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David Rivera Betancour. Aspects of Carrollian physics. High Energy Physics - Theory [hep-th]. Institut Polytechnique de Paris, 2023. English. NNT : 2023IPPAX146 . tel-04587347

HAL Id: tel-04587347

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INSTITUT
POLYTECHNIQUE
DE PARIS

NNT : 2023IPPAX146

Thèse de doctorat



Aspects of Carrollian physics

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'École Polytechnique

École doctorale n°626 Ecole Doctorale IP Paris (EDIPP)
Spécialité de doctorat : Physique

Thèse présentée et soutenue à Palaiseau, le December 6, 2023, par

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Acknowledgements

First of all, I am really grateful to my thesis supervisor Marios Petropoulos for his guidance and patience during all these years. I learned from him not only about physics but also about other good simple things in life like nice French and Greek cuisine, good wines and beers, among other things.

Special thanks to the members of my jury Christos Charmousis, Xavier Bekaert, Olivera Miskovic, Blaise Gouteraux and Anastasios Petkou for accepting being part of my committee and read this thesis.

I would like to thanks to my parents Nelson Rivera and Cecilia Betancour, as well as all the members of my family for supporting me in all my decisions. In particular my choice of pursuing an unconventional career. Without them I could never have made it this far.

Thanks to my colleagues and friends: Konstantinos Siampos, Rodrigo Olea, Matthieu Vilatte, Simon Pekar, Andrea Campoleoni, Arnaud Delfante, Nehal Mittal, Giorgos Anastasiou, Ignacio Araya, Javier Moreno and Gastón Giribet, as well as Anastasios Petkou and Olivera Miskovic. None of the papers that I am part of would have been published without their efforts.

Last but not least, I am grateful to the professors, postdocs and students of the CPHT: Guillaume Bossard, Emilian Dudas, Blaise Goutéraux, Hervé Partouche, Andrea Puhm, Gabriele Casagrande, Victor Franken, Adrien Loty, Yorgo Pano, Filippo Sottovia, Sangmin Choi, David Ramirez, Mikel Sanchez, Ashish Shukla. I will not forget their hospitality in the lab and the nice discussions and good times out of the academy.

This thesis was funded by the Agencia Nacional de Investigación y Desarrollo (ANID), Becas Chile, Scholarship No. 72200301, from 2019 to 2023. Also special thanks to the program Erasmus+ of the Institut Polytechnique de Paris as well as the Kapodistrian University of Athens and the Aristotle University for hosting me with this fellowship.

Abstract

Nous étudions divers aspects et applications gravitationnelles de la dynamique carrollienne. Les systèmes carrolliens se manifestent lorsque la vitesse de la lumière s'annule. Nous construisons des équations dynamiques générales carrolliennes et galiléennes valables pour des géométries carrolliennes/newtoniennes arbitraires, courbes et dépendantes du temps, l'accent étant mis sur l'hydrodynamique. La présence d'un courant $U(1)$ est aussi prise en compte. Cette démarche suit deux approches : basée sur l'invariance de l'action sous Carroll/Galilée et Weyl, ou par la prise d'une limite de grande/petite vitesse de la lumière dans le tenseur énergie-impulsion relativiste et le courant $U(1)$. La dynamique est régie par la conservation d'un ensemble de moments qui résultent soit de la variation de l'action par rapport aux différentes composantes de la géométrie carrollienne/newtonienne, soit apparaissent à différents ordres dans le développement en c du tenseur énergie-impulsion. Ces deux approches concordent, mais il est montré que la procédure de limite est plus riche en raison de la possibilité d'embrasser des situations avec des degrés de liberté supplémentaires. En fait, c'est cette liberté qui nous permet de déterminer dans quelles conditions l'invariance hydrodynamique est préservée lorsqu'on prend la limite de grande/petite vitesse de la lumière. Nous montrons que dans la limite galiléenne standard l'invariance hydrodynamique est perdue, mais récupérée en ajoutant deux degrés de liberté supplémentaires dans le développement en c du courant de chaleur et des courants $U(1)$. Dans le cas carrollien, l'invariance sous changement de repère hydrodynamique survit lorsque le comportement du tenseur énergie-impulsion est inspiré des fluides carrolliens holographiques. Nous présentons enfin l'analyse des courants associés aux isométries carrolliennes/galiléennes. Dans le cas carrollien/galiléen, ces courants ne sont pas toujours conservés et des conditions supplémentaires doivent être imposées.

La dérivation présentée pour la dynamique carrollienne transcende les fluides. Nous étudions le champ scalaire conforme sur une géométrie carrollienne générale et analysons les extensions carrolliennes de la théorie de Chern-Simons gravitationnelle à trois dimensions. Dans cette analyse, on découvre des dynamiques électriques et magnétiques codées à différents ordres en puissances de la vitesse de la lumière de l'action relativiste parente. Deux actions supplémentaires apparaissent, nommées paramagnétique et paraélectrique, respectivement.

Dans l'esprit de la dualité jauge/gravité plate, nous étudions la dynamique des espace-temps asymptotiquement plats d'un point de vue carrollien. Nous montrons que les espace-temps à Ricci nul sont exprimés dans une jauge covariante vis-à-vis du bord nul. Cette jauge est une extension de la jauge de Newman-Unti, valable pour constante cosmologique finie ou nulle. Le cas plat correspond à une limite carrollienne au bord. L'espace de solutions à Ricci nul résultant est constitué d'un ensemble infini de données carrolliennes. On y trouve la géométrie conforme carrollienne, les moments en nombre fini de la théorie du bord et un nombre infini de tenseurs arbitraires, obtenus en développant le tenseur énergie-impulsion relativiste d'origine en série de Laurent. Tous obéissent aux équations de bilan de flux carrolliennes. Pour les solutions de type Petrov algébrique, cette structure carrollienne au bord permet de déterminer les charges gravitationnelles usuelles. Nous retrouvons le développement multipolaire de la masse et du moment angulaire pour la famille de Kerr-Taub-NUT. Nous étudions enfin comment le groupe d'Ehlers de type Möbius agit sur les données du bord nul. Pour les espace-temps stationnaires, ce groupe se manifeste comme une transformation locale des observables carrolliennes du bord nul. Pour la solution de Kerr-Taub-NUT par exemple, la transformation de la masse/nut est une rotation de l'énergie/Cotton.

Abstract

The purpose of this thesis is to study aspects of Carrollian dynamics and its application to gravity with zero cosmological constant. Carrollian systems arise as the vanishing speed of light (k) limit of Lorentzian theories. Here, general Carrollian and Galilean dynamical equations valid for arbitrary curved and time dependent Carrollian/Newton-Cartan geometries are constructed, with focus on fluid mechanics. In both cases the presence of a $U(1)$ current is considered. The latter is done in two approaches: Carrollian/Galilean and Weyl invariance of the action, and by taking a large/small- k limit of the relativistic energy-momentum tensor and the $U(1)$ current. In both cases, the dynamic is given by the conservation of a set of momenta that arise either as the variation of the action with respect to the different pieces of the Carrollian/Newton-Cartan geometry or appear at different orders in the k -expansion of the energy-momentum tensor. Although these two approaches agree, the limiting procedure is shown to be richer due to the possibility of capturing more general situations with extra degrees of freedom. In fact, it is this freedom that allows us to find under which conditions hydrodynamic-frame invariance is preserved when taking the large/small- k limit. We show that, although in the standard Galilean limit hydrodynamic-frame invariance is lost, it is recovered by adding two extra degrees of freedom in the large- k expansion of the heat current and $U(1)$ currents. In the Carrollian case, hydrodynamic-frame invariance survives when the behavior of the energy-momentum tensor is guided by holographic Carrollian fluid results. We also present the analysis of the currents generated by Carrollian/Galilean isometries. In the Carrollian/Galilean instances, these currents are not guaranteed to be conserved and additional conditions must be imposed.

The presented derivation for Carrollian dynamics is not valid only for fluids. The investigation of the scalar field on a general Carrollian spacetime is also presented, as well as the analysis of three dimensional Carrollian gravitational Chern-Simons extensions. In this analysis one finds electric and magnetic dynamics that are encoded at different orders in powers of the speed of light of the relativistic action. We furthermore unravel two more Carrollian Chern-Simons actions, dubbed paramagnetic and paraelectric, respectively.

In relation to a possible flat version of the gauge/gravity duality, we also study some aspects of Ricci-flat dynamics from a Carrollian perspective. We show that Ricci-flat

spacetimes can be expressed in a gauge covariant with respect to the null boundary. This gauge is an extension of the Newman-Unti gauge which is valid for asymptotically anti-de Sitter and flat spacetimes. The flat instance is reached as the vanishing cosmological constant limit of the anti-de Sitter case, which corresponds to a Carrollian limit at the boundary. Therefore, the resulting Ricci-flat solution space is reconstructed in terms of an infinite set of boundary Carrollian data. These are composed by the Carrollian conformal geometry, a finite set of momenta of the theory hosted at the boundary, and an infinite number of arbitrary tensors, obtained by expanding the original energy-momentum tensor in Laurent series, which obey Carrollian flux balance equations. We take advantage of the latter to define gravitational charges by using Carrollian boundary techniques and restricting the spacetime to the algebraically special Petrov type. With this construction we recover the mass and angular momentum multipolar expansion for the Kerr-Taub-NUT family. We also learn how the hidden Ehlers Möbius group acts on the boundary data at null infinity. We find that, for stationary spacetimes, this group is manifested as a local transformation for the Carrollian geometry and the boundary Carrollian observables. We reproduce the mass/nut rotation as the energy/Cotton rotation for the Kerr-Taub-NUT solution.

List of publications

During the completion of this thesis, the following articles were published or submitted:

- [1] O. Miskovic, R. Olea, P. M. Petropoulos, D. Rivera-Betancour and K. Siampos, *Chern–Simons action and the Carrollian Cotton tensors*, arXiv:2310.19929
(accepted for publication in JHEP)
- [2] A. Campoleoni, A. Delfante, S. Pekar, P. M. Petropoulos, D. Rivera-Betancour and M. Vilatte, *Flat from anti de Sitter*, JHEP12(2023)078.
- [3] N. Mittal, P. M. Petropoulos, D. Rivera-Betancour and M. Vilatte, *Ehlers, Carroll, charges and dual charges*, JHEP **07** (2023), 065.
- [4] D. Rivera-Betancour and M. Vilatte, *Revisiting the Carrollian scalar field*, Phys. Rev. D **106** (2022) no.8, 085004.
- [5] A. C. Petkou, P. M. Petropoulos, D. R. Betancour and K. Siampos, *Relativistic fluids, hydrodynamic frames and their Galilean versus Carrollian avatars*, JHEP **09** (2022), 162.
- [6] G. Anastasiou, I. J. Araya, J. Moreno, R. Olea and D. Rivera-Betancour, *Renormalized holographic entanglement entropy for quadratic curvature gravity*, Phys. Rev. D **104** (2021) no.8, 086003.
- [7] G. Anastasiou, J. Moreno, R. Olea and D. Rivera-Betancour, *Shape dependence of renormalized holographic entanglement entropy*, JHEP **09** (2020), 173.
- [8] G. Giribet, O. Miskovic, R. Olea and D. Rivera-Betancour, *Topological invariants and the definition of energy in quadratic gravity theory*, Phys. Rev. D **101** (2020) no.6, 064046.

In the present thesis, I will elaborate on the content of the articles [1–5]. All these papers are appended for completeness.

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Nomenclature

(A, B, C, \dots) are $d + 2$ spacetime indices labeling $\{r, t, \mathbf{x}\}$.

$(\mu, \nu, \alpha, \dots)$ are $d + 1$ spacetime indices labeling $\{t, \mathbf{x}\}$.

(i, j, k, \dots) are d spatial indices labeling $\{\mathbf{x}\}$.

\hat{D}_t is the temporal Carroll covariant derivative in d space dimensions.

$\hat{\mathcal{D}}_i$ is the spatial Weyl-Carroll covariant derivative in d space dimensions.

$\hat{\mathcal{D}}_t$ is the temporal Weyl-Carroll covariant derivative in d space dimensions.

$\hat{\nabla}_i$ is the spatial Carroll covariant derivative in d space dimensions.

\mathcal{D}_μ is the Weyl covariant derivative in $d + 1$ spacetime dimensions.

∇_A is the Levi-Civita connection in $d + 2$ spacetime dimensions.

∇_μ is the Levi-Civita connection in $d + 1$ spacetime dimensions.

k is the speed of light.

In four spacetime dimensions: ϵ_{ABCD} is the Levi-Civita symbol in four dimensions.

In three spacetime dimensions: $\epsilon_{\mu\nu\alpha}$ is the Levi-Civita symbol in three dimensions.

In two space dimensions: ϵ_{ij} is the Levi-Civita symbol in two dimensions.

Chapter 1

Prologue

The discussion on Carrollian physics started first with the works of Lévy-Leblond [9] and Sen Gupta [10], where a new group of symmetries (by then) called Carroll group is obtained as the Inönü-Wigner contraction [11] of the Poincaré group. As opposed to the non-relativistic contraction that one can use to find the Galilean group, the Carroll group is obtained through a vanishing speed of light limit. For many years this peculiar contraction was thought as a mere mathematical curiosity since no reasonable physical systems seemed to be compatible with these symmetries. Indeed, in that limit, which we call in this thesis as Carrollian limit (also known as ultralocal limit), the light cone collapses into a single line in the time axis making space absolute. As a consequence all movement or signal propagation is forbidden, even if particles have nonzero momentum. Despite of the latter Carrollian physics has attracted interest from many different fields, specially its conformal extension. For instance, it has lead to the discovery of new geometric structures which we now know as Carrollian manifolds [12–21] specified by a degenerate metric and a vector field that generates its kernel. Carrollian manifolds have been seen to appear by taking the vanishing speed of light limit of a pseudo-Riemannian manifold, or as the geometry of null hypersurfaces embedded in Lorentzian spacetimes in one dimension higher.

It has also been shown in [22, 23] that the conformal extension of the Carroll group is isomorphic to the famous Bondi-Metzner-Sachs (BMS) group which rules the asymptotic symmetries of Ricci-flat spacetimes at the null boundary [24–26]. The latter, together with the fact that null geometries possesses Carrollian structures, might suggest a vanishing cosmological version of the gauge/gravity [27, 28] duality, such as the renowned anti-de Sitter/Conformal field theories (AdS/CFT) correspondence that relates gravity in asymptotically AdS spacetimes with the dynamics of a CFT defined on its timelike conformal boundary. In Ricci-flat spacetimes the conformal boundary is null (Carrollian) and therefore, if a similar holographic duality exists in this case, the theory living at the boundary of

Ricci-flat spacetimes should be a Carrollian conformal field theory (CCFT) defined on a Carrollian spacetime. This possibility of having a holographic correspondence for gravity with vanishing cosmological constant and CCFTs is a recent and hot topic that is still under scrutiny. Progress on the latter have been done in [29–37]. In this line, and inspired by the fluid/gravity correspondence [38–41] that relates asymptotically AdS spacetimes with relativistic conformal fluids in one dimension less, a duality between Ricci-flat spacetimes and Carrollian conformal fluids defined at null infinity was proposed in [42]. In particular it was shown that any Ricci-flat spacetime of algebraic special Petrov type can be reconstructed in terms of the Carrollian geometric data and the Carrollian fluid variables, which in the case worked out in [42], correspond only to the energy density of the Carrollian hydrodynamic system. We will see in the Chapter 6 of this thesis that the reconstruction of Ricci-flat spacetimes in terms of a set of Carrollian boundary data is not restricted to the algebraically special spacetimes but applies for general asymptotically flat spacetimes. In the general situation the number of Carrollian data necessary for the reconstruction of the spacetime becomes infinite and part of these degrees of freedom will be constraint to satisfy flux-balance equations [2]. Further investigations on Carrollian fluids have been carried out in [43–45, 5, 46, 47]. There are more recent applications as, for example, the emergence of Carrollian dynamics in black hole horizons [48–53], possible applications to condensed matter systems [54, 55], cosmology and dark energy [56] and even the appearance of Carrollian symmetries in tensionless string theory [57]. In particular, the first half of this thesis is devoted to the study of Carrollian and Galilean fluid dynamics.

Hydrodynamics has been an important topic in physics, thoroughly investigated and applied in many fields. Still, many questions remain to be answered. For instance we have the ambiguity of choosing the hydrodynamic frame when dealing with relativistic fluids. This is what is known as hydrodynamic frame invariance and it reflects the freedom to choose freely the velocity of the fluid due to the impossibility to distinguish the mass flow from the energy flow in relativistic hydrodynamic systems. The most common choices are the Landau-Lifshitz frame, which consist in the formulation of non-perfect relativistic hydrodynamics without heat current [58], and the Eckart frame for the description of non-perfect fluids with heat current and imposing the perfect matter/charge current ($U(1)$ current) [59]. These hydrodynamic frame choices have some inconveniences when the formulation is implemented in the truncated constitutive relations (the non-perfect pieces of the energy-momentum tensor and the matter/charge current are expressed in terms of a gradient expansion of the hydrodynamic variables, namely temperature, chemical potential and velocity). These problems come as instability of the thermal equilibrium state and superluminal propagation

of signals for non-gravitating relativistic fluids. A deep presentation of the latter can be found in [60–63] and for recent progress in the subject, see [64–74].

The emergence of hydrodynamic-frame invariance is also related in a more different context, being fluid gravity correspondence. In asymptotically AdS Einstein spaces, the asymptotic isometries are related to the symmetries of a fictitious fluid flowing on its conformal boundary [75–77]. In that case, the conformal boundary is timelike and the dual fluid is relativistic. If one performs a local Lorentz transformation on the velocity of the dual fluid, its action is reflected as part of the diffeomorphism on the gravitational side. One could wonder what happens now if we move to asymptotically flat spacetimes with a holographic Carrollian fluid dual. Do we still have hydrodynamic frame invariance in the Carrollian boundary fluid that can be reflected in the bulk as diffeomorphisms? In a different context, irrespective of any link to gravity, one can also wonder why there is no hydrodynamic frame invariance when reaching Galilean fluids from a non-relativistic limit of relativistic hydrodynamics, or under which conditions we can keep this property.

On the one hand we have Galilean fluid dynamics. The latter typically arises from the non-relativistic limit of the relativistic fluid equations. It was first formulated in Euclidean space with absolute time. Then, with the pass of the years, it was generalized to admit more general situations describing non-perfect fluids flowing on curved and time dependent spaces [78–84], with fluid equations covariant under Galilean diffeomorphisms given by the transformation $t' = t'(t)$ and $\mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$ [44, 85, 5] that respects the time/space splitting, the time being absolute.

On the other hand, Carrollian fluids can be understood as fluids "flowing" on a Carrollian manifold. As opposed to the Galilean situation, one way to obtain the Carrollian fluid equations is through the vanishing speed of light limit of the relativistic hydrodynamic equations. This approach was first discussed in [44] where the authors obtained the set of Carrollian fluid equations in the absence of a $U(1)$ current, and are fully covariant under the diffeomorphisms $t' = t'(t, \mathbf{x})$ and $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$, namely the Carrollian diffeomorphisms that respects the time/space splitting, with the space being absolute.

One of the goals of this thesis is to extend the analysis of [44] for the construction of the Galilean and Carrollian fluid equations in the presence of a conserved $U(1)$ current using two distinct approaches: The first one is based in the diffeomorphism invariance of the action (Carrollian or Galilean). In this approach the dynamic is given in terms of conservation laws for a set momenta [86] that are obtained through the variation of the action with respect to the geometry. The second approach consists on a limiting procedure of the relativistic fluid equations. This provides the set of conservation equations that the momenta must satisfy, the latter constructed in terms of kinematic and thermodynamic variables. This

procedure will allow us to study the fate of hydrodynamic-frame invariance in the Galilean and Carrollian limits. We will see that the absence or presence of this property known in relativistic non-perfect fluids, depends on the behavior of the fluid degrees of freedom with respect to the scaling of the velocity of light. For example in the Galilean limit, we will see that the only way to keep this property is by considering the presence of additional pieces in the heat current and the non-perfect part of $U(1)$ current that appear at $\mathcal{O}(k^2)$ and $\mathcal{O}(1)$ order respectively.

Additionally, we will study the construction of charges associated to Galilean and Carrollian isometries. We will see that these charges are not necessarily conserved and we need to impose additional conditions for them to be conserved: strong Carroll isometries or the absence of some of the momenta.

One can also consider the Carrollian contraction of other Lorentz-invariant theories. It was shown in [14] that electromagnetism in four spacetime dimensions allows for two different Carrollian contractions, one named as electric contraction, and the other named as magnetic contraction. In the electric contraction the dynamics is dominated by time derivatives while space gradients dominate the magnetic dynamics. It was then shown in [87] that these two contractions also apply to any Lorentz invariant theory where the guideline was the Hamiltonian formalism. There, the difference between electric and magnetic dynamics was found to rely on a different scaling of the fields with respect to the speed of light. In this thesis we will investigate the (conformally coupled) scalar field and the gravitational Chern-Simons action on Carrollian backgrounds. Our guideline will be the same employed in the case of Carrollian fluids, namely Carrollian diffeomorphisms and Weyl invariance, and a small- k expansion of the actions and equations of motion. At each power in the speed of light, different Carrollian dynamics appear for both theories: electric and magnetic, and for the Carrollian Chern-Simons we will find two additional actions at the $1/k^3$ and k^3 powers in the expansion. We call them paraelectric and paramagnetic Carroll-Chern-Simons actions.

Coming back to the discussion on the relation between Ricci-flat spacetimes and Carrollian fluid dynamics, there is yet another property of gravity that deserves to be studied in this context. This is the action of hidden symmetries, present in vacuum Einstein's equations, onto the boundary configuration. Hidden symmetries have been studied for a long time in the context of relativistic gravity theories. The latter was first started by Ehlers in [88], where it was shown that in the presence of an isometry Einstein's equations are invariant under Möbius transformation. This can be seen after a reduction of the Ricci-flat spacetime along the orbits of the Killing vector that generates its isometries. The latter opened the door for solution-generating techniques applicable to vacuum Einstein's equations [89] and then generalized to more general situations with more commuting Killing vectors [90–92]

and bigger hidden symmetries. In the same spirit as the mapping between the asymptotic symmetries of the bulk and the conformal symmetries of the boundary, one could wonder what would be the action of the Ehlers group on the conformal boundary. As mentioned previously, Ricci-flat spacetimes can be reconstructed in terms of Carrollian data. The latter requires an infinite set of functions that enters in the metric in a gauge that we will call "covariant Newmann-Unti gauge". Then, what does the Ehlers transformation do to the Carrollian boundary data? Part of this thesis will be devoted to answer this question. To this end we will work in the resumable case (algebraically special class) with solutions that possess a timelike Killing vector whose congruence coincides with the boundary Carrollian fiber ∂_t (stationary spacetimes).

In gravity it is possible to construct charges associated to the asymptotic symmetries group (BMS_4). In fact, in Ricci-flat spacetimes one has an infinite set of charges (not necessarily conserved). This is due to the infinite degrees of freedom necessary to reconstruct the solution, as well as the infinite generators of the asymptotic symmetry group. These "subleading" charges come also as electric-magnetic pairs and have been widely studied in the literature [93–100]. Here we will perform the construction of the charges from a purely boundary perspective where they are built using the boundary dynamics together with Carrollian (conformal) isometries. In this construction the Cotton tensor of the boundary (and its descendants in the flat limit) will play a prominent role and in fact, in relation to the discussion on the boundary action of the Ehlers group, we will see that part of the Ehlers group will correspond to an algebraic transformation that mixes the Cotton density and the energy density of the boundary Carrollian fluid. In the bulk this is a mixing between the Bondi mass (electric mass) and the nut (magnetic mass).

This thesis is based on the published works [1–5]. Some parts of this thesis are a transcription of these articles while other parts are written specifically for this thesis to facilitate the presentation of the results. The structure of this thesis goes as follows:

In Chapter 2 we will remind the necessary tools to go through the analysis and results presented in this thesis. This includes the basics of relativistic hydrodynamics in the presence of a matter current and an overview on hydrodynamic-frame invariance. We will present two frames in which this analysis can be performed. One is the Zermelo frame which will be suitable for taking the Galilean limit, while the other will be the Papapetrou-Randers frame suitable for taking the Carrollian limit. A review about the Ehlers group and the Geroch method will be also given which will be used in Chapter 6.

In Chapter 3 we derive the most general Galilean fluid equations on arbitrary curved and time dependent backgrounds. The latter will be first based on Galilean covariance (Galilean diffeomorphisms). In this context we will also investigate on the (non-) conservation laws

associated to Galilean isometries. We then move to the limiting procedure. The latter corresponds to taking $k \rightarrow \infty$ in the Zermelo frame. This will allow us to go through the analysis of hydrodynamic-frame invariance and find under which conditions it survives the limit. This Chapter is partly based on the work [5].

In Chapter 4, following the steps for the Galilean case, we derive the most general Carrollian fluid equations. The latter is derived from Carrollian covariance (Carrollian diffeomorphisms) and a limiting procedure by taking $k \rightarrow 0$ in the Papapetrou-Randers frame. We also study the (non-) conservation laws associated to Carrollian isometries, as well as the fate of hydrodynamic-frame invariance in the $k \rightarrow 0$ limit. The results of this Chapter are based on [5].

In Chapter 5 we will derive the Carrollian extensions of the (conformally coupled) scalar field in arbitrary dimensions and the Carrollian versions of the gravitational Chern-Simons theory in three spacetime dimensions. All this by following the steps presented in Chapter 4. The derivations presented here are based on the works [1, 4].

In Chapter 6 we will study some aspects on the relation between Ricci-flat spacetimes and Carrollian dynamics. The latter includes the reconstruction of Ricci-flat spacetimes, the relation between bulk and boundary isometries, charges and dual charges and stationary Ricci-flat solutions. We then show how the Ehlers group of symmetries that are present in Einstein's field equations translates to the boundary as a local transformation on the Carrollian geometric data as well as the Carrollian fluid data. The results included in this Chapter are based on our work [3, 2].

Finally in Chapter 7 a summary of the results obtained in this thesis are presented, as well as possible future directions.

Chapter 2

Introductory material

2.1 Relativistic hydrodynamics

2.1.1 Energy-momentum and matter conservation

Fluid mechanics can be understood as the effective description of the dynamics of matter with off-equilibrium thermodynamics at large enough length and time scales. This means that a small portion of the fluid is big enough compared to the distances between the molecules so one does not need to worry about the dynamics of the individual molecules. Deviations from thermal equilibrium are small and thus the global thermal equilibrium temperature, fluid velocity and chemical potential are promoted to slowly varied in time and space. It is reasonable to assume local thermal equilibrium and use the laws of thermodynamics (see App. A).

Part of the hydrodynamic description is governed by local conservation laws. When there are no external forces, springs or sinks, these conservation laws are recast in the form of $d + 2$ equations given by the vanishing divergence of the energy-momentum tensor and the matter current,

$$\nabla_{\mu} T^{\mu\nu} = 0, \tag{2.1}$$

$$\nabla_{\mu} J^{\mu} = 0. \tag{2.2}$$

Here we assume the $d + 1$ spacetime is equipped with a metric $g_{\mu\nu}$ with Lorentzian signature $(-, +, \dots, +)$. Both, the energy-momentum tensor and the matter current can be decomposed along the congruence u^{μ} , which corresponds to the velocity field of the fluid and it is

normalized as $u_\mu u^\mu = -k^2$. Here k is the light velocity.¹ This decomposition reads

$$T^{\mu\nu} = (\varepsilon + p) \frac{u^\mu u^\nu}{k^2} + p g^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\nu}{k^2} + \frac{u^\nu q^\mu}{k^2}, \quad (2.3)$$

$$J^\mu = \rho_0 u^\mu + j^\mu. \quad (2.4)$$

The heat current q^μ and the viscous stress tensor $\tau^{\mu\nu}$ are purely transverse, implying

$$u^\mu q_\mu = 0, \quad u^\mu \tau_{\mu\nu} = 0, \quad u^\mu T_{\mu\nu} = -q_\nu - \varepsilon u_\nu, \quad \varepsilon = \frac{1}{k^2} T_{\mu\nu} u^\mu u^\nu \quad (2.5)$$

Similarly, we have for the non-perfect piece of the matter current j_μ

$$u^\mu j_\mu = 0, \quad \rho_0 = -\frac{1}{k^2} u^\mu J_\mu. \quad (2.6)$$

The kinematic properties of the fluid² are given by the irreducible decomposition of the covariant derivative of the velocity field, namely

$$\nabla_\nu u_\mu = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{d} \Theta h_{\mu\nu} - \frac{1}{k^2} a_\mu u_\nu, \quad (2.7)$$

with

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \quad (2.8)$$

$$\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)} + \frac{1}{k^2} u_{(\mu} a_{\nu)} - \frac{1}{d} \Theta h_{\mu\nu}, \quad (2.9)$$

$$\omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]} + \frac{1}{k^2} u_{[\mu} a_{\nu]}, \quad (2.10)$$

the acceleration, the expansion, the shear and the vorticity, with $h_{\mu\nu}$ and $U_{\mu\nu}$ the projectors onto the space transverse and longitudinal to the velocity field,

$$h_{\mu\nu} = \frac{u_\mu u_\nu}{k^2} + g_{\mu\nu}, \quad U_{\mu\nu} = -\frac{u_\mu u_\nu}{k^2}. \quad (2.11)$$

In the decomposition of $T^{\mu\nu}$ and J^μ at hand, different variables are present:

- ε is the energy density per unit of proper volume and ρ_0 is the matter per unit of proper volume, both as measured by an observer co-moving at velocity u^μ ;

¹We use k instead of c for the light velocity in order to avoid possible confusions with the Cotton density (D.11), which is defined here with the letter c .

²Our conventions for (anti-) symmetrization are $A_{(\mu\nu)} = \frac{1}{2} (A_{\mu\nu} + A_{\nu\mu})$ and $A_{[\mu\nu]} = \frac{1}{2} (A_{\mu\nu} - A_{\nu\mu})$.

- p is the local-equilibrium thermodynamic pressure. The latter obeys an equation of state of the form $p = p(T, \mu_0)$, where T and μ_0 are the local temperature and chemical potential;
- the tensors j^μ , q^μ and $\tau^{\mu\nu}$ capture the non-perfect nature of the fluid. In other words, yields the information about the physics of the out of equilibrium state. These tensors are usually expressed as a gradient expansion of the thermodynamic variables (T , μ_0 and u^μ) giving the constitutive relations at each order in the expansion.

Besides the conservation of the energy-momentum tensor, the matter current and the equation of state $p = p(T, \mu_0)$, the description of a hydrodynamic system is accompanied with the second law of thermodynamics, also known as the non-decreasing entropy flux principle. The latter is realized by the condition

$$\nabla_\mu S^\mu \geq 0, \quad (2.12)$$

where S^μ is the entropy current. The latter does not have a microscopic definition but it is also built order by order in the derivative expansion³. The generic form of the entropy current is

$$S^\mu = \frac{1}{T} (pu^\mu - T^{\mu\nu}u_\nu - \mu_0 J^\mu) + R^\mu = \Sigma^\mu + R^\mu, \quad (2.13)$$

where Σ^μ is the universal piece of the current, and R^μ depends on the specific off-equilibrium thermodynamic theory. Using (2.5), (2.6), (A.3) and (A.5), Σ^μ is rewritten as

$$\Sigma^\mu = \sigma u^\mu - \frac{\mu_0}{T} j^\mu + \frac{1}{T} q^\mu = \frac{\sigma}{\rho_0} J^\mu - \frac{\mathfrak{w}}{T\rho_0} \left(j^\mu - \frac{\rho_0}{\mathfrak{w}} q^\mu \right) \quad (2.14)$$

with σ the entropy, \mathfrak{w} the relativistic enthalpy, and μ_0 the relativistic chemical potential.

It is convenient, both for the relativistic dynamics and for its Galilean or Carrollian limits, to consider the longitudinal and transverse projections of (2.1), possibly combined with (2.2) and the thermodynamic laws (A.5), (A.6) and (A.8) in order to trade the energy for the entropy. For the longitudinal projection, we find

$$-u_\nu \nabla_\mu T^{\mu\nu} = u(\varepsilon) + \left(\mathfrak{w} + \frac{\tau}{d} \right) \Theta + \tau^{\mu\nu} \sigma_{\mu\nu} + \nabla_\nu q^\nu + \frac{a_\nu q^\nu}{k^2}, \quad (2.15)$$

$$= T \nabla_\nu \Sigma^\nu + \frac{\tau}{d} \Theta + \tau^{\mu\nu} \sigma_{\mu\nu} + T j^\nu \partial_\nu \frac{\mu_0}{T} + q^\nu \left(\frac{\partial_\nu T}{T} + \frac{a_\nu}{k^2} \right), \quad (2.16)$$

where $\tau = \tau^{\mu\nu} g_{\mu\nu}$ is the relativistic non-equilibrium pressure and $u(f)$ stands for $u^\mu \partial_\mu f$.

³The first order is often referred to as classical irreversible thermodynamics, the second extended irreversible thermodynamics, etc.

The conservation of the energy-momentum tensor and the matter current are also the result of the symmetries of the effective action $S = \int d^{d+1}x \sqrt{-g} \mathcal{L}$ of our hydrodynamic system. Indeed, from this perspective the energy momentum tensor is defined as the variation of the action with respect to the metric,

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}, \quad (2.17)$$

while the matter current is defined through the variation of the action with respect to a $U(1)$ gauge field B_μ

$$J^\mu = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta B_\mu}. \quad (2.18)$$

If the action is invariant under diffeomorphisms generated by $\xi = \xi^\mu(t, \mathbf{x}) \partial_\mu$ that transform the geometry as

$$\delta_\xi g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu}, \quad (2.19)$$

where \mathcal{L}_ξ is the Lie derivative along the vector ξ and whose action on the metric is

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\nu\rho} \partial_\mu \xi^\rho = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \quad (2.20)$$

the variation $\delta_\xi S$ is expressed as

$$\begin{aligned} \delta_\xi S &= \frac{1}{2} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} T^{\mu\nu} \delta_\xi g_{\mu\nu} \\ &= \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \xi_\nu \nabla_\mu T^{\mu\nu} - \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \nabla_\mu (T^{\mu\nu} \xi_\nu). \end{aligned} \quad (2.21)$$

Therefore, the vanishing of $\delta_\xi S$ implies the conservation of the energy-momentum tensor. Here, the last term in (2.21) is a boundary term.

Similarly, the conservation of the matter current (2.2) comes from the invariance of the action under the $U(1)$ transformation

$$\delta_\Lambda B_\mu = -\partial_\mu \Lambda \quad (2.22)$$

with $\Lambda = \Lambda(t, \mathbf{x})$. Hence, the variation $\delta_\Lambda S$ yields

$$\begin{aligned} \delta_\Lambda S &= \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} J^\mu \delta_\Lambda B_\mu \\ &= \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \Lambda \nabla_\mu J^\mu - \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \nabla_\mu (J^\mu \Lambda), \end{aligned} \quad (2.23)$$

from which we can infer the conservation of the matter current by the requirement $\delta_\Lambda S = 0$. Again, the last term in (2.23) is nothing but a boundary term.

When the vector field ξ generates an isometry of the spacetime, we can derive conservation laws associated to them. In such case ξ is a Killing vector which satisfies the Killing equation

$$\mathcal{L}_\xi g_{\mu\nu} = 0. \quad (2.24)$$

Hence, through a Noether procedure it is possible to define a current as

$$I_\mu = \xi^\nu T_{\mu\nu} \quad (2.25)$$

that is divergence free

$$\nabla_\mu I^\mu = 0. \quad (2.26)$$

Then we can use the Stokes and Gauss theorems

$$\int_{\mathcal{W}} d^{d+1}x \sqrt{-g} \nabla_\mu I^\mu = \oint_{\partial\mathcal{W}} *I, \quad (2.27)$$

where \mathcal{W} is a domain inside \mathcal{M} and $*I$ is the \mathcal{M} -Hodge dual of $I = I_\mu dx^\mu$ ($\varepsilon_{01\dots d} = 1$). One can infer that

$$Q_I = \int_{\Sigma_d} *I \quad (2.28)$$

is a conserved quantity. Here, Σ_d is an arbitrary spacelike hypersurface embedded in \mathcal{M} .

The same construction can be made for the presence of conformal isometries. In that case the energy-momentum tensor is trace-free $T^\mu{}_\mu$ and the conformal Killing vectors satisfy

$$\mathcal{L}_\xi g_{\mu\nu} = \frac{2}{d+1} \nabla_\rho \xi^\rho g_{\mu\nu}. \quad (2.29)$$

We will come back to this subject in chapters 3 and 4 when discussing about conservation laws associated to Carrollian and Galilean (conformal) isometries.

2.1.2 Hydrodynamic-frame invariance

The freedom to chose a hydrodynamic-frame is rooted in the redundancy of the fluid data q^μ and j^μ (no distinction between energy and mass flow). For instance, in Eckart's formulation of relativistic fluids [59], the frame is chosen such that the matter current remains perfect ($j^\mu = 0$), while in the Landau and Lifshitz formulation the heat current q^μ is set to zero. It is more commonly known in the literature that the freedom to chose a hydrodynamic-frame relies on the freedom to redefine the thermodynamic variables together with the velocity field,

such that the energy-momentum tensor, the matter current and the entropy current remain invariant. That is, we make

$$T(x) \rightarrow T'(x), \quad \mu_0(x) \rightarrow \mu'_0(x), \quad u^\mu(x) \rightarrow u'^\mu(x), \quad (2.30)$$

provided we modify accordingly $\varepsilon(x)$, $p(x)$, $\rho_0(x)$, $q^\mu(x)$, $\tau^{\mu\nu}(x)$ and $j^\mu(x)$ so that

$$T^{\mu\nu} \rightarrow T'^{\mu\nu}, \quad J^\mu \rightarrow J'^\mu, \quad S^\mu \rightarrow S'^\mu. \quad (2.31)$$

The latter is actually a gauge symmetry, associated with a local Lorentz transformation of the velocity field u^μ , meaning that the velocity field has no first principle definition in relativistic hydrodynamics. The way a hydrodynamic-frame transformation works goes as follows:

Suppose we transform the fluid velocity through a local Lorentz transformation, namely

$$\mathbf{u} \rightarrow \mathbf{u} + \delta\mathbf{u}, \quad \delta\mathbf{u} \cdot \mathbf{u} = 0. \quad (2.32)$$

We require that the energy-momentum tensor remains invariant under such transformation, namely $\delta T^{\mu\nu} = 0$. This implies that the various pieces within the decomposition of the energy-momentum tensor (2.3) must transform as⁴

$$\delta\varepsilon = -\frac{2}{k^2} q^\mu \delta u_\mu, \quad (2.33)$$

$$\delta q_\nu = \frac{u_\nu}{k^2} q^\mu \delta u_\mu - \mathfrak{w} \delta u_\nu - \tau_{\nu\mu} \delta u^\mu, \quad (2.34)$$

$$\begin{aligned} \delta(p h_{\mu\nu} + \tau_{\mu\nu}) &= \frac{p}{k^2} (u_\mu \delta u_\nu + u_\nu \delta u_\mu) + \frac{1}{k^2} (u_\mu \tau_{\nu\rho} + u_\nu \tau_{\mu\rho}) \delta u^\rho \\ &\quad - \frac{1}{k^2} (\delta u^\mu q^\nu + \delta u^\nu q^\mu). \end{aligned} \quad (2.35)$$

In the same way, the requirement $\delta J^\mu = 0$ implies

$$\delta\rho_0 = -\frac{1}{k^2} J_\mu \delta u^\mu \quad (2.36)$$

$$\delta j_\mu = \frac{1}{k^2} u_\mu J_\nu \delta u^\nu - \rho_0 \delta u_\mu. \quad (2.37)$$

⁴We cannot disentangle at this stage the transverse components p and $\tau_{\mu\nu}$, as their separation relies on thermodynamics (p is the equilibrium pressure, $\tau_{\mu\nu}$ stands for the non-equilibrium stress and its trace is the non-equilibrium pressure).

The entropy current and the thermodynamic relations are also hydrodynamic-frame invariant, in particular the Gibbs-Duhem equation

$$\mathfrak{w} = T\sigma + \mu_0\rho_0 \Leftrightarrow p = Ts\rho_0 + \mu\rho_0 - e\rho_0. \quad (2.38)$$

and the equation of state. Although not part of our main goal, the latter provides the necessary tools for computing the local Lorentz transformations of the thermodynamic observables σ , p , T and μ_0 , all in terms of δu^μ . The latter will not be analyzed here but for a more detailed discussion on the subject is given in [65–74, 101].

The ambiguity in the definition of the fluid velocity, and hence in the temperature and chemical potential, appears only when viscosity and heat conduction are present. Indeed, when q^μ , j^μ and $\tau^{\mu\nu}$ vanish it is possible to consider the fluid moving with a precise velocity (u^μ is aligned with the matter current and it is also an eigenvector of the energy-momentum tensor) and define a frame co-moving with the fluid [58, 102]. In fact, if one performs a local hydrodynamic-frame transformation as given in (2.32), it will not preserve the perfect form of the energy-momentum tensor and the matter current. In that case all those transformations will generate non-perfect components in $T^{\mu\nu}$ and J^μ .

2.1.3 Weyl-invariant relativistic fluids

We now turn our attention to fluids which are invariant under Weyl transformations. Those fluid configurations, which are known as conformal fluids, are important within the fluid/gravity duality since they appear as the holographic duals of AAdS spacetimes [40, 38, 39] as well as their Carrollian version known as Conformal Carrollian fluids, which are the holographic duals of Ricci-flat spacetimes [42].

Weyl transformations act on the background metric and fluid velocity as

$$g_{\mu\nu} \rightarrow \mathcal{B}^{-2}g_{\mu\nu}, \quad u^\mu \rightarrow \mathcal{B}u^\mu, \quad (2.39)$$

with $\mathcal{B} \equiv \mathcal{B}(t, \mathbf{x})$. More general tensors are Weyl-covariant of weight w if they rescale with some power of \mathcal{B} . We can define a Weyl-covariant derivative \mathcal{D}_μ that maintains the canonical Weyl transformation of a Weyl-covariant tensor. This is achieved by the inclusion of a Weyl connection one-form⁵

$$A = \frac{1}{k^2} \left(\mathbf{a} - \frac{\Theta}{d} \mathbf{u} \right), \quad (2.40)$$

⁵The explicit form of A is obtained by demanding $\mathcal{D}_\mu u^\mu = 0$ and $u^\lambda \mathcal{D}_\lambda u_\mu = 0$

which transform as $A \rightarrow A - d \ln \mathcal{B}$. The Weyl covariant derivative is metric-compatible, implying

$$\mathcal{D}_\rho g_{\mu\nu} = 0, \quad (2.41)$$

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu) f = w f F_{\mu\nu}, \quad (2.42)$$

where the action on a weight- w scalar f is

$$\mathcal{D}_\lambda f = \nabla_\lambda f + w A_\lambda f, \quad (2.43)$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2.44)$$

is the Weyl curvature (weight-0). For concreteness, the action of \mathcal{D}_λ on a weight- w form v_μ and a weight- w tensor $t_{\mu\nu}$ is

$$\mathcal{D}_\lambda v_\mu = \nabla_\lambda v_\mu + (w+1) A_\lambda v_\mu + A_\mu v_\lambda - g_{\mu\lambda} A^\rho v_\rho, \quad (2.45)$$

$$\begin{aligned} \mathcal{D}_\lambda t_{\mu\nu} &= \nabla_\lambda t_{\mu\nu} + (w+2) A_\lambda t_{\mu\nu} + A_\mu t_{\lambda\nu} + A_\nu t_{\mu\lambda} - g_{\mu\lambda} A^\rho t_{\rho\nu} \\ &\quad - g_{\nu\lambda} A^\rho t_{\mu\rho}. \end{aligned} \quad (2.46)$$

As usual, one defines the Weyl-covariant Riemann tensor through the commutator of the Weyl-derivatives acting on vectors

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu) v^\rho = \mathcal{R}^\rho{}_{\sigma\mu\nu} v^\sigma + w v^\rho F_{\mu\nu} \quad (2.47)$$

(v^ρ are the components of a weight- w vector) and the usual subsequent quantities. The Weyl-covariant Ricci (weight 0) and scalar (weight 2) curvatures read

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + (d-1) \left(\nabla_\nu A_\mu + A_\mu A_\nu - g_{\mu\nu} A_\lambda A^\lambda \right) + g_{\mu\nu} \nabla_\lambda A^\lambda - F_{\mu\nu}, \quad (2.48)$$

$$\mathcal{R} = R + 2d \nabla_\lambda A^\lambda - d(d-1) A_\lambda A^\lambda. \quad (2.49)$$

We can also define the Weyl-invariant Schouten tensor as

$$\mathcal{S}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{4} \mathcal{R} g_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} + \nabla_\nu A_\mu + A_\mu A_\nu - \frac{1}{2} A_\lambda A^\lambda g_{\mu\nu} - F_{\mu\nu} \quad (2.50)$$

and the Weyl invariant tensor

$$S_{\mu\nu} = 2u_{(\mu} \mathcal{D}_\lambda \left(\sigma_{\nu)}^\lambda + \omega_{\nu)}^\lambda \right) - \frac{\mathcal{R}}{2} u_\mu u_\nu + (\sigma_{\mu\lambda} + \omega_{\mu\lambda}) \left(\sigma_\nu^\lambda + \omega_\nu^\lambda \right) \quad (2.51)$$

which are going to be very useful when discussing on the Ricci-flat/Carrollian fluid duality in Chapter 6.

When the dimension of the spacetime is three ($d = 2$), which is the dimension we will use in Chapter 6, we can define other Weyl-covariant velocity-related quantities as

$$\begin{aligned}\mathcal{D}_\mu u_\nu &= \nabla_\mu u_\nu + \frac{1}{k^2} u_\mu a_\nu - \frac{\Theta}{2} h_{\mu\nu} \\ &= \sigma_{\mu\nu} + \omega_{\mu\nu},\end{aligned}\tag{2.52}$$

$$\mathcal{D}_\nu \omega^\nu_\mu = \nabla_\nu \omega^\nu_\mu,\tag{2.53}$$

$$\mathcal{D}_\nu \eta^\nu_\mu = 2q u_\mu,\tag{2.54}$$

$$u^\lambda \mathcal{R}_{\lambda\mu} = \mathcal{D}_\lambda (\sigma^\lambda_\mu - \omega^\lambda_\mu) - u^\lambda F_{\lambda\mu},\tag{2.55}$$

$$u^\lambda \mathcal{D}_\lambda h_{\mu\nu} = 0,\tag{2.56}$$

of weights $-1, 1, 0, 1$ and -1 (the last one shows that the operator $u^\lambda \mathcal{D}_\lambda$ respects transversality). Here $\eta_{\mu\nu}$ is defined as ⁶

$$\eta_{\mu\nu} = -\frac{u^\rho}{k} \eta_{\rho\mu\nu},\tag{2.57}$$

obeying

$$\eta_{\mu\sigma} \eta_\nu^\sigma = h_{\mu\nu}, \quad \eta^{\mu\nu} \eta_{\mu\nu} = 2.\tag{2.58}$$

The scalar q is of weight 1 and defines the vorticity strength as

$$q^2 = \frac{1}{2k^4} \omega_{\mu\nu} \omega^{\mu\nu}.\tag{2.59}$$

The two-index tensor $\eta_{\mu\nu}$ defines a duality map within the space of symmetric, transverse (with respect to u) and traceless tensors. For a transverse tensor V^μ , its dual is given as

$$*V^\mu = \eta^\nu_\mu V_\nu,\tag{2.60}$$

which is also transverse.

Similarly with a symmetric, transverse and traceless tensor $W_{\mu\nu}$, we define its dual as

$$*W_{\mu\nu} = \eta^\rho_\mu W_{\rho\nu},\tag{2.61}$$

which is symmetric, transverse and traceless. As a reminder, the above is valid for three dimensional spacetimes.

⁶ $\eta_{\rho\mu\nu} = \sqrt{-g} \varepsilon_{\rho\mu\nu}$.

The fluid dynamics captured by (2.1) and (2.2) is Weyl-invariant as long as the energy-momentum tensor and the matter current are Weyl-covariant such that

$$\nabla_\mu T^{\mu\nu} = \mathcal{D}_\mu T^{\mu\nu}, \quad \nabla_\mu J^\mu = \mathcal{D}_\mu J^\mu. \quad (2.62)$$

The latter demands for the conformal weights of $T_{\mu\nu}$ and J_μ to be $d - 1$, and $T_{\mu\nu}$ must be traceless. For the energy-momentum tensor, these are consequences of the Weyl invariance of the action. Indeed, given the infinitesimal Weyl transformation

$$\delta_{\mathcal{B}} g_{\mu\nu} = -2 \ln \mathcal{B} g_{\mu\nu}, \quad (2.63)$$

one can see that

$$\delta_{\mathcal{B}} \mathcal{S} = - \int d^{d+1}x \sqrt{-g} \ln \mathcal{B} T_\mu{}^\mu, \quad (2.64)$$

implying $T_\mu{}^\mu = 0$ for a Weyl invariant action. The weight of the energy momentum-tensor can be inferred from (5.7).

In the decomposition (2.3) the latter condition reads $-\varepsilon + dp + \tau^\mu{}_\mu = 0$, usually splitted into the conformal global-equilibrium equation of state plus a condition on the piece associated with dynamical irreversible phenomena:

$$\varepsilon = dp, \quad \tau^\mu{}_\mu = 0. \quad (2.65)$$

The conformal weight of the gauge field B_μ , conjugate to J^μ , is $w = 0$ and it is inferred from (2.18) in order to comply with the expected weight for J_μ . A summary of the weights of the various physical quantities are given in the Tab. 2.1.

weight	observables
$d + 1$	ε, p
d	q_μ, ρ_0
$d - 1$	$\tau_{\mu\nu}, j_\mu$

Table 2.1 Conformal weights.

2.2 Fluids in Zermelo coordinates

2.2.1 Zermelo frame

There are different parametrizations one can chose for the metric of a pseudo-Rimannian manifold of $d + 1$ dimensions depending on what one is trying to achieve. For instance, one can always assume the Arnowitt–Deser–Misner form of the metric

$$ds^2 = -\Omega^2 k^2 dt^2 + a_{ij} (dx^i - w^i dt) (dx^j - w^j dt) \quad (2.66)$$

with a_{ij} , w^i and Ω functions of $x = (kt, \mathbf{x}) = \{x^\mu, \mu = 0, 1, \dots, d\}$ and \mathbf{x} stands for $\{x^1, \dots, x^d\}$. These choice of gauge is well-suited to take the Galilean limit [44]. Indeed, Galilean diffeomorphisms

$$t' = t'(t), \quad \mathbf{x}' = \mathbf{x}'(t, \mathbf{x}) \quad (2.67)$$

have Jacobian

$$J(t) = \frac{\partial t'}{\partial t}, \quad j^i(t, \mathbf{x}) = \frac{\partial x^{i'}}{\partial t}, \quad J_j^i(t, \mathbf{x}) = \frac{\partial x^{i'}}{\partial x^j}, \quad (2.68)$$

and the transformation of the metric components are reduced as

$$a'_{ij} = a_{kl} J^{-1k}{}_i J^{-1l}{}_j, \quad w'^k = \frac{1}{J} \left(J_i^k w^i + j^k \right), \quad \Omega' = \frac{\Omega}{J}. \quad (2.69)$$

Then a Galilean limit for (2.66) exists under the condition that Ω depends only on time t , making (2.66) stable under Galilean diffeomorphisms. We call Zermelo metrics [103] to the parameterization given by (2.66) with the restriction $\Omega = \Omega(t)$, whose corresponding Newton-Cartan structure reached in the $k \rightarrow \infty$ limit is torsion free [104]. The latter is equipped with an absolute, invariant Newtonian time $\int dt \Omega(t) = \int dt' \Omega'(t')$.

With the above gauge, any tensor component with an upper time index transform as a Galilean density and thus is a scalar upon multiplication by Ω . In the same way the components with lower spatial indices transform as Galilean tensors. As an example, the transformation of the $d + 1$ vector components V^μ under a Galilean diffeomorphism leads to⁷

$$V'^0 = J V^0, \quad V'_i = V_k J^{-1k}{}_i. \quad (2.70)$$

⁷When the indices are inverted, the transformations are of the connection type: $V'^i = J_k^i V^k + J^i V^0$, $V'_0 = \frac{1}{J} (V_0 - V_j J^{-1j}{}_k J^k)$. For those, the tensorial structure is restored at the infinite- k limit, where indices are lowered and raised with a_{ij} and its inverse.

As we mentioned previously, the fluid is flowing with a velocity field u with normalization $\|u\|^2 = -k^2$. The latter is described as

$$u^\mu = \frac{dx^\mu}{d\tau} \Rightarrow u^0 = \gamma k, \quad u^i = \gamma v^i, \quad (2.71)$$

where the Lorentz factor γ is expressed in the Zermelo frame as⁸

$$\gamma = \frac{dt}{d\tau} = \frac{1}{\Omega \sqrt{1 - \left(\frac{\mathbf{v}-\mathbf{w}}{k\Omega}\right)^2}}. \quad (2.72)$$

Under a Galilean diffeomorphism (2.68), the transformation of u^μ (see footnote 7) induces the transformation

$$v'^k = \frac{1}{J} \left(J_i^k v^i + j^k \right), \quad (2.73)$$

meaning the velocity v^i transform as a connection.

We can also extract the fundamental data for the non-perfect part of the matter current j_i , heat current q_i and the stress tensor τ_{kl} by the conditions (2.5) and (2.6). Their other components in Zermelo coordinates can be expressed as

$$\begin{aligned} j^0 &= \frac{(v^i - w^i) j_i}{k\Omega^2}, & q^0 &= \frac{(v^i - w^i) q_i}{k\Omega^2}, & \tau^{00} &= \frac{(v^k - w^k)(v^l - w^l) \tau_{kl}}{k^2 \Omega^4}, \\ \tau^0_j &= \frac{(v^k - w^k) \tau_{kj}}{k\Omega^2}, \end{aligned} \quad (2.74)$$

which transform as tensors under Galilean diffeomorphisms.

2.2.2 Hydrodynamic-frame transformations and invariants

The fluid velocity is parameterized in (2.71) with d components v^i . We can thus formulate the relativistic hydrodynamic-frame transformations in terms of an arbitrary local transformation on v^i , that is, $\delta v^i(x)$. In the Zermelo frame, the Lorentz factor transforms as

$$\delta \gamma = \frac{\gamma^3}{k^2} \delta v^i (v_i - w_i), \quad (2.75)$$

hence, the transformation of the velocity field u is given by

$$\delta u = \gamma \delta v^i \left(\partial_i + \frac{\gamma^2}{k^2} (v_i - w_i) \left(\partial_t + v^k \partial_k \right) \right). \quad (2.76)$$

⁸Expressions as \mathbf{v}^2 stand for $a_{ij} v^i v^j$, not to be confused with $\|u\|^2 = g_{\mu\nu} u^\mu u^\nu$.

Using Eqs. (2.33), (2.34) and (2.35), together with (2.74) and (2.76), we find

$$\delta \varepsilon = -2 \frac{\gamma}{k^2} \delta v^i q_i, \quad (2.77)$$

$$\delta q_i = \gamma \delta v^k \left(\frac{\gamma}{k^2} (v_i - w_i) q_k - \mathfrak{w} h_{ki} - \tau_{ki} \right), \quad (2.78)$$

$$\begin{aligned} \delta (p h_{ij} + \tau_{ij}) &= \frac{\gamma^2}{k^2} \delta v^k \left((v_i - w_i) (\tau_{jk} + p h_{jk}) + (v_j - w_j) (\tau_{ik} + p h_{ik}) \right) \\ &\quad - \frac{\gamma}{k^2} \delta v^k (q_i h_{jk} + q_j h_{ik}), \end{aligned} \quad (2.79)$$

where (see (2.11))

$$h_{ik} = a_{ik} + \frac{\gamma^2}{k^2} (v_i - w_i) (v_k - w_k). \quad (2.80)$$

When a matter current is available, the above is completed with (2.36), which gives

$$\delta \rho_0 = -\frac{\gamma}{c^2} \delta v^i j_i \quad (2.81)$$

and

$$\delta j_i = \delta v^k \left(\frac{\gamma^2}{c^2} j_k (v_i - w_i) - \gamma \rho_0 h_{ki} \right). \quad (2.82)$$

The transformations at hand are such that the energy–momentum tensor and the matter current remain invariant. The latter define therefore invariants, which are simply the energy density, the heat current, the stress tensor, the matter density and the matter non-perfect current in a privileged frame, that we will call “at rest” or “proper”. In this privileged frame we have then

$$T^{00} = \frac{\varepsilon_r}{\Omega^2}, \quad T_i^0 = \frac{1}{k\Omega} q_{ri}, \quad T_{ij} = p_r a_{ij} + \tau_{rij} \quad (2.83)$$

with trace

$$T_\mu{}^\mu = -\varepsilon_r + d p_r + a^{ij} \tau_{rij}, \quad (2.84)$$

and

$$J^0 = \frac{k}{\Omega} \rho_{0r}, \quad J_i = j_{ri}. \quad (2.85)$$

We can express the above invariants in terms of the fluid data in the general frame as

$$\varepsilon_r = \varepsilon\gamma^2\Omega^2 + \frac{2}{k^2}\gamma q_i (v^i - w^i) + (ph_{ij} + \tau_{ij}) \frac{(v^i - w^i)(v^j - w^j)}{k^2\Omega^2}, \quad (2.86)$$

$$q_{ri} = \varepsilon\gamma^2\Omega(v_i - w_i) + \gamma\Omega q_j \left(\delta_i^j + \frac{(v^j - w^j)(v_i - w_i)}{k^2\Omega^2} \right) + (ph_{ij} + \tau_{ij}) \frac{v^j - w^j}{\Omega}, \quad (2.87)$$

$$p_r a_{ij} + \tau_{rij} = \frac{\varepsilon\gamma^2}{k^2} (v_i - w_i)(v_j - w_j) + \frac{\gamma}{k^2} (q_i(v_j - w_j) + q_j(v_i - w_i)) + ph_{ij} + \tau_{ij}, \quad (2.88)$$

and

$$\rho_{0r} = \rho_0\Omega\gamma + j_i \frac{v^i - w^i}{k^2\Omega}, \quad (2.89)$$

$$j_{ri} = j_i + \rho_0\gamma(v^i - w^i). \quad (2.90)$$

It should be stressed that the above quantities are hydrodynamic-frame invariant but also covariant under Galilean diffeomorphisms. This latter property will be useful when considering the Galilean limit.

2.2.3 Killings and conserved currents

Consider a Killing field on \mathcal{M} satisfying (2.24)

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \xi^{\hat{t}} e_{\hat{t}} + \xi^{\hat{i}} e_{\hat{i}}, \quad (2.91)$$

where we have introduced a somewhat more convenient frame and coframe

$$e_{\hat{t}} = v = \frac{1}{\Omega} (\partial_t + w^j \partial_j), \quad e_{\hat{i}} = \partial_i, \quad \theta^{\hat{t}} = \mu = \Omega dt, \quad \theta^{\hat{i}} = dx^i - w^i dt, \quad (2.92)$$

so that the metric (2.66) reads

$$ds^2 = -k^2 (\theta^{\hat{t}})^2 + a_{ij} \theta^{\hat{i}} \theta^{\hat{j}}. \quad (2.93)$$

Hence

$$\xi^{\hat{t}} = \xi^t \Omega, \quad \xi^{\hat{i}} = \xi^i - \xi^t w^i, \quad \xi_{\hat{t}} = -k^2 \xi^{\hat{t}}, \quad \xi_{\hat{i}} = a_{ij} \xi^{\hat{j}} = \xi_i. \quad (2.94)$$

With these, the components of the conserved current (2.25) are⁹

$$I^0 = \frac{k}{\Omega} \iota_{0r}, \quad I_i = i_{ri}, \quad (2.95)$$

where

$$\iota_{0r} = \frac{1}{k^2} \xi^{\hat{i}} q_{ri} - \xi^{\hat{i}} \varepsilon_r, \quad (2.96)$$

$$i_{ri} = \xi^{\hat{j}} (p_r a_{ij} + \tau_{rij}) - \xi^{\hat{i}} q_{ri}. \quad (2.97)$$

The associated conserved charge is obtained using (2.28):

$$\begin{aligned} Q_I &= \int_{\Sigma_d} \sqrt{a} \iota_{0r} (dx^1 - w^1 dt) \wedge \dots \wedge (dx^d - w^d dt) \\ &\quad - \int_{\Sigma_d} \sqrt{a} \sum_{i=1}^d (dx^1 - w^1 dt) \wedge \dots \wedge a^{ij} i_{rj} \Omega dt \wedge \dots \wedge (dx^d - w^d dt), \end{aligned} \quad (2.98)$$

where $a^{ij} i_{rj} \Omega dt$ is the i th factor in the exterior product of the last term.

2.3 Fluids in Papapetrou-Randers coordinates

2.3.1 Papapetrou-Randers frame

Another alternative gauge for a $d + 1$ -dimensional pseudo-Riemannian manifold is given by

$$ds^2 = -k^2 (\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j, \quad (2.99)$$

where all functions depend on time and space. The latter is known as the Papapetrou–Randers frame, and oppositely to Zermelo frame, this form is stable under Carrollian diffeomorphisms

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \quad (2.100)$$

with Jacobian

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J_j^i(\mathbf{x}) = \frac{\partial x'^i}{\partial x^j}. \quad (2.101)$$

⁹We use the standard decomposition $I^\mu = \iota_0 u^\mu + i^\mu$ with $u^\mu i_\mu = 0$ and $\iota_0 = -u^\mu I_\mu$, and introduce ι_{0r} as a proper density, following the footsteps of the energy–momentum tensor and the matter current, Eqs. (2.83) and (2.85).

Under Carrollian diffeomorphisms, Ω and the d -dimensional metric a_{ij} transform as in (2.69), namely

$$\Omega' = \frac{\Omega}{J}, \quad a'^{ij} = J_k^i J_l^j a^{kl}, \quad (2.102)$$

while b^i transform as a connection

$$b'_k = \left(b_i + \frac{\Omega}{J} j_i \right) J^{-1i}{}_k. \quad (2.103)$$

In the Papapetrou–Randers frame, any tensor component with a lower time index transforms as a Carrollian density and provides a scalar upon division by Ω . For the components with upper spatial indices, we find Carrollian tensors. Again, taking the example with the $d + 1$ vector components V^μ , we see that it follows the following transformation rules under Carrollian diffeomorphism:

$$V'_0 = \frac{V_0}{J}, \quad V'^i = V^k J_k^i. \quad (2.104)$$

Here, our fluid velocity is characterized once again by $u^0 = \gamma k$ and $u^i = \gamma v^i$ with normalization $-k^2$ but this time, as we are going to see soon, we chose an appropriate parameterization for the velocity v^i . Papapetrou–Randers coordinates are well suited for taking Carrollian limits. For instance, in the $k \rightarrow 0$ limit of (2.99) one recovers a Carrollian structure with degenerate metric a_{ij} . Additionally, as mentioned previously, in a Carrollian limit particles are forbidden of all movement meaning that the velocity v^i has to be set to zero in this limit. The same must happen if v^i is now the velocity of an infinitesimal portion of a fluid. The latter means that its scaling has to be of the form

$$v^i = k^2 \Omega \beta^i + \mathcal{O}(k^4). \quad (2.105)$$

Here β^i is a Carrollian kinematic quantity of inverse velocity dimension. Therefore its definition has to be such that it transforms as a Carrollian vector. To that end the parameterization of v^i must be

$$v^i = \frac{k^2 \Omega \beta^i}{1 + k^2 \beta^j b_j} \Leftrightarrow \beta^i = \frac{v^i}{k^2 \Omega \left(1 - \frac{v^j b_j}{\Omega} \right)}, \quad (2.106)$$

from which we can see that β^i transforms as a Carrollian vector¹⁰

$$\beta'^i = J_j^i \beta^j. \quad (2.107)$$

¹⁰This is easily proven by observing that $\beta_i + b_i = -\frac{\Omega u_i}{k u_0}$. We define as usual $b^i = a^{ij} b_j$, $\beta_i = a_{ij} \beta^j$, $v_i = a_{ij} v^j$, $\mathbf{b}^2 = b_i b^i$, $\boldsymbol{\beta}^2 = \beta_i \beta^i$ and $\mathbf{b} \cdot \boldsymbol{\beta} = b_i \beta^i$.

Now in this parameterization the Lorentz factor reads

$$\gamma = \frac{1 + k^2 \boldsymbol{\beta} \cdot \mathbf{b}}{\Omega \sqrt{1 - k^2 \boldsymbol{\beta}^2}}. \quad (2.108)$$

In Papapetrou–Randers frame (2.99), the fundamental hydrodynamic variables are naturally chosen as j^i , q^j and τ^{kl} . Using the transversality conditions (2.5) we find

$$j_0 = -k\Omega\beta_i j^i, \quad q_0 = -k\Omega\beta_i q^i, \quad \tau_{00} = k^2\Omega^2\beta_k\beta_l\tau^{kl}, \quad \tau^i_0 = -k\Omega\beta_k\tau^{ik}. \quad (2.109)$$

These are all Carrollian tensors (or densities).

2.3.2 Hydrodynamic-frame transformations and invariants

Following the same pattern as for the Zermelo frame, we investigate the hydrodynamic-frame transformations, namely, local Lorentz transformations captured here in the d components $\delta\beta^i(x)$. We obtain

$$\delta u^0 = k\delta\gamma, \quad \delta u^i = k^2 \frac{h^{ik}\delta\beta_k}{\sqrt{1 - k^2\boldsymbol{\beta}^2}} \quad (2.110)$$

with

$$\delta\gamma = k^2\gamma\delta\beta^i \left(\frac{b_i}{1 + k^2\boldsymbol{\beta} \cdot \mathbf{b}} + \frac{\beta_i}{1 - k^2\boldsymbol{\beta}^2} \right), \quad (2.111)$$

and

$$h^{ik} = a^{ik} + \frac{k^2\beta^i\beta^k}{1 - k^2\boldsymbol{\beta}^2}. \quad (2.112)$$

Using the general transformation rules (2.33), (2.34) and (2.35) together with (2.109) and (2.110) we find (\mathfrak{w} is the relativistic enthalpy (A.3))¹¹

$$\delta\varepsilon = -2 \frac{q^i\delta\beta_i}{\sqrt{1 - k^2\boldsymbol{\beta}^2}}, \quad (2.113)$$

$$\delta q^i = \frac{k^2\delta\beta_k}{\sqrt{1 - k^2\boldsymbol{\beta}^2}} \left(\frac{q^k\beta^i}{\sqrt{1 - k^2\boldsymbol{\beta}^2}} - \mathfrak{w}h^{ki} - \tau^{ki} \right), \quad (2.114)$$

$$\begin{aligned} \delta(ph^{ij} + \tau^{ij}) &= \frac{k^2\delta\beta_k}{1 - k^2\boldsymbol{\beta}^2} \left(\beta^i (ph^{jk} + \tau^{jk}) + \beta^j (ph^{ik} + \tau^{ik}) \right) \\ &\quad - \frac{\delta\beta_k}{\sqrt{1 - k^2\boldsymbol{\beta}^2}} (q^i h^{jk} + q^j h^{ik}). \end{aligned} \quad (2.115)$$

¹¹Notice in passing $q_\mu \delta u^\mu = k^2 \frac{q^i \delta\beta_i}{\sqrt{1 - k^2\boldsymbol{\beta}^2}}$.

Similarly, using Eq. (2.36), we obtain

$$\delta\rho_0 = -\frac{j^i\delta\beta_i}{\sqrt{1-k^2\boldsymbol{\beta}^2}}, \quad (2.116)$$

and

$$\delta j^i = \frac{k^2\delta\beta_k}{\sqrt{1-k^2\boldsymbol{\beta}^2}} \left(\frac{j^k\beta^i}{\sqrt{1-k^2\boldsymbol{\beta}^2}} - \rho_0 h^{ki} \right). \quad (2.117)$$

The energy–momentum tensor is by definition invariant under hydrodynamic-frame transformations. This invariance can be characterized in terms of three canonical objects, which are the energy density ε_r , the heat current q_r^i and the stress tensor τ_r^{ij} , in the rest frame

$$T_{00} = \varepsilon_r \Omega^2, \quad T_0^i = -\frac{\Omega}{k} q_r^i, \quad T^{ij} = p_r a^{ij} + \tau_r^{ij} \quad (2.118)$$

with trace

$$T_\mu{}^\mu = -\varepsilon_r + d p_r + a_{ij} \tau_r^{ij}, \quad (2.119)$$

and

$$J_0 = -k\Omega\rho_{0r}, \quad J^i = j_r^i. \quad (2.120)$$

We can relate these invariants to the fluid data in an arbitrary frame encoded in β^i . They read

$$\varepsilon_r = \frac{\varepsilon}{1-k^2\boldsymbol{\beta}^2} + \frac{2\beta_i q^i}{\sqrt{1-k^2\boldsymbol{\beta}^2}} + k^2\beta_i\beta_j (ph^{ij} + \tau^{ij}), \quad (2.121)$$

$$q_r^i = \frac{k^2\varepsilon\beta^i}{1-k^2\boldsymbol{\beta}^2} + \frac{q^j}{\sqrt{1-k^2\boldsymbol{\beta}^2}} (\delta_j^i + k^2\beta^i\beta_j) + k^2\beta_j (ph^{ij} + \tau^{ij}), \quad (2.122)$$

$$p_r a^{ij} + \tau_r^{ij} = \frac{k^2\varepsilon\beta^i\beta^j}{1-k^2\boldsymbol{\beta}^2} + \frac{\beta^i q^j + \beta^j q^i}{\sqrt{1-k^2\boldsymbol{\beta}^2}} + ph^{ij} + \tau^{ij}, \quad (2.123)$$

and similarly for the components of the matter current we have

$$\rho_{0r} = \frac{\rho_0}{\sqrt{1-k^2\boldsymbol{\beta}^2}} + \beta_i j^i, \quad (2.124)$$

$$j_r^i = j^i + \frac{k^2\rho_0\beta^i}{\sqrt{1-k^2\boldsymbol{\beta}^2}}. \quad (2.125)$$

2.3.3 Killings and conserved currents

Once again, in the presence of isometries, the Killing field takes the generic form (2.91), that satisfies (2.24) on \mathcal{M} in Papapetrou–Randers coordinates, where the frame and coframe are

now¹²

$$e_{\hat{t}} = v = \frac{1}{\Omega} \partial_t, \quad e_{\hat{i}} = \partial_i + \frac{b_i}{\Omega} \partial_t, \quad \theta^{\hat{t}} = \mu = \Omega dt - b_i dx^i, \quad \theta^{\hat{i}} = dx^i, \quad (2.126)$$

so that the metric (2.99) becomes (2.93). The Killing components are

$$\xi^{\hat{t}} = \xi^t \Omega - \xi^i b_i, \quad \xi^{\hat{i}} = \xi^i, \quad \xi_{\hat{t}} = -k^2 \xi^{\hat{t}}, \quad \xi_{\hat{i}} = a_{ij} \xi^{\hat{j}} = \xi_i + \xi_i b_i, \quad (2.127)$$

and thus, the components of the conserved current (2.25) read

$$I_0 = -k\Omega \iota_{0r}, \quad I^i = i_r^i, \quad (2.128)$$

where

$$\iota_{0r} = \frac{1}{k^2} \xi_i q_r^i - \xi^{\hat{t}} \epsilon_r, \quad (2.129)$$

$$i_r^i = \xi_{\hat{j}} (p_r a^{ij} + \tau_r^{ij}) - \xi^{\hat{t}} q_r^i. \quad (2.130)$$

Using (2.28), one can express the conserved charge in the Papapetrou–Randers frame as

$$Q_I = \int_{\Sigma_d} \sqrt{a} \iota_{0r} dx^1 \wedge \dots \wedge dx^d - \int_{\Sigma_d} \sqrt{a} \sum_{i=1}^d dx^1 \wedge \dots \wedge i_r^i (\Omega dt - b_j dx^j) \wedge \dots \wedge dx^d, \quad (2.131)$$

where in the exterior product of the second term, $i_r^i (\Omega dt - b_j dx^j)$ is the i th factor.

2.4 Ehlers group and the Geroch method in General Relativity

In anticipation to the upcoming discussion on aspects of the Ricci-flat/Carrollian duality in Chapter 6, we now turn our attention to the dynamic of gravitational fields.

The most successful theory for the description of gravity is General Relativity. In this theory, gravitational forces are understood as the deformation of the geometry of the spacetime sourced by the presence of matter/energy. The dynamical equations are given by

¹²Later on $e_{\hat{t}}$ will be alternatively displayed as $\hat{\partial}_t$.

the Einstein's equations¹³

$$R_{AB} - \frac{1}{2}Rg_{AB} + \Lambda g_{AB} = 8\pi GT_{AB}^{(\phi)}, \quad (2.132)$$

where R_{AB} is the Ricci-tensor, R is the scalar curvature¹⁴, g_{AB} is the metric of the spacetime, Λ is the cosmological constant, and $T_{AB}^{(\phi)}$ is the energy-momentum tensor of the matter content. Here $A, B, \dots \in \{0, \dots, 3\}$

In the absence of a cosmological constant Λ and matter, vacuum Einstein's field equations are reduced to the vanishing of the Ricci tensor, namely $R_{AB} = 0$. It has been known for long that these equations are invariant under constant scalings of the metric and the action of infinitesimal diffeomorphisms [105]. The latter are not the only symmetries of Einstein's field equations and in fact more "hidden symmetries" arise upon dimensional reduction [106–112]. Here we focus on Ricci-flat spacetimes and we review the Geroch generalization of Ehlers work following [89, 92].

Consider a four-dimensional pseudo-Riemannian manifold (\mathcal{M}, g) possessing a timelike isometry¹⁵ generated by the Killing vector field ξ . For the latter we can define its norm

$$\lambda = \xi^A \xi_A, \quad (2.133)$$

and twist

$$w_A = \eta_{ABCD} \xi^B \nabla^C \xi^D, \quad (2.134)$$

where $\eta_{ABCD} = \sqrt{-g} \epsilon_{ABCD}$ ($\epsilon_{0123} = 1$). For a Ricci-flat spacetime,¹⁶ one can show that the one-form twist $w = w_A dx^A$ is closed, implying

$$w = d\omega \quad (2.135)$$

with ω a scalar function.

A three-dimensional space \mathcal{S} can be defined as the quotient $\mathcal{M}/\text{orb}(\xi)$. This coset space is not necessarily a subspace of \mathcal{M} unless ξ is hypersurface-orthogonal, which implies zero

¹³Einstein's equations are obtained by varying the Einstein-Hilbert action $S_{\text{EH}} = \int_{\mathcal{M}} d^{d+2}x \sqrt{g} (R - 2\Lambda)$ with respect to the field g_{AB} .

¹⁴The Ricci tensor is the subtrace of the Riemann tensor, namely $R_{BC} = R^A{}_{BAC}$ with the Riemann tensor defined as $R^A{}_{BCD} = \partial_C \Gamma^A_{BD} - \partial_D \Gamma^A_{BC} + \Gamma^A_{EC} \Gamma^E_{BD} - \Gamma^A_{ED} \Gamma^E_{BC}$, with Γ^A_{BC} the Christoffel symbol. The scalar curvature is the trace of the Ricci tensor, that is $R = g^{AB} R_{AB}$.

¹⁵The Geroch procedure is valid for both, timelike and spacelike isometries but here we consider the first to avoid unnecessary multiplication of indices without shedding more light on our purpose.

¹⁶This property actually holds more generally for Einstein spacetimes [113].

twist with \mathcal{S} the orthogonal hypersurface. We can define a metric on the space \mathcal{S} which is induced by g of \mathcal{M} . The latter reads

$$h_{AB} = g_{AB} - \frac{\xi_A \xi_B}{\lambda}, \quad (2.136)$$

which defines the projector onto \mathcal{S} as

$$h_A^B = \delta_A^B - \frac{\xi^B \xi_A}{\lambda}. \quad (2.137)$$

The fully anti-symmetric tensor for (2.136) is $\eta_{ABC} = \frac{-1}{\sqrt{-\lambda}} \eta_{ABCD} \xi^D$.

Tensors of \mathcal{M} that satisfy

$$\xi^{A_1} T_{A_1 \dots A_p}{}^{B_1 \dots B_q} = 0 \quad \text{and} \quad \mathcal{L}_\xi T_{A_1 \dots A_p}{}^{B_1 \dots B_q} = 0, \quad (2.138)$$

namely, tensors that are transverse and invariant with respect to ξ , are in one-to-one correspondence with tensors on \mathcal{S} . If T is a tensor of \mathcal{S} , we can define the covariant derivative \mathcal{D} as the following projection onto \mathcal{S} ,

$$\mathcal{D}_C T_{A_1 \dots A_p}{}^{B_1 \dots B_q} = h_C^L h_{A_1}^{M_1} \dots h_{A_p}^{M_p} h_{N_1}^{B_1} \dots h_{N_q}^{B_q} \nabla_L T_{M_1 \dots M_p}{}^{N_1 \dots N_q} \quad (2.139)$$

with ∇ the Levi-Civita connection on (\mathcal{M}, g) . The latter sets a relationship between the Riemann tensor on \mathcal{M} and the Riemann tensor on \mathcal{S} through a generalized Gauss-Codazzi equation given by

$$\mathcal{R}_{ABCD} = h_{[A}^P h_{B]}^Q h_{[C}^R h_{D]}^S (R_{PQRS} + \frac{2}{\lambda} (\nabla_P \xi_Q \nabla_R \xi_S + \nabla_P \xi_R \nabla_Q \xi_S)). \quad (2.140)$$

Here the calligraphic letters refer to curvature tensors of \mathcal{S} .

The Ricci-flat dynamics for g_{AB} is recast in the present framework in terms of¹⁷

$$\tilde{h}_{AB} = \lambda h_{AB}, \quad (2.141)$$

as well as ω and λ viewed as fields on \mathcal{S} , packaged in

$$\tau = \omega + i\lambda, \quad (2.142)$$

¹⁷With our conventions, this metric is definite-negative.

and obeying the equations¹⁸

$$\tilde{\mathcal{R}}_{AB} = -\frac{2}{(\tau - \bar{\tau})^2} \tilde{\mathcal{D}}_{(A} \tau \tilde{\mathcal{D}}_{B)} \bar{\tau}, \quad (2.143)$$

$$\tilde{\mathcal{D}}^2 \tau = \frac{2}{\tau - \bar{\tau}} \tilde{\mathcal{D}}_M \tau \tilde{\mathcal{D}}_N \tau \tilde{h}^{MN}. \quad (2.144)$$

Equation (2.143) is obtained through (2.140), while (2.144) is obtained by a direct computation of the \mathcal{S} -Laplacian acting on τ . Here $\tilde{\mathcal{D}}_A$ and $\tilde{\mathcal{R}}_{AB}$ are the Levi–Civita covariant derivative and the Ricci tensor associated with the metric \tilde{h}_{AB} displayed in (2.141).

Equations (2.143)–(2.144) are the reduction of Einstein’s equations onto \mathcal{S} and they yield two important properties. The first one is that they are invariant under the transformation keeping \tilde{h}_{AB} unaltered and mapping τ into

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R}). \quad (2.145)$$

From the latter one can infer the transformation of the triplet $(\lambda, \omega, h_{AB})$ as

$$h'_{AB} = [(\omega\gamma + \delta)^2 + \gamma^2\lambda^2] h_{AB}, \quad (2.146)$$

$$\lambda' = \frac{\lambda}{(\omega\gamma + \delta)^2 + \gamma^2\lambda^2}, \quad (2.147)$$

$$\omega' = \frac{(\omega\alpha + \beta)(\omega\gamma + \delta) + \alpha\gamma\lambda^2}{(\omega\gamma + \delta)^2 + \gamma^2\lambda^2}. \quad (2.148)$$

This is the first instance where a hidden group of symmetries, being $SL(2, \mathbb{R})$, arises upon reduction with respect to an isometry [88]. The second property is that, after the action of this group of symmetries which gives $\tau' = \omega' + i\lambda'$ and $h'_{AB} = \frac{1}{\lambda'} \tilde{h}_{AB}$ as another solution of (2.143) and (2.144), it is possible to reverse the reduction process in order to find a new four-dimensional Ricci-flat spacetime with an isometry [89, 92]. To this end, one has to show that the \mathcal{S} -two-form defined as

$$F'_{AB} = \frac{1}{(-\lambda')^{3/2}} \eta'_{ABC} \mathcal{D}^C \omega' \quad (2.149)$$

is closed. Thus, locally

$$F' = d\eta'. \quad (2.150)$$

¹⁸Equations (2.143) can be obtained by varying a three-dimensional sigma-model action defined on \mathcal{S} .

Then one can promote the one-form field η' , which is defined on \mathcal{S} , to a field on \mathcal{M} by adding the necessary exact piece such that its normalization is

$$\xi^A \eta'_A = 1. \quad (2.151)$$

The above defines a new Killing field on \mathcal{M}

$$\xi' = \lambda' \eta' \quad (2.152)$$

and then, a new four-dimensional metric is given by¹⁹

$$g'_{AB} = h'_{AB} + \frac{\xi'_A \xi'_B}{\lambda'}. \quad (2.153)$$

Before closing this section it is worth to mention some remarks. The $SL(2, \mathbb{R})$ group is hidden from the four-dimensional perspective of the Einstein's field equations but it appears explicitly in the three-dimensional reduced equations materialized here in (2.143) and (2.144) (3d sigma model). The latter can be used to generate new solutions, but actually only part of the group generates genuine new Ricci-flat solutions. A good illustration of the latter is the concrete example of Schwarzschild–Taub–NUT solutions with mass M and nut charge n . The subgroup of rotations $\begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \in SO(2) \subset SL(2, \mathbb{R})$ induces rotations of angle 2χ in the parameter space (M, n) , while non-compact transformations $\begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix} \in N \subset SL(2, \mathbb{R})$ generates a constant scaling of the metric given by $(M, n) \rightarrow (M/\alpha, n/\alpha)$.

¹⁹The consequence of Möbius transformations on the Weyl tensor has been investigated in Ref. [114].

Chapter 3

Galilean fluids

Having reviewed the basics on relativistic hydrodynamics and their different parameterizations, we are now ready to study the Galilean and Carrollian versions of fluid dynamics. In this chapter we will start our analysis with Galilean fluids. This includes a brief review on Newton-Cartan geometries, Galilean covariance, the derivation of Galilean fluid dynamics, hydrodynamic-frame invariance and the construction of Noether charges. The same analysis will be implemented for the case of Carrollian fluids in the subsequent chapter.

3.1 Galilean covariance and Newton-Cartan structures

3.1.1 Basics of Newton-Cartan

When describing non-relativistic fluid dynamics, the most natural geometric framework is torsionless Newton-Cartan, see [115–117, 17, 104, 18, 19] for a complete presentation of the subject. These geometries belong to a wide family which includes Bargmann and Leibnizian structures. Here we consider a manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ equipped with coordinates (t, \mathbf{x}) and a degenerate cometric

$$\partial_a^2 = a^{ij} \partial_i \partial_j, \quad i, j \dots \in \{1, \dots, d\}, \quad (3.1)$$

as well as a clock form

$$\mu = \Omega dt. \quad (3.2)$$

The above has a dual vector which is referred to as the field of observers and is given by

$$\mathbf{v} = \frac{1}{\Omega} (\partial_t + w^j \partial_j), \quad (3.3)$$

satisfying $\nu(\mu) = 1$. Here, a^{ij} and w^i are general functions of time and space whereas $\Omega = \Omega(t)$. The space independence of Ω makes the clock form defined in (3.2) exact, ensuring the torsionless nature of the Newton-Cartan manifold. This guarantees the existence of an absolute time $\int dt \Omega(t) = \int dt' \Omega'(t')$, invariant under Galilean diffeomorphisms (2.67).

