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# Having A Part $2 \times 2 = 4$ Times Over : Towards A Mereology Of Slots

Cédric Tarbouriech

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Cédric Tarbouriech. Having A Part  $2 \times 2 = 4$  Times Over : Towards A Mereology Of Slots. Library and information sciences. Université Paul Sabatier - Toulouse III, 2023. English. NNT : 2023TOU30316 . tel-04585081

**HAL Id: tel-04585081**

**<https://theses.hal.science/tel-04585081>**

Submitted on 23 May 2024

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# Doctorat de l'Université de Toulouse

préparé à l'Université Toulouse III - Paul Sabatier

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Avoir une partie  $2 \times 2 = 4$  fois : Vers une Méréologie des  
Slots

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Thèse présentée et soutenue, le 13 décembre 2023 par

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## École doctorale

EDMITT - Ecole Doctorale Mathématiques, Informatique et Télécommunications de Toulouse

## Spécialité

Informatique et Télécommunications

## Unité de recherche

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Having A Part  $2 \times 2 = 4$  Times Over:  
Towards A Mereology Of Slots

Cédric Tarbouriech



# Résumé/Abstract

La méréologie est la discipline qui s'intéresse aux relations entre une partie et son tout et entre parties au sein d'un même tout. Selon la théorie la plus communément utilisée, appelée « méréologie classique extensionnelle », une entité ne peut être partie d'une autre entité qu'une seule fois. Par exemple, votre cœur n'est qu'une seule fois partie de votre corps.

Ce principe a été remis en question par certains travaux antérieurs. En effet, il n'est pas possible de décrire la structure méréologique de certaines entités, telles que les universaux structurés ou les types de mots, dans le cadre de la méréologie classique extensionnelle. Ces entités peuvent avoir plusieurs fois la même partie. Par exemple, l'universel de molécule d'eau ( $H_2O$ ) a comme partie l'universel d'atome d'hydrogène ( $H$ ) deux fois, alors qu'une molécule d'eau particulière a comme parties deux atomes d'hydrogène distincts.

Dans ce travail, nous suivons la piste ouverte par Karen Bennett en 2013. Bennett a ébauché une nouvelle méréologie qui permet de représenter la structure méréologique de ces entités. Dans sa théorie, être une partie d'une entité, c'est remplir un « slot » de cette entité. Ainsi, dans le mot « patate », la lettre « a » est partie du mot deux fois, parce qu'elle occupe deux « slots » de ce mot : le deuxième et le quatrième.

La proposition de Bennett est innovante en cela qu'elle offre un cadre général, qui n'est pas restreint à un type d'entités. Toutefois, la théorie souffre de plusieurs problèmes. D'abord, elle est limitée : de nombreuses notions de méréologie classique n'y ont pas d'équivalent, telles que la somme méréologique ou l'extensionnalité. Ensuite, parce que la théorie, par son axiomatique, provoque des problèmes de comptage. Ainsi, l'universel d'électron n'est partie que sept fois de l'universel de méthane, au lieu des dix fois qui sont attendues.

Nous avons proposé une solution dont le principe est que les slots doivent être dupliqués autant de fois que nécessaire pour obtenir un comptage correct. Cette duplication est opérée grâce à un mécanisme appelé « contextualisation », qui permet de copier les slots en rajoutant un contexte supplémentaire. Ainsi, nous avons établi une théorie permettant de représenter des entités qui peuvent avoir plusieurs fois la même partie tout en évitant les problèmes de comptage.

Nous avons développé une méréologie des slots sur la base de cette théorie, c'est-à-dire une théorie représentant des relations méréologiques entre slots. Ainsi, nous avons pu développer les diverses notions présentes en méréologie classique, telles que la supplémentation, l'extensionnalité, la somme et la fusion méréologiques.

Cette proposition fournit une méréologie très expressive et logiquement bien fondée qui permettra d'explorer, dans de futurs travaux, des questions complexes soulevées dans la littérature scientifique. En effet, certaines entités ne peuvent pas être différenciées par leurs seules structures méréologiques, mais requièrent de représenter des relations additionnelles entre leurs parties. Notre théorie méréologique offre des outils et des pistes permettant d'explorer de telles questions.

Mereology is the discipline concerned with the relationships between a part and its whole and between parts within a whole. According to the most commonly used theory, “classical extensional mereology”, an entity can only be part of another one once. For example, your heart is part once of your body.

Some earlier works have challenged this principle. Indeed, it is impossible to describe the mereological structure of certain entities, such as structural universals and word types, within the framework of classical extensional mereology. These entities may have the same part several times over. For example, the universal of water molecule ( $H_2O$ ) has as part the universal of hydrogen atom ( $H$ ) twice, while a particular water molecule has two distinct hydrogen atoms as parts.

In this work, we follow the track opened by Karen Bennett in 2013. Bennett sketched out a new mereology to represent the mereological structure of these entities. In her theory, to be a part of an entity is to fill a “slot” of that entity. Thus, in the word “potato”, the letter “o” is part of the word twice because it occupies two “slots” of that word: the second and the sixth.

Bennett’s proposal is innovative in offering a general framework that is not restricted to one entity type. However, the theory has several problems. Firstly, it is limited: many notions of classical mereology have no equivalent, such as mereological sum or extensionality. Secondly, the theory’s axiomatics give rise to counting problems. For example, the electron universal is only part of the methane universal seven times instead of the expected ten times.

We have proposed a solution based on the principle that slots must be duplicated as often as necessary to obtain a correct count. This duplication is achieved through a mechanism called “contextualisation”, which allows slots to be copied by adding context. In this way, we have established a theory for representing entities that may have the same part multiple times while avoiding counting problems.

We have developed a mereology of slots based on this theory, which is a theory representing mereological relationships between slots. In this way, we have developed the various notions present in classical mereology, such as supplementation, extensionality, mereological sum and fusion.

This proposal provides a very expressive and logically sound mereology that will enable future work to explore complex issues raised in the scientific literature. Indeed, some entities cannot be differentiated by their mereological structures alone but require the representation of additional relationships between their parts. Our mereological theory offers tools and avenues to explore such questions.

# Remerciements

Avant toute chose, je tiens à remercier Annaïs, Naïmo et Pauline. Ces quatre ans ont été un véritable voyage, durant lequel je me suis perdu. Alors que j'étais au plus bas, et qu'en finir me semblait être la solution la plus adaptée, iels m'ont montré qu'il existait une voie où les choses se passeraient mieux. Je vous dois, en partie, la vie. Merci.

Je souhaite également remercier mes encadrant·es de thèse: Laure, Jean-François et Adrien qui m'ont guidé à travers ces quatre ans. Leurs conseils et support m'ont permis de réaliser un travail dont je suis fier. Ces quatre années ont été enrichissantes tant les sujets et les discussions ont été riches et diverses. Leur encadrement, rigoureux et bienveillant, m'a permis de progresser et d'apporter aux connaissances de l'humanité ma brique personnelle.

Je souhaite également remercier les rapporteurs et autres membres du jury qui liront mon travail. J'espère que cette lecture vous sera agréable, et que vous pourrez glâner, au gré des pages qui viennent, des connaissances utiles et enrichissantes.

D'abord, je remercie mes parents, Sandrine et Laurent, mon frère, Nicolas, et ma famille pour leur soutien inconditionnel.

Je remercie également les enseignant·es et étudiant·es de l'IUT où j'ai donné mes cours pendant ces quatre ans. L'IUT a été pour moi un sanctuaire, où j'ai trouvé du repos et l'énergie de continuer durant ces quatre ans. Ça aura été l'endroit où j'ai commencé mon cursus universitaire et où je l'aurai fini. En particulier, je souhaite remercier Fabienne et Hervé, que j'ai eu comme enseignant·es à mes débuts, puis comme collègues et guides.

Je remercie également les collègues et doctorant·es du labo qui m'ont permis de passer d'agréables moments. En particulier, je remercie Élise, qui m'a offert un nombre incalculable de thés, m'a guidé dans mes premières années de doctorant, m'a aidé à ranger, nettoyer et aménager notre bureau pour en faire un endroit plus accueillant et m'a fait découvrir les stroopwafels.

Je remercie également Damia, avec qui j'ai fait mon cursus universitaire depuis l'IUT jusqu'à la thèse. Elle a été l'une des premières avec qui j'ai discuté pendant des heures de méréologie et d'autres sujets. J'espère que nos discussions dureront encore longtemps.

Je remercie afin mes ami·es, où qu'ils soient. En particulier, je remercie mes ami·es de Toulouse, avec qui je suis depuis le lycée, voire le collègue. Par peur d'en oublier un·e, je ne vous nomme pas, mais je sais que vous vous reconnaitrez. Je remercie aussi les ami·es que je me suis fait durant mes voyages sur Internet, et en particulier en contribuant aux projets Wikimedia. Je remercie en particulier Amélie, Anne et Florian qui m'ont écouté parler de parties de chaise (et parfois d'autres sujets) pendant des heures. Là aussi, il y a trop de gens pour ne pas en oublier, que ça soit des membres de Wikimedia France, des contributeur·ices de Wikipédia, du Wiktionnaire ou d'ailleurs, ou des membres des sans pagEs. Vous m'avez apporté beaucoup et avez contribué à faire de la personne que je suis ce qu'elle est. En particulier, je remercie Antoine, Lucas et Natacha.





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# Chapter 1

## Introduction

Combien de lettres y a-t-il dans le mot « potato » ? Six ? Ou quatre peut-être ? La réponse à donner dépend de ce que l'on compte quand on demande « combien de lettres ? ». Lorsque nous répondons « six », nous comptons le nombre de caractères dans le mot, c'est-à-dire le nombre de motifs d'encre sur le papier : il y a six motifs, donc il y a six lettres. Certains de ces motifs peuvent se ressembler, ils sont quand même distincts. Lorsque nous répondons « quatre », nous comptons le nombre de symboles utilisés parmi un ensemble prédéfini : l'alphabet latin. Parmi les 26 symboles qu'il contient, quatre sont utilisés. Il y a donc plusieurs réponses possibles à la question initialement posée. Toutefois, affirmer que le mot « potato » contient à la fois quatre et six lettres porte a minima à confusion, voire s'avère contradictoire. Comment donc exprimer le fait qu'il existe deux réponses possibles ? Cette distinction est généralement réalisée grâce aux concepts de « type » et de « token ». Les diverses apparitions du mot « potato » dans cette introduction sont des tokens distincts d'un même type. De façon similaire, les apparitions de la lettre « o » dans chaque apparition du mot « potato » sur cette page sont des tokens distincts (les motifs d'encre) d'un même type (cette lettre spécifique dans l'alphabet latin de 26 lettres).

La question semble résolue : un token « potato » du type “POTATO” contient six tokens de quatre types distincts et le type “POTATO” contient les quatre types “A”, “O”, “P”, “T”. Toutefois, décrire le type “POTATO” comme contenant les quatre types, sans en dire plus, ne nous semble pas convaincant : les types “ATOP” et “TOPATO” contiennent également ces quatre types, et uniquement eux. Comment expliquer que “POTATO” soit distinct de “ATOP” et “TOPATO” ? Avec la seule description faite de ces types, ce n'est pas possible. Considérons un token de chacun de ces types et donnons-en une description. Le token « potato » contient six tokens de lettres de quatre types distincts. Le token « atop » contient quatre tokens de quatre types. Enfin, le token « topato » contient six tokens de quatre types. C'est grâce aux multiplicités des tokens de lettres que nous pouvons distinguer les deux premiers tokens. Pour distinguer le premier et le troisième, il faut utiliser l'ordre des tokens de lettres. Carrara and Smid (2022b) utilisent le même raisonnement pour définir trois critères d'identité des types de mots.

Les universaux structurés posent le même problème. Commençons par introduire les universaux. Ces entités sont des entités communes à toutes leurs instances. Tous les chevaux sont similaires dans une certaine mesure ; pour les philosophes réalistes, c'est l'universel CHEVAL qui explique ce fait : tous les individus de chevauxinstancient cet universel. Dans cette thèse, nous ne prendrons pas part au débat philosophique entre nominalistes et réalistes quant à l'existence de ces universaux (voir Armstrong (1978b,a)). Toutefois, s'ils existent, alors cette thèse propose un moyen d'en représenter la structure. Dans les années 1980, des philosophes, tels que David Lewis (1986), David Armstrong (1986), John Bigelow (1986, 1989), Robert Pargetter (1989) et Peter Forrest (1986) ont discuté de l'existence des universaux structurés, c'est-à-dire des universaux construits à

partir d'autres universaux.

L'exemple historiquement utilisé est celui de l'universel MÉTHANE, construit à partir des universaux CARBONE et HYDROGÈNE. Ainsi, selon une conception réaliste, les atomes de carbone sont des instances de l'universel CARBONE, les atomes d'hydrogène sont des instances de l'universel HYDROGÈNE et les molécules de méthane sont des instances de l'universel MÉTHANE. Si les universaux structurés existent, alors, de la même façon qu'une instance de MÉTHANE est faite d'une instance de CARBONE et de quatre instances d'HYDROGÈNE, l'universel de MÉTHANE est fait des universaux de CARBONE et d'HYDROGÈNE. Toutefois, alors qu'une instance de MÉTHANE est faite de quatre instances distinctes d'HYDROGÈNE, il n'existe qu'un seul universel d'HYDROGÈNE. Le même raisonnement que celui développé plus haut sur les tokens de mots et de lettres existe : prenons l'universel de BUTANE, dont les instances ont quatre atomes de carbone et dix atomes d'hydrogène. Ainsi, ce qui distingue une instance de MÉTHANE d'une instance de BUTANE est la multiplicité des atomes qui les composent. Considérons enfin l'universel d'ISOBUTANE, dont les instances ont les mêmes nombres d'atomes que celles du BUTANE. Ce qui distingue une instance de BUTANE d'une instance d'ISOBUTANE est la façon dont les atomes sont liés : les atomes d'une molécule de BUTANE sont disposés en chaîne, alors que ceux d'une molécule d'ISOBUTANE ont une forme de tétraèdre.

Enfin, s'il fallait donner d'autres exemples d'entités qui peuvent posséder plusieurs fois la même partie, on pourrait considérer celui du mur temporel d'Effingham and Robson (2007), dans lequel un mur est construit avec la même brique utilisée cent fois grâce à une succession de voyages temporels, ou encore les *courbes temporelles fermées* de Gilmore (2007) avec lesquelles un atome d'hydrogène pourrait coexister avec une version antérieure de lui-même, voire former une molécule de dihydrogène.

Ces problèmes de représentation sont des problèmes méréologiques : ils concernent la manière dont une entité et ses parties sont liées. La méréologie est la discipline qui s'intéresse aux relations de partie : « aux relations entre une partie et un tout, et aux relations entre des parties d'un même tout » (Varzi, 2019). La version communément acceptée parmi les théories méréologiques est appelée méréologie classique extensionnelle (CEM ; Classical Extensional Mereology). Elle contient diverses relations, comme les relations de partie ou de chevauchement. De plus, elle contient deux familles de principes, dits de décomposition et de composition. Le premier groupe de principes traite de la façon dont une entité peut-être décomposée en parties. Par exemple, l'un de ces principes, l'extensionnalité, affirme que deux choses qui ont les mêmes parties sont identiques. Le second groupe traite des principes qui régissent la composition de choses afin d'obtenir des entités plus complexes. Par exemple, la somme méréologique binaire est la relation qui lie deux parties à la chose composée d'elles et uniquement elles : ainsi, la somme des verres et du cadre forme mes lunettes.

La méréologie classique extensionnelle n'est pas compatible avec les entités telles que les types de mots et de lettres ou les universaux structurés. En effet, plusieurs principes de cette théorie s'opposent au fait que des entités puissent avoir plusieurs fois la même partie. Le principe « *Parts Just Once* » (*Parties Juste une Fois* en français) énoncé par Effingham and Robson (2007) affirme qu'« un objet composite ne peut avoir le même objet comme partie propre plusieurs fois ». Ce principe est en opposition directe avec ce que nous souhaitons développer : une théorie méréologique où un objet composite pourrait avoir la même partie propre plusieurs fois. Toutefois, ce principe ne peut pas être formulé en méréologie classique extensionnelle à cause du terme « plusieurs fois »<sup>1</sup>. Il existe

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<sup>1</sup>Cette théorie est généralement exprimée en logique classique (du premier ordre) et admet une relation de partie binaire  $P$ . Par conséquent,  $P(x, y) \wedge P(x, y)$  (avec  $P$  la relation de partie) se simplifie en logique classique en  $P(x, y)$  et ne permet pas de prendre en compte le nombre d'occurrences. De plus, une modification de la relation  $P$  en relation ternaire pour  $y$  inclure un nombre d'occurrences nous fait sortir du cadre traditionnel de la méréologie.

par contre deux théorèmes de la méréologie classique extensionnelle sur la composition méréologique. Le premier, appelé « *Idempotence* » affirme que « la somme de  $x$  avec lui-même est égale à  $x$  ». Le second, appelé « *Subpotence* » par Cotnoir and Varzi (2021), affirme que « si  $x$  est une partie de  $y$ , alors la somme de  $x$  et de  $y$  est identique à  $y$  »<sup>2</sup>. Par la suite, ces entités, non-idempotentes sous la somme méréologique, seront nommées des « *entités non-idempotentes* ».

Le constat que la méréologie classique extensionnelle n'est pas adaptée à toutes les entités n'est pas nouveau. Toutefois, les réactions face à ce constat diffèrent. Ainsi, Lewis (1986), après avoir introduit l'exemple de l'universel MÉTHANE présenté plus haut, conclut qu'une approche méréologique des universaux structurés n'est pas possible parce que cela nécessiterait qu'une entité puisse avoir la même partie plusieurs fois. Néanmoins, d'autres acceptent l'idée de modifier la méréologie classique extensionnelle afin qu'elle puisse prendre en compte les entités non-idempotentes. Parmi ces théories, on peut citer notamment la conception par occurrences de Wetzel (2009), la méréologie à slots de Bennett (2013), la méréologie abélienne de Cotnoir (2015) ou l'utilisation d'une logique plurale par Carrara and Smid (2022b)<sup>3</sup>.

Ainsi, l'objectif de cette thèse est d'établir une théorie générale qui permet de représenter tout type d'entités non-idempotentes. Cette théorie devra également permettre de correctement compter les multiplicités de chaque partie. Enfin, parce qu'il s'agira d'une théorie méréologique, nous attendons de cette théorie qu'elle possède les notions couramment admises par une théorie méréologique, à savoir les relations basiques de partie, de partie propre et de chevauchement, ainsi que des principes de décomposition tels que la supplémentation ou l'extensionnalité, mais également des principes de composition, tels que la somme ou la fusion.

Dans cette thèse, nous avons sélectionné, comme base de solution à notre problème, la méréologie à slots proposée par Bennett (2013). Bien qu'elle soit une ébauche, comme la décrit Bennett, cette théorie est novatrice en cela qu'elle apporte un cadre général pour représenter les entités non-idempotentes: de fait, elle n'est pas restreinte à certaines classes d'entités. Elle a suscité de nombreux commentaires (Fisher (2013), Cotnoir (2015), Garbacz (2016)) et a inspiré plusieurs théories manipulant et adaptant la notion de slot : Barton et al. (2020b), Sattig (2021), Barton et al. (2022). Nous avons pu nous-même identifier certaines difficultés et en proposer des corrections : Tarbouriech et al. (2021) et Tarbouriech et al. (2024).

La méréologie à slots analyse la relation de partie en utilisant deux primitives. Ces deux primitives, tirées d'une comparaison avec les ontologies des rôles, sont *est un slot de* et *remplit*. Ainsi, pour que l'entité  $a$  soit une partie de l'entité  $b$ , il faut que  $a$  remplisse un slot de  $b$ . Bien que cette proposition apporte de nouvelles possibilités de représentation, elle présente plusieurs problèmes. Fisher (2013), Cotnoir (2015) et Garbacz (2016) ont émis plusieurs critiques et identifié plusieurs problèmes d'ordre philosophique ou axiomatique. Toutefois, nous avons identifié des problèmes plus pratiques : nous avons montré que sous sa forme originale, cette théorie, qui permet de représenter des entités non-idempotentes, ne permet pas de compter correctement le nombre d'occurrences des parties. Cependant, comme nous le verrons, on peut l'adapter et la développer afin qu'elle remplisse cet objectif.

Dans ce travail, nous nous sommes restreints à la méréologie. Nous avons exploré les différentes notions méréologiques qui pouvaient être définies en se basant sur la méréologie

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<sup>2</sup>Lorsqu'il introduit le principe de Subpotence pour la première fois, Cotnoir (2015) le nomme « Idempotence ». Toutefois, dans le livre *Mereology*, écrit avec Varzi, ce principe est appelé « Subpotence ». Étant donné que l'idempotence désigne classiquement la propriété  $x + x = x$ , nous utilisons ici le terme d'« Idempotence » dans son sens traditionnel, et désignons le principe introduit par Cotnoir (2015) par « Subpotence ».

<sup>3</sup>Ces théories sont respectivement présentées dans la Section 2.6.1, le Chapitre 3, la Section 4.1 et la Section 4.2.

à slots. Cependant, nous verrons en discussion que notre proposition théorique pourrait être étendue, dans de futurs travaux, pour intégrer des relations non-méréologiques qui permettraient notamment de distinguer BUTANE et ISOBUTANE. Notre objectif est ici de proposer un socle robuste pour de telles extensions futures. De plus, cet objectif est abordé en utilisant un cadre de représentation de logique du premier ordre, fréquent en ontologie appliquée.

Cette thèse s’articule selon trois parties, découpées en chapitres :

- La Partie I constitue un état de l’art permettant de comprendre les sujets abordés dans les parties suivantes. Le Chapitre 2 de cette partie présente, en se basant sur les travaux récents de Cotnoir and Varzi (2021), les notions de méréologie classique extensionnelle utilisées dans la suite de cette thèse. La dernière partie de ce chapitre présente certaines entités non-idempotentes : universaux structurés et types de mots. La littérature étant vaste sur ces entités, cette section se concentre, après une présentation générale de ces entités, sur leurs aspects méréologiques. En effet, ces entités ne constituent pas le cœur de ce travail et ne sont utilisées que comme sources de questions et d’exemples. Le Chapitre 3 présente la méréologie à slots développée par Bennett (2013), ainsi que les critiques présentes dans la littérature. De plus, nous proposons dans ce chapitre notre propre critique de cette théorie, en présentant les problèmes de comptage. Enfin, le Chapitre 4 de cet état de l’art présente diverses théories qui apportent des compléments intéressants pour traiter la question qui nous intéresse.
- La Partie II constitue notre proposition comme solution à la question qui nous intéresse. Dans le Chapitre 5, adapté de l’article (Tarbouriech et al., 2021), une correction et extension de la théorie de Bennett est présentée. Cette correction et extension théorique, nommée « *Copy-Slot Mechanism* », permet de corriger les problèmes de comptage identifiés dans la théorie de Bennett. Toutefois, cette proposition introduit un nouveau problème de comptage. Afin de remédier à ce problème, nous avons établi les critères permettant d’évaluer notre théorie. Ainsi, le Chapitre 6 développe ces prérequis en y intégrant le choix d’exploiter la théorie de Bennett, ainsi que divers éléments tirés des autres théories présentées dans le Chapitre 4. Le Chapitre 7 présente les deux outils mis en place pour nous aider à développer une théorie permettant de corriger les problèmes de comptage identifiés dans la théorie de Bennett, sans en créer de nouveaux. Le premier outil utilisé est Alloy, qui est un outil pour explorer les modèles et trouver des contre-exemples à une théorie. Le second outil est Coq, qui est un assistant de preuves. Il nous a permis de vérifier les preuves de tous les théorèmes de la théorie. Les trois chapitres suivants constituent ensuite la proposition théorique aboutie. L’essentiel du contenu de ces chapitres a été publié dans l’article (Tarbouriech et al., 2024). Le Chapitre 8 aborde la notion de slot en tentant de la définir, ainsi que d’autres questions se rapportant à cette notion. Le Chapitre 9 présente le cœur de la proposition : une méréologie des slots<sup>4</sup>. Afin de régler les problèmes de comptage, cette théorie introduit une notion de contextualisation des slots. Cette notion est ensuite utilisée pour définir les notions méréologiques de partie, de chevauchement, ainsi que les principes de composition et de décomposition. Toutes ces relations méréologiques sont définies uniquement entre des slots, d’où le terme de « méréologie des slots ». Cette nouvelle approche permet de dépasser les limitations rencontrées par Bennett dans la définition de certains de ses axiomes et théorèmes (notamment la supplémentation). Ce chapitre est

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<sup>4</sup>Les noms en français des deux théories sont très semblables. Toutefois, « méréologie à slots » désigne la théorie de Bennett (« *Slot Mereology* » en anglais), alors que « méréologie des slots » désigne notre proposition (« *Mereology Of Slots* » en anglais).



illustré avec des exemples tirés des entités non-idempotentes, comme l’universel de molécule de dihydrogène. Enfin, le Chapitre 10 propose d’utiliser la théorie sur des cas particuliers, pour en démontrer certains aspects et la façon dont elle règle les problèmes évoqués.

- La Partie III est la partie de conclusion. Le Chapitre 11 propose des pistes de réflexion sur de futurs travaux qui pourraient compléter les limites identifiées. Enfin, le Chapitre 12 conclut cette thèse en synthétisant notre proposition.
- L’Annexe A fournit une liste des formules utilisées dans ce document, sauf celles présentées dans le Chapitre 4.

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How many letters are there in the word “potato”? Six? Or maybe just four? The answer depends on what we count when we ask: “How many letters?”. By answering “six”, we count the number of characters in the word, i.e. the number of ink patterns on the paper: there are six patterns, thus six letters. Some of these patterns might look similar, but they are nonetheless distinct. By answering “four”, we count the number of symbols used among a predefined set: the Latin alphabet. Four symbols appear in the word, among the 26 symbols the Latin alphabet contains. Therefore, there are multiple valid answers to the initial question. However, affirming that the word “potato” contains four and six letters is at least confusing, if not contradictory. How do we express the fact that these two answers are reasonable? It can be done with “types” and “tokens”. The multiple occurrences of the word “potato” in this introduction are distinct tokens of the same type. Similarly, the multiple occurrences of the letter “o” in the word “potato” are distinct tokens (the ink patterns) of the same type (this very letter in the 26-symbol Latin alphabet).

The question seems solved: a word token “potato” of the word type “POTATO” has six letter tokens of four distinct letter types, and the word type “POTATO” contains the four letter types “A”, “O”, “P”, “T”. Nevertheless, describing the word type “POTATO” as containing four letter types without saying more does not seem convincing: the word types “ATOP” and “TOPATO” also contain these four letter types, and only them. How do we explain that “POTATO” is distinct from “ATOP” and “TOPATO”? With the sole description given above, it is not achievable. Consider a token of each of these types and describe each of them. The word token “potato” contains six letter tokens of four letter types. The word token “atop” contains four letter tokens of four letter types. Finally, the word token “topato” contains six letter tokens of four letter types. We can distinguish the first and the second tokens thanks to the multiplicities of letter tokens. Furthermore, we can distinguish the first and the third one due to the order of the letter tokens. Carrara and Smid (2022b) use the same reasoning to establish three identity criteria for word types.

Structural universals are similarly problematic. Universals are shared entities to all of their instances: all horses are similar in some way, and for realistic philosophers, the universal HORSE explains this similarity. All horses instantiate this universal. In this manuscript, we will not enter into the philosophical debate between nominalists and realists concerning the existence of universals (see Armstrong (1978b,a)). Nonetheless, this thesis proposes a framework to represent their structures *if they exist*. In the 1980s, philosophers like David Lewis (1986), David Armstrong (1986), John Bigelow (1986, 1989), Robert Pargetter (1989) and Peter Forrest (1986) discussed the existence of structural universals, i.e. universals built from other universals.

The historical example used in the literature is the universal of METHANE, composed of the universals of CARBON and HYDROGEN. Therefore, according to a realist conception, carbon atoms are instances of the universal CARBON, hydrogen atoms are instances of the universal HYDROGEN and methane molecules are instances of the universal METHANE.

If structural universals exist, then, in a similar way that methane instances are made of a carbon instance and four hydrogen instances, the universal METHANE is made of the universals CARBON and HYDROGEN. However, while an instance of METHANE is made of four distinct instances of HYDROGEN, there is only one universal HYDROGEN. The same reasoning can be developed as the one presented above for word and letter tokens. Consider the universal BUTANE, whose instances have four carbon atoms and ten hydrogen atoms. What distinguishes an instance of METHANE and an instance of BUTANE is the multiplicity of the atoms that compose them. Consider now the universal ISOBUTANE, whose instances have the same number of atoms as those of Butane. What distinguishes them is how these atoms are bonded: in a butane molecule, carbon atoms are in a chain, while they form a tetraedron in an isobutane molecule.

Finally, if one requires more examples of entities that can have the same part multiple times, we could consider the example of the temporal wall of Effingham and Robson (2007), in which the same brick, used a hundred times thanks to time travel, composes a wall. Alternatively, close timelike curves of Gilmore (2007), thanks to which a hydrogen atom could coexist with a prior version of itself and bond with this prior version to make a dihydrogen molecule.

These problems of representation are mereological: they are about how an entity and its parts relate. Mereology is the study of parthood relations: “of the relations of part to whole and the relations of part to part within a whole”. The commonly accepted version is called Classical Extensional Mereology, or CEM. It introduces various relations, like relations of parthood or overlap. Moreover, it contains two families of principles, known as decomposition and composition principles. The first group of these principles deals with how an entity decomposes into parts. For example, one of these principles, extensionality, states that two things are equal if they have the same parts. The second group deals with how entities are composed to make bigger entities. For example, the binary sum is the relation that relates two entities to the thing composed of these two entities and only them; thus, my glasses are the sum of their lenses and their frame.

Classical extensional mereology is incompatible with entities such as word and letter types or structured universals. Several principles of this theory oppose the fact that entities can have the same part more than once. The “*Parts Just Once*” principle enunciated by Effingham and Robson (2007) asserts that “a composite object cannot have the same object as its proper part many times over”. This principle directly opposes what we want to develop: a mereological theory in which a composite object could have the same proper part several times over. However, this principle is not expressible in classical extensional mereology because of the term “many times over”.<sup>5</sup> On the other hand, there are two theorems of classical extensional mereology on mereological composition. The first, called “*Idempotence*”, asserts that “the sum of  $x$  with itself is equal to  $x$ ”. The second, called “*Subpotence*” by Cotnoir and Varzi (2021), asserts that “if  $x$  is part of  $y$ , then the sum of  $x$  and  $y$  is identical to  $y$ ”.<sup>6</sup> Hereafter, these entities, non-idempotent under the mereological sum, will be referred to as “*Non-idempotent entities*”.

The observation that classical extensional mereology is unsuitable for all entities is not new. However, reactions to this observation differ. Thus, Lewis (1986), after introducing the example of the universal METHANE presented above, concludes that a mereological

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<sup>5</sup>This theory is generally expressed in classical (first-order) logic and admits a binary part relation  $P$ . Consequently,  $P(x, y) \wedge P(x, y)$  (with  $P$  the part relation) simplifies in classical logic to  $P(x, y)$  and does not allow for the number of occurrences. Moreover, modifying the relation  $P$  into a ternary relation to include a number of occurrences takes us outside the traditional framework of mereology

<sup>6</sup> When he first introduced the principle of Subpotence, Cotnoir (2015) called it “Idempotence”. However, in the book *Mereology*, written with Varzi, this principle is called “Subpotence”. Since “Idempotence” classically refers to the property  $x + x = x$ , we use the term “Idempotence” here in its traditional sense and refer to the principle introduced by Cotnoir (2015) as “*Subpotence*”.

approach to structured universals is impossible because it would require an entity to have the same part more than once. Nevertheless, others accept the idea of modifying classical extensional mereology to consider non-idempotent entities. These theories include the occurrence-based design of Wetzel (2009), the slot mereology of Bennett (2013), the abelian mereology of Cotnoir (2015) or the use of plural logic by Carrara and Smid (2022b).<sup>7</sup>

Thus, this thesis aims to establish a general theory that allows the representation of non-idempotent entities. This theory must also enable each part's multiplicities to be counted correctly. Finally, because it will be a mereological theory, we expect it to possess the notions commonly accepted by a mereological theory, namely the fundamental relations of part, proper part and overlap, as well as decomposition principles such as supplementation or extensionality, but also composition principles, such as sum or fusion.

In this thesis, we have selected the slot mereology proposed by Bennett (2013) as the basis for a solution to our problem. Although it is a sketch, as Bennett describes it, this theory is innovative in providing a general framework for representing non-idempotent entities; thus, it is not restricted to certain classes of entities. It has prompted numerous comments (Fisher (2013), Cotnoir (2015), Garbacz (2016)) and inspired several theories manipulating and adapting the notion of slot: Barton et al. (2020b), Sattig (2021), Barton et al. (2022). We identified specific difficulties and proposed corrections: Tarbouriech et al. (2021) and Tarbouriech et al. (2024).

Slot mereology analyses the part relation using two primitives. These two primitives, drawn from a comparison with role ontologies, are *is a slot of* and *fills*. Thus, for  $a$  to be a part of  $b$ ,  $a$  must fill a slot of  $b$ . Although this proposal brings new representation possibilities, it raises several problems. Fisher (2013), Cotnoir (2015) and Garbacz (2016) have raised several criticisms and identified several philosophical or axiomatic problems. However, we have identified more practical problems: We have shown that in its original form, this theory, which allows non-idempotent entities to be represented, does not allow the number of occurrences of parts to be counted correctly. However, as we shall see, it can be adapted and developed to meet this objective.

In this work, we have restricted ourselves to mereology. We have explored the various mereological notions that could be defined based on slot mereology. However, we will see in the discussion that our theoretical proposal could be extended in future work to integrate non-mereological relations that would notably allow us to distinguish BUTANE and ISOBUTANE. Our aim here is to propose a robust foundation for such future extensions. Furthermore, this objective is addressed using a first-order logic representation framework, which is common in applied ontology.

This thesis is structured in three parts, divided into chapters:

- Part I constitutes a state of the art enabling the understanding of the topics covered in the following parts. Chapter 2 presents, based on the recent work of Cotnoir and Varzi (2021), the notions of extensional classical mereology used in the remainder of this thesis. The last part of this chapter presents some non-idempotent entities: structured universals and word types. As the literature on these entities is vast, this section focuses, after a general presentation of these entities, on their mereological aspects. Indeed, these entities do not constitute the core of this work and are only used as sources of questions and examples. Chapter 3 presents the slot-based mereology developed by Bennett (2013) and the criticisms present in the literature. In addition, we offer our own critique of this theory in this chapter, presenting counting problems. Finally, Chapter 4 of this state of the art presents various theories that provide exciting additions to address the question of interest to us.
- Part II constitutes our proposal as a solution to the question at hand. In Chap-

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<sup>7</sup>These theories are respectively presented in Sections 2.6.1, 4.1 and 4.2 and Chapter 3.

ter 5, adapted from the article (Tarbouriech et al., 2021), a correction and extension of Bennett’s theory is presented. This theoretical correction and extension, named “*Copy-Slot Mechanism*”, corrects the counting problems identified in Bennett’s theory. However, this proposal introduces a new counting problem. To remedy this problem, we have established criteria for evaluating our theory. Thus, Chapter 6 develops these prerequisites by integrating the choice of exploiting Bennett’s theory and various elements drawn from the other theories presented in Chapter 4. Chapter 7 presents the two tools to help us develop a theory to correct the counting problems identified in Bennett’s theory without creating new ones. The tools used are Alloy, a tool for exploring models and finding counterexamples to a theory, and Coq, a proof assistant. It enabled us to check the proofs of all the theorems in the theory. The following three chapters constitute the final theoretical proposal. Most of the content of these chapters has been published in the article (Tarbouriech et al., 2024). Chapter 8 tackles the notion of a slot by attempting to define it and other questions relating to it. Chapter 9 presents the heart of the proposal: a mereology of slots.<sup>8</sup> To solve counting problems, this theory introduces a notion of slot contextualization. This notion is then used to define the mereological notions of part and overlap and the principles of composition and decomposition. All these mereological relationships are defined only between slots, hence the term “mereology of slots”. This new approach makes it possible to overcome the limitations encountered by Bennett in defining some of her axioms and theorems (notably supplementation). The chapter is illustrated with examples drawn from non-idempotent entities, such as the universal DIHYDROGEN. Finally, Chapter 10 proposes to use the theory on particular cases to demonstrate certain aspects of it and how it solves the problems evoked.

- Part III is the concluding section. Chapter 11 first proposes our work, offering food for thought on future work that might complement the limitations identified. Finally, Chapter 12 concludes this thesis by offering a summary of our proposal.
- Appendix A provides a list of formulas presented in this work, excepted those of Chapter 4.

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<sup>8</sup>In this thesis, “*Slot Mereology*” will refer to Bennett’s theory, whereas “*Mereology of Slots*” refers to our proposal.

**Part I**

**State Of The Art**



# Chapter 2

## Classical Mereology

In this chapter, I present the mereological notions that will be used later on. Mereology aims at explaining what parthood relations are; more precisely, to explain the relations between a whole and its parts, and between parts within a whole. This chapter does not aim at offering a complete account of all mereological notions. This section is mainly based on Varzi (2019) and Cotnoir and Varzi (2021).

Mereology finds applications, with applied ontology, in many sciences, including computer science. In particular, it is used, among other things, in knowledge representation, conceptual modelling, ontology engineering or semantic web. For further details, see, for example, Guarino et al. (1996); Polkowski (2014); Keet (2006); Bjørner (2018).

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### 2.1 Historical Background

The term “mereology” was originally coined, in Polish as “mereologia”, by Stanisław Leśniewski to designate one of the three components of his formal systems. But the study of the mereological relations, like the parthood relation, is much older. Some mereological questions were already addressed by Greek thinkers (Presocratics), the Chinese school

of logicians *Mínghjiā* or the Hindu philosophy school *Vaiśeṣika*, all around 6<sup>th</sup> and 5<sup>th</sup> centuries BCE. It was also a matter of interest for medieval philosophers and Scholastics.

Systematic study of the parthood relation was only started at the beginning of the twentieth century, with Edmund Husserl and Leśniewski. The former (Husserl, 1901) studied it as an essential part of the general framework he was developing for formal ontology. The latter (Leśniewski, 1916, 1931) studied mereology as an alternative to set theory. The theory of Leśniewski, unlike Husserl’s, is a theory meant to be complete, and may be considered as “the first fully worked out example of a mereological theory” (Cotnoir and Varzi, 2021, p. 10).

However, Leśniewski’s work is in Polish, and thus was not accessible to non-speakers of Polish. In 1940, Leonard and Goodman (1940) developed *Calculus of Individuals*, which is, in fact, essentially equivalent to Leśniewski’s Mereology. This common theory is known as *classical mereology*.

In 1987, Simons published *Parts: A Study in Ontology* in which he gives a “connected account of the various kinds of mereology [...] which exists, widely scattered, in the literature”. In his work, he exposes the philosophical problems of Leonard, Goodman and Leśniewski theories and proposes a way to correct these problems. His work, as a seminal work on mereology, serves as a basis for many of the publications in formal ontology that have followed.

## 2.2 Basic Relations

Multiple axiomatisations of a mereological theory are possible, depending on the choice of the primitive relation: parthood, proper parthood, overlap or disjointness. As my proposal uses the parthood relation to define those other relations (although, as we will see, my proposal does not consider parthood as a primitive), I will present classical mereology using the parthood relation as the primitive. Refer to (Cotnoir and Varzi, 2021, pp. 44–55) for a general presentation of alternatives. More specifically, refer to Leśniewski (1916) and Simons (1987) for proper-parthood-based theories, to Goodman (1951) and Leonard and Goodman (1940) for theories based on overlap or disjointness.

### 2.2.1 Parthood

In classical mereology, *parthood* is the relation that relates an entity with the object it is a part of. For example, the handle is part of the door and my lunch time of tomorrow is part of my occupations of tomorrow. Contrary to the proper parthood relation, parthood is formalised as a reflexive relation. Therefore, as stated by Axiom 2.1, every entity is part of itself. The relation is also transitive (e.g. my right hand is part of my right arm which is part of my body, therefore my right hand is part of my body) as stated by Axiom 2.2. Finally, the parthood relation is antisymmetrical, i.e. two entities that are part of each other are in fact the very same entity, as posited by Axiom 2.3. These axioms characterise the parthood relation as a partial order. The theory generated by these three axioms is called *ground mereology* by Varzi (1996).

**Axiom 2.1** (Parthood Reflexivity).

$$\forall x(P(x, x))$$

**Axiom 2.2** (Parthood Transitivity).

$$\forall x, y, z(P(x, y) \wedge P(y, z) \rightarrow P(x, z))$$

**Axiom 2.3** (Parthood Antisymmetry).

$$\forall x, y(P(x, y) \wedge P(y, x) \rightarrow x = y)$$



## 2.2.2 Proper Parthood

The *proper parthood* relation is similar to the parthood relation, but without reflexivity, i.e. an entity cannot be a proper part of itself. When we use the word “part” in an everyday talk, it generally refers to proper parthood. The examples given in the previous section are examples of proper parts. Definition 2.1 follows Varzi (2019).

**Definition 2.1** (Proper Parthood).

$$PP(x, y) \triangleq P(x, y) \wedge x \neq y$$

As consequences of Definition 2.1 and Axioms 2.1 to 2.3, the proper parthood relation is irreflexive, transitive and asymmetric, as stated by Theorems 2.1 to 2.3.<sup>1</sup>

**Theorem 2.1** (Proper Parthood Irreflexivity).

$$\forall x(\neg PP(x, x))$$

**Theorem 2.2** (Proper Parthood Transitivity).

$$\forall x, y, z(PP(x, y) \wedge PP(y, z) \rightarrow PP(x, z))$$

**Theorem 2.3** (Proper Parthood Asymmetry).

$$\forall x, y(PP(x, y) \rightarrow \neg PP(y, x))$$

## 2.2.3 Overlap

The *overlap* relation holds between entities that have a part in common, as defined by Definition 2.2. Following (Cotnoir and Varzi, 2021, p. 55), sub-relations of overlap exist, as pictured in Figure 2.1, namely *proper overlap* and *improper overlap*. The former relation, defined by Definition 2.3, holds between overlapping entities that are not part of each other. The latter relation, defined by Definition 2.4, holds between a whole and one of its parts. Finally, if two entities do not overlap, i.e. there is no shared part, they are *disjoint*, as defined by Definition 2.5. All these relations are symmetric, as stated by Theorems 2.4 to 2.7. The overlap and improper overlap relations are reflexive, whereas the proper overlap and the disjointness relations are irreflexive as stated by Theorems 2.8 to 2.11.

**Definition 2.2** (Overlap).

$$O(x, y) \triangleq \exists z(P(z, x) \wedge P(z, y))$$

**Definition 2.3** (Proper Overlap).

$$PO(x, y) \triangleq O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)$$

**Definition 2.4** (Improper Overlap).

$$IO(x, y) \triangleq P(x, y) \vee P(y, x)$$

**Definition 2.5** (Disjointness).

$$D(x, y) \triangleq \neg O(x, y)$$

---

<sup>1</sup>The proofs of theorems and lemmas presented in the state of the art are not presented here, as they can be found in the literature about mereology.

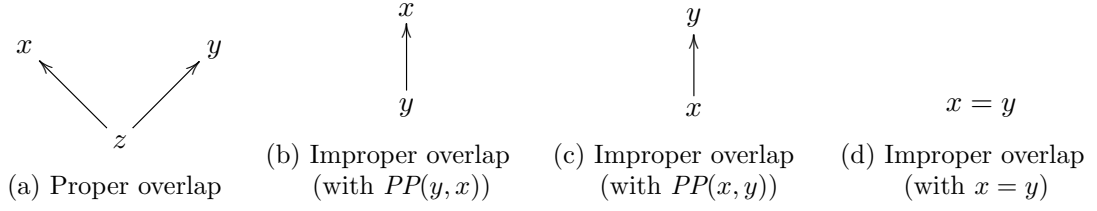


Figure 2.1: Different kinds of overlap between  $x$  and  $y$  (arrows represent the proper parthood relation)

**Theorem 2.4** (Overlap Symmetry).

$$\forall x, y (O(x, y) \rightarrow O(y, x))$$

**Theorem 2.5** (Proper Overlap Symmetry).

$$\forall x, y (PO(x, y) \rightarrow PO(y, x))$$

**Theorem 2.6** (Improper Overlap Symmetry).

$$\forall x, y (IO(x, y) \rightarrow IO(y, x))$$

**Theorem 2.7** (Disjointness Symmetry).

$$\forall x, y (D(x, y) \rightarrow D(y, x))$$

**Theorem 2.8** (Overlap Reflexivity).

$$\forall x (O(x, x))$$

**Theorem 2.9** (Proper Overlap Irreflexivity).

$$\forall x (\neg PO(x, x))$$

**Theorem 2.10** (Improper Overlap Reflexivity).

$$\forall x (IO(x, x))$$

**Theorem 2.11** (Disjointness Irreflexivity).

$$\forall x (\neg D(x, x))$$

## 2.3 Decomposition Principles

In mereological theories, the decomposition principles express how an entity decomposes into parts. Those principles range from weak principles, like company — i.e. if a whole has a proper part, then it has another one — to strong ones, like complementation — i.e. if an entity  $y$  is not part of  $x$ , then there is something that is the sum of all the parts of  $y$  that are disjoint from  $x$ . After having presented the notions of *mereological atom* in Section 2.3.1, the decomposition principles are presented from the weakest to the strongest in Sections 2.3.2 to 2.3.4.

### 2.3.1 Atoms

In mereological theories, *atoms*, sometimes called *simples*, are entities that do not have proper parts, as defined by Definition 2.6.

