which only have the global conformal symmetry. The analysis in this part of the thesis also applies to 2d non-local unitary CFTs.

Recently the conformal bootstrap approach has become a powerful tool in the study of strongly coupled systems [15]. On the numerical side, it gives precise predictions of experimentally measurable quantities, such as the critical exponents of the 3d Ising model [7–10], $O(N)$ model [11–14] and other critical systems. The functional methods, which are used in the numerical approach, can be realized analytically in low dimensions, and lead to insights into low dimensional CFTs and S-matrices [32–35, 38]. While the basic CFT assumptions are made

¹It does not mean that the Lorentzian correlators cannot be functions in other regions. For example, the correlators of generalized free fields are functions aside from light-cone singularities. Here we are talking about the minimal domain of Lorentzian correlators which can be derived from general principles of QFT.

²Often one uses Minkowski space to denote $\mathbb{R}^{3,1}$ only. While in this part of the thesis, we use this terminology for $\mathbb{R}^{d-1,1}$ and general d .

³The set of four-point configurations (x_1, x_2, x_3, x_4) with $|\rho|, |\bar{\rho}| < 1$ is a subset of (x_1, x_2, x_3, x_4) with $|q|, |\bar{q}| < 1$. Since the q -variable argument is based on the Virasoro symmetry which is only true in 2d [5], we cannot apply it to the case of $d \geq 3$.

in the Euclidean signature, many attempts have been made to study the bootstrap equations in the Lorentzian signature [3, 17–20, 22, 23]. In the conformal bootstrap approach, for crossing equations to be valid in the sense of functions, there should be at least two convergent OPE channels. To play the bootstrap game for four-point functions in the Lorentzian signature, it is important to know the convergent domains of various OPE channels. This provides an additional motivation for our work.

The main goal of this part of the thesis is to give complete tables of Lorentzian four-point configurations with the information about convergence in the sense of analytic functions in various OPE channels. In this part of the thesis we will mostly focus on four-point functions of identical scalar operators. Our techniques can be immediately generalized to the case of non-identical scalar operators. The four-point functions of spinning operators require extra work because of tensor structures. In this part of the thesis, we will only make some comments on the case of spinning operators. One may also be interested in the convergence of OPE in the sense of distributions [26]. We leave the discussions of distributional properties to the series of papers [40, 41].

The outline of this part of the thesis is as follows. In chapter 17 we introduce the main problem and provide a quick summary of the main results in this part of the thesis. In chapter 18 we give criteria of OPE convergence in s-, u- and t-channels. In chapter 19, we make a classification of the Lorentzian four-point configurations. All configurations in the same class have the same convergent OPE channels. All information on the OPE convergence properties can be looked up in appendix C.2. In appendix C.3 we review some classical results from Wightman QFT, and compare them with CFT four-point functions. In chapter 20 we generalize our results to the case of non-identical scalar operators and make some comments on the case of spinning operators. In chapter 21 we make conclusions and point out some open questions related to this part of the thesis.

Chapter 17

Main problem and summary of results

17.1 Main problem

We start from CFT in the Euclidean signature. Let $x_k = (\tau_k, \mathbf{x}_k)$ denote the k -th point in the Euclidean space ($k = 1, 2, 3, 4$), where $\tau_k = x_k^0$ is the temporal variable and $\mathbf{x}_k = (x_k^1, x_k^2, \dots, x_k^{d-1}) \in \mathbb{R}^{d-1}$ represents the vector of spatial variables. Lorentzian points are given by Wick rotating the temporal variables: $\tau = it$ where t is a real number. To get Lorentzian four-point functions we need to analytically continue the Euclidean four-point functions to the Lorentzian regime. We define the Wick rotation of the four-point function as follows:

Step 1.

We construct the analytic continuation of the Euclidean CFT four-point function $G_4(c)$ to the forward tube \mathcal{T}_4 , recall the definition

$$\mathcal{T}_4 = \left\{ c = (x_1, x_2, x_3, x_4) \in \mathbb{C}^{4d} \mid x_k = (\tau_k, \mathbf{x}_k), \operatorname{Re}(\tau_k) - \operatorname{Re}(\tau_{k+1}) > |\operatorname{Im}(\mathbf{x}_k) - \operatorname{Im}(\mathbf{x}_{k+1})| \right\}$$

Step 2.

For a Lorentzian configuration $c_L = (x_1, x_2, x_3, x_4)$, where $x_k = (t_k, \mathbf{x}_k)$. Lorentzian CFT four-point function is defined by the limit

$$G_4^L(c_L) := \lim_{\substack{\epsilon_k, \mathbf{y}_k \rightarrow 0 \\ \epsilon_k - \epsilon_{k+1} > |\mathbf{y}_k - \mathbf{y}_{k+1}|}} G_4(\epsilon_k + it_k, \mathbf{x}_k + i\mathbf{y}_k). \quad (17.1.1)$$

The above definition is consistent with Wightman QFT, where the Lorentzian four-point function is the boundary value of the Wightman four-point function from its domain of complex coordinates [102]. The domain of the four-point Wightman function includes \mathcal{T}_4 , so the limit (17.1.1) gives the Wightman four-point distribution when such a limit exists. We review the properties of Wightman functions in appendix C.3.

Our starting point is Euclidean CFT axioms (see section 7.3) instead of Wightman axioms. However it is proved in part II that Euclidean CFT axioms imply Wightman axioms at the level of four-point functions. So step 1 is finished, and the limit in step 2 exists in the sense of tempered distributions.

The CFT four-point function can be computed via three OPE channels (s, t and u). The union of their domains of convergence determines the minimal domain of analyticity of the four-point function, including much more configurations than the general Wightman four-point function. In this part, we would like to study the following problem:

- In which Lorentzian regions does the Lorentzian CFT four-point function, defined by (17.1.1), have a convergent operator product expansion in the sense of **functions**?

The goal of this part of the thesis is to determine the OPE convergence properties of four-point functions at all possible Lorentzian configurations.

17.1.1 Summary of results

In this subsection, we provide a quick summary of the main results for readers who wish to know the general ideas of this part of the thesis before going into the technical details. Readers will find here:

- The criteria of OPE convergence of the Lorentzian CFT four-point function G_4^L in s-, t- and u-channels (chapter 18). G_4^L is defined to be the boundary value of analytically continued Euclidean four-point function (see eq. (17.1.1)). One can imagine that the OPE convergence properties of G_4^L rely on the behavior of cross-ratio variables along the analytic continuation path. In the end we will see that for any fixed Lorentzian four-point configuration, one can check the criteria using any analytic continuation path in the forward tube \mathcal{T}_4 (starting from a Euclidean four-point configuration), and the conclusion does not depend on the choice of the path.
- A classification of the Lorentzian four-point configurations (chapter 19). The Lorentzian configurations are classified into a finite number of classes according to the range of cross-ratio variables (z, \bar{z}) (section 19.1) and the causal orderings (section 19.2). In each class, all configurations have the same OPE convergence properties (section 19.3).

Then the problem is reduced to checking convergence properties in a finite number of cases. We use time-reversal symmetry to further reduce the problem to fewer cases (see section 19.4). The conclusion of OPE convergence properties is lengthy because there are many cases (although finite) to check even after reduction, so we leave this part to appendix C.2. We share the Mathematica code for readers who wish to reproduce and check our results (see the auxiliary file on the arXiv webpage of [74]).

Chapter 18

Lorentzian CFT four-point function

18.1 Some preparations

In part II, we have shown that the s-channel expansion of the CFT four-point function is always convergent in the forward tube \mathcal{T}_4 , thus it performs the analytic continuation of the Euclidean four-point function to \mathcal{T}_4 (step 1 done). Moreover, the s-channel expansion is convergent in the sense of distributions in Minkowski space. At some of the Minkowski four-point configurations, the CFT correlator is well-defined as an analytic function. In this part, we would like to study the OPE convergence properties in the sense of functions in Minkowski space. We will call a configuration $c = (x_1, x_2, x_3, x_4)$ a *Lorentzian (four-point) configuration* if all its points lie in Minkowski space.

Lemma 10.0.1 shows that for any four-point configuration $c = (x_1, x_2, x_3, x_4)$ in the forward tube \mathcal{T}_4 , x_{ij}^2 is always non-zero. However if the four points of c are in Minkowski space, x_{ij}^2 vanishes when x_i and x_j are light-like separated. The CFT four-point function may have singularities at these configurations.

By lemma 10.0.1, 11.3.1 and the definition of z, \bar{z} variables (see eqs. (11.0.2) and (11.2.1)), we know that for all configurations in \mathcal{T}_4 , the cross-ratio variables z and \bar{z} never belong to $\{0\} \cup [1, +\infty)$. However, since the Lorentzian four-point configurations are in the closure of \mathcal{T}_4 , instead of \mathcal{T}_4 itself, some Lorentzian configurations will have z or \bar{z} in $\{0\} \cup [1, +\infty)$. These exceptional configurations are where s-channel expansion does not converge. We will see later that such configurations make up a large proportion of all Lorentzian configurations.

A priori we can also use the t- and u-channel expansions to construct the analytic continuation of the four-point function, starting from the t- and u-channel versions of eqs. (11.0.1) and (11.0.2):

$$G_4^E(c_E) = \frac{g(u_t, v_t)}{(x_{14}^2 x_{23}^2)^\Delta} = \frac{g(u_u, v_u)}{(x_{13}^2 x_{24}^2)^\Delta}, \quad (18.1.1)$$

where

$$u_t = v, \quad v_t = u, \quad u_u = \frac{1}{u}, \quad v_u = \frac{v}{u}. \quad (18.1.2)$$

In Euclidean space, these expansions should give the same result in their common domain of convergence. This

consistency condition is called *crossing symmetry* [124, 125]. It can be written in the following graphical way:

$$\sum_{\mathcal{O}} \text{s-channel} = \sum_{\mathcal{O}} \text{t-channel} = \sum_{\mathcal{O}} \text{u-channel} .$$

We want to remark that only the s-channel expansion could be used to extend the Euclidean CFT four-point function to the whole forward tube \mathcal{T}_4 , since lemma 11.3.1 holds only for the s-channel. We can use t- and u-channel expansion to analytically continue the four-point function to part of \mathcal{T}_4 , but not to the whole \mathcal{T}_4 . We will also consider t- and u-channel expansions because there are Lorentzian configurations where the s-channel expansion does not converge, but the t- or u-channel expansion converges (in fact such configurations make up a large part of the whole Lorentzian configuration space).

18.2 Excluding light-cone singularities

When $x_{ij}^2 = 0$ for some x_i, x_j pair, since at least one of the scaling factors $(x_{ij}^2 x_{kl}^2)^{-\Delta_{\mathcal{O}}}$ in eqs. (11.0.1) and (18.1.1) is infinity, we expect the four-point function to be infinity. The configurations which contain at least one light-like x_i, x_j pair are called light-cone singularities.

One example, for which the correlation functions are divergent at light-cone singularities, is the generalized free field (GFF)

$$G_4^{\text{GFF}}(c) = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta}} + \frac{1}{(x_{14}^2 x_{23}^2)^{\Delta}} + \frac{1}{(x_{13}^2 x_{24}^2)^{\Delta}} . \quad (18.2.1)$$

Since we are interested in the Lorentzian configurations where the four-point functions are genuine functions for all unitary CFTs, we only consider the configurations which are not light-cone singularities. In other words, we will only consider the following set of Lorentzian configurations:

$$\mathcal{D}_L := \left\{ (x_1, x_2, x_3, x_4) \mid x_k = (it_k, \mathbf{x}_k), \forall k; \quad x_{ij}^2 \neq 0, \forall i \neq j \right\} . \quad (18.2.2)$$

18.3 Criteria of OPE convergence

Now that $x_{ij}^2 \neq 0$ for all configurations in \mathcal{D}_L , all the cross-ratios defined in (11.0.2) and (18.1.2) are finite and non-zero, which implies

$$z, \bar{z} \neq 0, 1, \infty . \quad (18.3.1)$$

So the real axis in the z, \bar{z} -space is divided into three parts:

$$(-\infty, 0) \cup (0, 1) \cup (1, +\infty) . \quad (18.3.2)$$

In this section, we are going to establish criteria of OPE convergence in s-, t- and u-channels. The three intervals in (18.3.2) will play important roles because each of them is the place where one OPE channel stops being convergent.

18.3.1 s-channel

The analytic continuation of G_4 (from Euclidean region to \mathcal{T}_4) was discussed in detail in section 11.3. So far, as already mentioned, we only used the s-channel expansion because of lemma 11.3.1. Actually by using the s-channel expansion, we are able to extend G_4 to a larger domain $\mathcal{T}^s \supset \mathcal{T}_4$ (the label “s” just means s-channel) according to the constraint $0 < |\rho|, |\bar{\rho}| < 1$ (or equivalently, $z, \bar{z} \neq \{0\} \cup [1, +\infty)$). \mathcal{T}^s contains some but not all Lorentzian configurations. In other words, the Lorentzian four-point function has convergent s-channel OPE on the set $\mathcal{T}^s \cap \mathcal{D}_L$.

By eqs. (11.2.4) and (18.3.1), we have $\rho, \bar{\rho} \neq 0, \pm 1$ for all configurations in \mathcal{D}_L . Because of lemma 11.3.1 and the continuity, all configurations in \mathcal{D}_L have $|\rho|, |\bar{\rho}| \leq 1$. To check the convergence of s-channel OPE, it suffices to check whether $|\rho|, |\bar{\rho}| \neq 1$ or not. Equivalently, it suffices to check whether $z, \bar{z} \notin (1, +\infty)$ or not.

Therefore, given a Lorentzian configuration $c_L \in \mathcal{D}_L$, we have the following criterion of s-channel OPE convergence:

Theorem 18.3.1. (s-channel OPE convergence) *If neither z nor \bar{z} computed from c_L belong to $(1, +\infty)$, then the Lorentzian four-point function G_4 is analytic at c_L and is given by the formula (11.3.11) in $d \geq 3$ or (11.3.15) in $d = 2$.*

18.3.2 t-channel and u-channel

We define the variables z_t, \bar{z}_t and z_u, \bar{z}_u by replacing u, v with u_t, v_t and u_u, v_u in eq. (11.2.1). By eq. (18.1.2), we choose proper solutions to the t- and u-channel versions of eq. (11.2.1), and get the following relations¹

$$z_t = 1 - z, \quad \bar{z}_t = 1 - \bar{z}, \quad z_u = 1/z, \quad \bar{z}_u = 1/\bar{z}. \quad (18.3.3)$$

Then we define the t- and u-channel versions of radial coordinates $\rho_t, \bar{\rho}_t, \rho_u, \bar{\rho}_u$ by replacing z, \bar{z} with z_t, \bar{z}_t and z_u, \bar{z}_u in eq. (11.2.4).

t-channel

Let us demonstrate how analytic continuation is performed using the t-channel expansion. The u-channel expansion argument will be similar. We first start from the Euclidean configuration space with time ordering:

$$\mathcal{D}_E = \left\{ c_E = (x_1, x_2, x_3, x_4) \in \mathbb{R}^{4d} \mid x_1^0 > x_2^0 > x_3^0 > x_4^0 \right\}. \quad (18.3.4)$$

Since the cross-ratio variables z and \bar{z} are always complex conjugate for Euclidean configurations, they are not real as long as $z \neq \bar{z}$. Then recall eqs. (11.3.2) and (11.3.3), z and \bar{z} are not real when $c_E \in \mathcal{D}_E \setminus \Gamma$. In particular, we have $z_t, \bar{z}_t \notin [1, +\infty)$ for all configurations in $\mathcal{D}_E \setminus \Gamma$, which allows us to choose $|\rho_t| = |\bar{\rho}_t| < 1$ to start with convergent t-channel expansion. Analogously to the s-channel expansion, the t-channel expansion is given by sticking $\rho_t = re^{i\theta}, \bar{\rho}_t = re^{-i\theta}$ into the series expansion (11.2.6).

Now let us enter the forward tube \mathcal{T}_4 via the t-channel expansion. First of all, for the t-channel expansion to be convergent, the configuration must satisfy $z_t, \bar{z}_t \notin [1, +\infty)$. Otherwise we will have $|\rho_t|$ or $|\bar{\rho}_t| = 1$ and the series expansion does not converge. Suppose we have a path γ in $\mathcal{T}_4 \setminus \Gamma$ such that $\gamma(0) \in \mathcal{D}_E \setminus \Gamma$, we can find a neighbourhood $U_\gamma \subset \mathcal{T}_4 \setminus \Gamma$ of the set $\{\gamma(s) \mid 0 \leq s \leq 1\}$ and perform the analytic continuation of z, \bar{z} in U_γ via eqs. (11.0.2) and (11.2.2).² Then we get the analytic continuation of z_t, \bar{z}_t in U_γ by the relation in eq. (18.3.3). If $z_t, \bar{z}_t \notin [1, +\infty)$ in U_γ , or equivalently, $|\rho_t|, |\bar{\rho}_t| < 1$ in U_γ , then the t-channel expansion of G_4 is convergent in U_γ , and gives the analytic continuation to U_γ . Since the start point $\gamma(0)$ is a Euclidean configuration, $U_\gamma \cap \mathcal{D}_E$ is an open subset of \mathcal{D}_E , where the temporal variables τ_k are independent real numbers. According to the crossing symmetry, the s- and t-channel expansions agree in $U_\gamma \cap \mathcal{D}_E$, so they also agree in U_γ , where τ_k are independent complex numbers. Furthermore, by taking the limit from $\mathcal{T}_4 \setminus \Gamma$ to Γ , we can also use the t-channel expansion to compute the four-point function for configurations in Γ with the constraint $|\rho_t|, |\bar{\rho}_t| < 1$, and the result also agrees with the s-channel expansion by continuity. So we conclude that

- Given a configuration c in \mathcal{T}_4 , the t-channel expansion gives the same analytic continuation of G_4 as the s-channel expansion if there exists a path γ in \mathcal{D} such that $\gamma(0) \in \mathcal{D}_E \setminus \Gamma$, $\gamma(1) = c$ and $z_t, \bar{z}_t \notin [1, +\infty)$ along γ .

¹The other solutions of z_t, \bar{z}_t (z_u, \bar{z}_u) differ from (18.3.3) by interchanging z_t and \bar{z}_t (z_u and \bar{z}_u), which will give the same conclusions of convergence properties in the t-channel (u-channel) expansion.

²As long as $\gamma(s) \notin \mathcal{D} \setminus \Gamma$ along the path γ , such a neighbourhood U_γ always exists.

Analogously, by replacing z_t, \bar{z}_t with z_u, \bar{z}_u , we have the similar conclusion for the u-channel expansion.

While lemma 11.3.1 holds for z, \bar{z} , it does not hold for z_t, \bar{z}_t or z_u, \bar{z}_u , which means that the t- and u-channel expansions may diverge in \mathcal{T}_4 . To know the convergence properties of t- and u-channel expansions, we need to know not only the values of $z_t, \bar{z}_t, z_u, \bar{z}_u$ of a configuration, but also the values of these variables along a path. For convenience we use the relation (18.3.3) to translate $z_t, \bar{z}_t, z_u, \bar{z}_u \notin (1, +\infty)$ to equivalent conditions in z, \bar{z} :

$$\begin{aligned} z_t, \bar{z}_t \notin (1, +\infty) &\Leftrightarrow z, \bar{z} \notin (-\infty, 0) \\ z_u, \bar{z}_u \notin (1, +\infty) &\Leftrightarrow z, \bar{z} \notin (0, 1) \end{aligned} \quad (18.3.5)$$

Since in each interval there is one OPE that stops converging, we call these three intervals **s-channel cut** $((1, +\infty))$, **t-channel cut** $((-\infty, 0))$ and **u-channel cut** $((0, 1))$ respectively. Then it suffices to compute the z, \bar{z} -curves along the path and count how many times they cross the cuts.

To give the final criteria of convergence properties in t-channel expansions, we define some quantities which count how z, \bar{z} -curves cross the t-channel cut $(-\infty, 0)$. Given a path γ defined as follows

$$\begin{aligned} \gamma : [0, 1] &\longrightarrow \bar{\mathcal{T}}_4, \\ \gamma(0) &\in \mathcal{D}_E \setminus \Gamma, \\ \gamma(s) &\in \mathcal{T}_4 \setminus \Gamma, \quad s < 1, \end{aligned} \quad (18.3.6)$$

if the variables z, \bar{z} at the final point $\gamma(1)$ satisfy $z, \bar{z} \notin (-\infty, 0)$, we define

$$\begin{aligned} n_t(\gamma) &:= \text{number of times } z \text{ crosses } (-\infty, 0) \text{ from above} \\ &\quad - \text{number of times } z \text{ crosses } (-\infty, 0) \text{ from below,} \\ \bar{n}_t(\gamma) &:= \text{number of times } \bar{z} \text{ crosses } (-\infty, 0) \text{ from above} \\ &\quad - \text{number of times } \bar{z} \text{ crosses } (-\infty, 0) \text{ from below,} \end{aligned} \quad (18.3.7)$$

and

$$N_t(\gamma) := n_t(\gamma) + \bar{n}_t(\gamma). \quad (18.3.8)$$

Let us consider the t-channel expansion. We claim that N_t is a path-independent quantity:

Lemma 18.3.2. *Given a configuration $c \in \bar{\mathcal{T}}_4$ with $z, \bar{z} \notin (-\infty, 0]$, N_t is independent of the choice of the path. Therefore, we can write $N_t(\gamma)$ as $N_t(c)$.*

Proof. Without loss of generality, we assume that $\text{Im}(z) > 0$ and $\text{Im}(\log \rho) \in (0, \pi)$ at the start point (resp. $\text{Im}(\bar{z}) < 0$ and $\text{Im}(\log \bar{\rho}) \in (-\pi, 0)$).³

Let γ be a path satisfying condition (18.3.6). By definition (18.3.7), we have

$$\begin{aligned} \text{Im}(\log \rho(c)) &= 2\pi n_t(\gamma) + \varphi(\gamma), \\ \text{Im}(\log \bar{\rho}(c)) &= 2\pi \bar{n}_t(\gamma) + \bar{\varphi}(\gamma), \end{aligned} \quad (18.3.9)$$

where $\varphi(\gamma), \bar{\varphi}(\gamma) \in (-\pi, \pi)$. By eqs. (11.2.1) and (11.2.4), choosing a different path γ can only exchange $\varphi(\gamma)$ and $\bar{\varphi}(\gamma)$. So $\varphi(\gamma) + \bar{\varphi}(\gamma)$ is path independent.

By the analysis in section 11.3, we know that $R(c) = \rho(c)\bar{\rho}(c)$ is an analytic function on the forward tube \mathcal{T}_4 . Together with the fact that \mathcal{T}_4 is simply connected, we conclude that $\log(\rho(c)) + \log(\bar{\rho}(c))$ is also an analytic function on \mathcal{T}_4 . Therefore, by (18.3.9) conclude that $N_t(\gamma) = n_t(\gamma) + \bar{n}_t(\gamma)$ is path independent. \square

Suppose c is a configuration in \mathcal{T}_4 with $z, \bar{z} \notin (-\infty, 0)$ and $N_t = 0$. By choosing an arbitrary path γ with conditions (18.3.6) and $\gamma(1) = c$, we get the paths of $\log \rho$ and $\log \bar{\rho}$ along γ . By (18.3.9) and $N_t = 0$, (for each fixed path) there exists a unique integer k such that

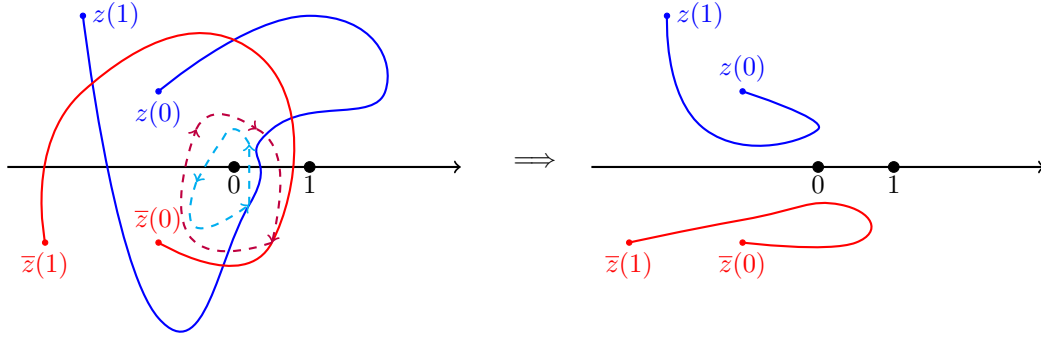
$$\text{Im}(\log \rho(c)) + 2k\pi, \text{Im}(\log \bar{\rho}(c)) - 2k\pi \in (-\pi, \pi). \quad (18.3.10)$$

³Choosing the convention $\text{Im}(z) < 0$ only exchanges n_t and \bar{n}_t , which does not effect N_t .

According to the s-channel expansion formula (11.3.6), the four-point function does not change if we replace $(\log \rho, \log \bar{\rho})$ with

$$\log \tilde{\rho} = \log \rho + 2ik\pi, \quad \log \tilde{\bar{\rho}} = \log \bar{\rho} - 2ik\pi. \quad (18.3.11)$$

In the z, \bar{z} space, transformation (18.3.11) corresponds to that z and \bar{z} move around zero in opposite directions, without touching the s-channel cut. Therefore, we can deform the z, \bar{z} -curves by letting z and \bar{z} go around zero in opposite directions. The following plot shows an example of $k = 1$. The blue and red solid curves on the left are the trajectories of z and \bar{z} along the path, which give $n_t = -1$ and $\bar{n}_t = 1$, i.e. $N_t = 0$. We add a clockwise loop (cyan dashed) to the z -curve and an anticlockwise loop (purple dashed) to the \bar{z} -curve. Then we continuously deform the curves without touching 0 and 1. In the end we arrive at the curves on the right.



Eq. (11.3.6) tells us that the left and the right curves give the same analytic continuation of the four-point function at the end point. Note that if z nor \bar{z} do not cross $(-\infty, 0)$ at all, then $|\rho_t|, |\bar{\rho}_t| < 1$ along the whole path, and the t-channel OPE is guaranteed to converge. So for the right z, \bar{z} -curves, we can safely use the t-channel expansion to compute the four-point function. Therefore, any $c \in \bar{\mathcal{T}}_4$ with $z, \bar{z} \notin (-\infty, 0)$ and $N_t = 0$ has convergent t-channel expansion.

For $c \in \bar{\mathcal{T}}_4$ such that $z, \bar{z} \notin (-\infty, 0)$ and $N_t = 0$, it can always be approached by the configurations in \mathcal{T}_4 with the same condition. Then the t-channel expansion still converges in the end by continuity. So we conclude that

Theorem 18.3.3. (t-channel OPE convergence) *Let $c \in \bar{\mathcal{T}}_4$ be a four-point configuration aside from light-cone singularities. If the cross-ratio variables $z(c), \bar{z}(c)$ do not belong to $(-\infty, 0)$, and furthermore if $N_t(c) = 0$, then the four-point function G_4 is analytic at c and can be computed using the t-channel expansion*

$$G_4(c) = \frac{g_t(c)}{(x_{23}^2 x_{14}^2)^\Delta}. \quad (18.3.12)$$

The function $g_t(c)$ is defined by replacing $\rho, \bar{\rho}$ with $\tilde{\rho}_t(c), \tilde{\bar{\rho}}_t(c)$ in the series expansion (11.3.6), and $\tilde{\rho}_t(c), \tilde{\bar{\rho}}_t(c)$ are the corresponding t-channel variables after transformation (18.3.11):

$$\rho \xrightarrow{(18.3.11)} \tilde{\rho} \rightarrow \tilde{z}_t \rightarrow \tilde{z}_t = 1 - z \xrightarrow{(11.2.4)} \tilde{\rho}_t = f(z_t). \quad (18.3.13)$$

u-channel

The argument for the u-channel expansion is similar. First of all, for the u-channel expansion to be convergent, the configuration should satisfy $z, \bar{z} \notin (0, 1)$, otherwise we will have $|\rho_u|$ or $|\bar{\rho}_u| = 1$. Given a configuration c with $z, \bar{z} \notin (0, 1)$, we choose a path γ satisfying condition (18.3.6) and $\gamma(1) = c$. We define

$$\begin{aligned} n_u(\gamma) &:= \text{number of times } z \text{ crosses } (0, 1) \text{ from above} \\ &\quad - \text{number of times } z \text{ crosses } (0, 1) \text{ from below,} \\ \bar{n}_u(\gamma) &:= \text{number of times } \bar{z} \text{ crosses } (0, 1) \text{ from above} \\ &\quad - \text{number of times } \bar{z} \text{ crosses } (0, 1) \text{ from below,} \end{aligned} \quad (18.3.14)$$

and

$$N_u(\gamma) := n_u(\gamma) + \bar{n}_u(\gamma). \quad (18.3.15)$$

Similarly to the t-channel case, $N_u(\gamma)$ does not depend on the choice of γ , so we can write it as $N_u(c)$. When $N_u(c) = 0$, we choose an arbitrary γ , compute the z, \bar{z} -curves, then deform the curves by moving z and \bar{z} around 0 in opposite directions. The deformed curves give the same analytic continuation of the four-point function at the end point. We can find a proper integer k in (18.3.11) such that the deformed curves do not touch the u-channel cut $(0, 1)$. Then we can safely use the u-channel expansion. Therefore, we conclude that

Theorem 18.3.4. (u-channel OPE convergence) *Let $c \in \bar{\mathcal{T}}_4$ be a four-point configuration aside from light-cone singularities. If the cross-ratio variables $z(c), \bar{z}(c)$ do not belong to $(0, 1)$, and furthermore if $N_u(c) = 0$, then the four-point function G_4 is analytic at c and can be compute using the u-channel expansion*

$$G_4(c) = \frac{g_u(c)}{(x_{13}^2 x_{24}^2)^\Delta}. \quad (18.3.16)$$

The function $g_u(c)$ is defined by replacing $\rho, \bar{\rho}$ with $\tilde{\rho}_u(c), \tilde{\bar{\rho}}_u(c)$ in the series expansion (11.3.6), and $\tilde{\rho}_u(c), \tilde{\bar{\rho}}_u(c)$ are the corresponding u-channel variables after transformation (18.3.11):⁴

$$\rho \xrightarrow{(18.3.11)} \tilde{\rho} \rightarrow \tilde{z} \rightarrow \tilde{z}_u = 1/z \xrightarrow{(11.2.4)} \tilde{\bar{\rho}}_u. \quad (18.3.17)$$

We would like to make four comments on the t-channel and u-channel expansions.

Remark 18.3.1. Theorems 18.3.3 and 18.3.4 cover the case of Lorentzian four-point functions because Lorentzian configurations lie on the boundary of \mathcal{T}_4 .

Remark 18.3.2. Theorem 18.3.3 and 18.3.4 cover the case when s-channel expansion is not convergent.

Remark 18.3.3. To compute the four-point function in t-channel or u-channel, it is important to know the arguments of $\tilde{\rho}_t, \tilde{\bar{\rho}}_t, \tilde{\rho}_u$ and $\tilde{\bar{\rho}}_u$. When $z, \bar{z} \notin (1, \infty)$, one can determine these arguments directly from the configuration itself, without looking at the path. The reason is that (up to homotopy of the path) there is only one possible way to connect start point $\tilde{z}(0)$ and end point $\tilde{z}(1)$, without touching the s- and t-(u)-channel cuts (since $\mathbb{C} \setminus \{(-\infty, 0] \cup [1, +\infty)\}$ and $\mathbb{C} \setminus [0, +\infty)$ are simply connected). When z or $\bar{z} \in (1, \infty)$, one can determine these arguments by looking at the behavior near the end of z -curve or \bar{z} -curve.

Remark 18.3.4. Unlike the s-channel case, even if we only want to check whether t- or u-channel expansion is convergent or not, we have to choose a path to compute N_t or N_u .

Before finishing this section, we want to remark that in practice, the condition (18.3.6) of the path γ can be relaxed in the way that γ is allowed to touch Γ :

$$\begin{aligned} \gamma &: [0, 1] \longrightarrow \bar{\mathcal{T}}_4, \\ \gamma(0) &\in \mathcal{D}_E \setminus \Gamma, \\ \gamma(s) &\in \mathcal{T}_4, \quad s < 1. \end{aligned} \quad (18.3.18)$$

Suppose we have a path γ which intersects with Γ . Let $\gamma(s_*) \in \Gamma$ be the first intersection point. At s_* we have $z(s_*) = \bar{z}(s_*)$, then $z(s), \bar{z}(s)$ become indistinguishable for $s > s_*$, so the quantities $n_t, \bar{n}_t, n_u, \bar{n}_u, \chi, \bar{\chi}$ are not well defined for γ . However, by manually choosing z, \bar{z} after each intersection, we still get two curves $z(s), \bar{z}(s)$: they may not be smooth at intersection points, but they are still continuous. By this trick we get $n_t, \bar{n}_t, n_u, \bar{n}_u, \chi, \bar{\chi}$, so that we are able to compute N_t, N_u and the four-point function. On the other hand, we can always deform γ to a path γ' , such that γ' has the same start and final points as γ but γ' does not intersect with Γ . By doing proper deformation, we can make γ' have the same $n_t, \bar{n}_t, n_u, \bar{n}_u, \chi, \bar{\chi}$ as selected on γ . Therefore, our manual selection will give the correct OPE convergence properties and the correct value of the four-point function.

18.4 Violating the conditions of theorem 18.3.1, 18.3.3 and 18.3.4

We would like to make a comment that theorem 18.3.1, 18.3.3 and 18.3.4 only give sufficient conditions for OPE convergence. For a Lorentzian configuration c_L which is not a light-cone singularity and which does not satisfy the

⁴In the u-channel case, the integer k in (18.3.11) should be such that the deformed z, \bar{z} -curves do not touch the u-channel cut $(0, 1)$.

conditions in these theorems, it does not mean that G_4 cannot be a function at c_L . It just means that for general CFT, we are not able to use the radial coordinates $\rho, \bar{\rho}$ ($\rho_t, \bar{\rho}_t, \rho_u, \bar{\rho}_u$) and the expansion (11.3.6) to prove the analyticity of G_4 at c_L . The four-point function still has a chance to be analytic at c_L . For example, the Euclidean four-point function of generalized free fields has analytic continuation to the whole Lorentzian region except for the light-cone singularities.

An interesting related open question is: can we relax the conditions in theorem 18.3.1, 18.3.3 and 18.3.4?

The s-channel condition

In theorem 18.3.1, we only assume the condition $z, \bar{z} \notin (1, +\infty)$ (equivalently, $|\rho|, |\bar{\rho}| < 1$). The Lorentzian configurations which violate this condition has $|\rho| = 1$ or $|\bar{\rho}| = 1$, then the proof of theorem 18.3.1 fails because in the proof we used the fact that the series expansion (11.3.6) is absolutely convergent when $|\rho|, |\bar{\rho}| < 1$.

We are interested in the Lorentzian configurations where the s-channel expansion is convergent for all unitary CFTs. For configurations with $|\rho| = 1$ or $|\bar{\rho}| = 1$, we may exhibit an explicit CFT four-point function, for which the s-channel expansion is divergent (then such configurations are ruled out). The generalized free field (GFF) theory is such an example. The GFF four-point function of identical scalar operators (with scaling dimension Δ) is defined by

$$(G_4)_{GFF}(x_1, x_2, x_3, x_4) = \frac{1}{[x_{12}^2 x_{34}^2]^\Delta} + \frac{1}{[x_{23}^2 x_{14}^2]^\Delta} + \frac{1}{[x_{13}^2 x_{24}^2]^\Delta}. \quad (18.4.1)$$

By (11.0.1), the conformal invariant part of $(G_4)_{GFF}$ is given by

$$g_{GFF}(\rho, \bar{\rho}) = 1 + \left(\frac{16\rho\bar{\rho}}{(1+\rho)^2(1+\bar{\rho})^2} \right)^\Delta + \left(\frac{16\rho\bar{\rho}}{(1-\rho)^2(1-\bar{\rho})^2} \right)^\Delta. \quad (18.4.2)$$

It has the series expansion

$$g_{GFF}(\rho, \bar{\rho}) = 1 + (16\rho\bar{\rho})^\Delta \sum_{m,n=0}^{\infty} \frac{(1+(-1)^{m+n})\Gamma(\Delta+m)\Gamma(\Delta+n)}{m!n!\Gamma(\Delta)^2} \rho^m \bar{\rho}^n \quad (18.4.3)$$

which diverges when $|\rho| = 1$ or $|\bar{\rho}| = 1$. It follows that theorem 18.3.1 cannot be extended to configurations with $|\rho| = 1$ or $|\bar{\rho}| = 1$ without extra assumptions on the theory. One such extra assumption will be mentioned in section 19.3.2 (locality of 2d CFT).

The t-channel and u-channel conditions

In theorem 18.3.3, we assumed two conditions: $z, \bar{z} \notin (-\infty, 0)$ and $N_t = 0$.

For Lorentzian configurations which violate the first condition, (analogously to the s-channel case) we can use GFF to conclude that these configurations do not have convergent t-channel expansion for some unitary CFTs.

When the second condition is violated, i.e. $N_t \neq 0$, according to our discussion in section 18.3.2, this is equivalent to putting \bar{z} in the first sheet of the complex plane with the t-channel cut $(-\infty, 0]$ and letting z cross the t-channel cut from above for N_t times. When z cross the t-channel cut for the first time, the corresponding radial variable ρ_t becomes greater than 1 in absolute value, so the t-channel expansion diverges. One may ask what happens if z cross the t-channel cut for even number of times (in these cases one has $|\rho_t| < 1$). Our answer is we do not know since the analytic continuation has already been interrupted in the second sheet (i.e. after z crossing the cut for the first time).

The arguments for theorem 18.3.4 (u-channel) are similar.

Chapter 19

Classifying the Lorentzian four-point configurations

In the previous chapter we gave the criteria of convergence properties of OPE in various channels for Lorentzian CFT four-point functions. These criteria say that given a Lorentzian configuration c_L , one can just start with an arbitrary Euclidean configuration in $\mathcal{D}_E \setminus \Gamma$ and choose an arbitrary path towards c_L , then decide if the conditions in theorem 18.3.1, 18.3.3 and 18.3.4 hold or not by watching the z, \bar{z} -curves (in theorem 18.3.1 one does not even have to choose a path).

However, it would be frustrating if we have to check the analytic continuation curves for all Lorentzian configurations in \mathcal{D}_L (recall definition (18.2.2)). We expect that these Lorentzian configurations can be classified such that for each class it suffices to choose one representative configuration to see if various OPE channels converge or not. There are two natural classification methods, one according to the range of z and \bar{z} , the other according to the causal orderings. We will show that combining these two methods leads to a complete classification for the convergence properties of Lorentzian CFT four-point functions.

19.1 z, \bar{z} of Lorentzian configurations

For all Lorentzian configurations, since x_{ij}^2 are real, the cross-ratios u, v are also real. By (11.2.2), there are only two possibilities for z, \bar{z} :

1. z, \bar{z} are independent real variables.
2. z, \bar{z} are complex conjugate to each other.

In addition, we have already excluded light-cone singularities in \mathcal{D}_L (recall its definition in eq. (18.2.2)), so the configurations in \mathcal{D}_L have $z, \bar{z} \neq 0, 1, \infty$. According to the range of the z, \bar{z} variables, we divide \mathcal{D}_L into four classes:

$$\mathcal{D}_L = S \sqcup T \sqcup U \sqcup E, \tag{19.1.1}$$

where the classes are defined as follows.

- Class S: configurations with $0 < z < 1, \bar{z} < 0$ or $z < 0, 0 < \bar{z} < 1$.
- Class T: configurations with $z > 1, 0 < \bar{z} < 1$ or $0 < z < 1, \bar{z} > 1$.
- Class U: configurations with $z > 1, \bar{z} < 0$ or $z < 0, \bar{z} > 1$.
- Class E: configurations with $z, \bar{z} < 0$ or $0 < z, \bar{z} < 1$ or $z, \bar{z} > 1$ or $z^* = \bar{z}$.

We use the name ‘‘S’’ (resp. ‘‘T’’, ‘‘U’’) because it corresponds to the configurations where only the s-channel (resp. t-channel, u-channel) expansion has a chance to converge. The name ‘‘E’’ means ‘‘Euclidean’’, since the variables z, \bar{z} in class E can be realized by the configurations with totally space-like separation. In addition, we divide the class E into four subclasses:

$$E = E_{su} \sqcup E_{st} \sqcup E_{tu} \sqcup E_{stu}, \tag{19.1.2}$$

where the subclasses are defined as follows.

- Subclass E_{su} : configurations with $z, \bar{z} < 0$.
- Subclass E_{st} : configurations with $0 < z, \bar{z} < 1$.
- Subclass E_{tu} : configurations with $z, \bar{z} > 1$.
- Subclass E_{stu} : configurations with $z^* = \bar{z}$ not real.

The subscripts in the above names indicate the possible convergent channels. Figure 19.1.1 shows the range of (z, \bar{z}) pair corresponding to each class/subclass. Let $P(\mathcal{C})$ denote the subset of (z, \bar{z}) pairs corresponding to class/subclass \mathcal{C} . Under identification $(z, \bar{z}) \sim (\bar{z}, z)$, $P(\mathcal{C})$ are connected subsets of \mathbb{C}/\mathbb{Z}_2 . $P(S), P(T), P(U), P(E)$ are disconnected from each other, but $P(E_{su}), P(E_{st})$ and $P(E_{tu})$ are connected to $P(E_{stu})$ (also note that $P(E_{su}), P(E_{st})$ and $P(E_{tu})$ are disconnected from each other).

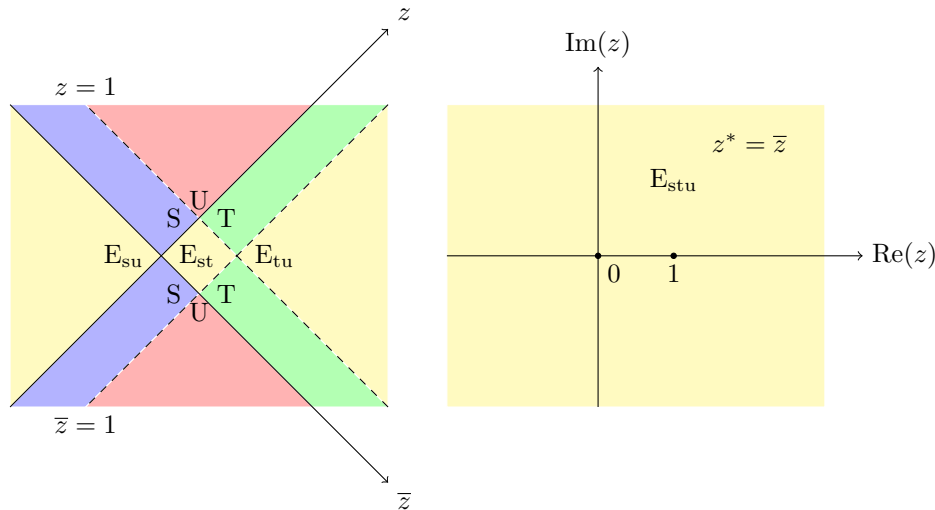


Figure 19.1.1: The corresponding range of (z, \bar{z}) pair of each class/subclass.

For each class/subclass, we immediately get some information about OPE convergence properties by theorem 18.3.1, 18.3.3 and 18.3.4 (see table 19.1).

class/subclass	s-channel	t-channel	u-channel
S	✓	✗	✗
T	✗		✗
U	✗	✗	
E _{st}	✓		✗
E _{su}	✓	✗	
E _{tu}	✗		
E _{stu}	✓		

Table 19.1: OPE convergence properties of classes/subclasses

In table 19.1, the check mark means that the sufficient conditions in theorem 18.3.1 or 18.3.3 or 18.3.4 holds, hence the corresponding channel is convergent. The cross mark means that the sufficient conditions do not hold, we cannot conclude that the corresponding channel is convergent or not (basically because one or both $\rho, \bar{\rho}$ variables are on the unit circles). The blank means that there is room for convergence but we need to check N_t, N_u conditions.

19.2 Causal orderings

In Minkowski space $\mathbb{R}^{d-1,1}$, causal ordering is a binary relation between two arbitrary points. Let $x_1 = (it_1, \mathbf{x}_1)$ and $x_2 = (it_2, \mathbf{x}_2)$ be two points in $\mathbb{R}^{d-1,1}$,¹ we say $x_1 \rightarrow x_2$ if x_2 is in the open forward light-cone of x_1 , or equivalently, $t_2 - t_1 > |\mathbf{x}_1 - \mathbf{x}_2|$.

By the triangle inequality, the causal ordering is transitive: if $x_1 \rightarrow x_2$ and $x_2 \rightarrow x_3$, then $x_1 \rightarrow x_3$.

Causal orderings are preserved by translations, Lorentz transformations and dilatations. But special conformal transformations may violate causal orderings. Given a pair of time-like separated points x_i, x_j in $\mathbb{R}^{d-1,1}$, there exists a special transformation such that the images x'_i, x'_j are space-like separated [126].

By “the causal ordering of a configuration $C = (x_1, x_2, x_3, x_4)$ ”, we will mean the directed graph (V, E) , where $V = \{1, 2, 3, 4\}$ is the set of indices and $E = \{(ij)\}$ is the set of arrows $i \rightarrow j$ encoding the causal orderings $x_i \rightarrow x_j$. For example, the causal ordering of the configuration

$$\begin{aligned}
 x_1 &= (0, 0, \dots, 0), \\
 x_2 &= (i, 0, \dots, 0), \\
 x_3 &= (2i, 0, \dots, 0), \\
 x_4 &= (3i, 0, \dots, 0),
 \end{aligned}
 \tag{19.2.1}$$

is given by

$$\begin{array}{ccc}
 1 & \longrightarrow & 2 \\
 \downarrow & \times & \downarrow \\
 4 & \longleftarrow & 3
 \end{array}
 \tag{19.2.2}$$

Since causal ordering is transitive, some arrows in the graph (19.2.2) are redundant and we will drop them. E.g. the graph

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4
 \tag{19.2.3}$$

¹Since in this part of the thesis our discussions start from the Euclidean signature, we use the Euclidean coordinates $x = (\epsilon + it, \mathbf{x})$. The Euclidean points correspond to $t = 0$ and the Lorentzian points correspond to $\epsilon = 0$.

represents the same causal ordering as (19.2.2). For simplicity, we will use the graphic notation with the least number of arrows like (19.2.3).

19.3 Classifying convergent OPE channels

We decompose the set \mathcal{D}_L according to the causal orderings of the configurations:

$$\mathcal{D}_L = \bigsqcup_{\alpha} \mathcal{D}_L^{\alpha}, \quad (19.3.1)$$

where each \mathcal{D}_L^{α} is the set of configurations with the same causal ordering, labelled by the index α .

19.3.1 Case $d \geq 3$

In $d \geq 3$, each \mathcal{D}_L^{α} in (19.3.1) is a connected component of \mathcal{D}_L . It is not hard to see that different \mathcal{D}_L^{α} are disconnected to each other. The proof that each \mathcal{D}_L^{α} is connected is given in appendix C.1.