The submanifold \mathcal{S} corresponds to the d -dimensional Newtonian space, with positive definite metric a_{ij} which corresponds to the inverse of a^{ij} . This is denoted as

$$d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i dx^j, \quad (3.4)$$

and observed from a frame with respect to which the locally inertial frame has velocity $\mathbf{w} = w^i \partial_i$. In our set up, an infinitesimal fluid element or a particle moves with velocity $\mathbf{v} = v^i \partial_i$ with $v^i = \frac{dx^i}{dt}$. Under Galilean diffeomorphisms (2.67) with Jacobian (2.68), the transformation rules are as in (2.69), (2.73), and

$$\partial'_t = \frac{1}{J} \left(\partial_t - j^k J^{-1i}{}_k \partial_i \right), \quad (3.5)$$

$$\partial'_j = J^{-1i}{}_j \partial_i. \quad (3.6)$$

The field of observers and the clock form remain invariant under Galilean diffeomorphisms, namely

$$\mu' = \mu, \quad \nu' = \nu. \quad (3.7)$$

Galilean tensors carry only spatial indices $i, j, \dots \in \{1, \dots, d\}$, which are lowered and raised with a_{ij} and a^{ij} . They are covariant under Galilean diffeomorphisms (2.67). In general, these tensors depend on time t and space \mathbf{x} .

We can define connections on this geometry, which define covariant time and space derivatives. These connections are not unique. There are different choices which are metric-compatible and torsion-free, but here we will make the specific choice that naturally emerges when this Newton-Cartan geometry is reached as an infinite- k limit of a pseudo-Rimannian manifold in the Zermelo frame (2.66). Using this choice we have a sharp separation between time and space materialized in \mathcal{S} for which we obtain the spatial connection as

$$\gamma^j{}_{jk} = \frac{a^{il}}{2} (\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}). \quad (3.8)$$

In the large k expansion, the latter appears as the leading order of the spatial component of the Christoffel symbol $\Gamma^i{}_{jk} = \gamma^i{}_{jk} + \mathcal{O}(1/k^2)$ in the Zermelo frame. The associated covariant derivative is spelled $\hat{\nabla}_i$, which has not to be confused with the spatial component of the Levi-Civita covariant derivative ∇_μ defined on the ascendent pseudo-Rimannian spacetime.

This connection is spatially torsionless, satisfying

$$\hat{t}^k_{ij} = 2\gamma^k_{[ij]} = 0, \quad (3.9)$$

and metric-compatible

$$\hat{\nabla}_i a_{jk} = 0. \quad (3.10)$$

We can define the curvature tensor on \mathcal{S} as usual, by using the commutation relation of covariant derivative $\hat{\nabla}_i$ as

$$[\hat{\nabla}_k, \hat{\nabla}_l] V^i = \left(\partial_k \gamma^i_{lj} - \partial_l \gamma^i_{kj} + \gamma^i_{km} \gamma^m_{lj} - \gamma^i_{lm} \gamma^m_{kj} \right) V^j = \hat{r}^i_{jkl} V^j. \quad (3.11)$$

We can also construct Galilean tensors from an object that is not a vectors but rather it transforms as a connections,

$$A^k = \frac{1}{J} \left(J^k_i A^i + j^k \right). \quad (3.12)$$

For instance, the combination

$$\frac{1}{\Omega} \hat{\nabla}^{(k} A^{l)} - \frac{1}{2\Omega} \partial_t a^{kl} = -\frac{1}{2\Omega} \left(\mathcal{L}_{\mathbf{A}} a^{kl} + \partial_t a^{kl} \right) \quad (3.13)$$

($\mathcal{L}_{\mathbf{A}}$ is the Lie derivative along $\mathbf{A} = A^i \partial_i$) and

$$\frac{1}{\Omega} \hat{\nabla}_{(k} A_{l)} + \frac{1}{2\Omega} \partial_t a_{kl} = \frac{1}{2\Omega} \left(\mathcal{L}_{\mathbf{A}} a_{kl} + \partial_t a_{kl} \right) \quad (3.14)$$

have tensorial transformation rules, and their trace transform as a Galilean scalar.¹ We can apply the above with \mathbf{w} or \mathbf{v} and define

$$\hat{\gamma}^w_{ij} = \frac{1}{\Omega} \left(\hat{\nabla}_{(i} w_{j)} + \frac{1}{2} \partial_t a_{ij} \right), \quad \hat{\gamma}^v_{ij} = \frac{1}{\Omega} \left(\hat{\nabla}_{(i} v_{j)} + \frac{1}{2} \partial_t a_{ij} \right), \quad (3.15)$$

where the upper indices refer to the vectors \mathbf{w} and \mathbf{v} , corresponding to the geometry and fluid respectively. Here $\hat{\gamma}^w_{ij}$ is purely geometrical and emerges in the large- k expansion of the Christoffel symbol of the relativistic-spacetime in the Zermelo frame, given by

$$\Gamma^0_{ij} = 1/(2k\Omega) \hat{\gamma}^w_{ij} + \mathcal{O}(0). \quad (3.16)$$

Both $\hat{\gamma}^w_{ij}$ and $\hat{\gamma}^v_{ij}$ coincide when the fluid is at rest in the locally inertial frame, namely for $v^i = w^i$. Using these tensors one is able to define the shear tensors and the expansion

¹Neither $\frac{1}{\Omega} \partial_t$ nor $\frac{1}{\Omega} \mathcal{L}_{\mathbf{A}}$ acting on Galilean tensors give separately tensors because of (3.5) and A^i transforming as (3.12).

associated to w^i and v^i . We have the geometric Galilean shear

$$\xi^w_{ij} = \frac{1}{\Omega} \left(\hat{\nabla}_{(i} w_{j)} + \frac{1}{2} \partial_t a_{ij} \right) - \frac{1}{d} a_{ij} \theta^w, \quad (3.17)$$

and the geometric Galilean expansion

$$\theta^w = \frac{1}{\Omega} \left(\partial_t \ln \sqrt{a} + \hat{\nabla}_i w^i \right), \quad (3.18)$$

while for the fluid velocity v^i we have the fluid Galilean shear

$$\xi^v_{ij} = \frac{1}{\Omega} \left(\hat{\nabla}_{(i} v_{j)} + \frac{1}{2} \partial_t a_{ij} \right) - \frac{1}{d} a_{ij} \theta^v, \quad (3.19)$$

and the fluid Galilean expansion

$$\theta^v = \frac{1}{\Omega} \left(\partial_t \ln \sqrt{a} + \hat{\nabla}_i v^i \right). \quad (3.20)$$

In the same way we defined the spatial covariant derivative, we can define a time and metric-compatible covariant derivative that emerges in the Galilean expansion of ∇_μ in the time direction. For a scalar function this is defined as

$$\frac{1}{\Omega} \frac{\hat{D}\Phi}{dt} = v(\Phi) = \frac{1}{\Omega} \partial_t \Phi + \frac{w^j}{\Omega} \partial_j \Phi, \quad (3.21)$$

whereas for vectors one finds

$$\begin{aligned} \frac{1}{\Omega} \frac{\hat{D}V^i}{dt} &= \frac{1}{\Omega} \partial_t V^i + \frac{w^j}{\Omega} \partial_j V^i - V^j \partial_j \frac{w^i}{\Omega} + \hat{\gamma}^{wi}_j V^j \\ &= \frac{1}{\Omega} (\partial_t V^i + \mathcal{L}_w V^i) + \hat{\gamma}^{wi}_j V^j. \end{aligned} \quad (3.22)$$

In general, using the Leibniz rule one finds

$$\frac{1}{\Omega} \frac{\hat{D}K^{i\dots j\dots}}{dt} = \frac{1}{\Omega} (\partial_t K^{i\dots j\dots} + \mathcal{L}_w K^{i\dots j\dots}) + \hat{\gamma}^{wi}_k K^{k\dots j\dots} + \dots - \hat{\gamma}^{wk}_j K^{i\dots k\dots} - \dots, \quad (3.23)$$

and as a consequence

$$\frac{1}{\Omega} \frac{\hat{D}a^{ij}}{dt} = \frac{1}{\Omega} \frac{\hat{D}a_{ij}}{dt} = 0. \quad (3.24)$$

When considering fluids, one can also introduce the more physical material derivative

$$\frac{d}{dt} = \partial_t + v^i \hat{\nabla}_i, \quad (3.25)$$

which produces a scalar density (or a scalar upon division by Ω) when acting on a scalar function. When acting on arbitrary tensors, it should be supported with the appropriate \mathbf{w} and/or \mathbf{v} terms in order to keep the tensorial transformation properties. There are several options to implement this but here we use the one defined in [44]. Hence, the material derivative acting on a Galilean vector reads

$$\frac{1}{\Omega} \frac{D V^i}{dt} = \frac{1}{\Omega} \frac{d V^i}{dt} - \frac{1}{\Omega} V^j \hat{\nabla}_j w^i, \quad \frac{1}{\Omega} \frac{D V_i}{dt} = \frac{1}{\Omega} \frac{d V_i}{dt} + \frac{1}{\Omega} V_j \hat{\nabla}_i w^j, \quad (3.26)$$

resulting in genuine tensors under Galilean diffeomorphisms. As opposed to (3.23), this time-covariant derivative is not metric compatible and its action on the metric a_{ij} gives

$$\frac{1}{\Omega} \frac{D a_{ij}}{dt} = 2 \hat{\gamma}^w_{ij}. \quad (3.27)$$

Space and time covariant derivatives do not commute with each other. The commutator acting on scalars and vectors are

$$\left[\frac{1}{\Omega} \frac{\hat{D}}{dt}, \hat{\nabla}_i \right] \Phi = -\hat{\gamma}^{wk}_i \partial_k \Phi, \quad (3.28)$$

$$\left[\frac{1}{\Omega} \frac{\hat{D}}{dt}, \hat{\nabla}_i \right] V^j = -\hat{\gamma}^{wk}_i \hat{\nabla}_k V^j + \hat{r}^j_{ik} V^k, \quad (3.29)$$

where

$$\hat{r}^j_{ik} = \frac{1}{\Omega} \left(\partial_t \gamma^j_{ik} + \hat{\nabla}_i \hat{\nabla}_k w^j - \hat{\nabla}_i \hat{\gamma}^{wj}_k + w^l \hat{r}^j_{kli} \right). \quad (3.30)$$

3.1.2 Conservation equations from symmetry

The conservation equations that describe the dynamics of our Galilean fluid can be derived from symmetry principles. Indeed, one can consider the effective action that describes the dynamics of a system defined on the geometry $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ discussed previously². In this case, the effective action is a functional of the geometric data a^{ij} , Ω and w^i as

$$S = \int dt d^d x \sqrt{a} \Omega \mathcal{L}. \quad (3.31)$$

²This is not necessary restricted to fluid dynamics.

Here we do not have the standard definition of a symmetric energy-momentum tensor. Instead, we can define a set of Galilean momenta as the variation of the action (3.31) with respect to the geometric data. These are

$$\Pi_{ij} = -\frac{2}{\sqrt{a}\Omega} \frac{\delta S}{\delta a^{ij}}, \quad (3.32)$$

$$P_i = -\frac{1}{\sqrt{a}\Omega} \frac{\delta S}{\delta w^i}, \quad (3.33)$$

$$\Pi = -\frac{1}{\sqrt{a}\Omega} \left(\Omega \frac{\delta S}{\delta \Omega} - \frac{w^i}{\Omega} \frac{\delta S}{\delta w^i} \right), \quad (3.34)$$

which correspond to the energy-stress tensor, momentum and energy density respectively. Therefore, the variation of the action in terms of the Galilean momenta reads

$$\delta S = - \int dt \Omega \int d^d x \sqrt{a} \left(\frac{1}{2} \Pi_{ij} \delta a^{ij} + P_i \delta w^i + \left(\Pi + \frac{w^i}{\Omega} P_i \right) \delta \ln \Omega \right). \quad (3.35)$$

The above set of momenta obey conservation equations, which arise as the consequence of the invariance of the action under Galilean diffeomorphisms. Galilean diffeomorphisms (2.67) are generated by vector fields on \mathcal{M} of the form

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \xi^{\hat{t}} \mathbf{v} + \xi^{\hat{i}} \partial_i \quad (3.36)$$

where the time component ξ^t depends only on time. We also used $\xi^{\hat{t}}(t) = \xi^t(t)\Omega(t)$ being a Galilean scalar, and $\xi^{\hat{i}}(t, \mathbf{x}) = \xi^i - \xi^t w^i$ corresponding to the components of a Galilean vector. The variation under diffeomorphisms is implemented through the Lie derivative³. Thus, for its action on the co-metric a^{ij} we find

$$-\delta_\xi a^{ij} = \mathcal{L}_\xi a^{ij} = -2 \left(\hat{\nabla}^{(i} \xi^{\hat{j})} + \hat{\gamma}^{w^{ij}} \xi^{\hat{t}} + \frac{1}{\Omega} w^{(i} a^{j)k} \partial_k \xi^{\hat{t}} \right), \quad (3.37)$$

where the last term drops for Galilean diffeomorphisms due to the space independence of $\xi^{\hat{t}}$. For the field of observers we have

$$\mathcal{L}_\xi \mathbf{v} = -\frac{1}{\Omega} \left(\partial_t \xi^{\hat{t}} + \mathcal{L}_w \xi^{\hat{t}} \right) \mathbf{v} - \frac{1}{\Omega} \left(\partial_t \xi^{\hat{i}} + \mathcal{L}_w \xi^{\hat{i}} \right) \partial_i, \quad (3.38)$$

³The minus sign in (3.37) is conventional.

from which, using (3.3), we infer

$$-\delta_\xi \Omega = \mathcal{L}_\xi \Omega = \partial_t \xi^{\hat{t}} + \mathcal{L}_w \xi^{\hat{t}}, \quad \delta_\xi w^i = -\mathcal{L}_\xi w^i = \partial_t \xi^{\hat{t}} + \mathcal{L}_w \xi^{\hat{t}}. \quad (3.39)$$

We can also find the action on the clock form. This is given by

$$\mathcal{L}_\xi \mu = \frac{1}{\Omega} \left(\partial_t \xi^{\hat{t}} + \mathcal{L}_w \xi^{\hat{t}} \right) \mu = \frac{1}{\Omega} \frac{\hat{D} \xi^{\hat{t}}}{dt} \mu = \mu \mu, \quad (3.40)$$

where we introduced

$$\mu(t, \mathbf{x}) = \frac{1}{\Omega} \frac{\hat{D} \xi^{\hat{t}}}{dt}, \quad (3.41)$$

not to be confused with the chemical potential nor the clock form.

With the latter we can now determine the diffeomorphic variation of the action. This reads

$$\begin{aligned} \delta_\xi S = & \int dt d^d x \sqrt{a} \Omega \left\{ -\xi^{\hat{t}} \left[\frac{1}{\Omega} \frac{\hat{D} \Pi}{dt} + \theta^w \Pi + \Pi_{ij} \hat{\gamma}^{wij} \right] \right. \\ & \left. + \xi^{\hat{t}} \left[\frac{1}{\Omega} \frac{\hat{D} P_i}{dt} + \theta^w P_i + P_j \hat{\gamma}^{wj}_i + \hat{\nabla}^j \Pi_{ij} \right] \right\} \\ & + \int dt d^d x \left\{ \partial_t \left(\sqrt{a} \left(\Pi \xi^{\hat{t}} - P_j \xi^{\hat{j}} \right) \right) \right. \\ & \left. + \partial_i \left(\sqrt{a} w^i \left(\Pi \xi^{\hat{t}} - P_j \xi^{\hat{j}} \right) - \sqrt{a} \Omega \Pi^i_j \xi^{\hat{j}} \right) \right\}. \end{aligned} \quad (3.42)$$

Invariance under Galilean diffeomorphisms requires the vanishing of $\delta_\xi S$. Here we ignore the boundary term in the last integral of (3.42). Therefore, from the condition $\delta_\xi S = 0$ we extract two conservation equations. The first one corresponds to the momentum equation

$$\boxed{\left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w \right) P_i + P_j \hat{\gamma}^{wj}_i + \hat{\nabla}^j \Pi_{ij} = 0}, \quad (3.43)$$

which is associated to $\xi^{\hat{t}}(t, \mathbf{x})$.

The procedure to obtain the energy equation is more subtle due to the space independence of $\xi^{\hat{t}}$. Indeed, for the vanishing of the first term in the integral of (3.42) it is enough to require

$$\boxed{\left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w \right) \Pi + \Pi_{ij} \hat{\gamma}^{wij} = -\hat{\nabla}_i \Pi^i}, \quad (3.44)$$

where Π^i is undetermined. The presence of $\hat{\nabla}_i \Pi^i$ in (3.44) comes as a boundary term in the variation of the action, i.e. $\sqrt{a} \Omega \xi^{\hat{t}} \hat{\nabla}_i \Pi^i = \partial_i \left(\sqrt{a} \Omega \xi^{\hat{t}} \Pi^i \right)$, which vanishes inside the integral

(3.42). As we going to see in the next sections, one can interpret Π^i as the energy current (also energy flux).

Besides Galilean covariance, one may also consider the action to be invariant under local $U(1)$ symmetry, which is parameterized by the function $\Lambda(t, \mathbf{x})$ and acts on the components of a gauge field $\mathbf{B} = B(t, \mathbf{x})dt + B_i(t, \mathbf{x})dx^i$ as

$$\delta_\Lambda B_i = -\partial_i \Lambda, \quad \delta_\Lambda B = -\partial_t \Lambda. \quad (3.45)$$

We can derive the conjugate momenta in the same way as we did in (3.32), (3.33) and (3.34). We obtain the matter density and the matter current as

$$\rho = -\frac{1}{\sqrt{a}} \frac{\delta S}{\delta B}, \quad (3.46)$$

$$N^i = \frac{1}{\Omega \sqrt{a}} \left(w^i \frac{\delta S}{\delta B} - \frac{\delta S}{\delta B_i} \right). \quad (3.47)$$

This allows to expressed the variation of the action as

$$\delta S = - \int dt d^d x \sqrt{a} (\rho \delta B + (\Omega N^i + \rho w^i) \delta B_i). \quad (3.48)$$

Therefore, under the action of a $U(1)$ symmetry, the variation of the action reads

$$\begin{aligned} \delta_\Lambda S &= \int dt d^d x \sqrt{a} (\rho \partial_t \Lambda + (\Omega N^i + \rho w^i) \partial_i \Lambda) \\ &= - \int dt d^d x \sqrt{a} \Omega \Lambda \left(\frac{1}{\Omega} \hat{\mathcal{D}} \rho + \theta^w \rho + \hat{\mathcal{V}}_i N^i \right) \\ &\quad + \int dt d^d x \{ \partial_t (\sqrt{a} \Lambda \rho) + \partial_i (\sqrt{a} \Lambda (\Omega N^i + \rho w^i)) \}, \end{aligned} \quad (3.49)$$

where the last integral in (3.49) are boundary terms. Condition $\delta_\Lambda S = 0$ leads to the Galilean continuity equation

$$\boxed{\left(\frac{1}{\Omega} \hat{\mathcal{D}} + \theta^w \right) \rho + \hat{\mathcal{V}}_i N^i = 0}, \quad (3.50)$$

where we have ignored the boundary terms in the variation of the action. The continuity equation can be alternatively presented in an integral form, using Stokes and Gauss theorem

as

$$\int_{\mathcal{W}} dt d^d x \Omega \sqrt{a} \left(\left(\frac{1}{\Omega} \hat{D} + \theta^w \right) \rho + \hat{\nabla}_i N^i \right) = \oint_{\partial \mathcal{W}} \sqrt{a} \rho (dx^1 - w^1 dt) \wedge \dots \wedge (dx^d - w^d dt) \\ - \oint_{\partial \mathcal{W}} \sqrt{a} \sum_{i=1}^d (dx^1 - w^1 dt) \wedge \dots \wedge N^i \mu \wedge \dots \wedge (dx^d - w^d dt) \quad (3.51)$$

where $\mathcal{W} \subset \mathcal{M} = \mathbb{R} \times \mathcal{S}$ and $N^i \mu$ is the i th factor in the exterior product of the last term. From this we obtain a conserved charge expressed as an integral over an arbitrary hypersurface Σ_d of $\mathcal{M} = \mathbb{R} \times \mathcal{S}$. It coincides with the relativistic Zermelo result captured e.g. in (2.98).

Although not compulsory, it is convenient to chose $\Sigma_d \equiv \mathcal{S}$ i.e. a constant- t hypersurface. We then find

$$Q_N = \int_{\mathcal{S}} d^d x \sqrt{a} \rho, \quad (3.52)$$

which fits the usual definition of charge in Galilean physics. Here, the conservation of the charges implies their independence with respect to the time t . Time-independence appears explicitly if one trades \mathcal{S} in the integral (3.52) with $\mathcal{V} \subset \mathcal{S}$. Assuming for simplicity that the boundary $\partial \mathcal{V}$ of that domain does not depend on t and using (3.50), the time evolution of the matter/charge content of \mathcal{V} is

$$\frac{1}{\Omega} \frac{d}{dt} \int_{\mathcal{V}} d^d x \sqrt{a} \rho = - \int_{\mathcal{V}} d^d x \partial_i \left(\sqrt{a} \left(N^i + \rho \frac{w^i}{\Omega} \right) \right) = - \int_{\partial \mathcal{V}} \star \left(\mathbf{N} + \rho \frac{\mathbf{W}}{\Omega} \right), \quad (3.53)$$

where \star stands for the d -dimensional \mathcal{S} -Hodge dual based on \sqrt{a} and on the antisymmetric symbol $\varepsilon_{i_1 \dots i_d}$ with $\varepsilon_{1 \dots d} = 1$. If the integral is performed over the entire \mathcal{S} it vanishes (assuming a reasonable asymptotic behaviour), and Q_N in (3.52) is conserved.

At this stage, (3.50) and the variables appearing there are independent of the energy and momentum equations given in (3.44) and (3.43). We will see later when discussing the limiting procedure that thermodynamics sets a relationship between the momentum P_i and the current N_i .

3.1.3 Isometries and (non-)conservation laws

In (pseudo-)Riemannian geometries, isometries are defined as diffeomorphisms generated by vector fields leaving the metric invariant, i.e. they satisfy (2.24). The same can be defined in Newton-Cartan geometries although in a different way due to the degeneracy of the co-metric. As mentioned previously, the fundamental objects that define a Newton-Cartan geometry are the co-metric and the clock form. Therefore, Galilean isometries are the diffeomorphisms that leave the co-metric and the clock form invariant. We call those vectors of the type (3.36)

as Galilean Killing fields and they are required to satisfy

$$\mathcal{L}_\xi a^{ij} = 0, \quad \mathcal{L}_\xi \mu = 0, \quad (3.54)$$

Using expressions (3.37) and (3.40) for Galilean diffeomorphisms ($\xi^{\hat{t}}$ is only t -dependent), we obtain the Galilean Killing equations

$$\hat{\nabla}^{(i} \xi^{\hat{j})} + \hat{\gamma}^{vij} \xi^{\hat{t}} = 0, \quad \frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{t}}}{dt} = 0. \quad (3.55)$$

The above equations are solved by an infinite number of solutions and they refer to the weak definition of Newton-Cartan geometries [23] defined only in terms of the degenerate co-metric and the clock form. A strong definition exists and it requires to define an affine connection that parallels transport the clock form and be metric-compatible. In that case, the conditions for having a Galilean isometry are supplemented by the invariance of the affine connection, reducing the set of solutions to a finite number [118].

Our Newton-Cartan structure is also accompanied with the field of observers (3.3) which is dual to the clock form. There latter is not required to be invariant and its transformation generated by the Galilean diffeomorphisms reads

$$\mathcal{L}_\xi \nu = -\frac{1}{\Omega} (\partial_t \xi^{\hat{t}} + \mathcal{L}_w \xi^{\hat{t}}) \partial_i, \quad (3.56)$$

for a generic Killing field ξ .

As an example, we can consider the Newton-Cartan manifold endowed with $a^{ij} = \delta^{ij}$, $\Omega = 1$ and constant w^i . The latter corresponds to the more familiar $\mathbb{R} \times \mathbb{E}_3$ spacetime, which is flat for the connection introduced earlier. Then, the Galilean Killing equations (3.55) are reduce to

$$\partial_t \xi^{\hat{t}} = 0 \quad (3.57)$$

and

$$\delta^{il} \partial_l \xi^{\hat{j}} + \delta^{jl} \partial_l \xi^{\hat{i}} = 0 \quad (3.58)$$

which are solved by an infinite number of solutions given by

$$\xi = \left(\Omega_i^j(t) x^i + Z^j(t) \right) \partial_j + T \partial_t \quad (3.59)$$

with T a constant and $\Omega_{ij} = \Omega_i^k \delta_{kj}$ antisymmetric. By imposing invariance of the affine connection (flat in this example) one fixes Ω_{ij} to be constant and generates the $\mathfrak{so}(d)$ rotations, while $Z^i = V^j t + X^j$ is the generator of the Galilean boosts and T the time translations. The

above form the $(d+2)(d+1)/2$ -dimensional Galilean algebra $\mathfrak{gal}(d+1)$ [118]. In this example, the transformation of the field of observers are found to be

$$\mathcal{L}_\xi \mathbf{v} = - \left(V^i + w^k \Omega_k^i \right) \partial_i \neq 0, \quad (3.60)$$

showing that the boosts produce a displacement in the field of observers. Here w^i describes the constant velocity of the original inertial frame which is shifted under Galilean boosts.

In the presence of isometries, one should be able to derive the conservation laws associated to them. This will take the Galilean form (3.50) with scalar κ and vector K^i . These pieces are built in terms of the momenta $\Pi, \Pi_i, \Pi_{ij}, P_i$ that satisfy (3.43) and (3.44), and the components of the Killing vector $\xi^{\hat{i}}$ and ξ^i . Their form are read off from the boundary terms in (3.42) from which we extract

$$\kappa = \xi^{\hat{i}} P_i - \xi^{\hat{i}} \Pi, \quad (3.61)$$

$$K_i = \xi^{\hat{j}} \Pi_{ij} - \xi^{\hat{i}} \Pi_i. \quad (3.62)$$

Then, the Galilean scalar

$$\mathcal{H} = \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w \right) \kappa + \hat{\nabla}_i K^i \quad (3.63)$$

should vanish on-shell making (3.63) a conservation equation for κ and K^i . Equations (3.62) and (3.63) also arise as the leading terms in the non-relativistic expansion of the relativistic Noether current $I^\mu = T^{\mu\nu} \xi_\nu$ and its conservation $\nabla_\mu I^\mu = 0$.

Using the conservation equations (3.43) and (3.44) we obtain the following result:

$$\mathcal{H} = -\frac{\Pi \hat{D} \xi^{\hat{i}}}{\Omega dt} + \frac{P_i}{\Omega} \left(\partial_t \xi^{\hat{i}} + \mathcal{L}_w \xi^{\hat{i}} \right) + \Pi_{ij} \left(\hat{\nabla}^i \xi^{\hat{j}} + \hat{\gamma}^{vij} \xi^{\hat{i}} \right) \quad (3.64)$$

$$= \frac{P_i}{\Omega} \left(\partial_t \xi^{\hat{i}} + \mathcal{L}_w \xi^{\hat{i}} \right). \quad (3.65)$$

The above equations shows that, contrary as one would expect, in Newton-Cartan space-times Galilean Killing vectors do not always define on-shell conservation laws for Galilean dynamics. This is due to the presence of P_i which is conjugate to w^i/Ω , and the fact that w^i transforms under diffeomorphisms according to (3.39), even if this diffeomorphism is generated by a Killing vector. Still, one can impose conservation by requiring additional constraints. For instance, for the Killing vector that satisfies

$$\mathcal{L}_\xi \mathbf{v} \equiv [\xi, \mathbf{v}] = 0, \quad (3.66)$$

which we call strong Galilean Killing vector, makes $\mathcal{K} = 0$. The generators of Galilean boosts in $\mathbb{R} \times \mathbb{E}_3$ do not satisfy this strong Killing condition.

One can also consider the cases when, in the presence of an isometry, the right-hand side of (3.65) is associated to a boundary term. In such case, κ and K^i can receive extra contributions from the boundary term and then an effective set of κ' and K'^i which are truly conserved can be constructed. Still, this does not seem to be true always. For example, in the flat space case we have

$$\mathcal{K} = P_i \left(V^i + w^k \Omega_k^i \right), \quad (3.67)$$

which is not conserved unless the momentum P_i is a potential flow (also called irrotational, see [58] §9). In that case P_i obeys

$$P_i = (\partial_t + w^j \partial_j) \phi_i + \partial_i \phi \quad (3.68)$$

for a set of functions $\phi(t, \mathbf{x})$ and $\phi_i(t, \mathbf{x})$. One can then construct an effective set $\kappa' = \kappa - \phi_i W^i$ and $K'^i = K^i - \phi W^i$ with $W^i = V^i + w^k \Omega_k^i$ such that they always satisfy the conservation equation $\mathcal{K}' = 0$. The motion of a free particle satisfies the above kind of conservation law since in that case the momentum P_i is a total derivative (typically denoted by \dot{x}). See App. E for more details. We will find a similar conclusion when studying the Carrollian case.

It is worth mentioning that when considering the non-relativistic expansion of the Noether current $I^\mu = T^{\mu\nu} \xi_\nu$ and the conservation law $\nabla_\mu I^\mu = 0$, we can have instances with multiple degrees of freedom appearing at each power in the expansion of k . In that case we have several currents associated to Killing fields, and some of them can be conserved without any additional constraint due to the accidental absence of P_i -like terms.

3.1.4 Weyl invariance, conformal isometries and (non-)conservation laws

We can consider fluids involving massless excitations for which their observables possess special scaling properties. We can introduce Weyl transformations acting as follow on the fundamental geometric data of a Newton–Cartan geometry:

$$a^{ij} \rightarrow \mathcal{B}^2 a^{ij}, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega, \quad w^i \rightarrow w^i, \quad w_i \rightarrow \frac{1}{\mathcal{B}^2} w_i. \quad (3.69)$$

Due to the space independence of Ω , the second term of (3.69) imposes $\mathcal{B} = \mathcal{B}(t)$. Requiring Weyl-invariance of the effective action S implies that the weights for the Galilean momenta in (3.32), (3.33), (3.34) are as follows: the energy–stress tensor Π_{ij} has weight $d - 1$, the

momentum P_i has weight d , and the energy density Π has weight $d + 1$. The energy flux Π_i introduced in (3.44) has also weight d . Furthermore, using (3.35) with $\delta_{\mathcal{B}}S = 0$ implies that

$$\Pi_i^i = \Pi. \quad (3.70)$$

On the matter sector, the gauge fields B and B_i are weight-zero, while ρ is weight- d and N_i , $d - 1$. Along the lines of [14, 23] a conformal isometry is generated by a vector field ξ which satisfies

$$\mathcal{L}_\xi a^{ij} = \lambda a^{ij}, \quad (3.71)$$

where

$$\lambda(t, \mathbf{x}) = -\frac{2}{d} \left(\hat{\nabla}_i \xi^i + \theta^w \xi^i \right) \quad (3.72)$$

is obtained by taking the trace of (3.71). This set of partial differential equations are insufficient for defining conformal Killing vectors. We require also the transformation of the clock form is proportional to itself as given in (3.40). It is then natural to tune μ and λ so that the scaling of the metric be twice the scaling of the clock form⁴ as

$$2\mu + \lambda = 0. \quad (3.73)$$

One can infer this scaling from the relativistic ascendant in Zermelo coordinates (2.66). Requiring $\mathcal{L}_\xi g_{\mu\nu} = -\lambda g_{\mu\nu}$ requires to set $\mu = -\lambda/2$. This is a consistent Weyl-covariant condition, leading to a reasonable set of conformal Killing fields.

Assuming Weyl invariance i.e. (3.70), and the presence of a conformal Killing field, the conservation equations (3.43) and (3.44) can be used to compute \mathcal{K} defined in (3.63), (3.61) and (3.62), giving rise to

$$\mathcal{K} = -\Pi \left(\frac{\lambda}{2} + \mu \right) + \frac{P_i}{\Omega} (\partial_t \xi^i + \mathcal{L}_w \xi^i). \quad (3.74)$$

The extra condition (3.73) which defines the conformal Killing vectors of Newton–Cartan spacetimes implies that

$$\mathcal{K} = \frac{P_i}{\Omega} (\partial_t \xi^i + \mathcal{L}_w \xi^i) \neq 0 \quad (3.75)$$

In the same way as the ordinary Killing fields, a conformal Killing vector does not guarantee a conservation law for Weyl-invariant Galilean dynamics.

⁴More generally, one considers $2\mu + z\lambda = 0$, where z is the dynamical exponent i.e. minus the conformal weight of Ω . Here, due to the close relationship of our Newton–Cartan spacetimes with relativistic ascendants, the weight of Ω is inherited from the latter and $z = 1$. One also defines the level $N = 2/z$, which appears in the conformal algebras emerging in flat Newton–Cartan spacetimes.

3.2 Galilean hydrodynamics as a non-relativistic limit

3.2.1 Non-relativistic expansion

In the previous section, we found the equations (3.43), (3.44) and (3.50) that describe the conservation properties of Galilean dynamics on a general curved and time-dependent space \mathcal{S} (the spatial section of a torsionless Newton–Cartan spacetime $\mathbb{R} \times \mathcal{S}$). These equations are valid for any theory whose action is invariant under Galilean diffeomorphisms, in the same way as in a relativistic theory, diffeomorphism invariance implies the conservation of the energy-momentum tensor and $U(1)$ invariance implies the conservation of the matter/charge current.

In fluid dynamics we need more information. For instance, their set of momenta must be expressed in terms of the velocity field \mathbf{v} , the heat current \mathbf{Q} , the stress tensor $\mathbf{\Sigma}$, and the local-equilibrium thermodynamic variables such as e , p , h , ρ , μ , T and s , that obey further thermodynamic laws (see App. A). The latter ultimately enters in the constitutive relations for the non-perfect contributions. A systematic approach to obtain the Galilean fluid equations is based on the large- k expansion of relativistic hydrodynamics in the Zermelo frame, which is the natural gauge where Galilean covariance is explicit. This method was first introduced in [44] where the authors assumed implicitly the Eckart frame without taking into account the role of the matter current and the chemical potential. This section is meant to give a generalization of this approach in the presence of a matter current together with a chemical potential. The latter will allow the analysis on the non-relativistic hydrodynamic-frame transformations, which relies in the behaviour of the various observables with respect to their scaling with the velocity of light.

In the Zermelo frame (2.66), the dependence with respect to the velocity of light in the geometric data is explicit. Hence, we can find the behaviour of all our kinematical quantities with respect to the velocity of light. In particular, for the velocity of the fluid we have

$$u^0 = \frac{k}{\Omega} + \mathcal{O}\left(\frac{1}{k}\right), \quad u_i = \frac{v_i - w_i}{\Omega} + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.76)$$

and

$$\sigma_{ij} = \xi_{ij}^v + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.77)$$

$$\Theta = \theta^v + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.78)$$

$$\omega_{ij} = \frac{1}{\Omega} (\partial_{[i}(v - w)_{j]}) + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.79)$$

are obtained from the non-relativistic expansion of (2.8), (2.9) and (2.10).

The non-relativistic limit of the thermodynamic variables are recalled in Appendix A. Few comments regarding the mass/charge density are in order here. In the introduction we defined the proper density ρ_0 as the projection of the current (2.4) onto the observer's velocity. For instance, an observer at rest with respect to the fluid, i.e. running with velocity \mathbf{u} , finds

$$\rho_0 = -\frac{1}{k^2} J_\mu u^\mu. \quad (3.80)$$

One can also define the density ρ which corresponds to the one that is measured by a fiducial observer in Zermelo frame. This observer defines a local inertial frame⁵ and its velocity in Zermelo frame is given by

$$\mathbf{u}_Z \equiv \mathbf{v} = \frac{1}{\Omega} (\partial_t + w^i \partial_i) \quad (3.81)$$

with vanishing acceleration

$$u_Z^\mu \nabla_\mu u_Z^\nu = 0. \quad (3.82)$$

Then, we obtain ρ as the projection of the current (2.4) onto the velocity (3.81). We find

$$\rho = -\frac{1}{k^2} J_\mu u_Z^\mu = \frac{\Omega}{k} J^0 = \Omega \gamma \rho_0 + j_i \frac{v^i - w^i}{k^2 \Omega}, \quad (3.83)$$

which coincides with the hydrodynamic-frame invariant ρ_{0r} introduced in (2.89) (matter/charge density in the rest frame). This expression agrees with Ref. [44] only in the Eckart frame, i.e. when $j_i = 0$.

We notice that the behaviour of ρ_0 in terms of ρ will depend on the behaviour of the non-perfect piece j_i . But, what could be the behaviour of the non-perfect current j_i , the heat current q_i and the stress tensor τ_{ij} with respect to k ? There is no a unique answer since it depends on the microscopic properties of the system, which are captured by the different transport coefficients appearing in the constitutive relations for j_i , q_i and τ_{ij} as derivative expansions. For instance, in first order hydrodynamics one has

$$\tau_{(1)ij} = -2\eta \sigma_{ij} - \zeta h_{ij} \Theta, \quad (3.84)$$

$$q_{(1)i} = -\kappa h_i^\nu \left(\partial_\nu T + \frac{T}{k^2} a_\nu \right), \quad (3.85)$$

$$j_{(1)i} = -h_i^\nu \sigma_T T \partial_\nu \frac{\mu_0}{T}, \quad (3.86)$$

⁵This is precisely why it was stated earlier that a frame with velocity $\mathbf{w} = w^i \partial_i$ was inertial in the Newton–Cartan geometry (3.3), (3.4).

where η , ζ and κ are the shear viscosity, bulk viscosity and heat conductivity respectively with $\kappa = \frac{\sigma_T w^2}{T \rho_0^2}$ (see e.g. [65], where σ_T is referred to as the charge conductivity).

In order to perform the non-relativistic limit of fluid dynamics we must make some assumptions regarding the scaling of the above quantities with respect to k . We can take the reasonable assumption that η , ζ and κ are of dominant order 1, which implies that σ_T is of order $1/k^4$ (due to w^2). Since μ_0 is of order k^2 (rest-mass contribution in (A.4)) one can conclude that τ_{ij} and q_i are of order 1, while j_i is of order $1/k^2$. These are the usual assumptions taken in the literature (see for instance [58, 102]). One can be a little bit more general and assume the behaviour of the non-perfect pieces as

$$\tau_{ij} = -\Sigma_{ij} + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.87)$$

$$q_i = k^2 r_i + k_i + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.88)$$

$$j_i = n_i + \frac{m_i}{k^2} + \mathcal{O}\left(\frac{1}{k^4}\right). \quad (3.89)$$

This choice will be made clear when discussing the hydrodynamic-frame invariance. For ordinary non-relativistic fluids, r_i and n_i are expected to vanish. We will see that their presence will disclose some interesting properties.

Inserting the expression (3.89) in (3.83), one finds⁶

$$\rho_0 = \rho - \frac{1}{k^2} \left(\frac{\rho}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 + \frac{\mathbf{n} \cdot (\mathbf{v} - \mathbf{w})}{\Omega} \right) + \mathcal{O}\left(\frac{1}{k^4}\right). \quad (3.90)$$

The latter can in turn be used inside (A.2) leading to

$$\varepsilon = k^2 \rho + \rho \left(e - \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \right) - \frac{\mathbf{n} \cdot (\mathbf{v} - \mathbf{w})}{\Omega} + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.91)$$

where the first term is the rest energy, the second is the internal energy corrected by the kinetic energy with respect to the local inertial frame, and the third is a contribution originating from the leading term in the matter current (3.89). One can notice from the above formulas that the presence of n_i amounts to the presence of a spring or a sink that create or consume matter. This will be confirmed when reaching the conservation equation. This situation is usually not considered except when discussing diffusion or similar phenomena (see for instance [58]).

⁶Although $\lim_{k \rightarrow \infty} \Omega \gamma = 1$, we must keep terms of order $1/k^2$ because of the rest mass contributions.

3.2.2 Galilean momenta

In the non-relativistic limit approach, all the momenta defined in (3.32), (3.33), (3.34), (3.46) and (3.47) will appear at different orders in the large- k expansion of the energy-momentum tensor and the matter current (2.83) and (2.85) in Zermelo frame, all of them expressed in terms of the fluid and thermodynamic variables. The latter includes the expression for the energy flux Π_i , which we found as an arbitrary function in (3.44) of the previous section.

Using Eqs. (2.86), (2.87), (2.88), (2.89), (2.90) and (3.83), we obtain

$$\rho_{0r} = \rho, \quad (3.92)$$

$$j_{ri} = N_i + \frac{1}{k^2} p_i + \mathcal{O}\left(\frac{1}{k^4}\right), \quad (3.93)$$

where we introduced the leading and subleading matter currents

$$N^i = \rho \frac{v^i - w^i}{\Omega} + n^i, \quad (3.94)$$

$$p_i = m_i - \frac{\mathbf{n} \cdot (\mathbf{v} - \mathbf{w})(v_i - w_i)}{\Omega^2}. \quad (3.95)$$

The subleading terms in (3.93) must be kept because they are multiplied in the expansions by the rest-mass term and contribute to the equations. As an anticipation for what is next, we can set

$$P^i = \rho \frac{v^i - w^i}{\Omega} + r^i. \quad (3.96)$$

One can recognize P_i (defined generically in (3.33) – indices raised with a^{ij}) as a slight extension of the usual fluid momentum, while the matter current N^i (introduced in (3.47)) is related to the former as

$$N^i = P^i + n^i - r^i. \quad (3.97)$$

In the standard non-relativistic hydrodynamics, the equality $N^i = P^i$ holds when $n^i = r^i$.

Similarly for the energy–momentum we have

$$\varepsilon_r = k^2 \rho + \Pi + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.98)$$

$$q_{ri} = k^2 P_i + \Pi_i + p_i + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.99)$$

$$p_r a_{ij} + \tau_{rij} = \Pi_{ij} + \mathcal{O}\left(\frac{1}{k^2}\right) \quad (3.100)$$

with

$$\Pi_{ij} = \rho \frac{(v_i - w_i)(v_j - w_j)}{\Omega^2} + p a_{ij} - \Sigma_{ij} + 2 \frac{(v_{(i} - w_{(i)} r_{j)})}{\Omega}, \quad (3.101)$$

$$\Pi = \rho \left(e + \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \right) + \frac{(2\mathbf{r} - \mathbf{n}) \cdot (\mathbf{v} - \mathbf{w})}{\Omega}, \quad (3.102)$$

the explicit expressions for (3.32) and (3.34), and

$$\begin{aligned} \Pi_i = & \rho \frac{v_i - w_i}{\Omega} \left(h + \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \right) - \frac{v^j - w^j}{\Omega} \Sigma_{ij} \\ & + \frac{r_i}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 + \frac{\mathbf{r} \cdot (\mathbf{v} - \mathbf{w})(v_i - w_i)}{\Omega^2} + k_i - m_i \end{aligned} \quad (3.103)$$

the explicit expression for the energy flux. The latter are the generalizations of the fluid energy-stress tensor, energy density and energy current defined in [58] for arbitrary torsionless Newton-Cartan geometries expressed in a covariant fashion. They all receive contributions from the n_i and r_i , absent in standard Galilean fluids. The combination

$$Q_i = k_i - m_i, \quad (3.104)$$

inside the energy current, appears as the Galilean heat current. The latter receives contributions from both the relativistic heat current q_i (3.88) and the non-perfect piece of the matter current j_i (3.89). For example, in the Landau–Lifshitz frame ($q_i = 0$) the Galilean heat current originates only from the relativistic non-perfect matter current, while in the Eckart frame ($j_i = 0$) the contribution to the Galilean heat current comes exclusively from q_i . One can also notice that, in the absence of a matter current, the Landau–Lifshitz frame is not well suited to take the Galilean limit since one will always arrive at a fluid with $Q_i = 0$. In that case the Eckart frame is the preferred choice [102].

3.2.3 Hydrodynamic equations

We can now perform the large- k expansion of the relativistic fluid equations (2.1) and (2.2). They read

$$k\Omega\nabla_\mu T^{\mu 0} = k^2 \left(\mathcal{E} + \hat{\nabla}_i (r^i - n^i) \right) + \hat{\nabla}_i p^i + \mathcal{E} + \mathcal{O} \left(\frac{1}{k^2} \right), \quad (3.105)$$

$$\nabla_\mu T^\mu_i = \mathcal{M}_i + \mathcal{O} \left(\frac{1}{k^2} \right), \quad (3.106)$$

$$\nabla_\mu J^\mu = \mathcal{C} + \frac{1}{k^2} \hat{\nabla}_i p^i + \mathcal{O} \left(\frac{1}{k^4} \right), \quad (3.107)$$

with

$$\mathcal{E} = \frac{1}{\Omega} \frac{\hat{\Delta}\Pi}{dt} + \theta^w \Pi + \Pi_{ij} \hat{\gamma}^{wj} + \hat{\nabla}_i \Pi^i, \quad (3.108)$$

$$\mathcal{M}_i = \frac{1}{\Omega} \frac{\hat{\Delta}P_i}{dt} + \theta^w P_i + P_j \hat{\gamma}^{wj}_i + \hat{\nabla}^j \Pi_{ij}, \quad (3.109)$$

$$\mathcal{C} = \frac{1}{\Omega} \frac{\hat{\Delta}\rho}{dt} + \theta^w \rho + \hat{\nabla}_i N^i. \quad (3.110)$$

Therefore, the Galilean fluid equations are extracted by the vanishing of (3.105), (3.106) and (3.107) at each order in the powers of k . For instance, from the order 1 of (3.107) we find the continuity equation⁷

$$\boxed{\mathcal{C} = 0}, \quad (3.111)$$

while the $1/k^2$ order gives a divergence free condition in the subleading matter current (3.95)

$$\boxed{\hat{\nabla}_i p^i = 0}. \quad (3.112)$$

From the k^2 order in (3.105) we obtain

$$\hat{\nabla}_i (r^i - n^i) = 0 \quad (3.113)$$

which, by using (3.97), can be rewritten as

$$\boxed{\hat{\nabla}_i (N^i - P^i) = 0}. \quad (3.114)$$

⁷This is the typical equation describing phenomena where several fluid components are present but are not separately conserved. Examples are diffusion or superfluid dynamics (e.g. [58]).

At the order 1 of (3.105) we find

$$\boxed{\mathcal{E} = 0}, \quad (3.115)$$

which is the fluid energy equation. Finally, (3.106) provides the fluid momentum equation

$$\boxed{\mathcal{M}_i = 0}. \quad (3.116)$$

We can also use (3.97) and (3.114) to recast the continuity equation as

$$\mathcal{C} = \frac{1}{\Omega} \frac{\hat{D}\rho}{dt} + \theta^w \rho + \hat{\nabla}_i p^i = \frac{1}{\Omega} \frac{d\rho}{dt} + \theta^v \rho + \hat{\nabla}_i r^i = \frac{1}{\Omega} \frac{d\rho}{dt} + \theta^v \rho + \hat{\nabla}_i n^i = 0. \quad (3.117)$$

The above results can be summarized as follows: the Galilean fluid equations (valid for arbitrary backgrounds) found above are fully covariant under Galilean diffeomorphisms and extend the standard hydrodynamic equations on flat Euclidean space with absolute time. The momentum equation (3.116) coincides with (3.43), whereas the energy equation (3.115) is (3.44). In the same way, the continuity equation (3.117) coincides with (3.50), once the divergences of the matter current and the fluid momentum are equal (3.114). This requirement is subsequent to the relationship (3.91), which finally relates the energy–momentum equations with the matter equation.

In general, (3.117) is not a conservation law. Integrating inside a static domain \mathcal{V} by using (3.53) we find

$$\frac{1}{\Omega} \frac{d}{dt} \int_{\mathcal{V}} d^d x \sqrt{a} \rho + \int_{\partial\mathcal{V}} \rho \frac{\star\mathbf{v}}{\Omega} = - \int_{\partial\mathcal{V}} \star\mathbf{n}. \quad (3.118)$$

Here, the density varies in time, not only because of the expansion or the contraction of \mathcal{V} captured by the term $\rho\theta^v$, but also due to the flux of \mathbf{n} through $\partial\mathcal{V}$. \mathbf{n} represents the flux of matter due to the presence of a sink or a spring. This matter loss or gain goes along with the heat loss or gain, as inferred in (3.113). In usual Galilean hydrodynamics the current r^i and n^i are absent, making the fluid momentum and the matter current identical and forbidding the presence of such springs and sinks. We will see below that this general Galilean fluid configurations possess an interesting property, which is broken in ordinary non-relativistic fluid dynamics: this is Galilean hydrodynamic-frame invariance.

3.2.4 The fate of hydrodynamic-frame invariance in the non-relativistic limit

As mentioned in the introduction, relativistic fluid equations are invariant under arbitrary local transformations of the velocity u , which is captured by⁸ $v^i \rightarrow v^i + \delta v^i(t, \mathbf{x})$, together with the appropriate transformations of all the other dynamical quantities such that the energy-momentum tensor and the matter current remain invariant. One could wonder if this property survives in the Galilean limit.

The natural answer to this question would be negative. The velocity field and the matter density are physical and observable quantities. A field redefinition of the velocity and the dynamical variables that keeps the fluid equations invariant without altering the meaning of the fields seems difficult.

This answer seems contradictory with the approach considered so far. After all the Galilean momenta that enters in the hydrodynamic equations comes from a relativistic hydrodynamic-frame invariant set of momenta. Naively, one would expect the Galilean momenta being also invariant under hydrodynamic-frame transformations but one has to be careful since those momenta are obtained after assuming some behavior with respect to k , and this behaviour may or may not be stable under velocity transformations.

In order to give a precise answer to the above question, we examine the transformations (2.77), (2.78), (2.79), (2.81) and (2.82) in the infinite- k limit, and their effect on the non-relativistic quantities introduced through (3.87), (3.88), (3.89), (3.90) and (3.91). The latter give rise to the following transformation rules of the fluid variables under the action of

⁸The infinitesimal local Lorentz transformations are parameterized with Lorentz boost and rotation generators, $V^i(t, \mathbf{x})$ and $\Omega^{ij}(t, \mathbf{x})$ – antisymmetric, as follows: $\delta v^i = V^i - V^j \frac{(v_j - w_j)(v^i - w^i)}{k^2 \Omega^2} + \Omega^{ij}(v_j - w_j)$. In the Galilean limit the general local velocity transformation is thus $\delta v^i = V^i + \Omega^{ij}(v_j - w_j)$ – Galilean boosts and rotations.

general local Galilean transformations:⁹

$$a_{ij}\delta p - \delta\Sigma_{ij} = -\frac{2}{\Omega}r_{(i}\delta v_{j)}, \quad (3.119)$$

$$\delta r_i = -\frac{\rho}{\Omega}\delta v_i, \quad (3.120)$$

$$\delta k_i = \frac{\delta v_i (\mathbf{v} - \mathbf{w}) \cdot \mathbf{n}}{\Omega} + \frac{\delta v^j}{\Omega} \left(\frac{v_i - w_i}{\Omega} r_j - \rho \frac{(v_i - w_i)(v_j - w_j)}{\Omega^2} - \rho h a_{ij} + \Sigma_{ij} \right), \quad (3.121)$$

$$\delta Q_i = \frac{\delta v^j}{\Omega} \left((r_j - n_j) \frac{v_i - w_i}{\Omega} - \rho h a_{ij} + \Sigma_{ij} \right), \quad (3.122)$$

$$\rho \delta e = (n_i - 2r_i) \frac{\delta v^i}{\Omega}, \quad (3.123)$$

$$\delta \rho = 0, \quad (3.124)$$

$$\delta n_i = -\frac{\rho}{\Omega}\delta v_i, \quad (3.125)$$

$$\delta m_i = \frac{\delta v^j}{\Omega} \left(n_j - \rho \frac{v_j - w_j}{\Omega} \right) \frac{v_i - w_i}{\Omega} + \frac{\delta v_i (\mathbf{v} - \mathbf{w}) \cdot \mathbf{n}}{\Omega}. \quad (3.126)$$

and thus

$$\delta (n^i - r^i) = 0. \quad (3.127)$$

The above transformations implies the invariance of the fundamental Galilean momenta, namely the fluid energy density, the fluid energy current, the fluid energy–stress tensor, the fluid momentum, the matter density and the matter current, that satisfy

$$\boxed{\delta\Pi = 0, \quad \delta\Pi^i = 0, \quad \delta\Pi^{ij} = 0, \quad \delta P^i = 0, \quad \delta\rho = 0, \quad \delta N^i = 0.} \quad (3.128)$$

The above results imply that the general Galilean fluid equations are invariant under an arbitrary local redefinition of the fluid velocity field $v^i(t, \mathbf{x})$.

Some comments are in order here. First, the non-relativistic matter density ρ is not sensitive to the velocity \mathbf{v} (as opposed to ρ_0). This is not the case for the ordinary fluid momentum P_i which depends explicitly on the velocity as $\rho \frac{v^i - w^i}{\Omega}$. In our configuration, the invariance of the momentum P_i defined in (3.96) is saved due to the presence of n^i . The same goes for the energy density Π , the energy current Π_i and the stress-energy tensor Π_{ij} whose invariance under local transformations of v^i are guaranteed thanks to the presence of n^i and r^i . This configuration is the one whose continuity equation does not express a truly conservation

⁹Notice the following useful formulas: $\delta\theta^v = \frac{1}{\Omega}\hat{\nabla}_i\delta v^i$ and $\delta\xi_{ij}^v = \frac{1}{\Omega}\left(\hat{\nabla}_{(i}\delta v_{j)} - \frac{1}{d}a_{ij}\hat{\nabla}_k\delta v^k\right)$, whereas $\delta\theta^w = \delta\xi_{ij}^w = 0$.

law due to the fluxes n^i and r^i . Hence, for truly conserved ordinary non-relativistic fluids the hydrodynamic-frame invariance is lost. For these fluids the behaviours of the relativistic heat and matter currents are $q_l = k_l + \mathcal{O}(1/k^2)$ and $j_l = \frac{m_l}{k^2} + \mathcal{O}(1/k^4)$. These behaviours are physical, but are unstable under velocity transformations.¹⁰

The summary of our findings goes as follows:

- As expected intuitively, ordinary non-relativistic hydrodynamics which enjoys conservation of matter are not hydrodynamic-frame invariant.
- Hydrodynamic-frame invariance is restored once we include additional matter and energy fluxes (due to a sink or spring), losing conservation of matter.

3.2.5 More general abstract equations

The large- k behaviours (3.92), (3.93), (3.98), (3.99) and (3.100) are the result of the physics behind these momenta, which are captured in the behaviours of the transport coefficients and reflected in the behaviour of (3.87), (3.88) and (3.89). We could even be more abstract and consider order- k^2 terms in the stress tensor τ_{ij} as we did in the heat and matter currents, and possibly further powers. In order to consider such situation we will forget for the moment the matter current. In that case the expansion of the components of the energy-momentum tensor in the rest-frame read

$$\begin{cases} \Omega^2 T^{00} = \varepsilon_r = k^2 \rho + \Pi + \mathcal{O}\left(\frac{1}{k^2}\right) \\ k\Omega T^0_i = q_{ri} = k^4 \tilde{P}_i + k^2 P_i + \Pi_i + \mathcal{O}\left(\frac{1}{k^2}\right) \\ T_{ij} = p_r a_{ij} + \tau_{rij} = k^2 \tilde{\Pi}_{ij} + \Pi_{ij} + \mathcal{O}\left(\frac{1}{k^2}\right). \end{cases} \quad (3.129)$$

¹⁰The requirements $r^i = 0$ or $n^i = 0$ are not compatible with the transformations (3.120) or (3.125). Observe however that a choice, stable under hydrodynamic-frame transformations, is $n_i = r_i$, thanks to (3.127). With this, $P^i = N^i$ and we are the closest possible to ordinary non-relativistic fluids, without genuine conservation though.

The latter produce a hierarchy of equations which are of the same sort of the ones that we already derived. We found

$$\begin{cases} \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w\right) \Pi + \Pi_{ij} \hat{\gamma}^{wij} + \hat{\nabla}_i \Pi^i = 0 \\ \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w\right) \rho + \tilde{\Pi}_{ij} \hat{\gamma}^{wij} + \hat{\nabla}_i P^i = 0 \\ \hat{\nabla}_j \tilde{P}^j = 0 \\ \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w\right) P_i + P_j \hat{\gamma}^{wj}_i + \hat{\nabla}^j \Pi_{ij} = 0 \\ \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w\right) \tilde{P}_i + \tilde{P}_j \hat{\gamma}^{wj}_i + \hat{\nabla}^j \tilde{\Pi}_{ij} = 0. \end{cases} \quad (3.130)$$

What we have here is a multiplication of degrees of freedom which comes naturally due to the existence of a parameter k that make it possible to organize a power expansion. In the analysis of the local symmetries of the action, the derivation of the above set of equations would be even more abstract since those extra momenta would need the inclusion of additional conjugate variables, such as \tilde{a}^{ij} , \tilde{w}^i , $\tilde{\Omega}$ etc. We will not go any further in this regard but we will keep the above structure of the equations in mind when we discuss the Carrollian case. For the latter, and as opposed to the Galilean situation, there is no physical intuition that we can use as a guideline (not a well understood thermodynamics). Only a small- k expansion applies, as suggested by the only known application of Carrollian fluids, which is flat holography [75–77, 42]. Then, the Carrollian equations that we can find will be express in a similar fashion as (3.130).

Multiplication of degrees of freedom occurs also in the matter sector. One could indeed assume that some matter current behaves like

$$\frac{\Omega}{k} J^0 = \iota_{0r} = k^2 \tilde{\kappa} + \kappa + \mathcal{O}\left(\frac{1}{k^2}\right), \quad I_i = i_{rk} = k^4 \tilde{K}_i + k^2 \tilde{K}_i + K_i + \mathcal{O}\left(\frac{1}{k^2}\right). \quad (3.131)$$

Using these expansions in the relativistic divergence of the matter current J^μ in Zermelo background we find

$$\nabla_\mu I^\mu = k^4 \tilde{\mathcal{H}} + k^2 \tilde{\mathcal{K}} + \mathcal{H} + \mathcal{O}\left(\frac{1}{k^2}\right) \quad (3.132)$$

with

$$\begin{cases} \tilde{\mathcal{H}} = \hat{\nabla}_j \tilde{K}^j \\ \tilde{\mathcal{K}} = \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w\right) \tilde{\kappa} + \hat{\nabla}_j \tilde{K}^j \\ \mathcal{H} = \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w\right) \kappa + \hat{\nabla}_j K^j, \end{cases} \quad (3.133)$$

which must vanish if $\nabla_\mu I^\mu = 0$.

The latter can also be used when dealing with the components of a relativistic conserved current (2.96) and (2.97) that is associated to isometries, where the components of the current are constructed in terms of the contraction of the energy–momentum tensor with the Killing field. We find

$$\begin{cases} \kappa = \xi^i P_i - \xi^i \Pi \\ \tilde{\kappa} = \xi^i \tilde{P}_i - \xi^i \rho \\ K_i = \xi^j \Pi_{ij} - \xi^i \Pi_i \\ \tilde{K}_i = \xi^j \tilde{\Pi}_{ij} - \xi^i P_i \\ \tilde{\tilde{K}}_i = -\xi^i \tilde{P}_i, \end{cases} \quad (3.134)$$

where κ and K_i are precisely as anticipated in (3.61) and (3.62). On-shell, namely assuming (3.130) and using (3.55), we find for (3.133)

$$\begin{cases} \tilde{\tilde{\mathcal{K}}} = 0 \\ \tilde{\mathcal{K}} = \frac{\tilde{P}_i}{\Omega} (\partial_i \xi^i + \mathcal{L}_{\mathbf{w}} \xi^i) \\ \mathcal{K} = \frac{P_i}{\Omega} (\partial_i \xi^i + \mathcal{L}_{\mathbf{w}} \xi^i), \end{cases} \quad (3.135)$$

in agreement with the result (3.65) for the last two. We can see that the first equation in (3.135) vanishes indicating that even if a Killing field does not guarantee the conservation of a Galilean current, a conservation might happen if the appropriate vector vanishes. The third equation for \mathcal{K} is the one that we already studied, which is non-conserved in general. The equation for \tilde{K} is conserved for the Galilean fluids studied previously ($\tilde{P}_i = 0$) but it does not bring any new conservation law. It is actually the continuity equation.

One might wonder how it is possible for the well defined conservation of the Noether current $\nabla_\mu I^\mu = 0$, with $I^\mu = \xi_\nu T^{\mu\nu}$ and ξ a Killing vector, break down in the infinite- k limit. The answer to that question is captured by the very definition of a Galilean Killing, which ultimately leaves non-vanishing terms in the divergence. The precise way this comes about is exposed in App. F.

3.3 Massless carriers and Weyl covariance

3.3.1 The fluid equations

We consider can now a fluid that is made of a collection of massless particles. These sort of systems could consider for example a gas of photons or some situations in condensed matter, as in graphene [119]. These hydrodynamic systems can also have a conserved current

and a chemical potential, but they will not be related with mass, but to some charge. Some examples of this sort are mentioned in App. A together with their basic thermodynamic properties.

The procedure to obtain their Galilean dynamics starting from relativistic hydrodynamics goes in the same way as we did previously, where the difference relies on the behaviour of the charge density and the energy density. Here, instead of (3.90), we can simply consider

$$\rho_0 = \rho + \mathcal{O}\left(\frac{1}{k^2}\right), \quad \varepsilon = \rho e + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.136)$$

where e is the energy per charge unit and ρ is the charge volume density – as opposed to proper volume – and $\varepsilon = \rho e$ the non-relativistic energy density (see also App. A). Since there is no matter density we do not have terms like the rest mass and kinetic energy contributions to ε . The goal is to find the fundamental variables as well as the dynamical equations, and probe the behaviour of the latter under Galilean hydrodynamic-frame transformations.

We can extract the large- k behaviour of the energy-momentum and the conserved current in the rest frame. Equations (2.86), (2.87), (2.88), (2.89) and (2.90), now give

$$\rho_{0r} = \rho, \quad j_{ri} = N_i + \mathcal{O}\left(\frac{1}{k^2}\right) \quad (3.137)$$

with Galilean charge current

$$N^i = \rho \frac{v^i - w^i}{\Omega} + n^i. \quad (3.138)$$

From the energy-momentum tensor we get

$$\varepsilon_r = \Pi + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.139)$$

$$q_{ri} = k^2 P_i + \Pi_i + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.140)$$

$$p_r a_{ij} + \tau_{rij} = \Pi_{ij} + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.141)$$

where

$$\Pi_{ij} = pa_{ij} - \Sigma_{ij} + 2 \frac{(v_{(i} - w_{(i)} r_{j)})}{\Omega}, \quad (3.142)$$

$$P^i = r^i, \quad (3.143)$$

$$\Pi = \varepsilon + \frac{2\mathbf{r} \cdot (\mathbf{v} - \mathbf{w})}{\Omega}, \quad (3.144)$$

$$\begin{aligned} \Pi_i &= ((\varepsilon + p)a_{ij} - \Sigma_{ij}) \frac{v^j - w^j}{\Omega} + \\ &+ \frac{r_i}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 + \frac{\mathbf{r} \cdot (\mathbf{v} - \mathbf{w})(v_i - w_i)}{\Omega^2} + k_i, \end{aligned} \quad (3.145)$$

are the explicit expressions for (3.32), (3.33) and (3.34), as well as for the energy current Π_i , which will appear in the energy equation (3.44). The Galilean heat current receives now a single contribution as

$$Q_l = k_l. \quad (3.146)$$

For the fluid under consideration, the structure of the conservation equations goes as

$$k\Omega \nabla_\mu T^{\mu 0} = k^2 \hat{\nabla}_j r^j + \mathcal{E} + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.147)$$

$$\nabla_\mu T^\mu_i = \mathcal{M}_i + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.148)$$

$$\nabla_\mu J^\mu = \mathcal{C} + \mathcal{O}\left(\frac{1}{k^2}\right), \quad (3.149)$$

with \mathcal{E} , \mathcal{M}_i and \mathcal{C} as in (3.108), (3.109), (3.110). At infinite k the hydrodynamic equations are again (3.115), (3.116), (3.111), and we recover Eqs. (3.43), (3.44) and (3.50), as expected, plus the extra equation

$$\hat{\nabla}_j r^j = 0, \quad (3.150)$$

which is absent when the vector r^j originating from the k^2 term of the relativistic heat current vanishes. The difference with respect to the massive case studied previously relies in the expression of the momenta, namely the energy–stress tensor, fluid current, fluid energy density and fluid energy current (the charge current is the same as the matter current before).

We can now combine the above results in order to reach the heat and next the entropy equations. Equivalently these are obtained as infinite- k limits of Eqs. (2.15) and (2.16). We

find

$$\begin{aligned} \mathcal{E} - \frac{v_i - w_i}{\Omega} \mathcal{M}_i &= \frac{1}{\Omega} \frac{d}{dt} \left(\varepsilon + \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \left(\varepsilon + p + \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} - \frac{\Sigma}{d} \right) \theta^v - \xi_{ij}^v \Sigma^{ij} \\ &+ \hat{\nabla}_i \left(Q^i - \frac{v^i - w^i}{\Omega} \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \frac{r^i}{\Omega} \frac{D}{dt} \frac{v_i - w_i}{\Omega} \\ &= 0. \end{aligned} \quad (3.151)$$

For the entropy equation there are two options. If no conserved charge current exists, the equation (3.111) is immaterial, the chemical potential vanishes and (A.19) gives $d\varepsilon = Td\sigma$, which can be substituted in (3.151). This happens e.g. for a gas of photons. If a conserved charge current is available then ε can be traded for ρe , $\varepsilon + p$ for ρh , σ for ρs , and using (A.18), (A.19) and (3.111) we obtain

$$\frac{1}{\Omega} \frac{d\rho e}{dt} = \frac{\rho T}{\Omega} \frac{ds}{dt} - \rho h \theta^v - h \hat{\nabla}_i n^i, \quad (3.152)$$

which can be inserted back in (3.151) to get

$$\begin{aligned} \frac{\rho T}{\Omega} \frac{ds}{dt} + \frac{1}{\Omega} \frac{d}{dt} \left(\frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \left(\frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} - \frac{\Sigma}{d} \right) \theta^v - \xi_{ij}^v \Sigma^{ij} - h \hat{\nabla}_i n^i \\ + \hat{\nabla}_i \left(Q^i - \frac{v^i - w^i}{\Omega} \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \frac{r^i}{\Omega} \frac{D}{dt} \frac{v_i - w_i}{\Omega} = 0. \end{aligned} \quad (3.153)$$

Given the above Galilean hydrodynamic equations, one may reconsider their behaviour under velocity local transformations. The absence of rest mass modifies the scalings with respect to the speed of light, and possibly the invariance properties. Bringing together the transformations (2.77), (2.78), (2.79), (2.81) and (2.82), and the scalings (3.87), (3.88), (3.89) and (3.136), we find in the infinite- k limit that: $\delta\rho$ and δn_i are still as in (3.124) and (3.125), while

$$a_{ij} \delta p - \delta \Sigma_{ij} = -\frac{2}{\Omega} r_{(i} \delta v_{j)}, \quad (3.154)$$

$$\delta r_i = 0, \quad (3.155)$$

$$\begin{aligned} \delta k_i &= \delta Q_i \\ &= \frac{\delta v^j}{\Omega} \left(\frac{v_i - w_i}{\Omega} r_j - \rho h a_{ij} + \Sigma_{ij} \right), \end{aligned} \quad (3.156)$$

$$\begin{aligned} \delta \varepsilon &= \rho \delta e \\ &= -2r_i \frac{\delta v^i}{\Omega}. \end{aligned} \quad (3.157)$$

These transformations guaranteed the invariance of the Galilean momenta as in (3.128), which thus implies that for fluids consisting of massless particles, the Galilean fluid equations given above are invariant under arbitrary hydrodynamic frame transformations.¹¹

Thanks to (3.155), the hydrodynamic-frame invariance holds even when $r_i = 0$, which is the physically interesting situation, following the previous discussion on the behaviour of the relativistic heat current. The momentum equation obtained from (3.109) simplifies in this case as

$$\mathcal{M}_i = \partial_i p - \nabla_j \Sigma_i^j = 0. \quad (3.158)$$

However, due to (3.125), hydrodynamic-frame invariance does not survive when n_i is required to vanish in the charge current, which is necessary for the continuity equation (3.110), (3.111) to be a genuine conservation. The latter does not happen for the case of fluids without conserved current (as a gas of photons). They are truly hydrodynamic-frame invariant in the Galilean regime with entropy equation (for the physical situation where $r_i = 0$)

$$\frac{T}{\Omega} \frac{d\sigma}{dt} + \left(\varepsilon + p - \frac{\Sigma}{d} \right) \theta^v - \xi^v_{ij} \Sigma^{ij} + \hat{\nabla}_i Q^i = 0. \quad (3.159)$$

3.3.2 Weyl invariance

Fluids involving massless excitation are compatible with Weyl symmetry. This was discussed in the first part of this chapter when presenting the basic features of Newton–Cartan geometry. We will now consider this property from the perspective of the large- k limit in Zermelo backgrounds.

The fundamental quantities of the Zermelo geometry (2.66) behave as follows under a Weyl transformation:

$$a_{ij} \rightarrow \frac{1}{\mathcal{B}^2} a_{ij}, \quad w^i \rightarrow w^i, \quad w_i \rightarrow \frac{1}{\mathcal{B}^2} w_i, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega, \quad (3.160)$$

and since Ω depends only on time, the last term in (3.160) imposes $\mathcal{B} = \mathcal{B}(t)$. The fluid velocity field u^μ has weight 1. This implies that the ordinary spatial fluid velocity v^i transforms as

$$v^i \rightarrow v^i, \quad v_i \rightarrow \frac{1}{\mathcal{B}^2} v_i. \quad (3.161)$$

Considering the latter, one could wonder for the existence of a Galilean Weyl-covariant derivative, that acts on Galilean Weyl-covariant tensors. Here we examine the infinite- k limit of the connection (2.40) and the corresponding Weyl-covariant derivative used in the relativistic case. For a more rigorous analysis of the latter, see e.g. [120, 116, 117]. This

¹¹Equations (3.151) and (3.153) are hydrodynamic-frame-invariant only on-shell.

splits into time and space Weyl derivatives, associated with time and space Weyl connections, inherited from the limit of the Weyl connection A given in (2.40). In this case we find

$$\lim_{k \rightarrow \infty} \Omega k A^0 = -\frac{\theta^v}{d}, \quad \lim_{k \rightarrow \infty} A_i = 0, \quad (3.162)$$

where we notice that there is no spatial Weyl connection in the Galilean limit. The ordinary Galilean spatial covariant derivative $\hat{\nabla}_i$ used here as the usual d -dimensional metric-compatible and torsionless covariant derivative with connection coefficients (3.8) (possibly time-dependent since generally $a_{ij} = a_{ij}(t, \mathbf{x})$) is already Weyl-covariant. This is because the Weyl rescaling with $\mathcal{B}(t)$ leaves the Christoffel symbols (3.8) unaltered.

It is not the same for the Galilean time covariant derivative $\frac{D}{dt}$ given in (3.26). The latter is not Weyl-covariant but it can be promoted to a Weyl-covariant Galilean time derivative \mathcal{D}_t thanks to θ^v , which transforms indeed as a connection. That is

$$\theta^v \rightarrow \mathcal{B} \theta^v - \frac{d}{\Omega} \partial_t \mathcal{B}. \quad (3.163)$$

Consequently, if $S^{ij\dots}_{kl\dots}$ are the components of a weight- w Galilean tensor, then

$$\frac{1}{\Omega} \mathcal{D}_t S^{ij\dots}_{kl\dots} = \left(\frac{1}{\Omega} \frac{D}{dt} + \frac{w}{d} \theta^v \right) S^{ij\dots}_{kl\dots} \quad (3.164)$$

are the components of Galilean tensor of weight $w + 1$. Observe that the components of the Galilean shear given in (3.19) is of weight -1 , namely it transform as

$$\xi^v_{ij} \rightarrow \frac{1}{\mathcal{B}} \xi^v_{ij}. \quad (3.165)$$

The Weyl transformation (3.163) holds equally for θ^w defined in (3.18), which also implies that the geometric Galilean shear ξ^w_{ij} defined in (3.17) transforms in the same way as (3.165). One can therefore introduce an alternative Galilean Weyl-covariant time derivative

defined in purely geometrical terms as¹²

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi = \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \frac{w}{d} \theta^w \right) \Phi, \quad (3.166)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t V_i = \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \frac{w+1}{d} \theta^w \right) V_i, \quad (3.167)$$

for weight- w Galilean scalars or forms, which can be generalized for any Galilean tensor by using the Leibniz rule.

With the above tools and imposing Weyl invariance (3.70), the fundamental fluid equations (3.43), (3.44) and (3.50) can be recast as

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t P_i + P_j \xi^{wj}{}_i + \hat{\nabla}^j \Pi_{ij} = 0, \quad (3.168)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \Pi_{ij} \xi^{wij} + \hat{\nabla}_i \Pi^i = 0, \quad (3.169)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \rho + \hat{\nabla}_i N^i = 0. \quad (3.170)$$

They are Weyl-covariant of weights $d+1$, $d+2$ and $d+1$.

When dealing with Galilean fluids, the Galilean momenta ρ , N_i , P_i , Π , Π_i and Π_{ij} emerge in the large- k expansion of ρ_{r0} , j_{ri} , ε_r , q_{ri} and $p_r a_{ij} + \tau_{rij}$ (see Eqs. (3.137), (3.139), (3.140), (3.141)). The weights inherited in this limiting procedure (the relativistic weights are available in Tab. 2.1) are in agreement with those previously defined through the effective-action definition of the momenta. These momenta are expressed in terms of the Galilean velocity v^i together with the usual fluid variables coming from the relativistic stress, heat current and charge current.

From the expressions (3.142), (3.143), (3.144), (3.145), we infer that the forms r_i , k_i (and thus Q_i) have weight d , while n_i and the Galilean stress Σ_{ij} have weight $d-1$. The Weyl condition¹³ (3.70) now reads $\varepsilon = dp - \Sigma$, with $\Sigma = a^{ij} \Sigma_{ij}$. This condition splits into the conformal equation of state

$$\varepsilon = dp, \quad (3.171)$$

¹²This sort of Weyl-covariant derivative is insensitive to the fluid velocity and is thus better suited for discussing hydrodynamic-frame invariance. Its relativistic ascendent is a Weyl connection A^Z constructed, as explained generally in footnote 5, with the vector field $u_Z = v$ defined in (3.3) (and used in Sec. 3.2), which has norm $-k^2$ in the Zermelo background (2.66). This connection exists irrespective of the fluid velocity: $A^Z = \frac{\theta^w}{d} \Omega dt$.

¹³Notice in passing that the Weyl-invariance requirement (3.70) determined from the effective action, is also the large- k expression of the relativistic condition $T_\mu{}^\mu = 0$

accompanied with the Weyl-invariance requirement

$$\Sigma \equiv \Sigma_{ij} a^{ij} = 0. \quad (3.172)$$

Other thermodynamic observables like e , T , μ or h have all weight 1, and s is weight zero.

It is worth mentioning that the above analysis is consistent because we have been referring to fluids with microscopic massless degrees of freedom, and we have thus used (3.137), (3.138), (3.139), (3.140), (3.141), (3.142), (3.143), (3.144), (3.145). If one had considered fluids with massive carriers, conflicts would have appeared in the conformal weights, as for example in (3.114) by setting a relationship among N^i and P^i , which in a Weyl-covariant system are expected to have different weights ($d+1$ and $d+2$).

Hydrodynamic equations (3.151) and (3.159) are recast as

$$\begin{aligned} \frac{1}{\Omega} \mathcal{D}_t \left(\varepsilon + \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \frac{r^i}{\Omega} \mathcal{D}_t \frac{v_i - w_i}{\Omega} - \xi^v_{ij} \Sigma^{ij} \\ + \hat{\nabla}_i \left(Q^i - \frac{v^i - w^i}{\Omega} \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) = 0, \end{aligned} \quad (3.173)$$

$$\begin{aligned} \frac{\rho T}{\Omega} \mathcal{D}_t s + \frac{1}{\Omega} \mathcal{D}_t \left(\frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \frac{r^i}{\Omega} \mathcal{D}_t \frac{v_i - w_i}{\Omega} - \xi^v_{ij} \Sigma^{ij} \\ - h \hat{\nabla}_i n^i + \hat{\nabla}_i \left(Q^i - \frac{v^i - w^i}{\Omega} \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) = 0. \end{aligned} \quad (3.174)$$

For more conventional conformal fluids with $r^i = 0$ and no conserved charge we find

$$\boxed{\frac{1}{\Omega} \mathcal{D}_t \varepsilon - \xi^v_{ij} \Sigma^{ij} + \hat{\nabla}_i Q^i = \frac{T}{\Omega} \mathcal{D}_t \sigma - \xi^v_{ij} \Sigma^{ij} + \hat{\nabla}_i Q^i = 0,} \quad (3.175)$$

which are Weyl-covariant of weight $d+2$. The Euler (transverse) equation (3.158) remains unchanged and can be expressed in terms of the energy thanks to (3.171), or further using (A.18). It reads

$$\boxed{\frac{1}{d} \partial_i \varepsilon - \hat{\nabla}^j \Sigma_{ij} = \frac{1}{(d+1)} \partial_i (T \sigma) - \hat{\nabla}^j \Sigma_{ij} = 0} \quad (3.176)$$

and it is Weyl-covariant of weight $d+1$.

Chapter 4

Carrollian fluids

4.1 Carroll structures and general Carrollian covariance

4.1.1 Basics on Carrollian structures

Carrollian structures can be thought as the dual of the Newton-Cartan structures. They consist of a $d + 1$ manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ equipped with a degenerate metric (as opposed to Newton-Cartan which has a degenerate co-metric) and vector field that generates the kernel. These manifolds are described in terms of fibre bundles with one-dimensional fibre and a d -dimensional base \mathcal{S} base space. These structures also appear as the vanishing- k limit of a pseudo-Riemannian geometry. A simple example is the ultra-relativistic limit of Minkowski spacetime which gives as a result the flat Carroll structure given by: $\mathcal{M} = \mathbb{R} \times \mathbb{R}^d$, $\mathbf{v} = \partial_t$ and $g_{\mu\nu}dx^\mu dx^\nu = \delta_{ij}dx^i dx^j$ with $g(\mathbf{v}) = 0$. The Carroll group [9, 10] is the isometry group of this flat Carroll structure. Here we are going to be slightly more general. We are not going to make any assumptions about isometries, but rather Carrollian diffeomorphisms (2.100), which have the advantage of preserving the time/space splitting.

For concreteness, our manifold \mathcal{M} will be equipped with coordinates (t, \mathbf{x}) and a degenerate metric of the form

$$d\ell^2 = a_{ij}(t, \mathbf{x})dx^i dx^j, \quad i, j \dots \in \{1, \dots, d\} \quad (4.1)$$

with kernel generated by

$$\mathbf{v} = \frac{1}{\Omega} \partial_t, \quad (4.2)$$

which defines a field of observers. The above coordinate system is adapted to the fiber/base splitting, which is respected by Carrollian diffeomorphisms (2.100). Additionally, just as the example of Minkowski spacetime, this Carrollian structure is reached when performing the

small- k limit of a pseudo-Riemannian spacetime parametrized in the Papapetrou-Randres gauge (2.99). This Carrollian structure also possesses an Ehresmann connection, which is a background gauge field $\mathbf{b} = b_i dx^i$, appearing in the dual of the kernel generator. The latter defines the clock form

$$\mu = \Omega dt - b_i dx^i, \quad (4.3)$$

The scale factor Ω and the gauge components b_i depend on t and \mathbf{x} .

Under Carrollian diffeomorphisms (2.100) (the Jacobian is defined in (2.101)), the transformation rules of the various geometric objects are as in (2.102), (2.103) as well as

$$\partial'_t = \frac{1}{J} \partial_t, \quad (4.4)$$

$$\partial'_j = J^{-1i}{}_j \left(\partial_i - \frac{J^i}{J} \partial_t \right), \quad (4.5)$$

$$\mu' = \mu, \quad (4.6)$$

$$\hat{\partial}'_i = J^{-1j}{}_i \hat{\partial}_j. \quad (4.7)$$

where

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega} \partial_t, \quad (4.8)$$

are the vector fields dual to the forms dx^i , also spelled e_i in (2.126).

In general, Carrollian tensors depend on time t and space \mathbf{x} . They carry indices $i, j, \dots \in \{1, \dots, d\}$, which are lowered and raised with a_{ij} and its inverse spatial co-metric a^{ij} , and transform covariantly under Carrollian diffeomorphisms (2.100) with Jacobian J^j_i and $J^{-1i}{}_j$ defined in (2.101). A Levi-Civita-Carroll connection can be introduced as

$$\hat{\gamma}^j_{jk} = \frac{a^{il}}{2} \left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk} \right). \quad (4.9)$$

This choice of connection is not unique (see [23, 117, 17]), but it is obtained naturally in the small- k expansion of a Levi-Civita connection in the Papapetrou-Randers coordinates (2.99). It defines a spatial Carrollian covariant derivative $\hat{\nabla}_i$ which transform covariantly under Carrollian diffeomorphisms.

The Levi-Civita-Carroll connection is spatially torsionless and metric-compatible, namely it satisfies

$$\hat{t}^k_{ij} = 2\hat{\gamma}^k_{[ij]} = 0, \quad \hat{\nabla}_i a_{jk} = 0. \quad (4.10)$$

The vectors $\hat{\partial}_i$ do not commute and define the Carrollian vorticity as

$$\left[\hat{\partial}_i, \hat{\partial}_j \right] = \frac{2}{\Omega} \varpi_{ij} \partial_t, \quad \varpi_{ij} = \partial_{[i} b_{j]} + b_{[i} \varphi_{j]} \quad (4.11)$$

with

$$\varphi_i = \frac{1}{\Omega} (\partial_t b_i + \partial_i \Omega). \quad (4.12)$$

the Carrollian acceleration. The exterior derivative of the clock-form is given by

$$d\mu = \varphi_i dx^i \wedge \mu - \varpi_{ij} dx^i \wedge dx^j. \quad (4.13)$$

So, the vanishing of the Carrollian acceleration and vorticity are sufficient and necessary conditions for μ to be closed and define a family of hypersurfaces inside $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ as $\tau(t, \mathbf{x}) = \text{const.}$, where locally $\mu = d\tau$.

From (4.4) we see that the time derivative operator $\frac{1}{\Omega} \partial_t$ transform nicely under Carrollian diffeomorphisms. However, it is not metric compatible due to the time dependence of the metric $a_{ij}(t, \mathbf{x})$. Still, we can define a new Carrollian time derivative that, besides having a good transformation property as $\frac{1}{\Omega'} \hat{D}'_t = \frac{1}{\Omega} \hat{D}_t$, it satisfies

$$\hat{D}_t a_{jk} = 0. \quad (4.14)$$

This is achieved by introducing a temporal Carrollian connection

$$\hat{\gamma}_{ij} = \frac{1}{2\Omega} \partial_t a_{ij} = \xi_{ij} + \frac{1}{d} a_{ij} \theta, \quad (4.15)$$

which is a genuine symmetric Carrollian tensor splitted into a symmetric and traceless part ξ_{ij} , called the Carrollian shear, and a traceful part $a_{ij} \theta$, where θ is the Carrollian expansion defined as

$$\theta = \frac{1}{\Omega} \partial_t \ln \sqrt{a}, \quad (4.16)$$

The action of \hat{D}_t on scalars is ∂_t

$$\hat{D}_t \Phi = \partial_t \Phi, \quad (4.17)$$

whereas on vectors or forms it acts as

$$\frac{1}{\Omega} \hat{D}_t V^i = \frac{1}{\Omega} \partial_t V^i + \hat{\gamma}^j_i V^j, \quad \frac{1}{\Omega} \hat{D}_t V_i = \frac{1}{\Omega} \partial_t V_i - \hat{\gamma}_i^j V_j. \quad (4.18)$$

The latter can be generalized for any tensor by the Leibniz rule and allows to demonstrate the metric compatibility (4.14).