**Definition 2.6** (Atom).

$$A(x) \triangleq \neg \exists y (PP(y, x))$$

### 2.3.2 Supplementation

When considering how an entity decomposes, one of the first principles that can be formulated is that if an entity  $y$  has a proper part  $x$ , then it must have another proper part  $z$ , different from  $x$ , called a *remainder*. In other words, it is never the case that something has only one proper part, e.g. my hand is not the only proper part of my arm. This principle is called *company* and is formalised by Axiom 2.4. In particular, it eliminates *finite linear orders*, such as depicted in Figure 2.2a, which do not correspond to a standard understanding of parthood. Even though this axiom excludes models such as depicted in Figure 2.2a, it does not exclude structures like those pictured in Figures 2.2b and 2.2c. To prevent Figure 2.2b from being a model, one can posit the Axiom 2.5 of *strong company*, i.e. an entity  $y$  that has a proper part  $x$  also has a proper part  $z$  which is not a part of  $x$ . Axiom 2.5 does prevent such  $z$  and  $x$  to be improper-overlapping. However, it does not prevent them to be proper-overlapping. The principle that does so is called *supplementation*, or *weak supplementation* in opposition to other forms of supplementation, and is formalised by Axiom 2.6. Note that Axiom 2.6 does not explicitly state that  $z$  is a proper part of  $y$ , but simply a part of  $y$ . However, it does imply that  $z$  is a proper part:  $x$  overlaps with  $y$  but not with  $z$ , therefore  $y$  and  $z$  are different. This axiom entails that an entity with proper parts has (at least) two distinct and disjunct proper parts.

**Axiom 2.4** (Company).

$$\forall x, y (PP(x, y) \rightarrow \exists z (PP(z, y) \wedge x \neq z))$$

**Axiom 2.5** (Strong Company).

$$\forall x, y (PP(x, y) \rightarrow \exists z (PP(z, y) \wedge \neg P(z, x)))$$

**Axiom 2.6** (Supplementation).

$$\forall x, y (PP(x, y) \rightarrow \exists z (P(z, y) \wedge \neg O(z, x)))$$

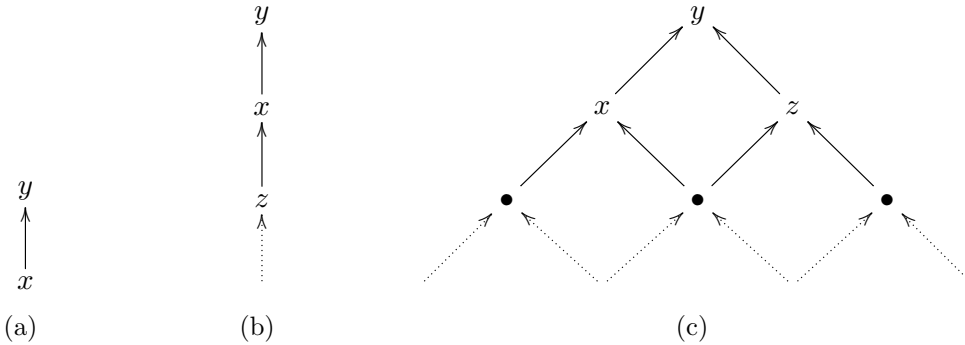


Figure 2.2: Three unsupplemented models (dotted arrows represent an infinite repetition of the pattern)

### 2.3.3 Strong Supplementation and Extensionality

A stronger version of the supplementation exists. It is called *strong supplementation* and formalised by Axiom 2.7. This axiom states that if an entity  $y$  is not a part of another entity  $x$ , then there is some part of  $y$  that does not overlap with  $x$ . This part can be  $y$  itself. This axiom excludes models such as pictured in Figure 2.3, where  $x$  and  $y$  are two entities both having as proper parts the same entities:  $w$  and  $z$ . Note that Axiom 2.6 is a theorem of ground mereology and strong supplementation (Axioms 2.1 to 2.3 and 2.7).

**Axiom 2.7** (Strong Supplementation).

$$\forall x, y (\neg P(y, x) \rightarrow \exists z (P(z, y) \wedge \neg O(z, x)))$$

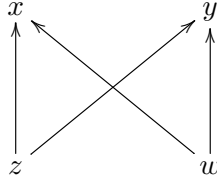


Figure 2.3: A model of supplementation, but not of strong supplementation

From ground mereology and strong supplementation, various theorems of *extensionality* can be proved. Extensionality is a restriction of Leibniz’s Law to a specific mereological relation.<sup>2</sup> Theorems 2.12 and 2.13 are two theorems of extensionality: the former uses the proper parthood relation, while the latter uses the overlap relation.

**Theorem 2.12** (PP-Extensionality).

$$\forall x, y [\exists z (PP(z, x) \vee PP(z, y)) \rightarrow (x = y \leftrightarrow \forall z (PP(z, x) \leftrightarrow PP(z, y)))]$$

**Theorem 2.13** (O-Extensionality).

$$\forall x, y (x = y \leftrightarrow \forall z (O(z, x) \leftrightarrow O(z, y)))$$

These theorems state multiple things: first, if two entities are identical, then they stand in the same mereological relation PP (respectively O) with the same entities. But this is a trivial consequence of the identity of those entities. Second, if two entities relate to exactly the same objects by the mereological relation PP (respectively O), then they are identical. There is a difference between those theorems: the former is conditioned to the existence of a proper part, but not the latter. This is because the overlap relation is reflexive: every entity overlap with at least itself. This is not the case for proper part: a mereological atom does not have any proper parts. In Theorem 2.12, if the premise was removed and  $x$  and  $y$  would be different atoms, then  $\forall z (PP(z, x) \leftrightarrow PP(z, y))$  would be vacuously true, and therefore, it could be concluded that  $x = y$ , which is a contradiction.

### 2.3.4 Complementation

Finally, there is an even stronger decomposition principle, called *complementation*, and formalised by Axiom 2.8. Although it starts like Axiom 2.7, its consequent is different. This axiom states that if  $y$  is not a part of  $x$ , then there is a  $z$  that is the composite object made of all the things that are parts of  $y$  and that do not overlap with  $x$ , i.e.  $z$  is the complement of  $x$  in  $y$ . Contrary to strong supplementation, which states that there is some remainder without saying anything about how all the remainders relate, the complementation axiom states that all the parts of  $y$  that do not overlap with  $x$  form a unique entity  $z$ . This axiom is stronger as it requires that all the remaining parts can be aggregated into something. As we do not have seen anything about aggregating parts yet, this discussion will be continued in the relevant section, i.e. Section 2.4. As an example of a strongly supplemented model that violates complementation, see Figure 2.4, where  $y$  is not a part of  $x$ , but there is no entity composed of the remaining parts, i.e.  $z$  and  $w$ .

<sup>2</sup>*Leibniz’s Law* can refer to two things: either the *Identity of Indiscernibles*—which states that two objects that have the same properties are identical ( $\forall x, y (\forall F (F(x) \leftrightarrow F(y)) \rightarrow x = y)$ )— or the conjunction of this principle with the *Indiscernibility of Identicals*—which states that two objects that are identical have the same properties ( $\forall x, y (x = y \rightarrow \forall F (F(x) \leftrightarrow F(y)))$ ). For more, see (Forrest, 2020). In literature about mereology, both the first principle, or the conjunction can be found to axiomatise extensionality: see (Cotnoir and Varzi, 2021) for the use of one principle and (Varzi, 2019) for the use of the conjunction.

**Axiom 2.8** (Complementation).

$$\forall x, y (\neg P(y, x) \rightarrow \exists z \forall w (P(w, z) \leftrightarrow (P(w, y) \wedge \neg O(z, x))))$$

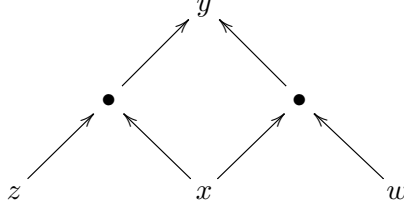


Figure 2.4: A strongly supplemented model violating complementation

## 2.4 Composition Principles

In the same way we express principles about how entities decompose into parts, it is possible to express principles to explain how entities compose into wholes. Those principles are called *composition principles*. This section first presents two of the composition principles, namely *binary sum* and *fusion*, and then the notion of *mereological universe*.

### 2.4.1 Binary Sum

Mereological sum is something we are used to: the blade and the handle make a knife, the roots, the trunk, the branches and the leaves make the tree, and my first name and my last name make my name. The literature about mereological sums contains multiple definitions (see for example Lewis (1991) and Casati and Varzi (1999)). Two of the commonly accepted definitions of sum are formalised by Definitions 2.7 and 2.8, where  $S_i(z, x, y)$  reads as “ $z$  is a sum <sub>$i$</sub>  of  $x$  and  $y$ ”.

**Definition 2.7** (Sum<sub>1</sub>).

$$S_1(z, x, y) \triangleq P(x, z) \wedge P(y, z) \wedge \forall w (P(w, z) \rightarrow O(w, x) \vee O(w, y))$$

**Definition 2.8** (Sum<sub>2</sub>).

$$S_2(z, x, y) \triangleq \forall w (O(z, w) \leftrightarrow O(w, x) \vee O(w, y))$$

By the first definition,  $z$  being a sum of  $x$  and  $y$  means that  $x$  and  $y$  are both parts of  $z$  and that every part of  $z$  either overlaps  $x$  or  $y$ . While  $P(x, z) \wedge P(y, z)$  ensures that  $z$  is above  $x$  and  $y$  (i.e. an *upper bound*),  $\forall w (P(w, z) \rightarrow O(w, x) \vee O(w, y))$  ensures that there is nothing else in between (i.e.  $z$  is a *minimal upper bound*). By the second definition,  $z$  being a sum of  $x$  and  $y$  means that, for every  $w$ ,  $w$  overlaps  $z$  iff  $w$  overlaps either  $x$  or  $y$ .

What about the existence of the sum <sub>$i$</sub> ? Under which circumstances does a sum between some  $x$  and  $y$  exist? To answer these questions, an axiom of existence is commonly accepted. It can be formalised as in Axiom Schema 2.9, where  $\xi$  is a suitable binary condition.

**Axiom Schema 2.9** (Existence of Sum <sub>$i$</sub> ).

$$\forall x, y (\xi(x, y) \rightarrow \exists z (S_i(z, x, y)))$$

Even though these two definitions are different, they are equivalent if the theory has the strong supplementation axiom Axiom 2.7. For more explanations about their differences in the absence of strong supplementation, see Varzi (2019). Furthermore, with strong supplementation, sum is also unique, thanks to extensionality (Theorem 2.13). Therefore, from now on assuming we are in the context of ground mereology with strong supplementation, we can speak of  $z$  as “the” sum of  $x$  and  $y$  and use some operator:  $x +_i y$  represents the sum between  $x$  and  $y$ . Finally, as long as  $x$  and  $y$  satisfy  $\xi$ , their sum <sub>$i$</sub>  is idempotent, commutative and associative.

Varzi (2019) gives four theorems to show how sum behaves with parthood: Theorems 2.14 to 2.17.

**Theorem 2.14.**

$$\forall x, y (P(x, x +_i y))$$

**Theorem 2.15.**

$$\forall x, y, z (P(x, y) \rightarrow P(x, y +_i z))$$

**Theorem 2.16.**

$$\forall x, y, z (P(x +_i y, z) \rightarrow P(x, z))$$

**Theorem 2.17.**

$$\forall x, y (P(x, y) \leftrightarrow x +_i y = y)$$

## 2.4.2 Fusion

Fusion (also called *generalised sum*) is a stronger composition principle that can be summarised as “infinitary sum”. While the sum can only deal with a finite amount of operands, fusion is defined to work with an infinite number of operands.

To generalise binary sum, the idea is to consider the generalised sum of all elements for which some predicate  $\phi$  holds. However, as the logical apparatus chosen in this work is first order logic, quantifying over predicates is not possible. Fusion is therefore defined using a definition schema. Two of the commonly accepted definitions of fusion are given by Definition Schemata 2.9 and 2.10, respectively named *Leśniewski fusion* and *Goodman fusion*.<sup>3</sup>

In these schemata,  $\phi$  is a formula in which the variable  $w$  should occur free.  $\phi$  itself is not a variable in our (first-order) language. For readability, we use a predicate-like notation, such as  $\phi(w)$ , even though  $\phi$  is not to be understood as a predicate variable. The formula  $F_{1\phi}z$  should read as “ $z$  is the fusion<sub>1</sub> of all the  $\phi$ -ers”, where the term “ $\phi$ -er” refers to an entity for which the formula  $\phi$  holds.

**Definition Schema 2.9 (Fusion<sub>1</sub>).**

$$F_{1\phi}(z) \triangleq \forall w(\phi(w) \rightarrow P(w, z)) \wedge \forall v(P(v, z) \rightarrow \exists w(\phi(w) \wedge O(v, w)))$$

**Definition Schema 2.10 (Fusion<sub>2</sub>).**

$$F_{2\phi}(z) \triangleq \forall v(O(v, z) \leftrightarrow \exists w(\phi(w) \wedge O(v, w)))$$

According to the first definition schema,  $z$  being the fusion of all entities for which  $\phi$  holds means that every  $\phi$ -er is a part of  $z$  and that every part of  $z$  overlaps with some  $\phi$ -er. This definition schema contains two parts, like the Definition 2.7 of binary sum: the left part makes  $z$  an upper bound while the right part makes  $z$  as minimal as possible.

<sup>3</sup>To read more about Leśniewski fusion, see Van Inwagen (1987) and Lewis (1991). For more about Goodman fusion, see Eberle (1970), Simons (1987) and Casati and Varzi (1999).

The second definition schema is similar to the Definition 2.8 of binary sum: it exploits the overlap relation to state that, for every  $v$ ,  $v$  overlaps  $z$  iff it overlaps some  $\phi$ -er.

Following Lewis (1991), take for example  $\phi$  as “being a cat”. If  $z$  is the fusion of the *phi*-ers (i.e. the things that are cats),  $z$  “is that large, scattered chunk of cat-stuff which is composed of all the cats there are, and nothing else”. Or, in other words,  $z$  is “the thing that overlaps all and only those things that overlap some cat”. To continue with the same example, Definitions 2.11 and 2.12 give an example of the definition schemata used with the predicate *Cat*.

**Definition 2.11** (Fusion<sub>1</sub> Of Cats).

$$F_{1Cat}(z) \triangleq \forall w(Cat(w) \rightarrow P(w, z)) \wedge \forall v(P(v, z) \rightarrow \exists w(Cat(w) \wedge O(v, w)))$$

**Definition 2.12** (Fusion<sub>2</sub> Of Cats).

$$F_{2Cat}(z) \triangleq \forall v(O(v, z) \leftrightarrow \exists w(Cat(w) \wedge O(v, w)))$$

As proved by (Cotnoir and Varzi, 2021, pp. 166–168), both definitions are equivalent as long as the parthood relation is reflexive and transitive, and the theory has strong supplementation.

Now that fusions are defined, let’s look at existence axioms. There are commonly two axiom schemata: one restricted and the other one unrestricted, as formalised by Axiom Schemata 2.10 and 2.11.

**Axiom Schema 2.10** (Restricted Existence of Fusion<sub>*i*</sub>).

$$\exists w(\phi(w)) \wedge \forall w(\phi(w) \rightarrow \psi(w)) \rightarrow \exists z(F_{i\phi}(z))$$

**Axiom Schema 2.11** (Unrestricted Existence of Fusion<sub>*i*</sub>).

$$\exists w(\phi(w)) \rightarrow \exists z(F_{i\phi}(z))$$

Both axioms have  $\exists w(\phi(w))$  as an antecedent. This is to guarantee that there is some  $\phi$ -er. Otherwise, it would give the existence of the fusion of no elements, thus creating a *null element*, which mereological theories reject. The difference between these two axioms is that in the first one, for a fusion  $z$  to exist, all the  $\phi$ -ers need to satisfy some condition  $\psi$ .

### 2.4.3 Universe

In mereological theories, the *universe* is the entity of which every other entity is a part. Varzi (2019) defines the universe as the fusion of everything that is part of itself, as presented by Definition 2.13. The parthood relation being reflexive, the universe is the fusion of everything. With the unicity of fusions, the universe is generally noted  $U$ .

**Definition 2.13** (The Universe).

$$U(z) \triangleq F_{iP(w,w)}(z)$$

## 2.5 Parts Just Once, Idempotence and Subpotence Principles

Effingham and Robson (2007) introduce the “Parts Just Once” principle as follows: “a composite object cannot have the same object as a proper part many times over” (Effingham and Robson, 2007, p. 636). This principle is not expressible in classical extensional

mereology due to the occurrence of “many times over” as Cotnoir (2013, 2015) explains it. Cotnoir (2015) uses similar principles, expressible in first-order logic: “Idempotence” and “Subpotence”. Idempotence, as generally admitted, is defined as  $x + x = x$ , where  $+$  refers to the binary sum. Thus, summing the same entity twice cannot result in anything else than the said entity. Cotnoir (2015) defines subpotence<sup>4</sup> as follows: “If  $x$  is a part of  $y$ , then the sum of  $x$  and  $y$  is identical to  $y$ .” (Cotnoir, 2015, p. 429).

## 2.6 Challenging Entities

In the previous sections, I presented classical extensional mereology. However, this theory has limitations. In particular, it cannot represent entities that do not comply with the *Parts Just Once*, *Idempotence* and *Subpotence* principles, presented in Section 2.5. In this section, I present some of these entities. In particular, I present structural universals, informational entities and types. Their analysis can provide useful knowledge about how those entities compose. The philosophical literature on these entities is vast and addresses questions such as their ontological nature or how they relate to other entities. I will not delve into those, and this section will be restricted to the analysis of some mereological issues raised by such entities.

### 2.6.1 Structural Universals

In this section, I give a presentation of the most common conception about structural universals, based on the analysis of the debate on structural universals proposed by Garbacz (2020), as well as the work of Fisher (2018). Some conceptions about structural universals will be mentioned without further deep exploration. Before presenting what structural universals are (or could be),<sup>5</sup> I will make an overview of the historical debate.

What are structural universals? First, they are universals, that is, abstract entities that explain resemblance between individuals. Consider three horses in a field. What explains the resemblance between these three individuals (or particulars) is the universal HORSE. These three horses are said to be *instances* of the universal. However, the existence of such universals is debated. Realism endorses it, while Nominalism or Conceptualism reject their existence. For more about this debate, see Armstrong (1978c,b); Macleod and Rubenstein (2023). Second, they are complex, that is, “somehow composed of other universals” (Garbacz, 2020, p. 4). The debate about the nature of the composition and the existence of structural universals dates, at least, from the 1970s and 1980s. Armstrong (1978a) introduces the notion of *structural properties*: “A property, S, is structural if and only if proper parts of particulars having S have some property or properties, T... not identical with S, and this state of affairs is, in part at least, constitutive of S” (Armstrong, 1978a, p. 69). This notion of structural properties is then later discussed by Armstrong (1986); Lewis (1986); Bigelow (1986); Forrest (1986); Bigelow and Pargetter (1989).

Lewis (1986) gives various conceptions of structural universals: the linguistic conception, the magical conception and the pictorial conception (and its variants). The linguistic conception asserts that a structural universal is “a set-theoretic construction out of simple universals, in just the way that a (parsed) linguistic expression can be taken as a set-theoretic construction out of its words” (Lewis, 1986, p. 31), which he rejects because “it presupposes simple universals and so fails to account for infinite complexity” (Fisher, 2018, p. 4). The magical conception asserts that a structural universal is a mereological atom (so it is not composed of other universals) that *involves* other universals. Lewis

<sup>4</sup>As explained in Footnote 6, Cotnoir first called it “idempotence”. However, in Cotnoir and Varzi (2021), the proposition is called “subpotence”.

<sup>5</sup>A part of the debate is about the very existence of such entities. However, this debate is not in the scope of my work. In the remainder, I will use present tense for brevity.



rejects this conception as it has an “unacceptable price in mystery” (Lewis, 1986, p. 42). As these two conceptions do not involve the parthood relation, I will not explore them further.

About the pictorial conception, Lewis says:

On the pictorial conception, a structural universal is isomorphic to its instances. The methane atom consists of one carbon atom and four hydrogen atoms, with the carbon bonded to each of the four hydrogens; the structural universal methane likewise consists of several parts, one for each of the five atoms, and one for each of the four bonds. (Lewis, 1986, p. 33)

I will take *isomorphism* in Lewis’ view to mean that if an instance of the universal  $X$  is composed of  $n$  instances of the universal  $Y$ , then  $X$  is composed  $n$  times of the universal  $Y$ . However, Lewis argues as follows:

Each methane molecule has not one hydrogen atom but four. So if the structural universal METHANE is to be an isomorph of the molecules that are its instances, it must have the universal HYDROGEN as a part not just once, but four times over. Likewise for BONDED, since each molecule has four bonded pair of atoms. But what can it mean for something to have a part four times over? What are there four of? There are not four of the universal HYDROGEN, or of the universal BONDED, there is only one. (Lewis, 1986, p. 34)

Lewis considered that something cannot be part of something else multiple times. Thus, he rejected the pictorial conception. He explored four variants of this pictorial conception. The fourth variant interestingly assumes the existence of entities he called *amphibians*. Amphibians are “like universals in the way they occur repeatedly” and “like particulars in the way they duplicate one another” (Lewis, 1986, p. 39). According to this conception, METHANE would contain four amphibians of HYDROGEN. Lewis rejected this variant, as amphibians seem too bizarre to be taken seriously.

Garbacz (2020) proposes an analysis of the debate over structural universals. As Lewis’ conceptions are sketchy, he proposes a formalisation for some of them: the pictorial conception, the magical conception and the amphibian variant of the pictorial conception, for which there are two formalisations. These formalisations are based on a mapping between the parts of an instance of the universal  $U$  and the parts of this universal.

Garbacz then gives the facets of structural universals, which seems accepted by all parties of the debate:

1. There are two disjoint domains of entities: individuals and universals, where the former instantiate the latter.
2. Universals are posited, rather than discovered, to serve some theoretical goal(s), for example [...] to explain various kinds of similarity between individuals [...]
3. The relation of parthood is primarily defined on individuals [...]
4. The relation of parthood or rather a counterpart thereof can be derivatively defined on universals [...] if individual  $x$  is part of individual  $y$  and if this mereological fact has to do with the universals  $x$  and  $y$  instantiate, say, respectively,  $V$  and  $U$ , then necessarily, every instance of  $V$  is part of some instance of  $U$ .<sup>6</sup>

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<sup>6</sup>The quantification and implication in the consequent seems in the wrong way, compared to Principle 2.2. Indeed, it concludes that every instance of  $V$  is part of some instance of  $U$ . But every instance of CARBON is not a part of an instance of METHANE. On the other hand, every instance of METHANE has an instance of CARBON as a part. I assume that the consequent should read “every instance of  $U$  has as part some instance of  $V$ ”.

5. If this account adequately represents the common assumptions of the debate in question, we are now in the position to provide a definition of structural universals. Here  $U$  is defined as a *structural universal* if and only if there exists (at least) one universal  $V$  such that the following principles are satisfied by  $U$  and  $V$ :

**Principle 2.1.** *It is possible that some instance of universal  $U$  has a part that is an instance of  $V$ .*

**Principle 2.2.** *If some instance of universal  $U$  has a part that is an instance of universal  $V$ , then necessarily, every instance of  $U$  has a part that is an instance of  $V$ .*

If universals  $U$  and  $V$  satisfy these principles, one can say that  $V$  is part, or better chunk, of  $U$ .

(Garbacz, 2020, pp. 9–10)

Garbacz’s Principle 2.2 is in line with my interpretation of Lewis’ isomorphism given above. Principle 2.2 asserts that as soon as an instance of  $U$  has a part that is an instance of  $V$ , it is the case for every instance of  $U$ . This may be problematic, as this view assumes that all instances of  $U$  have the same mereological structure. While, to some extent, this seems understandable for some entities, it might be problematic for others, like cats or bikes and the universals CAT or BIKE, as they can be tail-amputated cats, Manx cats (naturally tailless cats) or bikes with a pennant. The Isomorphism principle described here assumes that the instances used to understand the structure of the structural universals are typical.<sup>7</sup> The examples used here, structural universals of molecules or string types, are not problematic in this respect. Indeed, all their instances are similar, at least at the level of details considered: a methane molecule could contain a carbon-14 atom, while another one could contain a carbon-12 atom. Those two carbons have a different number of neutrons, like some cats have tail and others do not.

Garbacz gives the contestable aspects of the debate, which I will not elaborate here. He finishes his work with a classification of the theories of structural universals, some of which will be presented in Chapters 3 and 4.

Wetzel (2009) proposes another conception, called the *Occurrence Conception*. First introduced to be used on word types (see Section 2.6.2), Wetzel generalises occurrences to other entities like structural universals. This conception is more detailed in Section 2.6.2. Wetzel’s occurrences are similar to Lewis’ amphibians. However, they are not the same thing: Lewis presented the amphibians of HYDROGEN as being all “all alike” (Lewis, 1986, p. 39), but occurrences of HYDROGEN are different. Davis (2014) and Garbacz (2020) discuss further the notion of occurrences.

## 2.6.2 String Types

As I explained in the introduction, the distinction type/token is a useful philosophical tool. But what are such entities exactly? In this section, I present what letter and string types are and the problems of individuation they raise.

The identity criteria of words types are discussed by many scholars (Bromberger (2011); Hawthorne and Lepore (2011); Irmak (2019); Kaplan (1990, 2011); Lando (2019); McCulloch (1991); Wetzel (1993, 2009)). There are, at least, two conceptions of word types,

<sup>7</sup>“Typicality reflects how typical or representative an individual is with respect to a concept” (Yeung and Leung, 2006, pp. 98–99). Typicality has been considered as key to understand “functional parthood” and related transitivity issues (Vieu, 2006) and the quantification and implication form of the formula relating parthood among universals and parthood among their instances is addressed by Bittner et al. (2004).

based on the identity criterion used to individuate them. Kaplan (1990, 2011) argues that the same word type can be spelled and pronounced in different ways: the word type “COLOR” can be written as “colour” (in Canada and United Kingdom, for example) and “color” (in the United States of America); “SCHEDULE” is pronounced in two different ways. This conception is known as *Kaplanian Word Types* (Carrara and Smid, 2022b) or the *Common Currency Conception* (Kaplan, 1990).

The other conception, called the *Orthographic Conception*, bases the identity criterion of words types on their spelling: if two tokens are spelled differently, they cannot be instances of the same word type, contrary to the Kaplanian conception. This conception “suggests that letter types are parts of word types” (Carrara and Smid, 2022b, p. 178). Indeed, it only takes into account the spelling of word types,<sup>8</sup> and excludes semantic and phonological properties of words. Thus, this conception is more interesting to us as it challenges mereological extensionalism: consider two word types, given in introduction: “POTATO” and “ATOP”. As letter types are unique (there are no multiple letter types “A”), both of these word types have the following parts: “A”, “O”, “P”, “T”. Thus, by extensionality, they must be identical, but they are not.

Multiple formalisations of the mereological structure of string types have been given. Wetzel (2009) suggests that, in addition to types and tokens, there are also *occurrences*. Consider the string type “MACAVITY, MACAVITY, THERE’S NO ONE LIKE MACAVITY”. In this type, Wetzel says that the string type “MACAVITY” occurs three times: there are three occurrences of the string type “MACAVITY”. Wetzel gives three individuating parameters: *what* is occurring, *in what* is it occurring, and *where* is it occurring. From there, she defines string types as functions. For example, the function of the string type “ATOP” would be:  $0 \mapsto \text{“A”}$ ,  $1 \mapsto \text{“T”}$ ,  $2 \mapsto \text{“O”}$  and  $3 \mapsto \text{“P”}$ . A detailed analysis of Wetzel’s work is proposed by Davis (2014), who also gives a slightly different approach.

Carrara and Smid (2022b) suggests that string types are pluralities of mereological sums. Their theory is presented in Section 4.2. In this theory, they present three identity criteria “individually necessary and jointly sufficient for the identity of word types” (Carrara and Smid, 2022b, p. 6).

String types are informational entities, some of which are involved in information content entities, like words and sentences (Barton et al., 2020b). However, informational entities are not limited to strings nor words: narrative fictions, musical scores, images, process plans also are informational content entities. Sanfilippo (2021) gives a state of the art of what are informational entities. Their mereological structure has been described by Barton et al. (2020a,b, 2021, 2022), based on the Slot Mereology of Bennett, presented in Chapter 3.

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<sup>8</sup>As only the spelling is taken into account, we will use, in the remainder, “string types” to designate word types under the orthographic conception. As we will see, some of the string types used as examples are not English words.



# Chapter 3

## The Slot Mereology of Bennett

This chapter presents the theory proposed by Bennett (2013), later called “Slot Mereology” by Fisher (2013), based on the decomposition of the parthood relation into two relations.

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### 3.1 Theory Presentation

Bennett’s theory is motivated as an answer to Lewis’ critique of the isomorphic variant of the pictorial conception of structural universals (see Section 2.6.1). The idea of Bennett is inspired by the decomposition between a role and a role player. As she argues, besides structural universals, there are some cases where two entities are linked by the same relation multiple times. The examples she gives are the *cousin* relation and the *three feet from* relation. If two persons are double cousins (i.e. children of pairs of siblings), then they are cousins two times. And by considering a sphere such that the shortest distance along the surface between two antipodal points is three feet, then these two points are three feet from each other as many times as there are three-foot-long arcs on the surface.

### 3.1.1 Definitions

Instead of a single primitive, Slot Mereology considers that an entity is a part of another one by filling one of its slots. It uses two primitives *fills* ( $F$ ) and *slot-of* ( $P_s$ ). The former relates an entity to the slots it fills. The latter relates slots to their owners. On these two primitives, Bennett defines five relations Parthood, Proper Parthood<sup>1</sup>, Overlap, Slot-overlap and Proper Parthood Slot.

Definition 3.1 states that  $a$  is a part of  $b$  by filling some slot of  $b$ .

**Definition 3.1** (Parthood).

$$P(a, b) \triangleq \exists s(P_s(s, b) \wedge F(a, s))$$

Definition 3.2 states that  $a$  is a proper part of  $b$  iff  $a$  is a part of  $b$  and  $b$  is not a part of  $a$ .

**Definition 3.2** (Proper Parthood).

$$PP(a, b) \triangleq P(a, b) \wedge \neg P(b, a)$$

Definition 3.3 states that  $a$  and  $b$  overlap by sharing a common part.

**Definition 3.3** (Overlap).

$$O(a, b) \triangleq \exists c(P(c, a) \wedge P(c, b))$$

Definition 3.4 states that  $a$  and  $b$  slot-overlap by sharing a common slot.

**Definition 3.4** (Slot-overlap).

$$O_s(a, b) \triangleq \exists s(P_s(s, a) \wedge P_s(s, b))$$

In Bennett's theory, two entities have two ways to overlap: either the same entity fills a slot of each entity, or the two entities own the same slot.

Finally, Definition 3.5 states that a slot  $s$  is a proper parthood slot (just "proper slot" in the following) of  $a$  iff  $s$  is owned but not filled by  $a$ .

**Definition 3.5** (Proper Parthood Slot).

$$PP_s(s, a) \triangleq P_s(s, a) \wedge \neg F(a, s)$$

### 3.1.2 Axioms

Axioms 3.1 to 3.3 state the difference between slots and fillers: a slot is something that is filled and owned, and it cannot own slots nor fill them.

**Axiom 3.1** (Only Slots are Filled).

$$F(a, s) \rightarrow \exists b(P_s(s, b))$$

**Axiom 3.2** (Slots Cannot Fill).

$$F(a, s) \rightarrow \neg \exists b(P_s(a, b))$$

---

<sup>1</sup>Bennett's definition of proper parthood varies from the one presented in Definition 2.1. In ground mereology, these two definitions are equivalent as far as Axiom 2.3 holds. For more information about their differences, see (Cotnoir and Varzi, 2021, pp. 65–68) and (Carrara and Smid, 2022a).

**Axiom 3.3** (Slots Don't Have Slots).

$$P_s(s, a) \rightarrow \neg \exists t(P_s(t, s))$$

Axiom 3.4 states that wholes have improper slots, that is slots filled by their owner.

**Axiom 3.4** (Improper Parthood Slots).

$$\exists s(P_s(s, a)) \rightarrow \exists t(P_s(t, a) \wedge F(a, t))$$

Axiom 3.5 states that slots of parts are inherited.

**Axiom 3.5** (Slot Inheritance).

$$[P_s(s, b) \wedge F(a, s) \wedge P_s(t, a)] \rightarrow P_s(t, b)$$

Axiom 3.6 states that if two fillers occupy each other slots, they are equal.

**Axiom 3.6** (Mutual Occupancy is Identity).

$$(P_s(s, b) \wedge F(a, s)) \wedge (P_s(t, a) \wedge F(b, t)) \rightarrow a = b$$

Axiom 3.7<sup>2</sup> states that a slot has only one filler.

**Axiom 3.7** (Single Occupancy).

$$P_s(s, a) \rightarrow \exists! b(F(b, s))$$

Axiom 3.8 states that if two entities  $a$  and  $b$  have a slot and  $b$  is not a part of  $a$ , then  $b$  has a slot that  $a$  does not own.

**Axiom 3.8** (Slot Strong Supplementation).

$$\exists s(P_s(s, a)) \wedge \exists t(P_s(t, b)) \rightarrow [\neg(\exists u(P_s(u, a) \wedge F(b, u))) \rightarrow \exists v(P_s(v, b) \wedge \neg P_s(v, a))]$$

### 3.1.3 Theorems

Theorems 3.1 to 3.6 show that  $F$  and  $P_s$  are irreflexive, asymmetric and transitive, which make them strict order relations.

**Theorem 3.1** (Filler-Irreflexivity).

$$\forall a(\neg F(a, a))$$

**Theorem 3.2** (Filler-Asymmetry).

$$\forall a, s(F(a, s) \rightarrow \neg F(s, a))$$

**Theorem 3.3** (Filler-Transitivity).

$$\forall a, b, c(F(a, b) \wedge F(b, c) \rightarrow F(a, c))$$

**Theorem 3.4** (Slot-Irreflexivity).

$$\forall s(\neg P_s(s, s))$$

---

<sup>2</sup>Following Garbacz (2016), I have corrected the typographical mistake in Bennett's axiom stating as a consequent  $\exists! b(F(s, b))$ . This cannot be as  $s$  is a slot, and therefore, it cannot fill.

**Theorem 3.5** (Slot-Asymmetry).

$$\forall a, s(P_s(s, a) \rightarrow \neg P_s(a, s))$$

**Theorem 3.6** (Slot-Transitivity).

$$\forall s, t, u(P_s(s, t) \wedge P_s(t, u) \rightarrow P_s(s, u))$$

Theorems 3.7 to 3.9 show that the parthood relation, defined by Definition 3.1, is transitive, anti-symmetric and (conditionally) reflexive, as expected from a parthood relation.

**Theorem 3.7** (Transitivity).

$$\forall a, b, c(P(a, b) \wedge P(b, c) \rightarrow P(a, c))$$

**Theorem 3.8** (Anti-Symmetry).

$$\forall a, b(P(a, b) \wedge P(b, a) \rightarrow a = b)$$

**Theorem 3.9** (Conditional Reflexivity).

$$\forall a(\exists s(P_s(s, a)) \rightarrow P(a, a))$$

Theorem 3.10 (resp. Theorem 3.11) states that anything that has a (resp. proper) part has a (resp. proper) slot. Both theorems originally contain a typographical mistake. The theorems are introduced as “everything that has a part has a parthood slot, and *vice versa* (ditto for proper parts)” (Bennett, 2013, p. 94).

**Theorem 3.10** (Parts  $\leftrightarrow$  Slots).

$$\forall a(\exists b(P(b, a)) \leftrightarrow \exists s(P_s(s, a)))$$

**Theorem 3.11** (Composites  $\leftrightarrow$  Slot-Composites).

$$\forall a(\exists b(PP(b, a)) \leftrightarrow \exists s(PP_s(s, a)))$$

Theorem 3.12 states that if  $a$  is a proper part of  $b$ , then there is a slot owned by  $b$ , but not by  $a$ . Bennett intended it as a slot version of Weak Supplementation (see Axiom 2.6). However, as presented in Section 3.2.1, Bennett’s theory makes the similarity with Weak Supplementation “viciously superficial” (Garbacz, 2016, p. 173).

**Theorem 3.12** (Slot Weak Supplementation).

$$\forall a, b(PP(a, b) \rightarrow \exists s(P_s(s, b) \wedge \neg P_s(s, a)))$$

Finally, Bennett introduces Theorem 3.13<sup>3</sup> as a slot version of PP-Extensionality. However, as showed by Garbacz (see Section 3.2.1), this is a not a theorem of the theory.

**Theorem 3.13** (Slot Extensionality).

$$\forall a, b[\exists c(PP(c, a)) \vee \exists c(PP(c, b)) \rightarrow (a = b \leftrightarrow \forall s(PP_s(s, a) \leftrightarrow PP_s(s, b)))]$$

---

<sup>3</sup>Following Garbacz (2016), I have corrected the the last quantifier, which was mistyped:  $\exists$  instead of  $\forall$ .



### 3.1.4 About Composition

In her theory, Bennett did not propose any sum operator. However, she made three remarks about sums.

First, she noted that a fully unrestricted axiom of unrestricted sum is not reasonable. Indeed, she argues that in her theory, slots cannot be parts of anything, so there cannot be sums involving slots.

Second, slots being put aside, she argues that sums of fillers cannot be unique. Bennett gives an example of two things  $a$  and  $b$  that might have multiple sums:  $a+b$ ,  $a+a+b$ . There are entities that have the same parts but not the same number of them. For example, METHANE and ETHANE ( $CH_4$  and  $C_2H_6$ ) have the same atom universals as parts, but in different quantities. Moreover, keeping our molecule universal examples, we could think of molecules that are not bonded in the same way. As pointed out by Lewis (1986) and McFarland (2018), butane and isobutane have the same chemical formula, i.e. the same parts ( $C_4H_{10}$ ), but are isomers, i.e. their parts are arranged in different ways. If a sum of fillers were to exist, it cannot be idempotent, as it would make METHANE and ETHANE the same entity. Nonetheless, even a non-idempotent sum would not be enough as it would not enable us to differentiate BUTANE and ISOBUTANE universals.

Third, she argues that a non-idempotent sum of fillers would entail that the world is junky (i.e. everything is a proper part of something else; see (Cotnoir and Varzi, 2021, pp. 220–229)), and thus, there is no universe. This is in opposition with classical mereology, whose unrestricted composition makes junks impossible (see Bohn (2009); Schaffer (2010); Cotnoir (2014)). She gives the following example:

“To see this, imagine a world with two simple fillers,  $a$  and  $b$ . Almost unrestricted composition entails that they have a fusion; call it  $c$ . But if *every* two or more things have a fusion, then every composite must fuse with each and all of its own proper parts. So  $c$  and  $a$  must compose something, as must  $c$  and  $b$ , and  $c$  and  $a$  and  $b$  (also known as  $d$ ).” (Bennett, 2013, p. 99)

### 3.1.5 Some Examples

#### 3.1.5.1 The Structural Universal Methane

Any methane molecule particular is composed of five atoms: one carbon and four hydrogens, each bound to the carbon atom. Moreover, any carbon atom particular has six electrons, and each hydrogen atom particular has one electron.<sup>4</sup> Using slot mereology, the universal METHANE is described as having five proper slots, one for the CARBON universal and four for the HYDROGEN universal. The current structure is described by Facts (3.1)<sup>5</sup> and pictured in Figure 3.1, where  $a \leftarrow \square^s$  represents  $P_s(s, a)$ ,  $\boxed{a}^s$  represents  $F(a, s)$  and  $\boxed{a}^s \circlearrowleft$  represents  $P_s(s, a) \wedge F(a, s)$  (the same notation will be used in following figures). Note that Axiom 3.4 entails that  $S_0$  exists and is an improper slot of METHANE.

$$\begin{array}{ll} P_s(S_i, \text{METHANE}) & 0 \leq i \leq 5 \\ F(\text{METHANE}, S_0) & F(\text{HYDROGEN}, S_i) \quad 2 \leq i \leq 5 \\ & F(\text{CARBON}, S_1) \end{array} \quad (3.1)$$

Let us represent the electrons of each atom. Each carbon atom has six electrons, and each hydrogen atom has one. So, in our case, the universal CARBON owns six slots filled

<sup>4</sup>Note that atoms also have other parts, such as nuclei, but for simplicity, we only represent electrons in this example. The reasoning we will develop for electrons also applies to other parts such as nuclei. Furthermore, the specific problems of identity raised by quantum mechanics are ignored (see (French, 2019)).

<sup>5</sup>The formulas that use index  $i$  are compact notations for several formulas.

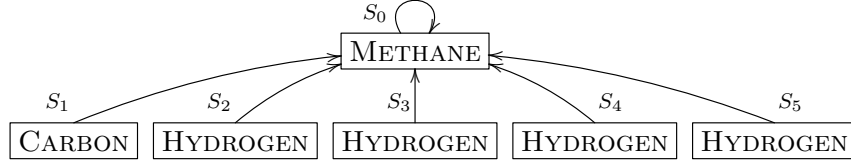


Figure 3.1: Representation of METHANE and its parts, CARBON and HYDROGEN

with ELECTRON, whereas the universal HYDROGEN owns only one, as described by Facts (3.2).

$$\begin{aligned} F(\text{ELECTRON}, S_{12}) \wedge P_s(S_{12}, \text{HYDROGEN}) \\ F(\text{ELECTRON}, S_i) \wedge P_s(S_i, \text{CARBON}) \quad 6 \leq i \leq 11 \end{aligned} \quad (3.2)$$

One thing to keep in mind is the fact that even if there are four slots of METHANE filled by HYDROGEN, there is only one universal of HYDROGEN, which has only one slot filled by ELECTRON. Bennett's Axiom 3.5 states that wholes inherit slots from their parts. In our case, this means that METHANE inherits from CARBON and HYDROGEN their slots filled by ELECTRON. METHANE inherits six slots from CARBON and only one slot from HYDROGEN. Considering that METHANE has no other slots filled by ELECTRON, METHANE has in total seven slots filled by ELECTRON.

According to Axioms 3.4 and 3.5, there are two additional slots, called  $S_{13}$  and  $S_{14}$ , that are improper slots of CARBON and HYDROGEN, respectively (see Facts (3.3)). For readability, we do not present the full mereological structure of METHANE here. Nonetheless, other examples are presented, with a full representation of the mereological structure, in Chapter 10.

$$\begin{aligned} P_s(S_{13}, \text{CARBON}) \quad P_s(S_{14}, \text{HYDROGEN}) \\ F(\text{CARBON}, S_{13}) \quad F(\text{HYDROGEN}, S_{14}) \end{aligned} \quad (3.3)$$

Regarding improper slots  $S_{13}$  and  $S_{14}$ , there are two possibilities: either those two slots are different from the ones previously mentioned, or some of them are identical to some of the previously mentioned slots. In this example, we chose the first possibility:  $S_{13}$  and  $S_{14}$  are different from all the other slots. All the slots of CARBON and HYDROGEN are inherited by METHANE, due to Axiom 3.5:  $P_s(S_i, \text{METHANE})$  ( $6 \leq i \leq 14$ ). Note that it is also the case for improper slots.

### 3.1.5.2 The Structural Universal TwoAdjoiningRooms

Consider the universal whose instances are two two-wall rooms<sup>6</sup> separated by an adjoining wall. We consider the universals WALL, ROOM and TWOADJOININGROOMS. The first one is atomic and only has an improper slot  $S_{\text{WALL}}$ . The second one has an improper slot  $S_{\text{ROOM}}$  and two slots  $S_1$  and  $S_2$  filled by WALL. Finally, the last one has an improper slot  $S_{\text{TAR}}$  and two slots  $S_3$  and  $S_4$  filled by ROOM. The mereological structures are pictured in Figure 3.2 where only improper slots and direct slots are represented.

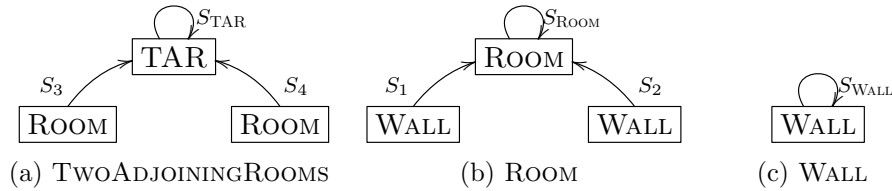


Figure 3.2: Direct slots of TwoAdjoiningRooms (TAR), ROOM and WALL

<sup>6</sup>Although our world does not abound with two-wall rooms, this makes for a simple example. One may imagine a semi-circular wall closed by a straight wall.

## 3.2 Critiques from the Literature

Various critiques, more or less developed, exist about Bennett’s theory. In the following sections, I present and analyze three of them, i.e. Garbacz’s, Fisher’s and Cotnoir’s.

### 3.2.1 Garbacz’s Critique

Garbacz (2016) shows why Bennett’s proof of Theorem 3.12 based on Axiom 3.8 is invalid: the premise of Theorem 3.12 ( $PP(a, b)$ ) does not imply the premises of Axiom 3.8 ( $\exists s(P_s(s, a)) \wedge \exists s(P_s(s, b))$ ). Indeed,  $a$  being a proper part of  $b$  implies that  $b$  has a slot, but tells nothing about  $a$  having a slot. Garbacz thus offers two solutions: weaken 3.8 or introduce a new axiom that states that every filler has a slot, which can be improper.

Garbacz notes that even though Bennett’s proof is not correct, Theorem 3.12 still is a theorem of the theory. Indeed, the Slot Weak Supplementation Theorem states that “if  $a$  is a proper part of  $b$ , then  $b$  has a parthood slot  $s$  that isn’t a parthood slot of  $a$ ” (Bennett, 2013, p. 97). As  $a$  is a proper part of  $b$ , it occupies one of the slots of  $b$ . According to Axiom 3.4,  $b$  has an improper slot (let’s call it  $t$ ). Slot  $t$  cannot be owned by  $a$ , otherwise, by Axiom 3.6 (Mutual Occupancy Is Identity),  $a$  and  $b$  would be equal, which would contradict the fact that  $a$  is a proper part of  $b$ . So, in every case, Slot Weak Supplementation is satisfied thanks to the existence of the improper slot of  $b$ . Note that if the slot  $s$  of  $b$  filled by  $a$  is not an improper slot of  $a$  (that is, it is not owned by  $a$ ), it is true that  $P_s(s, b) \wedge \neg P_s(s, a)$ , and thus the Slot Weak Supplementation is satisfied. As Garbacz noted, “instead of being a form of supplementation, [Theorem 3.12] is a consequence of the specific form of the reflexivity of slot parthood: each object that has slots fills a slot in itself”.