Since \mathcal{D}_L^{α} is connected, with the identification $(z, \bar{z}) \sim (\bar{z}, z)$, the set of corresponding (z, \bar{z}) pairs is a connected subset of $\mathbb{C}^2/\mathbb{Z}_2$. Recalling our classification in section 19.1, we conclude that

Lemma 19.3.1. *For $d \geq 3$, all configurations with the same causal ordering belong to the same class S, T, U, E (recall their definitions in section 19.1).*

By the lemma, we can assign class S, T, U and E to each causal ordering of the configurations. In addition, if \mathcal{D}_L^{α} is in class E, we subdivide \mathcal{D}_L^{α} according to the subclasses of class E. We summarize these relations in figure 19.3.1.

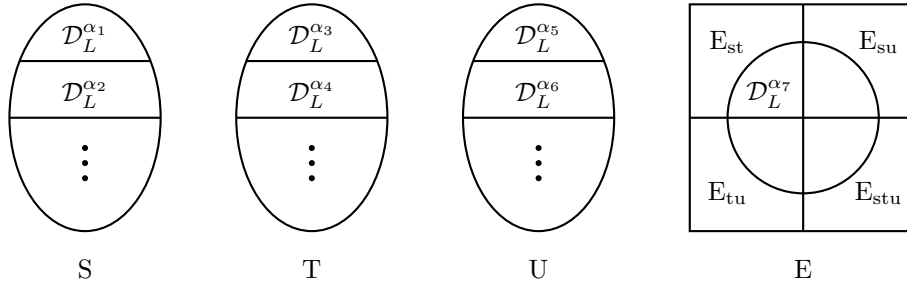


Figure 19.3.1: The class S, T, U and E are subdivided according causal orderings. For each \mathcal{D}_L^{α} in class E, \mathcal{D}_L^{α} is subdivided according to subclasses.

Now we are ready to state the classification of convergent OPE channels for Lorentzian CFT four-point functions.

Theorem 19.3.2. *Let G_4^L be the Lorentzian four-point function which is defined by the Wick rotation (17.1.1) from a Euclidean unitary CFT in $d \geq 3$. Let α be a causal ordering and let \mathcal{D}_L^{α} be the set of all configurations with this causal ordering.*

- If \mathcal{D}_L^{α} is in class S, then all configurations in \mathcal{D}_L^{α} only have convergent s-channel expansion for G_4^L .
- If \mathcal{D}_L^{α} is in class T, then all configurations in \mathcal{D}_L^{α} have the same N_t .
- If \mathcal{D}_L^{α} is in class U, then all configurations in \mathcal{D}_L^{α} have the same N_u .
- If \mathcal{D}_L^{α} is in class E, then
 - All configurations in $\mathcal{D}_L^{\alpha} \cap E_{st}$ have the convergent s-channel expansion and the same N_t .
 - All configurations in $\mathcal{D}_L^{\alpha} \cap E_{su}$ have the convergent s-channel expansion and the same N_u .

- All configurations in $\mathcal{D}_L^\alpha \cap E_{tu}$ have the same N_t, N_u .
- All configurations in $\mathcal{D}_L^\alpha \cap E_{stu}$ have the convergent s-channel expansion and the same N_t, N_u .

Proof. Let us check the conclusions case by case.

Case 1: \mathcal{D}_L^α is in class S.

The s-channel convergence follows from theorem 18.3.1. For other cases, the s-channel arguments are the same, and we will only focus on N_t and N_u .

Case 2: \mathcal{D}_L^α is in class T.

It remains to show that N_t is a constant in \mathcal{D}_L^α . For any $c_L, c'_L \in \mathcal{D}_L^\alpha$, since \mathcal{D}_L^α is connected, there exists a path γ_1 which connects c_L and c'_L :

$$\begin{aligned} \gamma_1 : [0, 1] &\longrightarrow \mathcal{D}_L^\alpha, \\ \gamma_1(0) &= c_L, \quad \gamma_1(1) = c'_L. \end{aligned} \tag{19.3.2}$$

Since $\gamma_1(s)$ are always configurations in class T, the corresponding z, \bar{z} never touch the interval $(-\infty, 0)$. So $n_t(\gamma_1) = \bar{n}_t(\gamma_1) = 0$, which implies $N_t(\gamma_1) = 0$. On the other hand, given a path γ_2 from $\mathcal{D}_E \setminus \Gamma$ to c_L , we get a path from $\mathcal{D}_E \setminus \Gamma$ to c'_L by connecting γ_1 and γ_2 . So we have

$$N_t(c'_L) = N_t(\gamma_1) + N_t(\gamma_2) = N_t(c_L). \tag{19.3.3}$$

In other words, N_t is a constant in \mathcal{D}_L^α .

Case 3: \mathcal{D}_L^α is in class U.

It remains to show that N_u is a constant in \mathcal{D}_L^α . The argument is similar to case 2.

Case 4: \mathcal{D}_L^α is in class E.

Suppose c_L, c'_L are two configurations in $\mathcal{D}_L^\alpha \cap E_{st}$. It remains to show that $N_t(c_L) = N_t(c'_L)$. Analogously to case 2, there exists a path γ_1 satisfying the condition (19.3.2), and it suffices to show that $N_t(\gamma_1) = 0$. Here it is different from case 2 because $\gamma_1(s)$ may go through the other subclasses of the class E, and the curves of z, \bar{z} may touch the interval $(-\infty, 0)$. In class E, the curves $z(s), \bar{z}(s)$ touch the interval $(-\infty, 0)$ only when $\gamma(s)$ enters the subclass E_{su} . However, $\gamma(s)$, which starts from E_{st} , must go through E_{stu} before entering E_{su} . When $\gamma(s)$ leaves E_{su} , it must go through E_{stu} again. Since in E_{stu} , the variables z, \bar{z} are complex conjugate to each other, the curves of z, \bar{z} must cross $(-\infty, 0)$ from opposite directions, e.g. see figure 19.3.2.

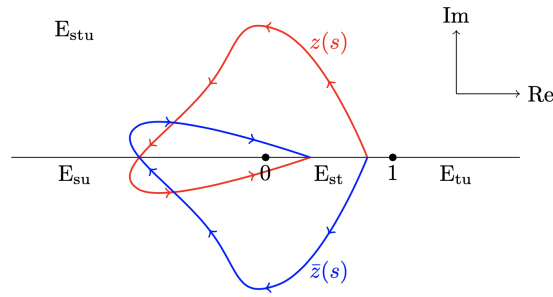


Figure 19.3.2: An example of $z(s), \bar{z}(s)$ along γ_1 in case 4.

So we get

$$n_t(\gamma_1) = -\bar{n}_t(\gamma_1), \tag{19.3.4}$$

which implies $N_t(\gamma_1) = 0$, hence $N_t(c_L) = N_t(c'_L)$.

The arguments for $\mathcal{D}_L^\alpha \cap E_{su}$, $\mathcal{D}_L^\alpha \cap E_{tu}$ and $\mathcal{D}_L^\alpha \cap E_{stu}$ are similar. \square

An immediate consequence of theorem 19.3.2 is that for a fixed causal ordering (say \mathcal{D}_L^α), each blank space in table 19.1 satisfy the all-or-none law: either check mark for all configurations in \mathcal{D}_L^α or cross mark for all configurations in \mathcal{D}_L^α . Therefore, if \mathcal{D}_L^α is in class S or T or U, then all its configurations have the same OPE convergence properties; if \mathcal{D}_L^α is in class E, then all its configurations in the same subclass have the same OPE convergence properties.²

19.3.2 Comments on the 2d case

In 2d unitary local CFTs, we have Al. Zamolodchikov's uniformizing variables q, \bar{q} [123]. The function g in eq. (11.0.1) has a convergent expansion in terms of Virasoro blocks, and Virasoro blocks have convergent series expansions in q, \bar{q} if $0 < |q|, |\bar{q}| < 1$, which includes the configurations with $0 \leq |\rho|, |\bar{\rho}| \leq 1$ except for ρ or $\bar{\rho} = \pm 1$. However, ρ or $\bar{\rho} = \pm 1$ only happens at light-cone singularities.³ So we conclude that in the Lorentzian signature, the s-channel OPE is always convergent aside from light-cone singularities [45].

The above CFT argument is valid only for 2d unitary local CFTs, where by local we mean there exists a stress tensor $T_{\mu\nu}(x)$, which has the mode expansion in Virasoro generators [5]. There are also non-local CFTs, e.g. the generalized free field theories. These non-local CFTs have only global conformal symmetry, for which we can only use $\rho, \bar{\rho}$ instead of q, \bar{q} .

We claim that the conclusions in theorem 19.3.2 are still true for 2d unitary CFT (here we only assume global conformal symmetry). Unlike the case $d \geq 3$, the sets \mathcal{D}_L^α are usually disconnected in 2d. This is because in 2d, there are two disconnected space-like separations. So we cannot copy the proof of theorem 19.3.2. However, any 2d configuration can be embedded into $d \geq 3$. Since our criteria of OPE convergence properties are based on counting how the analytic continuation curves of z, \bar{z} cross the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, +\infty)$, which is dimension independent, the 2d path gives the same counting of N_t, N_u as in $d \geq 3$. Therefore, theorem 19.3.2 also covers the 2d case.

The only little difference is that in the 2d case, the Lorentzian four-point configurations only have real z, \bar{z} . This follows from (11.3.13) and (11.3.14). So the subclass E_{stu} , where z, \bar{z} are not real, does not exist in 2d.

19.4 Time reversals

In theorem 19.3.2, we have classified the Lorentzian configurations in \mathcal{D}_L into a finite number of cases. For each case, we will have to choose a representative configuration and a path from $\mathcal{D}_E \setminus \Gamma$, then check if conditions of theorem 18.3.1, 18.3.3 and 18.3.4 hold. Actually, there are some further simplifications which will reduce the number of checks to perform. We are going to show that different \mathcal{D}_L^α which are related by time reversals have the same convergent OPE channels.

We define two time reversals:

$$\begin{aligned}\theta_E &: (\epsilon + it, \mathbf{x} + i\mathbf{y}) \mapsto (-\epsilon + it, \mathbf{x} - i\mathbf{y}) \\ \theta_L &: (\epsilon + it, \mathbf{x} + i\mathbf{y}) \mapsto (\epsilon - it, \mathbf{x} - i\mathbf{y})\end{aligned}\tag{19.4.1}$$

They correspond to the time reversals in Euclidean and Lorentzian space. Under time reversals, x_{ij}^2 takes its complex conjugate

$$(\theta_E x_i - \theta_E x_j)^2 = (\theta_L x_i - \theta_L x_j)^2 = [(x_i - x_j)^2]^*\tag{19.4.2}$$

Given a configuration $c = (x_1, x_2, x_3, x_4)$, we define the time reversals of the configuration by (notice the change of order of points in $\theta_E c$)

$$\begin{aligned}\theta_E c &= (\theta_E x_4, \theta_E x_3, \theta_E x_2, \theta_E x_1), \\ \theta_L c &= (\theta_L x_1, \theta_L x_2, \theta_L x_3, \theta_L x_4).\end{aligned}\tag{19.4.3}$$

Then the following properties are easily checked:

²By configurations having the same OPE convergence properties, we mean that in each OPE channel, all or none of these configurations have the convergent expansion for the four-point function.

³If ρ or $\bar{\rho} = 1$, then $v = 0$. If ρ or $\bar{\rho} = -1$, then u or $v = \infty$. Thus, for any configuration with ρ or $\bar{\rho} = \pm 1$, there exists at least one x_i, x_j pair such that $(x_i - x_j)^2 = 0$.

- The sets \mathcal{T}_4 , \mathcal{D}_E and \mathcal{D}_L are preserved by θ_E and θ_L .
- Under the transformation $c \mapsto \theta_E c$ or $c \mapsto \theta_L c$, the conformal invariants $u, v, z, \bar{z}, \rho, \bar{\rho}$ become their complex conjugates.

Suppose we have a path γ from $\mathcal{D}_E \setminus \Gamma$ to \mathcal{D}_L . Then $\theta_E \gamma$ and $\theta_L \gamma$ are still paths from $\mathcal{D}_E \setminus \Gamma$ to \mathcal{D}_L . The curves of z, \bar{z} are reflected with respect to the real axis, which implies

$$\begin{aligned} N_t(\theta_E \gamma), N_t(\theta_L \gamma) &= -N_t(\gamma), \\ N_u(\theta_E \gamma), N_u(\theta_L \gamma) &= -N_u(\gamma). \end{aligned} \quad (19.4.4)$$

By theorem 18.3.1, 18.3.3 and 18.3.4, we conclude that

- Different Lorentzian configurations which are related by θ_E, θ_L have the same convergent OPE channels.

By lemma 19.3.1 and theorem 19.3.2, we translate the above results to the level of causal orderings:

- If two different sets $\mathcal{D}_L^\alpha, \mathcal{D}_L^\beta$ are related by θ_E, θ_L , then they belong to the same class (S, T, U, E).
- If two different sets $\mathcal{D}_L^\alpha, \mathcal{D}_L^\beta$ are in class S or T or U and are related by θ_E, θ_L , then they have the same convergent OPE channels.
- If two different sets $\mathcal{D}_L^\alpha, \mathcal{D}_L^\beta$ are in class E and are related by θ_E, θ_L , then their intersections with each subclass have the same convergent OPE channels.

Given a Lorentzian configuration $c = (x_1, x_2, x_3, x_4)$, θ_E interchanges $x_1 \leftrightarrow x_4$ and $x_2 \leftrightarrow x_3$. At the level of causal orderings, θ_E is the permutation of indices

$$1 \leftrightarrow 4, \quad 2 \leftrightarrow 3, \quad (19.4.5)$$

with all the arrows kept fixed. For example, under θ_E we have

$$1 \longrightarrow 2 \begin{cases} \nearrow 3 \\ \searrow 4 \end{cases} \xrightarrow{\theta_E} 4 \longrightarrow 3 \begin{cases} \nearrow 2 \\ \searrow 1 \end{cases}. \quad (19.4.6)$$

Under θ_L , the Lorentzian configuration $x_k = (it_k, \mathbf{x}_k)$ is mapped to $c' = (x'_1, x'_2, x'_3, x'_4)$ with

$$x'_k = \theta_L x_k = (-it_k, \mathbf{x}_k), \quad k = 1, 2, 3, 4. \quad (19.4.7)$$

So the operator ordering does not change but the causal ordering is reversed. For example, under θ_L we have

$$1 \longrightarrow 2 \begin{cases} \nearrow 3 \\ \searrow 4 \end{cases} \xrightarrow{\theta_L} 1 \longleftarrow 2 \begin{cases} \nwarrow 3 \\ \swarrow 4 \end{cases}. \quad (19.4.8)$$

By definitions (19.4.1) and (19.4.3), we have the following properties for θ_E, θ_L :

$$\theta_E^2 = id, \quad \theta_L^2 = id, \quad \theta_E \theta_L = \theta_L \theta_E. \quad (19.4.9)$$

So the group generated by θ_E, θ_L is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Under the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -actions, the orbit of a given causal ordering contains 1 or 2 or 4 causal orderings. In each orbit, it suffices to check the OPE convergence properties of only one causal ordering and make the same conclusions for other causal orderings. This simplifies our work.

19.5 The table of four-point causal orderings

Given two Lorentzian configurations (x_1, \dots, x_n) and (y_1, \dots, y_n) , we say that they are in the same causal type if there is a permutation $\sigma \in S_n$ such that $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ has the same causal ordering as (y_1, \dots, y_n) or $(\theta_L y_1, \dots, \theta_L y_n)$.

In table 19.2, we give a classification of four-point causal orderings according to the causal types. The vertices labelled by a, b, c, d can be any permutation of 1, 2, 3, 4. In the end we will give one table about OPE convergence properties for each causal type in table 19.2.

Table 19.2: Classification of four-point causal orderings

Type No.	causal ordering	θ_L time reversal
1	$a \longrightarrow b \longrightarrow c \longrightarrow d$	same
2	$a \longrightarrow b \begin{matrix} \nearrow c \\ \searrow d \end{matrix}$	$\begin{matrix} c \searrow \\ d \nearrow \end{matrix} b \longrightarrow a$
3	$a \begin{matrix} \longrightarrow b \longrightarrow c \\ \searrow d \end{matrix}$	$c \longrightarrow b \begin{matrix} \longrightarrow a \\ \nearrow d \end{matrix}$
4	$a \begin{matrix} \nearrow b \\ \searrow c \end{matrix} \begin{matrix} \longrightarrow d \\ \longrightarrow d \end{matrix}$	same
5	$a \begin{matrix} \nearrow b \\ \longrightarrow c \\ \searrow d \end{matrix}$	$\begin{matrix} b \searrow \\ c \longrightarrow \\ d \nearrow \end{matrix} a$
6	$a \longrightarrow b \longrightarrow c$ d	same
7	$a \begin{matrix} \nearrow b \\ \searrow c \end{matrix}$ d	$\begin{matrix} b \searrow \\ c \nearrow \end{matrix} a$ d
8	$a \begin{matrix} \nearrow b \\ \searrow d \end{matrix}$ $c \longrightarrow d$	same
9	$a \longrightarrow b$ c d	same
10	$a \begin{matrix} \nearrow c \\ \longrightarrow d \end{matrix}$ $b \begin{matrix} \longrightarrow c \\ \longrightarrow d \end{matrix}$	same
11	$a \longrightarrow b$ $c \longrightarrow d$	same
12	$a \quad b \quad c \quad d$	same

Each causal type thus represents at most $4! \times 2 = 48$ causal orderings ($4!$ for possible assignments of 1, 2, 3, 4 $\rightarrow a, b, c, d$ and $\times 2$ for two columns). This maximal number is realized in type 3, while there are less causal orderings in other types because of symmetries of the graphs.⁴

⁴It may happen that the causal ordering of a configuration does not change under time reversals θ_E, θ_L or permutations of the points.

often second column is equivalent to the first and because of little group (see appendix C.2.0.3).

It makes sense to do this grouping of causal orderings into causal types for two reasons:

- causal orderings related by θ_E and θ_L action (and which thus have same OPE convergence properties) belong to the same causal type.
- if we know class/subclass of \mathcal{D}_L^α for one α in a given causal type, it is easy to determine the class/subclass of any other \mathcal{D}_L^α in the same causal type (see appendix C.2.0.1).

19.6 Examples

The tables which classify the OPE convergence properties will be particularly large, we leave them in appendix C.2. Readers can pick the cases they are interested in. To make it easy for readers to check, we also share the Mathematica code which contain the OPE convergence results of all causal orderings, see the auxiliary file on the arXiv webpage of [74]. In this section we only give some examples.

The Lorentzian four-point correlation functions defined in eq.(9.0.8) are either time-ordered ($t_1 > t_2 > t_4 > t_4$) or out-of-time-order (not $t_1 > t_2 > t_4 > t_4$). The time-ordered correlators have applications in scattering theories [127, 128], and the out-of-time-order correlation functions have applications in the study of many-body systems [129–136]. An example in part I shows the existence of out-of-time-order correlators which do not have a convergent OPE channel (see appendix A in [73]).⁵ Our first example is to show that not all time-ordered correlators have a convergent OPE channel.

Then we will discuss two other examples from AdS/CFT. One is the Regge kinematics [46, 47], the other is related to the bulk-point singularities [45].

19.6.1 A time-orderd correlation function

Let us consider the following two-dimensional configuration:

$$x_1 = (0, 0), \quad x_2 = (-0.1i, 1), \quad x_3 = (-2i, -1.5), \quad x_4 = (-2.1i, 1.5). \quad (19.6.1)$$

The four-point function $G_4^L(x_1, x_2, x_3, x_4)$ at the configuration (19.6.1) is time-ordered. The causal ordering of (19.6.1) is given by

$$\begin{array}{c}
 & & & & 2 \\
 & & & \nearrow & \\
 4 & & & & \\
 & & & \searrow & \\
 & & & & 1 \\
 & & & \nearrow & \\
 3 & & & &
 \end{array} , \quad (19.6.2)$$

which is of causal type 8 in table 19.2. A quick way to know the OPE convergence property is to look up the table of OPE convergence in appendix C.2.8. The causal ordering (19.6.2) corresponds to the label “(4231)” in table C.10. We see from table C.10 that there is no convergent OPE channel for this causal ordering.

Let us also choose a start point in $\mathcal{D}_E \setminus \Gamma$ and a path to compute the z, \bar{z} -curves, and directly check the OPE convergence properties. Figure 19.6.1 shows the z, \bar{z} -curves along the path.⁶

A typical example is the configuration whose points are totally space-like separated.

⁵By “a configuration do not have a convergent OPE channel” we mean the configuration do not satisfy the conditions of theorem 18.3.1, 18.3.3 and 18.3.4.

⁶We choose the start point $x_1^E = (0, -0.8)$, $x_2^E = (-1, -0.2)$, $x_3^E = (-2, -0.6)$ and $x_4^E = (-3, -0.3)$. The path is given by the straight line.

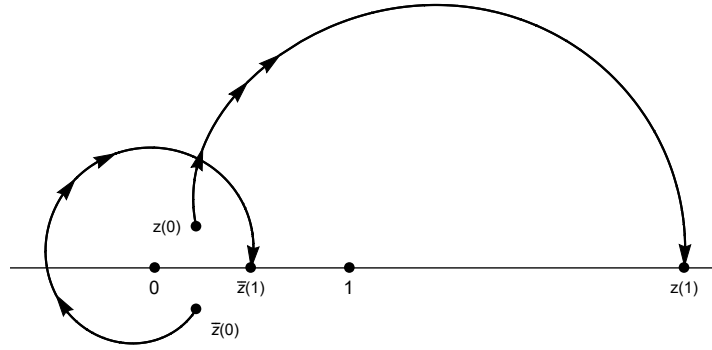


Figure 19.6.1: z, \bar{z} -curves of the configuration (19.6.1).

We see from figure 19.6.1 that $z > 1$, $0 < \bar{z} < 1$ at the final point, which implies that the configuration (19.6.1) is in class T. The curve of z variable crosses $(-\infty, 0)$ from below, which gives $N_t = -1$. So the t-channel OPE (the only undetermined case by table 19.1) is not convergent. Thus, as already mentioned, there is no convergent OPE channel for the four-point function at the configuration (19.6.1).

This example shows that not all time-ordered correlation functions have a convergent OPE channel.

19.6.2 Regge kinematics

The second example is the Lorentzian four-point function in the Regge regime [46–48]. Let x_1, x_4 and x_2, x_3 pairs be time-like separated, while other pairs be space-like separated (see figure 19.6.2).

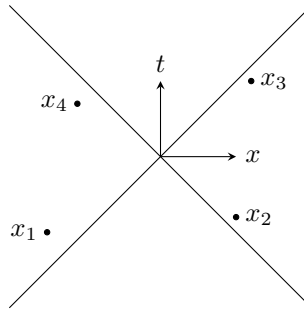


Figure 19.6.2: Regge kinematics.

It is well known that the four-point function at Regge kinematics only has convergent t-channel expansion [46]. Here we just review this result. The causal ordering of the Regge kinematics is given by

$$\begin{aligned} 1 &\longrightarrow 4 \\ 2 &\longrightarrow 3 \end{aligned} \tag{19.6.3}$$

The Regge kinematics belongs to causal type 11 in table 19.2. Let us look up this causal ordering in appendix C.2.11. The causal ordering (19.6.3) corresponds to the label “(1423)” in table C.13. We see that only t-channel OPE is convergent.

We would like to also choose a representative configuration and a path to compute the curves of z, \bar{z} . The plot is given by figure 19.6.3.⁷

⁷We choose the Euclidean configuration $x_1 = 0$, $x_2 = (-1, 0, 0, 0)$, $x_3 = (-2, 0.9, 0, 0)$, $x_4 = (-4, 0, 0, 0)$ and the representative Lorentzian configuration $y_1 = 0$, $y_2 = (0, 0, 0.6, 0)$, $y_3 = (2i, 0, 0, 0.7)$, $y_4 = (2i, -0.05, 0, -3)$. We choose the path to be the straight line between them.

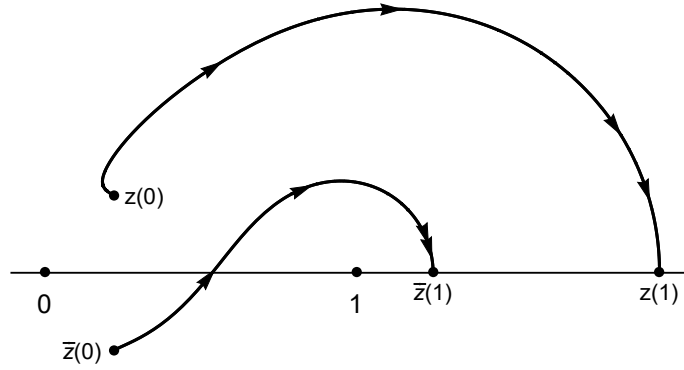


Figure 19.6.3: The plot of z, \bar{z} -curves of the Regge kinematics.

We see from figure 19.6.3 that $z, \bar{z} > 1$ at the final point,⁸ which implies that the Regge kinematics is in class E. In fact the Regge kinematics can only be in the subclass E_{tu} , where $z, \bar{z} > 1$ [46], so only t- and u-channel expansions have a chance to converge. We see from figure 19.6.3 that the \bar{z} -curve crosses $(0, 1)$ from below, and z, \bar{z} -curves do not cross $(-\infty, 0)$. So we get

$$N_t = 0, \quad N_u = -1, \tag{19.6.4}$$

which implies that only the t-channel expansion is convergent.

19.6.3 Causal ordering of bulk-point singularities

The third example is as follows. Let x_1, x_2 and x_3, x_4 pairs be space-like separated. We put the x_1, x_2 pair in the open backward light-cone of some base point and x_3, x_4 pair in the open forward light-cone of the base point (see figure 19.6.4).

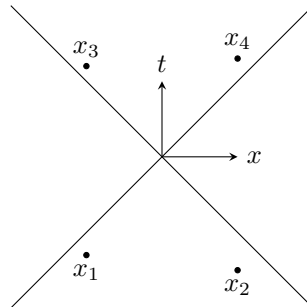


Figure 19.6.4: Configuration of example 3.

Such configurations have the causal ordering

$$\begin{array}{c}
 1 \\
 \swarrow \quad \searrow \\
 \quad 3 \quad \rightarrow 4 \\
 \nwarrow \quad \nearrow \\
 2
 \end{array}
 . \tag{19.6.5}$$

The causal ordering (19.6.5) is of causal type 10 in table 19.2. We look up the OPE convergence properties in appendix C.2.10. The causal ordering (19.6.5) corresponds to the label “(1234)” in table C.12. We see that this causal ordering is in class E, which has four subclasses. From table C.12 we also see that the configurations with the

⁸The definition of z, \bar{z} in [46] is different from this part of the thesis. In their work, $0 < z, \bar{z} < 1$ at Regge kinematics, while in this part, $z, \bar{z} > 1$. One can compare the definitions and get the relation of z, \bar{z} between [46] and our work: $z \rightarrow 1/z, \bar{z} \rightarrow 1/\bar{z}$.

causal ordering (19.6.5) exist in each subclass. We wish to consider the subclass E_{ut} , where $z, \bar{z} > 1$. In table C.12, we see that the configurations with causal ordering (19.6.5) and in subclass E_{ut} have no convergent OPE channels.

Let us also choose a representative configuration to check this result. We want to remark that such case does not exist in 2d (see appendix C.2.10 for the proof). We choose the following three-dimensional configuration

$$x_1 = (0, 0, 0), \quad x_2 = (0, 1, 0), \quad x_3 = (i, 0.2, 0.5), \quad x_4 = (i, 0.5, 0.8). \quad (19.6.6)$$

Figure 19.6.5 shows the plot of z, \bar{z} -curves.

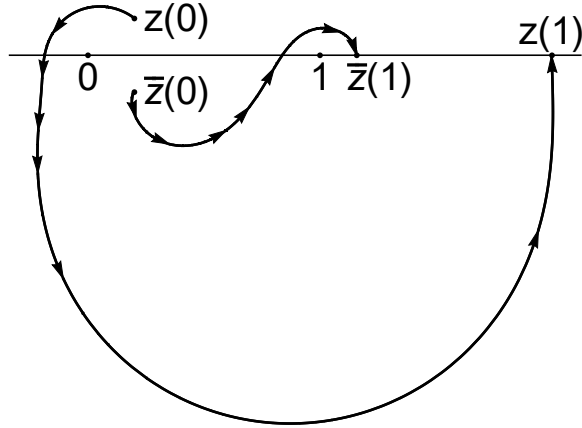


Figure 19.6.5: The plot of z, \bar{z} -curves of the configuration (19.6.6).

We see that along the path, z crosses the interval $(-\infty, 0)$ and \bar{z} crosses the interval $(0, 1)$. We get

$$N_t = 1, \quad N_u = -1. \quad (19.6.7)$$

which implies that the t- and u-channel expansions do not converge.

We conclude that there is no convergent OPE channel for the causal ordering (19.6.5) with $z, \bar{z} > 1$.

Here we give a hint why this example is related to the bulk-point singularities in AdS/CFT [45]. The bulk-point singularities are not exactly the configurations in Minkowski space $\mathbb{R}^{d-1,1}$, instead they are configurations on the Minkowski cylinder $\mathbb{R} \times S^{d-1}$ [25]. The Minkowski space can be embedded into a patch of the Minkowski cylinder in a Weyl equivalent way, this patch is called the Poincaré patch [30]. The Minkowski cylinder also admits a causal ordering which is equivalent to the causal ordering of the Minkowski space in the Poincaré patch [137, 138]. One can show the following facts:

- The bulk-point singularities have the causal ordering (19.6.5) and $z, \bar{z} > 1$.
- One can find a path from an arbitrary bulk-point singularity to a configuration in the Poincaré patch, such that the path preserves the causal ordering (19.6.5) and does not touch the light-cone singularities in the Minkowski cylinder.
- The CFT four-point function in the Poincaré patch is the same as the CFT four-point function in the Minkowski space up to a scaling factor.⁹

Based on the above facts, the OPE convergence properties of the bulk-point singularities are exactly the same as this example: there is no convergent OPE channel. We will revisit this case in section 23.6, where the similar classification results will be given for CFT four-point functions in the Minkowski cylinder. Our result does not contradict the two-dimensional result in [45] (see the beginning of section 19.3.2) because here we only use the global conformal symmetry instead of the Virasoro symmetry.

⁹The definition of the CFT four-point function on the Minkowski cylinder is similar to Minkowski space. We replace the planar time variables τ_k by the cylindrical time variables. Then do Wick rotations.

Chapter 20

Generalization to non-identical scalar or spinning operators

20.1 Generalization to the case of non-identical scalar operators

In this section we show that the arguments in section 18 and 19 are valid for CFT four-point functions of non-identical scalar operators. We start from a Euclidean CFT four-point function

$$G_{1234}(c) = \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle, \quad c = (x_1, x_2, x_3, x_4). \quad (20.1.1)$$

The scalar operators \mathcal{O}_i in (20.1.1) have scaling dimensions Δ_i . By conformal symmetry, (20.1.1) can be factorized as

$$G_{1234}(c) = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}} (x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}} \left(\frac{x_{24}^2}{x_{14}^2} \right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2} \right)^{\frac{\Delta_3-\Delta_4}{2}} g_{1234}(\rho, \bar{\rho}) \quad (20.1.2)$$

As we discussed in chapter 10, the prefactor multiplying $g_{1234}(\rho, \bar{\rho})$ has analytic continuation to the forward tube $\overline{\mathcal{T}}_4$. In section 4.1 and 4.2, we showed that $g_{1234}(\rho, \bar{\rho})$ is an analytic function (with branch cut) on the polydisc $|\rho|, |\bar{\rho}| < 1$. Using the same argument as in section 11.3, we composite $g_{1234}(\rho, \bar{\rho})$ and $(\rho(c), \bar{\rho}(c))$. This procedure performs the analytic continuation of $g_{1234}(c)$, as a function of x_k 's, to $\overline{\mathcal{T}}_4$.

The remaining steps are the same as the case of identical scalar operators. Since our criteria of OPE convergence only concern the properties of cross-ratios, which are purely geometric, we conclude that for the case of four-point functions with non-identical scalar operators, the OPE convergence properties are the same as the case of identical scalar operators.

20.2 Comments on the case of spinning operators

Before finishing this section we want to make some comments on the case of four-point functions with spinning operators:

$$G_{1234}^{a_1 a_2 a_3 a_4}(c) = \langle \mathcal{O}_1^{a_1}(x_1) \mathcal{O}_2^{a_2}(x_2) \mathcal{O}_3^{a_3}(x_3) \mathcal{O}_4^{a_4}(x_4) \rangle, \quad c = (x_1, x_2, x_3, x_4). \quad (20.2.1)$$

where $\mathcal{O}_i^{a_i}$ are primary operators with scaling dimensions Δ_i and $SO(d)$ -representation ρ_i . a_i are the indices for the spin representations ρ_i . In the Euclidean signature, the Jacobian of any conformal transformation f in $SO(1, d+1)$ can be factorized as

$$J_\nu^\mu(x) := \frac{\partial f^\mu(x)}{\partial x^\nu} = \Omega(x) \mathcal{R}_\nu^\mu(x), \quad (20.2.2)$$

where $\Omega(x) > 0$ is a scaling factor and \mathcal{R} is a rotation matrix. The four-point function $G_{1234}^{a_1 a_2 a_3 a_4}$ is invariant if we replace all $\mathcal{O}_i^{a_i}(x)$ in (20.2.1) with

$$\mathcal{O}_i^{a_i}(x) \rightarrow \Omega(x)^{\Delta_i} [\rho_i (\mathcal{R}(x)^{-1})]_{b_i}^{a_i} \mathcal{O}_i^{b_i}(f(x)). \quad (20.2.3)$$

If we choose the conformal transformation f to be the one which maps (x_1, x_2, x_3, x_4) to its $\rho, \bar{\rho}$ -configuration $(x'_1 = \rho, x'_2 = -\rho, x'_3 = -1, x'_4 = 1$ in the (01)-plane), then by conformal invariance we get

$$G_{1234}^{a_1 a_2 a_3 a_4}(c) = \frac{1}{(x_{12}^2)^{\frac{\Delta_1 + \Delta_2}{2}} (x_{34}^2)^{\frac{\Delta_3 + \Delta_4}{2}}} \left(\frac{x_{24}^2}{x_{14}^2} \right)^{\frac{\Delta_1 - \Delta_2}{2}} \left(\frac{x_{14}^2}{x_{13}^2} \right)^{\frac{\Delta_3 - \Delta_4}{2}} \times T_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4}(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4) g_{1234}^{b_1 b_2 b_3 b_4}(\rho, \bar{\rho}) \quad (20.2.4)$$

where \mathcal{R}_k is the rotation matrix of f at x_k , and T is a function of rotation matrices, which is determined by the representations ρ_i of the spinning operators \mathcal{O}_i .

The analytic continuation of the x_{ij}^2 prefactor is trivial. Analogously to the scalar case, the function $g_{1234}^{b_1 b_2 b_3 b_4}(\rho, \bar{\rho})$ has series expansion (11.2.6), which is convergent in the polydisc $|\rho|, |\bar{\rho}| < 1$. Then $g_{1234}^{b_1 b_2 b_3 b_4}(\rho(c), \bar{\rho}(c))$, as a function of the four-point configuration c , has analytic continuation to the forward tube \mathcal{T}_4 .

The main difficulty is that there is the function $T(\mathcal{R}_k)$ in (20.2.4) because of the non-trivial representations of the spinning operators. If we think of the above conformal transformation f as a conformal-group-valued function of (x_1, x_2, x_3, x_4) ,¹ then \mathcal{R}_k are also functions of (x_1, x_2, x_3, x_4) . In general, the entries of \mathcal{R}_k have singularities in \mathcal{T}_4 , e.g. at configurations where $z = \bar{z}$ [52, 89]. In a word, it is not obvious that $T(\mathcal{R}_k)$ in (20.2.4) is under control. Some extra work is required for a good estimate on the object

$$T_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4}(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4) g_{1234}^{b_1 b_2 b_3 b_4}(\rho, \bar{\rho})$$

In this part of the thesis we do not study the correlators of spinning operators. In chapter 22 we will justify the analytic continuation of the spinning four-point functions for almost all four-point functions of spinning operators. Here we say ‘‘almost all’’ because there will be a very small set of exceptional cases that we do not know how to deal with, see section 22.3.3. For the spinning cases where we manage to justify the analytic continuation, the classification results will be the same as the scalar case. We leave this part of the generalization for future work.

¹Since such a conformal transformation f is not unique, the definition of this group-valued function depends on how we construct it.

Chapter 21

Conclusions and outlooks

In this part of the thesis we studied the convergence properties of various OPE channels for Lorentzian CFT four-point functions of scalar operators in $d \geq 2$, assuming global conformal symmetry. Our analysis is based on the convergence properties of OPE in the Euclidean unitary CFTs. We classified the Lorentzian four-point configurations according to their causal orderings and the range of the variables z, \bar{z} . The Lorentzian correlators are analytic functions in a neighbourhood of some Lorentzian configuration as long as there exists at least one convergent OPE channel in the sense of functions. We showed that the convergence properties of various OPE channels are fully determined by the causal orderings and the range of z, \bar{z} of the four-point configurations. The CFT four-point functions are analytic in a very big domain, including configurations with totally space-like separations and configurations with some other causal orderings. All the results of OPE convergence properties are given in Appendix C.2.

Before ending, we would like to point out some related open questions.

1. We mainly used the radial coordinates $\rho, \bar{\rho}$ in our analysis. We have seen that by using the q, \bar{q} -variables in 2d, the domain of CFT four-point functions are larger than the domain derived by using the $\rho, \bar{\rho}$ -variables. A natural question is
 - For CFTs with only global conformal symmetry, are there any other coordinates which allow us to extend G_4 to some other domains which are not covered by using radial variables?

Our conjecture is that there are no such coordinates.

2. Our results provide some safe Lorentzian regions where conformal bootstrap approach can be applied. One can use bootstrap equations to analyze the four-point functions at Lorentzian configurations with at least two convergent OPE channels. It is also interesting to play with crossing symmetry at Lorentzian configurations with
 - One convergent OPE channel in the sense of functions, another one in the sense of distributions.
 - Two convergent OPE channels in the sense of distributions.¹

The above situations are closely related to the topics on analytic functional bootstrap when the functionals touch the boundaries of the regions with convergent OPE [32–36, 38, 39].

3. There are also Lorentzian configurations with no convergent OPE channels. For these cases we do not know whether the general four-point correlators are genuine functions or not. We may need other techniques to handle these situations. For example, there are questions similar to appendix C.3.2.4:
 - One can derive a complex domain \mathcal{D}^{stu} which is the union of the domains of three OPE channels. Then what is \mathcal{D}^{stu} and what is its envelope of holomorphy $H(\mathcal{D}^{stu})$? Does $H(\mathcal{D}^{stu})$ contain more Lorentzian configurations than those provided by the results in this part of the thesis?

¹The similar idea was proposed recently by Gillioz et al, see [139], section 5.

Once we are able to construct $H(\mathcal{D}^{stu})$, one can ask

- Given a Lorentzian configuration $c_L \in \mathcal{D} \setminus H(\mathcal{D}^{stu})$, can we find a CFT example such that the four-point function is divergent at c_L ?
4. One can also consider higher-point correlation functions in CFTs. A natural question is:
- For $n \geq 5$, what is the Lorentzian domain of G_n in the sense of functions?

Part IV

Preview on some generalizations

Chapter 22

Spinning four-point functions

This chapter is aiming for the generalization to four-point functions of general bosonic operators, i.e. operators with $SO(d)$ -spins (fermionic operators are excluded since they carry half-integer $Spin(d)$ -spins). We only focus on the temperedness property in the Lorentzian signature, because the derivation of the other Wightman axioms are similar to the scalar case. According to the strategy of the scalar case (see chapter 9), proving temperedness consists of two steps: analytically continuing the correlation function to the forward tube and proving the power-law bound.

Let \mathcal{O}^a denote a bosonic primary operator with $SO(d)$ -spin indices a . Here we use the tensor indices $a = (\mu_1, \dots, \mu_l)$, as described in section 7.2. A general Euclidean CFT four-point function of bosonic primary operators has partial wave expansion:

$$G_{1234}^{a_1 a_2 a_3 a_4}(c_E) := \langle \mathcal{O}_1^{a_1}(x_1) \mathcal{O}_2^{a_2}(x_2) \mathcal{O}_3^{a_3}(x_3) \mathcal{O}_4^{a_4}(x_4) \rangle = \sum_{\mathcal{O}} G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}(c_E), \quad (22.0.1)$$

where $c_E = (x_1, x_2, x_3, x_4)$ is the Euclidean configuration and the sum over the primary operators that appear in the OPE. According to the OPE convergence assumption in the Euclidean CFT axioms, the expansion (22.0.1) is convergent as long as there exists a sphere S^{d-1} which separates the x_1, x_2 pair from the x_3, x_4 pair, e.g. see figure 22.0.1.

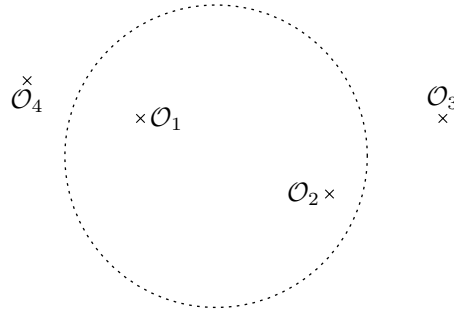


Figure 22.0.1: An example of a sphere separating $\mathcal{O}_1, \mathcal{O}_2$ from $\mathcal{O}_3, \mathcal{O}_4$.

Consider the configurations with complex coordinates: $x_k = (\varepsilon_k + it_k, \mathbf{x}_k + i\mathbf{y}_k)$. We would like to analytically continue the four-point function to the forward tube \mathcal{T}_4 (i.e. $\varepsilon_k - \varepsilon_{k+1} > |\mathbf{y}_k - \mathbf{y}_{k+1}|$, $k = 1, 2, 3$). The analytic continuation and the power-law bound for the scalar case were done in section 11.3 and 11.5. We would like to briefly review the key strategies in the scalar case, and then see the extra subtleties that arise in the case of spinning operators.

22.1 Revisit the scalar case

Given any configuration $c = (x_1, x_2, x_3, x_4)$ in \mathcal{T}_4 , we can always find a conformal transformation f which maps c to the “ ρ -configuration” $c_\rho = (x'_1, x'_2, x'_3, x'_4)$ with

$$x'_1 = -x'_2 = \left(\frac{\rho + \bar{\rho}}{2}, \frac{\rho - \bar{\rho}}{2i}, 0, \dots, 0 \right), \quad x'_3 = -x'_4 = (-1, 0, \dots, 0). \quad (22.1.1)$$

For scalar 4pt functions, conformal invariance implies

$$G_{1234}(c) = \Omega_f(x_1)^{\Delta_1} \Omega_f(x_2)^{\Delta_2} \Omega_f(x_3)^{\Delta_3} \Omega_f(x_4)^{\Delta_4} G_{1234}(c_\rho), \quad (22.1.2)$$

where $\Omega_f(x) = \det(\partial f^\mu / \partial x^\nu)^{1/d}$. This formula is just a writting of eq. (11.10.1) in section 11.10. These two equations are matched by

$$G_{1234}(c_\rho) = \frac{1}{2^{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4} (\rho \bar{\rho})^{\frac{\Delta_1 + \Delta_2}{2}}} \left(\frac{(1 + \rho)(1 + \bar{\rho})}{(1 - \rho)(1 - \bar{\rho})} \right)^{\frac{\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4}{2}} g_{1234}(c_E). \quad (22.1.3)$$

In section 11.10, the analytic continuation of $G_{1234}(c)$ relies on the fact that both the x_{ij}^2 -prefactor and $g_{1234}(c)$ has the analytic continuation as a function of c . We would like to reinterpret the analytic continuation procedure, using the following facts: under the constraints $f(x_k) = x'_k$ ($k = 1, 2, 3, 4$), the scale factors $\Omega_f(x_k)$ do not depend on the choice of f and they are analytic in Cartesian coordinates of c . To see this we use the formula

$$x'_{ij}{}^2 = \Omega_f(x_i) \Omega_f(x_j) x_{ij}^2, \quad (22.1.4)$$

which implies

$$\begin{aligned} \Omega_f(x_1)^2 &= \frac{x_{34}^2}{x_{13}^2 x_{14}^2} \times \frac{(1 - \rho^2)(1 - \bar{\rho}^2)}{4}, \\ \Omega_f(x_2)^2 &= \frac{x_{34}^2}{x_{23}^2 x_{24}^2} \times \frac{(1 - \rho^2)(1 - \bar{\rho}^2)}{4}, \\ \Omega_f(x_3)^2 &= \frac{x_{12}^2}{x_{13}^2 x_{23}^2} \times \frac{(1 - \rho^2)(1 - \bar{\rho}^2)}{4\rho\bar{\rho}}, \\ \Omega_f(x_4)^2 &= \frac{x_{12}^2}{x_{14}^2 x_{24}^2} \times \frac{(1 - \rho^2)(1 - \bar{\rho}^2)}{4\rho\bar{\rho}}. \end{aligned} \quad (22.1.5)$$

We see from (22.1.5) that $\Omega(x_k)$'s only depend on c and they are analytic functions on \mathcal{T}_4 .¹ So we will use the notation

$$\Omega_k(c) := \Omega_f(x_k). \quad (22.1.6)$$

On the other hand, $G_{1234}(c_\rho)$ has convergent series expansion in $\rho, \bar{\rho}$, which defines an analytic function on \mathcal{T}_4 . Therefore, the RHS of (22.1.2) is a product of five analytic functions on \mathcal{T}_4 , so it performs the analytic continuation of G_{1234} to \mathcal{T}_4 .

Furthermore, we can show that $\Omega_k(c)$ and $G_{1234}(c_\rho)$ have power-law bounds when $c \in \mathcal{T}_4$. So $G_{1234}(c)$, as a product of these factors, also has a power-law bound on \mathcal{T}_4 . Then by Vladimirov's theorem (theorem 9.0.1), G_{1234} is a tempered distribution in the Lorentzian regime.

22.2 Subtlety in the spinning case, strategy

Now let us turn to the spinning case. The spinning-field analog of eq. (22.1.2) is given by

$$G_{1234}^{a_1 a_2 a_3 a_4}(c) = \left(\prod_{i=1}^4 \Omega_i(c)^{\Delta_i} \rho_i (R_f(x_i)^{-1})^{a_i}{}_{b_i} \right) G_{1234}^{b_1 b_2 b_3 b_4}(c_\rho), \quad (22.2.1)$$

¹The argument for analyticity is as follows. First, it is obvious that the x_{ij}^2 factors are analytic functions on \mathcal{T}_4 . The analyticity of the $\rho, \bar{\rho}$ factors follows from the facts that $\rho, \bar{\rho}$ always appear symmetrical and $0 < |\rho|, |\bar{\rho}| < 1$ on \mathcal{T}_4 .

where $\Omega_i(c)$ is the scale factor and $(R_f)^\mu_\nu(x) := \Omega_f(x_i)^{-1} \partial f^\mu / \partial x^\nu$ is the rotation matrix. For $a_i = (\mu_1, \dots, \mu_{l_i})$ and $b_i = (\nu_1, \dots, \nu_{l_i})$, the matrix $\rho_i(R)^{a_i}_{b_i}$ is the tensor product of rotation matrices

$$\rho_i(R)^{a_i}_{b_i} = R^{\mu_1}_{\nu_1} R^{\mu_2}_{\nu_2} \dots R^{\mu_{l_i}}_{\nu_{l_i}}. \quad (22.2.2)$$

As reviewed in the previous section, $\Omega_k(c)^{\Delta_k}$ and $G_{1234}^{b_1 b_2 b_3 b_4}(c_\rho(c))$ in eq. (22.2.1) are analytic functions on \mathcal{T}_4 . However, in $d \geq 3$, the rotation matrices $R_f(x_i)$'s are not analytic in c , mainly because the conformal transformation f has $\text{SO}(d-2)$ or $\text{SO}(d-1)$ ambiguity.² So we do not know how to perform the analytic continuation using the RHS of eq. (22.2.1).