The commutators of Carrollian covariant spatial derivatives define further Carrollian tensors (Φ and V^i are a Carrollian scalar and a Carrollian vector). One finds¹

$$\left[\hat{\nabla}_i, \hat{\nabla}_j \right] \Phi = \varpi_{ij} \frac{2}{\Omega} \partial_t \Phi, \quad (4.19)$$

$$\left[\hat{\nabla}_k, \hat{\nabla}_l \right] V^i = \left(\hat{\partial}_k \hat{\gamma}_{lj}^i - \hat{\partial}_l \hat{\gamma}_{kj}^i + \hat{\gamma}_{km}^i \hat{\gamma}_{lj}^m - \hat{\gamma}_{lm}^i \hat{\gamma}_{kj}^m \right) V^j + \left[\hat{\partial}_k, \hat{\partial}_l \right] V^i \quad (4.20)$$

$$= \hat{R}^i{}_{jkl} V^j + \varpi_{kl} \frac{2}{\Omega} \hat{D}_t V^i. \quad (4.21)$$

Similarly, time and space derivatives do not commute, resulting in

$$\left[\frac{1}{\Omega} \hat{D}_t, \hat{\nabla}_i \right] V^j = \varphi_i \left(\left(\frac{1}{\Omega} \hat{D}_t + \theta \right) V^j - \hat{\gamma}_k^j V^k \right) - \hat{\gamma}_i^k \hat{\nabla}_k V^j - d \hat{r}_{ik}^j V^k \quad (4.22)$$

with²

$$\hat{r}_{ik}^j = \frac{1}{d} \left(\theta \varphi_i \delta_k^j + \hat{\nabla}_i \hat{\gamma}_k^j - \frac{1}{\Omega} \partial_t \hat{\gamma}_{ik}^j \right) \quad (4.23)$$

and

$$\hat{r}^j{}_{jk} = \hat{r}_k = \frac{1}{d} \left(\hat{\nabla}_j \hat{\gamma}_k^j - \hat{\partial}_k \theta \right), \quad (4.24)$$

further Carrollian curvature tensors.

4.1.2 Diffeomorphism invariance and Carrollian dynamics

Similarly as we did in the Galilean case, we can consider a dynamical system on a Carrollian manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$, that is described by an action $S = \int dt d^d x \sqrt{a} \Omega \mathcal{L}$ which is a functional of a_{ij} , Ω and b_i . The variation of this action with respect to the Carrollian geometry defines a set of Carrollian momenta [86, 121] (analogous to the definition of a relativistic energy-momentum tensor). They are defined by

$$\Pi^{ij} = \frac{2}{\sqrt{a} \Omega} \frac{\delta S}{\delta a_{ij}}, \quad (4.25)$$

$$\Pi^i = \frac{1}{\sqrt{a} \Omega} \frac{\delta S}{\delta b_i}, \quad (4.26)$$

$$\Pi = -\frac{1}{\sqrt{a}} \left(\frac{\delta S}{\delta \Omega} + \frac{b_i}{\Omega} \frac{\delta S}{\delta b_i} \right), \quad (4.27)$$

¹In [44] an alternative tensor was defined as $\hat{r}^i{}_{jkl} = \hat{R}^i{}_{jkl} + 2\hat{\gamma}_j^i \varpi_{kl}$ with $\left[\hat{\nabla}_k, \hat{\nabla}_l \right] V^i = \hat{r}^i{}_{jkl} V^j + \varpi_{kl} \frac{2}{\Omega} \partial_t V^i$.

²Notice that $\frac{1}{\Omega} \partial_t \hat{\gamma}_{ik}^j = \left(\hat{\nabla}_i + \varphi_i \right) \hat{\gamma}_k^j + \left(\hat{\nabla}_k + \varphi_k \right) \hat{\gamma}_i^j - \left(\hat{\nabla}^j + \varphi^j \right) \hat{\gamma}_{ik}^j$ is a Carrollian tensor, even though $\hat{\gamma}_{ik}^j$ is not.

with $\frac{\delta S}{\delta \Omega} = -\sqrt{a} (\Pi + b_i \Pi^i)$. These are the energy–stress tensor, the energy current and the energy density.

Diffeomorphisms are generated by vector fields of the form (2.91)

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \left(\xi^t - \xi^i \frac{b_i}{\Omega} \right) \partial_t + \xi^i \left(\partial_i + \frac{b_i}{\Omega} \partial_t \right) = \xi^{\hat{t}} \frac{1}{\Omega} \partial_t + \xi^i \hat{\partial}_i. \quad (4.28)$$

Additionally, Carrollian diffeomorphisms (2.100) impose $\xi^i = \xi^i(\mathbf{x})$. Variation under diffeomorphisms is implemented through the Lie derivative, which acts on the metric a_{ij} as

$$-\delta_\xi a_{ij} = \mathcal{L}_\xi a_{ij} = 2\hat{\nabla}_{(i} \xi^k a_{j)k} + 2\xi^{\hat{t}} \hat{\gamma}_{ij}. \quad (4.29)$$

For the field of observers we have

$$\mathcal{L}_\xi \mathbf{v} = - \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i \right) \mathbf{v} = \mu \mathbf{v}, \quad (4.30)$$

while for the clock form μ we find

$$\mathcal{L}_\xi \mu = \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i \right) \mu + \left((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right) dx^i. \quad (4.31)$$

From the latter we can extract the transformation of Ω and the Ehresmann connection b_i . They read

$$-\delta_\xi \ln \Omega = \frac{1}{\Omega} \mathcal{L}_\xi \Omega = \frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i, \quad (4.32)$$

$$-\delta_\xi b_i = \mathcal{L}_\xi b_i = b_i \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_j \xi^j \right) - (\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} + 2\xi^j \varpi_{ji}. \quad (4.33)$$

Using the above diffeomorphic transformation rules, we can compute the variation of the action. We find

$$\delta_\xi S = \int dt d^d x \sqrt{a} \Omega \left(\frac{1}{2} \Pi^{ij} \delta_\xi a_{ij} + \Pi^i \delta_\xi b_i - \frac{1}{\Omega} (\Pi + b_i \Pi^i) \delta_\xi \Omega \right). \quad (4.34)$$

Then, using (4.29) and (4.33) with $\xi^i = \xi^i(\mathbf{x})$, we obtain

$$\begin{aligned} \delta_\xi S = & \int dt d^d x \sqrt{a} \Omega \left\{ -\xi^{\hat{i}} \left[\left(\frac{1}{\Omega} \partial_t + \theta \right) \Pi + \left(\hat{\mathbf{V}}_i + 2\varphi_i \right) \Pi^i + \Pi^{ij} \hat{\gamma}_{ij} \right] \right. \\ & + \xi^i \left[\left(\hat{\mathbf{V}}_j + \varphi_j \right) \Pi^j_i + 2\Pi^j \varpi_{ji} + \Pi \varphi_i \right] \left. \right\} \\ & + \int dt d^d x \left\{ \partial_t \left[\sqrt{a} \left(\xi^{\hat{i}} \left(\Pi + b_i \Pi^i \right) - \xi^j b_i \Pi^i_j \right) \right] \right. \\ & \left. + \partial_i \left[\sqrt{a} \Omega \left(\xi^{\hat{i}} \Pi^i - \xi^j \Pi^i_j \right) \right] \right\}. \end{aligned} \quad (4.35)$$

Invariance under Carrollian diffeomorphisms implies $\delta_\xi S = 0$ (we ignore boundary terms, which correspond to the last integral in (4.35)). The vanishing of the variation of the action give rise to two equations that have to be satisfied. The first one is associated to $\xi^{\hat{i}}$ and corresponds to the energy equation given by

$$\boxed{\left(\frac{1}{\Omega} \partial_t + \theta \right) \Pi + \left(\hat{\mathbf{V}}_i + 2\varphi_i \right) \Pi^i + \Pi^{ij} \hat{\gamma}_{ij} = 0.} \quad (4.36)$$

The second one is associated to ξ^i and corresponds to the momentum equation, expressed as

$$\boxed{\left(\hat{\mathbf{V}}_j + \varphi_j \right) \Pi^j_i + 2\Pi^j \varpi_{ji} + \Pi \varphi_i = - \left(\frac{1}{\Omega} \partial_t + \theta \right) P_i.} \quad (4.37)$$

The right hand side of (4.37) arise due to the time independence of ξ^i . Hence, in the second line of (4.35) one is free to add $\sqrt{a} \Omega \xi^i \left(\frac{1}{\Omega} \partial_t + \theta \right) P_i = \partial_t \left(\sqrt{a} \xi^i P_i \right)$, which is a boundary term and vanishes inside the integral.

We call the new vector P^i as momentum (analogous to what we have in the Galilean case). We can see from this analysis that the momentum P^i is not defined through a variation of the action with respect to some conjugate variable³.

It is worth mentioning the similarity that the above equations (4.36) and (4.37) share with the Galilean ones given by (3.44) and (3.43). This is superficial because first, when dealing with genuine fluid equations, the various momenta are expressed in term of kinematical and thermodynamic parameters, which are completely different in both cases. For instance, there is no velocity in the Carrollian case. The second major difference is the fact that the conservation equations in both instances are the result of symmetries (Carrollian and Galilean diffeomorphisms) which are distinct.

³The momentum P^i is obtained through variation with respect to an additional variable in [46].

4.1.3 $U(1)$ local invariance and conservation law

We can also consider an action on the Carrollian manifold which is invariant under a local $U(1)$ transformation associated with a gauge field $B = B(t, \mathbf{x})dt + B_i(t, \mathbf{x})dx^i$ as in (3.45). Then, an additional conservation law is available, although it is not as useful as in the Galilean case. In the Galilean instance we have that the thermodynamic law (A.2) sets a relationship between a conserved charge and the energy. The latter is invalidated in the vanishing- k limit, and plays no subsequent role in the fluid dynamics.

The conjugate momenta are again the charge density and the charge current:

$$\rho = \frac{1}{\sqrt{a}} \left(\frac{\delta S}{\delta B} - \frac{b_i}{\Omega} \frac{\delta S}{\delta B_i} \right), \quad (4.38)$$

$$N^i = \frac{1}{\Omega \sqrt{a}} \frac{\delta S}{\delta B_i} \quad (4.39)$$

with $\frac{\delta S}{\delta B} = \sqrt{a}(\rho + b_i N^i)$. The gauge variation of the action gives

$$\begin{aligned} \delta_\Lambda S &= \int dt d^d x \sqrt{a} ((\rho + b_i N^i) \delta_\Lambda B + \Omega N^i \delta_\Lambda B_i) \\ &= - \int dt d^d x \sqrt{a} ((\rho + b_i N^i) \partial_t \Lambda + \Omega N^i \partial_i \Lambda) \\ &= \int dt d^d x \sqrt{a} \Omega \Lambda \left(\frac{1}{\Omega} \partial_t \rho + \theta \rho + (\hat{\nabla}_i + \varphi_i) N^i \right) \\ &\quad - \int dt d^d x \{ \partial_t (\sqrt{a} \Lambda (\rho + b_i N^i)) + \partial_i (\sqrt{a} \Lambda \Omega N^i) \}. \end{aligned} \quad (4.40)$$

The $U(1)$ invariance of the action S leads to a Carrollian continuity equation, expressed here as

$$\boxed{\left(\frac{1}{\Omega} \partial_t + \theta \right) \rho + (\hat{\nabla}_i + \varphi_i) N^i = 0.} \quad (4.42)$$

Using Stokes and Gauss theorems and the Carrollian continuity equation (4.42) we find

$$\begin{aligned} \int_{\mathcal{W}} dt d^d x \Omega \sqrt{a} \left(\left(\frac{1}{\Omega} \partial_t + \theta \right) \rho + (\hat{\nabla}_i + \varphi_i) N^i \right) &= \oint_{\partial \mathcal{W}} \sqrt{a} \rho dx^1 \wedge \dots \wedge dx^d \\ &\quad - \oint_{\partial \mathcal{W}} \sqrt{a} \sum_{i=1}^d dx^1 \wedge \dots \wedge N^i \mu \wedge \dots \wedge dx^d, \end{aligned} \quad (4.43)$$

where $\mathcal{W} \subset \mathcal{M} = \mathbb{R} \times \mathcal{S}$ and $N^i \mu$ (μ given in (4.3)) is the i th factor in the exterior product of the last term. Assuming a good behaviour for the fields, a conserved charge exists and can be expressed as an integral over an arbitrary space-like hypersurface Σ_d of $\mathcal{M} = \mathbb{R} \times \mathcal{S}$.

This conserved charge is identical to the relativistic Papapetrou–Randers result obtained e.g. in (2.131). In a similar way as in the Galilean instance, it is suitable to choose $\Sigma_d \equiv \mathcal{S}$ i.e. a constant- t hypersurface, and the charge then reads⁴

$$Q_N = \int_{\mathcal{S}} d^d x \sqrt{a} (\rho + b_i N^i). \quad (4.44)$$

Following the Galilean steps, time independence reveals by replacing \mathcal{S} in (4.57) with $\mathcal{V} \subset \mathcal{S}$, where the boundary $\partial\mathcal{V}$ does not depend on t . Using (4.42), the time evolution of the matter/charge content of \mathcal{V} is the following:

$$\frac{d}{dt} \int_{\mathcal{V}} d^d x \sqrt{a} (\rho + b_i N^i) = - \int_{\mathcal{V}} d^d x \partial_i (\sqrt{a} \Omega N^i) = - \int_{\partial\mathcal{V}} \Omega \star \mathbf{N}. \quad (4.45)$$

If \mathcal{V} is extended to the whole \mathcal{S} the time dependence fades and we find that Q_N is conserved.

4.1.4 Isometries and (non-)conservation

Carrollian isometries are generated by Killing fields given by (4.28), which are required to satisfy

$$\mathcal{L}_\xi a_{ij} = 0, \quad \mathcal{L}_\xi \mathbf{v} = 0, \quad (4.46)$$

since the metric (4.1) and the field of observers (4.2) are the fundamental geometric data in the spacetimes at hand (see [14, 22, 23, 20]). For Carrollian diffeomorphisms ($\xi^i \equiv \xi^{\hat{i}}$ is only \mathbf{x} -dependent), equations (4.29) and (4.30) lead to

$$\boxed{\hat{\nabla}_{(i} \xi^k a_{j)k} + \xi^{\hat{i}} \hat{\gamma}_{ij} = 0, \quad \frac{1}{\Omega} \partial_t \xi^{\hat{i}} + \varphi_i \xi^{\hat{i}} = 0.} \quad (4.47)$$

The above are the equations that reflect the invariance of a weak Carroll structure [22] and possesses an infinite set of solutions. As we discussed previously in the Newton-Cartan instance, we could choose a strong Carroll structure which is equipped with a field-of-observers-compatible and metric-compatible connection (as our choice in (4.9)), required to be invariant under Carrollian isometries. This restricts the solution space of (4.47).

Contrary to the field of observers and the metric of the base space, the Ehresmann connection is not required to be invariant. For a Carrollian Killing field ξ , using (4.47) inside

⁴It should be noticed that the presence of b_i apparently breaks the manifest covariance, since according to (2.103) the form of the integrand is respected only by coordinate transformations such that $t' = t'(t)$ i.e. a subset of Carrollian diffeomorphisms. This actually just translates a feature of the hypersurface chosen for computing the charge, which is otherwise an absolute constant. If the clock form is closed (see (4.13)), locally $\mu = d\tau$ one may alternatively choose the integration hypersurface Σ_τ as $\tau(t, \mathbf{x}) = \text{constant}$. In this instance, we obtain $Q_N = \int_{\Sigma_\tau} d^d x \sqrt{a} \rho$. Nevertheless, all choices of space-like hypersurface Σ_d lead to the same charge.

(4.31) we find

$$\mathcal{L}_\xi \mu = \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{i}} - 2\xi^j \varpi_{ji} \right) dx^i. \quad (4.48)$$

In order to have a good illustration of the above we can consider a flat Carroll spacetime, consisting with a Carroll structure with $a^{ij} = \delta^{ij}$ the d -dimensional Euclidean metric, $\Omega = 1$ and constant b_i . The equations in (4.47) possess an infinite number of solutions for the Killing field. This read

$$\xi = \left(\Omega_i^j x^i + X^j \right) \partial_j + f(\mathbf{x}) \partial_t \quad (4.49)$$

with constant and antisymmetric $\Omega_{ij} = \Omega_i^k \delta_{kj}$ generating the rotations in $\mathfrak{so}(d)$, constant X^j for the space translations, and an arbitrary function of space $f(\mathbf{x})$. If we additionally required the affine connection of the strong Carroll structure to remain invariant under ξ , then the function $f(\mathbf{x})$ is restricted to be linear, e.g. $f = T - B_i x^i$ with T generating time translations and B_i being the Carroll boosts. The total number of solutions is now $(d+2)(d+1)/2$, which is the dimension of the Carroll algebra $\text{cart}(d+1)$. Also, for the Ehresmann connection we find

$$\mathcal{L}_\xi b_i = - \left(B_i + \Omega_i^j b_j \right) \neq 0, \quad (4.50)$$

showing a constant shift in the Ehresmann connection.

In principle we should manage to obtain the conservation laws associated to the Carrollian isometries. The latter ultimately will be of the form of (4.42) with a Carrollian scalar κ and a Carrollian vector K^i determined from the components of the Carrollian Killing fields $\xi^{\hat{i}}$ and $\xi^{\hat{j}}$, and from the Carrollian momenta, namely the energy density Π , the energy current Π^i and the energy–stress tensor Π^{ij} , as well as the momentum P^i , and satisfying the conservation equations (4.37) and (4.36). Assuming that such a conservation exists, the Carrollian scalar

$$\mathcal{H} = \left(\frac{1}{\Omega} \partial_t + \theta \right) \kappa + \left(\hat{\nabla}_i + \varphi_i \right) K^i \quad (4.51)$$

should vanish on-shell. In the above, κ and K^i are obtained from the on-shell boundary terms of $\delta_\xi S$ (see (4.35))⁵. They read

$$\kappa = \xi^i P_i - \xi^{\hat{i}} \Pi, \quad (4.52)$$

$$K^i = \xi^j \Pi_j^i - \xi^{\hat{j}} \Pi^i. \quad (4.53)$$

⁵We can also obtain κ and K^i from the small- k expansion of the relativistic-current components (2.128), (2.129), (2.130).

The scalar \mathcal{K} can be determined using the conservation equations (4.37) and (4.36):

$$\begin{aligned} \mathcal{K} &= -\Pi \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i \right) - \Pi^i \left((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right) \\ &\quad + \Pi^i_j \left(\hat{\nabla}_i \xi^j + \xi^{\hat{t}} \hat{\gamma}_i^j \right) \end{aligned} \quad (4.54)$$

$$= -\Pi^i \left((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right). \quad (4.55)$$

The above result shows that in Carroll structures, a Killing field does not guarantee an on-shell conservation law for Carrollian dynamics.⁶

This result is similar to what we found in the Galilean case. Here, the energy current Π^i is conjugate to b_i (4.26) and b_i does transform under diffeomorphisms (see (4.33)), even if we are dealing with an isometry. Still, equation (4.55) suggest that a conservation law exists for a restricted version of the Killing vectors such that

$$\mathcal{L}_\xi \mu = 0. \quad (4.56)$$

We call strong Carrollian Killing vectors to the ones that satisfy the above condition together with the first two Carrollian Killing equations given in (4.47).⁷ In general, these strong Carroll isometries produce conserved currents, in agreement with general Noether's theorem. Another instance where conservation is satisfied is when the right-hand side of (4.55) originates from a boundary term (as for the corresponding Galilean equation (3.65)). In that instance, conservation would occur with an effective Carrollian current given by κ' and K^i , both getting corrections from boundary-term contributions.

Another possibility is requiring the vanishing of the energy flux Π^i . To see this we can go back to the simple example of flat Carroll spacetime. In that case one can notice that Carrollian boosts in flat Carroll structures do not satisfy the extra condition $\mathcal{L}_\xi \mu = 0$ (see (4.50)) since it creates a constant shift in the Ehresmann connection. This implies that no conservation is necessarily associated with them. Then, in order to have a conservation law associated to Carroll boost invariance, the vanishing of the energy flux Π^i is required.

We can construct charges associated with the current (κ, \mathbf{K}) , which is going to be important when discussing gravitational charges in Chapter 6. These charges are defined as an integral at fixed t over the hypersurface $\Sigma_d \equiv \mathcal{S}$

$$Q_K = \int_{\mathcal{S}} d^d x \sqrt{a} (\kappa + b_i K^i), \quad (4.57)$$

⁶The observation of this phenomenon was first introduced in [86].

⁷The Jacobi identity is used to show that the commutator of two ξ_s obeying $\mathcal{L}_\xi \mu = 0$, satisfies the same condition.

and obey the time evolution

$$\frac{dQ_K}{dt} = \int_{\mathcal{S}} d^d x \sqrt{a} \Omega \mathcal{K} - \int_{\partial \mathcal{S}} *K \Omega. \quad (4.58)$$

The last term is of boundary type with $*K$ the \mathcal{S} -Hodge dual of $K_i dx^i$. Generally, one can ignore it owing to adequate fall-off or boundary conditions on the fields. Again, only if the current satisfies $\mathcal{K} = 0$, the charge is conserved.

4.1.5 Weyl invariance, conformal isometries, and (non-)conservation

Carrollian manifolds can be covariant with respect to Weyl transformations. They act on their geometric data as

$$a_{ij} \rightarrow \frac{1}{\mathcal{B}^2} a_{ij}, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega, \quad b_i \rightarrow \frac{1}{\mathcal{B}} b_i, \quad (4.59)$$

where $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$ is an arbitrary function. When the action is Weyl-invariant, the Carrollian momenta Π^{ij} , Π^i and Π defined in (4.25), (4.26) and (4.27) have conformal weights $d + 3$, $d + 2$ and $d + 1$. The momentum P^i appearing in (4.37) has also weight $d + 2$, and in the matter sector, assuming the gauge field B and B_i be weight-zero, we can find from (4.38) and (4.39) that the weights of the density ρ and the matter current N^i are d and $d + 1$.

Requiring Weyl invariance for the effective action, namely $\delta_{\mathcal{B}} S = 0$, expression (4.34) implies that

$$\Pi_i^i = \Pi. \quad (4.60)$$

We will see later in this chapter how to implement Weyl covariance through the appropriate covariant derivatives for time and space, dubbed Weyl-Carroll derivatives.

Following [13, 14, 22, 23, 20] a conformal isometry is generated by a vector field ξ satisfying

$$\mathcal{L}_{\xi} a_{ij} = \lambda a_{ij}, \quad (4.61)$$

and

$$\mathcal{L}_{\xi} v = \mu v, \quad (4.62)$$

where

$$\lambda(t, \mathbf{x}) = \frac{2}{d} \left(\hat{\nabla}_i \xi^i + \theta \xi^{\hat{t}} \right), \quad (4.63)$$

and (see (4.30))

$$\mu(t, \mathbf{x}) = - \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i \right). \quad (4.64)$$

The latter is accompanied with the extra condition $2\mu + \lambda = 0$.

Assuming the existence of a conformal isometry, the conservation equations (4.36) and (4.37) can be used for computing the Carrollian scalar \mathcal{K} (4.51) with (4.52), (4.53) and (4.60). This reads

$$\mathcal{K} = \Pi \left(\frac{\lambda}{2} + \mu \right) - \Pi^i \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{i}} - 2\xi^j \varpi_{ji} \right). \quad (4.65)$$

One can see that even if $2\mu + \lambda = 0$ is satisfied, the Carrollian scalar \mathcal{K} does not vanish in general. The latter shows that a conformal Killing field does not generically provide a conservation law in Weyl-invariant Carrollian dynamics.

As an example, one can consider again the standard flat Carroll spacetime ($a^{ij} = \delta^{ij}$, $\Omega = 1$ and constant b_i). Equations (4.61) and (3.73) are satisfied by an infinite number of solutions, which for a strong Carroll structure read [22, 23, 20]

$$\xi = Y^j(\mathbf{x}) \partial_j + \left(T(\mathbf{x}) + \frac{t}{d} \partial_i Y^i \right) \partial_t \quad (4.66)$$

with $T(\mathbf{x})$ an arbitrary function generating the supertranslations and $Y^i(\mathbf{x}) \partial_j$ being the conformal Killing fields of Euclidean d -dimensional space, generating $\mathfrak{so}(d+1, 1)$. This is the conformal Carroll algebra $\mathfrak{ccarr}(d+1) \equiv \mathfrak{so}(d+1, 1) \times \text{supertranslations}$, known to be isomorphic to the Bondi–van der Burg–Metzner–Sachs algebra BMS_{d+2} [22, 23].

The clock form is found to transform as

$$\mathcal{L}_\xi \mu = \left(\partial_i (T - Y^j b_j) + \frac{b_i}{d} \partial_j Y^j + \frac{t}{d} \partial_i \partial_j Y^j \right) dx^i. \quad (4.67)$$

The associated current is not conserved unless $\partial_j Y^j = C_0$ and $T = T_0 + Y^j b_j - \frac{C_0}{d} b_i x^i$, thus linear in x^i (C_0 and T_0 are constants). This excludes the d special conformal transformations of $\mathfrak{so}(d+1, 1)$ and leaves the supertranslations with the time translation as unique freedom, leading to a symmetry subgroup of finite dimension $\frac{d^2}{2} + \frac{d}{2} + 2$.

4.2 Carrollian limit of relativistic hydrodynamics

We now study the vanishing- k limit of relativistic fluid dynamics on a pseudo-Riemannian manifold, parameterized in the Papapetrou-Randers gauge (2.99). As in the Galilean instance, Papapetrou-Randers gauge makes the k -dependence in the geometry explicit. The fluid velocity is parameterized with the Carrollian vector β^i introduced in (2.106), which in the

small- k expansion behaves as

$$v^i = k^2 \Omega \beta^i + \mathcal{O}(k^4). \quad (4.68)$$

The fluid velocity vanishes at zero k as expected, but still the kinematical parameter β^i with dimensions of inverse velocity will remain as a Carrollian-fluid variable (they will appear in the momenta). The behaviour of the components of the fluid congruence then read

$$u_0 = -k\Omega + \mathcal{O}(k^3), \quad u^i = k^2 \beta^i + \mathcal{O}(k^4). \quad (4.69)$$

We can also compute the small- k expansion of the kinematical quantities associated to u^i . For instance, the expansion and the shear behave as

$$\Theta = \frac{1}{\Omega} \partial_t \ln \sqrt{a} + \mathcal{O}(k^2) = \theta + \mathcal{O}(k^2), \quad (4.70)$$

$$\sigma^{ij} = -\frac{1}{\Omega} \left(\frac{1}{2} \partial_t a^{ij} + \frac{1}{d} a^{ij} \partial_t \ln \sqrt{a} \right) + \mathcal{O}(k^2) = \xi^{ij} + \mathcal{O}(k^2) \quad (4.71)$$

with ξ^{ij} and θ defined for a Carrollian manifold in (4.15) and (4.16).

In order to obtain the Carrollian fluid equations in the limiting procedure, we need to handle the behaviour of the energy-momentum tensor at small k . Therefore we need the behaviour of the energy density ε , the pressure p , the heat current q^i and the stress tensor τ^{ij} . We also need the behaviour of the current j^i if present. To this end we consider an ansatz motivated by the Carrollian fluid that appears in the Ricci-flat/Carrollian fluid duality [42]. In this ansatz we have^{8 9}

$$\varepsilon = \varepsilon + \mathcal{O}(k^2), \quad (4.72)$$

$$p = p + \mathcal{O}(k^2), \quad (4.73)$$

$$q^i = Q^i + k^2 \pi^i + \mathcal{O}(k^4), \quad (4.74)$$

$$\tau^{ij} = -\Xi^{ij} + \mathcal{O}(k^2). \quad (4.75)$$

This is similar to the Galilean counterpart (3.87) and (3.88), with the difference that the energy is now of order 1, as for the case of a massless-carriers Galilean fluid.

For the components of the $U(1)$ conserved current, we find

$$\rho_0 = \chi + \mathcal{O}(k^2), \quad j^k = n^k + \mathcal{O}(k^2). \quad (4.76)$$

⁸In the literature one can find a different behaviour for the energy-momentum pieces. See for instance [56] where q^i is assumed to behave as $q^i = k^2 \pi^i + \mathcal{O}(k^4)$.

⁹We are using the same symbols for the leading terms in the energy density and pressure. This choice is made here for simplicity since we will not use their subleading orders in this thesis.

Here ρ_0 is the proper density, namely the density measured by an observer with velocity u^μ . We could also consider a fiducial observer, who would play here the role of u_Z in Zermelo frame. This reads

$$u_{\text{PR}} = \frac{1}{\Omega} \partial_t. \quad (4.77)$$

For this observer, the fluid density is $-\frac{1}{k^2} J_\mu u_{\text{PR}}^\mu = \frac{k}{\Omega} J_0$, which coincides with ρ_{0r} given in (2.124).

4.2.1 Carrollian momenta and conservation equations

With the data (4.72), (4.73), (4.74), (4.75), the invariant pieces of the relativistic energy–momentum tensor defined in (2.118), (2.121), (2.122) and (2.123) read

$$q_r^i = \Pi^i + k^2 P^i + \mathcal{O}(k^4), \quad (4.78)$$

$$\varepsilon_r = \Pi + \mathcal{O}(k^2), \quad (4.79)$$

$$p_r a^{ij} + \tau_r^{ij} = \Pi^{ij} + \mathcal{O}(k^2) \quad (4.80)$$

with

$$\Pi = \varepsilon + 2\beta_i Q^i, \quad \Pi^i = Q^i, \quad \Pi^{ij} = Q^i \beta^j + \beta^i Q^j + p a^{ij} - \Xi^{ij}, \quad (4.81)$$

and

$$P^i = \pi^i + \beta^i \left(\varepsilon + p + \beta_k Q^k \right) - \beta_k \Xi^{ki} + \frac{\beta^2}{2} Q^i. \quad (4.82)$$

Equation (2.1) with the energy–momentum tensor at hand implies the following expressions

$$\frac{k}{\Omega} \nabla_\mu T^{\mu}_0 = \mathcal{E} + \mathcal{O}(k^2), \quad (4.83)$$

$$\nabla_\mu T^{\mu i} = \frac{1}{k^2} \left\{ \left(\frac{1}{\Omega} \hat{D}_t + \theta \right) \Pi^i + \Pi^j \hat{\gamma}_j^i \right\} + \mathcal{G}^i + \mathcal{O}(k^2) \quad (4.84)$$

are zero at each order in the small- k expansion with

$$\mathcal{E} = - \left(\frac{1}{\Omega} \hat{D}_t + \theta \right) \Pi - \left(\hat{\nabla}_i + 2\varphi_i \right) \Pi^i - \Pi^{ij} \hat{\gamma}_{ij}, \quad (4.85)$$

$$\mathcal{G}_j = \left(\hat{\nabla}_i + \varphi_i \right) \Pi^i_j + 2\Pi^i \varpi_{ij} + \Pi \varphi_j + \left(\frac{1}{\Omega} \hat{D}_t + \theta \right) P_j + P^i \hat{\gamma}_{ij}. \quad (4.86)$$

We recover the Carrollian momenta conservation equations (4.36) and (4.37) with the addition of an extra equation for the energy current

$$\left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\Pi^i + \Pi^j\hat{\gamma}_j^i = 0. \quad (4.87)$$

Equation (4.87) was absent when we derived the Carrollian conservation equations from the symmetry analysis. Indeed, this equation appears as a boundary term that could not be obtained through a variational principle, i.e.

$$\Omega\sqrt{a}\xi_i \left[\left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\Pi^i + \Pi^j\hat{\gamma}_j^i \right] = \partial_t(\sqrt{a}\Pi_i\xi^i) \quad (4.88)$$

due to the time independence of $\xi^i(\mathbf{x})$. This is similar to what happens with the momentum P^i . It appears as an arbitrary function in the boundary term of $\delta_\xi S$ when working in the framework of a Carrollian manifold with Carrollian diffeomorphisms, but in the limiting procedure appears explicitly and is expressed in terms of the kinematical and ‘‘thermodynamic–transport’’ observables – β^i and ε , p , Q^i , π^i , Ξ^{ij} in (4.82). Hence, getting the Carrollian dynamics as a vanishing- k limit of relativistic hydrodynamics is richer. From the relativistic ascendant prespective, (4.87) is a vestige of the original full diffeomorphism relativistic invariance.

We can also consider the case where a conserved $U(1)$ current (not necessarily associated to mass conservation) is present in the relativistic ascendant. If that is the case, we find, after inserting (4.76) inside (2.124) and (2.125),

$$\rho_{0r} = \rho + \mathcal{O}(k^2), \quad (4.89)$$

$$j_r^i = N^i + \mathcal{O}(k^2), \quad (4.90)$$

with

$$\rho = \chi + \beta_i n^i, \quad N^i = n^i, \quad (4.91)$$

the matter Carrollian momenta explicitly determined in terms of β^i , χ and n^i . We can now compute the divergence of (2.4) in the Papapetrou–Randers background (2.99). The result is

$$\nabla_\mu J^\mu = \mathcal{J} + \mathcal{O}(k^2) \quad (4.92)$$

with

$$\mathcal{J} = \left(\frac{1}{\Omega}\partial_t + \theta\right)\rho + \left(\hat{\nabla}_j + \varphi_j\right)N^j, \quad (4.93)$$

and demanding the conservation, we recover the Carrollian continuity equation (4.42).

4.2.2 Hydrodynamic-frame invariance

When relativistic hydrodynamics was introduced in Chapter 2, we saw that under an arbitrary local transformation of the velocity field u , the fluid equations remain invariant providing the appropriate transformations of fluid variables such that the energy-momentum tensor and the $U(1)$ current remain invariant¹⁰. The question now is what happens in the Carrollian limit.

From the analysis in the Galilean limit we learned that answering this question requires some care, in particular with the assumed behaviour of the various observables with respect to the speed of light (see (4.72), (4.73), (4.74), and (4.75)), which may or may not be stable under Carrollian (or Galilean) hydrodynamic-frame transformations. In the Carrollian instance we do not have physical intuition (as opposed to the Galilean case) for the latter. However, there are concrete results from flat holography [76, 77] suggesting that Carrollian hydrodynamic-frame invariance exist as a local boundary symmetry, which is reflected in the bulk as a diffeomorphism transformation.

We start by analysing the fluid equations (4.85), (4.86) and (4.87). On one hand, all the operators acting in the momenta are independent of the kinematic vector β^i . On the other hand the momenta P_i , Π , Π_i and Π_{ij} appear as the coefficients at each order in the small- k expansion of the hydrodynamic-frame invariant relativistic momenta q_r , ε_r and $p_r a^{ij} + \tau_r^{ij}$. Then, in order to make any conclusion about the fate of hydrodynamic-frame invariance in the Carrollian limit, we must examine the stability of the scaling properties captured in (4.72), (4.73), (4.74) and (4.75). Using the transformation rules set in the Papapetrou–Randers frame (2.113), (2.114) and (2.115) we find the following transformations:

$$\delta\eta = -2\delta\beta_i Q^i, \quad (4.94)$$

$$\delta Q^i = 0, \quad (4.95)$$

$$\delta\pi^i = \delta\beta_j (\Xi^{ij} - (\eta + \varpi)a^{ij} + \beta^i Q^j), \quad (4.96)$$

$$\delta(\Xi^{ij} - \varpi a^{ij}) = \delta\beta_k (Q^i a^{jk} + Q^j a^{ik}). \quad (4.97)$$

With the above transformation rules, the Carrollian densities, fluxes and the energy-stress tensor defined in (4.81) and (4.82) remain invariant, namely

$$\delta\Pi = 0, \quad \delta\Pi^i = 0, \quad \delta\Pi^{ij} = 0, \quad \delta P^i = 0, \quad (4.98)$$

¹⁰In the Papapetrou–Randers frame, the local transformations (2.32) are captured by $\beta^i \rightarrow \beta^i + \delta\beta^i(t, \mathbf{x})$ (see Eqs. (2.71)), (2.106), (2.108)) parameterized as $\delta\beta^i = B^i - k^2 B^j \beta_j \beta^i + \Omega^{ij} \beta_j$. Infinitesimal Lorentz boosts are associated with $B^i(t, \mathbf{x})$, while infinitesimal rotations go along with the antisymmetric $\Omega^{ij}(t, \mathbf{x})$. In the Carrollian limit, the general transformation, which captures Carrollian boosts and rotations, reads: $\delta\beta^i = B^i + \Omega^{ij} \beta_j$.

which implies the hydrodynamic-frame invariance of the Carrollian fluid equations (4.85), (4.86) and (4.87).

The same procedure applies to study the hydrodynamic-frame transformation of the Carrollian continuity equation (4.93). This demands an analysis on the behaviour of the charge/mass density ρ_0 and the current j^i in the small- k expansion. The transformation rules in Papapetrou–Randers frame are (2.116) and (2.117), whereas the invariant relativistic momenta (4.89), (4.90) should be used together with the small- k behaviour (4.76). We find

$$\delta\chi = -\delta\beta_i n^i, \quad (4.99)$$

$$\delta n^i = 0. \quad (4.100)$$

Using (4.91), we show that

$$\delta\rho = 0, \quad \delta N^i = 0. \quad (4.101)$$

This result demonstrates the invariance of (4.93). Therefore, Carrollian fluid dynamics, consisting in the energy, momentum and energy current equations (4.85), (4.86) and (4.87) together with the Carrollian continuity equation (4.93), is hydrodynamic-frame invariant.

4.2.3 Weyl-invariant Carrollian fluids

Weyl-invariant fluids are important in flat holography since, in the Ricci-flat/Carrollian fluid duality, they appear as the holographic duals hosted at the conformal null boundary [42]. On a pseudo-Riemannian manifold with Papapetrou–Randers coordinates, the action of the Weyl transformations, generated by $\mathcal{B}(t, \mathbf{x})$, in the fundamental geometric data is given as in (4.59). From the zero- k limit of the tracelessness condition of the relativistic energy-momentum tensor, we find the Weyl-invariance condition (4.60) by using using (2.119), (4.78), (4.79) and (4.80). The set of Carrollian momenta (4.81), (4.82), (4.91) are expressed in terms of fluid variables such as the inverse velocity β^i , the energy density ε , the pressure p as well as Q_i , π_i and Ξ_{ij} . Their conformal weights are¹¹ $1, d+1, d+1, d, d$ and $d-1$ and similarly the weights of χ and n^i are d and $d+1$. Condition (4.60) reads $\varepsilon = dp - \Xi^i_i$, which (as in the Galilean case) is splitted as

$$\varepsilon = dp, \quad \Xi^i_i = 0. \quad (4.102)$$

¹¹We mentioned in Sec. 4.1 that Π^{ij} , Π^i , P^i and Π have conformal weights $d+3, d+2, d+2$ and $d+1$, whereas the density ρ and the matter current N^i have weights d and $d+1$.

We define Weyl–Carroll covariant time and space derivatives using θ and φ_i defined in (4.16) and (4.12), which transform as connections

$$\theta \rightarrow \mathcal{B}\theta - \frac{d}{\Omega} \partial_t \mathcal{B}, \quad \varphi_i \rightarrow \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad (4.103)$$

as opposed to the Carrollian shear ξ_{ij} (4.15) and Carrollian vorticity ϖ_{ij} (4.11), which are Weyl-covariant of weight -1 . The action of the Carrollian Weyl-covariant time derivative on a weight- w function Φ is

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi = \frac{1}{\Omega} \hat{D}_t \Phi + \frac{w}{d} \theta \Phi = \frac{1}{\Omega} \partial_t \Phi + \frac{w}{d} \theta \Phi, \quad (4.104)$$

and this is a scalar of weight $w + 1$. On a weight- w vector, the action is

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t V^l = \frac{1}{\Omega} \hat{D}_t V^l + \frac{w-1}{d} \theta V^l = \frac{1}{\Omega} \partial_t V^l + \frac{w}{d} \theta V^l + \xi^l_i V^i. \quad (4.105)$$

These are the components of a Carrollian vector of weight $w + 1$. The action of the time Weyl–Carroll covariant derivative on the metric is

$$\hat{\mathcal{D}}_t a_{kl} = 0. \quad (4.106)$$

For a weight- w scalar function Φ , we introduce the space Weyl-covariant Carrollian derivative

$$\hat{\mathcal{D}}_j \Phi = \hat{\partial}_j \Phi + w \varphi_j \Phi, \quad (4.107)$$

which has the same conformal weight. Similarly, for a vector with weight- w and components V^l we have

$$\hat{\mathcal{D}}_j V^l = \hat{\nabla}_j V^l + (w-1) \varphi_j V^l + \varphi^l V_j - \delta_j^l V^i \varphi_i. \quad (4.108)$$

The Weyl–Carroll spatial derivative does not modify the weight of the tensor it acts on. The action on any other tensor is obtained using the Leibniz rule. For instance, for a rank-two tensor we have

$$\hat{\mathcal{D}}_j t_{kl} = \hat{\nabla}_j t_{kl} + (w+2) \varphi_j t_{kl} + \varphi_k t_{jl} + \varphi_l t_{kj} - a_{jl} t_{ki} \varphi^i - a_{jk} t_{il} \varphi^i. \quad (4.109)$$

Moreover, it is also metric-compatible, meaning

$$\hat{\mathcal{D}}_j a_{kl} = 0. \quad (4.110)$$

Time and space Weyl–Carroll covariant derivatives do not commute. Various geometric tensors can be defined through their commutators, such as the Weyl–Carroll curvatures (see App. C), which also arise in the small- k expansion of the relativistic Weyl curvature tensors introduced in (2.47), (2.48), (2.49), and evaluated in a Papapetrou–Randers background.

Then, the Carrollian equations (4.36), (4.37) and (4.87) for a Weyl-invariant fluid can be recast as

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \hat{\mathcal{D}}_i \Pi^i + \Pi^{ij} \xi_{ij} = 0, \quad (4.111)$$

$$\hat{\mathcal{D}}_i \Pi^i_j + 2\Pi^i \omega_{ij} + \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \delta_j^i + \xi^i_j \right) P_i = 0, \quad (4.112)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi_j + \Pi_i \xi^i_j = 0. \quad (4.113)$$

These equations are Weyl-covariant of weights $d+2$, $d+1$ and $d+1$ (P_i is weight- d). They are also manifestly hydrodynamic-frame invariant. The same goes for the matter sector, (4.42) which is recast

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \rho + \hat{\mathcal{D}}_j N^j = 0, \quad (4.114)$$

and is Weyl-covariant of weight $d+1$ and hydrodynamic-frame invariant.

4.2.4 More degrees of freedom

The main ingredient to obtain Carrollian fluid dynamics from the vanishing- k limit of relativistic hydrodynamics was the assumption on the behaviour of the energy, pressure, heat current and stress tensor given in (4.72), (4.73), (4.74) and (4.75) respectively. In the Galilean case, we considered the situation involving extra degrees of freedom satisfying extended systems of fluid equations which were reached at the infinite- k limit of the relativistic theory. In the Galilean case, the latter was of limited use but it may play a more prominent role when studying Carrollian fluids in the context of flat holography, where additional over-leading terms are needed in the components of the boundary energy-momentum tensor [42].

Now we consider the behaviour for the fluid variables as

$$\varepsilon = \frac{\zeta}{k^2} + \varepsilon + \mathcal{O}(k^2), \quad (4.115)$$

$$p = \frac{\phi}{k^2} + p + \mathcal{O}(k^2), \quad (4.116)$$

$$q^i = \frac{\psi^i}{k^2} + Q^i + k^2 \pi^i + \mathcal{O}(k^4), \quad (4.117)$$

$$\tau^{ij} = -\frac{\Sigma^{ij}}{k^2} - \Xi^{ij} + \mathcal{O}(k^2), \quad (4.118)$$

while for the matter sector we consider

$$\rho_0 = \frac{\omega}{k^2} + \chi + \mathcal{O}(k^2), \quad j^k = \frac{m^k}{k^2} + n^k + \mathcal{O}(k^2). \quad (4.119)$$

With these scalings, the expansion of the components of the energy–momentum tensor (2.121), (2.122), (2.123) becomes

$$\begin{cases} \varepsilon_r = \frac{\tilde{\Pi}}{k^2} + \Pi + \mathcal{O}(k^2), \\ q_r^i = \frac{\tilde{\Pi}^i}{k^2} + \Pi^i + k^2 P^i + \mathcal{O}(k^4), \\ p_r a^{ij} + \tau_r^{ij} = \frac{\tilde{\Pi}^{ij}}{k^2} + \Pi^{ij} + \mathcal{O}(k^2), \end{cases} \quad (4.120)$$

with Carrollian momenta given by

$$\begin{cases} \tilde{\Pi}^i = \psi^i \\ \Pi^i = Q^i - \beta_j (\Sigma^{ij} - \phi a^{ij}) + \beta^i (\zeta + \beta_j \psi^j) + \frac{\beta^2}{2} \psi^i \\ P^i = \pi^i - \beta_j (\Xi^{ij} - p a^{ij}) + \beta^i (\varepsilon + \beta_j Q^j) + \frac{\beta^2}{2} (Q^i + \frac{3\beta^2}{4} \psi^i) + \beta^i \beta^2 (\zeta + \phi + \frac{1}{2} \beta_j \psi^j) \\ \tilde{\Pi} = \zeta + 2\beta_i \psi^i \\ \Pi = \varepsilon + 2\beta_i Q^i - \beta_i \beta_j (\Sigma^{ij} - \phi a^{ij}) + \beta^2 (\zeta + \beta_i \psi^i) \\ \tilde{\Pi}^{ij} = \psi^i \beta^j + \beta^i \psi^j + \phi a^{ij} - \Sigma^{ij} \\ \Pi^{ij} = Q^i \beta^j + \beta^i Q^j + p a^{ij} - \Xi^{ij} + \beta^i \beta^j (\zeta + \phi) + \frac{\beta^2}{2} (\psi^i \beta^j + \beta^i \psi^j), \end{cases} \quad (4.121)$$

Similarly, the matter current (Eqs. (2.124) and (2.125)) exhibits the following:

$$\rho_{0r} = \frac{\tilde{\rho}}{k^2} + \rho + \mathcal{O}(k^2), \quad j_r^k = \frac{\tilde{N}^k}{k^2} + N^k + \mathcal{O}(k^2) \quad (4.122)$$

with

$$\begin{cases} \tilde{N}^j = m^j \\ N^j = n^j + \beta^j \omega \\ \tilde{\rho} = \omega + \beta_k m^k \\ \rho = \chi + \beta_k n^k + \frac{\beta^2}{2} \omega. \end{cases} \quad (4.123)$$

We use the above expansion in the conservation of the relativistic of the energy-momentum tensor in the Papapetrou–Randers frame (2.99). We find

$$\frac{k}{\Omega} \nabla_\mu T^\mu_0 = \frac{\mathcal{F}}{k^2} + \mathcal{E} + \mathcal{O}(k^2), \quad (4.124)$$

$$\nabla_\mu T^{\mu i} = \frac{\mathcal{X}^i}{k^4} + \frac{\mathcal{H}^i}{k^2} + \mathcal{G}^i + \mathcal{O}(k^2). \quad (4.125)$$

From the divergence of the matter current we obtain

$$\nabla_\mu J^\mu = \frac{\mathcal{N}}{k^2} + \mathcal{J} + \mathcal{O}(k^2). \quad (4.126)$$

In these expressions, \mathcal{E} , \mathcal{G}_j and \mathcal{J} are again given in (4.85), (4.86) and (4.93), while the new expressions are

$$\mathcal{F} = - \left(\frac{1}{\Omega} \hat{D}_t + \theta \right) \tilde{\Pi} - \left(\hat{V}_i + 2\varphi_i \right) \tilde{\Pi}^i - \tilde{\Pi}^{ij} \hat{\gamma}_{ij}, \quad (4.127)$$

$$\mathcal{H}_j = \left(\hat{V}_i + \varphi_i \right) \tilde{\Pi}^i_j + 2\tilde{\Pi}^i \varpi_{ij} + \tilde{\Pi} \varphi_j + \left(\frac{1}{\Omega} \hat{D}_t + \theta \right) \Pi_j + \Pi^i \hat{\gamma}_{ij}, \quad (4.128)$$

$$\mathcal{X}_j = \left(\frac{1}{\Omega} \hat{D}_t + \theta \right) \tilde{\Pi}_j + \tilde{\Pi}^i \hat{\gamma}_{ij}, \quad (4.129)$$

and

$$\mathcal{N} = \left(\frac{1}{\Omega} \partial_t + \theta \right) \tilde{\rho} + \left(\hat{V}_j + \varphi_j \right) \tilde{N}^j. \quad (4.130)$$

Therefore, the Carrollian equations are given by $\mathcal{E} = \mathcal{F} = \mathcal{G}_j = \mathcal{H}_j = \mathcal{X}_j = 0$, and $\mathcal{J} = \mathcal{N} = 0$ which describe the matter sector. All these equations are invariant under hydrodynamic-frame transformations. This is because the differential operators are geometric and thus invariant, and also because the momenta $\tilde{\Pi}$, Π , $\tilde{\Pi}^i$, Π^i , P^i , $\tilde{\Pi}^{ij}$, Π^{ij} , $\tilde{\rho}$, ρ , \tilde{N}^i , N^i are hydrodynamic-frame invariant, as shown in App. B.

4.2.5 Weyl-invariant Carrollian fluids

For this kind of systems with extra degrees of freedom, the analysis based on an effective action would required to know the complete set of variables conjugate to the momenta Π , Π^i , Π^{ij} and $\tilde{\Pi}$, $\tilde{\Pi}^i$, $\tilde{\Pi}^{ij}$, which is bigger than a_{ij} , b_i and Ω . Still, we can impose Weyl invariance by studying the zero- k limit of $T^\mu_\mu = 0$. Weyl invariance is here easier to impose as a zero- k

limit by using (2.119) with (4.120) $T^\mu{}_\mu = 0$. We find

$$T^\mu{}_\mu = \frac{1}{k^2} (\tilde{\Pi}_i{}^i - \tilde{\Pi}) + \Pi_i{}^i - \Pi + \mathcal{O}(k^2) = 0, \quad (4.131)$$

which lead to

$$\tilde{\Pi}_i{}^i = \tilde{\Pi}, \quad \text{and} \quad \Pi_i{}^i = \Pi. \quad (4.132)$$

Using the explicit expressions of the momenta (4.121), the above conditions can be recast in terms of Carrollian fluid observables. They are splitted in the same way as in (2.65), giving global equilibrium equations of state and conditions for the dissipative terms. They read

$$\zeta = d\phi, \quad \Sigma_i{}^i = 0, \quad \varepsilon = dp, \quad \Xi_i{}^i = \beta_i \beta_j \Sigma^{ij}. \quad (4.133)$$

The momenta $\tilde{\Pi}$, $\tilde{\Pi}^i$, $\tilde{\Pi}^{ij}$ have the same conformal weights as the set Π , Π^i , Π^{ij} . This implies that the extra variables ζ , ϕ , ψ_i and Σ_{ij} are of weight $d+1$, $d+1$, d and $d-1$, while for ω and m^i we find the weights d and $d+1$.

We use the definition of the time and space Weyl–Carroll covariant derivative to recast the Carrollian fluid equations $\mathcal{E} = \mathcal{F} = \mathcal{G}_j = \mathcal{H}_j = \mathcal{X}_j = 0$ and $\mathcal{J} = \mathcal{N} = 0$ in a Weyl covariant fashion as

$$\mathcal{J} = \frac{1}{\Omega} \hat{\mathcal{D}}_t \rho + \hat{\mathcal{D}}_j N^j, \quad (4.134)$$

$$\mathcal{N} = \frac{1}{\Omega} \hat{\mathcal{D}}_t \tilde{\rho} + \hat{\mathcal{D}}_j \tilde{N}^j, \quad (4.135)$$

and

$$\mathcal{E} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi - \hat{\mathcal{D}}_i \Pi^i - \Pi^{ij} \xi_{ij}, \quad (4.136)$$

$$\mathcal{F} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \tilde{\Pi} - \hat{\mathcal{D}}_i \tilde{\Pi}^i - \tilde{\Pi}^{ij} \xi_{ij}, \quad (4.137)$$

$$\mathcal{G}_j = \hat{\mathcal{D}}_i \Pi^i{}_j + 2\Pi^i \omega_{ij} + \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \delta_j^i + \xi^i{}_j \right) P_i, \quad (4.138)$$

$$\mathcal{H}_j = \hat{\mathcal{D}}_i \tilde{\Pi}^i{}_j + 2\tilde{\Pi}^i \omega_{ij} + \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \delta_j^i + \xi^i{}_j \right) \Pi_i, \quad (4.139)$$

$$\mathcal{X}_j = \frac{1}{\Omega} \hat{\mathcal{D}}_t \tilde{\Pi}_j + \tilde{\Pi}_i \xi^i{}_j. \quad (4.140)$$

Part of the above are the set of equations¹² present in flat holography, relating Ricci-flat spacetimes and Carrollian conformal fluids. See for example [75–77, 42]. They are covariant

¹²They are also known in the literature as flux balance equations. See for instance [35, 37].

under Carrollian diffeomorphisms, covariant under Weyl rescalings and invariant under hydrodynamic-frame transformations, which are local Carroll transformations corresponding to local Carroll boosts and rotations.

4.2.6 Isometries and conformal isometries

As a last application of the above results, we can consider the behaviour of Noether currents associated to (conformal) isometries. We insert the behaviour (4.120) inside the components (2.128), (2.129) and (2.130) of the relativistic Noether current. We get

$$-\frac{1}{k\Omega}I_0 = \iota_{0r} = \frac{\tilde{\kappa}}{k^4} + \frac{\tilde{\kappa}}{k^2} + \kappa + \mathcal{O}(k^2), \quad I^k = i_r^k = \frac{\tilde{K}^k}{k^2} + K^k + \mathcal{O}(k^2) \quad (4.141)$$

with (remember that $\xi^i \equiv \xi^{\hat{i}}$ is a function of \mathbf{x} only for Carrollian diffeomorphisms)

$$\begin{cases} \kappa = \xi^i P_i - \xi^{\hat{i}} \Pi \\ \tilde{\kappa} = \xi^i \Pi_i - \xi^{\hat{i}} \tilde{\Pi} \\ \tilde{\tilde{\kappa}} = \xi^i \tilde{\tilde{\Pi}}_i \\ K^i = \xi^j \Pi_j^i - \xi^{\hat{i}} \Pi^i \\ \tilde{K}^i = \xi^j \tilde{\tilde{\Pi}}_j^i - \xi^{\hat{i}} \tilde{\tilde{\Pi}}^i. \end{cases} \quad (4.142)$$

Inserting (4.141) in the relativistic divergence of the Noether current I^μ in Papapetrou–Randers background we find an extension of (4.51) in the form

$$\nabla_\mu I^\mu = \frac{\tilde{\mathcal{K}}}{k^4} + \frac{\tilde{\mathcal{K}}}{k^2} + \mathcal{K} + \mathcal{O}(k^2) \quad (4.143)$$

with

$$\begin{cases} \tilde{\mathcal{K}} = \left(\frac{1}{\Omega}\partial_t + \theta\right) \tilde{\tilde{\kappa}} = 0 \\ \tilde{\mathcal{K}} = \left(\frac{1}{\Omega}\partial_t + \theta\right) \tilde{\kappa} + \left(\hat{\nabla}_i + \varphi_i\right) \tilde{K}^i = -\tilde{\Pi}^i \left(\left(\hat{\partial}_i - \varphi_i\right) \xi^{\hat{i}} - 2\xi^j \varpi_{ji}\right) \\ \mathcal{K} = \left(\frac{1}{\Omega}\partial_t + \theta\right) \kappa + \left(\hat{\nabla}_i + \varphi_i\right) K^i = -\Pi^i \left(\left(\hat{\partial}_i - \varphi_i\right) \xi^{\hat{i}} - 2\xi^j \varpi_{ji}\right). \end{cases} \quad (4.144)$$

The above equations hold when ξ^i and $\xi^{\hat{i}}$ are Carrollian Killings satisfying (4.47) (or conformal Killings satisfying (3.73), (4.61), (4.63), (4.64)) and the equations of motion $\mathcal{E} = \mathcal{F} = \mathcal{G}_j = \mathcal{H}_j = \mathcal{X}_j = 0$ are satisfied (see equations (4.136), (4.137), (4.138), (4.139), (4.140) and (4.132) for the Weyl-covariant situation). We can see from (4.144) that one set of the three current pieces is always conserved, while the other two are not.

We see that a multiplication of degrees of freedom induced by additional pieces in the behaviour of the relativistic energy–momentum tensor and the matter current, implies a multiplication of the Noether currents (here three but possibly more if we consider more terms in (4.120)) associated to the (conformal) isometries of the spacetime. In general, these currents are not conserved unless we consider a more restricted set of (conformal) isometries that satisfy $\mathcal{L}_\xi \mu = \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{i}} - 2\xi^j \varpi_{ji} \right) dx^i = 0$. The other alternative to render the currents conserved is by imposing the vanishing of the appropriate piece in expansion of q_T^i . This is why $\tilde{\mathcal{H}} = 0$.

Chapter 5

Carrollian scalar field and gravitational Chern-Simons

The two different approaches that we used to derive Carrollian fluid dynamics can also be applied to other theories. Here we will investigate the Carrollian reduction of the scalar field and the gravitational Chern-Simons theory and find their Carrollian dynamics.

5.1 Carrollian scalar field

5.1.1 Small- k expansion of the relativistic scalar field

Two different situations arise when considering the dynamics of a scalar field defined on a Carrollian spacetime. These are given by the Lagrangian densities

$$\mathcal{L}_e = \frac{1}{2} \left(\frac{1}{\Omega} \partial_t \Phi \right)^2 - V_e(\Phi), \quad (5.1)$$

$$\mathcal{L}_m = -\frac{1}{2} a^{ij} \hat{\partial}_i \Phi \hat{\partial}_j \Phi - V_m(\Phi), \quad (5.2)$$

and they enter in the Carrollian action as $S_C = \int_{\mathcal{M}} dt d^d x \sqrt{a} \Omega \mathcal{L}$. The indices “e” and “m” stand for electric and magnetic and in both cases the action is Carrollian diffeomorphism invariant. They refer to the origin of these actions from the ultralocal limit of the original relativistic theory [87, 56]. Indeed starting from the action of a relativistic scalar field on a Papapetrou–Randers background (2.99)

$$S = - \int_{\mathcal{M}} dt d^d x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) \right), \quad (5.3)$$

and assuming the expansion of the potential $V(\Phi)$ in terms of the speed of light as

$$V(\Phi) = \frac{1}{k^2}V_e(\Phi) + V_m(\Phi) + \mathcal{O}(k^2), \quad (5.4)$$

we find

$$S = \frac{1}{k^2}S_e + S_m + \mathcal{O}(k^2) \quad (5.5)$$

with S_e and S_m the Carrollian actions with Lagrangian densities (5.1) and (5.2). The assumption given in (5.4) makes possible to reach these two Carrollian diffeomorphism invariant actions.

Due to the form of the metric (2.99), and to its behaviour under Carrollian diffeomorphisms, the decomposition of any relativistic tensor as a small- k expansion, provides a Carrollian tensor for each term. If we insist in reaching a single Carrollian tensor at vanishing k , then an appropriate rescaling by some power of k^2 is necessary in order to select one out of two options, if only two options are available as in the above scalar-field action (see [87], were this procedure is illustrated in Hamiltonian formalism and for flat spacetime).

5.1.2 The case of conformally coupled scalar field

For a relativistic curved spacetime in $d + 1$ dimensions we can consider the following potential:

$$V(\Phi) = \frac{d-1}{8d}R\Phi^2. \quad (5.6)$$

For a scalar field Φ of weight $w = \frac{d-1}{2}$, this is a conformal coupling. Here the relativistic energy–momentum tensor for (5.3) with (5.6) has the form $(\nabla_\mu \Phi = \partial_\mu \Phi)$

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ &= \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2}g_{\mu\nu} \nabla_\alpha \Phi \nabla^\alpha \Phi + \frac{d-1}{4d} (G_{\mu\nu} \Phi^2 + g_{\mu\nu} \square \Phi^2 - \nabla_\mu \nabla_\nu \Phi^2) \\ &= \mathcal{D}_\mu \Phi \mathcal{D}_\nu \Phi - \frac{1}{2}g_{\mu\nu} \mathcal{D}_\alpha \Phi \mathcal{D}^\alpha \Phi + \frac{d-1}{4d} \left(\left(\mathcal{R}_{(\mu\nu)} - \frac{\mathcal{R}}{2} g_{\mu\nu} \right) \Phi^2 + g_{\mu\nu} \mathcal{D}_\alpha \mathcal{D}^\alpha \Phi^2 - \mathcal{D}_{(\mu} \mathcal{D}_{\nu)} \Phi^2 \right), \end{aligned} \quad (5.7)$$

where $G_{\mu\nu}$ is the Einstein tensor, $\mathcal{R}_{\mu\nu}$ and \mathcal{R} the Weyl-covariant Ricci and scalar defined in (2.48) and (2.49), together with the Weyl-covariant derivative \mathcal{D}_μ . This energy–momentum tensor is traceless when Φ is on-shell, and Weyl-covariant of weight $d - 1$. The action is Weyl-invariant (up to boundary terms), whereas the equations of motion can be recast as

$$-\mathcal{D}_\mu \mathcal{D}^\mu \Phi + \frac{d-1}{4d} \mathcal{R} \Phi = 0. \quad (5.8)$$

As a consequence of diffeomorphism invariance, the energy–momentum tensor obeys a Weyl-covariant conservation equation given as

$$\nabla_\mu T^{\mu\nu} = \mathcal{D}_\mu T^{\mu\nu} = 0. \quad (5.9)$$

The interest for studying relativistic conformally coupled scalar fields is originally found in inflationary models of cosmology.¹ On the Carrollian side the motivation goes along the lines of the flat extension of gauge/gravity holographic correspondence, where the boundary is null infinity (endowed with a Carrollian structure).

The decomposition of the Ricci scalar R is

$$R = \frac{1}{k^2} \left(\frac{2}{\Omega} \partial_t \theta + \frac{1+d}{d} \theta^2 + \xi_{ij} \xi^{ij} \right) + \hat{r} - 2\hat{\nabla}_i \varphi^i - 2\varphi^i \varphi_i + k^2 \varpi_{ij} \varpi^{ij}, \quad (5.10)$$

which leads to the following expansion for the potential (5.6) as

$$V(\Phi) = \frac{1}{k^2} V_e(\Phi) + V_m(\Phi) + k^2 V_{\text{nd}}(\Phi) \quad (5.11)$$

with

$$V_e(\Phi) = \frac{d-1}{8d} \left(\frac{2}{\Omega} \partial_t \theta + \frac{1+d}{d} \theta^2 + \xi_{ij} \xi^{ij} \right) \Phi^2, \quad (5.12)$$

$$V_m(\Phi) = \frac{d-1}{8d} \left(\hat{r} - 2\hat{\nabla}_i \varphi^i - 2\varphi^i \varphi_i \right) \Phi^2, \quad (5.13)$$

$$V_{\text{nd}}(\Phi) = \frac{d-1}{8d} \varpi_{ij} \varpi^{ij} \Phi^2. \quad (5.14)$$

In the last expression the index “nd” stands for “non-dynamical.” The reason is that when the expression (5.11) of the potential is used in the relativistic action (5.3), it produces the Carrollian electric and magnetic actions – with some boundary terms dropped here², as well

¹See e.g. [122] where more references are displayed.

²On the relativistic side we find: $\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{d-1}{8d} R \Phi^2 = \frac{1}{2} \mathcal{D}^\mu \Phi \mathcal{D}_\mu \Phi + \frac{d-1}{8d} \mathcal{R} \Phi^2 - \frac{d-1}{4\sqrt{-g}} \partial_\mu (\sqrt{-g} A^\mu \Phi^2)$.

as a third action with no kinetic terms for Φ .³

$$S_e = \int dt d^d x \sqrt{a} \Omega \left(\frac{1}{2} \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 - \frac{d-1}{8d} \xi_{ij} \xi^{ij} \Phi^2 \right), \quad (5.15)$$

$$S_m = \int dt d^d x \sqrt{a} \Omega \left(-\frac{1}{2} \hat{\mathcal{D}}_i \Phi \hat{\mathcal{D}}^i \Phi - \frac{d-1}{8d} \hat{\mathcal{R}} \Phi^2 \right), \quad (5.16)$$

$$S_{\text{nd}} = - \int dt d^d x \sqrt{a} \Omega \frac{d-1}{8d} \varpi_{ij} \varpi^{ij} \Phi^2 \quad (5.17)$$

The Carrollian equations of motion for the two non-trivial cases are as follows:

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi + \frac{d-1}{4d} \xi_{ij} \xi^{ij} \Phi = 0 \quad \text{electric}, \quad (5.18)$$

$$-\hat{\mathcal{D}}_i \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} \hat{\mathcal{R}} \Phi = 0 \quad \text{magnetic}. \quad (5.19)$$

These equations are Weyl-covariant of weight $w = \frac{d+1}{2}$.

5.1.3 Carrollian set of momenta

As we have shown in the previous chapter, we can compute the set of momenta that are conjugate to the Carrollian geometric pieces. This give rise to an energy–stress tensor Π^{ij} , an energy flux Π^i and an energy density Π , defined as in (4.25), (4.26) and (4.27)⁴ with conformal weights $d+3$, $d+2$ and $d+1$. Weyl invariance for the action translates into condition (4.60) which is valid on-shell (as the tracelessness of the relativistic energy–momentum tensor).

We can also introduce the momentum P_i (weight d) which does not appear upon variation of the action. Still, as we saw in the previous chapter, it enters the conservation equations that mirror the Carrollian diffeomorphism invariance. For Weyl-invariant dynamics these are given by equations (4.111) and (4.112).

The conservation equations are satisfied when the field Φ is on-shell, and this allows to determine the momentum.

³These results coincide with those obtained for $d=2$ in Ref. [123], where the authors proceed with a thorough investigation of the possible Weyl-compatible terms. The kinetic terms of the electric and magnetic actions, (5.15) and (5.16), can also be compared to the corresponding results of [87]. They also agree up to the magnetic constraint introduced in Ref. [87], which would read here $\Pi_m^i = 0$ (see (5.21)). The latter guarantees the invariance of the action under local Carrollian boosts, which we have not required a priori – Carrollian invariance features here the covariance under Carrollian diffeomorphisms (2.100) of a theory defined on a Carrollian spacetime (4.1) and (4.2).

⁴See also [121].

Using Eqs. (4.25), (4.26) and (4.27), we obtain the following energy and momenta for the Carrollian electric and magnetic actions:

$$\begin{cases} \Pi_e^{ij} = \frac{a^{ij}}{2} \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 + \frac{d-1}{4d} \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t (\xi^{ij} \Phi^2) - a^{ij} \left(\frac{1}{2} \xi_{lk} \xi^{lk} \Phi^2 + \frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi^2 \right) \right) \\ \Pi_e^i = 0 \\ \Pi_e = \frac{1}{2} \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 - \frac{d-1}{8d} \xi_{ij} \xi^{ij} \Phi^2, \end{cases} \quad (5.20)$$

$$\begin{cases} \Pi_m^{ij} = \hat{\mathcal{D}}^i \Phi \hat{\mathcal{D}}^j \Phi - \frac{a^{ij}}{2} \hat{\mathcal{D}}_l \Phi \hat{\mathcal{D}}^l \Phi + \frac{d-1}{4d} \left(\left(\hat{\mathcal{R}}^{(ij)} - \frac{\hat{\mathcal{R}}}{2} a^{ij} \right) \Phi^2 + a^{ij} \hat{\mathcal{D}}_l \hat{\mathcal{D}}^l \Phi^2 - \hat{\mathcal{D}}^i \hat{\mathcal{D}}^j \Phi^2 \right) \\ \Pi_m^i = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} \left(\hat{\mathcal{D}}^i \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi^2 - \hat{\mathcal{D}}_j (\xi^{ij} \Phi^2) \right) \\ \Pi_m = \frac{1}{2} \hat{\mathcal{D}}_i \Phi \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} \left(\frac{\hat{\mathcal{R}}}{2} \Phi^2 - \hat{\mathcal{D}}_i \hat{\mathcal{D}}^i \Phi^2 \right). \end{cases} \quad (5.21)$$

For the non-dynamical action we find

$$\begin{cases} \Pi_{\text{nd}}^{ij} = \frac{d-1}{4d} \left(2\varpi^{li} \varpi_l^j - \frac{a^{ij}}{2} \varpi_{lk} \varpi^{lk} \right) \Phi^2 \\ \Pi_{\text{nd}}^i = \frac{d-1}{4d} \hat{\mathcal{D}}_j (\varpi^{ji} \Phi^2) \\ \Pi_{\text{nd}} = \frac{3(d-1)}{8d} \varpi_{ij} \varpi^{ij} \Phi^2. \end{cases} \quad (5.22)$$

They all obey (4.60), and conservation equations (4.111) and (4.112) are satisfied with the electric momenta, assuming the field be on-shell, namely obeying (5.18), and deliver the electric momentum

$$P_e^i = \Pi_m^i. \quad (5.23)$$

In a similar fashion for the magnetic dynamics, and using the equation of motion (5.19), we obtain

$$P_m^i = \Pi_{\text{nd}}^i. \quad (5.24)$$

One might be puzzled at this stage by the interplay of Eqs. (5.23) and (5.24), which seem to entail amongst electric, magnetic and non-dynamics. There is no doubt that electric and magnetic Carrollian scalar dynamics resulting from \mathcal{L}_e and \mathcal{L}_m are distinct, and can be studied separately, on any Carrollian background. Likewise, the action $\mathcal{L}_{\text{nd}} = -V_{\text{nd}}$ is also Carrollian-invariant, but is non-dynamical. What sets a deeper link between these dynamics, which is not visible when treating them directly in the Carrollian framework, is that they all emerge in the “small- k expansion” of a unique relativistic theory for the scalar field. This was one possible guideline for obtaining the Carrollian scalar theories. It can also be applied to the relativistic energy–momentum tensor a small- k expansion⁵ such that the Carrollian momenta appear at each power of k in the components of the energy–momentum tensor. The

⁵The wording “expansion” is an abuse because the result is exact here.

latter reads

$$\begin{cases} T^{ij} = \frac{1}{k^2}\Pi_e^{ij} + \Pi_m^{ij} + k^2\Pi_{nd}^{ij} \\ -\frac{k}{\Omega}T_0^i = \Pi_m^i + k^2\Pi_{nd}^i \\ \frac{1}{\Omega^2}T_{00} = \frac{1}{k^2}\Pi_e + \Pi_m + k^2\Pi_{nd}. \end{cases} \quad (5.25)$$

We can expand the relativistic conservation of energy–momentum (5.9) and recollect the Carrollian conservation equations for the electric, the magnetic and the non-dynamical cases. In this process Eqs. (4.111) and (4.112) arise for each case at a different k -order, and their momenta P_e^i and P_m^i are naturally determined in terms of the next-order energy fluxes. This explains the above results (5.23) and (5.24).

Now, for these Carrollian electric and magnetic dynamics, together with their set of momenta, one can easily obtain the Noetherian currents and charges associated to (conformal) isometries. The latter is explained in detail in [4], which is appended to this thesis, where we compute the conformal isometric charges for the case of a scalar field propagating on the null boundary of four-dimensional Robinson-Trautman spacetime.

5.2 Gravitational Chern-Simons and its Carrollian extensions

5.2.1 Relativistic Cotton tensor and gravitational Chern-Simons action

The Cotton tensor is a three index object defined on Riemannian manifolds of arbitrary dimension and partly antisymmetric that measures the deviation of a given geometry from conformal flatness. In three dimensions, which is the framework of this section, one can dualize this tensor and get the two-index symmetric tensor version as

$$C_{\mu\nu} = \eta_{\mu}{}^{\rho\sigma}\nabla_{\rho}\left(R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}\right). \quad (5.26)$$

Here $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$ is the metric with signature $(-++)$, $\eta_{\mu\nu\sigma} = \sqrt{-g}\varepsilon_{\mu\nu\sigma}$ ($\varepsilon_{012} = 1$), ∇_{ρ} the associated Levi–Civita connection and $R_{\nu\sigma}$ are the components of the Ricci tensor with scalar R . The combination of the Ricci tensor and Ricci scalar inside the derivative defines the Schouten tensor in three dimensions.

The Cotton tensor is Weyl-covariant, and conserved as a consequence of the second Bianchi identity

$$\nabla_{\rho}C^{\rho}{}_{\nu} = 0, \quad (5.27)$$

for any $g_{\mu\nu}$. In fact, the Cotton tensor appears as the energy–momentum tensor of the gravitational Chern–Simons action as

$$C^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{CS}}}{\delta g_{\mu\nu}} \quad (5.28)$$

where the gravitational Chern-Simons action is given by

$$S_{\text{CS}} = \frac{1}{2k} \int_{\mathcal{M}} \text{Tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right), \quad (5.29)$$

whit ω is the Levi–Civita connection one-form defined as $\omega^\mu{}_\nu = \Gamma_{\rho\nu}^\mu dx^\rho$. In this picture S_{CS} is a functional of the metric and of its derivatives.

As we did for the relativistic scalar field, the idea is to derive the different Carrollian Chern-Simons actions and their corresponding dynamics, namely the different Carrollian Cotton momenta at different powers in the small- k expansion. For the latter we work with the relativistic Chern-Simons action in a Papapetrou-Randers background (2.99) whose k dependence is explicit and allows to perform a Carrollian reduction for every relativistic tensor. On top of that, the Papapetrou–Randers metric (2.99) allows for the use of a convenient although non-orthonormal Cartan mobile frame $\{e_{\hat{0}} = \frac{1}{k\Omega} \partial_t, e_i = \hat{\partial}_i\}$ and coframe $\{\theta^{\hat{0}} = -k\mu, \theta^i = dx^i\}$. The hatted indices $\{\hat{0}, \hat{i}\}$ are meant to distinguish this frame from the coordinate coframe $\{\theta^0 = dx^0 = kdt, \theta^i = dx^i\}$ and frame $\{e_0 = \frac{1}{k} \partial_t, e_i = \partial_i\}$. In order to make the notation simpler and consistent with the conventions used in this thesis so far, we will keep the hat exclusively on the time direction. For the Carrollian side, we will rather use $\{e_{\hat{i}} = \frac{1}{\Omega} \partial_t, e_i = \hat{\partial}_i\}$ and $\{\theta^{\hat{i}} = -\mu, \theta^i = dx^i\}$, and ignore the hat on the spatial indices.

The relativistic (affine) connection one-form elements $\omega^\mu{}_\nu = \Gamma_{\rho\nu}^\mu \theta^\rho$ with $\Gamma_{\rho\nu}^\mu$ the Levi–Civita connection coefficients read:

$$\omega_{\hat{0}i} = k(\varphi_i \mu + \varpi_{ij} dx^j) - \frac{1}{k} \hat{\gamma}_{ij} dx^j, \quad (5.30)$$

$$\omega_{ij} = (k^2 \varpi_{ij} - \hat{\gamma}_{ij}) \mu + a_{il} \hat{\gamma}_{jk}^l dx^k \quad (5.31)$$

with $\omega_{\hat{0}i} = -\omega_{i\hat{0}}$. These expressions contain the Carrollian vorticity and acceleration defined in (4.11) and (4.12), as well as the Carroll-Levi-Civita connection coefficients and the extrinsic curvature of the spatial section \mathcal{S} defined in (4.9) and (4.15) respectively.

The above enter the Carrollian connection adopted here:

$$\hat{\omega}^{\hat{i}}_{\hat{i}} = \hat{\omega}^{\hat{i}}_{\hat{i}} = \hat{\omega}^{\hat{i}}_{\hat{i}} = 0, \quad \hat{\omega}^i_j = -\hat{\gamma}^i_j \mu + \hat{\gamma}^i_{jk} dx^k \quad (5.32)$$

(the hat signals this connection is Carrollian as opposed to the Riemannian displayed in (5.30) and (5.31)).

It is worth mentioning that the Carrollian connection (5.32) has been designed so as to define a parallel transport that respects the time-and-space splitting, as mentioned in the previous chapter, embracing distinct time and space Carrollian covariant derivatives $\frac{1}{\Omega}\hat{D}_t$ and $\hat{\nabla}_i$ being both metric compatible.

5.2.2 Carrollian expansion and its dynamics

The Carrollian descendants of the Riemann and Ricci tensors in the Papapetrou-Randers background (2.99) are obtained by using the prescription already used previously for the relativistic fluids and the scalar field. In a first step, this consists in reducing the relativistic tensors with respect to Carrollian diffeomorphisms. Next, one expands the latter in powers of k , and at each power one finds Carrollian tensors that transform covariantly with respect to Carrollian diffeomorphisms. Following this prescription for the Riemann curvature two-form of (2.99), which is generally defined as

$$\mathcal{R}^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu = \frac{1}{2}R^\mu{}_{\nu\rho\lambda} \theta^\rho \wedge \theta^\lambda \quad (5.33)$$

we find

$$\begin{aligned} \mathcal{R}^{\hat{0}}{}_i &= k \left[k^2 * \varpi^2 a_{ik} + \hat{\nabla}_{(i} \varphi_k) + \varphi_i \varphi_k - 2 * \varpi \eta_{j(i} \hat{\gamma}_k)^j - \frac{1}{k^2} \left(\frac{1}{\Omega} \hat{D}_t \hat{\gamma}_{ik} + \hat{\gamma}_{ij} \hat{\gamma}_k^j \right) \right] \mu \wedge dx^k \\ &\quad - k \left[\hat{\partial}_i * \varpi + 2\varphi_i * \varpi - \frac{1}{k^2} \eta^{mn} \hat{\nabla}_m \hat{\gamma}_{ni} \right] \frac{1}{2} \eta_{kl} dx^k \wedge dx^l, \end{aligned} \quad (5.34)$$

$$\begin{aligned} \mathcal{R}^i{}_j &= \hat{\mathcal{R}}^i{}_j - \eta^i{}_j \left[k^2 \left(\hat{\partial}_k * \varpi + 2\varphi_k * \varpi \right) + \eta^{mn} \varphi_m \hat{\gamma}_{nk} \right] \mu \wedge dx^k \\ &\quad + \eta^i{}_j \left[3k^2 * \varpi^2 + \frac{1}{2k^2} \eta^{mn} \eta^{rs} \hat{\gamma}_{mr} \hat{\gamma}_{ns} \right] \frac{1}{2} \eta_{kl} dx^k \wedge dx^l, \end{aligned} \quad (5.35)$$

where various Carrollian tensors emerge besides the Carrollian curvature two-form defined as

$$\hat{\mathcal{R}}^{\hat{i}}{}_j = 0, \quad \hat{\mathcal{R}}^i{}_j = \hat{R}^i{}_{kj} \mu \wedge dx^k + \frac{1}{2} \hat{R}^i{}_{jkl} dx^k \wedge dx^l. \quad (5.36)$$

No torsion is available for the relativistic Levi-Civita connection at hand.⁶

We can obtain the Cotton Carrollian relatives by following the above pattern. The reduction of $C^{\mu\nu}$ is straightforward: Carrollian scalars and vectors emerge from $C^{\hat{0}\hat{0}}$ and $C^{\hat{0}i}$, while $C^{ij} - \frac{C^{\hat{0}\hat{0}}}{2} a^{ij}$ leads to symmetric and traceless Carrollian tensors. Their decomposition

⁶The torsion two-form is defined as $\mathcal{T}^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = \frac{1}{2} T^\mu{}_{\nu\rho} \theta^\nu \wedge \theta^\rho$

in powers of k is

$$\frac{1}{k}C^{\hat{0}\hat{0}} = k^2c_{(-1)} + c_{(0)} + \frac{c_{(1)}}{k^2} + \frac{c_{(2)}}{k^4}, \quad (5.37)$$

$$C^{\hat{0}i} = k^2\psi^i + \chi^i + \frac{z^i}{k^2}, \quad (5.38)$$

$$\frac{C^{\hat{0}\hat{0}}a^{ij}}{2k} - \frac{C^{ij}}{k} = \Psi^{ij} + \frac{X^{ij}}{k^2} + \frac{Z^{ij}}{k^4}. \quad (5.39)$$

With this, any Carrollian structure supplied with the connection at hand, is naturally endowed with ten Weyl-covariant Carrollian Cotton descendants which are defined in App. D.

As for the conservation equation (5.27), we can obtain its Carrollian decompositions as (see App. D)

$$\nabla_\rho C^{\rho}_{\hat{0}} = k^2 \mathcal{D}_{\text{Cot}} + \mathcal{E}_{\text{Cot}} + \frac{\mathcal{F}_{\text{Cot}}}{k^2} + \frac{\mathcal{W}_{\text{Cot}}}{k^4} = 0, \quad (5.40)$$

and

$$\frac{1}{k}\nabla_\rho C^{\rho i} = k^2 \mathcal{I}_{\text{Cot}}^i + \mathcal{G}_{\text{Cot}}^i + \frac{\mathcal{H}_{\text{Cot}}^i}{k^2} + \frac{\mathcal{X}_{\text{Cot}}^i}{k^4} = 0. \quad (5.41)$$

All these identities are Weyl-covariant with

$$\mathcal{D}_{\text{Cot}} = -\frac{1}{\Omega}\hat{\mathcal{D}}_t c_{(-1)} - \hat{\mathcal{D}}_i \psi^i, \quad (5.42)$$

$$\mathcal{E}_{\text{Cot}} = -\frac{1}{\Omega}\hat{\mathcal{D}}_t c_{(0)} - \hat{\mathcal{D}}_i \chi^i + \Psi_{ij}\xi^{ij}, \quad (5.43)$$

$$\mathcal{F}_{\text{Cot}} = -\frac{1}{\Omega}\hat{\mathcal{D}}_t c_{(1)} - \hat{\mathcal{D}}_i z^i + X_{ij}\xi^{ij}, \quad (5.44)$$

$$\mathcal{W}_{\text{Cot}} = -\frac{1}{\Omega}\hat{\mathcal{D}}_t c_{(2)} + Z_{ij}\xi^{ij}, \quad (5.45)$$

and

$$\mathcal{I}_{\text{Cot}}^i = \frac{1}{2}\hat{\mathcal{D}}^i c_{(-1)} + 2*\omega*\psi^i, \quad (5.46)$$

$$\mathcal{G}_{\text{Cot}}^i = \frac{1}{2}\hat{\mathcal{D}}^i c_{(0)} - \hat{\mathcal{D}}_j \Psi^{ij} + 2*\omega*\chi^i + \frac{1}{\Omega}\hat{\mathcal{D}}_t \psi^i + \psi_j \xi^{ij}, \quad (5.47)$$

$$\mathcal{H}_{\text{Cot}}^i = \frac{1}{2}\hat{\mathcal{D}}^i c_{(1)} - \hat{\mathcal{D}}_j X^{ij} + 2*\omega*z^i + \frac{1}{\Omega}\hat{\mathcal{D}}_t \chi^i + \chi_j \xi^{ij}, \quad (5.48)$$

$$\mathcal{X}_{\text{Cot}}^i = \frac{1}{2}\hat{\mathcal{D}}^i c_{(2)} - \hat{\mathcal{D}}_j Z^{ij} + \frac{1}{\Omega}\hat{\mathcal{D}}_t z^i + z_j \xi^{ij}. \quad (5.49)$$

Interpreting the Cotton Carrollian descendants is possible along the same lines as for the ordinary Riemannian Cotton tensor. The main differences are that a Carrollian geometry has a fibre-bundle structure and a wider freedom for its affine connection. This blurs to some

extent the concept of conformal flatness, which is the feature emerging when the Cotton vanishes in three-dimensional Riemannian manifolds, and more options emerge.

Vanishing geometric shear When $\xi_{ij} = 0$ the time dependence in the metric a_{ij} is factorized as $a_{ij}(t, \mathbf{x}) = e^{2\sigma(t, \mathbf{x})} \bar{a}_{ij}(\mathbf{x})$. Moreover, in two dimensions $\bar{a}_{ij}(\mathbf{x})$ is necessarily proportional to δ_{ij} , hence choosing complex coordinates, the metric on the two-dimensional surface \mathcal{S} is recast as

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta} \quad (5.50)$$

with $P = P(t, \zeta, \bar{\zeta})$ a real function. Consequently, a subset of the Carroll–Cotton tensors vanish, as it is inferred from Eqs. (D.17), (D.18), (D.21) and (D.24), namely $c_{(1)}$, $c_{(2)}$, z_i and Z_{ij} .