Moreover, Garbacz (2016) showed that Theorem 3.13 is not a theorem of Bennett’s theory. Indeed, Garbacz gives a model of the theory, pictured in Figure 3.3, in which all Bennett’s axioms are satisfied, but not Theorem 3.13. Consequently, Garbacz proposed three revisions of Axiom 3.8 to be able to prove Theorem 3.13.

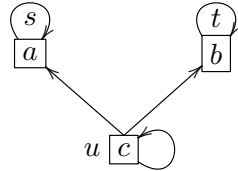


Figure 3.3: Garbacz’s counter-example

Garbacz’s final revision of 3.8 is exposed in (3.4). Intuitively, it means that if  $b$  has a proper slot and  $b$  is not a part of  $a$ , then there is a proper slot of  $b$  that is not a proper slot of  $a$ .

$$\forall a, b[\exists s(PP_s(s, b)) \rightarrow (\neg\exists t(P_s(t, a) \wedge F(b, t)) \rightarrow \exists u(PP_s(u, b) \wedge \neg PP_s(u, a))] \quad (3.4)$$

### 3.2.2 Fisher’s Critique

Fisher (2013, 2018) analyses the ontological cost of Bennett’s slot mereology and explains why slot mereology cannot account for the composition of structural universals.

Fisher first compares the ideological costs of slot mereology and CEM, using the number of primitives as the comparison criterion: the fewer primitives there are, the cheaper it is. Thus, his first argument is the following:

“In comparing CEM with slot-mereology, it follows that (*ceteris paribus*) the extra primitive is a *prima facie* reason to reject slot-mereology. If we do not

have an argument that defeats this *prima facie* reason, we ought to reject slot-mereology.” (Fisher, 2013, p. 759)

In Fisher’s argument, I think that the “*ceteris paribus*” is of great importance. Indeed, CEM and slot mereology are not equal, besides the number of primitives: slot mereology has more expressive power than CEM. Even though, as we have seen in Section 3.1.4, slot mereology cannot account for the difference between BUTANE and ISOBUTANE universals, it can account for METHANE and ETHANE universals, whereas CEM cannot. Therefore, it seems that the *ceteris paribus* clause is wrong here. So the extra primitive is not a *prima facie* reason to reject slot mereology, contrary to what Fisher argues.

The next Fisher’s argument I want to discuss about is his objection based on Theorem 3.13. His argument is the following:

“Notice that *being butane* has *being hydrogen* ten times over, *being carbon* four times over, and *bonded* thirteen times over. It therefore has twenty-seven parthood slots, each filled by its respective universal. But *being isobutane* also has *being hydrogen* ten times over, *being carbon* four times over, and *bonded thirteen* times over. Therefore, it also has twenty-seven parthood slots. Given *Slot Extensionality* ([Theorem 3.13]), ‘composite objects are identical just in case they have exactly the same proper parthood slots’ [(Bennett, 2013, p. 97)]. Therefore, *being isobutane* and *being butane* are identical. [...] according to slot-mereology they are identical given that they have the same number of slots.” (Fisher, 2013, p. 760)

Besides the fact that Slot Extensionality is not a theorem of Bennett’s slot mereology (see Section 3.2.1), it seems that Fisher used a stronger version of the theorem. Indeed, the theorem states that “composite objects are identical just in case they have exactly the same proper parthood slots” (Bennett, 2013, p. 97). However, Fisher seems to use a version that states that composite objects are identical just in case that, for every existing filler, they have the same number of slots filled by this filler. According to Bennett’s Slot Extensionality, it would require, for BUTANE and ISOBUTANE to be identical, that both owns the exact same proper slots. Nonetheless, as already analysed by (Masolo and Vieu, 2018, p. 73), nothing enforces that BUTANE and ISOBUTANE have a common slot, besides inherited slots (see Section 3.3), let alone that they share all their proper slots. However, it is true that the mereological structures of BUTANE and ISOBUTANE are similar, not to say almost identical. I think this is an expected result, as the difference between them is a structural one, and not mereological one. They are different not because they have different parts, but because their parts are not bonded in the same way.

### 3.2.3 Cotnoir’s Critique

Cotnoir (2015) presents some remarks and open questions about Bennett’s theory. First, following Fisher (2013), he notes the ideological cost involved in ontological commitments to slots. In particular, he asks “[h]ow exactly are we to think of slots?” (Cotnoir, 2015, p. 435). Then he notes that Bennett does not provide any mereological sum or fusion on parts. Cotnoir’s remarks complete Bennett’s remarks on composition, presented in Section 3.1.4.

“Slots cannot be parts of slots; nor are they parts of the objects they are slots for. As a result, it is very difficult to tell when looking at some parts a and b what their sum should be. We need to know what slot-structure is present in the whole before we can determine what the relevant sum is. How many slots need to be filled? There are many possible answers to this question,

each of which determines at least one distinct object, but usually many more. Suppose, for example, the whole has three slots. Then there are six possible sums aab, aba, baa, abb, bab, and bba, assuming it matters to the identity of the object which part fills which slot. Even if we identify objects like aba and baa, we still fail to have it that mereological sum is unique (i.e. the extensionality of sums fails). Perhaps this is to be expected; but it would be nice to have a theory of composition that explained this. However, given the complexities of composition Bennett does not develop any theory of it. These are open questions a fully developed non-idempotent mereology would need to answer.” (Cotnoir, 2015, p. 436)

### 3.3 My Critique

#### 3.3.1 About the 8<sup>th</sup> Axiom

Bennett’s Axiom 3.8 is an adaptation of classical strong supplementation into slot mereology. It is used in the proofs of Slot Weak Supplementation (3.12) and Slot Extensionality (3.13). However, as we have seen above, Garbacz (2016) showed that this move has multiple problems, as 3.12 does not capture the idea of Weak Supplementation and 3.13 actually is not a theorem of Bennett’s theory. Therefore, Garbacz proposed a revision of the theory. We can even go further than Garbacz’s reasoning by noticing that Axiom 3.8 is actually a theorem of the theory. Here follows a proof that Axiom 3.8 is a theorem of Bennett’s theory.

⌊ *Proof.* Let  $a$  and  $b$  be two fillers such that both have a slot. Suppose that there are no slot owned by  $a$  and filled by  $b$ , i.e. suppose that  $b$  is not a part of  $a$ . Let us call this H1. We want to prove that there is some slot that is a slot of  $b$  but not a slot of  $a$ . From the assumptions,  $b$  has a slot. Thus, by Axiom 3.4, there is a slot  $u$  that is an improper slot of  $b$ . Suppose that  $u$  is owned by  $a$ : contradiction with H1. ⌋ Therefore,  $u$  is owned by  $b$ , but not by  $a$ . □

#### 3.3.2 Counting Problems

“Counting how many times filler A has filler B as a part” means counting the number of appearances of B in A. But what counts as a genuine appearance? As Bennett’s theory includes improper slots, we can define two counting criteria. The first counting criterion, **C1**, enables us to count the number of different slots owned by A that are filled by B, whether they are also owned by B (and are thus improper slots of B) or not. The second counting criterion, **C2**, enables us to count the number of different slots owned by A that are filled by B and that are not owned by B (that is, that are not improper slots of B). Note that it means that, according to **C2**, A is part of itself zero times, even though A has improper slots. Therefore **C2** is only relevant when counting proper parts.

The results obtained with the two criteria will be compared following the isomorphism principle proposed by Lewis (1986), presented in Section 2.6.1: a methane molecule has as parts one carbon atom particular, four hydrogen atom particulars and ten electron particulars.<sup>7</sup> Thus, we expect from a mereological theory adequate for non-idempotent

<sup>7</sup>As noticed by a reviewer of (Tarbouriech et al., 2024), in some common understanding of chemistry, only atoms are considered as bona fide parts of molecules, and electrons are only considered when analysing bonding between atoms. The chemistry examples of this work mainly serve to illustrate the formal theory, in particular parthood transitivity, and should not be considered as providing a detailed account of mereology in chemistry. If the ELECTRON universal is not to be considered as a part of molecule universals, further work might be needed to address parthood transitivity issues among universals, similarly as when applying classical mereology to represent parthood relations among concrete particulars (see, e.g., (Vieu, 2006)).

entities and counting criterion to lead to the methane molecule universal METHANE having as parts CARBON once, HYDROGEN four times and ELECTRON ten times.

Bennett’s theory raises two issues concerning the countability of parts: a first one stems from the existence of improper slots, and a second one from slots of parts. Ultimately, as we will see, both are caused by the slot inheritance axiom (Axiom 3.5).

### 3.3.2.1 Improper Slot Problem

Let us say we want to count the number of times HYDROGEN is part of METHANE. METHANE has five different slots filled with HYDROGEN: four direct slots<sup>8</sup> ( $S_2$  to  $S_5$ ) and one inherited slot ( $S_{14}$ ), the latter being the improper slot of HYDROGEN itself. HYDROGEN is part of METHANE five times according to **C1** and four times according to **C2**.<sup>9</sup> By comparing those results with the expected result when we count how many hydrogen particulars belong to a particular of methane molecule, we can state that criterion **C2** leads to a correct result, whereas criterion **C1** leads to an incorrect result.

Among the possible models of the slot mereology, Figure 3.4 illustrates three models worthy of interest. In these models,  $A$  and  $B$  are different and  $B$  is part of  $A$ . Here is an informal description of the three models:

- (a)  $A$  has only one slot  $S_1$  that is filled by  $B$  and this slot is not owned by  $B$ ;
- (b)  $A$  has only one slot  $S_1$  that is filled by  $B$  and this slot is also owned by  $B$  (and thus, is an improper slot of  $B$ );
- (c)  $A$  has exactly two different slots  $S_1$  and  $S_2$  that are filled by  $B$ . One of these slots (say  $S_2$ ) is also owned by  $B$ .

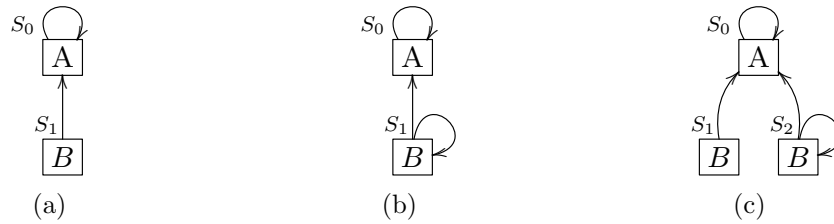


Figure 3.4: Three possible models of the slot mereology.

From the facts represented in Figure 3.4, we can deduce the following facts:

- in model (a), according to both **C1** and **C2**,  $B$  is part of  $A$  once;
- in model (b), by **C1**,  $B$  is part of  $A$  once. However, by **C2**,  $B$  is part of  $A$ , but zero times ;
- in model (c), by **C1**,  $B$  is part of  $A$  twice. However, by **C2**,  $B$  is part of  $A$  once.<sup>10</sup>

According to the counting criteria, we get different results for models (b) and (c). Since it is obviously absurd for  $B$  to be part of  $A$ , but zero times, **C2** is inappropriate. Thus, we have shown that neither **C1** (by the METHANE example) nor **C2** (by examples of Figure 3.4) can be compatible with Bennett’s theory.

<sup>8</sup> I define direct slots for Bennett’s theory as in Definition 3.6.

**Definition 3.6** (Direct Slot).  $DP_s(s, a) \triangleq P_s(s, a) \wedge \neg \exists b (PP(b, a) \wedge P_s(s, b))$

<sup>9</sup>Note that different results from the ones presented are possible if a different representation of the slot structure of the methane molecule is used.

<sup>10</sup> $S_1$  and  $S_2$  are filled by the very same universal. In contrast to Bennett’s figures in which slots can be drawn one inside another, we chose to separate them, even if it implies to repeat the filler.

### 3.3.2.2 Parts of Parts Problem

The second problem stems from the parts of the parts. Let's say we want to count how many times ELECTRON is part of METHANE. If we do so, we will find six slots inherited from CARBON (namely  $S_6, S_7, S_8, S_9, S_{10}, S_{11}$ ) and one slot inherited from HYDROGEN (namely  $S_{12}$ ), that is, seven slots in total. **C1** would thus lead to METHANE having ELECTRON as a part seven times, whereas **C2** would lead to it having ELECTRON as a part at most seven times (depending on whether some of those slots also are improper slots of ELECTRON). This result is different from the expected result of ELECTRON being part of METHANE ten times.

### 3.3.2.3 Conclusion on the Two Counting Problems

These two problems are in fact caused by the same axiom of slot inheritance (Axiom 3.5), which i) makes improper slots inheritable and ii) does not make the subparts inheritable multiple times. Since this axiom was presumably introduced by Bennett to allow parthood transitivity, we will need, in the proposals that will be presented in Chapters 5, 8 and 9, to replace it by alternative axioms that do not lead to the same problems, while still ensuring parthood transitivity.

### 3.3.3 About the Overlap Relation

In her theory, Bennett defines two overlap relations: the overlap on fillers (Definition 3.3) and the overlap on slots (Definition 3.4). The first relation is the classic overlap relation of mereology. However, in the examples we are interested in, this relation is not that relevant. Indeed, if we consider the structural universals of organic molecules, all those universals are overlapping, because they all have as a part the universal of CARBON. The second relation introduced by Bennett, at first sight, seems more interesting: two fillers are slot-overlapping if they share a slot. However, the relevance of this relation is also questionable: remember that in Bennett's theory, the improper slots are also inherited. So, in our examples of universals of organic molecules, each of them inherits the improper slot of the CARBON universal. One could posit that the universal of CARBON does not have an improper slot. However, as soon as CARBON is assumed to have parts (for instance ELECTRON), it does have an improper slot. Therefore, all universals of organic molecules slot-overlap with each other. Once again, the problem here ultimately lies in the Slot Inheritance Axiom 3.5.

A second problem with Bennett's overlap is about model ambiguity. Consider the two strings "xyz" and "xyyz". The mereological structure pictured in Figure 3.5 is compatible with Bennett's theory, where X can be either "xyz" or "xyyz". "xy" and "yz" are overlapping in the general sense of having a common part (namely "y"). But from the mereological structure pictured on this figure, there is no way to know whether "xy" and "yz" are overlapping in the sense of sharing the same occurrence of "y" (as in the whole "xyz") or not (as in the whole "xyyz"). While a model in which there are two different slots filled by "y", one owned by "xy" and the other by "yz", can exclusively represent "xyyz", we cannot find a model exclusive to "xyz" that contains slots filled by "xy", "y", and "yz". The model pictured in Figure 3.5 seems to be such a model for "xyz", but it is ambiguous, as just showed.

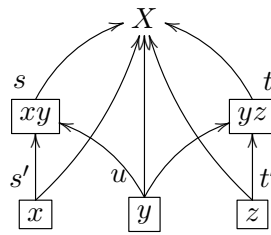


Figure 3.5: Bennett's slot-overlap (and overlap) between  $xy$  and  $yz$



# Chapter 4

## Alternative Theories

This chapter presents other theories existing in the literature to represent non-idempotent entities. They do not serve as the basis of the solutions I will develop in Part II. However, they provide other views on the representation of non-idempotent entities. Some existing theories that explore the representation of non-idempotent entities are not discussed in this section. Wetzel (2009) and Barton et al. (2020a,b, 2021, 2022) have been addressed in Section 2.6. Mormann (2012) and Masolo and Vieu (2018) use states of affairs and graph theory to represent mereological structures, which is too far from our aim to use first order logic.

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## 4.1 Cotnoir – Multisets

### 4.1.1 Theory Presentation

As Cotnoir (2015) writes it, his aim was “to explore and develop a mereology for which [subpotence] and *Parts Just Once* can fail.” (Cotnoir, 2015, p. 445). These principles have been presented in Section 2.5. Cotnoir proposes what he called a non-idempotent mereological theory, using multiset theory (see (Blizard, 1989) for more). Multisets are similar to sets, except that each element is associated to some multiplicity. A multiset  $A$  can be characterised by its multiplicity map  $m_A : D \rightarrow \mathbb{N}$ , where  $D$  is a class of multisets, such that  $m_A(x) > 0$  iff  $x \in A$ . The multiplicity of  $x$  in  $A$  is assumed to be unique. Note that multisets are extensional, as they are defined by the multiplicities of their elements.

Cotnoir defines parthood and proper parthood with Definitions 4.1.1 and 4.1.2. The parthood relation is reflexive, transitive and antisymmetric.

**Definition 4.1.1 (PART).**  $A \subseteq B$  iff  $m_A(x) \leq m_B(x)$ , for all  $x \in D$

**Definition 4.1.2** (PROPER PART).  $A \subset B$  iff  $A \subseteq B \wedge A \neq B$

Cotnoir also defines other notions of parthood, namely *whole parthood* and *full parthood*, and the notion of root with Definitions 4.1.3 to 4.1.5. For example, the mset  $\{a, a\}$  is a whole part of  $\{a, a, b\}$  and  $\{a, b\}$  is a full part of  $\{a, a, b\}$ . As noted by Cotnoir, “full parts are just parts that have the same root” (Cotnoir, 2015, p. 442). Cotnoir also gives an example with structural universals: “the structural universal WATER is a full part, but not a whole part, of the universal HYDROGEN PEROXIDE” (Cotnoir, 2015, p. 442). Presumably, Cotnoir considers that WATER contains HYDROGEN twice and OXYGEN once, while HYDROGEN PEROXIDE contains HYDROGEN and OXYGEN twice.

**Definition 4.1.3** (WHOLE PART).  $A$  is a whole part of  $B$  iff for all  $x \in D$ , if  $m_A(x) > 0$ , then  $m_A(x) = m_B(x)$

**Definition 4.1.4** (ROOT). The root of an mset  $A$  is the set  $A^* = \{x \in D \mid m_A(x) > 0\}$

**Definition 4.1.5** (FULL PART).  $A$  is a full part of  $B$  iff  $A \subseteq B$  and  $A^* = B^*$

Cotnoir proposes a first notion of composition, namely *sum*, with *product* as its dual, defined by Definitions 4.1.7 and 4.1.8. With this notion, he also defines overlap in a non-classical way, as presented by Definition 4.1.6. Indeed, using the classical definition of overlap based on the parthood relation, it would imply that everything overlaps with everything else, as every msets have the empty mset as a part.

**Definition 4.1.6** (OVERLAP).  $A$  and  $B$  overlap iff  $A \cap B \neq \emptyset$

**Definition 4.1.7** (SUM).  $A \cup B$  is the mset defined by  $m_{A \cup B}(x) = \max(m_A(x), m_B(x))$

**Definition 4.1.8** (PRODUCT).  $A \cap B$  is the mset defined by  $m_{A \cap B}(x) = \min(m_A(x), m_B(x))$

Cotnoir also introduces, with Definition 4.1.9, the notion of powerset as a way to have the exact multiplicities of each part. For example, the powerset of  $\{x, x, y\}$  is  $\{\emptyset, \{x\}, \{x, x\}, \{y\}, \{x, y\}, \{x, y, x\}, \{x, x, y\}\}$ . In this case,  $\{x\}$  being in the powerset twice means that  $\{x\}$  is part of  $\{x, x, y\}$  twice.

**Definition 4.1.9** (POWERMSET). The powerset of an mset  $X$ ,  $\tilde{\wp}(X)$ , is the multiset containing (multiplicities of) all parts of  $X$

Cotnoir introduces the notions of merge and complements with Definitions 4.1.10 and 4.1.11.

**Definition 4.1.10** (MERGE).  $A \uplus B$  is the mset defined by  $m_{A \uplus B}(x) = m_A(x) + m_B(x)$

**Definition 4.1.11** (COMPLEMENTS). If  $A \not\subseteq B$  then  $A - B$  is the mset defined by  $m_{A - B}(x) = m_A(x) - m_{A \cap B}(x)$ , for all  $x \in D$

## 4.1.2 My Critique

Cotnoir’s aim was to “to explore and develop a mereology for which idempotence and *Parts Just Once* can fail.” (Cotnoir, 2015, p. 445), which he managed to achieve (see below). However, using multisets has some consequences that I would like to discuss.

#### 4.1.2.1 Mereological Relations

Cotnoir's theory provides a partial order relation that handles multiplicities defined on multisets. However, the definition of the parthood relation entails uncommon results:

- the empty mset being a subset of all msets, it is a part of everything, which is a generally unwanted result in mereological theories. However, Cotnoir provides a non-classical definition of overlap that takes this into account. Therefore, the overlap does not hold between any pair of msets.
- the structural universal WATER is a part of the structural universal HYDROGEN PEROXIDE.

In his paper, Cotnoir only states that WATER is part of HYDROGEN PEROXIDE, but he does not state whether OXYGEN and HYDROGEN are parts of WATER and HYDROGEN PEROXIDE. Is it possible to be a part and a member of the same mset? The structural universal WATER is the mset  $\{H, H, O\}$ .<sup>1</sup> To be a part is to be a subset. The subsets of WATER are  $\emptyset, \{H\}, \{H\}, \{O\}, \{H, H\}, \{H, O\}, \{H, O\}, \{H, H, O\}$ . Therefore, if H and O are subsets of WATER, each of them must be identical with one of the members of the powerset. The mset  $\{H, H, O\}$  is WATER itself. Furthermore, the multiset theory possesses an *axiom of foundation* (see (Blizard, 1989, pp. 47–48)) that rules out the cases where msets are members of themselves and where pairs of msets are mutually members of each other. Therefore  $\{H\}$ ,  $\{O\}$ ,  $\{H, H\}$  and  $\{H, O\}$  are not identical to H and O. Finally, the only mset remaining among the ones cited above is the empty set. But if both HYDROGEN and OXYGEN are unintuitively the empty set, they are equal, which is an unwanted conclusion. Therefore, it is not possible for HYDROGEN and OXYGEN to be both members and parts of WATER.

If HYDROGEN and OXYGEN are not parts of WATER, Cotnoir's mereology states that WATER is part of HYDROGEN PEROXIDE, which is too far from what is commonly expected from a mereological theory. Finally, maybe the problem comes from the definition of parthood.

Let us suppose that HYDROGEN and OXYGEN are not members of WATER. What could be the members of these three msets? One possible formalisation is to consider that there is some  $o$  and  $h$  such that OXYGEN is  $\{o\}$ , HYDROGEN is  $\{h\}$  and WATER is  $\{h, h, o\}$ . This way, HYDROGEN and OXYGEN are parts of WATER. However, this raises the question of the nature of  $h$  and  $o$ . Furthermore, Cotnoir states that the members of an mset are msets. Thus, this raises the question of what would be the nature of the elements of  $h$  and  $o$ , and how to avoid infinite regression without inducing unwanted chemical facts: if HYDROGEN contains the empty mset  $n$  times and OXYGEN contains the empty mset  $m$  times, one is part of the other.

A possible solution to make compatible being parts and members is to define being part as being member. This way, HYDROGEN and OXYGEN would be parts of WATER and WATER would not be a part of HYDROGEN PEROXIDE. However, while the subset relation is a partial order (reflexive, antisymmetric and transitive), it is not the case for the member relation, which is irreflexive, asymmetric and non-transitive. Thus, the aim to obtain a theory similar to ground mereology, i.e. based on a partial order, is not reached.

#### 4.1.2.2 Multiplicities

In Cotnoir's theory, counting the multiplicities of parts can be done using the powerset, i.e. the mset containing all subsets of an mset. Consider the example of the structural

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<sup>1</sup>The chemical symbols H and O are used for brevity while expressing msets. They respectively refer to HYDROGEN and OXYGEN.

universal TWOADJOININGROOMS presented in Section 3.1.5.2. Using Cotnoir’s theory, the structural universal ROOM might be represented by the mset  $\{W, W, W, W\}$  where  $W$  is the universal WALL.<sup>2</sup> Similarly, the universal TWOADJOININGROOMS might be represented by the mset  $\{W, W, W, W, W, W, W\}$ , containing  $W$  seven times as there is a common wall. The powerset of TWOADJOININGROOMS contains  $\{W, W, W, W\}$  35 times.<sup>3</sup> In other words, ROOM is a part of TWOADJOININGROOMS 35 times, instead of the expected two times. Finding a representation that would avoid this problem is a challenge for Cotnoir’s theory.

#### 4.1.2.3 Decomposition Principles

Msets are extensional, i.e. if two msets have exactly the same members with the same multiplicities, they are the same mset. From this, one could deduce mereological extensionality: two msets that have the same parts with the same multiplicities have the same powerset, and two msets that have the same powerset are identical. Therefore, two msets that have the same parts with the same multiplicities are identical. However, as mentioned in Section 2.6.1, some structural universals have the same parts and are different, like BUTANE and ISOBUTANE. Thus, it is not clear how the theory could be extended to handle such cases.

Cotnoir introduces a notion of complement. However, as he mentions, there are a lot of counter-intuitive and non-classical results: the sum of an mset and its complement does not necessarily equal the universe, and their product is not necessarily the empty set (i.e. an mset and its complement can overlap). We can also add to that that an mset can be its own complement. For example, the complement of  $\{a\}$  in  $\{a, a\}$  is  $\{a\}$  itself.

#### 4.1.2.4 Composition Principles

Let us see how Cotnoir’s theory deals with providing some composition operations that handle multiplicities. Consider once again the example of the universal TWOADJOININGROOMS. It is the mset  $\{W, W, W, W, W, W, W\}$ , containing  $W$  seven times, as there is a common wall. Of the operations proposed by Cotnoir, the sum  $\text{ROOM} \cup \text{ROOM}$  is equal to  $\{W, W, W, W\}$  (i.e. ROOM itself) and the merge  $\text{ROOM} \oplus \text{ROOM}$  is  $\{W, W, W, W, W, W, W, W, W\}$  (i.e. the mset containing WALL eight times). Therefore, there are no operations in Cotnoir’s theory that make possible a composition with partial overlap.

#### 4.1.2.5 Conclusion On Cotnoir’s Theory

For the reasons explained in the above sections, Cotnoir’s theory does not seem to be an adequate solution to represent non-idempotent entities. While Bennett’s theory has some counting problems, it is axiomatised in first-order logic, unlike Cotnoir’s mereology. Cotnoir admits the lack of axiomatisation as a current limitation of his theory. As first-order logic is more compatible with standard works in applied ontology, Bennett’s theory seems to be a better choice.

<sup>2</sup>Taking into account the results of the previous discussion, ROOM could be represented by  $\{w, w, w, w\}$  and WALL by  $\{w\}$ , but this would not change the argumentation developed here.

<sup>3</sup>Cotnoir gives the following instructions to obtain the powerset: “first imagine distinguishing between each individual ‘instance’ of a given element in X; second, take the classical powerset; and third, undo all the ‘distinctions’ you made in step one.” (Cotnoir, 2015, p. 442). If we distinguish each ‘instance’ with a subscript number, TWOADJOININGROOMS equals  $\{W_1, W_2, W_3, W_4, W_5, W_6, W_7\}$ . The number of sets of size 4 is  $\binom{7}{4}$ , i.e. 35.

## 4.2 Carrara and Smid – Plural Logic

### 4.2.1 Theory Presentation

Carrara and Smid (2022b) claim that the two main controversies of mereology are about the condition of existence and the uniqueness of fusions. The latter is about the extensionality of parthood (see Section 2.3.3 for more). As explained in Section 2.6.2, word types challenge classical mereology. In their article, they “investigate an approach to word types that is compatible with classical mereology, and in particular with extensionality” (Carrara and Smid, 2022b, p. 3). As alternative proposals, such as the ones of Bennett (2013) or Cotnoir (2015) we have just seen, they propose a theory compatible with classical mereology. To do so, they use *plural logic* (see Yi (2005, 2006); Oliver and Smiley (2016) for more about plural logics).

Carrara and Smid use the overlap relation<sup>4</sup> as a primitive for classical theory, and define parthood, proper parthood and fusion as in Definitions 4.2.1 to 4.2.3.  $x$  is a part of  $y$  if everything that overlaps with  $x$  overlaps with  $y$ . Proper parthood is defined as in Definition 2.1. Finally,  $y$  is a fusion of the  $\phi$ -ers iff for every entity, it overlaps with  $y$  iff it overlaps with a  $\phi$ -er, i.e. this is Goodman fusion (see Definition Schema 2.10). The fusion of two things (the binary sum) is noted  $x + y$ .

**Definition 4.2.1** (Parthood).

$$P(x, y) \triangleq \forall z(O(z, x) \rightarrow O(z, y))$$

**Definition 4.2.2** (Proper Parthood).

$$PP(x, y) \triangleq P(x, y) \wedge \neg P(y, x)$$

**Definition 4.2.3** (Fusion).

$$F(y, \phi) \triangleq \forall z(O(z, y) \leftrightarrow \exists x(\phi(x) \wedge O(z, x)))$$

With these three definitions, they use Axioms 4.2.1 to 4.2.3. The first one posits the equivalence between overlap and having a common part. The second states that if there is some  $\phi$ -er, then the fusion of  $\phi$ -ers exists. Finally, the last one is equivalent to the O-Extensionality, that is, two entities are identical if every entity that overlaps with one overlaps with the other one.

**Axiom 4.2.1** (Shared Part).

$$\forall x, y(O(x, y) \leftrightarrow \exists z(P(z, x) \wedge P(z, y)))$$

**Axiom 4.2.2** (Unrestricted Composition).

$$\exists x(\phi(x)) \rightarrow \exists y(F(y, \phi))$$

**Axiom 4.2.3** (Extensionality).

$$\forall x, y(\forall z(O(z, x) \leftrightarrow O(z, y)) \rightarrow x = y)$$

Carrara and Smid also use some basic plural logic, which is based on the relation  $\preceq$ , where  $x \preceq X$  is read “ $x$  is one of the  $X$ ’s”. Capital letters refer to pluralities. They use Axioms 4.2.4 to 4.2.6 as axioms. The first axiom is an extensionality principle. The second one states that every plurality has some member. Finally, the third one states

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<sup>4</sup>Carrara and Smid use infix operators as relations. I replace the infix operators by (binary) predicates for homogeneity.

that if there is some  $\phi$ -er, then there is a plurality that contains all the  $\phi$ -ers, and only them. Axiom 4.2.6 is syntactically close from Axiom 4.2.2. The latter states that if a  $\phi$ -er exists, then the fusion of the  $\phi$ -ers exists. The former states that if a  $\phi$ -er exists, then the plurality whose members are  $\phi$ -ers exists. The difference is that in the plurality, there are  $\phi$ -ers and only  $\phi$ -ers, while in the fusion of the  $\phi$ -ers, there are also parts of  $\phi$ -ers (if some of them are not mereological atoms) and sums of  $\phi$ -ers.

**Axiom 4.2.4** (Plural Identity).

$$\forall X, Y (X = Y \leftrightarrow \forall z (z \preceq X \leftrightarrow z \preceq Y))$$

**Axiom 4.2.5** (Not Empty).

$$\forall X (\exists x (x \preceq X))$$

**Axiom 4.2.6** (Comprehension).

$$\exists x (\phi x) \rightarrow \exists Y (\forall x (x \preceq Y \leftrightarrow \phi x))$$

From their analysis of word types identity, Carrara and Smid define three identity conditions, namely Kind, Number and Order. They take these principles' logical inverses to be "individually necessary and jointly sufficient for the identity of word types" (Carrara and Smid, 2022b, p. 6).

**(Kind)** *If two words do not have all the same letters, they are distinct.*

**(Number)** *If the number of letter instances in one word is different from the number of letter instances in another, they are distinct.*

**(Order)** *If the order of letters in one word is different from the order of letters in another word, they are distinct.*

From there, they define word types as a pluralities of fusions of a letter type (the plurality of all letter types is  $\mathcal{A}$ ) and an integer (the plurality of integers is  $\mathbb{N}$ ) to represent the position of the letter in the word type.<sup>5</sup> For example, the word type 'LATTE' is the plurality of 'L'+1, 'A'+2, 'T'+3, 'T'+4 and 'E'+5, and 'LATE' is 'L'+1, 'A'+2, 'T'+3 and 'E'+4. They also define the two relations of  $\mathbb{N}$ -overlap and  $\mathcal{A}$ -overlap, respectively noted  $O_{\mathbb{N}}$  and  $O_{\mathcal{A}}$ , as is Definition 4.2.4. In other words, two entities  $\mathbb{N}$ -overlap (respectively  $\mathcal{A}$ -overlap) iff they have the same integer part (resp. the same letter-type part). Finally, they formulate the theorem of Identity Theorem which captures the identity criteria expressed by Kind, Number and Order.

**Definition 4.2.4** ( $\mathbb{N}$ - and  $\mathcal{A}$ -Overlap).  $\forall x_1, x_2 \preceq \mathbb{N}, y_1, y_2 \preceq \mathcal{A}$ , let  $z_1 = x_1 + y_1$  and  $z_2 = x_2 + y_2$ . Then

$$O_{\mathbb{N}}(z_1, z_2) \triangleq x_1 = x_2$$

$$O_{\mathcal{A}}(z_1, z_2) \triangleq y_1 = y_2$$

**Theorem 4.2.1** (Identity Theorem).

$$\begin{aligned} & [\forall z_1 \preceq W_1 \exists z_2 \preceq W_2 (O_{\mathbb{N}}(z_1, z_2) \wedge O_{\mathcal{A}}(z_1, z_2)) \wedge \\ & \forall z_2 \preceq W_2 \exists z_1 \preceq W_1 (O_{\mathbb{N}}(z_1, z_2) \wedge O_{\mathcal{A}}(z_1, z_2))] \\ & \leftrightarrow W_1 = W_2 \end{aligned}$$

---

<sup>5</sup>Said otherwise, word types are functions from a set  $X$  to a set  $Y$  such that there is a total ordering on  $X$ , and for each element of each element of  $X$  (or a subset of  $X$ ), it relates to a single element in  $Y$ , and if a member  $x$  of  $X$  relates to a member of  $Y$ , then all members of  $X$  that are smaller than  $x$  also relates to a member of  $Y$ .

While not having mereological relations between letter types and word types, Carrara and Smid successfully define a way to count the number of times a letter type is in a word type. This is defined with Definition 4.2.5. This definition states that for a letter type  $y$ , if there are only  $n$  distinct fusions in the word type that have  $y$  as their letter part, then  $n$  is the number of times the letter type  $y$  is in the word type.

**Definition 4.2.5** (Number of Letter Instances). *If  $\exists z_1, \dots, z_n \preceq W[(z_1 \neq z_2 \wedge z_1 \neq z_3 \wedge \dots \wedge z_{n-1} \neq z_n) \wedge y < z_1 \wedge (O_{\mathcal{A}}(z_1, z_2) \wedge O_{\mathcal{A}}(z_2, z_3) \wedge \dots \wedge O_{\mathcal{A}}(z_{n-1}, z_n)) \wedge \forall v \preceq W(P(y, v) \rightarrow v = z_1 \vee \dots \vee v = z_n)]$ , then  $n$  is the number of instances of the letter  $y$  in  $W$ , i.e.  $\#W^y = n$ .*

### 4.2.2 My Critique

Carrara and Smid do not define mereological relations between letter types and word types. However, such relations are definable: one could state that the letter type ‘A’ is a part of the word type ‘LATTE’ because it is a part of one of the fusions of the word type. But this is defining the parthood relation in an alternative way, while one of the requirements of Carrara and Smid was to keep classical mereology.

My biggest concern is that this theory seems adequate for word types (and more generally for entities whose structure is in one dimension), however, this does not seem easily extendable to represent other non-idempotent entities, such as universals of molecules.





**Part II**

**A Theory Proposal**



# Chapter 5

## Copy-Slot Mechanism

In this chapter, I present an adaptation of the *Copy-Slot Mechanism* presented in (Tarbouriech et al., 2021). This was a first attempt to fix counting problems.

This chapter contains three parts. The counting problems identified in sections Sections 3.3.2.1 and 3.3.2.2 will be respectively addressed in the first two sections. As a reminder, the first problem, called the **Improper Slot Problem**, comes from the inheritance of improper slots by entities other than their filler. The second problem, called the **Parts of Parts Problem**, comes from the inheritance of parts of parts in such a way that they are counted only once.

As it will be presented, these two counting problems will be identified. However, a new counting problem will emerge. This problem is described in Section 5.3. A solution will be proposed in Chapter 9.

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### 5.1 Constraining Improper Slots Further

Bennett does not state explicitly why she admitted in her theory the possibility for fillers to have improper slots. However, we can assume that they were introduced to satisfy conditional reflexivity (Theorem 3.9) as they are not used for any other purpose in Bennett’s paper. Hence, we could impose that an improper slot should not be owned by anything else than the filler it is an improper slot of, with Axiom 5.1.

**Axiom 5.1** (Improper Slots are only owned by their Filler).

$$\forall a, s (P_s(s, a) \wedge F(a, s) \rightarrow \forall b (P_s(s, b) \rightarrow a = b))$$

Using Axiom 5.1 and Definitions 3.1 and 3.2, we can deduce Theorem 5.1<sup>1</sup>, which states that a proper part of  $x$  is a part of  $x$  by filling a slot it does not own itself.

**Theorem 5.1** (Improper Slots Don’t Contain Proper Parts).

$$\forall x, y (PP(y, x) \rightarrow \exists s (P_s(s, x) \wedge F(y, s) \wedge \neg P_s(s, y)))$$

---

<sup>1</sup>In (Tarbouriech et al., 2021), there is a typographical mistake:  $s$  is quantified twice, once in  $\forall$  and once with  $\exists$ .

⌈ *Proof.* Let  $x$  and  $y$  be two fillers such that  $y$  is a proper part of  $x$ . By definition of proper parthood (Definition 3.2), we know that  $P(y, x) \wedge \neg P(x, y)$ , which leads to  $\exists s(P_s(s, x) \wedge F(y, s))$ .  
 To complete the proof, we need to show that  $s$  is not a slot of  $y$ . Suppose that  $s$  is a slot of  $y$ . We have  $P_s(s, y) \wedge F(y, s)$ , which, according to Axiom 5.1, leads to  $x = y$ . However, we know that  $\neg P(x, y)$ : contradiction. Therefore, we have  $\neg P_s(s, y)$ .  $\square$

This would make sure that every proper part of a filler fills a slot of this filler which it does not own. Therefore, models like model (b) in Figure 3.4 are excluded.

Axiom 5.1 and Bennett’s slot inheritance Axiom 3.5 and Definition 3.1 lead together to Formula 5.1, stating that every part which has an improper slot is identical to its whole, which is way too restrictive, and leads to trivial models only. For this reason, the slot inheritance axiom should be revised.

$$\forall x, y, s(P(y, x) \wedge P_s(s, y) \wedge F(y, s) \rightarrow x = y) \quad (5.1)$$

⌈ *Proof.* Let  $x$  and  $y$  be two fillers such that  $y$  is part of  $x$ . Let  $s$  be an improper slot of  $y$ . According to axiom (3.5),  $s$  is also a slot of  $x$ . Therefore, according to axiom (A5.1),  $x = y$ .  $\square$

We revise Axiom 3.5 by accepting instead that if  $x$  is a part of  $y$  and  $s$  is a proper slot of  $x$ , then  $s$  is also a slot of  $y$ . That is, we restrict slot inheritance to proper slots, by the following Axiom 5.2.

**Axiom 5.2** (Proper Slot Inheritance).

$$\forall a, b, s(P(a, b) \wedge P_s(s, a) \wedge \neg F(a, s) \rightarrow P_s(s, b))$$

The proof of parthood transitivity Theorem 3.7 relies on Axiom 3.5. As we replaced this axiom by Axiom 5.2, we need to prove that parthood is still transitive, which is done by Theorem 5.2:

**Theorem 5.2** (Parthood Transitivity).

$$\forall a, b, c(P(a, b) \wedge P(b, c) \rightarrow P(a, c))$$

⌈ *Proof.* Let  $a, b$  and  $c$  be three fillers such that  $a$  is a part of  $b$  and  $b$  is a part of  $c$ .  
 If  $a = b$  then  $P(a, c)$ . Let’s suppose now that  $a \neq b$ .  
 By definition of  $P$  (Definition 3.1), there is a slot  $s$  such that  $P_s(s, b) \wedge F(a, s)$ .  
 By unicity of the filler and  $a \neq b$ ,  $b$  does not fill  $s$ . Then, since  $b$  is a part of  $c$ , by Axiom 5.2  $s$  is a slot of  $c$ . Since  $a$  fills  $s$ ,  $a$  is a part of  $c$  by definition of parthood (Definition 3.1).  
 Thus, in all cases,  $a$  is a part of  $c$ .  $\square$

There are still two things to discuss: the generalisation of improper slots to all fillers, and the possibility for a filler to have multiple improper slots.

Bennett’s Axiom 3.4 ensures that anything that possesses a slot has an improper slot. Bennett justifies the conditional reflexivity by stating that “the reflexivity of parthood is restricted to things that have parthood slots. That’s because [Axiom 3.3] and the definition of parthood entail that parthood slots cannot have parts at all” (Bennett, 2013, p. 94). This certainly justifies why slots are not part of themselves, but it does not justify why fillers without slots are not part of themselves. We found no other justification for this. Garbacz (2016) made the same observation. We therefore generalize Bennett’s Axiom 3.4 by adding that every filler has an improper slot, in line with what Garbacz proposed in his axiom (GA9). We thus add Axiom 5.3 to the theory.

**Axiom 5.3** (Additional Improper Parthood Slots).

$$\forall a, s(F(a, s) \rightarrow \exists t(P_s(t, a) \wedge F(a, t)))$$

We can broaden the theorem of conditional reflexivity (3.9) as Theorem 5.3.

**Theorem 5.3** (General Conditional Reflexivity).

$$\forall a(\exists s(P_s(s, a) \vee F(a, s)) \rightarrow \text{part\_of}(a, a))$$

[*Proof.* This is a trivial consequence of Definition 3.1 and Axioms 3.4 and 5.3. □

The last point to discuss about improper slots is that in Bennett’s theory, an entity can have several improper slots. Remember that we determine the number of slots of a universal by considering the number of parts of a particular that would instantiate this universal, because of the Isomorphism principle (see Section 2.6.1) and the counting criteria (see Section 3.3.2). For example, a particular of METHANE has arguably itself as a part only once. From this viewpoint, METHANE should have a unique improper slot. More generally, we add the following Axiom 5.4 asserting that a thing has only one improper slot.

**Axiom 5.4** (Only One Improper Slot per Filler).

$$\forall a, s, t(P_s(s, a) \wedge F(a, s) \wedge P_s(t, a) \wedge F(a, t) \rightarrow s = t)$$

With these new axioms, we can reconsider the METHANE universal. The mereological structure of METHANE is described by Facts 5.2 for the proper slots as well as Facts 5.3 for the improper slots.

$$\begin{array}{l|l} P_s(S_i, \text{METHANE}) & 1 \leq i \leq 12 \\ P_s(S_{12}, \text{HYDROGEN}) & \\ F(\text{HYDROGEN}, S_i) & 2 \leq i \leq 5 \end{array} \quad \left| \quad \begin{array}{l} P_s(S_i, \text{CARBON}) & 6 \leq i \leq 11 \\ F(\text{CARBON}, S_1) & \\ F(\text{ELECTRON}, S_i) & 6 \leq i \leq 12 \end{array} \right. \quad (5.2)$$

$$\begin{array}{l} P_s(S_0, \text{METHANE}) & F(\text{METHANE}, S_0) \\ P_s(S_{13}, \text{CARBON}) & F(\text{CARBON}, S_{13}) \\ P_s(S_{14}, \text{HYDROGEN}) & F(\text{HYDROGEN}, S_{14}) \\ P_s(S_{15}, \text{ELECTRON}) & F(\text{ELECTRON}, S_{15}) \end{array} \quad (5.3)$$

If we count how many times HYDROGEN is part of METHANE, the result is four times, for both counting criteria **C1** and **C2**, defined in Section 3.3.2. In the remainder, we will no longer refer to counting criteria **C1** and **C2** for proper parts, as they are equivalent in the new theory, since improper slots are not inheritable anymore. Also, as every filler has now a unique improper slot, improper slots will no longer be represented on figures in the remainder of the paper.

## 5.2 Parts of Parts

### 5.2.1 Pre-Formal Idea

The second problem comes from the inheritance by the whole of the proper slots owned by its parts. As shown with the slots filled by ELECTRON in section 3.3.2.2, these slots are not inherited the correct number of times. To solve this problem, we will propose a different system. This system will rest on the pre-formal intuition that slots should not be inherited but *copied*.

To implement this idea, we drop Axioms 3.5 and 5.2 altogether and “simulate” a (controlled) slot inheritance using slots that we will call “copy-slots”. If a whole is a part of a bigger whole, its entire structure is copied using copy-slots. These copy-slots are slots that have the same filler as the slot they are copied from. Improper slots are not copied.

We can represent from which slot a copy-slot is copied from with the relation *copied-from*, noted *CF*. We can also represent through which HYDROGEN-filled slots the copy-slots (filled by ELECTRON) are copied thanks to another relation: *copied-through*, noted *CT*, between a copy-slot and a slot. Those two relations are represented in Figure 5.1, where  $\square_{\leftarrow}^{S_i} - \square_{\leftarrow}^{S_j}$  represents  $CT(S_j, S_i)$  and  $\square_{\leftarrow}^{S_i} \cdots \square_{\leftarrow}^{S_j}$  represents  $CF(S_j, S_i)$ . On this figure,  $S_3$  is a copy-slot owned by  $A$  which is copied from  $S_2$  through  $S_1$ .

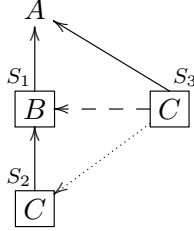


Figure 5.1: Example of application of *CF* and *CT*.

Let's see what would happen on METHANE by using this pre-formal idea. CARBON fills one slot of METHANE. Therefore, its structure is present only once: METHANE has six copy-slots filled by ELECTRON. HYDROGEN fills four slots of METHANE. Hence, its structure is repeated four times: METHANE has four additional copy-slots filled by ELECTRON. With this structure repetition, METHANE has now ten copy-slots filled by ELECTRON, which is the correct number. That is, among the ten copy-slots of METHANE filled by ELECTRON, six of them are copied from the six electron-filled slots of CARBON and four of them are the result of copying four times from the one electron-filled slot of HYDROGEN.