In $d = 2$, both $\text{SO}(d-2)$ and $\text{SO}(d-1)$ are trivial groups. So given a four-point configuration c , there is a unique conformal transformation f which maps c to its corresponding ρ -configuration c_ρ . Using proper coordinates to characterize f , one can show that f is an analytic function of c in the forward tube, so the rotation matrices induced by f are also analytic functions of c . Then we are able to use eq. (22.2.1) to construct the analytic continuation of the four-point function, and show its power-law bound in the two-dimensional case (see appendix D.1 for details).

In summary, eq. (22.2.1) is still available for the analytic continuation in $d = 2$, but no longer useful in $d \geq 3$. For the analytic continuation in $d \geq 3$, we would like to take a different strategy:

Step 1. Show directly that each conformal partial wave $G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}(c)$ on the RHS of (22.0.1) has analytic continuation to \mathcal{T}_4 .

Step 2. Show that the partial wave expansion (22.0.1) is convergent in $\mathcal{D}_4 \subset \mathcal{T}_4$. Recall that \mathcal{D}_4 is defined to be the \mathcal{T}_4 configurations with real spatial components.³

The main point here is that the non-analyticity of the rotation matrices are packaged into the conformal partial waves, so we can just bypass this difficulty. We will see that once step 1 is done, we can reduce step 2 to the case of $d = 2$, where everything is already well-established.

22.3 Step 1: analytic continuation of the conformal partial wave

We would like to use the following integral representation to perform the analytic continuation of the conformal partial wave in x_i 's, inspired by Mack ([26], eq. (8.7)):

$$G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}(c) = \int \frac{d^d p}{(2\pi)^d} (\widehat{G}_{\mathcal{O}})_{ab}(p) \widehat{B}_{\mathcal{O}, 12}^{a_1 a_2}(p, x_1, x_2) \overline{\widehat{B}_{\mathcal{O}, \overline{43}}^{a_4 a_3}(p, x_4, x_3)}. \quad (22.3.1)$$

Here $(\widehat{G}_{\mathcal{O}})_{ab}(p)$ is the Fourier transform of the two-point function

$$(\widehat{G}_{\mathcal{O}})_{ab}(p) = \int d^d x \langle \mathcal{O}_a^\dagger(0) \mathcal{O}_b(x) \rangle e^{ip \cdot x}, \quad (x = (it, \mathbf{x}), p = (E, \mathbf{p}), x \cdot p = -Et + \mathbf{p} \cdot \mathbf{x},) \quad (22.3.2)$$

the function $\widehat{B}_{\mathcal{O}, ij}^{a_i a_j}(p, x_i, x_j)$ is the OPE kernel in the momentum space:⁴

$$[\mathcal{O}_i(x_i) \mathcal{O}_j(x_j)]_{\mathcal{O}} = \int \frac{d^d p}{(2\pi)^d} \widehat{B}_{\mathcal{O}, ij}^{a_i a_j}(p, x_i, x_j) \widehat{\mathcal{O}}_a^\dagger(-p), \quad (22.3.4)$$

²By eq. (22.1.1), c_ρ stays in the two-dimensional plane when $\rho \neq \bar{\rho}$, or in the one-dimensional line when $\rho = \bar{\rho}$. It is invariant under rotations g of the other $d-2$ or $d-1$ dimensions. Therefore, any such $g \circ f$ maps c to c_ρ .

³We consider \mathcal{D}_4 instead of \mathcal{T}_4 because the reflection-symmetric configurations in \mathcal{D}_4 can be mapped into the two-dimensional subspace in a nice way, which will be useful to prove convergence of the partial wave expansion (22.0.1).

⁴In [26], Mack formulated the operator product expansion in the Lorentzian position space:

$$\mathcal{O}_i(x_i) \mathcal{O}_j(x_j) |0\rangle = \sum_{\mathcal{O}} \int \frac{d^d x}{(2\pi)^d} B_{\mathcal{O}, ij}^{a_i a_j}(x, x_i, x_j) \mathcal{O}_a^\dagger(x) |0\rangle. \quad (22.3.3)$$

This is the reason why we use the notation \widehat{B} in the momentum space. In fact we have already mentioned it in section 13.3.

and the complex version of the reflection operator θ is defined by

$$x = (\varepsilon + it, \mathbf{x} + i\mathbf{y}) \rightarrow x^\theta = (-\varepsilon + it, \mathbf{x} - i\mathbf{y}). \quad (22.3.5)$$

Regardless of the convergence issue, one can formally check that (22.3.1) should be correct:

$$\begin{aligned} G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}(c) &= \int \frac{d^d p}{(2\pi)^d} \widehat{B}_{\mathcal{O}, 12}^{a_1 a_2}(p, x_1, x_2) \langle \widehat{\mathcal{O}}_a^\dagger(-p) \mathcal{O}_3^{a_3}(x_3) \mathcal{O}_4^{a_4}(x_4) \rangle \quad \text{by (22.3.4)} \\ &= \int \frac{d^d p}{(2\pi)^d} \widehat{B}_{\mathcal{O}, 12}^{a_1 a_2}(p, x_1, x_2) \overline{\langle \widehat{\mathcal{O}}_a(p) \mathcal{O}_3^{\dagger a_3}(x_3^\theta) \mathcal{O}_4^{\dagger a_4}(x_4^\theta) \rangle} \quad \text{by (7.3.7)} \\ &= \int \frac{d^d p d^d q}{(2\pi)^{2d}} \widehat{B}_{\mathcal{O}, 12}^{a_1 a_2}(p, x_1, x_2) \overline{\widehat{B}_{\mathcal{O}, 43}^{b a_4 a_3}(q, x_4^\theta, x_3^\theta) \langle \widehat{\mathcal{O}}_a(p) \widehat{\mathcal{O}}_b^\dagger(-q) \rangle} \quad \text{by (22.3.4)} \\ &= \int \frac{d^d p d^d q}{(2\pi)^{2d}} \widehat{B}_{\mathcal{O}, 12}^{a_1 a_2}(p, x_1, x_2) \overline{\widehat{B}_{\mathcal{O}, 43}^{b a_4 a_3}(q, x_4^\theta, x_3^\theta) \langle \widehat{\mathcal{O}}_a^\dagger(-p) \widehat{\mathcal{O}}_b(q) \rangle} \quad \text{by (7.3.6)} \\ &= \int \frac{d^d p d^d q}{(2\pi)^{2d}} \widehat{B}_{\mathcal{O}, 12}^{a_1 a_2}(p, x_1, x_2) \overline{\widehat{B}_{\mathcal{O}, 43}^{b a_4 a_3}(q, x_4^\theta, x_3^\theta) (\widehat{G}_{\mathcal{O}})_{ab}(q)} (2\pi)^d \delta(p - q) \quad \text{by (22.3.2)} \\ &= \int \frac{d^d p}{(2\pi)^d} (\widehat{G}_{\mathcal{O}})_{ab}(p) \widehat{B}_{\mathcal{O}, 12}^{a_1 a_2}(p, x_1, x_2) \overline{\widehat{B}_{\mathcal{O}, 43}^{b a_4 a_3}(p, x_4^\theta, x_3^\theta)}. \end{aligned} \quad (22.3.6)$$

To make the integral representation (22.3.1) valid in rigorous sense, we would like to show that it can be viewed as an inner product in an L^2 -space, where the inner product is defined by

$$\langle f, g \rangle_{\mathcal{O}} := \int \frac{d^d p}{(2\pi)^d} (\widehat{G}_{\mathcal{O}})_{ab}(p) f^a(p) \overline{g^b(p)}. \quad (22.3.7)$$

In addition, to show that eq. (22.3.1) performs the analytic continuation, one needs to show that (a) the OPE kernel $\widehat{B}(p, x_i, x_j)$ is analytic in x_i and x_j in the regime

$$\varepsilon_i - \varepsilon_j > |\mathbf{y}_i - \mathbf{y}_j|, \quad \varepsilon_j > |\mathbf{y}_j|, \quad (x = (\varepsilon + it, \mathbf{x} + i\mathbf{y})); \quad (22.3.8)$$

(b) $\widehat{B}(p, x_i, x_j)$ and its derivatives in x have sufficiently good upper bound, such that the derivatives commute with the integration. (c) the r.h.s. of eq. (22.3.1) is indeed the conformal partial wave in the Euclidean regime.

Based on the above ideas, we would like to show the followings things:

- The momentum-space two-point function $\widehat{G}_{\mathcal{O}}$ is a good measure.
For \mathcal{O}^a above unitarity bound, i.e. $\Delta \geq \Delta_*(\rho)$, $(\widehat{G}_{\mathcal{O}})_{ab}(p)$ is a positive tempered measure, means that (a) $(\widehat{G}_{\mathcal{O}})_{ab}(p) \xi^a \bar{\eta}^b d^d p$ is a tempered measure for any constant tensors ξ, η , (b) $(\widehat{G}_{\mathcal{O}})_{ab}(p) \xi^a \bar{\xi}^b d^d p$ is a positive tempered measure for any constant tensor ξ . (In fact (b) implies (a).)
- The OPE kernel $\widehat{B}_{\mathcal{O}, 12}$ is well-defined.
Define the Fourier-transform of the three-point function with respect to the third point:

$$\widehat{G}_{12\mathcal{O}}^{a_1 a_2 a}(x_1, x_2, p) = \int d^d x \langle \mathcal{O}_1^{a_1}(x_1) \mathcal{O}_2^{a_2}(x_2) \mathcal{O}^a(it, \mathbf{x}) \rangle e^{ip \cdot x}, \quad (22.3.9)$$

The OPE kernel $\widehat{B}_{\mathcal{O}, 12}$ is defined by the decomposition

$$\widehat{G}_{12\mathcal{O}}^{a_1 a_2 a}(x_1, x_2, p) = \widehat{B}_{\mathcal{O}, 12}^{b a_1 a_2}(p, x_1, x_2) (\widehat{G}_{\mathcal{O}})_b^a(p), \quad (22.3.10)$$

$\widehat{B}_{\mathcal{O}, 12}$ has a crucial property: it is in the form of $(x_{12}^2)^{-\sigma} \times$ entire function of x_1, x_2, p .

- Some upper bounds of the OPE kernel and its derivatives.
For $x_1 = (\varepsilon_1 + it_1, \mathbf{x}_1 + i\mathbf{y}_1), x_2 = (\varepsilon_2 + it_2, \mathbf{x}_2 + i\mathbf{y}_2)$ under conditions $\varepsilon_1 - \varepsilon_2 > |\mathbf{y}_1 - \mathbf{y}_2|, \varepsilon_2 > |\mathbf{y}_2|$, in a small neighbourhood \mathcal{U} of (x_1, x_2) we have the following bound

$$\left| \frac{\partial^{|\alpha|+|\beta|}}{\partial x_2^{(\alpha)} \partial x_{12}^{(\beta)}} \widehat{B}_{\mathcal{O}, 12}^{a_1 a_2}(p, x_1, x_2) \right| \leq K(\mathcal{U}) e^{-E\varepsilon_2 + \mathbf{p} \cdot \mathbf{y}_2} [1 + |p|]^\gamma \quad (22.3.11)$$

for $p = (E, \mathbf{p})$ in the closed forward light cone \overline{V}_+ . Here $K(\mathcal{U}), \gamma < +\infty$.

Let us first see how the above three properties lead to the analytic continuation of the conformal partial wave. By translation invariance, we can always shift the configuration $c = (x_1, x_2, x_3, x_4)$ from the forward tube to the regime where

$$\begin{aligned} \varepsilon_1 - \varepsilon_2 &> |\mathbf{y}_1 - \mathbf{y}_2|, & \varepsilon_2 &> |\mathbf{y}_2|, \\ \varepsilon_3 - \varepsilon_4 &> |\mathbf{y}_3 - \mathbf{y}_4|, & \varepsilon_3 &< -|\mathbf{y}_3|, \end{aligned} \quad (22.3.12)$$

so the OPE kernels $\widehat{B}_{\mathcal{O},12}^{a_1 a_2}(p, x_1, x_2)$ and $\widehat{B}_{\mathcal{O},43}^{b_1 b_2}(p, x_4, x_3)$ satisfy the condition for the upper bound (22.3.11).

By Vladimirov's theorem and Lorentz invariance, we know that $\widehat{G}_{ab}(p)$ is a tempered distribution which is supported in the forward light-cone. Now since $\widehat{G}_{ab}(p)$ is also a measure, together with the exponential-decay bounds (22.3.11), we see that the r.h.s. of eq. (22.3.1) is an integral of a tempered measure against two exponential-decay functions over the forward light-cone, so it is convergent. Then since the derivatives of the OPE kernels also satisfy the exponential-decay bounds, uniform in a small neighbourhood of the configuration c , we can interchange the derivatives (in x_i 's) and the integration (in p) according to Lebesgue's dominated convergence theorem. So the r.h.s. of eq. (22.3.1) defines an analytic function on the complex domain (22.3.12). Then by translation invariance, its domain is easily extended to the forward tube.

Now we have to argue that this analytic function agrees with the Euclidean conformal partial wave in the Euclidean regime. This is actually done formally in eq. (22.3.6). We claim that using the assumption of OPE convergence in Euclidean, it can be done in a rigorous way but the idea is basically the same as eq. (22.3.6). We leave the details to [78].

We conclude that the conformal partial wave $G_{1234,\mathcal{O}}(c)$ has analytic continuation to $c \in \mathcal{T}_4$, performed by the integral representation (22.3.1). If the configuration c satisfies the condition (22.3.12), then by positivity of the measure $\widehat{G}_{\mathcal{O}}(p)$, we have the Cauchy-Schwarz inequality:

$$|G_{1234,\mathcal{O}}^{a_1 a_2 a_3 a_4}(c)|^2 \leq G_{12\bar{1},\mathcal{O}}^{a_1 a_2 a_2 a_1}(c_{12}) G_{4\bar{3},\mathcal{O}}^{a_4 a_3 a_3 a_4}(c_{34}), \quad (22.3.13)$$

where $c_{12} = (x_1, x_2, x_2^\theta, x_1^\theta)$ and $c_{34} = (x_4^\theta, x_3^\theta, x_3, x_4)$.

It remains to prove the required properties of $(\widehat{G}_{\mathcal{O}})_{ab}(p)$ and $\widehat{B}_{\mathcal{O},12}(p, x_1, x_2)$. The following subsections present a sketch of the proof. We claim that the properties of $\widehat{G}_{\mathcal{O}}$ is true. But for the properties of $\widehat{B}_{\mathcal{O},12}$, we only manage to show that they are true for generic $\Delta_{\mathcal{O}}$, and there a discrete set of exceptional $\Delta_{\mathcal{O}}$ for which we do not have a proof.

22.3.1 Properties of $(\widehat{G}_{\mathcal{O}})_{ab}$

We would like to show that $\frac{d^d p}{(2\pi)^d} (\widehat{G}_{\mathcal{O}})_{ab}(p)$ is a positive tempered measure. This is actually true for general QFT two-point functions. The claim is that given any QFT two-point function satisfying Wightman axioms (W0) - (W2) in Lorentzian or OS axioms (OS0) - (OS2) in Euclidean,⁵ the corresponding momentum-space two-point function is a positive tempered measure.⁶ This claim is a corollary of *Bochner-Schwartz theorem*:

Theorem 22.3.1. (*Bochner-Schwartz, see [110], p125.*) *For a generalized function f taken from $\mathcal{D}'(\mathbb{R}^d)$ to be positive definite, which means that*

$$\int d^d x f(x-y) \varphi(x) \overline{\varphi(y)} \geq 0, \quad \forall \varphi \in \mathcal{D}, \quad (22.3.14)$$

it is necessary and sufficient that it be a Fourier transform of a nonnegative tempered measure: $f = \mathcal{F}[\mu]$, $\mu \in \mathcal{S}'(\mathbb{R}^d)$, $\mu \geq 0$.

⁵At the level of two-point functions, without any extra assumptions, one can use the argument in [1] to show that Wightman axioms (W0) - (W2) are equivalent to OS axioms (OS0) - (OS2).

⁶The conclusion consists of three ingredients: (a) measure; (b) positive; (c) temperedness. The concept of "measure" is in the usual mathematical sense. By "positive" we means that for any constant tensor ξ^a in $V_{\rho_{\mathcal{O}}}$, the $\text{SO}(d)$ representation space of \mathcal{O} , the measure $(\widehat{G}_{\mathcal{O}})_{ab}(p) \xi^a \bar{\xi}^b d^d p$ is positive. By "tempered" we mean that the integral $\int (\widehat{G}_{\mathcal{O}})_{ab}(p) \varphi(p) d^d p$ defines a tempered distribution, where φ is a Schwartz test function.

Remark 22.3.1. Recall that \mathcal{S} is the Schwarz space and \mathcal{D} is the space of compactly supported Schwartz test functions, we always have $\mathcal{D} \subset \mathcal{S}$, thus $\mathcal{S}' \subset \mathcal{D}'$. In the case of Wightman QFT, f is the two-point function, which is a tempered distribution, so $f \in \mathcal{D}'$ is satisfied.

Remark 22.3.2. To show that the diagonal components \widehat{G}_{aa} are tempered measures, one just need to set $f(p) = (\widehat{G}_{\mathcal{O}})_{ab}(p)\xi^a\bar{\xi}^b$ with proper ξ in eq. (22.3.14). For the off-diagonal components, one need to apply the Cauchy-Schwarz inequality

$$\left| \int \frac{d^d p}{(2\pi)^d} (\widehat{G}_{\mathcal{O}})_{ab}(p) \varphi(p) \bar{\eta}(p) \right|^2 \leq \int \frac{d^d p}{(2\pi)^d} (\widehat{G}_{\mathcal{O}})_{aa}(p) |\varphi(p)|^2 \times \int \frac{d^d p}{(2\pi)^d} (\widehat{G}_{\mathcal{O}})_{bb}(p) |\eta(p)|^2 \quad (22.3.15)$$

and choose proper test functions $\varphi, \eta \in \mathcal{S}(\mathbb{R}^d)$.

22.3.2 Properties of $\widehat{B}_{\mathcal{O},12}$.

We would like to explain why $\widehat{B}_{\mathcal{O},12}$ is an entire function of p , and why it has an exponentially decaying upper bound in the light cone, leaving the technical details to the upcoming paper [78].

22.3.2.1 Analyticity

Let us first consider the scalar case, i.e. no spin indices. The three-point function $G_{12\mathcal{O}}(x_1, x_2, x)$ is fixed up to a constant factor

$$G_{12\mathcal{O}}(x_1, x_2, x) = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta}{2}} ((x_1-x)^2)^{\frac{\Delta_1-\Delta_2+\Delta}{2}} ((x_2-x)^2)^{\frac{\Delta_2-\Delta_1+\Delta}{2}}}, \quad (22.3.16)$$

where $\Delta_1, \Delta_2, \Delta$ are the scaling dimensions of $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}$. We put $\mathcal{O}(x)$ in Minkowski space and take Fourier transform with respect to x . This gives $\widehat{G}_{12\mathcal{O}}(x_1, x_2, p)$, the scalar version of eq. (22.3.9). By explicit computation [26], one can show that $\widehat{G}_{12\mathcal{O}}$ has the following integral representation

$$\begin{aligned} \widehat{G}_{12\mathcal{O}}(x_1, x_2, p) &= \frac{2\pi^{d/2+1}}{\Gamma(\frac{\Delta+\Delta_1-\Delta_2}{2})\Gamma(\frac{\Delta-\Delta_1+\Delta_2}{2})} \frac{(-p^2/4)^{\frac{\Delta-d/2}{2}}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-d/2}{2}}} \theta_{V_+}(p) \\ &\times \int_0^1 du \left(\frac{u}{1-u} \right)^{\frac{\Delta_1-\Delta_2}{2}} [u(1-u)]^{\frac{d-4}{4}} e^{ip \cdot (x_2 + ux_{12})} J_{\Delta-d/2}([-u(1-u)x_{12}^2 p^2]^{1/2}), \end{aligned} \quad (22.3.17)$$

where J_ν is the Bessel function of the first kind. Under normalization $\langle \mathcal{O}(0)\mathcal{O}(x) \rangle = \frac{1}{(x^2)^\Delta}$, the momentum-space two-point function of \mathcal{O} (defined in eq. (22.3.2)) is given by

$$\widehat{G}_{\mathcal{O}}(p) = \frac{2\pi^{d/2+1}}{\Gamma(\Delta+1-d/2)\Gamma(\Delta)} \left(-\frac{1}{4}p^2 \right)^{\Delta-d/2} \theta_{V_+}(p). \quad (22.3.18)$$

By comparing (22.3.17) and (22.3.18), we see that $\widehat{G}_{12\mathcal{O}}$ is factorized into

$$\widehat{G}_{12\mathcal{O}}(x_1, x_2, p) = \widehat{G}_{\mathcal{O}}(p) \widehat{B}_{\mathcal{O},12}(p, x_1, x_2). \quad (22.3.19)$$

By expanding the Bessel function in eq. (22.3.17) into power series and integrating over u term by term, we will get a power series for $\widehat{G}_{12\mathcal{O}}$. Then after removing the $\widehat{G}_{\mathcal{O}}$ -factor, we get the power series for $\widehat{B}_{\mathcal{O},12}$:

$$\begin{aligned} \widehat{B}_{\mathcal{O},12}(p, x_1, x_2) &= \frac{e^{ip \cdot x_2}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta}{2}}} F(\Delta_1, \Delta_2, \Delta; ip \cdot x_{12}, x_{12}^2 p^2/4), \\ F(\Delta_1, \Delta_2, \Delta; x, y) &= \sum_{m,n=0}^{\infty} \frac{\left(\frac{\Delta+\Delta_1-\Delta_2}{2} \right)_{m+n} \left(\frac{\Delta-\Delta_1+\Delta_2}{2} \right)_n}{(\Delta)_{m+2n} (\Delta-d/2+1)_n} \frac{x^m y^n}{m!n!}. \end{aligned} \quad (22.3.20)$$

When $\Delta, \Delta - \frac{d-2}{2} \notin -\mathbb{N}$, the power series above is absolutely convergent for all $x, y \in \mathbb{C}$, hence $\widehat{B}_{\mathcal{O},12}(p, x_1, x_2)$ is $(x_{12}^2)^{\frac{\Delta-\Delta_1-\Delta_2}{2}}$ \times an entire function of x_1, x_2, p .

Now let us consider the three-point function of spinning operators, $\langle \mathcal{O}_1^{a_1} \mathcal{O}_2^{a_2} \mathcal{O}_3^{a_3} \rangle$, which is kinematically determined by the scaling dimensions Δ_k and the $\text{SO}(d)$ representations ρ_k :

$$G_{123}^{a_1 a_2 a_3}(x_1, x_2, x_3) = \sum_{\ell_1, \ell_2, \ell_3} T_{\ell_1, \ell_2, \ell_3}^{a_1 a_2 a_3}(x_k) G_3(x_k; \Delta_k + \ell_k), \quad (22.3.21)$$

where $T_{\ell_1, \ell_2, \ell_3}^{a_1 a_2 a_3}(x_k)$'s are homogeneous polynomials of x_k 's, ℓ_k 's are integers and $G_3(x_k; \Delta_k + \ell_k)$'s are the scalar three-point functions with the scaling dimensions $\Delta_k + \ell_k$. This sum is finite, and each set of ℓ_k 's is determined by a totally symmetric traceless representation that appears in $\rho_1 \otimes \rho_2 \otimes \rho_3$. Since T 's are polynomials, we have

$$\widehat{G}_{123}^{a_1 a_2 a_3}(x_1, x_2, p) = \sum_{\ell_1, \ell_2, \ell_3} T_{\ell_1, \ell_2, \ell_3}^{a_1 a_2 a_3}(x_1, x_2, i\partial_p) \widehat{G}_3(x_1, x_2, p; \Delta_k + \ell_k). \quad (22.3.22)$$

We claim that $\widehat{B}_{3,12}^{a_3 a_1 a_2}(p, x_1, x_2)$ has the form⁷

$$\widehat{B}_{3,12}^{a_3 a_1 a_2}(p, x_1, x_2) = \sum_{\ell_1, \ell_2, \ell_3} c_{\ell_1, \ell_2, \ell_3} T_{\ell_1, \ell_2, \ell_3}^{a_1 a_2 a_3}(x_1, x_2, i\partial_p) \widehat{B}_3(p, x_1, x_2; \Delta_3 - \ell_3, \Delta_1 + \ell_1, \Delta_2 + \ell_2), \quad (22.3.23)$$

where \widehat{B}_3 is the scalar OPE kernel. This claim is based on three facts:

1. For fixed $\text{SO}(d)$ spins (ρ_1, ρ_2, ρ_3) , there are only finitely many independent tensor structures $T_{\ell_1, \ell_2, \ell_3}$ allowed by conformal invariance.
2. $T_{\ell_1, \ell_2, \ell_3}$ only depends on $(\rho_1, \rho_2, \rho_3, \ell_1, \ell_2, \ell_3)$. In particular, it does not depend on scaling dimensions.
3. If $\widehat{B}_{3,12}^{a_3 a_1 a_2}(p, x_1, x_2)$ has the form (22.3.23), then $\widehat{G}_{123}^{a_1 a_2 a_3}(x_1, x_2, p) (= \widehat{B}_{3,12}^{b_3 a_1 a_2}(p, x_1, x_2) (\widehat{G}_{\mathcal{O}_3})_{b_3}^{a_3}(p))$ satisfies the conformal Ward identities for scaling dimensions $(\Delta_1, \Delta_2, \Delta_3)$ and $\text{SO}(d)$ spins (ρ_1, ρ_2, ρ_3) .

The above three facts imply that under (22.3.10), the space spanned by the basis $\{T_{\ell_1, \ell_2, \ell_3}^{a_1 a_2 a_3}(x_1, x_2, i\partial_p) \widehat{B}_3(x_1, x_2, p; \Delta_1 + \ell_1, \Delta_2 + \ell_2, \Delta_3 - \ell_3)\}$ the space of $\widehat{G}_{123}^{a_1 a_2 a_3}(x_1, x_2, p)$ (the space of tensor structures allowed by conformal invariance).

Since $\widehat{B}_{123}(x_1, x_2, p)$ is an entire function of p in the scalar case except when $\Delta_3 \in (-\mathbb{N}) \cup (\frac{d-2}{2} - \mathbb{N})$, by eq. (22.3.23), it is also an entire function of p in the spinning case because it is the derivatives of entire functions, except when $\Delta_3 - \ell_3 \in (-\mathbb{N}) \cup (\frac{d-2}{2} - \mathbb{N})$. We will comment on these exceptional cases in section 22.3.3.

22.3.2.2 Exponential decay bound

In this section, we would like to give a sketch of the proof of the upper bound (22.3.11).

For convenience, we think of $\widehat{B}_{\mathcal{O},12}^{a_1 a_2}(p, x_1, x_2)$ as a function of x_{12}, x_2 and p . By eqs. (22.3.11) and (22.3.23), $\partial_{x_2}^{(\mu)} \partial_{x_{12}}^{(\nu)} \widehat{B}_{\mathcal{O},12}^{a_1 a_2}(p, x_1, x_2)$ is a finite sum of terms in the following form:

$$\frac{e^{ip \cdot x_2}}{(x_{12}^2)^{\frac{\Delta_1 + \Delta_2 - \Delta + \ell_1 + \ell_2 + \ell + |\nu|}{2}}} \times (\text{polynomial of } x_2, x_{12} \text{ and } p) \times F^{(k,l)}(\Delta_1 + \ell_1, \Delta_2 + \ell_2, \Delta - \ell; x, y), \quad (22.3.24)$$

where $F^{(k,l)}(x, y) = \partial_x^k \partial_y^l F(x, y)$. So we see that the exponential decay factor in eq. (22.3.11) comes from $e^{ip \cdot x_2}$. To prove the upper bound (22.3.11), it remains to show that the extra factors in eq. (22.3.24) has a power-law bound in p for p in the forward light cone.

When x_1 and x_2 are Lorentzian points, $e^{ip \cdot x_2}$ becomes a phase factor, then the upper bound (22.3.11) becomes a power-law bound. Mack claimed this upper bound in his paper [26], where it says that the temperedness of $G_{12\mathcal{O}}$ and the decomposition $\widehat{G}_{12\mathcal{O}} = \widehat{B}_{\mathcal{O},12} \widehat{G}_{\mathcal{O}}$ imply the power-law bound of $\widehat{B}_{\mathcal{O},12}$ in p for p in the forward light cone

⁷Here we remind readers that in \widehat{B} , the scaling dimension Δ_3 is shifted by $-\ell_3$ instead of $+\ell_3$.

(see [26], p181, the last paragraph). From my own understanding, the underlying argument of this claim is that if an analytic function is also a tempered distribution, then it should be power-law bounded. However this is not true. A toy counterexample is the function

$$f(t) = \frac{d}{dt} \sin(e^t). \quad (22.3.25)$$

We see that $f(t)$ is an analytic function as well as a tempered distribution, but it does not have a power-law bound. For this reason, we regard the upper bound (22.3.11) as an unknown result, and we need some careful computation to justify it.

For the technical reason, we divide the forward light-cone into two regimes: $0 \leq -p^2 \leq m^2$ and $-p^2 \geq m^2$, where $m^2 > 0$ is an arbitrary positive cutoff.

Case $0 \leq -p^2 \leq m^2$

We take the (k, l) -th derivative and sum over m in the series expansion (22.3.20), which gives

$$F^{(k,l)}(x, y) = \sum_{n=0}^{\infty} \frac{(\alpha)_{n+k+l}(\beta)_{n+l}}{n!(\alpha + \beta)_{2n+k+2l}(\alpha + \beta - \frac{d-2}{2})_{n+l}} {}_1F_1(\alpha + n + k + l; \alpha + \beta + 2n + k + 2l; x) y^n. \quad (22.3.26)$$

When $x \rightarrow \infty$, each term in series (22.3.26) has the following asymptotic behavior

$$\begin{aligned} [F^{(k,l)}(x, y)]_n &= \frac{(\beta)_{n+l}\Gamma(\alpha + \beta)}{n!\Gamma(\alpha)(\alpha + \beta)_{n+l}} e^x x^{-\beta-n-l} y^n \left[1 + O\left(\frac{1}{x}\right) \right] \\ &\quad + \frac{(\alpha)_{n+k+l}\Gamma(\alpha + \beta)}{n!\Gamma(\beta)(\alpha + \beta)_{n+l}} (-x)^{-\alpha-n-k-l} y^n \left[1 + O\left(\frac{1}{x}\right) \right]. \end{aligned} \quad (22.3.27)$$

For $x_{12} = (\varepsilon + it, \mathbf{x} + i\mathbf{y})$ such that $\varepsilon > |\mathbf{y}|$ and $p \in \bar{V}_+$, one can show that $x = ip \cdot x_{12}$ is in the regime where

$$\operatorname{Re}(x) \leq 0, \quad \left| \frac{\operatorname{Im}(x)}{\operatorname{Re}(x)} \right| \leq \frac{\sqrt{2(t^2 + \mathbf{x}^2)}}{\varepsilon - |\mathbf{y}|}. \quad (22.3.28)$$

Then e^x in the first term of (22.3.27) decays exponentially in p when p goes to ∞ in \bar{V}_+ . In addition, since $0 \leq -p^2 \leq m^2$, $y = \frac{x_{12} p^2}{4}$ is bounded. Therefore, each term in series (22.3.26) has the following power-law bound in p :

$$[F^{(k,l)}(x, y)]_n \leq C_n^{(k,l)}(x_{12}, m) (1 + |p|)^{-\alpha-n-k-l}, \quad p^0 \geq 0, \quad 0 \leq -p^2 \leq m^2, \quad (22.3.29)$$

where $C_n^{(k,l)}(x_{12}, m)$ is some constant.

We split above sum into two parts:

$$F^{(k,l)}(x, y) = \left(\sum_{n=0}^N + \sum_{n=N+1}^{\infty} \right) [\dots] := F_N^{(k,l)} + F_{>N}^{(k,l)}. \quad (22.3.30)$$

$F_N^{(k,l)}$ has a power-law bound in p as described above. For $F_{>N}^{(k,l)}$, we choose N to be sufficiently large such that $\alpha + N + k + l, \beta + N + l > 0$. Then using the following property of the hypergeometric function:

$$|{}_1F_1(\alpha; \alpha + \beta; x)| \leq \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta)} \frac{1}{[-\operatorname{Re}(x)]^\alpha}, \quad \alpha, \beta > 0, \quad \operatorname{Re}(x) < 0, \quad (22.3.31)$$

one can show that

$$|F_{>N}^{(k,l)}(x, y)| \leq C_N^{(k,l)}(x_{12}, m) [1 + |p|]^{-\alpha-k-l-N-1}, \quad p^0 \geq 0, \quad 0 \leq -p^2 \leq m^2, \quad (22.3.32)$$

where $C_N^{(k,l)}(x_{12}, m)$ is some constant. Combining the upper bounds of $F_N^{(k,l)}$ and $F_{>N}^{(k,l)}$, we get an estimate

$$|F^{(k,l)}(x, y)| \leq C^{(k,l)}(x_{12}, m) [1 + |p|]^{-\alpha-k-l}, \quad p^0 \geq 0, \quad 0 \leq -p^2 \leq m^2. \quad (22.3.33)$$

The constant $C^{(k,l)}(x_{12}, m)$ is uniformly bounded for x_{12} in a small neighbourhood.

We would like make a comment on why the above estimate of $F^{(k,l)}$ is not available for p in the whole forward light-cone. In the estimate (22.3.32) of $F_{>N}^{k,l}(x, y)$ for $0 \leq -p^2 \leq m^2$, we implicitly used the condition $|y| \leq |x_{12}^2|m^2/4$, and the y -dependence of this estimate is absorbed into the constant $C_n^{(k,l)}(x_{12}, m)$. If there is no restriction $0 \leq -p^2 \leq m^2$, then $y \sim p^2$ is not bounded, and the estimate of $F_{>N}$ will contain the factor $e^{|y/\text{Re}(x)|}$, which grows exponentially in p towards some direction in the forward light-cone. This is the reason why we have to introduce the cutoff m^2 .

Case $-p^2 \geq m^2$

In this case, we would like to use the inversion of (22.3.19):

$$\widehat{B}_{\mathcal{O},12}(p, x_1, x_2) = \widehat{G}_{12\mathcal{O}}(x_1, x_2, p)\widehat{G}_{\mathcal{O}}(p)^{-1}, \tag{22.3.34}$$

and use eq. (22.3.23) to produce \widehat{B} of spinning operators. Then the estimate of $\partial_{x_2}^{(\alpha)}\partial_{x_{12}}^{(\beta)}\widehat{B}_{\mathcal{O},12}^{a_1 a_2}(p, x_1, x_2)$ is reduced to the estimate of $\widehat{G}_{12\mathcal{O}}(x_1, x_2, p)$, $\widehat{G}_{\mathcal{O}}(p)^{-1}$ and their derivatives in the scalar case.

By (22.3.18), $\widehat{G}_{\mathcal{O}}(p)^{-1}$ is well-defined except for $\Delta \in (-\mathbb{N}) \cup (\frac{d-2}{2} - \mathbb{N})$, which are poles of the Gamma functions. Since the singularities at $p^2 = 0$ are excluded by the condition $-p^2 \geq m^2$, $\widehat{G}(p)^{-1}$ and its derivatives have power-law upper bounds in p . Then it suffices to show that $\widehat{G}_{12\mathcal{O}}(x_1, x_2, p)$ and its derivatives have an exponential decay bounds in p for $-p^2 \geq m^2$.

For convenience we introduce the notation $\alpha = \frac{\Delta+\Delta_1-\Delta_2}{2}$, $\beta = \frac{\Delta-\Delta_1+\Delta_2}{2}$. Since $G_{12\mathcal{O}}(x_1, x_2, x)$ (as a function of x) is a product of two two-point functions, $\widehat{G}_{12\mathcal{O}}(x_1, x_2, p)$ is the convolution of two 2pt functions (up to $x_{12}^\#$)

$$\widehat{G}_{12\mathcal{O}}(x_1, x_2, p) = \frac{C_\alpha C_\beta}{(x_{12}^2)^\sigma} \int \frac{d^d q}{(2\pi)^d} (-q^2)^{\alpha-d/2} (-(p-q)^2)^{\beta-d/2} e^{iq \cdot x_1 + i(p-q) \cdot x_2} \theta_{V_+}(q)\theta_{V_+}(p-q), \tag{22.3.35}$$

where

$$C_\alpha = \frac{\pi^{d/2+1}}{2^{2\alpha-d-1}\Gamma(\alpha+1-d/2)\Gamma(\alpha)}. \tag{22.3.36}$$

The above integral is over a finite domain because of the θ -functions. When $\alpha, \beta > d/2 - 1$, the integral is absolutely convergent and is bounded from above by some polynomial of p .

When α or $\beta \leq d/2 - 1$, $\widehat{G}_{12\mathcal{O}}$ is regularized by the analytic continuation in α and β . To show the general feature of this regulation, we would like to introduce the following toy example (from [93], chapter 3):

$$I(\alpha) = \int_0^1 x^\alpha \varphi(x), \tag{22.3.37}$$

where φ is a smooth function. This integral is convergent and analytic in α when $\text{Re}(\alpha) > -1$. To analytically continue $I(\alpha)$ to $\text{Re}(\alpha) \leq -1$, we split the integral into two parts:

$$I(\alpha) = \int_0^1 x^\alpha \left[\varphi(x) - \sum_{n=0}^N \frac{\varphi^{(n)}(0)}{n!} x^n \right] + \sum_{n=0}^N \frac{\varphi^{(n)}(0)}{n!} \int_0^1 x^{\alpha+n}. \tag{22.3.38}$$

The first term defines an analytic function of α in the region $\text{Re}(\alpha) > -1 - N$. The second term can be evaluated explicitly when $\text{Re}(\alpha) > -1$

$$\sum_{n=0}^N \frac{\varphi^{(n)}(0)}{n!} \int_0^1 x^{\alpha+n} = \sum_{n=0}^N \frac{\varphi^{(n)}(0)}{n!} \frac{1}{\alpha+n+1}. \tag{22.3.39}$$

Each term has analytic continuation to the whole complex plane, with a pole at some special point. By increasing N , we see that $I(\alpha)$ has analytic continuation to $\alpha \in \mathbb{C} \setminus \{-1, -2, \dots\}$.

Back to the three-point function, the regulation of the integral (22.3.35) is quite similar to the above toy example. The only complexity is that we need to deal with more integral variables and more singularities (including single and double light-cone singularities). Here we only summarize the main results, and leave the technical details to [78]:

- For $-p^2 > 0$ fixed, the integral in (22.3.35) has analytic continuation to $\alpha, \beta \in \mathbb{C} \setminus (-\mathbb{N}) \cup (\frac{d-2}{2} - \mathbb{N})$. It has poles at these exceptional points, but they are canceled out by the zeros of the prefactor $C_\alpha C_\beta$. Therefore, $\widehat{G}_{12\mathcal{O}}(x_1, x_2, p)$ is an entire function of α and β .⁸
- When $-p^2 \geq m^2$, $\widehat{G}_{12\mathcal{O}}(x_1, x_2, p)$ is bounded from above by $\text{const} \times |e^{ip \cdot x_2}| \times (\text{polynomial of } p \text{ and } x_{12})$. The constant coefficient is finite and only depends on x_{12} and m .
- The same is true for the derivatives of $\widehat{G}_{12\mathcal{O}}(x_1, x_2, p)$.

For $\widehat{B}_{\mathcal{O},12}$, there is an extra factor $\Gamma(\Delta)\Gamma(\Delta - \frac{d-2}{2})$ which comes from the two-point function $\widehat{G}_{\mathcal{O}}$ in eq. (22.3.19). So $\widehat{B}_{\mathcal{O},12}$ may have poles at $\Delta \in (-\mathbb{N}) \cup (\frac{d-2}{2} - \mathbb{N})$, which agrees with our computation result (22.3.20). Therefore, the conclusions in the case $-p^2 \geq m^2$ are the same as $0 \leq -p^2 \leq m^2$.

We identify Δ in the scalar case to $\Delta_3 - \ell_3$ in the spinning case, according to eq. (22.3.23). We conclude that

Proposition 22.3.2. *Let $\widehat{B}_{3,12}^{a_3 a_1 a_2}(p, x_1, x_2)$ be the OPE kernel defined by eq. (22.3.23). Then*

(a) *As a function of Δ_i , $\widehat{B}_{3,12}^{a_3 a_1 a_2}(p, x_1, x_2)$ has analytic continuation to*

$$\left\{ (\Delta_1, \Delta_2, \Delta_3) \in \mathbb{C}^3 \left| \Delta_3 - \ell_3 \notin (-\mathbb{N}) \cup \left(\frac{d-2}{2} - \mathbb{N} \right) \right. \right\},$$

and may have simple or double poles at $\Delta_3 - \ell_3 \in (-\mathbb{N}) \cup (\frac{d-2}{2} - \mathbb{N})$. Here ℓ_3 is the same as in eq. (22.3.23).

(b) *For each regular $(\Delta_1, \Delta_2, \Delta_3)$, the function $\widehat{B}_{123}^{a_1 a_2 a_3}(x_1, x_2, p)$ and its derivatives have the form*

$$(x_{12})^\# \times \text{entire function of } x_1, x_2 \text{ and } p. \quad (22.3.40)$$

They have the following bound in the light cone:

$$\left| \frac{\partial^{|\lambda|+|\mu|} \widehat{B}_{123}^{a_1 a_2 a_3}}{\partial x_2^{(\lambda)} \partial x_{12}^{(\mu)}} \right| \leq C_{(\lambda),(\mu)}(x_{12}) |\text{polynomial of } p| \times |e^{ip \cdot x_2}|. \quad (22.3.41)$$

The constant $C_{(\lambda),(\mu)}(x_{12})$ is finite as long as $x_{12} = (t - i\varepsilon, \mathbf{x} - i\mathbf{y})$ satisfies $\varepsilon > |\mathbf{y}|$. For each x_{12} we can choose a small neighbourhood of it such that $C_{(\lambda),(\mu)}(x_{12})$ is uniformly bounded in the neighbourhood.

22.3.3 Comments on the exceptional cases

The OPE kernel $\widehat{B}_{3,12}$ may be singular when $\Delta_3 - \ell_3 \in (-\mathbb{N}) \cup (\frac{d-2}{2} - \mathbb{N})$, because of poles of $\Delta = \Delta_3 - \ell_3$ in the power-series (22.3.20). So far we do not have a proof of analyticity and the exponential-decay bound for these cases. Here we just makes some comments on why we believe such a proof exist.

In section 13.3.1, we related the OPE kernel $\widehat{B}_{3,12}(p, x_1, x_2)$ to the Euclidean OPE kernel $C_3(x_1, x_2, x_0, \partial_0)$ (see eq. (13.3.14) with “3” replaced by “ χ ”). Since we assume OPE convergence in the Euclidean signature, we expect that $\widehat{B}_{3,12}(p, x_1, x_2)$ is finite when x_1, x_2 are Euclidean points.

By Euclidean OPE convergence, we also expect that the integral representation (22.3.1) converges when c is a Euclidean four-point configuration. This gives us a hint that \widehat{B}_3 's should have good upper bound to make (22.3.1) convergent. We hope that this upper bound does not change much when x_k 's are moved to the complex regime.

For the above reasons, we expect that the conclusions of proposition 22.3.2 is also true for these exceptional values of Δ_3 .

⁸This conclusion agrees with the series expansion (22.3.20). If we multiply \widehat{B}_3 by $(\Gamma(\Delta)\Gamma(\Delta - \frac{d-2}{2}))^{-1}$, which comes from the two-point function $\widehat{G}_{\mathcal{O}}$ in eq. (22.3.19), we will get an entire function of Δ_1, Δ_2 and Δ .

22.4 Step 2: convergence of the partial wave expansion in \mathcal{D}_4

We would like to show that the partial wave expansion (22.0.1), as a sum of analytic functions, converges to an analytic function on the domain \mathcal{D}_4 (the subset of \mathcal{T}_4 with real spatial components). By translation invariance, we can always act with a time translation on $c \in \mathcal{D}_4$, such that the temporal components of the new configuration satisfies

$$\epsilon_1 > \epsilon_2 > 0 > \epsilon_3 > \epsilon_4. \quad (22.4.1)$$

Then using the Cauchy-Schwarz inequality (22.3.13), we bound the sum in the r.h.s. of eq. (22.0.1) as follows:

$$\begin{aligned} |G_{1234}^{a_1 a_2 a_3 a_4}(c)| &\leq \sum_{\mathcal{O}} |G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}(c)| \\ &\leq \sum_{\mathcal{O}} \sqrt{G_{122\bar{1}, \mathcal{O}}^{a_1 a_2 a_2 a_1}(c_{12}) G_{4334, \mathcal{O}}^{a_4 a_3 a_3 a_4}(c_{34})} \\ &\leq \sqrt{\left(\sum_{\mathcal{O}} G_{122\bar{1}, \mathcal{O}}^{a_1 a_2 a_2 a_1}(c_{12}) \right) \left(\sum_{\mathcal{O}'} G_{4334, \mathcal{O}'}^{a_4 a_3 a_3 a_4}(c_{34}) \right)}. \end{aligned} \quad (22.4.2)$$

So it suffices to show that the partial wave expansion is convergent at reflection-symmetric configurations in \mathcal{D}_4 . A nice property of such configurations is that they are mapped to two-dimensional configurations by a composition of translation and real rotation, preserving the reflection symmetry:

$$\begin{aligned} x_1 &= (\epsilon_1 + it_1, \mathbf{x}_1), \quad x_2 = (\epsilon_2 + it_2, \mathbf{x}_2), \quad x_3 = (-\epsilon_2 + it_2, \mathbf{x}_2), \quad x_4 = (-\epsilon_1 + it_1, \mathbf{x}_1); \\ \implies x'_1 &= (\epsilon_1, 0), \quad x'_2 = (\epsilon_2 + it_2 - it_1, |\mathbf{x}_2 - \mathbf{x}_1|, 0), \quad x'_3 = (-\epsilon_2 + it_2 - it_1, |\mathbf{x}_2 - \mathbf{x}_1|, 0), \quad x'_4 = (-\epsilon_1, 0). \end{aligned}$$

By Poincaré invariance, the partial waves at configuration $c_{12} = (x_1, x_2, x_2^\theta, x_1^\theta)$ is equal to a finite linear combination of the partial waves at $c'_{12} = (x'_1, x'_2, x_2^\theta, x_1^\theta)$:

$$G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}(c_{12}) = \rho_1(R)_{b_1}^{a_1} \rho_2(R)_{b_2}^{a_2} \rho_3(R)_{b_3}^{a_3} \rho_4(R)_{b_4}^{a_4} G_{1234, \mathcal{O}}^{b_1 b_2 b_3 b_4}(c'_{12}), \quad (22.4.3)$$

where the coefficients $\rho_1(R)_{b_1}^{a_1} \rho_2(R)_{b_2}^{a_2} \rho_3(R)_{b_3}^{a_3} \rho_4(R)_{b_4}^{a_4}$ are less than or equal to 1 since R is a real rotation matrix.