Vanishing Carrollian vorticity From $*\varpi = 0$ we find

$$d\mu = \varphi \wedge \mu. \quad (5.51)$$

Using Fröbenius criterion we are instructed that μ is proportional to an exact form. We can choose appropriately the time coordinate so that the Ehresmann connection b vanishes, leading to

$$\mu = -\Omega(t, \mathbf{x}) dt. \quad (5.52)$$

The vanishing Carrollian Cotton tensors are now $c_{(-1)}$, $c_{(0)}$, ψ_i and Ψ_{ij} .

Vanishing Carrollian shear and vorticity This merges the two previous situations and the Carrollian structure is of the form

$$d\ell^2 = \frac{2}{P(t, \zeta, \bar{\zeta})^2} d\zeta d\bar{\zeta}, \quad \mu = -\Omega(t, \zeta, \bar{\zeta}) dt. \quad (5.53)$$

Despite the factorization of the metric and of the clock form, not all Carroll–Cotton tensors are zero. We find indeed from (C.8), (C.5),

$$\hat{\mathcal{R}}_\zeta = \frac{1}{\Omega} \partial_t \partial_\zeta \ln(\Omega P), \quad \hat{\mathcal{A}} = 0, \quad \hat{\mathcal{K}} = 2P^2 \partial_\zeta \partial_{\bar{\zeta}} \ln(\Omega P) \quad (5.54)$$

($\hat{\mathcal{R}}_{\bar{\zeta}}$ is the complex conjugate of $\hat{\mathcal{R}}_\zeta$), and using (D.20), (D.23) we obtain for the Carroll–Cotton

$$\begin{cases} \chi_\zeta = \frac{i}{2} \hat{\mathcal{D}}_\zeta \hat{\mathcal{K}} = \frac{i}{\Omega^2} \partial_\zeta \left((\Omega P)^2 \partial_\zeta \partial_{\bar{\zeta}} \ln(\Omega P) \right) \\ X_{\zeta\zeta} = i \hat{\mathcal{D}}_\zeta \hat{\mathcal{R}}_\zeta = \frac{i}{\Omega (\Omega P)^2} \partial_\zeta \left((\Omega P)^2 \partial_t \partial_\zeta \ln(\Omega P) \right), \quad X_{\zeta\bar{\zeta}} = 0 \end{cases} \quad (5.55)$$

with $\chi_{\bar{\zeta}} = \bar{\chi}_{\zeta}$ and $X_{\bar{\zeta}\bar{\zeta}} = \bar{X}_{\zeta\zeta}$. The tensors in (5.55) vanish if ΩP is constant. This result could have been anticipated by noticing that the Papapetrou–Randers (see Eq. (2.99)) pseudo-Riemannian ascendant of (5.53) is conformally flat provided ΩP be constant.

So in short, in Carrollian geometry, “conformal flatness” concerns separately the base and the fibre of the bundle $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ with distinct vanishing Carroll–Cotton tensors.

Now, expanding the Chern–Simons action (5.29) in powers of k in the Papapetrou–Randers background (2.99) equipped with Levi–Civita connection (5.30) and (5.31), delivers four distinct Carrollian avatars of the Chern–Simons dynamics,

$$S_{\text{CS}} = k^3 S_{\text{CCS}}^{\text{pm}} + k S_{\text{CCS}}^{\text{m}} + \frac{1}{k} S_{\text{CCS}}^{\text{e}} + \frac{1}{k^3} S_{\text{CCS}}^{\text{pe}}, \quad (5.56)$$

possessing four sets of Weyl-covariant Carrollian momenta, obeying four sets of conformal Carrollian conservation equations (4.111), (4.112). As for the relativistic gravitational Chern–Simons action, these Carroll–Chern–Simons descendants are in general anomalous under Carrollian diffeomorphisms and Weyl transformations with topological anomalies. The completed analysis of this feature is given in detail in [1], which is appended to this thesis. The Carrollian momenta and Carrollian conservation equations are precisely those recovered in Sec. 5.2.1 when decomposing the Riemannian Cotton tensor and its divergence. This is summarized as follows.

Paramagnetic Carroll–Chern–Simons This comes from the k^3 -order term in the Chern–Simons action and it reads

$$S_{\text{CCS}}^{\text{pm}} = 4 \int_{\mathcal{M}} dt d^2x \sqrt{a} \Omega * \bar{\omega}^3. \quad (5.57)$$

The associated momenta are $\Pi = 2c_{(-1)}$, $\Pi^i = 2\psi^i$, $\Upsilon^{ij} = 0$ ⁷ and Eqs. (4.136), (4.138) are now

$$\mathcal{D}_{\text{Cot}} = 0, \quad \mathcal{I}_{\text{Cot}}^i = 0, \quad (5.58)$$

see (5.42) and (5.46). From these equations we infer that the momentum P^i vanishes.

Magnetic Carroll–Chern–Simons The next order provides

$$S_{\text{CCS}}^{\text{m}} = \frac{1}{2} \int_{\mathcal{M}} \mu \wedge \left(\hat{\omega}^i{}_j \wedge dx^k \hat{\mathcal{D}}_k * \bar{\omega} + * \bar{\omega} d \hat{\omega}^i{}_j \right) \eta^j{}_i + \int_{\mathcal{M}} dt d^2x \sqrt{a} \Omega \left[* \bar{\omega} \hat{\nabla}_i \varphi^i - \varphi^i \hat{\mathcal{D}}_i * \bar{\omega} \right]. \quad (5.59)$$

⁷Here Υ^{ij} corresponds to the traceless part of Π^{ij} .

Now $\Pi = 2c_{(0)}$, $\Pi^i = 2\chi^i$, $\Upsilon^{ij} = -2\Psi^{ij}$ and Eqs. (4.111), (4.112) are

$$\mathcal{E}_{\text{Cot}} = 0, \quad \mathcal{G}_{\text{Cot}}^i = 0, \quad (5.60)$$

see (D.27) and (D.31), from which $P^i = 2\psi^i$.

Electric Carroll–Chern–Simons The order $1/k$ goes as

$$\begin{aligned} S_{\text{CCS}}^e = & \frac{1}{2} \int_{\mathcal{M}} \text{Tr} \left(\hat{\omega} \wedge d\hat{\omega} + \frac{2}{3} \hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega} \right) + \int_{\mathcal{M}} dt d^2x \sqrt{a} \Omega \left[\phi^i \eta^{kl} \hat{\nabla}_k \hat{\gamma}_{li} \right. \\ & \left. + * \varpi \eta^{kl} \eta^{ij} \hat{\gamma}_{ki} \hat{\gamma}_{lj} - * \varpi \left(\frac{1}{\Omega} \hat{D}_t \hat{\gamma}_i^j + 2 \hat{\gamma}_{ij} \hat{\gamma}^{ij} \right) + \hat{\gamma}_k^j \eta^{kl} \left(\hat{\nabla}_{(l} \phi_{i)} + \phi_l \phi_i \right) \right] \end{aligned} \quad (5.61)$$

with $\Pi = 2c_{(1)}$, $\Pi^i = 2z^i$, $\Upsilon^{ij} = -2X^{ij}$ and Eqs. (4.111), (4.112) are now

$$\mathcal{F}_{\text{Cot}} = 0, \quad \mathcal{H}_{\text{Cot}}^i = 0, \quad (5.62)$$

see (5.44) and (D.33). The momentum is $P^i = 2\chi^i$.

Paraelectric Carroll–Chern–Simons Finally, the lowest power in k gives

$$S_{\text{CCS}}^{\text{pe}} = - \int_{\mathcal{M}} dt d^2x \sqrt{a} \eta^{kl} \hat{\gamma}_k^i \hat{D}_t \hat{\gamma}_{il} = - \int_{\mathcal{M}} dt d^2x \sqrt{a} * \xi^{ij} \hat{\mathcal{D}}_t \xi_{ij} \quad (5.63)$$

leading to $\Pi = 2c_{(2)}$, $\Pi^i = 0$ and $\Upsilon^{ij} = -2Z^{ij}$. Equations (4.111) and (4.112) reduce to

$$\mathcal{W}_{\text{Cot}} = 0, \quad \mathcal{X}_{\text{Cot}}^i = 0, \quad (5.64)$$

see (5.45) and (5.49), which exhibit $P^i = 2z^i$.

The names given to the four Carroll–Chern–Simons actions follow the pattern already used e.g. for the scalar field in [4, 124, 87].

Chapter 6

Carrollian perspective of Einstein dynamics

In chapter 4 we derived the most general hydrodynamic equations of Carrollian fluids defined on an arbitrary Carrollian background. It has been argued that these exotic dynamics emerges when null hypersurfaces are under consideration, see for instance [20]. Then, it is natural to expect that physical systems that are constrained to live on null hypersurfaces will be of Carrollian nature. Two examples of the latter have been known in gravity: black hole horizon dynamics [48, 50, 51] and flat holography [35, 37, 42]. In this chapter we will focus in the emergence of Carrollian physics in the context of Ricci-flat/Carrollian fluid duality, which maps Einstein dynamics to Carrollian fluid dynamics defined at the null boundary of Ricci-flat spacetimes.

6.1 Bulk reconstruction of Ricci-flat spacetimes

6.1.1 Covariant Newman-Unti gauge

In general, four dimensional Ricci-flat spacetimes are expressed as an infinite series expansion in powers of the radial coordinate. The latter is performed in a designated gauge which is usually chosen to be Bondi or Newmann-Unti gauge. After the partial resolution of Einstein equations and assuming the appropriate fall-off conditions for the metric components, one is able to expressed the solution space in terms of an infinite set of functions that depend on time and the angles and satisfy a set of evolution (or constraint) equations¹. These set of equations capture the evolution of the Bondi mass and the angular momentum aspect, sourced by the news tensor encoding gravitational radiation [126]. Although powerful, Bondi

¹A nice review of the later is given in [125].

and Newmann-Unti gauges lack of a three dimensional boundary interpretation, which is necessary for our purpose here. In this direction one could wonder if it is possible to construct a gauge for the spacetime such that it is manifestly covariant with respect to the three dimensional boundary.

The latter is known to be possible for Einstein spacetimes with negative cosmological constant in the context of the AdS/CFT correspondence. Asymptotically AdS spacetimes can be expressed in the Fefferman-Graham gauge² [128, 129]. Here, the timelike conformal boundary is a three dimensional pseudo-Riemannian spacetime and the bulk solution is expressed as an expansion in the radial coordinate, parameterized order by order in terms of boundary tensors, covariant with respect to the boundary geometry. All these tensorial objects are expressed in terms of two independent boundary tensors: the boundary metric and the boundary energy-momentum tensor, which is covariantly conserved with respect to the boundary Levi-Civita connection. Its covariance with respect to the conformal boundary makes this choice of gauge suitable for holographic applications in the framework of AdS/CFT correspondence.

In the Ricci-flat spacetimes the geometry of the boundary is completely different from the asymptotically AdS case. The conformal boundary is null and therefore, it is endowed with a Carrollian geometry whose structure is given by (4.1) and (4.2), with a coordinate system adapted to the space/time splitting and respected by Carrollian diffeomorphisms

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}). \quad (6.1)$$

The Fefferman-Graham gauge is only valid for asymptotically AdS spacetimes and there is no extension for Ricci-flat spacetimes (no smooth vanishing Λ limit). Of course one can consider the Bondi or Newmann-Unti gauges to study asymptotically flat³ spacetimes, but one disadvantage of this is the fact that they fail in being covariant with respect to the boundary geometry since in those instances the geometry is locked⁴. An alternative is to consider a generalization of the latter (gauge relaxation) that belongs to the Eddington-Finkelstein type, namely possessing a light-like radial direction. The latter was first introduced in the particular context of fluid/gravity correspondence [38–41, 130–133] relating asymptotically AdS spacetimes with the dynamics of a relativistic conformal fluid in one dimension less. So this gauge applies to both, asymptotically AdS and Ricci-flat spacetimes, and it is not restricted to be used in the context of fluid/gravity duality. In this section we are not going to

²A construction of a Weyl covariant extension of the latter can be found in [127].

³Bondi and Newman-Unti gauges are also valid for asymptotically (A)dS spacetimes, see [126]

⁴Usually on those gauges the boundary the boundary geometry consists on base space restricted to be the round sphere and with no Ehresmann connection.

derive in full detail this alternative boundary covariant gauge but rather we will study on the key ideas behind this bulk reconstruction and its properties. A fully detailed derivation of the latter is given in [2].⁵

In this gauge the asymptotically AdS bulk spacetime is reconstructed in terms of the boundary data given by: the boundary geometry with metric $g_{\mu\nu}$, a boundary congruence (or velocity field) u^μ , and the energy-momentum tensor splitted in the energy density ε as well, heat current q^μ , and the viscous stress tensor $\tau^{\mu\nu}$. Notice that for the boundary data we are using hydrodynamic nomenclature. The holographic fluid interpretation is only valid when constitutive relations exist for the out of equilibrium quantities such as the heat current and the viscous stress tensor⁶. Regardless of the latter, we consider here the heat current and stress tensor as independent and arbitrary variables that may or may not give rise to hydrodynamic modes, and consider these as the data of an "abstract fluid". Then the reconstruction of the spacetime goes as follows: the above data enter in the bulk geometry in an expansion of inverse powers of the radial coordinate. This is guided by the gauge conditions

$$\boxed{g_{rr} = 0 \quad \text{and} \quad g_{r\mu} dx^\mu = \frac{\mathbf{u}}{k^2},} \quad (6.2)$$

and Weyl covariance implemented by the inclusion of the Weyl connection A_μ as defined in (2.40). Here $k^2 = 1/\ell^2$, being ℓ^2 the AdS radius. The allowed boundary terms that enters in the line element are selected by the resolution of Einstein's equations. We find that the reconstruction of the spacetime is of the form

$$\begin{aligned} ds_{\text{bulk}}^2 &= 2\frac{\mathbf{u}}{k^2}(dr + rA) + r^2 ds^2 + r\mathcal{C}_{\mu\nu} dx^\mu dx^\nu + \frac{1}{k^4} S_{\mu\nu} dx^\mu dx^\nu \\ &+ \sum_{s=1}^{\infty} \frac{1}{r^s} \left(f_{(s)} \frac{\mathbf{u}^2}{k^4} + 2\frac{\mathbf{u}}{k^2} f_{(s)\mu} dx^\mu + f_{(s)\mu\nu} dx^\mu dx^\nu \right), \end{aligned} \quad (6.3)$$

where $S = S_{\mu\nu} dx^\mu dx^\nu$ is defined in (2.51). The tensor $\mathcal{C}_{\mu\nu}$ is the shear of the affine null geodesic congruence tangent to ∂_r . We will refer to the latter as "Bondi shear". The Bondi shear is not free but, thanks to Einstein's equations, it is set to be related to the shear of the boundary congruence \mathbf{u} by

$$k^2 \mathcal{C}_{\mu\nu} = -2\sigma_{\mu\nu}, \quad (6.4)$$

⁵In [2] we worked in the orthonormal frame rather than the natural frame used in this Chapter.

⁶As a starting point on this framework, the authors in [39] consider a boosted planar AdS_{d+2} black hole, parameterized with the black hole temperature T and the velocity u^μ of the black, which is mapped to a perfect conformal fluid living on $\mathbb{R}^{d,1}$.

So it is transverse and traceless with respect to $g_{\mu\nu}$ and u^μ . As we can see here, the Bondi shear in the situation at hand (asymptotically AdS spacetime) is not dynamical, which reflects the absence of gravitational radiation. We will see soon that, when reaching the asymptotically flat instance through the vanishing k limit, $\mathcal{C}_{\mu\nu}$ will become a dynamical and free piece of data that encodes part of the information about gravitational radiation. Anticipating the latter, we can construct a boundary covariant, symmetric and traceless news-like tensor as

$$\mathcal{N}_{\mu\nu} = u^\lambda \mathcal{D}_\lambda \mathcal{C}_{\mu\nu}. \quad (6.5)$$

In (6.3), the functions $f_{(s)}$ are boundary scalars, $f_{(s)\mu}$ are boundary transverse vectors, and $f_{(s)\mu\nu}$ are boundary symmetric and transverse tensors. The conformal weights of these objects are $s+2$, $s+1$ and s respectively. Again, Einstein's equations dictate the form of these objects. In particular, for $s=1$ we find

$$f_{(1)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(1)\mu} dx^\mu + f_{(1)\mu\nu} dx^\mu dx^\nu = \frac{8\pi G}{k^4} \left(\varepsilon u^2 + \frac{4}{3} u \Delta q + \frac{2k^2}{3} \Delta \tau \right) \quad (6.6)$$

with $\Delta q = \Delta q_\mu dx^\mu$ and $\Delta \tau = \Delta \tau_{\mu\nu} dx^\mu dx^\nu$ defined as

$$\Delta q_\mu = q_\mu - \frac{1}{8\pi G} *c_\mu, \quad \Delta \tau_{\mu\nu} = \tau_{\mu\nu} + \frac{1}{8\pi G k^2} *c_{\mu\nu}, \quad (6.7)$$

where q_μ is the boundary heat current, $\tau_{\mu\nu}$ the boundary viscous stress tensor⁷, and c_ν and $c_{\mu\nu}$ are the vector and traceless-symmetric tensor parts of the Cotton tensor (see the Appendix D for decomposition of the Cotton tensor in three dimensions).

For the $s=2$ order we find

$$f_{(2)} = \frac{8\pi G}{3k^2} \left(\sigma_{\alpha\beta} \Delta \tau^{\alpha\beta} + \mathcal{D}_\alpha \Delta q^\alpha \right) + cq, \quad (6.8)$$

$$f_{(2)\mu} = -\frac{8\pi G}{3k^4} \sigma_{\mu\alpha} \Delta q^\alpha + \frac{4\pi G}{k^2} \left(h_{\mu\alpha} \mathcal{D}_\beta \Delta \tau^{\alpha\beta} + \frac{8}{3} q * \Delta q_\mu \right), \quad (6.9)$$

$$f_{(2)\mu\nu} = -\frac{4\pi G}{k^4} \left(\frac{4}{3} u^\alpha \mathcal{D}_\alpha \Delta \tau_{\mu\nu} + \frac{2}{3} h_{\mu\alpha} h_{\nu\gamma} \mathcal{D}^{(\alpha} \Delta q^{\gamma)} - \frac{1}{3} h_{\mu\nu} h^{\alpha\gamma} \mathcal{D}_\alpha \Delta q_\gamma + 2 \sigma_{(\mu}^\alpha \Delta \tau_{\nu)\alpha} \right) - \frac{1}{2k^4} \left(8\pi G \varepsilon \sigma_{\mu\nu} - c * \sigma_{\mu\nu} \right) + \frac{32\pi G}{3k^2} q * \Delta \tau_{\mu\nu}. \quad (6.10)$$

⁷Here we keep the heat current and the viscous stress tensor arbitrary. When dealing with genuine boundary fluids, these are also expressed in a derivative expansion of temperature and fluid velocity, together with the corresponding transport coefficients.

With the above, a subset of Einstein's equations are solved up to relevant order as⁸

$$\mathcal{E}_{rr} = \mathcal{O}\left(\frac{1}{r^7}\right), \quad \mathcal{E}_{r0} = \mathcal{O}\left(\frac{1}{r^5}\right), \quad \mathcal{E}_{ri} = \mathcal{O}\left(\frac{1}{r^5}\right), \quad \mathcal{E}_{ij} = \mathcal{O}\left(\frac{1}{r^3}\right). \quad (6.11)$$

The remaining equations give

$$\frac{1}{8\pi G}\mathcal{E}_{tt} = \frac{1}{r^2}\nabla_\mu T^\mu_t + \mathcal{O}\left(\frac{1}{r^3}\right), \quad \frac{1}{8\pi G}\mathcal{E}_{ti} = \frac{1}{r^2}\nabla_\mu T^\mu_i + \mathcal{O}\left(\frac{1}{r^3}\right). \quad (6.12)$$

The latter implies that, for a conserved boundary energy-momentum tensor, equations in (6.12) are solved up to $1/r^3$ order.

This gauge for the spacetime has the advantage of being valid also for asymptotically flat spacetime. The latter can be reached by taking the $k \rightarrow 0$ limit of (6.3). This vanishing cosmological constant limit acts in the conformal boundary fluid as a Carrollian limit (seen in the previous chapter) and was first discussed in [42] for the reconstruction of algebraically special Ricci-flat spacetimes. So, this has two effects in the three dimensional boundary: (i) in the $k \rightarrow 0$ limit the timelike conformal boundary turns to a null conformal boundary; (ii) in this limiting procedure the boundary energy momentum tensor is expanded in powers of k , where at each power of k different Carrollian data appear that obey different evolution equations. We take the expansion as

$$\varepsilon = \sum_{n \geq 0} \varepsilon_{(n)} k^{2n}, \quad (6.13)$$

$$q^i = \frac{\zeta^i}{k^2} + Q^i + k^2 \pi^i + \sum_{n \geq 2} k^{2n} \pi_{(n)}^i, \quad (6.14)$$

$$\tau^{ij} = -\frac{\zeta^{ij}}{k^4} - \frac{\Sigma^{ij}}{k^2} - \Xi^{ij} - k^2 E^{ij} - \sum_{n \geq 2} k^{2n} E_{(n)}^{ij}. \quad (6.15)$$

Notice that we have terms with inverse powers of k . These terms might induced some combinations of Carrollian tensors that are multiplied by factors like $1/k^n$, which diverge when taking the vanishing- k limit. Therefore, extra conditions have to be imposed in order to avoid these blow ups. Some of these conditions are

$$\zeta^i = \frac{{}^*z^i}{8\pi G}, \quad Q^i = \frac{{}^*\chi^i}{8\pi G}, \quad \zeta^{ij} = \frac{{}^*Z^{ij}}{8\pi G}, \quad \Sigma^{ij} = \frac{{}^*X^{ij}}{8\pi G}, \quad \Xi^{ij} = \frac{{}^*\Psi^{ij}}{8\pi G}. \quad (6.16)$$

The above enforces five pieces of the energy momentum tensor to be completely determined by the Carrollian geometry through the Cotton descendants (see App. D). Additionally, other

⁸Here $\mathcal{E}_{AB} = R_{AB} - \frac{1}{2}Rg_{AB} - 3k^2g_{AB}$

conditions will imply to satisfy some flux balance equations of some Carrollian dynamical fields, as we will see soon.

Now we have all the ingredients to expressed any Ricci-flat spacetime in what we call covariant Newmann-Unti gauge. The latter is reached here as a vanishing cosmological constant limit of an asymptotically AdS spacetime with line element (6.3). We find, up to $\mathcal{O}(1/r^2)$ order,

$$\begin{aligned}
ds_{\text{Ricci-flat}}^2 &= \mu \left[2dr + \left(2r\varphi_i - 2*\hat{\mathcal{D}}_i*\varpi - \hat{\mathcal{D}}_j\mathcal{C}_i^j \right) dx^i - \left(r\theta + \mathcal{K} \right) \mu \right] \\
&+ \left(r^2 + *\varpi^2 + \frac{\mathcal{C}_{kl}\mathcal{C}^{kl}}{8} \right) dl^2 + \left(r\mathcal{C}_{ij} + *\varpi*\mathcal{C}_{ij} \right) dx^i dx^j \\
&+ \frac{1}{r} \left(8\pi G \varepsilon_{(0)} \mu^2 - \frac{4}{3} \mu N_i dx^i - \frac{16\pi G}{3} E_{ij} dx^i dx^j \right) \\
&+ \frac{1}{r^2} \left(2*\varpi v - \hat{\mathcal{D}}_i N^i \right) \mu^2 - \frac{1}{r^2} \mu \left(\frac{1}{3} N^j \mathcal{C}_{ji} + \frac{8}{3} *\varpi*N_i + 8\pi G \hat{\mathcal{D}}_j E_i^j \right) dx^i \\
&+ \frac{1}{r^2} \left(\frac{16\pi G}{3} F_{ij} - 4\pi G \mathcal{C}_{(i}{}^k E_{j)k} \right) dx^i dx^j + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{6.17}
\end{aligned}$$

where the star designates a $d = 2$ Carrollian Hodge duality as defined in Appendix D, and the tensor F_{ij} is defined as ⁹

$$F_{ij} = \frac{1}{\Omega} \hat{\mathcal{D}}_t E_{(2)ij} - \frac{1}{2} \hat{\mathcal{D}}_{(i} \pi_{(2)j)} + \frac{3}{8} \varepsilon_{(1)} \mathcal{C}_{ij} - \frac{3}{8\pi G} *\varpi^3 *\mathcal{C}_{ij}. \tag{6.18}$$

The gauge conditions for this line element are

$$\boxed{g_{rr} = 0 \quad \text{and} \quad g_{r\mu} dx^\mu = \mu,} \tag{6.19}$$

with $\mu = -\Omega dt + b_i dx^i$ the clock form of the boundary Carrollian geometry. As one can see from the gauge conditions (6.19), the latter does not correspond neither to Bondi (since there is no determinant condition) nor Newman-Unti¹⁰ ($g_{rt} = -1$ and $g_{ri} = 0$). Here the time coordinate t corresponds to a retarded time and coincides at the boundary with the Carrollian time used in (4.1) and (4.2).

All the tensors entering expression (6.17) are defined on the null conformal boundary, which is Carrollian, and can be sorted as follows.

⁹We define the symmetric and traceless part of a Carrollian two-tensor s_{ab} as $s_{(ab)} = s_{(ab)} - \frac{1}{d} s_c^c \delta_{ab}$ (here $d = 2$).

¹⁰In our modified Newman-Unti gauge, the affinely parameterized null geodesic congruence tangent to ∂_r has in general expansion and twist due to Ω and b_i , making it non-hypersurface orthogonal as opposed to what happen in the usual Newman-Unti gauge.

Carrollian geometry The geometry of the null boundary is part of the bulk solution space, parameterized by a_{ij} , b_i and Ω . The latter provides different Carrollian quantities and attributes such as Carrollian connections, Carrollian curvature tensors, Carrollian Cotton descendants (D.12), (D.13), (D.14), among others. a_{ij} , b_i and Ω are free data whose only restriction is the vanishing of the Carrollian shear $\xi_{ij} = 0$ which arises as a consequence of Einstein's equations.

Bulk shear Additionally we have the symmetric and traceless boundary dynamic shear $\mathcal{C}_{ij}(t, \mathbf{x})$ that we call Bondi shear. As we mentioned previously, in the flat limit, the Bondi shear decouples and becomes a completely free function and sources the evolution equation of other tensorial data (as we will see for the Einstein's/Carrollian fluid equations). The dynamic shear carries part of the information of the bulk gravitational radiation through the symmetric and traceless Bondi-like news tensor¹¹

$$\hat{\mathcal{N}}_{ij} = \frac{1}{\Omega} \hat{\mathcal{D}}_t \mathcal{C}_{ij}. \quad (6.20)$$

The latter makes the shear and news tensor be defined as truly boundary conformal Carroll-covariant tensors of weights -1 and 0 respectively.

Carrollian fluid The boundary Carrollian fluid of asymptotically flat spacetimes corresponds to the descendant of a relativistic conformal fluid dual to asymptotically AdS spacetimes with conserved energy momentum tensor $T^{\mu\nu}$. As mentioned previously, our Carrollian fluid is reached at the vanishing velocity light limit of the relativistic hydrodynamic equation $\nabla_\mu T^{\mu\nu} = 0$, which corresponds to a vanishing cosmological constant limit in the bulk. This Carrollian fluid is described in terms of the energy density $\varepsilon_{(0)}$, two heat currents Q^i and π^i , and two the symmetric and traceless stress tensors Σ^{ij} and Ξ^{ij} .¹² These fluid data enter in the Carrollian momenta of the fluid dynamics defined in the previous chapter as

$$\Pi = \varepsilon_{(0)}, \quad \Pi^i = Q^i, \quad P^i = \pi^i, \quad \tilde{\Pi}^{ij} = -\Sigma^{ij}, \quad \Pi^{ij} = \frac{\varepsilon_{(0)}}{2} a^{ij} - \Xi^{ij}. \quad (6.21)$$

The boundary Carrollian fluid is not free, as opposed to its relativistic ascendant, but it is sourced by the shear, news and the Cotton descendants. These Carrollian fluid

¹¹These definitions do not exactly coincide with the original shear and news defined in BMS gauge. They vanish in Robinson–Trautmann spacetimes expressed in the gauge at hand, which is their defining gauge, and correspond to a radiating solution.

¹²The other two pieces ζ^i and ζ^{ij} are set to zero thanks to the condition $\xi_{ij} = 0$.

equations comes from the Einstein's equations $R_{tt} = 0$ and $R_{ti} = 0$. They read

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \hat{\mathcal{D}}_i \Pi^i = \frac{1}{16\pi G} \left[\hat{\mathcal{D}}_i \hat{\mathcal{D}}_j \hat{\mathcal{N}}^{ij} + \mathcal{E}^{ij} \hat{\mathcal{D}}_i \hat{\mathcal{R}}_j + \frac{1}{2} \mathcal{E}_{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_t \hat{\mathcal{N}}^{ij} \right] \quad (6.22)$$

$$\begin{aligned} \hat{\mathcal{D}}_j \Pi^{ij} + \frac{1}{\Omega} \hat{\mathcal{D}}_t P^i + 2 * \varpi * \Pi^i &= \frac{1}{16\pi G} \left[\mathcal{E}^{ij} \hat{\mathcal{D}}_j \hat{\mathcal{K}} + * \mathcal{E}^{ij} \hat{\mathcal{D}}_j \hat{\mathcal{A}} - 4 * \varpi * \mathcal{E}^{ij} \hat{\mathcal{R}}_j \right. \\ &\quad \left. - \frac{1}{2} \hat{\mathcal{D}}^j \left(\hat{\mathcal{D}}_j \hat{\mathcal{D}}_k \mathcal{E}^{ik} - \hat{\mathcal{D}}^i \hat{\mathcal{D}}^k \mathcal{E}_{jk} \right) + \mathcal{E}^{ij} \hat{\mathcal{D}}^k \hat{\mathcal{N}}_{jk} \right. \\ &\quad \left. + \frac{1}{2} \hat{\mathcal{D}}^j \left(\mathcal{E}^{ik} \hat{\mathcal{N}}_{jk} \right) - \frac{1}{4} \hat{\mathcal{D}}^i \left(\mathcal{E}^{jk} \hat{\mathcal{N}}_{jk} \right) \right] \quad (6.23) \end{aligned}$$

with Π , Π^i , Π^{ij} , $\tilde{\Pi}^{ij}$, P^i as in Eqs. (6.21). The above Carrollian equations are often referred to as flux-balance equations [35, 37] and describe the evolution of the independent momenta $\Pi = \varepsilon_{(0)}$ and $P^i = \pi^i$. These two independent data are related to the Bondi mass and angular momentum aspects, $M(t, \mathbf{x})$ and $N^i(t, \mathbf{x})$ as

$$8\pi G \varepsilon_{(0)} = 2M + \frac{1}{4} \mathcal{E}^{jk} \hat{\mathcal{N}}_{jk}, \quad (6.24)$$

$$8\pi G \pi^i = * \psi^i - N^i \quad (6.25)$$

with ψ^i defined in (D.20). The other two fluid equations for the fluid configuration at hand are the ones defined in (4.137) and (4.139) with vanishing $\tilde{\Pi}$ and $\tilde{\Pi}^i$, and are satisfied due to geometric identities of the Cotton descendants at zero ξ_{ij} (see App. D).

We can notice that in this fluid configuration $\Pi^i = Q^i \neq 0$. The presence of a non vanishing energy flux breaks local Carroll boost invariance on the boundary Carrollian fluid associated to Ricci-flat spacetimes. This breaking give also account to gravitational radiation, which in the boundary-covariant gauge designed in (6.17) it is not only encoded in the news tensor (6.20) but also in the Carrollian energy flux $\Pi^i = Q^i = \frac{1}{8\pi G} * \chi^i$ and the Carrollian stress $\tilde{\Pi}^{ij} = -\Sigma^{ij} = -\frac{1}{8\pi G} * X^{ij}$. As an example we can consider Robinson-Trautman spacetimes, where the gravitational radiation is exclusively captured in χ^i and X^{ij} .

All the above fluid related tensors appear at every order in the $1/r$ expansion of the bulk spacetime, in the same spirit as in the case of bulk asymptotically AdS spacetimes (in the Fefferman-Graham or in the modified Newman-Unti gauge).

Magnetic mass aspect Here we define a magnetic mass aspect which is denoted by v and appears in the bulk line element (6.17) at order $1/r^2$. Its definition relies on the vanishing k limit of the relativistic Cotton density c associated to the timelike conformal

boundary. The latter read

$$\mathbf{v} = \frac{1}{2} \lim_{k \rightarrow 0} c = \frac{1}{2} c_{(0)} - \frac{1}{4} \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j * \mathcal{C}^{ij} - \frac{1}{8} \mathcal{C}_{ij} * \hat{\mathcal{N}}^{ij}. \quad (6.26)$$

We can subtract the radiative contribution of (6.26) to define the nut aspect

$$N = \mathbf{v} + \frac{1}{8} \mathcal{C}_{ij} * \hat{\mathcal{N}}^{ij} = \frac{1}{2} c_{(0)} - \frac{1}{4} \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j * \mathcal{C}^{ij} \quad (6.27)$$

that satisfies the following evolution equation:

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t N = -\frac{1}{2} \hat{\mathcal{D}}_i \mathcal{X}^i - \frac{1}{4} \left(\hat{\mathcal{D}}_i \hat{\mathcal{D}}_j * \hat{\mathcal{N}}^{ij} - * \mathcal{C}^{ij} \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j \right). \quad (6.28)$$

The above equation is derived by inserting (6.27) into the Bianchi identity (D.27), and therefore is purely geometrical.

Further non-fluid degrees of freedom While asymptotically AdS spacetimes are reconstructed in terms of a finite set of boundary data (boundary metric and energy-momentum tensor), Ricci-flat spacetimes contain an infinite set of arbitrary degrees of freedom besides the geometric and fluid data. These correspond to an infinite number of Carrollian tensors that appear at all orders in the radial expansion, and obey Carrollian evolution (or flux balance) equations. These are dubbed "Chthonian" degrees of freedom. Part of this infinite set of Chthonian degrees of freedom that we find up to order $1/r^3$ are the dynamic shear \mathcal{C}_{ij} that appear at $\mathcal{O}(0)$, the symmetric and traceless tensors E_{ij} and $E_{(2)ij}$, and the transverse vector $\pi_{(2)i}$. As we mentioned before, some of the conditions necessary to avoid divergences in the line element due to the $k \rightarrow 0$ limit correspond to flux-balance equations. The flux-balance equation for E_{ij} appears in this way and we find

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t E_{ij} = \frac{3}{16\pi G} \left(-\frac{1}{3} \hat{\mathcal{D}}_{(i} N_{j)} - 2\pi G \varepsilon_{(0)} \mathcal{C}_{ij} + \frac{\mathbf{v}}{2} * \mathcal{C}_{ij} \right). \quad (6.29)$$

The flux-balance equations for $E_{(2)ij}$ and $\pi_{(2)i}$ are expected to appear at $1/r^3$ in the same way.

All the above make (6.17) fully covariant with respect to the Carrollian boundary. The latter can also be supplemented with the boundary-fluid hydrodynamic frame invariance by relaxing even more the gauge. This has been done in three dimensional bulk in [76, 77, 134, 75].

6.1.2 Resumming the series expansion

There are some cases where we can resumme the series expansion given in (6.17). Resummation occurs when additional conditions on the boundary data are imposed, enforcing specific features for the bulk Weyl tensor. These conditions are the following:

1. the vanishing of the dynamic shear $\mathcal{C}_{ij}(t, \mathbf{x})$, implying the relation $M = 4\pi G \varepsilon_{(0)}$;
2. all other non-Carrollian-fluid related degrees of freedom are discarded, as e.g. E_{ij} , $E_{(2)ij}$ and $\pi_{(2)i}$;
3. N^i in (6.25) is set to zero. This fixes the Carrollian momentum P^i with the Carrollian Cotton descendant ψ_i as

$$\pi^i = \frac{1}{8\pi G} * \psi^i. \quad (6.30)$$

With the above conditions, the degrees of freedom necessary to reconstruct the bulk spacetime are reduced to the ones describing the boundary Carrollian geometry (metric, field of observers and Erhesmann connection), and the Carrollian fluid energy density (Bondi mass). Then, the line element (6.17) is now resummed into an exact Ricci-flat spacetime of algebraically special type¹³ whose expression is

$$\begin{aligned} ds_{\text{res.}}^2 = & \mu \left[2dr + 2 \left(r\varphi_j - *\hat{\mathcal{D}}_j * \bar{\omega} \right) dx^j - \left(r\theta + \mathcal{K} \right) \mu \right] + \rho^2 d\ell^2 \\ & + \frac{\mu^2}{\rho^2} \left[8\pi G \varepsilon_{(0)} r + *\bar{\omega} c_{(0)} \right] \end{aligned} \quad (6.31)$$

with

$$\rho^2 = r^2 + *\bar{\omega}^2. \quad (6.32)$$

Here, Ricci-flatness is guaranteed by the Carrollian fluid equations (4.111) and (4.112) which, due to the vanishing of \mathcal{C}_{ij} , are genuine Carrollian conservation equations for the momenta, without forcing terms in the right hand side. The set of momenta are now given by

$$\begin{aligned} \Pi = \varepsilon_{(0)}, \quad \Pi^i = \frac{1}{8\pi G} * \chi^i, \quad P^i = \frac{1}{8\pi G} * \psi^i, \quad \tilde{\Pi}^{ij} = -\frac{1}{8\pi G} * X^{ij}, \\ \Pi^{ij} = \frac{\varepsilon_{(0)}}{2} a^{ij} - \frac{1}{8\pi G} * \Psi^{ij}. \end{aligned} \quad (6.33)$$

The Carrollian Cotton tensors defined in App. D obey identical set of equations and the geometric shear remains vanishing. The latter makes possible to further simplify equations

¹³See details and examples in [42].

(4.111) and (4.112) into

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \varepsilon_{(0)} + \frac{1}{8\pi G} \hat{\mathcal{D}}_i * \chi^i = 0, \quad (6.34)$$

$$\hat{\mathcal{D}}_j \varepsilon_{(0)} - \frac{1}{8\pi G} * \hat{\mathcal{D}}_j c_{(0)} = 0, \quad (6.35)$$

where $c_{(0)}$ and χ^i are given in geometric terms in (D.15) and (D.20), and $\varepsilon_{(0)}$ given by the Bondi mass, as stressed in item 1.

One can anticipate, by looking at (6.34) and (6.35) as well as (D.27), that the energy density $\varepsilon_{(0)}$ and the Carrollian Cotton scalar density $c_{(0)}$ play dual roles. We will formulate this concretely in the coming sections when discussing the boundary action of the Ehlers group. Still, by anticipating this observation, we can introduce the following Carrollian complex scalar $\hat{\tau}(t, \mathbf{x})$ and vector $\hat{\chi}^j(t, \mathbf{x})$:

$$\hat{\tau} = -c_{(0)} + 8\pi i G \varepsilon_{(0)}, \quad (6.36)$$

$$\hat{\chi}^j = \chi^j - i * \chi^j. \quad (6.37)$$

The above allows to recast the aforementioned equations as¹⁴

$$\begin{aligned} \frac{1}{\Omega} \hat{\mathcal{D}}_t \hat{\tau} &= \hat{\mathcal{D}}_j \hat{\chi}^j, & \hat{\mathcal{D}}_j \hat{\tau} \hat{\mathcal{D}}^j \hat{\tau} &= 0, \\ \hat{\mathcal{D}}_j \hat{\tau} \hat{\mathcal{D}}^j \hat{\tau} &= 8 \left(2 * \varpi * \chi_i + \frac{1}{\Omega} \hat{\mathcal{D}}_i \Psi_i - \hat{\mathcal{D}}^j \Psi_{ji} \right) \left(2 * \varpi * \chi^i + \frac{1}{\Omega} \hat{\mathcal{D}}_i \Psi^i - \hat{\mathcal{D}}_k \Psi^{ki} \right). \end{aligned} \quad (6.38)$$

We can also act with a second spatial derivative on (6.35) and use (C.1) to obtain

$$\hat{\mathcal{D}}^j \hat{\mathcal{D}}_j \hat{\tau} = 2i \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t * \varpi \hat{\tau} - \hat{\mathcal{A}} \hat{\tau} \right). \quad (6.39)$$

It is worth mentioning that the fluid equations (6.34) and (6.34) coincide with equations (29.15) and (29.16) of [137]¹⁵. Indeed, complicated Einstein's equations as the latter can be easily tamed into simple fluid equations supplemented by a sort of "self duality requirements" (relationship between Cotton and fluid data), by using the null boundary analysis studied in this section, plus Carrollian geometric tools. The latter suggests that Carrollian covariance is definitely the natural language for studying on properties of asymptotically flat spacetimes.

¹⁴The first equation in (6.38) is a flux-balance equation, driven exclusively by the Cotton vector $\hat{\chi}^j$ displayed in (6.37). The loss phenomenon concerns both the mass aspect $\varepsilon_{(0)}$ and the "magnetic-mass aspect" v , as captured in Eqs. (76) and (80) of [135] – see also App. D of [136].

¹⁵For that purpose, the following identifications are necessary: $b_\zeta = -L$, $*\varpi = -\Sigma$, $\hat{\tau} = 2(M + im)$, $\Omega = 1$, $t = u$, whereas their radial coordinate is $\tilde{r} = r - r_0$ with $r_0(t, \zeta, \bar{\zeta})$ the origin in the affine parameter of the geodesic congruence tangent to ∂_r .

The solution space of the resummed Ricci-flat spacetime given by (6.31) is restricted to be of algebraically special Petrov type. This is guaranteed thanks to the Goldberg-Sachs theorem of the null, geodesic congruence tangent to ∂_r , which in the resummed case, it is shear-free. The latter is part of the canonical null tetrad introduced in [42] and [137], which is parallelly transported along the null direction ∂_r (thanks to the affine nature of r). In complex coordinates ζ and $\bar{\zeta}$, as given in Appendix D, the null tetrad is expressed in a Carrollian fashion as

$$\begin{cases} \mathbf{k} = \partial_r \\ \mathbf{l} = \frac{1}{2} \left(\frac{8\pi G \mathcal{E}_{(0)} r^{+\ast} \bar{\omega} c_{(0)}}{\rho^2} - r\theta - \hat{\mathcal{K}} \right) \partial_r + \mathbf{v} \\ \mathbf{m} = \frac{P}{r - i\ast\bar{\omega}} \left(\hat{\partial}_{\bar{\zeta}} + \left(\ast \hat{\mathcal{D}}_{\bar{\zeta}} \ast \bar{\omega} - r\varphi_{\bar{\zeta}} \right) \partial_r \right) \end{cases} \quad (6.40)$$

with the usual relations $\mathbf{k} \cdot \mathbf{l} = -1$, $\mathbf{m} \cdot \bar{\mathbf{m}} = 1$ and $ds_{\text{res.}}^2 = -2\mathbf{k}\mathbf{l} + 2\mathbf{m}\bar{\mathbf{m}}$. In general, \mathbf{k} is a multiplicity-two principal null direction of the Weyl tensor, and by using the tetrad (6.40) we find the following Weyl complex scalars:¹⁶

$$\begin{cases} \Psi_0 = \Psi_1 = 0 \\ \Psi_2 = \frac{i\hat{t}}{2(r - i\ast\bar{\omega})^3} \\ \Psi_3 = \frac{iP\chi_{\zeta}}{(r - i\ast\bar{\omega})^2} + \mathcal{O}(1/(r - i\ast\bar{\omega})^3) \\ \Psi_4 = \frac{iX_{\zeta}}{r - i\ast\bar{\omega}} + \mathcal{O}(1/(r - i\ast\bar{\omega})^2). \end{cases} \quad (6.41)$$

All Ψ s are spelled using the Carrollian descendants of the boundary Cotton tensor – as well as their derivatives in the higher-order terms.

6.2 Bulk versus boundary isometries

In this chapter, the spacetimes that we are going to consider have at least one Killing vector field. Along the lines of a three dimensional boundary interpretation of Ricci-flat spacetimes, it is natural to wonder how bulk isometries are manifested on the boundary geometry, and vice versa, how one could design bulk isometries from purely boundary considerations.

For the sake of clarity of the present analysis, we consider vector fields that do not have components in ∂_r , and whose other components depend only on the boundary coordinates t and \mathbf{x} . We could be more general and consider radial dependence by assuming a series expansion in powers of r in every component, but this would make the calculations more

¹⁶In the instance of Petrov type D solutions, neither Ψ_3 nor Ψ_4 are necessarily zero, because \mathbf{l} is not always a principal null direction. One can reach another null tetrad through a Lorentz transformation such that \mathbf{l}' be a principal direction of multiplicity two whereas $\mathbf{k}' \propto \mathbf{k}$, and hence $\Psi'_3 = \Psi'_4 = 0$.

complicated without adding more value to our simple but concrete conclusion: bulk isometries are mapped onto boundary Carrollian isometries generated by strong Killing vectors that satisfy (4.56). For the same reason, from now on, we will work with algebraically special Ricci-flat solutions which are given by the line element (6.31).

In order to reach the aforementioned conclusion in a simple way, we can adopt bulk Cartan frame and co-frame aligned with the fiber (4.2) and the clock form (4.3) of the boundary Carrollian structure. In this formalism we have

$$\begin{aligned} e_{\hat{t}} &\equiv v = \frac{1}{\Omega} \partial_t, & e_{\hat{i}} &\equiv \hat{\partial}_i = \partial_i + \frac{b_i}{\Omega} \partial_t, & e_{\hat{r}} &\equiv \partial_r, \\ \theta^{\hat{t}} &\equiv -\mu = \Omega dt - b_i dx^i, & \theta^{\hat{i}} &\equiv dx^i, & \theta^{\hat{r}} &\equiv dr. \end{aligned} \quad (6.42)$$

Then, the components for the resummed bulk metric (6.31) read

$$\begin{aligned} g_{\hat{t}\hat{t}} &= \frac{1}{\rho^2} (8\pi G\mathcal{E}r + *\varpi c) - r\theta - \hat{\mathcal{K}}, & g_{\hat{t}\hat{i}} &= *\hat{\mathcal{D}}_i *\varpi - r\varphi_i, & g_{\hat{t}\hat{r}} &= -1, \\ g_{\hat{r}\hat{i}} &= 0, & g_{\hat{r}\hat{r}} &= 0, & g_{\hat{i}\hat{j}} &= \rho^2 a_{ij}. \end{aligned} \quad (6.43)$$

Assuming the bulk Killing vector of the form

$$\xi = \xi^t(t, \mathbf{x}) \partial_t + \xi^k(t, \mathbf{x}) \partial_k = \xi^{\hat{t}}(t, \mathbf{x}) v + \xi^k(t, \mathbf{x}) \hat{\partial}_k, \quad (6.44)$$

where $\xi^{\hat{t}} = \Omega \xi^t - \xi^k b_k$, we can determine the Lie derivative of the metric in terms of boundary Carrollian objects as

$$\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{r\hat{t}} = \mu, \quad \mathcal{L}_\xi g_{ri} = v_i, \quad (6.45)$$

$$\mathcal{L}_\xi g_{ij} = 2\rho^2 \left(\hat{\nabla}_{(i} \xi^k a_{j)k} + \xi^{\hat{t}} \hat{\gamma}_{ij} \right) - 2g_{\hat{t}(i} v_{j)} + a_{ij} \xi (*\varpi^2), \quad (6.46)$$

$$\begin{aligned} \mathcal{L}_\xi g_{\hat{t}\hat{i}} &= -g_{\hat{t}\hat{i}} \mu - g_{\hat{t}\hat{r}} v_i - r \left(\xi(\varphi_i) + \varphi_j \hat{\partial}_i \xi^j \right) \\ &\quad + \xi \left(*\hat{\mathcal{D}}_i *\varpi \right) + \left(*\hat{\mathcal{D}}_j *\varpi \right) \hat{\partial}_i \xi^j + \rho^2 a_{ij} \frac{1}{\Omega} \partial_t \xi^j \end{aligned} \quad (6.47)$$

and

$$\mathcal{L}_\xi g_{\hat{t}\hat{t}} = -2g_{\hat{t}\hat{t}} \mu + 2g_{\hat{t}\hat{i}} \frac{1}{\Omega} \partial_t \xi^i - \xi \left(\frac{1}{\rho^2} (8\pi G\mathcal{E}r + *\varpi c) - r\theta - \hat{\mathcal{K}} \right), \quad (6.48)$$

with $\mu(t, \mathbf{x})$ given by (4.64) and

$$v_i(t, \mathbf{x}) = - \left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{t}} + 2\xi^j \varpi_{ji}. \quad (6.49)$$

Now, since we are considering Killing fields that are independent of the radial coordinate, the Lie derivative of the metric must vanish at every order in the r expansion. This gives as a result several conditions for the components of ξ . The first one is

$$\partial_t \xi^i = 0, \quad (6.50)$$

which is expected for the generators of Carrollian diffeomorphisms. We also find, as a result of the bulk Killing equations, the conditions

$$\hat{\nabla}_{(i} \xi^k a_{j)k} + \xi^i \hat{\gamma}_{j} = 0, \quad (6.51)$$

$$\frac{1}{\Omega} \partial_t \xi^i + \varphi_i \xi^i = 0, \quad (6.52)$$

$$-\left(\hat{\partial}_i - \varphi_i\right) \xi^i + 2\xi^j \varpi_{ji} = 0, \quad (6.53)$$

which corresponds to the Carrollian Killing equations plus the extra condition for strong Carrollian Killing vectors. Thus, the bulk Killing field (6.44) is mapped onto a strong Carrollian Killing vector. There are additional conditions that appear as the vanishing of the Lie derivatives of other Carrollian tensors, but those conditions are guaranteed due to the strong Killing requirements.

6.3 Tower of charges from the boundary perspective

The asymptotic symmetry group of Ricci-flat spacetimes (BMS_4) allows to define an infinite set of charges. A bulk construction of such leading and subleading gravitational charges has been presented in [95, 94, 93, 97, 98] based on the asymptotic expansion of the Barnich-Brandt charges [138]. These infinite set of charges (and dual charges) are not generally conserved due to the presence of gravitational radiation.

The Carrollian approach for Ricci-flat spacetimes that was presented in the previous sections allow us for an alternative construction of the gravitational charges and dual charges from purely boundary considerations. That is, from the Carrollian dynamics emerging on the boundary due to bulk Einstein's equations, and the Carrollian conformal isometry group (isomorphic to BMS_4) available in the boundary theory. These charges are not conserved in general due to gravitational radiation, which in this approach is encoded in the Carrollian Cotton descendants (D.13), (D.14), and the Bondi shear \mathcal{C}_{ij} .

Before presenting the Carrollian version of the bulk charges, two observations are worth mentioning regarding Ricci-flat metric in either its general form (6.17) or its resummed version (6.31). On the one hand, at every $1/r^{2s+1}$ order in the metric expansion, Einstein's

equations reveal Carrollian dynamics obeying (4.136), (4.137), (4.138) and (4.139) with momenta $\Pi_{(s)}$, $\Pi_{(s)}^i$, $P_{(s)}^i$, $\tilde{\Pi}_{(s)}^{ij}$ and $\Pi_{(s)}^{ij}$, and possibly a non-vanishing right hand side sourced by the dynamic shear and news tensor. For every set of momenta, together with the Carrollian conformal Killing vectors (D.34), we can construct a set of currents $\{\kappa_{(s)}, K_{(s)}^i, \tilde{\kappa}_{(s)}, \tilde{K}_{(s)}^i\}$ following (4.142) and a set of charges $\{Q_{(s)T,Y}, \tilde{Q}_{(s)T,Y}\}$ by using (4.57). The conservation or non-conservation of these charges depend on $\mathcal{K}_{(s)}$ and $\tilde{\mathcal{K}}_{(s)}$ (see (4.144)). For $s = 0$ ($1/r$ order), we obtain the set of Carrollian fluid momenta given in (6.21) and their corresponding charges are leading. The $s \geq 1$ set of momenta lead to subleading charges. These leading and subleading charges are referred as electric charges since their conservation, if any, are on-shell.

On the other hand, at every $1/r^{2s+2}$, the metric expansion reveals again Carrollian dynamics but this time associated to the Cotton tensor descendants (see App. D for the expansion of the Cotton pieces). That is, the set of equations (4.136), (4.137), (4.138) and (4.139) with Cotton momenta $\Pi_{\text{Cot}(s)}$, $\Pi_{\text{Cot}(s)}^i$, $P_{\text{Cot}(s)}^i$, $\Pi_{\text{Cot}(s)}^{ij}$, $\tilde{\Pi}_{\text{Cot}(s)}^{ij}$, which allows to construct the set of currents $\{\kappa_{\text{Cot}(s)}, K_{\text{Cot}(s)}^i, \tilde{\kappa}_{\text{Cot}(s)}, \tilde{K}_{\text{Cot}(s)}^i\}$, and charges $\{Q_{\text{Cot}(s)T,Y}, \tilde{Q}_{\text{Cot}(s)T,Y}\}$ that we labelled as magnetic since their conservation, if any, are realized off-shell.

Additionally, due to the relationship between the fluid and the Cotton data ((6.16) in the general instance and (6.33) for the resumable instance), the two towers of charges $\tilde{Q}_{(s)T,Y}$ and $\tilde{Q}_{\text{Cot}(s)T,Y}$ are equivalent to each other.

In this section we will restrict ourselves to the resumable case, namely the algebraically special class of solutions, where the number of degrees of freedom that parameterize the spacetime is finite and the construction of the subleading charges relies on the expansion of $1/\rho^2$. The latter brings two sets of charges.

6.3.1 Electric tower

The electric tower of charges are made out of s -th momenta $\Pi_{(s)}$, $\Pi_{(s)}^i$, $P_{(s)}^i$, $\tilde{\Pi}_{(s)}^{ij}$ and $\Pi_{(s)}^{ij}$ which, thanks to the $1/\rho^2$ expansion of the bulk metric, are nothing but equal to the set given in (6.33) multiplied by a factor of $*\varpi^{2s}$. They read

$$\begin{aligned} \Pi_{(s)} &= *\varpi^{2s} \varepsilon_{(0)}, & \Pi_{(s)}^i &= \frac{*\varpi^{2s}}{8\pi G} *\chi^i, & P_{(s)}^i &= \frac{*\varpi^{2s}}{8\pi G} *\psi^i, & \tilde{\Pi}_{(s)}^{ij} &= -\frac{*\varpi^{2s}}{8\pi G} *X^{ij}, \\ \Pi_{(s)}^{ij} &= *\varpi^{2s} \left(\frac{\varepsilon_{(0)}}{2} a^{ij} - \frac{1}{8\pi G} *\Psi^{ij} \right), \end{aligned} \quad (6.54)$$

where one can easily see that the $s = 0$ corresponds to the leading ones.

Therefore, the latter makes possible to define the set of s -th current pieces, $\kappa_{(s)}$, $K_{(s)}^i$, $\tilde{\kappa}_{(s)}$ and $\tilde{K}_{(s)}^i$ as the multiplication of (4.142) with $*\varpi^{2s}$.

These set of Carrollian currents are not conserved in general and, using the Bianchi identities given in (D.6), (D.7) and (D.8), we find the divergences (4.144), which contribute the time evolution of the charges computed as in (4.57), and (4.58) as

$$\tilde{\mathcal{K}}_{(s)} = -s * \bar{\omega}^{2s-2} \left(* \bar{\omega} \hat{\mathcal{A}} \tilde{\kappa} + \frac{1}{3} \tilde{K}^i * \psi_i \right), \quad (6.55)$$

$$\mathcal{K}_{(s)} = -\frac{* \bar{\omega}^{2s}}{8\pi G} * \chi^i \left(\hat{\mathcal{D}}_i \xi^{\hat{i}} - 2 \xi^j \bar{\omega}_{ji} \right) - s * \bar{\omega}^{2s-2} \left(* \bar{\omega} \hat{\mathcal{A}} \kappa + \frac{1}{3} K^i * \psi_i \right) \quad (6.56)$$

with

$$\begin{cases} \kappa = \frac{1}{8\pi G} \xi^i * \psi_i - \xi^{\hat{i}} \varepsilon_{(0)} \\ \tilde{\kappa} = \frac{1}{8\pi G} \xi^i * \chi_i \\ K^i = \frac{\varepsilon_{(0)}}{2} \xi^i - \frac{1}{8\pi G} \left(\xi^j * \Psi^i_j + \xi^{\hat{i}} * \chi^i \right) \\ \tilde{K}^i = -\frac{1}{8\pi G} \xi^j * X^i_j, \end{cases} \quad (6.57)$$

and the Killing components $\xi^{\hat{i}}$ and ξ^i defined as in (D.34) following (4.28).

From (6.55), we can easily see that the set $\{\tilde{\kappa}_{(0)}, \tilde{K}_{(0)}^i\}$ give rise to a set of always conserved charges $\tilde{Q}_{(0),T,Y}$. These charges are given as the following surface integral over \mathcal{S}

$$\begin{aligned} \tilde{Q}_{(0),T,Y} &= \int_{\mathcal{S}} d^2x \sqrt{a} (\tilde{\kappa} + b_j \tilde{K}^j) \\ &= -\frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} * \xi^i (\chi_i - b_j X^j_i), \end{aligned} \quad (6.58)$$

and are purely geometrical since they only depend on the geometry of the null conformal boundary (encoded in the Cotton pieces).

From (6.56) we can see that the set $\{\kappa_{(0)}, K_{(0)}^i\}$ allows to define the set of leading charges $Q_{(0),T,Y}$ as the surface integral

$$\begin{aligned} Q_{(0),T,Y} &= \int_{\mathcal{S}} d^2x \sqrt{a} (\kappa + b_j K^j) \\ &= -\frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} \xi^{\hat{i}} (8\pi G \varepsilon_{(0)} + b_i * \chi^i) \\ &\quad + \frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left(* \psi_i + 4\pi G \varepsilon_{(0)} b_i - b_j * \Psi^j_i \right). \end{aligned} \quad (6.59)$$

The latter not only depend on the boundary geometry, but also on the energy density of the boundary fluid. These charges are conserved in the presence of strong Carrollian Killing fields. They could also be conserved for specific spacetime configurations. The rest of the

electric tower of subleading charges for all s are constructed upon multiplying the integrands of (6.58) and (6.59) by $*\varpi^{2s}$ (due to the expansion in $1/\rho^2$).

6.3.2 Magnetic tower

Similar to the electric case, here we have the s -th magnetic momenta $\Pi_{\text{Cot}(s)}$, $\Pi_{\text{Cot}(s)}^i$, $P_{\text{Cot}(s)}^i$, $\tilde{\Pi}_{\text{Cot}(s)}^{ij}$ and $\Pi_{\text{Cot}(s)}^{ij}$ whose definition in the configuration at hand is given by

$$\begin{aligned}\Pi_{\text{Cot}(s)} &= *\varpi^{2s} c_{(0)}, & \Pi_{\text{Cot}(s)}^i &= *\varpi^{2s} \chi^i, & P_{\text{Cot}(s)}^i &= *\varpi^{2s} \psi^i, & \tilde{\Pi}_{\text{Cot}(s)}^{ij} &= -*\varpi^{2s} X^{ij}, \\ \Pi_{\text{Cot}(s)}^{ij} &= *\varpi^{2s} \left(\frac{c_{(0)}}{2} a^{ij} - \Psi^{ij} \right).\end{aligned}\quad (6.60)$$

Again, the same factor appears in all magnetic currents $\kappa_{\text{Cot}(s)}$, $K_{\text{Cot}(s)}^i$, $\tilde{\kappa}_{\text{Cot}(s)}$ and $\tilde{K}_{\text{Cot}(s)}^i$ built out of the leading ones ($s = 0$), namely

$$\begin{cases} \kappa_{\text{Cot}} = \xi^i \psi_i - \xi^{\hat{t}} c_{(0)} \\ \tilde{\kappa}_{\text{Cot}} = \xi^i \chi_i \\ K_{\text{Cot}}^i = \frac{c_{(0)}}{2} \xi^i - \xi^j \Psi_j^i - \xi^{\hat{t}} \chi^i \\ \tilde{K}_{\text{Cot}}^i = -\xi^j X_j^i. \end{cases}\quad (6.61)$$

The divergence of the magnetic currents (4.144) read

$$\tilde{\mathcal{H}}_{\text{Cot}(s)} = -s * \varpi^{2s-2} \left(* \varpi \hat{\mathcal{D}} \tilde{\kappa}_{\text{Cot}} + \frac{1}{3} \tilde{K}_{\text{Cot}}^i * \psi_i \right), \quad (6.62)$$

$$\begin{aligned}\mathcal{H}_{\text{Cot}(s)} &= -*\varpi^{2s} \chi^i \left(\hat{\mathcal{D}}_i \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right) \\ &\quad -s * \varpi^{2s-2} \left(* \varpi \hat{\mathcal{D}} \kappa_{\text{Cot}} + \frac{1}{3} K_{\text{Cot}}^i * \psi_i \right).\end{aligned}\quad (6.63)$$

The evolution of the magnetic charges are also determined by the above divergences following (4.57) and (4.58). For the set $\{\tilde{\kappa}_{\text{Cot}(0)}, \tilde{K}_{\text{Cot}(0)}^i\}$ we find the geometric charges

$$\begin{aligned}\tilde{Q}_{\text{Cot}(0)T,Y} &= \int_{\mathcal{I}} d^2x \sqrt{a} \left(\tilde{\kappa}_{\text{Cot}} + b_j \tilde{K}_{\text{Cot}}^j \right) \\ &= \int_{\mathcal{I}} d^2x \sqrt{a} \xi^i \left(\chi_i - b_j X_j^i \right),\end{aligned}\quad (6.64)$$

which are always conserved.

From the set $\{\kappa_{\text{Cot}(0)}, K_{\text{Cot}(0)}^i\}$, we build the charges

$$\begin{aligned} Q_{\text{Cot}(0)T,Y} &= \int_{\mathcal{S}} d^2x \sqrt{a} \left(\kappa_{\text{Cot}} + b_j K_{\text{Cot}}^j \right) \\ &= - \int_{\mathcal{S}} d^2x \sqrt{a} \xi^{\hat{i}} (c_{(0)} + b_i \chi^i) + \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left(\psi_i + \frac{c_{(0)}}{2} b_i - b_j \Psi_i^j \right), \end{aligned} \quad (6.65)$$

which are conserved off-shell for strong Carrollian Killings. It is also possible to have other conserved magnetic charges in other specific configurations. Again, the rest of the magnetic tower of subleading charges for all s are constructed upon multiplying the integrands of (6.64) and (6.65) by $*\varpi^{2s}$.

So far we have obtained the tower of electric and magnetic charges for the resumable metric (6.31). Several observations are worth mentioning here regarding this towers of charges. The first one is the fact that the set of geometrical charges $\tilde{Q}_{(s),T,Y}$ coincide with the set $\tilde{Q}_{\text{Cot}(s),T,Y}$ for any s . Indeed, in $d = 2$, if ξ^i are the spatial components of a conformal Killing vector, so are $*\xi^i$.¹⁷ Therefore, the set of ξ^i are identical to the set of $*\xi^i$. Because of the latter, we call the aforementioned charges $\{\tilde{Q}_{(s),T,Y}\} \equiv \{\tilde{Q}_{\text{Cot}(s),T,Y}\}$ "self dual". The other two set of charges correspond to the electric $\{Q_{(s),T,Y}\}$ and magnetic $\{Q_{\text{Cot}(s),T,Y}\}$ ones, which due to the insertion of $*\varpi^{2s}$ in the integrands (6.59) and (6.65), the subleading towers have the status of multipolar moments (see the original works [139–142] as well as [143] for a modern perspective). In particular, we will see in the next section that, when considering stationary spacetimes, $Q_{(s),T,Y}$ and $Q_{\text{Cot}(s),T,Y}$ reproduce the mass and angular momentum multipolar expansion for the Kerr-Taub-NUT family.

Among the above charges associated to Carrollian Conformal Killings (D.34), which are not always conserved, one can find the ones that are associated to bulk isometries, if present. As discussed in Sec. 6.2, these isometries are mapped to the boundary as strong Carrollian isometries. These generates two electric and two magnetic towers of charges $\{Q_{(s)}, \tilde{Q}_{(s)}, Q_{\text{Cot}(s)}, \tilde{Q}_{\text{Cot}(s)}\}$ from which the four leading charges $\{Q_{(0)}, \tilde{Q}_{(0)}, Q_{\text{Cot}(0)}, \tilde{Q}_{\text{Cot}(0)}\}$ are always conserved, although part of them may be trivial.

6.4 Time-independent solutions

6.4.1 Some classes of stationary spacetimes

The final goal of this chapter is to analyze the action of the Ehlers hidden symmetry group of Einstein's dynamics onto the Carrollian boundary dynamics. To that end, a big part of the

¹⁷The proof of this statement is straightforward in complex coordinates, see footnote 6.

above will be based on the Ehlers-Geroch reduction procedure which relies on the presence of isometries.¹⁸ For the sake of the forthcoming analysis on the Ehlers group, from now on we will restrict ourselves to spacetimes of algebraically special Petrov type possessing a timelike Killing vector, which can be generally of the form (6.44). In stationary spacetimes, this Killing field remains timelike in the asymptotic region. The latter allows us to choose our Killing vector such that it coincides with the boundary vector v defined in (6.42). This implies that on the conformal boundary, our timelike Killing vector coincides with the fiber of the boundary Carrollian structure. Moreover, we can set $\Omega = 1$ through a Weyl rescaling, bringing the timelike Killing vector to the simple form ∂_t (a detailed description of this setting is given in [144]). In this set up the solution space is much more reduced and in particular does not consider spacetimes where the timelike Killing vector becomes spacelike in the asymptotic region, as the Plebański–Demiański family (like the C-metric) [145] (see also [137, 146]), which is algebraically special of Petrov type D.¹⁹ We could generalize our analysis by considering more general Killing fields at the cost of making the algebraic structure of the spacetime unstable under Ehlers transformation, which increases the technical difficulty of the problem. Because of the latter and in order to make the discussion clear we will restrict our analysis of the Ehlers-Geroch procedure for algebraically special Ricci-flat spacetimes with a timelike isometry generated by the Killing vector $\xi = \partial_t$, namely truly stationary Ricci-flat spacetimes whose algebraic structure are stable under the action of the Ehlers group.

With this choice, every Carroll tensor in the metric (6.31) becomes independent of the time. For instance, time independence implies $\theta = 0$ and $\varphi_i = 0$ since we have set $\Omega = 1$. The time independence of the spacetime also makes all the Weyl-Carroll-covariant derivatives to become ordinary Levi-Civita derivatives. Thus, the only non-vanishing Carrollian tensors are given in complex coordinates with $P = P(\zeta, \bar{\zeta})$ (see App. D) as

$$*\varpi = \frac{iP^2}{2} \left(\partial_\zeta b_{\bar{\zeta}} - \partial_{\bar{\zeta}} b_\zeta \right), \quad (6.66)$$

$$\hat{\mathcal{K}} = \hat{K} = K = \Delta \ln P, \quad (6.67)$$

$$c_{(0)} = (\Delta + 2K) *\varpi, \quad (6.68)$$

$$\chi_\zeta = \frac{i}{2} \partial_\zeta K, \quad \chi_{\bar{\zeta}} = -\frac{i}{2} \partial_{\bar{\zeta}} K, \quad (6.69)$$

$$\psi_\zeta = 3i \partial_\zeta *\varpi^2, \quad \psi_{\bar{\zeta}} = -3i \partial_{\bar{\zeta}} *\varpi^2, \quad (6.70)$$

$$\Psi_{\zeta\zeta} = \frac{1}{P^2} \partial_\zeta (P^2 \partial_\zeta *\varpi), \quad \Psi_{\bar{\zeta}\bar{\zeta}} = \frac{1}{P^2} \partial_{\bar{\zeta}} (P^2 \partial_{\bar{\zeta}} *\varpi), \quad (6.71)$$

¹⁸This procedure can be done for timelike and spacelike isometries. For the case of null isometries see [114].

¹⁹Their Weyl components are given in Eq. (6.41) – see also footnote 16.

where $\Delta f = 2P^2 \partial_{\bar{\zeta}} \partial_{\zeta} f$. In addition, we also have the energy density $\varepsilon_{(0)}(\zeta, \bar{\zeta})$ (Bondi mass aspect), as well as another scalar given by

$$\varpi = \frac{P^2}{2} \left(\partial_{\zeta} b_{\bar{\zeta}} + \partial_{\bar{\zeta}} b_{\zeta} \right), \quad (6.72)$$

which is $\frac{1}{2} \nabla_i b^i$ and should not be confused with the two-form Carrollian vorticity $\varpi = \frac{1}{2} \varpi_{ij} dx^i \wedge dx^j$, i.e. the Hodge-dual of the scalar $*\varpi = -\frac{1}{2} \nabla_i *b^i$ expressed explicitly in (6.66). These two real twist scalars can be combined into the complex Carrollian twist

$$\hat{\varpi} = *\varpi + i\varpi. \quad (6.73)$$

In this stationary configuration, the equations of motion (6.34), (6.35) are recast as

$$\Delta K = 0, \quad (6.74)$$

$$\partial_{\zeta} \hat{\tau} = 0. \quad (6.75)$$

The first equation implies that the curvature K is required to be a harmonic function, that is, it is of the form

$$K(\zeta, \bar{\zeta}) = \frac{1}{2} \left(\hat{k}(\zeta) + \hat{\bar{k}}(\bar{\zeta}) \right), \quad (6.76)$$

which puts some limitations on the function $P(\zeta, \bar{\zeta})$ (still not all solutions for P in this instance are known). We can also define the imaginary part of \hat{k} (for future use) as another harmonic function

$$K^*(\zeta, \bar{\zeta}) = \frac{1}{2i} \left(\hat{k}(\zeta) - \hat{\bar{k}}(\bar{\zeta}) \right). \quad (6.77)$$

The second equation (6.75), together with (6.36), show that the Cotton density $-c_{(0)}$ is the real part of an arbitrary holomorphic function $\hat{\tau}(\zeta)$, while the energy density $8\pi G\varepsilon_{(0)}$ corresponds to the imaginary part of the latter. Thanks to (6.39), which in the stationary case reduces to

$$\Delta \hat{\tau} = 0, \quad (6.78)$$

both the Cotton and the energy density are required to be harmonic functions. Having $c_{(0)}$ and K , it is always possible to determine the functions b_{ζ} and $b_{\bar{\zeta}}$ by using (6.68).

Although this chapter is not devoted to solve Einstein's equations, we can illustrate and sort some the classes of solutions based on the curvature K (the ones we will see can be found in various chapters of [137, 146]). This separation goes as follows:

Non-constant K This is the general case for an arbitrary harmonic curvature K . This is the most obscure situation since, as we mentioned previously, very few known solutions of P s that possess a non-constant harmonic K are known in the literature²⁰. Still, if we assume we have one, together with its holomorphic function $\hat{k}(\zeta)$, and specifying the choice for the arbitrary function $\hat{\tau}(\zeta)$, we can solve (6.68) whose expression is given by (6.66) with Ehresmann connection

$$b_\zeta(\zeta, \bar{\zeta}) = \frac{i\hat{\tau}(\bar{\zeta})}{P^2(\zeta, \bar{\zeta})\partial_{\bar{\zeta}}\hat{k}(\bar{\zeta})}. \quad (6.79)$$

Constant K This is the most common situation in the literature (the Kerr-Taub-NUT family belongs to this class). It implies that $\hat{k}(\zeta)$ is also constant and the above solution for b_i is invalid. The constant K situation captures three instances: spherical, flat or hyperbolic foliations, which corresponds to $K = 1$, $K = 0$ and $K = -1$ respectively. Here the metric function P is parameterized as

$$P(\zeta, \bar{\zeta}) = A\zeta\bar{\zeta} + B\zeta + \bar{B}\bar{\zeta} + D \quad (6.80)$$

with A, D arbitrary real constants and B an arbitrary complex constant. The curvature K reads

$$K = 2(AD - B\bar{B}). \quad (6.81)$$

Within the constant K class, two cases must be treated separately.

$K \neq 0$

As we mentioned previously, thanks to (6.78) the Cotton density is restricted to be an harmonic function, namely it is expressed as $c_{(0)}(\zeta, \bar{\zeta}) = -\frac{\hat{\tau}(\zeta) + \hat{\tau}(\bar{\zeta})}{2}$. Then, equation (6.68) is solved for

$$*\omega(\zeta, \bar{\zeta}) = \frac{c_{(0)}(\zeta, \bar{\zeta})}{2K} + i \left(\bar{f}(\bar{\zeta})\partial_{\bar{\zeta}} \ln P(\zeta, \bar{\zeta}) - f(\zeta)\partial_{\zeta} \ln P(\zeta, \bar{\zeta}) + \frac{1}{2} \left(\partial_{\zeta} f(\zeta) - \partial_{\bar{\zeta}} \bar{f}(\bar{\zeta}) \right) \right), \quad (6.82)$$

²⁰An example of solution with non-constant harmonic K is given in [137] which (up to holomorphic coordinate transformations) $K = -3(\zeta + \bar{\zeta})$ realized with $P = (\zeta + \bar{\zeta})^{3/2}$

where $f(\zeta)$ is an arbitrary holomorphic function. The vorticity defined above is reached with the Ehresmann connection ($\hat{\tau}_0$ is a real constant)

$$b_\zeta(\zeta, \bar{\zeta}) = -\frac{\bar{\zeta}(\hat{\tau}_0 + i\hat{\tau}(\zeta))}{2K(B\zeta + D)P(\zeta, \bar{\zeta})} + \frac{\bar{f}(\bar{\zeta})}{P^2(\zeta, \bar{\zeta})}. \quad (6.83)$$

$K = 0$

In this case $A = B = 0$ so that $P = D$. Thus, given $c_{(0)}(\zeta, \bar{\zeta}) = -\frac{\hat{\tau}(\zeta) + \hat{\tau}(\bar{\zeta})}{2}$ and solving (6.68) we get

$$*\omega(\zeta, \bar{\zeta}) = \frac{i}{2}(Z(\zeta) - \bar{Z}(\bar{\zeta})) - \frac{1}{4P^2} \left(\bar{\zeta} \int^\zeta dz \hat{\tau}(z) + \zeta \int^{\bar{\zeta}} d\bar{z} \hat{\tau}(\bar{z}) \right), \quad (6.84)$$

with $Z(\zeta)$ is an arbitrary holomorphic function, and

$$b_\zeta(\zeta, \bar{\zeta}) = \frac{1}{P^2} \int^{\bar{\zeta}} d\bar{z} \bar{Z}(\bar{z}) - \frac{\bar{\zeta}^2}{4P^4} \int^\zeta dz (\hat{\tau}_0 + i\hat{\tau}(z)), \quad (6.85)$$

with $\hat{\tau}_0$ is a real integration constant.

The last two cases have in common the instance where $c_{(0)} = K = 0$, realized with vanishing $\hat{\tau}$ and constant P .

As stated before, in this thesis we are not going to solve and analyze Einstein's equations in full generality. Still, we will emphasize the subclass composed by the Kerr-Taub-NUT family. In this class, the curvature K is constant and reached with $B = 0$. Therefore two instances arises in this subclass: vanishing and non vanishing K , which are realized by vanishing and non-vanishing A respectively.

- For non-vanishing K , the holomorphic function $\hat{\tau}$ is

$$\hat{\tau} = 2i(M + iKn), \quad (6.86)$$

where M is the mass and n the nut charge, with both constants. The holomorphic function $f(\zeta)$ reads

$$f(\zeta) = ia\zeta \quad (6.87)$$

with a the Kerr angular velocity. We find the Ehresmann connection by using (6.82) and (6.83) with $\hat{\tau}_0 = 2M$. It reads

$$b_\zeta(\zeta, \bar{\zeta}) = -i\bar{\zeta} \left(\frac{a}{P^2} - \frac{n}{DP} \right) \quad (6.88)$$

which gives the vorticity

$$*\varpi(\zeta, \bar{\zeta}) = n + a - \frac{2Da}{P}, \quad (6.89)$$

where $P = A\zeta\bar{\zeta} + D$ and $K = 2AD$.

- In the case of $K = 0$ (i.e. $P = D$ constant), we can find the Ehresmann connection by using (6.84) and (6.85) with $\hat{\tau}_0 = 2M$.²¹ Here

$$\hat{\tau} = 2iM \quad (6.90)$$

and the holomorphic function $Z(\zeta)$ is given as

$$Z = ia. \quad (6.91)$$

The latter leads to

$$b_\zeta(\zeta, \bar{\zeta}) = -i\frac{\bar{\zeta}a}{P^2} \quad (6.92)$$

with vorticity

$$*\varpi = -a. \quad (6.93)$$

In this case there is no magnetic charge, namely no nut charge.²²

6.4.2 Multipolar expansion of the charges from the boundary perspective

We conclude this section on stationary spacetimes with the analysis of the gravitational charges associated to the isometries of the spacetime. As we saw in the previous section, gravitational charges can be computed exclusively from the null conformal boundary by computing the boundary charges generated by the strong Carrollian Killing vectors. These techniques are still in an early stage since a contact with standard methods still need to be elaborated. Despite the latter, we will see that in the case at hand, the Carrollian approach for the construction of the charges enables us to find the gravitational mass and angular momentum multipolar expansion.

For a time independent Ricci-flat solution, a timelike isometry is generated by simply the Killing vector $\xi = \partial_t$, which is mapped on the null conformal boundary as a strong Carrollian

²¹Both for vanishing and non-vanishing K , $\hat{\tau}_0$ has been tuned to ensure that M does not appear in b_ζ , displayed in (6.88) and (6.92). There is no principle behind this choice, it is simply in line with standard conventions for the Kerr–Taub–NUT family. As a consequence, ϖ defined in (6.72) vanishes.

²²Despite the absence of magnetic charges, the solution at hand belongs formally to the Taub–NUT family (see Ref. [146], §12.3.2).

Killing with components $\xi^{\hat{t}} = 1$ and $\xi^i = 0$. Then, the pairs κ , K^i , and κ_{Cot} , K_{Cot}^i are given by²³

$$\kappa = -\varepsilon_{(0)}, \quad K^i = -\frac{1}{8\pi G} * \mathcal{X}^i, \quad \kappa_{\text{Cot}} = -c_{(0)}, \quad K_{\text{Cot}}^i = -\mathcal{X}^i. \quad (6.94)$$

The above allows to compute the pair of leading charges ($s = 0$) by using (6.59) and (6.65). We find²⁴

$$Q_{\text{em}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (8\pi G \varepsilon_{(0)} + \varpi K), \quad Q_{\text{mm}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (-c_{(0)} + * \varpi K), \quad (6.95)$$

up to boundary terms and a factor $-8\pi G$ in the first charge. These two charges are nicely combined into

$$Q_{\text{m}} = Q_{\text{mm}} + i Q_{\text{em}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + \hat{\varpi} K). \quad (6.96)$$

Here the indices refer to magnetic mass (mm) and electric mass (em).

Now, the subleading electric and magnetic charges are obtained by the insertion of $*\varpi^{2s}$ in the current components (6.94), defining thus the subleading pairs $\{\kappa_{(s)}, K_{(s)}^i, \kappa_{\text{Cot}(s)}, K_{\text{Cot}(s)}^i\}$. Then, the higher- s mass multipole moment is defined as

$$Q_{\text{m}(s)} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + \hat{\varpi} K) * \varpi^{2s}. \quad (6.97)$$

In the instance of the $K = 1$ with Kerr–Taub–NUT solution displayed in (6.86), (6.87), (6.88), (6.89) with $A = 1/2$ and $D = 1$, the electric-magnetic mass multipole expansion is given by

$$Q_{\text{m}(s)} = 4\pi i (M + in) \left(\frac{(n+a)^{2s+1} - (n-a)^{2s+1}}{a(2s+1)} \right), \quad (6.98)$$

where M is the Bondi mass (here constant), n the NUT charge and a the angular velocity.

The isometry group of stationary Ricci-flat spacetimes might also contain the $U(1)$ group generated by additional rotational Killing fields²⁵. For the class of time independent

²³The pairs $\tilde{\kappa}$, \tilde{K}^i , and $\tilde{\kappa}_{\text{Cot}}$, \tilde{K}_{Cot}^i vanish in the configuration at hand.

²⁴The integrals can be performed by setting $\zeta = Ze^{i\Phi}$, where $0 \leq \Phi < 2\pi$ and $Z = \sqrt{2} \tan \frac{\Theta}{2}$, $0 < \Theta < \pi$ for \mathbb{S}^2 ; $Z = \frac{R}{\sqrt{2}}$, $0 < R < +\infty$ for \mathbb{E}_2 ; $Z = \sqrt{2} \tanh \frac{\Psi}{2}$, $0 < \Psi < +\infty$ for \mathbb{H}_2 .

²⁵This is a consequence of the rigidity theorem for stationary spacetimes [147] which is best presented in [148, 149].

spacetimes with constant curvature K , the available spacelike Killing vectors are

$$\xi_1 = i(\zeta \partial_\zeta - \bar{\zeta} \partial_{\bar{\zeta}}), \quad (6.99)$$

$$\xi_2 = \frac{i}{2\sqrt{|AD|}} \left((D - A\zeta^2) \partial_\zeta - (D - A\bar{\zeta}^2) \partial_{\bar{\zeta}} \right), \quad (6.100)$$

$$\xi_3 = \frac{1}{2\sqrt{|AD|}} \left((D + A\zeta^2) \partial_\zeta + (D + A\bar{\zeta}^2) \partial_{\bar{\zeta}} \right), \quad (6.101)$$

From the above, only ξ_1 can be mapped onto the null boundary as a strong Carrollian Killing vector, namely, ξ_1 satisfies the strong Carrollian Killing equations (4.61), (4.62) and (4.56), while ξ_2 and ξ_3 do not.

Therefore, we can find the tower of rotational electric and magnetic charges by using (6.59) and (6.65) with ξ_1 . For the case of $K = 1$ we have

$$Q_{r(s)} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} 6\zeta \bar{\zeta} \left(\frac{n+iM}{P^2} (a-nP) \left(n+a-\frac{2a}{P} \right)^{2s} - \frac{2a}{P^2} \left(n+a-\frac{2a}{P} \right)^{2s+1} \right) \quad (6.102)$$

with $P = 1 + \frac{1}{2}\zeta\bar{\zeta}$. This charge is non vanishing as long as the rotational parameter or the NUT charge are present. For $s = 0$ we find the leading rotational charge as

$$Q_{r(0)} = -8\pi [a(n+iM) + 3n(n-iM)]. \quad (6.103)$$

Expressions (6.98) and (6.103) are in line with the results obtained in Refs. [139–142] (see also [150], where the electric part of $Q_{r(0)}$ is given) using standard methods relying on the bulk dynamics. They provide conserved moments since the divergences (6.56) and (6.63) vanish.

6.5 Ehlers, Geroch and Carroll

Having developed the machinery for a Carrollian approach for the dynamics of Ricci-flat spacetimes, we are now ready to unravel the action of the hidden Ehlers group (2.145) on the boundary Carrollian data. To this end, we will focus on algebraically special Ricci-flat spacetimes whose metric is written in the gauge (6.31) (the resumable class), and possesses a timelike isometry generated by the Killing field $\xi = \partial_t$, with $\Omega = 1$.

The Geroch reduction is then performed along the orbits of ∂_t following the steps presented in Sec. 2.4. For the latter we have to compute the norm λ (2.133) and the twist potential ω (2.135) associated to ∂_t , such that we can build τ as defined in (2.142). The idea

is to compute the aforementioned quantities by using the resumable covariant Newman-Unti gauge (6.31) such that λ , ω , τ and h_{AB} are expressed in terms of the boundary data at each power of the $1/r$ expansion. In this way, by using the transformation laws given by (2.145), one would be able to get the transformation for each boundary Carrollian geometric and fluid quantity, encoded at different orders in the transformation of τ . Now, in order for the latter procedure to make sense, one has to take into account the non-local nature of the Ehlers-Geroch procedure. As we saw in Sec. 2.4, after the action of the Ehlers transformation on $\tau \rightarrow \tau'$ and $h_{AB} \rightarrow h'_{AB}$, a new Ricci-flat spacetime can be found by an oxidation procedure following (2.150) and (2.152). In this way the new Ricci-flat solution is given by (2.153). Nothing guarantees that the new solution will belong to the same class as the seed solution, and actually it could also be expressed in a completely different gauge. Because of this, and in order to obtain the transformations of the Carrollian data using the strategy suggested earlier, we will impose that the new Ricci-flat solution, obtained by a Geroch procedure, remain in the same gauge, and thus it will be also algebraically special. This requirement will make the radial coordinate to transform by a shift (as we will see later) such that the latter is true.