## 5.2.2 Axiomatizing Copy-Slots

According to our pre-formal idea, the first thing to do is to get rid of the revised slot inheritance axiom (Axiom 5.2) (as well as the original Axiom 3.5) and endorse an axiom<sup>2</sup> of anti-inheritance (Axiom 5.5).<sup>3</sup>

**Axiom 5.5** (Anti-Inheritance).

$$\forall a, b, s, t ([a \neq b \wedge P_s(s, b) \wedge F(a, s) \wedge P_s(t, a)] \rightarrow \neg P_s(t, b))$$

Since Axiom 5.2 was previously used to prove parthood transitivity (Theorem 3.7), we need to accept new axioms involving copy-slots that would enable to prove (Theorem 3.7). To illustrate how copy-slots work, let's use a simpler example: the HELIUMDIMER universal, whose mereological structure is pictured in Figure 5.2a.

Note that since a particular of HELIUMDIMER has four particulars of ELECTRON as parts, the universal HELIUMDIMER should have four slots filled by ELECTRON. And this is indeed the case in our theory. As a matter of fact, the HELIUM universal has two slots filled by ELECTRON, which are each copied twice, through each of the two slots of HELIUMDIMER filled by HELIUM.

More generally, we can say that there are as many copy-slots as there are possible pairs of slots  $(s, t)$  such that the first element is the slot through which the copy-slot copies, called "path-slot", and the second element, called "source-slot", is the slot from which the copy-slot is copied. In the case of the HELIUMDIMER, those pairs are  $(S_1, S_3)$ ,  $(S_1, S_4)$ ,

<sup>2</sup>In (Tarbouriech et al., 2021), there is a missing premise:  $a \neq b$ , thus reducing the possible models to the empty model and the model with only one filler, but no slot.

<sup>3</sup>This implies that parthood is discrete (see (Masolo and Vieu, 1999)), which is not an issue for the examples considered here.

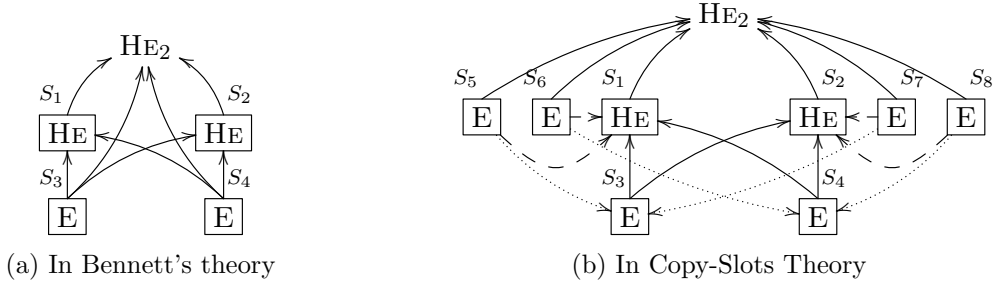


Figure 5.2: The mereological structure of HELIUMDIMER represented in Bennett's and Copy-Slots Theories. HE<sub>2</sub> is HELIUMDIMER, HE is HELIUM and E is ELECTRON. (A dotted arrow represents the *CF* relation. A dashed arrow represents the *CT* relation.)

( $S_2, S_3$ ) and ( $S_2, S_4$ ). This is what Axiom 5.6 describes. Axiom 5.7 imposes that the copy-slot has the same filler as its source-slot. Figure 5.2b pictures how copy-slots work with the HELIUMDIMER universal.

**Axiom 5.6** (Existence of a Unique Copy-Slot for each Whole and Path-Slot, Source-Slot Pair).

$$\forall a, b, s, t (PP_s(s, a) \wedge F(b, s) \wedge PP_s(t, b) \rightarrow \exists! u (P_s(u, a) \wedge CT(u, s) \wedge CF(u, t)))$$

**Axiom 5.7** (Copied Slot has the Same Filler as its Source).

$$\forall s, t (CF(t, s) \rightarrow \exists a (F(a, s) \wedge F(a, t)))$$

Axioms 5.6 and 5.7 and Definition 3.1 are sufficient to prove the theorem of transitivity (Theorem 3.7):

□ *Proof.* Let  $x, y$  and  $z$  be three fillers such that  $x$  is a part of  $y$  and  $y$  is a part of  $z$ .  
 If  $x = y$  then  $x$  is a part of  $z$ . So let's suppose that  $x \neq y$ .  
 By definition of  $P$  (Definition 3.1), there are two slots  $s$  and  $t$  such that  $P_s(s, y) \wedge F(x, s) \wedge P_s(t, z) \wedge F(y, t)$ .  
 According to Axiom 5.6, there is a slot  $u$  of  $z$  copied from  $s$  through  $t$ .  
 □ By Axiom 5.7, since  $x$  fill  $s$ ,  $x$  also fills  $u$ . Therefore,  $x$  is a part of  $z$ . □

As pictured in Figure 5.1, any copy-slot is owned by the same filler as the path-slot it copies through is. Axiom 5.8 ensures that both slots are owned by the same filler. Also, any pair of path-slot and source-slot are related by a filler: the path-slot is filled by it, and the source-slot is owned by it. Axiom 5.9 ensures that the path-slot and the source-slot are related.

**Axiom 5.8** (Same Owner).

$$\forall a, s, t (PP_s(t, a) \wedge CT(t, s) \rightarrow PP_s(s, a))$$

**Axiom 5.9** (Copy Constrains Structure).

$$\forall s, t, u (CT(u, s) \wedge CF(u, t) \rightarrow \exists a (F(a, s) \wedge PP_s(t, a)))$$

Finally, both relations *CF* and *CT* are constrained to be functional, by Axioms 5.10 and 5.11.<sup>4</sup> This ensures that a copy-slot is only related to one pair. Otherwise, we cannot be sure that counting yields a proper result; if the same copy-slot is used for multiple pairs of slots filled with CARBON and ELECTRON, METHANE will not have the right number of ELECTRON parts.

<sup>4</sup>These axioms are discussed in the context of overlap in section 5.3.

**Axiom 5.10** (Functionality Of  $CF$ ).

$$\forall s, t, u (CF(s, t) \wedge CF(s, u) \rightarrow t = u)$$

**Axiom 5.11** (Functionality Of  $CT$ ).

$$\forall s, t, u (CT(s, t) \wedge CT(s, u) \rightarrow t = u)$$

The resulting theory, with Axioms 3.1 to 3.4, 3.6, 3.7, 5.1 and 5.3 to 5.11, along with Definitions 3.1 to 3.6, is sufficient to prove that the problem of counting inherited slots is solved. The proof below focuses on the representative case of a filler  $a$  that has  $b$  as a part  $m$  times, where  $b$  itself has  $c$  as a part  $n$  times. It shows that  $a$  has  $c$  as a part  $m \times n$  times.

*Proof.* Let  $a$ ,  $b$  and  $c$  be three different fillers, let  $s_1, \dots, s_m$  be different slots of  $a$  filled by  $b$  and let  $z_1, \dots, z_n$  be different slots of  $b$  filled by  $c$ .

We want to make sure that  $a$  has exactly one slot filled by  $c$  for each pair  $(s_i, z_j)$  of slots.

Let's first prove that  $a$  has at least one slot filled by  $c$  for each pair. Let  $(s_i, z_j)$  and  $(s_k, z_l)$  be two different pairs. Since  $PP_s(s_i, a) \wedge F(b, s_i) \wedge PP_s(z_j, b)$ , according to axiom (A5.6), there is a copy-slot  $v$  such that  $P_s(v, a)$ ,  $CT(v, s_i)$  and  $CF(v, z_j)$ . Assume that  $v$  is also the copy-slot for the pair  $(s_k, z_l)$ , i.e.,  $CF(v, z_l)$  and  $CT(v, s_k)$ . According to axioms (A5.10) and (A5.11), both relations  $CF$  and  $CT$  are functional. Therefore, we deduce that  $s_i = s_k$  and  $z_j = z_l$ , making the two pairs the same: contradiction. Hence, there is a different slot of  $a$  filled by  $c$  for each pair. Due to the unicity in axiom (A5.6), there is at most one slot for each pair.

We can conclude that  $a$ , having at least and at most one slot for each pair, has exactly the right number of slots filled by  $c$ .  $\square$

Let's illustrate this on the METHANE universal. Facts (5.4) describe the mereological structure without improper slots nor copy-slots; note that Facts (5.3) still hold in addition. Facts (5.5) describe the copy-slots. With copy-slots, METHANE has exactly ten slots ( $S_{16}$  to  $S_{25}$ ) filled by ELECTRON, which is the expected result.

$$\begin{array}{l|l} P_s(S_i, \text{METHANE}) & 1 \leq i \leq 5 \\ P_s(S_{12}, \text{HYDROGEN}) & \\ F(\text{HYDROGEN}, S_i) & 2 \leq i \leq 5 \end{array} \left| \begin{array}{l|l} P_s(S_i, \text{CARBON}) & 6 \leq i \leq 11 \\ F(\text{CARBON}, S_1) & \\ F(\text{ELECTRON}, S_i) & 6 \leq i \leq 12 \end{array} \right. \quad (5.4)$$

$$\begin{array}{l|l} P_s(S_i, \text{METHANE}) & 16 \leq i \leq 25 \\ CT(S_i, S_1) & 16 \leq i \leq 21 \\ CF(S_i, S_{i-10}) & 16 \leq i \leq 21 \end{array} \left| \begin{array}{l|l} F(\text{ELECTRON}, S_i) & 16 \leq i \leq 25 \\ CT(S_i, S_{i-20}) & 22 \leq i \leq 25 \\ CF(S_i, S_{12}) & 22 \leq i \leq 25 \end{array} \right. \quad (5.5)$$

### 5.3 Critique on Copy-Slot Mechanism

In this chapter, we removed Bennett's Slot Inheritance (Axiom 3.5), adopted an anti-inheritance axiom, made more accurate what improper slots are, and developed the "copy-slot mechanism" as a solution for the two counting problems. The idea we developed is that slots were no longer inherited but rather that the mereological structure of fillers is duplicated every time the filler occupies a slot, i.e., the slots of the filler are duplicated to be slots of the whole. The original slot and the duplicated slot have the same content: only their owners change. We introduced two relations  $CF$  and  $CT$  to implement this copy mechanism. Consider the structure of HELIUMDIMER, presented in Figure 5.2: the structure of the HELIUM universal (namely slots  $S_3$  and  $S_4$ ) is repeated twice: once for each of the occurring slot of the HELIUM universal (namely slots  $S_1$  and  $S_2$ ). By redefining improper slots, we fixed the first problem (**Improper Slot Problem**). With the copy-slot mechanism, we fixed the second counting problem (**Parts of Parts Problem**).

However, the copy-slot mechanism has a flaw. This flaw leads to an excessive duplication of slots, and by extension to a wrong counts of parts. Figures 5.3a and 5.3b represent



the same structure of the string “cats”. The first one does not have copy-slots, the second does.<sup>5</sup> By looking at the second figure, we see that “cats” owns two slots  $S_7$  and  $S_{13}$  filled by “c”, whereas it should only have one. This is because of the multiple levels in the mereological structure:  $S_7$  is a slot copied from the structure of “cat”, while  $S_{13}$  is copied from the structure of “ca”. The copy-slot mechanism misses a feature ensuring unicity.

As a solution, we will propose a new system. Even though the relations  $CF$  and  $CT$  are discarded, the core idea of the copy-slot mechanism is preserved. This new system also works by considering that the mereological structure of each filler should be duplicated as many times as possible, but not more than necessary, as we will see in Chapter 9.

But before describing our proposal in details in Chapters 8 and 9, we need to introduce requirements to guide us and tools to help us. This is done in Chapters 6 and 7.

$$\begin{array}{llll}
 P_s(S_3, cat) & F(c, S_3) & CF(S_3, S_1) & CT(S_3, S_5) \\
 P_s(S_4, cat) & F(a, S_4) & CF(S_4, S_2) & CT(S_4, S_5) \\
 P_s(S_7, cats) & F(c, S_7) & CF(S_7, S_3) & CT(S_7, S_9) \\
 P_s(S_8, cats) & F(a, S_8) & CF(S_8, S_4) & CT(S_8, S_9) \\
 P_s(S_{10}, cats) & F(ca, S_{10}) & CF(S_{10}, S_5) & CT(S_{10}, S_9) \\
 P_s(S_{11}, cats) & F(t, S_{11}) & CF(S_{11}, S_6) & CT(S_{11}, S_9) \\
 P_s(S_{13}, cats) & F(c, S_{13}) & CF(S_{13}, S_1) & CT(S_{13}, S_{10}) \\
 P_s(S_{14}, cats) & F(a, S_{14}) & CF(S_{14}, S_2) & CT(S_{14}, S_{10})
 \end{array} \tag{5.6}$$

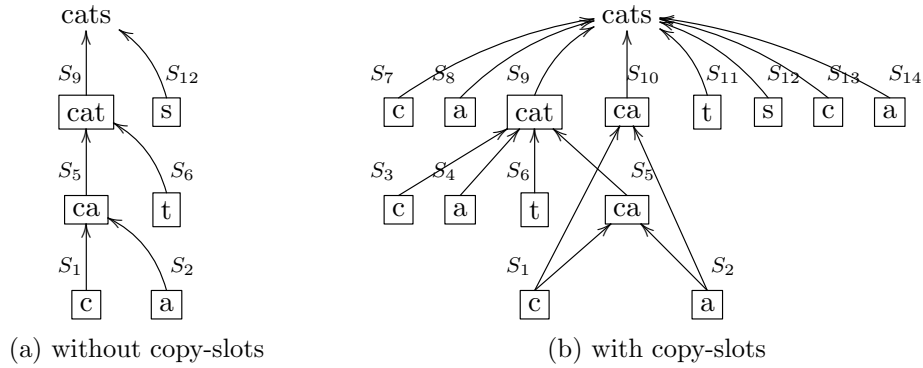


Figure 5.3: A partial view on the mereological structure of “cats” without and with copy-slots

<sup>5</sup>Only entities useful for the current reasoning are pictured in the figures. It is possible to consider other entities, such as “at” or “ct”, but they are not relevant here.



# Chapter 6

## Requirements

In this chapter, I develop requirements to evaluate the theory and to guide its development in order to fill potential theoretical gaps.

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### 6.1 Mereological Relations

Consider the structural universal of DIHYDROGEN or the string type “aa”. Both entities are composed twice of the same entity: respectively the universal HYDROGEN and the letter “a”. However, classical mereology cannot represent such a fact. For example,  $P(x, y) \wedge P(x, y)$  is equivalent to  $P(x, y)$  and does not state that  $x$  is part of  $y$  twice. Therefore, the parthood relation has to be changed in such a way that parthood multiplicity (i.e. how many times  $x$  is a part of  $y$ ) can be formalised.

In the next sections, I will assume that parthood multiplicity can be formalised. We would like such a theory to provide classical features of mereology, such as partial order, supplementation or extensionality.

Furthermore, the theory should provide a way to count how many times  $x$  is part of  $y$ . I explained in Sections 3.3 and 4.1.2 how Cotnoir’s and Bennett’s theories fail to provide a correct account of how many times an entity is part of another one – although for different reasons. Note that, a relation that just states that  $x$  is part of  $y$  multiple times is not enough: consider a ternary relation, such as *part-of- $n$ -times*( $x, y, 2$ ) meaning that  $x$  is part of  $y$  two times. One could then think of changing parthood transitivity into an axiom involving a multiplication: *part-of- $n$ -times*( $x, y, 2$ )  $\wedge$  *part-of- $n$ -times*( $y, z, 2$ )  $\rightarrow$  *part-of- $n$ -times*( $x, z, 4$ ). However, this would not account for cases where one of the occurrences of  $x$  of one occurrence of  $y$  in  $z$  overlaps with one of the occurrence of  $x$  of the other occurrence of  $y$  in  $z$ . Then,  $x$  should only be part of  $z$  only thrice. A multiplication would only provide an upper bound of the number of times that  $x$  can be part of  $z$ .

### 6.2 Decomposition Principles

Do the decomposition principles, presented in Section 2.3, still make sense? In fact, even the weakest principle, the Company axiom, is not compatible with non-idempotent entities. It states that if an entity has a proper part, it has another distinct proper part, or in other

words, no entity has only one proper part. The Company axiom is not compatible with entities such as DIHYDROGEN<sup>1</sup> or ‘AA’.

A quick objection to this observation is that the Company axiom is satisfied whenever a proper part has itself some proper part - consider proper parts of HYDROGEN, like ELECTRON. However, what could be the proper parts of ‘A’? The strokes used to write a letter depend on the font and are arguably not parts of the letter type. If ‘A’ is a mereological atom, i.e. it does not have any proper parts, then the Company axiom does not hold for ‘AA’.

The second axiom of decomposition introduced in Section 2.3.2 is Strong Company, which states that if an entity has a proper part, it has another proper part that is not a part of the first one. ELECTRON being a proper part of HYDROGEN, DIHYDROGEN violates the Strong Company axiom.

If we consider a molecule with two different atoms as a part, it is also problematic. Consider the CARBONMONOXIDE universal, composed of CARBON and OXYGEN. Those two latter universals are different from one another, and none of them is part of the other one. Therefore, Company and Strong Company axioms hold in this case. However, consider now the Supplementation axiom, which states that for any entity  $y$ , for each of its proper parts  $x$ , there is another proper part  $z$  which does not overlap with  $x$ . In the case of CARBONMONOXIDE, if we consider that CARBON and OXYGEN are structural universals with both ELECTRON as a part, they share a part, i.e. they are overlapping. Therefore, the CARBONMONOXIDE universal is a counter-example to Supplementation.

If we restrict the universe to universals of molecules, of atoms and ELECTRON, we get what mereological theories typically reject, namely a mereological equivalent to the empty set of set theory: ELECTRON would be part of anything else. Therefore, the Strong Supplementation axiom, that states that if  $y$  is not a part of  $x$ , then there is something in  $y$  that does not overlap with  $x$ , is also violated.

Finally, it is interesting to understand how some classical mereological theorems fail and what are the implications of these failures. PP-Extensionality (respectively O-Extensionality) theorem states that two entities that have the same proper parts (resp. overlaps with the same entities) are the same entity. What this means is that for two entities to be different, there need to be something that makes the difference. But consider the universals of BUTANE and ISOBUTANE, or the string types “cab” and “bac”. Those entities have the same parts with the same multiplicities, but arranged in a different way. However, how the parts are arranged is not a concern of mereology. These arrangements are not encoded in the mereological relations. Therefore, according to PP-Extensionality, BUTANE and ISOBUTANE should be the same entity. This conclusion also holds for “cab” and “bac”.

As it was showed, classical mereology fails to provide a way to represent entities that can have the same part multiple times. What can be done? If we want a theory encompassing some decomposition principles, it should capture the spirit of these principles, instead of copying their syntax. It will be addressed in Chapter 9.

### 6.3 Composition Principles

What about Composition Principles? The first principle introduced in Section 2.4 is the sum, i.e. the idea of adding multiple things together to obtain something bigger. Among the properties of the sum are idempotence and unicity.

Idempotence of sum means that the sum of an entity with itself is identical to this entity. For example, the sum of your computer and itself is nothing else than itself. There is not something bigger, composed twice of your computer. Now, consider the HELIUM universal. Add HELIUM with itself. What is the result? In other words, what is the entity

---

<sup>1</sup>In this example, the chemical bond relation is not considered as a part of the HELIUMDIMER universal.

that has HELIUM as a part twice? The first answer seems to be HELIUMDIMER. But what about idempotence? Maybe the answer is HELIUM?

Unicity of sum means that the result of the sum is unique. Consider the following example: the sum of CARBON four times and HYDROGEN ten times can be either BUTANE or ISOBUTANE have these parts in a correct amount. Here, there is no unique sum in general.

## 6.4 Establishing Requirements

Classical mereology does not provide a suitable framework to represent entities that can have the same part multiple times. What kind of mereology would enable to do so? To analyse it, let's first remember that the core theory of mereology is usually considered as being a partial order relation and some supplementation principle. However, one could expect a more elaborate theory that includes other relations, advanced decomposition principles and some composition principles.

First, the theory must provide a partial order relation that can represent the multiplicities of parts, as some entities can have the same part multiple times. On this basis, the theory could provide relevant relations, such as proper part and overlap.

Second, the theory must be able to express not only the fact that an entity has a part multiple times, but also how many times it has it as a part. These counts must be correctly handled in mereological relations and operations, like parthood inheritance or sum. For example, if  $x$  has  $y$  as a part twice, and  $y$  has  $z$  as a part twice,  $x$  must have  $z$  as a part four times - or less in case of overlap.

Third, the theory must provide some decomposition principles that capture the spirit of the classical ones, while taking into account that some entities are parts more than once. The weakest one we have seen is the Company axiom, which states that nothing has only one proper part.

Fourth, the theory must also provide some composition principles that take into account multiplicities (to differentiate methane and butane) and non idempotence of sum (to differentiate butane and isobutane).



# Chapter 7

## Tools

This chapter presents two tools I used in order to obtain a more reliable theory: Alloy and Coq. Alloy is a sample and counter-example finder, used to test the consequences of axioms. I used it in three different ways: finding samples, finding counter-examples and as an heuristic tool. These three uses are explained in the following sections. The second tool, Coq, is a proof assistant. I used it to prove each of my theorem candidates in order to be sure that they are theorems of the theory. I explain how I used it in the following sections. The last section of this chapter presents use examples of these tools with Bennett’s 8th Axiom (Axiom 3.8) and 14th Theorem (Theorem 3.13).

All the code written with both of these tools is available at <https://github.com/CedricTarbouriech/PhD> under the license GNU General License v3.0. Among Alloy files, there are different implementations in Alloy of Bennett’s theory. The differences between them are explained in Section 7.1.3. There are also an implementation of the extension of Garbacz (2016) and implementations of the two theories I proposed: the Copy-Slot Mechanism, presented in Chapter 5 and the Mereology of Slots, presented in Chapter 9. Among Coq files, there is a file with Bennett’s theory. The proofs in this file do not necessarily follow Bennett’s proofs. In particular, Axiom 3.8 is presented as an theorem and Theorem 3.12 is proved with Axiom 3.4. Finally, there are files for my two theories.

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## 7.1 Alloy

### 7.1.1 Presentation

“Alloy is a language for describing structures and a tool for exploring them. It has been used in a wide range of applications from finding holes in security mechanisms to designing telephone switching networks.

An Alloy model<sup>1</sup> is a collection of constraints that describes (implicitly) a set of structures, for example: all the possible security configurations of a web application, or all the possible topologies of a switching network. Alloy’s tool, the Alloy Analyzer, is a solver that takes the constraints of a model and finds structures that satisfy them. It can be used both to explore the model by generating sample structures, and to check properties of the model by generating counterexamples. Structures are displayed graphically, and their appearance can be customized for the domain at hand.

At its core, the Alloy language is a simple but expressive logic based on the notion of relations, and was inspired by the Z specification language and Tarski’s relational calculus. Alloy’s syntax is designed to make it easy to build models incrementally, and was influenced by modeling languages (such as the object models of OMT and UML). Novel features of Alloy includes many new rich subtype facilities for factoring out common features and a uniform and powerful syntax for navigation expressions.” (*About Alloy*, from Alloy’s website)

Even though Alloy<sup>2</sup> is designed for software modeling, I used it to develop my theory while exploring the axioms. In addition to finding samples and counter-examples, I used it as a tool to get proof hints. These uses are described in Section 7.1.2 and illustrated with examples.

### 7.1.2 Uses And Examples

I used Alloy in three different ways: to find samples, counter-examples and hints for theorem proofs. They are presented in the following sections. Listing 7.1 gives a bit of Alloy code that will be used in the following sections. Note that not all the code is in this listing. It can be found on Github.

```
1 // The signatures Slot and Filler represent respectively slots and fillers.
2 // A slot can relate to fillers with the relation slot_of.
3 // A filler can relate to slots with the relation fills.
4 // The two relations are sets of couples.
5 sig Slot { slot_of : set Filler }
6 sig Filler { fills : set Slot }
7
8 // Three predicates:
9 // * The two primitives F and P_s are defined with operations on sets:
10 // x fills s means that the couple (x, s) is in the set fills.
11 // * The relation P is defined in Alloy like it is defined in the theory.
12 pred F(a : Filler, s : Slot) { a -> s in fills }
13 pred Ps(s : Slot, a : Filler) { s -> a in slot_of }
14 pred P(a, b : Filler) { some s : Slot | Ps[s, b] and F[a, s] }
15
16 // Improper Parthood Slots
```

<sup>1</sup>In Alloy, the word “model” is used with a different meaning than in logic. In the remainder of this section, “model” will refer to Alloy model. Logic models will be referred to as “samples”, except when referred as “model of the axiomatic theory”.

<sup>2</sup><https://alloytools.org>



```

17 fact A4 {
18   all a : Filler | (some s : Slot | Ps[s, a]) implies
19     (some t : Slot | Ps[t, a] and F[a, t])
20 }

```

Listing 7.1: Signatures, Predicates and an Axiom

In Alloy, *signatures* are sets of entities of some kind: the signature **Slot** is the set containing all the slots of a sample.

### 7.1.2.1 Finding Samples

The first way I use Alloy was to find samples. After having implemented axioms of the theory, I used the command **run** to ask Alloy to generate samples. The command **run** can be parametered to specify some additional conditions for the samples. For example, Listing 7.2 shows two calls to the **run** command. The first call asks for samples in which there are three distinct fillers  $a$ ,  $b$  and  $c$ , such that  $a$  and  $b$  are parts of  $c$ . The second call asks for samples in which there are three distinct fillers  $a$ ,  $b$  and  $c$ , such that  $a$  is a part of  $b$  and  $b$  is a part of  $c$ . Figure 7.1 shows a sample Alloy found for each of the run.

```

1 run { some disj a, b, c : Filler | P[a, c] and P[b, c] } for 3
2
3 run { some disj a, b, c : Filler | P[a, b] and P[b, c] } for 3

```

Listing 7.2: Two Calls of the **run** Command

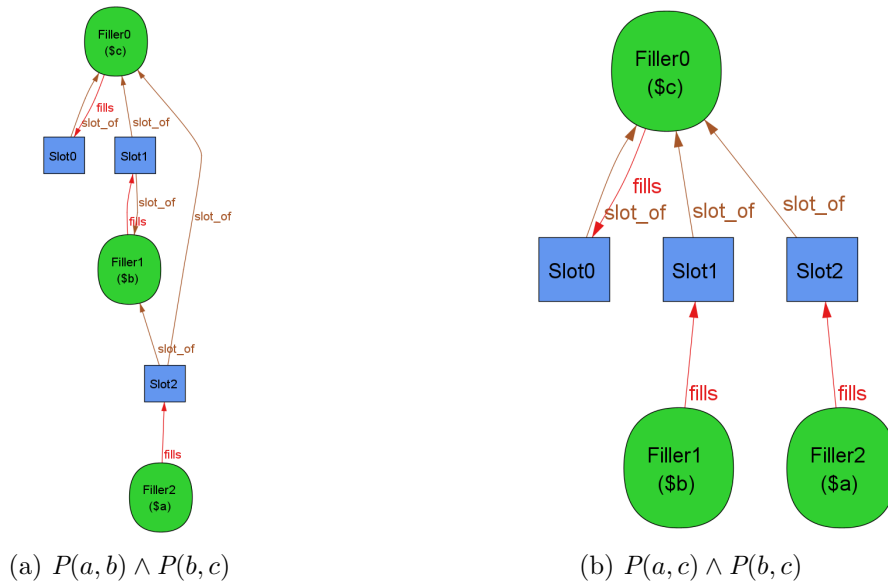


Figure 7.1: Two Samples Found by Alloy

Finally, while Alloy is useful to see the various ways axioms influence samples, it is also a good indicator of problematic axiomatisations. As explained in Footnote 2 on page 60, the Axiom 5, presented in (Tarbouriech et al., 2021, p. 10), is problematic. It corresponds to Axiom 5.5 and was missing  $x \neq y$  in its premises. This created a contradiction whenever there was a slot involved: the incorrect version of axiom states that improper slots are not owned by their owners. Thus, the only two models of the axiomatic theory existing were the empty model, containing no entities, and the model with only one filler. This problem was detected thanks to Alloy, as, once the axiom was added, it only found two samples.

### 7.1.2.2 Finding Counter-Examples

We know that Bennett’s 14th Theorem (Slot Extensionality Theorem 3.13) is not a theorem of her theory, as explained in Section 3.2.1. The present section shows how to exploit Alloy to find this result. After implementing Bennett’s definitions and axioms in Alloy, we could ask Alloy to check this theorem as done in Listing 7.3. The command **check** checks whether the given assertion (here, an implementation in Alloy of Bennett’s Theorem 3.13) has counter-examples, with, at most, 3 slots and 3 fillers.

```

1 check T14 {
2   all a, b : Filler |
3     ((some c : Filler | PP[c, a]) or (some c : Filler | PP[c, b])) implies
4     ((a = b) iff (all s : Slot | PPs[s, a] iff PPs[s, b]))
5 } for 3

```

Listing 7.3: Assertion **T14** And **check** Command

Alloy finds a counter-example, pictured in Figure 7.2. In this counter-example, Alloy shows which entities are the  $a$  and  $b$  of the assertion. Moreover, this counter-example is also the one presented by Garbacz (2016), pictured in Figure 3.3.

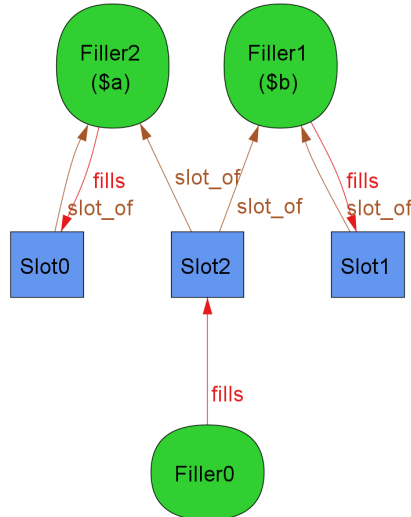


Figure 7.2: The First Counter-Example Proposed By Alloy

### 7.1.2.3 Giving Hints For Proofs

In this section, I show how I found out that Bennett’s 8th Axiom (Slot Strong Supplementation Axiom 3.8) is not an axiom, but a theorem. Alloy gave me an intuition about this, that I finally proved using Coq. The proof is explained in Section 7.2.2. In this section, I used a different implementation of the theory: see Section 7.1.3 for more explanations.

Using Alloy, I first posited Axioms 3.1 to 3.7 as facts. They are called A1-A7 in Alloy. Then I wrote an assertion corresponding to Bennett’s Axiom 3.8 and asked Alloy to search for counter-examples with a high maximum number (20) of entities, as in Listing 7.4.

```

1 fact { A1 A2 A3 A4 A5 A6 A7 }
2
3 assert A8 {
4   all a, b : Entity |
5     ((some s : Entity | Ps[s, a]) and (some t : Entity | Ps[t, b]))
6     implies
7     ((no u : Entity | Ps[u, a] and F[b, u]) implies

```

```

8   (some v: Entity | Ps[v, b] and not Ps[v, a]))
9 }
10
11 check A8 for 20

```

Listing 7.4: Assertion **A8** And **check** Command

From my tests, 20 entities is already a significant number as most of the counter-examples are generally found with less entities. This is called the *Small Scope Hypothesis* by Jackson (2012): “if an assertion is invalid, it probably has a small counterexample” (Jackson, 2012, p. 143). As presented in Section 3.2.1, Garbacz showed that Theorem 3.12 does not need Axiom 3.8, contrary to Bennett’s proof, but just Axiom 3.4. Following my intuition, I removed Axiom 3.4 from the facts (and just keep A1-A3 and A5-A7) and asked Alloy to find a counter-example, which it did nearly instantly (even with a very little scope). In order to strengthen my intuition, I then removed all axioms but Axiom 3.4 and asked Alloy to find a counter-example with a scope of 20, which it did not find.

### 7.1.3 Various Implementations

Alloy is a powerful tool that offers various ways to implement a theory. In this section, I want to discuss three implementations I did, and the consequences they have.

#### 7.1.3.1 Number Of Signatures

Bennett did not axiomatically partition the entities in her theory into slots and fillers. There are entities, and these entities can play a slot-role or a filler-role. When it comes to Alloy implementation, the choice is possible: either create one signature **Entity** whose instances can relate to other instances with **slot\_of** and **fills**, or create two signatures **Slot** and **Filler** whose instances relate to each other instances with, respectively, **slot\_of** and **fills**. These two implementations are presented in Listing 7.5.

Furthermore, when declaring a relation in a signature, we should give a cardinality: a slot relates to some (i.e. at least zero) fillers with **slot\_of**. While the **slot\_of** relation is a n to n relation (a slot can have multiple owners and a filler can have multiple slots), the relation **fills** is a 1 to n relation (a filler can fill multiple slots, but a slot is only filled by exactly one filler). This can be done in Alloy, but it requires that the relation is defined within the **Slot** signature, as showed in Listing 7.5.

```

1 // First implementation: only one signature
2 sig Entity { slot_of : set Entity, fills : set Entity }
3
4 // Second implementation: two signatures, with a relation each
5 sig Slot { slot_of : set Filler }
6 sig Filler { fills : set Slot }
7
8 // Third implementation: two signatures, with both relations in Slot
9 sig Slot { slot_of : set Filler, filled_by : one Filler }
10 sig Filler {}

```

Listing 7.5: Three Different Implementations of the Signatures

The choice made for the implementation of the signature(s) influences the implementation of the remaining parts. First, when implementing relations, the types of the parameters change, as showed in Listing 7.6. When using a relation, Alloy checks if the given argument is of the correct type. This is important for some axioms. Indeed, consider the variable  $a$  in Axiom 3.2. In the premise,  $a$  occupies the “filler-place” of the relation. In

the conclusion,  $a$  occupies the “slot-place” of the relation. While this works with the one-signature implementation, it does not with the two-signatures one: Alloy detects that the relation is given a filler while it expects a slot. The same reasoning holds for Axiom 3.3. These two axioms are useful to clearly differentiate slots and fillers: nothing can be both. In Alloy, signatures cannot share instances. Therefore, the two-signatures implementation does not need these two axioms. For the same reason, Theorems 3.1 to 3.6, whose proofs rely on contradiction and vacuously true premises, cannot be implemented in the two-signatures implementation.

```

1 // First implementation: only one signature
2 pred F(a, s : Entity) { a→s in fills }
3 pred Ps(s, a : Entity) { s→a in slot_of }
4
5 // Second implementation: two signatures
6 pred F(a : Filler, s: Slot) { s in a.fills }
7 pred Ps(s : Slot, a: Filler) { a in s.slot_of }

```

Listing 7.6: Two Different Implementations of the Relations

Furthermore, in the second two-signatures implementation, Axiom 3.7 is not needed. Indeed, this axiom posits the uniqueness of the filler, which is implemented with cardinality of the relation. Finally, when using the commands **run** and **check**, the choice on the number of signatures is important.

First, these commands can be parametered with the scope size (the number of entities). In Listing 7.4, the scope size is set to 20, which means that Alloy can use at most 20 entities. However, it does not tell Alloy whether it should have at most 10 fillers and 10 slots, or 5 fillers and 15 slots. On the other hand, with a two-signatures implementation, we can tell Alloy how many of each we want.

Second, while it seems that the two-signatures implementation is better than the one-signature one, I think there is an advantage to sometimes use a one-signature implementation. In the process I described in Section 7.1.2.3, in order to activate/deactivate axioms, they cannot be implemented directly in signatures. Otherwise, I would have to modify the signatures each time I wanted to see the consequences of an axiom, instead of just deactivating it.

Using two-signatures implementation seems more efficient in execution time, as some constraints are directly implemented in the very structure of signatures, instead of stated as facts. However, the one-signature offers more flexibility to explore the theory. As I do not have extensively analysed the impacts of the implementation choice, I will not go further on the track of efficiency.

### 7.1.3.2 Three Coding Styles

Alloy provides three different coding styles to implement theories: the *predicate calculus* style, the *navigation expression* style, and the *relation calculus* style. Listing 7.7 shows the same constraint expressed in the three different styles. This example is from (Jackson, 2012, pp. 33–34). The example is the implementation of an address book, that maps names to addresses. This constraint states that each name is mapped to at most one address.

```

1 // Predicate calculus style
2 all n: Name, d, d': Address | n→d in address and n→d' in address implies d = d'
3
4 // Navigation expression style
5 all n: Name | lone n.address
6

```

```

7 // Relational calculus style
8 no ~address.address - iden

```

Listing 7.7: The Same Constraint in Three Different Coding Styles

The first style is verbose and the third style often cryptic. Therefore, the second style is often preferred. To stay as syntactically close as possible from the theory, I first used the predicate calculus style. However, I also tried to implement the theory in other coding styles. Listing 7.8 shows two different implementations of Axioms 3.5 and 3.6 (in this listing, the uses of **P**s and **F**, defined in Listing 7.6, are replaced by their definitions in order to facilitate comparisons).

```

1 // First implementation: predicate calculus style
2 fact A5 {
3   all a, b : Filler, s, t : Slot |
4     (b in s.slot_of and s in a.fillls and a in t.slot_of) implies b in t.slot_of
5 }
6 fact A6 {
7   all a, b : Filler, s, t : Slot | ((b in s.slot_of and s in a.fillls) and
8     (a in t.slot_of and t in b.fillls)) implies a = b
9 }
10
11 // Second implementation: relation calculus style
12 fact A5 { slot_of.fillls.slot_of in slot_of }
13 fact A6 { fillls.slot_of & ~(fillls.slot_of) in iden }

```

Listing 7.8: Axioms 3.5 and 3.6 in Different Styles

In order to compare, I asked Alloy to check Theorem 3.12. Between the two implementations, everything was identical but the implementations of Axioms 3.5 and 3.6. The first implementation generates a problem to solve that contains 289370 variables and 326475 clauses. On a mean of ten runs, it was solved in about 1211 milliseconds. The second implementation generates a problem to solve that contains 19610 variables and 36810 clauses. On a mean of ten runs, it was solved in about 52 milliseconds. Although the two implementations are logically equivalent, the latter generates 15 times fewer variables, 9 times fewer clauses and solves the problem 23 times faster. As I did not extensively analyse the efficiency of the implementations, but only did simple tests, I will not go further in the efficiency comparisons.

## 7.2 Coq

### 7.2.1 Presentation

*Coq* is a proof assistant with which students, researchers, or engineers can express specifications and develop programs that fulfill these specifications. This tool is well adapted to develop programs for which absolute trust is required: for example, in telecommunication, transportation, energy, banking, etc. In these domains the need for programs that rigorously conform to specifications justifies the effort required to verify these programs formally. [...]

The *Coq* system is not only interesting to develop safe programs. It is also a system with which mathematicians can develop proofs in a very expressive logic, often called *higher-order logic*. These proofs are built in an interactive manner with the aid of automatic search tools when possible. The application domains are very numerous, for instance logic, automata theory, computational linguistics and algorithms [...].

This system can also be used as a logical framework to give the axioms of new logics and to develop proofs in these logics. For instance, it can be used to implement reasoning systems for modal logics, temporal logics, resource-oriented logics, or reasoning systems on imperative programs.

(Bertot and Castéran, 2013, p. 1)

I used Coq as it is presented in the third paragraph above. After defining the relations and the axioms, I proved each of the theorems. Coq is a really helpful tool as it helps keep track of the goals to prove at each step of a proof. In addition to prove the theorems presented in this thesis, it helped me to understand some of the complex concepts I had to deal with. In particular, it helped me strengthen my intuitions about which formula could be theorems, which were not and which axioms were needed for non-theorems to become theorems of the theory. I used the work of Bertot and Castéran (2013) as a reference, as well as questions on StackOverflow,<sup>3</sup> a Q&A website for developers.

### 7.2.2 Use Example

To give some examples of Coq, Listing 7.9 is the code used to prove that Axiom 3.8 is a theorem. I commented the code directly in the listing. From there, even though it was not yet proved, I had the intuition that Axiom 3.8 might be a theorem whose proof only relies on Axiom 3.4. Listing 7.9 shows the proof in Coq. I commented the proof to make it easily understandable to anyone that does not know how Coq works. Furthermore, the proof shows that the only axiom needed is Axiom 3.4.

```

1 Parameter Entity : Set.
2
3 (* The two primitives of the theory take two entities
4  * and return a logical proposition. *)
5 Parameter F : Entity → Entity → Prop.
6 Parameter Ps : Entity → Entity → Prop.
7
8 (* The relation P is defined as taking x and y
9  * and returning the proposition  $\exists s(F(x,s) \wedge P_s(s,y))$ . *)
10 Definition P x y := exists s, F x s ∧ Ps s y.
11
12 (* Coq Implementation of Axiom 3.4. *)
13 (*  $\forall a(\exists s(P_s(s,a) \rightarrow \exists t(P_s(t,a) \wedge F(a,t)))$  *)
14 (* Improper Parthood Slots *)
15 Axiom A4 : forall x, (exists y, Ps y x) → (exists z, Ps z x ∧ F x z).
16
17 Theorem BA8 : forall a b, (exists s, Ps s a) ∧ (exists t, Ps t b) →
18   (~ (exists u, Ps u a ∧ F b u) → (exists v, Ps v b ∧ ~Ps v a)).
19 Proof.
20   (* Let a and b be such that s is a slot of a and t a slot of b. *)
21   (* Call nPba the hypothesis that b is not a part of a. *)
22   intros a b ((s & Pssa) & (t & Pstb)) nPba.
23   (* According to Axiom 3.4, b has an improper slot u. *)
24   pose proof (whole_improper_slots b (ex_intro _ t (Pstb))) as (u & Psub & Fbu).
25   (* We want to prove that there is a slot owned by b, but not a. *)
26   (* I affirm that this slot is u. We already know it is owned by b. *)
27   exists u; split; auto.
28   (* We only have to prove that it is not owned by a. *)
29   (* Suppose it is the case: call this hypothesis Psua. *)
30   intros Psua.
31   (* According to nPba, there are no slot owned by a and filled by b. *)

```

<sup>3</sup><https://stackoverflow.com/>

```
32 | destruct nPba.  
33 | (* However,  $u$  does, because of hypothesis  $P_{sua}$ : contradiction. *)  
34 | exists u; split; assumption.  
35 | (* Therefore,  $u$  is not owned by  $a$ . *)  
36 | Qed.
```

Listing 7.9: Proof Of Axiom 3.8 Commented





# Chapter 8

## Slot Characterisation

In this chapter, I propose a better characterisation of what slots are by addressing different topics: ownership, properness and dependencies. This characterisation is based on Definitions 3.1 to 3.5 and Axioms 3.1 to 3.4, 3.6 and 3.7.

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---

### 8.1 Definition

Bennett defines a slot as “a location in a mereological nexus” and “an aspect of the mereological structure of a whole” (Bennett, 2013, p. 87). We restrict those definitions by stating that a slot is a holistic aspect of the mereological structure of a whole, i.e., a contextual mereological location in a unique whole. Therefore, it depends existentially on this whole, as explained below in Section 8.4. We add to the theory the unary predicate  $S$ , that states that something is a slot, defined by Definition 8.1.

**Definition 8.1** (Slot).

$$S(s) \triangleq \exists a(P_s(s, a))$$

### 8.2 Ownership

In our theory, we consider slots as inner elements of wholes that characterise different contexts in which parts occur.<sup>1</sup> As inner elements of a whole, that is, as elements of its mereological structure, they are not shareable with other wholes, as imposed by Axiom 8.1 below. This is in contradiction with Bennett’s axiomatisation since in her theory, slots can have multiple owners because of slot inheritance.

**Axiom 8.1** (Single Owner).

$$\forall a, b, s(P_s(s, a) \wedge P_s(s, b) \rightarrow a = b)$$

---

<sup>1</sup>To follow the analogy with roles that inspired Bennett’s theory, those contexts can be seen as relational roles. In this view, we consider that slots correspond to fully saturated roles, such as “President of the United States”, as opposed to the unsaturated role “President”. For more on role saturation, see Masolo et al. (2004).

We showed in Section 3.3.2 that slot inheritance is the origin of counting problems. By imposing that slots have a single owner, slot inheritance is no longer possible, as demonstrated by Theorem 8.1 below. Therefore, we reject Bennett’s Slot Inheritance 3.5. Consequently, as parthood transitivity Theorem 3.7 is the only theorem of Bennett’s theory that was proved with 3.5, we will need to show that it is still a theorem of our theory when the relevant axioms will be added.

**Theorem 8.1** (Anti-Inheritance).

$$\forall a, b, s, t ([a \neq b \wedge P_s(s, b) \wedge F(a, s) \wedge P_s(t, a)] \rightarrow \neg P_s(t, b))$$

⌈ *Proof.* Let  $a$  and  $b$  be two different fillers ( $a \neq b$ ) and  $s$  and  $t$  two slots such that  $a$  is a part of  $b$  by filling its slot  $s$  ( $P_s(s, b) \wedge F(a, s)$ ) and  $t$  is a slot of  $a$  ( $P_s(t, a)$ ). We want to prove that  $t$  cannot be a slot of  $b$  ( $\neg P_s(t, b)$ ). Suppose that  $t$  is a slot of  $b$  ( $P_s(t, b)$ ). Because  $t$  is a slot of  $a$  and  $b$ , we know by A8.1 that  $a = b$ . Contradiction. Therefore,  $t$  cannot be a slot of  $b$ .  $\square$

In the remainder, we will sometimes call slots “occurring contexts”. Consider the two following definite descriptions: “the universal of hydrogen that fills  $s_2$ ” and “the universal of hydrogen that fills  $s_3$ ”. Those two descriptions are misleading. By reading them, one could assume the existence of two different universals of hydrogen, which does not make sense. Indeed, what is intended here is to refer to two occurrences of the same universal, HYDROGEN, in the slots  $s_2$  and  $s_3$ . Those two different occurrences exist because there are two different slots. Those slots are “occurring contexts” of the universal within a given universal, the owner of the slots. Those occurring contexts are not intrinsic properties of the universals filling the slots. An occurring context characterises the internal structures of an entity larger than the filler of this context.

We add two relations, namely  $SO$  and  $SF$ , respectively defined by Definition 8.2 and Definition 8.3. These relations state that two slots have the same owner, for the former, and the same filler, for the latter. They are trivially reflexive, symmetric, and because the owner and the filler of a slot are unique, transitive.