We claim that the conformal partial wave expansion (22.0.1) is absolutely convergent if c is a two-dimensional configuration in the forward tube. The main reason for this claim is that in the two-dimensional case, we can explicitly write down the conformal transformation which maps c to c_ρ , and estimate the corresponding rotation matrices. We leave the proof to appendix D.1.

Continue the estimate (22.4.2):

$$\begin{aligned} \sum_{\mathcal{O}} \left| G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}(c_{12}) \right| &\leq \sum_{b_1, b_2, b_3, b_4} \left| \rho_1(R)_{b_1}^{a_1} \rho_2(R)_{b_2}^{a_2} \rho_3(R)_{b_3}^{a_3} \rho_4(R)_{b_4}^{a_4} \right| \sum_{\mathcal{O}} \left| G_{1234, \mathcal{O}}^{b_1 b_2 b_3 b_4}(c'_{12}) \right| \\ &\leq \sum_{b_1, b_2, b_3, b_4} \sum_{\mathcal{O}} \left| G_{1234, \mathcal{O}}^{b_1 b_2 b_3 b_4}(c'_{12}) \right|. \end{aligned} \quad (22.4.4)$$

The last line is a finite sum of absolutely convergent series, thus it is convergent. The estimate of the c_{34} -part is similar. This finishes the proof of the convergence of the partial wave expansion (22.0.1) for $c \in \mathcal{D}_4$. Thus $G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}(c)$ is an analytic function for $c \in \mathcal{D}_4$.

In appendix D.1, we also show that the four-point function $G_{1234, \mathcal{O}}^{a_1 a_2 a_3 a_4}(c)$ has a power-law bound if c is in the two-dimensional subspace of \mathcal{T}_4 . Then the power-law bound follows using the above Cauchy-Schwarz argument.

Chapter 23

CFT four-point functions in the Minkowski cylinder

In this chapter we discuss the generalization to the scalar four-point functions in the Minkowski cylinder, which has the topology $\mathbb{R} \times S^{d-1}$. The Minkowski cylinder is known to be the boundary of Lorentzian AdS_{d+1} in the framework of $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence [30]. We use coordinates (t, Ω) to describe its points, where t is a real number and $\Omega = (\Omega^1, \dots, \Omega^d)$ is a d -dimensional unit vector (i.e. $\sum_i (\Omega^i)^2 = 1$).

The Minkowski cylinder can be viewed as an infinite-sheet version of the compactified Minkowski space,¹ with covering map

$$\begin{aligned} x_M^0 &= \frac{\sin t}{\cos t + \Omega^d}, \\ x_M^i &= \frac{\Omega^i}{\cos t + \Omega^d}, \quad i = 1, 2, \dots, d-1. \end{aligned} \tag{23.0.1}$$

This covering map is conformal. One can show that the Minkowski metric is pulled back to the cylinder by

$$ds_M^2 = -(dx_M^0)^2 + \sum_{i=1}^{d-1} (dx_M^i)^2 = \frac{1}{(\cos t + \Omega^d)^2} (-dt^2 + d\Omega^2). \tag{23.0.2}$$

Therefore, these two Lorentzian spaces share the same conformal algebra. But the topologies of their conformal groups are different.

The Minkowski space is conformally embedded into the *Poincaré patch* of the Minkowski cylinder [25]:

$$-\pi < t < \pi, \quad \cos t + \Omega^d > 0. \tag{23.0.3}$$

The Minkowski cylinder admits a causal ordering

$$(t_1, \Omega_1) > (t_2, \Omega_2) \quad \Leftrightarrow \quad t_1 - t_2 > \arccos(\Omega_1 \cdot \Omega_2), \tag{23.0.4}$$

which agrees with the Minkowski-space causal ordering in the Poincaré patch and is preserved by any finite conformal transformation [138]. In contrast, in Minkowski space, the special conformal transformations can violate the causal ordering. Because of this subtlety, in the early days, people realized that one cannot naively postulate global conformal invariance in the framework of Wightman QFT because it will violate the causality condition.² There are two resolutions to the puzzle. The first one is to assume Wightman axioms in Lorentzian but global conformal

¹Here we include the points at infinity in Minkowski space, which correspond to $\cos t + \Omega^d = 0$.

²By global conformal invariance we mean that the correlators are invariant under any finite conformal transformation. Then given any two local operators $\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)$ which are time-like separated, one can find a special conformal transformation which maps x_1, x_2 to a space-like separated pair. As a consequence of the global conformal invariance and the causality condition, $[\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)] = 0$ for all possible x_1, x_2 .

invariance in the regime where Wightman distributions are well-defined analytic functions, e.g. in the Euclidean space. Then after Wick rotation, the Lorentzian conformal invariance only holds in the infinitesimal way (the same as we have justified in part II, see section 11.6). This way of formulating conformal invariance is called the hypothesis of “weak conformal invariance” [140]. The second one is to put CFT in the Minkowski cylinder, where the conformal group action is globally well-defined and does not contradict anything [25].

Assuming Wightman axioms and Euclidean conformal invariance, Lüscher and Mack tried to show that the correlators in Minkowski space (viewed as the Poincaré patch) can be extended to the whole Minkowski cylinder. Similarly to the Osterwalder-Schrader theorem, L&M started from the Euclidean CFT correlators (whose existence is easily justified using Wightman axioms [121]) and tried to perform the analytic continuation in cylinder temporal variables. However, they did not manage to show the temperedness in the whole Lorentzian regime. Therefore, we still consider this as an open problem.

In the following sections, we will show that the Euclidean CFT axioms imply Wightman axioms in the Minkowski cylinder. We will only demonstrate the proof of temperedness, since the proof of other Wightman axioms are the same as part II.

23.1 Euclidean cylinder and its Wick rotation

We define the hyperbolic coordinates $(\sigma, \Omega) \in \mathbb{R} \times S^{d-1}$, where $\Omega = (\Omega^1, \dots, \Omega^d)$ is under constraint $\sum_{i=1}^d (\Omega^i)^2 = 1$. The relation of the Cartesian coordinates $(x^0, x^1, \dots, x^{d-1})$ and the hyperbolic coordinates is given in [25]:

$$\begin{aligned} x^0 &= \frac{\sinh \sigma}{\cosh \sigma + \Omega^d}, \\ x^i &= \frac{\Omega^i}{\cosh \sigma + \Omega^d}, \quad i = 1, 2, \dots, d-1. \end{aligned} \quad (23.1.1)$$

By eq. (23.1.1), one can show that the Euclidean flat metric is conformally equivalent to a cylinder metric:

$$ds^2 = \sum_{i=0}^{d-1} (dx^i)^2 = \frac{1}{(\cosh \sigma + \Omega^d)^2} (d\sigma^2 + d\Omega^2). \quad (23.1.2)$$

Also, the Euclidean distance between two points x_1, x_2 is given by

$$(x_1 - x_2)^2 = \frac{2(\cosh(\sigma_1 - \sigma_2) - \Omega_1 \cdot \Omega_2)}{(\cosh \sigma_1 + \Omega_1^d)(\cosh \sigma_2 + \Omega_2^d)} \quad (23.1.3)$$

In the context of CFT, the hyperbolic coordinates correspond to the so-called *N-S (North-South pole) quantization picture* [82]. In the N-S picture, the Euclidean flat space is foliated into slices of constant- σ spheres, characterized by

$$(x^0 - \coth \sigma)^2 + \mathbf{x}^2 = \frac{1}{\sinh^2 \sigma}, \quad (23.1.4)$$

see fig. 23.1.1.

We would like to make some remarks on why hyperbolic coordinates are good.

Good reflection operator

In the Euclidean flat space, the reflection of a point is defined by $x = (x^0, \mathbf{x}) \mapsto x^\theta = (-x^0, \bar{\mathbf{x}})$. By eq. (23.1.1), the reflection is written in terms of hyperbolic coordinates as follows

$$x = (\sigma, \Omega) \mapsto x^\theta = (-\sigma, \Omega) \quad (23.1.5)$$

So it turns out that the flat space reflection corresponds to the cylinder reflection. Based on this observation, the formulations of the reflection positivity condition are exactly the same for the Euclidean flat space and Euclidean

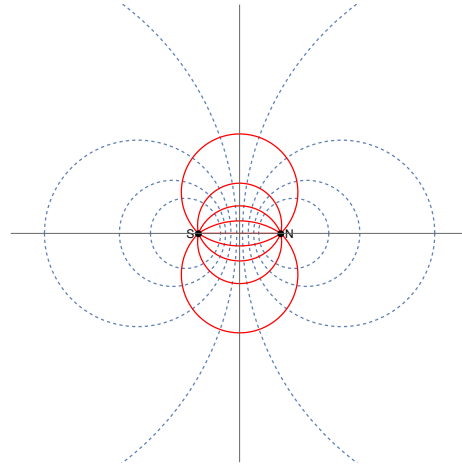


Figure 23.1.1: N-S quantization picture. The dashed blue circles are the constant- σ slices, and the red curves from the south pole to the north pole are the flows induced by ∂_σ .

cylinder.

Good Wick rotation

If we take imaginary cylinder time, i.e. $\sigma = it$, then by eq. (23.1.1), the corresponding flat space coordinates are given by

$$\begin{aligned} x^0 &= i \frac{\sin t}{\cos t + \Omega^d}, \\ x^i &= \frac{\Omega^i}{\cos t + \Omega^d}, \quad i = 1, 2, \dots, d-1. \end{aligned} \quad (23.1.6)$$

We see that the imaginary cylinder time corresponds to the imaginary temporal variable and real spatial variables in the flat space. This is exactly eq. (23.0.1) under substitution $x^0 = ix_M^0$, $x^i = x_M^i$. So the Wick rotation of under the hyperbolic coordinates agrees perfectly with the Wick rotation under the Cartesian coordinates.

23.2 CFT in the Euclidean cylinder

For a scalar primary operator with scaling dimension Δ , we establish the correspondence between the flat-space operator and the cylinder operator:

$$\mathcal{O}_{cyl}(\sigma, \Omega) = (\cosh \sigma + \Omega^d)^{-\Delta} \mathcal{O}_{flat}(x). \quad (23.2.1)$$

The scale factor comes from eq. (23.1.2). Under this definition, the correlation functions of \mathcal{O}_{cyl} are conformally invariant with respect to the cylinder metric $(ds^2)_{cyl} = d\sigma^2 + d\Omega^2$.

For the correlators in the flat space, the Wick rotation is defined in two steps: analytic continuation + taking the limit to the Lorentzian regime (see step 1 and step 2 in chapter 9). For cylinder correlators, we define the following sets of configurations:

$$\begin{aligned} (\mathcal{D}_n)_{cyl} &= \left\{ (x_1, \dots, x_n) \in (\mathbb{C} \times S^{d-1})^n \mid x_i = (\sigma_i, \Omega_i), \quad \text{Re}(\sigma_i) > \text{Re}(\sigma_{i+1}) \text{ for } i = 1, 2, \dots, n-1 \right\}. \\ (\mathcal{D}_n^E)_{cyl} &= \left\{ c \in (\mathcal{D}_n)_{cyl} \mid \text{Im}(\sigma_i) = 0 \text{ for } i = 1, 2, \dots, n \right\}. \end{aligned} \quad (23.2.2)$$

Then the Wick rotation of cylinder correlators is defined similarly by:

Step 1. One finds an extension G_n^E from $(\mathcal{D}_n^E)_{cyl}$ to a function G_n such that the following condition is satisfied:

$$G_n \text{ is defined on } (\mathcal{D}_n)_{cyl}, \text{ is analytic in variables } \sigma_k \text{ and is continuous in } \Omega_k. \quad (23.2.3)$$

Step 2. Let $\sigma_k = \epsilon_k + it_k$ be the complex (Euclidean) cylinder temporal variable. Lorentzian correlators are defined as the limits of G_n from inside $(\mathcal{D}_n)_{cyl}$ by sending $\epsilon_i \rightarrow 0$:

$$G_n^L(x_1^L, \dots, x_n^L) = \lim_{\epsilon_i \rightarrow 0} G_n(x_1, \dots, x_n), \quad x_k^L = (t_k, \Omega_k), \quad k = 1 \dots n. \quad (23.2.4)$$

Now let us focus on the four-point function. By conformal invariance, the four-point function of \mathcal{O}_{cyl} can be written as

$$\langle \mathcal{O}_{cyl}(x_1) \mathcal{O}_{cyl}(x_2) \mathcal{O}_{cyl}(x_3) \mathcal{O}_{cyl}(x_4) \rangle = \frac{g(\rho, \bar{\rho})}{2^{2\Delta} [\cosh(\sigma_1 - \sigma_2) - \Omega_1 \cdot \Omega_2]^\Delta [\cosh(\sigma_3 - \sigma_4) - \Omega_3 \cdot \Omega_4]^\Delta}, \quad (23.2.5)$$

where the conformally invariant factor $g(\rho, \bar{\rho})$ is the same as eq. (11.0.1) in the Euclidean flat space.

The analytic continuation procedure of the cylinder four-point function is analogous to section 11.3. Firstly, one can show that in the complex regime, $\cosh(\sigma_1 - \sigma_2) - \Omega_1 \cdot \Omega_2 \neq 0$ as long as $\text{Re}(\sigma_1) > \text{Re}(\sigma_2)$.³ So the prefactor in eq. (23.2.5) has analytic continuation to $(\mathcal{D}_n)_{cyl}$. Secondly, we have the series expansion (11.2.6) (or (11.3.6)) for $g(\rho, \bar{\rho})$. Using the same argument in section 11.3, the analytic continuation problem is reduced to showing that $z, \bar{z} \neq [1, +\infty)$, or equivalently, $|\rho|, |\bar{\rho}| < 1$ for configurations in $(\mathcal{D}_n)_{cyl}$. Then we need power-law bounds of the prefactor $[\cosh(\sigma_i - \sigma_j) - \Omega_i \cdot \Omega_j]^{-\Delta}$ and the cross-ratio quantities $(1 - |\rho|)^{-1}, (1 - |\bar{\rho}|)^{-1}$ to show that the limit to the Lorentzian regime (step 2) defines a tempered distribution.

The estimate of $(\cosh(\sigma_1 - \sigma_2) - \Omega_1 \cdot \Omega_2)^{-1}$ is trivial:

$$\left| \frac{1}{\cosh(\epsilon + it) - \Omega_1 \cdot \Omega_2} \right| \leq \text{const} \left(1 + \frac{1}{\epsilon} \right)^2 e^{-\epsilon}. \quad (23.2.6)$$

We will give a proof of $|\rho|, |\bar{\rho}| < 1$ and the power-law bound of $\rho, \bar{\rho}$ in the next section. Our strategy will be the same as chapter 12.

23.3 Cauchy-Schwarz type inequality, power-law bound of $\rho, \bar{\rho}$

In chapter 12, we derived Cauchy-Schwarz type inequality (12.0.3) of the radial cross-ratio variables, using the Cauchy-Schwarz inequality of conformal blocks that appear in the generalized free theory. The same argument applies to the cylinder case. We let $(\mathcal{D}_4^{(0)})_{cyl}$ denote the set of cylinder configurations such that

$$\begin{aligned} c &= (x_1, x_2, x_3, x_4), \\ x_j &= (\sigma_j, \Omega_j), \quad \sigma_j = \epsilon_j + it_j, \\ \epsilon_1 &> \epsilon_2 > 0 > \epsilon_3 > \epsilon_4. \end{aligned} \quad (23.3.1)$$

Each $c \in (\mathcal{D}_4^{(0)})_{cyl}$ has its corresponding reflection-symmetric configurations, given by

$$c_{12} = (x_1, x_2, x_2^\theta, x_1^\theta), \quad c_{34} = (x_4^\theta, x_3^\theta, x_3, x_4). \quad (23.3.2)$$

Using the conformal block argument in section (12.3), we get

$$\max\{|\rho(c)|, |\bar{\rho}(c)|\}^2 \leq \max\{|\rho(c_{12})|, |\bar{\rho}(c_{12})|\} \times \max\{|\rho(c_{34})|, |\bar{\rho}(c_{34})|\} \quad (23.3.3)$$

in the region where $|\rho(c_{12})|, |\bar{\rho}(c_{12})|, |\rho(c_{34})|, |\bar{\rho}(c_{34})| < 1$. To make eq. (23.3.3) true for all $c \in (\mathcal{D}_4^{(0)})_{cyl}$, we need an estimate on the $\rho, \bar{\rho}$ of reflection-symmetric configurations.

Let us consider the configuration c_{12} . We compute u, v for c_{12} :

$$u = \frac{|\cosh(\epsilon_1 - \epsilon_2 + i(t_1 - t_2)) - \Omega_1 \cdot \Omega_2|^2}{|\cosh(\epsilon_1 + \epsilon_2 + i(t_1 - t_2)) - \Omega_1 \cdot \Omega_2|^2}, \quad v = \frac{(\cosh(2\epsilon_1) - 1)(\cosh(2\epsilon_2) - 1)}{|\cosh(\epsilon_1 + \epsilon_2 + i(t_1 - t_2)) - \Omega_1 \cdot \Omega_2|^2}. \quad (23.3.4)$$

³Using the identity $\cosh(\epsilon + it) = \cosh \epsilon \cos t + i \sinh \epsilon \sin t$, we see that $\cosh(\epsilon + it) \notin [-1, 1]$ as long as $\epsilon \neq 0$.

We see that (a) u, v are real, (b) $0 < u < 1$, (c) $v > 0$. Since $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$, we have $0 < z, \bar{z} < 1$.⁴ So we conclude that $|\rho|, |\bar{\rho}| < 1$ for all reflection-symmetric configurations in $(\mathcal{D}_4^{(0)})_{cyl}$, thus eq. (23.3.3) holds for all $c \in (\mathcal{D}_4^{(0)})_{cyl}$. One can also check this by explicit computation of z, \bar{z} , using eq. (11.2.2):

$$z, \bar{z} = \frac{\cosh^2 \epsilon_1 + \cosh^2 \epsilon_2 + \cos^2 t + \cos^2 \theta - 2 \cosh \epsilon_1 \cosh \epsilon_2 \cos t \cos \theta \pm 2 \sinh \epsilon_1 \sinh \epsilon_2 \sin t \sin \theta - 2}{(\cosh(\epsilon_1 + \epsilon_2) \cos t - \cos \theta)^2 + \sinh^2(\epsilon_1 + \epsilon_2) \sin^2 t}, \quad (23.3.5)$$

$(t = t_1 - t_2, \quad \cos \theta = \Omega_1 \cdot \Omega_2.)$

From this we get

$$1 - z, 1 - \bar{z} = \frac{2 \sinh \epsilon_1 \sinh \epsilon_2 (\cosh(\epsilon_1 + \epsilon_2) - \cos(t \pm \theta))}{(\cosh(\epsilon_1 + \epsilon_2) \cos t - \cos \theta)^2 + \sinh^2(\epsilon_1 + \epsilon_2) \sin^2 t}, \quad (23.3.6)$$

which implies the power-law bound⁵

$$\frac{1}{1 - z(c_{12})}, \frac{1}{1 - \bar{z}(c_{12})} \leq \text{const} \left(1 + \frac{1}{\epsilon_1}\right) \left(1 + \frac{1}{\epsilon_2}\right) \leq \text{const} \left(1 + \frac{1}{\epsilon_2}\right)^2. \quad (23.3.7)$$

Then by eq. (12.2.3) we get an estimate for ρ -variables

$$\frac{1}{1 - \rho(c_{12})}, \frac{1}{1 - \bar{\rho}(c_{12})} \leq \text{const} \left(1 + \frac{1}{\epsilon_2}\right). \quad (23.3.8)$$

For a general configuration $c \in (\mathcal{D}_4)_{cyl}$, we shift the coordinates by time translation:

$$x_k = (\epsilon_k + it_k, \Omega_k) \rightarrow x'_k = \left(\epsilon_k - \frac{\epsilon_2 + \epsilon_3}{2} + it_k, \Omega_k\right), \quad (23.3.9)$$

so the new configuration $c' = (x'_1, x'_2, x'_3, x'_4)$ belongs to $(\mathcal{D}_4^{(0)})_{cyl}$ and it has the same $\rho, \bar{\rho}$ as c . Combining eqs. (23.3.3) and (23.3.8), we get an estimate of ρ -variables:

$$\frac{1}{1 - \rho(c)}, \frac{1}{1 - \bar{\rho}(c)} \leq \text{const} \left(1 + \frac{1}{\epsilon_2 - \epsilon_3}\right) \quad (c \in (\mathcal{D}_4)_{cyl}). \quad (23.3.10)$$

This is the power-law bound we want.

23.4 Wightman axioms in the Minkowski cylinder

Now we briefly demonstrate the Wightman axioms in the Minkowski cylinder, using the same argument as chapter 11. In the cylinder case, the spherical coordinates Ω_k 's are treated as function variables because these directions are compactified.

Temperedness

By eqs. (23.2.6), (23.3.10) and (11.2.6), the cylinder four-point function $G_4(c)$ has analytic continuation to the complex domain $(\mathcal{D}_4)_{cyl}$ (defined in eq. (23.2.2)), and satisfies a power-law bound in $\epsilon_k - \epsilon_{k+1}$ (recall that the temporal variables are $\sigma_k = \epsilon_k + it_k$). Then by the first part of theorem 9.0.1, $G_4^L(t_k, \Omega_k)$ (defined by the limit (23.2.4)) is a tempered distribution in the temporal variables t_k , and is continuous in spherical coordinates Ω_k .

⁴By $u, v > 0$ there are only three possibilities (a) $z, \bar{z} < 0$, (b) $0 < z, \bar{z} < 1$, (c) $z, \bar{z} > 1$. The last one is ruled out by $u < 1$. By continuity we set $t_1 = t_2$ and $\Omega_1 = \Omega_2$, which gives $v < 1$, so the first possibility is ruled out.

⁵A naive estimate may give $(1 + 1/\epsilon_2)^4$. By a careful estimate, one can get an upper bound

$$\frac{(\cosh(\epsilon_1 + \epsilon_2) \cos t - \cos \theta)^2 + \sinh^2(\epsilon_1 + \epsilon_2) \sin^2 t}{\cosh(\epsilon_1 + \epsilon_2) - \cos(t \pm \theta)} \leq \text{const} e^{\epsilon_1 + \epsilon_2},$$

which is regular at small ϵ . So the only power-law singularities come from $(\sinh \epsilon_1 \sinh \epsilon_2)^{-1}$.

Since the power-law bound is uniform for truncated conformal block expansions, the conformal block expansion converges in the sense of tempered distributions (by the same argument as theorem 13.1.1).

Conformal invariance

Using the same argument as section 11.6, one can show that $G_4^L(t_k, \Omega_k)$ satisfies the conformal Ward identities, i.e. the infinitesimal cylinder conformal symmetry holds. Since the conformal group action is globally well-defined on the Minkowski cylinder, the conformal Ward identities imply the global conformal symmetry of $G_4^L(t_k, \Omega_k)$. In other words, for any finite conformal transformation $x' = f(x)$, the following identity holds:

$$J(x_1)^{\Delta_0} J(x_2)^{\Delta_0} J(x_3)^{\Delta_0} J(x_4)^{\Delta_0} G_4^L(x'_1, x'_2, x'_3, x'_4) = G_4^L(x_1, x_2, x_3, x_4), \quad (23.4.1)$$

where $J(x) = \det(\partial f^\mu / \partial x^\nu)^{1/d}$ is the local scale factor.

Unitarity

Since the Euclidean flat-space reflection coincides with the Euclidean cylinder reflection (23.1.5), the conjugation of the Euclidean cylinder operator is given by

$$(\mathcal{O}_{cyl}(\sigma, \Omega))^\dagger = \mathcal{O}_{cyl}(-\bar{\sigma}, \Omega). \quad (23.4.2)$$

We define the Minkowski cylinder operator by $\mathcal{O}^L(t, \Omega) = \mathcal{O}_{cyl}(it, \Omega)$. Then we have

$$(\mathcal{O}^L(t, \Omega))^\dagger = \mathcal{O}^L(t, \Omega). \quad (23.4.3)$$

Since the reflection positivity condition in the Euclidean cylinder is formulated in the same way as in the Euclidean space, the unitarity (Wightman positivity) condition in the Minkowski cylinder is also formulated in the same way as in the Minkowski space, i.e. eqs. (7.1.11) and (7.1.12).

Microcausality

The microcausality condition in the Minkowski cylinder is with respect to the causal ordering (23.0.4). Inside the four-point function,

$$[\mathcal{O}^L(t_k, \Omega_k), \mathcal{O}^L(t_{k+1}, \Omega_{k+1})] = 0 \quad \text{if} \quad |t_k - t_{k+1}| < \arccos(\Omega_k \cdot \Omega_{k+1}). \quad (23.4.4)$$

Similarly to the argument in section 11.9.1, it suffices to show that $|\rho|, |\bar{\rho}| < 1$ for the following three types of configurations:

$$k = 1: \epsilon_1 = \epsilon_2 > \epsilon_3 > \epsilon_4 \text{ and } |t_1 - t_2| < \arccos(\Omega_1 \cdot \Omega_2);$$

$$k = 2: \epsilon_1 > \epsilon_2 = \epsilon_3 > \epsilon_4 \text{ and } |t_2 - t_3| < \arccos(\Omega_2 \cdot \Omega_3);$$

$$k = 3: \epsilon_1 > \epsilon_2 > \epsilon_3 = \epsilon_4 \text{ and } |t_2 - t_3| < \arccos(\Omega_2 \cdot \Omega_3).$$

Using eq. (23.3.6), we see that the condition for $|\rho|, |\bar{\rho}| < 1$ can actually be relaxed to $\epsilon_1, \epsilon_2 > \epsilon_3, \epsilon_4$. This applies to the cases of $k = 1$ and $k = 3$.

For $k = 2$, we shift the configuration to $\epsilon_2 = \epsilon_3 = 0$, i.e. $\epsilon_1 > 0 > \epsilon_4$. To analyze the ρ -variables, we map the cylinder configuration to the flat space by eq. (23.1.1) (the cross-ratios do not change). Then by eqs. (23.1.3) and (23.1.6), x_2 and x_3 are mapped to two space-like separated points in Minkowski space. According to the $k = 2$ argument in section 11.9.1, it suffices to show that the flat-space images of x_1 and x_4 satisfy

$$\text{Re}(x_1^0) > |\text{Im}(\mathbf{x}_1)|, \quad \text{Re}(x_4^0) < -|\text{Im}(\mathbf{x}_4)|. \quad (23.4.5)$$

By (23.1.1), we have

$$\begin{aligned} \operatorname{Re}(x_1^0) &= \frac{\sinh \epsilon_1 (\cosh \epsilon_1 + \cos t_1 \Omega_1^d)}{(\cosh \epsilon_1 \cos t_1 + \Omega_1^d)^2 + \sinh^2 \epsilon_1 \sin^2 t_1}, \\ \operatorname{Im}(x_1^i) &= -\frac{\sinh \epsilon_1 \sin t_1 \Omega_1^i}{(\cosh \epsilon_1 \cos t_1 + \Omega_1^d)^2 + \sinh^2 \epsilon_1 \sin^2 t_1}, \quad i = 1, \dots, d-1, \end{aligned} \quad (23.4.6)$$

which implies that

$$\begin{aligned} \operatorname{Re}(x_1^0) - |\operatorname{Im}(\mathbf{x}_1)| &= \frac{\sinh \epsilon_1 (\cosh \epsilon_1 + \cos t_1 \Omega_1^d - |\sin t_1| \sqrt{1 - (\Omega_1^d)^2})}{(\cosh \epsilon_1 \cos t_1 + \Omega_1^d)^2 + \sinh^2 \epsilon_1 \sin^2 t_1} \\ &\geq \frac{\sinh \epsilon_1 (\cosh \epsilon_1 - 1)}{(\cosh \epsilon_1 \cos t_1 + \Omega_1^d)^2 + \sinh^2 \epsilon_1 \sin^2 t_1} > 0. \end{aligned} \quad (23.4.7)$$

The computation for x_4 is similar. So the condition (23.4.5) holds when $\epsilon_1 > 0 > \epsilon_4$. Now we are in the same situation as the $k = 2$ case in section 11.9.1, so we conclude that $|\rho|, |\bar{\rho}| < 1$ in this case. This finishes the proof of eq. (23.4.4).

Cluster property

Recall that the cluster property talks about the asymptotic behavior of the correlator when the two clusters of operators are separated very far from each other in the space-like direction, there is no cluster property for the cylinder CFT since the spatial directions are compactified.

Spectral condition

As a consequence of the power-law bound and the second part of theorem 9.0.1, $G_4^L(t_k, \Omega_k)$ has the Fourier transform (in the sense of distributions) in temporal variables:

$$G_4^L(t_k, \Omega_k) = \int \frac{dE_1 dE_2 dE_3}{(2\pi)^3} F(E_1, E_2, E_3; \Omega_k) e^{-i(E_1 t_1 + E_2 t_2 + E_3 t_3)}, \quad (23.4.8)$$

where F is a tempered distribution in E_k 's and continuous in Ω_k 's. F is supported in the region $E_1, E_2, E_3 \geq 0$.

23.5 Matching the four-point functions in the flat space and in the cylinder

When we define the correlators in the Minkowski space and in the Minkowski cylinder, the Wick rotations start from the same correlation function but they are performed in different coordinates. As we mentioned at the beginning of this chapter, the Minkowski space is conformally embedded into the Poincaré patch of the Minkowski cylinder. Then a question naturally arises: do the correlators in the Poincaré patch agree with the ones in the Minkowski space, up to the scale factors in eq. (23.2.1)? In this section, we claim that the answer is yes.

To show this claim, we would like to introduce the following lemma:

Lemma 23.5.1. *For any n -point configuration $c = (x_1, x_2, \dots, x_n)$, if all the points of c are in the Poincaré patch, i.e. $x_k = (it_k, \Omega_k)$ is under condition (23.0.3), then there exists a collection of ϵ_k 's such that*

1. $\epsilon_1 > \epsilon_2 > \dots > \epsilon_n > 0$.
2. The the cylinder path $c(s)$ with

$$x_k(s) = ((1-s)\epsilon_k + ist_k, \Omega_k), \quad 0 \leq s < 1, \quad k = 1, 2, \dots, n \quad (23.5.1)$$

is mapped into the forward tube \mathcal{T}_n in the flat space, i.e. $\operatorname{Re}(x_k^0(s)) - \operatorname{Re}(x_{k+1}^0(s)) > |\mathbf{x}_k(s) - \mathbf{x}_{k+1}(s)|$ for $k = 1, 2, \dots, n-1$.

Proof. Let $\eta_1 > \eta_2 > \dots > \eta_n > 0$ be a collection of positive numbers, we choose ϵ_k 's to be

$$\epsilon_k = \lambda \eta_k \quad (23.5.2)$$

with a positive real number λ . For sufficiently small λ , we have

$$\begin{aligned} \operatorname{Re}(x_k^0(s)) &= \frac{\sinh((1-s)\lambda\eta_k) [\cosh((1-s)\lambda\eta_k) + \cos(st_k)\Omega_k^d]}{(\cosh((1-s)\lambda\eta_k) \cos(st_k) + \Omega_k^d)^2 + \sinh^2((1-s)\lambda\eta_k) \sin^2(st_k)} \\ &= \frac{(1-s)\lambda\eta_k [1 + \Omega_k^d \cos(st_k)]}{[\cos(st_k) + \Omega_k^d]^2} (1 + O(\lambda^2 \eta_k^2)), \\ \operatorname{Im}(x_k^i(s)) &= - \frac{\sinh((1-s)\lambda\eta_k) \sin(st_k)\Omega_k^i}{(\cosh((1-s)\lambda\eta_k) \cos(st_k) + \Omega_k^d)^2 + \sinh^2((1-s)\lambda\eta_k) \sin^2(st_k)} \\ &= - \frac{(1-s)\lambda\eta_k \Omega_k^i \sin(st_k)}{[\cos(st_k) + \Omega_k^d]^2} (1 + O(\lambda^2 \eta_k^2)). \end{aligned} \quad (23.5.3)$$

As long as $x_k = (it_k, \Omega_k)$ is in the Poincaré patch, the denominators in eq. (23.5.3) are always bounded from below by $\cos(st_k) + \Omega_k^d$, which is strictly positive. To make the path mapped into the forward tube, we require that

$$\left(\frac{\eta_k [1 + \Omega_k^d \cos(st_k)]}{[\cos(st_k) + \Omega_k^d]^2}, -\frac{\eta_k \Omega_k^i \sin(st_k)}{[\cos(st_k) + \Omega_k^d]^2} \right) \succ \left(\frac{\eta_{k+1} [1 + \Omega_{k+1}^d \cos(st_{k+1})]}{[\cos(st_{k+1}) + \Omega_{k+1}^d]^2}, -\frac{\eta_{k+1} \Omega_{k+1}^i \sin(st_{k+1})}{[\cos(st_{k+1}) + \Omega_{k+1}^d]^2} \right) \quad (23.5.4)$$

for all $s \in [0, 1]$ and $k = 1, 2, \dots, n-1$. Here “ $x \succ y$ ” means that x is in the future light-cone of y . The above condition is easily satisfied by choosing $\eta_1 \gg \eta_2 \gg \dots \gg \eta_n$. \square

Since the Wick rotations of the flat-space correlator and the cylinder correlator start from the same Euclidean correlator (up to the scale factor), by choosing the analytic continuation path constructed in lemma 23.5.1 and by the uniqueness of analytic continuation, we see that the CFT correlator in the Poincaré patch is the same as in the Minkowski space, up to the scale factors.

Remark 23.5.1. The result in this section is valid for all n -point correlation functions. In the case of four-point functions, one can check it explicitly by computing the phase factors of ρ -variables.

23.6 OPE convergence in the sense of functions

In part III, we classify the four-point configurations in the Minkowski space and show the OPE convergence properties of each configuration class in appendix C.2. Our analysis was basically computing the range of cross-ratio z, \bar{z} and the numbers $N_t, N_{\bar{t}}$ that count how many times the analytic continuation path crosses the t-channel and u-channel cuts.

The same analysis works for the four-point configurations in the Minkowski cylinder. However, there are infinitely many such classes in the cylinder case because each configuration in the Minkowski space corresponds to infinitely many preimages in the Minkowski cylinder, and the four-point function may take different values at different preimages. For this reason, we cannot finish this work directly.

In this section, we would like to argue that the problem of classifying cylinder four-point configurations can be reduced to the cases of configurations the Poincaré patch.

For convenience, we use $x = (t, \Omega)$, instead of $x = (it, \Omega)$, to denote Minkowski cylinder points. The cross-ratios are given by the formula

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad (23.6.1)$$

where

$$x_{ij}^2 = \lim_{\epsilon \rightarrow 0^+} [\operatorname{ch}[\epsilon + i(t_i - t_j)] - \Omega_i \cdot \Omega_j], \quad (i < j). \quad (23.6.2)$$

From above we can compute the corresponding radial variables $\rho, \bar{\rho}$ and their phases.

By eq. (23.2.5), the four-point function in the Minkowski cylinder has the following s-channel expansion

$$G_4^L(c) = \frac{g(\rho, \bar{\rho})}{2^{2\Delta} [\cos(t_1 - t_2) - \Omega_1 \cdot \Omega_2]^\Delta [\cos(t_3 - t_4) - \Omega_3 \cdot \Omega_4]^\Delta}, \quad (23.6.3)$$

where $g(\rho, \bar{\rho})$ given by the power series (11.2.6). Analogous to eq. (18.1.1), we also have the t-channel and u-channel expansions. The above formula and its t-/u-channel version imply that in the Minkowski cylinder, the light-cone singularities are the configurations where

$$\cos(t_i - t_j) = \Omega_i \cdot \Omega_j \quad (23.6.4)$$

for some x_i, x_j pair. In the following discussion we exclude the light-cone singularities.

Given a four-point configuration $c = (x_1, x_2, x_3, x_4)$, if all x_k 's are in the Poincaré patch, then according to section 23.5, the OPE convergence properties are the same as in the Minkowski space.

Now we consider the general case, when x_k 's are not necessarily in the Poincaré patch. Without loss of generality we assume that

$$\cos(t_k) + \Omega_k^d \neq 0, \quad (k = 1, 2, 3, 4.) \quad (23.6.5)$$

Otherwise we can always shift c by a small time translation, which does not change the four-point function.

The cylinder conformal group has a \mathbb{Z} -center, generated by the following map

$$x = (t, \Omega) \mapsto Tx := (t + \pi, -\Omega). \quad (23.6.6)$$

Lemma 23.6.1. *Let $c = (x_1, x_2, x_3, x_4)$ be a four-point configuration in the Minkowski cylinder, $n = (n_1, n_2, n_3, n_4)$ and $T^n c = (T^{n_1} x_1, T^{n_2} x_2, T^{n_3} x_3, T^{n_4} x_4)$. Assuming that $x_{ij}^2 \neq 0$ for all $i < j$, we have*

$$\log(u(T^n c)) = \log(u(c)) + 2\pi i(n_3 - n_2), \quad \log(v(T^n c)) = \log(v(c)). \quad (23.6.7)$$

Given a (Euclidean) cylinder point $x = (\sigma, \Omega) \in \mathbb{R} \times S^{d-1}$, we define. The cross-ratios are given by the formula

Proof. By eq. (23.6.2) we have

$$\log[(T^{n_i} x_i - T^{n_j} x_j)^2] = \log(x_{ij}^2) + \pi i(n_i - n_j), \quad (i < j). \quad (23.6.8)$$

Then summing over all the x_{ij}^2 contributions in eq. (23.6.1), we get (23.6.7). \square

Remark 23.6.1. The causal ordering of $T^n c$ may be different from the causal ordering of c .

Remark 23.6.2. In eq. (23.6.7), there is no contribution from n_1 and n_4 . Therefore, shifting the first or the last operator by the center of the conformal group does not change $g(\rho, \bar{\rho})$ in eq. (23.6.3). However it produces some overall phase factor via the prefactor $(x_{12}^2 x_{34}^2)^{-\Delta}$.

Let N_t and N_u count the total (signed) number that z and \bar{z} cross the t- and u-channel cuts from above during the analytic continuation from Euclidean. Then lemma 23.6.1 implies

$$N_t(T^n c) = N_t(c) + n_3 - n_2, \quad N_u(T^n c) = N_u(c) - n_3 + n_2. \quad (23.6.9)$$

Therefore, the OPE classification problem in Minkowski cylinder is reduced to the same problem in Minkowski space, via the following procedure:

- Find $n = (n_1, n_2, n_3, n_4)$ such that $T^{-n} c$ is in the Poincaré patch.
- Find the causal ordering of $T^{-n} c$, compute $N_t(c), N_u(c)$ by eq. (23.6.9). (Here we assume that $N_t(T^{-n} c), N_u(T^{-n} c)$ are already known as the Minkowski space results.)
- Determine the OPE convergence properties in s-, t- and u-channels.

Example: bulk-point singularity

A bulk-point singularity is the configuration $c = (x_1, x_2, x_3, x_4)$ with

$$x_1 = (0, \Omega), \quad x_2 = (0, -\Omega), \quad x_3 = (\pi, \Omega'), \quad x_4 = (\pi, -\Omega'). \quad (23.6.10)$$

According to the above described scheme, we first find the proper n such that $T^{-n}c$ is in the Poincaré patch. The result is $n = (0, 0, 1, 1)$, and $T^{-n}c$ is a totally space-like configuration because all its points are in the time slice $t = 0$. By explicit computation we get

$$u(c) = \frac{4}{(1 + \Omega \cdot \Omega')^2}, \quad v(c) = \left(\frac{1 - \Omega \cdot \Omega'}{1 + \Omega \cdot \Omega'} \right)^2,$$

which implies that $z(c) = \bar{z}(c) = \frac{2}{1 + \Omega \cdot \Omega'} > 1$. So the s-channel OPE does not converge. For t- and u-channel OPE, since we know that in the Minkowski space, the totally space-like configuration with $z = \bar{z} > 1$ has $N_t = N_u = 0$, by (23.6.9) we get

$$N_t(c) = 1, \quad N_u(c) = -1. \quad (23.6.11)$$

So neither t- nor u-channel OPE converges. We conclude that there is no convergent OPE channel for bulk-point singularities.

As we hinted in section 19.6.3 that in the Minkowski space, the causal ordering (19.6.5) with $z, \bar{z} > 1$ has the same OPE convergence properties as the bulk-point singularities. Here we confirmed that indeed $z, \bar{z} > 1$ for bulk-point singularities. Also, the results of N_t and N_u here are the same as eq. (19.6.7) there.

Conclusions and Outlooks

In this thesis we have presented studies of functional and distributional properties of CFT four-point functions. An important feature of our analysis is that we only rely on Euclidean assumptions made in the modern conformal bootstrap approach, without assuming Wightman nor Osterwalder-Schrader axioms.

In part [I](#), we studied the CFT four-point functions in the cross-ratio space. We start with the well-known conformal block expansion of the CFT four-point function, which is convergent when the radial cross-ratios $\rho, \bar{\rho}$ are in the open unit disk. Using Vladimirov's theorem, we showed that both the correlation functions and conformal blocks are tempered distributions on the boundary $|\rho| = |\bar{\rho}| = 1$, i.e. they make sense when being smeared with a smooth test function. We showed that the conformal block expansion converges in the sense of distributions. In other words, the pairing of the correlation function with a test function can be evaluated by summing over the contributions from conformal blocks.

In the Lorentzian signature, there are four-point configurations where the correlation functions only converge in the distributional sense. In these cases, we can only compute the correlation function by pairing it with various test functions. Therefore, these results are important for understanding the nature of the correlation functions in the Lorentzian signature.

The results of part [I](#) also gives us a hint at a uniform description of the bootstrap functionals that in the conformal bootstrap. From the distribution point of view, the bootstrap functionals correspond to test functions and the swapping property corresponds to OPE convergence in the sense of distributions. We managed to rewrite some bootstrap functionals (e.g. derivatives at $z = \bar{z} = 1/2$) in terms of smooth test functions. However, the space of smooth test functions is too small to enough to contain all the bootstrap functionals. In the future research, it is would be interesting to find a sufficiently large space of test functions, which contains all the bootstrap functionals. A better understanding of this space will be helpful in deepening the analytical understanding of conformal bootstrap.

In part [II](#), we studied the relation of the modern Euclidean CFT axioms to the more traditional Osterwalder-Schrader and Wightman axioms. In our version of Euclidean CFT axioms, we only assumed a minimal set of assumptions, including real analyticity, conformal invariance, unitarity condition of two-point functions, reality constraints of three-point functions and a very weak form of convergent OPE. We showed that at least for two-, three- and four-point functions, the Euclidean CFT axioms imply both OS and Wightman axioms. In particular, an important conclusion is that the CFT four-point functions are tempered distributions in Lorentzian signature, and the conformal invariance holds in the infinitesimal sense. We also showed that the Lorentzian CFT four-point functions have convergent s-channel OPE in the sense of distributions, generalizing our results in part [I](#) and giving a derivation of classic Lorentzian CFT assumptions (by Mack [\[25\]](#)) from Euclidean axioms.

We noticed that the radial cross-ratios satisfy $|\rho|, |\bar{\rho}| < 1$ for four-point configurations in the forward tube, and $|\rho|, |\bar{\rho}| \leq 1$ in the Lorentzian regime. This agrees with the range of $\rho, \bar{\rho}$ we considered in part [I](#).

In part [III](#), we revisited the classic problem about the domain of analyticity of QFT correlators. We focus on CFT four-point functions which assume the Euclidean CFT axioms as in the previous parts. We established criteria of OPE convergence in s-, t- and u-channels. In Lorentzian signature, we gave a complete classification of the four-point configurations according to their causal orderings and range of cross-ratio variables z, \bar{z} . We checked the OPE convergence properties case by case according to the classification and gave the tables of results in appendix [C.2](#). Our results show that the domain of a general CFT four-point function is much bigger than general QFT four-point functions. On the other hand, there are still many regions of Lorentzian four-point configurations where there is no convergent OPE channel.

In our analysis, we used variables $\rho, \bar{\rho}$ to probe OPE convergence properties. It would be interesting if there are some other coordinates that can be used to discover more domain of CFT four-point functions. It is also an open question whether the old results in multi-variable complex analysis, such as envelope of analyticity, can be applied to this problem.

In part IV, we briefly previewed on the generalizations to two directions. The first generalization is about the CFT four-point functions with arbitrary bosonic spinning operators. We showed that for generic internal scaling dimensions, the conformal partial wave has analytic continuation to the forward tube, and it satisfies a Cauchy-Schwarz type inequality. There is a discrete set of exceptional internal scaling dimensions, for which we do not know whether the same conclusions hold. Based on the existence of conformal partial waves in the Euclidean signature and its continuity as a function of internal scaling dimension, we conjecture that the answer is yes if the scaling dimension is above the unitarity bound. The analyticity and power-law bound of the spinning four-point functions follow from these two properties of the conformal partial waves. The remaining derivation of the Wightman axioms are the same as the scalar case.

The second generalization is about the scalar CFT four-point functions in the Minkowski cylinder. Using the same technique as the scalar four-point function in the flat space, we proved Wightman axioms, conformal invariance and OPE convergence in the sense of distributions. Since the conformal transformations are globally well-defined in the Minkowski cylinder, the conformal invariance of the cylinder four-point functions hold not only in the infinitesimal form, but also in the finite form. We showed that up to scaling prefactors, the n -point functions in the Poincaré patch agrees with n -point functions in the Minkowski space.

We also generalized the OPE classification results in part III to the cylinder case. Using the center elements of the cylinder conformal group, we can map a generic cylinder four-point configuration into the Poincaré patch. Using such maps, we show that for the OPE convergence properties in the sense of functions, the classification of the four-point configurations in the Minkowski cylinder can be reduced to the case of the Poincaré patch, i.e. the results in part III.