To begin with, we will adopt the Cartan frame as defined in (6.42), leading to the bulk metric

$$\begin{aligned} g_{\hat{t}\hat{t}} &= \frac{1}{\rho^2} (8\pi G\mathcal{E}_{(0)}r + *\overline{\omega}c_{(0)}) - K, & g_{\hat{t}\hat{i}} &= *\partial_i*\overline{\omega}, & g_{\hat{t}\hat{r}} &= -1, \\ g_{\hat{r}\hat{i}} &= 0, & g_{\hat{r}\hat{r}} &= 0, & g_{\hat{i}\hat{j}} &= \rho^2 a_{ij} \end{aligned} \quad (6.104)$$

obtained using (6.43), and assuming t -independence. In this expression $*\overline{\omega}$, K and $c_{(0)}$ are given in (6.66), (6.67) and (6.68). The Killing 1-form is then expressed as

$$\xi = \left(K - \frac{1}{\rho^2} (8\pi G\mathcal{E}_{(0)}r + *\overline{\omega}c_{(0)}) \right) \mu + *\partial_i*\overline{\omega}dx^i - dr. \quad (6.105)$$

Then, the norm is given as

$$\lambda = \frac{8\pi G\mathcal{E}_{(0)}r + *\overline{\omega}c_{(0)}}{\rho^2} - K. \quad (6.106)$$

For the twist we use equation (2.134), expressed here as

$$\mathbf{w} = -\star(\xi \wedge d\xi), \quad (6.107)$$

where “ \star ” stands for the four-dimensional Hodge duality. The latter one-form is exact on-shell, and its potential is expressed here as

$$\omega = \frac{8\pi G\varepsilon_{(0)} \star \bar{\omega} - c_{(0)} r}{\rho^2} + K^*, \quad (6.108)$$

where K and K^* satisfies

$$\partial_i K + \eta_i^j \partial_j K^* = 0. \quad (6.109)$$

Here, on-shellness is implemented through the boundary dynamics as summarized in Sec. 6.4, i.e. in the equations (6.75), (6.76) and (6.77).

Inserting the above results into (2.142) and using (6.36), we find

$$\tau = \frac{\hat{\tau}}{r + i \star \bar{\omega}} - i \hat{k}, \quad (6.110)$$

with $\hat{k} = K + iK^*$.

In the same way, we can write the Geroch reduced and rescaled metric $\tilde{h}_{AB} = \lambda h_{AB}$ as

$$\tilde{h}_{AB} dx^A dx^B = - \left(dr - \star \partial_k \star \bar{\omega} dx^k \right)^2 + \lambda \rho^2 a_{ij} dx^i dx^j, \quad (6.111)$$

which, in holomorphic and anti-holomorphic coordinates as introduced in App. D, reads

$$\tilde{h}_{AB} dx^A dx^B = - \left(dr - i \partial_\zeta \star \bar{\omega} d\zeta + i \partial_{\bar{\zeta}} \star \bar{\omega} d\bar{\zeta} \right)^2 + \frac{(\tau - \bar{\tau})(r + i \star \bar{\omega})(r - i \star \bar{\omega})}{i P^2} d\zeta d\bar{\zeta}. \quad (6.112)$$

The above τ and \tilde{h}_{AB} are solution of the reduced Einstein's equations given in (2.143), just as it should.

Now, we can expressed the complex scalar τ as an asymptotic series, namely we expand the latter in powers of $1/r$

$$\tau = \sum_{n=0}^{+\infty} \frac{\tau^{(n)}}{r^n}, \quad (6.113)$$

where, by using the definition of (6.110), at each power in $1/r^n$ we find

$$\begin{aligned} \tau^{(0)} &= -i \hat{k} \\ &= K^* - iK, \end{aligned} \quad (6.114)$$

$$\begin{aligned} \tau^{(1)} &= \hat{\tau} \\ &= -c_{(0)} + i8\pi G\varepsilon_{(0)}, \end{aligned} \quad (6.115)$$

and

$$\tau^{(n+1)} = \tau^{(1)} (-i * \varpi)^n. \quad (6.116)$$

From Sec. 2.4, we know that the transformation of τ under the Ehlers group is as given by (2.145). Additionally, if we require the new Ricci-flat solution generated by the Ehlers transformation to be given in the same gauge, the form of the transformed τ' should follow the structure

$$\tau' = \frac{\hat{\tau}'}{r' + i * \varpi'} - i \hat{k}', \quad (6.117)$$

where $r = r' + R$, with $R \equiv R(t, \mathbf{x})$. The latter can also be expanded in powers of $1/r$, so that we have

$$\tau' = \tau'^{(0)} + \frac{\tau'^{(1)}}{r'} - \frac{i * \varpi' \tau'^{(1)}}{r'^2} + \mathcal{O}\left(\frac{1}{r'^3}\right). \quad (6.118)$$

Then, taking the transformation rule (2.145) together with the asymptotic expansions (6.113) and (6.118), we find the following behaviour:

$$\begin{aligned} \tau' &= \frac{\alpha \hat{k} + i\beta}{\gamma \hat{k} + i\delta} - \frac{\hat{\tau}}{(\gamma \hat{k} + i\delta)^2} \frac{1}{r'} \\ &\quad - \frac{\hat{\tau}}{(\gamma \hat{k} + i\delta)^2} \left(\frac{\delta R + \gamma(\hat{\tau} - i \hat{k} R)}{\gamma \hat{k} - \delta} - i * \varpi \right) \frac{1}{r'^2} + \mathcal{O}\left(\frac{1}{r'^3}\right). \end{aligned} \quad (6.119)$$

From the above, the derivation of the Ehlers transformation of the Carrollian data is straightforward. At order r^0 we find the transformation of the complex curvature scalar

$$\hat{k}' = i \frac{\alpha \hat{k} + i\beta}{\gamma \hat{k} + i\delta}, \quad (6.120)$$

which leads to the transformation of $P(\zeta, \bar{\zeta})$ as

$$P' = \frac{P}{|\gamma \hat{k} + i\delta|}. \quad (6.121)$$

The r^{-1} order reveals the transformation of $\hat{\tau}$. The latter reads

$$\hat{\tau}' = -\frac{\hat{\tau}}{(\gamma \hat{k} + i\delta)^2}. \quad (6.122)$$

It is easy to see that the r^{-2} order will tell us the transformation rule of $\tau'^{(2)} = -i * \varpi' \hat{\tau}'$. Moreover, by using the transformation (6.122), we are just left with the relation

$$-i * \varpi' = \frac{\delta R + \gamma(\hat{\tau} - i\hat{k}R)}{\gamma\hat{k} - \delta} - i * \varpi. \quad (6.123)$$

In order for the latter transformation to be consistent, the right hand side must be purely imaginary. This requirement enables to fix $R(t, x)$, which in turn gives the following transformation for the radial coordinate:

$$r' = r + \frac{i}{2} \left(\frac{\gamma\hat{\tau}}{\gamma\hat{k} + i\delta} - \frac{\gamma\hat{\tau}}{\gamma\hat{k} - i\delta} \right). \quad (6.124)$$

Then we can find the transformation of the vorticity by using the latter in (6.123). We find

$$* \varpi' = \frac{\gamma}{2} \left[\frac{\hat{\tau}}{\gamma\hat{k} + i\delta} + \frac{\hat{\tau}}{\gamma\hat{k} - i\delta} \right] + * \varpi. \quad (6.125)$$

These transformation rules leave (6.111) invariant (as it should) and they are local transformation, providing a direct transformation for the boundary geometry given in (6.121). The transformation of the energy density can be obtained more explicitly from (6.122) using (6.36). We find

$$8\pi G \varepsilon'_{(0)} = \frac{8\pi G \varepsilon_{(0)} \left((\gamma K^* + \delta)^2 - \gamma^2 K^2 \right) - 2c_{(0)} \gamma K (\gamma K^* + \delta)}{\left(\gamma^2 K^2 + (\gamma K^* + \delta)^2 \right)^2}. \quad (6.126)$$

Similarly, the transformation of the Cotton density is given as

$$c'_{(0)} = \frac{c_{(0)} \left((\gamma K^* + \delta)^2 - \gamma^2 K^2 \right) + 16\pi G \varepsilon_{(0)} \gamma K (\gamma K^* + \delta)}{\left(\gamma^2 K^2 + (\gamma K^* + \delta)^2 \right)^2}. \quad (6.127)$$

All these rules are compatible with equations (6.67) and (6.68) and moreover, the transformations of the rest Carrollian Cotton tensors can be reached by using the above results combined with Eqs. (6.69), (6.70) and (6.71).

The transformation of the Ehresmann connection can be inferred by applying the transformation rules derived here into the expressions (6.79), (6.83) and (6.85), depending on the nature of the scalar curvature K . In order to do so, for example in the constant \hat{k} instance, we can notice that A , B , \bar{B} and D transform with a factor $1/|\gamma\hat{k} + i\delta|$ such that the transformation

rule (6.121) is satisfied. In the same way, by looking at the transformation (6.125), we can conclude that $f(\zeta)$ is invariant under the action of the Ehlers group, while $Z(\zeta)$ transforms as

$$Z'(\zeta) = Z(\zeta) + i \frac{\gamma \hat{\tau}(\zeta)}{\gamma \hat{k} + i \delta}. \quad (6.128)$$

Once the Ehlers transformations of the Carrollian quantities are found, the reconstruction of the new Ricci-flat spacetime is straightforward. The latter is done by using the resumable metric (6.31) expressed with the transformed Carrollian data (except for the boundary set of coordinates $\{t, \zeta, \bar{\zeta}\}$). This is equivalent to the oxidation procedure followed in (2.150) and (2.152) that brings the three dimensional dynamics captured by τ' and h'_{AB} to four dimensions captured by the new metric g'_{AB} defined as in (2.153) with

$$\eta' = -\mu' - \frac{1}{\lambda'} \left(dr' - i \partial_{\zeta} * \varpi' d\zeta + i \partial_{\bar{\zeta}} * \varpi' d\bar{\zeta} \right), \quad \mu' = -dt + b'_{\zeta} d\zeta + b'_{\bar{\zeta}} d\bar{\zeta}. \quad (6.129)$$

The new Killing vector of g'_{AB} is obtained by the relation $\xi' = \lambda' \eta'$, which in the case at hand is again ∂_t .

Before we conclude, let us take the example of the Kerr-Taub-NUT family treated at the end of Sec. 6.4. This solution is reached by the choices of $P = \frac{1}{2} \zeta \bar{\zeta} + 1$, $K = 1$ and $K^* = 0$ (this was not explicitly demanded). From the $SL(2, \mathbb{R})$ hidden group, only one subgroup generates genuine new Ricci-flat solutions. The latter corresponds to the subgroup of rotations

$$\begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \in SL(2, \mathbb{R}). \quad (6.130)$$

The latter induces a rotations in the $(\varepsilon_{(0)}, c_{(0)})$ plane as

$$c'_{(0)} = c_{(0)} \cos 2\chi - 8\pi G \varepsilon_{(0)} \sin 2\chi, \quad (6.131)$$

$$8\pi G \varepsilon'_{(0)} = 8\pi G \varepsilon_{(0)} \cos 2\chi + c_{(0)} \sin 2\chi. \quad (6.132)$$

The above is the boundary manifestation of the rotation in the (M, n) plane discussed at the end of Sec. 2.4.

Chapter 7

Summary of the results and outlook

In this thesis we passed through the study of fluid dynamics on Galilean and Carrollian backgrounds, to the investigation of some aspects of flat holography in connection to Carrollian dynamics. From the fluid mechanics perspective, we constructed the most general fluid equations on both, Galilean and Carrollian backgrounds, by using two distinct approaches: one based on the symmetries of the theory, and the other as the large and vanishing speed of light limits.

In the first approach we based our analysis on the local symmetries of the action, where in particular we focused on Carrollian and Galilean diffeomorphisms, Weyl and U(1) invariance. As a consequence of diffeomorphism invariance, the fluid equations are presented as conservation laws of momenta which are constructed upon variation of the action with respect to the geometry, except for two additional variables that enters the equations as arbitrary functions that are not conjugate to any geometric data. These correspond to the energy flux Π_i in the Galilean case, and the momentum P^i in the Carrollian counterpart. The fluid equations are also supplemented with the conservation of a matter/charge current, arising as a consequence of U(1) invariance of the action. In the Galilean instance, it corresponds to the continuity equation.

The second approach relied on the limiting procedure of the relativistic fluid equations by taking $k \rightarrow \infty$ in the Galilean case, and $k \rightarrow 0$ for the Carrollian case. The latter was performed in the appropriate set of coordinates for the relativistic geometry, being Zermelo and Papapetrou-Randers for the Galilean and Carrollian cases respectively. This procedure allowed us to find all the Galilean and Carrollian fluid equations in terms of momenta, which only in this approach are expressed in terms of kinematical (the velocity v^i in the Galilean limit and the inverse velocity β_i in the Carrollian limit) and thermodynamical quantities. A huge advantage of this approach with respect to the symmetry based one is that it makes possible to capture additional degrees of freedom by taking into account over-leading terms

in the small/large k expansion of the heat and matter currents, as well as the viscous stress tensor. This feature helped us to unravel the fate of hydrodynamic-frame invariance in the Galilean and Carrollian limits. Indeed, hydrodynamic-frame transformations are sensitive to the expansion of the observables with respect to the light velocity. As we saw in Sec. 3.2, by working the Galilean limit of relativistic fluid dynamics with the standard behaviour at large k of the fluid observables, hydrodynamic frame invariance is lost in the Navier-Stokes equations. The only exception to this feature are Galilean fluids with massless energy carriers where the absence of the rest mass energy term saves hydrodynamic-frame invariance. The other possibility is when we considered extra degrees of freedom captured in the k^2 contribution of q_i and the k^0 contribution of j_i (alternative behaviour). In that situation we saw that hydrodynamic-frame invariance is restored at the price of giving up the continuity equation as a conservation law. The resulting dynamics in this case is comparable with diffusion processes or super fluids, where all the constituent are not conserved simultaneously.

For the Carrollian limit, the behaviour of the energy-momentum tensor and the $U(1)$ current at small velocity of light is not obvious. There is no a concrete answer to what thermodynamic is in a Carrollian theory due to the absence of motion (no kinetic theory), and therefore we do not have hints on how ε , p , q^μ , j^μ and $\tau^{\mu\nu}$ should behave at small k . For that reason, we worked with the ansatz given in Eqs. (4.72), (4.73), (4.74) and (4.75), that was guided by Carrollian fluids that are holographic duals to Ricci-flat spacetimes [42]. The latter allowed us to handle the Carrollian limit of relativistic hydrodynamics, and in particular we found that hydrodynamic-frame invariance persists for Carrollian fluid dynamics. This is principally due to the fact that the energy density is not decomposed into a rest plus kinetic contribution, just as in the Galilean case with massless carriers. In the flat holography side, hydrodynamic-frame invariance is reflected in the bulk spacetime as residual bulk diffeomorphisms. This is something that has been shown for the case of 3d bulk/2d boundary in [75–77].

When considering Galilean/Carrollian isometries, an analogous set of currents, κ and K^i , can be constructed as different combinations of momenta and Galilean/Carrollian Killing vectors, reminiscent of the relativistic ascendant where the current is constructed as the contraction of the energy-momentum tensor and the Killing vector. These currents satisfy equations that, as we shown in Sections 3.1 and 4.1, do not correspond to conservation laws in general. Different extra conditions must be imposed in order for these currents to be truly conserved. One is by restricting the isometries to be "strong" Galilean and Carroll isometries, which requires an extra Killing equations such as (3.66) in the Galilean case, and (4.56) in the Carrollian counterpart. This extra condition is incompatible with Galilean/Carrollian boosts. The second option is to impose the vanishing of P_i (Galilean) and Π^i (Carrollian). In

fact, the vanishing of the energy flux Π^i in the Carrollian theory has been considered to be an important condition in order for the theory to be Carroll boost invariant [43, 56], which is the case in this situation.

The approach and some of the results obtained from the Carrollian extension of fluid mechanics were also used to analyze the dynamics of (conformally coupled) scalar fields and the gravitational Chern-Simons on arbitrary Carrollian backgrounds. In the small- k expansion procedure, we find electric and magnetic dynamics as well as paraelectric and paramagnetic dynamics for the Carroll Chern-Simons action at different powers of k , whose equations of motions are Carrollian diffeomorphisms and Weyl covariant.

Now, in the context of the Ricci-flat/Carroll duality perspective, we constructed an appropriate gauge for the four dimensional bulk spacetime that features covariance with respect to the null conformal boundary. We called this gauge as the covariant Newman-Unti gauge and it is reached by relaxing the Newman-Unti gauge conditions as described in (6.19). This gauge is not only valid for asymptotically flat spacetimes but also for asymptotically AdS spaces, where the gauge conditions are the ones in (6.2) and the spacetime is reconstructed in terms of tensors covariant with respect to the timelike conformal boundary and built in terms of the boundary geometry and the components of the boundary energy-momentum tensor. The advantage of this gauge with respect to the well known Feffermann-Graham gauge (in the AAdS case) is that it accepts a smooth limit for vanishing cosmological constant, here parameterized in terms of k^2 . Then, the line element (6.17) can be reached as the $k \rightarrow 0$ of (6.3), where the bulk spacetime (Ricci-flat) is reconstructed order by order in the $1/r$ expansion in terms of the Carrollian geometry, the Carrollian fluid data plus other infinite set of Carrollian tensors reminiscent of the small- k expansion of the boundary energy-momentum tensor.

Resummation of (6.17) is possible and occurs when imposing conditions in the Carrollian boundary, as explained in Sec. 6.1. This was the arena for the discussion on the towers of gravitational charges and the Ehlers hidden symmetry of Einstein's equations, all from the Carrollian boundary perspective. Indeed, the Carrollian approach of Ricci-flat spacetimes allowed us for the construction of the gravitational charges based on the strong Carrollian isometries of the boundary. The latter defined towers of electric and magnetic towers of charges, constructed in terms of Carrollian momenta of the boundary fluid, and the components of the Carrollian Killing vectors. In the electric tower, the charges are constructed in terms of the energy density, the Cotton heat currents and Cotton stress, and only the leading charges are generally conserved, which happens on-shell. For the magnetic tower we found that those are built in terms of the Cotton density, Cotton currents and Cotton stress, and the conservation of the leading charges occur off-shell. In particular, for the Kerr-Taub-NUT

family, the definition of charges presented here were shown to reproduce the mass and angular momentum multipolar expansion, associated to the isometries generated by $\xi = \partial_t$ and $\xi = \partial_\phi$ respectively.

When a Ricci-flat spacetime possesses an isometry, one can show that the Einstein equations are invariant under Möbius transformations (Ehlers group). The latter is revealed by a dimensional reduction along the orbits of the Killing vector that generates the isometry. These transformations can be used to generate new solutions, by starting from simpler seed spacetimes. We exhibited the action of these transformations from a boundary-to-bulk perspective. Indeed, when considering Ricci-flat spacetimes of the resumable class with a timelike isometry generated by ∂_t , the Ehlers group of transformations acts in the Carrollian boundary as a local transformation that mixes the geometric boundary data with the dynamical data (in this case the energy density of the boundary Carrollian fluid). These transformed boundary data also enable to find new Ricci-flat solutions by reconstructing the spacetime in the resumable covariant Newman-Unti gauge with the new boundary variables. Moreover, in the configurations considered in Sec. 6.5 (resumable metric with Killing vector $\xi = \partial_t$) the new Ricci-flat spacetimes remain in the same gauge, and therefore it belongs to the algebraically special Petrov class, once the appropriate transformation in the radial coordinate is considered.

The work presented here allows to explore interesting directions within the applications of Carrollian physics, and actually some of the results presented here are first steps in understanding some aspects of the so called "flat holography". For instance, when we constructed the relationship between the Ehlers transformations acting on Ricci-flat spacetimes with their manifestation in the boundary, we assumed that the bulk spacetime was of the algebraically special class and moreover, we restricted to time independent solutions. This excludes a big portion of the solution space, including radiating solutions. It would be interesting to complete the picture of the bulk-to-null boundary analysis of Ehlers transformations for more general Ricci-flat spacetimes with more general isometries. This includes to take into account what the action of the Ehlers group is on the infinite extra degrees of freedom that parameterized the bulk spacetime, such as the bulk shear (among others) that encodes gravitational radiation.

In the same way, the completion of the boundary construction of the towers of gravitational charges requires to go beyond the resumable case, as well as its comparison with the existing bulk approaches, such as the Godazgar-Godazgar-Pope definition of the charges. In this regard one can wonder where the Newman-Penrose charges are in this boundary-to-bulk set up. In some bulk approaches these charges appear in the tower of BMS subleading

charges at $1/r^3$. A Carrollian approach of the latter will necessary need to go beyond $1/r^2$ order in the covariant Newman-Unti gauge.

Besides the achievements and the concrete follow ups this thesis has yield, some fundamental questions remain open and deserve special emphasis. The Carrollian techniques we have developed have enabled us to recast many of the results accumulated in the sixties in a language and a perspective typical of the holographic playground. Ricci-flat spacetimes are reconstructed in terms of boundary Carrollian degrees of freedom, in a similar way as the reconstruction of asymptotically anti-de Sitter spacetimes in terms of boundary conformal-field-theory degrees of freedom. Although it would be inaccurate to call this bulk-from-boundary reconstruction holography, its nature surely points toward the existence of a possible flat extension of the standard AdS gauge/gravity correspondence, fueled in particular by our recent observation on the central role of the energy-momentum tensor. To support these ideas, a long route remains to be paved, and a wealth of questions to be answered. What would the fundamental observables be in the dual Carrollian field theory? What is the role played by the replicas of the energy–momentum tensor, namely the Chthonian data? Where does the celestial holography programme [151] sit in the more general Carrollian perspective? The celestial holography program relates gravitational S-matrix elements with correlation functions of a two-dimensional seemingly exotic CFT. Can we do better using boundary Carrollian field theory? ¹ Supposing even we succeed, given the infinite number of data necessary to reconstruct asymptotically flat spacetimes, would this correspondence still qualify for holographic? Although the approach presented here does not provide clues for answering these questions, our results on how to reach flat from AdS gives some confidence in the vanishing- Λ limit of the AdS/CFT correspondence.

¹Investigations on this direction have been done in [35, 37].

Appendix A

Thermodynamics

The Appendices A, B, C, D, E and F were extracted from [5, 3] and have been adapted to the conventions used in this thesis.

Relativistic thermodynamics

We remind here the usual observables of global-equilibrium thermodynamics. These are supposed to make sense also in local-equilibrium thermodynamics, as for fluids where the absence of short wave-length modes is assumed. In this case they depend on time and space and refer to measurements performed by an observer comoving with respect to the fluid. Matter conservation is generically (but not necessarily) akin to the existence of massive carriers in conserved number.

- The temperature T .
- The mass density ρ_0 per unit proper volume.
- The entropy per unit proper volume σ , and the entropy per unit mass s (specific entropy):

$$\sigma = s\rho_0. \tag{A.1}$$

- The relativistic internal energy density per unit proper volume ε , which contains the rest mass, and the specific energy per unit mass e :

$$\varepsilon = (e + k^2) \rho_0. \tag{A.2}$$

- The pressure p and the relativistic enthalpy \mathfrak{w} per unit proper volume:

$$\mathfrak{w} = \varepsilon + p. \tag{A.3}$$

- The relativistic chemical potential per unit mass (specific chemical potential) μ_0 . This contains the rest-mass contribution, as opposed to μ :

$$\mu_0 = \mu + k^2. \quad (\text{A.4})$$

These quantities obey the grand-potential equation, sometimes referred to as the Gibbs–Duhem equation:

$$\mathfrak{w} = T\sigma + \mu_0\rho_0 \Leftrightarrow p = Ts\rho_0 + \mu\rho_0 - e\rho_0. \quad (\text{A.5})$$

The energy density is a functional of two thermodynamic variables: $\varepsilon = \varepsilon(\sigma, \rho_0)$. The first law of thermodynamics reads:

$$d\varepsilon = Td\sigma + \mu_0d\rho_0 \Leftrightarrow de = Tds - pd\left(\frac{1}{\rho_0}\right). \quad (\text{A.6})$$

The Gibbs–Duhem equation allows to exhibit the dependence of the enthalpy per unit proper volume $\mathfrak{w} = \mathfrak{w}(\sigma, p, \rho_0)$

$$d\mathfrak{w} = Td\sigma + dp + \mu_0d\rho_0, \quad (\text{A.7})$$

whereas a double Legendre transformation on ε infers the dependence of the grand potential $p = p(T, \mu_0)$

$$dp = \sigma dT + \rho_0 d\mu_0 = s\rho_0 dT + \rho_0 d\mu. \quad (\text{A.8})$$

We would like to mention the situations where no massive degrees of freedom are present in the microscopic theory.¹ A gas of photons is the prime example but other instances exist in condensed matter, in particular when fermions are involved, as in graphene (see e.g. [119]). In the latter case, as opposed to the gas of photons, there is a conserved quantity. So ρ_0 is non-vanishing, but it is not a mass density; $\varepsilon = e\rho_0$ and $\mu_0 = \mu$, without rest-mass contribution. These systems can be conformally invariant, and in that case the dependence $p = p(T, \mu)$ is

$$p = T^D f(\mu T) \quad (\text{A.9})$$

in $D = d + 1$ spacetime dimensions.²

Coming back to a system with massless carriers and no conserved charge, as for the gas of photons, the above thermodynamic relationships simplify by setting $\mu = 0$ and dropping the rest-mass terms. Specific quantities are no longer significant in this instance. Fluid dynamics

¹This happens effectively in the usual ultra-relativistic limit, meant to be relevant microscopically at high temperature or high pressure.

²The precise bearing between conformal invariance, absence of mass and existence of conserved currents is subtle and tight to the microscopic theory.

of such systems does not involve any conserved current.³ The basic laws are summarized as follows:

$$\left\{ \begin{array}{l} \mathfrak{w} = T\sigma \\ d\mathfrak{w} = Td\sigma + dp \\ d\mathfrak{e} = Td\sigma \\ dp = \sigma dT, \end{array} \right. \quad (\text{A.10})$$

and when the system is furthermore conformal, $p \propto T^D$.

Several conserved charges might exist simultaneously in a thermodynamic system. They would each be associated with a density and a chemical potential. Only one, if any, would however enter the energy density (A.2).

Non-relativistic limit

The thermodynamic variables introduced earlier in the relativistic theory such as ρ_0 , ε , μ etc. are referring to a comoving observer. Measurements performed by another observer, be this an inertial observer in special relativity or some fiducial observer in a general gravitational background, are more relevant for the Galilean framework, but are not equal and should be spelled *stricto sensu* with some distinctive index. Their differences, however, are of order $1/k^2$ and vanish in the infinite- k limit. In order to avoid inflation in notation, we will keep the same symbols, e , T , p , s , μ , with the exception of ρ_0 , which becomes ρ for the fiducial observer. The $1/k^2$ corrections amongst ρ_0 and ρ (see (3.83)) play no role in thermodynamics, but are indispensable in recovering Navier–Stokes equations as the Galilean limit of the relativistic hydrodynamic equations.

In non-relativistic thermodynamics, it is customary to introduce the specific volume (not to be confused with the velocity)

$$v = \frac{1}{\rho}, \quad (\text{A.11})$$

as well as specific enthalpy $h = h(s, p)$ as

$$h = e + pv, \quad (\text{A.12})$$

which also enters in

$$\mu = h - Ts. \quad (\text{A.13})$$

³This instance was discussed in the precise framework of relativistic fluid dynamics in [58] §134, footnote 1 and exercise 2.

Using these definitions and the various relativistic laws mentioned above, we find the standard expressions:

$$dh = Tds + vdp, \quad (\text{A.14})$$

$$d(e\rho) = \rho Tds + h d\rho, \quad (\text{A.15})$$

$$de = Tds - p dv, \quad (\text{A.16})$$

$$d\mu = -s dT + v dp. \quad (\text{A.17})$$

Before closing this chapter, let us quote that Galilean thermodynamics can accommodate fluids with massless energy carriers, as long as the macroscopic velocity is small compared to k – although at the microscopic level the dynamics is ultra-relativistic. Again, a conserved current may or may not exist. In such a case a current is available, ρ is the charge density with⁴ $\varepsilon = e\rho$ the internal energy density and μ the chemical potential. The basic relationships are now

$$\mathfrak{w} = p + \varepsilon = T\sigma + \mu\rho \Leftrightarrow p = T s\rho + \mu\rho - e\rho, \quad (\text{A.18})$$

and

$$\left\{ \begin{array}{l} d\mathfrak{w} = T d\sigma + dp + \mu d\rho \\ d\varepsilon = T d\sigma + \mu d\rho \\ dp = \sigma dT + \rho d\mu. \end{array} \right. \quad (\text{A.19})$$

Equations (A.11), (A.12), (A.13), (A.14), (A.15), (A.16), (A.17) remain also valid, together with (A.9) in case of conformal invariance.

If no conserved charge is present, the chemical potential vanishes (as does $d\mu$) and the relevant equations are expressed with \mathfrak{w} , ε and σ rather than $h\rho$, $e\rho$ or $s\rho$.

Carrollian thermodynamics

Carrollian thermodynamics is poorly understood. In most parts of this work dealing with the fluid equations, we have kept the energy density ε and the pressure p unaltered in the limit of vanishing velocity of light. Neither have we introduced any temperature, nor discussed an entropy equation, and when a conserved current was assumed (as eluded in [101]), no relationship was established or set among energy and conserved-charge densities. This is minimalistic by default. Indeed, the shrinking of the light cone and the absence of particle motion or signal propagation, raise fundamental questions regarding the origin – and even

⁴Notice the distinction: $\varepsilon = \lim_{k \rightarrow \infty} \varepsilon = e\rho$, $\sigma = \lim_{k \rightarrow \infty} \sigma = s\rho$. In order to avoid multiplication of symbols, we keep $\mathfrak{w} = h\rho$, p and μ both for the relativistic quantities and for their Galilean limits.

the definition – of energy, pressure, entropy, temperature and thermalization processes. Even the kinematic parameter of the fluid is an inverse velocity, which could point towards the dynamics of instantonic space-filling branes, as mentioned in [44]. Obviously, this sort of objects are tachyonic – like those introduced later in [56] – and we feel uneasy advocating any sort of kinetic theory for setting up thermodynamic laws and deviations from equilibrium.

Appendix B

Carrollian momenta and hydrodynamic-frame invariance

In Sec. 4.2.4 we obtained the Carrollian fluid equations under k^2 -scaling assumptions involving more degrees of freedom than the standard ones: (4.115), (4.116), (4.117) and (4.118) for the relativistic energy ε , pressure p , heat current q^i and stress tensor τ^{ij} , and similarly for the matter sector with the matter density ρ_0 and the non-perfect current j^i in (4.119). These equations involve the Carrollian momenta $\tilde{\Pi}$, Π , $\tilde{\Pi}^i$, Π^i , P^i , $\tilde{\Pi}^{ij}$, Π^{ij} , $\tilde{\rho}$, ρ , \tilde{N}^i , N^i , which were given in (4.121) and (4.123).

The aim of the present appendix is to show that these are hydrodynamic-frame-invariant. Although hydrodynamic-frame invariance is built-in for the relativistic momenta (2.121), (2.122), (2.123), (2.124) and (2.125), it is not guaranteed to persist in the vanishing- k limit because it can be incompatible with the presumed small- k behaviour of the physical observables. This happens in the Galilean (infinite- k) limit, as we have witnessed in Sec. 3.2 for the standard case i.e. with $n^i = 0$ non-relativistic fluids, because $\delta n^i \propto \delta v^i$ (see Eq. (3.125)). Here it turns out to hold and in order to prove that we use the relativistic transformation rules (2.113), (2.114), (2.115), (2.116) and (2.117) in the Papapetrou–Randers frame (2.99). Using (4.115), (4.116), (4.117), (4.118) and (4.119), and expanding we find

$$\begin{cases} \delta m^j = 0 \\ \delta n^j = -\delta\beta^j\omega + \delta\beta_k m^k \beta^j \\ \delta\omega = -\delta\beta_k m^k \\ \delta\chi = -\delta\beta_k \left(n^k + \frac{\beta^2}{2} m^k \right), \end{cases} \quad (\text{B.1})$$

and

$$\left\{ \begin{array}{l} \delta\psi^i = 0 \\ \delta Q^i = \delta\beta_j (\Sigma^{ij} - \phi a^{ij}) - \delta\beta^i \zeta + \delta\beta_j \psi^j \beta^i \\ \delta\pi^i = \delta\beta_j (\Xi^{ij} - (\eta + \varpi) a^{ij}) + \frac{\beta^2}{2} \delta\beta_j (\Sigma^{ij} - (\zeta + \phi) a^{ij}) + \\ \delta\beta_j \beta^i (\beta^j (\zeta + \phi) + Q^j) + \beta^2 \beta^i \delta\beta^j \psi^j \\ \delta\zeta = -2\delta\beta_i \psi^i \\ \delta\eta = -2\delta\beta_i \left(Q^i + \frac{\beta^2}{2} \psi^i \right) \\ \delta\Sigma^{ij} - \delta\phi a^{ij} = \psi^i \delta\beta^j + \psi^j \delta\beta^i \\ \delta\Xi^{ij} - \delta\varpi a^{ij} = Q^i \delta\beta^j + Q^j \delta\beta^i + \delta\beta_k (\Sigma^{ki} \beta^j + \Sigma^{kj} \beta^i) + \frac{\beta^2}{2} (\psi^i \delta\beta^j + \psi^j \delta\beta^i) \\ + \delta\beta_k \beta^k (\psi^i \beta^j + \psi^j \beta^i) + \beta^i \beta^j \delta\phi. \end{array} \right. \quad (\text{B.2})$$

It is straightforward to show that the variations of all momenta (4.121) and (4.123) vanish.

Appendix C

More on Weyl-Carroll covariance

In Chapter 3 we saw that, in the presence of Weyl-Carroll covariance, we can define time and space Weyl-Carroll derivatives. These derivatives enable to define other Weyl-Carroll covariant objects through their commutators. They read

$$\left[\hat{\mathcal{D}}_i, \hat{\mathcal{D}}_j \right] \Phi = \frac{2}{\Omega} \bar{\omega}_{ij} \hat{\mathcal{D}}_t \Phi + w \Omega_{ij} \Phi, \quad (\text{C.1})$$

$$\left[\hat{\mathcal{D}}_k, \hat{\mathcal{D}}_l \right] V^i = \left(\hat{\mathcal{R}}^i{}_{jkl} - 2\xi^i{}_j \bar{\omega}_{kl} \right) V^j + \bar{\omega}_{kl} \frac{2}{\Omega} \hat{\mathcal{D}}_t V^i + w \Omega_{kl} V^i, \quad (\text{C.2})$$

where

$$\begin{aligned} \hat{\mathcal{R}}^i{}_{jkl} &= \hat{r}^i{}_{jkl} - \delta_j^i \varphi_{kl} - a_{jk} \hat{\nabla}_l \varphi^i + a_{jl} \hat{\nabla}_k \varphi^i + \delta_k^i \hat{\nabla}_l \varphi_j - \delta_l^i \hat{\nabla}_k \varphi_j \\ &\quad + \varphi^i (\varphi_k a_{jl} - \varphi_l a_{jk}) - (\delta_k^i a_{jl} - \delta_l^i a_{jk}) \varphi_m \varphi^m + (\delta_k^i \varphi_l - \delta_l^i \varphi_k) \varphi_j, \end{aligned} \quad (\text{C.3})$$

$$\Omega_{ij} = \varphi_{ij} - \frac{2}{d} \bar{\omega}_{ij} \theta, \quad (\text{C.4})$$

and $\varphi_{ij} = \hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i$, are weight-0 Weyl-covariant tensors. Taking traces of (C.3) we get

$$\hat{\mathcal{R}}_{ij} = \hat{\mathcal{R}}^k{}_{ikj}, \quad \hat{\mathcal{R}} = a^{ij} \hat{\mathcal{R}}_{ij} \quad (\text{C.5})$$

with

$$\hat{\mathcal{R}} = \hat{r} + (d-1) \left(2\hat{\nabla}_i \varphi^i - (d-2) \varphi_i \varphi^i \right), \quad (\text{C.6})$$

of weights zero and 2. The Weyl-covariant Carroll-Ricci tensor is not symmetric, $\hat{\mathcal{R}}_{[ij]} = -\frac{d}{2} \Omega_{ij}$, and a weight-1 curvature form also appears with

$$\left[\frac{1}{\Omega} \hat{\mathcal{D}}_t, \hat{\mathcal{D}}_i \right] \Phi = w \hat{\mathcal{R}}_i \Phi - \xi^j{}_i \hat{\mathcal{D}}_j \Phi, \quad (\text{C.7})$$

where

$$\hat{\mathcal{R}}_i = \frac{1}{\Omega} \partial_i \varphi_i - \frac{1}{d} (\hat{\partial}_i + \varphi_i) \theta. \quad (\text{C.8})$$

Appendix D

Carrollian geometry in three dimensions and Carrollian Cotton tensors

For $d = 2$, the \mathcal{S} -Hodge duality is induced by¹ $\eta_{ij} = \sqrt{a}\varepsilon_{ij}$. This duality is involutive on Carrollian vectors as well as on two-index symmetric and traceless Carrollian tensors:

$$*V_i = \eta^l{}_i V_l, \quad *W_{ij} = \eta^l{}_i W_{lj}. \quad (\text{D.1})$$

This fully antisymmetric form can be used to recast some of the Carrollian curvatures introduced previously. The Carroll–Ricci tensor is decomposed as

$$\hat{r}_{ij} = \hat{s}_{ij} + \hat{K}a_{ij} + \hat{A}\eta_{ij} \quad (\text{D.2})$$

with

$$\hat{s}_{ij} = 2 * \varpi * \xi_{ij}, \quad \hat{K} = \frac{1}{2} a^{ij} \hat{r}_{ij} = \frac{1}{2} \hat{r}, \quad \hat{A} = \frac{1}{2} \eta^{ij} \hat{r}_{ij} = * \varpi \theta, \quad * \varpi = \frac{1}{2} \eta^{ij} \varpi_{ij}. \quad (\text{D.3})$$

Similarly

$$\hat{\mathcal{R}}_{ij} = \hat{s}_{ij} + \hat{\mathcal{K}}a_{ij} + \hat{\mathcal{A}}\eta_{ij}, \quad (\text{D.4})$$

where we have introduced two weight-2 Weyl-covariant scalar Gauss–Carroll curvatures:

$$\hat{\mathcal{K}} = \frac{1}{2} a^{ij} \hat{\mathcal{R}}_{ij} = \hat{K} + \hat{\nabla}_k \varphi^k, \quad \hat{\mathcal{A}} = \frac{1}{2} \eta^{ij} \hat{\mathcal{R}}_{ij} = \hat{A} - * \varphi, \quad (\text{D.5})$$

¹We use here the conventions of Ref. [42], namely $\varepsilon_{12} = -1$, convenient when using complex coordinates $\{\zeta, \bar{\zeta}\}$. Notice that $\eta^{il}\eta_{jl} = \delta_j^i$ and $\eta^{ij}\eta_{ij} = 2$.

and $*\varphi = \frac{1}{2}\eta^{ij}\varphi_{ij}$. These obey Carroll-Bianchi identities:

$$\frac{2}{\Omega}\hat{\mathcal{D}}_t*\varpi + \hat{\mathcal{A}} = 0, \quad (\text{D.6})$$

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t\hat{\mathcal{K}} - a^{ij}\hat{\mathcal{D}}_i\hat{\mathcal{R}}_j - \hat{\mathcal{D}}_i\hat{\mathcal{D}}_j\xi^{ij} = 0, \quad (\text{D.7})$$

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t\hat{\mathcal{A}} + \eta^{ij}\hat{\mathcal{D}}_i\hat{\mathcal{R}}_j = 0. \quad (\text{D.8})$$

Besides the various curvature tensors, which are second derivatives of the metric and the Ehresmann connection, one defines third-derivative tensors, the descendants of the relativistic Cotton tensor.

In a three dimensional pseudo-Riemannian geometry, the Cotton tensor is typically defined as

$$C_{\mu\nu} = \eta_{\mu}^{\rho\sigma}\nabla_{\rho}\left(R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}\right), \quad (\text{D.9})$$

with $\eta_{\mu\nu\rho} = \sqrt{-g}\varepsilon_{\mu\nu\rho}$ a fully anti-symmetric tensor. The Cotton tensor measures the deviation of the spacetime from being conformally flat. In the definition (D.9), it is symmetric, Weyl-covariant of weight 1, and identically conserved, namely it satisfies

$$\mathcal{D}_{\mu}C^{\mu\nu} = \nabla_{\mu}C^{\mu\nu} = 0. \quad (\text{D.10})$$

Similar to the relativistic energy-momentum tensor, the Cotton tensor can be decomposed as

$$\frac{1}{k}C_{\mu\nu} = \frac{3c}{2}\frac{u_{\mu}u_{\nu}}{k^2} + \frac{c}{2}g_{\mu\nu} - \frac{c_{\mu\nu}}{k^2} + \frac{u_{\mu}c_{\nu}}{k^2} + \frac{u_{\nu}c_{\mu}}{k^2}, \quad (\text{D.11})$$

with c the Cotton scalar density of weight 3, $c_{\mu\nu}$ the symmetric and traceless Cotton stress of weight 1, and the Cotton current c_{μ} of weight 2. The Cotton stress and the Cotton current are purely transverse, obeying $u^{\mu}c_{\mu\nu} = 0$ and $u^{\mu}c_{\mu} = 0$.

The Cotton tensor components can be expressed in terms of a power expansion of k , packaging at every order its Carrollian-covariant versions of the Cotton pieces. We have

$$c = c_{(-1)}k^2 + c_{(0)} + \frac{c_{(1)}}{k^2} + \frac{c_{(2)}}{k^4}, \quad (\text{D.12})$$

$$c^i = k^2\psi^i + \chi^i + \frac{z^i}{k^2}, \quad (\text{D.13})$$

$$c^{ij} = k^2\Psi^{ij} + X^{ij} + \frac{Z^{ij}}{k^2} \quad (\text{D.14})$$

where each of these Cotton pieces is given as

$$c_{(-1)} = 8 * \varpi^3, \quad (\text{D.15})$$

$$c_{(0)} = \left(\hat{\mathcal{D}}_i \hat{\mathcal{D}}^i + 2 \hat{\mathcal{K}} \right) * \varpi, \quad (\text{D.16})$$

$$c_{(1)} = \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j * \xi^{ij}, \quad (\text{D.17})$$

$$c_{(2)} = * \xi_{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_t \xi^{ij}, \quad (\text{D.18})$$

$$\psi^i = 3 \eta^{ji} \hat{\mathcal{D}}_j * \varpi^2, \quad (\text{D.19})$$

$$\chi^i = \frac{1}{2} \eta^{ji} \hat{\mathcal{D}}_j \hat{\mathcal{K}} + \frac{1}{2} \hat{\mathcal{D}}^i \hat{\mathcal{A}} - 2 * \varpi \left(\hat{\mathcal{R}}^i + 2 \hat{\mathcal{D}}_j \xi^{ij} \right) + 3 \hat{\mathcal{D}}_j (* \varpi \xi^{ij}), \quad (\text{D.20})$$

$$z^i = \frac{1}{2} \eta^{ij} \hat{\mathcal{D}}_j \xi^2 - \hat{\mathcal{D}}_j \frac{1}{\Omega} \hat{\mathcal{D}}_t * \xi^{ij} - * \xi^i_j \hat{\mathcal{D}}_k \xi^{jk}, \quad (\text{D.21})$$

$$\Psi^{ij} = -2 * \varpi^2 * \xi^{ij} + \hat{\mathcal{D}}^i \hat{\mathcal{D}}^j * \varpi - \frac{1}{2} \delta^{ij} \hat{\mathcal{D}}_k \hat{\mathcal{D}}^k * \varpi - \eta^{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_t * \varpi^2, \quad (\text{D.22})$$

$$\begin{aligned} X^{ij} &= \frac{1}{2} \eta^{ki} \hat{\mathcal{D}}_k \left(\hat{\mathcal{R}}^j + \hat{\mathcal{D}}_l \xi^{jl} \right) + \frac{1}{2} \eta^{kj} \hat{\mathcal{D}}^i \left(\hat{\mathcal{R}}_k + \hat{\mathcal{D}}^l \xi_{kl} \right) \\ &\quad - \frac{3}{2} \hat{\mathcal{A}} \xi^{ij} - \hat{\mathcal{K}} * \xi^{ij} + 3 * \varpi \frac{1}{\Omega} \hat{\mathcal{D}}_t \xi^{ij}, \end{aligned} \quad (\text{D.23})$$

$$Z^{ij} = 2 * \xi^{ij} \xi^2 - \frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t * \xi^{ij}, \quad (\text{D.24})$$

where we defined

$$\xi^2 = \frac{1}{2} \xi^{ij} \xi_{ij} \Leftrightarrow \xi^{ik} \xi_k^j = \xi^2 a^{ij}. \quad (\text{D.25})$$

As a consequence of the relativistic conservation of the Cotton tensor, its Carrollian descendants obey Carrollian conservation equations, similar to (4.136), (4.138) and (4.139).

These equations are

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t c_{(-1)} + \hat{\mathcal{D}}_i \psi^i = 0, \quad (\text{D.26})$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t c_{(0)} + \hat{\mathcal{D}}_i \chi^i - \Psi_{ij} \xi^{ij} = 0, \quad (\text{D.27})$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t c_{(1)} + \hat{\mathcal{D}}_i z^i - X_{ij} \xi^{ij} = 0, \quad (\text{D.28})$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t c_{(2)} - Z_{ij} \xi^{ij} = 0, \quad (\text{D.29})$$

$$\frac{1}{2} \hat{\mathcal{D}}_t c_{(-1)} + 2 * \bar{\omega} * \psi_i = 0, \quad (\text{D.30})$$

$$\frac{1}{2} \hat{\mathcal{D}}_t c_{(0)} - \hat{\mathcal{D}}^j \Psi_{ij} + 2 * \bar{\omega} * \chi_i + \frac{1}{\Omega} \hat{\mathcal{D}}_t \psi_i + \xi_{ij} \psi^j = 0, \quad (\text{D.31})$$

$$\frac{1}{2} \hat{\mathcal{D}}_t c_{(1)} - \hat{\mathcal{D}}^j X_{ij} + 2 * \bar{\omega} * z_i + \frac{1}{\Omega} \hat{\mathcal{D}}_t \chi_i + \xi_{ij} \chi^j = 0, \quad (\text{D.32})$$

$$\frac{1}{2} \hat{\mathcal{D}}_t c_{(2)} - \hat{\mathcal{D}}^j Z_{ij} + \frac{1}{\Omega} \hat{\mathcal{D}}_t z_i + \xi_{ij} z^j = 0. \quad (\text{D.33})$$

These are purely geometrical identities valid in three-dimensional Carrollian structures given by (4.1) and (4.2).

When the geometric Carrollian shear vanishes, the time dependence in the metric is factorized as $a_{ij}(t, \mathbf{x}) = e^{2\sigma(t, \mathbf{x})} \bar{a}_{ij}(\mathbf{x})$. One then shows [42, 20] that the Carrollian conformal isometry group is the semi-direct product of the conformal group of $\bar{a}_{ij}(\mathbf{x})$ with the infinite-dimensional supertranslation group. The former is generated by $Y^i(\mathbf{x})$, the latter by $T(\mathbf{x})$, and the Carrollian conformal Killing fields read:

$$\xi_{T,Y} = \left(T(\mathbf{x}) - Y^i(\mathbf{x}) \hat{\partial}_i C(t, \mathbf{x}) + \frac{1}{2} C(t, \mathbf{x}) \bar{\nabla}_i Y^i(\mathbf{x}) \right) \frac{e^{\sigma(t, \mathbf{x})}}{\Omega} \partial_t + Y^i(\mathbf{x}) \hat{\partial}_i \quad (\text{D.34})$$

with

$$C(t, \mathbf{x}) \equiv \int^t d\tau e^{-\sigma(\tau, \mathbf{x})} \Omega(\tau, \mathbf{x}). \quad (\text{D.35})$$

This result is valid in any dimension. At $d = 2$, $\bar{a}_{ij}(\mathbf{x})$ is conformally flat and $Y^i(\mathbf{x})$ generate $\mathfrak{so}(3, 1)$.²

The conservation of the Carrollian Cotton momenta (6.60) makes it possible to define two infinite towers of Carrollian Cotton charges $Q_{\text{Cot}T,Y}$ and $\tilde{Q}_{\text{Cot}T,Y}$ following (4.57), based on the Carrollian Cotton currents κ_{Cot} , K_{Cot}^i , $\tilde{\kappa}_{\text{Cot}}$ and \tilde{K}_{Cot}^i (see (6.61)). According to (4.144), the

²The $\mathfrak{so}(3, 1)$ factor can also be promoted to superrotations (double Virasoro) if we give up the absolute regularity requirement.

latter are always conserved,³ whereas the former are only if $\chi^i v_i = -\chi^i \left(\hat{\mathcal{D}}_i \xi^{\hat{i}} - 2\xi^j \varpi_{ji} \right) = 0$. This occurs for special geometries ($\chi^i = 0$) or for the subset of strong Carrollian conformal Killing fields ($v_i = 0$).

In $d = 2$, it is convenient to use complex spatial coordinates ζ and $\bar{\zeta}$. With the permission of the authors of [42], we reproduce here the appendix of that reference, summarizing the useful formulas in this coordinate system. Using Carrollian diffeomorphisms (2.100), the metric (4.1) of the Carrollian geometry on the two-dimensional surface \mathcal{S} can be recast in conformally flat form,

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta} \quad (\text{D.36})$$

with $P = P(t, \zeta, \bar{\zeta})$ a real function, under the necessary and sufficient condition that the Carrollian shear ξ_{ij} displayed in (4.15) vanishes. We will here assume that this holds and present a number of useful formulas for Carrollian and conformal Carrollian geometry. These geometries carry two further pieces of data: $\Omega(t, \zeta, \bar{\zeta})$ and

$$\mathbf{b} = b_\zeta(t, \zeta, \bar{\zeta}) d\zeta + b_{\bar{\zeta}}(t, \zeta, \bar{\zeta}) d\bar{\zeta} \quad (\text{D.37})$$

with $b_{\bar{\zeta}}(t, \zeta, \bar{\zeta}) = \bar{b}_\zeta(t, \zeta, \bar{\zeta})$. Our choice of orientation is inherited from the one adopted for the relativistic boundary with $a_{\zeta\bar{\zeta}} = 1/P^2$ and⁴

$$\eta_{\zeta\bar{\zeta}} = -\frac{i}{P^2}. \quad (\text{D.38})$$

The first-derivative Carrollian tensors are the acceleration (4.11), the expansion (4.15) and the scalar vorticity (4.11), (D.3):

$$\varphi_\zeta = \partial_t \frac{b_\zeta}{\Omega} + \hat{\partial}_\zeta \ln \Omega, \quad \varphi_{\bar{\zeta}} = \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \ln \Omega, \quad (\text{D.39})$$

$$\theta = -\frac{2}{\Omega} \partial_t \ln P, \quad * \varpi = \frac{i\Omega P^2}{2} \left(\hat{\partial}_\zeta \frac{b_{\bar{\zeta}}}{\Omega} - \hat{\partial}_{\bar{\zeta}} \frac{b_\zeta}{\Omega} \right) \quad (\text{D.40})$$

with

$$\hat{\partial}_\zeta = \partial_\zeta + \frac{b_\zeta}{\Omega} \partial_t, \quad \hat{\partial}_{\bar{\zeta}} = \partial_{\bar{\zeta}} + \frac{b_{\bar{\zeta}}}{\Omega} \partial_t. \quad (\text{D.41})$$

³The conformal Killing fields (D.34), (3.2) depend explicitly on time. Inside the charges they define, when conserved, this time dependence is confined, on-shell, in a boundary term, and hence drops – see concrete examples in [4].

⁴This amounts to setting $\sqrt{a} = i/P^2$ in coordinate frame and $\varepsilon_{\zeta\bar{\zeta}} = -1$. The volume form reads $d^2x\sqrt{a} = \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}$.

Curvature scalars and vector are second-derivative (see (D.3), (4.24)):⁵

$$\hat{K} = P^2 \left(\hat{\partial}_{\bar{\zeta}} \hat{\partial}_{\zeta} + \hat{\partial}_{\zeta} \hat{\partial}_{\bar{\zeta}} \right) \ln P, \quad \hat{A} = iP^2 \left(\hat{\partial}_{\bar{\zeta}} \hat{\partial}_{\zeta} - \hat{\partial}_{\zeta} \hat{\partial}_{\bar{\zeta}} \right) \ln P, \quad (\text{D.42})$$

$$\hat{r}_{\zeta} = \frac{1}{2} \hat{\partial}_{\zeta} \left(\frac{1}{\Omega} \partial_t \ln P \right), \quad \hat{r}_{\bar{\zeta}} = \frac{1}{2} \hat{\partial}_{\bar{\zeta}} \left(\frac{1}{\Omega} \partial_t \ln P \right), \quad (\text{D.43})$$

and we also quote:

$$*\varphi = iP^2 \left(\hat{\partial}_{\zeta} \varphi_{\bar{\zeta}} - \hat{\partial}_{\bar{\zeta}} \varphi_{\zeta} \right), \quad (\text{D.44})$$

$$\hat{\nabla}_k \varphi^k = P^2 \left[\hat{\partial}_{\zeta} \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \partial_t \frac{b_{\zeta}}{\Omega} + \left(\hat{\partial}_{\zeta} \hat{\partial}_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}} \hat{\partial}_{\zeta} \right) \ln \Omega \right]. \quad (\text{D.45})$$

Regarding conformal Carrollian tensors we remind the weight-2 curvature scalars (D.5):

$$\hat{\mathcal{K}} = \hat{K} + \hat{\nabla}_k \varphi^k, \quad \hat{\mathcal{A}} = \hat{A} - *\varphi, \quad (\text{D.46})$$

and the weight-1 curvature one-form (C.8):

$$\hat{\mathcal{R}}_{\zeta} = \frac{1}{\Omega} \partial_t \varphi_{\zeta} - \frac{1}{2} \left(\hat{\partial}_{\zeta} + \varphi_{\zeta} \right) \theta, \quad \hat{\mathcal{R}}_{\bar{\zeta}} = \frac{1}{\Omega} \partial_t \varphi_{\bar{\zeta}} - \frac{1}{2} \left(\hat{\partial}_{\bar{\zeta}} + \varphi_{\bar{\zeta}} \right) \theta. \quad (\text{D.47})$$

In the resumable case the non-vanishing three-derivative Cotton descendants are displayed as: one scalar

$$c_{(0)} = \left(\hat{\mathcal{D}}_t \hat{\mathcal{D}}^t + 2\hat{\mathcal{K}} \right) *\varpi \quad (\text{D.48})$$

of weight 3 (* ϖ is of weight 1), two vectors

$$\chi_{\zeta} = \frac{i}{2} \hat{\mathcal{D}}_{\zeta} \hat{\mathcal{K}} + \frac{1}{2} \hat{\mathcal{D}}_{\zeta} \hat{\mathcal{A}} - 2*\varpi \hat{\mathcal{R}}_{\zeta}, \quad \chi_{\bar{\zeta}} = -\frac{i}{2} \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{K}} + \frac{1}{2} \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{A}} - 2*\varpi \hat{\mathcal{R}}_{\bar{\zeta}}, \quad (\text{D.49})$$

$$\psi_{\zeta} = 3i \hat{\mathcal{D}}_{\zeta} *\varpi^2, \quad \psi_{\bar{\zeta}} = -3i \hat{\mathcal{D}}_{\bar{\zeta}} *\varpi^2, \quad (\text{D.50})$$

of weight 2, and two symmetric and traceless tensors

$$X_{\zeta\zeta} = i \hat{\mathcal{D}}_{\zeta} \hat{\mathcal{R}}_{\zeta}, \quad X_{\bar{\zeta}\bar{\zeta}} = -i \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{R}}_{\bar{\zeta}}, \quad (\text{D.51})$$

$$\Psi_{\zeta\zeta} = \hat{\mathcal{D}}_{\zeta} \hat{\mathcal{D}}_{\zeta} *\varpi, \quad \Psi_{\bar{\zeta}\bar{\zeta}} = \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{D}}_{\bar{\zeta}} *\varpi, \quad (\text{D.52})$$

⁵We also quote for completeness (useful *e.g.* in Eq. (D.46)):

$$\hat{K} = K + P^2 \left[\partial_{\zeta} \frac{b_{\bar{\zeta}}}{\Omega} + \partial_{\bar{\zeta}} \frac{b_{\zeta}}{\Omega} + \partial_t \frac{b_{\zeta} b_{\bar{\zeta}}}{\Omega^2} + 2 \frac{b_{\bar{\zeta}}}{\Omega} \partial_{\zeta} + 2 \frac{b_{\zeta}}{\Omega} \partial_{\bar{\zeta}} + 2 \frac{b_{\zeta} b_{\bar{\zeta}}}{\Omega^2} \partial_t \right] \partial_t \ln P$$

with $K = 2P^2 \partial_{\bar{\zeta}} \partial_{\zeta} \ln P$ the ordinary Gaussian curvature of the two-dimensional metric (D.36).

of weight 1. Notice that in holomorphic coordinates a symmetric and traceless tensor S_{ij} has only diagonal entries: $S_{\zeta\bar{\zeta}} = 0 = S_{\bar{\zeta}\zeta}$.

We also remind for convenience some expressions for the determination of Weyl–Carroll covariant derivatives. If Φ is a weight- w scalar function

$$\hat{\mathcal{D}}_{\zeta}\Phi = \hat{\partial}_{\zeta}\Phi + w\varphi_{\zeta}\Phi, \quad \hat{\mathcal{D}}_{\bar{\zeta}}\Phi = \hat{\partial}_{\bar{\zeta}}\Phi + w\varphi_{\bar{\zeta}}\Phi. \quad (\text{D.53})$$

For weight- w form components V_{ζ} and $V_{\bar{\zeta}}$ the Weyl–Carroll derivatives read:

$$\hat{\mathcal{D}}_{\zeta}V_{\zeta} = \hat{\nabla}_{\zeta}V_{\zeta} + (w+2)\varphi_{\zeta}V_{\zeta}, \quad \hat{\mathcal{D}}_{\bar{\zeta}}V_{\bar{\zeta}} = \hat{\nabla}_{\bar{\zeta}}V_{\bar{\zeta}} + (w+2)\varphi_{\bar{\zeta}}V_{\bar{\zeta}}, \quad (\text{D.54})$$

$$\hat{\mathcal{D}}_{\zeta}V_{\bar{\zeta}} = \hat{\nabla}_{\zeta}V_{\bar{\zeta}} + w\varphi_{\zeta}V_{\bar{\zeta}}, \quad \hat{\mathcal{D}}_{\bar{\zeta}}V_{\zeta} = \hat{\nabla}_{\bar{\zeta}}V_{\zeta} + w\varphi_{\bar{\zeta}}V_{\zeta}, \quad (\text{D.55})$$

while the Carrollian covariant derivatives are simply:

$$\hat{\nabla}_{\zeta}V_{\zeta} = \frac{1}{P^2}\hat{\partial}_{\zeta}(P^2V_{\zeta}), \quad \hat{\nabla}_{\bar{\zeta}}V_{\bar{\zeta}} = \frac{1}{P^2}\hat{\partial}_{\bar{\zeta}}(P^2V_{\bar{\zeta}}), \quad (\text{D.56})$$

$$\hat{\nabla}_{\zeta}V_{\bar{\zeta}} = \hat{\partial}_{\zeta}V_{\bar{\zeta}}, \quad \hat{\nabla}_{\bar{\zeta}}V_{\zeta} = \hat{\partial}_{\bar{\zeta}}V_{\zeta}. \quad (\text{D.57})$$

Finally,

$$\hat{\mathcal{D}}_k\hat{\mathcal{D}}^k\Phi = P^2\left(\hat{\partial}_{\zeta}\hat{\partial}_{\bar{\zeta}}\Phi + \hat{\partial}_{\bar{\zeta}}\hat{\partial}_{\zeta}\Phi + w\Phi\left(\hat{\partial}_{\zeta}\varphi_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}}\varphi_{\zeta}\right) + 2w\left(\varphi_{\zeta}\hat{\partial}_{\bar{\zeta}}\Phi + \varphi_{\bar{\zeta}}\hat{\partial}_{\zeta}\Phi + w\varphi_{\zeta}\varphi_{\bar{\zeta}}\Phi\right)\right). \quad (\text{D.58})$$

Using complex coordinates, we can recast the conformal Killing vectors of a shear-free Carrollian spacetime \mathcal{M} in three dimensions, given in Eqs. (D.34) and (3.2). These are expressed in terms of an arbitrary real function $T(\zeta, \bar{\zeta})$, which encodes the supertranslations, and the conformal Killing vectors of flat space $d\bar{\ell}^2 = 2d\zeta d\bar{\zeta}$. The latter are of the form $Y^{\zeta}(\zeta)\partial_{\zeta} + Y^{\bar{\zeta}}(\bar{\zeta})\partial_{\bar{\zeta}}$, reached with any combination of $\ell_m + \bar{\ell}_m$ or $i(\ell_m - \bar{\ell}_m)$, where⁶

$$\ell_m = -\zeta^{m+1}\partial_{\zeta}, \quad \bar{\ell}_m = -\bar{\zeta}^{m+1}\partial_{\bar{\zeta}}, \quad (\text{D.59})$$

obeying the Witt \oplus Witt algebra:

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}, \quad (\text{D.60})$$

⁶Notice that combining (D.1) and (D.38), we find $*(\ell_m + \bar{\ell}_m) = -i(\ell_m - \bar{\ell}_m)$.

and referred to as superrotations. Usually one restricts to $\mathfrak{so}(3, 1)$, generated by $n = 0, \pm 1$. The conformal Killing fields of \mathcal{M} are thus

$$\xi_{T,Y} = \left(T - \left(Y^\zeta \hat{\partial}_\zeta + Y^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}} \right) C + \frac{C}{2} \left(\partial_\zeta Y^\zeta + \partial_{\bar{\zeta}} Y^{\bar{\zeta}} \right) \right) \frac{1}{P} \mathbf{v} + Y^\zeta \hat{\partial}_\zeta + Y^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}} \quad (\text{D.61})$$

with

$$C(t, \zeta, \bar{\zeta}) \equiv \int^t d\tau P(\tau, \zeta, \bar{\zeta}) \Omega(\tau, \zeta, \bar{\zeta}). \quad (\text{D.62})$$

The structure $\mathfrak{so}(3, 1) \ltimes$ supertranslations – or $(\text{Witt} \oplus \text{Witt}) \ltimes$ supertranslations – is recovered in

$$[\xi_{T,Y}, \xi_{T',Y'}] = \xi_{M_Y(T') - M_{Y'}(T), [Y, Y']} \quad (\text{D.63})$$

with

$$M_Y(f) = \left(Y^\zeta \hat{\partial}_\zeta + Y^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}} \right) f - \frac{f}{2} \left(\partial_\zeta Y^\zeta + \partial_{\bar{\zeta}} Y^{\bar{\zeta}} \right). \quad (\text{D.64})$$

Appendix E

Free motion

Our results on the failure of conservation laws associated with some Galilean or Carrollian (conformal) Killing vector fields are generic and rooted to the nature of the underlying geometries. The same phenomenon occurs when studying free-particle motion in Newton–Cartan spacetimes, or instantonic branes on Carrollian structures (see [44] for motivations on the latter paradigm). For concreteness we will illustrate here the former case.

The stage is set with an action

$$S[\mathbf{x}] = \int_{\mathcal{E}} dt \Omega(t) \mathcal{L}(t, \mathbf{x}, \mathbf{v}), \quad (\text{E.1})$$

where $L = \Omega \mathcal{L}$ is the Lagrangian – as opposed to the Lagrangian density. The generalized Lagrange momenta are

$$p_i = \frac{\partial \mathcal{L}}{\partial v^i} \quad (\text{E.2})$$

and the energy $E = \Omega \mathcal{E}$ with

$$\mathcal{E} = \frac{p_i v^i}{\Omega} - \mathcal{L}. \quad (\text{E.3})$$

The equations of motion are Euler–Lagrange

$$\frac{1}{\Omega} \dot{p}_i - \frac{\partial \mathcal{L}}{\partial x^i} = 0. \quad (\text{E.4})$$

The dot stands for the total derivative along the trajectory, which can act also as $\partial_t + v^i \partial_i$ on any tensor, and should not be confused with d/dt defined in (3.25) unless they act on scalars (cf. ordinary vs. covariant spatial derivative).

Consider Galilean diffeomorphisms generated by

$$\xi = \xi^t \partial_t + \xi^i \partial_i, \quad (\text{E.5})$$

where $\xi^t = \xi^t(t)$ and

$$\xi^{\hat{t}} = \xi^t \Omega, \quad \xi^{\hat{i}} = \xi^i - \xi^t w^i, \quad \xi_{\hat{t}} = -c^2 \xi^{\hat{t}}, \quad \xi_{\hat{i}} = a_{ij} \xi^{\hat{j}} = \xi_i. \quad (\text{E.6})$$

Their effect on the dynamical variables is

$$\begin{cases} t \rightarrow t + \xi^t \\ x^i \rightarrow x^i + \xi^i \\ v^i \rightarrow v^i + \partial_t \xi^i + v^j \partial_j \xi^i - v^i \partial_t \xi^t. \end{cases} \quad (\text{E.7})$$

On the one hand, the invariance of the action is characterized as follows:

$$\delta S = 0 \Leftrightarrow \Omega \delta \mathcal{L} + \mathcal{L} \partial_t \xi^{\hat{t}} = \frac{d\phi}{dt}, \quad (\text{E.8})$$

where $\phi = \phi(t, \mathbf{x})$ is an arbitrary function, that needs not be zero. On the other hand, one can determine the on-shell variation of the Lagrangian density:

$$\delta \mathcal{L} = -\frac{\mathcal{L}}{\Omega} \partial_t \xi^{\hat{t}} + \frac{1}{\Omega} \frac{d}{dt} \left(p_i \xi^i - \mathcal{E} \xi^{\hat{t}} \right). \quad (\text{E.9})$$

The simplest of Noether's theorems states that

$$\delta S = 0 \Leftrightarrow p_i \xi^i - \mathcal{E} \xi^{\hat{t}} - \phi = \text{constant of motion}. \quad (\text{E.10})$$

Suppose now that the motion is free on a Newton–Cartan spacetime featured by a^{ij} , w^i and Ω . The Lagrangian density is

$$\mathcal{L} = \frac{1}{2\Omega^2} a_{ij} (v^i - w^i) (v^j - w^j) \quad (\text{E.11})$$

with

$$p_i = \frac{1}{\Omega} (v_i - w_i). \quad (\text{E.12})$$

Euler–Lagrange equations read:

$$\left(\frac{1}{\Omega} \hat{D} + p^j \hat{\nabla}_j \right) p_i + p_j \hat{\gamma}^{wj}_i = 0. \quad (\text{E.13})$$

As an aside remark, the latter equation is the infinite- k limit of the spatial component of the geodesic equation $u^\mu \nabla_\mu u_i = 0$, in a Zermelo background. The time component $u^\mu \nabla_\mu u^0 = 0$ leads to

$$\frac{1}{2\Omega} \frac{dp_i p^i}{dt} + p_i p_j \hat{\gamma}^{wij} = 0, \quad (\text{E.14})$$

which is the energy equation, obtained by contracting (E.13) with p^i .

We can now compute the generic variation of (E.11) under Galilean diffeomorphisms acting as (E.7). We obtain the following:

$$\delta \mathcal{L} = p_i p_j \left(\hat{\nabla}^{(i} \xi^{\hat{j})} + \xi^{\hat{i}} \hat{\gamma}^{wij} \right) - p_i p^i \frac{1}{\Omega} \frac{\hat{D} \xi^{\hat{i}}}{dt} + p_i \left(\frac{1}{\Omega} \frac{\hat{D} \xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi}{}_j \xi^{\hat{j}} \right). \quad (\text{E.15})$$

If ξ is a Killing field it satisfies (3.55), the first two terms drop and

$$\delta \mathcal{L} = p_i \left(\frac{1}{\Omega} \frac{\hat{D} \xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi}{}_j \xi^{\hat{j}} \right) \quad (\text{E.16})$$

does not vanish, exactly as in the Galilean fluid dynamics in the presence of an isometry. This betrays the break down of conservation, unless the right-hand side of Eq. (E.16) happens to be of the form (E.9), in which case Noether's theorem applies in its version (E.10).

As already emphasized repeatedly, this pattern works the same way in all situations we have met, involving Galilean or Carrollian dynamics. In flat spacetimes (either Galilean or Carrollian) boosts belong invariably to the class of isometries with non-vanishing Lagrangian variation (see (3.60) and (4.50)). There is not much we could extract from this in fluid dynamics (except for the case of flat-space potential flows), but for Galilean free-particle motion on flat spacetime ($a_{ij} = \delta_{ij}$, $\Omega = 1$, w^i constants) the situation is simpler. We find indeed:

$$\delta \mathcal{L} = (\dot{x}^i - w^i) \left(V_i + w^k \Omega_{ki} \right) = \frac{d}{dt} \left(x^i V_i - w^i V_i t + w^k x^i \Omega_{ki} \right). \quad (\text{E.17})$$

In this particular case, (E.10) applies and gives the general constant of motion as (see also (3.59))

$$V_i (\dot{x}^i t - x^i) - \frac{T}{2} (\dot{\mathbf{x}}^2 - \mathbf{w}^2) + X_i (\dot{x}^i - w^i) + \Omega_{ij} x^i \dot{x}^j. \quad (\text{E.18})$$

The boosts V^i do not generate any useful first integral (the initial position x_0^i), as opposed to time translation T , space translations X^i and rotations Ω_{ij} , which lead to energy, momentum and angular momentum conservations.

Appendix F

From conservation to (non)-conservation

Galilean law from infinite speed of light

Our starting point is a pseudo-Riemannian spacetime in Zermelo frame (2.66)

$$ds^2 = -\Omega^2 k^2 dt^2 + a_{ij} (dx^i - w^i dt) (dx^j - w^j dt) \quad (\text{F.1})$$

with an energy–momentum tensor $T^{\mu\nu}$ obeying $\nabla_\mu T^{\mu\nu} = 0$, and a vector field

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \xi^{\hat{t}} \mathbf{e}_{\hat{t}} + \xi^{\hat{i}} \mathbf{e}_{\hat{i}}, \quad (\text{F.2})$$

where the frame and coframe are defined as in (2.92), and

$$\xi^{\hat{t}} = \xi^t \Omega, \quad \xi^{\hat{i}} = \xi^i - \xi^t w^i, \quad \xi_{\hat{t}} = -k^2 \xi^{\hat{t}}, \quad \xi_{\hat{i}} = a_{ij} \xi^{\hat{j}} = \xi_i. \quad (\text{F.3})$$

We define a current as in (2.25), $I_\mu = T_{\mu\nu} \xi^\nu$, and compute its on-shell divergence, using Eqs. (2.83):

$$\begin{aligned} \nabla_\mu I^\mu &= -\frac{1}{2} T_{\mu\nu} \mathcal{L}_\xi g^{\mu\nu} = -\frac{\varepsilon_r}{\Omega} \frac{\hat{D} \xi^{\hat{t}}}{dt} + (p_r a_{ij} + \tau_{rij}) \left(\hat{\nabla}^i \xi^{\hat{j}} + \xi^{\hat{t}} \hat{\gamma}^{wij} \right) \\ &\quad + \frac{1}{k^2} q_{ri} \left(\frac{1}{\Omega} \frac{\hat{D} \xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi}{}_j \xi^{\hat{j}} - k^2 a^{ij} \partial_j \xi^{\hat{t}} \right). \end{aligned} \quad (\text{F.4})$$

This result is relativistic, expressed with Galilean derivatives though. It vanishes if

$$\begin{cases} \frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} = 0 \\ \hat{\nabla}^{(i} \xi^{\hat{j})} + \xi^{\hat{i}} \hat{\gamma}^{wj} = 0 \\ \frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi} \xi^{\hat{j}} - k^2 a^{ij} \partial_j \xi^{\hat{i}} = 0, \end{cases} \quad (\text{F.5})$$

which are simply the conditions for ξ be a Killing field of the pseudo-Riemannian manifold.

We would like now to consider the infinite- k limit of (F.4). At the first place, we must provide the behaviour of ε_r , q_{ri} and $p_r a_{ij} + \tau_{rij}$ for large k . This is typically of the form ¹

$$\begin{cases} \varepsilon_r = \Pi + \mathcal{O}(1/k^2) \\ q_{ri} = k^2 P_i + \Pi_i + \mathcal{O}(1/k^2) \\ p_r a_{ij} + \tau_{rij} = \Pi_{ij} + \mathcal{O}(1/k^2), \end{cases} \quad (\text{F.6})$$

and (F.4) becomes:

$$\begin{aligned} \nabla_\mu I^\mu &= -\frac{\Pi \hat{D}\xi^{\hat{i}}}{\Omega dt} + \Pi_{ij} \left(\hat{\nabla}^i \xi^{\hat{j}} + \xi^{\hat{i}} \hat{\gamma}^{wj} \right) + \left(P_i + \frac{\Pi_i}{k^2} \right) \left(\frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi} \xi^{\hat{j}} - k^2 a^{ij} \partial_j \xi^{\hat{i}} \right) \\ &+ \mathcal{O}\left(\frac{1}{k^2}\right). \end{aligned} \quad (\text{F.7})$$

For this expression to remain finite at infinite k , we must impose that²

$$\partial_j \xi^{\hat{i}} = 0, \quad (\text{F.8})$$

¹More general behaviours have appeared in (3.98), (3.99), (3.100), or in (3.129). These choices wouldn't change our present argument though.

²One may refine the limiting procedure for the Killing fields, and reach the Galilean diffeomorphisms as $\xi^{\hat{i}}(t, \mathbf{x}) = \xi_G^{\hat{i}}(t) + \frac{1}{k^2} v(t, \mathbf{x}) + \mathcal{O}\left(\frac{1}{k^4}\right)$. This would alter equation (F.11) as $\frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} - \hat{\gamma}^{wj} \xi_j - \partial_i v = 0$. Similarly the arbitrary function $v(t, \mathbf{x})$ would also appear in the large- k expansions of equations (2.96) and (2.97), altering the Galilean currents (3.134). Ultimately, this would have no incidence on our conclusions about the interplay between Galilean isometries and conservation. It may nevertheless provide a complementary view on the large- k contraction of the relativistic diffeomorphisms, possibly in line with the approach followed in [152], where a further duality relationship has been established among leading Galilean and subleading Carrollian contributions (see footnote 4), and vice-versa.

which is the requirement that ξ generates a Galilean diffeomorphism. Conservation holds in the limit if expression (F.7) vanishes, which is again a threefold condition:

$$\frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} = 0, \quad (\text{F.9})$$

$$\hat{\nabla}^{(i}\xi^{\hat{j})} + \xi^{\hat{i}}\hat{\gamma}^{wij} = 0, \quad (\text{F.10})$$

$$\frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi}{}_j \xi^{\hat{j}} = 0. \quad (\text{F.11})$$

Equations (F.9) and (F.10) are nothing but (3.55) i.e. the definition of a Galilean Killing field. Equation (F.11) is an extra condition, absent for generic Galilean isometries. The latter do not guarantee the existence of a conserved Galilean current unless (F.11) is satisfied. The break down of the conservation is read off in

$$\lim_{k \rightarrow \infty} \nabla_{\mu} I^{\mu} = P_i \left(\frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi}{}_j \xi^{\hat{j}} \right) = \frac{P_i}{\Omega} (\partial_t \xi^{\hat{i}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{i}}), \quad (\text{F.12})$$

which agrees with (3.65) or (3.135). As stressed in Sec. 3.1.3, the failure might be only apparent, if the term $\frac{P_i}{\Omega} (\partial_t \xi^{\hat{i}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{i}})$ turns out to be a boundary term, that would then contribute the Galilean current.

Carrollian law from zero speed of light

Here we will consider a pseudo-Riemannian spacetime in Papapetrou–Randers frame (2.99)

$$ds^2 = -k^2 (\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j. \quad (\text{F.13})$$

We assume a conserved energy–momentum tensor $T^{\mu\nu}$ and a vector field as in (F.2) with

$$\xi^{\hat{i}} = \xi^i \Omega - \xi^i b_i, \quad \xi^{\hat{i}} = \xi^i, \quad \xi_{\hat{i}} = -k^2 \xi^{\hat{i}}, \quad \xi_{\hat{i}} = a_{ij} \xi^{\hat{j}} = \xi_i + \xi_i b_i. \quad (\text{F.14})$$

The frame and coframe are defined in (2.126).

We now compute the on-shell divergence of the current (2.25) $I_{\mu} = T_{\mu\nu} \xi^{\nu}$, using Eqs. (2.118):

$$\begin{aligned} \nabla_{\mu} I^{\mu} = \frac{1}{2} T^{\mu\nu} \mathcal{L}_{\xi} g_{\mu\nu} = & -\varepsilon_r \left(\frac{1}{\Omega} \partial_t \xi^{\hat{i}} + \varphi_i \xi^{\hat{i}} \right) + (p_r a^{ij} + \tau_r^{ij}) \left(\hat{\nabla}_i \xi_{\hat{j}} + \xi^{\hat{i}} \hat{\gamma}_{ij} \right) \\ & - q_r^i \left((\hat{\partial}_i - \varphi_i) \xi^{\hat{i}} - 2 \xi^{\hat{j}} \varpi_{ji} - \frac{1}{k^2 \Omega} a_{ij} \partial_t \xi^{\hat{j}} \right). \quad (\text{F.15}) \end{aligned}$$

Although expressed with Carrollian derivatives, this is relativistic and vanishes if

$$\begin{cases} \frac{1}{\Omega} \partial_t \xi^{\hat{i}} + \varphi_i \xi^i = 0 \\ \hat{\nabla}_i(\xi_{\hat{j}}) + \xi^{\hat{i}} \hat{\gamma}_{ij} = 0 \\ \left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{i}} - 2\xi^j \varpi_{ji} - \frac{1}{k^2 \Omega} a_{ij} \partial_t \xi^j = 0. \end{cases} \quad (\text{F.16})$$

These conditions define a Killing field ξ on a pseudo-Riemannian manifold.

We would like now to consider the zero- k limit of (F.15). We must provide the behaviour of ε_r , q_r^i and $p_r a^{ij} + \tau_r^{ij}$ for small k , which is typically of the form (4.78), (4.79), (4.80)³

$$\begin{cases} \varepsilon_r = \Pi + \mathcal{O}(k^2) \\ q_r^i = \Pi^i + k^2 P^i + \mathcal{O}(k^4) \\ p_r a^{ij} + \tau_r^{ij} = \Pi^{ij} + \mathcal{O}(k^2). \end{cases} \quad (\text{F.17})$$

Equation (F.15) reads now:

$$\begin{aligned} \nabla_\mu J^\mu &= -\Pi \left(\frac{1}{\Omega} \partial_t \xi^{\hat{i}} + \varphi_i \xi^i \right) + \Pi^{ij} \left(\hat{\nabla}_i \xi_{\hat{j}} + \xi^{\hat{i}} \hat{\gamma}_{ij} \right) \\ &\quad - \left(\Pi^i + k^2 P^i \right) \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{i}} - 2\xi^j \varpi_{ji} - \frac{1}{k^2 \Omega} a_{ij} \partial_t \xi^j \right) + \mathcal{O}(k^2). \end{aligned} \quad (\text{F.18})$$

Finiteness at zero k , demands⁴

$$\partial_t \xi^i = 0, \quad (\text{F.19})$$

hence ξ generates a Carrollian diffeomorphism. Conservation holds in the limit if expression (F.18) vanishes. This occurs if

$$\frac{1}{\Omega} \partial_t \xi^{\hat{i}} + \varphi_i \xi^i = 0, \quad (\text{F.20})$$

$$\hat{\nabla}_{(i} \xi_{\hat{j})} + \xi^{\hat{i}} \hat{\gamma}_{ij} = 0, \quad (\text{F.21})$$

$$\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{i}} - 2\xi^j \varpi_{ji} = 0. \quad (\text{F.22})$$

³More general behaviours have appeared in (4.120). The latter would not change our present conclusions though.

⁴Mirroring footnote 2, an option is to set $\xi^i(t, \mathbf{x}) = \xi_{\text{C}}^i(\mathbf{x}) + k^2 v^i(t, \mathbf{x}) + \mathcal{O}(k^4)$. With this, equation (F.22) becomes $\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{i}} - 2\xi_{\text{C}}^j \varpi_{ji} - \frac{1}{\Omega} a_{ij} \partial_t v^j = 0$, and further work would be necessary on equations. (2.129), (2.130) and (4.142), that would not alter our final conclusions, but could shed light on the small- k contraction of general diffeomorphisms.

Equations (F.20) and (F.21) are as in (4.47) i.e. the definition of a Carrollian Killing field. Equation (F.22) is an extra condition, absent for generic Carrollian isometries, which therefore do not guarantee the existence of a conserved Carrollian current. The disruption to the conservation is measured as

$$\lim_{k \rightarrow 0} \nabla_{\mu} I^{\mu} = -\Pi^i \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{i}} - 2\xi^j \varpi_{ji} \right), \quad (\text{F.23})$$

in agreement with (4.55) or (4.144).

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Published and submitted articles

As mentioned at the beginning of this thesis, during the completion of the doctoral program a series of articles were published or submitted. All this articles are attached here.

CHERN–SIMONS ACTION AND THE CARROLLIAN COTTON TENSORS

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ABSTRACT

In three-dimensional pseudo-Riemannian manifolds, the Cotton tensor arises as the variation of the gravitational Chern–Simons action with respect to the metric. It is Weyl-covariant, symmetric, traceless and covariantly conserved. Performing a reduction of the Cotton tensor with respect to Carrollian diffeomorphisms in a suitable frame, one discloses four sets of Cotton Carrollian relatives, which are conformal and obey Carrollian conservation equations. Each set of Carrollian Cotton tensors is alternatively obtained as the variation of a distinct Carroll–Chern–Simons action with respect to the degenerate metric and the clock form of a strong Carroll structure. The four Carroll–Chern–Simons actions emerge in the Carrollian reduction of the original Chern–Simons ascendant. They inherit its anomalous behaviour under diffeomorphisms and Weyl transformations. The extremums of these Carrollian actions are commented and illustrated.

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1 Prologue

THE COTTON TENSOR is defined on Riemannian manifolds of arbitrary dimension, carries three indices and is partly antisymmetric. In three dimensions, which will be our framework, this tensor was introduced by Émile Cotton in 1899 [1] and was formulated as a two-index symmetric tensor related to the previous by Hodge duality:

$$C_{\mu\nu} = \eta_{\mu}^{\rho\sigma} \nabla_{\rho} \left(R_{\nu\sigma} - \frac{R}{4} g_{\nu\sigma} \right). \quad (1)$$

Here $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$ is the metric with signature $(-++)$, $\eta_{\mu\nu\sigma} = \sqrt{-g} \epsilon_{\mu\nu\sigma}$ ($\epsilon_{012} = 1$), ∇_{ρ} the associated Levi–Civita connection and $R_{\nu\sigma}$ are the components of the Ricci tensor with scalar R . The combination of the latter objects inside the parentheses defines the Schouten tensor in three dimensions.