**Definition 8.2** (Same Owner).

$$SO(s, t) \triangleq \exists a (P_s(s, a) \wedge P_s(t, a))$$

**Definition 8.3** (Same Filler).

$$SF(s, t) \triangleq \exists a (F(a, s) \wedge F(a, t))$$

Now that slots only have one owner, the two counting criteria **C1** and **C2**, defined in Section 3.3.2, are equivalent when counting proper parts. Note that with anti-inheritance, the distinction between direct and non-direct slots (see Footnote 8 on page 44) is not relevant anymore: all slots are direct (but see Section 9.1 to see how this distinction is recovered).

Since slot inheritance was instrumental in Bennett’s theory to derive important theorems, we will adopt instead what we call an operation of “contextualisation” between slots. Consider the following example: some entity  $a$  has a slot  $s$  filled by  $b$ , and  $b$  owns a slot  $t$  filled by  $c$ . In Bennett’s theory, with slot inheritance, the slot  $t$  will also be a slot of  $a$ . Here, instead of inheriting  $t$ , we posit that  $a$  has a slot  $u$  filled by  $c$ , which is a copy of  $t$ , *contextualised* by  $s$ . Contextualisation will fix the flaw of the copy-slot mechanism, presented in Section 5.3, as we will see in Section 9.1.

But first, let’s address some questions about slots’ nature that are barely touched in Bennett’s paper: are slots existentially dependent on their owners? On their fillers? And what are improper slots?

### 8.3 Properness

Bennett's theory proposes Definition 3.5 to define proper slots. The definition states that some slot  $s$  is a proper slot of  $a$  if it is a slot of  $a$  and  $a$  does not fill  $s$ . Therefore, an improper slot is a slot owned and filled by the same entity, as defined by Definition 8.4.

**Definition 8.4** (Improper Slot).

$$IP_s(s, a) \triangleq P_s(s, a) \wedge F(a, s)$$

In Bennett's theory, the properness of a slot is a property relative to an owner of this slot. Therefore, a slot can be an improper slot of a filler  $a$  and, at the same time, a proper slot of a filler  $b$ . However, in our theory, Axiom 8.1 makes it impossible for a slot to have multiple owners. Therefore, proper and improper slots form a partition of slots, i.e. slots are either proper or improper, but not both, as expressed by Lemma 8.2, where  $\oplus$  is the XOR connective.

**Lemma 8.2** (Either Proper or Improper).

$$\forall s[S(s) \rightarrow \exists! a(PP_s(s, a) \oplus IP_s(s, a))]$$

*Proof.* Let  $s$  be a slot.  $s$  has an owner  $a$  (Definition 8.1), and this owner is unique (Axiom 8.1). Either  $a$  fills  $s$ , and therefore  $s$  is an improper slot, or  $a$  does not fill  $s$ , and therefore  $s$  is a proper slot of  $a$ .  $\square$

Bennett does not explicitly state why she admitted improper slots. However, as she clearly aims to mimic classical extensional mereology, she needs some sort of parthood reflexivity and improper parthood relation. Improper slots make the parthood relation conditionally reflexive (3.9), and we note that they are not used for any other purpose in Bennett's paper. Therefore, we can presume that improper slots are introduced only for the conditional reflexivity of parthood.

Using Axiom 8.1 and Definitions of  $P$  (3.1) and  $PP$  (3.2), we can deduce Lemma 8.3, which states that a proper part of  $a$  is filler of a proper slot of  $a$ . This ensures that every proper part of a whole fills a slot of this whole, without owning this slot. This excludes models like the one pictured in Figure 3.4b.

**Lemma 8.3** (Proper Parts iff Proper Slots).

$$\forall a, b(PP(b, a) \leftrightarrow \exists s(PP_s(s, a) \wedge F(b, s)))$$

*Proof.* Let  $a$  and  $b$  be two fillers.

Let us first suppose that  $b$  is a proper part of  $a$ . By definition of proper parthood 3.2,  $P(b, a) \wedge \neg P(a, b)$ . From  $P(b, a)$  and definition 3.1, we deduce the existence of a slot  $s$  such that  $P_s(s, a) \wedge F(b, s)$ . We want to prove that 1)  $s$  is a proper slot of  $a$  and that 2)  $b$  fills  $s$ , which is already in the hypotheses.  $s$  is a slot of  $a$ . If  $a$  does not fill  $s$ , then  $s$  is a proper slot of  $a$ . Suppose that  $a$  fills  $s$ . According to 3.7,  $a = b$ . Contradiction:  $b$  is a proper part of  $a$ . Therefore  $a$  does not fill  $s$ .

Let us now suppose that there is some  $s$  that is a proper slots of  $a$  and that is filled by  $b$ . We want to prove that  $b$  is a part of  $a$  and that  $a$  is not a part of  $b$ .  $b$  fills a slot of  $a$ . Thus, by 3.1,  $b$  is a part of  $a$ . Suppose now that  $a$  is a part of  $b$ . By antisymmetry (3.8),  $a = b$ . Thus  $a$  fills  $s$ : contradiction as  $s$  is a proper slot of  $a$ . Therefore  $a$  is not a part of  $b$ .  $\square$

As explained in Section 5.1, Bennett's Axiom 3.4 ensures that anything that possesses a slot has an improper slot. In this previous section, we generalise improper slots to every fillers, whether they own or fills a slot. This was done by Axiom 5.3. We add axiom Axiom 8.2 to the current theory and broaden Bennett's Theorem of conditional reflexivity 3.9 as Lemma 8.4.

**Axiom 8.2** (Additional Improper Slot).

$$\forall a, s (F(a, s) \rightarrow \exists t (IP_s(t, a)))$$

**Lemma 8.4** (General Conditional Reflexivity).

$$\forall a, s (P_s(s, a) \vee F(a, s) \rightarrow P(a, a))$$

*Proof.* This is a trivial consequence of 3.1, 3.4 and Axiom 8.2. □

In Bennett's theory, nothing excludes that an entity has an improper slot several times. Remember that we determine the number of slots of a universal by considering the number of parts of a particular that would instantiate this universal (Isomorphism principle and counting criteria, see Sections 2.6.1 and 3.3.2). For example, a particular of METHANE has arguably itself as a part only once. From this viewpoint, METHANE should have a unique improper slot. More generally, we add the following Axiom 8.3 asserting that a filler has at most one improper slot.

**Axiom 8.3** (Unique Improper Slot per Filler).

$$\forall a, s, t (IP_s(s, a) \wedge IP_s(t, a) \rightarrow s = t)$$

With Axiom 3.6, Bennett introduces anti-symmetry, i.e., if  $a$  is part of  $b$  by filling its slot  $s$  and  $b$  is part of  $a$  by filling its slot  $t$ , then  $a = b$ . With Axiom 8.3, we can assert that, besides  $a = b$ , we also have  $s = t$ , as expressed by Theorem 8.5.

**Theorem 8.5** (Mutual Occupancy is Slot Identity).

$$\forall a, b, s, t (P_s(s, b) \wedge F(a, s) \wedge P_s(t, a) \wedge F(b, t) \rightarrow s = t)$$

*Proof.* Let  $s$  and  $t$  be two slots, and  $a$  and  $b$  two fillers such that  $(P_s(s, b) \wedge F(a, s)) \wedge (P_s(t, a) \wedge F(b, t))$ . By 3.6, we know that  $a = b$ . Therefore, by Definition 8.4,  $s$  and  $t$  are improper slots of  $a$  ( $IP_s(s, a) \wedge IP_s(t, a)$ ). By Axiom 8.3,  $s = t$ . □

To sum up, so far we removed slot inheritance by removing Bennett's Axiom 3.5 and clarified what slots are in our theory, i.e., non-shareable elements of mereological structures. We also made clear the differences between proper and improper slots. We kept axioms Axioms 3.1 to 3.4 and Axioms 3.6 to 3.8 in our theory, which will be used, with our Axioms 8.1 to 8.3 as a basis for the operation of contextualisation between slots and mereology of slots that we will now present.

As explained above, the removal of Bennett's Axiom 3.5 implies that we will need to show that Theorem 3.7 is a theorem of our theory. Nonetheless, other Bennett's theorems are not dependent on Axiom 3.5 and still hold with the axioms we consider in the remainder of the paper, as do all of Theorems 8.1 and 8.5 and Lemmas 8.2 to 8.4 demonstrated in this section.

## 8.4 Existential Dependencies

Even though Bennett explores the existential dependencies of entities on their slots and parts, she says nothing about the existential dependencies of slots on their owners and fillers.

In Bennett's theory, slots can be related to fillers by two relations:  $P_s$  and  $F$ . Therefore, a specific slot can be owned by some fillers and filled by some other (or identical) fillers.

Does a slot existentially depend on its owner? Or on its filler? And if so, what is the nature of this dependency?

Bennett's theory asserts that all slots have an owner (Axiom 3.1). In our theory, a slot is an inner element of a single whole (Axiom 8.1). Therefore, the facts that slots always have an owner and are elements of their single owner's mereological structure suggest that slots existentially depend on their unique owner. Thus, although we do not formally account for the modal aspect of dependence in our theory, we can assume that slots are specifically dependent on their owner: the existence of a specific slot depends on the existence of its owner. If the owner would disappear, the slot would also cease to exist.

We just pointed out that slots are inner elements of their owner. They have another important characteristic in Bennett's theory: they are always filled (Axiom 3.7). Bennett also points out that the content of a slot can change over time: a slot can have two different fillers at two different times. Thus, a slot does not specifically depend on its filler. This raises the question of whether a slot generically depends on its fillers across time, which we will not address here as we leave the complex issue of mereological change for further work.



# Chapter 9

## Mereology Of Slots

In this chapter, I present the theory proposed in Tarbouriech et al. (2024). This theory uses the idea developed in the Copy-Slot Mechanism: the slots should be duplicated as many times as needed. This theory fixes the counting problems identified in Bennett’s theory and in the Copy-Slot Mechanism. After establishing a strong basis that enables to count correctly, this chapter presents an exploration of the mereology of slots.

I define mereological relations similar to those of classical mereology and show and prove that we get similar definitions and theorems, in particular decomposition and principle theorems. This theory is based on Definitions 3.1 to 3.5 and 8.1 to 8.4 and Axioms 3.1 to 3.4, 3.6, 3.7 and 8.1 to 8.3.

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### 9.1 Slot Contextualisation

Now that we further explained what slots are in our theory, let us look at our solution to make parthood transitive. We keep the core idea developed with the copy-slot mechanism: the mereological structure of a filler should be duplicated as many times as the filler occupies a slot. In this paper, as seen in Chapter 8, the idea is to use slots as occurring contexts of parts. Instead of using the relations  $CF$  and  $CT$ , we want a theory in which we can assert about the Figure 5.2b that  $S_5$  and  $S_6$  are respectively the results of  $S_3$  and  $S_4$  being contextualised by  $S_1$ . In the same way, we want to say that  $S_7$  and  $S_8$  are the results of  $S_3$  and  $S_4$  being contextualised by  $S_2$ .

In Figure 9.1,  $b$  is a part of  $a$  by filling  $s$ , and  $c$  is part of  $b$  by filling  $t$ . In this configuration, we want  $b$ ’s parts to also be parts of  $a$ , i.e., we want  $c$  to be a part of  $a$ . The slot  $t$  cannot be inherited by  $a$ , as it would be in Bennett’s theory:  $t$  is a slot of  $b$  and only  $b$ . However,  $c$  should fill a slot of  $a$  because it fills a slot of  $b$ , which itself fills a slot of  $a$ . For this, we introduce the relation of *slot contextualisation*. In our example, slot contextualisation relates a slot  $u$  of  $a$  to  $s$  and  $t$ . We say that  $u$  is the “contextualisation of  $t$  by  $s$ ”. The relation of slot contextualisation is abbreviated  $CoS$ :  $CoS(u, s, t)$  means that  $u$  is the result of the contextualisation of  $t$  by  $s$ . Furthermore, we want this relation

to only hold between slots, we want slots  $u$  and  $s$  to have the same owner and we want slots  $u$  and  $t$  to have the same filler. From now on,  $\square \leftarrow \bullet \dashrightarrow \square$  represents, in figures, the contextualisation with the arrow head, the black dot and the arrow tail respectively linked to  $u$ ,  $s$  and  $t$  in  $CoS(u, s, t)$ .

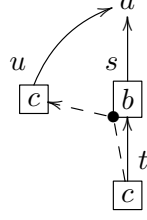


Figure 9.1: An example of contextualisation

As already mentioned, the contextualisation is a relation between three slots, as imposed by Axiom 9.1.

**Axiom 9.1** (Domains of Contextualisation).

$$\forall s, t, u (CoS(u, s, t) \rightarrow S(u) \wedge S(s) \wedge S(t))$$

The existence of the contextualisation is implied by the existence of some filler  $a$  such that one of the slots is a slot of  $a$ , and the other one is filled by  $a$ . This condition of existence is captured by the predicate  $Cb$ , defined by Definition 9.1:  $Cb(t, s)$  reads “ $t$  is contextualisable by  $s$ ”. Axiom 9.2 states that a slot  $t$  is contextualisable by a slot  $s$  if and only if a contextualisation  $u$  of  $t$  by  $s$  exists.

**Definition 9.1** (Contextualisable by).

$$Cb(t, s) \triangleq \exists a (F(a, s) \wedge P_s(t, a))$$

**Axiom 9.2** (Contextualisable iff Contextualisation Exists).

$$\forall s, t [Cb(t, s) \leftrightarrow \exists u (CoS(u, s, t))]$$

We also impose, with Axiom 9.3, that the contextualisation is unique. Consequently, we will use the notation  $s \circ t$  as a binary function that gives the resulting slot of  $t$  contextualised by  $s$ , in other words, the contextualisation of  $t$  by  $s$ .

**Axiom 9.3** (Unicity of Contextualisation).

$$\forall s, t, u, v (CoS(u, s, t) \wedge CoS(v, s, t) \rightarrow u = v)$$

We will now use the notation  $u = s \circ t$  instead of the statement  $CoS(u, s, t)$ , and in the context in which  $t$  is contextualisable by  $s$ , we will simply use  $s \circ t$ .<sup>1</sup>

We can deduce from this that if  $s \circ t$  and  $t \circ s$  exist, then  $s = t$ . In other words, in the general case where  $s$  and  $t$  are different, if the contextualisation  $s \circ t$  exists, then  $t \circ s$  does not. This is expressed by Theorem 9.1.

**Theorem 9.1** (Symmetric Contextualisation is Slot Identity).

$$\forall s, t, u, v (u = s \circ t \wedge v = t \circ s \rightarrow s = t)$$

<sup>1</sup>In proofs, we will often omit reference to Axiom 9.2 for this last use.



⌈*Proof.* Let  $s, t, u$  and  $v$  be four slots such that  $u = s \circ t$  and  $v = t \circ s$ . By Axiom 9.2, we know there are  $a$  and  $b$  such that  $F(a, s) \wedge P_s(t, a)$  and  $F(b, t) \wedge P_s(s, b)$ . By T8.5,  $s = t$ .  $\square$

Let us now introduce Axioms 9.4 and 9.5 to prevent some unwanted contextualisations. Axiom 9.4 ensures that if  $s \circ t$  and  $s \circ u$  are equal, then  $t = u$ , i.e. it ensures that when an entity  $a$  is part of another entity  $c$  by filling a slot  $s$ , there are as many slots of  $a$  as there are slots of  $c$  that are the results of a contextualisation by  $s$ . Otherwise, it would be possible for the contextualisations of two different slots of the same entity  $a$  to be identical, as pictured in Figure 9.2a. In such a case,  $a$  could have two slots filled by  $b$  which by contextualisation could lead to only one slot of  $c$  filled by  $b$ ; this would obviously not lead to the correct result when counting how many times  $b$  is part of  $c$ .

In addition, we would like to ensure that a contextualisation of an improper slot  $s$  by some slot  $t$  results in  $t$ . Otherwise, in the example pictured by Figure 9.2b,  $a$  would be part of  $c$  three times instead of once. However, it is possible to adopt a weaker axiom, more similar to Axiom 9.4 and to use it to demonstrate such a proposition. This adopted axiom is Axiom 9.5 and the resulting theorem is Theorem 9.3, presented further below.

**Axiom 9.4** (Injectivity to the Left).

$$\forall s, t, u, v (v = s \circ t \wedge v = s \circ u \rightarrow t = u)$$

**Axiom 9.5** (Injectivity to the Right).

$$\forall s, t, u, v (v = t \circ s \wedge v = u \circ s \wedge \exists a (IP_s(s, a)) \rightarrow t = u)$$

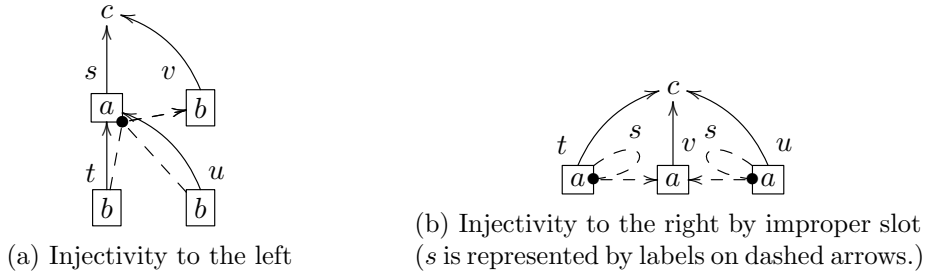


Figure 9.2: Injectivities of Contextualisation: unwanted models with  $t \neq u$

Axiom 9.5 ensures that if  $s$  is an improper slot and  $v = t \circ s = u \circ s$ , then  $u = t$ . However, this axiom does not ensure that  $t = v$ . For this, we will introduce Axiom 9.6, an axiom of associativity. There is moreover another motivation to introduce it. Consider the example given in Figure 5.3, reinterpreted using contextualisation. We know that  $S_7 = S_9 \circ S_3$  and  $S_{13} = S_{10} \circ S_1$ . Among the slots used in these contextualisations, some are also results of contextualisation:  $S_3 = S_5 \circ S_1$  and  $S_{10} = S_9 \circ S_5$ . By replacing  $S_3$  and  $S_{10}$  in the first two equations, we get  $S_7 = S_9 \circ (S_5 \circ S_1)$  and  $S_{13} = (S_9 \circ S_5) \circ S_1$ . If we want to avoid slot duplication, as exposed in the copy-slot mechanism, we want that  $S_7 = S_{13}$ , i.e.,  $S_9 \circ (S_5 \circ S_1) = (S_9 \circ S_5) \circ S_1$ . This is enabled by Axiom 9.6, which intuitively states that if  $t \circ u$  and  $s \circ (t \circ u)$  exist, then  $s \circ t$  and  $(s \circ t) \circ u$  exist and  $s \circ (t \circ u) = (s \circ t) \circ u$ , and vice versa.

**Axiom 9.6** (Contextualisation Associativity).

$$\forall s, t, u, v [\exists w (v = s \circ w \wedge w = t \circ u) \leftrightarrow \exists x (v = x \circ u \wedge x = s \circ t)]$$

This axiom means that there are two “paths” from  $u$  to  $v = (s \circ t) \circ u = s \circ (t \circ u)$ , as pictured respectively by Figures 9.3a and 9.3b.<sup>2</sup> In Figure 9.3a,  $t$  is contextualised by

<sup>2</sup>Figure 9.3 is subdivided into two figures for readability: they are two partial views of the same model.

$s$  to get  $(s \circ t)$ , filled by the same filler as  $t$ , which owns  $u$ . Thus,  $(s \circ t) \circ u$  exists. In Figure 9.3b,  $u$  is contextualised by  $t$  to get  $(t \circ u)$ . And this slot, owned by the filler of  $s$ , is contextualised by  $s$  to get  $s \circ (t \circ u)$ . Associativity implies that  $(s \circ t) \circ u$  and  $s \circ (t \circ u)$  are identical.

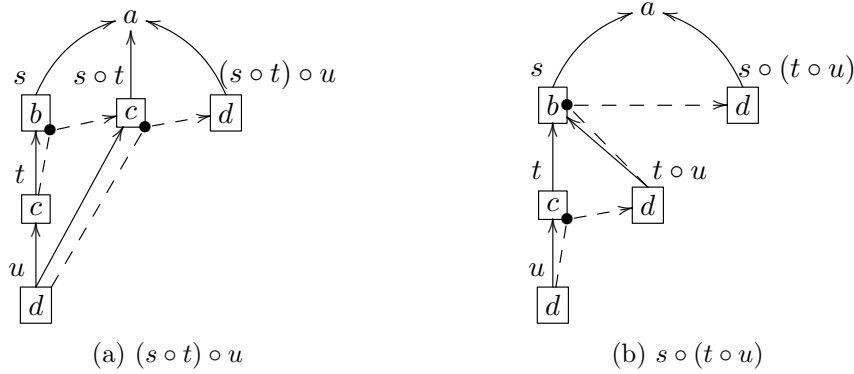


Figure 9.3: Associativity:  $(s \circ t) \circ u = s \circ (t \circ u)$

Slots of various properness can be contextualised together. We can distinguish three specific non-disjoint cases: 1) both slots are improper slots (Left-and-Right-Improper Contextualisation, see Theorem 9.2); 2) the first operand is an improper slot (Improper Slot Is Right Neutral Element, see Theorem 9.3); 3) the second operand is an improper slot (Improper Slot Is Left Neutral Element, see Theorem 9.4).

The only slots that can be contextualised by themselves are improper slots. This is because of the condition expressed by Axiom 9.2 (Contextualisable iff Contextualisation Exists). Therefore, a slot  $s$  is improper iff  $s = s \circ s$  is true, as expressed by Theorem 9.2.

**Theorem 9.2** (Left-and-Right-Improper Contextualisation).

$$\forall s(\exists a(IP_s(s, a)) \leftrightarrow s = s \circ s)$$

*Proof.* Left-to-right: let  $a$  be a filler and  $s$  its improper slot ( $IP_s(s, a)$ ). By definition, this means that  $P_s(s, a) \wedge F(a, s)$ . According to Axiom 9.2, the slot  $s \circ s$  exists. By Axiom 9.1, we know that  $s \circ s$  is owned by some  $b$  ( $P_s(s \circ s, b)$ ). And by Single Occupancy Axiom (3.7), we know that there is a  $c$  that fills  $s \circ s$  ( $F(c, s \circ s)$ ). We want to show that  $s = s \circ s$ . We will show that  $a = b = c$ , and conclude that, as there can only be one improper slot per filler,  $s = s \circ s$ .

Let us first show that  $a = c$ . We know by Axiom 8.2 that  $c$  has an improper slot  $t$  ( $P_s(t, c) \wedge F(c, t)$ ). With  $F(c, s \circ s) \wedge P_s(t, c)$ , we know that  $(s \circ s) \circ t$  exists. By Contextualisation Associativity (Axiom 9.6),  $s \circ t$  exists. Thus, by Axiom 9.2, there is a  $d$  that fills  $s$  and owns  $t$  ( $F(d, s) \wedge P_s(t, d)$ ). However, we know that  $s$  is already filled by  $a$  and that a slot can only be filled by one filler (3.7), therefore  $a = d$ . We also know that  $t$  is owned by  $c$  and  $d$ , i.e.  $a$ . However, there can only be one owner (Axiom 8.1), therefore,  $a = c = d$ . One of the consequences is that  $s = t$ , as  $s$  and  $t$  are improper slots of  $a$  and that there is only one improper slot per filler.

Let us prove that  $a = b$ . We know that  $s \circ (s \circ s)$  exists. Therefore by Axiom 9.2, there is a  $e$  that fills  $s$  and owns  $s \circ s$ . By 3.7 (Single Occupancy),  $a = e$ . By Axiom 8.1 (Single Owner),  $e = b$ . So  $a = b$ .

We know that  $s$  and  $s \circ s$  are filled and owned by  $a$ , i.e. they are  $a$ 's improper slots. However, there is only one improper slot per filler. Therefore,  $s = s \circ s$ .

Right-to-left: let  $s$  be a slot such that  $s = s \circ s$ . By Axiom 9.2, we know that there is an  $a$  such that  $P_s(s, a) \wedge F(a, s)$ , which is, by definition,  $IP_s(s, a)$ .  $\square$

Theorems 9.3 and 9.4 show that improper slots act as neutral elements when contextualisation is defined. The converses, i.e. neutral elements of contextualisation are improper slots, will be demonstrated below by Theorems 9.9 and 9.10.

**Theorem 9.3** (Improper Slot Is Right Neutral Element).

$$\forall a, s, t(IP_s(s, a) \wedge F(a, t) \rightarrow t = t \circ s)$$

⌈ *Proof.* Let  $s$  and  $t$  be two slots and  $a$  be a filler such that  $s$  is the improper slot of  $a$  ( $IP_s(s, a)$ ) and  $a$  fills  $t$  ( $F(t, a)$ ). According to Theorem 9.2,  $s = s \circ s$ . According to Axiom 9.2, there is a  $u$  such that  $u = t \circ s$ . So  $u = t \circ (s \circ s)$ . By associativity (Axiom 9.6), we know that  $u = (t \circ s) \circ s$ . So  $u = u \circ s$ . Finally, as  $t \circ s = u \circ s$ , by Axiom 9.5,  $t = u$ . Thus  $t = t \circ s$ .  $\square$

**Theorem 9.4** (Improper Slot Is Left Neutral Element).

$$\forall a, s, t (IP_s(s, a) \wedge P_s(t, a) \rightarrow t = s \circ t)$$

⌈ *Proof.* Let  $s$  and  $t$  be slots, and  $a$  a filler, such that  $s$  is the improper slot of  $a$  ( $IP_s(s, a)$ ) and  $t$  is a slot of  $a$  ( $P_s(t, a)$ ). With Theorem 9.2, we know that  $s = s \circ s$ . By Axiom 9.2 (Contextualisable iff Contextualisation Exists), there is a  $u$  such that  $u = s \circ t$ . So  $u = (s \circ s) \circ t$ , which, by Contextualisation Associativity (Axiom 9.6), gives  $u = s \circ (s \circ t)$ . So  $u = s \circ u$ . Finally, as  $s \circ t = s \circ u$ , by Axiom 9.4,  $t = u$ . Thus  $t = s \circ t$ .  $\square$

Theorem 9.5 states that if a slot  $s$  is the result of a contextualisation of some slot by  $t$  and  $t$  is the result of a contextualisation of some slot by  $s$ , then  $s = t$ .

**Theorem 9.5** (Mutual Contextualisation is Identity).

$$\forall s, t, u, v (s = t \circ u \wedge t = s \circ v \rightarrow s = t)$$

⌈ *Proof.* Let  $s, t, u$  and  $v$  be slots such that  $s = t \circ u$  and  $t = s \circ v$ . By replacing  $s$  by  $t \circ u$  in  $t = s \circ v$ , we know that  $t = (t \circ u) \circ v$ . By Contextualisation Associativity (Axiom 9.6), we know  $t = t \circ (u \circ v)$ . With the same reasoning, we know  $s = s \circ (v \circ u)$ . With Theorem 9.1,  $u = v$ . So, by replacing  $v$  by  $u$ , we know that  $s = t \circ u$  and  $t = t \circ (u \circ u)$ .  $u$  being an improper slot, we know by Theorem 9.2 that  $u = u \circ u$ . So  $t = t \circ u$ . Therefore by Unicity of Contextualisation (Axiom 9.3),  $s = t$ .  $\square$

As said when introducing contextualisation, the resulting slot and the contextualising slot should have the same owner. This is guaranteed by Theorem 9.6. Furthermore, the resulting slot and the contextualised slot have the same filler, as expressed by Theorem 9.7.

**Theorem 9.6** (Contextualisation Same Owner).

$$\forall u, s, t (u = s \circ t \rightarrow SO(u, s))$$

⌈ *Proof.* Let  $u, s$  and  $t$  be such that  $u = s \circ t$ . By Axiom 9.1, we know that  $u, s$  and  $t$  are slots, i.e. they have an owner. We call  $a$  the owner of  $u$  ( $P_s(u, a)$ ). The owners of  $s$  and  $t$  are not useful for this proof, so we ignore them. By 3.4, we know that  $a$  as an improper slot  $v$  ( $IP_s(v, x)$ ). By Improper Slot Is Left Neutral Element (Theorem 9.4), we know that  $u = v \circ u$ . So  $u = v \circ (s \circ t)$ . By Contextualisation Associativity (Axiom 9.6),  $u = (v \circ s) \circ t$ . As there is a contextualisation  $v \circ s$ , we know by Axiom 9.2 that there is a  $b$  that fills  $v$  ( $F(b, v)$ ) and owns  $s$  ( $P_s(s, b)$ ). The slot  $v$  is filled by  $a$  and  $b$ , which gives, by 3.4, that  $a = b$ . This means that  $s$  and  $u$  are both owned by  $a$ , i.e. they have the same owner.  $\square$

**Theorem 9.7** (Contextualisation Same Filler).

$$\forall u, s, t (u = s \circ t \rightarrow SF(u, t))$$

⌈ *Proof.* Let  $u, s$  and  $t$  be such that  $u = s \circ t$ . By Axiom 9.1, we know that  $u, s$  and  $t$  are slots. With 3.7, we know there is a  $a$  that fills  $t$ . We want to prove that  $a$  also fills  $u$ . By Axiom 8.2,  $a$  has an improper slot  $v$ . According to Axiom 9.2,  $t \circ v$  exists. According to Axiom 9.2, there is a  $b$  that fills  $s$  and owns  $t$ . According to Theorem 9.6,  $t \circ v$  and  $t$  have the same owner. This owner being unique (Axiom 8.1),  $t \circ v$  is owned by  $b$ . Therefore,  $s \circ (t \circ v)$  exists. By Contextualisation Associativity (Axiom 9.6),  $s \circ (t \circ v) = (s \circ t) \circ v = u \circ v$ . So, by Axiom 9.2, there is something that fills  $u$  and owns  $v$ . With Axiom 8.1, there is only one owner, namely  $a$ . Therefore,  $u$  is filled by  $a$ . As  $a$  fills  $t$  and  $u$ , by Definition 8.3,  $SF(u, t)$ .  $\square$

We mentioned earlier that because we removed Slot Inheritance Axiom 3.5, we would need to prove Parthood Transitivity Theorem 3.7 using our new theory. Here is a proof of it, using contextualisation.

**Theorem 9.8** (Parthood Transitivity).

$$\forall a, b, c (P(a, b) \wedge P(b, c) \rightarrow P(a, c))$$

*Proof.* Let  $a$ ,  $b$  and  $s$  be three fillers such that  $a$  is a part of  $b$  and  $b$  is a part of  $c$ . We want to prove that  $a$  is a part of  $c$ , i.e. that there is some slot owned by  $c$  and filled by  $a$ . By definition of parthood (3.1), there are two slots  $s$  and  $t$  such that  $F(a, s) \wedge P_s(s, b)$  and  $F(b, t) \wedge P_s(t, c)$ . According to Axiom 9.2, as  $b$  fills  $t$  and owns  $s$ , there is a slot  $u$  such that  $u = t \circ s$ . By Theorem 9.7,  $u$  and  $s$  have the same filler, i.e.  $a$ . By Theorem 9.6,  $u$  and  $t$  have the same owner, i.e.  $c$ . Therefore,  $u$  being filled by  $a$  and owned  $c$ ,  $a$  is a part of  $c$ .  $\square$

As already mentioned before, Theorems 9.9 and 9.10 show that neutral elements are improper slots.

**Theorem 9.9** (Right Neutral Element Is Improper Slot).

$$\forall s, t (t = t \circ s \rightarrow \exists a (IP_s(s, a) \wedge F(a, t)))$$

*Proof.* Let  $s$  and  $t$  be two slots such that  $t = t \circ s$ . According to Axiom 9.2 (Contextualisable iff Contextualisation Exists), there is a filler  $a$  that fills  $t$  and owns  $s$ . According to Theorem 9.7 (Contextualisation Same Filler) and 3.7 (Single Occupancy),  $s$  and  $t$  have the same filler, i.e.  $a$ . The slot  $s$  is filled and owned by  $a$ , so by Definition 8.4,  $s$  is the improper slot of  $a$ .  $\square$

**Theorem 9.10** (Left Neutral Element Is Improper Slot).

$$\forall s, t (t = s \circ t \rightarrow \exists a (IP_s(s, a) \wedge P_s(t, a)))$$

*Proof.* Let  $s$  and  $t$  be slots such that  $t = s \circ t$ . According to Axiom 9.2 (Contextualisable iff Contextualisation Exists), there is a filler  $a$  that fills  $s$  and owns  $t$ . According to Theorem 9.6 (Contextualisation Same Owner) and Axiom 8.1 (Single Owner),  $s$  and  $t$  have the same owner, i.e.  $a$ . The slot  $s$  is filled and owned by  $a$ , so by Definition 8.4,  $s$  is the improper slot of  $a$ .  $\square$

Theorem 9.11 shows that if  $s$  and  $t$  are contextualisable by  $u$ , then for all  $v$ ,  $s$  is a contextualisation of  $t$  by  $v$  iff  $(u \circ s)$  is a contextualisation of  $(u \circ t)$  by  $v$ .

**Theorem 9.11** (Contextualisation Stable under Contextualisation).

$$\forall s, t, u (Cb(s, u) \wedge Cb(t, u) \rightarrow \forall v (s = t \circ v \leftrightarrow u \circ s = (u \circ t) \circ v))$$

*Proof.* Let  $s$ ,  $t$  and  $u$  be slots such that  $s$  and  $t$  are contextualisable by  $u$ . Let  $v$  be a slot. We want to prove that  $s = t \circ v \leftrightarrow u \circ s = (u \circ t) \circ v$ .

Left-to-right: suppose that  $s = t \circ v$ . Let us prove that  $u \circ s = (u \circ t) \circ v$ . First, let us prove that  $(u \circ t) \circ v$  exists. For it to exist, according to Axiom 9.2, there must be some filler that fills  $u \circ t$  and owns  $v$ . According to Axiom 9.2,  $s = t \circ v$  implies that there is a filler  $a$  that fills  $t$  and owns  $v$ . Let us show that  $a$  fills  $u \circ t$ . According to Theorem 9.7 (Contextualisation Same Filler),  $u \circ t$  and  $t$  have the same filler, i.e.  $a$ . Therefore, as there is a filler that fills  $u \circ t$  and owns  $v$ ,  $(u \circ t) \circ v$  exists.

Let us now prove that  $(u \circ t) \circ v$  equals  $u \circ s$ . By Axiom 9.6 (Contextualisation Associativity),  $(u \circ t) \circ v = u \circ (t \circ v)$ . Using the fact that  $s = t \circ v$ , we get that  $(u \circ t) \circ v = u \circ s$ .

Right-to-left: suppose that  $u \circ s = (u \circ t) \circ v$ . Let us prove that  $s = t \circ v$ . By Axiom 9.6 (Contextualisation Associativity),  $u \circ s = u \circ (t \circ v)$ . By Axiom 9.4 (Injectivity to the Left),  $s = t \circ v$ .  $\square$

Definition 3.6 defines what a direct slot is on the basis of Bennett’s theory, in particular by using multiple owners for the same slot. However, since A8.1 prevents any slot from having multiple owners, this definition is not relevant anymore. Direct slots can be redefined using slot contextualisation: a direct slot is a slot that is not the result of the contextualisation of a proper slot by another proper slot; put differently, a direct slot  $s$  can be equal to  $t \circ u$  only if  $t$ ,  $u$ , or both are improper slots (which implies that if one of them is not an improper slot, it is identical to  $s$ ).

**Definition 9.2** (Direct Slot — With Contextualisation).

$$DP'_s(s, a) \triangleq P_s(s, a) \wedge \forall t, u [s = t \circ u \rightarrow (\exists b(IP_s(t, b))) \vee (\exists c(IP_s(u, c)))]$$

We defined the *slot contextualisation* relation and operator in order to recontextualise parts when they should be inherited. After stating the domain and existential conditions of contextualisation, we explained why it should be unique and associative. After exploring special cases of contextualisation, we demonstrated expected properties, such as, in contextualisation  $u = s \circ t$ ,  $u$  and  $s$  have the same owner, and  $u$  and  $t$  have the same filler. We also proved that parthood transitivity holds. Exploiting this relation of slot contextualisation, we will define in the next sections basic mereological relations between slots.

## 9.2 Slot Parthood

We first define slot (general) parthood, then slot proper parthood.

If  $u = s \circ t$ , as in Figure 9.1, the two slots  $s$  and  $u$  of the same owner  $a$  cannot be seen as representing two non-overlapping locations in the mereological structure of  $a$ . In fact,  $u = s \circ t$  counts as a copy of  $t$ , which is a slot of the filler of  $s$ . Matching the parthood relation between  $c$  (the filler of  $t$ ) and  $b$  (the filler of  $s$ ), we propose to consider  $u$  (the contextualisation of  $t$  by  $s$ ) as a part of  $s$  and define a parthood relation between slots (*PoS*) on the basis of the contextualisation relation, as expressed by Definition 9.3. Lemma 9.12 gives the domain and the range of the relation. As we will see, this will enable the characterisation of a classical mereology among slots of the same owner.

**Definition 9.3** (Part of Slot).

$$PoS(u, s) \triangleq \exists t (u = s \circ t)$$

**Lemma 9.12** (*PoS* Domain and Range).

$$\forall s, t (PoS(s, t) \rightarrow S(s) \wedge S(t))$$

[*Proof.* By Definition 9.3 and Axiom 9.1. □

The relation *PoS* is conditionally reflexive, anti-symmetrical and transitive, as respectively expressed by Theorems 9.13 to 9.15. Except for the restriction to slots in reflexivity, these are the properties expected from a parthood relation, such as in classical mereology (Varzi, 2019).

**Theorem 9.13** (Conditional *PoS* Reflexivity).

$$\forall s (S(s) \rightarrow PoS(s, s))$$

⌈*Proof.* Let  $s$  be a slot and  $a$  its owner ( $P_s(s, a)$ ). By 3.7, we know that there is a  $b$  that fills  $s$  ( $F(b, s)$ ). By Axiom 8.2,  $b$  has an improper slot  $t$  ( $P_s(t, b) \wedge F(b, t)$ ). By Axiom 9.2 (Contextualisable iff Contextualisation Exists) and Theorem 9.3 (Improper Slot Is Right Neutral Element), we know that  $s \circ t = s$ . By Definition 9.3,  $PoS(s, s)$ .  $\square$

**Theorem 9.14** (*PoS Anti-Symmetry*).

$$\forall s, t (PoS(s, t) \wedge PoS(t, s) \rightarrow s = t)$$

⌈*Proof.* Derives directly from Theorem 9.5 (Mutual Contextualisation is Identity).  $\square$

**Theorem 9.15** (*PoS Transitivity*).

$$\forall s, t, u (PoS(s, t) \wedge PoS(t, u) \rightarrow PoS(s, u))$$

⌈*Proof.* Let  $s, t$  and  $u$  be slots such that  $PoS(s, t)$  and  $PoS(t, u)$ . By Definition 9.3, there are  $v$  and  $w$  such that  $s = t \circ v$  and  $t = u \circ w$ . By replacing  $t$  in the expression of  $s$ , we get  $s = (u \circ w) \circ v$ . According to Axiom 9.6, the contextualisation is associative, therefore  $s = u \circ (w \circ v)$ . By Definition 9.3,  $PoS(s, u)$ .  $\square$

Two slots standing in a *PoS* relation have the same owner, as expressed by Theorem 9.16. Furthermore, all slots are slots of  $a$  iff they are slot-parts of their  $a$ 's improper slot, as expressed by Theorem 9.17.

**Theorem 9.16** (*PoS Same Owner*).

$$\forall s, t (PoS(s, t) \rightarrow SO(s, t))$$

⌈*Proof.* Derives directly from Theorem 9.6 (Contextualisation Same Owner).  $\square$

**Theorem 9.17** (Slots iff Slot-Parts of Improper Slot).

$$\forall a, s (IP_s(s, a) \rightarrow \forall t (P_s(t, a) \leftrightarrow PoS(t, s)))$$

⌈*Proof.* Let  $a$  be a filler and  $s$  its improper slot ( $IP_s(s, a)$ ). Let  $t$  be a slot.  
 Left-to-right: suppose that  $t$  is a slot of  $a$ . By Theorem 9.4 (Improper Slot Is Left Neutral Element),  $t = s \circ t$ . Therefore, by Definition 9.3,  $PoS(t, s)$ .  
 Right-to-left: suppose that  $t$  is a slot-part of  $s$ . By Theorem 9.16,  $s$  and  $t$  have the same owner. Therefore,  $t$  is a slot of  $a$ .  $\square$

The mereology we are currently defining has an uncommon characteristic. Indeed, the relation of slot-parthood *PoS* is locally restricted. As showed by Theorems 9.16 and 9.17, this relation can only hold between slots of the same owner and all slots of a filler are parts of its improper slots.<sup>3</sup> Thus, there are as many separated mereological structures as slot-owners. Two slots owned by different entities cannot be mereologically related. Similarly, we will later introduce other mereological relations that can also only hold between slots of the same owner (Theorems 9.23, 9.29 and 9.40).

Moreover, the mereological structure of slots and the mereological structure of fillers constrain each other, as expressed by Theorem 9.18. For any pair of slots  $t$  and  $s$ , if  $t$ , filled by  $b$ , is a slot-part of  $s$ , filled by  $a$ , then  $b$  is a part of  $a$ . However, if  $b$  is a part

<sup>3</sup>As we will see in Section 9.6 and especially with Theorem 9.57, it means that improper slots are local universes. For a given owner, there is nothing bigger than its improper slot, i.e. the improper slot is not a proper part of something. Therefore, this mereological theory does not have junks, as defined in (Varzi, 2019, § 4.2) or (Cotnoir and Varzi, 2021, pp. 220–229).

of  $a$ , it is not true that for every pair of slots respectively filled by  $b$  and  $a$ , the first slot is a part of the second one. As a matter of fact, these two slots do not necessarily fulfil the contextualisation conditions. As an example, consider two molecule universals: METHANE and CARBONDIOXIDE. Both have the universal CARBON as a part, as it fills a slot of each. The slot of METHANE filled by CARBON is not a slot-part of the improper slot of CARBONDIOXIDE. Reciprocally, the slot of CARBONDIOXIDE filled by CARBON is not a slot-part of the improper slot of METHANE. What we can state is that if  $b$  is a part of  $a$ , then, there are (at least) two slots respectively filled by  $b$  and  $a$  with the first one being a part of the second; in particular, this will be satisfied with the improper slot of  $a$ .

**Theorem 9.18** (Slot Structure and Filler Structure constrain Each Other).

$$\forall a, b (\exists s, t (PoS(t, s) \wedge F(a, s) \wedge F(b, t)) \leftrightarrow P(b, a))$$

⌈ *Proof.* Left-to-right: let  $a$  and  $b$  be two fillers, and  $s$  and  $t$  be two slots, such that  $F(b, t)$ ,  $F(a, s)$  and  $PoS(t, s)$ . By Definition 9.3, there is a  $u$  such that  $t = s \circ u$ . By Theorem 9.7 (Contextualisation Same Filler) and 3.7 (Single Occupancy),  $t$  and  $u$  have the same filler, namely  $b$ . By Axioms 9.2 (Contextualisable iff Contextualisation Exists) and 3.7, we know that there is an entity filling  $s$  and owning  $u$ , and this entity is  $a$ . Therefore,  $P_s(u, a)$  and  $F(b, u)$ . By 3.1, this means that  $P(b, a)$ .

Right-to-left: let  $a$  be a filler and  $b$  a part of  $a$  ( $P(b, a)$ ). According to 3.4, there is a  $s$  that is the improper slot of  $a$  ( $IP_s(s, a)$ ). By the Parthood Definition (3.1), there is a  $t$  such that  $F(b, t) \wedge P_s(t, a)$ . According to Theorem 9.17,  $t$  is a slot-part of  $s$ . □

Theorem 9.19 states that  $PoS$  is stable under contextualisation, i.e.  $u$  is a slot-part of  $t$  iff  $s \circ u$  is a slot-part of  $s \circ t$ .

**Theorem 9.19** ( $PoS$  Stable under Contextualisation).

$$\forall s, t, u (Cb(t, s) \wedge Cb(u, s) \rightarrow (PoS(u, t) \leftrightarrow PoS(s \circ u, s \circ t)))$$

⌈ *Proof.* Let  $s, t, u$  such that  $t$  and  $u$  are contextualisable by  $s$ .

Left-to-right: suppose that  $u$  is a slot-part of  $t$  ( $PoS(u, t)$ ). Therefore, there is a  $v$  such that  $u = t \circ v$ . Thus, by Theorem 9.11,  $s \circ u = s \circ (t \circ v)$ . By Contextualisation Associativity (Axiom 9.6),  $s \circ u = (s \circ t) \circ v$ . By Definition 9.3, this means that  $PoS(s \circ u, s \circ t)$ .

Right-to-left: suppose that  $s \circ u$  is a slot-part of  $s \circ t$  ( $PoS(s \circ u, s \circ t)$ ). Therefore, there is a  $v$  such that  $s \circ u = (s \circ t) \circ v$ . By associativity,  $s \circ u = s \circ (t \circ v)$ , i.e. by Theorem 9.11,  $u = t \circ v$ . By Definition 9.3, this means that  $PoS(u, t)$ . □

With the relation  $PoS$ , we can define the relation of proper parthood between slots  $PPoS$ , as expressed in Definition 9.4. This definition follows that of proper parthood in classical mereology, presented in Section 2.2.2.

**Definition 9.4** (Proper Part of Slot).

$$PPoS(s, t) \triangleq PoS(s, t) \wedge s \neq t$$

As is the case in classical mereology, this relation is irreflexive, asymmetrical and transitive, as expressed by Theorems 9.20 to 9.22. Furthermore, two slots in a  $PPoS$ -relation share the same owner, as stated by Theorem 9.23.