Appendix A

Appendices of Part I

A.1 Lorentzian 4pt correlator with no convergent OPE channel

In this section we will give an example of a Lorentzian 4pt configuration in which there's no convergent OPE channel. For simplicity let's consider the correlators of identical scalar operators. Recall that, in a general QFT, Lorentzian correlators can be recovered from Euclidean correlators by analytic continuation. Starting from a configuration of Euclidean points $x_i = (\tau_i, \mathbf{x}_i)$ with ordered times

$$\tau_1 > \tau_2 > \dots > \tau_n, \quad (\text{A.1.1})$$

we analytically continue each time variable as $\tau_i = \epsilon_i + it_i$ and take the limit $\epsilon_i \rightarrow 0$, preserving the ordering of real parts. The result is interpreted as the Lorentzian correlator at (Lorentzian) points $y_i = (t_i, \mathbf{x}_i)$. Schematically:

$$\langle 0 | \phi(t_1, \mathbf{x}_1) \dots \phi(t_n, \mathbf{x}_n) | 0 \rangle := \lim_{\substack{\epsilon_i \rightarrow 0 \\ \epsilon_1 > \dots > \epsilon_n}} \langle \phi(\epsilon_1 + it_1, \mathbf{x}_1) \dots \phi(\epsilon_n + it_n, \mathbf{x}_n) \rangle \quad (\text{A.1.2})$$

Now we will apply this to a 4pt function in a CFT. In a CFT, this analytic continuation can be performed starting from Eq. (2.0.1). We just complexify all Euclidean times as described above, and then take the limit. It is easy to see (exercise) that the distances x_{ij}^2 do not vanish in this process, except perhaps at the very end if the Lorentzian points y_i are lightlike separated. We will be interested in the case when all points are spacelike or timelike separated. So the prefactor in Eq. (2.0.1) is thus analytically continued (notice that there is an interesting phase for timelike separation).

In order to analytically continue the factor $g(u, v)$, we will use the existence of the conformal block expansion (2.0.5) which as mentioned there is convergent for $|\rho|, |\bar{\rho}| < 1$ ("OPE convergence region"). Concretely, we are instructed to compute u, v corresponding to complexified Euclidean times, then evaluate z, \bar{z} defined by (2.0.4), which gives

$$z, \bar{z} = \frac{1}{2}(1 + u - v \pm \sqrt{(1 + u - v)^2 - 4u}), \quad (\text{A.1.3})$$

then evaluate the corresponding $\rho, \bar{\rho}$ via (2.0.3), and finally stick these into the expansion (2.0.5). This procedure defines an analytic function of τ_i as long as $|\rho|, |\bar{\rho}| < 1$.¹ The question then is if this condition will hold all along the analytic continuation curve needed to recover the Lorentzian correlator, including the endpoint. If this happens, Lorentzian correlator can be computed by summing up a convergent expansion, in particular it is non-singular.

Above we describe how to use the s -channel expansion for the analytic continuation. A priori we can also use the t - and u -channels for this purpose, starting from the t - and u -channel versions of Eq. (2.0.1):

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \frac{1}{(x_{23}^2)^{\Delta_\phi} (x_{14}^2)^{\Delta_\phi}} g(u_t, v_t) = \frac{1}{(x_{13}^2)^{\Delta_\phi} (x_{24}^2)^{\Delta_\phi}} g(u_u, v_u). \quad (\text{A.1.4})$$

¹Note that even though z, \bar{z} will have a branch point when $(1 + u - v)^2 - 4u = 0$, the function $g(u, v)$ is symmetric under the interchange of z, \bar{z} and will remain analytic as a function of complexified Euclidean times.

The cross ratios u_t, v_t are obtained from u, v via $x_1 \leftrightarrow x_3$, and u_u, v_u via $x_2 \leftrightarrow x_3$. The functions $g(u_t, v_t), g(u_u, v_u)$ can be computed via the corresponding conformal block expansions with their own regions of analyticity set by the conditions $|\rho_t|, |\bar{\rho}_t| < 1$ and $|\rho_u|, |\bar{\rho}_u| < 1$.

It is not a priori clear and requires a separate analysis, which OPE channel, if any, is convergent for a given Lorentzian configuration. The answer turns out to depend, generically, only on the causal structure of the configuration (who is timelike, who is spacelike). The OPE can stop converging in two ways: either at the end point of the analytic continuation, or somewhere along the way. As we will show in [40], for the s -channel we always have $|\rho|, |\bar{\rho}| \leq 1$, so OPE converges along the way but may diverge at the end point. For other channels the OPE may start diverging already along the way.

We will give an exhaustive discussion of these phenomena, for all possible causal structures, in a later publication [74]. Here we will just give an extreme example of a configuration where all channels diverge.

Consider the causal ordering

$$y_3 \rightarrow y_1 \rightarrow y_4 \rightarrow y_2, \quad (\text{A.1.5})$$

where $y_i \rightarrow y_j$ means that y_i is in the past open lightcone of y_j . We pick some points (it_i, \mathbf{x}_i) corresponding to this causal ordering, as well as some initial Euclidean times ϵ_i satisfying the ordering $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$, and consider a curve of complexified points corresponding to these initial and final positions. E.g. we can use linear interpolation:

$$x_i(\theta) = ((1 - \theta)\epsilon_i + \theta it_i, \mathbf{x}_i), \quad \theta \in [0, 1]. \quad (\text{A.1.6})$$

We choose the initial point with $|\rho|, |\bar{\rho}| < 1$, and we would like to see if this condition stays true along this curve. For this it is enough to evaluate z, \bar{z} and see if they cross the cut $[1, +\infty)$ which corresponds to $|\rho| = 1$. This is how the check is carried out in practice for the s -channel. For the t - and u -channel, we have the same check in terms of z_t, \bar{z}_t and z_u, \bar{z}_u . But in fact we have relations

$$z_t = 1 - z, \quad z_u = 1/z \quad (\text{A.1.7})$$

and similarly for \bar{z} . These relations map the $[1, +\infty)$ cut on $(-\infty, 0]$ and $[0, 1]$, respectively. Thus we don't have to redo the analysis for z_t, \bar{z}_t and z_u, \bar{z}_u separately, we just have to watch if the s -channel z, \bar{z} crosses these additional cuts to conclude about the convergence of the t - and u -channel OPEs.

In practice, we just pick some numerical values for the initial and final points (respecting the orderings), plot the curves $z(\theta), \bar{z}(\theta)$ and see what they do. For the causal ordering (A.1.5), we get the plot shown in Fig. A.1.1. To draw the plot we picked numerical values:

$$\begin{aligned} \epsilon_1 = 4, & & \epsilon_2 = 3, & & \epsilon_3 = 2, & & \epsilon_4 = 0, \\ y_1 = (2, 0, 0, 0), & & y_2 = (20, 0, 0, 0), & & y_3 = (0, 0.9, 0, 0), & & y_4 = (3, 0, 0, 0), \end{aligned} \quad (\text{A.1.8})$$

where $y_i = (t_i, \mathbf{x}_i)$. Any other initial point $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$ and the final point corresponding to the ordering (A.1.5) gives rise to a topologically equivalent configuration of curves.

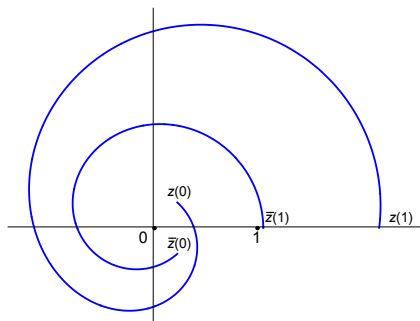


Figure A.1.1: The curves $z(\theta)$ and $\bar{z}(\theta)$ for the causal ordering $y_3 \rightarrow y_1 \rightarrow y_4 \rightarrow y_2$.

We see that the curves $z(\theta), \bar{z}(\theta)$ touch the $[1, \infty)$ cut at $\theta = 1$ but do not cross it at the intermediate values of θ . This means that $|\rho| = |\bar{\rho}| = 1$ at the corresponding Lorentzian configuration. Furthermore, both curves cross the

t -channel cut $(-\infty, 0]$, which according to the above discussion means $|\rho_t| > 1$, $|\bar{\rho}_t| > 1$. One of the two curves also crosses the u -channel cut $[0, 1]$, which means $|\rho_u| > 1$. We conclude that the Lorentzian configuration under study is outside the region of OPE convergence of any of the three channels.

The given recipe to determine which channels diverge would require some care in situations when a curve crosses a cut and then goes back, or when the $z(\theta)$ and $\bar{z}(\theta)$ curves cross the same cut in opposite directions. We will discuss these subtleties and their interpretation in [74]. In the given example they do not occur, so our conclusion that all three channels diverge is robust.

Another comment is in order concerning the 2d CFT case. In this case, the region of analyticity of 4pt functions is larger than $|\rho|, |\bar{\rho}| < 1$, being instead given by the condition $|q|, |\bar{q}| < 1$ [44] where q is Al. Zamolodchikov's uniformizing variable. Using this variable, one can show that the Lorentzian 4pt function in a 2d CFT is analytic for all possible causal orderings away from null cone singularities [45].

A.2 Proof of lemma 3.5.3

To prove the first inequality,² we start by constructing a map $\tilde{\varphi}(w)$ from \mathbb{D} into \mathbb{D} which satisfies $\tilde{\varphi}(0) = 0$. This is achieved by a fractional linear transformation as follows

$$\tilde{\varphi}(w) = \frac{\varphi(w) - \varphi(0)}{1 - \varphi(w)\overline{\varphi(0)}}. \tag{A.2.1}$$

Now, Schwarz lemma implies that $|\tilde{\varphi}(w)| \leq |w|$ and so $1 - |\tilde{\varphi}(w)| \geq 1 - |w|$. At the same time, we find

$$1 - |\tilde{\varphi}(w)|^2 = \frac{(1 - |\varphi(w)|^2)(1 - |\varphi(0)|^2)}{(1 - \varphi(w)\overline{\varphi(0)})(1 - \varphi(0)\overline{\varphi(w)})} \leq C(1 - |\varphi(w)|), \quad C = 2\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \tag{A.2.2}$$

where the first equality follows by a short computation from (A.2.1), and to get the inequality we bounded some factors using $|\varphi(w)| \leq 1$. Furthermore, since $1 - |\tilde{\varphi}(w)|^2 = (1 - |\tilde{\varphi}(w)|)(1 + |\tilde{\varphi}(w)|) \geq 1 - |\tilde{\varphi}(w)|$, we find

$$1 - |w| \leq 1 - |\tilde{\varphi}(w)| \leq 1 - |\tilde{\varphi}(w)|^2 \leq C(1 - |\varphi(w)|). \tag{A.2.3}$$

To prove the second inequality, it will be important that $\varphi(w)$ is one-to-one and that $\varphi(w) \neq 0$.³ Under these conditions the function $\frac{1}{\varphi(w)}$ is holomorphic and one-to-one. Such functions from \mathbb{D} onto a subset of \mathbb{C} are called univalent, or schlicht [142]. The shifted and rescaled function

$$h(w) = -\frac{\varphi(0)^2}{\varphi'(0)} \left(\frac{1}{\varphi(w)} - \frac{1}{\varphi(0)} \right), \tag{A.2.4}$$

is then also univalent, and in addition satisfies normalization conditions $h(0) = 0$ and $h'(0) = 1$. A basic result about normalized univalent functions is the Growth Theorem ([142], Theorem 2.6)

$$|h(w)| \leq \frac{|w|}{(1 - |w|)^2}. \tag{A.2.5}$$

This immediately implies the second bound in (3.5.3).

A.3 Comments on the proof of theorem 4.3.1

Compared to theorem 3.2.1, theorem 4.3.1 has only two essentially new ingredients. First, we now have the freedom of choosing $v \in V$ so we want to show that this choice doesn't matter, and second, we have to prove that the

²See [141], Exercise 6.3 for similar arguments. For this result it's only important that $|\phi(w)| \leq 1$. That it's one-to-one and avoids the cut does not matter.

³It won't be important that it avoids the rest of the cut.

boundary value is holomorphic in w . Without these two ingredients, the proof of section 3.3 goes through without any essential modifications.

Let us briefly recall the main steps of that proof, but now in the context of theorem 4.3.1.⁴ First, for a Schwartz test function $f(x)$ we define

$$L_v(w, \epsilon) = \int d^d x g(w, x + iv\epsilon) f(x). \quad (\text{A.3.1})$$

Using integration by parts, we show that

$$\partial_\epsilon^k L_v(w, \epsilon) = (-i)^k \int d^d x g(w, x + iv\epsilon) v^{\mu_1} \cdots v^{\mu_k} \partial_{\mu_1} \cdots \partial_{\mu_k} f(x). \quad (\text{A.3.2})$$

We then use this identity and the slow-growth condition on g (4.3.1) to bound

$$|\partial_\epsilon^k L_v(w, \epsilon)| \leq \frac{C_k}{\epsilon^{2K}}. \quad (\text{A.3.3})$$

for some $C_k > 0$ that is proportional to some semi-norm of f . In what follows, it will be important to us how C_k depends on v . It is easy to see that

$$|\partial_\epsilon^k L_v(w, \epsilon)| \leq \frac{C'_k \|v\|_\infty^k \|v\|_2^{-2K}}{\epsilon^{2K}}. \quad (\text{A.3.4})$$

for some $C'_k > 0$ that is independent of v . Furthermore, since the bound (4.3.1) is independent of w , C'_k is also independent of w .⁵ Then we use the obvious analogue of (3.3.5) starting from sufficiently large k to conclude

$$|\partial_\epsilon L_v(w, \epsilon)| \leq C \|v\|_\infty^k \|v\|_2^{-2K} \quad (\text{A.3.5})$$

for some $C > 0$ proportional to a semi-norm of f . This immediately implies that

$$L_v(w, \epsilon) = - \int_\epsilon^{\epsilon_0} \partial_\epsilon L_v(w, \epsilon) + L_v(w, \epsilon_0) \quad (\text{A.3.6})$$

is continuous down to $\epsilon = 0$ and that thus defined $L_v(w, 0)$ depends continuously on f in $\mathcal{S}(\mathbb{R}^d)$. The slight refinements that we made to the bound (A.3.5), i.e. observing that it holds uniformly in w and exhibiting its dependence on v , allow us to make the following statement: the limit $L_v(w, \epsilon) \rightarrow L_v(w, 0)$ is reached uniformly on compact sets $\mathcal{K} \subset U$ in w and on compact sets $\mathcal{V} \subset V$ in v (recall that V doesn't contain 0). This statement is the key in proving that the limit is independent of $v \in V$ and is holomorphic in w .

The fact that $L_v(w, 0)$ is holomorphic in w is now indeed straightforward, since $L_v(w, \epsilon)$ is holomorphic in w for $\epsilon > 0$.⁶ To prove that it is independent of v requires a bit more work. Take $v_1, v_2 \in V$ and write

$$\begin{aligned} L_{v_1}(w, \epsilon) - L_{v_2}(w, \epsilon) &= \int d^d x (g(w, x + iv_1\epsilon) - g(w, x + iv_2\epsilon)) f(x) \\ &= \int d^d x \int_0^1 dt \partial_t g(w, x + iv(t)\epsilon) f(x), \quad v(t) = tv_1 + (1-t)v_2 \\ &= -i\epsilon \int_0^1 dt \int d^d x g(w, x + iv(t)\epsilon) (v_1 - v_2) \cdot \partial f(x) \\ &= -i\epsilon \int_0^1 dt \tilde{L}_{v(t)}(w, \epsilon). \end{aligned} \quad (\text{A.3.7})$$

⁴Our proof is an adaptation of the proof of theorem 7.2.6 in [50].

⁵This holds on compact subsets $\mathcal{K} \subset U$, see footnote 4.

⁶The standard argument is as follows. Suppose holomorphic functions h_n converge uniformly to some function h . Then, first of all, h is continuous because h_n are and the limit is uniform. Second, the uniform limit can be exchanged with contour integration. Since integrals of h_n over closed curves are 0, so are the integrals of h . By Morera's theorem, this implies holomorphicity of h .

where $\tilde{L}_v(w, \epsilon)$ is defined as $L_v(w, \epsilon)$ but with $(v_1 - v_2) \cdot \partial f(x)$ instead of $f(x)$. Since $(v_1 - v_2) \cdot \partial f(x)$ is also a test function, we have that by the same arguments as the above, $\tilde{L}_v(w, \epsilon)$ converges to a finite limit $\tilde{L}_v(w, 0)$ uniformly in v on compacts of V . This implies that the integral

$$\int_0^1 dt \tilde{L}_{v(t)}(w, \epsilon) \tag{A.3.8}$$

has a finite limit as $\epsilon \rightarrow 0$, and thus

$$L_{v_1}(w, \epsilon) - L_{v_2}(w, \epsilon) = -i\epsilon \int_0^1 dt \tilde{L}_{v(t)}(w, \epsilon) \rightarrow 0. \tag{A.3.9}$$

Appendix B

Appendices of Part II

B.1 Lorentzian CFT literature

Recent years have seen an explosion of the uses of Lorentzian CFT, motivated in particular by the conformal bootstrap applications. In this appendix we will mention some of these works, and comment on their underlying assumptions. See also [143] for a modern pedagogical introduction to Lorentzian CFT.

Conformal collider bounds. One of the first “modern” Lorentzian CFT results was obtained in [16]. This work considered a thought experiment, creating a CFT state via a (smeared) local operator and measuring energy coming out at null infinity in a particular direction, integrated over time. On physical grounds, one expects $\langle \Psi | \int dx^- T_{--} | \Psi \rangle \geq 0$ for any state (“averaged null energy condition” - ANEC). One interesting case is of 3-point functions $\langle \mathcal{O}^\dagger T_{\mu\nu} \mathcal{O} \rangle$ where \mathcal{O} has nontrivial spin, when there are several independent OPE coefficients multiplying different tensor structures allowed by conformal symmetry. In this case ANEC implies that certain linear combinations of these OPE coefficients must be non-negative (“conformal collider bounds”). Interference effects can be used to strengthen conformal collider bounds to provide explicit lower bounds [144], while combining conformal collider bounds with stress-tensor Ward identities leads to constraints on operator dimensions which are sometimes stronger than standard unitarity bounds [145]. See below for work aiming to justify ANEC, or to derive conformal conformal bounds directly without using ANEC.

Light-cone bootstrap. Refs. [17, 18] were the first to notice that some bootstrap constraints become more visible in the Lorentzian signature. These references pioneered the “analytic light-cone bootstrap” which studies conformal four point functions in the regime of $0 < z, \bar{z} < 1$ real, i.e. in the kinematics of Fig. 15.0.1 when the point x_2 is spacelike with respect to x_1, x_3 . By studying the light cone limit $z \rightarrow 0$ at fixed \bar{z} of one OPE channel and requiring that it should be reproduced by the crossed channel, they argued that, in any CFT for $d > 2$, the OPE should contain a series of operators of arbitrarily large spin and twist asymptoting to a particular value. The original argument has some caveats (see the discussion in [146], App. F) and a mathematically rigorous proof is lacking. It would be nice to provide such a proof, given the extreme importance of the light-cone bootstrap in the modern bootstrap literature. There is little doubt that the light-cone bootstrap results are correct. Numerical bootstrap studies of the critical 3d Ising and the $O(2)$ models [53, 147] have found the series of operators predicted by the light-cone bootstrap see also [148]). Ref. [149] used the light-cone bootstrap to derive the conformal collider bounds of [16] without using ANEC.

Causality constraints. Refs. [3, 19, 20] pioneered the study of causality constraints for CFT 4-point functions. In particular Ref. [3] pointed out that the z, \bar{z} and $\rho, \bar{\rho}$ expansions are sufficient to construct Lorentzian 4-point functions for many kinematic configurations and show local commutativity (i.e. that spacelike-separated operators commute). See also note 22. These techniques led to a proof of ANEC [20]. As mentioned in footnote 21, some steps in these papers are not completely rigorous. See App. B.1.1 below for a more detailed review of [3].

Bulk point singularity. Ref. [45] studied the CFT 4-point function on the Lorentzian cylinder focusing on “bulk-point” configurations which correspond to scattering events in AdS/CFT [150–154]. Using a local AdS dual description, one may suspect that the 4-point function should be singular at such configurations. However, on the

boundary CFT side, one does not see this singularity in perturbation theory in $d = 2$ and $d = 3$ dimensions [45]. In $d = 2$, Ref. [45] showed non-perturbatively that the CFT 4-point function is analytic everywhere away from light cones (in particular regular at bulk-point configurations). This assumes Virasoro symmetry and unitarity and uses Zamolodchikov’s q -variables [123]. What happens non-perturbatively in $d \geq 3$ (or in $d = 2$ in the absence of the local stress tensor) is still an open problem. Note that at bulk-point configurations, the ρ -expansion of the CFT 4-point function does not absolutely converge in s-channel (as $|\rho| = |\bar{\rho}| = 1$ there) and diverges in t-,u-channels [74]. In this paper we only considered the CFT 4-point functions in flat space, but by the same strategy we will show in [41] that the Wightman axioms also hold for CFT 4-point functions on Lorentzian cylinder. In particular, this will show that the CFT 4-point functions are well defined at bulk-point configurations in the sense of tempered distributions (but it will not settle the question of their analyticity there).

Lorentzian inversion formula. Ref. [22] introduced an analogue of Froissart-Gribov formula in the context of conformal field theory, which is now known as the Lorentzian inversion formula (LIF). This formula computes the OPE data of a scalar 4-point function in terms of a Lorentzian integral of this 4-point function. The OPE data is extracted in the form of a function $C(\Delta, \ell)$. For integer ℓ , the function $C(\Delta, \ell)$ encodes the scaling dimensions of exchanged primary operators of spin ℓ in the positions of poles in Δ , and the corresponding OPE coefficients are encoded in residues. LIF has many interesting properties, such as analyticity in ℓ , and suppression of double-twist operators when a cross-channel conformal block expansion is used under the integral. The original derivation of [22] was done in cross-ratio space. The formula was re-derived in position space in [23]. The derivation was further simplified and generalized in [89].

Among other applications, LIF has been used to systematize and extend many of the results of light-cone bootstrap (see, e.g., [147, 155–161]). Similarly to light-cone bootstrap, this application is not completely rigorous simply due to the fact that LIF expresses $C(\Delta, \ell)$ in terms of an integral, and the local operators correspond to singularities of $C(\Delta, \ell)$. In other words, the integral has no chance of converging near the values of Δ, ℓ relevant to local operators, except perhaps for leading-twist operators (see [148] for steps in this direction). This necessarily makes any conclusions about anomalous dimensions of local operators reliant on additional assumptions. These are easy to justify in some perturbative expansions, but in non-perturbative setting do not appear to have been solidly understood.

Light-ray operators. Ref. [89] generalized LIF to external operators with spin and uncovered an interesting relation to Knapp-Stein intertwining operators, especially to what they called the light transform. They interpreted the analyticity of LIF in ℓ in terms of families of non-local non-integer-spin operators, the light-ray operators. These operators are defined for generic complex ℓ and reduce to light-transforms (null integrals) of local operators for integer spins. More recently, light-ray operators have been used to understand an OPE for event-shape observables such as energy-energy correlators in CFT [21, 162, 163] (see also [164, 165]). The light-ray operators correspond to poles in Δ of $C(\Delta, \ell)$, and the issues with convergence of LIF described above prevent a simple rigorous proof of their non-perturbative existence. (E.g., for generic ℓ , $C(\Delta, \ell)$ could have cuts or a natural boundary of analyticity in Δ .) It would be interesting to find such a proof. In addition to clarifying the nature of light-ray operators, it would probably also have a bearing on the light-cone bootstrap results discussed above.

Conformal Regge theory provides a way to understand Minkowski correlators in Regge limit, and was developed in Refs. [46–48, 166]. Regge limit in CFT is a limit of a 4-point function in Lorentzian signature in which \mathcal{O}_2 approaches the “image of \mathcal{O}_3 in the next Poincaré patch,” in 4-point function with the ordering

$$\langle \mathcal{O}_4 \mathcal{O}_3 \mathcal{O}_2 \mathcal{O}_1 \rangle. \tag{B.1.1}$$

The operators \mathcal{O}_1 and \mathcal{O}_4 are kept spacelike separated, with \mathcal{O}_1 in past of \mathcal{O}_2 and \mathcal{O}_4 in the future of \mathcal{O}_3 .¹ The image of \mathcal{O}_3 in the next Poincaré patch is the first point on Minkowski cylinder where all future-directed null geodesics from \mathcal{O}_3 meet. A lot of interest in Regge limit comes from its interpretation as bulk high-energy scattering through AdS/CFT. Kinematically, this limit is somewhat similar to the $\mathcal{O}_2 \rightarrow \mathcal{O}_3$ limit because \mathcal{O}_3 and its image in the next Poincaré patch transform in the same way under conformal group. For example, the cross-ratios $z_t, \bar{z}_t \rightarrow 0$ in Regge limit. (Here by z_t, \bar{z}_t we mean the cross-ratios for t-channel $\mathcal{O}_2 \times \mathcal{O}_3$.) However, they do so after \bar{z}_t crosses the cut $[1, \infty)$, and so in terms of $\rho_t, \bar{\rho}_t$ we have $\rho_t \rightarrow 0$ and $\bar{\rho}_t \rightarrow \infty$. Therefore, the $\mathcal{O}_2 \times \mathcal{O}_3$ OPE is divergent. Conformal Regge theory gives a way to resum the $\mathcal{O}_2 \times \mathcal{O}_3$ OPE in a way that exhibits a dominant contribution from a “Reggeon” exchange, which is an example of a light-ray operator. Justification for

¹In a symmetric version of the limit, which is related to the one described here by a conformal transformation, the operators \mathcal{O}_1 and \mathcal{O}_4 approach each other’s images in the same way as \mathcal{O}_2 and \mathcal{O}_3 do.

this resummation, which involves analytic continuation of OPE data in spin, comes from LIF (which historically was understood after Conformal Regge theory was established). In the context of our paper, it would be interesting to understand whether such resummations can be made rigorous enough (in axiomatic sense) and used to prove that Minkowski correlators are functions in regions where so far only temperedness has been proven.² For this it might not be necessary to understand the Reggeon or more general light-ray operators, since the resummation procedure can be stopped at a point where the correlator is expressed as an integral of $C(\Delta, \ell)$ over a region where LIF converges. See [119] for progress on these questions.

Works of Gillioz, Luty et al. Papers by this group of authors are characterized by the systematic use of momentum space in Lorentzian CFT. So, Refs. [92, 107] computed Lorentzian momentum space 3-point functions (3 scalars and scalar-scalar-spin ℓ) by solving the conformal Ward identities. In momentum space, it's also possible to form conformal blocks by gluing 3-point functions [107]. See also notes 2, 3.

Ref. [139] carried out this program quite explicitly in 2d CFT, with an eye towards eventual conformal bootstrap applications. They stressed that the momentum conformal block expansion generally converges only in the sense of distributions—one of the first mentions of distributional convergence in the modern CFT literature. For some momenta configurations, they argued that the momentum conformal blocks can be pointwise bounded by the position conformal blocks with an appropriately chosen real $z \in (0, 1)$. For such configurations the momentum expansion converges in the ordinary sense of functions. The same work also proposes a bootstrap equation in the momentum space, obtained by transforming the local commutativity constraint multiplied by a test function selecting configurations with a spacelike pair of points (however, examples of test functions chosen in [139] may be too singular).

Refs. [167–169] studied the Fourier transform of the time-ordered Minkowski 4-point function in relation to various interesting physics questions. Note that, as mentioned in the conclusions, time-ordered Minkowski CFT 4-point functions have not yet been rigorously defined as a distributions. The Fourier transform depends on 4 momenta p_i , and to reduce functional complexity it is interesting to take some or all of these momenta lightlike, $p_i^2 \rightarrow 0$.

So, Ref. [167] considered the Fourier transform of the connected time-ordered 4-point function $\langle \mathcal{T}\{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\} \rangle_c$. Here they worked with operators of scaling dimension $\Delta_i > d/2$ for which the Fourier transform is expected to have a finite limit as $p_i^2 \rightarrow 0$.³ Ref. [167] proposed a Lorentzian CFT analogue of the optical theorem:

$$\text{Im } \mathcal{M}_{1234}(s, t) = \sum_{\mathcal{O} \neq 1} f_{12\mathcal{O}} f_{43\mathcal{O}}^* \mathcal{N}_{\mathcal{O}}(q) \langle \overline{\mathcal{T}}\{\widehat{\mathcal{O}}_1(p_1)\widehat{\mathcal{O}}_2(p_2)\} \mathcal{O}^\dagger(0) \rangle \langle \mathcal{O}(0) \mathcal{T}\{\widehat{\mathcal{O}}_3(p_3)\widehat{\mathcal{O}}_4(p_4)\} \rangle, \quad (\text{B.1.2})$$

where \mathcal{M}_{1234} is proportional to the Fourier transform of $\langle \mathcal{T}\{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\} \rangle_c$, f_{ijk} is the same as in (7.3.8), $\mathcal{N}_{\mathcal{O}}(q)$ is some normalization factor at $q = p_1 + p_2 (= -p_3 - p_4)$, and $\widehat{\mathcal{O}}$ denotes the Fourier transform of \mathcal{O} . Eq. (B.1.2) is supposed to hold in the following kinematic region in the momentum space:

$$p_i^2 = 0, \quad s = (p_1 + p_2)^2 > 0, \quad t = (p_1 + p_3)^2 \leq 0,$$

and was derived from a combinatorial operator identity

$$\sum_{k=0}^n (-1)^k \sum_{\sigma \in S_n} \frac{1}{k!(n-k)!} \overline{\mathcal{T}}\{\mathcal{O}_{\sigma_1}(x_{\sigma_1}) \dots \mathcal{O}_{\sigma_k}(x_{\sigma_k})\} \mathcal{T}\{\mathcal{O}_{\sigma_{k+1}}(x_{\sigma_{k+1}}) \dots \mathcal{O}_{\sigma_n}(x_{\sigma_n})\} = 0, \quad (\text{B.1.3})$$

summing over all permutations, with $\mathcal{T}(\overline{\mathcal{T}})$ time ordering (anti-time ordering). Note that the use of this identity may be not fully safe in the distributional context, as it arises from a non-smooth partition of unity.

The CFT optical theorem (B.1.2) was used in Ref. [167] to study the scale anomalies that appear in a specific class of CFT correlation functions. In fact, unlike Wightman functions which are conformally invariant distributions, time-ordered correlator distributions may, for certain scaling dimensions, contain pieces which violate scale invariance. Thus the scale anomaly describes the violation of dilatation Ward identities, and in position space it is an ultralocal term, located at coincident points. In Fourier space, scale anomaly translates into a nonzero imaginary part of $\mathcal{M}_{1234}(s, t)$ at $t = 0$. Eq. (B.1.2) then computes the scale anomaly coefficient through a positive definite sum rule

²In the classic Regge limit there is a channel in which the OPE converges regularly, but it is possible that some causal orderings can be relaxed while keeping the resummation procedure valid.

³We thank Marc Gillioz for explanations of his work and in particular of the distinction between the high dimension case discussed here and the low dimension case below.

(in particular predicts that it is positive). These scale anomalies also appear in the Euclidean signature, and a similar sum rule for anomaly coefficients can also be found in Ref. [167]. However, the Euclidean sum rule is not positive definite unlike the Lorentzian case.

Ref. [167] tested the above ideas for the scalar 4-point function of external dimensions $\Delta = 3d/4$. Ref. [168] then studied the more interesting case of the stress tensor 4-point function $\langle \mathcal{T}\{T_{\mu_1\nu_1}T_{\mu_2\nu_2}T_{\mu_3\nu_3}T_{\mu_4\nu_4}\} \rangle$ whose scale anomaly is proportional to the stress-tensor 2-point function coefficient c_T . Assuming that the $t \rightarrow 0$ limit is finite, the CFT optical theorem expresses c_T as a sum of positive contributions of all operators in the $T \times T$ OPE apart from the identity (the stress tensor contribution is known, proportional to c_T , and can be moved to the l.h.s.). The contributions from the scalars and the spin-2 operators are computed explicitly in [168].

The more recent Ref. [169] studied instead the Fourier transform of the time-ordered 4-point function (or Euclidean 4-point function) in the opposite case of the low external dimensions $\Delta_\phi < d/2$. Unlike in [167, 168], in this low dimension case the Fourier transform is singular as $p_i^2 \rightarrow 0$, and one obtains a finite quantity multiplying it by $(p_i^2)^{d/2-\Delta_\phi}$ before taking the limit, a CFT analogue of LSZ reduction. Doing so, they defined a ‘‘CFT scattering amplitude’’ $A(s, t, u)$ ($p_i^2 \rightarrow 0$ for all i) and a closely related ‘‘form factor’’ $F(s, t, u)$ where $p_i^2 \rightarrow 0$ only for $i = 1, 2, 3$. Because the limit $p_i^2 \rightarrow 0$ has to be taken one momentum at a time, crossing symmetry is not obvious. Ref. [169] also gave an alternative derivation, starting from the Mellin representation of the CFT 4-point function, where crossing symmetry of $F(s, t, u)$ and $A(s, t, u)$ follows from the crossing symmetry of the Mellin amplitude. In the future, crossing symmetric quantities $A(s, t, u)$ and $F(s, t, u)$ may turn out useful in a bootstrap analysis. It should be stressed that the results of [169] in no way contradict the usual lore that there are no S-matrices in interacting CFTs. In spite of the name adopted in [169], the existence of the quantity $A(s, t, u)$ does not imply that we can set up a wave-packet scattering experiment in a CFT. Wave-packets would quickly diffuse before reaching the interaction region, the singularity of $(p_i^2)^{\Delta_\phi-d/2}$ being a cut rather than a pole.

B.1.1 Review of Hartman et al [3]

Relating different orderings via analytic continuations. Here we will comment on some of the results of [3] in more details. The first part of this paper considers the Lorentzian CFT 4-point functions with operators $\mathcal{O}_1, \mathcal{O}_3, \mathcal{O}_4$ fixed at zero time and the spatial positions $0, \hat{e}_1$ and ∞ , while the operator \mathcal{O}_2 is inserted at Minkowski position $t_2\hat{e}_0 + y_2\hat{e}_1$. They consider four different operator orderings

$$\langle \mathcal{O}_2\mathcal{O}_1\mathcal{O}_3\mathcal{O}_4 \rangle, \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \rangle, \langle \mathcal{O}_3\mathcal{O}_2\mathcal{O}_1\mathcal{O}_4 \rangle, \langle \mathcal{O}_1\mathcal{O}_3\mathcal{O}_2\mathcal{O}_4 \rangle \quad (\text{B.1.4})$$

in the region of $0 < y_2 < 1/2$ and t_2 positive. As t_2 is increased from zero, the operator \mathcal{O}_2 , initially spacelike with respect to all other insertions, crosses the light cone first of \mathcal{O}_1 and then of \mathcal{O}_3 . With $z, \bar{z} = y_2 \pm t_2$, denoting $G(z, \bar{z}) = \langle \mathcal{O}_2\mathcal{O}_1\mathcal{O}_3\mathcal{O}_4 \rangle$, they give the following prescription to compute the correlators for the other orderings (Ref. [3], Eq. (3.22)):

$$\begin{aligned} \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4 \rangle &= G(z, \bar{z})_{z \rightarrow e^{-2\pi i}z}, \\ \langle \mathcal{O}_3\mathcal{O}_2\mathcal{O}_1\mathcal{O}_4 \rangle &= G(z, \bar{z})_{(\bar{z}-1) \rightarrow e^{-2\pi i}(\bar{z}-1)}, \\ \langle \mathcal{O}_1\mathcal{O}_3\mathcal{O}_2\mathcal{O}_4 \rangle &= G(z, \bar{z})_{z \rightarrow e^{-2\pi i}z, (\bar{z}-\bar{z}_0) \rightarrow e^{-2\pi i}(\bar{z}-\bar{z}_0)}. \end{aligned} \quad (\text{B.1.5})$$

Their justification of this prescription relied on some presumed analyticity properties of $G(z, \bar{z})$ which, to our knowledge, have never been shown in a general QFT context. Nevertheless we will see below that for CFTs Eq. (B.1.5) turns out to be true (with $\bar{z}_0 = 1$).

The real parameter \bar{z}_0 was introduced in [3] as the position of the first \bar{z} singularity of $G(z, \bar{z})_{z \rightarrow e^{-2\pi i}z}$. Their goal was to show that $\bar{z}_0 \geq 1$, which using (B.1.5) then implies local commutativity $\langle \mathcal{O}_1[\mathcal{O}_2, \mathcal{O}_3]\mathcal{O}_4 \rangle = 0$ for $\bar{z} < 1$ i.e. when \mathcal{O}_2 is spacelike to \mathcal{O}_3 . In our paper (Sec. 11.9) we presented a different way to understand and derive local commutativity which is closer to the classic literature: as we have reviewed there, it is a robust consequence of the existence of the analytic continuation to the forward tube, which we constructed.

Let’s see what it would take to justify (B.1.5). We define the function $G(z, \bar{z})$ by the $\mathcal{O}_2 \times \mathcal{O}_1$ OPE expansions, which converges for $|\rho|, |\bar{\rho}| < 1$, i.e. as long as z, \bar{z} stay away from $z, \bar{z} \in (+1, \infty)$. The points $z = 0$ and $\bar{z} = 0$ are branch points singularities with cuts which we put along the negative real axis. Note that the contours in (B.1.5) are all within the analyticity domain of $G(z, \bar{z})$. So the prescription (B.1.5) is meaningful. We still have to see if it

agrees with the rigorous definition which computes the Minkowski 4-point function by analytically continuing from the Euclidean region staying in the forward tube corresponding to the chosen operator ordering. We will see that it will indeed agree, but showing it for the last ordering will be subtle.

For definiteness we will focus on the region $t_2 > 1 - y_2$ i.e. $\bar{z} > 1$, where \mathcal{O}_2 crossed both light cones. The end point of the analytic continuation contour is always the same while the initial point depends on the operator ordering. E.g. for the ordering $\langle \mathcal{O}_2 \mathcal{O}_1 \mathcal{O}_3 \mathcal{O}_4 \rangle$ we have to pick initial Euclidean times $\varepsilon_2 > \varepsilon_1 > \varepsilon_3$, while for $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle$ we have $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$ etc. For any of these orderings, we denote

$$z_1, \bar{z}_1 = \pm i\varepsilon_1, \quad z_2, \bar{z}_2 = y_2 \pm i(\varepsilon_2 + it_2), \quad z_3, \bar{z}_3 = 1 \pm i\varepsilon_3 \tag{B.1.6}$$

and compute (in the limit $z_4, \bar{z}_4 = \infty$)

$$z = \frac{z_1 - z_2}{z_1 - z_3}, \quad \bar{z} = \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_1 - \bar{z}_3}. \tag{B.1.7}$$

We are interested in the curves which z, \bar{z} trace as the Euclidean times are scaled to zero and the Minkowski time t_2 from 0 to its final value. For the first three orderings the resulting curves are shown in Fig. B.1.1.

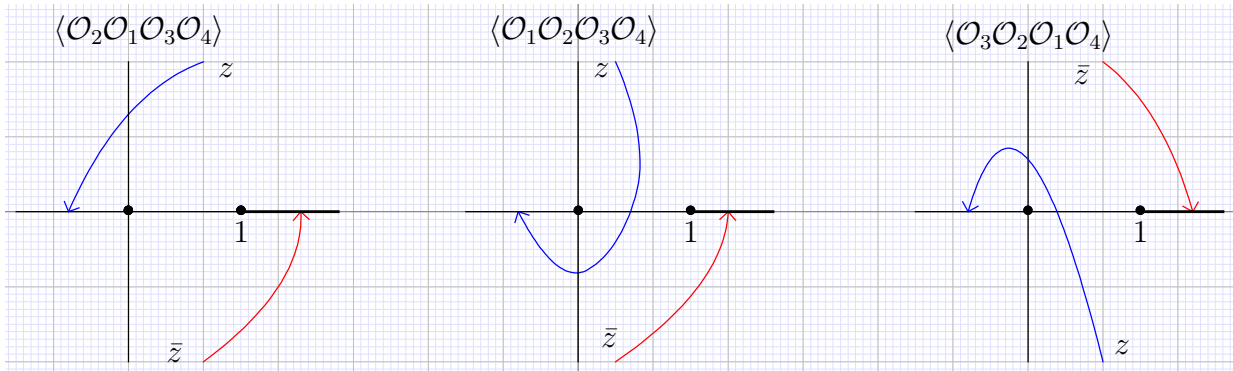


Figure B.1.1: z, \bar{z} curves for the analytic continuation from the Euclidean; the first 3 orderings.

We see that in all these cases, the curves lie in the analyticity domain of $G(z, \bar{z})$ i.e. they don't cross $(1, +\infty)$. For the first two cases this was guaranteed by our results that $|\rho|, |\bar{\rho}| < 1$ for the s-channel OPE expansion. For the third case it was not guaranteed but it also turns out to be true, by inspection. We also see that in all these 3 cases, the curves go around $z = 0$ and $\bar{z} = 1$ in agreement with (B.1.5).⁴

For the fourth ordering $\langle \mathcal{O}_1 \mathcal{O}_3 \mathcal{O}_2 \mathcal{O}_4 \rangle$ when we have to assign $\varepsilon_1 > \varepsilon_3 > \varepsilon_2$, the analytic continuation inside the forward tube gives the z, \bar{z} curves shown in Fig. B.1.2, while prescription (B.1.5) would correspond to Fig. B.1.3.

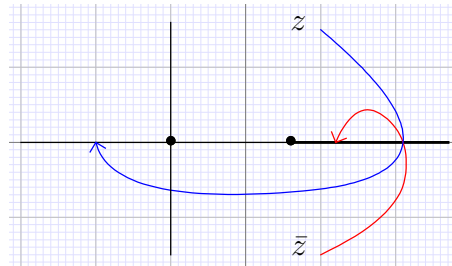


Figure B.1.2: z, \bar{z} curves for the analytic continuation from the Euclidean for the $\langle \mathcal{O}_1 \mathcal{O}_3 \mathcal{O}_2 \mathcal{O}_4 \rangle$ ordering.

The two figures are clearly not the same. Moreover the curves in the first figure cross $(+1, \infty)$ where the definition of $G(z, \bar{z})$ via the $\mathcal{O}_1 \times \mathcal{O}_2$ channel OPE expansion stops converging. Can we show that the analytic continuation in Fig. B.1.2 exists and that it agrees with the one in Fig. B.1.3?

⁴To see this more clearly in the third case, deform the curves continuously moving the initial z into the upper half plane and the initial \bar{z} into the lower half plane.

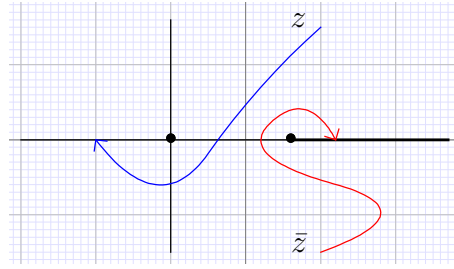


Figure B.1.3: z, \bar{z} curves for computing the $\langle \mathcal{O}_1 \mathcal{O}_3 \mathcal{O}_2 \mathcal{O}_4 \rangle$ ordering via (B.1.5).

For this, let us bring in the $\mathcal{O}_2 \times \mathcal{O}_3$ OPE expansion, which correspond to expanding in $z_t = 1 - z, \bar{z}_t = 1 - \bar{z}$ or in the corresponding $\rho_t, \bar{\rho}_t$. In the Euclidean region the two expansions agree. The $\mathcal{O}_2 \times \mathcal{O}_3$ expansion converges away from $z_t, \bar{z}_t \in (+1, \infty)$ i.e. $z, \bar{z} \in (-\infty, 0)$. Thus the curves in both Figs. B.1.2, B.1.3 lie within the range of analyticity of the $\mathcal{O}_2 \times \mathcal{O}_3$ expansion, so we can compare the analytic continuations. Since only integer spins occur in the expansion, the analytic continuation does not change under $\rho_t \rightarrow e^{2\pi i} \rho_t, \bar{\rho}_t \rightarrow e^{-2\pi i} \bar{\rho}_t$ (such arguments were systematically exploited in [74]).⁵ So let us add extra loops to the blue and the red curves in the opposite directions around 1, see Fig. B.1.4. Adding the loops and deforming the curves continuously (the first step is shown in Fig. B.1.4) we can bring them to those in Fig. B.1.3. This finishes the proof that the prescription (B.1.5) is correct also for the fourth ordering.

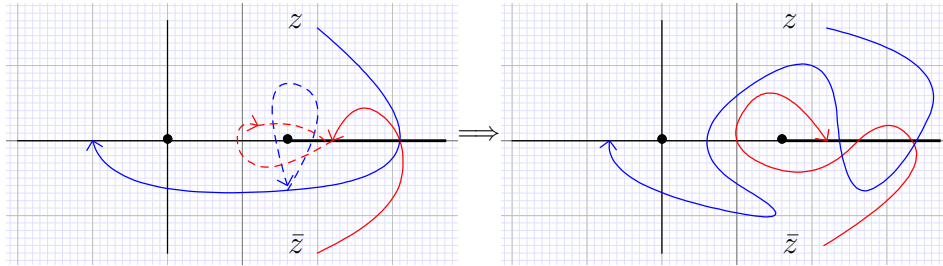


Figure B.1.4: Deforming the z, \bar{z} curves for the $\langle \mathcal{O}_1 \mathcal{O}_3 \mathcal{O}_2 \mathcal{O}_4 \rangle$ ordering.

Positivity constraints. We wish to comment on another result of [3]: an argument for positivity of certain conformal block expansion coefficients. We present the argument exchanging the role of s and t channels w.r.t. [3]. Let $G(z, \bar{z}) = 1 + \dots$ be the holomorphic function defined by the s-channel OPE expansion (i.e. $(z\bar{z})^{\Delta_1 + \Delta_2}$ times the $G(z, \bar{z})$ discussed above). We will define a certain analytic continuation of the function $G(z, \bar{z})$. Let us start with \bar{z} and z close to zero, \bar{z} in the upper half plane and z in the lower half plane. In this range all three channels s,t,u converge. We wish to analytically continue $G(z, \bar{z})$ by taking z through $(1, +\infty)$ and bring it back close to zero, in the upper half plane (see Fig. B.1.5), while we don't touch \bar{z} . This analytic continuation can be performed using the t-channel or u-channel expansions, with the same result (but not the s-channel since it stops converging on $(1, +\infty)$). We denote the result of this analytic continuation by $\widehat{G}(z, \bar{z})$, with $\text{Im } z, \text{Im } \bar{z} > 0$.

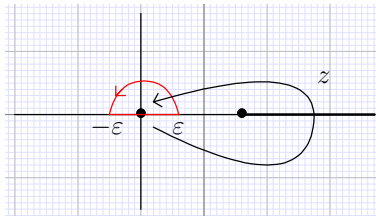


Figure B.1.5: Definition of $\widehat{G}(z, \bar{z})$. Red: integration contour in Eq. (B.1.11).

Although $\widehat{G}(z, \bar{z})$ is so defined with both z, \bar{z} in the upper half plane, it has continuous limits when they both approach positive real axis, or both approach negative real axis, since in the first case the t-channel and in the

⁵Sometimes this property is called “Euclidean single-valuedness”.

second case the u-channel remains convergent. We will be interested in the situation when $\mathcal{O}_1 = \mathcal{O}_2$, $\mathcal{O}_3 = \mathcal{O}_4$. In this case all expansion coefficients in the t and u channels are positive. This implies that the Euclidean correlator $G_E(z, \bar{z})$ will be positive for real $z, \bar{z} > 0$ (using t-channel) and for real $z, \bar{z} < 0$ (using u-channel). The difference between $G_E(z, \bar{z})$ and $\widehat{G}(z, \bar{z})$, for real $z, \bar{z} > 0$ or $z, \bar{z} < 0$ is that in the first case z, \bar{z} approach the real axis from the opposite sides while in the second case from the same one. When we take z through $(1, +\infty)$ cut, this only brings in some phases in the t and u-channel expansion of $\widehat{G}(z, \bar{z})$ with respect to $G_E(z, \bar{z})$. This implies that we have a bound for real $z, \bar{z} > 0$ or $z, \bar{z} < 0$:

$$|\widehat{G}(z, \bar{z})| \leq G_E(z, \bar{z}) \quad (\text{B.1.8})$$

In what follows $\widehat{G}(z, \bar{z})$ will be used as a holomorphic function with z, \bar{z} in the upper half plane satisfying the bound (B.1.8) on its boundary, while $G_E(z, \bar{z})$ will be used only with real z, \bar{z} .