The Cotton tensor is Weyl-covariant, and conserved as a consequence of the first Bianchi identity and the absence of Weyl tensor

$$\nabla_{\rho} C^{\rho}_{\nu} = 0, \quad (2)$$

irrespective of the dynamics on $g_{\mu\nu}$. In fact, the Cotton tensor emerges as the “energy–momentum” tensor of the gravitational Chern–Simons action:

$$C_{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{CS}}}{\delta g^{\mu\nu}}, \quad (3)$$

with

$$S_{\text{CS}} = \frac{1}{2c} \int_{\mathcal{M}} \text{Tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right), \quad (4)$$

where ω is the Levi–Civita connection one-form. In this picture S_{CS} is a functional of the metric and of its derivatives.

There are numerous instances where the Cotton tensor is encountered and plays a fine role in gravitational physics. In four-dimensional asymptotically anti-de Sitter spacetimes, the Schouten tensor of the conformal boundary appears explicitly at a subleading order in the Fefferman–Graham expansion of the bulk metric, after the boundary metric and before the boundary energy–momentum tensor. The bound-

ary Cotton tensor itself arises as the leading term of the Fefferman–Graham expansion of the bulk Weyl tensor. More explicitly, the Schouten tensor appears as a gauge field associated with conformal boosts and the Cotton tensor as the corresponding field strength in non-linearly realized conformal group on the boundary of AdS₄ (super)gravity [2]. Its presence reveals that the boundary *is not* conformally flat or equivalently that the bulk is asymptotically *locally* anti-de Sitter. Alongside, the Chern–Simons action appears under specific circumstances as the leading-order effective action of the boundary theory, and can serve alternatively for amending the standard boundary conditions imposed in anti-de Sitter holography. These and other interesting properties, such as the role of the Cotton tensor and the occurrence of the Chern–Simons action in gravitational electric–magnetic duality, can be found in Refs. [3–9].

In an effort to design a bulk gauge that would be covariant with respect to the conformal boundary, as the Fefferman–Graham gauge is, but at the same time be regular for vanishing cosmological constant, as opposed to Fefferman–Graham, a modified version of the Newman–Unti gauge was reached [10–12], inspired by fluid/gravity correspondence [13, 14]. In this gauge, the Cotton tensor appears explicitly in the bulk metric, and its deeper role in the spacetime reconstruction — also recognized in [15] — is more transparent.

The attempts for generalizing the gravitational holographic principle to asymptotically flat spacetimes have abundantly fueled the interest for Carrollian geometries [16, 17], namely for structures equipped with a degenerate metric, as are null infinities. In this framework, one naturally wonders how the Cotton tensor materializes within the various curvature attributes, what sort of dynamics it conveys, and which role it plays in the bulk reconstruction from boundary data — now defined at null infinity. Some of these questions were accurately answered in the seminal work [18], exhibiting some of the *Carrollian Cotton descendants*, their dynamics inherited from (2), as well as their occurrence in the flat exegesis of the modified/covariantized Newman–Unti gauge. Further properties have been more recently elaborated in [19], in relation to the null-boundary manifestation of Ehlers’ hidden Möbius group, or in defining towers of gravito-magnetic charges exclusively from a Carrollian boundary perspective.

IN THE WORKS CITED EARLIER, the analysis of the Carrollian Cotton descendants was circumscribed to Carrollian geometries with vanishing geometric shear, a requirement imposed by bulk Ricci flatness. However, reaching the ultimate radiation-flux-balance equations for asymptotically flat spacetimes in a limiting procedure from anti-de Sitter requires to start with a non-zero shear, as recently demonstrated in [20]. The purpose of the present note is to present a comprehensive picture of the Carrollian Cotton tensors, while providing at the same time the Carrollian descendants for the Chern–Simons action, which are met in various facets of flat-asymptotic symmetries — see e.g. [21, 22].

Our strategy can be summarized as follows: choose an adapted frame, expand in powers of c^2 and read off the possible Carrollian dynamics (actions and equations of motion). These usually appear in electric and magnetic, plus some secondary occasionally non-dynamical versions. As for their Riemannian ascendant, diffeomorphism or local Lorentz/Carroll/Weyl invariances call for a cautious inspection involving boundary terms. This is part of our agenda.

2 Carrollian Cotton: intrinsic features and zero- c limit

CARROLL BASICS are available in [23–34]. The underlying geometries consist of a $d + 1$ -dimensional manifold $\mathcal{M} = \mathbb{R} \times \mathbb{S}$ equipped with a degenerate metric and its kernel vector field. We will be here restricted to $d = 2$ and adopt a metric of the form

$$d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i dx^j, \quad i, j \dots \in \{1, 2\}. \quad (5)$$

The kernel of the metric is the *field of observers*, here

$$\mathbf{v} = \frac{1}{\Omega} \partial_t. \quad (6)$$

The dual *clock form* obeying $\mu(\mathbf{v}) = -1$ reads

$$\mu = -\Omega dt + b_i dx^i, \quad (7)$$

and incorporates an *Ehresmann connection*, which is the background gauge field $\mathbf{b} = b_i dx^i$. Notice that (5) is not the most general degenerate metric, which could a priori have components along dt , supplying ∂_i components to the field of observers (this option is sometimes chosen — see e.g. [20, 35]). Our choice allows for a natural splitting of time and space coordinates, along the fibre and the base of the Carrollian fibre bundle respectively, invariant under Carrollian diffeomorphisms $t' = t'(t, \mathbf{x})$ and $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$. This remains nonetheless compatible with general covariance. Restoring explicitly the latter is possible without any conflict with the dynamics at work here, at the expense of obfuscating the distinction of the Carrollian framework with ordinary Riemannian situations.

The vector fields dual to the forms dx^i are

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega} \partial_t. \quad (8)$$

They transform covariantly under Carrollian diffeomorphisms.¹ More generally, Carrollian tensors depend on time t and space \mathbf{x} . The metric being degenerate the spacetime indices cannot be lowered or raised. This inconvenience can be handled by introducing a pseudo-inverse [25], but we take instead advantage of the time-and-space splitting mentioned previously, and consider tensors carrying only spatial indices i, j, \dots lowered and raised with a_{ij} and its inverse a^{ij} .² These transform covariantly under Carrollian diffeomorphisms. The time index is omitted and the corresponding object is a Carrollian scalar. Details can be found, e.g., in the appendices of Ref. [19], of which a minimal selection will be hosted in the present publication.

¹Defining the Jacobians as $J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}$, $j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}$, $J_j^i(\mathbf{x}) = \frac{\partial x^{i'}}{\partial x^j}$, the transformations follow: $a^{ij'} = J_k^i J_l^j a^{kl}$, $\Omega' = \frac{\Omega}{J}$, $b'_k = \left(b_i + \frac{\Omega}{J} j_i\right) J^{-1i}_k$ (connection-like transformation) and consequently $\mathbf{v}' = \mathbf{v}$, $\mu' = \mu$.

²Working with an orthonormal Cartan frame is yet another option adopted in [20]. In this perspective, the Carrollian tensors carry again spatial indices solely, but are now organized in representations of the d -dimensional orthogonal local group, subgroup of the local Carroll group, and are raised or lowered with the identity.

A STRONG CARROLL STRUCTURE comes with a metric-compatible and field-of-observers-compatible connection, which is not unique due to the metric degeneracy. Here, we use the connection inspired from an ascendant pseudo-Riemannian spacetime equipped with Papapetrou–Randers metric

$$ds^2 = -c^2 (\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j. \quad (9)$$

We should stress that in this metric the dependence with respect to the velocity of light c is explicit, leading thus to (5) at zero c . This feature plays a pivotal role in our subsequent analysis, since the Carrollian Cotton descendants, the corresponding Chern–Simons actions and the conservation equations are reached respectively from Eqs. (3), (4) and (2) by expanding in c . Such an expansion would receive extra contributions if every function in (9) were itself provisioning various powers of c . This would not alter the general form of the Carrollian equations, but the details would be different.³

The relativistic Papapetrou–Randers metric (9) infers a convenient although non-orthonormal Cartan mobile frame $\{e_{\hat{0}} = \frac{1}{c\Omega}\partial_t, e_i = \hat{\partial}_i\}$ and coframe $\{\theta^{\hat{0}} = -c\mu, \theta^i = dx^i\}$. The hatted indices $\{\hat{0}, \hat{i}\}$ are meant to distinguish this frame from the coordinate coframe $\{\theta^0 = dx^0 = cdt, \theta^i = dx^i\}$ and frame $\{e_0 = \frac{1}{c}\partial_t, e_i = \partial_i\}$. In order to avoid cluttering of symbols and comply with the conventions used, e.g., in Ref. [19, 36, 37], we will keep the hat exclusively on the time direction. For the Carrollian side, we will rather use $\{e_i = \frac{1}{\Omega}\partial_t, e_i = \hat{\partial}_i\}$ and $\{\theta^i = -\mu, \theta^i = dx^i\}$, and ignore the hat on the spatial indices.

The relativistic (affine) connection one-form elements $\omega^\mu{}_\nu = \Gamma^\mu_{\rho\nu}\theta^\rho$ with $\Gamma^\mu_{\rho\nu}$ the Levi–Civita connection coefficients read:

$$\omega_{\hat{0}i} = c(\varphi_i\mu + \varpi_{ij}dx^j) - \frac{1}{c}\hat{\gamma}_{ij}dx^j, \quad (10)$$

$$\omega_{ij} = (c^2\varpi_{ij} - \hat{\gamma}_{ij})\mu + a_{il}\hat{\gamma}_{jk}^l dx^k \quad (11)$$

with $\omega_{\hat{0}i} = -\omega_{i\hat{0}}$. These expressions disclose two Carrollian tensors, the vorticity and the acceleration

$$\varpi_{ij} = \partial_{[i}b_{j]} + b_{[i}\varphi_{j]}, \quad \varphi_i = \frac{1}{\Omega}(\partial_t b_i + \partial_i\Omega), \quad (12)$$

revealed inside

$$d\mu = \varpi + \varphi \wedge \mu = \varpi_{ij}dx^i \wedge dx^j + \varphi_i dx^i \wedge \mu. \quad (13)$$

We have also introduced the symmetric symbols

$$\hat{\gamma}_{jk}^i = \frac{a^{il}}{2}(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}) \quad (14)$$

dubbed Carroll–Levi–Civita connection coefficients, and the Carrollian tensors

$$\hat{\gamma}_{ij} = \frac{1}{2\Omega}\partial_t a_{ij} = \xi_{ij} + \frac{1}{d}a_{ij}\theta, \quad \theta = \frac{1}{\Omega}\partial_t \ln \sqrt{a}, \quad (15)$$

³Keeping c explicit in the Papapetrou–Randers metric has been the ruling pattern in Refs. [36, 37]. Alternative approaches for designing descendant Carrollian dynamics can be found e.g. in [38, 39].

which define the geometric Carrollian shear (traceless) ξ_{ij} and the Carrollian expansion θ ($\hat{\gamma}_{ij}$ is the extrinsic curvature of the spatial section \mathcal{S}). All these enter the *Carrollian connection* adopted here:

$$\hat{\omega}^i{}_{\hat{t}} = \hat{\omega}^i{}_i = \hat{\omega}^i{}_{\hat{t}} = 0, \quad \hat{\omega}^i{}_j = -\hat{\gamma}^i{}_j \mu + \hat{\gamma}^i{}_{jk} dx^k \quad (16)$$

(the hat signals this connection is Carrollian as opposed to the Riemannian displayed in (10) and (11)).

It is important to stress that the Carrollian connection (16) has been designed so as to *define a parallel transport that respects the time-and-space splitting* mentioned above, embracing distinct time and space Carrollian covariant derivatives $\frac{1}{\Omega} \hat{D}_t$ and $\hat{\nabla}_i$. Both are metric-compatible: $\frac{1}{\Omega} \hat{D}_t a_{ij} = \hat{\nabla}_k a_{ij} = 0$.

For $d = 2$, the \mathcal{S} -Hodge duality is induced by $\eta_{ij} = \sqrt{a} \epsilon_{ij}$. This duality is involutive on Carrollian vectors as well as on two-index symmetric and traceless Carrollian tensors:

$$*V_i = \eta^l{}_i V_l, \quad *W_{ij} = \eta^l{}_i W_{lj}. \quad (17)$$

In particular⁴

$$*\varpi = \frac{1}{2} \eta^{ij} \varpi_{ij} \Leftrightarrow \varpi_{ij} = *\varpi \eta_{ij}. \quad (18)$$

W_{EYL} TRANSFORMATIONS act as

$$a_{ij} \rightarrow \frac{1}{\mathcal{B}^2} a_{ij}, \quad b_i \rightarrow \frac{1}{\mathcal{B}} b_i, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega \quad (19)$$

with $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$. The expansion θ and the acceleration φ_i transform as connections under these rescalings:

$$\varphi_i \rightarrow \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \theta \rightarrow \mathcal{B} \theta - \frac{d}{\Omega} \partial_t \mathcal{B}. \quad (20)$$

They play the role of Weyl connections in time and space Carroll–Weyl covariant metric-compatible derivatives.⁵ The weights of ϖ_{ij} and ξ_{ij} are -1 ; hence the connection transforms as follows:

$$\begin{cases} \omega^{\hat{0}}{}_i \rightarrow \frac{1}{\mathcal{B}} \left(\omega^{\hat{0}}{}_i + c \hat{\partial}_i \ln \mathcal{B} \mu - \frac{1}{c} \frac{1}{\Omega} \partial_t \ln \mathcal{B} a_{ij} dx^j \right) \\ \omega^i{}_j \rightarrow \omega^i{}_j - \delta^i{}_j d \ln \mathcal{B} + \eta^i{}_j * \hat{\partial}_k \ln \mathcal{B} dx^k, \end{cases} \quad (21)$$

and similarly for the Carrollian connection:

$$\hat{\omega}^i{}_j \rightarrow \hat{\omega}^i{}_j - \delta^i{}_j d \ln \mathcal{B} + \eta^i{}_j * \hat{\partial}_k \ln \mathcal{B} dx^k. \quad (22)$$

⁴We use here the conventions of Ref. [18], namely $\epsilon_{12} = -1$, convenient when using complex coordinates $\{\zeta, \bar{\zeta}\}$, where $d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta}$ with $P = P(t, \zeta, \bar{\zeta})$ a real function. In this case $\eta_{\zeta\bar{\zeta}} = -i/P^2$, $\sqrt{a} = i/P^2$ and the volume form reads $\frac{1}{2} \eta_{ij} dx^i \wedge dx^j = d^2x \sqrt{a} = \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}$. Notice that $\eta^{il} \eta_{jl} = \delta^i{}_j$ and $\eta^{ij} \eta_{ij} = 2$ so that $\mu \wedge dx^i \wedge dx^j = -\eta^{ij} dt d^2x \sqrt{a} \Omega = i \eta^{ij} dt \frac{d\zeta \wedge d\bar{\zeta}}{P^2} \Omega$.

⁵On the one hand, the Carrollian covariant derivatives act as ordinary derivatives on scalars, whereas on Carrollian vectors V^i we obtain: $\frac{1}{\Omega} \hat{D}_t V^i = \frac{1}{\Omega} \partial_t V^i + \hat{\gamma}^i{}_j V^j$ and $\hat{\nabla}_j V^i = \hat{\partial}_j V^i + \hat{\gamma}^i{}_{jk} V^k$. On the other hand Weyl–Carroll covariant derivatives on weight- w fields are as follows: $\frac{1}{\Omega} \hat{D}_t \Phi = \frac{1}{\Omega} \partial_t \Phi + \frac{w}{d} \theta \Phi$, $\frac{1}{\Omega} \hat{D}_t V^l = \frac{1}{\Omega} \partial_t V^l + \frac{w-1}{d} \theta V^l$ (both of weight $w+1$), and $\hat{D}_j \Phi = \hat{\partial}_j \Phi + w \varphi_j \Phi$, $\hat{D}_j V^l = \hat{\nabla}_j V^l + (w-1) \varphi_j V^l + \varphi^l V_j - \delta^l{}_j V^i \varphi_i$ (with unaltered weights). More details on the latter are given in [19, 36, 37].

CURVATURE TENSORS are defined through the action of covariant-derivative commutators on various fields. Considering for instance a Carrollian vector field V^i , we obtain for $d = 2$:⁶

$$\left[\hat{\nabla}_i, \frac{1}{\Omega} \hat{D}_t \right] V^j = \hat{R}^j{}_{ik} V^k + \hat{\gamma}_i{}^k \hat{\nabla}_k V^j - \varphi_i \frac{1}{\Omega} \hat{D}_t V^j, \quad (23)$$

$$[\hat{\nabla}_k, \hat{\nabla}_l] V^i = \hat{R}^i{}_{jkl} V^j + * \varpi \eta_{kl} \frac{2}{\Omega} \hat{D}_t V^i, \quad (24)$$

from which we further define

$$\hat{R}^k{}_{ikj} = \hat{R}_{ij} = \hat{K} a_{ij}. \quad (25)$$

One reaches the same tensors by computing the Carrollian curvature two-form starting from the connection (16):⁷

$$\hat{\mathcal{R}}^i{}_j = 0, \quad \hat{\mathcal{R}}^i{}_j = \hat{R}^i{}_{kj} \mu \wedge dx^k + \frac{1}{2} \hat{R}^i{}_{jkl} dx^k \wedge dx^l. \quad (26)$$

It should be emphasized that the last terms in the commutators (23), (24) (and (28), (29) below) betray the presence of torsion in the Carroll (or the Carroll–Weyl) connection adopted here. This torsion is encoded in the tensors φ_i , ϖ_{ij} , θ and ξ_{ij} :

$$\hat{\mathcal{T}}^i = \varphi_i \mu \wedge dx^i - * \varpi \eta_{ij} dx^i \wedge dx^j, \quad \hat{\mathcal{T}}^i = \hat{\gamma}^i{}_j dx^j \wedge \mu. \quad (27)$$

Introducing torsion is the price to pay for maintaining metric compatibility, while ensuring the time-and-space splitting and a non-trivial interplay between the base and the fibre. As a bonus, the field of observers is parallelly transported, $\hat{\nabla}_\nu v = 0$, and the time fibres are geodesics for the Carrollian manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$.

Likewise, we obtain the curvature tensors for the Carroll–Weyl connection:

$$\left[\hat{D}_i, \frac{1}{\Omega} \hat{D}_t \right] V^j = \hat{S}^j{}_{ik} V^k - (w-1) \hat{\mathcal{R}}_i V^j + \xi_i{}^k \hat{D}_k V^j, \quad (28)$$

$$[\hat{D}_k, \hat{D}_l] V^i = \hat{S}^i{}_{jkl} V^j - (w-1) \hat{A} \eta_{kl} V^i + * \varpi \eta_{kl} \frac{2}{\Omega} \hat{D}_t V^i, \quad (29)$$

where w stands for the weight of the vector field V^j . Furthermore, setting

$$\hat{\mathcal{R}}_i = \frac{1}{\Omega} \partial_t \varphi_i - \frac{1}{2} (\hat{\partial}_i + \varphi_i) \theta, \quad \hat{A} = * \varpi \theta - \eta^{ij} \hat{\nabla}_i \varphi_j \quad (30)$$

⁶We set $\hat{R}_{ijk} = (\hat{\nabla}_i + \varphi_i) \hat{\gamma}_{jk} - (\hat{\nabla}_k + \varphi_k) \hat{\gamma}_{ij} = -\hat{R}_{kji}$ and $\hat{R}_{ijkl} = \hat{K} (a_{ik} a_{jl} - a_{il} a_{jk})$, handier than \hat{r}_{ijk} and \hat{r}_{ijkl} previously introduced in Refs. [19, 36, 37].

⁷The torsion and curvature two-forms are $\mathcal{T}^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = \frac{1}{2} T^\mu{}_{\nu\rho} \theta^\nu \wedge \theta^\rho$ and $\mathcal{R}^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\rho \wedge \omega^\rho{}_\nu = \frac{1}{2} R^\mu{}_{\nu\rho\lambda} \theta^\rho \wedge \theta^\lambda$.

defines unambiguously the Carroll–Weyl curvature tensors $\hat{\mathcal{S}}^j_{ik}$ and $\hat{\mathcal{S}}^i_{jkl}$.⁸ Tracing finally we obtain

$$\hat{\mathcal{S}}^k_{ikj} = \hat{\mathcal{S}}_{ij} = \hat{\mathcal{K}}a_{ij}, \quad \hat{\mathcal{K}} = \hat{K} + \hat{\nabla}_k \varphi^k. \quad (31)$$

In summary, the independent Carroll–Weyl curvature tensors in $d = 2$ are $\hat{\mathcal{K}}$, \hat{A} and $\hat{\mathcal{R}}_i$ of weights 2, 2 and 1. They obey

$$\frac{2}{\Omega} \hat{\mathcal{D}}_t * \varpi + \hat{A} = 0, \quad (32)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \hat{\mathcal{K}} - a^{ij} \hat{\mathcal{D}}_i \hat{\mathcal{R}}_j - \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j \xi^{ij} = 0, \quad (33)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \hat{A} + \eta^{ij} \hat{\mathcal{D}}_i \hat{\mathcal{R}}_j = 0, \quad (34)$$

the last two being the Carroll–Weyl–Bianchi identities.

R IEMANN, RICCI AND CARROLLIAN DESCENDANTS of the Riemannian metric (9) are yield using the prescription already exploited in Refs. [18,19] — or [36,37] for general energy–momentum tensors. In a first step, this consists in reducing the relativistic tensors with respect to Carrollian diffeomorphisms. Next, one expands the latter in powers of c , and at each power bona-fide Carrollian tensors emerge. Following this prescription for the Riemann curvature two-form of (9) we find

$$\begin{aligned} \mathcal{R}^{\hat{0}}_i &= c \left[c^2 * \varpi^2 a_{ik} + \hat{\nabla}_{(i} \varphi_{k)} + \varphi_i \varphi_k - 2 * \varpi \eta_{j(i} \hat{\gamma}_{k)}^j - \frac{1}{c^2} \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \hat{\gamma}_{ik} + \hat{\gamma}_{ij} \hat{\gamma}_{ik}^j \right) \right] \mu \wedge dx^k \\ &\quad - c \left[\hat{\partial}_i * \varpi + 2\varphi_i * \varpi - \frac{1}{c^2} \eta^{mn} \hat{\nabla}_m \hat{\gamma}_{ni} \right] \frac{1}{2} \eta_{kl} dx^k \wedge dx^l, \end{aligned} \quad (35)$$

$$\begin{aligned} \mathcal{R}^i_j &= \hat{\mathcal{R}}^i_j - \eta^i_j \left[c^2 \left(\hat{\partial}_k * \varpi + 2\varphi_k * \varpi \right) + \eta^{mn} \varphi_m \hat{\gamma}_{nk} \right] \mu \wedge dx^k \\ &\quad + \eta^i_j \left[3c^2 * \varpi^2 + \frac{1}{2c^2} \eta^{mn} \eta^{rs} \hat{\gamma}_{mr} \hat{\gamma}_{ns} \right] \frac{1}{2} \eta_{kl} dx^k \wedge dx^l, \end{aligned} \quad (36)$$

where various Carrollian tensors emerge besides the Carrollian curvature two-form read off in (26). No torsion is available for the relativistic Levi–Civita connection at hand.

The Riemannian scalar curvature R reads:⁹

$$\frac{R}{2} = c^2 * \varpi^2 + \hat{K} - \left(\hat{\nabla}_k + \varphi_k \right) \varphi^k + \frac{1}{c^2} \left(\xi^2 + \frac{3}{4} \theta^2 + \frac{1}{\Omega} \partial_t \theta \right), \quad (37)$$

where $\xi^2 = \frac{1}{2} \xi^{ij} \xi_{ij}$. It captures various Carrollian curvature scalars, which infer Carrollian avatars of the Einstein–Hilbert action $\int_{\mathcal{M}} \frac{dx^3}{c} \sqrt{-g} R$. We will not elaborate on this aspect of Carrollian gravitational dynamics that would deserve a thorough comparison with Refs. [38,39].

C OTTON CARROLLIAN RELATIVES are reached following the above pattern. The reduction of $C^{\mu\nu}$ is

⁸Again those are slightly different from $\hat{\mathcal{R}}^j_{ik}$ and $\hat{\mathcal{R}}^i_{jkl}$ used earlier in [19,36,37]: $\hat{\mathcal{S}}^j_{ik} = \hat{\mathcal{D}}^j \xi_{ik} - \hat{\mathcal{D}}_k \xi^j_i + \delta^j_i \hat{\mathcal{R}}_k - a_{ik} \hat{\mathcal{R}}^j$ and $\hat{\mathcal{S}}^i_{jkl} = \hat{\mathcal{K}} (a_{ik} a_{jl} - a_{il} a_{jk})$.

⁹For arbitrary d we find: $R = c^2 \varpi^{ij} \varpi_{ij} + \hat{R} - 2 \left(\hat{\nabla}_k + \varphi_k \right) \varphi^k + \frac{1}{c^2} \left(\xi^{ij} \xi_{ij} + \frac{d+1}{d} \theta^2 + \frac{2}{\Omega} \partial_t \theta \right)$.

straightforward: Carrollian scalars and vectors emerge from $C^{\hat{0}\hat{0}}$ and $C^{\hat{0}i}$, while $C^{ij} - \frac{C^{\hat{0}\hat{0}}}{2}a^{ij}$ leads to symmetric and traceless Carrollian tensors. They are readily decomposed in powers of c as follows:

$$\frac{1}{c}C^{\hat{0}\hat{0}} = c^2\gamma + \varepsilon + \frac{\zeta}{c^2} + \frac{\tau}{c^4}, \quad (38)$$

$$C^{\hat{0}i} = c^2\psi^i + \chi^i + \frac{z^i}{c^2}, \quad (39)$$

$$\frac{C^{\hat{0}\hat{0}}a^{ij}}{2c} - \frac{C^{ij}}{c} = \Psi^{ij} + \frac{X^{ij}}{c^2} + \frac{Z^{ij}}{c^4}. \quad (40)$$

With this, any Carrollian structure supplied with the connection at hand, is naturally endowed with ten Weyl-covariant Carrollian Cotton descendants. These are

- four weight-3 scalars:

$$\gamma = 8 * \varpi^3, \quad \varepsilon = (\hat{\mathcal{D}}_l \hat{\mathcal{D}}^l + 2\hat{\mathcal{K}}) * \varpi, \quad \zeta = \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j * \xi^{ij}, \quad \tau = * \xi_{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_t \xi^{ij}; \quad (41)$$

- three weight-2 forms:

$$\psi_i = 3\eta_{ji} \hat{\mathcal{D}}^j * \varpi^2, \quad (42)$$

$$\chi_i = \frac{1}{2}\eta_{ji} \hat{\mathcal{D}}^j \hat{\mathcal{K}} + \frac{1}{2}\hat{\mathcal{D}}_i \hat{\mathcal{A}} - 2 * \varpi (\hat{\mathcal{R}}_i + 2\hat{\mathcal{D}}^j \xi_{ij}) + 3\hat{\mathcal{D}}^j (* \varpi \xi_{ij}), \quad (43)$$

$$z_i = \frac{1}{2}\eta_{ij} \hat{\mathcal{D}}^j \xi^2 - \hat{\mathcal{D}}^j \frac{1}{\Omega} \hat{\mathcal{D}}_t * \xi_{ij} - * \xi_{ij} \hat{\mathcal{D}}_k \xi^{jk}; \quad (44)$$

- three weight-1 traceless and symmetric two-index covariant tensors:

$$\Psi_{ij} = -2 * \varpi^2 * \xi_{ij} + \hat{\mathcal{D}}_i \hat{\mathcal{D}}_j * \varpi - \frac{1}{2}a_{ij} \hat{\mathcal{D}}^k \hat{\mathcal{D}}_k * \varpi - \eta_{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_t * \varpi^2, \quad (45)$$

$$X_{ij} = \frac{1}{2}\eta_{ki} \hat{\mathcal{D}}^k (\hat{\mathcal{R}}_j + \hat{\mathcal{D}}^l \xi_{jl}) + \frac{1}{2}\eta_{kj} \hat{\mathcal{D}}_i (\hat{\mathcal{R}}^k + \hat{\mathcal{D}}^l \xi^{kl}) - \frac{3}{2}\hat{\mathcal{A}}\xi_{ij} - \hat{\mathcal{K}} * \xi_{ij} + 3\frac{* \varpi}{\Omega} \hat{\mathcal{D}}_t \xi_{ij}, \quad (46)$$

$$Z_{ij} = 2 * \xi_{ij} \xi^2 - \frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t * \xi_{ij}. \quad (47)$$

As for the conservation equation (2), it supplies the following Carrollian decompositions:

$$\nabla_\rho C^\rho_{\hat{0}} = c^2 \mathcal{D}_{\text{Cot}} + \mathcal{E}_{\text{Cot}} + \frac{\mathcal{F}_{\text{Cot}}}{c^2} + \frac{\mathcal{W}_{\text{Cot}}}{c^4} = 0, \quad (48)$$

and

$$\frac{1}{c} \nabla_\rho C^{\rho i} = c^2 \mathcal{J}_{\text{Cot}}^i + \mathcal{G}_{\text{Cot}}^i + \frac{\mathcal{H}_{\text{Cot}}^i}{c^2} + \frac{\mathcal{X}_{\text{Cot}}^i}{c^4} = 0. \quad (49)$$

All identities are Weyl-covariant with

$$\mathcal{D}_{\text{Cot}} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \gamma - \hat{\mathcal{D}}_i \psi^i, \quad (50)$$

$$\mathcal{E}_{\text{Cot}} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \varepsilon - \hat{\mathcal{D}}_i \chi^i + \Psi_{ij} \xi^{ij}, \quad (51)$$

$$\mathcal{F}_{\text{Cot}} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \zeta - \hat{\mathcal{D}}_i z^i + X_{ij} \xi^{ij}, \quad (52)$$

$$\mathcal{W}_{\text{Cot}} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \tau + Z_{ij} \xi^{ij}, \quad (53)$$

and

$$\mathcal{F}_{\text{Cot}}^i = \frac{1}{2} \hat{\mathcal{D}}^i \gamma + 2 * \varpi * \psi^i, \quad (54)$$

$$\mathcal{E}_{\text{Cot}}^i = \frac{1}{2} \hat{\mathcal{D}}^i \varepsilon - \hat{\mathcal{D}}_j \Psi^{ij} + 2 * \varpi * \chi^i + \frac{1}{\Omega} \hat{\mathcal{D}}_t \psi^i + \psi_j \xi^{ij}, \quad (55)$$

$$\mathcal{H}_{\text{Cot}}^i = \frac{1}{2} \hat{\mathcal{D}}^i \zeta - \hat{\mathcal{D}}_j X^{ij} + 2 * \varpi * z^i + \frac{1}{\Omega} \hat{\mathcal{D}}_t \chi^i + \chi_j \xi^{ij}, \quad (56)$$

$$\mathcal{X}_{\text{Cot}}^i = \frac{1}{2} \hat{\mathcal{D}}^i \tau - \hat{\mathcal{D}}_j Z^{ij} + \frac{1}{\Omega} \hat{\mathcal{D}}_t z^i + z_j \xi^{ij}. \quad (57)$$

INTERPRETING THE COTTON CARROLLIAN DESCENDANTS is possible along the same lines as for the ordinary Riemannian Cotton tensor. The main differences are that a Carrollian geometry has a fibre-bundle structure and a wider freedom for its affine connection. This blurs to some extent the concept of conformal flatness, which is the feature emerging when the Cotton vanishes in three-dimensional Riemannian manifolds, and more options emerge.

Vanishing geometric shear From Eq. (15), when $\xi_{ij} = 0$ the time dependence in the metric a_{ij} is factorized: $a_{ij}(t, \mathbf{x}) = e^{2\sigma(t, \mathbf{x})} \bar{a}_{ij}(\mathbf{x})$. Moreover, in two dimensions $\bar{a}_{ij}(\mathbf{x})$ is necessarily proportional to δ_{ij} , hence choosing complex coordinates, the metric on the two-dimensional surface \mathcal{S} is recast as

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta} \quad (58)$$

with $P = P(t, \zeta, \bar{\zeta})$ a real function. Consequently, a subset of the Carroll–Cotton tensors vanish, as it is inferred from Eqs. (41), (44), (47): ζ , τ , z_i and Z_{ij} .

Vanishing Carrollian vorticity In the Eq.(13), for $*\varpi = 0$ we find :

$$d\mu = \varphi \wedge \mu. \quad (59)$$

Using Fröbenius criterion we are instructed that μ is proportional to an exact form. We can choose appropriately the time coordinate so that the Ehresmann connection \mathfrak{b} vanishes, leading to

$$\mu = -\Omega(t, \mathbf{x}) dt. \quad (60)$$

The vanishing Carrollian Cotton tensors are now γ , ε , ψ_i and Ψ_{ij} (see (41), (42), (45)).

Vanishing Carrollian shear and vorticity This merges the two previous situations and the Carrollian structure is of the form

$$d\ell^2 = \frac{2}{P(t, \zeta, \bar{\zeta})^2} d\zeta d\bar{\zeta}, \quad \mu = -\Omega(t, \zeta, \bar{\zeta}) dt. \quad (61)$$

Despite the factorization of the metric and of the clock form, not all Carroll–Cotton tensors are zero. We find indeed from (30), (31),

$$\hat{\mathcal{R}}_\zeta = \frac{1}{\Omega} \partial_t \partial_\zeta \ln(\Omega P), \quad \hat{A} = 0, \quad \hat{\mathcal{K}} = 2P^2 \partial_\zeta \partial_{\bar{\zeta}} \ln(\Omega P) \quad (62)$$

($\hat{\mathcal{R}}_{\bar{\zeta}}$ is the complex conjugate of $\hat{\mathcal{R}}_\zeta$), and using (43), (46) we obtain for the Carroll–Cotton:

$$\begin{cases} \chi_\zeta = \frac{i}{2} \hat{\mathcal{D}}_\zeta \hat{\mathcal{K}} = \frac{i}{\Omega^2} \partial_\zeta ((\Omega P)^2 \partial_\zeta \partial_{\bar{\zeta}} \ln(\Omega P)) \\ X_{\zeta\zeta} = i \hat{\mathcal{D}}_\zeta \hat{\mathcal{R}}_\zeta = \frac{i}{\Omega} \frac{1}{(\Omega P)^2} \partial_\zeta ((\Omega P)^2 \partial_t \partial_\zeta \ln(\Omega P)), \quad X_{\zeta\bar{\zeta}} = 0 \end{cases} \quad (63)$$

with $\chi_{\bar{\zeta}} = \bar{\chi}_\zeta$ and $X_{\bar{\zeta}\bar{\zeta}} = \bar{X}_{\zeta\zeta}$. The tensors in (63) vanish if ΩP is constant. This result could have been anticipated by noticing that the Papapetrou–Randers (see Eq. (9)) pseudo-Riemannian ascendant of (61) is conformally flat provided ΩP be constant.

Concluding, in Carrollian geometry, “conformal flatness” concerns separately the base and the fibre of the bundle $\mathcal{M} = \mathbb{R} \times S$ with distinct vanishing Carroll–Cotton tensors.

3 Carroll–Chern–Simons actions and transformation properties

GENERAL-COVARIANT ACTIONS $S = \frac{1}{c} \int_{\mathcal{M}} d^{d+1}x \sqrt{-g} \mathcal{L}$ on Riemannian spacetimes \mathcal{M} lead to covariantly conserved energy–momentum tensors $T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}$. When the action is furthermore Weyl-invariant, $T^{\mu\nu}$ is Weyl-covariant of weight $d + 3$ with $T^\mu{}_\mu = 0$ and $\mathcal{D}_\mu T^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0$.

The Chern–Simons action (4) is *not* invariant under frame transformations. The latter may be induced by diffeomorphisms in coordinate frames or be more general (e.g., local Lorentz transformations on orthonormal frames, when applicable):

$$\theta \rightarrow \theta' = \Lambda \theta, \quad (64)$$

where θ is a column matrix encoding all θ^μ s and Λ a square matrix with entries $\Lambda^\mu{}_\nu$. With this, genuine tensors are invariant — their components transform with $\Lambda^\mu{}_\nu$ and $\Lambda^{-1\mu}{}_\nu$ though. The torsion (if any) and curvature two-forms transform homogeneously

$$\mathcal{T}' = \Lambda \mathcal{T}, \quad \mathcal{R}' = \Lambda \mathcal{R} \Lambda^{-1}, \quad (65)$$

whereas the connection one-form acquires an extra non-homogeneous piece:¹⁰

$$\omega' = \Lambda (\omega - X) \Lambda^{-1}, \quad X = \Lambda^{-1} d\Lambda. \quad (66)$$

Using

$$d\omega' = \Lambda (d\omega + X \wedge \omega + \omega \wedge X - X \wedge X) \Lambda^{-1} \quad (67)$$

we find generally

$$\text{Tr} \left(\omega' \wedge d\omega' + \frac{2}{3} \omega' \wedge \omega' \wedge \omega' \right) = \text{Tr} \left(\omega \wedge d\omega + \frac{2}{3} \omega \wedge \omega \wedge \omega \right) + d\text{Tr} (X \wedge \omega) - \frac{1}{3} \text{Tr} (X \wedge dX). \quad (68)$$

Let us focus on infinitesimal transformations of the form

$$\Lambda = \mathbb{1} + \Xi \quad \Rightarrow \quad X = d\Xi. \quad (69)$$

Under the latter, the Chern–Simons action (4) transforms as

$$\delta_{\Xi} S_{\text{CS}} = \frac{1}{2c} \oint_{\partial\mathcal{M}} \text{Tr} (d\Xi \wedge \omega) = -\frac{1}{2c} \oint_{\partial\mathcal{M}} \text{Tr} (\Xi d\omega), \quad (70)$$

where we have assumed that $\partial\partial\mathcal{M} = \emptyset$. The last expression can be alternatively expressed in terms of the curvature $\mathcal{R} = d\omega + \omega \wedge \omega$, and provides the anomaly, be it gravitational, Lorentz or mixed, depending on the frame and the transformation performed.

In view of the Carrollian applications, we would like to elaborate on the behaviour (70) in Papapetrou–Randers coframe $\{\theta^{\hat{0}} = -c\mu, dx^i\}$, for Carrollian diffeomorphisms mapping $\{t, \mathbf{x}\}$ onto $\{t'(t, \mathbf{x}), \mathbf{x}'(\mathbf{x})\}$. This guarantees the form stability of the metric (9) and of the time-and-space splitting advertised earlier. Under these transformations we find — see footnote 1:

$$\Lambda^{\hat{0}}_{\hat{0}} = 1, \quad \Lambda^{\hat{0}}_j = 0, \quad \Lambda^i_{\hat{0}} = 0, \quad \Lambda^i_j = J^i_j = \frac{\partial x^{i'}}{\partial x^j}. \quad (71)$$

Carrollian diffeomorphisms are generated by vector fields $\xi = \xi^{\hat{0}}(t, \mathbf{x}) \frac{1}{c\Omega} \partial_t + \xi^i(\mathbf{x}) \hat{\partial}_i$ so that

$$\Xi^{\hat{0}}_{\hat{0}} = 0, \quad \Xi^{\hat{0}}_j = 0, \quad \Xi^i_{\hat{0}} = 0, \quad \Xi^i_j = \partial_j \xi^i. \quad (72)$$

With this, in the present framework, the Chern–Simons variation (70) reads:

$$\delta_{\xi} S_{\text{CS}} = -\frac{1}{2c} \oint_{\partial\mathcal{M}} \partial_j \xi^i d\omega^j_i. \quad (73)$$

¹⁰In terms of connection coefficients the transformation reads: $\Gamma^{\rho'}_{\mu\nu} = \Lambda^{-1\alpha}_{\mu} \Lambda^{-1\beta}_{\nu} \Lambda^{\rho}_{\gamma} \Gamma^{\gamma}_{\alpha\beta} - \Lambda^{-1\alpha}_{\mu} e_{\alpha}{}^{\rho} (\Lambda^{\rho}_{\gamma}) \Lambda^{-1\gamma}_{\nu}$. For coordinate-frame transformations induced by diffeomorphisms, Λ is the Jacobian matrix J with entries $\frac{\partial x^{\mu'}}{\partial x^{\nu}}$.

Assuming a Levi–Civita connection (10), (11), and using Eq. (16), Eq. (73) is recast as:

$$\delta_{\xi} S_{\text{CS}} = -\frac{c}{2} \oint_{\partial\mathcal{M}} \partial_j \xi^i d(\varpi^j_i \mu) - \frac{1}{2c} \oint_{\partial\mathcal{M}} \partial_j \xi^i d\dot{\omega}^j_i. \quad (74)$$

Being topological, the anomalous contributions (74) neither compromise the covariant conservation of the Cotton tensor nor spoil its conformal properties. Regarding this last statement, one should notice that Weyl rescalings are also anomalous. In the Papapetrou–Randers frame, using the Levi–Civita connection (21) for infinitesimal transformations $\mathcal{B} = 1 + \lambda$, we find:

$$\delta_{\lambda} S_{\text{CS}} = \frac{1}{c} \oint_{\partial\mathcal{M}} dx^i \wedge \left(\frac{1}{2} da_{ij} a^{jk} \hat{\partial}_k \lambda + (\varpi_{ij} dx^j + \varphi_i \mu) \frac{1}{\Omega} \partial_t \lambda \right). \quad (75)$$

A more detailed analysis of the associated boundary dynamics should shed some light on the Weyl anomaly, on a similar fashion as in [40].

CONSIDER NOW A DYNAMICAL SYSTEM ON A CARROLLIAN MANIFOLD $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ described with an action $S = \int_{\mathcal{M}} dt d^d x \sqrt{a} \Omega \mathcal{L}$, functional of a_{ij} , Ω and b_i . The associated *Carrollian momenta*, which replace the corresponding relativistic energy–momentum tensor $T^{\mu\nu}$ are now (see [41, 42])

$$\begin{cases} \Pi^{ij} = \frac{2}{\sqrt{a} \Omega} \frac{\delta S}{\delta a_{ij}} \\ \Pi^i = \frac{1}{\sqrt{a} \Omega} \frac{\delta S}{\delta b_i} \\ \Pi = -\frac{1}{\sqrt{a}} \left(\frac{\delta S}{\delta \Omega} + \frac{b_i}{\Omega} \frac{\delta S}{\delta b_i} \right). \end{cases} \quad (76)$$

These are the *energy–stress tensor*, the *energy current* and the *energy density*. Invariance under Carrollian diffeomorphisms impose two conservation equations [37], respectively in time and space directions

$$\left(\frac{1}{\Omega} \partial_t + \theta \right) \Pi + \left(\hat{\nabla}_i + 2\varphi_i \right) \Pi^i + \Pi^{ij} \hat{\gamma}_{ij} = 0, \quad (77)$$

$$\left(\hat{\nabla}_j + \varphi_j \right) \Pi^j_i + 2\Pi^j \varpi_{ji} + \Pi \varphi_i + \left(\frac{1}{\Omega} \partial_t + \theta \right) P_i = 0, \quad (78)$$

involving a fourth Carrollian momentum dubbed *momentum current*, P_i . In the present formalism based on the Carrollian data (5) and (7) the latter is not defined directly through a variation of the action with respect to some conjugate variable. It is however inevitable, and this can be verified whenever a microscopic action is available in terms of fundamental fields as, e.g., in [43].¹¹

When the Carrollian action is Weyl-invariant Π^{ij} , Π^i , P^i and Π are Weyl-covariant with weights $d+3$, $d+2$, $d+2$ and $d+1$, and obey the trace condition:

$$\Pi^{ij} = \Upsilon^{ij} + \frac{\Pi}{d} a^{ij}, \quad \Upsilon^i_i = 0. \quad (79)$$

¹¹In fact, one could chose a different description for the degenerate metric $d\ell^2$, where dx^i are traded for $dx^i - w^i dt$. The Carrollian momenta conjugate to the variables w^i would then be P_i — see e.g. [44].

Equations (77) and (78) are thus traded for

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \hat{\mathcal{D}}_i \Pi^i + \Upsilon^{ij} \xi_{ij} = 0, \quad (80)$$

$$\frac{1}{d} \hat{\mathcal{D}}_j \Pi + \hat{\mathcal{D}}_i \Upsilon^i_j + 2\Pi^i \varpi_{ij} + \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \delta^i_j + \xi^i_j \right) P_i = 0. \quad (81)$$

EXPANDING THE CHERN–SIMONS ACTION (4) in powers of c in the Papapetrou–Randers background (9) equipped with Levi–Civita connection (10) and (11), delivers four distinct Carrollian avatars of the Chern–Simons dynamics,

$$S_{\text{CS}} = c^3 S_{\text{CCS}}^{\text{pm}} + c S_{\text{CCS}}^{\text{m}} + \frac{1}{c} S_{\text{CCS}}^{\text{e}} + \frac{1}{c^3} S_{\text{CCS}}^{\text{pe}}, \quad (82)$$

possessing four sets of Weyl-covariant Carrollian momenta of the type (76), (79), obeying four sets of conformal Carrollian conservation equations (80), (81). As we have already anticipated, these Carroll–Chern–Simons actions are in general anomalous under Carrollian diffeomorphisms and Weyl transformations with topological anomalies. The Carrollian momenta and Carrollian conservation equations are precisely those recovered in Sec. 2 when decomposing the Riemannian Cotton tensor and its divergence. This is summarized as follows.

Paramagnetic Carroll–Chern–Simons This stems out of the c^3 -order term in the Chern–Simons action:

$$S_{\text{CCS}}^{\text{pm}} = 4 \int_{\mathcal{M}} dt d^2x \sqrt{a} \Omega * \varpi^3. \quad (83)$$

The associated momenta are $\Pi = 2\gamma$ in (41), $\Pi^i = 2\psi^i$ in (42), $\Upsilon^{ij} = 0$ and Eqs. (80), (81) are now

$$\mathcal{D}_{\text{Cot}} = 0, \quad \mathcal{J}_{\text{Cot}}^i = 0, \quad (84)$$

see (50) and (54). From these equations and comparison with (81) we infer that the momentum P^i vanishes. For the paramagnetic Carroll–Chern–Simons action¹² $\delta_\xi S_{\text{CCS}}^{\text{pm}} = 0$ in agreement with (74), where no term of order c^3 is present. This action is also manifestly Weyl-invariant, in line with (75).

Magnetic Carroll–Chern–Simons The c -order provides

$$S_{\text{CCS}}^{\text{m}} = \frac{1}{2} \int_{\mathcal{M}} \mu \wedge \left(\dot{\omega}^i_j \wedge dx^k \hat{\mathcal{D}}_k * \varpi + * \varpi d\dot{\omega}^i_j \right) \eta^j_i + \int_{\mathcal{M}} dt d^2x \sqrt{a} \Omega \left[* \varpi \hat{\mathcal{V}}_i \varphi^i - \varphi^i \hat{\mathcal{D}}_i * \varpi \right]. \quad (85)$$

Now $\Pi = 2\varepsilon$ in (41), $\Pi^i = 2\chi^i$ in (43), $\Upsilon^{ij} = -2\Psi^{ij}$ in (45) and Eqs. (80), (81) are

$$\mathcal{E}_{\text{Cot}} = 0, \quad \mathcal{G}_{\text{Cot}}^i = 0, \quad (86)$$

¹²The variation vanishes for covariant actions based on genuine scalars Φ as $S[\Phi] = \int_{\mathcal{M}} dt d^2x \sqrt{a} \Omega \Phi$.

see (51), (55) and (81), from which $P^i = 2\psi^i$, Eq. (42). The Carrollian-diffeomorphism transformation of (85) is performed with the help of the rules for the connection (66) and its exterior differential (67) in the infinitesimal version (69) — μ , φ and $*\varpi$ are invariant, whereas η^j_i transforms ordinarily. Only the first integral contributes with

$$\delta_\xi S_{\text{CCS}}^{\text{m}} = -\frac{1}{2} \oint_{\partial\mathcal{M}} \partial_j \xi^i d(\varpi^j_i \mu). \quad (87)$$

Unsurprisingly this expression coincides with the magnetic (c) order in (74).

Although the action (85) contains explicitly the Carrollian Weyl connection φ , it turns out to be Weyl-invariant, as expected from (75), which features only the order $1/c$.

Electric Carroll–Chern–Simons The order $1/c$ is as follows:

$$\begin{aligned} S_{\text{CCS}}^{\text{e}} &= \frac{1}{2} \int_{\mathcal{M}} \text{Tr} \left(\hat{\omega} \wedge d\hat{\omega} + \frac{2}{3} \hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega} \right) + \int_{\mathcal{M}} dt d^2x \sqrt{a} \Omega \left[\varphi^i \eta^{kl} \hat{\nabla}_k \hat{\gamma}_{li} \right. \\ &\quad \left. + * \varpi \eta^{kl} \eta^{ij} \hat{\gamma}_{ki} \hat{\gamma}_{lj} - * \varpi \left(\frac{1}{\Omega} \hat{D}_t \hat{\gamma}^i_i + 2 \hat{\gamma}_{ij} \hat{\gamma}^{ij} \right) + \hat{\gamma}^i_k \eta^{kl} \left(\hat{\nabla}_{(l} \varphi_{i)} + \varphi_{l} \varphi_i \right) \right] \end{aligned} \quad (88)$$

with $\Pi = 2\zeta$ in (41), $\Pi^i = 2z^i$ in (44), $\Upsilon^{ij} = -2X^{ij}$ in (46) and Eqs. (80), (81) are now

$$\mathcal{F}_{\text{Cot}} = 0, \quad \mathcal{H}_{\text{Cot}}^i = 0, \quad (89)$$

see (52) and (56). Comparing with Eq. (81) we find the momentum $P^i = 2\chi^i$, given in (43). As for the magnetic case, the electric Carroll–Chern–Simons action transforms under Carrollian diffeomorphisms. Only the first integral in (88) is anomalous and its transformation is readily determined using the generic expression (73) with the connection $\hat{\omega}$:

$$\delta_\xi S_{\text{CCS}}^{\text{e}} = -\frac{1}{2} \oint_{\partial\mathcal{M}} \partial_j \xi^i d\hat{\omega}^j_i = -\frac{1}{2} \oint_{\partial\mathcal{M}} \partial_j \xi^i \hat{\mathcal{R}}^j_i + \frac{1}{2} \oint_{\partial\mathcal{M}} \partial_j \xi^i \hat{\omega}^j_k \wedge \hat{\omega}^k_i. \quad (90)$$

This agrees with the order- $1/c$ term in the variation (74), which is indeed the electric order. Under a Weyl rescaling, the behaviour of (88) is

$$\delta_\lambda S_{\text{CCS}}^{\text{e}} = \oint_{\partial\mathcal{M}} dx^i \wedge \left(\frac{1}{2} da_{ij} a^{jk} \hat{\partial}_k \lambda + (\varpi_{ij} dx^j + \varphi_i \mu) \frac{1}{\Omega} \partial_t \lambda \right), \quad (91)$$

also read off in (75), which contains a single order in c . This boundary term does not affect the field equations, which are Weyl-covariant.

Paraelectric Carroll–Chern–Simons Finally, the order $1/c^3$ gives

$$S_{\text{CCS}}^{\text{pe}} = - \int_{\mathcal{M}} dt d^2x \sqrt{a} \eta^{kl} \hat{\gamma}^i_k \hat{D}_t \hat{\gamma}_{il} = - \int_{\mathcal{M}} dt d^2x \sqrt{a} * \xi^{ij} \hat{D}_t \xi_{ij} \quad (92)$$

leading to $\Pi = 2\tau$ in (41), $\Pi^i = 0$ and $Y^{ij} = -2Z^{ij}$ in (47). Equations (80) and (81) reduce to

$$\mathcal{W}_{\text{Cot}} = 0, \quad \mathcal{X}_{\text{Cot}}^i = 0, \quad (93)$$

see (53) and (57), which exhibit using (81) $P^i = 2z^i$, Eq. (44). In the case at hand the action is Carroll-diffeomorphism invariant, hence $\delta_\xi S_{\text{CCS}}^{\text{pe}} = 0$ in agreement with (74), which does not feature any order- $1/c^3$ term. It is also Weyl-invariant.

THE PRESENT ANALYSIS CALLS FOR SEVERAL COMMENTS. We have reached four Carroll–Chern–Simons actions following a precise guideline, which consists in choosing a Papapetrou–Randers background for a relativistic theory that enables a subsequent Carrollian reduction organized in powers of c . The names given to the four Carroll–Chern–Simons actions follow the pattern already used for the scalar field or the Einstein–Hilbert action in Refs. [38, 39, 43–45].¹³ As pointed out in the introduction, besides electric and magnetic actions, non-dynamical replicas sometimes appear. This phrasing seems less appropriate in the present context because we are dealing anyhow with topological actions, and the terms paramagnetic and paraelectric are better qualified. It should be added in passing that the various patterns used for reaching Carrollian actions — algebra design, algebra gauging, strict zero- c limit or our reduction method — sometimes deliver different though equally consistent results (see also e.g. [47–51]). Specific approaches for the search of Carrollian topological theories might also be devised, like Carrollian analogues of topological Riemannian terms [52]. In our scheme, the topological nature of the Carroll–Chern–Simons descendants seems pledged by the original pseudo-Riemannian Chern–Simons action. A Hamiltonian analysis would be suitable to address this aspect, together with the counting of degrees of freedom in every instance, but it lies beyond the scope of the present note.

A noticeable feature of ours is that all but the paraelectric dynamics break off-shell local Carroll-boost invariance because of non-vanishing energy fluxes Π^i . The latter emerge in the expansion of the “heat current” $C^{\hat{0}i}$ and originate from the original relativistic theory.¹⁴ Examples of this sort are numerous. The magnetic Carrollian scalar dynamics naturally accommodates such a current, as opposed to the electric instance, where it is absent [43]. One may choose to discard it at the expense of facing a constrained theory [44, 45]. When the Carrollian manifold is the null-infinity conformal hypersurface in asymptotically flat spacetimes, such currents are unavoidably generated by outgoing (or incoming) gravitational radiation [18–20]. No alternative exists in those cases, other than accepting the physical consequences, which are the non-conservation of charges [19, 37].

EXTREMUMS of the pseudo-Riemannian Chern–Simons action have vanishing Cotton tensor. These are conformally flat three-dimensional pseudo-Riemannian spacetimes. In the Carrollian framework, the nature of the extremums is different for each of the four available dynamics. We will not perform a systematic resolution of the equations in each instance, but rather provide some generic fea-

¹³It was suggested in [46] to use type I and II for electric and magnetic, respectively; an alternative option could be time-like and space-like.

¹⁴Not every Lorentz-boost-invariant relativistic theory is expected to lead to a Carroll-boost-invariant Carrollian relative, as explained in the footnote 40 of [19].

tures.

The paramagnetic Carroll–Chern–Simons dynamics (83) requires geometries with $\gamma = \psi^i = 0$, hence (using (41), (42)) with vanishing vorticity $*\varpi$. These geometries are equipped with a clock form (60). Since the dynamics at hand is Weyl-invariant, Ω can be set to 1 by a Weyl transformation, which further simplifies the clock form to $\mu = -dt$, thus leading to zero φ_i and \hat{A} . The fibre of the Carrollian manifold is trivial but the basis metric $a_{ij}(t, \mathbf{x})$ is utterly arbitrary.

Extremums of the magnetic Carroll–Chern–Simons action (85) are reached with $\varepsilon = \psi^i = \Psi^{ij} = \chi^i = 0$. The three first equations are satisfied with $*\varpi = 0$ (see (41), (42) and (45)). This might not be the most general solution, but it has the virtue of setting $b_i = 0$ and $\hat{\partial}_i = \partial_i$, and as explained for the paramagnetic action, $\Omega = 1$ and $\varphi_i = \hat{A} = 0$. The last magnetic equation $\chi_i = 0$ imposes thus (see (43)) $\hat{\mathcal{K}} = \hat{K} = K$ be a function of time only. This is a severe constraint on the metric $a_{ij}(t, \mathbf{x})$, which admits non trivial solutions besides those reached by introducing a factorized time dependence on a two-dimensional metric with constant curvature having thus zero shear.¹⁵

The paraelectric instance (92) selects $\tau = z^i = Z^{ij} = 0$. Using (41), (44) and (47) a typical solution emerges with zero shear ξ_{ij} , corresponding to three-dimensional Carrollian bundles with a base metric proportional to the Euclidean metric as in (58), and arbitrary clock form and Ehresmann connection.

Finally, the electric Carroll–Chern–Simons dynamics (88) leads to $\zeta = z^i = X^{ij} = \chi^i = 0$. The two first equations are satisfied with $\xi_{ij} = 0$ (see Eqs. (41), (44)) — we do not exclude less restrictive solutions. This simplifies the base metric as in the paraelectric case, but further conditions remain stemming out of (43) and (46). In holomorphic coordinates defined in (58) these conditions read:

$$\begin{cases} \chi_\zeta = \frac{i}{2} \hat{\mathcal{D}}_\zeta \hat{\mathcal{K}} + \frac{1}{2} \hat{\mathcal{D}}_\zeta \hat{A} - 2 * \varpi \hat{\mathcal{R}}_\zeta = 0 \\ X_{\zeta\zeta} = i \hat{\mathcal{D}}_\zeta \hat{\mathcal{R}}_\zeta = 0, \end{cases} \quad (94)$$

and set an interplay between the base and the fibre. Non-trivial solutions exist as the time-independent instance with $\Omega = \Omega(\zeta, \bar{\zeta})$, $b_i = b_i(\zeta, \bar{\zeta})$ and $a_{ij} = a_{ij}(\zeta, \bar{\zeta})$. There $\theta = 0$, $\varphi_\zeta = \partial_\zeta \ln \Omega$, $\hat{\mathcal{R}}_\zeta = 0$, $\hat{A} = 0$, $X_{\zeta\zeta} = 0$, while $*\varpi = \frac{i\Omega P^2}{2} \left(\partial_\zeta \frac{b_\zeta}{\Omega} - \partial_{\bar{\zeta}} \frac{b_{\bar{\zeta}}}{\Omega} \right)$ and $\chi_\zeta = \frac{i}{2\Omega^2} \partial_\zeta \left(\Omega^2 \hat{\mathcal{K}} \right)$. The vanishing of the latter delivers the family of shear-free Carrollian manifolds that define the null boundaries of stationary algebraically special asymptotically flat spacetimes, as displayed in detail in Refs. [18, 19].

To conclude this section, we observe that for Carrollian manifolds with closed clock form μ (i.e. vanishing vorticity and acceleration, following (13)), the paramagnetic and magnetic Carroll–Chern–Simons actions vanish, whereas the electric action simplifies as $S_{\text{CCS}}^e = \frac{1}{2} \int_{\mathcal{M}} \text{Tr} \left(\hat{\omega} \wedge d\hat{\omega} + \frac{2}{3} \hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega} \right)$. Demanding the absence of torsion, Eq. (27), makes furthermore the paraelectric vanish, and trivializes to some extent the electric action by removing time derivatives, and effectively downgrading the geometry to its two-dimensional traits with no time dependence.

¹⁵An example is $d\ell^2 = k(t) e^{b(y)} dx^2 - \frac{1}{4K(t)} (b'(y))^2 dy^2$ with arbitrary $b(t)$, positive $k(t)$ and $K(t) < 0$. For this metric the shear does not vanish.

4 In short

In the present note, we have considered strong Carroll structures equipped with a Carrollian connection designed to respect the time-and-space splitting inferred by the chosen Papapetrou–Randers frame. In this setup, we have reached four distinct Carroll–Chern–Simons actions, expanding the original ascendant pseudo-Riemannian Chern–Simons in powers of c , which amounts to performing a reduction under Carrollian diffeomorphisms. The variation of these actions with respect to the Carrollian geometric data, i.e., the metric and the clock form, yield four sets of three Carrollian Cotton tensors, which obey Carrollian conservation equations.

Two out of the four Carroll–Chern–Simons actions are truly invariant under Carrollian diffeomorphisms and Weyl transformations: the paramagnetic and paraelectric avatars. Extremizing the former leads to Carrollian manifolds with trivial fibres, while the extremums of the latter capture Carrollian geometries with conformally Euclidean spatial sections and arbitrary clock forms.

The magnetic and electric actions are more intriguing. At the first place Carroll diffeomorphisms are broken by boundary terms, and so are Weyl transformations for the electric case. In the pseudo-Riemannian framework, and from a holographic perspective, these phenomena are likened to anomalies. Our understanding of Carrollian dynamics is still too poor, let alone holography on boundaries of Carrollian spacetimes, to venture into such interpretations. The question is however relevant.

The magnetic extremums embrace paramagnetic Carrollian manifolds with purely time-dependent base curvature. Examples of such spaces do exist, but conceivable applications remain unexplored. Extremizing the electric action reveals Carrollian spacetimes with conformally Euclidean spatial sections and a non-trivial interplay between the base and the fibre. These geometries are tailor-made for describing null infinity in Ricci-flat four-dimensional spacetimes.

Along with an effort to further understand the general solutions, the properties and the applications of the Carroll–Chern–Simons actions at hand, one should not dismiss the importance of generalizing our approach, possibly by considering more general Carrollian connections, or alternative techniques for disclosing Carrollian dynamics. The physics at null infinity of asymptotically flat spacetimes can provide a possible playground for challenging potential findings, another being that of black-hole horizons, as already mentioned. Among the standard tools for analyses of this sort, one finds the search for charges and conservation, encoded in the (conformal) Carroll group.

Acknowledgements

We would like to thank our colleagues Andrea Campoleoni, Sangmin Choi, Simon Pekar, Anastasios Petkou and Matthieu Vilatte for useful discussions. The work of David Rivera-Betancour was funded by Becas Chile (ANID) Scholarship No. 72200301. Marios Petropoulos thanks Olivera Mišković and Rodrigo Olea for financial support and hospitality in Instituto de Física, Pontificia Universidad Católica de Valparaíso and Departamento de Ciencias Físicas, Universidad Andrés Bello, Santiago. Olivera Mišković and Rodrigo Olea thank the Centre de Physique Théorique of the Ecole Polytechnique for hospitality

during the completion of this work. This work has been funded in part by Anillo Grant ANID/ACT210100 *Holography and its Applications to High Energy Physics, Quantum Gravity and Condensed Matter Systems* and FONDECYT Regular Grants 1190533, 1230492 and 1231779. David Rivera-Betancour thanks the programme Erasmus+ of the Institut Polytechnique de Paris as well as the Aristotle University of Thessaloniki and the Kapodistrian University of Athens for hosting him with this fellowship. The *Third Carroll Workshop* held in the Aristotelian University of Thessaloniki in October 2023 is also acknowledged for providing an utmost creative framework, where ideas related to the present work have been exchanged.

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Flat from anti de Sitter

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ABSTRACT: Ricci-flat solutions to Einstein's equations in four dimensions are obtained as the flat limit of Einstein spacetimes with negative cosmological constant. In the limiting process, the anti-de Sitter energy-momentum tensor is expanded in Laurent series in powers of the cosmological constant, endowing the system with the infinite number of boundary data, characteristic of an asymptotically flat solution space. The governing flat Einstein dynamics is recovered as the limit of the original energy-momentum conservation law and from the additional requirement of the line-element finiteness, providing at each order the necessary set of flux-balance equations for the boundary data. This analysis is conducted using a covariant version of the Newman-Unti gauge designed for taking advantage of the boundary Carrollian structure emerging at vanishing cosmological constant and its Carrollian attributes such as the Cotton tensor.

KEYWORDS: AdS-CFT Correspondence, Classical Theories of Gravity

ARXIV EPRINT: [2309.15182](https://arxiv.org/abs/2309.15182)

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1 Introduction

The solution space of Einstein’s equations and the corresponding asymptotic symmetries are severely altered by the presence of a cosmological constant Λ .¹ Firstly, asymptotically flat spacetimes support incoming and outgoing gravitational radiation, which are harder to accommodate in asymptotically anti-de Sitter — unless leaky boundary conditions are assumed [2]. Secondly, the number of free functions on the boundary characterizing the solution space is finite for non-zero Λ and infinite for $\Lambda = 0$. Hence, investigating the holographic description of Ricci-flat spacetimes from the limit of Einstein spacetimes with non-vanishing cosmological constant seems at best a futile task, limited to special cases like three spacetime dimensions.

The purpose of the present work is to reconsider this statement in four dimensions and show that expanding the anti-de Sitter energy-momentum tensor in Laurent series in $k^2 = -\Lambda/3$, one recovers the full Ricci-flat solution space in a $1/r$ -expansion together with its evolution dynamics captured in flux-balance equations.

In order to perform the above analysis, a choice of gauge is required, as usual. Baring in mind potential further developments towards flat holography, it is desirable to privilege null infinity in the asymptotically flat instance, which plays the role of a conformal boundary hosting all independent functions of the solution space, often referred to as degrees of freedom in the following. The null boundary is a three-dimensional Carrollian manifold and it is therefore convenient to select a bulk gauge making the corresponding boundary general and Weyl covariance manifest. This has prompted to choose a modified version of the Newman-Unti gauge [3]. This gauge is built upon an incomplete gauge fixing that is expected to lead to an enhancement of the asymptotic symmetries with respect to more customary gauges like the Bondi one, in analogy with what has been observed in three bulk

¹See [1] for a recent review and further reading suggestions.

dimensions in several examples of incomplete gauge fixings [4–13]. We will not pursue this direction here, although it has attracted recent interest — see, e.g., [14, 15] where in the latter reference the gauge under consideration was dubbed “differential Newman-Unti” — and we will focus on the comparison between the space of solutions of Einstein’s equations in the asymptotically anti-de Sitter and flat cases in the chosen gauge. Reference [3] has set the stage for the gauge we will describe here, although it was originally circumscribed to the restricted class of algebraically special solutions (for AdS, see [16–22]).

In a nutshell, the starting point is the anti-de Sitter case, where the solution space admittedly consists of the boundary metric and the boundary energy-momentum tensor, which is covariantly conserved as a consequence of bulk Einstein’s equations. The analogue of the “Bondi shear” (sometimes referred to as “dynamical shear” later) is not an independent piece of data because Einstein’s equations require it be proportional with a k -dependent factor to the geometric shear of the gauge congruence — already part of the solution space. We move to the asymptotically flat instance by sending k to zero, expanding the energy-momentum tensor in powers of k^2 , trading the geometric shear for the dynamical shear along the lines of [23], and requiring the bulk line element to remain finite. This imposes further evolution equations for the new degrees of freedom at every order in the radial $1/r$ expansion, which supplement the energy-momentum conservation in the zero- k limit. The resulting infinite set defines the flux-balance equations, which can otherwise be obtained directly by solving Einstein’s equations without cosmological constant.

Besides reaching the correct boundary Ricci-flat dynamics and tracing the AdS origin of the asymptotically flat solution space, the method presented here delivers Carroll-covariant flux-balance equations revealing the entire freedom for the choice of the boundary Carrollian geometry. The pattern involves the general and Weyl-covariant gauge mentioned earlier, which naturally incorporates the Cotton tensor of the anti-de Sitter boundary, or its Carrollian emanations in the asymptotically flat situation (see [24]). This tensor carries information on the gravitational radiation and plays a pivotal role for the description of magnetic charges [25].

We begin our presentation by defining the covariant Newman-Unti gauge for asymptotically anti-de Sitter spaces. Along the way, we review its boundary Weyl covariance, as well as a useful decomposition of the boundary energy-momentum and Cotton tensors. We then provide an on-shell expression of the line element up to order $1/r^2$, where r is a null radial coordinate, infinite at the conformal boundary. The flat limit, following the prescription summarized above, is carried out in the upcoming section, after a precise setting of the boundary Carrollian structure consecutive to the zero- Λ limit. We show how the new degrees of freedom resulting from the expansion of the anti-de Sitter energy-momentum tensor are sorted out, how they enter the metric and how flux-balance equations are reached. Two appendices complement the main exposition, bringing about the necessary tools of Carrollian geometry in arbitrary frames (strong Carroll structures, connections, curvatures) as well as showing how such structures can be attained from pseudo-Riemannian geometries at vanishing speed of light. A presentation of the Carrollian descendants of the Cotton tensor closes this article.

2 Einstein spacetimes in covariant Newman-Unti gauge

Choosing a boundary-covariant gauge. Due to the Fefferman-Graham ambient metric construction [26–28], asymptotically locally anti-de Sitter four-dimensional spacetimes are determined by a set of independent boundary data, namely a three-dimensional metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ and a rank-two tensor $T = T_{\mu\nu}dx^\mu dx^\nu$, symmetric ($T_{\mu\nu} = T_{\nu\mu}$), traceless ($T^\mu{}_\mu = 0$) and conserved:²

$$\nabla^\mu T_{\mu\nu} = 0. \tag{2.1}$$

This construction is reached by setting a homonymous gauge, imposing fall-offs and solving Einstein’s equations order by order in powers of the radial space-like coordinate.³ At every order in this expansion, the line element is determined by a tensor $G_{\mu\nu}^{(s)}$, fixed by Einstein’s equations in terms of $g_{\mu\nu} = G_{\mu\nu}^{(-2)}$, $T_{\mu\nu} = \frac{3k}{16\pi G} G_{\mu\nu}^{(1)}$ and their derivatives (here the conformal boundary is located at $\rho \rightarrow +\infty$):

$$ds_{\text{Einstein}}^2 = \frac{d\rho^2}{(k\rho)^2} + \sum_{s \geq -2} \frac{1}{(k\rho)^s} G_{\mu\nu}^{(s)}(x) dx^\mu dx^\nu. \tag{2.2}$$

The conservation (2.1) is itself a consequence of Einstein’s dynamics.

Fefferman-Graham’s gauge is covariant with respect to the three-dimensional pseudo-Riemannian boundary \mathcal{M} . It can also be modified so as to make it Weyl-covariant [30–33]. However, it does not admit a smooth vanishing- k limit. Alternative gauges are Bondi or Newman-Unti [34–36], valid regardless of the cosmological constant, but not covariant with respect to the boundary. Let us consider for concreteness the Newman-Unti gauge with radial coordinate r .⁴ The line element reads:

$$ds_{\text{bulk}}^2 = \frac{V}{r} du^2 - 2dudr + G_{ij} \left(dx^i - U^i du \right) \left(dx^j - U^j du \right), \tag{2.3}$$

where V , U_i and G_{ij} are functions of *all* coordinates. They are treated as power series of r , possibly including logarithms,⁵ with coefficients depending on boundary coordinates x : the retarded time u and the angles \mathbf{x} .

The bulk metric (2.3) is stable neither under boundary diffeomorphisms $x \rightarrow x'$, nor under Weyl rescalings $r \rightarrow r\mathcal{B}(x)$, and these are the features we would like to implement in a “covariantized” version of the gauge at hand. To this end, we trade $-k^2 du$ for a boundary one-form $u = u_\mu dx^\mu$, which is an invariant object dual to a time-like vector field normalized at $-k^2$. As it will become manifest in section 3, where the timelike conformal boundary

²The covariant derivative ∇ stands for the boundary Levi-Civita connection. Indices $\mu, \nu, \dots \in \{0, 1, 2\}$ fill in the boundary natural frame, whereas $i, j, \dots \in \{1, 2\}$ are associated with spatial sections.

³Residual symmetries constrain the possible terms entering each order in the radial expansion, thus simplifying the process of solving Einstein’s equations. The constraints imposed by the boundary Weyl symmetry were studied in the Fefferman-Graham gauge in [29] and will play an important role too in the covariant Newman-Unti gauge discussed in the following.

⁴Both in Bondi and Newman-Unti, ∂_r is tangent to a null geodesic congruence. In Newman-Unti gauge this congruence is affinely parameterized, in contrast to Bondi. This enables to parallelly transport a canonical null tetrad and make contact with Newman-Penrose formalism [37].

⁵Logarithms also appear in the Fefferman-Graham gauge when the boundary dimension is even [38, 39].

will become a null manifold equipped with a Carrollian structure in the limit $k \rightarrow 0$, our parameterization has been chosen such that $k = \sqrt{-\Lambda/3}$ plays the role of effective boundary velocity of light. Therefore, the previous substitution amounts to choosing a time-like boundary congruence that could be interpreted as a hydrodynamic velocity field, if the boundary energy-momentum tensor were associated with a fluid. This is not necessarily so because the hydrodynamic regime requires constitutive relations, which are not obeyed everywhere in the Einstein solution space. The subspace where this happens is the realm of fluid/gravity correspondence [16, 17]. For convenience, we will nonetheless refer to u as the “velocity field” and decompose the energy-momentum tensor accordingly.

Introducing a boundary congruence provides also the appropriate tool for addressing Weyl invariance. In the spirit of the Fefferman-Graham ambient construction, the bulk geometry should be insensitive to a Weyl rescaling of the boundary metric (weight -2) and of the boundary velocity form (weight -1)

$$ds^2 \rightarrow \frac{ds^2}{\mathcal{B}^2}, \quad u \rightarrow \frac{u}{\mathcal{B}}, \tag{2.4}$$

which should be reabsorbed into a redefinition of the radial coordinate: $r \rightarrow r\mathcal{B}$. This requires to introduce a Weyl connection one-form $A = A_\mu dx^\mu$ transforming as

$$A \rightarrow A - d \ln \mathcal{B}, \tag{2.5}$$

and suggests the following amendment to the Newman-Unti gauge

$$-dudr \rightarrow \frac{u}{k^2}(dr + rA), \tag{2.6}$$

which is indeed Weyl-invariant, as well as being boundary-general-covariant.

We can follow the suggested pattern and recast (2.3), ignoring the logarithms,⁶ avoiding the demarcation of angular and time directions, and reorganizing the expansion in terms of boundary tensors according to their transversality with respect to the congruence u as well as their conformal weights:

$$ds_{\text{bulk}}^2 = 2\frac{u}{k^2}(dr + rA) + r^2 ds^2 + r\mathcal{C}_{\mu\nu} dx^\mu dx^\nu + \frac{1}{k^4} \mathcal{F}_{\mu\nu} dx^\mu dx^\nu + \sum_{s=1}^{\infty} \frac{1}{r^s} \left(f_{(s)} \frac{u^2}{k^4} + 2\frac{u}{k^2} f_{(s)\mu} dx^\mu + f_{(s)\mu\nu} dx^\mu dx^\nu \right). \tag{2.7}$$

In this expression $f_{(s)}$ are *boundary scalars*, $f_{(s)\mu}$ *boundary transverse vectors*, $f_{(s)\mu} u^\mu = 0$, and $f_{(s)\mu\nu}$ *boundary symmetric and transverse tensors*, $f_{(s)\mu\nu} u^\mu = 0$. Their conformal weights are $s + 2$, $s + 1$ and s . The r^2 term defines the boundary metric ds^2 , which is a

⁶Logarithms of the radial coordinate might or might not be required depending on the gauge chosen to investigate the space of solutions. In some cases, like, e.g., when choosing the Fefferman-Graham gauge in odd spacetime dimensions, they are necessary to reconstruct the full solution space. In other cases, like, e.g., in asymptotically flat spacetimes in the Bondi gauge, they describe an independent sector of the solution space that might be added or not to the polynomial expansion. Our case fits in the latter class and performing a thorough investigation of the larger space of solutions including logarithms and analyzing its interplay with residual symmetries (see, e.g., [2, 23, 40–42]) is not part of our present agenda.

free piece of data in the spirit of [9–11, 13, 23]. As long as the bulk metric (2.7) is off-shell, the boundary symmetric tensors $\mathcal{F}_{\mu\nu}$ (weight 0) and $\mathcal{C}_{\mu\nu}$ (weight -1) have no reason to be transverse with respect to u . The latter is the shear of the affine null geodesic congruence tangent to ∂_r , aka “Bondi” shear.⁷ Imposing Einstein’s equations will determine all the boundary tensors introduced so far in terms of basic independent functions that define the solution space. As we will see, this set of functions includes u as well as the boundary metric ds^2 and a rank-two symmetric, traceless and conserved tensor coinciding with the energy-momentum tensor of the Fefferman-Graham gauge.

Before moving on to Einstein’s equations, a few comments are worth making to appreciate the covariant Newman-Unti gauge (2.7). Introducing a normalized but otherwise arbitrary boundary congruence amounts to the on-set of two degrees of freedom, i.e. to a relaxation of the original Newman-Unti gauge fixing. Incomplete gauge fixings might produce enhancements of asymptotic symmetries and materialize in extra charges — not always integrable or conserved. They have been investigated mostly in three bulk dimensions [4–13], where the introduction of an arbitrary congruence⁸ combined with the freedom of choosing the boundary metric restores the boundary local Lorentz symmetry and its realization as bulk diffeomorphisms, augmenting the asymptotic symmetry group [9–13]. Following [10], an elegant way of taming this information without redundancy is to express the boundary metric in an arbitrary orthonormal Cartan coframe,⁹

$$ds^2 = \eta_{AB}\theta^A\theta^B = -(\theta^{\hat{0}})^2 + \delta_{ab}\theta^a\theta^b, \tag{2.8}$$

and set

$$u = -k\theta^{\hat{0}}. \tag{2.9}$$

The dual frame vectors are $\{e_A\} = \{e_{\hat{0}}, e_a\}$ with $\theta^B(e_A) = \delta_A^B$. A possible parameterization of the frame, which we will not use explicitly though, is displayed in eqs. (A.41), (A.42) and (A.43).

We will not delve into the analysis of asymptotic symmetries in the present note. Due to the partial relaxation of the gauge this complementary task is more intricate and deserves a separate and thorough treatment [9–15].

In order to proceed with the covariant Newman-Unti gauge (2.7) and impose Einstein’s equations, it is desirable to list the available tensors with the correct conformal weights at

⁷Strictly speaking, the Bondi shear is defined in the BMS gauge (in the expansion $G_{ij} = r^2 q_{ij} + \mathcal{O}(r)$ the two-dimensional metric q_{ij} is fixed to be the round sphere) with a prominent role in the asymptotically flat instance. Normally it is related to the one introduced here by an inhomogeneous transformation.

⁸Whenever the energy-momentum tensor empowers a fluid interpretation, the congruence at hand is interpreted as the fluid lines and its arbitrariness portrays the relativistic hydrodynamic-frame invariance [43–45]. This feature is however strictly *local* because the bulk diffeomorphisms associated with the boundary hydrodynamic-frame transformations are possibly charged. These properties have been thoroughly investigated in two boundary dimensions [9–11, 13] and would undoubtedly deserve a generalization in higher dimensions, which is outside our scope here. It would better fit a broader study where frame orthonormality would be downsized, probing general boundary linear transformations.

⁹We use $A, B, \dots \in \{\hat{0}, \hat{1}, \hat{2}\}$ as boundary “flat” indices with $a, b, \dots \in \{\hat{1}, \hat{2}\}$. The parameterization of the coframe in terms of 8 arbitrary functions, suitable for the Carrollian limit, is provided in appendix A, eq. (A.43).

each order s of the radial expansion. To achieve this, we need to cope with Weyl covariance and decompose the energy-momentum tensor with respect to the chosen congruence.

Kinematics, Weyl covariance and transverse duality. Covariantization with respect to Weyl transformations requires to introduce a connection one-form $A = A_A \theta^A$, built on the congruence $u = u_A \theta^A$:

$$A = \frac{1}{k^2} \left(a - \frac{\Theta}{2} u \right), \quad (2.10)$$

which transforms as anticipated in (2.5). In this expression $a = a_A \theta^A$ and Θ are the acceleration and expansion of the congruence u , defined together with the shear and the vorticity as¹⁰

$$a_A = u^B \nabla_B u_A, \quad \Theta = \nabla_A u^A, \quad (2.11)$$

$$\sigma_{AB} = \nabla_{(A} u_{B)} + \frac{1}{k^2} u_{(A} a_{B)} - \frac{1}{2} \Theta h_{AB}, \quad (2.12)$$

$$\omega_{AB} = \nabla_{[A} u_{B]} + \frac{1}{k^2} u_{[A} a_{B]}, \quad (2.13)$$

where h_{AB} is the projector onto the space transverse to the velocity field:

$$h_{AB} = \frac{u_A u_B}{k^2} + \eta_{AB} \quad (2.14)$$

(remember we work in an orthonormal Cartan mobile frame — metric displayed in (2.8)). The above vectors are transverse, whereas the tensors are transverse and traceless.

The Weyl connection A enters the Weyl covariant derivative \mathcal{D}_A acting on a weight- w tensor as e.g. a scalar Φ :

$$\mathcal{D}_A \Phi = e_A(\Phi) + w A_A \Phi, \quad (2.15)$$

or a form v_A :

$$\mathcal{D}_B v_A = \nabla_B v_A + w A_B v_A + A_A v_B - \eta_{AB} A^C v_C. \quad (2.16)$$

The resulting tensors have weight $w + 1$.¹¹ The form field u has weight -1 i.e. u_A are weight-zero, whereas ω_{AB} and σ_{AB} have all weight 1. The explicit form of A (2.10) is obtained by demanding

$$\mathcal{D}_A u^A = 0 \quad \text{and} \quad u^C \mathcal{D}_C u_A = 0. \quad (2.17)$$

¹⁰Our conventions for (anti-) symmetrization are: $A_{(AB)} = \frac{1}{2} (A_{AB} + A_{BA})$ and $A_{[AB]} = \frac{1}{2} (A_{AB} - A_{BA})$.

¹¹Special caution is advised in comparing the present expressions with those appearing e.g. in refs. [3, 45, 46], where a natural frame was used. When dealing with Weyl covariance in orthonormal frame, the metric components have weight zero. Hence for any tensor, covariant and contravariant components have the same weights. The coframe form elements, however, have weight -1 , whereas the frame vectors have weight $+1$. If a weight- w tensor has p contravariant and q covariant indices, its Weyl-covariant derivative reads:

$$\begin{aligned} \mathcal{D}_C K_{B\dots}^{A\dots} &= \nabla_C K_{B\dots}^{A\dots} + (w + p - q) A_C K_{B\dots}^{A\dots} \\ &+ (\eta_{CD} A^A - \delta_C^A A_D - \delta_D^A A_C) K_{B\dots}^{D\dots} + \dots \\ &- (\eta_{CB} A^D - \delta_C^D A_B - \delta_B^D A_C) K_{D\dots}^{A\dots} - \dots \end{aligned}$$

and this has now weight $w + 1$.

The Weyl covariant derivative is metric-compatible with effective torsion:

$$\mathcal{D}_C \eta_{AB} = 0, \tag{2.18}$$

$$(\mathcal{D}_A \mathcal{D}_B - \mathcal{D}_B \mathcal{D}_A) \Phi = w \Phi F_{AB}, \tag{2.19}$$

where

$$F = \frac{1}{2} F_{AB} \theta^A \wedge \theta^B = dA \tag{2.20}$$

is Weyl-invariant (F_{AB} are weight-2). Metric compatibility and (2.17) imply

$$u^C \mathcal{D}_C h_{AB} = 0, \tag{2.21}$$

inferring that the operator $u^C \mathcal{D}_C$ respects transversality.

Commuting the Weyl-covariant derivatives acting on vectors, one defines the Weyl covariant Riemann tensor

$$(\mathcal{D}_A \mathcal{D}_B - \mathcal{D}_B \mathcal{D}_A) V^C = \mathcal{R}^C{}_{DAB} V^D + (w + 1) V^C F_{AB} \tag{2.22}$$

(V^C are weight- w whereas $V = V^C e_C$ has weight $w + 1$) and the usual subsequent quantities. In three (boundary) spacetime dimensions, the covariant Ricci and the scalar (both weight-2) curvatures read:

$$\mathcal{R}_{AB} = R_{AB} + \nabla_B A_A + A_A A_B + \eta_{AB} (\nabla_C A^C - A_C A^C) - F_{AB}, \tag{2.23}$$

$$\mathcal{R} = R + 4 \nabla_A A^A - 2 A_A A^A, \tag{2.24}$$

where R_{AB} is the Ricci tensor of the boundary Levi-Civita connection and R the corresponding scalar curvature. The Weyl-invariant Schouten tensor is

$$\mathcal{S}_{AB} = \mathcal{R}_{AB} - \frac{1}{4} \mathcal{R} \eta_{AB} = R_{AB} - \frac{1}{4} R \eta_{AB} + \nabla_B A_A + A_A A_B - \frac{1}{2} A_C A^C \eta_{AB} - F_{AB}. \tag{2.25}$$

It is customary to introduce the vorticity two-form

$$\omega = \frac{1}{2} \omega_{AB} dx^A \wedge dx^B = \frac{1}{2} \left(du + \frac{1}{k^2} u \wedge a \right), \tag{2.26}$$

as well as the Hodge dual of this form, which is proportional to u :

$$k \gamma u = \star \omega \quad \Leftrightarrow \quad k \gamma u_A = \frac{1}{2} \epsilon_{ABC} \omega^{BC}. \tag{2.27}$$

In this expression γ is a scalar of weight 1.