**Theorem 9.20** ( $PPoS$  Irreflexivity).

$$\forall s (\neg PPoS(s, s))$$

[*Proof.* Derives directly from Definition 9.4. □

**Theorem 9.21** (*PPoS Asymmetry*).

$$\forall s, t (PPoS(s, t) \rightarrow \neg PPoS(t, s))$$

[*Proof.* Derives directly from Definition 9.4 and Theorem 9.14. □

**Theorem 9.22** (*PPoS Transitivity*).

$$\forall s, t, u (PPoS(s, t) \wedge PPoS(t, u) \rightarrow PPoS(s, u))$$

[*Proof.* Derives directly from Definition 9.4 and Theorems 9.14 and 9.15. □

**Theorem 9.23** (*PPoS Same Owner*).

$$\forall s, t (PPoS(s, t) \rightarrow SO(s, t))$$

[*Proof.* Derives directly from Definition 9.4 and Theorem 9.16. □

A slot can be in relation *PPoS* with either a proper or an improper slot. These two possibilities are pictured in Figure 9.4. Figure 9.4a is identical to Figure 9.1, adding the *PPoS*(*u*, *s*). Figure 9.4b highlights an important feature of contextualisation by improper slots: every proper slot of a filler *a* is a slot-proper-part of *a*'s improper slot, as expressed by Theorem 9.24.



(a) *PPoS*(*u*, *s*): proper part of proper slot ( $u = s \circ t$ ) (b) *PPoS*(*t*, *s*): proper part of improper slot ( $t = s \circ t$ )

Figure 9.4: *PPoS* relation according to whole properness (dotted arrows represent the *PPoS* relation)

**Theorem 9.24** (Proper Slots iff Proper Parts Of Improper Slot).

$$\forall a, s (IP_s(s, a) \rightarrow \forall t (PP_s(t, a) \leftrightarrow PPoS(t, s)))$$

[*Proof.* Let *a* be a filler and *s* its improper slot ( $IP_s(s, a)$ ). Let *t* be a slot. We want to prove that  $PP_s(t, a) \leftrightarrow PPoS(t, s)$ .

Left-to-right: suppose *t* is a proper slot of *a* ( $PP_s(t, a)$ ). We want to prove  $PPoS(t, s) \wedge s \neq t$ . With Theorem 9.17 (Slots iff Slot-Parts of Improper Slot),  $PPoS(t, s)$ . *s* and *t* cannot be identical, as the same slot would be proper and improper, which is not possible according to Lemma 8.2 (Either Proper or Improper). Thus  $s \neq t$ .

Right-to-left: suppose *t* is proper slot-part of *s* ( $PPoS(t, s)$ ). We want to prove that *t* is a proper slot of *a*, i.e. it is a slot of *a*, but it is not filled by *a*. By Theorem 9.23, *s* and *t* have the same owner, thus  $P_s(t, a)$ . Suppose that *t* is filled by *a*, making *t* an improper slot of *a*. By Axiom 8.3,  $s = t$ . However, as  $PPoS(t, s)$ ,  $s \neq t$ : contradiction. Therefore, *t* is not filled by *a*. □



Theorem 9.25 shows that proper parthood relations between slots and proper parthood relations between fillers constrain each other.

**Theorem 9.25** (Slot Structure and Filler Structure constrain Each Other — Proper Part).

$$\forall a, b(\exists s, t(PPoS(t, s) \wedge F(a, s) \wedge F(b, t)) \leftrightarrow PP(b, a))$$

*Proof.* Let  $a$  and  $b$  be two fillers.

Left-to-right: let  $s$  and  $t$  be two slots such that  $t$  is a slot-proper-part of  $s$ ,  $a$  fills  $s$  and  $b$  fills  $t$ . According to Theorem 9.18,  $b$  is a part of  $a$ . According to 3.2, we have to prove that  $a$  is not a part of  $b$  to finish the proof. Suppose that  $a$  is a part of  $b$ . With 3.8,  $P$  is anti-symmetric. Therefore,  $a = b$ . According to Definition 9.4, as  $PPoS(t, s)$ ,  $PoS(t, s)$  and  $s \neq t$ . Furthermore, with Definition 9.3, there is some  $u$  such that  $t = s \circ u$ . With Axiom 9.2 and 3.7, there is a single entity that fills  $s$  and owns  $u$ , i.e.  $a$ . With Theorem 9.7 and 3.7,  $t$  and  $u$  are filled by the same entity, i.e.  $a$ . The slot  $u$ , being owned and filled by  $a$ , is its improper slot. By Theorem 9.3,  $s = s \circ u$ . Finally, with Axiom 9.4,  $s = t$ : contradiction. Therefore  $a$  is not a part of  $b$ , and thus,  $PP(b, a)$ .

Right-to-left: suppose that  $b$  is a proper part of  $a$ . According to 3.2,  $b$  is a part of  $a$  and  $a$  is not a part of  $b$ . According to Theorem 9.18 and  $P(b, a)$ , there are two slots  $s$  and  $t$  such that  $s$  is a slot-part of  $t$ ,  $a$  fills  $s$  and  $b$  fills  $t$ . According to Definition 9.4, we have to prove  $s \neq t$  to fulfil the proof. Suppose that  $s = t$ . Therefore, because  $F(a, s)$ ,  $F(b, t)$  and Single Occupancy (3.7),  $a = b$ . Contradiction:  $b$  is a proper part of  $a$  and they cannot be identical. Therefore,  $s \neq t$ .  $\square$

Theorem 9.26 states that  $PPoS$  is stable under contextualisation, i.e. if  $u$  is a slot-proper-part of  $t$ , then  $s \circ u$  is a slot-proper-part of  $s \circ t$ .

**Theorem 9.26** ( $PPoS$  Stable under Contextualisation).

$$\forall s, t, u(Cb(t, s) \wedge Cb(u, s) \rightarrow (PPoS(u, t) \leftrightarrow PPoS(s \circ u, s \circ t)))$$

*Proof.* Let  $s$ ,  $t$  and  $u$  be slots such that  $t$  and  $u$  are contextualisable by  $s$ .

Left-to-right: suppose that  $u$  is a slot-proper-part of  $t$ . By Definition 9.4,  $u$  is a slot-part of  $t$  and is different from  $t$ . We want to prove that  $s \circ u$  is a slot-proper-part of  $s \circ t$ , i.e. that  $s \circ u$  is a slot-part of  $s \circ t$  and that  $s \circ t \neq s \circ u$ . By Theorem 9.19 ( $PoS$  Stable under Contextualisation), we know that  $PoS$  is stable under contextualisation. Therefore,  $s \circ u$  is a slot-proper-part of  $s \circ t$ . Suppose now that  $s \circ t = s \circ u$ . By Axiom 9.4 (Injectivity to the Left), we infer that  $t = u$ . Contradiction. Therefore,  $s \circ t \neq s \circ u$ , and  $PPoS(s \circ u, s \circ t)$ .

Right-to-left: suppose that  $s \circ u$  is a slot-proper-part of  $s \circ t$ . We want to prove that  $u$  is a slot-proper-part of  $t$ , i.e. that  $u$  is a slot-part of  $t$  and that  $u$  and  $t$  are different. By Definition 9.4,  $s \circ u$  is a slot-part of  $s \circ t$ . Therefore, by Theorem 9.19,  $u$  is a slot-part of  $t$ . We want to prove  $u \neq t$ . By Definition 9.4,  $s \circ u \neq s \circ t$ . Suppose that  $u = t$ . Thus  $s \circ t \neq s \circ t$ : contradiction. Therefore,  $u \neq t$ .  $\square$

With slot contextualisation, we defined the relations of slot-parthood and slot-proper-parthood. Those relations have the same properties as the relations of ground mereology, as presented in Sections 2.2.1 and 2.2.2: slot-parthood is (conditionally) reflexive, antisymmetric and transitive, while slot-proper-parthood is irreflexive, asymmetric and transitive. We also showed that both relations also have the same-owner property and are stable under contextualisation. Finally, we showed that slot-mereological structure and filler-mereological structure are linked and partially constrain each other. With these relations, we can enrich further the mereology of slots, by defining the slot-overlap relation.

### 9.3 Overlap of Slots

With the slot parthood relation  $PoS$ , we can define the relation of overlap between slots  $OoS$ , as expressed in Definition 9.5, following the classical definition of overlap (see Section 2.2.3).

**Definition 9.5** (Overlap of Slots).

$$OoS(s, t) \triangleq \exists u(PoS(u, s) \wedge PoS(u, t))$$

Figure 9.5 pictures two configurations in which  $s$  and  $t$  slot-overlap on  $u$ : in Figure 9.5a,  $s$  and  $t$  are filled by the same entity  $b$ . On the contrary, in Figure 9.5b,  $s$  and  $t$  have different fillers, namely  $b$  and  $d$ . Unlike Bennett’s slot-overlap, this overlap of slots is less pervasive as it is restricted to slots of the same whole and corresponds more closely to our preformal intuition.

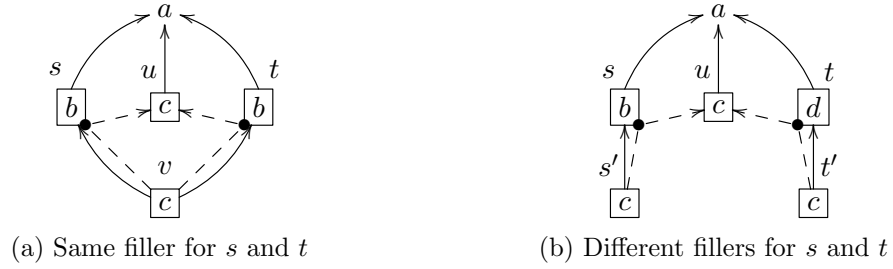


Figure 9.5: Slot-overlap between  $s$  and  $t$

With our relation of overlap of slots and with the existence of contextualisations, the model ambiguity depicted in Figure 3.5 with Bennett’s theory disappears: we have either the model in Figure 9.6a for the string “xyz” or the model in Figure 9.6b for the string “xyyz” (slots for parts “x” and “z” in “xy” and “yz” and their contextualisations are omitted for readability). The model in Figure 9.6a cannot represent “xyyz” as there is only one slot filled by “y” in the whole.

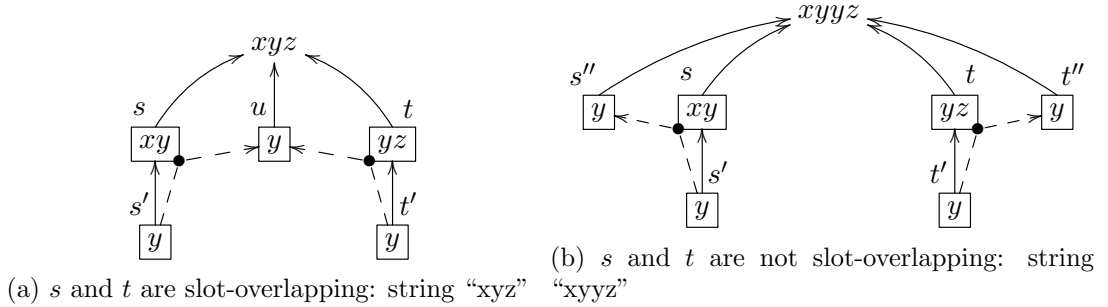


Figure 9.6: Non-ambiguous models for “xyz” and “xyyz”

This relation is conditionally reflexive and symmetrical, as expressed by Theorems 9.27 and 9.28. Except for the premise in Theorem 9.27, these theorems are common theorems of classical mereology.

**Theorem 9.27** (Conditional  $OoS$  Reflexivity).  $\forall s(S(s) \rightarrow OoS(s, s))$

[Proof. Relation  $PoS$  is reflexive, therefore, by definition  $OoS$  is also reflexive. □

**Theorem 9.28** ( $OoS$  Symmetry).  $\forall s, t(OoS(s, t) \rightarrow OoS(t, s))$

[Proof. According to the Definition 9.5 and the commutativity of the AND logical operator. □

Two overlapping slots have the same owner, as showed by Theorem 9.29.

**Theorem 9.29** ( $OoS$  Same Owner).  $\forall s, t(OoS(s, t) \rightarrow SO(s, t))$

⌈ *Proof.* Let  $s$  and  $t$  be two slots such that  $OoS(s, t)$ . According to Definition 9.5, we know  $\exists u(PoS(u, s) \wedge PoS(u, t))$ . With Theorem 9.6 (Contextualisation Same Owner) and Axiom 8.1 (Single Owner),  $SO(s, t)$ .  
 ⌋

As expressed by Lemma 9.30, if two slots overlap, they also overlap with the slots the other one is part of.

**Lemma 9.30** (Overlap with Part Implies Overlap with Whole).

$$\forall s, t, u (OoS(u, t) \wedge PoS(t, s) \rightarrow OoS(u, s))$$

⌈ *Proof.* Let  $s, t$  and  $u$  be three slots such that  $OoS(u, t) \wedge PoS(t, s)$ . By Definition 9.5, we know that there is a  $v$  such that  $PoS(v, u) \wedge PoS(v, t)$ . By Theorem 9.15 ( $PoS$  Transitivity), we know that  $PoS(v, s)$ .  
 ⌋ Therefore, there is a common slot-part between  $u$  and  $s$ , namely  $v$ . By definition,  $OoS(u, s)$ .  $\square$

Lemma 9.31 expresses the fact that all the slots of any filler  $a$  overlap with  $a$ 's improper slot.

**Lemma 9.31** (Slot-Overlap With Improper Slot).  $\forall a, s, t (IP_s(s, a) \wedge P_s(t, a) \rightarrow OoS(s, t))$

⌈ *Proof.* Let  $a$  be a filler,  $s$  its improper slot ( $IP_s(s, a)$ ) and  $t$  one of its slots ( $P_s(t, a)$ ). According to Theorem 9.13 (Conditional  $PoS$  Reflexivity),  $PoS(t, t)$ . According to Theorem 9.17 (Slots iff Slot-Parts of Improper Slot),  $PoS(t, s)$ . By Definition 9.5,  $OoS(s, t)$ .  $\square$

From the definition of  $OoS$ , we can deduce that every slot  $s$  that is a slot-part of a slot  $t$  overlaps with that slot, as stated by Lemma 9.32.

**Lemma 9.32** ( $PoS$  Implies  $OoS$ ).  $\forall s, t (PoS(s, t) \rightarrow OoS(s, t))$

⌈ *Proof.* Let  $s$  and  $t$  be slots such that  $PoS(s, t)$ . We want to prove that there is a slot that is a slot-part of  $s$  and  $t$ . The slot  $s$  fulfils these requirements. It is a slot-part of  $t$  by hypothesis. And the slot-part relation being reflexive, it is also a slot-part of itself.  $\square$

Theorem 9.33 states that  $OoS$  is stable under contextualisation, i.e. if  $t$  and  $u$  overlap, then  $s \circ t$  and  $s \circ u$  overlap.

**Theorem 9.33** ( $OoS$  Stable under Contextualisation).

$$\forall s, t, u (Cb(t, s) \wedge Cb(u, s) \rightarrow (OoS(t, u) \leftrightarrow OoS(s \circ t, s \circ u)))$$

⌈ *Proof.* Let  $s, t$  and  $u$  be slots such that  $t$  and  $u$  are contextualisable by  $s$ .

Left-to-right: suppose that  $u$  slot-overlaps with  $t$  ( $OoS(t, u)$ ). By Definition of  $OoS$  (Definition 9.5), we know there is a  $v$  such that  $PoS(v, t) \wedge PoS(v, u)$ . By Theorem 9.16 ( $PoS$  Same Owner),  $v$  has the same owner as  $t$ , thus  $v$  is contextualisable by  $s$ . By Theorem 9.19 ( $PoS$  Stable under Contextualisation) and  $PoS(v, t) \wedge PoS(v, u)$ , we deduce  $PoS(s \circ v, s \circ t) \wedge PoS(s \circ v, s \circ u)$ . Therefore, by Definition 9.5,  $OoS(s \circ t, s \circ u)$ .

Right-to-left: suppose that  $s \circ t$  overlaps with  $s \circ u$ . By Definition D9.5, there is a  $v'$  such that  $PoS(v', s \circ t)$  and  $PoS(v', s \circ u)$ . By Definition 9.3, there is  $v_t$  and  $v_u$  such that  $v' = (s \circ t) \circ v_t = (s \circ u) \circ v_u$ . By Contextualisation Associativity (Axiom 9.6),  $s \circ (t \circ v_t) = s \circ (u \circ v_u)$ . Thus, by Axiom 9.4,  $t \circ v_t = u \circ v_u$ . Let us call  $v$  the slot equal to  $t \circ v_t$  and  $u \circ v_u$ . By Definition 9.3,  $v$  is a slot-part of  $t$  and  $u$ . Thus,  $t$  and  $u$  overlap.  $\square$

With Theorem 9.18, we showed cross-constraints between  $PoS$  relations between slots and  $P$  relations between fillers. Theorem 9.34 expresses a constraint from the  $OoS$  relations between slots towards the  $O$  relations between fillers. However, unlike Theorem 9.18, we did not find interesting constraints from  $O$  relations between fillers towards  $OoS$  relations between slots. Indeed, if two entities  $a$  and  $b$  overlap, it means that they have a common part  $c$ , but in two different slots  $s$  and  $t$ , as slots cannot have multiple owners. Nothing ensures that there are two slots  $u$  and  $v$  such that  $u \circ s = v \circ t$ .

**Theorem 9.34** (Slot-Overlap Constrains Overlap between Fillers).

$$\forall a, b, s, t (OoS(s, t) \wedge F(a, s) \wedge F(b, t) \rightarrow O(a, b))$$

*Proof.* Let  $s$  and  $t$  be two slots and  $a$  and  $b$  their respective fillers. Suppose that  $s$  and  $t$  slot-overlap ( $OoS(s, t)$ ). By Definition 9.5, it means that there is a slot  $u$  that is a slot-part of both  $s$  and  $t$ . By Lemma 9.12 and 3.7, there is some filler  $c$  that fills  $u$ . By Theorem 9.18,  $c$  is a part of  $a$  and  $b$ . Therefore, by 3.3,  $a$  and  $b$  are overlapping.  $\square$

In our theory, as a slot has a unique owner, Bennett's relation of slot-overlap between fillers, presented in Section 3.3.3, collapses to the identity on fillers. Fortunately, the slot-mereological structure we have introduced enables us to grasp a more adequate notion of overlap.

We defined a slot-overlap relation, which has the same properties as in classical mereology: (conditionally) reflexive and symmetric. Furthermore, this relation has the same-owner property and is stable under contextualisation. Those mereological relations  $PoS$  and  $OoS$  will now be used to introduce supplementation principles in the next section.

## 9.4 Slot Supplementation Principles

First of all, let us note that the main motivation behind Bennett's slot mereology and ours is to represent entities that can have a same part multiple times. Therefore, we do not want supplementation over fillers. Indeed, the DIHYDROGEN molecule universal has as proper part the HYDROGEN atom universal (twice), but there is no proper part of DIHYDROGEN that does not overlap with HYDROGEN.

Our Single Owner Axiom (Axiom 8.1) makes Garbacz's revisions of Bennett's Axiom 3.8, presented in Section 3.2.1, vacuously true theorems. Therefore, we have to find a new axiomatic formulation for slot strong supplementation. Like in classical mereology, we have introduced here notions of parthood, proper parthood and overlap relations between slots. This means that we can adapt the Strong Supplementation axiom of classical mereology (see Section 2.3.3), as expressed by Axiom 9.7 below. We previously explained that in our mereology, the mereological relations can only hold between slots of the same owner. However, in the Slot Strong Supplementation Axiom, we need not impose that  $s$  and  $t$  have the same owner. Indeed, if they do not have the same owner, none of the possible slot-parts of  $t$  can overlap with  $s$ , and in particular,  $t$  does not overlap with  $s$ . Thus, the axiom is trivially true in that case.

**Axiom 9.7** (Slot Strong Supplementation).

$$\forall s, t [S(s) \wedge S(t) \rightarrow (\neg PoS(t, s) \rightarrow \exists u (PoS(u, t) \wedge \neg OoS(u, s)))]$$

From this axiom, we can deduce a theorem of Slot Weak Supplementation, expressed by T9.35. This theorem states that if a whole has a proper slot, then it must have another slot that does not overlap with the first one.

**Theorem 9.35** (Slot Weak Supplementation).

$$\forall s, t (PPoS(s, t) \rightarrow \exists u (PoS(u, t) \wedge \neg OoS(u, s))$$

*Proof.* Let  $s$  and  $t$  be two slots such that  $PPoS(s, t)$ , i.e.  $PoS(s, t) \wedge s \neq t$ . From  $PoS(s, t)$ , and  $PoS$  Anti-Symmetry (Theorem 9.14), we deduce  $\neg PoS(t, s)$ . From  $\neg PoS(t, s)$  and Slot Strong Supplementation (Axiom 9.7), we deduce  $\exists u (PoS(u, t) \wedge \neg OoS(u, s))$ .  $\square$

We can also prove theorems of *OoS*-Extensionality (Theorem 9.36) and *PPoS*-Extensionality (Theorem 9.37). For Theorem 9.36, the premise  $S(s) \wedge S(t)$  is necessary: without it,  $\forall u, OoS(s, u) \leftrightarrow OoS(t, u)$  would be vacuously true for every pair of fillers, leading to all fillers being equal.

**Theorem 9.36** (*OoS*-Extensionality).

$$\forall s, t [S(s) \wedge S(t) \rightarrow (\forall u, OoS(s, u) \leftrightarrow OoS(t, u)) \rightarrow s = t]$$

*Proof.* Let  $s$  and  $t$  be two slots such that  $\forall u, OoS(s, u) \leftrightarrow OoS(t, u)$  (let us call this H1). We want to prove that  $s = t$ . Suppose that  $s$  is different from  $t$  ( $s \neq t$ ). Either  $PoS(t, s)$  or  $\neg PoS(t, s)$ .

Suppose first that  $PoS(t, s)$ . By Definition 9.4, we know  $PPoS(t, s)$ . According to the Slot Weak Supplementation Theorem 9.35, there is a  $u$  such that  $PoS(u, s) \wedge \neg OoS(u, t)$ . With  $PoS(u, s)$  and Lemma 9.32 (*PoS* Implies *OoS*), we know that  $OoS(u, s)$ . By H1, we deduce that  $OoS(u, t)$ . Contradiction:  $u$  does and does not overlap  $t$ . Suppose now that  $\neg PoS(t, s)$ . By Slot Strong Supplementation Axiom 9.7, there is an  $u$  such that  $PoS(u, t) \wedge \neg OoS(u, s)$ . By the same reasoning, we also get a contradiction:  $u$  does and does not overlap  $s$ .

Thus,  $s = t$ . □

**Theorem 9.37** (*PPoS*-Extensionality).

$$\forall s, t [\exists u (PPoS(u, s) \vee PPoS(u, t)) \rightarrow (\forall u, PPoS(u, s) \leftrightarrow PPoS(u, t)) \rightarrow s = t]$$

*Proof.* Let  $s$  and  $t$  be two slots such that one of them as a slot-proper-part  $u$ , and that  $\forall u, PPoS(u, s) \leftrightarrow PPoS(u, t)$  (let us call this H1). We want to prove that  $s = t$ . Suppose that  $s$  is different from  $t$ . Either  $PoS(t, s)$  or  $\neg PoS(t, s)$ .

Suppose first that  $PoS(t, s)$ . Therefore, by Definition of *PPoS* (Definition 9.4),  $PPoS(t, s)$ , and by H1,  $PPoS(t, t)$ . Contradiction, as *PPoS* is irreflexive (Theorem 9.20).

Suppose thus that  $\neg PoS(t, s)$ . According to Slot Strong Supplementation (Axiom 9.7), there is a  $v$  such that  $PoS(v, t) \wedge \neg OoS(v, s)$ . Suppose that  $t$  and  $v$  are different. By definition,  $PPoS(v, t)$ , and by H1,  $PPoS(v, s)$ . However,  $v$  does not overlap  $s$  ( $\neg OoS(v, s)$ ). Contradiction. Thus  $t = v$ , and  $\neg OoS(t, s)$ . However,  $u$  is a slot-part of  $s$  and  $t$ : by hypothesis, it is a slot-part of one of them, and by H1, it is also a slot-part of the other one. Therefore,  $OoS(t, s)$ . Contradiction.

Thus  $s = t$ . □

We added the Slot Strong Supplementation axiom and proved Slot Weak Supplementation and *OoS*-Extensionality. Thanks to our mereological relations among slots, this axiom and these theorems are syntactically similar or identical to those of classical mereology. Compared to Bennett's theory, our supplementation correctly captures the spirit of supplementation and extensionality. In the next section, we will develop further our mereology of slots and introduce the sum of slots.

## 9.5 Slot-Sum

Bennett did not propose any sum operator in her theory. But she made three remarks presented in Section 3.1.4. The first one was that a fully unrestricted axiom of unrestricted sum would not be reasonable, as it would lead to sum of slots and fillers. Therefore, we follow Bennett here: unrestricted sum is off the table. The second remark was that sums of fillers cannot be unique. As we do not propose any sums of fillers, there is nothing more to add here. Finally, with the third remark, she argues that, in classical extensional mereology, Strong Supplementation entails that all "composites" of  $a$  and  $b$  are identical. Importantly, our axiom of Slot Strong Supplementation does not entail the same conclusion. Indeed, in our theory, different entities which own different slots may have the same entities as part, whether with the same multiplicities (like BUTANE or ISOBUTANE) or not (like METHANE and ETHANE). We agree with Bennett on her entire reasoning

about sums of fillers. However, there is one point where our visions diverge: Bennett's theory does not have a parthood relation over slots, but our theory does. In the following section, I will show how sums over slots can be introduced in the theory.

We introduce the relation  $SoS_1$  to capture the intuition of sum of slots.  $SoS_1(u, s, t)$  means that  $u$  is a sum of  $s$  and  $t$ . We first define  $SoS_1$  and  $SoS_2$  following Definitions 2.7 and 2.8, presented in Section 2.4.1. The definitions are given by Definitions 9.6 and 9.7. We will show that these two definitions are equivalent, as proved by Theorem 9.39. The domains of  $SoS_1$  is given by Lemma 9.38.

**Definition 9.6** (Sum with Parthood).

$$SoS_1(u, s, t) \triangleq PoS(s, u) \wedge PoS(t, u) \wedge \forall v (PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$$

**Definition 9.7** (Sum with Overlap).

$$SoS_2(u, s, t) \triangleq \forall v (OoS(u, v) \leftrightarrow OoS(s, v) \vee OoS(t, v))$$

**Lemma 9.38** (Domains of Sum).

$$\forall s, t, u (SoS_1(u, s, t) \rightarrow S(s) \wedge S(t) \wedge S(u))$$

$\lceil$  *Proof.* Trivially by Definition 9.6 and Lemma 9.12.  $\square$

**Theorem 9.39** ( $SoS_1$  and  $SoS_2$  are Equivalent).

$$\forall s, t [SO(s, t) \rightarrow \forall u (SoS_1(u, s, t) \leftrightarrow SoS_2(u, s, t))]$$

$\lceil$  *Proof.* Let  $s$  and  $t$  be two slots with the same owner. Let  $u$  be a slot.

Left-to-right: suppose  $SoS_1(u, s, t)$ . By Definition 9.6,  $s$  and  $t$  are slot-parts of  $u$  and  $\forall v (PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$  (let us call this H). We want to prove  $\forall v (OoS(u, v) \leftrightarrow OoS(s, v) \vee OoS(t, v))$ . Let  $w$  be a slot. We want to prove that 1)  $OoS(u, w) \rightarrow OoS(s, w) \vee OoS(t, w)$  and 2)  $OoS(s, w) \vee OoS(t, w) \rightarrow OoS(u, w)$ .

1) Suppose that  $u$  and  $w$  overlap. By Definition 9.5, there is a  $x$  that is slot-part of  $u$  and  $w$ . According to H,  $x$  overlaps  $s$  or  $t$ . Suppose that  $x$  overlaps  $s$ . Therefore, by Lemma 9.30 (Overlap with Part Implies Overlap with Whole),  $OoS(s, x)$  and  $PoS(x, w)$ , we deduce that  $OoS(s, w)$ . Similarly, we can show that if  $x$  overlaps  $t$ , then  $w$  overlaps  $t$ .

2) Suppose that  $s$  overlaps  $w$ . By Lemma 9.30 and  $PoS(s, u)$ , we deduce that  $w$  and  $u$  overlap. The same reasoning holds if we suppose that  $t$  and  $w$  overlap:  $w$  overlap  $u$ .

Right-to-left: suppose  $SoS_2(u, s, t)$ , i.e.  $\forall v (OoS(v, u) \leftrightarrow OoS(s, v) \vee OoS(t, v))$  (let us call this H). We want to prove that 1)  $s$  and  $t$  are slot-parts of  $u$  and that 2)  $\forall v (PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$ .

1) Let us prove that  $s$  is a slot-part of  $u$ . Suppose that that  $s$  is not a slot-part of  $u$ . Therefore, by the Slot Strong Supplementation (Axiom 9.7), there is a  $v$  that is a slot-part of  $s$  and that does not overlap  $u$ .  $v$  being a slot-part of  $s$ , both are overlapping, by Lemma 9.32. According to H, as  $v$  overlaps  $s$ , it overlaps  $u$ . Contradiction: by Slot Strong Supplementation, it cannot overlap  $u$ . Therefore,  $s$  is a slot-part of  $u$ . A similar reasoning proves that  $t$  is slot-part of  $u$ .

2) Let  $v$  be a slot-part of  $u$ . By Lemma 9.32,  $v$  and  $u$  are overlapping. With H,  $OoS(v, s) \vee OoS(v, t)$ .  $\square$

$SoS_1$  and  $SoS_2$  being equivalent, we will use  $SoS_1$  in the remainder. However, every true proposition involving  $SoS_1$  is also true when substituting  $SoS_2$  for  $SoS_1$ . Axiom 9.8 is the only axiom we will accept involving the sum relation. It ensures that a sum of two slots exists if they have the same owner. The addition of this axiom will be enough to derive classical properties of binary sum, presented in Section 2.4.1.

**Axiom 9.8** (Sum Existence).

$$\forall s, t (SO(s, t) \rightarrow \exists u (SoS_1(u, s, t)))$$

Slots can only be summed within the same mereological structure, i.e. if they have the same owner. Therefore the resulting slot also has the same owner. This is stated by Theorem T9.40.

**Theorem 9.40** (Sum Same Owner).

$$\forall s, t, u [SoS_1(u, s, t) \rightarrow \exists a (P_s(u, a) \wedge P_s(s, a) \wedge P_s(t, a))]$$

*Proof.* Let  $u, s$  and  $t$  be slots such that  $SoS_1(u, s, t)$ . By Definition 9.6, we know  $PoS(s, u) \wedge PoS(t, u)$ . By Theorem 9.16 ( $PoS$  Same Owner) and Axiom 8.1 (Single Owner), we know that  $s, t$  and  $u$  have the same owner.  $\square$

Slot Strong Supplementation ensures the unicity of the sum, as showed by Theorem 9.41. The unicity being proved, we will use the operator  $+$  to represent sums of slots:  $s + t$  is a binary function that returns the sum of  $s$  and  $t$ . Furthermore, we will use the notation  $u = s + t$  instead of  $SoS_1(u, s, t)$ .

**Theorem 9.41** (Sum Unicity).

$$\forall s, t, u, v (SoS_1(u, s, t) \wedge SoS_1(v, s, t) \rightarrow u = v)$$

*Proof.* Let  $s, t, u$  and  $v$  be four slots such that  $u$  and  $v$  are both sums of  $s$  and  $t$ . We want to prove  $u = v$ . According to  $OoS$ -Extensionality (Theorem 9.36), if we prove  $\forall w (OoS(u, w) \leftrightarrow OoS(v, w))$ , then we have  $u = v$ . Both sums being equivalent, let us use the Definition 9.7: we know that  $\forall w (OoS(u, w) \leftrightarrow OoS(s, w) \vee OoS(t, w))$  and  $\forall w (OoS(v, w) \leftrightarrow OoS(s, w) \vee OoS(t, w))$ . Hence,  $\forall w (OoS(u, w) \leftrightarrow OoS(v, w))$ , and  $u = v$ .  $\square$

The sum of slots is idempotent and commutative, as expressed by Theorems 9.42 and 9.43. Those properties are standard results of mereology, preseted in Section 2.4.1 (Propositions 41, 42 and 43 of Varzi (2019)).

**Theorem 9.42** (Sum Idempotence).

$$\forall s (S(s) \rightarrow s + s = s)$$

*Proof.* Let  $s$  be a slot and  $a$  its owner ( $P_s(s, a)$ ). We have to prove that  $s + s = s$ , which means, by Definition 9.7,  $\forall v (OoS(v, s) \leftrightarrow OoS(v, s) \vee OoS(v, s))$ , which is tautologically true. Thus  $s + s = s$ .  $\square$

**Theorem 9.43** (Sum Commutativity).

$$\forall s, t (SO(s, t) \rightarrow s + t = t + s)$$

*Proof.* By Definition 9.7 and the commutativity of the OR operand.  $\square$

The following explores the various existing theorems resulting from the use of the slot-sum and contextualisation operators with the relations  $PoS$  and  $OoS$ . We consider the  $\circ$  operator to have a higher priority than the  $+$  operator. Therefore  $s \circ t + u$  has to be interpreted as  $(s \circ t) + u$ . The first four lemmas (Lemmas 9.44 to 9.47) are similar to Theorems 2.14 to 2.17, presented in Section 2.4.1. They illustrate how slot-sum and  $PoS$  work together. Lemma 9.44 states that if  $s + t$  exists, then  $s$  is a slot-part of  $s + t$ . Lemma 9.45 states that if  $t + u$  exists and  $s$  is a slot-part of  $t$ , then  $s$  is a slot-part of  $t + u$ . Lemma 9.46 states that if  $s + t$  is a slot-part of  $u$ , then  $s$  is a slot-part of  $u$ . And Lemma 9.47 states that  $s$  is a slot-part of  $t$  if and only if  $s + t = t$ .

**Lemma 9.44.**

$$\forall s, t (SO(s, t) \rightarrow PoS(s, s + t))$$

⌈ *Proof.* By Definition 9.6. □

**Lemma 9.45.**

$$\forall t, u [SO(t, u) \rightarrow \forall s (PoS(s, t) \rightarrow PoS(s, t + u))]$$

⌈ *Proof.* Let  $s, t$  and  $u$  be three slots such that  $t$  and  $u$  have the same owner and that  $s$  is part of  $t$ . By Lemma 9.44,  $t$  is slot-part of  $t + u$ . Thus, by  $PoS$  Transitivity (Theorem 9.15),  $PoS(s, t + u)$ . □

**Lemma 9.46.**

$$\forall s, t [SO(s, t) \rightarrow \forall u (PoS(s + t, u) \rightarrow PoS(s, u))]$$

⌈ *Proof.* Let  $s, t$  and  $u$  be three slots such that  $s$  and  $t$  have the same owner and  $s + t$  is slot-part of  $u$ . By Definition 9.6,  $PoS(s, s + t)$ . By  $PoS$  Transitivity (Theorem 9.15),  $PoS(s, u)$ . □

**Lemma 9.47.**

$$\forall s, t (PoS(s, t) \leftrightarrow s + t = t)$$

⌈ *Proof.* Left-to-right: let  $s$  be a slot-part of  $t$  ( $PoS(s, t)$ ). We want to prove  $s + t = t$ , that is, by Definition 9.7,  $\forall v (OoS(v, t) \leftrightarrow OoS(v, s) \vee OoS(v, t))$ , which is tautologically true. Thus  $s + t = t$ .  
Right-to-left: let  $s$  and  $t$  be slots such that  $s + t = t$ . By Definition 9.6,  $PoS(s, t)$ . □

The following Theorem 9.48 describes the behaviour of overlap combined with sums. It states that something overlaps the contextualisation of the sum of two operands by a slot  $s$  iff it overlaps the contextualisation of one of the operands by  $s$ .

**Theorem 9.48** (Overlaps the Contextualised Sum iff Overlaps one of the Contextualised Operands).

$$\forall s, t, u [Cb(t, s) \wedge Cb(u, s) \rightarrow \forall v (OoS(v, s \circ (t + u)) \leftrightarrow OoS(v, s \circ t) \vee OoS(v, s \circ u))]$$

⌈ *Proof.* Let  $s, t$  and  $u$  be slots such that  $t$  and  $u$  are contextualisable by  $s$ . By Axiom 9.2 (Contextualisable iff Contextualisation Exists),  $s \circ t$  and  $s \circ u$  exist. By Axiom 9.8 (Sum Existence),  $t + u$  exists, and by Theorem 9.40,  $t + u$  is a slot of  $a$ . Finally, by Axiom 9.2,  $s \circ (t + u)$  exists. Let  $v$  be a slot.

Left-to-right: suppose that  $v$  and  $s \circ (t + u)$  overlap. We want to prove that  $v$  overlaps  $s \circ t$  or  $s \circ u$ . By Definition 9.5 ( $OoS$ ), there is a  $w$  that is a slot-part of  $v$  and  $s \circ (t + u)$ .  $w$  being a slot-part of  $s \circ (t + u)$ , we know by Definition 9.3 ( $PoS$ ), that there is a  $w_1$  such that  $w = (s \circ (t + u)) \circ w_1$ , i.e., by Contextualisation Associativity (Axiom 9.6),  $w = s \circ ((t + u) \circ w_1)$ . Let  $w_2$  be  $(t + u) \circ w_1$ . By definition,  $w_2$  is a slot-part of  $t + u$ . The definition of sum (Definition 9.6) states that every slot-part of  $t + u$  overlaps  $t$  or  $u$ . So  $w_2$  overlaps (1)  $t$  or (2)  $u$ .

(1) If  $w_2$  overlaps  $t$ : with Theorem 9.33 ( $OoS$  Stable under Contextualisation), as we know  $OoS(w_2, t)$ , we know that  $OoS(s \circ w_2, s \circ t)$ , i.e.  $OoS(w, s \circ t)$ . By definition of  $OoS$ , there is a  $w_3$ , slot-part of  $w$  and  $s \circ t$ . Let us prove that  $w_3$  is a slot-part of  $v$ . We know that  $w_3$  is slot-part of  $w$ , and that  $w$  is slot-part of  $v$ , therefore, by  $PoS$  Transitivity (Theorem 9.15),  $w_3$  is slot-part of  $v$ . Thus  $OoS(v, s \circ t)$ .

(2) By the same reasoning applied to  $u$ ,  $OoS(v, s \circ u)$ .

That is, in every cases,  $OoS(v, s \circ t) \vee OoS(v, s \circ u)$ .

Right-to-left: suppose that  $v$  overlaps  $s \circ t$ , i.e. there is a  $w$  slot-part of  $v$  and  $s \circ t$ . We want to prove that  $v$  and  $s \circ (t + u)$  overlap, i.e. that there is a shared slot-part. Let us prove that this shared slot-part is  $w$ . As  $w$  being a slot-part of  $v$  is one of the hypotheses, we only have to prove that  $w$  is a slot-part of  $s \circ (t + u)$ . As  $w$  is a slot-part of  $s \circ t$ , we know that there is a  $w'$  such that  $w = (s \circ t) \circ w'$ . With Contextualisation Associativity (Axiom 9.6), we get  $w = s \circ (t \circ w')$ .  $t$  being a slot-part of  $t + u$ , there is a  $t'$  such that  $t = (t + u) \circ t'$ . So, we can write, by replacing  $t$  by  $(t + u) \circ t'$  in  $w = s \circ (t \circ w')$ , that  $w = s \circ (((t + u) \circ t') \circ w')$ . Finally, by Associativity, we get that  $w = (s \circ (t + u)) \circ (t' \circ w')$ , which is, by Definition 9.3, that  $w$  is a slot-part of  $s \circ (t + u)$ . A similar reasoning leads to the fact that if  $OoS(v, s \circ u)$ , then  $OoS(v, s \circ (t + u))$ . □

Let us focus on the distributivity of contextualisation over the sum. In the first step, we prove that left distributivity holds, presented by Theorem 9.49. In the second step, we discuss why right distributivity is unwanted in our theory.



**Theorem 9.49** (Left Distributivity).

$$\forall s, t, u [Cb(t, s) \wedge Cb(u, s) \rightarrow (s \circ (t + u) = s \circ t + s \circ u)]$$

*Proof.* Let  $s, t$  and  $u$  be slots such that  $t$  and  $u$  are contextualisable by  $s$ . We want to prove that  $s \circ (t + u)$  and  $s \circ t + s \circ u$  are identical. To do so, we will use the *OoS-Extensionality* (Theorem 9.36). Therefore, we have to prove that for every  $v$  that overlaps one, it also overlaps the other one ( $\forall v (OoS(v, s \circ (t + u)) \leftrightarrow OoS(v, s \circ t + s \circ u))$ ). Let  $v$  be a slot.

Left-to-right: suppose  $v$  overlaps  $s \circ (t + u)$ . We want to prove that  $v$  overlaps with  $s \circ t + s \circ u$ . By Theorem 9.48, we know that  $v$  overlaps  $s \circ t$  or  $s \circ u$ . In both cases, by Definition 9.7, we know that  $v$  overlaps  $s \circ t + s \circ u$ .

Right-to-left: suppose that  $v$  overlaps  $s \circ t + s \circ u$ . We want to prove that  $v$  overlaps with  $s \circ (t + u)$ . By Definition 9.7, we know that  $v$  overlaps  $s \circ t$  or  $s \circ u$ . In both cases, by Theorem 9.48, we know that  $v$  overlaps  $s \circ (t + u)$ .

We proved that  $\forall v (OoS(v, s \circ (t + u)) \leftrightarrow OoS(v, s \circ t + s \circ u))$ , therefore by *OoS-Extensionality*,  $s \circ (t + u) = s \circ t + s \circ u$ .  $\square$

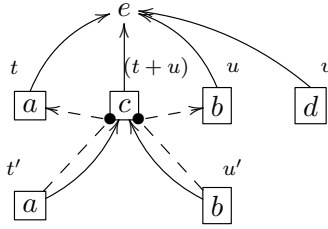


Figure 9.7: Counter-example to right distributivity

In this theory, we do not want right distributivity of contextualisation over the sum. Take the example pictured by Figure 9.7. Consider the sum of  $t$  and  $u$ , i.e. the slot  $(t + u)$ . This slot is filled by an entity  $c$ , that has two slots  $t'$  and  $u'$ , such that  $t = (t + u) \circ t'$  and  $u = (t + u) \circ u'$ . From the first equality, if we admit right distributivity,  $t \circ t'$  and  $u \circ t'$  would exist and  $t$  would be identical to  $t \circ t' + u \circ t'$ . However, neither  $t \circ t'$  nor  $u \circ t'$  exist, as the conditions of existence are not fulfilled: there is no entity that fills  $t$  or  $u$  and owns  $t'$ .

In fact, the only case where the right distributivity holds, i.e.  $(s + t) \circ u = s \circ u + t \circ u$  (call this H) is true, is when  $s = t$ . If so,  $s + t = s + s = s$ , and by replacing  $t$ , we get  $s \circ u = s \circ u + s \circ u$ . The sum being idempotent, this is a tautology. However, it is possible to go further: if  $(s + t) \circ u, s \circ u$  and  $t \circ u$  exist (so without positing H) then  $s = t$ . This is the result of Theorem 9.51. Before proving it, consider Lemma 9.50 that shows that if  $s$  and a sum of  $s$  and another slot are filled by the same filler, they are equal.

**Lemma 9.50** (Same Filler of Operand and Sum Implies Identity).

$$\forall s, t (SF(s, s + t) \rightarrow s = s + t)$$

*Proof.* Let  $s$  and  $t$  be two slots such that they have the same filler. Let us call this filler  $a$ . By Definition 9.6,  $s$  is a slot-part of  $s + t$ . So by Definition 9.3, there is a slot  $s'$  such that  $s = (s + t) \circ s'$ . According to Theorem 9.7 (Contextualisation Same Filler) and 3.7 (Single Occupancy),  $s$  and  $s'$  have the same filler, i.e.  $a$ . According to Axioms 9.2 (Contextualisable iff Contextualisation Exists) and 3.7, there is something that fills  $s + t$  and owns  $s'$ , i.e.  $a$ . The slot  $s'$  is owned and filled by  $a$ . So by Definition 8.4,  $s'$  is the improper slot of  $a$ . Finally, by Theorem 9.3 (Improper Slot Is Right Neutral Element), as  $s'$  is an improper slot, we deduce, from  $s = (s + t) \circ s'$ , that  $s = s + t$ .  $\square$

**Theorem 9.51** (Right Distributivity Is Trivial).

$$\forall s, t, u (Cb(s + t, u) \wedge Cb(s, u) \wedge Cb(t, u) \rightarrow s = t)$$

⌈ *Proof.* Let  $s$ ,  $t$  and  $u$  be three slots such that  $(s + t) \circ u$ ,  $s \circ u$  and  $t \circ u$  exist. We want to prove that  $s = t$ . According to Axiom 9.2 (Contextualisable iff Contextualisation Exists), there are three fillers  $a$ ,  $b$  and  $c$  such that  $a$  fills  $s + t$  and owns  $u$ ,  $b$  fills  $s$  and owns  $u$  and  $c$  fills  $t$  and owns  $u$ . Furthermore, by Axiom 8.1 (Single Owner),  $a = b = c$ . According to Lemma 9.50, as  $s$  and  $s + t$  are filled by the same filler,  $s = s + t$ . In the same way, we deduce that  $t = s + t$ . Therefore,  $s = t$ .  $\square$

We show with Lemma 9.52 that if  $s$  and  $t$  are slot-parts of  $u$ , then  $s + t$  is a slot-part of  $u$ .

**Lemma 9.52** (Sum is Slot-Part if Operands are Slot-Parts).

$$\forall s, t, u (PoS(s, u) \wedge PoS(t, u) \rightarrow PoS(s + t, u))$$

⌈ *Proof.* Let  $s$ ,  $t$  and  $u$  be three slots such that  $s$  and  $t$  are slot-parts of  $u$  ( $PoS(s, u) \wedge PoS(t, u)$ ), i.e. there are  $s'$  and  $t'$  such that  $s = u \circ s'$  and  $t = u \circ t'$ . According to Theorem 9.16 ( $PoS$  Same Owner) and Axiom 8.1 (Single Owner),  $s$ ,  $t$  and  $u$  share the same owner. Therefore, according to Axiom 9.8 (Sum Existence),  $s + t$  exists. We know that  $s + t = (u \circ s') + (u \circ t')$ . With Left Distributivity (Theorem 9.49), we get that  $s + t = u \circ (s' + t')$ , i.e. that  $PoS(s + t, u)$ .  $\square$

We then prove that slot-sum is associative, as demonstrated by Theorem 9.53.