In particular, since $G_E \approx 1$ for z, \bar{z} near zero up to small corrections, Eq. (B.1.8) says that \widehat{G} is bounded, for small real $z, \bar{z} > 0$ or $z, \bar{z} < 0$, by 1 up to small corrections. This argument can be generalized to show that $\widehat{G}(z, \eta z)$ for $\eta > 0$ real and z near zero in the upper half plane is bounded by a constant.⁶

We now pass to the non-rigorous part of the argument. Although the s-channel stops converging when crossing $(1, +\infty)$, Ref. [3] proposed that, in the regime $|\bar{z}| \ll |z| \ll 1$, the behavior $\widehat{G}(z, \bar{z})$ can nevertheless be predicted from the s-channel expansion, by organizing it in $\bar{z}^{\tau/2}$ where $\tau = \frac{1}{2}(\Delta - \ell)$ is the twist. The typical term is

$$\bar{z}^{\tau/2} k_{\frac{1}{2}(\Delta+\ell)}(z), \quad (\text{B.1.9})$$

where $k_h(x) = {}_2F_1(h, h, 2h, x)$ is the collinear conformal block. This is the same expansion as used in the light-cone bootstrap, which has its own problems of rigor, but here it is proposed to use it after $z - 1 \rightarrow e^{2\pi i}(z - 1)$. Under this continuation the collinear conformal block, which has a $\log(1 - z)$ behavior near $z = 1$, picks up an imaginary piece which, for z small, behaves as $\sim iz^{1-\frac{1}{2}(\Delta+\ell)}$ (see [3], Eq. (4.28)). Considering $\bar{z} = \eta z$, $\eta \ll 1$, $|z| \ll 1$, we then have, according to the proposal of Ref. [3],

$$\widehat{G}(z, \eta z) \approx 1 - B(\Delta, \ell) p_{\Delta, \ell} \times i \frac{\eta^{\tau/2}}{z^{\ell-1}}, \quad (\text{B.1.10})$$

where Δ, ℓ are the dimension and spin of the leading twist operator (which may e.g. be the stress tensor), $p_{\Delta, \ell}$ its conformal block coefficient, and $B(\Delta, \ell) \geq 0$ some explicitly known constant. The spin ℓ is even since we are assuming $\mathcal{O}_1 = \mathcal{O}_2$. Eq. (B.1.10) assumes that the limit $\eta \rightarrow 0$ is taken before $z \rightarrow 0$.⁷

Now, let us consider the holomorphic function $f(z) = 1 - \widehat{G}(z, \eta z)$, and integrate $z^{\ell-2} f(z)$ along the contour shown in Fig. B.1.5.⁸ We have

$$\int_{\text{arc}} z^{\ell-2} f(z) dz + \int_{-\varepsilon}^{\varepsilon} x^{\ell-2} f(x) dx = 0. \quad (\text{B.1.11})$$

Using (B.1.10) and that the integral over the arc of $1/z$ is πi , we get in particular:⁹

$$\pi B(\Delta, \ell) p_{\Delta, \ell} \approx \eta^{-\tau/2} \int_{-\varepsilon}^{\varepsilon} x^{\ell-2} \text{Re}[1 - \widehat{G}(z, \eta z)] dx \approx \eta^{-\tau/2} \int_{-\varepsilon}^{\varepsilon} x^{\ell-2} \text{Re}[G_E(z, \eta z) - \widehat{G}(z, \eta z)] dx, \quad (\text{B.1.12})$$

where in the final step we replaced 1 by $G_E(z, \eta z)$. Since $G_E(z, \eta z)$ has a rigorously convergent expansion for small z , it satisfies the bound:

$$G_E(z, \bar{z}) = 1 + O(\eta^{\tau/2}). \quad (\text{B.1.13})$$

⁶Let $z = re^{i\varphi}$, $r \ll 1$. We consider $0 \leq \varphi \leq \pi/2$, when the argument uses the t-channel, the case $\pi/2 \leq \varphi \leq \pi$ is analogous using the u-channel. The key point is that the ρ variable in the t-channel $\rho_t \approx 1 - \sqrt{r}e^{i\varphi/2}$, $|\rho_t| \approx 1 - \sqrt{r} \cos(\varphi/2)$. This allows to compare the function $\widehat{G}(z, \eta z)$ to $G_E(z', \eta z')$ with real $z' = r \cos^2(\varphi/2)$, times a factor $\sim (z\bar{z})^{\Delta_1}/(z'\bar{z}')^{\Delta_1}$ from the crossing kernel, which is bounded by a constant.

⁷In fact, in the opposite limit $z \rightarrow 0$ for fixed η , Eq. (B.1.10) would violate the discussed above rigorous bound that $\widehat{G}(z, \eta z)$ is bounded by a constant. There is no paradox because that's not the limit we are supposed to be taking.

⁸We can also take an intermediate step of adding a small semicircle of radius ε' around zero, but since \widehat{G} is bounded for small z , the limit $\varepsilon' \rightarrow 0$ is not problematic.

⁹Note that the quantity $\text{Re}[G_E(z, \eta z) - \widehat{G}(z, \eta z)]$ appearing in (B.1.12) is essentially the double discontinuity considered in [22]. Similarly, Eq. (B.1.12) can be formally obtained from the Lorentzian inversion formula of [22] by expanding the integrand in a light-cone limit. We thank Tom Hartman for pointing this out.

So the last replacement was legitimate if e.g. $\ell \geq 2$. By (B.1.8), the r.h.s. of (B.1.12) is a positive quantity. This equation then implies that $p_{\Delta, \ell}$ must be positive as well.

As already mentioned, the weak point of this argument is that the s-channel expansion stops converging when we cross $(1, +\infty)$. It is therefore not at all obvious that analytic continuations of the individual conformal block expansion terms have anything to do with the asymptotics of $\widehat{G}(z, \bar{z})$. Ref. [3] was of course aware of this, and provided some arguments, inspired by the light-cone bootstrap, why nevertheless the asymptotics from the leading twist terms can be trusted. We don't know how to make those arguments rigorous. It would be interesting to understand if asymptotics (B.1.10) can be justified using just Euclidean CFT axioms of Sec. 7.3 or requires additional assumptions. The same question also looms over the proofs of ANEC [20] and ANEC commutativity [21] which involved similar ‘‘light-cone limit on the second sheet’’ considerations.

B.2 OS axioms for higher-point functions

In this appendix we discuss the modifications necessary to derive from the Euclidean CFT axioms the OS axioms (positivity and cluster property) for n -point functions with $n > 4$, compared to the $n \leq 4$ case considered in Sec. 8.1. As we explain below, it appears that there is no simple proof of OS positivity for $n > 4$ from the Euclidean CFT axioms of Sec. 7.3. Since the reason for this is rather technical, let us first discuss the conceptual implications of this.

Ideally, one would like to have a set of Euclidean CFT axioms that would imply Wightman axioms (and therefore also OS axioms) and also be powerful enough to derive all the usual CFT lore such as OPEs, radial quantization, operator-state correspondence, crossing symmetry, etc. These statements, as we saw in the main text, make sense and can be non-trivial even when we restrict our attention to n -point functions with bounded n .

In particular, we have found that the axioms we formulated in Sec. 7.3 achieve the above goal for $n \leq 4$. Extending our results to $n > 4$ using the same strategy would require a solution to two problems: first, we need to derive OS axioms (specifically, positivity and cluster property) for $n > 4$, and, second, we need to prove that OS axioms together with the OPE imply Wightman axioms.

Conceptually, it seems plausible that OS axioms + OPE imply Wightman axioms for $n > 4$ because we expect that for $n > 4$ there is again an OPE channel which is convergent in the entire forward tube (i.e. the one given by taking the OPE in the same order as the operators appear in the Wightman ordering). This question clearly merits further study but is beyond the scope of this paper.

However, it is less clear to us how to even attempt a derivation of OS positivity for $n > 4$ from Euclidean CFT axioms of Sec. 7.3. Let us first explain why this is the case, and then we will discuss the possible modifications to these CFT axioms.

Suppose we want to prove the positivity

$$\langle \Psi | \Psi \rangle \geq 0, \tag{B.2.1}$$

where Ψ is a state created by a product of three local operators, $|\Psi\rangle = |\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\rangle$.

To prove this positivity the natural idea would be to use the OPE expansion repeatedly for the two copies of Ψ and then use the positivity of the 2-point function. However, for this we need our OPE approximation for $\langle \Psi |$ to be conjugate to our approximation for $|\Psi\rangle$. This is non-trivial to achieve because we have to perform the OPEs one at a time. For example, we can first construct an approximation of $|\Psi\rangle$ in terms of a state $|\Psi'\rangle$, created by single operator insertions, such that

$$|\langle \Psi | \Psi \rangle - \langle \Psi | \Psi' \rangle| < \varepsilon. \tag{B.2.2}$$

Similarly, we can construct a state $\langle \Psi'' |$ such that

$$|\langle \Psi'' | \Psi' \rangle - \langle \Psi | \Psi' \rangle| < \varepsilon \tag{B.2.3}$$

and thus

$$|\langle \Psi'' | \Psi' \rangle - \langle \Psi | \Psi \rangle| < 2\varepsilon. \tag{B.2.4}$$

These approximations are possible by the repeated use of the OPE (8.1.2). Note, however, that since the OPE axiom is formulated for correlation functions, the number of terms we have to include in the OPE for a given ε

depends on the correlation function in which the OPE is performed. It then follows that the state $\langle \Psi'' |$ depends on $|\Psi'\rangle$ (because in order to construct it we use the OPE in the correlation function $\langle \Psi | \Psi' \rangle$) and is in general different from it. It is therefore not obvious that $\langle \Psi'' | \Psi' \rangle \geq 0$, which is what we would like to use in order to prove $\langle \Psi | \Psi \rangle \geq 0$ with the help of the above inequalities.

In the case when $n = 4$ and $|\Psi\rangle$ is created by 2 operators we were able to solve this difficulty. This was because in this case the only difference between $|\Psi'\rangle$ and $\langle \Psi'' |$ can be in the number of OPE terms included in the approximation, and we were able to use an orthogonality property of the 2-point function to show $\langle \Psi'' | \Psi' \rangle = \langle \Psi' | \Psi' \rangle$ by arguing that we can assume that $\langle \Psi'' |$ contains more terms than $|\Psi'\rangle$ and that those terms which are in $\langle \Psi'' |$ but not in $|\Psi'\rangle$ do not contribute to the product $\langle \Psi'' | \Psi' \rangle$.

This argument does not work in the case at hand, $|\Psi\rangle = |\varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\rangle$. The reason for this is that in order to construct $|\Psi'\rangle$ or $\langle \Psi'' |$ we need to perform two OPE's in each case. For example, the first one can be $\varphi_1 \times \varphi_2 = \sum_k \mathcal{O}_k$ and the second one can be $\varphi_3 \times \mathcal{O}_k$. Both OPE's have to be truncated at some point, and while the truncation of the second OPE affects only the set of terms that are present in $|\Psi'\rangle$ or $\langle \Psi'' |$, where we truncate the first $\varphi_1 \times \varphi_2$ OPE affects the *coefficients* of these terms. Since now $|\Psi'\rangle$ and $\langle \Psi'' |$ contain terms with differing coefficients, we cannot use orthogonality to argue $\langle \Psi'' | \Psi' \rangle = \langle \Psi' | \Psi' \rangle$ anymore. There is no way to ensure that $\varphi_1 \times \varphi_2$ OPEs are truncated in the same way in the construction of both states because the truncation in $\langle \Psi'' |$ depends, through our OPE axiom, on $|\Psi'\rangle$, and thus might happen to be always at a higher order than the truncation used for $|\Psi'\rangle$.

This all is to say that due to a rather technical reason it appears that there is no *simple* proof of OS positivity of higher-point functions from the Euclidean CFT axioms as stated in Sec. 7.3. Importantly, this doesn't mean that there is no proof at all. Indeed, the Euclidean CFT axioms are sufficient to derive the standard crossing-symmetry equations for 4-point functions. It could happen that in all solutions to these crossing-symmetry equations the OPE coefficients have such asymptotics that a stronger form of the OPE axiom holds and allows us to prove the OS positivity for $n > 4$. However, it is not clear how to implement this line of reasoning in practice.

It is therefore interesting to look for a stronger version of Euclidean CFT axioms. We discuss below some simple modifications of the OPE axiom which avoid the above problem and allow to prove OS positivity for higher-point functions.

Morally, we want some kind of statement of uniformity for the convergence rate of the OPE: it should make $|\Psi''\rangle$ above independent of the truncation made in $|\Psi'\rangle$, as long as this truncation is done at a sufficiently high order. This would allow us to make both truncations at a high order and ensure $\langle \Psi'' | \Psi' \rangle = \langle \Psi' | \Psi' \rangle \geq 0$.

One option is to assume a stronger form of the OPE, which allows us to perform two OPE's simultaneously. Specifically, we can assume that the double sum

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \dots \rangle = \sum_{k,l} \langle \mathcal{O}_k \mathcal{O}_l \dots \rangle, \quad (\text{B.2.5})$$

is convergent, where we wrote the two OPEs schematically as $\mathcal{O}_1 \mathcal{O}_2 = \sum_k \mathcal{O}_k$ and $\mathcal{O}_3 \mathcal{O}_4 = \sum_l \mathcal{O}_l$. Convergence of the double sum means that

$$\left| \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \dots \rangle - \sum_{k,l} \langle \mathcal{O}_k \mathcal{O}_l \dots \rangle \right| < \varepsilon, \quad (\text{B.2.6})$$

when the sums are truncated in a way that includes some ε -dependent finite set of terms, but is otherwise arbitrary. In particular, both sums can be truncated in the same way, and this solves the problem that we encountered above. A disadvantage of this approach is that it is unclear how to derive this axiom from OS axioms and the usual single OPE axiom (however, a heuristic argument based on cutting the Euclidean path integral can be made). This is somewhat subtle and is related to the question of whether the path integral over a spherical layer ($r_1 < r < r_2$) with operator insertions in the interior represents a bounded operator. We can't say with confidence whether or not this is the case.

Another option is to assume resummed repeated OPE, i.e. that the following sum converges, schematically,

$$\langle \mathcal{O}_1 \dots \mathcal{O}_m \mathcal{O}_{m+1} \dots \mathcal{O}_n \rangle = \sum_k c_k \langle \mathcal{O}_k \mathcal{O}_{m+1} \dots \mathcal{O}_n \rangle, \quad (\text{B.2.7})$$

where the coefficients c_k are chosen so that

$$\langle \mathcal{O}_1 \dots \mathcal{O}_m \mathcal{O}_k^\theta \rangle = c_k \langle \mathcal{O}_k \mathcal{O}_k^\theta \rangle, \quad (\text{B.2.8})$$

assuming $\langle \mathcal{O}_k \mathcal{O}_l^\theta \rangle \propto \delta_{kl}$. This version of the axiom is essentially the statement that one-operator states form a basis of the CFT Hilbert space, formulated without explicitly introducing the Hilbert space. In other words, above we are approximating the state $\langle \mathcal{O}_1 \dots \mathcal{O}_m |$ in terms of an orthonormal basis of states $\langle \mathcal{O}_k |$, and the coefficients are computed by inner products. This form of the axiom is easy to derive from OS + convergent OPE, and also easily allows us to solve our problem by using the same strategy as in the case $n = 4$. However, it does appear to be an overly strong assumption, making our axioms not very different from assuming OS + convergent OPE outright.

Finally, an interesting prospect might be, instead of formulating an entirely new set of axioms, to add an *asymptotic* OPE axiom (and conformal invariance) to OS axioms. It is likely that using logic very similar to that of Mack [26], which we reviewed in Sec. 13.3, one can prove that (OS axioms)+(asymptotic OPE)+(conformal invariance) imply convergent OPE.

B.3 Details on Vladimirov's theorem

B.3.1 Limit in the sense of distributions

Let us start with a reminder of what the limit in the sense of tempered distributions means. Let $f(u)$, $u = (t_k, \mathbf{x}_k) \equiv (t_1, \mathbf{x}_1, \dots, t_n, \mathbf{x}_n) \in \mathbb{R}^{nd}$, be a Schwartz test function, i.e. an infinitely differentiable function decreasing at infinity faster than any power together with all its derivatives. This can be also stated as finiteness of all Schwartz norms:

$$|f|_N = \sup_{u \in \mathbb{R}^{nd}, |\alpha| \leq N} (1 + u^2)^{N/2} |\partial_u^\alpha f| < \infty \quad \forall N \geq 0. \quad (\text{B.3.1})$$

That the limit (9.0.8) exists in the sense of distributions means two requirements. First, that the r.h.s. of (9.0.8) has a finite limit integrated against any f as above:

$$(G_n^M, f) := \lim_{\epsilon_k \rightarrow 0} \int dt d\mathbf{x} G_n(\epsilon_k + it_k, \mathbf{x}_k) f(t_k, \mathbf{x}_k) \quad \text{exists for any Schwartz } f. \quad (\text{B.3.2})$$

The G_n^M defined by this equation is a linear functional on the Schwartz space. The second requirement is that this functional should be continuous (and thus is itself a tempered distribution). Continuity means that it should be bounded by one of the norms (B.3.1) with a sufficiently large N , i.e.:

$$|(G_n^M, f)| \leq C |f|_{N_*}, \quad (\text{B.3.3})$$

with f -independent C and N_* .

Note that by Eq. (B.3.3), G_n^M can be extended from the Schwartz space to a larger space of test functions, which are required to be differentiable only N_* times and have a finite $|f|_{N_*}$. Parameter N_* thus characterizes regularity of the distribution G_n^M . The proof of Theorem 9.0.1 will determine N_* in terms of A_n and B_n , see Eq. (B.3.6).

B.3.2 Proof of Theorem 9.0.1

Unfortunately, we do not know a reference where Theorem 9.0.1 is stated and proved succinctly in the form we need. Such results are considered standard in the theory of several complex variables. For similar statements see [27], Chapter 5, and [24], Theorem 2-10. For the convenience of the reader, we present here a proof based on these sources.

The usefulness of Vladimirov's theorems for establishing distributional properties of CFT correlators was recognized in our recent work part I. There, we considered expansions of the CFT 4-point function $g(\rho, \bar{\rho})$ in terms of conformally invariant cross-ratios $\rho, \bar{\rho}$. It is well known that such expansions converge in the interior of the unit disk $|\rho|, |\bar{\rho}| < 1$. Using Vladimirov's theorems, we showed in part I that they also converge on the boundary of this disk, in the sense

of distributions. In this paper we are interested in CFT correlators as functions of positions x_k , not of cross-ratios, but the basic principle is the same as in part I: a powerlaw bound on an holomorphic function near a boundary implies temperedness of the limiting distribution.

By translation invariance it's enough to study the function G_n expressed in terms of the differences $y_k = x_k - x_{k+1}$ which we denote by $\mathcal{G}(y)$, $y = (y_1, \dots, y_{n-1})$. We also denote $y_k = (y_k^0, \mathbf{y}_k)$, $y_k^0 = \varepsilon_k + is_k$, $\varepsilon_k > 0$, $\mathbf{y}_k \in \mathbb{R}^{d-1}$.

Consider first the case when all ε_k go to zero together along a fixed direction: $\varepsilon_k = rv_k$ where $r \rightarrow 0$ and $v = (v_k)$ is a vector with positive components. Later on we will show that the limit continues to exist if $\varepsilon_k \rightarrow 0$ independently (as well as the more general statement about the limit from inside the forward tube).

So, let us prove that $\mathcal{G}(y)$ has a limit as $r \rightarrow 0$ which is a tempered distribution in variables s_k, \mathbf{y}_k . As in (B.3.2), we fix a Schwartz test function f and consider the integral (we will omit index k on $\varepsilon, s, v, \mathbf{y}$ if no confusion may arise)

$$h(r) = \int ds d\mathbf{y} \mathcal{G}(rv + is, \mathbf{y}) f(s, \mathbf{y}). \quad (\text{B.3.4})$$

The problem is analogous to theorems used in part I, so we will be brief. As in section 9.0.1 and appendix A.3, using analyticity in y^0 , integration by parts, and the powerlaw bound one can show that derivatives of h in r satisfy the bound:

$$|\partial_r^j h(r)| \leq \frac{C}{r^{A_n}} |f|_N, \quad (\text{B.3.5})$$

where $|f|_N$ is a Schwartz norm (B.3.1) of a sufficiently large order N depending on j and B_n . The constant A_n is the same as in (9.0.9), in particular the same A_n works for all j . In what follows we only need this equation for finitely many j (up to $[A_n] + 1$). Using the Newton-Leibniz formula in the r direction several times, one then proves that the same bound as (B.3.5) holds in fact without $1/r^{A_n}$ singularity in the r.h.s. It then follows that, first of all, $\lim_{r \rightarrow 0} h(r)$ exists, and second, it is a continuous linear functional of f , that is, a distribution. The limit holds uniformly when the components v_k vary on any fixed compact interval contained in $(0, +\infty)$. Its v -independence is shown exactly as in part I, Eq. (C.7). Let us denote the limiting distribution $\mathcal{G}(is, \mathbf{y}) \equiv \mathcal{G}^M(s, \mathbf{y})$.

It is of some interest to know the precise regularity of the distribution \mathcal{G}^M (i.e. how many derivatives the test function must have to be pairable with \mathcal{G}^M) and the rate of its growth at infinity. Following the above argument in detail, one can show the following bound which contains this information:

$$|(\mathcal{G}^M, f)| \leq \text{Const.} \int ds d\mathbf{y} (1 + |s| + |\mathbf{y}|)^{B_n} \max_{|\alpha| \leq [A_n] + 1} |\partial_s^\alpha f(s, \mathbf{y})|. \quad (\text{B.3.6})$$

This in particular implies (B.3.3) with $N_* = \max([A_n] + 1, B_n + nd + 1)$.

Parts 2,3 of Theorem 9.0.1 are new compared to part I, since such questions do not arise in the cross-ratio space.

Lorentz invariance is easy to show, as follows. Rotation invariance of G_n^E implies that $\mathcal{G}(y)$ satisfies for real y the differential equations

$$\{y^a \partial_{y^b} - y^b \partial_{y^a}\} \mathcal{G}(y) = 0, \quad a, b \in \{0, 1, \dots, d-1\} \quad (\text{B.3.7})$$

(as usual $y = (y_k)$, summation in k understood). By the uniqueness of analytic continuation, these equations continue to hold for complex y_0 . That's the only place where we use real-analyticity in the spatial direction.¹⁰ By taking the limit $\varepsilon \rightarrow 0$ in (B.3.7), we recover precisely the differential equations expressing the Lorentz invariance of \mathcal{G}^M . Let us explain in more detail how the limit is taken and why it exists. Consider for definiteness $a = 0, b = 1$, other cases being similar. Eq. (B.3.7) then says $\{(\varepsilon + it)\partial_{y^1} + iy^1 \partial_t\} \mathcal{G}(\varepsilon + it, \mathbf{y}) = 0$, in the sense of functions, and hence integrating by parts in the sense of distributions acting on test functions $\varphi(t, \mathbf{y})$:

$$(\mathcal{G}_\varepsilon, \{(\varepsilon + it)\partial_{y^1} + iy^1 \partial_t\} \varphi) = 0, \quad (\text{B.3.8})$$

where we denoted $\mathcal{G}_\varepsilon(t, \mathbf{y}) = \mathcal{G}(\varepsilon + it, \mathbf{y})$. Now we take the limit $\varepsilon \rightarrow 0$. We know that (a) $\mathcal{G}_\varepsilon \rightarrow \mathcal{G}^M$ in the sense of distributions, and also that (b) $|(\mathcal{G}_\varepsilon, \varphi)|$ is uniformly bounded as $\varepsilon \rightarrow 0$ by some Schwartz norm of φ . By (b) the term $(\mathcal{G}_\varepsilon, \varepsilon \varphi)$ in (B.3.8) drops out when $\varepsilon \rightarrow 0$, and by (a) the rest tends to $(\mathcal{G}^M, \{it\partial_{y^1} + iy^1 \partial_t\} \varphi)$. So we conclude that $(\mathcal{G}^M, \{it\partial_{y^1} + iy^1 \partial_t\} \varphi) = 0$ which expresses invariance of \mathcal{G}^M under the 01 Lorentz transformation.

¹⁰With some extra tricks, it's possible to replace it by the assumption of mere continuity in \mathbf{y} , as in [2], Theorem 4.3. In the CFT applications we have in mind, real analyticity appears a more natural assumption.

Let us proceed to show the rest of Parts 2,3. It will be crucial that \mathcal{G} can be written as a “Fourier-Laplace transform”:

$$\mathcal{G}(\varepsilon + is, \mathbf{y}) = \int \frac{dE d\mathbf{p}}{(2\pi)^{d(n-1)}} g(E, \mathbf{p}) e^{-(\varepsilon+is)E - i\mathbf{p}\mathbf{y}}, \quad (\text{B.3.9})$$

where $g(E, \mathbf{p})$, $E \in \mathbb{R}^{n-1}$, $\mathbf{p} \in (\mathbb{R}^{d-1})^{n-1}$ is a tempered distribution, called “spectral function”, supported at $E \geq 0$ (by which we mean all $E_k \geq 0$) [later this will be improved to $E \geq |\mathbf{p}|$]. We are omitting the indices, thus $\varepsilon E = \sum_k \varepsilon_k E_k$, etc. The equality in (B.3.9) is understood in the sense of distributions, with the r.h.s. being the inverse Fourier transform of the tempered distribution $g(E, \mathbf{p})e^{-\varepsilon E}$. In other words, what this means is that

$$\int ds d\mathbf{y} \mathcal{G}(\varepsilon + is, \mathbf{y}) f(-s, -\mathbf{y}) = \int \frac{dE d\mathbf{p}}{(2\pi)^{d(n-1)}} g(E, \mathbf{p}) e^{-\varepsilon E} \widehat{f}(E, \mathbf{p}), \quad (\text{B.3.10})$$

for any Schwartz test function f , and \widehat{f} its Fourier transform.

Let us show (B.3.9). Notice first that for every $\varepsilon > 0$ we can write

$$\mathcal{G}(\varepsilon + is, \mathbf{y}) = \int \frac{dE d\mathbf{p}}{(2\pi)^{d(n-1)}} g_\varepsilon(E, \mathbf{p}) e^{-isE - i\mathbf{p}\mathbf{y}}, \quad (\text{B.3.11})$$

where g_ε is the Fourier transform of $\mathcal{G}(\varepsilon + is, \mathbf{y})$ with respect to s, \mathbf{y} . This Fourier transform exists as a tempered distribution, since $\mathcal{G}(\varepsilon + is, \mathbf{y})$ is itself a tempered distribution in s, \mathbf{y} (being a real-analytic function, bounded by a power at infinity). In addition, $\mathcal{G}(\varepsilon + is, \mathbf{y})$ is differentiable in ε and s and satisfies the Cauchy-Riemann equations. From here it's easy to show that g_ε as a distribution is differentiable in ε and satisfies the differential equations:

$$\frac{\partial g_\varepsilon}{\partial \varepsilon_k} + E_k g_\varepsilon = 0 \quad (k = 1, \dots, d-1). \quad (\text{B.3.12})$$

From here we conclude that

$$g(E, \mathbf{p}) := g_\varepsilon(E, \mathbf{p}) e^{\varepsilon E} \quad (\text{B.3.13})$$

is an ε -independent distribution. Substituting $g_\varepsilon(E, \mathbf{p}) = g(E, \mathbf{p}) e^{-\varepsilon E}$ into (B.3.11), we obtain (B.3.9). Note that since g and g_ε are related by an exponential factor, we can so far only claim that g is defined as a distribution on test functions of compact support. Let us show next that it is in fact tempered (i.e. extends to Schwartz test functions).

To this end, consider the inverse of (B.3.11):

$$g_\varepsilon(E, \mathbf{p}) = g(E, \mathbf{p}) e^{-\varepsilon E} = \int ds d\mathbf{y} \mathcal{G}(\varepsilon + is, \mathbf{y}) e^{isE + i\mathbf{p}\mathbf{y}}, \quad (\text{B.3.14})$$

and integrate it against a compactly supported test function $\varphi(E, \mathbf{p})$. We get (compare (B.3.10)):

$$\int dE d\mathbf{p} g(E, \mathbf{p}) e^{-\varepsilon E} \varphi(E, \mathbf{p}) = \int ds d\mathbf{y} \mathcal{G}(\varepsilon + is, \mathbf{y}) \widehat{\varphi}(-s, -\mathbf{y}). \quad (\text{B.3.15})$$

As $\varepsilon \rightarrow 0$, the l.h.s. tends to the pairing (g, φ) . Using Part 1 of the theorem, the r.h.s. tends in the same limit to $\int ds d\mathbf{y} \mathcal{G}^M(s, \mathbf{y}) \widehat{\varphi}(-s, -\mathbf{y})$ which exists in the sense of tempered distributions and so is bounded by some Schwartz-space norm $|\widehat{\varphi}|_N$. We get

$$|(g, \varphi)| \leq \text{const} \cdot |\widehat{\varphi}|_N \leq \text{const} \cdot |\varphi|_{N'}, \quad (\text{B.3.16})$$

where in the second inequality we used that the Fourier transform is continuous in the Schwartz space. This inequality, valid for any compactly supported φ , means that g extends to a tempered distribution on the whole Schwartz space. The representation (B.3.9) is thus established.

Next let us show that g is supported at $E \geq 0$. For this we will pass to the large ε limit in (B.3.13). Supposing that $E_k < 0$ for some k , the factor $e^{E\varepsilon}$ in (B.3.13) decreases exponentially as the corresponding $\varepsilon_k \rightarrow +\infty$. On the other hand $g_\varepsilon(E, \mathbf{p})$ is bounded in the same limit by a power of ε , because it's the Fourier transform of $\mathcal{G}(\varepsilon + is, \mathbf{y})$ which satisfies a powerlaw bound.¹¹ This implies that $g(E, \mathbf{p}) = 0$ unless $E \geq 0$.¹²

Consider then the following lemma, proven analogously to, and easier than, Lemma B.3.2 below.

¹¹This is the only place where we use the powerlaw bound on $\mathcal{G}(\varepsilon + is, \mathbf{y})$ for large rather than small ε .

¹²If unhappy with this intuitive reasoning, the argument may be made more rigorous in its integrated version: show that g vanishes on test functions supported in the complement of $E \geq 0$.

Lemma B.3.1. *Let $g(E, \mathbf{p})$ be a tempered distribution supported at $E \geq 0$, and consider the distribution $g(E, \mathbf{p})e^{-\varepsilon E}$ ($\varepsilon > 0$). This distribution, being initially defined by this formula on compactly supported test functions, extends to a tempered distribution, and moreover $g(E, \mathbf{p})e^{-\varepsilon E} \rightarrow g(E, \mathbf{p})$ as $\varepsilon \rightarrow 0$, in the sense of tempered distributions.*

Let us now take the $\varepsilon \rightarrow 0$ limit on both sides of (B.3.9) (or, which is the same, (B.3.10)). The l.h.s. has a limit by Part 1, while the r.h.s. has a limit by Lemma B.3.1. We obtain that $\mathcal{G}^M(s, \mathbf{y})$ and $g(E, \mathbf{p})$ are related by the Fourier transform:

$$\mathcal{G}^M(s, \mathbf{y}) = \int \frac{dE d\mathbf{p}}{(2\pi)^{d(n-1)}} g(E, \mathbf{p}) e^{-isE - i\mathbf{p}\mathbf{y}}. \quad (\text{B.3.17})$$

We can now complete the proof of Part 2, namely to show the spectral condition. Above we proved that $\mathcal{G}^M(s, \mathbf{y})$ is Lorentz invariant. Since $g(E, \mathbf{p})$ is its Fourier transform, it is also Lorentz invariant, and in particular its support must be a Lorentz-invariant set. We also know that $\text{supp } g \subset \{E \geq 0\}$. These two facts together imply that $\text{supp } g$ must be contained in the product of the forward null cones, i.e. $g(E, \mathbf{p}) = 0$ unless each $E_k \geq |\mathbf{p}_k|$, which is the spectral condition.

Part 3 follows by standard Wightman theory arguments. Namely, let us write $(iy_k^0, \mathbf{y}_k) = \xi_k + i\eta_k$ where $\xi_k, \eta_k \in \mathbb{R}^{1, d-1}$ and $\eta_k = (\text{Re } y_k^0, \text{Im } \mathbf{y}_k) \succ 0$. The extension to the forward tube is given by the equation (with $p = (E, \mathbf{p})$)

$$\int dp g(p) e^{i(p, \xi)} e^{(p, \eta)}, \quad (\text{B.3.18})$$

which reduces to (B.3.9) for real \mathbf{y}_k . It is holomorphic by Part (c) of the following lemma, while Parts (a,b) imply that this extension has the same limit as (B.3.9).

Lemma B.3.2. *Let $g(p)$ be a tempered supported at $p \succeq 0$ (closed forward light cone). Consider the distribution $g_\eta(p) = g(p)e^{-(p, \eta)}$, initially defined by this formula on compactly supported test functions. Then*

- (a) g_η for $\eta \succ 0$ extends to a tempered distribution;
- (b) $g_\eta \rightarrow g$ as $\eta \rightarrow 0$ from inside the forward light cone $\eta \succ 0$, in the sense of tempered distributions;
- (c) The Fourier transform $\widehat{g}_\eta(\xi)$ of the distribution $g_\eta(p)$ is a holomorphic function of $\xi + i\eta$ for $\eta \succ 0$.

Proof. Let $\omega(p)$ be a C^∞ function which is identically 1 on the forward light cone $\overline{V_+}$, and zero as soon as $\text{dist}(p, \overline{V_+}) \geq 1$ where dist is the Euclidean distance. We can choose this function so that all its derivatives are uniformly bounded by a constant depending only on the derivative order: $|\omega^{(\alpha)}(p)| \leq C_\alpha$ for any p .

Consider the family of C^∞ functions parametrized by $\xi, \eta \in \mathbb{R}^{1, d-1}$:

$$\Omega_{\xi, \eta}(p) = e^{i(p, \xi)} e^{(p, \eta)} \omega(p). \quad (\text{B.3.19})$$

It is not hard to check that $\Omega_{\xi, \eta}$ is a Schwartz function for $\eta \succ 0$ and any ξ .

Let us define g_η paired with a Schwartz function $\varphi(p)$ via

$$(g_\eta, \varphi) = (g, \Omega_{0, \eta} \varphi). \quad (\text{B.3.20})$$

We know that $\Omega_{0, \eta} \varphi$ is a Schwartz function for $\eta \succ 0$, so this definition makes sense. Furthermore it is not hard to check that $\Omega_{0, \eta} \varphi \rightarrow \omega \varphi$ in the Schwartz space topology as $\eta \rightarrow 0, \eta \succ 0$. This proves Parts (a),(b).

Next, let us define

$$F(\xi, \eta) = (g, \Omega_{\xi, \eta}), \quad \xi, \eta \in \mathbb{R}^{1, d-1}. \quad (\text{B.3.21})$$

We know that $\Omega_{\xi, \eta}$ is a Schwartz function for $\eta \succ 0$, so $F(\xi, \eta)$ is a function. Moreover it is not hard to show that the family $\Omega_{\xi, \eta}$ is continuous and continuously differentiable in the Schwartz space topology. It also obviously satisfies the Cauchy-Riemann equations: $(\partial_\xi + i\partial_\eta)\Omega_{\xi, \eta} = 0$. This implies that $F(\xi, \eta)$ is a holomorphic function in $\xi + i\eta$. It remains to show that $F(\xi, \eta) = \widehat{g}_\eta(\xi)$. It's enough to check this integrated against a compactly supported test function $\chi(\xi)$:

$$\int F(\xi, \eta) \chi(\xi) d\xi = \int (g, \Omega_{\xi, \eta}) \chi(\xi) d\xi = \left(g, \int d\xi \chi(\xi) \Omega_{\xi, \eta} \right) = (g, \Omega_{0, \eta} \widehat{\chi}) = (g_\eta, \widehat{\chi}) = (\widehat{g}_\eta, \chi). \quad (\text{B.3.22})$$

The proof is complete. □

B.4 Intuition about Lemma 9.1.1

The proof of Lemma 9.1.1 in Sec. 9.1 was by contradiction. To help intuition, we will give here a constructive argument of a special case of Lemma 9.1.1, namely $d = 1$ and $n = 2$. I.e. we will show how any Schwartz function $f \in \mathcal{S}(\mathbb{R})$ can be approximated by Schwartz functions g which for $E \geq 0$ agree with Laplace transform:

$$\mathcal{L}(\varphi)(E) = \int_0^\infty dt \varphi(t) e^{-Et}, \quad (\text{B.4.1})$$

$\varphi \in C_0^\infty(\mathbb{R}_+)$ (compactly supported with support strictly inside $(0, +\infty)$), while for $E < 0$, $g(E)$ is extended arbitrarily. Recall that the Schwartz space topology is given by the family of norms

$$|f|_n = \sup_{E \in \mathbb{R}, m \leq n} (1 + E^2)^{n/2} |f^{(m)}(E)|, \quad (\text{B.4.2})$$

and we need to find a sequence $\{g_r\}_{r=1}^\infty$ such that $|f - g_r|_n \rightarrow 0$ as $r \rightarrow \infty$ for any n (we stress that one sequence g_r should work for any n).

We will also consider the Schwartz space $\mathcal{S}(\overline{\mathbb{R}_+})$, consisting of C^∞ functions on $E \geq 0$ (not necessarily vanishing at $E = 0$) with topology given by the family of norms $|f|_{n,+}$ defined by the same equations as (B.4.2) but with sup taken over $E \geq 0$. It will be sufficient to arrange that for any n

$$|f - \mathcal{L}(\varphi_r)|_{n,+} \rightarrow 0 \quad (r \rightarrow \infty). \quad (\text{B.4.3})$$

This is because there exists an extension operator which takes a function $h \in \mathcal{S}(\overline{\mathbb{R}_+})$ and provides a function $\mathcal{E}(h) \in \mathcal{S}(\mathbb{R})$ such that $\mathcal{E}(h) = h$ for $E \geq 0$ (which is why it called an extension operator), and in addition

$$|\mathcal{E}(h)|_n \leq C_n |h|_{n,+} \quad (\text{B.4.4})$$

for all n with some finite constants C_n independent of h . E.g., Seeley's linear extension operator [170, 171] has this property. Then, given (B.4.3), we put

$$g_r = f + \mathcal{E}(\mathcal{L}(\varphi_r) - f), \quad (\text{B.4.5})$$

which, on the one hand satisfies $g_r(E) = \mathcal{L}(\varphi_r)(E)$ for $E \geq 0$ and on the other hand by (B.4.3) and (B.4.4) has $|g_r - f|_n \leq C_n |\mathcal{L}(\varphi_r) - f|_{n,+} \rightarrow 0$ which is what we need.

So let us focus on satisfying (B.4.3). By a map $x = \frac{1}{1+E}$ the half-line $[0, +\infty)$ is mapped to the interval $(0, 1]$ and the function $f(E)$ is mapped to a function $F(x) = f\left(\frac{1}{x} - 1\right)$ which is a C^∞ function vanishing at $x = 0$ faster than any power of x . For any ε and any N we can find, by the Weierstrass theorem, a polynomial $Q(x)$ such that

$$|F^{(N)}(x) - Q(x)| \leq \varepsilon \quad (0 \leq x \leq 1). \quad (\text{B.4.6})$$

Let $P(x)$ be the polynomial such that $P^{(N)}(x) = Q(x)$ and $P(0) = \dots = P^{(N-1)}(0) = 0$. Then $P(x) = O(x^N)$ and it is not hard to see that

$$|F^{(n)}(x) - P^{(n)}(x)| \leq \varepsilon x^{N-n} \quad (0 \leq x \leq 1). \quad (\text{B.4.7})$$

We also put $p(E) = P\left(\frac{1}{1+E}\right)$. From $f(E) = F\left(\frac{1}{1+E}\right)$ we know that

$$|f^{(n)}(E)| \leq B_n \max_{m \leq n} \left| F^{(m)}\left(\frac{1}{1+E}\right) \right|. \quad (\text{B.4.8})$$

So combining this with Eq. (B.4.7), and going up to $n = N/2$ we may conclude that

$$|f - p|_{N/2,+} \leq B'_N \varepsilon. \quad (\text{B.4.9})$$

Now, by construction p has the form

$$p(E) = \sum_{N \leq n \leq M} a_n \frac{1}{(1+E)^n}. \quad (\text{B.4.10})$$

Since $\frac{1}{(1+E)^n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-(1+E)t} dt$, we see that $p(E)$ is the Laplace transform of a function $\psi(t)$:

$$p = \mathcal{L}(\psi), \quad \psi(t) = \sum_{N \leq n \leq M} \frac{a_n}{(n-1)!} t^{n-1} e^{-t}. \tag{B.4.11}$$

Now we can finish the argument as follows. For $r = 1, 2, 3, \dots$ we apply the above argument with $N = 2r$ and $\varepsilon = 1/(B'_N r)$ to find ψ_r such that, by (B.4.9),

$$|f - \mathcal{L}(\psi_r)|_{r,+} \leq 1/r. \tag{B.4.12}$$

The function ψ_r is not in $C_0^\infty(0, \infty)$ although $\psi_r^{(k)} = 0$ for $k = 0 \dots 2r - 2$, and it vanishes at ∞ exponentially. We can therefore approximate ψ_r by a $C_0^\infty(0, \infty)$ function φ_r so that $|\psi_r - \varphi_r|_{2r-2,+}$ is arbitrarily small, where the order $2r - 2$ of the norm is related to the order of the vanishing of ψ_r at $t = 0$. Furthermore we have the following lemma:

Lemma B.4.1. *Let χ be a C^∞ function on $[0, +\infty)$ which exponentially vanishes at infinity and*

$$\chi^{(k)}(0) = 0, \quad k = 0 \dots n - 1. \tag{B.4.13}$$

Then, with some constant D_n independent of χ ,

$$|\mathcal{L}(\chi)|_{n,+} \leq D_n |\chi|_{n+2,+}. \tag{B.4.14}$$

Proof. We use the following elementary properties of Laplace transform:

$$\begin{aligned} \left(\frac{d}{dE}\right)^m \mathcal{L}(\chi)(E) &= \mathcal{L}[\chi(t)(-t)^m](E), \\ E^n \mathcal{L}(\chi)(E) &= \mathcal{L}[\chi^{(n)}(t)](E), \end{aligned} \tag{B.4.15}$$

where the second equation is derived by integration by parts and is valid under (B.4.13) and exponential decay. So we have (where \lesssim denotes \leq with some n -dependent but function-independent constant)

$$|\mathcal{L}(\chi)|_{n,+} \lesssim \sum_{m=0}^n \sup_{E \geq 0} (1 + E^n) |\mathcal{L}(\chi)^{(m)}(E)| \leq \sum_{m=0}^n \sup_{E \geq 0} |\mathcal{L}[\chi(t)t^m](E)| + |\mathcal{L}[(\chi(t)t^m)^{(n)}](E)|, \tag{B.4.16}$$

Using further the elementary bound $|\mathcal{L}(f)(E)| \lesssim \sup_{t \geq 0} |(1 + t^2)f(t)|$ we deduce (B.4.14). □

We use this lemma with $n = r$ and $\chi = \psi_r - \varphi_r$, which satisfies $\chi^{(k)} = 0$ up to $k = 2r - 2 \geq r - 1$, so (B.4.13) is satisfied. By (B.4.14), we have

$$|\mathcal{L}(\psi_r) - \mathcal{L}(\varphi_r)|_{r,+} \leq D_r |\psi_r - \varphi_r|_{r+2,+} \leq D_r |\psi_r - \varphi_r|_{2r-2,+} \tag{B.4.17}$$

as long as $r \geq 4$ so that $2r - 2 \geq r + 2$. As mentioned above $|\psi_r - \varphi_r|_{2r-2,+}$ can be made arbitrarily small. Combining with (B.4.12), we can arrange so that $|f - \mathcal{L}(\varphi_r)|_{r,+} \leq 2/r \rightarrow 0$ as $r \rightarrow \infty$, which in particular implies (B.4.3).

Appendix C

Appendices of Part III

C.1 Connectedness of \mathcal{D}_L^α in $d \geq 3$

In this section we are going to show that in $d \geq 3$, each \mathcal{D}_L^α in (19.3.1) is connected.

Observe first of all that all \mathcal{D}_L^α of the same causal type in table 19.2 have the same connectedness property (this is obvious because they are related by renumbering points), so it suffices to prove the connectedness property for one \mathcal{D}_L^α in each causal type.

Given a \mathcal{D}_L^α , we define $(\mathcal{D}_L^\alpha)_3$ to be the set of all three-point Lorentzian configurations which have the causal ordering of the first three points of the configurations in \mathcal{D}_L^α . Then there is a natural projection from \mathcal{D}_L^α to $(\mathcal{D}_L^\alpha)_3$:

$$\begin{aligned} \pi : \mathcal{D}_L^\alpha &\longrightarrow (\mathcal{D}_L^\alpha)_3, \\ (x_1, x_2, x_3, x_4) &\mapsto (x_1, x_2, x_3), \end{aligned} \tag{C.1.1}$$

Then \mathcal{D}_L^α has the following decomposition

$$\mathcal{D}_L^\alpha = \bigcup_{(x_1, x_2, x_3) \in (\mathcal{D}_L^\alpha)_3} \{x_4 \mid (x_1, x_2, x_3, x_4) \in \mathcal{D}_L^\alpha\}. \tag{C.1.2}$$

For each causal type in table 19.2, we want to show that there exists a \mathcal{D}_L^α in this causal type such that

1. For fixed $(x_1, x_2, x_3) \in (\mathcal{D}_L^\alpha)_3$, the set $F_{x_1, x_2, x_3}^\alpha = \{x_4 \mid (x_1, x_2, x_3, x_4) \in \mathcal{D}_L^\alpha\}$ is non-empty and connected.
2. $(\mathcal{D}_L^\alpha)_3$ is connected.

C.1.1 Step 1

For a fixed three-point configuration (x_1, x_2, x_3) , the set $F_{x_1, x_2, x_3}^\alpha = \{x_4 \mid (x_1, x_2, x_3, x_4) \in \mathcal{D}_L^\alpha\}$ is non-empty and connected if one of the following conditions holds as a consequence of causal ordering imposed by \mathcal{D}_L^α :

1. $x_i \rightarrow x_4$ for $i = 1, 2, 3$.
2. $x_4 \rightarrow x_i$ for $i = 1, 2, 3$.
3. x_4 is space-like separated from all of x_1, x_2, x_3 .