In three spacetime dimensions and in the presence of a vector field u , one naturally defines a fully antisymmetric two-index tensor:¹²

$$\hat{\eta}_{AB} = -\frac{u^C}{k} \epsilon_{CAB}, \tag{2.28}$$

obeying

$$\hat{\eta}_{AC} \hat{\eta}_B{}^C = h_{AB}, \quad \hat{\eta}^{AB} \hat{\eta}_{AB} = 2. \tag{2.29}$$

¹²This hatted two-index tensor should not be confused with Minkowski metric.

With this tensor the vorticity reads:

$$\omega_{AB} = k^2 \gamma \hat{\eta}_{AB}. \quad (2.30)$$

The two-index tensor $\hat{\eta}_{AB}$ defines a duality map within the space of symmetric, transverse (with respect to u) and traceless tensors. If V^A is transverse, so is

$$*V^A = \hat{\eta}^B{}_A V_B. \quad (2.31)$$

Similarly with a symmetric, transverse and traceless tensor W_{AB} :

$$*W_{AB} = \hat{\eta}^C{}_A W_{CB} \quad (2.32)$$

is symmetric, transverse and traceless.

The energy-momentum tensor and the Cotton tensor. Given a normalized congruence $\|u\|^2 = -k^2$ we can decompose the energy-momentum tensor as in hydrodynamics:

$$T_{AB} = (\varepsilon + p) \frac{u^A u_B}{k^2} + p \eta_{AB} + \tau_{AB} + \frac{u^A q_B}{k^2} + \frac{u^B q_A}{k^2}, \quad (2.33)$$

where

$$\varepsilon = \frac{1}{k^2} T_{AB} u^A u^B \quad (2.34)$$

is the energy density and p the analogue of a perfect stress. The symmetric viscous stress tensor τ_{AB} and the heat current q_A are purely transverse:

$$u^A \tau_{AB} = 0, \quad u^A q_A = 0, \quad q_B = -\varepsilon u_B - u^A T_{AB}. \quad (2.35)$$

In three dimensions, a conformal energy-momentum tensor has weight-1 covariant components in the coordinate basis, and weight-3 components in the orthonormal frame. Consequently, the pressure and energy density, the heat-current q_A and the viscous stress tensor τ_{AB} have all weight 3. Furthermore, since the splitting of the stress tensor into p and τ_{AB} is arbitrary, we choose to implement the absence of trace as

$$\varepsilon = 2p, \quad \tau_A{}^A = 0. \quad (2.36)$$

Due to the absence of trace, the conservation equation (2.1) can be traded for

$$\mathcal{D}_C T^C{}_B = 0. \quad (2.37)$$

In the gauge under consideration, the energy-momentum tensor comes along with the boundary Cotton tensor. They both enter the bulk metric, playing dual, electric versus magnetic, roles in various instances, as e.g. in the bulk Weyl tensor. The Cotton tensor is generically a three-index tensor with mixed symmetries.¹³ In three dimensions, which is

¹³From the bulk viewpoint, the boundary energy-momentum and Cotton tensors play dual roles. Notice that the energy-momentum tensor in (2.33) has an extra factor of k with respect to the Cotton tensor in (2.40), due to their different dimensions.

the case for our boundary geometry, the Cotton tensor can be dualized into a two-index, symmetric and traceless tensor:¹⁴

$$C_{AB} = \epsilon_A^{CD} \mathcal{D}_C (F_{BD} + \mathcal{S}_{BD}) = \epsilon_A^{CD} \nabla_C \left(R_{BD} - \frac{R}{4} \eta_{BD} \right), \quad (2.38)$$

where we recall that F_{BD} and \mathcal{S}_{BD} are respectively the Weyl curvature and the Weyl-covariant Schouten tensor defined in (2.20) and (2.25). The Cotton tensor $C_{AB} \theta^A \theta^B$ is Weyl-covariant of weight 1, and is *identically* conserved:

$$\mathcal{D}_C C^C{}_B = \nabla_C C^C{}_B \equiv 0, \quad (2.39)$$

sharing thereby all properties of the energy-momentum tensor.

Following (2.33) we can decompose the Cotton tensor into longitudinal, transverse and mixed components with respect to the congruence u :

$$\frac{1}{k} C_{AB} = \frac{3c}{2} \frac{u_A u_B}{k^2} + \frac{c}{2} \eta_{AB} - \frac{c_{AB}}{k^2} + \frac{u_{ACB}}{k^2} + \frac{u_{BCA}}{k^2}. \quad (2.40)$$

Such a decomposition naturally defines the weight-3 *Cotton scalar density*

$$c = \frac{1}{k^3} C_{AB} u^A u^B, \quad (2.41)$$

as the longitudinal component. The symmetric and traceless *Cotton stress tensor* c_{AB} and the *Cotton current* c_A (also weight-3) are purely transverse:

$$c_A{}^A = 0, \quad u^A c_{AB} = 0, \quad u^A c_A = 0, \quad (2.42)$$

and obey

$$c_{AB} = -k h^C{}_A h^D{}_B C_{CD} + \frac{ck^2}{2} h_{AB}, \quad c_B = -c u_B - \frac{u^A c_{AB}}{k}. \quad (2.43)$$

One can use the definition (2.38) to further express the Cotton density, current and stress tensor as ordinary or Weyl derivatives of the curvature. We find

$$c = \frac{1}{k^2} u^B \hat{\eta}^{DC} \mathcal{D}_C (\mathcal{S}_{BD} + F_{BD}), \quad (2.44)$$

$$c_B = \hat{\eta}^{CD} \mathcal{D}_C (\mathcal{S}_{BD} + F_{BD}) - c u_B, \quad (2.45)$$

$$c_{AB} = -h^E{}_A \left(k \epsilon_B{}^{CD} - u_B \hat{\eta}^{CD} \right) \mathcal{D}_C (\mathcal{S}_{ED} + F_{ED}) + \frac{ck^2}{2} h_{AB}. \quad (2.46)$$

Solving Einstein's equations. Einstein's equations are¹⁵

$$\mathcal{E}_{MN} \equiv R_{MN}^{\text{bulk}} - \frac{1}{2} R^{\text{bulk}} g_{MN}^{\text{bulk}} - 3k^2 g_{MN}^{\text{bulk}} = 0, \quad (2.47)$$

and we must probe them in the covariant Newman-Unti gauge. Assuming a boundary metric given in (2.8), the bulk line element (2.7) reads:

$$\begin{aligned} ds_{\text{bulk}}^2 = & 2 \frac{u}{k^2} (dr + rA) + r^2 ds^2 + r \mathcal{C}_{AB} \theta^A \theta^B + \frac{1}{k^4} \mathcal{F}_{AB} \theta^A \theta^B \\ & + \sum_{s=1}^{\infty} \frac{1}{r^s} \left(f_{(s)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(s)A} \theta^A + f_{(s)AB} \theta^A \theta^B \right), \end{aligned} \quad (2.48)$$

¹⁴We use a plain font for the Cotton C_{AB} versus a curly font for the shear \mathcal{C}_{AB} .

¹⁵We use $M, N, \dots \in \{r, \text{boundary}\}$ as bulk indices.

where all boundary tensors are now defined in the orthonormal frame at hand.¹⁶ The summation over A and B in the last terms of (2.48) is actually reduced to a summation over the transverse components a and b thanks to the transversality of $f_{(s)A}$ and $f_{(s)AB}$ with respect to the velocity field (2.9).

We will limit here the analysis to the order $1/r^2$, which is sufficient for illustrating accurately later the asymptotically flat pattern.

Order r The important output here is that the Bondi shear \mathcal{C}_{AB} is not free, but settled by the shear of the congruence u , which is of geometric nature:

$$k^2 \mathcal{C}_{AB} = -2\sigma_{AB}. \quad (2.49)$$

On shell, the Bondi shear is thus manifestly *traceless and transverse* with respect to u . Anticipating the usage of the present formalism in describing general solutions of vacuum Einstein's equations, we also introduce a *news tensor* (similarly defined in arbitrary dimension). As opposed to the usual definitions, the present tensor is *boundary-covariant, Weyl-invariant, symmetric, traceless and transverse*:

$$\mathcal{N}_{AB} = u^C \mathcal{D}_C \mathcal{C}_{AB}. \quad (2.50)$$

Equation (2.49) will be assumed when moving to the next orders.

Order 1 Unsurprisingly from the Fefferman-Graham experience, we learn that \mathcal{F}_{AB} is related to the boundary Weyl-invariant Schouten tensor displayed in eq. (2.25):¹⁷

$$\begin{aligned} \mathcal{F}_{AB} &= 2u^C \left(\mathcal{L}_{C(A} + F_{C(A)} \right) u_{B)} + \mathcal{D}_A u^C \mathcal{D}_B u^C, \\ &= 2u_{(A} \mathcal{D}_C \left(\sigma_{B)}^C + \omega_{B)}^C \right) - \frac{\mathcal{R}}{2} u_A u_B + \left(\sigma^2 + k^4 \gamma^2 \right) h_{AB} + 2\omega_{(A}^C \sigma_{B)C}, \end{aligned} \quad (2.51)$$

where

$$\gamma^2 = \frac{1}{2k^4} \omega_{AB} \omega^{AB}, \quad \sigma^2 = \frac{1}{2} \sigma_{AB} \sigma^{AB} \quad (2.52)$$

(γ was defined alternatively in eq. (2.27)). At this stage, the only independent and free data are those defining the boundary geometry (as stressed in (2.9), the congruence u is aligned with the observers at rest with respect to (2.8)).

Orders $1/r$ and $1/r^2$ At order $1/r$ new information is expected to come up in the form of a boundary conformal energy-momentum tensor. In contrast with the Fefferman-Graham gauge, the energy-momentum enters through its decomposition with respect to the congruence u , i.e. ε , q_A and τ_{AB} , rather than T_{AB} . Furthermore, it comes accompanied with the transverse-dual of the Cotton current and stress, $*c_A$ and $*c_{AB}$, see eqs. (2.31), (2.32) and (2.45), (2.46) — yet another motivation to split the

¹⁶In this frame \mathcal{C}_{AB} has weight one, $f_{(s)}$, $f_{(s)A}$ and $f_{(s)AB}$ have all weight $s+2$ whereas \mathcal{F}_{AB} is weight-2.

¹⁷The tensor defined in (2.51) is slightly different from the analogous tensor S_{AB} introduced in [3], eq. (2.42). It contains extra shear terms. The reason is that in ref. [3], when writing (2.41), the authors wanted to stress that shear terms were present, but ultimately the shear was vanishing. The present definition accounts for all shear terms.

energy-momentum tensor as discussed earlier. This trait is new, both compared to the Fefferman-Graham gauge, where the Cotton tensor does not appear explicitly at any order, and with respect to standard Bondi or Newman-Unti gauges, where it is present in disguise.¹⁸

The functions to be determined are $f_{(1)}$, $f_{(1)A}$ and $f_{(1)AB}$, which must have conformal weight 3. This leaves little freedom, given the available tensors. We find:

$$f_{(1)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(1)A} \theta^A + f_{(1)AB} \theta^A \theta^B = \frac{8\pi G}{k^4} \left(\varepsilon u^2 + \frac{4}{3} u \Delta q + \frac{2k^2}{3} \Delta \tau \right) \quad (2.53)$$

with $\Delta q = \Delta q_A \theta^A$ and $\Delta \tau = \Delta \tau_{AB} \theta^A \theta^B$ defined as

$$\Delta q_A = q_A - \frac{1}{8\pi G} {}^*c_A, \quad \Delta \tau_{AB} = \tau_{AB} + \frac{1}{8\pi G k^2} {}^*c_{AB}. \quad (2.54)$$

The functions ε , q_A and τ_{AB} , which merely parameterize the line element at this stage, can be packaged in a symmetric and traceless tensor T_{AB} as in (2.33), (2.36) and, as we shall see shortly in (2.60), Einstein's equations demand the conservation of T_{AB} , thus completing its identification as the boundary energy-momentum tensor as in the Fefferman-Graham gauge.

We now ought to focus on the $1/r^2$ contribution to the line element (2.48), i.e., on

$$f_{(2)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(2)A} \theta^A + f_{(2)AB} \theta^A \theta^B, \quad (2.55)$$

where $f_{(2)}$, $f_{(2)A}$ and $f_{(2)AB}$ must have conformal weight 4. The analogy with the Fefferman-Graham expansion suggests that no new free boundary functions should appear without spoiling Einstein's equations. Indeed, upon imposing (2.49) and (2.51), one finds

$$\begin{cases} \mathcal{E}_{rr} &= -\frac{3}{r^5} \eta_{AB} f_{(1)}^{AB} - 6 \left(\eta_{AB} f_{(2)}^{AB} + \frac{3}{2k^2} \sigma_{AB} f_{(1)}^{AB} \right) \frac{1}{r^6} + \mathcal{O} \left(\frac{1}{r^7} \right) \\ k \mathcal{E}_{r\hat{0}} &= \left(-f_{(2)} - 2k^2 \eta_{AB} f_{(2)}^{AB} + \frac{1}{2} h_{AB} \mathcal{D}^A f_{(1)}^B - \frac{5}{2} \sigma_{AB} f_{(1)}^{AB} + c\gamma \right) \frac{1}{r^4} + \mathcal{O} \left(\frac{1}{r^5} \right) \\ \mathcal{E}_{ra} &= \left(2f_{(2)a} - \frac{3}{2} h_{aB} \mathcal{D}_C f_{(1)}^{BC} + \frac{1}{k^2} (\sigma_{aB} + 4\omega_{aB}) f_{(1)}^B \right) \frac{1}{r^4} + \mathcal{O} \left(\frac{1}{r^5} \right) \\ \mathcal{E}^{ab} &= \left(-f_{(2)} h^{ab} + c\gamma h^{ab} + 4\omega_C {}^a f_{(1)}^{bC} + 2k^2 \hat{\eta}_C {}^a \hat{\eta}_D {}^b f_{(2)}^{CD} - 2u^C \mathcal{D}_C f_{(1)}^{ab} \right. \\ &\quad \left. + \hat{\eta}_C {}^a \hat{\eta}_D {}^b \mathcal{D}^{(C} f_{(1)}^{D)} + \frac{1}{k^2} (c \hat{\eta}_C {}^a \sigma^{Cb} - f_{(1)} \sigma^{ab}) + 4\sigma_C {}^a f_{(1)}^{bC} \right) \frac{1}{r^2} + \mathcal{O} \left(\frac{1}{r^3} \right) \end{cases} \quad (2.56)$$

for the often referred to as constraint Einstein's equations. These equations fix algebraically all terms at the $1/r^2$ order in the expansion of the bulk metric, thus confirming the absence of any new free function. When rewritten in terms of the basic

¹⁸One could not stress enough the profound versatility of the boundary Cotton tensor. Together with the boundary energy-momentum tensor, they control the asymptotic behaviour of the bulk Weyl tensor, the electric versus magnetic gravitational characteristics, the duality issues, and are natural ingredients in Newman-Penrose formalism. In the flat instance and in the current gauge, the Cotton tensor contributes to the gravitational radiation along with the Bondi shear. A recent presentation of some of these features is available in ref. [25].

quantities parameterizing the space of solutions, the three coefficients in (2.55) read:

$$f_{(2)} = \frac{8\pi G}{3k^2} \left(\sigma_{CD} \Delta \tau^{CD} + \mathcal{D}_C \Delta q^C \right) + c\gamma, \quad (2.57)$$

$$f_{(2)A} = -\frac{8\pi G}{3k^4} \sigma_{AC} \Delta q^C + \frac{4\pi G}{k^2} \left(h_{AC} \mathcal{D}_D \Delta \tau^{CD} + \frac{8}{3} \gamma * \Delta q_A \right), \quad (2.58)$$

$$f_{(2)AB} = -\frac{4\pi G}{k^4} \left(\frac{4}{3} u^C \mathcal{D}_C \Delta \tau_{AB} + \frac{2}{3} h_{AC} h_{BD} \mathcal{D}^{(C} \Delta q^{D)} - \frac{1}{3} h_{AB} h^{CD} \mathcal{D}_C \Delta q_D \right. \\ \left. + 2\sigma_{(A}^C \Delta \tau_{B)C} \right) - \frac{1}{2k^4} (8\pi G \varepsilon \sigma_{AB} - c * \sigma_{AB}) + \frac{32\pi G}{3k^2} \gamma * \Delta \tau_{AB}. \quad (2.59)$$

These expressions contain all possible combinations of the shear and of the vorticity together with adequately projected Weyl covariant derivatives of the energy-momentum and Cotton tensors,¹⁹ carrying the right tensorial structure and conformal weight. Substituting eqs. (2.57), (2.58), (2.59) into the remaining Einstein's equations (2.47) one obtains:

$$\frac{k}{8\pi G} \mathcal{E}_{\hat{0}\hat{0}} = \frac{1}{r^2} \mathcal{D}_B T^B_{\hat{0}} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad \frac{k}{8\pi G} \mathcal{E}_{\hat{0}a} = \frac{1}{r^2} \mathcal{D}_B T^B_a + \mathcal{O}\left(\frac{1}{r^3}\right) \quad (2.60)$$

(since T_{AB} is traceless, $\mathcal{D}_A \equiv \nabla_A$, the Levi-Civita boundary connection for the frame metric η_{AB}). The omitted terms contain the tensors $f_{(3)}$, $f_{(3)A}$ and $f_{(3)AB}$. This confirms that no additional constraints are imposed on the quantities parameterizing the solution space identified at the previous orders, i.e., the velocity field, the boundary metric (frame in the present formalism) and the boundary energy-momentum tensor.

Higher orders and possible resummation The above pattern can be repeated ad nauseam at the cost of a substantial growth in admissible terms. The third order would be interesting as it is expected to host the Newman-Penrose charges in the flat limit. This is beyond our motivations, but raises the issue of resummability under conditions of the series (2.7). This question is usually immaterial in Bondi or Newman-Unti gauges, where due to the absence of boundary vorticity²⁰ simple solutions such as Kerr's are embodied in the form of infinite series. In the covariant Newman-Unti gauge, the explicit appearance of the boundary Cotton tensor allows to tune the bulk Weyl tensor and select algebraically special Einstein spacetimes, for which the series is resumable. This is achieved by imposing

$$\sigma_{AB} = 0, \quad \Delta q_A = 0, \quad \Delta \tau_{AB} = 0, \quad (2.61)$$

which implies that

$$f_{(s)A} = 0, \quad f_{(s)AB} = 0 \quad (2.62)$$

¹⁹The covariant Newman-Unti gauge has often been referred to as the *derivative-expansion gauge* for this reason. This was borrowed from the original fluid/gravity literature, where the derivative expansion was inspired by the fluid constitutive relations.

²⁰An explicit realization of the boundary frame of Newman-Unti gauge is displayed in (A.41), (A.42) with $\Delta_i = 0$, as mentioned earlier in this section. The boundary vorticity always vanishes then as it is proven by comparing (A.45) with (A.46).

and

$$f_{(2s+1)} = (-)^s 8\pi G \varepsilon \gamma^{2s}, \quad f_{(2s+2)} = (-)^s c \gamma^{2s+1}. \quad (2.63)$$

The boundary metric is still a free variable, but only the energy density $\varepsilon(x)$ remains from the energy-momentum tensor, whose heat current and stress are fixed by those of the Cotton:

$$q_A = \frac{1}{8\pi G} *c_A, \quad \tau_{AB} = -\frac{1}{8\pi G k^2} *c_{AB}. \quad (2.64)$$

As a consequence, assuming that (2.1) is satisfied, one finds

$$ds_{\text{res. Einstein}}^2 = 2 \frac{u}{k^2} (dr + rA) + r^2 ds^2 + \frac{\mathcal{F}}{k^4} + \frac{u^2}{k^4 \rho^2} (8\pi G \varepsilon r + c\gamma) \quad (2.65)$$

with

$$\rho^2 = r^2 + \gamma^2 \quad (2.66)$$

and $\mathcal{F} = \mathcal{F}_{AB} \theta^A \theta^B$ given in (2.51) imposing zero shear. The Petrov analysis of (2.65) has been discussed in refs. [3, 20].

3 The flat avatars

First things first. Handling the flat limit is a triptych. At the first place stands the boundary geometry, which becomes Carrollian as the time-like conformal boundary of asymptotically anti-de Sitter spacetimes is traded for the null infinity of their asymptotically flat relatives. Secondly, the energy-momentum tensor should be expanded in Laurent series with respect to k^2 and embrace all extra degrees of freedom of the flat solution space. Finally comes the bulk line element that should remain finite in the zero- k limit, imposing to this end constraints and evolution equations on the functions defining the solution space, besides the Carrollian limit of the already available eqs. (2.1).

Given the relativistic boundary metric and the velocity field, (2.8) and (2.9), the starting point of our analysis is the bulk line element (2.48), which we reproduce here bearing in mind the transversality properties:

$$ds_{\text{bulk}}^2 = 2 \frac{u}{k^2} (dr + rA) + r^2 ds^2 + r \mathcal{C}_{ab} \theta^a \theta^b + \frac{1}{k^4} \mathcal{F}_{AB} \theta^A \theta^B + \sum_{s=1}^{\infty} \frac{1}{r^s} \left(f_{(s)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(s)a} \theta^a + f_{(s)ab} \theta^a \theta^b \right). \quad (3.1)$$

The Carrollian limit of the boundary geometry is reached as follows:²¹

$$\mu = \lim_{k \rightarrow 0} \frac{u}{k^2} = -\lim_{k \rightarrow 0} \frac{\theta^{\hat{0}}}{k}, \quad \hat{\theta}^a = \lim_{k \rightarrow 0} \theta^a, \quad (3.2)$$

so that the Carrollian degenerate metric spells

$$d\ell^2 = \lim_{k \rightarrow 0} ds^2 = \delta_{ab} \hat{\theta}^a \hat{\theta}^b. \quad (3.3)$$

²¹Carrollian quantities will often be distinguished with hats. However, in order to avoid cluttering of indices and symbols, we do make the distinction amongst relativistic and Carrollian attributes, only when it is necessary. This will not be the case e.g. for the Bondi shear and news.

For the frame vectors, the prescription is

$$\mathbf{v} = \lim_{k \rightarrow 0} \mathbf{u} = \lim_{k \rightarrow 0} k e_{\hat{0}}, \quad \hat{e}_a = \lim_{k \rightarrow 0} e_a. \quad (3.4)$$

It should be stressed that the limit may not be necessary, because the parameterization of the diads θ^a in terms of the natural-coframe components dx^μ can be chosen so as not to depend on k , and that $\hat{\theta}^0$ could simply be $dx^0 = k du$ in which case $\mu = -du$ (some further examples are displayed in appendix A, eqs. (A.3), (A.4), (A.7) and eqs. (A.41), (A.42), (A.43)). This will be definitely our viewpoint here.

The kernel of the degenerate metric (3.3) is the *field of observers* \mathbf{v} , and μ is its dual *clock form* embracing also the Ehresmann connection, as explained in appendix A. All these obey

$$\mu(\mathbf{v}) = -1, \quad \hat{\theta}^a(\hat{e}_b) = \delta_b^a, \quad \hat{\theta}^a(\mathbf{v}) = 0, \quad \mu(\hat{e}_a) = 0. \quad (3.5)$$

The Carrollian geometric data are part of the solution space of Ricci-flat spacetimes in the flat covariant Newman-Unti gauge. Compared to the standard flat Newman-Unti gauge, the extra piece of data is the clock form μ , which echoes the velocity congruence of the AdS relative. More accurately, the additional piece of information carried by the covariant Newman-Unti gauge is the boundary vorticity $*\varpi$, as discussed in appendix A, footnote 36.

The vanishing- k limit the AdS-boundary Weyl connection Λ is readily reached due to its k -independence. As described explicitly in appendix A, one effortlessly expresses Λ in Carrollian terms, eq. (A.53):

$$\Lambda = \varphi_a \hat{\theta}^a - \frac{\theta}{2} \mu \quad (3.6)$$

with φ_a and θ given in (A.13) or (A.46) and (A.19). Therefore, the first two terms in (3.1) have a well-defined limit without the need of imposing Einstein's equations.

The next term in (3.1) plays an essential role in gravitational physics. Indeed, Einstein's equation (2.49), reproduced here for the spatial components — the only non-zero due transversality combined with our choice of congruence \mathbf{u} ,

$$k^2 \mathcal{C}_{ab} = -2\sigma_{ab}, \quad (3.7)$$

implies that $\sigma_{ab} = 0$ at vanishing k . As explained in eq. (A.52), the latter translates in Carrollian terms into

$$\xi_{ab} = 0, \quad (3.8)$$

where ξ_{ab} is defined in (A.19) as the traceless component of the extrinsic curvature. On the one hand, the geometrical shear ξ_{ab} of the boundary Carrollian geometry *must vanish* — an extrinsic-curvature condition for the conformal null boundary. On the other hand, the dynamical shear \mathcal{C}_{ab} *is free* and carries information on the gravitational radiation. No equation will constrain it or make it evolve, but it will source the evolution of other degrees of freedom.

In summary, till the order r , the Ricci-flat bulk metric reads:

$$ds_{\text{Ricci-flat}}^2 \Big|_r = \mu \left[2dr + r \left(2\varphi_a \hat{\theta}^a - \theta \mu \right) \right] + r^2 d\ell^2 + r \mathcal{C}_{ab} \hat{\theta}^a \hat{\theta}^b, \quad (3.9)$$

where $\mathcal{C}_{ab}(u, \mathbf{x})$ is an arbitrary traceless Carrollian tensor, referred to as the Bondi shear (cf. the footnote 7). The Bondi news is another traceless Carrollian tensor obtained as the Carrollian limit of eq. (2.50):

$$\hat{\mathcal{N}}_{ab} = \hat{\mathcal{D}}_v \mathcal{C}_{ab}. \quad (3.10)$$

In order to pursue the study of the next orders, we must be careful with the zero- k limit. Both in the line element and in the conservation equations (2.1), the geometric shear $\sigma_{ab} = \xi_{ab}$ must be substituted by $-\frac{k^2}{2}\mathcal{C}_{ab}$ on account of eq. (3.7) before the limit is taken. Often this won't have any effect and the term in consideration will drop. Sometimes, however, due to the presence of negative powers of k , finite terms will survive or divergences will impose further requirements.

The first and simplest application of the rule just stated concerns the order-1 term $\frac{\mathcal{F}}{k^4} = \frac{1}{k^4} \mathcal{F}_{AB} \theta^A \theta^B$. Expressing (2.51) in Carrollian terms we find:

$$\begin{aligned} \frac{\mathcal{F}}{k^4} &= \frac{\xi^2}{k^4} d\ell^2 + \frac{1}{k^2} \left(3\xi^2 \mu^2 + 2\hat{\mathcal{D}}_b \xi^b{}_a \mu \hat{\theta}^a - 2 * \varpi * \xi_{ab} \hat{\theta}^a \hat{\theta}^b \right) \\ &\quad + * \varpi^2 d\ell^2 - 2 * \hat{\mathcal{D}}_a * \varpi \mu \hat{\theta}^a - \hat{\mathcal{K}} \mu^2 - 5k^2 * \varpi^2 \mu^2 \\ &= \left(\frac{\mathcal{C}^2}{4} + * \varpi^2 \right) d\ell^2 - \hat{\mathcal{K}} \mu^2 - \hat{\mathcal{D}}_b \mathcal{C}^b{}_a \mu \hat{\theta}^a - 2 * \hat{\mathcal{D}}_a * \varpi \mu \hat{\theta}^a \\ &\quad + * \varpi * \mathcal{C}_{ab} \hat{\theta}^a \hat{\theta}^b + k^2 \left(\frac{3}{4} \mathcal{C}^2 - 5 * \varpi^2 \right) \mu^2 \end{aligned} \quad (3.11)$$

with $\mathcal{C}^2 = \frac{1}{2} \mathcal{C}^{ab} \mathcal{C}_{ab}$ and quantities like ξ^2 , $*\varpi$, $\hat{\mathcal{K}}$ defined in appendix B. The asterisk stands for the relativistic congruence-transverse or Carrollian-basis duality introduced in eqs. (2.28), (2.29), (2.31), (2.32) or (B.1), (B.2). Some terms drop in the zero- k limit but no divergence occurs and *we are left with a piece in the line element, which now contains explicitly the Bondi shear*:

$$\lim_{k \rightarrow 0} \frac{\mathcal{F}}{k^4} = \left(\frac{\mathcal{C}^2}{4} + * \varpi^2 \right) d\ell^2 - \hat{\mathcal{K}} \mu^2 - \hat{\mathcal{D}}_b \mathcal{C}^b{}_a \mu \hat{\theta}^a - 2 * \hat{\mathcal{D}}_a * \varpi \mu \hat{\theta}^a + * \varpi * \mathcal{C}_{ab} \hat{\theta}^a \hat{\theta}^b. \quad (3.12)$$

Before moving on to the next order, which uncovers the method of expanding the anti-de Sitter energy-momentum tensor as a mean of reconstructing Ricci-flat spacetimes, it is fair to give credit to the authors of ref. [23], where the pioneering idea of substituting the Bondi for the geometric shear with the accompanying power of the cosmological constant was initiated.

Order $1/r$ and the advent of the energy-momentum tensor. Let us assume that in the course of the bulk flat limit, the boundary energy-momentum tensor is analytic in k^2 . It can thus be represented as a Laurent series about $k = 0$:

$$\varepsilon = \sum_{n \in \mathbb{Z}} k^{2n} \varepsilon_{(n)}, \quad (3.13)$$

$$q^a = \sum_{n \geq 2} \frac{\zeta_{(n)}^a}{k^{2n}} + \frac{\zeta^a}{k^2} + Q^a + k^2 \pi^a + \sum_{n \geq 2} k^{2n} \pi_{(n)}^a, \quad (3.14)$$

$$\tau^{ab} = - \sum_{n \geq 3} \frac{\zeta_{(n)}^{ab}}{k^{2n}} - \frac{\zeta^{ab}}{k^4} - \frac{\Sigma^{ab}}{k^2} - \Xi^{ab} - k^2 E^{ab} - \sum_{n \geq 2} k^{2n} E_{(n)}^{ab}. \quad (3.15)$$

Each function in these series (some have been singled out for reasons that will be clarified later) is a Carrollian tensor (scalar, vector, or symmetric and traceless two-tensor) that is possibly one of the boundary degrees of freedom, which we call *Chthonian* to recall they encode the asymptotically flat Einstein dynamics probing the bulk metric *in depth* from the boundary. These tensors are expected to obey flux-balance equations, which are Carrollian avatars of vacuum Einstein's equations, and that we will attain using anti-de Sitter dynamics and imposing a regular behaviour at zero k .

As an introductory statement, it is important to stress that we have no proof for the proclaimed analyticity. The latter is a working framework, resulting in a consistent description of asymptotically flat spacetimes, and this end justifies the means. The rules are simple: insert (3.13), (3.14), (3.15) in the line element (3.1) at each order, and impose regularity at $k = 0$ after trading ξ_{ab} for $-\frac{k^2}{2}\mathcal{C}_{ab}$. This process starts with $1/r$, since this is the first term sensitive to the energy-momentum tensor, but the substitution of \mathcal{C}_{ab} will be performed systematically, in the line element, in Einstein's equations, or in the further definition of the complex mass aspect, without raising any order ambiguity.

At order $1/r$ we should probe (2.53), which spells

$$f_{(1)}\frac{u^2}{k^4} + 2\frac{u}{k^2}f_{(1)a}\theta^a + f_{(1)ab}\theta^a\theta^b = 8\pi G \left(\varepsilon\mu^2 + \frac{4}{3}\mu\frac{\Delta q_a}{k^2}\hat{\theta}^a + \frac{2}{3}\frac{\Delta\tau_{ab}}{k^2}\hat{\theta}^a\hat{\theta}^b \right) \quad (3.16)$$

with ε given in (3.13) and

$$\begin{aligned} \frac{\Delta q^a}{k^2} &= \sum_{n \geq 2} \frac{\zeta_{(n)}^a}{k^{2n+2}} + \frac{1}{k^4} \left(\zeta^a - \frac{*z^a}{8\pi G} \right) + \frac{1}{k^2} \left(Q^a - \frac{*X^a}{8\pi G} \right) + \left(\pi^a - \frac{*v^a}{8\pi G} \right) \\ &\quad + \sum_{n \geq 2} k^{2n-2} \pi_{(n)}^a, \end{aligned} \quad (3.17)$$

$$\begin{aligned} \frac{\Delta\tau^{ab}}{k^2} &= - \sum_{n \geq 3} \frac{\zeta_{(n)}^{ab}}{k^{2n+2}} - \frac{1}{k^6} \left(\zeta^{ab} - \frac{*Z^{ab}}{8\pi G} \right) - \frac{1}{k^4} \left(\Sigma^{ab} - \frac{*X^{ab}}{8\pi G} \right) - \frac{1}{k^2} \left(\Xi^{ab} - \frac{*Psi^{ab}}{8\pi G} \right) \\ &\quad - E^{ab} - \sum_{n \geq 2} k^{2n-2} E_{(n)}^{ab}, \end{aligned} \quad (3.18)$$

where we have used (3.14), (3.15), the definitions (2.54) of Δq^a and $\Delta\tau^{ab}$, as well as the Carrollian Cotton tensors z^a , χ^a , ψ^a , Z^{ab} , X^{ab} , Ψ^{ab} displayed in (B.13), (B.14). Finiteness in the flat limit sets up two sorts of requirements on the Carrollian descendants of the energy-momentum tensor.

- Infinite subsets of Laurent coefficients are required to vanish:

$$\begin{cases} \varepsilon_{(n)} = 0 & \forall n < 0 \\ \zeta_{(n)}^a = 0 & \forall n \geq 2 \\ \zeta_{(n)}^{ab} = 0 & \forall n \geq 3; \end{cases} \quad (3.19)$$

- Five Laurent coefficients are locked in terms of the Carroll Cotton tensors defined in (B.17), (B.18), (B.20), (B.21), (B.22):

$$\zeta^a = \frac{*z^a}{8\pi G}, \quad Q^a = \frac{*X^a}{8\pi G}, \quad \zeta^{ab} = \frac{*Z^{ab}}{8\pi G}, \quad \Sigma^{ab} = \frac{*X^{ab}}{8\pi G}, \quad \Xi^{ab} = \frac{*Psi^{ab}}{8\pi G}. \quad (3.20)$$

Hence a finite subset of energy-momentum Carrollian descendants *are not independent but are instead of geometric nature, determined by the boundary Carroll structure via its Cotton tensor.*

No more constraints show on the Chthonian degrees of freedom at $1/r$ order.

Defining

$$N^a = *\psi^a - 8\pi G\pi^a, \tag{3.21}$$

we recast the order- $1/r$ term (3.16) in the flat limit as:

$$\begin{aligned} \lim_{k \rightarrow 0} \left(f_{(1)} \frac{u^2}{k^4} + 2 \frac{u}{k^2} f_{(1)a} \theta^a + f_{(1)ab} \theta^a \theta^b \right) &= 8\pi G \varepsilon_{(0)} \mu^2 - \frac{4}{3} \mu N_a \hat{\theta}^a - \frac{16\pi G}{3} E_{ab} \hat{\theta}^a \hat{\theta}^b \\ &\equiv \hat{f}_{(1)} \mu^2 + 2\mu \hat{f}_{(1)a} \hat{\theta}^a + \hat{f}_{(1)ab} \hat{\theta}^a \hat{\theta}^b. \end{aligned} \tag{3.22}$$

The latter expression calls for two remarks. Firstly, the Carrollian tensors $\varepsilon_{(n \geq 1)}$, $\pi_{(n \geq 2)}^a$ and $E_{(n \geq 2)}^{ab}$ are absent. We should refrain from interpreting this as a sign that those aren't genuine degrees of freedom. Some of them ought to appear in the line element in the next orders and therefore participate in the dynamics. Only when one is guaranteed that a Laurent coefficient is absent from the line element at any order, can we declare it is irrelevant and set it consistently to zero. The order- $1/r^2$ analysis will significantly underpin this statement.

Secondly comes an important question: *what is the dynamics of the boundary degrees of freedom $\varepsilon_{(0)}$, N_a and E_{ab} that remain in the $1/r$ term of the bulk line element?* Ensuing our philosophy, this dynamics is encoded (i) in the zero- k limit of anti-de Sitter Einstein's equations and (ii) in the finiteness requirement of the line element. The latter has already been exploited at the order under consideration, while the former is the energy-momentum conservation (2.1) on which we will elaborate now. Our treatment consists in the four steps summarized below.

1. In the frame at use, we consider the relativistic energy-momentum tensor conservation equations (2.1) recast in Carrollian terms as in appendix A, eqs. (A.59) and (A.60), which we redisplay here for convenience:

$$\mathcal{L} = \hat{\mathcal{D}}_v \varepsilon + \hat{\mathcal{D}}_a q^a + \xi_{ab} \tau^{ab} = 0, \tag{3.23}$$

$$\mathcal{T}^a = \frac{1}{d} \hat{\mathcal{D}}^a \varepsilon + \hat{\mathcal{D}}_b \tau^{ab} + 2q_b \varpi^{ba} + \frac{1}{k^2} \left(\hat{\mathcal{D}}_v q^a + \xi^{ab} q_b \right) = 0. \tag{3.24}$$

2. We insert in these equations the variables ε , q^a and τ^{ab} in their expanded forms (3.13), (3.14) and (3.15), taking into account the finiteness requirements (3.19) and (3.20).
3. The requirements (3.19) and (3.20) bring the Cotton tensor inside the boundary energy-momentum conservation equations $\mathcal{L} = 0$ and $\mathcal{T}^a = 0$. At this stage we must exploit the Cotton identities $\{\mathcal{D}_{\text{Cot}} = 0, \mathcal{I}_{\text{Cot}}^a = 0\}$, $\{\mathcal{E}_{\text{Cot}} = 0, \mathcal{G}_{\text{Cot}}^a = 0\}$, $\{\mathcal{F}_{\text{Cot}} = 0, \mathcal{H}_{\text{Cot}}^a = 0\}$ and $\{\mathcal{W}_{\text{Cot}} = 0, \mathcal{X}_{\text{Cot}}^a = 0\}$ set in appendix B — see

eqs. (B.25), (B.26), (B.27), (B.28), (B.29), (B.30), (B.31), (B.32) — and recast for our needs as:

$$\hat{\mathcal{D}}_b * \Psi^{ab} + 2 * \varpi \chi^a = \hat{\mathcal{D}}_v * \psi^a + \frac{1}{2} * \hat{\mathcal{D}}^a c_{(0)} - * \psi_b \xi^{ab}, \quad (3.25)$$

$$\hat{\mathcal{D}}_b * X^{ab} + 2 * \varpi z^a - \hat{\mathcal{D}}_v * \chi^a = \frac{1}{2} * \hat{\mathcal{D}}^a c_{(1)} - * \chi_b \xi^{ab}, \quad (3.26)$$

$$\hat{\mathcal{D}}_b * Z^{ab} - \hat{\mathcal{D}}_v * z^a = \frac{1}{2} * \hat{\mathcal{D}}^a c_{(2)} - * z_b \xi^{ab}. \quad (3.27)$$

With this we reach the following:

$$\begin{aligned} \mathcal{L} = & k^2 \hat{\mathcal{D}}_a \pi^a + \sum_{n \geq 2} k^{2n} \hat{\mathcal{D}}_a \pi_{(n)}^a - \xi_{ab} \left(k^2 E^{ab} + \sum_{n \geq 2} k^{2n} E_{(n)}^{ab} \right) \\ & + \hat{\mathcal{D}}_v \varepsilon_{(0)} + \sum_{n \geq 1} k^{2n} \hat{\mathcal{D}}_v \varepsilon_{(n)} - \frac{1}{8\pi G} \left(* \Psi^{ab} \xi_{ab} - \hat{\mathcal{D}}_a * \chi^a \right) \\ & - \frac{1}{8\pi G k^2} \left(* X^{ab} \xi_{ab} - \hat{\mathcal{D}}_a * z^a \right) - \frac{1}{8\pi G k^4} * Z^{ab} \xi_{ab}, \end{aligned} \quad (3.28)$$

and

$$\begin{aligned} \mathcal{T}^a = & -\hat{\mathcal{D}}_b \left(k^2 E^{ab} + \sum_{n \geq 2} k^{2n} E_{(n)}^{ab} \right) + 2 * \varpi \left(k^2 * \pi^a + \sum_{n \geq 2} k^{2n} * \pi_{(n)}^a \right) + \frac{1}{2} \hat{\mathcal{D}}^a \varepsilon_{(0)} \\ & + \frac{1}{2} \sum_{n \geq 1} k^{2n} \hat{\mathcal{D}}^a \varepsilon_{(n)} + \hat{\mathcal{D}}_v \left(\pi^a + \sum_{n \geq 2} k^{2n-2} \pi_{(n)}^a \right) + \xi^a_b \left(\pi^b + \sum_{n \geq 2} k^{2n-2} \pi_{(n)}^b \right) \\ & - \frac{1}{8\pi G} \left(\hat{\mathcal{D}}_v * \psi^a + \frac{1}{2} * \hat{\mathcal{D}}^a c_{(0)} - * \psi_b \xi^{ab} \right) \\ & - \frac{1}{8\pi G k^2} \left(\frac{1}{2} * \hat{\mathcal{D}}^a c_{(1)} - * \chi_b \xi^{ab} \right) - \frac{1}{8\pi G k^4} \left(\frac{1}{2} * \hat{\mathcal{D}}^a c_{(2)} - * z_b \xi^{ab} \right). \end{aligned} \quad (3.29)$$

4. Lastly we express the geometric Carrollian shear as $\xi_{ab} = -\frac{k^2}{2} \mathcal{C}_{ab}$ inside eqs. (3.28) and (3.29). This is a juggernaut due to the heavy presence of ξ_{ab} in the Carrollian Cotton tensors $c_{(1)}$, $c_{(2)}$, χ^a , z^a , Ψ^{ab} , X^{ab} and Z^{ab} — see their definitions (B.15), (B.17), (B.18), (B.20), (B.21) and (B.22). This operation regularizes the otherwise singular behaviour of the last lines in (3.28) and (3.29) at vanishing k , which instead produce a wealth of finite terms, *all rooted in the Carrollian Cotton tensor*.

The flat limit of the boundary energy-momentum conservation can now be safely taken and yields:

$$\lim_{k \rightarrow 0} \mathcal{L} = \hat{\mathcal{D}}_v \varepsilon_{(0)} + \frac{1}{8\pi G} \hat{\mathcal{D}}_a * \chi^a - \frac{1}{16\pi G} \left(\hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \hat{\mathcal{N}}^{ab} + \mathcal{C}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b + \frac{1}{2} \mathcal{C}_{ab} \hat{\mathcal{D}}_v \hat{\mathcal{N}}^{ab} \right), \quad (3.30)$$

and

$$\begin{aligned}
\lim_{k \rightarrow 0} \mathcal{T}^a &= \frac{1}{2} \hat{\mathcal{D}}_b \left(\delta^{ab} \varepsilon_{(0)} + \frac{1}{8\pi G} \eta^{ab} c_{(0)} \right) + \hat{\mathcal{D}}_v \left(\pi^a - \frac{1}{8\pi G} * \psi^a \right) \\
&+ \frac{1}{16\pi G} \left[\mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + * \mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{A}} - 4 * \varpi * \mathcal{E}^{ab} \hat{\mathcal{R}}_b \right. \\
&- \frac{1}{2} \hat{\mathcal{D}}^b \left(\hat{\mathcal{D}}_b \hat{\mathcal{D}}_c \mathcal{E}^{ac} - \hat{\mathcal{D}}^a \hat{\mathcal{D}}^c \mathcal{E}_{bc} \right) \\
&\left. + \mathcal{E}^{ab} \hat{\mathcal{D}}^c \hat{\mathcal{N}}_{bc} + \frac{1}{2} \hat{\mathcal{D}}^b \left(\mathcal{E}^{ac} \hat{\mathcal{N}}_{bc} \right) - \frac{1}{4} \hat{\mathcal{D}}^a \left(\mathcal{E}^{bc} \hat{\mathcal{N}}_{bc} \right) \right]. \quad (3.31)
\end{aligned}$$

Equations (3.30) and (3.31) are one of our main achievements and deserve further discussion. We would like to insist that there is neither magic nor ambiguity in reaching them. We have followed a plain zero- k limit informed about the regularity conditions (3.20), which involve the Carrollian Cotton tensor and its identities, and instructed with Einstein's equation $\xi_{ab} = -\frac{k^2}{2} \mathcal{C}_{ab}$. Although long and technical, the method reveals the central role of the Cotton tensor: *all terms responsible for the gravitational radiation, involving among others the shear and the news tensors, originate from the Carrollian Cotton tensors*. Because of the vanishing ξ_{ab} , only six of those remain — see appendix B: $c_{(-1)}$, $c_{(0)}$, ψ^a , eqs. (B.15), (B.16), and χ^a , Ψ^{ab} , X^{ab} given in (B.35), (B.36), (B.37) for vanishing Carrollian shear. They obey eqs. (B.38), (B.39), (B.40), (B.41) and (B.42).²² This means in particular that once the bulk flat limit is reached i.e. the boundary Carroll structure has no geometric shear, z^a and Z^{ab} vanish. The Carrollian energy-momentum tensors ζ^a and ζ^{ab} do also vanish by virtue of (3.20). Only Q^a , Σ^{ab} and Ξ^{ab} survive, and (3.30), (3.31) lead to an alternative writing of the conservation equations:

$$\hat{\mathcal{D}}_v \varepsilon_{(0)} + \hat{\mathcal{D}}_a Q^a = \frac{1}{16\pi G} \left(\hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \hat{\mathcal{N}}^{ab} + \mathcal{E}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{R}}_b + \frac{1}{2} \mathcal{E}_{ab} \hat{\mathcal{D}}_v \hat{\mathcal{N}}^{ab} \right) \quad (3.32)$$

and

$$\begin{aligned}
\frac{1}{2} \hat{\mathcal{D}}^a \varepsilon_{(0)} - \hat{\mathcal{D}}_b \Xi^{ab} + 2 * \varpi * Q^a + \hat{\mathcal{D}}_v \pi^a &= -\frac{1}{16\pi G} \left[\mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + * \mathcal{E}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{A}} - 4 * \varpi * \mathcal{E}^{ab} \hat{\mathcal{R}}_b \right. \\
&- \frac{1}{2} \hat{\mathcal{D}}^b \left(\hat{\mathcal{D}}_b \hat{\mathcal{D}}_c \mathcal{E}^{ac} - \hat{\mathcal{D}}^a \hat{\mathcal{D}}^c \mathcal{E}_{bc} \right) + \mathcal{E}^{ab} \hat{\mathcal{D}}^c \hat{\mathcal{N}}_{bc} \\
&\left. + \frac{1}{2} \hat{\mathcal{D}}^b \left(\mathcal{E}^{ac} \hat{\mathcal{N}}_{bc} \right) - \frac{1}{4} \hat{\mathcal{D}}^a \left(\mathcal{E}^{bc} \hat{\mathcal{N}}_{bc} \right) \right]. \quad (3.33)
\end{aligned}$$

This latter form discloses a Carrollian conservation of the type (B.33), (B.34) *with a right-hand side though*. This is thus a *flux-balance equation*, where the source is maintained by the bulk gravitational radiation encoded in the shear and the news. Notice that the above momentum π^a coincides with P^a in (B.34) and is dynamical, whereas the traceless

²²The Carrollian Cotton identities described in appendix B as $\{\mathcal{D}_{\text{Cot}} = 0, \mathcal{I}_{\text{Cot}}^a = 0\}$, $\{\mathcal{E}_{\text{Cot}} = 0, \mathcal{G}_{\text{Cot}}^a = 0\}$, $\{\mathcal{F}_{\text{Cot}} = 0, \mathcal{H}_{\text{Cot}}^a = 0\}$ and $\{\mathcal{W}_{\text{Cot}} = 0, \mathcal{X}_{\text{Cot}}^a = 0\}$ are obtained from the relativistic conservation (2.39) expressed as (B.23) and (B.24). We *must not* insert in these equations $\xi_{ab} = -\frac{k^2}{2} \mathcal{C}_{ab}$ before taking the flat limit as they are agnostic about bulk Einstein's equations. The Cotton identities at hand are of boundary-geometric nature, and when the Carrollian shear vanishes, they just become simpler by setting $\xi_{ab} = 0$.

Carrollian stress Ξ^{ab} is $-\Upsilon^{ab}$ in (B.33), (B.34), and is dictated by the Cotton due to (3.20); similarly Q^a here is the energy flux Π^a of (B.33), (B.34), also locked by the Cotton in (3.20).

Even in absence of Bondi shear \mathcal{C}_{ab} and news $\hat{\mathcal{N}}_{ab}$, the presence of a non-vanishing energy flux betrays the breaking of local Carroll boost invariance (see the end of appendix B) in the boundary Carrollian dynamics associated with Ricci-flat spacetimes. This breaking accounts for bulk gravitational radiation, which in the covariant Newman-Unti gauge does not originate solely in the news (3.10) but is also carried by the Carrollian energy flux $\Pi^a = Q^a = \frac{1}{8\pi G} * \chi^a$. In Robinson-Trautman spacetimes and in the present gauge, the gravitational radiation is exclusively rooted in this Carrollian Cotton descendant — see ref. [3].

Observe in passing the Carrollian Cotton identities (B.39) and (B.41), which we replicate here for convenience:

$$\hat{\mathcal{D}}_b c_{(0)} + \hat{\mathcal{D}}_a \chi^a = 0, \tag{3.34}$$

$$\frac{1}{2} \hat{\mathcal{D}}^a c_{(0)} - \hat{\mathcal{D}}_b \Psi^{ab} + 2 * \varpi * \chi^a + \hat{\mathcal{D}}_b \psi^a = 0. \tag{3.35}$$

They play dual roles with respect to eqs. (3.32) and (3.33), because the energy density $\varepsilon_{(0)}$ carries information on the mass of the source, while $c_{(0)}$ endorses its nut charge (monopole-like magnetic mass) (see e.g. [25] for a recent discussion on these electric-magnetic dual observables). The two sets of equations are dissymmetric though: eq. (3.34) for instance is driven exclusively by the Cotton vector χ^a — as opposed to its Carroll-dual $*\chi^a$ entering the electric-mass equation (3.32) through $Q^a = \frac{1}{8\pi G} * \chi^a$. Even though loss phenomena concern both the electric and the magnetic masses, as captured e.g. in eqs. (76) and (80) of [47] — see also appendix D of [48], the time evolution of the nut is not affected by \mathcal{C}_{ab} and $\hat{\mathcal{N}}_{ab}$, whereas that of the mass is, in line with an important distinction between these aspects raised in [49].

A useful exercise, which we will not undertake here, would be to set up a precise dictionary between the gauge at hand and the more conventional Newman-Unti or Bondi gauges, regarding the radiation observables. We can nonetheless take a few steps towards this end using the Carrollian tensor N^a introduced in (3.21), reminiscent of the Bondi *angular-momentum aspect*,²³ and a Bondi *mass aspect*

$$M = 4\pi G \varepsilon_{(0)} - \frac{1}{8} \mathcal{C}^{ab} \hat{\mathcal{N}}_{ab}. \tag{3.36}$$

This definition is reached from eq. (2.39) of [23] valid in anti-de Sitter, at $k = 0$.²⁴ What distinguishes the energy density $4\pi G \varepsilon_{(0)}$ and the mass M is a radiative contribution.

We can attempt to define a magnetic-mass aspect starting from anti-de Sitter, where the behaviour of the bulk Weyl tensor in the gauge used here exhibits the complex-mass

²³As for the shear and the news, the physics conveyed by N^a in the covariant Newman-Unti gauge, is slightly different compared to the standard angular-momentum aspect. For the Kerr geometry, as an example, in the gauge at hand $N^a = 0$ and the angular momentum is carried by the Carrollian vorticity, as opposed to plain Newman-Unti gauge, where the boundary vorticity is absent (see eq. (A.20)). This hints towards the recent progress in defining a supertranslation-invariant angular momentum and comparing the multiple routes to it (see e.g. [42, 50–53]).

²⁴It coincides with (42) of [47] upon identifying \mathcal{M} of this reference with our $4\pi G \varepsilon_{(0)}$.

combination $\tau = -c + 8\pi i G \varepsilon$ (see [22]) with ε the AdS-boundary energy density and c the Cotton scalar (longitudinal component with respect to the congruence u). We thus define the *complex mass aspect* of Ricci-flat spacetimes in covariant Newman-Unti gauge as

$$\hat{\tau} = \lim_{k \rightarrow 0} \tau = -2\nu + 8\pi i G \varepsilon_{(0)}, \quad (3.37)$$

where

$$\nu = \frac{1}{2} \lim_{k \rightarrow 0} c = \frac{1}{2} c_{(0)} - \frac{1}{4} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \mathcal{C}^{ab} - \frac{1}{8} \mathcal{C}_{ab} * \hat{\mathcal{N}}^{ab} \quad (3.38)$$

is the *magnetic-mass aspect* reached using (B.12) and (B.15) upon substitution of $\xi_{ab} = -\frac{k^2}{2} \mathcal{C}_{ab}$. Subtracting the radiative contribution as in (3.36), we define the *nut aspect*

$$N = \nu + \frac{1}{8} \mathcal{C}_{ab} * \hat{\mathcal{N}}^{ab} = \frac{1}{2} c_{(0)} - \frac{1}{4} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \mathcal{C}^{ab}, \quad (3.39)$$

where $c_{(0)} = (\hat{\mathcal{D}}_a \hat{\mathcal{D}}^a + 2\hat{\mathcal{K}}) * \varpi$ is one of the four Carroll Cotton scalars displayed in (B.15).²⁵ Following the case of asymptotically AdS spacetimes quoted earlier, the behaviour of the bulk Weyl tensor in the Ricci-flat instance does also depend on the complex mass aspect $\hat{\tau}$, and we find indeed

$$\Psi_2 = \frac{i\hat{\tau}}{2r^3} + \mathcal{O}(1/r^4). \quad (3.40)$$

The higher-order missing terms in (3.40) are absent in the resumable, algebraically special solutions discussed in refs. [20, 22, 25]. Unsurprisingly, this expression coincides with eq. (68c) of [47].

With the above definitions, eqs. (3.34), (3.32) and (3.33) become:²⁶

$$\hat{\mathcal{D}}_\nu N = -\frac{1}{2} \hat{\mathcal{D}}_a \chi^a - \frac{1}{4} \left(\hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \hat{\mathcal{N}}^{ab} - * \mathcal{C}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \right), \quad (3.41)$$

$$\hat{\mathcal{D}}_\nu M = -\frac{1}{2} \hat{\mathcal{D}}_a * \chi^a + \frac{1}{4} \left(\hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \hat{\mathcal{N}}^{ab} + \mathcal{C}^{ab} \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b - \frac{1}{2} \hat{\mathcal{N}}_{ab} \hat{\mathcal{N}}^{ab} \right), \quad (3.42)$$

$$\begin{aligned} \hat{\mathcal{D}}_\nu N^a - \hat{\mathcal{D}}^a M + * \hat{\mathcal{D}}^a N &= \frac{1}{2} \left[\mathcal{C}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + * \mathcal{C}^{ab} \hat{\mathcal{D}}_b \hat{\mathcal{A}} - 4 * \varpi * \mathcal{C}^{ab} \hat{\mathcal{D}}_b - \frac{1}{2} * \hat{\mathcal{D}}^a \hat{\mathcal{D}}_b \hat{\mathcal{D}}_c * \mathcal{C}^{bc} \right. \\ &\quad \left. - \frac{1}{2} \hat{\mathcal{D}}^b \left(\hat{\mathcal{D}}_b \hat{\mathcal{D}}_c \mathcal{C}^{ac} - \hat{\mathcal{D}}^a \hat{\mathcal{D}}^c \mathcal{C}_{bc} \right) + \mathcal{C}^{ab} \hat{\mathcal{D}}^c \hat{\mathcal{N}}_{bc} + \frac{1}{2} \hat{\mathcal{D}}^b \left(\mathcal{C}^{ac} \hat{\mathcal{N}}_{bc} \right) \right]. \end{aligned} \quad (3.43)$$

The first equation phrases the loss process of the nut aspect sustained by the Carroll-dual news $*\hat{\mathcal{N}}_{ab}$ and the Carroll Cotton current χ^a . It is actually a *geometric identity* associated with the Carroll structure — as is (3.35), which could have been reexpressed as well in terms of the nut aspect. The last two flux-balance equations (3.42) and (3.43) for the electric-mass and angular-momentum aspects are *genuinely dynamical* and coincide with eqs. (2.53) and (2.50) of ref. [23], where the approach to asymptotic flatness via a limit

²⁵Our definitions for ν and N match with $-\tilde{\mathcal{M}}$ and $-\tilde{M}$ of [47], eqs. (53) and (55), for $c_{(0)} = 0$ (no magnetic monopole mass). This condition pertains to the use of the Bondi gauge in the quoted reference, where no Ehresmann connection exists and thus $*\varpi$ vanishes (as φ_a).

²⁶All these computations call for abundant use of the Weyl-covariant-derivative commutators presented in the appendix, eqs. (A.31), (A.32), (A.33), (A.36), (A.37) and (A.38).

of vanishing cosmological constant was proposed, or else with (4.50) and (4.49) of [54], obtained in a plain Ricci-flat context.²⁷

Order $1/r^2$ and the next flux-balance equation. The Carrollian symmetric and traceless two-tensor E_{ab} , descendant of the AdS-boundary stress, enters the line element at order $1/r$. However, the fundamental Carrollian energy-momentum conservation equations (3.30) and (3.31) fail to capture its dynamics. In a direct search of Ricci-flat spacetimes, Einstein's equations bring their share at each order and this is how the flux-balance equations emerge for the Chthonian degrees of freedom as E_{ab} . In the present method, Einstein's equations have already been imposed at the considered order. The bulk metric including the term (2.55) with the $f_{(2)}$ s as in (2.57) (2.58) and (2.59) is thus on-shell — assuming (2.1) is satisfied. However, this term is due to exhibit divergences at vanishing k . Removing them will impose conditions involving the Chthonian degrees of freedom as well as their longitudinal derivatives appearing explicitly in (2.59). This is how flat *flux-balance equations are recovered in the transition from anti-de Sitter to asymptotically flat spacetimes*, and this is another laudable achievement of this note.

The protocol is by now well established: we ought to follow the four steps enumerated earlier, starting with any tensor $f_{(2)}$ — and later on with other $f_{(s)}$. Let us open the study with the scalar contribution $f_{(2)}$, eq. (2.57). With little effort we find:

$$\lim_{k \rightarrow 0} f_{(2)} = 2 * \varpi \nu - \frac{1}{3} \hat{\mathcal{D}}_a N^a \equiv \hat{f}_{(2)}. \quad (3.44)$$

Next we consider the transverse vector $f_{(2)a} \theta^a$ in (2.58):

$$\lim_{k \rightarrow 0} f_{(2)a} = -\frac{1}{6} N^b \mathcal{C}_{ba} - \frac{4}{3} * \varpi * N_a - 4\pi G \hat{\mathcal{D}}_b E^b{}_a \equiv \hat{f}_{(2)a}. \quad (3.45)$$

Neither the limit (3.44) nor (3.45) introduce any new Chthonian degree of freedom or impose any further condition on their evolution. As we will now see, the situation is different for the transverse tensor (2.59) $f_{(2)ab} \theta^a \theta^b$. Using the numerous tools developed in this work, we find:²⁸

$$\begin{aligned} f_{(2)ab} = & \frac{1}{k^2} \left(\frac{16\pi G}{3} \hat{\mathcal{D}}_\nu E_{ab} + \frac{1}{3} \hat{\mathcal{D}}_{\langle a} N_{b \rangle} + 2\pi G \varepsilon_{(0)} \mathcal{C}_{ab} - \frac{\nu}{2} * \mathcal{C}_{ab} \right) \\ & + 2\pi G \left(\frac{8}{3} \hat{\mathcal{D}}_\nu E_{(2)ab} - \frac{4}{3} \hat{\mathcal{D}}_{\langle a} \pi_{(2)b \rangle} + \varepsilon_{(1)} \mathcal{C}_{ab} - 2\mathcal{C}_{(a}{}^c E_{b)c} \right) - 2 * \varpi^3 * \mathcal{C}_{ab} \\ & + \mathcal{O}(k^2). \end{aligned} \quad (3.46)$$

This result meets our expectations and allows us to draw significant conclusions.

²⁷In the quoted section 2.5 of [23] $\mu = -du$ so that $\varphi_a = \varpi_{ab} = 0$ (Bondi gauge with $\exp 2\beta_0 = 1$). Furthermore our definition of N^a is slightly different: $N_{\text{here}}^a = N_{\text{there}}^a + \frac{1}{4} (\mathcal{C}^{ab} \hat{\nabla}^c \mathcal{C}_{bc} + \frac{3}{8} \hat{\nabla}^a (\mathcal{C}^{bc} \mathcal{C}_{bc}))$ with $\hat{\nabla}^c$ being actually the ordinary two-dimensional Levi-Civita connection due to the absence of Ehresmann connection in [23] (see eq. (A.29) where the Carroll-Weyl covariant derivative reduces to the ordinary one when φ_a vanishes). This definition is in line with that of [55]. Likewise $\mathcal{N}_{\text{here}}^{ab} = N_{\text{TF there}}^{ab} - \frac{l_{\text{there}}}{2} \mathcal{C}^{ab}$ with $l_{\text{there}} = \theta_{\text{here}}$ and for further use we also quote that $E_{\text{here}}^{ab} = -\frac{3}{16\pi G} (\mathcal{E}_{\text{there}}^{ab} - \frac{1}{16} \mathcal{C}^{ab} \mathcal{C}^{cd} \mathcal{C}_{cd})$. The comparison with ref. [54] is reviewed in [23].

²⁸We define the symmetric and traceless part of a Carrollian two-tensor s_{ab} as $s_{(ab)} = s_{(ab)} - \frac{1}{d} s_c{}^c \delta_{ab}$ (here $d = 2$).

- The flat limit is singular unless the order- $1/k^2$ contribution to $f_{(2)ab}$ is absent i.e.

$$\hat{\mathcal{D}}_{\nu} E_{ab} = \frac{3}{16\pi G} \left(-\frac{1}{3} \hat{\mathcal{D}}_{\langle a} N_{b \rangle} - 2\pi G \varepsilon_{(0)} \mathcal{C}_{ab} + \frac{\nu}{2} * \mathcal{C}_{ab} \right), \quad (3.47)$$

which is the sought-after Carrollian *flux-balance equation for E_{ab}* , later referred to as $\text{FBE}_{(1)} = 0$. This equation matches with eq. (4e) of [47].²⁹

- Assuming eq. (3.47) is fulfilled, the limit can be taken

$$\begin{aligned} \lim_{k \rightarrow 0} f_{(2)ab} &= \frac{16\pi G}{3} \left(\hat{\mathcal{D}}_{\nu} E_{(2)ab} - \frac{1}{2} \hat{\mathcal{D}}_{\langle a} \pi_{(2)b \rangle} + \frac{3}{8} \varepsilon_{(1)} \mathcal{C}_{ab} - \frac{3}{4} \mathcal{C}_{(a} {}^c E_{b)c} \right) - 2 * \varpi^3 * \mathcal{C}_{ab} \\ &\equiv \hat{f}_{(2)ab}, \end{aligned} \quad (3.48)$$

and provides the last piece of the order- $1/r^2$ term in the Ricci-flat line element.

- New Chthonian degrees of freedom enter the bulk metric at this order: $E_{(2)ab}$, $\pi_{(2)a}$ and $\varepsilon_{(1)}$ in the form of a symmetric and traceless Carrollian tensor

$$F_{ab} = \hat{\mathcal{D}}_{\nu} E_{(2)ab} - \frac{1}{2} \hat{\mathcal{D}}_{\langle a} \pi_{(2)b \rangle} + \frac{3}{8} \varepsilon_{(1)} \mathcal{C}_{ab} - \frac{3}{8\pi G} * \varpi^3 * \mathcal{C}_{ab}. \quad (3.49)$$

Their dynamics is unknown at this stage but will be unravelled in the course of the analysis at order $1/r^3$.

We will close this paragraph exhibiting the explicit Ricci flat metric at the considered order. To this end we use the results (3.1), (3.9), (3.12), (3.22), (3.44), (3.45), (3.48) and (3.49):

$$\begin{aligned} ds_{\text{Ricci-flat}}^2 &= \mu \left[2dr + \left(2r\varphi_a - 2 * \hat{\mathcal{D}}_a * \varpi - \hat{\mathcal{D}}_b \mathcal{C}_a^b \right) \hat{\theta}^a - \left(r\theta + \hat{\mathcal{K}} \right) \mu \right] \\ &+ \left(r^2 + * \varpi^2 + \frac{\mathcal{C}^2}{4} \right) d\ell^2 + \left(r \mathcal{C}_{ab} + * \varpi * \mathcal{C}_{ab} \right) \hat{\theta}^a \hat{\theta}^b \\ &+ \frac{1}{r} \left(8\pi G \varepsilon_{(0)} \mu^2 - \frac{4}{3} \mu N_a \hat{\theta}^a - \frac{16\pi G}{3} E_{ab} \hat{\theta}^a \hat{\theta}^b \right) \\ &+ \frac{1}{r^2} \left(2 * \varpi \nu - \frac{1}{3} \hat{\mathcal{D}}_a N^a \right) \mu^2 - \frac{1}{r^2} \mu \left(\frac{1}{3} N^b \mathcal{C}_{ba} + \frac{8}{3} * \varpi * N_a + 8\pi G \hat{\mathcal{D}}_b E_a^b \right) \hat{\theta}^a \\ &+ \frac{1}{r^2} \left(\frac{16\pi G}{3} F_{ab} - 4\pi G \mathcal{C}_{(a} {}^c E_{b)c} \right) \hat{\theta}^a \hat{\theta}^b + \mathcal{O}(1/r^3). \end{aligned} \quad (3.50)$$

This solution to vacuum Einstein's equations is built upon the following boundary Carrollian data: (i) a generic Carrollian structure with geometric shear $\xi_{ab} = 0$ (but arbitrary Ehresmann connection providing φ_a and $*\varpi$); (ii) a dynamical shear \mathcal{C}_{ab} , utterly free; (iii) an energy density $\varepsilon_{(0)}$ i.e. a Bondi mass M , a heat current N_a aka the Bondi angular momentum aspect and a stress E_{ab} , all satisfying the flux-balance equations (3.42), (3.43) and (3.47);³⁰ (iv) three more degrees of freedom $E_{(2)ab}$, $\pi_{(2)a}$ and $\varepsilon_{(1)}$ encoded in F_{ab} (3.49) with evolution equations yet to be uncovered.

²⁹For this we use the dictionary for ref. [47] set up in footnotes 24 and 25, together with the relations $-16\pi G E_{\text{here}}^{ab} = \mathcal{T}_{\text{there}}^{ab}$ and $N_{\text{here}}^a = \mathcal{P}_{\text{there}}^a$. Observe that eqs. (58) and (63) of [47] are also compatible with further quantities introduced in [23] and mentioned in footnote 27.

³⁰As pointed out earlier the nut aspect N — equivalently the magnetic mass ν — is in essence part of the Carrollian structure and its evolution equation (3.41) is a geometric identity in disguise.

Recursion and the fate of Chthonian degrees of freedom. All this has been achieved as the limit of vanishing cosmological constant within general asymptotically anti-de Sitter Einstein spacetimes, where infinitely many flat degrees of freedom originate from the Laurent expansion of the anti-de Sitter boundary energy-momentum tensor about $k^2 = -\Lambda/3 = 0$ and constrained through evolution equations. Compared to the anti-de Sitter solution space, the extra — Chthonian — functions are $\{\chi_{(n \geq 2)}\} \equiv \{\varepsilon_{(n-1 \geq 1)}, \pi_{(n \geq 2)}^a, E_{(n \geq 2)}^{ab}\}$. It is natural to wonder whether these are truly independent functions. Answering this question demands a higher-order analysis but some simple considerations allow to infer that $\varepsilon_{(n-1)}, \pi_{(n)}^a$ and $E_{(n)}^{ab}$ could be repackaged in a single symmetric traceless tensor $F_{(n)}^{ab}$, having the expected conformal weight.

Indeed, one should recall that the $\chi_{(n)}$ s are all weight-3 and contribute to the $f_{(s)}$ s (of weight $s + 2$) through an appropriate number of longitudinal or transverse Weyl-covariant derivatives $u^C \mathcal{D}_C$ or \mathcal{D}_a , powers of vorticity ω_{ab} or shear σ_{ab} , all raising the weight by one unit (in the Carrollian limit, the latter two bring a factor k^2 with $*\varpi \hat{\eta}_{ab}$ or \mathcal{C}_{ab}). The analysis of Einstein's equations $\mathcal{E}_{rr}, \mathcal{E}_{r\hat{0}}, \mathcal{E}_{ra}$ and \mathcal{E}_{ab} in the radial expansion exhibits a remarkable recursion structure for the $f_{(s)}, f_{(s)}^a$ and $f_{(s)}^{ab}$ — for $s = 2$ these equations are sorted in (2.56). The latter are given in terms of quantities of order $s - 1$ along with one transverse Weyl derivative, one power of vorticity, or one power of shear. Furthermore, the scalar and the vector do not involve any net power of k^2 , whereas the tensor does: $f_{(s)}^{ab} = \frac{1}{k^2} [u^C \mathcal{D}_C f_{(s-1)}^{ab} + \dots]$. This shows, on the one hand, that the scalar and vector contributions to the line element remain finite in the Carrollian limit, and do not impose any supplementary constraint. On the other hand, flux-balance equations originate exclusively from the two-index term.

Owing to the fact that $\chi = \sum_{m \geq 2} k^{2m} \chi_{(m)}$, the Chthonian degrees of freedom $\chi_{(m)}$ persist in the Carrollian limit of the f s if a power of k^2 equal to or more negative than $-m$ is inherited from the AdS solution. Combined with the above recursive pattern, where in particular negative powers appear solely in the tensor $f_{(s)}^{ab}$, this suggests that once a combination of $\chi_{(m)}$ has emerged, such as $F^{ab} \equiv F_{(2)}^{ab}$ in eq. (3.49) for $\chi_{(2)}$ inside (3.46), only this precise expression will appear in the subsequent orders, along with more derivatives, powers of shear and vorticity, and increasing negative powers of k^2 . For instance, this occurs for $F_{(1)}^{ab} \equiv E_{(1)}^{ab} \equiv E^{ab}$ in $f_{(1)}^{ab}$ as in eq. (3.22), $f_{(2)}^{ab}$ as in eq. (3.46), and likewise in higher orders.

This scheme has two consequences. The first is that at order $(s + 1)$, one new flux-balance equation $\text{FBE}_{(s)} = 0$ emerges, for the previously determined combination $F_{(s)}^{ab}$ of the Chthonian functions $\chi_{(s)}$, as it should for global evolution consistency. Schematically this property is captured in the following:

$$f_{(s+1)}^{ab} = \sum_{n=1}^{s-1} \frac{c_{(s,n)}}{k^{2(s-n+1)}} \hat{\mathcal{D}}_v^{s-n} \text{FBE}_{(n)} + \frac{c_{(s,s)}}{k^2} \text{FBE}_{(s)} + \hat{f}_{(s+1)}^{ab} + \mathcal{O}(k^2), \quad (3.51)$$

with $c_{(s,n)}$ some immaterial coefficients, the new equation being $\text{FBE}_{(s)} = 0$. The second consequence is that the triplet $\chi_{(s)}$ counts as a single Chthonian degree of freedom materialized in $F_{(s)}^{ab}$, the one appearing in the line element and obeying a flux-balance equation

revealed at the next order in $1/r$. The following generic structure of the solutions underpins the above reasoning.³¹

- $\hat{f}_{(2s+1)}$ contains $(-)^s 8\pi G \varepsilon_{(0)} * \varpi^{2s}$ and $\hat{\mathcal{D}}_a \hat{f}_{(2s)}^a$;
- $\hat{f}_{(2s+2)}$ contains $(-)^s 2\nu * \varpi^{2s+1}$ and $\hat{\mathcal{D}}_a \hat{f}_{(2s+1)}^a$;
- $\hat{f}_{(s+1)}^a$ contains $\hat{\mathcal{D}}_b \hat{f}_{(s)}^{ab}$, $*\varpi * \hat{f}_{(s)}^a$, $\mathcal{C}^a_b \hat{f}_{(s)}^b$;
- $\hat{f}_{(s+1)}^{ab} = c_{(s+1,s+1)} F_{(s+1)}^{ab} +$ tensors based on objects of order s ;
- $F_{(s+1)}^{ab}$ contains $\hat{\mathcal{D}}_\nu^s E_{(s+1)}^{ab}$, $\hat{\mathcal{D}}_\nu^{s-1} \hat{\mathcal{D}}^{(a} \pi_{(s+1)}^{b)}$, $\varepsilon_{(s)} * \varpi^{s-1} \mathcal{C}^{ab}$, \dots ;
- $\text{FBE}_{(s)} = 0$ is of the form $\hat{\mathcal{D}}_\nu F_{(s)}^{ab} = \{ \hat{\mathcal{D}}^{(a} \hat{f}_{(s)}^{b)}, \mathcal{C}^{ab} \hat{f}_{(s)}, \dots \}$,

where the dots stand for other possible admissible terms. As anticipated, the actual Chthonian degrees of freedom capturing the flat dynamics are the emerging $F_{(s)}^{ab}$, which should be substituted for the $(s-1)$ th derivatives of $E_{(s)}^{ab}$, $\pi_{(s)}^a$ and $\varepsilon_{(s-1)}$ delivered by the anti-de Sitter energy-momentum tensor.

A legitimate question one may finally ask in view of our analysis pertains to the existence of other, possibly infinite, sets of Chthonian data originating from a Laurent expansion of the AdS boundary metric (see e.g. [45, 56]). Direct exploration of Ricci-flat solution spaces does not seem to support such an expectation, but a definite answer requires a thorough investigation, which would bring us far from our main goal.

The flat resummation. The anti-de Sitter resumable instance presented in eq. (2.65) can be realized in the flat limit, as it was shown in [3]. In this case all Chthonian functions should vanish, together with N^a , E^{ab} and the shear \mathcal{C}^{ab} , leading ultimately to

$$\begin{aligned} ds_{\text{res. Ricci-flat}}^2 = \mu \left[2dr + \left(2r\varphi_a - 2 * \hat{\mathcal{D}}_a * \varpi \right) \hat{\theta}^a - \left(r\theta + \hat{\mathcal{X}} \right) \mu \right] \\ + \left(r^2 + * \varpi^2 \right) d\ell^2 + \frac{1}{r^2 + * \varpi^2} \left(8\pi G \varepsilon_{(0)} r + * \varpi c_{(0)} \right) \mu^2. \end{aligned} \quad (3.52)$$

This captures all algebraically special Ricci-flat spacetimes provided $\varepsilon_{(0)}$ obeys (3.32) and (3.33) which now read:

$$\hat{\mathcal{D}}_\nu \varepsilon_{(0)} + \frac{1}{8\pi G} \hat{\mathcal{D}}_a * \chi^a = 0, \quad (3.53)$$

$$\hat{\mathcal{D}}_a \varepsilon_{(0)} - \frac{1}{8\pi G} * \hat{\mathcal{D}}_a c_{(0)} = 0. \quad (3.54)$$

Equations (3.53) and (3.54) coincide with eqs. (29.16) and (29.15) of [57].³² The latter are rather complicated and it is remarkable they are tamed into simple conservation equations

³¹We remind that $F_{(1)}^{ab} \equiv E_{(1)}^{ab} \equiv E^{ab}$, $\pi_{(1)}^a \equiv \pi^a$, and $F_{(2)}^{ab} \equiv F^{ab}$.

³²For that purpose, the following identifications are necessary, in Papapetrou-Randers frame and complex coordinates $\mathbf{x} = \{ \zeta, \bar{\zeta} \}$ with $d\ell^2 = \frac{2}{P^2(u, \zeta, \bar{\zeta})} d\zeta d\bar{\zeta}$, $\nu = \frac{1}{\Omega} \partial_u$, $\mu = -\Omega du + b_a dx^a$ and $\hat{e}_a = \hat{\partial}_a = \partial_a + \frac{b_a}{\Omega} \partial_u$, $\hat{\theta}^a = dx^a$: $\Omega = 1$, $b_\zeta = -L$, $*\varpi = -\Sigma$, $\hat{\tau} = 2(M + im)$, whereas their radial coordinate is $\tilde{r} = r - r_0$ with $r_0(u, \zeta, \bar{\zeta})$ the origin in the affine parameter of the geodesic congruence tangent to ∂_r .

such as (3.53) and (3.54). Reaching this conclusion would have been inconceivable without the null boundary perspective and the Carrollian tools, which are the appropriate language for asymptotically flat spacetimes.

The algebraically-special nature of the metric (3.52) is proven using the Goldberg-Sachs theorem with the null, geodesic, and shear-free in the resummed instance, bulk congruence tangent to ∂_r . The latter is part of the canonical null tetrad parallelly transported along ∂_r (thanks to the affine nature of r) introduced in [3], which coincides with that of [57], eq. (29.13a), as well as with the original ref. [37]. In complex celestial-sphere coordinates ζ and $\bar{\zeta}$ (see footnote 32) the null tetrad reads:

$$\begin{cases} \mathbf{k} = \partial_r \\ \mathbf{l} = \frac{1}{2} \left(\frac{8\pi G \varepsilon_{(0)} r + * \varpi c_{(0)}}{r^2 + * \varpi^2} - r\theta - \hat{\mathcal{H}} \right) \partial_r + \mathbf{v} \\ \mathbf{m} = \frac{P}{r - i * \varpi} \left(\hat{\partial}_{\bar{\zeta}} + (* \hat{\mathcal{D}}_{\bar{\zeta}} * \varpi - r \varphi_{\bar{\zeta}}) \partial_r \right) \end{cases} \quad (3.55)$$

with $\mathbf{k} \cdot \mathbf{l} = -1$, $\mathbf{m} \cdot \bar{\mathbf{m}} = 1$ and $ds_{\text{res. Ricci-flat}}^2 = -2\mathbf{k}\mathbf{l} + 2\mathbf{m}\bar{\mathbf{m}}$. Generically, \mathbf{k} is a multiplicity-two principal null direction of the Weyl tensor, and using the tetrad at hand we find the following Weyl complex scalars: $\Psi_0 = \Psi_1 = 0$ and

$$\Psi_2 = \frac{i\hat{r}}{2(r - i * \varpi)^3}, \quad \Psi_3 = \frac{iP\chi\zeta}{(r - i * \varpi)^2} + \mathcal{O}(1/(r - i * \varpi)^3), \quad \Psi_4 = \frac{iX_{\zeta}^{\bar{\zeta}}}{r - i * \varpi} + \mathcal{O}(1/(r - i * \varpi)^2). \quad (3.56)$$

Observe that neither Ψ_3 nor Ψ_4 vanish in the instance of Petrov type D solutions, because \mathbf{l} is *not* a principal null direction. Another tetrad is reached with a Lorentz transformation suitably adjusted for \mathbf{l}' be a principal direction of multiplicity two whereas $\mathbf{k}' \propto \mathbf{k}$, and $\Psi'_3 = \Psi'_4 = 0$. Unsurprisingly, all Ψ s are spelled using the Carrollian descendants of the boundary Cotton tensor — as well as their derivatives in the higher-order terms.

4 Outlook

Asymptotically anti-de Sitter and flat spacetimes subject to Einstein equations are distinguished mainly by two features. The first is gravitational radiation escaping at or arriving from null infinity in the flat instance, which is absent under the usual boundary conditions for anti-de Sitter. The second concerns the data required for a faithful depiction of these geometries and of their dynamics imposed by Einstein's equations: a finite versus an infinite number for asymptotically AdS or flat.

In spite of the sharp distinctness of the solution spaces with non-vanishing and zero cosmological constant, the latter can be smoothly reached from the former in a procedure that is the core of this work. It can be outlined in three steps, performed along with the process of sending Λ to zero, which simultaneously transmutes the pseudo-Riemannian conformal boundary of anti-de Sitter into a Carrollian descendant, carrying akin information.

- Bondi's shear \mathcal{C}_{AB} is substituted on-shell for the geometric shear σ_{AB} .
- The anti-de Sitter boundary energy-momentum tensor T_{AB} is Laurent-expanded in powers of $k^2 = -\Lambda/3$ about $k^2 = 0$. This supplies an infinite number of replicas, which account for the awaited flat, Chthonian, degrees of freedom.

- The evolution (flux-balance) equations of the — now Carrollian — degrees of freedom are reached using both the limit of the conservation of the energy and momentum, as well as the requirement of finiteness for the line element in the flat limit. The latter (i) selects the Chthonian variables $\varepsilon_{(n\geq 1)}$, $\pi_{(n\geq 2)}^a$ and $E_{(n\geq 2)}^{ab}$ besides the standard energy density $\varepsilon_{(0)}$, momentum π^a and stress E^{ab} , and we have argued that genuine degrees of freedom are only the $F_{(n)}^{ab}$ s; (ii) freezes a few other components of the expanded energy-momentum tensor in terms of the boundary Carrollian Cotton tensors $(\zeta^a, Q^a, \zeta^{ab}, \Sigma^{ab}, \Xi^{ab})$; (iii) delivers the Chthonian dynamics, which is not captured by the energy-momentum conservation but echoes flat Einstein equations.

The technical *tour de force* of our exploration shouldn't shadow the conceptual aftermath of our findings. These bring back the boundary energy and momentum at the center of the asymptotically flat bulk reconstruction, besides the Bondi shear, under the form of a Carrollian energy density, momentum and stress, together with an infinite tower of replicas of the latter. Speculating over a flat extension of AdS gauge/gravity duality, and owing to the key role played by the energy-momentum tensor in the latter, one is led to several unescapable questions. What would the fundamental observables be in the dual Carrollian field theory? What role would the replicas of the energy-momentum sector play? What is the interplay between the Chthonian and the shear/news sector, which has been investigated in celestial holography? Could this correspondence still be qualified as holographic — given the seemingly infinite number of necessary data? Our approach does not yet provide any cue for answering these questions, though it hands some confidence in the zero- Λ limit, that could be inquired within the AdS/CFT correspondence. This last point is probably the deepest our analysis conveys.

This is the big picture. Other questions merit equal attention, starting with the ones related to symmetries and charges. What are the asymptotic symmetries in a partially unfixed gauge like the covariant Newman-Unti introduced here? What sort of charges does this extension carry? What is the precise combination of vorticity and angular-momentum aspect that would define the physical angular momentum? How would logarithmic terms in the radial expansion alter the analysis? In answering these questions, one could follow recent works such as, e.g., [10, 13, 15] as well as [42, 50–52, 58]. In particular, one should address the Weyl invariance in conjunction with the boundary local Lorentz (or Carroll) gauge invariance inherited from the onset of a velocity congruence or a clock form in the boundary pseudo-Riemannian (or Carrollian) structure. In a similar fashion as the one presented in this work, a careful analysis would allow to embrace both the anti-de Sitter and flat cases.

Regarding the charges, a thorough comparison of our method with Newman-Penrose's would be a valuable practice, reasonably accessible thanks to the affinely parameterised radial congruence ∂_r present in the (covariant) Newman-Unti gauge. In the first place, this would allow to extract the famous ten non-vanishing Newman-Penrose conserved charges — we know that these are carried by the Chthonian stress tensor descendant F_{ab} .³³ Secondly,

³³Contact with the Newman-Penrose formalism beyond the algebraically special resumable metrics mentioned at the end of section 3 starts with $\Psi_0^0 \propto iE_\zeta^{\bar{\zeta}}$, $\Psi_0^1 \propto iF_\zeta^{\bar{\zeta}}$, $\Psi_1^0 \propto iN_\zeta$, $\Psi_2^0 \propto i\hat{r}$ (see (3.40)),

one could recast these charges following their general Carrollian definition, as described in refs. [25, 59], giving credit to this full-fledged boundary method for the charge computation. Lastly, one may deepen concepts such as subleading charges or electric versus magnetic charges and possible dualities involving the Carrollian Cotton tensors, as recently undertaken in [25] from the Carrollian standpoint in the limited framework of resumable, algebraically special Ricci-flat spacetimes, and more generally discussed in refs. [60–75].