**Theorem 9.53** (Sum Associativity).

$$\forall s, t, u [SO(s, t) \wedge SO(t, u) \rightarrow (s + t) + u = s + (t + u)]$$

⌈ *Proof.* Let  $s$ ,  $t$  and  $u$  be three slots that have the same owner.  $SoS_1$  and  $SoS_2$  being equivalent, let us use  $SoS_2$ . By Definition 9.7,  $(s + t) + u$  is:

- (a)  $\forall v (OoS(v, s + t) \leftrightarrow OoS(v, s) \vee OoS(v, t))$
- (b)  $\forall v (OoS(v, (s + t) + u) \leftrightarrow OoS(v, s + t) \vee OoS(v, u))$

By using (a) to partially rewrite (b), we get  $\forall v (OoS(v, (s + t) + u) \leftrightarrow OoS(v, s) \vee OoS(v, t) \vee OoS(v, u))$  (1).

Similarly, from  $s + (t + u)$ , we get  $\forall v (OoS(v, s + (t + u)) \leftrightarrow OoS(v, s) \vee OoS(v, t) \vee OoS(v, u))$  (2).

From (1) and (2), we obtain  $\forall v (OoS(v, (s + t) + u) \leftrightarrow OoS(v, s + (t + u)))$ , which, according to Theorem 9.36, entails  $(s + t) + u = s + (t + u)$ .  $\square$

We can prove that sums are stable under contextualisation, i.e. that for some slots  $u$ ,  $s$  and  $t$  that can be contextualised by  $v$ , it is true that  $u = s + t$  iff  $v \circ u = v \circ s + v \circ t$ , as showed by Theorem 9.54.

**Theorem 9.54** (Sum Stable under Contextualisation).

$$\forall s, t, u, v [Cb(u, v) \wedge Cb(s, v) \wedge Cb(t, v) \rightarrow (u = s + t \leftrightarrow (v \circ u) = (v \circ s) + (v \circ t))]$$

⌈ *Proof.* Let  $s$ ,  $t$ ,  $u$  and  $v$  be slots such that  $u$ ,  $s$  and  $t$  are contextualisable by  $v$ . By Definition 9.1 and Axiom 8.1, there is a unique  $a$  that is the filler of  $v$  and the owner of  $u$ ,  $s$  and  $t$ . According to Theorem 9.19 ( $PoS$  Stable under Contextualisation), we have  $PoS(s, u) \leftrightarrow PoS(v \circ s, v \circ u)$  and  $PoS(t, u) \leftrightarrow PoS(v \circ t, v \circ u)$  (call them H1 and H2). We want to prove that  $u = s + t \leftrightarrow (v \circ u) = (v \circ s) + (v \circ t)$ .

Using Definition 9.6, it is  $[PoS(s, u) \wedge PoS(t, u) \wedge \forall v' (PoS(v', u) \rightarrow OoS(s, v') \vee OoS(t, v'))] \leftrightarrow [PoS(v \circ s, v \circ u) \wedge PoS(v \circ t, v \circ u) \wedge \forall v' (PoS(v', v \circ u) \rightarrow OoS(v \circ s, v') \vee OoS(v \circ t, v'))]$ . Using H1 and H2, it can be rewritten as  $[PoS(v \circ s, v \circ u) \wedge PoS(v \circ t, v \circ u) \wedge \forall v' (PoS(v', u) \rightarrow OoS(s, v') \vee OoS(t, v'))] \leftrightarrow [PoS(v \circ s, v \circ u) \wedge PoS(v \circ t, v \circ u) \wedge \forall v' (PoS(v', v \circ u) \rightarrow OoS(v \circ s, v') \vee OoS(v \circ t, v'))]$ , which can be simplified into  $\forall v' (PoS(v', u) \rightarrow OoS(s, v') \vee OoS(t, v')) \leftrightarrow \forall v' (PoS(v', v \circ u) \rightarrow OoS(v \circ s, v') \vee OoS(v \circ t, v'))$

Left-to-right: suppose that  $\forall v' (PoS(v', u) \rightarrow OoS(s, v') \vee OoS(t, v'))$  (call this H). Let  $w'$  be a slot such that  $PoS(w', v \circ u)$ , i.e. there is a  $x$  such that  $w' = (v \circ u) \circ x$ . We want to prove  $OoS(v \circ s, w') \vee OoS(v \circ t, w')$ . We have  $w' = (v \circ u) \circ x$ . By Axiom 9.6, we get  $w' = v \circ (u \circ x)$ . Let  $w$  be  $u \circ x$ . Given  $w' = v \circ w$ , by Theorem 9.33, the goal  $OoS(v \circ s, v \circ w) \vee OoS(v \circ t, v \circ w)$  is equivalent to  $OoS(s, w) \vee OoS(t, w)$ . By Definition 9.3,  $w = u \circ x$  gives  $PoS(w, u)$ . By H and  $PoS(w, u)$ , we prove the goal  $OoS(s, w) \vee OoS(t, w)$ .

Right-to-left: suppose that  $\forall v' (PoS(v', v \circ u) \rightarrow OoS(v \circ s, v') \vee OoS(v \circ t, v'))$  (call this H). We want to prove that  $\forall v' (PoS(v', u) \rightarrow OoS(s, v') \vee OoS(t, v'))$ . Let  $w$  be a slot such that  $PoS(w, u)$ . Let

us prove that  $OoS(s, w) \vee OoS(t, w)$ . According to Theorem 9.16,  $u$  and  $w$  have the same owner, i.e.  $a$ . Therefore,  $w$  is contextualisable by  $v$ . By Theorem 9.33, the goal  $OoS(s, w) \vee OoS(t, w)$  is equivalent to  $OoS(v \circ s, v \circ w) \vee OoS(v \circ t, v \circ w)$ . By Theorem 9.19,  $Pos(w, u)$  is equivalent to  $Pos(v \circ w, v \circ u)$ , which, by H, gives  $OoS(v \circ s, v \circ w) \vee OoS(v \circ t, v \circ w)$ , i.e. the goal.  $\square$

We added a new relation of slot-sum, syntactically based on Varzi’s 39<sub>2</sub>. After axiomatising the existence condition of the sum of slots, we highlighted the properties of the sum of slots: unicity, same-owner, idempotence, commutativity, left distributivity, associativity and stability under contextualisation. In the next section, we will take a step further by defining the fusion of slots.

## 9.6 Slot-Fusion

After having axiomatised the sum of slots, we look at mereological fusion, in the sense presented in Section 2.4.2. This has two goals. First, fusion is a classical construct of mereological theories: adding it to our theory extends the coverage of mereological concepts. Second, we expect the fusion to make provable one of our pre-formal intuitions, left unproved until now: the fact that an improper slot is the “union”<sup>4</sup> of all the slots of its filler.

While binary sum can be understood as an operation over a finite collection of entities where the operands must be given in an extensional way (e. g.  $a + b + c$ ), fusion operates over a collection of entities given in an intensional way (e. g. the fusion of all the cats, or the fusion of all the slots of a given filler), such a collection possibly being infinite.

As is the case with sum, there are multiple definitions of fusion in the literature (see Section 2.4.2, (Varzi, 2019) or (Cotnoir and Varzi, 2021, pp. 161–163)). While other definitions exist,<sup>5</sup> we choose to use the definition of the so-called Leśniewski fusions, as given by Definition Schema 9.8.

What does it mean for  $z$  to be the fusion of all  $\phi$ -ers? According to the definition schema, which contains two conjuncts,  $z$  is the fusion of all  $\phi$ -ers iff 1) every  $\phi$ -er is a slot-part of  $z$  and 2) every slot-part of  $z$  overlaps with some  $\phi$ -er.

**Definition Schema 9.8** (Fusion of Slots).

$$FoS_{\phi}(z) \triangleq \forall w(\phi(w) \rightarrow Pos(w, z)) \wedge \forall v(Pos(v, z) \rightarrow \exists w(\phi(w) \wedge OoS(v, w)))$$

In the literature, multiple formulas enforcing the existence of fusion are often taken as axioms. The Unrestricted Existence, presented in (9.1), ensures the existence of the fusion of the  $\phi$ -ers, as long as there is at least one  $\phi$ -er. This condition avoids positing the existence of the fusion of a collection of zero elements.

$$\exists w(\phi(w)) \rightarrow \exists z(FoS_{\phi}(z)) \tag{9.1}$$

This formula is problematic for multiple reasons: it enables the fusion of fillers, the fusion of slots and fillers, and the fusion of slots that do not have the same owner. Indeed, as explored in Section 9.2 and as stated by Theorems 9.16 and 9.29, our theory’s mereological relations only holds between slots that have the same owner. Therefore, we conclude that Unrestricted Existence is not desirable to our theory.

The Restricted Existence, presented in (9.2), generally contains another premise, which states that all  $\phi$ -ers must satisfy some condition  $\psi$ .

<sup>4</sup>The word “union” is used here to avoid using both the word “sum”, since our binary sum is not applicable to any (possibly infinite) number of slots, as well as the word “fusion” which is yet to be formally defined.

<sup>5</sup>Other existing definitions are equivalent to the definition we choose under some conditions: for more details, see Section 2.4.2 and (Cotnoir and Varzi, 2021, pp. 160–174).

$$(\exists w(\phi(w)) \wedge \forall w(\phi(w) \rightarrow \psi(w)) \rightarrow \exists z(FoS_\phi(z)) \quad (9.2)$$

In our case, we identified two conditions: each  $\phi$ -er must be a slot, and all  $\phi$ -ers must have the same owner. While a formula of the form  $\forall w(\phi(w) \rightarrow \psi(w))$  can easily take into account that every  $w$  is a slot, we do not see what formula can say that all slots have the same owner. Therefore, we propose a slightly modified version as follows:

$$\exists w(\phi(w)) \wedge \forall w(\phi(w) \rightarrow S(w)) \wedge \exists w(\phi(w) \wedge \forall v(\phi(v) \rightarrow SO(w, v))) \rightarrow \exists z(FoS_\phi(z))$$

This formula states that 1) there is a  $\phi$ -er, 2) every  $\phi$ -er is a slot, and 3) there is a  $\phi$ -er that has the same owner as all other  $\phi$ -ers. This formula can be simplified as in Axiom Schema 9.9, i. e. if there is a  $\phi$ -er and every  $\phi$ -er have the same owner, then the fusion of these  $\phi$ -ers exists.

**Axiom Schema 9.9** (Fusion Existence).

$$\exists w(\phi(w) \wedge \forall v(\phi(v) \rightarrow SO(v, w))) \rightarrow \exists s(FoS_\phi(s))$$

The fusion of the  $\phi$ -ers is unique, as proved by Theorem Schema 9.55.

**Theorem Schema 9.55** (Fusion Unicity).

$$\exists w(\phi(w)) \rightarrow \forall s, t(FoS_\phi(s) \wedge FoS_\phi(t) \rightarrow s = t)$$

$\lceil$  *Proof.* Consider  $w$  such that  $\phi(w)$  holds. Let  $s$  and  $t$  be fusions of the  $\phi$ -ers. We want to show that  $s$  and  $t$  are equal. To do so, let us use *OoS-Extensionality*: we now have to prove that  $s$  and  $t$  are slots and that  $\forall u, OoS(s, u) \leftrightarrow OoS(t, u)$ . By the first conjunct of Definition Schema 9.8 and  $\phi(w)$ ,  $w$  is a slot-part of  $s$  and  $t$ . According to Lemma 9.12 (*PoS Domain and Range*),  $s$  and  $t$  are slots. We still have to prove  $\forall u, OoS(s, u) \leftrightarrow OoS(t, u)$ .

Let  $u$  be a slot. The proof is similar from left-to-right and right-to-left: suppose that  $s$  (resp.  $t$ ) and  $u$  are overlapping. Therefore, by Definition 9.5, there is some slot  $a$  such that  $a$  is a slot-part of  $s$  (resp.  $t$ ) and  $u$ . According to the second conjunct of Definition Schema 9.8, as  $a$  is a slot-part of  $s$  (resp.  $t$ ), there exists some  $\phi$ -er  $b$  that overlaps with  $a$ , i.e. there is some slot  $c$  that is a slot-part of  $a$  and  $b$ . According to the first conjunct of Definition Schema 9.8, as  $b$  is a  $\phi$ -er, it is a slot-part of  $t$  (resp.  $s$ ). At this point, we know that  $c$  is a slot-part of  $a$  and  $b$ , which are respectively parts of  $u$  and  $t$  (resp.  $s$ ). Therefore, by  $\lceil$  *PoS Transitivity*,  $c$  is a common slot-part of  $u$  and  $t$  (resp.  $s$ ), i.e.  $u$  and  $t$  (resp.  $s$ ) are overlapping.  $\square$

Theorem 9.56 shows that slot-sum, defined in Definition 9.6, is equivalent to the fusion of a collection of two slots. This means that binary slot-sum is a particular of fusion, and therefore that Axiom 9.8 is an instantiation of Axiom Schema 9.9 and so redundant.

**Theorem 9.56** (Sum is a special Case of Fusion).

$$\forall s, t, u(FoS_{w=s \vee w=t}(u) \leftrightarrow SoS_1(u, s, t))$$

$\lceil$  *Proof.* Let  $s, t$  and  $u$  be slots. We want to prove that  $FoS_{w=s \vee w=t}(u) \leftrightarrow SoS_1(u, s, t)$ .

Left-to-right: suppose that  $u$  is the fusion of  $\phi$ -ers such that  $\phi$  is  $w = s \vee w = t$ . By Definition Schema 9.8,  $\forall w(w = s \vee w = t) \rightarrow PoS(w, u)$  (call this H1) and  $\forall v(PoS(v, u) \rightarrow \exists w((w = s \vee w = t) \wedge OoS(v, w)))$  (call this H2).

We want to prove that  $SoS_1(u, s, t)$ , i.e.  $PoS(s, u)$ ,  $PoS(t, u)$  and  $\forall v(PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$ .  $PoS(s, u)$  and  $PoS(t, u)$  are trivial using H1: if  $w$  is  $s$  or  $t$ , then  $PoS(w, u)$ . Let us prove that  $\forall v(PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$ . Let  $v$  be a slot such that  $PoS(v, u)$ . By H1, there is  $w$  such that  $(w = s \vee w = t) \wedge OoS(v, w)$ . By rewriting it, we got  $(w = s \wedge OoS(v, w)) \vee (w = t \wedge OoS(v, w))$ , i.e.  $OoS(s, v) \vee OoS(t, v)$ .

Right-to-left: suppose that  $u = s + t$ , i.e.  $PoS(s, u)$ ,  $PoS(t, u)$  and  $\forall v(PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$  (call this H). We want to prove that  $u$  is the fusion of the  $\phi$ -ers, i.e.  $\forall w(w = s \vee w = t) \rightarrow PoS(w, u)$  (call this G1) and  $\forall v(PoS(v, u) \rightarrow \exists w((w = s \vee w = t) \wedge OoS(v, w)))$  (call this G2).

G1: let  $w$  be a slot such that  $w = s \vee w = t$ . We want to prove that  $PoS(w, u)$ . This is trivial, given that both  $s$  and  $t$  are slot-parts of  $u$ .

G2: let  $v$  be a slot such that  $PoS(v, u)$ . We want to prove that there is some  $w$  such that  $(w = s \vee w = t) \wedge OoS(v, w)$ , or, rewritten the same way it was previously,  $OoS(v, s) \vee OoS(v, t)$ . This is trivial, given that  $PoS(v, u)$  and H.  $\square$

Theorem 9.57 shows that if an entity  $a$  has some slot, its improper slot is the sum of its slots. The proposition  $\forall t(IP_s(t, a) \leftrightarrow FoS_{P_s(w, a)}(t))$  is equivalent to the current consequent of the theorem, as both improper slot and fusion are unique, by Axiom 8.3 and Theorem Schema 9.55.

**Theorem 9.57** (Improper Slot is Fusion of Filler's Slots).

$$\forall a(\exists s(P_s(s, a)) \rightarrow \exists t(IP_s(t, a) \wedge FoS_{P_s(w, a)}(t)))$$

*Proof.* Let  $a$  be a filler and  $s$  a slot of  $a$ . We want to prove that the improper slot of  $a$  is the fusion of  $a$ 's slots. Because  $a$  owns  $s$ , we deduce by 3.4 that there is a slot  $t$  that is the improper slot of  $a$ . Let us prove that  $t$  is the fusion of  $a$ 's slots. According to Fusion's Definition Schema 9.8,  $t$  is the fusion of  $a$ 's slots iff 1)  $\forall w(P_s(w, a) \rightarrow PoS(w, t))$  and 2)  $\forall v(PoS(v, t) \rightarrow \exists w(P_s(w, a) \wedge OoS(v, w)))$ .

1) Let  $w$  be a slot of  $a$ . According to Theorem 9.17 (Slots iff Slot-Parts of Improper Slot),  $w$  is a slot-part of  $a$ 's improper slot, i.e.  $w$  is a slot-part of  $t$ .

2) Let  $v$  be a slot such that  $v$  is a slot-part of  $t$ . Let us prove that there is a slot that is a slot of  $a$  and that slot-overlaps with  $v$ . As  $a$ 's improper slot,  $t$  is a slot of  $a$ . Furthermore, as  $v$  is a slot-part of  $t$ , by Lemma 9.32, it overlaps with  $t$ .

In conclusion, if  $a$  has a slot, its improper slot is the fusion of its slots.  $\square$

Finally, Theorem Schema 9.58 shows that fusion is stable under contextualisation. It states that if  $s'$  is the contextualisation of  $s$  by  $t$  and that every  $\phi$ -er can be contextualised by  $t$ , then  $s$  is the fusion of the  $\phi$ -ers iff  $s'$  is the fusion of the contextualisations by  $t$  of the  $\phi$ -ers.

**Theorem Schema 9.58** (Fusion Stable under Contextualisation).

$$\forall s, t[Cb(s, t) \wedge \forall w(\phi(w) \rightarrow Cb(w, t)) \rightarrow (FoS_\phi(s) \leftrightarrow FoS_{\exists w'(w=t \circ w' \wedge \phi(w'))}(t \circ s))]$$

*Proof.* Let  $s$  and  $t$  be slots such that  $s$  is contextualisable by  $t$  and  $\forall w(\phi(w) \rightarrow Cb(w, t))$  (call this H). We want to prove  $FoS_\phi(s) \leftrightarrow FoS_{\exists w'(w=t \circ w' \wedge \phi(w'))}(t \circ s)$ .

Left-to-right: suppose that  $s$  is the fusion of the  $\phi$ -ers. By Definition Schema 9.8,  $\forall w(\phi(w) \rightarrow PoS(w, s))$  and  $\forall v(PoS(v, s) \rightarrow \exists w(\phi(w) \wedge OoS(v, w))$  (call them H1 and H2). We want to prove that  $FoS_{\exists w'(w=t \circ w' \wedge \phi(w'))}(t \circ s)$ , i.e. that  $\forall w(\exists w'(w = t \circ w' \wedge \phi(w')) \rightarrow PoS(w, t \circ s))$  and  $\forall v(PoS(v, t \circ s) \rightarrow \exists w(\exists w'(w = t \circ w' \wedge \phi(w')) \wedge OoS(v, w)))$  (call them G1 and G2).

G1: let  $w$  and  $w'$  be such that  $w = t \circ w'$  and  $\phi(w')$ . We want to prove  $PoS(w, t \circ s)$ . According to H1, as  $w'$  is a  $\phi$ -er, then  $PoS(w', s)$ . Given that  $w = t \circ w'$  and  $PoS(w', s)$ , by Theorem 9.19 ( $PoS$  Stable under Contextualisation), we get  $PoS(w, t \circ s)$ .

G2: let  $v$  be such that  $PoS(v, s')$ , i.e. by Definition 9.3, there is  $v'$  such that  $v = s' \circ v'$ . We want to prove  $\exists w(\exists w'(w = t \circ w' \wedge \phi(w')) \wedge OoS(v, w))$ . By Axiom 9.6,  $v = (t \circ s) \circ v' = t \circ (s \circ v')$ . Let  $v''$  be  $s \circ v'$ . By Definition 9.3,  $PoS(v'', s)$ . By H2, we get that there is  $w''$  such that  $\phi(w'')$  and  $OoS(v'', w'')$ . Given that  $w''$  is a  $\phi$ -er, by H1,  $PoS(w'', s)$ . By Theorem 9.16, there is some  $a$  such that  $P_s(w'', a)$  and  $P_s(s, a)$ . With Axiom 9.2 and  $s' = t \circ s$ , there is some  $b$  such that  $F(b, t)$  and  $P_s(s, b)$ . Because of Axiom 8.1 (Single Owner),  $a = b$ . As  $a$  owns  $w''$  and fills  $t$ , by Axiom 9.2, there is a  $w'''$  such that  $w''' = t \circ w''$ . Our goal is  $\exists w(\exists w'(w = t \circ w' \wedge \phi(w')) \wedge OoS(v, w))$ . There are  $w'''$  and  $w''$  such that  $w''' = t \circ w''$  and  $\phi(w'')$ . We only need to prove that  $OoS(v, w''')$ . Given that  $v = t \circ v''$  and  $w''' = t \circ w''$ , by Theorem 9.33, we need to prove that  $OoS(v'', w'')$ . This is already proven.

Right-to-left: suppose that  $s'$  is the fusion of the contextualisations by  $t$  of the  $\phi$ -ers. By Definition Schema 9.8,  $\forall w(\exists w'(w = t \circ w' \wedge \phi(w')) \rightarrow PoS(w, s'))$  and  $\forall v(PoS(v, s') \rightarrow \exists w(\exists w'(w = t \circ w' \wedge \phi(w')) \wedge OoS(v, w)))$  (call them H1 and H2). We want to prove that  $FoS_\phi(s)$ , i.e.  $\forall w(\phi(w) \rightarrow PoS(w, s))$  and  $\forall v(PoS(v, s) \rightarrow \exists w(\phi(w) \wedge OoS(v, w))$  (call them G1 and G2).

G1: let  $w$  be a  $\phi$ -er. We want to prove that  $PoS(w, s)$ . With H, there is a  $w'$  such that  $w' = t \circ w$ . Given that  $s' = t \circ s$  and  $w' = t \circ w$ , by Theorem 9.19, we get that  $PoS(w, s) \leftrightarrow PoS(w', s')$ . Therefore, to prove  $PoS(w, s)$ , it is enough to prove  $PoS(w', s')$ . This can be proven using H1:  $w$  is a  $\phi$ -er such that  $w' = t \circ w$ , therefore  $PoS(w', s')$ .

G2: let  $v'$  be a slot such that it is a slot-part of  $s$ , i.e. there is a  $v$  such that  $v' = s \circ v$ . We want to prove that there is a  $\phi$ -er that overlaps with  $v'$ .  $s$  being contextualisable by  $t$ , let  $v''$  be  $v' = t \circ v' = s' \circ v = t \circ s \circ v$ .

By Definition 9.3,  $v''$  is a slot-part of  $s'$ . Therefore, with H2, there are  $w$  and  $w'$  such that  $w = t \circ w'$ ,  $w'$  is a  $\phi$ -er and  $v''$  overlaps with  $w$ . As  $w'$  is a  $\phi$ -er, let us prove that it overlaps with  $v'$  to complete the proof. As  $OoS(v'', w)$ , i.e.  $OoS(t \circ v', t \circ w')$ , by Theorem 9.33,  $OoS(v', w')$ .  $\square$

## 9.7 Evaluating The Theory

In Chapter 6, I presented requirements to evaluate the theory. In this section, I propose to do so.

The first requirement is about providing a partial order relation that can represent multiplicities of parts. This relation was provided by Bennett's Slot Mereology. This theory has a parthood relation that is partial order relation. Furthermore, by defining this parthood relation with the two relations *slot-of*  $P_s$  and *fills*  $F$ , Bennett provided a way to represent non-idempotent entities.

The second requirement is about being able to count correctly the multiplicities of parts. This is ensured by contextualisation, as the axiomatisation of contextualisation removes counting problems. This is showed in Chapter 10.

The third and fourth requirements are about providing decomposition and composition principles. While providing such principles is not possible using a parthood relation between fillers, it is possible for slots. Therefore, the *slot-part-of* relation ( $Pos$ ) is used to formalise supplementation and extensionality, for decomposition principles, and sum and fusion, for composition principles.

# Chapter 10

## Illustrations

In this chapter, I apply the mereology of slots to more realistic examples, in particular those used in Section 3.3.2, where we exposed the counting problems. Section 10.1 explores the DIHYDROGEN universal as a first and basic example. Section 10.2 illustrates how overlap works with the universal of two rooms with an adjoining wall. Finally, Section 10.3 shows how to deal with overduplications, illustrated with the string type “cats”. The first example will also be used to give examples of interesting properties of slot-sums. The examples will be presented step by step: we will first show the mereological structures with only direct slots; then we will apply contextualisation to represent the full structure.

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### 10.1 The Dihydrogen Universal

This section illustrates the theory using the DIHYDROGEN. The DIHYDROGEN universal has two proper slots  $S_1$  and  $S_2$  filled by the HYDROGEN universal and one improper slot  $S_{H_2}$ . The HYDROGEN universal has two proper slots  $S_3$  and  $S_4$ , respectively filled by the ELECTRON and the HYDROGENNUCLEUS universals, and one improper slot  $S_H$ . Finally, both ELECTRON and HYDROGENNUCLEUS only have an improper slot (as before, we consider HYDROGENNUCLEUS as a mereological atom). Their four mereological structures are pictured in Figure 10.1, where only improper and direct slots are represented.

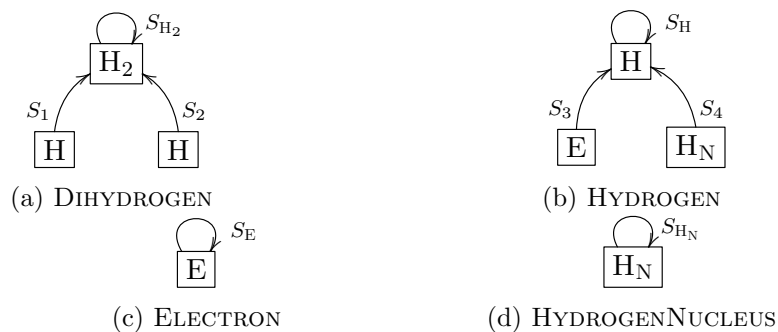


Figure 10.1: Direct slots of DIHYDROGEN ( $H_2$ ), HYDROGEN ( $H$ ), ELECTRON ( $E$ ) and HYDROGENNUCLEUS ( $H_N$ )

A partial view on the mereological structure of DIHYDROGEN is pictured in Figure 10.2.

To simplify the figure, neither the improper slots nor the contextualisations in which those improper slots are involved are displayed. Some sums, like  $S_5 + S_7$  are not represented.

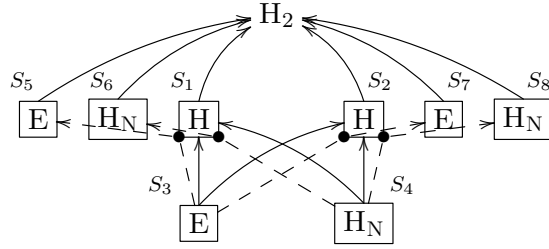


Figure 10.2: A partial view on the mereological structure of DIHYDROGEN

In this example, the DIHYDROGEN universal has HYDROGEN as a part twice, and ELECTRON and HYDROGENNUCLEUS two times each, as expected. Furthermore, there are no overduplications. However, this partial view does not represent all the slots of the DIHYDROGEN universal. In Figure 10.3, all the slots of DIHYDROGEN are represented. The

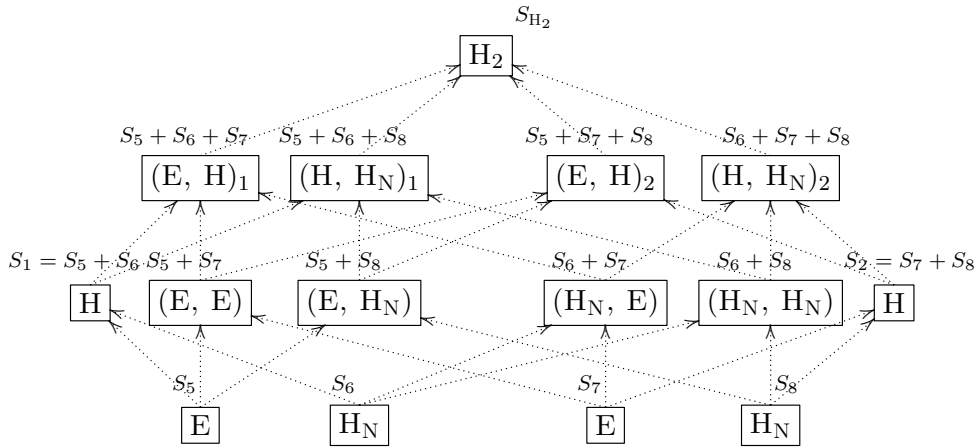


Figure 10.3: A semi-lattice of sums of slots (dotted arrows represent the  $PPoS$  relation)

relation pictured by dotted arrows is not  $P_s$ , but  $PPoS$ . Slots linked to the same slot all sum together to that targeted slot. For example, summing the slots  $S_1$ ,  $S_5 + S_7$  and  $S_6 + S_7$  results in slot  $S_5 + S_6 + S_7$ . Some slots are filled by entities whose nature is unknown. For example, slot  $S_5 + S_8$  is filled by something that has ELECTRON and HYDROGENNUCLEUS as parts once each.<sup>1</sup> However, this filler of  $S_5 + S_8$  is not HYDROGEN. Indeed, we can adapt Bennett (2017)'s argumentation for her *Spatial Separation* case. She argued convincingly that the electron of one hydrogen atom and the nucleus of another hydrogen atom do not sum up to an hydrogen atom. Similarly, in the current theory, a slot filled by ELECTRON from one contextualisation of HYDROGEN ( $S_1$ ) and a slot filled by HYDROGENNUCLEUS from a different contextualisation of HYDROGEN ( $S_2$ ) do not sum up to a slot filled by HYDROGEN, as they do not come from the same contextualisation.<sup>2</sup>

Also, as it is unknown whether the filler of  $S_5 + S_8$  and the one of  $S_6 + S_7$  are identical, indexes have been added to differentiate them. Those numbers only have a differentiating

<sup>1</sup>Our theory does not include any mereological sum on fillers. Such a sum would be by no means trivial, as developed in Section 9.5. Therefore, the content of these slots in this example is simply written as a list of their parts. This notation does not belong to the language of our theory.

<sup>2</sup>This conclusion is supported by the fact that for  $S_5 + S_8$  to be filled by HYDROGEN,  $S_5 + S_8$  must be the improper slot of HYDROGEN ( $S_H$ ) or one of its contextualisation. But it does not seem that we can rewrite  $S_5 + S_8 = (S_1 \circ S_3) + (S_2 \circ S_4)$  using  $S_H$ . Contrast this with  $S_5 + S_6 = S_1 \circ S_3 + S_1 \circ S_4$ . By Left Distributivity, it is  $S_1 \circ (S_3 + S_4)$ , i.e.  $S_1 \circ S_H$ . The latter being a contextualisation of  $S_H$ , it is filled by HYDROGEN.



use, and not an ordering one. The sum of all the slots of DIHYDROGEN is its improper slot  $S_{H_2}$ .

## 10.2 The TwoAdjoiningRooms Universal

Consider the universal TWOADJOININGROOMS, presented in Section 3.1.5.2. There are four slots resulting of the contextualisation of a proper slot of ROOM by a proper slot of TWOADJOININGROOMS, namely  $S_5$  to  $S_8$ . Those contextualisations are pictured in Figure 10.4. Improper slots and contextualisations involving them are not displayed.

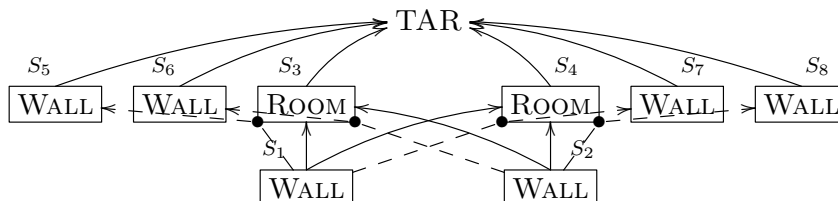


Figure 10.4: Full structure of TWOADJOININGROOMS

If we want a universal isomorphic to its instances, the two occurrences of the universal ROOM within the universal TWOADJOININGROOMS are supposed to overlap. To do so, a slot resulting of a contextualisation of some slot by  $S_3$  has to be the same as a slot resulting of a contextualisation of some slot by  $S_4$ . Let us say that  $S_6 = S_7$ . Figure 10.5 represents the correct structure of TWOADJOININGROOMS, with  $S_3$  and  $S_4$  overlapping on  $S_6$ .

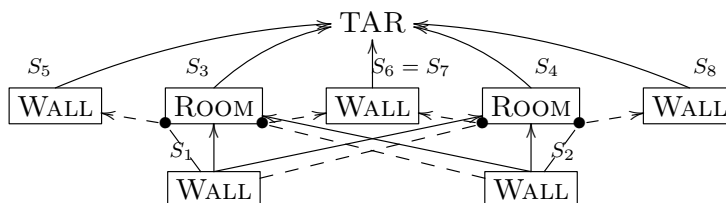


Figure 10.5: Full structure of TWOADJOININGROOMS with overlapping slots

## 10.3 The String Type “cats”

For this example, let us recursively decompose the string “cats” into two parts: all the letters except the last one and the last letter, i.e. “cat” and “s”. The string “cat” is itself decomposed into “ca” and “t”. Finally, “ca” is decomposed into “c” and “a”. The four letters only own an improper slot each, respectively  $S_c$ ,  $S_a$ ,  $S_t$  and  $S_s$ . The string “ca” has an improper slot  $S_{ca}$  and two proper slots  $S_1$  and  $S_2$  respectively filled by “c” and “a”. The string “cat” has an improper slot  $S_{cat}$  and two direct proper slots  $S_3$  and  $S_4$  filled by “ca” and “t”. Finally, the string “cats” has an improper slot  $S_{cats}$  and two direct proper slots  $S_5$  and  $S_6$  filled by “cat” and “s”. The mereological structures are pictured in Figure 10.6 where only improper slots and direct slots are represented. In this figure and the following ones, some slots are not pictured. For example, we can consider that “cats” also decomposes into “ca” and “ts”. However, for simplicity, we do not explore every possibility here.

To explain the role of Axiom 9.6 (Contextualisation Associativity), in Figure 10.7, we first do not accept it and show a model with overduplicated slots. Then, we show in Figure 10.8 how overduplications are solved thanks to the adoption of Axiom 9.6. In

these two last figures, contextualisations are not represented as usual. For readability, the black dots used in all previous figures to represent the contextualising slots are replaced by labels between square brackets on the dashed arrows ( $\square \leftarrow [s] - \square$ ).

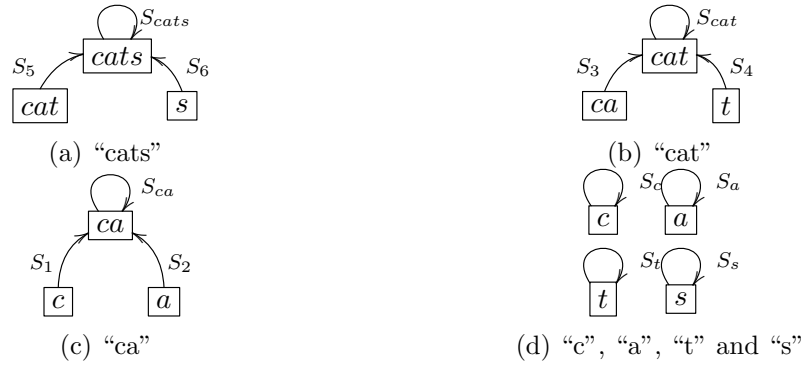


Figure 10.6: Some relevant direct and improper slots of the string types

As in the previous examples, improper slots and contextualisations involving them are not displayed. The indirect slots of “cat” are  $S_7$  and  $S_8$  respectively equal to  $S_3 \circ S_1$  and  $S_3 \circ S_2$ . Those of “cats” are  $S_9 = S_5 \circ S_3$ ,  $S_{10} = S_9 \circ S_1$ ,  $S_{11} = S_9 \circ S_2$ ,  $S_{12} = S_5 \circ S_7$ ,  $S_{13} = S_5 \circ S_8$ , and  $S_{14} = S_5 \circ S_4$ . These are all the possible contextualisations by the proper slots initially considered.

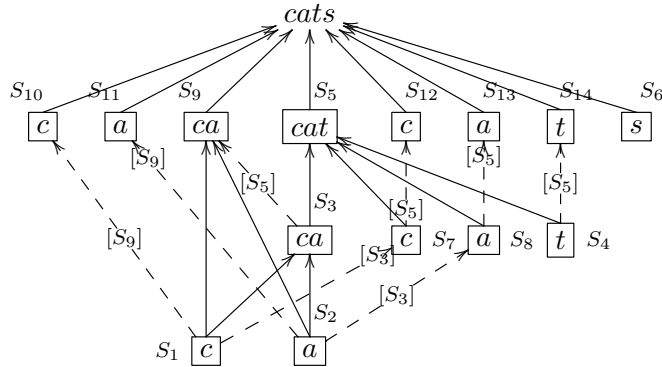


Figure 10.7: A partial view of the structure of “cats” without the Associativity Axiom

However, in Figure 10.7, “cats” has two slots filled by “c” and two slots filled by “a”, where it should only have one of each. This is because Contextualisation Associativity (Axiom 9.6) is not accepted here. If we accept it,  $S_{10} = (S_5 \circ S_3) \circ S_1 = S_5 \circ (S_3 \circ S_1) = S_{12}$  and  $S_{11} = (S_5 \circ S_3) \circ S_2 = S_5 \circ (S_3 \circ S_2) = S_{13}$ , as pictured in Figure 10.8.

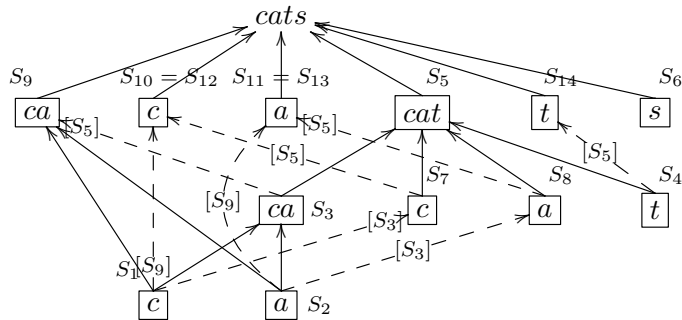


Figure 10.8: A partial view of the structure of “cats” with the Associativity Axiom

**Part III**

**Discussion, Perspectives And  
Conclusion**



# Chapter 11

## Discussion And Perspectives

In this chapter, I discuss different questions that emerged during this work. Some are inspired by reviewers questions or by discussions with my supervisors. Others are my own reflections.

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### 11.1 On the Primitive Choice

One of the many questions in mereology concerns the choice of the primitive used to axiomatise the theory. As I explained in Chapter 2, there are, at least, as many axiomatisations as relations: each mereological relation (part, proper part, overlap, etc.) has been used as a primitive for some mereological theories. Therefore, when it comes to the mereology of slots, it is natural to wonder whether a similar theory would have been possible using a different primitive relation than contextualisation. As a matter of fact, one of the reviewers of (Tarbouriech et al., 2024) raised this question. In this section I explain why I think it is not relevant to use one of the classical mereological relations among slots as a primitive, and why we should keep the contextualisation relation as a primitive. Although the theory has three primitives, I will not discuss removing the primitives  $F$  and  $P_s$ .

Changing the primitive would rest on a misunderstanding of what the contextualisation relation is about. As it happens, contextualisation serves as a basis to define all the mereological relations between slots. However, this is not its first use. The contextualisation relation is a way to ensure that parthood between fillers is transitive, and that the parts are inherited with the right cardinality: its motivation was to fix the counting problems (at least the **Parts of Parts Problem** of Bennett’s theory, and the **Overduplication Problem** of the Copy-Slot Mechanism). Removing this primitive would require to find another fix to the counting problems.

## 11.2 Being Part of Itself Multiple Times

In Bennett’s slot mereology and in the mereology of slots, to occupy a slot is equivalent to be a part of the slot’s owner. To occupy two slots of the same owner is to be twice a part of this owner. Until now, only proper parts can be parts of the same thing multiple times. Is it possible for an entity to be a part of itself multiple times?

Consider the *Sierpiński carpet*. To create it, take a square. Divide it in 9 same-size squares in a 3-by-3 grid and remove the central subsquare. Apply the same procedure recursively to the remaining eight squares. In this fractal, the eight subsquares are similar to the bigger square they are in, with just a change of scale. Consider now the universal SIERPIŃSKICARPET, whose instances are Sierpiński carpets. If we keep the *Isomorphism* principle (presented in Section 2.6.1), the universal has itself as parts eight times.

Consider another example, inspired by wall-building examples of Kearns (2011) and Effingham and Robson (2007). Kearns (2011) tells the story of Jane as follows:

Jane builds a wall, using various bricks (including, as we shall see, one very special brick). Jane then shrinks this wall down to the size of a brick using her superpowers (she has superpowers, by the way) and then travels back in time with the mini-wall to before the wall was built. She then uses the mini-wall as one of the bricks to build the wall. Kearns (2011)

Let me change the end of the story. Instead of using the mini-wall once, Jane uses one of her superpowers to duplicate the special brick, as many times as needed. She then rebuilds the wall with many copies of the shrunken wall. In this case, the wall is part of itself multiple times.

Mereology of slots does not provide a way to represent such relations. An entity is a part of itself because it fills its improper slot. Thus, according to the counting criterion, it is part of itself once. To be able to represent “being part of itself multiple times”, the entity should thus have multiple improper slots, which is forbidden by Axiom 8.3. But removing this axiom is not an easy task. Indeed, in the theory, improper slots have some important properties. First, they are the sum of the entity’s slots. If an entity can have multiple improper slots, then it means that the sum of slots is not idempotent. From there, a lot of things must be changed in the mereology of slots (for example, supplementation and extensionality principles must be abandoned). This is clear when considering how much had to be changed to classical mereology in order for fillers to have multiple parts. This cannot be a solution, as it would just push the problem one step further.

It seems there are two ways to be part of itself. When considering a reflexive form of parthood, the SIERPIŃSKICARPET is part of itself once, in the same way that METHANE (or any other entity) is part of itself once: the SIERPIŃSKICARPET is a *self-part* once. On the other hand, the SIERPIŃSKICARPET is part of the SIERPIŃSKICARPET multiple times, in the same way that HYDROGEN is part of METHANE multiple times. However, by definition of proper part, SIERPIŃSKICARPET cannot be a proper part of itself, in the same way that HYDROGEN is a proper part of METHANE. A possibility would be to build a theory that admits a new type of slot – let’s call it *self-slot* – that would be what makes parthood reflexive (this self-slot might also be the sum of all the slots of its filler); on the other hand, improper slots would be slots filled by their owner and a filler could have multiple improper slots. Therefore, SIERPIŃSKICARPET would be a self-part only once, but an improper part (in the sense of filling an improper slot) multiple times.

However, by definition, improper slots are what are filled and owned by the same thing. Self-slots will, supposedly, also be filled and owned by the same thing. What would differentiate self-slots among improper slots? It does not seem easy to differentiate them with definitions. Thus, the theory might need more primitives to assert which slots are

self-slots and which slots are improper slots. But new primitives increase the ontological cost of the theory.

I do not have a fully developed theory that would account for entities such as the Sierpiński Carpet that are part of themselves multiple times, but have provided above a few pointers towards a possible solution.

### 11.3 Non-Mereological Relations Between Slots

The proposed theory provides a framework to describe entities that have the same part multiple times. In doing so, it makes it possible to explain the difference between METHANE, presented in Section 3.1.5.1, and ETHANE.<sup>1</sup> The first reason is that they own different slots, and in particular different improper slots. However, this reason does not satisfy me, as I do not think there is a difference between stating that “they are different” and “their improper slots, which are unique and intrinsic to them, are different”. The difference is asserted, it does not come from a general principle, like extensionality. The second reason is that they have a different number of slots. We could define appropriate predicates to characterise this distinction. For example, having a single slot filled by CARBON and having exactly two slots filled by CARBON.<sup>2</sup> Consider now BUTANE and ISOBUTANE, two universals whose instances have six carbon atoms and ten hydrogen atoms as parts. Only the first reason is valid in their case, as they do not have a different number of slots: they have the same number of slots, filled by the same thing (except, of course, their improper slots). Nothing, except asserting it, can prove their difference.

What can make a difference? If we analyse the mereology of slots using the three criteria of string types identity of Carrara and Smid (2022b) presented in Section 4.2, we can see that the mereology of slots satisfies **Kind** and **Number**. However, **Order** is not satisfied. Indeed, there is nothing in the theory that enables us to represent the non-mereological relations, such as precedence for letter types, or bonding for molecule universals. In the literature, multiple authors consider that the bonding relation is a filler itself. Consider METHANE. In the representation I used, presented in Section 3.1.5.1, METHANE has five direct proper slots: four are filled by HYDROGEN and one is filled by CARBON. Some authors, like Lewis (1986); Bennett (2013), consider that there should also be four slots filled by the bonding relation, as there are four chemical bondings in a methane molecule particular. However, this is not satisfying, as it does not help in satisfying the **Order** criterion. There are thirteen bondings in any butane particular, and same in any isobutane particular. Adding thirteen slots to BUTANE and ISOBUTANE does nothing for the **Order** criterion. Furthermore, in the literature, the relation is described as a dyadic universal BONDING. Therefore, while METHANE has four slots filled by BONDING, there is no information about which entities are linked by this relation.

The relation could not hold between the fillers. The universal CARBON is not bonded to the universal HYDROGEN in a general way. The bonding exists in the context of being parts of METHANE. Furthermore, the relation cannot fill a slot. As slots are occurring contexts of their fillers, it makes sense to link the slots with the relations. Thus, if the slot  $S_1$  of METHANE, filled by CARBON, relates with the bonding relation to the slot  $S_2$  filled by HYDROGEN, then it means that the part-of-METHANE occurrence of CARBON relates to a part-of-METHANE occurrence of HYDROGEN.