For the first case, F_{x_1, x_2, x_3}^α is given by the intersection of the open forward light-cones of x_1, x_2, x_3 , which is non-empty. Since cones are convex, F_{x_1, x_2, x_3}^α is also convex, thus connected. The connectedness for the second case

follows from a similar argument. For the third case, we use the fact that the connectedness property does not change under Poincaré transformations, which allows us to move x_1 to 0 by translation

$$x_k \mapsto x_k - x_1, \quad k = 1, 2, 3, 4, \tag{C.1.3}$$

and then move x_2, x_3 onto a 2d subspace by a Lorentz transformation. We enumerate all possible three-point causal orderings

$$a \longrightarrow b \longrightarrow c, \quad \begin{array}{c} a \longrightarrow b \\ \longrightarrow c \end{array}, \quad \begin{array}{c} b \longrightarrow a \\ \longrightarrow c \end{array}, \quad \begin{array}{c} a \\ b \longrightarrow c \\ \longrightarrow c \end{array}, \tag{C.1.4}$$

and check case by case that in $d \geq 3$ we can always move the extra point x_4 from any position to ∞ , preserving the constraint that x_4 is space-like to a, b, c . This observation implies that F_{x_1, x_2, x_3}^α is connected for the third case.

In table 19.2, we find a \mathcal{D}_L^α satisfying one of the above conditions for some but not all causal types. That's why the connectedness of \mathcal{D}_L^α is not so obvious. The exceptional cases are causal type 8, 10 and 11, for which we need to discuss case by case. Without loss of generality we set $a = 1, b = 2, c = 3, d = 4$ (comparing with table 19.2) in the following discussion.

Type 8. By translations, Lorentz transformations and dilatations we fix the configurations to

$$x_1 = 0, \quad x_2 = (x^0, x^1, 0, \dots, 0), \quad x_3 = (0, 1, 0, \dots, 0). \tag{C.1.5}$$

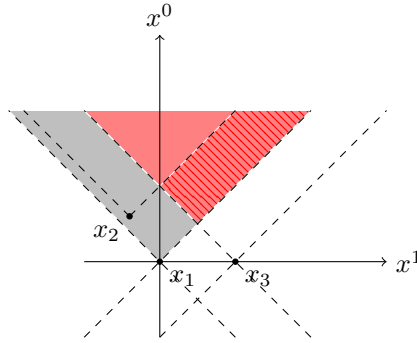


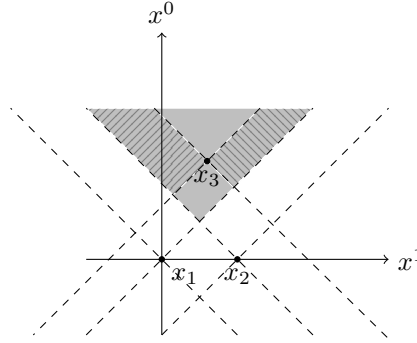
Figure C.1.1: Type 8

Then x_2 is in the open forward light-cone of x_1 , but out of the light-cones of x_3 (see the grey region in figure C.1.1), and x_4 is in the intersection of open forward light-cones of x_1 and x_3 (see the red region in figure C.1.1). Once x_2 is fixed somewhere in the grey region, the space of allowed positions for x_4 is given by the red region minus the forward light-cone of x_2 , so the remaining region for x_4 , which is F_{x_1, x_2, x_3}^α , is the red dashed region in figure C.1.1. Figure C.1.1 shows the 2d situation but a similar 3d figure shows that F_{x_1, x_2, x_3}^α is non-empty and connected in 3d, thus also non-empty and connected in higher d (because we can always find a spatial rotation which preserves x_1, x_2, x_3 and maps x_4 to $(x, y, z, 0, \dots, 0)$).

Type 10. By translations, Lorentz transformations and dilatations we fix the configurations to

$$x_1 = 0, \quad x_2 = (0, 1, 0, \dots, 0), \quad x_3 = (x^0, x^1, 0, \dots, 0). \tag{C.1.6}$$

Figure C.1.2: Type 10



The remaining x_3, x_4 pair are in the intersection of the open forward light-cones of x_1, x_2 , i.e. the grey region in figure C.1.2. Once x_3 is fixed, by the constraint that x_3, x_4 are space-like separated, F_{x_1, x_2, x_3}^α is given by the grey dashed region in figure C.1.2, which is obviously non-empty. This region is topologically the same as \mathbb{R}^d minus the light-cones of x_3 , thus connected when $d \geq 3$.

Type 11. By translations, Lorentz transformations and dilatations we fix the configurations to

$$x_1 = 0, \quad x_2 = (i, 0, \dots, 0), \quad x_3 = (x^0, x^1, 0, \dots, 0) \tag{C.1.7}$$

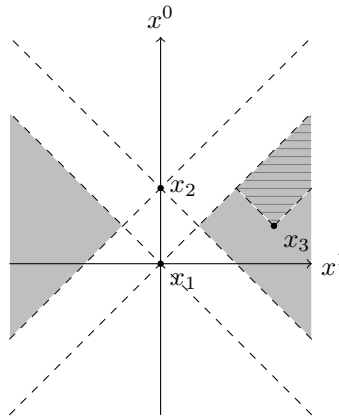


Figure C.1.3: Type 11

The remaining x_3, x_4 pair are in the grey region in figure C.1.3. One can see that in $d \geq 3$ the grey region is topologically the same as the triangle slice (see one of the grey triangle in figure C.1.3) times S^{d-2} , which is connected. Once x_3 is fixed, F_{x_1, x_2, x_3}^α is given by the forward light-cone of x_3 in the grey region (see the grey dashed region in figure C.1.3), which is connected.

C.1.2 Step 2

By slightly improving the argument in step 1, we claim that for the representative set \mathcal{D}_L^α (which we chose in step 1) of each causal type in table 19.2, the map (C.1.1) is surjective. In other words, for each three-point configuration in $(\mathcal{D}_L^\alpha)_3$, its preimage in \mathcal{D}_L^α is non-empty.

This claim is true for the cases which satisfy one of the conditions at the beginning of section C.1.1 because for these cases we can always find an x_4 which is very far away from x_1, x_2 and x_3 . This claim is also true for the exceptional cases because from the figure C.1.1, C.1.2 and C.1.3 we see that the remaining region for x_4 is always non-empty.

Now it remains to show that $(\mathcal{D}_L^\alpha)_3$, which is the set of all three-point configurations with a fixed causal ordering, is connected. For each causal type in (C.1.4), we choose

$$x_1 = b, \quad x_2 = c, \quad x_3 = a. \quad (\text{C.1.8})$$

Analogously to the four-point case we define a projection

$$\begin{aligned} \pi : (\mathcal{D}_L^\alpha)_3 &\longrightarrow (\mathcal{D}_L^\alpha)_2, \\ (x_1, x_2, x_3) &\mapsto (x_1, x_2). \end{aligned} \quad (\text{C.1.9})$$

Then we decompose $(\mathcal{D}_L^\alpha)_3$ into

$$(\mathcal{D}_L^\alpha)_3 = \bigcup_{(x_1, x_2) \in (\mathcal{D}_L^\alpha)_2} \{x_3 \mid (x_1, x_2, x_3) \in (\mathcal{D}_L^\alpha)_3\}. \quad (\text{C.1.10})$$

By comparing (C.1.4) and (C.1.8), we find each $(\mathcal{D}_L^\alpha)_3$ satisfies one of the following conditions:

1. $x_i \rightarrow x_3$ for $i = 1, 2$.
2. $x_3 \rightarrow x_i$ for $i = 1, 2$.
3. x_3 is space-like separated from both of x_1, x_2 .

This observation implies that for each $(\mathcal{D}_L^\alpha)_3$:

- For any fixed $(x_1, x_2) \in (\mathcal{D}_L^\alpha)_2$, the set $\{x_3 \mid (x_1, x_2, x_3) \in (\mathcal{D}_L^\alpha)_3\}$ is connected.
- $(\mathcal{D}_L^\alpha)_2 = \pi((\mathcal{D}_L^\alpha)_3)$ contains all two-point configurations with the corresponding causal ordering.

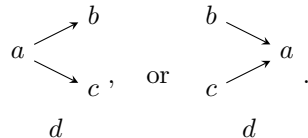
It remains to show that in $d \geq 3$, the set of two-point configurations with a given causal ordering is connected. This is trivial.

C.2 Tables of OPE convergence

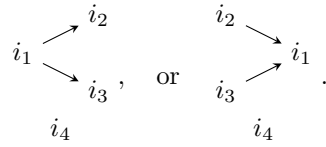
In this appendix we will give 12 tables of the results about convergence properties of three OPE channels: one table for one causal type. For each causal type we will give a template graph with points a, b, c, d . Given a Lorentzian configuration $c_L = (x_1, x_2, x_3, x_4) \in \mathcal{D}_L$, the way to look up the tables is as follows.

1. Compute the causal ordering of c_L , draw the graph of this causal ordering. Find the corresponding type number (say type X) in table 19.2.
2. Go to the section of causal type X. Compare the causal ordering of c_L with the template causal ordering of causal type X at the beginning of appendix C.X. Match the points i_1, i_2, i_3, i_4 with $(abcd)$. We will get a sequence $(i_1 i_2 i_3 i_4)$.
3. Look up the convergence properties of $(i_1 i_2 i_3 i_4)$ in the table of causal type X.

For example, consider the following template causal ordering



Then $(i_1 i_2 i_3 i_4)$ means



In appendix C.2.1 we will explain in detail how to make the table of OPE convergence for type 1 causal ordering. The procedure is similar for the other causal types, so we will only give the results for them. Before we start, we would like to introduce some tricks in appendix C.2.0.1, C.2.0.2 and C.2.0.3. They will be helpful in making the tables.

C.2.0.1 S_4 -action

There is a natural S_4 -action on the space of four-point configurations. Let $\sigma \in S_4$ be a symmetry group element:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \end{pmatrix}. \tag{C.2.1}$$

Let $C = (x_1, x_2, x_3, x_4)$ be a four-point configuration such that $x_{ij}^2 \neq 0$ for all x_i, x_j pairs. We define the action

$$\sigma \cdot C = (x'_1, x'_2, x'_3, x'_4), \quad x'_k = x_{\sigma^{-1}(k)}. \tag{C.2.2}$$

By computing z, \bar{z} of $\sigma \cdot C$ and comparing with z, \bar{z} of C , we get a natural S_4 -action on z, \bar{z} :

$$\begin{aligned} w_\sigma : \mathbb{C} \setminus \{0, 1\} &\longrightarrow \mathbb{C} \setminus \{0, 1\}, \\ z &\mapsto w_\sigma(z), \\ \bar{z} &\mapsto w_\sigma(\bar{z}), \end{aligned} \tag{C.2.3}$$

where $w_\sigma(z), w_\sigma(\bar{z})$ are the variables z, \bar{z} computed from $\sigma \cdot C$. We have the following properties:

- $\{w_\sigma\}_{\sigma \in S_4}$ belong to a set of 6 fractional linear transformation forming a group which is isomorphic to S_3 . The map $\sigma \mapsto w_\sigma$ is a group homomorphism from S_4 to S_3 (i.e. $w_{\sigma_1} \circ w_{\sigma_2} = w_{\sigma_1 \sigma_2}$).
- The S_4 -action on \mathcal{D}_L permutes classes S,T,U among themselves.
- The S_4 -action on \mathcal{D}_L permutes subclasses E_{su}, E_{st}, E_{tu} among themselves.
- The S_4 -action on \mathcal{D}_L preserves the subclass E_{stu} .

Let us denote σ by $[\sigma(1)\sigma(2)\sigma(3)\sigma(4)]$. We summarize the above properties in table C.1.

Table C.1: The list of w_σ and the S_4 transformation between classes and subclasses.

σ	$w_\sigma(z)$	S	T	U	E_{su}	E_{st}	E_{tu}	E_{stu}
[1234], [2143], [3412], [4321]	z	S	T	U	E_{su}	E_{st}	E_{tu}	E_{stu}
[2134], [1243], [4312], [3421]	$\frac{z}{z-1}$	S	U	T	E_{st}	E_{su}	E_{tu}	E_{stu}
[3214], [4123], [1432], [2341]	$1 - z$	T	S	U	E_{tu}	E_{st}	E_{su}	E_{stu}
[1324], [2413], [3142], [4231]	$\frac{1}{z}$	U	T	S	E_{su}	E_{tu}	E_{st}	E_{stu}
[2314], [1423], [4132], [3241]	$\frac{1}{1-z}$	T	U	S	E_{st}	E_{tu}	E_{su}	E_{stu}
[3124], [4213], [1342], [2431]	$1 - \frac{1}{z}$	U	S	T	E_{tu}	E_{su}	E_{st}	E_{stu}

Suppose a configuration c_L gives the template causal ordering of a causal type, which means that c_L corresponds to the sequence (1234). For $\sigma = [i_1 i_2 i_3 i_4]$, we get a configuration $c'_L = \sigma \cdot C$ by eq. (C.2.2). The causal ordering of c'_L is in the same causal type as c_L . By comparing the causal orderings of c_L and c'_L , we see that the sequence of c'_L is exactly $(i_1 i_2 i_3 i_4)$. Therefore, given a causal type, if we know the class/subclass of the template causal ordering, by looking up table C.1 we decide the classes/subclasses of the other causal ordering in the same causal type. Then by looking up table 19.1, we immediately get a part of the OPE convergence properties for each causal ordering.

By using the above trick, the problem of determining the classes/subclasses of causal orderings belong is reduced to determining the class/subclass of the template causal ordering in each causal type. In appendix C.2.0.2, we will introduce a trick to determine the classes/subclasses of the template causal orderings.

C.2.0.2 Lorentzian conformal frame

Our goal in this subsection is to give a systematic way to determine the class/subclass of \mathcal{D}_L^α , where α is a fixed causal ordering.

Recalling lemma 19.3.1, all configurations in \mathcal{D}_L^α belong to the same class. We can choose a particular configuration $c_L \in \mathcal{D}_L^\alpha$ and compute z, \bar{z} of c_L , then we immediately know the class (not subclass) of \mathcal{D}_L^α .

If \mathcal{D}_L^α belongs to class S/T/U, then we are done. The rest of this subsection is for the case that \mathcal{D}_L^α belongs to class E. If \mathcal{D}_L^α belongs to class E, then according to theorem 19.3.2, we need to check the OPE convergence properties for the intersection of \mathcal{D}_L^α and each subclass of class E as long as the intersection is non-empty. We will find that only the type 1, 5, 6, 10, 11, 12 causal orderings in table 19.2 belong to class E.¹ In the tables of OPE convergence properties of causal type 5, 10 and 12, we give the results of all subclasses for each causal ordering (see table C.7, C.12 and C.14); while in the tables of causal type 1, 6 and 11, we only give the results of one subclass for each causal ordering (see table C.3, C.8 and C.13). We claim that our tables are complete, based on the following lemma.

Lemma C.2.1. *Given a fixed causal ordering α , if α is in causal type 1/6/11, then \mathcal{D}_L^α only belongs to one of the three subclasses E_{st}, E_{su}, E_{tu} .*

The basic tool we use to prove the above lemma is the Lorentzian conformal frame. The Lorentzian conformal frame is similar to the Euclidean conformal frame (11.2.3). Given a Lorentzian configuration c_L , its conformal frame configuration c'_L is a Lorentzian configuration which has one of the following forms

$$\begin{aligned} 1. & x'_1 = 0, x'_2 = (ia, b, 0, \dots, 0), x'_3 = (i, 0, \dots, 0), x'_4 = \infty. \\ 2. & x'_1 = 0, x'_2 = (ib, a, 0, \dots, 0), x'_3 = (0, 1, 0, \dots, 0), x'_4 = \infty. \\ 3. & x'_1 = 0, x'_2 = (0, a, b, 0, \dots, 0), x'_3 = (0, 1, 0, \dots, 0), x'_4 = \infty. \end{aligned} \tag{C.2.4}$$

c'_L and c_L are related by a Lorentzian conformal transformation. Computing the cross-ratios from (C.2.4), we see that: for the first and second cases $z = a + b, \bar{z} = a - b$; for the third case $z = a + ib, \bar{z} = a - ib$. Analogously to the Euclidean conformal frame, the Lorentzian conformal frame configuration is unique up to a reflection $b \mapsto -b$, which corresponds to interchanging z and \bar{z} .

Let us describe how we map a four-point configuration to the conformal frame by conformal transformations. Let $c_L = (x_1, x_2, x_3, x_4)$ be a Lorentzian configuration. We will go from c_L to c'_L in a few steps, and each step is a conformal transformation. The configuration after the k -th step is denoted by $c_L^{(k)}$.

Step 1. We move x_1 to 0 by translation. The configuration $c_L^{(1)}$ after the first step is given by

$$c_L^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)} \right) = (0, x_2 - x_1, x_3 - x_1, x_4 - x_1). \tag{C.2.5}$$

This step preserves the causal ordering.

Step 2. We move x_4 to ∞ by special conformal transformation

$$x'^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2x \cdot b + x^2 b^2}, \quad b^\mu = \frac{(x_4 - x_1)^\mu}{(x_4 - x_1)^2}. \tag{C.2.6}$$

¹This can be easily done by choosing one particular configuration and compute z, \bar{z} for each template causal ordering, and by the fact that the S_4 -action preserves class E (as discussed in appendix C.2.0.1).

$x_1 = 0$ is preserved by special conformal transformation. This step may change the causal ordering. Under general conformal transformations, x_{ij}^2 transforms as

$$\begin{aligned} (x'_i - x'_j)^2 &= \Omega(x_i)\Omega(x_j)(x_i - x_j)^2, \\ (ds'^2 &= \Omega(x)^2 ds^2), \end{aligned} \quad (\text{C.2.7})$$

and for special conformal transformation (C.2.6), the scaling factor at $x_k^{(1)} = x_k - x_1$ is given by

$$\Omega\left(x_k^{(1)}\right) = \frac{(x_4 - x_1)^2}{(x_4 - x_k)^2}, \quad k = 1, 2, 3, 4. \quad (\text{C.2.8})$$

Let $c_L^{(2)} = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)})$ be the configuration after step 2. For any configuration c_L in \mathcal{D}_L (where the light-cone singularities are excluded), by (C.2.7) and (C.2.8) we have

$$\left(x_i^{(2)} - x_j^{(2)}\right)^2 \neq 0, \quad i, j = 1, 2, 3, \quad (\text{C.2.9})$$

Furthermore, the sign of $\left(x_i^{(2)} - x_j^{(2)}\right)^2$ is determined by the causal ordering of c_L . So we know if each $x_i^{(2)}, x_j^{(2)}$ pair of $c_L^{(2)}$ is space-like or time-like. The information we do not know a priori from (C.2.7) and (C.2.8) is the causal orderings of time-like $x_i^{(2)}, x_j^{(2)}$ pairs (who is in the future of whom).²

Step 3. We move x_3 to its final position by some composition of Lorentz transformations, dilatations and time reversal θ_L (these conformal transformations preserve $x_1 = 0$ and $x_4 = \infty$). Lorentz transformations and dilatations preserve causal orderings, and time reversal only reverse causal orderings (i.e. x_i, x_j pairs change from time-like to time-like, or from space-like to space-like). There are two possibilities after step 2: $x_3^{(2)}$ could be space-like or time-like to $x_1^{(2)}$. If $x_1^{(2)}, x_3^{(2)}$ are time-like, then $x_3^{(3)}$ is put at $(i, 0, \dots, 0)$. If $x_1^{(2)}, x_3^{(2)}$ are space-like, then $x_3^{(3)}$ is put at $(0, 1, 0, \dots, 0)$. Therefore, $c_L^{(3)}$ is in one of the following forms:

1. $x_1^{(3)} = 0, x_3^{(3)} = (i, 0, \dots, 0), x_4^{(3)} = \infty$.
2. $x_1^{(3)} = 0, x_3^{(3)} = (0, 1, 0, \dots, 0), x_4^{(3)} = \infty$.

Step 4. We move x_2 to somewhere in the (01)-plane or (12)-plane by Lorentz transformations in the little group of $x_3^{(3)}$. If $x_3^{(3)} = (i, 0, \dots, 0)$, then we move x_2 to the (01)-plane by rotation, i.e. $x_2^{(4)} = (ia, b, 0, \dots, 0)$. If $x_3^{(3)} = (0, 1, 0, \dots, 0)$, then we move x_2 onto the (01)-plane or the (12)-plane, determined as follows:

- If $x_2^{(3)} = (i\beta_1, a, \beta_2, \dots, \beta_{d-1})$ with $(\beta_1)^2 \geq (\beta_2)^2 + \dots + (\beta_{d-1})^2$, then x_2 is put in the (01)-plane, i.e. $x_2^{(4)} = (ib, a, 0, \dots, 0)$ and $b^2 = (\beta_1)^2 - (\beta_2)^2 - \dots - (\beta_{d-1})^2$.
- If $x_2^{(3)} = (i\beta_1, a, \beta_2, \dots, \beta_{d-1})$ with $(\beta_1)^2 \leq (\beta_2)^2 + \dots + (\beta_{d-1})^2$, then x_2 is put in the (01)-plane, i.e. $x_2^{(4)} = (0, a, b, 0, \dots, 0)$ and $b^2 = (\beta_2)^2 + \dots + (\beta_{d-1})^2 - (\beta_1)^2$.

In the end, $c'_L = c_L^{(4)}$ has one of the forms in eq. (C.2.4). Moreover, the above discussion provides us with the following fact:

- the sign of $(x'_{ij})^2$ of c'_L is going to be the same for all configurations $c_L \in \mathcal{D}_L^\alpha$ in each causal ordering α .

Now back to our question. Suppose \mathcal{D}_L^α is in class E. Let $c'_L = (x'_1, x'_2, x'_3, x'_4)$ be the conformal frame configuration of $c_L \in \mathcal{D}_L^\alpha$. Comparing the range of the (a, b) pair in (C.2.4) with the range of the (z, \bar{z}) pair in class E (see section 19.1) and using the above fact, we see that there are only two possibilities for c'_L :

²Of course for any particular configuration we can just compute $c_L^{(2)}$ and then determine its causal ordering.

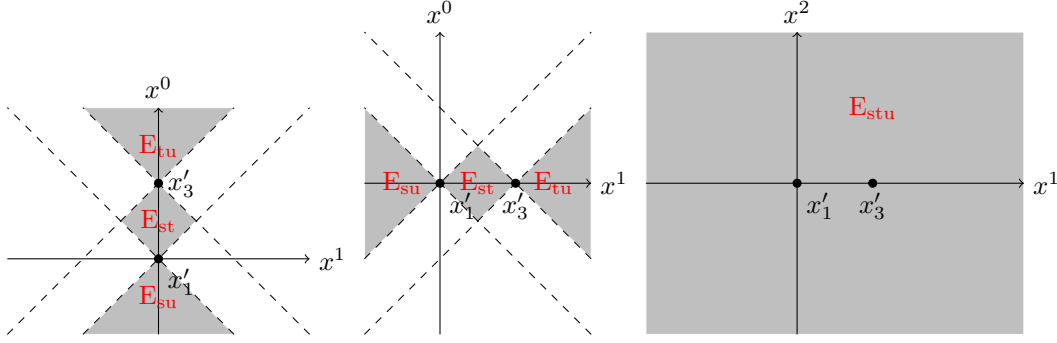


Figure C.2.1: The conformal frame configurations realized by configurations in class E.

1. $(x'_{13})^2, (x'_{12})^2, (x'_{23})^2 < 0$ for all $c_L \in \mathcal{D}_L^\alpha$. All possible c'_L are given by the grey region in the first picture of figure C.2.1. In this case z, \bar{z} are real, so we have

$$\mathcal{D}_L^\alpha = (\mathcal{D}_L^\alpha \cap E_{st}) \sqcup (\mathcal{D}_L^\alpha \cap E_{su}) \sqcup (\mathcal{D}_L^\alpha \cap E_{tu}). \quad (\text{C.2.10})$$

Because the z, \bar{z} ranges corresponding to E_{st}, E_{su}, E_{tu} are disconnected from each other in figure C.2.1 and because \mathcal{D}_L^α is connected in $d \geq 3$, only one of the above intersections is non-empty. We conclude that such \mathcal{D}_L^α only belongs to one of the three subclasses E_{st}, E_{su}, E_{tu} . This conclusion remains valid also in 2d, because 2d configurations can be embedded into higher d.

2. $(x'_{13})^2, (x'_{12})^2, (x'_{23})^2 > 0$ for all $c_L \in \mathcal{D}_L^\alpha$. All possible conformal frame configurations are given by the grey region in the second and third pictures of figure C.2.1. In this case we have

$$\mathcal{D}_L^\alpha = (\mathcal{D}_L^\alpha \cap E_{st}) \sqcup (\mathcal{D}_L^\alpha \cap E_{su}) \sqcup (\mathcal{D}_L^\alpha \cap E_{tu}) \sqcup (\mathcal{D}_L^\alpha \cap E_{stu}), \quad (\text{C.2.11})$$

which means that the configurations in \mathcal{D}_L^α may appear in all subclasses.

To see which of these possibilities is realized, it is enough to know the sign of $(x'_{13})^2$. To finish the proof of lemma C.2.1, it remains to check that $(x'_{13})^2 < 0$ for all type 1, 6, 11 causal orderings (so that possibility 1 is realized for c'_L). Since all causal orderings of a fixed causal type can be realized by permuting the indices of the template causal ordering and such S_4 -action permutes subclasses E_{st}, E_{su}, E_{tu} among themselves (see section C.2.0.1), it suffices to check that $(x'_{13})^2 < 0$ for the template causal orderings of causal type 1, 6, 11. Moreover, since we are only interested in the sign of $(x'_{13})^2$, we can use the following formula:

$$\text{Sign} \left((x'_{13})^2 \right) = \text{Sign} (x_{13}^2 x_{14}^2 x_{34}^2). \quad (\text{C.2.12})$$

(C.2.12) follows from (C.2.7), (C.2.8) and the fact that step 3,4 (of constructing the conformal frame) preserve the sign of $(x_{ij}^{(k)})^2$. Now let us do the check.

Type 1. The template causal ordering is given by

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4, \quad (\text{C.2.13})$$

which gives $x_{13}^2, x_{14}^2, x_{34}^2 < 0$, hence $(x'_{13})^2 < 0$.

Type 6. The template causal ordering is given by

$$1 \longrightarrow 2 \longrightarrow 3, \quad (\text{C.2.14})$$

4

which gives $x_{13}^2 < 0$ and $x_{14}^2, x_{34}^2 > 0$, hence $(x'_{13})^2 < 0$.

Type 11. The template causal ordering is given by

$$\begin{array}{l} 1 \longrightarrow 2 \\ 3 \longrightarrow 4 \end{array} \quad (\text{C.2.15})$$

which gives $x_{13}^2, x_{14}^2 > 0$ and $x_{34}^2 < 0$, hence $(x'_{13})^2 < 0$.

So we finished the proof of lemma C.2.1. As a consistency check one can also compute the sign of $(x'_{13})^2$ for type 5,10,12 causal orderings and find that $(x'_{13})^2 > 0$ for these cases.

For type 1, 6, 11 causal orderings, to determine the subclasses, it remains to determine the subclass of one particular configuration of the template causal ordering in each type, then determine the subclasses of other causal orderings by looking up table C.1.

We would like to remark that Lorentzian conformal frame is just a way to figure out the range of z, \bar{z} . There is no claim that correlation functions at c_L, c'_L agree. As mentioned in section C.3.3.3, the global conformal invariance does not hold in a general Lorentzian CFT.

C.2.0.3 Symmetry of the graph

Usually, we go from one causal ordering to another by permuting the indices. For example, given the causal ordering $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$, by permuting 1, 4 we get another causal ordering $4 \rightarrow 2 \rightarrow 3 \rightarrow 1$.

However, a causal ordering may have a non-trivial little group.³ For example, consider the following causal ordering

$$a \longrightarrow b \begin{matrix} \nearrow c \\ \searrow d \end{matrix} . \tag{C.2.16}$$

The causal ordering (C.2.16) does not change if we interchange c and d , so it has a non-trivial little group \mathbb{Z}_2 .

The little group is unique up to an isomorphism for all causal orderings in the same causal type. Let G be the little group of one causal type and let $|G|$ be the order of G . Then the total number of causal orderings in this type is given by $24/|G|$. So in the table of each causal type, we will only list $24/|G|$ sequences $(i_1 i_2 i_3 i_4)$. Below the tables, we will point out the sequences which give the same causal ordering.

C.2.1 Type 1

The type 1 causal ordering is given by

$$a \rightarrow b \rightarrow c \rightarrow d. \tag{C.2.17}$$

We let (C.2.17) be the template causal ordering, then the causal ordering $i_1 \rightarrow i_2 \rightarrow i_3 \rightarrow i_4$ is labelled by the sequence $(i_1 i_2 i_3 i_4)$. Any permutation of the indices in (C.2.17) will change the causal ordering, so the little group of the type 1 causal orderings is trivial. We have to list 24 causal orderings in the table.

Under time reversal θ_L , $a \rightarrow b \rightarrow c \rightarrow d$ is mapped to $d \rightarrow c \rightarrow b \rightarrow a$, which is equivalent to the permutation $\theta_L : a \leftrightarrow d, b \leftrightarrow c$. This action is causal-type specific. In addition, we have θ_E action which is always given by $\theta_E : 1 \leftrightarrow 4, 2 \leftrightarrow 3$ (see eq. (19.4.5)). Using $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by these permutations, we divide 24 type 1 causal orderings into 8 orbits:

1. (1234), (4321).
2. (1243), (4312), (3421), (2134).
3. (1324), (4231).
4. (1423), (3241), (4132), (2314).
5. (1342), (2431), (4213), (3124).
6. (1432), (2341), (4123), (3214).

³By little group of a causal ordering, we mean the permutations of the indices which do not change this causal ordering.

7. (2143), (3412).

8. (2413), (3142).

As discussed in section 19.4, all the causal orderings in each orbit have the same convergent OPE channels.

Let us consider (1234), or equivalently the causal ordering $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$. We first pick a particular configuration and compute z, \bar{z} . Here we choose

$$x_1 = 0, \quad x_2 = (i, 0, \dots, 0), \quad x_3 = (2i, 0, \dots, 0), \quad x_4 = (3i, 0, \dots, 0) \quad (\text{C.2.18})$$

and get $z = \bar{z} = \frac{1}{4}$, which is in the range corresponding to subclass E_{st} . By lemma C.2.1, the whole (1234) is in subclass E_{st} .

All other $(i_1 i_2 i_3 i_4)$ causal orderings can be obtained by applying permutation $\sigma = [i_1 i_2 i_3 i_4]$ to the template ordering (1234). Using table C.1, we can easily determine the subclasses of all other $(i_1 i_2 i_3 i_4)$ sequences (look at the column having E_{st} on top). Then by looking up table 19.1, we get some OPE convergence properties of type 1 causal orderings, which are summarized in table C.2.

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (4321)	E_{st}	✓		✗
(1243), (3421), (4312), (2134)	E_{su}	✓	✗	
(1324), (4231)	E_{tu}	✗		
(1423), (3241), (4132), (2314)	E_{tu}	✗		
(1342), (2431), (4213), (3124)	E_{su}	✓	✗	
(1432), (2341), (4123), (3214)	E_{st}	✓		✗
(2143), (3412)	E_{st}	✓		✗
(2413), (3142)	E_{tu}	✗		

Table C.2: The classes/subclasses of type 1 causal orderings

It remains to complete the rest of table C.2. For this we choose a representative configuration for each orbit of causal orderings and choose a path to compute the curves of z, \bar{z} , then decide the convergence properties of t- and u-channel expansions. In practice this is done numerically, by plotting the curves of z, \bar{z} and staring at the plots to determine N_t, N_u (as in examples in section 19.6). The final results of OPE convergence properties in the three channels are shown in table C.3, where we use red to indicate the new marks.

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (4321)	E_{st}	✓	✓	✗
(2143), (3412)	E_{st}	✓	✗	✗
(1432), (2341), (4123), (3214)	E_{st}	✓	✓	✗
(1324), (4231)	E_{tu}	✗	✓	✗
(2413), (3142)	E_{tu}	✗	✗	✗
(1423), (3241), (4132), (2314)	E_{tu}	✗	✓	✗
(1342), (2431), (4213), (3124)	E_{su}	✓	✗	✗
(1243), (3421), (4312), (2134)	E_{su}	✓	✗	✗

Table C.3: OPE convergence properties of type 1 causal orderings

C.2.2 Type 2

The type 2 causal ordering is given by

$$\begin{array}{c}
 a \longrightarrow b \begin{array}{l} \nearrow c \\ \searrow d \end{array} , \quad \begin{array}{l} c \longrightarrow b \\ \nearrow d \end{array} \longrightarrow a .
 \end{array}
 \tag{C.2.19}$$

We choose a particular configuration for (1234):

$$x_1 = 0, \quad x_2 = (i, 0, \dots, 0), \quad x_3 = (2i, 0.5, 0, \dots, 0), \quad x_4 = (2i, -0.5, 0, \dots, 0),
 \tag{C.2.20}$$

and get $z = -\frac{8}{5}$, $\bar{z} = \frac{4}{9}$, which is in the range corresponding to class S. So we conclude that (1234) is in class S. The remaining steps are the same as appendix C.2.1. The results of OPE convergence properties in the three channels are shown in table C.4.

Table C.4: OPE convergence properties of type 2 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (4321)	S	✓	✗	✗
(1324), (4231)	U	✗	✗	✗
(1423), (4132)	T	✗	✓	✗
(2134), (3421)	S	✓	✗	✗
(3124), (2431)	U	✗	✗	✗
(2314), (3241)	T	✗	✓	✗

There are only 12 causal orderings because $(i_1i_2i_3i_4)$ and $(i_1i_2i_4i_3)$ are the same causal ordering (the little group is \mathbb{Z}_2).

C.2.3 Type 3

The type 3 causal ordering is given by

$$\begin{array}{c}
 a \longrightarrow b \longrightarrow c \\
 \searrow d
 \end{array}
 , \quad
 \begin{array}{c}
 c \longrightarrow b \longrightarrow a \\
 \nearrow d
 \end{array}
 .
 \tag{C.2.21}$$

We choose a particular configuration for (1234):

$$x_1 = 0, \quad x_2 = (i, 0, \dots, 0), \quad x_3 = (2i, 0, \dots, 0), \quad x_4 = (1.5i, 1, 0, \dots, 0),
 \tag{C.2.22}$$

and get $z = \frac{1}{6}$, $\bar{z} = \frac{3}{2}$, which is in the range corresponding to class T. So we conclude that (1234) is in class T. The results of OPE convergence properties in the three channels are shown in table C.5:

Table C.5: OPE convergence properties of type 3 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (4321)	T	✗	✓	✗
(1243), (4312)	U	✗	✗	✓
(1324), (4231)	T	✗	✓	✗
(1342), (4213)	S	✓	✗	✗
(1423), (4132)	U	✗	✗	✓
(1432), (4123)	S	✓	✗	✗
(2134), (3421)	U	✗	✗	✗
(2143), (3412)	T	✗	✗	✗
(2314), (3241)	U	✗	✗	✗
(2341), (3214)	S	✓	✗	✗
(2413), (3142)	T	✗	✗	✗
(2431), (3124)	S	✓	✗	✗

C.2.4 Type 4

The type 4 causal ordering is given by

$$\begin{array}{c}
 & & b & & \\
 & \nearrow & & \searrow & \\
 a & & & & d \\
 & \searrow & & \nearrow & \\
 & & c & &
 \end{array} . \tag{C.2.23}$$

We choose a particular configuration for (1234):

$$x_1 = 0, \quad x_2 = (i, 0.5, 0, \dots, 0), \quad x_3 = (i, -0.5, 0, \dots, 0), \quad x_4 = (2i, 0, \dots, 0), \tag{C.2.24}$$

and get $z = \frac{1}{9}$, $\bar{z} = 9$, which is in the range corresponding to class T. So we conclude that (1234) is in class T. The results of OPE convergence properties in the three channels are shown in table C.6:

Table C.6: OPE convergence properties of type 4 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (4321)	T	✗	✓	✗
(1243), (4312), (2134), (3421)	U	✗	✗	✗
(1342), (4213), (2341), (3214)	S	✓	✗	✗
(2143), (3412)	T	✗	✗	✗

Here we use the fact that $(i_1 i_2 i_3 i_4)$ and $(i_1 i_3 i_2 i_4)$ are the same causal ordering (the little group is \mathbb{Z}_2).

C.2.5 Type 5

The type 5 causal ordering is given by

$$\begin{array}{c}
 & & b & & \\
 & \nearrow & & \searrow & \\
 a & & & & a \\
 & \searrow & & \nearrow & \\
 & & c & & \\
 & & d & &
 \end{array} . \tag{C.2.25}$$

We choose a particular configuration for (1234):

$$x_1 = 0, x_2 = (i, 0.5, 0, \dots, 0), x_3 = (i, 0, \dots, 0), x_4 = (i, -0.5, \dots, 0), \tag{C.2.26}$$

and get $z = \frac{1}{4}$, $\bar{z} = \frac{3}{4}$, which is in the range corresponding to subclass E_{st} . So (1234) is in class E. We would like to find a particular configuration in each subclass of class E. The little group of this causal type is S_3 , which corresponds to permutations among b, c, d in (C.2.25). By looking up table C.1, we see that permuting x_2, x_3 in (C.2.26) gives E_{tu} and permuting x_3, x_4 gives E_{su} . To realize E_{stu} we choose the following configuration in (1234):

$$x_1 = 0, x_2 = (i, 0.5, 0, \dots, 0), x_3 = (i, -0.5, 0, \dots, 0), x_4 = (i, 0, 0.5, 0, \dots, 0), \tag{C.2.27}$$

and get $z = i$, $\bar{z} = -i$, which is indeed in the range corresponding to subclass E_{stu} . So we conclude that the configurations of (1234) do appear in all subclasses of class E in $d \geq 3$, while they only appear in subclasses E_{st}, E_{su}, E_{tu} in 2d.⁴

The results of OPE convergence properties in the three channels are shown in table C.7:

Table C.7: OPE convergence properties of type 5 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (4321)	E_{st}	✓	✓	✗
	E_{su}	✓	✗	✓
	E_{tu}	✗	✓	✓
	E_{stu}	✓	✓	✓
(2134), (3124)	E_{st}	✓	✗	✗
	E_{su}	✓	✗	✗
	E_{tu}	✗	✗	✗
	E_{stu}	✓	✗	✗

Here we use the fact that for $(i_1 i_2 i_3 i_4)$, any permutation of 2, 3, 4 does not change the causal ordering (the little group is S_3).

C.2.6 Type 6

The type 6 causal ordering is given by

$$a \longrightarrow b \longrightarrow c \\ \qquad \qquad \qquad d \tag{C.2.28}$$

We choose a particular configuration for (1234):

$$x_1 = 0, x_2 = (i, 0, \dots, 0), x_3 = (2i, 0, \dots, 0), x_4 = (i, 2, \dots, 0), \tag{C.2.29}$$

and get $z = \frac{1}{4}$, $\bar{z} = \frac{3}{4}$, which is in the range corresponding to subclass E_{st} . By lemma C.2.1, the whole (1234) is in subclass E_{st} . The results of OPE convergence properties in the three channels are shown in table C.8.

⁴We used two dimensions in (C.2.26) and three dimensions in (C.2.27). On the other hand, as mentioned at the end of section 19.3.2, subclass E_{stu} does not exist in 2d because z, \bar{z} can only be real.

Table C.8: OPE convergence properties of type 6 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (4321), (3214), (2341)	E_{st}	✓	✓	✗
(1243), (4312), (4213), (1342)	E_{su}	✓	✗	✓
(1324), (4231), (2314), (3241)	E_{tu}	✗	✓	✗
(1423), (4132), (2413), (3142)	E_{tu}	✗	✗	✓
(1432), (4123), (3412), (2143)	E_{st}	✓	✗	✗
(2134), (3421), (3124), (2431)	E_{su}	✓	✗	✗

C.2.7 Type 7

The type 7 causal ordering is given by



We choose a particular configuration for (1234):

$$x_1 = 0, \quad x_2 = (i, 0.5, 0, \dots, 0), \quad x_3 = (i, -0.5, 0, \dots, 0), \quad x_4 = (0, 2, 0, \dots, 0), \tag{C.2.31}$$

and get $z = \frac{7}{15}$, $\bar{z} = 9$, which is in the range corresponding to class T. So we conclude that (1234) is in class T. The results of OPE convergence properties in the three channels are shown in table C.9.

Table C.9: OPE convergence properties of type 7 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (4321)	T	✗	✓	✗
(1243), (4312)	U	✗	✗	✓
(1342), (4213)	S	✓	✗	✗
(2134), (3421)	U	✗	✗	✗
(2143), (3412)	T	✗	✗	✗
(2341), (3124)	S	✓	✗	✗

Here we use the fact that $(i_1 i_2 i_3 i_4)$ and $(i_1 i_3 i_2 i_4)$ are the same causal ordering (the little group is \mathbb{Z}_2).

C.2.8 Type 8

The type 8 causal ordering is given by



We choose a particular configuration for (1234):

$$x_1 = 0, \quad x_2 = (i, -0.5, 0, \dots, 0), \quad x_3 = (0, 1, 0, \dots, 0), \quad x_4 = (i, 0.5, 0, \dots, 0), \tag{C.2.33}$$

and get $z = \frac{1}{4}$, $\bar{z} = \frac{9}{4}$, which is in the range corresponding to class T. So we conclude that (1234) is in class T. The results of OPE convergence properties in the three channels are shown in table C.10:

Table C.10: OPE convergence properties of type 8 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (4321)	T	✗	✗	✗
(1243), (4312), (3421), (2134)	U	✗	✗	✓
(1342), (4213), (2431), (3124)	S	✓	✗	✗
(2143), (3412)	T	✗	✗	✗
(1324), (4231)	T	✗	✗	✗
(1432), (4123), (2341), (3214)	S	✓	✗	✗
(1423), (4132), (3241), (2314)	U	✗	✗	✗
(2413), (3142)	T	✗	✗	✗

C.2.9 Type 9

The type 9 causal ordering is given by

$$\begin{array}{c}
 a \longrightarrow b \\
 c \phantom{} \\
 \phantom{} d
 \end{array}
 \tag{C.2.34}$$

We choose a particular configuration for (1234):

$$x_1 = 0, \quad x_2 = (i, 0, \dots, 0), \quad x_3 = (0, 2, 0, \dots, 0), \quad x_4 = (0, 3, 0, \dots, 0),
 \tag{C.2.35}$$

and get $z = -\frac{1}{8}$, $\bar{z} = \frac{1}{4}$, which is in the range corresponding to class S. So we conclude that (1234) is in class S. The results of OPE convergence properties in the three channels are shown in table C.11:

Table C.11: OPE convergence properties of type 9 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (4312), (2134), (3412)	S	✓	✗	✗
(1324), (4213), (3124), (2413)	U	✗	✗	✓
(1423), (4123)	T	✗	✓	✗
(2314), (3214)	T	✗	✓	✗

Here we use the fact that $(i_1 i_2 i_3 i_4)$ and $(i_1 i_2 i_4 i_3)$ are the same causal ordering (the little group is \mathbb{Z}_2).

C.2.10 Type 10

The type 10 causal ordering is given by

$$\begin{array}{c}
 a d \\
 \\
 b d
 \end{array}
 \tag{C.2.36}$$

We choose a particular configuration for (1234):

$$x_1 = 0, \quad x_2 = (0, 1, 0, \dots, 0), \quad x_3 = (2i, 0, \dots, 0), \quad x_4 = (2i, 1, \dots, 0),
 \tag{C.2.37}$$

and get $z = \bar{z} = \frac{1}{4}$, which is in the range corresponding to subclass E_{st} . So (1234) is in class E. We would like to find a particular configuration in each subclass of class E. The little group of this causal type is $\mathbb{Z}_2 \times \mathbb{Z}_2$, which is generated by $a \leftrightarrow b$ and $c \leftrightarrow d$ in (C.2.36). By looking up table C.1, we see that by acting with the little group on configuration (C.2.37), we can get E_{su} , but we cannot get E_{tu} . The underlying fact is that the 2d configurations of (1234) do not appear in E_{tu} (it is obvious that 2d configurations do not appear in E_{stu}). Let us show this fact. In 2d Minkowski space we can use the light-cone coordinates:

$$z_k = t_k + \mathbf{x}_k, \quad \bar{z}_k = t_k - \mathbf{x}_k, \quad (x_k = (it_k, \mathbf{x}_k)). \quad (\text{C.2.38})$$

The causal ordering (C.2.36) implies

$$\begin{aligned} z_3, z_4 > z_1, z_2, \quad \bar{z}_3, \bar{z}_4 > \bar{z}_1, \bar{z}_2, \\ (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) < 0, \quad (z_3 - z_4)(\bar{z}_3 - \bar{z}_4) < 0. \end{aligned} \quad (\text{C.2.39})$$

Since the little group $\mathbb{Z}_2 \times \mathbb{Z}_2$ of (1234) preserves E_{tu} (see table C.1), by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action, it suffices to show that E_{tu} configurations do not exist when

$$\begin{aligned} z_3, z_4 > z_1, z_2, \quad \bar{z}_3, \bar{z}_4 > \bar{z}_1, \bar{z}_2, \\ z_1 - z_2 < 0, \quad \bar{z}_1 - \bar{z}_2 > 0, \\ z_3 - z_4 < 0, \quad \bar{z}_3 - \bar{z}_4 > 0. \end{aligned} \quad (\text{C.2.40})$$

In this case the computation is straightforward:

$$\begin{aligned} z &= \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)} < \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_1)(z_4 - z_2)} = 1, \\ \bar{z} &= \frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_3 - \bar{z}_4)}{(\bar{z}_3 - \bar{z}_1)(\bar{z}_4 - \bar{z}_2)} < \frac{(\bar{z}_4 - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}{(\bar{z}_3 - \bar{z}_1)(\bar{z}_4 - \bar{z}_2)} = 1. \end{aligned} \quad (\text{C.2.41})$$

So we conclude that the 2d configurations in (1234) have $z, \bar{z} < 1$, i.e. (1234) does not intersect with subclass E_{tu} in 2d. To find a E_{tu} configuration in (1234) we need to construct it in $d \geq 3$. We choose the 3d configuration (19.6.6) and get $z \approx 1.1$, $\bar{z} \approx 6.3$, which is in the range corresponding to subclass E_{tu} .

To realize E_{stu} we choose the following configuration in (1234):

$$x_1 = 0, \quad x_2 = (0, 0.5, 0, \dots, 0), \quad x_3 = (2i, 0, 0.5, 0, \dots, 0), \quad x_4 = (i, 0.5, 0, \dots, 0), \quad (\text{C.2.42})$$

and get $z \approx 0.33 + 0.24i$, $\bar{z} = 0.33 - 0.24i$, which is in the range corresponding to subclass E_{stu} . So we conclude that the configurations of (1234) do appear in all subclasses of class E in $d \geq 3$, while they only appear in subclasses $E_{\text{st}}, E_{\text{su}}$ in 2d.

The results of OPE convergence properties in the three channels are shown in table C.12:

Table C.12: OPE convergence properties of type 10 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (3412)	E_{st}	✓	✗	✗
	E_{su}	✓	✗	✗
	E_{tu}	✗	✗	✗
	E_{stu}	✓	✗	✗
(1324), (2413)	E_{st}	✓	✗	✗
	E_{su}	✓	✗	✗
	E_{tu}	✗	✗	✗
	E_{stu}	✓	✗	✗
(1423), (2314)	E_{st}	✓	✓	✗
	E_{su}	✓	✗	✓
	E_{tu}	✗	✓	✓
	E_{stu}	✓	✓	✓

Here we use the fact $(i_1i_2i_3i_4)$, $(i_2i_1i_3i_4)$, $(i_1i_2i_4i_3)$ and $(i_2i_1i_4i_3)$ are the same causal ordering (the little group is $\mathbb{Z}_2 \times \mathbb{Z}_2$).

C.2.11 Type 11

The type 11 causal ordering is given by

$$\begin{aligned} a &\longrightarrow b \\ c &\longrightarrow d \end{aligned} \tag{C.2.43}$$

We choose a particular configuration for (1234):

$$x_1 = 0, \quad x_2 = (i, 0, \dots, 0), \quad x_3 = (0, 2, \dots, 0), \quad x_4 = (i, 2, \dots, 0), \tag{C.2.44}$$

and get $z = \bar{z} = \frac{1}{4}$, which is in the range corresponding to subclass E_{st} . By lemma C.2.1, the whole (1234) is in subclass E_{st} . The results of OPE convergence properties in the three channels are shown in table C.13.

Table C.13: OPE convergence properties of type 11 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234), (2143)	E_{st}	✓	✗	✗
(1243), (2134)	E_{su}	✓	✗	✓
(1324), (3142)	E_{tu}	✗	✗	✓
(1342), (3124)	E_{su}	✓	✗	✓
(1423), (3241)	E_{tu}	✗	✓	✗
(1432), (2341)	E_{st}	✓	✓	✗

Here we use the fact that $(i_1i_2i_3i_4)$ and $(i_3i_4i_1i_2)$ are the same causal ordering (the little group is \mathbb{Z}_2).

C.2.12 Type 12

The type 12 causal ordering is given by

$$a \quad b \quad c \quad d. \tag{C.2.45}$$

One can show that this causal type belongs to class E. In fact this is the “Euclidean” case (that’s why this class is called class E), and there is no need to check numerically for this type. Suppose we have a totally space-like separated configuration (x_1, x_2, x_3, x_4) , where $x_k = (it_k, \mathbf{x}_k)$. We can reach this configuration via the path

$$x_k(s) = \begin{cases} ((1 - 2s)\epsilon_k, 2s\mathbf{x}_k) & s \in [0, 1/2] \\ ((2s - 1)it_k, \mathbf{x}_k) & s \in [1/2, 1] \end{cases} \tag{C.2.46}$$

Along the path all the x_i, x_j pairs are space-like separated. As a result, the totally space-like separated configurations always have $N_t = N_u = 0$ (as long as N_t, N_u are well-defined). On the other hand, there is no doubt that all subclasses of class E can be realized in $d \geq 3$.⁵ We summarize the OPE convergence properties of this type in table C.14.

⁵One can realize one of the three subclasses E_{st}, E_{su}, E_{tu} in 2d, and then realize the other two subclasses by S_4 -action. For subclass E_{stu} , one can put four points in the hyperplane with $t = 0$.

Table C.14: OPE convergence properties of type 12 causal orderings

causal ordering	class/subclass	s-channel	t-channel	u-channel
(1234)	E_{st}	✓	✓	✗
	E_{su}	✓	✗	✓
	E_{tu}	✗	✓	✓
	E_{stu}	✓	✓	✓

Here we use the fact that there is only 1 causal ordering in this type (the little group is S_4).

C.3 Wightman functions: a brief review

In this section we will review some classical results about regular points (points where Wightman distributions are genuine functions) in a general QFT [24, 104, 121]. For simplicity let us still consider a scalar theory in the Minkowski space, which is characterized by a collection of Lorentzian correlators:

$$\mathcal{W}_n(x_1, x_2, \dots, x_n) = \langle 0 | \phi(x_1) \phi(x_2) \dots \phi(x_n) | 0 \rangle \quad (\text{C.3.1})$$

where x_i are Lorentzian coordinates.⁶ We will introduce the Wightman axioms for QFTs, and then review the domain of correlation functions which can be derived from Wightman axioms. In the end, we will compare these classical results with our results for CFT four-point functions.

This section is logically independent from the rest of part III. Here we assume Wightman axioms while in the rest we did not. The only connection is to justify the definition of Wick rotation (steps 1 and 2 in chapter 9).

C.3.1 Wightman axioms for Lorentzian correlators

We assume the Wightman axioms for correlators $\{\mathcal{W}_n\}$:

(W1) Temperedness.

\mathcal{W}_n is a tempered distribution (called Wightman distribution). It becomes a complex number after being smeared with rapidly decreasing test functions f_n :

$$\mathcal{W}_n(f_n) = \int f(x_1, \dots, x_n) \mathcal{W}_n(x_1, \dots, x_n) dx_1 \dots dx_n \quad (\text{C.3.2})$$

The Fourier transform $\widehat{\mathcal{W}}_n$ of \mathcal{W}_n is well defined since the space of rapidly decreasing test functions (Schwartz space) is closed under Fourier transform [172]. One has the definition $\widehat{\mathcal{W}}_n(f_n) := \mathcal{W}_n(\widehat{f}_n)$.

(W2) Poincaré invariance.

The correlators transform invariantly under action of the Poincaré group:

$$\mathcal{W}_n(g \cdot x_1, \dots, g \cdot x_n) = \mathcal{W}_n(x_1, \dots, x_n) \quad (\text{C.3.3})$$

for all $n \geq 0$ and g in the Poincaré group.

(W3) Unitarity.

The vector space generated by the states of the form

$$\Psi(\underline{f}) = \sum_{n \geq 0} \int f_n(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n) | 0 \rangle dx_1 \dots dx_n \quad (\text{C.3.4})$$

⁶In the rest of part III, we the Lorentzian points were denoted by (it_k, \mathbf{x}_k) . Only in this section we use the notation $x_k = (t_k, \mathbf{x}_k)$.

has a non-negative inner product. Here \underline{f} is an arbitrary finite sequence of complex valued Schwartz functions: $\underline{f} = (f_0, f_1, f_2, \dots)$ and f_n denotes the Schwartz function with n Lorentzian points as variables. If we assume that $\phi(x)$ are Hermitian operators, i.e. $[\phi(x)]^\dagger = \phi(x)$, then the unitarity condition is written as

$$\sum_{n,m} \int \overline{f_n(x_1, \dots, x_n)} f_m(y_1, \dots, y_m) \mathcal{W}_{n+m}(x_n, \dots, x_1, y_1, \dots, y_m) dx dy \geq 0 \quad (\text{C.3.5})$$

(W4) Spectral condition.

The open forward light-cone V_+ is defined by the collection of vectors $x \in \mathbb{R}^d$ such that

$$x^0 > \sqrt{\sum_{\mu \geq 1} (x^\mu)^2}. \quad (\text{C.3.6})$$

In a general QFT we have self-adjoint momentum operators P^μ . The spectral condition says that the spectrum of $P = (P^0, P^1, \dots, P^{d-1})$ is inside the closed forward light-cone \overline{V}_+ , and the normalized eigenvector of $P = 0$ is unique (up to a phase factor), denoted by $|0\rangle$.

We define the reduced correlators W_{n-1} by

$$W_n(x_2 - x_1, \dots, x_{n+1} - x_n) = \mathcal{W}_{n+1}(x_1, \dots, x_{n+1}) \quad (\text{C.3.7})$$

Since \mathcal{W}_{n+1} is a translational invariant tempered distribution, W_n is well defined and is also a tempered distribution. The spectral condition implies that the Fourier transforms \widehat{W}_n of W_n is supported in the forward light-cone. That is to say, $\widehat{W}_n(p_1, \dots, p_n) \neq 0$ only if all the momentum variables p_k are inside \overline{V}_+ .

(W5) Microscopic causality.

$\mathcal{W}_n(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = \mathcal{W}_n(x_1, \dots, x_{k+1}, x_k, \dots, x_n)$ if x_k and x_{k+1} are space-like separated.

C.3.2 Wightman functions and their domains

C.3.2.1 Forward tube

Let us consider the “reduced correlator” W_n defined in eq. (C.3.7). W_n has Fourier transform

$$W_n(x_1, \dots, x_n) = \int \frac{dp_1}{(2\pi)^d} \dots \frac{dp_n}{(2\pi)^d} \widehat{W}_n(p_1, \dots, p_n) e^{-i(p_1 \cdot x_1 + \dots + p_n \cdot x_n)}, \quad (\text{C.3.8})$$

where \widehat{W}_n is also a tempered distribution, and the Lorentzian inner product is defined by $p \cdot x = -p^0 x^0 + p^1 x^1 + \dots + p^{d-1} x^{d-1}$. In general, W_n is not a function if x_k are real. However, if we replace x_k with complex coordinates $x_k \rightarrow z_k = x_k + iy_k$, because of the spectral condition (W4), $W_n(z_1, \dots, z_n)$ is indeed a function if the imaginary parts of z_k belong to V_+ . The argument is as follows. Suppose $y_k \in V_+$ for all $k = 1, 2, \dots, n$, then there exists a Schwartz function $f(p_1, \dots, p_n)$ in the momentum space such that $f(p_1, \dots, p_n) = e^{-i(p_1 \cdot z_1 + \dots + p_n \cdot z_n)}$ when all the momentum variables p_k are inside the closure of forward light-cone.⁷ Since \widehat{W}_n is supported in \overline{V}_+ , $\widehat{W}_n(f)$ is exactly in the form of (C.3.8) with x_k replaced by z_k . So $W_n(z_1, \dots, z_n)$ is a well-defined complex number when $Im(z_k) \in V_+$ for all k .

Furthermore, since $-i(p_k)_\mu f(p_1, \dots, p_n)$ is also a Schwartz test function, we have

$$\begin{aligned} \frac{\partial}{\partial z_k^\mu} W_n(z_1, \dots, z_n) &= \widehat{W}_n[-i(p_k)_\mu f], \\ k &= 1, \dots, n \text{ and } \mu = 0, \dots, d-1. \end{aligned} \quad (\text{C.3.9})$$

⁷The crucial point is that if all y_k are inside the forward light-cone, then $f(p_1, \dots, p_n)$ decays exponentially fast when some p_k goes to infinity inside the forward light-cone.

As a result $W_n(z_1, \dots, z_n)$ is an analytic function inside the “forward tube”, denoted as I_n

$$I_n = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^{nd} \mid \text{Im}(z_k) \in V_+, k = 1, 2, \dots, n \right\}, \quad (\text{C.3.10})$$

The distribution $W_n(x_1, \dots, x_n)$ is the boundary value of the analytic function $W_n(z_1, \dots, z_n)$ on the forward tube I_n :

$$W_n(x_1, \dots, x_n) = \lim_{\substack{y \rightarrow 0 \\ y \in V_+^n}} W_n(x_1 + iy_1, \dots, x_n + iy_n). \quad (\text{C.3.11})$$

C.3.2.2 Bargmann-Hall-Wightman theorem, extended tube

Now let us use the Lorentz invariance (W2) to analytically continue W_n to a larger domain. By (W2), W_n is invariant under the action of real Lorentz group $SO^+(1, d-1)$.⁸

$$W_n(g \cdot x_1, \dots, g \cdot x_n) = W_n(x_1, \dots, x_n), \quad \forall g \in SO^+(1, d-1). \quad (\text{C.3.12})$$

The Lorentz transformations preserve the inner product $p_k \cdot x_k$ and the measure dx_k , so the Fourier transform \widehat{W}_n is also Lorentz invariant

$$W_n(g \cdot p_1, \dots, g \cdot p_n) = W_n(p_1, \dots, p_n), \quad \forall g \in SO^+(1, d-1). \quad (\text{C.3.13})$$

Since $W_n(z_1, \dots, z_n)$ is defined by the Fourier transform (C.3.8) (replace x_k with z_k), we have

$$W_n(g \cdot z_1, \dots, g \cdot z_n) = W_n(z_1, \dots, z_n), \quad \forall g \in SO^+(1, d-1). \quad (\text{C.3.14})$$

Here we remark that the real Lorentz group actions preserve the forward tube I_n .

An important observation is that (C.3.14) remains true if we replace the real Lorentz group $SO^+(1, d-1)$ by the proper complex Lorentz group $L_+(\mathbb{C})$.⁹ Given an arbitrary $g \in L_+(\mathbb{C})$, we define

$$W_n^g(z_1, \dots, z_n) := W_n(g^{-1} \cdot z_1, \dots, g^{-1} \cdot z_n) \quad (\text{C.3.15})$$

for $(z_1, \dots, z_n) \in gI_n$. The Bargmann-Hall-Wightman theorem [173] tells us that if we choose different complex Lorentz group elements g_1, g_2 , the functions $W_n^{g_1}$ and $W_n^{g_2}$ coincide in the domain $g_1I_n \cap g_2I_n$. So $W_n(z_1, \dots, z_n)$ has analytic continuation to the “extended forward tube”, denoted by \tilde{I}_n :

$$\tilde{I}_n := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^{nd} \mid (g \cdot z_1, \dots, g \cdot z_n) \in I_n \text{ for some } g \in L_+(\mathbb{C}) \right\}. \quad (\text{C.3.16})$$

Here we only give the idea of the proof. It suffices to show that for any $g \in L_+(\mathbb{C})$, the function W_n^g coincides with W_n in the domain $gI_n \cap I_n$. Since I_n and gI_n are two convex sets, their intersection $gI_n \cap I_n$ is also convex, thus connected. So it suffices to show that W_n^g coincides with W_n in the neighbourhood of one point. This is obvious for g near the identity element, but the proof for an arbitrary g is based on the fact that the set $\{g \in L_+(\mathbb{C}) \mid gI_n \cap I_n \neq \emptyset\}$ is connected, which follows from the group structure of the complex Lorentz group $L_+(\mathbb{C})$ (for more details, see [102]).

C.3.2.3 Jost points

While I_n does not contain Lorentzian points (i.e. points with $\text{Im}(z_k) = 0$ for all k), \tilde{I}_n contains a region of Lorentzian points. These points are called Jost points [121], and they are defined by the configurations (x_1, \dots, x_n) such that the following cone

$$\left\{ \lambda_1 x_1 + \dots + \lambda_n x_n \mid \lambda_k \geq 0 \text{ for all } k, \sum_{k=1}^n \lambda_k > 0 \right\} \quad (\text{C.3.17})$$

contains only space-like points (see [102], the theorem on page 81 and the corollary on page 82).

⁸By $SO^+(1, d-1)$ we mean the connected component of the identity element in $O(1, d-1)$.

⁹Let d be the spacetime dimension. The complex Lorentz group $L(\mathbb{C})$ is defined by the set of all $d \times d$ complex matrices M such that $M^t \eta M = \eta$. Here $\eta = \text{diag}(-1, 1, \dots, 1)$ is the matrix of Lorentzian inner product, and M^t is the transpose of M . $L_+(\mathbb{C})$ is the subgroup of $L(\mathbb{C})$ with constraint $\det M = 1$. $L_+(\mathbb{C})$ is connected, unlike the real case where we need to introduce the constraints “proper”, “orthochronous” for connectedness.

C.3.2.4 Microscopic causality, envelope of holomorphy

Now let us go back to $\mathcal{W}_n(x_1, \dots, x_n)$ via (C.3.7). We define \mathcal{J}_n as the set of (x_1, \dots, x_n) such that $(x_2 - x_1, \dots, x_n - x_{n-1})$ are Jost points. The configurations in \mathcal{J}_n have totally space-like separations. To see this we rewrite $x_i - x_j$ ($i > j$) as

$$x_i - x_j = (x_i - x_{i-1}) + (x_{i-1} - x_{i-2}) + \dots + (x_{j+1} - x_j), \quad (\text{C.3.18})$$

which is in the form of (C.3.17). By the definition of Jost points, we have $(x_i - x_j)^2 > 0$ for $i \neq j$. It is obvious that \mathcal{J}_2 contains all totally space-like configurations. For $n \geq 3$, \mathcal{J}_n does not contain all totally space-like configurations.

Since Jost points are the configurations with totally space-like separations, by the microscopic causality condition (W5), $\mathcal{W}_n(x_1, \dots, x_n)$ is also regular at (x_1, \dots, x_n) if there exists a permutation $\sigma \in S_n$ such that

$$(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) \in \mathcal{J}_n. \quad (\text{C.3.19})$$

Then the equation $\mathcal{W}_n(x_1, \dots, x_n) = \mathcal{W}_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ in \mathcal{J}_n can be analytically continued to

$$\mathcal{W}_n(z_1, \dots, z_n) = \mathcal{W}_n(z_{\sigma(1)}, \dots, z_{\sigma(n)}), \quad (z_2 - z_1, \dots, z_n - z_{n-1}) \in \tilde{I}_{n-1}. \quad (\text{C.3.20})$$

Therefore, \mathcal{W}_n has analytic continuation to the following domain of complex coordinates (z_1, \dots, z_n) :

$$\mathcal{U}_n = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^{nd} \mid \exists \sigma \in S_n \text{ s.t. } (z_{\sigma(2)} - z_{\sigma(1)}, \dots, z_{\sigma(n)} - z_{\sigma(n-1)}) \in \tilde{I}_{n-1} \right\}. \quad (\text{C.3.21})$$

\mathcal{U}_n have the following properties:¹⁰

1. In 2d, \mathcal{U}_n contains all totally space-like configurations.
2. \mathcal{U}_3 contains all totally space-like configurations.
3. In $d \geq 3$ and $n \geq 4$, \mathcal{U}_n does not contain all totally space-like configurations [102].

To show the first property, we use an analogous version of the complex coordinates (11.3.14):

$$w_k = \mathbf{x}_k + t_k, \quad \bar{w}_k = \mathbf{x}_k - t_k. \quad (\text{C.3.22})$$

Given an arbitrary totally space-like configuration, we have

$$(w_j - w_k)(\bar{w}_j - \bar{w}_k) > 0, \quad (j \neq k) \quad (\text{C.3.23})$$

which implies $w_j > w_k, \bar{w}_j > \bar{w}_k$ or $w_j < w_k, \bar{w}_j < \bar{w}_k$. So we can find a permutation σ such that

$$w_{\sigma(k)} < w_{\sigma(k+1)}, \quad \bar{w}_{\sigma(k)} < \bar{w}_{\sigma(k+1)}, \quad k = 1, 2, \dots, n-1. \quad (\text{C.3.24})$$

We see from (C.3.24) that $w_{\sigma(k+1)} - w_{\sigma(k)}, \bar{w}_{\sigma(k+1)} - \bar{w}_{\sigma(k)}$ are positive, so the cone (C.3.17) generated from these vectors only contain points with positive components. Thus the configuration $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ is in \mathcal{J}_n , or equivalently, (x_1, \dots, x_n) is in \mathcal{U}_n .

To show the second property, we consider the totally space-like three-point configurations in the following form:

$$x_1 = 0, \quad x_2 = (0, 1, 0), \quad x_3 = (a, b, c). \quad (\text{C.3.25})$$

To check that all totally space-like configurations are in \mathcal{U}_3 , it suffices to check the totally space-like configurations in the form of (C.3.25) because \mathcal{U}_n is Poincaré invariant and scale invariant, and any totally space-like configuration can be mapped to a configuration in the form of (C.3.25) by Poincaré transformations and dilatation. If $(x_1, x_2, x_3) \in \mathcal{J}_3$

¹⁰We were unable to track properties 1,2 in prior literatures. Readers are welcome to provide us with references.

then we are done. Suppose $(x_1, x_2, x_3) \notin \mathcal{J}_3$, which means that there exists a positive λ such that $\lambda(x_2 - x_1) + (x_3 - x_2)$ is a null vector:

$$a^2 = (b - 1 + \lambda)^2 + c^2. \quad (\text{C.3.26})$$

The above equation implies $a^2 \geq c^2$, so there exists a Lorentz transformation which maps the configuration (C.3.25) to

$$x'_1 = 0, \quad x'_2 = (0, 1, 0), \quad x'_3 = (a', b, 0). \quad (\text{C.3.27})$$

We see that the problem is reduced to the 2d case. According second property, the configuration (C.3.27) is in \mathcal{U}_3 . So we conclude that all totally space-like configurations are in \mathcal{U}_3 .

To show the third property, it suffices to give a counterexample for $d = 3$ and $n = 4$:

$$\begin{aligned} x_1 &= (1 - \epsilon, 1, 1), & x_2 &= (1 - \epsilon, -1, -1), \\ x_3 &= (\epsilon - 1, 1, -1), & x_4 &= (\epsilon - 1, -1, 1). \end{aligned} \quad (\text{C.3.28})$$

where $\epsilon > 0$ is small. (C.3.28) is a totally space-like configuration but it does not belong to \mathcal{U}_4 (see [102], p. 89).

\mathcal{W}_n has analytic continuation from \mathcal{U}_n to its envelope of holomorphy $H(\mathcal{U}_n)$ [174], which is defined by the following property:

- Any holomorphic function on \mathcal{U}_n has analytic continuation to $H(\mathcal{U}_n)$.

A theorem proved by Ruelle [29] shows that $H(\mathcal{U}_n)$ contains all totally space-like configurations.

We conclude that Wightman distributions $\mathcal{W}_n(x_1, \dots, x_n)$ are analytic functions at all totally space-like configurations.

C.3.3 Comparison with CFT

C.3.3.1 Justifying the definition of Wick rotation

In section 11.3, we showed that the CFT four-point function is analytic in the forward tube \mathcal{T}_4 .¹¹ Consider the points $(x_2 - x_1, x_3 - x_2, x_4 - x_3)$ where $(x_1, \dots, x_4) \in \mathcal{T}_4$. The notation we use in the rest of part III is $x_k = (\epsilon_k + it_k, \mathbf{x}_k)$, so we have

$$x_{k+1} - x_k = (\epsilon_{k+1} - \epsilon_k + it_{k+1} - it_k, \mathbf{x}_{k+1} - \mathbf{x}_k + i\mathbf{y}_{k+1} - i\mathbf{y}_k). \quad (\text{C.3.29})$$

By translating (C.3.29) to the notation in this section (see footnote 6), we have

$$x_{k+1} - x_k = (t_{k+1} - t_k, \mathbf{x}_{k+1} - \mathbf{x}_k) + i(\epsilon_k - \epsilon_{k+1}, 0). \quad (\text{C.3.30})$$

We see from (C.3.30) that the points $(x_2 - x_1, x_3 - x_2, x_4 - x_3)$ are in the forward tube I_3 if $(x_1, x_2, x_3, x_4) \in \mathcal{D}$ (because $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$). Recalling that the Wightman distributions $\mathcal{W}_n(x_1, \dots, x_n)$ can be obtained by taking the limit of the analytic functions $\mathcal{W}_n(z_1, \dots, z_n)$ from the domain $\{(z_2 - z_1, \dots, z_n - z_{n-1}) \in I_{n-1}\}$, we see that our definition of Wick rotation (17.1.1) is consistent with Wightman QFT.

C.3.3.2 Osterwalder-Schrader theorem

In fact we use the same analytic continuation path as in the Osterwalder-Schrader (OS) theorem [1]. The OS theorem shows that under certain conditions, a Euclidean QFT can be Wick rotated to a Wightman QFT.

¹¹Here we are abusing terminology “forward tube”. The forward tube $(x_1, x_2, x_3, x_4) \in \mathcal{T}_4$ corresponds to the forward tube $(x_2 - x_1, x_3 - x_2, x_4 - x_3) \in I_3$.

In part III we focus on the domain of analyticity of the four-point function, and we do not explore the distributional properties of it. To show that the limit (17.1.1) of the CFT four-point functions defines a Wightman four-point distribution, one needs to deal with the four-point function not only at regular points (where $|\rho|, |\bar{\rho}| < 1$ in s- or t- or u-channel), but also at all the other Lorentzian configurations where ρ and/or $\bar{\rho}$ is 1 in absolute value (this needs a lot of extra work). Readers can go to [40], where we show that Wick rotating a Euclidean CFT four-point function indeed results in a Wightman four-point distribution.

Let us contrast our construction and the OS construction. Our construction extends G_4 in a CFT to domain \mathcal{D} using only information about the four-point function itself. The OS construction can extend G_4 (in fact any G_n) to domain \mathcal{D} in a general QFT. But the price to pay is that analytic continuation involves infinitely many steps and needs information about higher-point functions [2, 31].

C.3.3.3 Domain of analyticity: Wightman axioms + conformal invariance

Let us summarize how the domains of Wightman functions are derived in Wightman QFT. We first use the temperedness property (W1), translational invariance (W2) and the spectral condition (W4) to show that the reduced Wightman distribution W_n is an analytic function in the forward tube I_n . Then we use the Lorentz invariance (W2) to show that W_n has analytic continuation to the “extended tube” \tilde{I}_n , which includes the set of Jost points. Finally we use the microscopic causality condition (W5) to show that \mathcal{W}_{n+1} has analytic continuation to all configurations with totally space-like separations.

The unitarity condition (W3) is not involved in the above argument, only the conditions (W1), (W2), (W4) and (W5) are used.

In the rest of thesis we explored the domain of CFT four-point functions by assuming unitarity, Euclidean conformal invariance and OPE (not assuming Wightman axioms). Here we would like to discuss a related but different situation:

- What is the domain of the four-point function if we only assume Wightman axioms and conformal invariance (not assuming OPE)?

We want to emphasize that global conformal invariance does not hold for general CFT in $\mathbb{R}^{d-1,1}$ because Lorentzian special conformal transformations may violate causal orderings. The precise meaning of conformal invariance here is the Euclidean global conformal invariance: we Wick rotate Wightman functions to the Euclidean signature, then the corresponding Euclidean correlation functions are invariant under Euclidean global conformal transformations. This assumption is called *weak conformal invariance* [25].

It is obvious that the Wightman function \mathcal{W}_4 is analytic in \mathcal{U}_4 (as discussed in section C.3.2). In section C.3.2, a crucial step is to extend the real Poincaré invariance to complex Poincaré invariance. Then the reduced Wightman function W_3 has analytic continuation from the forward tube I_3 to the extended forward tube \tilde{I}_3 . Now that we assume weak conformal invariance, given any Euclidean conformal transformation g and Euclidean configuration $C = (x_1, x_2, x_3, x_4)$, we have

$$\begin{aligned} \mathcal{W}_4(C) &= \Omega(x_1)^\Delta \Omega(x_2)^\Delta \Omega(x_3)^\Delta \Omega(x_4)^\Delta \mathcal{W}_4(g \cdot C), \\ g \cdot C &= (g \cdot x_1, g \cdot x_2, g \cdot x_3, g \cdot x_4), \end{aligned} \tag{C.3.31}$$

where $\Omega(x)$ is the scaling factor of the conformal transformation. It is not hard to show that given any configuration C (which can be non-Euclidean) in the domain of \mathcal{W}_4 , eq. (C.3.31) holds for g in a neighbourhood of the identity element in the Euclidean conformal group (this neighbourhood depends on configuration). Then we can show that for a fixed C , eq. (C.3.31) holds not only in a neighbourhood of identity element in the Euclidean conformal group, but also in a neighbourhood in the complex conformal group.¹² Therefore, by using (C.3.31) with g in the complex conformal group, we can extend \mathcal{W}_4 to a bigger domain than \mathcal{U}_4 (recall definition (C.3.21)).¹³ We say that configurations $C = (x_1, x_2, x_3, x_4), C' = (x'_1, x'_2, x'_3, x'_4)$ are conformally equivalent if there exists a path $g(s)$ in the complex conformal group such that

¹²The complex conformal group is generated by translations $x \rightarrow x + a$, rotations $x \rightarrow \exp[\alpha^{\mu\nu} M_{\mu\nu}] x$, dilatations $x \rightarrow e^\tau x$, special conformal translations $x \rightarrow x'^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2b \cdot x + b^2 x^2}$ with complex parameters.

¹³This can be called conformal extension of Bargmann-Hall-Wightman theory. We were unable to find this idea in prior literature.

- $g(0) = \text{id}$, and $g(1) \cdot C_1 = C_2$.
- The scaling factors $\Omega(x_1), \Omega(x_2), \Omega(x_3), \Omega(x_4)$ along $g(s)$ do not go to 0 or ∞ .

We define

$$\mathcal{U}_4^c = \{C = (x_1, x_2, x_3, x_4) \in \mathbb{C}^{4d} \mid C \text{ is conformally equivalent to some } C' \in \mathcal{U}_4\}, \quad (\text{C.3.32})$$

where the superscript “c” in \mathcal{U}_4^c means “conformal”. Naively, one may expect that \mathcal{W}_4 has analytic continuation to \mathcal{U}_4^c by (C.3.31).¹⁴ However, for each conformally equivalent C, C' pair in \mathcal{U}_4^c , the path $g(s)$ in the complex conformal group is not unique, which means choosing different $g(s)$ may give different analytic continuation. In other words, \mathcal{W}_4 may not be a single-valued function on \mathcal{U}_4^c .

There is one very simple partial case which is guaranteed not to lead to multi-valuedness. It is the case when the whole curve $C(s) = g(s) \cdot C = (x_1(s), x_2(s), x_3(s), x_4(s))$ has point differences $(x_2(s) - x_1(s), x_3(s) - x_2(s), x_4(s) - x_3(s))$ in the forward tube I_3 , except for the end point $g(1) \cdot C$.

In this thesis we do not fully explore the Lorentzian domain of \mathcal{W}_4 by using the above method. We left it for future work. Here we only give a simple example which shows that by assuming weak conformal invariance in Wightman QFT, the domain of the Wightman function contains more Lorentzian configurations than totally space-like configurations.

We would like to show that the following Lorentzian configurations are in the domain of (conformally invariant) \mathcal{W}_4 :

$$x_k = (t_k, \mathbf{x}_k), \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 4. \quad (\text{C.3.33})$$

We act with complex dilatation on (C.3.33):

$$x'_k = e^{i\alpha} x_k. \quad (\text{C.3.34})$$

Then the point differences are given by $x'_{jk} = x_{jk} \cos \alpha + ix_{jk} \sin \alpha$. By the causal ordering in (C.3.33), we have $(x'_2 - x'_1, x'_3 - x'_2, x'_4 - x'_3) \in I_3$ when $0 < \alpha \leq 1$. On the other hand, the scaling factors of complex dilatations (C.3.34) are constants: $\Omega(x) = e^{i\alpha}$, which are finite and non-zero. Thus, we can use the above-mentioned simple partial case. We conclude that the Lorentzian configurations in the form of (C.3.33) are in the domain of (conformally invariant) \mathcal{W}_4 .

C.3.3.4 Domain of analyticity: unitarity + conformal invariance + OPE

Now back to our CFT construction in this thesis. We assumed unitarity, conformal invariance and OPE, but did not assume Wightman axioms.

With the help of OPE, we are able to control the upper bound of the CFT four-point function more efficiently [43]. It seems to be rather difficult to apply the unitarity condition (C.3.5) in a general Wightman QFT because it is a non-linear constraint. While, in the expansion (11.3.6), we are able to use the unitarity condition for CFT four-point functions.

The domain of the Lorentzian CFT four-point function G_4 contains all configurations where there exists at least one convergent OPE channel. The results in appendix C.2 show that the domain of G_4 contains much richer set of causal orderings than just the totally space-like causal ordering obtainable from Wightman axioms alone.

One interesting point is that if we act with conformal transformations on the configurations which have point differences in the forward tube I_3 (let us call this set \mathcal{I}_4), we can get at most Lorentzian configurations whose cross-ratios can be realized by configurations in \mathcal{I}_4 because conformal transformations do not change cross-ratios.¹⁵ However, if we additionally assume OPE, then it is not necessary that the cross-ratios of the Lorentzian configurations can be realized by configurations in \mathcal{I}_4 (the only requirement is that $|\rho|, |\bar{\rho}| < 1$ in the corresponding OPE channel). It would be interesting to figure out:

¹⁴As a next step, one could consider the Lorentzian configurations in envelope of holomorphy $H(\mathcal{U}_4^c)$.

¹⁵The similar idea has been used to look for the domain of analyticity of the Wightman functions in a general QFT. E.g. G. Källén explored the domain of the Wightman four-point function by studying configurations (x_1, x_2, x_3, x_4) such that the Poincaré invariants $x_{ij} \cdot x_{kl}$ can be realized by configurations in \mathcal{I}_4 [175].

- can we get more Lorentzian configurations by assuming unitarity + conformal invariance + OPE than by assuming Wightman axioms + conformal invariance?

We left it for future work.

Appendix D

Appendices of Part IV

D.1 Some properties of 2d conformal transformations

In this appendix we consider conformal transformations which preserves the 2d subspace $\mathbb{R}^2 \subset \mathbb{R}^d$.

The 2d Euclidean conformal group is $\text{SO}(1, 3) \simeq \text{PSL}(2; \mathbb{C})$. To construct the conformal transformation we will use complex coordinates (z_k, \bar{z}_k) :

$$z_k = x_k^0 + ix_k^1, \quad \bar{z}_k = x_k^0 - ix_k^1. \quad (\text{D.1.1})$$

The 2d conformal transformations are given by

$$z'_k = f(z_k), \quad \bar{z}'_k = \bar{f}(\bar{z}_k), \quad (\text{D.1.2})$$

where f and \bar{f} are Möbius transformations. In the Euclidean region, f and \bar{f} are complex conjugate to each other, which means

$$[f(z)]^* = \bar{f}(z^*). \quad (\text{D.1.3})$$

When we analytically continue the correlation functions to complex x_k 's, the conformal transformations, which map (x_1, x_2, x_3, x_4) to ρ -configurations (22.1.1), belong to the complex conformal group $\text{SO}(1, 3)_{\mathbb{C}}$, where f and \bar{f} are independent Möbius transformations.

By (D.1.1) and (D.1.2) we have $\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \Omega(x) R^{\mu}_{\nu}(x)$, where

$$\begin{aligned} \Omega(x_k) &= \sqrt{f'(z_k) \bar{f}'(\bar{z}_k)}, \\ R(x_k) &= \begin{pmatrix} \cos \theta_k & -\sin \theta_k \\ \sin \theta_k & \cos \theta_k \end{pmatrix}, \quad e^{i\theta_k} = \sqrt{\frac{f'(z_k)}{\bar{f}'(\bar{z}_k)}}. \end{aligned} \quad (\text{D.1.4})$$

D.1.1 Mapping (z_1, z_2, z_3, z_4) to $(\rho, -\rho, -1, 1)$

Let us first consider the Möbius transformation which maps (z_1, z_2, z_3, z_4) to $(\rho, -\rho, -1, 1)$. Such an f can be constructed via composing two intermediate maps:

$$f = f_2 \circ f_1. \quad (\text{D.1.5})$$

f_1 maps (z_1, z_2, z_3, z_4) to $(0, z, 1, \infty)$, where $z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$:

$$f_1(w) = \frac{z_3 - z_4}{z_3 - z_1} \frac{w - z_1}{w - z_4}. \quad (\text{D.1.6})$$

f_2 maps $(0, z, 1, \infty)$ to $(\rho, -\rho, -1, 1)$, where $\rho = \frac{z}{(1 + \sqrt{1 - z})^2}$:

$$f_2(w) = \frac{aw + b}{cw + d}, \quad (\text{D.1.7})$$

with

$$\begin{aligned} a, c &= \frac{1}{\sqrt{2}(1-z)^{1/4}}, \\ b &= -\frac{1}{\sqrt{2}}[(1-z)^{-1/4} - (1-z)^{1/4}], \\ d &= -\frac{1}{\sqrt{2}}[(1-z)^{-1/4} + (1-z)^{1/4}]. \end{aligned} \quad (\text{D.1.8})$$

Then the Jacobian of f is given by

$$f'(w) = -\frac{(z_3 - z_4)(z_1 - z_4)}{(z_3 - z_1)} \frac{2\sqrt{1-z}}{\left[\frac{z_3 - z_4}{z_3 - z_1}(w - z_1) - (1 + \sqrt{1-z})(w - z_4) \right]^2}. \quad (\text{D.1.9})$$

In particular

$$\begin{aligned} f'(z_1) &= -\frac{z_3 - z_4}{(z_3 - z_1)(z_1 - z_4)} \frac{2\sqrt{1-z}}{(1 + \sqrt{1-z})^2}, \\ f'(z_2) &= -\frac{z_3 - z_4}{(z_3 - z_1)(z_1 - z_4)(z_2 - z_4)^2} \frac{2}{\sqrt{1-z}(1 + \sqrt{1-z})^2}, \\ f'(z_3) &= -\frac{z_1 - z_4}{(z_3 - z_1)(z_3 - z_4)} \frac{2}{\sqrt{1-z}}, \\ f'(z_4) &= -2 \frac{z_3 - z_1}{(z_3 - z_4)(z_1 - z_4)} \sqrt{1-z}. \end{aligned} \quad (\text{D.1.10})$$

Analogously we can map $(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$ to $(\bar{\rho}, -\bar{\rho}, -1, 1)$ by a conformal transformation \bar{f} and compute the Jacobian. The results are similar:

$$\begin{aligned} \bar{f}'(\bar{z}_1) &= -\frac{\bar{z}_3 - \bar{z}_4}{(\bar{z}_3 - \bar{z}_1)(\bar{z}_1 - \bar{z}_4)} \frac{2\sqrt{1-\bar{z}}}{(1 + \sqrt{1-\bar{z}})^2}, \\ \bar{f}'(\bar{z}_2) &= -\frac{\bar{z}_3 - \bar{z}_4}{(\bar{z}_3 - \bar{z}_1)(\bar{z}_1 - \bar{z}_4)(\bar{z}_2 - \bar{z}_4)^2} \frac{2}{\sqrt{1-\bar{z}}(1 + \sqrt{1-\bar{z}})^2}, \\ \bar{f}'(\bar{z}_3) &= -\frac{\bar{z}_1 - \bar{z}_4}{(\bar{z}_3 - \bar{z}_1)(\bar{z}_3 - \bar{z}_4)} \frac{2}{\sqrt{1-\bar{z}}}, \\ \bar{f}'(\bar{z}_4) &= -2 \frac{\bar{z}_3 - \bar{z}_1}{(\bar{z}_3 - \bar{z}_4)(\bar{z}_1 - \bar{z}_4)} \sqrt{1-\bar{z}}. \end{aligned} \quad (\text{D.1.11})$$

Eqs. (D.1.1), (D.1.4), (D.1.10) and (D.1.11) imply that in 2d, $\Omega(x_k)$ and $R(x_k)$ are analytic functions of the complex coordinates x_i^μ (where $i \in \{1, 2, 3, 4\}$ and $\mu \in \{0, 1\}$) in the region

$$\text{Re}(z_1) > \text{Re}(z_2) > \text{Re}(z_3) > \text{Re}(z_4), \quad \text{Re}(\bar{z}_1) > \text{Re}(\bar{z}_2) > \text{Re}(\bar{z}_3) > \text{Re}(\bar{z}_4). \quad (\text{D.1.12})$$

D.1.2 Extension to higher dimensions

We consider 01-plane which is a 2d subspace in \mathbb{R}^d . Any 2d Möbius transformation $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ can be naturally extended to a higher dimensional conformal transformation $\tilde{f}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ as follows:

- f has a unique decomposition $f = \exp(a \cdot P) \exp(b \cdot K) \exp(\omega^{01} M_{01}) \exp(\lambda D)$ in 2d. Here

$$\begin{aligned} a &= (a^0, a^1), \\ b &= (b^0, b^1). \end{aligned}$$

- Then $\tilde{f} = \exp(a \cdot P) \exp(b \cdot K) \exp(\omega^{01} M_{01}) \exp(\lambda D)$ in higher d.

An important observation is that

Lemma D.1.1. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a 2d Möbius transformation and $\tilde{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be its higher dimensional extension as described above. Then in the 2d subspace, the scaling factors $\Omega_f(x), \tilde{\Omega}_f(x)$ and the rotation matrices $R_f(x), \tilde{R}_f(x)$ induced by f, \tilde{f} satisfy the following relations:*

$$\tilde{\Omega}(x)_f = \Omega_f(x), \quad \tilde{R}_f(x) = \begin{pmatrix} R_f(x) & 0 \\ 0 & \text{Id}_{d-2} \end{pmatrix}, \quad \forall x \in \mathbb{R}^2. \quad (\text{D.1.13})$$

where Id_{d-2} is the $(d-2)$ -by- $-(d-2)$ identity matrix.

Proof. (D.1.13) is trivial for the case when f is generated by translations, rotations and dilatations in \mathbb{R}^2 . So it remains to check special conformal transformations, for which it suffices to check inversion:

$$x'^\mu = \frac{x^\mu}{x^2}. \quad (\text{D.1.14})$$

The Jacobian of (D.1.14) is given by

$$\frac{\partial x'^\mu}{\partial x^\nu} = \frac{\delta^\mu_\nu}{x^2} - \frac{2x^\mu x_\nu}{(x^2)^2} = \frac{1}{x^2} \left(\delta^\mu_\nu - \frac{2x^\mu x_\nu}{x^2} \right), \quad (\text{D.1.15})$$

which implies that $\Omega_{\text{inv}}(x) = \frac{1}{x^2}$ and $(R_{\text{inv}})^\mu_\nu(x) = \delta^\mu_\nu - \frac{2x^\mu x_\nu}{x^2}$. Here R_{inv} belongs to $O(d)$ instead of $\text{SO}(d)$ because inversion does not belong to the connected conformal group $\text{SO}(1, d+1)$.

When x is in the (01)-plane, i.e. $x^\mu = 0$ for $\mu \notin \{0, 1\}$, we see that

- $\Omega_{\text{inv}}(x) = \frac{1}{x^2}$ agrees with the scaling factor in 2d.
- When $\mu, \nu \in \{0, 1\}$, $(R_{\text{inv}})^\mu_\nu(x) = \delta^\mu_\nu - \frac{2x^\mu x_\nu}{x^2}$ which agrees with the rotation matrix in 2d.
- When $\mu \in \{0, 1\}, \nu \notin \{0, 1\}$ or $\mu \notin \{0, 1\}, \nu \in \{0, 1\}$, $(R_{\text{inv}})^\mu_\nu(x) = 0$.
- When $\mu, \nu \in \{0, 1\}$, $(R_{\text{inv}})^\mu_\nu = \delta^\mu_\nu$.

Therefore the inversion map satisfies (D.1.13). □

Remark D.1.1. Lemma D.1.1 can be easily generalized to complex conformal transformations.

D.1.3 CFT four-point function in the two-dimensional subspace

We consider the four-point function $G_{1234}^{a_1 a_2 a_3 a_4}(c) = \langle \mathcal{O}_1^{a_1}(x_1) \mathcal{O}_2^{a_2}(x_2) \mathcal{O}_3^{a_3}(x_3) \mathcal{O}_4^{a_4}(x_4) \rangle$ with $c = (x_1, x_2, x_3, x_4)$ staying in the (01)-plane (i.e. $x_k^\mu = 0$ for $\mu = 2, 3, \dots, d-1$).

By the arguments in the previous two subsections, we can find a d -dimensional complex conformal transformation \tilde{f} , which extends the two-dimensional conformal transformation f acting on the (01)-plane and maps c to the ρ -configuration c_ρ , defined in eq. (22.1.1). Then the relation of $G_{1234}^{a_1 a_2 a_3 a_4}(c)$ to $G_{1234}^{b_1 b_2 b_3 b_4}(c_\rho)$ is given in eq. (22.2.1).

By lemma D.1.1, the rotation matrices induced by \tilde{f} is block diagonal: $R_{\tilde{f}}(x_k) = \text{diag}(R_f(x_k)_{2 \times 2}, \text{Id}_{d-2})$. By the discussions in section D.1.1, the entries of $R_f(x_k)$ are analytic functions of the (01)-plane coordinates. By eqs. (D.1.4), (D.1.10) and (D.1.11), these entries have power-law bounds in the (01)-plane coordinates. The extra entries of the rotation matrices are either 0 or 1.

The analyticity and power-law bound of $\Omega_{\tilde{f}}(x_k)$ and $G_{1234}(c_\rho)$ have been reviewed in section 22.1. Together with the results of the rotation matrices here, we conclude that

Theorem D.1.2. *In the unitary CFT, given a four-point function $G_{1234}^{a_1 a_2 a_3 a_4}(c) = \langle \mathcal{O}_1^{a_1}(x_1) \mathcal{O}_2^{a_2}(x_2) \mathcal{O}_3^{a_3}(x_3) \mathcal{O}_4^{a_4}(x_4) \rangle$ with $c = (x_1, x_2, x_3, x_4)$ staying in the (01)-plane, then*

- (a) $G_{1234}^{a_1 a_2 a_3 a_4}(c)$ (as a function of the (01)-plane coordinates) has analytic continuation to the forward tube \mathcal{T}_4 , with x_k 's in the complex (01)-plane;
- (b) $G_{1234}^{a_1 a_2 a_3 a_4}(c)$ satisfies power-law bound

$$|G_{1234}^{a_1 a_2 a_3 a_4}(c)| \leq C \left(1 + \max_k \frac{1}{\operatorname{Re}(z_k - z_{k+1})}\right)^\alpha \left(1 + \max_k \frac{1}{\operatorname{Re}(\bar{z}_k - \bar{z}_{k+1})}\right)^\alpha \left(1 + \max_k |x_k - x_{k+1}|\right)^\beta, \quad (\text{D.1.16})$$

for c in this region. Here the coordinates (z_k, \bar{z}_k) are the same as in eq. (D.1.1).

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RÉSUMÉ

Les théories des champs conformes (*CFTs*) en signature euclidienne satisfont plusieurs règles bien acceptées, telles que l'invariance conforme et la convergence de l'expansion du produit d'opérateurs (*OPE*) en signature euclidienne. De nos jours, il est courant de supposer l'existence des fonctions de corrélation d'une *CFT* et d'assumer diverses propriétés en signature lorentzienne. Certaines de ces propriétés peuvent représenter des hypothèses supplémentaires, et leur validité reste incertaine dans les *CFT* de physique statistique familières telles que le modèle d'Ising critique en trois dimensions. Dans cette thèse, nous clarifions qu'au niveau des fonctions de corrélation à quatre points, les axiomes *CFT* euclidiens impliquent les axiomes standards de la théorie quantique des champs tels que les axiomes d'Osterwalder-Schrader (en signature euclidienne) et les axiomes de Wightman (en signature lorentzienne).

MOTS CLÉS

théorie des champs conformes, rotation de Wick, axiomes de Wightman

ABSTRACT

Conformal field theories (*CFTs*) in Euclidean signature satisfy well-accepted rules, such as conformal invariance and the convergent Euclidean operator product expansion (*OPE*). Nowadays, it is common to assume that *CFT* correlators exist and have various properties in the Lorentzian signature. Some of these properties may represent extra assumptions, and it is an open question if they hold for familiar statistical-physics *CFTs* such as the critical 3d Ising model. In this thesis, we clarify that at the level of four-point functions, the Euclidean *CFT* axioms imply the standard quantum field theory axioms such as Osterwalder-Schrader axioms (in Euclidean) and Wightman axioms (in Lorentzian).

KEYWORDS

conformal field theory, Wick Rotation, Wightman axioms