Using non-mereological relations between slots seems possible. In fact, it is already the case. Consider the relations Same Filler (*SF*) and Same Owner (*SO*). They are not, *per se*, mereological relations. In addition to differentiate entities, like isomers, non-mereological

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<sup>1</sup>The chemical formula of ETHANE is  $C_2H_6$ : molecules of ethane have two carbon atoms and six hydrogen atoms as parts.

<sup>2</sup>But in first-order logic, one cannot define a generic predicate of having a different number of slots.

relations between slots could be of great interest. For example, one could extend the expressivity of the theory by adding Allen (1983) interval algebra between slots (which could be applied to string types, for example). However, there are still many questions to answer. Here are some: is this enough to satisfy the **Order** criterion? are these relations stable under contextualisation, like mereological relations are? can a relation hold between slots of different fillers? In the next subsections, I share some of my thoughts about these questions.

### 11.3.1 Satisfying the Order Criterion

With non-mereological relations between slots, there would be a difference between BUTANE and ISOBUTANE, as one would have a slot filled by CARBON linked by a bonding relation to three CARBON-filled slots, and the other would not. However, this might not be enough. BUTANE and ISOBUTANE are *structural isomers*, which means that they have the same parts, but their structure is different. But, there are other types of isomers, called *stereoisomers* or *spatial isomers*. These molecules have the same molecular formula, their parts relates in the same way, the only difference being the position in space of their parts. Therefore, in the case of spatial isomers, the bonding relation is not enough to differentiate them. Thus, only taking into account the bonding relation is not satisfying for the **Order** criterion. It may be interesting to consider relations with higher arity (for example, ternary relations) between slots.

### 11.3.2 Stability Under Contextualisation

Consider the universal METHANE, its two slots  $S_1$  and  $S_2$ , respectively filled by CARBON and HYDROGEN and the relation linking these two slots. Consider now the universal PAIROFMETHANE which is the universal instantiated by any pair of methane molecules. Consider that METHANE fills the slots  $S_3$  and  $S_4$  of PAIROFMETHANE. In this case, the slots  $S_3 \circ S_1$ ,  $S_3 \circ S_2$ ,  $S_4 \circ S_1$  and  $S_4 \circ S_2$  exist. It seems reasonable to say that  $S_3 \circ S_1$  and  $S_3 \circ S_2$  are linked by the bonding relation, as they are contextualised from the same occurrence of METHANE, i.e. the one that occurs thanks to  $S_3$ . In the same way,  $S_4 \circ S_1$  and  $S_4 \circ S_2$  are linked by the bonding relation, as they are both parts of the  $S_4$ -occurrence of METHANE. With the same reasoning, it seems reasonable to say that  $S_3 \circ S_1$  and  $S_4 \circ S_2$  are not linked by the bonding relation. While it seems to be the case that bonding is stable under contextualisation, the generalisation is not trivial, as it depends on the relation. Therefore, I will not push further the analysis.

### 11.3.3 Relations Between Slots of Different Owners

If two slots have different owners, they might be different occurring contexts. Consider one slot  $s$  of the universal WATER filled by HYDROGEN and the slot  $t$  of METHANE filled by HYDROGEN. It is not clear how these two occurrences of HYDROGEN can relate. However, the slots  $s$  and  $t$  relate by the  $SF$  (same filler) relation. This subject is tricky and need a deeper investigation.

## 11.4 Using the Mereology of Slots for Idempotent Entities

We have used slot mereology to represent the structure of non-idempotent entities, such as structural universals and string types. The common feature between these entities is the fact that they can have the same entity as a part multiple times. However, we haven't discussed the relevance of using the slot mereologies for entities that do not have this property, like material individuals. It might indeed be considered as desirable to have



a single mereological theory for both universals and particulars. But one can wonder whether it is feasible.

We do not see any feature of the theory that would make impossible its use to represent the structure of material individuals. However, an important advantage of slot mereologies is to represent various contexts of appearance of an entity, as discussed in Chapter 8. This is not the case of material individuals, as such individuals can only appear in one mereological context at a given time.

Unless slots are reused for something else than mereological purposes, for instance to represent roles, as suggested by Bennett, slot mereologies appear unnecessary for material individuals.



## Chapter 12

# Conclusion

Dans cette thèse, nous avons basé notre travail sur la méréologie à slots proposée par Bennett. Notre objectif était de fournir un cadre théorique permettant de représenter les entités non-idempotentes. Ces entités, qui peuvent avoir plusieurs fois la même partie, étaient un défi pour la méréologie classique extensionnelle en cela qu'elles n'étaient pas compatibles avec plusieurs des principes de cette méréologie, comme le « *Parts Just Once* », l'*Idempotence* ou la *Subpotence*. Notre proposition, en plus de permettre la représentation de telles entités, permet d'assurer un comptage exact du nombre de parties d'une entité.

La méréologie à slots, sur laquelle nous avons basé notre travail, possède plusieurs problèmes. En particulier, nous avons identifié deux problèmes dits de comptage : lorsque la méréologie à slots est utilisée pour représenter la structure d'entités complexes, des erreurs de comptage dans le nombre de parties qu'ont ces entités surviennent. Ces deux problèmes sont liés à l'axiomatisation de la théorie. Le premier vient de l'héritage des slots impropres, ces slots qui permettent à la relation de partie d'être réflexive. Le second problème vient de l'héritage des slots des parties. Au final, les deux problèmes trouvent leur origine dans un seul axiome de la théorie de la méréologie à slots : l'axiome d'héritage des slots.

Afin de résoudre ces problèmes, nous nous sommes débarrassés de cet axiome. Toutefois, sa présence dans la théorie était cruciale : c'est cet axiome qui assure la transitivité de la relation de partie. Afin de restaurer cette transitivité, il nous a fallu développer une nouvelle théorie.

Dans un premier temps, nous avons proposé le mécanisme de copie des slots. Ce mécanisme repose sur l'idée que les slots des parties d'un tout doivent être copiés autant de fois que nécessaire pour assurer un juste comptage des parties. Le Chapitre 5 restitue ce mécanisme, qui a fait l'objet d'une publication (Tarbouriech et al., 2021). Bien que ce mécanisme permette de corriger les problèmes de la théorie de Bennett et rétablisse la transitivité de la relation de partie, il comporte des limitations et produit de nouveaux problèmes de comptage.

Dans un second temps, nous avons développé une nouvelle solution, proposée dans (Tarbouriech et al., 2024), qui repose sur l'idée déjà présente dans le mécanisme des slots copiés : les slots doivent être copiés autant que nécessaire pour assurer un juste comptage des parties. Après avoir clairement défini ce qu'était un slot impropre, nous avons proposé l'opération de contextualisation. Cette opération, munie, entre autres, d'axiomes d'existence, d'unicité et d'associativité, a permis de restaurer la transitivité de la relation de partie, tout en évitant les limitations et problèmes de comptage identifiés avec le mécanisme des slots copiés.

À ce stade, cette extension de la théorie de Bennett corrigeait les problèmes et permettait de compter correctement le nombre de parties d'une entité. Toutefois, il s'agissait

d’une théorie méréologique restreinte au strict minimum : bien qu’elle admettait des axiomes assurant la réflexivité (conditionnelle), la transitivité et l’anti-symétrie de la relation de partie entre fillers, elle n’admettait pas de principes de composition ou de décomposition. En effet, le seul axiome de la théorie allant dans ce sens, le Supplémentation Forte des Slots, est en fait un théorème dérivant directement de l’axiome d’existence des slots impropres.

Afin de pallier ce problème, nous avons défini, basée sur l’opération de contextualisation, une relation de partie entre slots. Après avoir prouvé que cette relation de partie entre slots possédait les propriétés requises (transitivité, réflexivité et anti-symétrie), nous avons développé une théorie méréologie syntaxiquement proche à la méréologie classique extensionnelle présentée, par exemple, par Varzi (2019).

Ainsi, cette méréologie comprend, en plus de la relation de partie entre slots, une relation de partie propre entre slots et une relation de chevauchement entre slots. De plus, nous avons introduit des principes de décomposition traditionnels, tels que la supplémentation et l’extensionnalité. Nous avons également introduit les principes de composition que sont la somme et la fusion, tout en montrant qu’ils agissent de façon similaire aux principes de méréologie classique. Par conséquent, les quatre critères, établis dans le Chapitre 6, sont remplis.

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In this thesis, we based our work on the slot-based mereology proposed by Bennett. We aimed to provide a theoretical framework for representing non-idempotent entities. These entities, which can have the same part several times, were a challenge for classical extensional mereology in that they were not compatible with several of the principles of this mereology, such as “*Parts Just Once*”, *Idempotence* or *Subpotence*. Our proposal not only allows the representation of such entities but also ensures an exact count of the number of parts of an entity.

The slot mereology on which we based our work has several problems. In particular, we have identified two so-called counting problems: when slot mereology is used to represent the structure of complex entities, counting errors in the number of parts these entities have to occur. These two problems are linked to the axiomatization of the theory. The first comes from the inheritance of improper slots, which allow the part relation to be reflexive. The second problem comes from the inheritance of part slots. Ultimately, both problems originate in a single axiom of the theory of slot mereology: the slot inheritance axiom.

In order to solve these problems, we got rid of this axiom. However, its presence in the theory was crucial: this axiom ensures the transitivity of the part relation. In order to restore this transitivity, we had to develop a new theory.

First, we proposed the slot copy mechanism. This mechanism is based on the idea that the slots of the parts of a whole must be copied as many times as necessary to ensure a fair count of the parts. Chapter 5 describes this mechanism, which has already been published in (Tarbouriech et al., 2021). Although this mechanism corrects the problems of Bennett’s theory and restores the transitivity of the part relation, it has limitations and produces new counting problems.

In a second step, we developed a new solution, proposed in (Tarbouriech et al., 2024), based on the idea already present in the copied slots mechanism: slots must be copied as much as necessary to ensure a fair counting of the parts. Having defined what improper slots are, we proposed the contextualization operation. This operation, equipped with, among other things, axioms of existence, uniqueness and associativity, restored the transitivity of the part relation while avoiding the limitations and counting problems identified with the copied slot mechanism.

At this stage, this extension of Bennett’s theory corrected the problems and enabled correct counting. However, it was a mereological theory restricted to the strict minimum: although it admitted axioms ensuring the (conditional) reflexivity, transitivity and anti-

symmetry of the part relation between fillers, it did not admit principles of composition or decomposition. The only axiom of the theory along these lines, Strong Slot Supplementation, is a theorem derived directly from the axiom of the existence of improper slots.

To overcome this problem, we have defined a part relation between slots based on the contextualization operation. After proving that this part relation between slots possesses the required properties (transitivity, reflexivity and anti-symmetry), we developed a theory of mereology syntactically close to the classical extensional mereology presented, for example, by Varzi (2019).

Thus, besides the part relation between slots, this mereology has a proper part relation between slots and an overlap relation between slots. In addition, we have introduced traditional decomposition principles, such as supplementation and extensionality. We have also introduced the compositional principles of sum and merge, showing that they act similarly to the principles of classical mereology. As a result, the four criteria set out in Chapter 5 have been met.



# Bibliography

- Allen, J. F. (1983). Maintaining knowledge about temporal intervals. *Communications of the ACM*, 26(11):832–843.
- Armstrong, D. M. (1978a). *A Theory of Universals: Universals & Scientific Realism*, volume 2. Cambridge University Press.
- Armstrong, D. M. (1978b). *Nominalism and Realism: Universals & Scientific Realism*, volume 1. Cambridge University Press.
- Armstrong, D. M. (1978c). *Universals: An Opinionated Introduction*. Cambridge University Press, New York.
- Armstrong, D. M. (1986). In defence of structural universals. *Australasian Journal of Philosophy*, 64(1):85–88.
- Barton, A., Tarbouriech, C., Vieu, L., and Éthier, J.-F. (2022). A mereological system for informational templates roles. In T. P. Sales, M. M. Hedblom et al, editor, *Proceedings of the Joint Ontology Workshops 2022 (JOWO 2022). 6th International Workshop on Foundational Ontologies (FOUST)*.
- Barton, A., Toyoshima, F., and Ethier, J.-F. (2020a). Clinical documents and their parts. In *Proceedings of the 11th International Conference on Biomedical Ontologies (ICBO 2020)*. CEUR, Vol. 2807.
- Barton, A., Toyoshima, F., Vieu, L., Fabry, P., and Ethier, J.-F. (2020b). The mereological structure of informational entities. In *Formal Ontology in Information Systems (FOIS 2020)*, page 201–215. IOS Press.
- Barton, A., Vieu, L., and Ethier, J.-F. (2021). Expressions, Utterances and Directive Slots. In Sanfilippo, E. M., Kutz, O., Troquard, N., Hahmann, T., Masolo, C., Hoehndorf, R., and Vita, R., editors, *5th Workshop on Foundational Ontology (FOUST 2021) @ JOWO 2021: The Joint Ontology Workshops*, volume 2969 of *CEUR Workshop Proceedings*, Bozen-Bolzano, Italy. CEUR-WS.org.
- Bennett, K. (2013). Having a part twice over. *Australasian Journal of Philosophy*, 91(1):83–103.
- Bennett, K. (2017). Part and whole, again. *Philosophical Issues*, 27:pp. 7–25.
- Bertot, Y. and Castéran, P. (2013). *Interactive theorem proving and program development: Coq’Art: the calculus of inductive constructions*. Springer Science & Business Media.
- Bigelow, J. (1986). Towards structural universals. *Australasian Journal of Philosophy*, 64(1):94–96.
- Bigelow, J. and Pargetter, R. (1989). A theory of structural universals. *Australasian Journal of Philosophy*, 67(1):1–11.

- Bittner, T., Donnelly, M., and Smith, B. (2004). Individuals, universals, collections: On the foundational relations of ontology. In Varzi, A. and Vieu, L., editors, *Formal Ontology in Information Systems. Proceedings of the Third International Conference*, pages 37–48. IOS Press.
- Bjørner, D. (2018). To every manifest domain a csp expression – a rôle for mereology in computer science. *Journal of Logical and Algebraic Methods in Programming*, 94:91–108.
- Blizard, W. D. (1989). Multiset theory. *Notre Dame Journal of formal logic*, 30(1):36–66.
- Bohn, E. D. (2009). An argument against the necessity of unrestricted composition. *Analysis*, 69(1):27–31.
- Bromberger, S. (2011). What are words? Comments on Kaplan (1990), on Hawthorne and LePore, and on the issue. *The Journal of Philosophy*, 108(9):486–503.
- Carrara, M. and Smid, J. (2022a). On Cotnoir’s two notions of proper parthood. *Philosophical Studies*, 179(9):2787–2795.
- Carrara, M. and Smid, J. (2022b). There is overlap in letters: a mereological definition of word types. *Synthese*, 200(2):1–17.
- Casati, R. and Varzi, A. C. (1999). *Parts and places: The structures of spatial representation*. MIT press.
- Cotnoir, A. (2015). Abelian mereology. *Logic and Logical Philosophy*.
- Cotnoir, A. J. (2013). Strange parts: The metaphysics of non-classical mereologies. *Philosophy Compass*, 8(9):834–845.
- Cotnoir, A. J. (2014). Universalism and Junk. *Australasian Journal of Philosophy*, 92(4):649–664.
- Cotnoir, A. J. and Varzi, A. C. (2021). *Mereology*. Oxford University Press.
- Davis, W. A. (2014). On Occurrences of Types in Types. *Australasian Journal of Philosophy*, 92(2):349–363.
- Eberle, R. A. (1970). *Nominalistic systems*. Reidel & Dordrecht.
- Effingham, N. and Robson, J. (2007). A mereological challenge to endurantism. *Australasian Journal of Philosophy*, 85(4):633–640.
- Fisher, A. R. J. (2013). Bennett on Parts Twice Over. *Philosophia*, 41(3):757–761.
- Fisher, A. R. J. (2018). Structural universals. *Philosophy Compass*, 13(10):e12518.
- Forrest, P. (1986). Neither magic nor mereology: A reply to lewis. *Australasian Journal of Philosophy*, 64(1):89–91.
- Forrest, P. (2020). The Identity of Indiscernibles. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Winter 2020 edition.
- French, S. (2019). Identity and Individuality in Quantum Theory. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Winter 2019 edition.



- Garbacz, P. (2016). Slot mereology revised. *Australasian Journal of Philosophy*, 95(1):171–177.
- Garbacz, P. (2020). An analysis of the debate over structural universals. In *Formal Ontology in Information Systems (FOIS 2020)*, pages 3–16. IOS Press.
- Gilmore, C. (2007). Time travel, coinciding objects, and persistence. *Oxford studies in metaphysics*, 3:177–198.
- Goodman, N. (1951). *The Structure of Appearance*. Harvard University Press.
- Guarino, N., Pribbenow, S., and Vieu, L., editors (1996). Modeling Parts and Wholes, *Data & Knowledge Engineering*, volume 20(3):257–434. Elsevier Science.
- Hawthorne, J. and Lepore, E. (2011). On words. *Journal of Philosophy*, 109(9):447–485.
- Husserl, E. (1901). Logische untersuchungen ii/2.
- Irmak, N. (2019). An ontology of words. *Erkenntnis*, 84(5):1139–1158.
- Jackson, D. (2012). *Software Abstractions: logic, language, and analysis*. MIT press.
- Kaplan, D. (1990). Words. *Proceedings of the Aristotelian society, supplementary volumes*, 64:93–119.
- Kaplan, D. (2011). Words on Words. *The Journal of Philosophy*, 108(9):504–529.
- Kearns, S. (2011). Can a thing be part of itself? *American Philosophical Quarterly*, 48(1):87–93.
- Keet, C. M. (2006). Introduction to part-whole relations: mereology, conceptual modelling and mathematical aspects. *KRDB Research Centre Technical Report KRDB06-3 (Tutorial reader)*, Faculty of Computer Science, Free University of Bozen-Bolzano, Italy, 2.
- Lando, G. (2019). A uniform, concretist metaphysics for linguistic types. *Metaphysica*, 20(2):195–221.
- Leonard, H. S. and Goodman, N. (1940). The calculus of individuals and its uses. *The journal of symbolic logic*, 5(2):45–55.
- Leśniewski, S. (1916). Podstawy ogólnej teorii mnogości. i. *Prace Polskiego Koła Naukowego w Moskwie*, 2:129–173.
- Leśniewski, S. (1931). O podstawach matematyki (on the foundations of mathematics). *Przegląd Filozoficzny*, 34:142–170.
- Lewis, D. (1986). Against structural universals. *Australasian Journal of Philosophy*, 64(1):25–46.
- Lewis, D. (1991). Parts of classes. *Mind*, 100(3):394–397.
- Macleod, M. C. and Rubenstein, E. M. (2023). Universals. In *Internet Encyclopedia of Philosophy*.
- Masolo, C. and Vieu, L. (1999). Atomicity vs. infinite divisibility of space. In Freksa, C. and Mark, D., editors, *Spatial Information theory. Proceedings of COSIT'99*, LNCS 1661, pages 235–250. Springer Verlag, Berlin.

- Masolo, C. and Vieu, L. (2018). Graph-based approaches to structural universals and complex states of affairs. In *Formal Ontology in Information Systems (FOIS 2018)*, pages 69–82. IOS Press.
- Masolo, C., Vieu, L., Bottazzi, E., Catenacci, C., Ferrario, R., Gangemi, A., and Guarino, N. (2004). Social roles and their descriptions. In *Proceedings of the 9th Int. Conf. on Principles of Knowledge Representation and Reasoning (KR 2004)*, pages 267–277.
- McCulloch, G. (1991). Making sense of words. *Analysis*, 51(2):73–79.
- McFarland, A. (2018). Causal powers and isomeric chemical kinds. *Synthese*, 195:1441–1457.
- Mormann, T. (2012). On the mereological structure of complex states of affairs. *Synthese*, 187:403–418.
- Oliver, A. and Smiley, T. (2016). *Plural Logic: Second Edition, Revised and Enlarged*. Oxford University Press.
- Polkowski, L. (2014). Mereology in engineering and computer science. *Mereology and the Sciences: Parts and Wholes in the Contemporary Scientific Context*, pages 217–291.
- Sanfilippo, E. M. (2021). Ontologies for information entities: State of the art and open challenges. *Applied ontology*, 16(2):111–135.
- Sattig, T. (2021). Part, slot, ground: foundations for neo-Aristotelian mereology. *Synthese*, 198(Suppl 11):2735–2749.
- Schaffer, J. (2010). Monism: The Priority of the Whole. *The Philosophical Review*, 119(1):31–76.
- Simons, P. (1987). *Parts: A Study in Ontology*. Clarendon Press, Oxford.
- Tarbouriech, C., Barton, A., Vieu, L., and Éthier, J.-F. (2021). The Counting Problem of Slot Mereology. In *Workshop on Foundational Ontology (FOUST V), Joint Ontology Workshops (JOWO 2021)*. CEUR Workshop Proceedings.
- Tarbouriech, C., Vieu, L., Barton, A., and Éthier, J.-F. (2024). From Slot Mereology To A Mereology Of Slots. *Applied Ontology*. To Appear.
- Van Inwagen, P. (1987). When are objects parts? *Philosophical Perspectives*, 1:21–47.
- Varzi, A. C. (1996). Parts, wholes, and part-whole relations: The prospects of mereotopology. *Data & Knowledge Engineering*, 20(3):259–286. Modeling Parts and Wholes.
- Varzi, A. C. (2019). Mereology. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Spring 2019 edition.
- Vieu, L. (2006). On the transitivity of functional parthood. *Applied Ontology*, 1(2):147–155.
- Wetzel, L. (1993). What are occurrences of expressions? *Journal of Philosophical Logic*, 22(2):215–219.
- Wetzel, L. (2009). *Types and tokens: On abstract objects*. MIT Press.
- Yeung, C.-m. A. and Leung, H.-f. (2006). Ontology with likeliness and typicality of objects in concepts. In *International Conference on Conceptual Modeling*, pages 98–111. Springer.

Yi, B.-U. (2005). The Logic and Meaning of Plurals. Part I. *Journal of Philosophical Logic*, 34(5):459–506.

Yi, B.-u. (2006). The Logic and Meaning of Plurals. Part II. *Journal of Philosophical Logic*, 35(3):239–288.



# Appendix A

## List Of Formulas

### A.1 Definitions

Number	Name	Definition
D2.1	Proper Parthood	$PP(x, y) \triangleq P(x, y) \wedge x \neq y$
D2.2	Overlap	$O(x, y) \triangleq \exists z(P(z, x) \wedge P(z, y))$
D2.3	Proper Overlap	$PO(x, y) \triangleq O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)$
D2.4	Improper Overlap	$IO(x, y) \triangleq P(x, y) \vee P(y, x)$
D2.5	Disjointness	$D(x, y) \triangleq \neg O(x, y)$
D2.6	Atom	$A(x) \triangleq \neg \exists y(PP(y, x))$
D2.7	Sum <sub>1</sub>	$S_1(z, x, y) \triangleq P(x, z) \wedge P(y, z) \wedge \forall w(P(w, z) \rightarrow O(w, x) \vee O(w, y))$
D2.8	Sum <sub>2</sub>	$S_2(z, x, y) \triangleq \forall w(O(z, w) \leftrightarrow O(w, x) \vee O(w, y))$
DS2.9	Fusion <sub>1</sub>	$F_{1\phi}(z) \triangleq \forall w(\phi(w) \rightarrow P(w, z)) \wedge \forall v(P(v, z) \rightarrow \exists w(\phi(w) \wedge O(v, w)))$
DS2.10	Fusion <sub>2</sub>	$F_{2\phi}(z) \triangleq \forall v(O(v, z) \leftrightarrow \exists w(\phi(w) \wedge O(v, w)))$
D2.11	Fusion <sub>1</sub> Of Cats	$F_{1Cat}(z) \triangleq \forall w(Cat(w) \rightarrow P(w, z)) \wedge \forall v(P(v, z) \rightarrow \exists w(Cat(w) \wedge O(v, w)))$
D2.12	Fusion <sub>2</sub> Of Cats	$F_{2Cat}(z) \triangleq \forall v(O(v, z) \leftrightarrow \exists w(Cat(w) \wedge O(v, w)))$
D2.13	The Universe	$U(z) \triangleq F_{iP(w,w)}(z)$
D3.1	Parthood	$P(a, b) \triangleq \exists s(P_s(s, b) \wedge F(a, s))$
D3.2	Proper Parthood	$PP(a, b) \triangleq P(a, b) \wedge \neg P(b, a)$
D3.3	Overlap	$O(a, b) \triangleq \exists c(P(c, a) \wedge P(c, b))$
D3.4	Slot-overlap	$O_s(a, b) \triangleq \exists s(P_s(s, a) \wedge P_s(s, b))$
D3.5	Proper Parthood Slot	$PP_s(s, a) \triangleq P_s(s, a) \wedge \neg F(a, s)$
D3.6	Direct Slot	$DP_s(s, a) \triangleq P_s(s, a) \wedge \neg \exists b(PP(b, a) \wedge P_s(s, b))$

Number	Name	Definition
D8.1	Slot	$S(s) \triangleq \exists a(P_s(s, a))$
D8.2	Same Owner	$SO(s, t) \triangleq \exists a(P_s(s, a) \wedge P_s(t, a))$
D8.3	Same Filler	$SF(s, t) \triangleq \exists a(F(a, s) \wedge F(a, t))$
D8.4	Improper Slot	$IP_s(s, a) \triangleq P_s(s, a) \wedge F(a, s)$
D9.1	Contextualisable by	$Cb(t, s) \triangleq \exists a(F(a, s) \wedge P_s(t, a))$
D9.2	Direct Slot — With Contextualisation	$DP'_s(s, a) \triangleq P_s(s, a) \wedge \forall t, u[s = t \circ u \rightarrow (\exists b(IP_s(t, b))) \vee (\exists c(IP_s(u, c)))]$
D9.3	Part of Slot	$PoS(u, s) \triangleq \exists t(u = s \circ t)$
D9.4	Proper Part of Slot	$PPoS(s, t) \triangleq PoS(s, t) \wedge s \neq t$
D9.5	Overlap of Slots	$OoS(s, t) \triangleq \exists u(PoS(u, s) \wedge PoS(u, t))$
D9.6	Sum with Parthood	$SoS_1(u, s, t) \triangleq PoS(s, u) \wedge PoS(t, u) \wedge \forall v(PoS(v, u) \rightarrow OoS(s, v) \vee OoS(t, v))$
D9.7	Sum with Overlap	$SoS_2(u, s, t) \triangleq \forall v(OoS(u, v) \leftrightarrow OoS(s, v) \vee OoS(t, v))$
DS9.8	Fusion of Slots	$FoS_\phi(z) \triangleq \forall w(\phi(w) \rightarrow PoS(w, z)) \wedge \forall v(PoS(v, z) \rightarrow \exists w(\phi(w) \wedge OoS(v, w)))$

## A.2 Axioms

Number	Description	Axiom
A2.1	Parthood Reflexivity	$P(x, x)$
A2.2	Parthood Transitivity	$P(x, y) \wedge P(y, z) \rightarrow P(x, z)$
A2.3	Parthood Antisymmetry	$P(x, y) \wedge P(y, x) \rightarrow x = y$
A2.4	Company	$PP(x, y) \rightarrow \exists z(PP(z, y) \wedge x \neq z)$
A2.5	Strong Company	$PP(x, y) \rightarrow \exists z(PP(z, y) \wedge \neg P(z, x))$
A2.6	Supplementation	$PP(x, y) \rightarrow \exists z(P(z, y) \wedge \neg O(z, x))$
A2.7	Strong Supplementation	$\neg P(y, x) \rightarrow \exists z(P(z, y) \wedge \neg O(z, x))$
A2.8	Complementation	$\neg P(y, x) \rightarrow \exists z \forall w(P(w, z) \leftrightarrow (P(w, y) \wedge \neg O(z, x)))$
AS2.9	Existence of Sum <sub>i</sub>	$\xi(x, y) \rightarrow \exists z(S_i(z, x, y))$
AS2.10	Restricted Existence of Fusion <sub>i</sub>	$\exists w(\phi(w)) \wedge \forall w(\phi(w) \rightarrow \psi(w)) \rightarrow \exists z(F_{i\phi}(z))$
AS2.11	Unrestricted Existence of Fusion <sub>i</sub>	$\exists w(\phi(w)) \rightarrow \exists z(F_{i\phi}(z))$
A3.1	Only Slots are Filled	$F(a, s) \rightarrow \exists b(P_s(s, b))$
A3.2	Slots Cannot Fill	$F(a, s) \rightarrow \neg \exists b(P_s(a, b))$
A3.3	Slots Don't Have Slots	$P_s(s, a) \rightarrow \neg \exists t(P_s(t, s))$
A3.4	Improper Parthood Slots	$\exists s(P_s(s, a)) \rightarrow \exists t(P_s(t, a) \wedge F(a, t))$
A3.5	Slot Inheritance	$[P_s(s, b) \wedge F(a, s) \wedge P_s(t, a)] \rightarrow P_s(t, b)$
A3.6	Mutual Occupancy is Identity	$(P_s(s, b) \wedge F(a, s)) \wedge (P_s(t, a) \wedge F(b, t)) \rightarrow a = b$
A3.7	Single Occupancy	$P_s(s, a) \rightarrow \exists! b(F(b, s))$
A3.8	Slot Strong Supplementation	$\exists s(P_s(s, a)) \wedge \exists t(P_s(t, b)) \rightarrow [\neg(\exists u(P_s(u, a) \wedge F(b, u))) \rightarrow \exists v(P_s(v, b) \wedge \neg P_s(v, a))]$

Number	Description	Axiom
A5.1	Improper Slots are only owned by their Filler	$P_s(s, a) \wedge F(a, s) \rightarrow \forall b(P_s(s, b) \rightarrow a = b)$
A5.2	Proper Slot Inheritance	$P(a, b) \wedge P_s(s, a) \wedge \neg F(a, s) \rightarrow P_s(s, b)$
A5.3	Additional Improper Part-hood Slots	$F(a, s) \rightarrow \exists t(P_s(t, a) \wedge F(a, t))$
A5.4	Only One Improper Slot per Filler	$P_s(s, a) \wedge F(a, s) \wedge P_s(t, a) \wedge F(a, t) \rightarrow s = t$
A5.5	Anti-Inheritance	$[a \neq b \wedge P_s(s, b) \wedge F(a, s) \wedge P_s(t, a)] \rightarrow \neg P_s(t, b)$
A5.6	Existence of a Unique Copy-Slot for each Whole and Path-Slot, Source-Slot Pair	$PP_s(s, a) \wedge F(b, s) \wedge PP_s(t, b) \rightarrow \exists! u(P_s(u, a) \wedge CT(u, s) \wedge CF(u, t))$
A5.7	Copied Slot has the Same Filler as its Source	$CF(t, s) \rightarrow \exists a(F(a, s) \wedge F(a, t))$
A5.8	Same Owner	$PP_s(t, a) \wedge CT(t, s) \rightarrow PP_s(s, a)$
A5.9	Copy Constrains Structure	$CT(u, s) \wedge CF(u, t) \rightarrow \exists a(F(a, s) \wedge PP_s(t, a))$
A5.10	Functionality Of $CF$	$CF(s, t) \wedge CF(s, u) \rightarrow t = u$
A5.11	Functionality Of $CT$	$CT(s, t) \wedge CT(s, u) \rightarrow t = u$
A8.1	Single Owner	$P_s(s, a) \wedge P_s(s, b) \rightarrow a = b$
A8.2	Additional Improper Slot	$F(a, s) \rightarrow \exists t(IP_s(t, a))$
A8.3	Unique Improper Slot per Filler	$IP_s(s, a) \wedge IP_s(t, a) \rightarrow s = t$
A9.1	Domains of Contextualisation	$CoS(u, s, t) \rightarrow S(u) \wedge S(s) \wedge S(t)$
A9.2	Contextualisable iff Contextualisation Exists	$Cb(t, s) \leftrightarrow \exists u(CoS(u, s, t))$
A9.3	Unicity of Contextualisation	$CoS(u, s, t) \wedge CoS(v, s, t) \rightarrow u = v$
A9.4	Injectivity to the Left	$v = s \circ t \wedge v = s \circ u \rightarrow t = u$
A9.5	Injectivity to the Right	$v = t \circ s \wedge v = u \circ s \wedge \exists a(IP_s(s, a)) \rightarrow t = u$
A9.6	Contextualisation Associativity	$\exists w(v = s \circ w \wedge w = t \circ u) \leftrightarrow \exists x(v = x \circ u \wedge x = s \circ t)$
A9.7	Slot Strong Supplementation	$S(s) \wedge S(t) \rightarrow (\neg PoS(t, s) \rightarrow \exists u(PoS(u, t) \wedge \neg OoS(u, s)))$
A9.8	Sum Existence	$SO(s, t) \rightarrow \exists u(SoS_1(u, s, t))$
AS9.9	Fusion Existence	$\exists w(\phi(w) \wedge \forall v(\phi(v) \rightarrow SO(v, w))) \rightarrow \exists s(FoS_\phi(s))$

### A.3 Theorems And Lemmas

Number	Description	Theorem/Lemma
T2.1	Proper Parthood Irreflexivity	$\neg PP(x, x)$
T2.2	Proper Parthood Transitivity	$PP(x, y) \wedge PP(y, z) \rightarrow PP(x, z)$
T2.3	Proper Parthood Asymmetry	$PP(x, y) \rightarrow \neg PP(y, x)$
T2.4	Overlap Symmetry	$O(x, y) \rightarrow O(y, x)$
T2.5	Proper Overlap Symmetry	$PO(x, y) \rightarrow PO(y, x)$
T2.6	Improper Overlap Symmetry	$IO(x, y) \rightarrow IO(y, x)$
T2.7	Disjointness Symmetry	$D(x, y) \rightarrow D(y, x)$
T2.8	Overlap Reflexivity	$O(x, x)$
T2.9	Proper Overlap Irreflexivity	$\neg PO(x, x)$
T2.10	Improper Overlap Reflexivity	$IO(x, x)$
T2.11	Disjointness Irreflexivity	$\neg D(x, x)$
T2.12	PP-Extensionality	$\exists z(PP(z, x) \vee PP(z, y)) \rightarrow (x = y \leftrightarrow \forall z(PP(z, x) \leftrightarrow PP(z, y)))$
T2.13	O-Extensionality	$x = y \leftrightarrow \forall z(O(z, x) \leftrightarrow O(z, y))$
T2.14	-	$P(x, x +_i y)$
T2.15	-	$P(x, y) \rightarrow P(x, y +_i z)$
T2.16	-	$P(x +_i y, z) \rightarrow P(x, z)$
T2.17	-	$P(x, y) \leftrightarrow x +_i y = y$
T3.1	Filler-Irreflexivity	$\neg F(a, a)$
T3.2	Filler-Asymmetry	$F(a, s) \rightarrow \neg F(s, a)$
T3.3	Filler-Transitivity	$F(a, b) \wedge F(b, c) \rightarrow F(a, c)$
T3.4	Slot-Irreflexivity	$\neg P_s(s, s)$
T3.5	Slot-Asymmetry	$P_s(s, a) \rightarrow \neg P_s(a, s)$
T3.6	Slot-Transitivity	$P_s(s, t) \wedge P_s(t, u) \rightarrow P_s(s, u)$
T3.7	Transitivity	$P(a, b) \wedge P(b, c) \rightarrow P(a, c)$
T3.8	Anti-Symmetry	$P(a, b) \wedge P(b, a) \rightarrow a = b$
T3.9	Conditional Reflexivity	$\exists s(P_s(s, a)) \rightarrow P(a, a)$
T3.10	Parts $\leftrightarrow$ Slots	$\exists b(P(b, a)) \leftrightarrow \exists s(P_s(s, a))$
T3.11	Composites $\leftrightarrow$ Slot-Composites	$\exists b(PP(b, a)) \leftrightarrow \exists s(PP_s(s, a))$
T3.12	Slot Weak Supplementation	$PP(a, b) \rightarrow \exists s(P_s(s, b) \wedge \neg P_s(s, a))$
T3.13	Slot Extensionality	$\exists c(PP(c, a)) \vee \exists c(PP(c, b)) \rightarrow (a = b \leftrightarrow \forall s(PP_s(s, a) \leftrightarrow PP_s(s, b)))$



Number	Description	Theorem/Lemma
T5.1	Improper Slots Don't Contain Proper Parts	$PP(y, x) \rightarrow \exists s(P_s(s, x) \wedge F(y, s) \wedge \neg P_s(s, y))$
T5.2	Parthood Transitivity	$P(a, b) \wedge P(b, c) \rightarrow P(a, c)$
T5.3	General Conditional Reflexivity	$\exists s(P_s(s, a) \vee F(a, s)) \rightarrow part\_of(a, a)$
T8.1	Anti-Inheritance	$[a \neq b \wedge P_s(s, b) \wedge F(a, s) \wedge P_s(t, a)] \rightarrow \neg P_s(t, b)$
L8.2	Either Proper or Improper	$S(s) \rightarrow \exists! a(PP_s(s, a) \oplus IP_s(s, a))$
L8.3	Proper Parts iff Proper Slots	$PP(b, a) \leftrightarrow \exists s(PP_s(s, a) \wedge F(b, s))$
L8.4	General Conditional Reflexivity	$P_s(s, a) \vee F(a, s) \rightarrow P(a, a)$
T8.5	Mutual Occupancy is Slot Identity	$P_s(s, b) \wedge F(a, s) \wedge P_s(t, a) \wedge F(b, t) \rightarrow s = t$
T9.1	Symmetric Contextualisation is Slot Identity	$u = s \circ t \wedge v = t \circ s \rightarrow s = t$
T9.2	Left-and-Right-Improper Contextualisation	$\exists a(IP_s(s, a)) \leftrightarrow s = s \circ s$
T9.3	Improper Slot Is Right Neutral Element	$IP_s(s, a) \wedge F(a, t) \rightarrow t = t \circ s$
T9.4	Improper Slot Is Left Neutral Element	$IP_s(s, a) \wedge P_s(t, a) \rightarrow t = s \circ t$
T9.5	Mutual Contextualisation is Identity	$s = t \circ u \wedge t = s \circ v \rightarrow s = t$
T9.6	Contextualisation Same Owner	$u = s \circ t \rightarrow SO(u, s)$
T9.7	Contextualisation Same Filler	$u = s \circ t \rightarrow SF(u, t)$
T9.8	Parthood Transitivity	$P(a, b) \wedge P(b, c) \rightarrow P(a, c)$
T9.9	Right Neutral Element Is Improper Slot	$t = t \circ s \rightarrow \exists a(IP_s(s, a) \wedge F(a, t))$
T9.10	Left Neutral Element Is Improper Slot	$t = s \circ t \rightarrow \exists a(IP_s(s, a) \wedge P_s(t, a))$
T9.11	Contextualisation Stable under Contextualisation	$Cb(s, u) \wedge Cb(t, u) \rightarrow \forall v(s = t \circ v \leftrightarrow u \circ s = (u \circ t) \circ v)$

Number	Description	Theorem/Lemma
L9.12	<i>PoS</i> Domain and Range	$PoS(s, t) \rightarrow S(s) \wedge S(t)$
T9.13	Conditional <i>PoS</i> Reflexivity	$\forall s(S(s) \rightarrow Pos(s, s))$
T9.14	<i>PoS</i> Anti-Symmetry	$Pos(s, t) \wedge Pos(t, s) \rightarrow s = t$
T9.15	<i>PoS</i> Transitivity	$Pos(s, t) \wedge Pos(t, u) \rightarrow Pos(s, u)$
T9.16	<i>PoS</i> Same Owner	$Pos(s, t) \rightarrow SO(s, t)$
T9.17	Slots iff Slot-Parts of Improper Slot	$IP_s(s, a) \rightarrow \forall t(P_s(t, a) \leftrightarrow Pos(t, s))$
T9.18	Slot Structure and Filler Structure constrain Each Other	$\exists s, t(Pos(t, s) \wedge F(a, s) \wedge F(b, t)) \leftrightarrow P(b, a)$
T9.19	<i>PoS</i> Stable under Contextualisation	$Cb(t, s) \wedge Cb(u, s) \rightarrow (Pos(u, t) \leftrightarrow Pos(s \circ u, s \circ t))$
T9.20	<i>PPoS</i> Irreflexivity	$\neg PPos(s, s)$
T9.21	<i>PPoS</i> Asymmetry	$PPoS(s, t) \rightarrow \neg PPos(t, s)$
T9.22	<i>PPoS</i> Transitivity	$PPoS(s, t) \wedge PPos(t, u) \rightarrow PPos(s, u)$
T9.23	<i>PPoS</i> Same Owner	$PPoS(s, t) \rightarrow SO(s, t)$
T9.24	Proper Slots iff Proper Parts Of Improper Slot	$IP_s(s, a) \rightarrow \forall t(PP_s(t, a) \leftrightarrow PPos(t, s))$
T9.25	Slot Structure and Filler Structure constrain Each Other — Proper Part	$\exists s, t(PPoS(t, s) \wedge F(a, s) \wedge F(b, t)) \leftrightarrow PP(b, a)$
T9.26	<i>PPoS</i> Stable under Contextualisation	$Cb(t, s) \wedge Cb(u, s) \rightarrow (PPoS(u, t) \leftrightarrow PPos(s \circ u, s \circ t))$
T9.27	Conditional <i>OoS</i> Reflexivity	$S(s) \rightarrow OoS(s, s)$
T9.28	<i>OoS</i> Symmetry	$OoS(s, t) \rightarrow OoS(t, s)$
T9.29	<i>OoS</i> Same Owner	$OoS(s, t) \rightarrow SO(s, t)$
L9.30	Overlap with Part Implies Overlap with Whole	$OoS(u, t) \wedge Pos(t, s) \rightarrow OoS(u, s)$
L9.31	Slot-Overlap With Improper Slot	$IP_s(s, a) \wedge P_s(t, a) \rightarrow OoS(s, t)$
L9.32	<i>PoS</i> Implies <i>OoS</i>	$Pos(s, t) \rightarrow OoS(s, t)$
T9.33	<i>OoS</i> Stable under Contextualisation	$Cb(t, s) \wedge Cb(u, s) \rightarrow (OoS(t, u) \leftrightarrow OoS(s \circ t, s \circ u))$
T9.34	Slot-Overlap Constrains Overlap between Fillers	$OoS(s, t) \wedge F(a, s) \wedge F(b, t) \rightarrow O(a, b)$

Number	Description	Theorem/Lemma
T9.35	Slot Weak Supplementation	$PPoS(s, t) \rightarrow \exists u(PoS(u, t) \wedge \neg OoS(u, s))$
T9.36	$OoS$ -Extensionality	$S(s) \wedge S(t) \rightarrow (\forall u, OoS(s, u) \leftrightarrow OoS(t, u)) \rightarrow s = t$
T9.37	$PPoS$ -Extensionality	$\exists u(PPoS(u, s) \vee PPoS(u, t))$ $\rightarrow (\forall u, PPoS(u, s) \leftrightarrow PPoS(u, t)) \rightarrow s = t$
L9.38	Domains of Sum	$SoS_1(u, s, t) \rightarrow S(s) \wedge S(t) \wedge S(u)$
T9.39	$SoS_1$ and $SoS_2$ are Equivalent	$SO(s, t) \rightarrow \forall u(SoS_1(u, s, t) \leftrightarrow SoS_2(u, s, t))$
T9.40	Sum Same Owner	$SoS_1(u, s, t) \rightarrow \exists a(P_s(u, a) \wedge P_s(s, a) \wedge P_s(t, a))$
T9.41	Sum Unicity	$SoS_1(u, s, t) \wedge SoS_1(v, s, t) \rightarrow u = v$
T9.42	Sum Idempotence	$S(s) \rightarrow s + s = s$
T9.43	Sum Commutativity	$SO(s, t) \rightarrow s + t = t + s$
L9.44	-	$SO(s, t) \rightarrow PoS(s, s + t)$
L9.45	-	$SO(t, u) \rightarrow \forall s(PoS(s, t) \rightarrow PoS(s, t + u))$
L9.46	-	$SO(s, t) \rightarrow \forall u(PoS(s + t, u) \rightarrow PoS(s, u))$
L9.47	-	$PoS(s, t) \leftrightarrow s + t = t$ $Cb(t, s) \wedge Cb(u, s) \rightarrow$
T9.48	Overlaps the Contextualised Sum iff Overlaps one of the Contextualised Operands	$\forall v(OoS(v, s \circ (t + u))$ $\leftrightarrow OoS(v, s \circ t) \vee OoS(v, s \circ u))$
T9.49	Left Distributivity	$Cb(t, s) \wedge Cb(u, s) \rightarrow$ $(s \circ (t + u) = s \circ t + s \circ u)$
L9.50	Same Filler of Operand and Sum Implies Identity	$SF(s, s + t) \rightarrow s = s + t$
T9.51	Right Distributivity Is Trivial	$Cb(s + t, u) \wedge Cb(s, u) \wedge Cb(t, u) \rightarrow s = t$
L9.52	Sum is Slot-Part if Operands are Slot-Parts	$PoS(s, u) \wedge PoS(t, u) \rightarrow PoS(s + t, u)$
T9.53	Sum Associativity	$SO(s, t) \wedge SO(t, u) \rightarrow (s + t) + u = s + (t + u)$
T9.54	Sum Stable under Contextualisation	$Cb(u, v) \wedge Cb(s, v) \wedge Cb(t, v) \rightarrow$ $(u = s + t \leftrightarrow (v \circ u) = (v \circ s) + (v \circ t))$
TS9.55	Fusion Unicity	$\exists w(\phi(w)) \rightarrow \forall s, t(FoS_\phi(s) \wedge FoS_\phi(t) \rightarrow s = t)$
T9.56	Sum is a special Case of Fusion	$\forall s, t, u(FoS_{w=s \vee w=t}(u) \leftrightarrow SoS_1(u, s, t))$
T9.57	Improper Slot is Fusion of Filler's Slots	$\exists s(P_s(s, a)) \rightarrow \forall t(IP_s(t, a) \leftrightarrow FoSP_s(w, a)(t))$
TS9.58	Fusion Stable under Contextualisation	$Cb(s, t) \wedge \forall w(\phi(w) \rightarrow \exists w'(w' = t \circ w)) \rightarrow$ $(FoS_\phi(s) \leftrightarrow FoS_{\exists w'(w=t \circ w' \wedge \phi(w'))}(t \circ s))$