Acknowledgments

The present investigation started in 2020 with discussions involving L. Ciambelli, C. Marteau, A. Petkou, R. Ruzziconi, K. Siampos and later A. Fiorucci. We would like to thank these colleagues as well as B. Oblak and A. Seraj. We also thank each other’s institutions for hospitality and financial support for the numerous collaboration visits. The work of A. Campoleoni, A. Delfante and S. Pekar was partially supported by the Fonds de la Recherche Scientifique — FNRS under grants F.4503.20, T.0022.19, FC.41161 and FC.36447. Starting October 1st the work of S. Pekar is funded by the *Fonds Friedmann* run by the *Fondation de l’École polytechnique*. The work of D. Rivera-Betancour was funded by Becas Chile (ANID) Scholarship No. 72200301. The work of M. Vilatte was supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the *First Call for H.F.R.I. Research Projects to support Faculty members and Researchers and the procurement of high-cost research equipment grant* (MIS 1524, Project Number: 96048). The graduate students D. Rivera-Betancour and M. Vilatte thank the programme *Erasmus+* of the Institut Polytechnique de Paris as well as the Kapodistrian University of Athens, the Aristotle University of Thessaloniki and the University of Mons for hosting them with these fellowships.

A Carrollian geometry in Cartan frame and arbitrary dimension

Frame and covariance. Carroll structures on $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ with a d -dimensional base \mathcal{S} were alluded to in section 3. They are equipped with a degenerate metric,

$$d\ell^2 = \delta_{ab} \hat{\theta}^a \hat{\theta}^b, \tag{A.1}$$

as well as a frame and a coframe, $\{\hat{e}_{\hat{u}} = \mathbf{v}, \hat{e}_a\}$ and $\{\hat{\theta}^{\hat{u}} = -\mu, \hat{\theta}^a\}$ obeying

$$\mu(\mathbf{v}) = -1, \quad \hat{\theta}^a(\hat{e}_b) = \delta_b^a, \quad \hat{\theta}^a(\mathbf{v}) = 0, \quad \mu(\hat{e}_a) = 0. \tag{A.2}$$

Here \mathbf{v} is the field of observers, kernel of the degenerate metric, and μ the clock form (see e.g. [76]).

$\Psi_3^0 \propto iP\chi_\zeta$ and $\Psi_4^0 \propto iX_\zeta^{\bar{\zeta}}$, where the adopted Carrollian frame is that of footnote 32. The higher-order terms will involve derivatives of the Cotton tensors, of the energy density, the momentum and the stress, as well as the infinite tower of Chthonian replicas $F_{(s)}^{ab}$.

A convenient parameterization in terms of $d + \frac{(d+1)(d+2)}{2}$ functions (i.e. 8 for $d=2$) is³⁴

$$\mathbf{v} = \gamma \left(\partial_u + v^i \partial_i \right) \quad \Leftrightarrow \quad \boldsymbol{\mu} = -\frac{du}{\gamma} + \Delta_i \left(dx^i - v^i du \right), \quad (\text{A.3})$$

$$\hat{e}_a = e_a^i \left(\partial_i + \gamma \Delta_i \left(\partial_u + v^j \partial_j \right) \right) \quad \Leftrightarrow \quad \hat{\theta}^a = e_a^i \left(dx^i - v^i du \right) \quad (\text{A.4})$$

with

$$\Gamma_{ij}^2 = \delta_{ab} e_a^i e_b^j \quad \Leftrightarrow \quad \delta_{ab} = e_a^i e_b^j \Gamma_{ij}^2 \quad (\text{A.5})$$

and

$$e_a^i e_a^j = \delta_i^j, \quad e_b^j e_a^j = \delta_a^b, \quad \delta^{ab} e_a^i \Gamma_{ij}^2 = e_b^j, \quad \delta_{ab} e_a^i \Gamma^{2ij} = e_b^j, \quad (\text{A.6})$$

where $(\Gamma^2)^{ik} \Gamma_{kj}^2 = \delta_j^i$. Consequently, the degenerate metric assumes the form³⁵

$$d\ell^2 = \Gamma_{ij}^2 \left(dx^i - v^i du \right) \left(dx^j - v^j du \right). \quad (\text{A.7})$$

In this specific parameterization, which generalizes that of [10] in arbitrary dimension, the bulk Newman-Unti gauge is recovered by setting $\Delta_i = 0$ in the boundary frame.³⁶

Carrollian tensors have commonly spacetime indices. In the Cartan frame (A.1), (A.2), their tensorial behaviour refers to the local Carroll group, as much as relativistic tensors in an orthonormal Cartan frame are tamed according to the local Lorentz group. Here, the metric being degenerate the spacetime indices cannot be lowered or raised. One way to manage this inconvenience is by introducing a pseudo-inverse [83]. Our strategy has been slightly different, and is hinged on *separating time and space*, since this is natural in Carrollian manifolds due to the fibre structure. In the frame at hand, the method boils down to considering tensors with solely spatial indices, organized in representations of the d -dimensional orthogonal local group, subgroup of the local Carroll group, and raised or lowered with δ^{ab} or δ_{ab} . The fibre null-time direction supports scalars without indices.³⁷ This approach is in line with the boundary reconstruction of Ricci-flat spacetimes, where the longitudinal/transverse decomposition of the fundamental tensors coincides with the time/space reduction of the Carrollian tensors.

A strong Carroll structure comes with a metric-compatible and field-of-observers-compatible connection, which is not unique due to the metric degeneracy. The connection we use defines a parallel transport that respects the time/space splitting mentioned above,

³⁴Here γ is an arbitrary function and must not to be confused with (2.27) which is related to the vorticity of the timelike congruence u .

³⁵The degenerate metric is often spelled $d\ell^2 = q_{\mu\nu} dx^\mu dx^\nu$ in the Carrollian literature, and $n = n^\mu \partial_\mu$ stands for the field of observers.

³⁶The presence of $\gamma \equiv \exp(-2\beta_0)$, which persists in the bulk line element as $-2 \exp(2\beta_0) du dr$, assesses a slight redefinition of the radial coordinate before reaching stricto sensu Newman-Unti gauge. We are cavalier with this detail because the counting from the point of view of the solution space matches: the contribution of the boundary geometry is $\frac{(d+1)(d+2)}{2}$. The same holds for the anti-de Sitter ascendant.

³⁷When working in natural frames, as in refs. [3, 9, 24, 25, 45, 46, 85], the tensor structure is based instead on diffeomorphisms. The time/space splitting sought for is realized in Papapetrou-Randers frame, i.e. setting $v^i = 0$ in the formulas (A.3), (A.4), (A.7), because this frame is stable under the Carrollian subset of diffeomorphisms, consisting of transformations $u \rightarrow u'(u, \mathbf{x})$ and $\mathbf{x} \rightarrow \mathbf{x}'(\mathbf{x})$. Carrollian tensors have again spatial indices and transform with the Jacobian matrices of Carrollian diffeomorphisms.

embracing distinct time and space Carrollian covariant derivatives $\hat{\nabla}_{\mathbf{v}}$ acting as a scalar and $\hat{\nabla}_a$ acting as a form. We set for this purpose

$$\hat{\nabla}_{\mathbf{v}}\mathbf{v} = 0, \quad \hat{\nabla}_{\mathbf{v}}\hat{e}_a = \hat{\gamma}_{[ab]}\delta^{bc}\hat{e}_c, \quad \hat{\nabla}_{\hat{e}_a}\mathbf{v} = 0, \quad \hat{\nabla}_{\hat{e}_a}\hat{e}_b = \hat{\gamma}_{ab}^c\hat{e}_c, \quad (\text{A.8})$$

from which we infer the resulting Carrollian affine connection one-form:³⁸

$$\hat{\omega}^{\hat{u}}_{\hat{u}} = \hat{\omega}^{\hat{u}}_b = \hat{\omega}^a_{\hat{u}} = 0, \quad \hat{\omega}^a_b = \delta^{ac}\hat{\gamma}_{[cb]}\mu + \hat{\gamma}_{cb}^a\hat{\theta}^c \quad (\text{A.9})$$

At this stage $\hat{\gamma}_{[ab]}$ and $\hat{\gamma}_{cb}^a$ are arbitrary, although anticipating the next step (metric compatibility), we have imposed antisymmetry for the former.

The covariant time and space derivatives act on Carrollian scalars as time and space directional derivatives. For Carrollian vectors $\zeta = \zeta^a\hat{e}_a$ and forms $\zeta = \zeta_a\hat{\theta}^a$ we obtain:

$$\hat{\nabla}_a\zeta^b = \hat{e}_a(\zeta^b) + \hat{\gamma}_{ac}^b\zeta^c \Leftrightarrow \hat{\nabla}_a\zeta_b = \hat{e}_a(\zeta_b) - \hat{\gamma}_{ab}^c\zeta_c, \quad (\text{A.10})$$

$$\hat{\nabla}_{\mathbf{v}}\zeta^a = \mathbf{v}(\zeta^a) - \hat{\gamma}^{[ab]}\zeta_b \Leftrightarrow \hat{\nabla}_{\mathbf{v}}\zeta_a = \mathbf{v}(\zeta_a) - \hat{\gamma}_{[ab]}\zeta^b. \quad (\text{A.11})$$

Under a frame transformation, $\hat{\gamma}_{[ab]}$ and $\hat{\gamma}_{cb}^a$ transform as connection coefficients, i.e. with inhomogeneous terms.

Field-of-observers-compatibility is built in (A.8). Metric-compatibility translates in $\hat{\omega}_{(ab)} = 0$. This imposes

$$\hat{\gamma}_{(a|c|b)} = 0, \quad (\text{A.12})$$

where the symmetrization acts on the two extreme indices. The latter can be utterly determined by further imposing the absence of torsion in the spatial section, $T_{ab}^c = 0$. In order to implement this we can use the following parameterization of the $d\hat{\theta}^A$ s:

$$d\mu - \varphi_a\hat{\theta}^a \wedge \mu - \varpi_{ab}\hat{\theta}^a \wedge \hat{\theta}^b = 0, \quad d\hat{\theta}^c + \hat{\gamma}^c_a\mu \wedge \hat{\theta}^a + \frac{1}{2}\hat{c}^c_{ab}\hat{\theta}^a \wedge \hat{\theta}^b = 0, \quad (\text{A.13})$$

or equivalently

$$[\mathbf{v}, \hat{e}_a] = \varphi_a\mathbf{v} - \hat{\gamma}^c_a\hat{e}_c, \quad [\hat{e}_a, \hat{e}_b] = 2\varpi_{ab}\mathbf{v} + \hat{c}^c_{ab}\hat{e}_c. \quad (\text{A.14})$$

We have again foreseen the following action by introducing $\hat{\gamma}_{ab}$ whose antisymmetric part already appears in the affine connection one-form. Hence, the extra condition of the absence of torsion in the spatial section combined with (A.12) delivers

$$\hat{\gamma}_{bc}^a = \frac{1}{2}(\hat{c}^a_{bc} + \hat{c}^a_c{}_b + \hat{c}^a_b{}_c). \quad (\text{A.15})$$

Let us also point out the useful integrability conditions $d^2\mu = d^2\hat{\theta}^a = 0$ associated with (A.13):

$$\begin{cases} \hat{\nabla}_{[c}\varpi_{ab]} = \varphi_{[c}\varpi_{ab]} \\ \hat{\nabla}_{\mathbf{v}}\varpi_{ab} + \varpi_a{}^c\hat{\gamma}_{(cb)} - \varpi_b{}^c\hat{\gamma}_{(ca)} = \hat{\nabla}_{[a}\varphi_{b]} \end{cases} \quad (\text{A.16})$$

³⁸Remember that $\omega^A_B = \Gamma^A_{CB}\theta^C$ with $\nabla_{e_A}e_B = \Gamma^C_{AB}e_C$. The torsion and curvature two-forms are $\mathcal{T}^C = d\theta^C + \omega^C_A \wedge \theta^A = \frac{1}{2}T^C_{AB}\theta^A \wedge \theta^B$ and $\mathcal{R}^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B = \frac{1}{2}R^A_{BCD}\theta^C \wedge \theta^D$. Torsion and curvature tensors can alternatively be determined using the commutator of covariant derivatives: $[\nabla_A, \nabla_B]W^C = R^C_{DAB}W^D - T^D_{AB}\nabla_D W^C$.

and

$$\begin{cases} \mathbf{v}(\hat{c}^a{}_{bc}) - \hat{\gamma}^a{}_d \hat{c}^d{}_{bc} - 2\hat{c}^a{}_{d[b} \hat{\gamma}^d{}_{c]} + 2\hat{\epsilon}_{[b}(\hat{\gamma}^a{}_{c]}) - 2\hat{\gamma}^a{}_{[b} \varphi_{c]} = 0 \\ \hat{\epsilon}_{[d}(\hat{c}^a{}_{bc])} - \hat{c}^a{}_{e[b} \hat{c}^e{}_{cd]} + 2\gamma^a{}_{[b} \varpi_{cd]} = 0. \end{cases} \quad (\text{A.17})$$

In summary, our strong Carroll connection is totally determined thanks to the information stored inside the second of eqs. (A.13), by requiring the time-and-space splitting and the absence of spatial torsion. The total torsion is non-zero though and we find:

$$\hat{\mathcal{T}}^{\hat{u}} = \varphi_a \mu \wedge \hat{\theta}^a - \varpi_{ab} \hat{\theta}^a \wedge \hat{\theta}^b, \quad \hat{\mathcal{T}}^a = \delta^{ab} \hat{\gamma}_{(bc)} \hat{\theta}^c \wedge \mu. \quad (\text{A.18})$$

The torsion is thus encoded in three Carrollian tensors (i.e. transforming homogeneously), featuring three properties of the null-time fibre materialized in \mathbf{v} : the *acceleration* φ_a , the *vorticity* ϖ_{ab} and the *extrinsic curvature* $\hat{\gamma}_{(ab)}$, which can be further decomposed into the *geometric shear* ξ_{ab} (traceless) and the *expansion* θ :

$$\hat{\gamma}_{(ab)} = \xi_{ab} + \frac{\theta}{d} \delta_{ab}. \quad (\text{A.19})$$

We could consistently set the Carrollian torsion to zero. From the bulk perspective, this would significantly impoverish the range of options the covariant Newman-Unti gauge offers for Ricci-flat spacetimes, as discussed in section 3. It is opportune to recall that in the frame-parameterization (A.3), (A.4), ordinary Newman-Unti gauge corresponds to $\Delta_i = 0$. In more intrinsic terms, this amounts to setting

$$d\mu = \varphi_a \hat{\theta}^a \wedge \mu \Leftrightarrow [\hat{e}_a, \hat{e}_b] = \hat{c}^c{}_{ab} \hat{e}_c \quad (\text{A.20})$$

i.e. to discarding the vorticity.

We can finally determine the curvature of the Carrollian connection under consideration using Cartan's formula, cf. footnote 38:

$$\hat{\mathcal{R}}^{\hat{u}}{}_{\hat{b}} = 0, \quad \hat{\mathcal{R}}^a{}_{\hat{b}} = \hat{R}^a{}_{cb} \mu \wedge \hat{\theta}^c + \frac{1}{2} \hat{R}^a{}_{bcd} \hat{\theta}^c \wedge \hat{\theta}^d \quad (\text{A.21})$$

with

$$\hat{R}^a{}_{bcd} = \hat{e}_c(\hat{\gamma}^a{}_{db}) - \hat{e}_d(\hat{\gamma}^a{}_{cb}) + \hat{\gamma}^e{}_{db} \hat{\gamma}^a{}_{ce} - \hat{\gamma}^e{}_{cb} \hat{\gamma}^a{}_{de} - \hat{c}^e{}_{cd} \hat{\gamma}^a{}_{eb} + 2\varpi_{cd} \hat{\gamma}_{[eb]} \delta^{ae}, \quad (\text{A.22})$$

$$\hat{R}^a{}_{cb} = (\hat{\nabla}^a + \varphi^a) \hat{\gamma}_{(bc)} - (\hat{\nabla}_b + \varphi_b) \hat{\gamma}_{(cd)} \delta^{ad}. \quad (\text{A.23})$$

One can trace the above and yield the Carroll-Ricci tensor and the Carroll scalar curvature:

$$\hat{R}_{cd} = \hat{R}^a{}_{cad}, \quad \hat{R} = \delta^{cd} \hat{R}_{cd}. \quad (\text{A.24})$$

Let us stress anew that the freedom in designing a Carrollian connection is rather wide — see [76–79] or [80–82] for a review — even when conditions like Levi-Civita are imposed, which we haven't. Our guideline has been to ensure that all information ultimately stored in the Carrollian frame, connection, torsion and curvature coincides with that of the relativistic, pseudo-Riemannian ascendant, as we will shortly see: φ_a , ϖ_{ab} , $\hat{\gamma}_{ab}$ and $\hat{c}^c{}_{ab}$.

As a final comment, we would like to mention that Carrollian geometries may have isometries and in particular conformal isometries. The latter play a central role when considering the null conformal boundary, as they mirror bulk asymptotic symmetries. A vector field $\xi = \xi^{\hat{u}}\mathbf{v} + \xi^a\hat{e}_a$ is a Carrollian Killing if the Lie derivative of the degenerate metric and of the field of observers vanishes. This requirement generates three conditions:

$$\begin{cases} \hat{\nabla}_{(a}\xi_{b)} + \xi^{\hat{u}}\hat{\gamma}_{(ab)} = 0 \\ \mathbf{v}(\xi^{\hat{u}}) + \xi^a\varphi_a = 0 \\ \hat{\nabla}_{\mathbf{v}}\xi_a - \hat{\gamma}_{(ab)}\xi^b = 0. \end{cases} \quad (\text{A.25})$$

In Papapetrou-Randers frame where $\mathbf{v} = \frac{1}{\Omega}\partial_u$ and the degenerate metric has no time legs, the last condition selects the Carrollian diffeomorphisms, $\partial_u\xi^i = 0$. In the Cartan frame at hand all diffeomorphisms are permitted; the Killing fields are nonetheless further constrained. As usual, strong Killing fields must also leave the clock form invariant, which implies

$$\hat{e}_a(\xi^{\hat{u}}) - \varphi_a\xi^{\hat{u}} + 2\varpi_{ab}\xi^b = 0. \quad (\text{A.26})$$

Bulk Killing fields of Ricci-flat spacetimes are mapped onto strong Killings of their null boundary [25].

Weyl covariance. Following the pattern adopted for the affine connection, we introduce here a Weyl connection that respects the time and space splitting, associated with two Weyl-covariant derivatives. These act on weight- w Carrollian tensors and deliver Carrollian tensors of weight $w + 1$.³⁹ The Weyl connection is encoded in θ and φ_a , see (A.13) and (A.19), and the Weyl-covariant derivatives are defined as follows:

- on scalars

$$\hat{\mathcal{D}}_{\mathbf{v}}\Phi = \mathbf{v}(\Phi) + \frac{w}{d}\theta\Phi, \quad \hat{\mathcal{D}}_a\Phi = \hat{e}_a(\Phi) + w\varphi_a\Phi; \quad (\text{A.27})$$

- on vectors $\mathbf{v} = v^a\hat{e}_a$

$$\hat{\mathcal{D}}_{\mathbf{v}}v^a = \hat{\nabla}_{\mathbf{v}}v^a + \frac{w}{d}\theta v^a, \quad \hat{\mathcal{D}}_av^b = \hat{\nabla}_av^b + w\varphi_av^b + \varphi^bv_a - \delta_a^bv^c\varphi_c; \quad (\text{A.28})$$

- on rank-2 tensors $\mathbf{t} = t_{ab}\hat{\theta}^a \otimes \hat{\theta}^b$:

$$\hat{\mathcal{D}}_ct_{ab} = \hat{\nabla}_ct_{ab} + w\varphi_ct_{ab} + \varphi_at_{cb} + \varphi_bt_{ac} - \delta_{ac}t_{db}\varphi^d - \delta_{cb}t_{ad}\varphi^d, \quad (\text{A.29})$$

$$\hat{\mathcal{D}}_{\mathbf{v}}t_{ab} = \hat{\nabla}_{\mathbf{v}}t_{ab} + \frac{w}{d}\theta t_{ab}. \quad (\text{A.30})$$

Using Leibniz' rule one obtains the generalization for any conformal tensor.

³⁹As already mentioned in footnote 11, when working in a Cartan frame the Weyl properties are slightly modified and there is no contradiction with the results displayed in refs. [3, 9, 24, 25, 45, 46, 85], where a Papapetrou-Randers frame was in use.

The Riemann-Carroll-Weyl curvature is a weight-2 tensor defined through the commutator of the Carrollian Weyl derivatives acting on Carrollian scalars Φ , vectors v^c or 2-tensors t^{cd} of weight w :⁴⁰

$$\left[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b\right] \Phi = 2\varpi_{ab} \hat{\mathcal{D}}_v \Phi + w\Omega_{ab} \Phi, \quad (\text{A.31})$$

$$\left[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b\right] v^c = \hat{\mathcal{S}}^c_{dab} v^d + 2\varpi_{ab} \hat{\mathcal{D}}_v v^c + w\Omega_{ab} v^c, \quad (\text{A.32})$$

$$\left[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b\right] t^{cd} = \hat{\mathcal{S}}^c_{eab} t^{ed} + \hat{\mathcal{S}}^d_{cab} t^{ce} + 2\varpi_{ab} \hat{\mathcal{D}}_v t^{cd} + w\Omega_{ab} t^{cd}, \quad (\text{A.33})$$

where

$$\Omega_{ab} = \hat{e}_a(\varphi_b) - \hat{e}_b(\varphi_a) - \hat{c}^c_{ab} \varphi_c - \frac{2}{d} \varpi_{ab} \theta \quad (\text{A.34})$$

is yet another weight-2 Carrollian tensor. From the Riemann-Weyl-Carroll tensor, we define

$$\hat{\mathcal{S}}_{cd} = \hat{\mathcal{S}}^a_{cad}, \quad \hat{\mathcal{R}} = \delta^{cd} \hat{\mathcal{S}}_{cd}, \quad (\text{A.35})$$

all weight-2.

We can further consider time and space derivatives:

$$\left[\hat{\mathcal{D}}_v, \hat{\mathcal{D}}_a\right] \Phi = -\xi^b_a \hat{\mathcal{D}}_b \Phi + w\hat{\mathcal{R}}_a \Phi, \quad (\text{A.36})$$

$$\left[\hat{\mathcal{D}}_v, \hat{\mathcal{D}}_a\right] v^b = -\hat{\mathcal{S}}^b_{ac} v^c - \xi^c_a \hat{\mathcal{D}}_c v^b + w\hat{\mathcal{R}}_a v^b, \quad (\text{A.37})$$

$$\left[\hat{\mathcal{D}}_v, \hat{\mathcal{D}}_a\right] t^{bc} = -\hat{\mathcal{S}}^b_{ad} t^{dc} - \hat{\mathcal{S}}^c_{ad} t^{bd} - \xi^d_a \hat{\mathcal{D}}_d t^{bc} + w\hat{\mathcal{R}}_a t^{bc}, \quad (\text{A.38})$$

revealing a clear pattern for any Carrollian conformal tensor. In these expressions

$$\hat{\mathcal{S}}^c_{ab} = -\hat{\mathcal{S}}^c_{ba} = \hat{\mathcal{D}}^c \xi_{ab} - \hat{\mathcal{D}}_b \xi^c_a + \delta^c_a \hat{\mathcal{R}}_b - \delta_{ab} \hat{\mathcal{R}}^c \quad (\text{A.39})$$

and $\hat{\mathcal{R}}_a$ are weight-two tensors. Note that in Cartan frame, both the shear ξ_{ab} and the vorticity ϖ_{ab} have weight one, regardless of the position of the indices. In natural frame ξ_{ij} and ϖ_{ij} have weight -1 , but raising an index augments the weight by two units.

Relation with a relativistic ascendant. A Carrollian manifold as described earlier can be reached from a pseudo-Riemannian geometry at zero velocity of light k . Following the pattern proposed in eqs. (3.2), (3.3) and (3.4), we can express the metric (2.8) of the pseudo-Riemannian ascendant as

$$ds^2 = \eta_{AB} \theta^A \theta^B = -\left(\theta^{\hat{0}}\right)^2 + \delta_{ab} \theta^a \theta^b = -k^2 \left(\hat{\theta}^{\hat{u}}\right)^2 + \delta_{ab} \hat{\theta}^a \hat{\theta}^b, \quad (\text{A.40})$$

where we have assumed that all k -dependence is explicit i.e. $\theta^a = \hat{\theta}^a$ while $\theta^{\hat{0}} = k\hat{\theta}^{\hat{u}}$. The relationship among the relativistic congruence (2.9) and the Carrollian fibre attributes, field of observers and clock form, is $\mathbf{v} = \mathbf{u} = \hat{e}_{\hat{u}}$ for the former and $\boldsymbol{\mu} = \frac{\mathbf{u}}{k^2} = -\hat{\theta}^{\hat{u}}$ for the latter.

⁴⁰The use of $\hat{\mathcal{S}}$ is unconventional for a curvature, but is intended to avoid confusion with a slightly different definition given as $\hat{\mathcal{R}}$ in [3, 25, 45, 46].

If the Carrollian frame, coframe and degenerate metric are parameterized as in eqs. (A.3), (A.4) and (A.7), then

$$e_{\hat{0}} = \frac{\gamma}{k} \left(\partial_u + v^i \partial_i \right) \quad \Leftrightarrow \quad \theta^{\hat{0}} = k \left(\frac{du}{\gamma} - \Delta_i \left(dx^i - v^i du \right) \right), \quad (\text{A.41})$$

$$e_a = e_a^i \left(\partial_i + \gamma \Delta_i \left(\partial_u + v^j \partial_j \right) \right) \quad \Leftrightarrow \quad \theta^a = e^a_i \left(dx^i - v^i du \right) \quad (\text{A.42})$$

and the relativistic metric reads:

$$\begin{aligned} ds^2 &= -k^2 \left(\frac{du}{\gamma} - \Delta_i \left(dx^i - v^i du \right) \right)^2 + \Gamma_{ij}^2 \left(dx^i - v^i du \right) \left(dx^j - v^j du \right), \\ &= -\frac{k^2}{\gamma^2} \left(du^2 - 2\gamma \Delta_i du \left(dx^i - v^i du \right) \right) + \left(\Gamma_{ij}^2 - k^2 \Delta_i \Delta_j \right) \left(dx^i - v^i du \right) \left(dx^j - v^j du \right), \end{aligned} \quad (\text{A.43})$$

where the normalized vector congruence is

$$\mathbf{u} = \gamma \left(\partial_u + v^i \partial_i \right). \quad (\text{A.44})$$

We will not explicitly operate with this frame, which coincides at $v^i = 0$ with the Papapetrou-Randers form employed in refs. [3, 9, 24, 25, 45, 46], where $\Omega = 1/\gamma$, $b_i = \Delta_i$ and $a_{ij} = \Gamma_{ij}^2$.

At $\Delta_i = 0$, one recovers the boundary frame of bulk Newman-Unti anti-de Sitter gauge (modulo a remark stated in footnote 36 and valid here), and

$$d\theta^{\hat{0}} = \varphi_a \theta^a \wedge \theta^{\hat{0}}, \quad (\text{A.45})$$

which resonates with the Carrollian relative (A.20). Hence the boundary vorticity vanishes following eq. (A.46) below.

The pseudo-Riemannian manifold is equipped with a Levi-Civita connection. We would like to express the latter in terms of the Carrollian tensors appearing in eqs. (A.9) and (A.13) or (A.14). The purpose of this exercise is to provide the suitable tools for reaching the $k \rightarrow 0$ limit in relativistic dynamical equations such as (2.1). We reckon that in the parameterization of $\{d\theta^A\} = \{d\theta^{\hat{0}}, d\theta^a\}$, eqs. (A.13) and (A.14), hold:

$$d\theta^{\hat{0}} - \varphi_a \theta^a \wedge \theta^{\hat{0}} + k \varpi_{ab} \theta^a \wedge \theta^b = 0, \quad d\theta^c + \frac{1}{k} \hat{\gamma}^c_a \theta^a \wedge \theta^{\hat{0}} + \frac{1}{2} \hat{c}^c_{ab} \theta^a \wedge \theta^b = 0. \quad (\text{A.46})$$

Thus the Levi-Civita affine connection one-form reads:

$$\begin{aligned} \omega_{ab} &= - \left(k \varpi_{ab} + \frac{1}{k} \hat{\gamma}_{[ab]} \right) \theta^{\hat{0}} + \delta_{ad} \hat{\gamma}_{cb}^d \hat{\theta}^c \\ &= \left(k^2 \varpi_{ab} + \hat{\gamma}_{[ab]} \right) \boldsymbol{\mu} + \delta_{ad} \hat{\gamma}_{cb}^d \hat{\theta}^c \\ &= k^2 \varpi_{ab} \boldsymbol{\mu} + \hat{\omega}_{ab}, \end{aligned} \quad (\text{A.47})$$

and

$$\omega^{\hat{0}}_a = \varphi_a \theta^{\hat{0}} - k \varpi_{ab} \theta^b + \frac{1}{k} \hat{\gamma}_{(ab)} \theta^b = -k \left(\varphi_a \boldsymbol{\mu} + \varpi_{ab} \hat{\theta}^b \right) + \frac{1}{k} \hat{\gamma}_{(ab)} \hat{\theta}^b \quad (\text{A.48})$$

with $\hat{\gamma}_{ab}^c$ as in (A.15). It has zero torsion and the curvature reads:

$$\begin{aligned} \mathcal{R}^{\hat{0}}_a &= \left[\frac{1}{k} \left(\hat{\nabla}_v \hat{\gamma}_{(ab)} + \hat{\gamma}_{(ac)} \hat{\gamma}_{(bd)} \delta^{cd} \right) - k \left(\varpi_a^c \hat{\gamma}_{(cb)} + \varpi_b^c \hat{\gamma}_{(ca)} + \hat{\nabla}_{(a} \varphi_{b)} + \varphi_a \varphi_b \right) \right. \\ &\quad \left. + k^3 \varpi_a^c \varpi_{bc} \right] \hat{\theta}^b \wedge \mu + \frac{1}{2} \left[\frac{1}{k} \left(\hat{R}_{bac} - \varphi_b \hat{\gamma}_{(ac)} + \varphi_c \hat{\gamma}_{(ab)} \right) \right. \\ &\quad \left. - k \left(\hat{\nabla}_a \varpi_{bc} + \varphi_a \varpi_{bc} + \varphi_b \varpi_{ac} - \varphi_c \varpi_{ab} \right) \right] \hat{\theta}^b \wedge \hat{\theta}^c, \end{aligned} \quad (\text{A.49})$$

$$\begin{aligned} \mathcal{R}^a_b &= \hat{\mathcal{R}}^a_b + \delta^{ad} \left[\varphi_d \hat{\gamma}_{(cb)} - \varphi_b \hat{\gamma}_{(cd)} + k^2 \left(\hat{\nabla}_c \varpi_{db} + \varphi_c \varpi_{db} + \varphi_d \varpi_{cb} - \varphi_b \varpi_{cd} \right) \right] \hat{\theta}^c \wedge \mu \\ &\quad + \frac{1}{2} \delta^{ae} \left[\frac{1}{k^2} \left(\gamma_{(ec)} \gamma_{(bd)} - \gamma_{(ed)} \gamma_{(bc)} \right) - \gamma_{(ec)} \varpi_{bd} + \gamma_{(ed)} \varpi_{bc} \right. \\ &\quad \left. - \gamma_{(bd)} \varpi_{ec} + \gamma_{(bc)} \varpi_{ed} + k^2 (2\varpi_{eb} \varpi_{cd} - \varpi_{ed} \varpi_{bc} + \varpi_{ec} \varpi_{bd}) \right] \hat{\theta}^c \wedge \hat{\theta}^d, \end{aligned} \quad (\text{A.50})$$

where we have used the Carrollian expressions available in (A.21), (A.22) and (A.23).

We would like now to make the contact with the Carrollian descendants. The relativistic congruence is $u = -k\hat{\theta}^0$ see (2.9). Given the connection, we can determine its kinematical properties: the expansion Θ , the acceleration a_A , the shear σ_{AB} and the vorticity ω_{AB} as defined in eqs. (2.11), (2.12), (2.13). The latter tensors are all transverse (and traceless for the last two) and have thus non-vanishing components in spatial directions only (indices a, b, \dots). We find

$$\Theta = \theta = \hat{\gamma}^c_c, \quad a_a = k^2 \varphi_a, \quad (\text{A.51})$$

and

$$\sigma_{ab} = \xi_{ab} = \hat{\gamma}_{(ab)} - \frac{\theta}{d} \delta_{ab}, \quad \omega_{ab} = k^2 \varpi_{ab}. \quad (\text{A.52})$$

We can furthermore determine the Weyl connection (2.10) (where we must trade the 2 for d)

$$A = \varphi_a \hat{\theta}^a - \frac{\theta}{d} \mu, \quad (\text{A.53})$$

and its curvature (2.20):

$$F = dA = \frac{1}{2} \Omega_{ab} \hat{\theta}^a \wedge \hat{\theta}^b + \hat{\mathcal{H}}_a \hat{\theta}^a \wedge \mu, \quad (\text{A.54})$$

where Ω_{ab} and $\hat{\mathcal{H}}_a$ are defined in eqs. (A.34) and (A.36) — explicitly

$$\hat{\mathcal{H}}_a = \hat{\nabla}_v \varphi_a + \xi_{ab} \varphi^b - \frac{1}{d} \hat{e}_a(\theta). \quad (\text{A.55})$$

All the above quantities are relativistic, but expressed in terms of the Carrollian descendants describing the properties of the manifold reached at vanishing- k .

We can finally convey the relativistic conservation equations (2.1) for an arbitrary energy-momentum tensor T^{AB} as in (2.33), stated in Carrollian language. Given the choice of congruence, the transverse heat current and stress tensor have only spatial components:

q^a and τ^{ab} . We then define as usual the longitudinal and transverse components of the conservation equations,

$$\mathcal{L} = -u^B \nabla_C T^C_B = -k \nabla_C T^C_{\hat{0}} = -\nabla_C T^C_{\hat{u}}, \quad \mathcal{T}^a = e^a_B \nabla_C T^{CB} = \nabla_C T^{Ca}, \quad (\text{A.56})$$

and explicitly find:

$$\mathcal{L} = \mathbf{v}(\varepsilon) + \theta \varepsilon + \left(\hat{\nabla}_a + 2\varphi_a \right) q^a + \left(\xi_{ab} + \frac{\theta}{d} \delta_{ab} \right) \left(\tau^{ab} + p \delta^{ab} \right), \quad (\text{A.57})$$

$$\mathcal{T}^a = \left(\hat{\nabla}_b + \varphi_b \right) \left(\tau^{ab} + p \delta^{ab} \right) + \varphi^a \varepsilon + 2q_b \varpi^{ba} + \frac{1}{k^2} \left(\hat{\nabla}_b q^a + \frac{d+1}{d} \theta q^a + \xi^{ab} q_b \right). \quad (\text{A.58})$$

In the conformal case, assuming thus $\varepsilon = dp$ and $\tau_a^a = 0$ and canonical conformal weights $d+1$ for ε , q^a and τ^{ab} (we are in Cartan' frame and the weights do not depend on the position of the indices), these equations are recast as:

$$\mathcal{L} = \hat{\mathcal{D}}_{\mathbf{v}} \varepsilon + \hat{\mathcal{D}}_a q^a + \xi_{ab} \tau^{ab}, \quad (\text{A.59})$$

$$\mathcal{T}^a = \frac{1}{d} \hat{\mathcal{D}}^a \varepsilon + \hat{\mathcal{D}}_b \tau^{ab} + 2q_b \varpi^{ba} + \frac{1}{k^2} \left(\hat{\mathcal{D}}_b q^a + \xi^{ab} q_b \right). \quad (\text{A.60})$$

As discussed extensively in refs. [45, 46], the outcome of the Carrollian limit depends on the behaviour of ε , q^a and τ^{ab} with respect to k . The equations at hand will be conceivably multiplied, leading to replicas. The same phenomenon occurs in the Galilean limit with the emergence of the continuity equation out of the relativistic longitudinal equation, besides the energy equation.

We would like to close this section with some formulas that are useful when considering the zero- k limit, leading in particular to the flux-balance equation (3.47). In the following, we reduce the Riemannian Levi-Civita and Weyl covariant derivatives in terms of the Carrollian connections introduced earlier.

Levi-Civita We will present the vector and the rank-two tensor:

$$\mathbf{V} = V^A \mathbf{e}_A \quad - \quad V^a \quad \text{provide the components of a Carrollian vector and} \\ V_{\hat{u}} = k V_{\hat{0}} = -k V^{\hat{0}} \quad \text{a Carrollian scalar}$$

$$\begin{cases} k^2 \nabla_{\hat{0}} V^{\hat{0}} = k \mathbf{v} \left(V^{\hat{0}} \right) + k^2 \varphi_a V^a \\ k \nabla_{\hat{0}} V^b = \hat{\nabla}_{\mathbf{v}} V^b + k V^{\hat{0}} \varphi^b + k^2 V^a \varpi_a^b \\ k \nabla_a V^{\hat{0}} = k \hat{e}_a \left(V^{\hat{0}} \right) + \left(\xi_{ab} + \frac{\theta}{d} \delta_{ab} + k^2 \varpi_{ab} \right) V^b \\ \nabla_a V^b = \hat{\nabla}_a V^b + \frac{1}{k} \left(\xi_a^b + \frac{\theta}{d} \delta_a^b + k^2 \varpi_a^b \right) V^{\hat{0}}; \end{cases} \quad (\text{A.61})$$

$$\mathbf{T} = T^{AB} \mathbf{e}_A \otimes \mathbf{e}_B \quad - \quad T^{ab} \quad \text{are farther interpreted as components of a Carrollian rank-} \\ \text{two tensor, } T_{\hat{u}}^a = k T_{\hat{0}}^a = -k T^{\hat{0}a} \quad \text{and } T_{\hat{u}}^a = k T_{\hat{0}}^a = -k T^{a\hat{0}} \quad \text{those of Carrollian}$$

vectors, while $T_{\hat{u}\hat{u}} = k^2 T_{\hat{0}\hat{0}} = k^2 T^{\hat{0}\hat{0}}$ gives a Carrollian scalar

$$\begin{cases} k^3 \nabla_{\hat{0}} T^{\hat{0}\hat{0}} = k^2 \mathbf{v} (T^{\hat{0}\hat{0}}) + k^3 \varphi_a (T^{a\hat{0}} + T^{\hat{0}a}) \\ k^2 \nabla_{\hat{0}} T^{b\hat{0}} = k \hat{\nabla}_{\mathbf{v}} T^{b\hat{0}} + k^2 \varphi^b T^{\hat{0}\hat{0}} + k^2 \varphi_a T^{ba} + k^3 \varpi_a^b T^{a\hat{0}} \\ k \nabla_{\hat{0}} T^{ab} = \hat{\nabla}_{\mathbf{v}} T^{ab} + k (\varphi^a T^{\hat{0}b} + \varphi^b T^{a\hat{0}}) + k^2 (T^{ac} \varpi_c^b + T^{cb} \varpi_c^a) \\ k \nabla_a T^{b\hat{0}} = k \hat{\nabla}_a T^{b\hat{0}} + (\xi_{ac} + \frac{\theta}{d} \delta_{ac} + k^2 \varpi_{ac}) T^{bc} + (\xi_a^b + \frac{\theta}{d} \delta_a^b + k^2 \varpi_a^b) T^{\hat{0}\hat{0}} \\ k^2 \nabla_a T^{\hat{0}\hat{0}} = k^2 \hat{\nabla}_a (T^{\hat{0}\hat{0}}) + k (\xi_{ac} + \frac{\theta}{d} \delta_{ac} + k^2 \varpi_{ac}) T^{c\hat{0}} + k (\xi_{ac} + \frac{\theta}{d} \delta_{ac} + k^2 \varpi_{ac}) T^{\hat{0}c} \\ \nabla_a T^{bc} = \hat{\nabla}_a T^{bc} + \frac{1}{k} (\xi_a^b + \frac{\theta}{d} \delta_a^b + k^2 \varpi_a^b) T^{\hat{0}c} + \frac{1}{k} (\xi_a^c + \frac{\theta}{d} \delta_a^c + k^2 \varpi_a^c) T^{b\hat{0}}; \end{cases} \quad (\text{A.62})$$

Weyl similarly:

$$\begin{aligned} \mathbf{V} &= V^A \mathbf{e}_A \\ \begin{cases} k^2 \mathcal{D}_{\hat{0}} V^{\hat{0}} = k \hat{\mathcal{D}}_{\mathbf{v}} V^{\hat{0}} \\ k \mathcal{D}_{\hat{0}} V^b = \hat{\mathcal{D}}_{\mathbf{v}} V^b + k^2 V^a \varpi_a^b \\ k \mathcal{D}_a V^{\hat{0}} = k \hat{\mathcal{D}}_a V^{\hat{0}} + (\xi_{ab} + k^2 \varpi_{ab}) V^b \\ \mathcal{D}_a V^b = \hat{\mathcal{D}}_a V^b + \frac{1}{k} (\xi_a^b + k^2 \varpi_a^b) V^{\hat{0}}; \end{cases} \end{aligned} \quad (\text{A.63})$$

$$\mathbf{T} = T^{AB} \mathbf{e}_A \otimes \mathbf{e}_B$$

$$\begin{cases} k^3 \mathcal{D}_{\hat{0}} T^{\hat{0}\hat{0}} = \hat{\mathcal{D}}_{\mathbf{v}} (k^2 T^{\hat{0}\hat{0}}) \\ k^2 \mathcal{D}_{\hat{0}} T^{\hat{0}b} = \hat{\mathcal{D}}_{\mathbf{v}} (k T^{\hat{0}b}) + k^3 T^{\hat{0}a} \varpi_a^b \\ k \mathcal{D}_{\hat{0}} T^{ab} = \hat{\mathcal{D}}_{\mathbf{v}} T^{ab} + k^2 (T^{cb} \varpi_c^a + T^{ac} \varpi_c^b) \\ k^2 \mathcal{D}_a T^{\hat{0}\hat{0}} = \hat{\mathcal{D}}_a (k^2 T^{\hat{0}\hat{0}}) + (\xi_{ab} + k^2 \varpi_{ab}) k T^{b\hat{0}} + (\xi_{ab} + k^2 \varpi_{ab}) k T^{\hat{0}b} \\ k \mathcal{D}_a T^{\hat{0}b} = \hat{\mathcal{D}}_a (k T^{\hat{0}b}) + (\xi_{ac} + k^2 \varpi_{ac}) T^{cb} + (\xi_a^b + k^2 \varpi_a^b) T^{\hat{0}\hat{0}} \\ \mathcal{D}_a T^{bc} = \hat{\mathcal{D}}_a T^{bc} + \frac{1}{k} (\xi_a^b + k^2 \varpi_a^b) T^{\hat{0}c} + \frac{1}{k} (\xi_a^c + k^2 \varpi_a^c) T^{b\hat{0}}. \end{cases} \quad (\text{A.64})$$

B The Carrollian Cotton tensors in three dimensions

The Cotton tensor introduced in section 2 can be decomposed in terms of Carrollian descendants, which obey Carrollian identities resulting from (2.39). In Papapetrou-Randers' frame and for vanishing ξ_{ab} a thorough exhibition is available in appendix C of [25]. The Carrollian Cotton tensor will be investigated from a more general viewpoint in [24]. Here we will summarize its properties in Cartan' frame with $\xi_{ab} \neq 0$. Prior to this presentation we need to spend some time on $d = 2$.

In three boundary spacetime dimensions, we pointed out that given a congruence \mathbf{u} , a transverse Hodge duality can be designed mapping transverse vectors to transverse vectors and symmetric, traceless and transverse two-tensors onto similar objects, eqs. (2.28), (2.29), (2.31), (2.32). This procedure is readily extended to a Carroll structure $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ and the duality coincides with the Hodge duality in the 2-dimensional basis \mathcal{S} : Carrollian vectors are mapped onto Carrollian vectors and Carrollian symmetric and

traceless two-tensors onto the same class. In the relativistic Cartan frame we use here the antisymmetric pseudo-tensor is ϵ_{ABC} or ϵ^{ABC} with $\epsilon_{\hat{0}\hat{1}\hat{2}} = -\epsilon^{\hat{0}\hat{1}\hat{2}} = 1$, whereas using (2.9) in (2.28) we find $\hat{\eta}_{AB} = -\epsilon_{\hat{0}AB}$ so that only $\hat{\eta}_{ab}$ is non-zero with $\hat{\eta}_{\hat{1}\hat{2}} = -1$. We adopt this convention for the Carrollian object, without introducing any further symbol. Now (2.29) translates into

$$\hat{\eta}_{ac}\hat{\eta}_b^c = \delta_{ab}, \quad \hat{\eta}^{ab}\hat{\eta}_{ab} = 2, \quad (\text{B.1})$$

and (2.31), (2.32) give

$$*v^a = \hat{\eta}^b_a v_b, \quad *w_{ab} = \hat{\eta}^c_a w_{cb}, \quad (\text{B.2})$$

for Carrollian vectors v^a and Carrollian symmetric, traceless tensors w_{ab} . We will often use the following identities, generalizable to any tensor:

$$**v^a = -v^a, \quad *v^a w_a = -v^a *w_a. \quad (\text{B.3})$$

The Carroll-Riemann, Carroll-Ricci and scalar (A.24) curvatures read:

$$\hat{R}_{abcd} = \hat{K}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \quad \hat{R}_{ab} = \hat{K}\delta_{ab}, \quad \hat{R} = 2\hat{K}. \quad (\text{B.4})$$

The Carroll-Weyl-Riemann and Ricci tensors, the Carroll-Weyl-Ricci scalar (see (A.35)) as well as the Carroll-Weyl tensor curvature (A.34) are

$$\hat{\mathcal{S}}_{abcd} = \hat{\mathcal{K}}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \quad \hat{\mathcal{S}}_{ab} = \hat{\mathcal{K}}\delta_{ab}, \quad \Omega_{ab} = -\hat{\mathcal{A}}\hat{\eta}_{ab}, \quad (\text{B.5})$$

expressed in terms of two weight-2 Weyl-covariant scalars:

$$\hat{\mathcal{K}} = \hat{K} + \hat{\nabla}_a \varphi^a, \quad \hat{\mathcal{A}} = *\varpi\theta - *\varphi \quad (\text{B.6})$$

with

$$*\varpi = \frac{1}{2}\hat{\eta}^{ab}\varpi_{ab} \Leftrightarrow \varpi_{ab} = \hat{\eta}_{ab}*\varpi, \quad (\text{B.7})$$

and

$$*\varphi = \frac{1}{2}\hat{\eta}^{ab}\varphi_{ab} \quad \text{where} \quad \varphi_{ab} = \hat{e}_a(\varphi_b) - \hat{e}_b(\varphi_a). \quad (\text{B.8})$$

These obey Carroll-Bianchi identities:

$$2\hat{\mathcal{D}}_\nu *\varpi + \hat{\mathcal{A}} = 0, \quad (\text{B.9})$$

$$\hat{\mathcal{D}}_\nu \hat{\mathcal{K}} - \hat{\mathcal{D}}_a \hat{\mathcal{K}}^a - \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b \xi^{ab} = 0, \quad (\text{B.10})$$

$$\hat{\mathcal{D}}_\nu \hat{\mathcal{A}} + \hat{\eta}^{ab}\hat{\mathcal{D}}_a \hat{\mathcal{K}}_b = 0. \quad (\text{B.11})$$

The Carroll reduction of the Cotton tensor is encrypted in the longitudinal, mixed and transverse components (2.41) and (2.43), which encompass several weight-3 Carrollian scalars, vectors and symmetric, traceless two-tensors, dubbed ‘‘Carrollian Cotton tensors.’’ In Cartan’ frame we obtain

$$c = c_{(-1)}k^2 + c_{(0)} + \frac{c_{(1)}}{k^2} + \frac{c_{(2)}}{k^4}, \quad (\text{B.12})$$

$$c^a = k^2\psi^a + \chi^a + \frac{z^a}{k^2}, \quad (\text{B.13})$$

$$c^{ab} = k^2\Psi^{ab} + X^{ab} + \frac{Z^{ab}}{k^2} \quad (\text{B.14})$$

with

- four Carroll scalars:

$$c_{(-1)} = 8 * \varpi^3, \quad c_{(0)} = \left(\hat{\mathcal{D}}_a \hat{\mathcal{D}}^a + 2 \hat{\mathcal{K}} \right) * \varpi, \quad c_{(1)} = \hat{\mathcal{D}}_a \hat{\mathcal{D}}_b * \xi^{ab}, \quad c_{(2)} = * \xi_{ab} \hat{\mathcal{D}}_v \xi^{ab}; \quad (\text{B.15})$$

- three Carroll vectors:

$$\psi^a = 3 \hat{\eta}^{ba} \hat{\mathcal{D}}_b * \varpi^2, \quad (\text{B.16})$$

$$\chi^a = \frac{1}{2} \hat{\eta}^{ba} \hat{\mathcal{D}}_b \hat{\mathcal{K}} + \frac{1}{2} \hat{\mathcal{D}}^a \hat{\mathcal{A}} - 2 * \varpi \left(\hat{\mathcal{R}}^a + 2 \hat{\mathcal{D}}_b \xi^{ab} \right) + 3 \hat{\mathcal{D}}_b \left(* \varpi \xi^{ab} \right), \quad (\text{B.17})$$

$$z^a = \frac{1}{2} \hat{\eta}^{ab} \hat{\mathcal{D}}_b \xi^2 - \hat{\mathcal{D}}_b \hat{\mathcal{D}}_v * \xi^{ab} - * \xi^a_b \hat{\mathcal{D}}_c \xi^{bc}, \quad (\text{B.18})$$

where we defined⁴¹

$$\xi^2 = \frac{1}{2} \xi^{ab} \xi_{ab} \Leftrightarrow \xi^{ac} \xi_c^b = \xi^2 \delta^{ab}; \quad (\text{B.19})$$

- three Carroll traceless and symmetric two-index tensors:

$$\Psi^{ab} = -2 * \varpi^2 * \xi^{ab} + \hat{\mathcal{D}}^a \hat{\mathcal{D}}^b * \varpi - \frac{1}{2} \delta^{ab} \hat{\mathcal{D}}_c \hat{\mathcal{D}}^c * \varpi - \hat{\eta}^{ab} \hat{\mathcal{D}}_v * \varpi^2, \quad (\text{B.20})$$

$$X^{ab} = \frac{1}{2} \hat{\eta}^{ca} \hat{\mathcal{D}}_c \left(\hat{\mathcal{R}}^b + \hat{\mathcal{D}}_d \xi^{bd} \right) + \frac{1}{2} \hat{\eta}^{cb} \hat{\mathcal{D}}^a \left(\hat{\mathcal{R}}_c + \hat{\mathcal{D}}^d \xi_{cd} \right) - \frac{3}{2} \hat{\mathcal{A}} \xi^{ab} - \hat{\mathcal{K}} * \xi^{ab} + 3 * \varpi \hat{\mathcal{D}}_v \xi^{ab}, \quad (\text{B.21})$$

$$Z^{ab} = 2 * \xi^{ab} \xi^2 - \hat{\mathcal{D}}_v \hat{\mathcal{D}}_v * \xi^{ab}. \quad (\text{B.22})$$

As for the conservation equations (2.39), expressing them as in (A.59), (A.60), they yield

$$\mathcal{L}_{\text{Cot}} = -k^3 \mathcal{D}_{\text{Cot}} - k \mathcal{E}_{\text{Cot}} - \frac{\mathcal{F}_{\text{Cot}}}{k} - \frac{\mathcal{W}_{\text{Cot}}}{k^3} = 0, \quad (\text{B.23})$$

$$\mathcal{T}_{\text{Cot}}^a = k^3 \mathcal{I}_{\text{Cot}}^a + k \mathcal{G}_{\text{Cot}}^a + \frac{\mathcal{H}_{\text{Cot}}^a}{k} + \frac{\mathcal{X}_{\text{Cot}}^a}{k^3} = 0 \quad (\text{B.24})$$

with

$$\mathcal{D}_{\text{Cot}} = -\hat{\mathcal{D}}_v c_{(-1)} - \hat{\mathcal{D}}_a \psi^a, \quad (\text{B.25})$$

$$\mathcal{E}_{\text{Cot}} = -\hat{\mathcal{D}}_v c_{(0)} - \hat{\mathcal{D}}_a \chi^a + \Psi_{ab} \xi^{ab}, \quad (\text{B.26})$$

$$\mathcal{F}_{\text{Cot}} = -\hat{\mathcal{D}}_v c_{(1)} - \hat{\mathcal{D}}_a z^a + X_{ab} \xi^{ab}, \quad (\text{B.27})$$

$$\mathcal{W}_{\text{Cot}} = -\hat{\mathcal{D}}_v c_{(2)} + Z_{ab} \xi^{ab} \quad (\text{B.28})$$

and

$$\mathcal{I}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(-1)} + 2 * \varpi * \psi^a, \quad (\text{B.29})$$

$$\mathcal{G}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(0)} - \hat{\mathcal{D}}_b \Psi^{ab} + 2 * \varpi * \chi^a + \hat{\mathcal{D}}_v \psi^a + \xi^{ab} \psi_b, \quad (\text{B.30})$$

$$\mathcal{H}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(1)} - \hat{\mathcal{D}}_b X^{ab} + 2 * \varpi * z^a + \hat{\mathcal{D}}_v \chi^a + \xi^{ab} \chi_b, \quad (\text{B.31})$$

$$\mathcal{X}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(2)} - \hat{\mathcal{D}}_b Z^{ab} + \hat{\mathcal{D}}_v z^a + \xi^{ab} z_b. \quad (\text{B.32})$$

⁴¹Many identities of this sort are useful: $\xi^{ac} * \xi_c^b = \xi^2 \hat{\eta}^{ab}$, $* \xi^{ac} * \xi_c^b = \xi^2 \delta^{ab}$, $\varpi^{ac} \varpi_c^b = * \varpi^2 \delta^{ab}$.

The four couples of equations $\{\mathcal{D}_{\text{Cot}} = 0, \mathcal{I}_{\text{Cot}}^a = 0\}$, $\{\mathcal{E}_{\text{Cot}} = 0, \mathcal{G}_{\text{Cot}}^a = 0\}$, $\{\mathcal{F}_{\text{Cot}} = 0, \mathcal{H}_{\text{Cot}}^a = 0\}$ and $\{\mathcal{W}_{\text{Cot}} = 0, \mathcal{X}_{\text{Cot}}^a = 0\}$ originate from the different orders in k in which the conservation of the Cotton tensor (2.39) decomposes. These are *purely geometrical identities fulfilled on any three-dimensional Carroll structure* $\mathcal{M} = \mathbb{R} \times \mathcal{S}$. Moreover, they are typical Carrollian conservation equations obtained as a consequence of general covariance applied to a Weyl-invariant action $S = -\frac{1}{2} \int_{\mathcal{M}} \hat{\eta}_{ab} \hat{\theta}^a \wedge \hat{\theta}^b \wedge \mu \mathcal{L}$ defined on $\mathcal{M} = \mathbb{R} \times \mathcal{S}$:

$$\hat{\mathcal{D}}_{\nu} \Pi + \hat{\mathcal{D}}_a \Pi^a + \Upsilon^a_b \xi_a^b = 0, \tag{B.33}$$

$$\frac{1}{2} \hat{\mathcal{D}}_a \Pi + \hat{\mathcal{D}}_b \Upsilon^b_a + 2 * \varpi * \Pi^a + \hat{\mathcal{D}}_{\nu} P_a + \xi_a^b P_b = 0. \tag{B.34}$$

The momenta Π , Π^a , P_b and $\Pi^a_b = \Upsilon^a_b + \frac{1}{2} \Pi \delta^a_b$ are defined as variations of the action with respect to the triad $\{\mu, \hat{\theta}^a\}$ (the explicit computation is accessible in ref. [45] for the Papapetrou-Randers frame,⁴² where the organizing pattern is the subgroup of Carrollian diffeomorphisms instead of the subgroup of local orthogonal transformations in the tangent space). These are the energy density, the energy flux, the momentum and the stress.

For Carroll structures with vanishing Carrollian shear, $\xi_{ab} = 0$, met e.g. at null infinity of asymptotically flat spacetimes, six out of the ten Carroll Cotton tensors survive: $c_{(-1)}$, $c_{(0)}$, ψ^a as in eqs. (B.15), (B.16) and χ^a , Ψ^{ab} , X^{ab} . Using eqs. (B.17), (B.20), (B.21) we find the simplified expressions of the latter:

$$\chi^a = \frac{1}{2} * \hat{\mathcal{D}}^a \mathcal{K} + \frac{1}{2} \hat{\mathcal{D}}^a \hat{\mathcal{A}} - 2 * \varpi \hat{\mathcal{K}}^a, \tag{B.35}$$

$$\Psi^{ab} = \hat{\mathcal{D}}^a \hat{\mathcal{D}}^b * \varpi - \frac{1}{2} \delta^{ab} \hat{\mathcal{D}}_c \hat{\mathcal{D}}^c * \varpi - \hat{\eta}^{ab} \hat{\mathcal{D}}_{\nu} * \varpi^2, \tag{B.36}$$

$$X^{ab} = \frac{1}{2} * \hat{\mathcal{D}}^a \hat{\mathcal{K}}^b + \frac{1}{2} \hat{\mathcal{D}}^a * \hat{\mathcal{K}}^b. \tag{B.37}$$

These tensors now obey

$$\mathcal{D}_{\text{Cot}} = -\hat{\mathcal{D}}_{\nu} c_{(-1)} - \hat{\mathcal{D}}_a \psi^a = 0, \tag{B.38}$$

$$\mathcal{E}_{\text{Cot}} = -\hat{\mathcal{D}}_{\nu} c_{(0)} - \hat{\mathcal{D}}_a \chi^a = 0, \tag{B.39}$$

and

$$\mathcal{I}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(-1)} + 2 * \varpi * \psi^a = 0, \tag{B.40}$$

$$\mathcal{G}_{\text{Cot}}^a = \frac{1}{2} \hat{\mathcal{D}}^a c_{(0)} - \hat{\mathcal{D}}_b \Psi^{ab} + 2 * \varpi * \chi^a + \hat{\mathcal{D}}_{\nu} \psi^a = 0, \tag{B.41}$$

$$\mathcal{H}_{\text{Cot}}^a = -\hat{\mathcal{D}}_b X^{ab} + \hat{\mathcal{D}}_{\nu} \chi^a = 0. \tag{B.42}$$

On a Carroll manifold in Cartan frame, the degenerate metric is invariant under local Carroll-group transformations. Invariance of the action under its local orthogonal subgroup is in line with a symmetric Π_{ab} ; invariance under local Carroll boosts demands $\Pi^a = 0$. This

⁴²Equations (B.33) and (B.34) were obtained for the first time in ref. [46]. They have been recently rediscussed in [84].

is not always met in Carrollian theories approached from relativistic theories at vanishing speed of light (see e.g. [85]) — alternatively it can be imposed by hand as in [86]. The Cotton tensor and the corresponding Chern-Simons dynamics [24] admirably illustrate this feature, which persists in the flux-balance equations of Ricci flat spacetimes, powered by gravitational radiation.

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Ehlers, Carroll, charges and dual charges

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ABSTRACT: We unravel the boundary manifestation of Ehlers' hidden Möbius symmetry present in four-dimensional Ricci-flat spacetimes that enjoy a time-like isometry and are Petrov-algebraic. This is achieved in a designated gauge, shaped in the spirit of flat holography, where the Carrollian three-dimensional nature of the null conformal boundary is manifest and covariantly implemented. The action of the Möbius group is local on the space of Carrollian boundary data, among which the Carrollian Cotton tensor plays a predominant role. The Carrollian and Weyl geometric tools introduced for shaping an appropriate gauge, as well as the boundary conformal group, which is BMS_4 , allow to define electric/magnetic, leading/subleading towers of charges directly from the boundary Carrollian dynamics and explore their behaviour under the action of the Möbius duality group.

KEYWORDS: Classical Theories of Gravity, Gauge-Gravity Correspondence, Global Symmetries, Space-Time Symmetries

ARXIV EPRINT: [2212.14062](https://arxiv.org/abs/2212.14062)

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1 Introduction

Hidden symmetries have a long history in relativistic theories of gravity, which started with the seminal work of Ehlers in the late fifties [1]. It was shown in this article that in the presence of an isometry, vacuum Einstein’s equations were invariant under Möbius transformations. This observation triggered an important activity in several directions. In line with the sixties’ renaissance of general relativity, it opened the way for solution-generating techniques applicable to vacuum Einstein’s equations [2, 3]. This was soon generalized to situations with more commuting Killing fields [4, 5] — and bigger hidden symmetry group, providing the system with remarkable and unexpected integrability properties [6–14]. The underlying deep origin for the above pattern was unravelled with the advent of higher-dimensional supergravity theories, and is rooted in the reduction mechanism. This has revealed a wide class of hidden groups, among which the exceptional play a prominent role (see e.g. [15–17], or [18] for a more recent presentation and further references).

The integrable sector of Einstein’s equations is only a tiny fraction of their solution space. Unveiling the latter, in conjunction with its asymptotic symmetries and conserved charges, has been in the very early agenda of general relativity. It shares features with

gauge theories because of general covariance, and has led Bondi to come out with his homonymous gauge, where a systematic resolution of Einstein’s equations is possible as an expansion in powers of a radial coordinate.¹ This delivers a set of functions of time and angular coordinates, obeying first-order time-evolution equations. For Ricci-flat spacetimes (asymptotically locally flat) this set is infinite, but it is finite for Einstein spacetimes with negative cosmological constant (asymptotically locally anti-de Sitter).

In modern language, the set of functions necessary for reconstructing the solution are said to be defined on the conformal boundary of the spacetime. In the asymptotically flat instance the conformal boundary is null infinity and features a Carrollian three-dimensional hypersurface.² Bulk Einstein dynamics is therefore traded for boundary effective Carrollian conformal field dynamics. This statement is accurate when discussing Einstein’s equations. Whether it could be promoted to a holographic principle akin to the better known AdS/CFT involving asymptotically anti-de Sitter spacetimes and conformal field theories defined on their time-like conformal boundary is a timely subject, currently under scrutiny.

How do hidden symmetries such as Ehlers’ act on the Carrollian boundary data? This is the central question we would like to address in the present work. The conformal symmetries of the boundary reflect the asymptotic symmetries of the bulk. These define for instance the BMS_4 algebra (Bondi-van der Burg-Metzner-Sachs [27–29]), which is isomorphic to the conformal Carroll algebra in three dimensions $\mathfrak{ccarr}(3)$ (see [30, 31]), and emerges upon appropriate fall-off conditions. From this perspective, wondering how the bulk hidden symmetries are embraced by the Carrollian boundary and what their interplay is with $BMS_4 \equiv \mathfrak{ccarr}(3)$, is both natural and relevant.

There is yet another motivation for pursuing this analysis. Following Geroch [2, 3], the action of some Ehlers subgroup is a duality rotation in the plane of gravitational electric and gravitational magnetic charges, as are e.g. the mass and the nut charge. Ricci-flat spacetimes possess in fact multiple infinities of charges (not necessarily conserved), incarnated in pairs of electric and magnetic representatives, and originating from the infinitely many independent “subleading” degrees of freedom necessary for reconstructing the bulk solution, as well as the infinitely many generators of the asymptotic symmetry group. This picture has been widely conveyed through the work of Godazgar-Godazgar-Pope [32–34] (see also [35–39]) and amply deserves to be reconsidered in the light of hidden symmetries. The remarkable fact is here that such an analysis can be conducted exclusively on the boundary, where the charges are constructed (see e.g. [40]) using the boundary dynamics combined with the three-dimensional-boundary Carrollian conformal isometries, the latter being always generated by the infinite-dimensional algebra $BMS_4 \equiv \mathfrak{so}(3, 1) \ltimes$ supertranslations [41]. Translating the Ehlers group on the null boundary forcedly exhibits a mapping among the infinite towers of charges, which is obscured in a bulk approach. The boundary Carrollian geometry provides the most suitable language for clarifying these properties.

¹Other canonical gauges are Fefferman-Graham or Newman-Unti — see e.g. refs. [19, 20] for a review and more complete reading suggestions on this subject.

²The original observation that triggered this “flat-holography” activity is described in refs. [21, 22]. A more systematic analysis in four dimensions was presented in [23], which set the foundations for a Carrollian description of the dual theory, and provides a more complete reference list. Up-to-date developments in this vein are refs. [24–26].

In the present work, we will analyse along the above lines the integrable sector of Ricci-flat spacetimes possessing a time-like Killing field, whose congruence coincides with the boundary Carrollian fiber. This sector is obtained by setting conditions on the boundary data, which ultimately guarantee that the infinite series in powers of the radial coordinates is resummed. The boundary conditions involve the Carrollian boundary Cotton tensor and the Carrollian boundary momenta (see [23]), which both enter the boundary computation of the charges associated with the solution at hand. They rephrase the special structure of bulk Weyl tensor³ and unsurprisingly lead to algebraic Ricci-flat spacetimes. Although this class leaves interesting cases aside, it captures the main feature of the Ehlers-group boundary manifestation. The latter turns out to be an *algebraic* transformation mixing e.g. the Carrollian Cotton scalar and the Bondi mass aspect. This mixing is transmitted to other boundary observables, including the charges through their boundary expression, and completes the picture of the bulk-and-boundary action of the hidden group. The action of the Möbius group, generically non-local on the four-dimensional Ricci-flat metric, is therefore *local* on the boundary — as it is on the three-dimensional sigma-model of the reduction along the bulk Killing congruence.

The starting point of our study is a reminder on the Ehlers group and the Geroch method, as they emerge in the reduction of Ricci-flat spacetimes along orbits of one-dimensional groups of motions. We next move and describe the bulk-to-boundary relationship for four-dimensional Ricci-flat spacetimes. This requires the use of a gauge (we call it “modified Newman-Unti” or “covariantized”) in which the three-dimensional Carrollian boundary geometry is ostensible. The bulk metric in this gauge is manifestly covariant with respect to the boundary Carrollian diffeomorphisms and to the boundary Weyl transformations. The Carrollian boundary dynamics induced by the bulk Einstein’s equations is the following item in our agenda, which further enables us to define sets of charges and dual charges — electric and magnetic. Finally, using the available tools for a Ricci-flat spacetime enjoying a time-like isometry, we translate the action of the bulk Möbius transformations onto the boundary observables. This analysis is performed for the integrable sector (resummable metrics) and is based on a specific class of time-like Killing fields.

Most of our investigation relies on rather unusual geometric tools, which have been developed recently in the framework of Carroll structures. We have sorted them out in a first appendix, valid for any dimension $d + 1$. Carrollian dynamics and conservation properties, necessary for describing the boundary perspective as inherited from bulk Einstein’s equations, is summarized in the second appendix. The third appendix is specific to three dimensions with emphasis on the Carrollian Cotton tensors.

³The interplay between the bulk Weyl tensor, expanded in powers of the radial coordinate, and the boundary Cotton plus energy-momentum tensors for Einstein spacetimes was disclosed in [42–44] — see also [45]. There is no rigorous similar statement for the Ricci-flat instance since the Carrollian relatives of the Cotton tensor have not yet been thoroughly investigated.

2 Ehlers and Geroch

We remind here Geroch' generalization of Ehlers' work following [2]. We consider a four-dimensional pseudo-Riemannian manifold (\mathcal{M}, g) possessing an isometry generated by a time-like⁴ Killing vector field ξ . The latter has norm and twist — here $A, B, \dots \in \{0, \dots, 3\}$:

$$\lambda = \xi^A \xi_A, \tag{2.1}$$

$$w_A = \eta_{ABCD} \xi^B \nabla^C \xi^D, \tag{2.2}$$

where $\eta_{ABCD} = \sqrt{-g} \epsilon_{ABCD}$ ($\epsilon_{0123} = 1$). Assuming the spacetime be Ricci-flat,⁵ one shows that the one-form $w = w_A dx^A$ is closed so locally exact, hence

$$w = d\omega \tag{2.3}$$

with ω a scalar function.

We define the three-dimensional space \mathcal{S} as the quotient $\mathcal{M}/\text{orb}(\xi)$. This coset space is not a subspace of \mathcal{M} unless ξ is hypersurface-orthogonal, which would imply zero twist with \mathcal{S} the orthogonal hypersurface. A natural metric on \mathcal{S} is induced by g of \mathcal{M} :

$$h_{AB} = g_{AB} - \frac{\xi_A \xi_B}{\lambda}, \tag{2.4}$$

which defines the projector onto \mathcal{S} as

$$h^B_A = \delta^B_A - \frac{\xi^B \xi_A}{\lambda}. \tag{2.5}$$

The fully antisymmetric tensor for (2.4) is $\eta_{ABC} = \frac{-1}{\sqrt{-\lambda}} \eta_{ABCD} \xi^D$.

Tensors of \mathcal{M} , transverse and invariant with respect to ξ , are in one-to-one correspondence with tensors on \mathcal{S} . If T is a tensor of \mathcal{S} , the covariant derivative \mathcal{D} defined following this correspondence,

$$\mathcal{D}_C T_{A_1 \dots A_p}{}^{B_1 \dots B_q} = h^L_C h^{M_1}_{A_1} \dots h^{M_p}_{A_p} h^{B_1}_{N_1} \dots h^{B_q}_{N_q} \nabla_L T_{M_1 \dots M_p}{}^{N_1 \dots N_q} \tag{2.6}$$

with ∇ the Levi-Civita connection on (\mathcal{M}, g) , coincides with the Levi-Civita connection on (\mathcal{S}, h) . This sets a relationship between the Riemann tensor on \mathcal{S} and the Riemann tensor on \mathcal{M} , generalizing thereby the Gauss-Codazzi equations to the instance where ξ is not-hypersurface orthogonal:

$$\mathcal{R}_{ABCD} = h^P_{[A} h^Q_{B]} h^R_{[C} h^S_{D]} \left(R_{PQRS} + \frac{2}{\lambda} (\nabla_P \xi_Q \nabla_R \xi_S + \nabla_P \xi_R \nabla_Q \xi_S) \right) \tag{2.7}$$

(the calligraphic letters refer to curvature tensors of \mathcal{S}).

The Ricci-flat dynamics for g_{AB} is recast in the present framework in terms of⁶

$$\tilde{h}_{AB} = \lambda h_{AB}, \tag{2.8}$$

⁴The described procedure goes through in the same fashion with space-like isometries, but keeping the two options would bring unnecessary multiplication of indices without shedding more light on our purpose.

⁵This property actually holds more generally for Einstein spacetimes [46].

⁶With our conventions, this metric is definite-negative.

as well as ω and λ viewed as fields on \mathcal{S} , packaged in

$$\tau = \omega + i\lambda, \tag{2.9}$$

and obeying the following equations:⁷

$$\begin{aligned} \tilde{\mathcal{R}}_{AB} &= -\frac{2}{(\tau - \bar{\tau})^2} \tilde{\mathcal{D}}_{(A} \tau \tilde{\mathcal{D}}_{B)} \bar{\tau}, \\ \tilde{\mathcal{D}}^2 \tau &= \frac{2}{\tau - \bar{\tau}} \tilde{\mathcal{D}}_M \tau \tilde{\mathcal{D}}_N \tau \tilde{h}^{MN}. \end{aligned} \tag{2.10}$$

The first results from (2.7), while the second is obtained by a direct computation of the \mathcal{S} -Laplacian acting on τ . Here $\tilde{\mathcal{D}}_A$ and $\tilde{\mathcal{R}}_{AB}$ are the Levi-Civita covariant derivative and the Ricci tensor associated with the metric \tilde{h}_{AB} displayed in (2.8).

Equations (2.10) feature two important properties. The first, due to Ehlers [1], is the invariance under transformations maintaining \tilde{h}_{AB} *unaltered* and mapping τ into

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2, \mathbb{R}). \tag{2.11}$$

This is the original instance where a *hidden* group, $\text{SL}(2, \mathbb{R})$, reveals upon reduction with respect to an isometry. The second, described by Geroch in [2, 3], is the method for reversing the reduction process, and finding a Ricci-flat four-dimensional spacetime with an isometry, starting from any solution of eqs. (2.10) encoded in $\omega' + i\lambda' = \tau'$ and $h'_{AB} = \frac{1}{\lambda'} \tilde{h}_{AB}$. To this end, one shows that the \mathcal{S} -two-form defined as

$$F'_{AB} = \frac{1}{(-\lambda')^{3/2}} \eta'_{ABC} \mathcal{D}^C \omega' \tag{2.12}$$

is *closed*. Thus, locally

$$F' = d\eta'. \tag{2.13}$$

The one-form field η' , defined on \mathcal{S} , can be promoted to a field on \mathcal{M} by adding the necessary exact piece such that its normalization be

$$\xi^A \eta'_A = 1. \tag{2.14}$$

This defines a new Killing field on \mathcal{M}

$$\xi' = \lambda' \eta' \tag{2.15}$$

and the new four-dimensional metric reads:⁸

$$g'_{AB} = h'_{AB} + \frac{\xi'_A \xi'_B}{\lambda'}. \tag{2.16}$$

⁷Equations (2.10) can be reached by varying a three-dimensional sigma-model action defined on \mathcal{S} . This is at the heart of many developments about integrability and hidden symmetries — see the already quoted literature for more information.

⁸The consequence of Möbius transformations on the Weyl tensor has been investigated in ref. [47].

Closing this executive reminder, we would like to add a remark. The $SL(2, \mathbb{R})$ is hidden from the four-dimensional perspective, but explicit in the three-dimensional sigma-model, materialized here in eqs. (2.10). Nevertheless, part of this group is in fact visible in four dimensions because it acts as four-dimensional diffeomorphisms; part is creating genuinely different Ricci-flat solutions. This can be illustrated in the concrete example of Schwarzschild-Taub-NUT solutions with mass M and nut charge n . The compact subgroup of rotations $\begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \in SO(2) \subset SL(2, \mathbb{R})$ induces rotations of angle 2χ in the parameter space (M, n) , while non-compact transformations $\begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix} \in N \subset SL(2, \mathbb{R})$ act homothetically, $(M, n) \rightarrow (M/\alpha, n/\alpha)$.

3 Ricci-flat spacetimes and Carrollian dynamics

3.1 Bulk reconstruction and resumable Ricci-flat metrics

Choosing a covariant gauge. Four-dimensional Ricci-flat metrics are generally obtained as expansions in powers of a radial coordinate, in a designated gauge, usually Bondi or Newman-Unti. Appropriate fall-offs are assumed, and the solution is expressed in terms of an infinite set of functions of time and angles, obeying some evolution equations, mirroring Einstein’s equations (see [19] for details and further references). Can one define a three-dimensional boundary, and describe covariantly this set of functions and their dynamics?

The answer to this question has been known to be positive for a long time in the case of Einstein spacetimes. It is best formulated in the Fefferman-Graham gauge [48, 49] — see also [50] for a Weyl-covariant extension of this gauge. The (conformal) boundary is a three-dimensional pseudo-Riemannian spacetime, and every order in the expansion brings a tensorial object with respect to the boundary geometry. All these are expressed in terms of two independent tensors: the first and second fundamental forms of the boundary, namely the boundary metric and the boundary energy-momentum tensor, which is covariantly conserved with respect to the associated Levi-Civita connection. This conservation translates those of Einstein’s equations that have not been used in the process of taming the expansion.

The boundary covariance of the Fefferman-Graham gauge makes it elegant and suitable for holographic applications in the framework of anti-de Sitter/conformal-field-theory correspondence. Setting up a gauge that is covariant with respect to the boundary is therefore desirable as part of the effort to unravel a similar duality for asymptotically flat spacetimes. In this case, the conformal boundary is at null infinity and is endowed with a Carrollian geometry [21, 22].

Carroll structures [30, 31, 41, 51–56] consist of a $d+1$ -dimensional manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ equipped with a degenerate metric. The kernel of the metric is a vector field called *field of observers*. We will adopt coordinates (t, \mathbf{x}) and a metric of the form

$$d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i dx^j, \quad i, j \dots \in \{1, \dots, d\} \tag{3.1}$$

with kernel

$$\mathbf{v} = \frac{1}{\Omega} \partial_t. \tag{3.2}$$

The coordinate system at hand is adapted to the space/time splitting. It is thus respected by Carrollian diffeomorphisms

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \tag{3.3}$$

with Jacobian

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J_j^i(\mathbf{x}) = \frac{\partial x'^i}{\partial x^j}. \tag{3.4}$$

The *clock form* is dual to the field of observers with $\mu(\mathbf{v}) = -1$:

$$\mu = -\Omega dt + b_i dx^i \tag{3.5}$$

(Ω and b_i depend on t and \mathbf{x}) and incorporates an *Ehresmann connection*, which is the background gauge field $\mathbf{b} = b_i dx^i$.⁹ Carrollian tensors depend on time t and space \mathbf{x} . They carry indices i, j, \dots lowered and raised with a_{ij} and its inverse a^{ij} , and transform covariantly under (3.3) with J_i^j and J^{-1j}_i defined in (3.4). The basics on Carrollian tensors and Carrollian covariant derivatives are summarized in appendix A. In the following we will focus on $d = 2$, corresponding to the three-dimensional conformal null boundary of a four-dimensional asymptotically flat spacetime, and further information on this instance is available in appendix C. The boundary Carrollian covariance is part of the bulk general covariance, as inherited in the boundary geometry.

Fefferman-Graham gauge is only valid for Einstein spacetimes, on the one hand. On the other hand, Bondi and Newman-Unti gauges, applicable to Ricci-flat spacetimes, are not covariant with respect to the boundary, because the spatial section of the three-dimensional null boundary is locked. An alternative, still of the Eddington-Finkelstein type i.e. with a light-like radial direction, was introduced in the framework of fluid/gravity correspondence [60, 61], and made more systematic in the subsequent works both in AdS [62–65] and for Ricci-flat spacetimes [23]. It is a sort of modified and slightly incomplete Newman-Unti gauge [57, 58, 66] (see also [67, 68] for other extensions of the Bondi or Newman-Unti gauges). The time coordinate t is actually a retarded time (usually spelled u) and coincides at the boundary with the Carrollian time used in (3.1), (3.2) and (3.5).

⁹A Carroll structure endowed with metric (3.1) and clock form (3.5) is naturally reached in the Carrollian limit ($c \rightarrow 0$) of a pseudo-Riemannian spacetime \mathcal{M} in Papapetrou-Randers gauge $ds^2 = -c^2 (\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j$, where all functions are x -dependent with $x \equiv (x^0 = ct, \mathbf{x})$. It should be noticed here that the degenerate metric could generally have components along dt , which would in turn give ∂_i components to the field of observers. In this instance, the above Carrollian diffeomorphisms (3.3) play no privileged role, and plain general covariance is at work — without affecting the dynamics presented in appendix B. This option is sometimes chosen (see e.g. [54] for a general approach, or [57–59] for an application to three-dimensional Minkowski spacetime), but it is always possible to single out the time direction supported by the fiber of the Carrollian structure, i.e. distinguish time and spatial sections with no conflict with general covariance.

We can summarize as follows the structure of the four-dimensional Ricci-flat solutions in the advertised gauge, up to order $1/r^2$ (G is four-dimensional Newton's constant):

$$\begin{aligned}
 ds_{\text{Ricci-flat}}^2 = & \mu \left[2dr - \left(r\theta + \hat{\mathcal{K}} \right) \mu + \left(2r\varphi_i - 2 * \hat{\mathcal{D}}_i * \varpi - \hat{\mathcal{D}}_j \mathcal{C}_i^j \right) dx^i \right] \\
 & + \mathcal{C}_{ij} \left(r dx^i dx^j - * \varpi * dx^i dx^j \right) + \left(r^2 + * \varpi^2 + \frac{\mathcal{C}_{kl} \mathcal{C}^{kl}}{8} \right) dl^2 \\
 & + \frac{1}{r} \left[8\pi G \varepsilon \mu^2 + \frac{32\pi G}{3} \left(\pi_i - \frac{1}{8\pi G} * \psi_i \right) dx^i \mu - \frac{16\pi G}{3} E_{ij} dx^i dx^j \right] \\
 & + \frac{1}{r^2} \left(* \varpi c \mu^2 + \dots \right) + \mathcal{O} \left(\frac{1}{r^3} \right), \tag{3.6}
 \end{aligned}$$

where the star designates a $d = 2$ Carrollian Hodge duality as defined in eq. (C.1).¹⁰ As anticipated, this expression is neither in Bondi gauge (no determinant condition — see [27, 28]), nor in Newman-Unti ($g_{rt} = -\Omega \neq -1$ and $g_{ri} = b_i \neq 0$, obtained using (3.5) — see [69]).¹¹ Delving into the details of this gauge would bring us outside the main purpose of the present work. We will rather explain the various ingredients appearing in the above expression and insist on their Carrollian-covariant nature. This includes the account of the required boundary data and the description of the evolution equations they obey so that the bulk metric be Ricci-flat.

All quantities entering expression (3.6) are defined on the conformal boundary and can be sorted as follows (see also the appendices for further information).

Carrollian geometry. The conformal boundary itself is part of the solution space. It is materialized in a_{ij} , b_i and Ω , accompanied with all attributes such as Carrollian connections and curvature tensors, Carrollian Cotton descendants etc. — see appendices A and C. These are free data, without evolution equations, except for the restriction of *vanishing Carrollian geometric shear as a consequence of Einstein's equations*: $\xi_{ij} = 0$.¹²

Shear. The *dynamic shear* is a symmetric and traceless Carrollian boundary tensor $\mathcal{C}_{ij}(t, \mathbf{x})$ not to be confused with the geometric shear $\xi_{ij}(t, \mathbf{x})$.¹³ It is a boundary emanation of the bulk ∂_r -congruence shear, and is completely free, although it sources the evolution equations of other tensorial data. The dynamic shear carries information on the bulk gravitational radiation through the symmetric and traceless *Bondi-like news*:

$$\hat{\mathcal{N}}_{ij} = \frac{1}{\Omega} \hat{\mathcal{D}}_i \mathcal{C}_{ij}. \tag{3.7}$$

¹⁰Referring to the complex coordinates introduced in appendix C, we chose the orientation as inherited from the parent Riemannian spacetime: $\eta_{0\zeta\bar{\zeta}} = \Omega \sqrt{a} \epsilon_{0\zeta\bar{\zeta}} = \frac{i\Omega}{P^2}$, where $x^0 = kt$.

¹¹In all quoted Eddington-Finkelstein type of gauges, ∂_r is tangent to a null geodesic congruence. In Newman-Unti and in modified Newman-Unti this congruence is affinely parameterized, in contrast to Bondi. In modified Newman-Unti gauge, as opposed to the others, ∂_r is *not* hypersurface-orthogonal. Indeed, the metric-dual form to ∂_r is μ , which has a twist because of Ω and b_i , the defining features of the gauge at hand: $\mu \wedge d\mu = * \varpi \eta_{ij} dx^i \wedge dx^j \wedge \mu$ (we have used eqs. (A.5) and (C.3)).

¹²In BMS gauge, one would set $b_i = 0$, $\Omega = 1$, and a_{ij} the round sphere.

¹³In Einstein spacetimes these two shears are proportional with the cosmological constant as a factor. In the asymptotically flat limit, the geometric shear is required to vanish, while the dynamic shear decouples.

With these definitions, the shear and the news are supported by genuine boundary conformal Carrollian-covariant tensors (weight -1 and 0), hence meeting the advertised expectations.¹⁴

Carrollian fluid. The boundary Carrollian fluid of Ricci-flat spacetimes is the descendant of the relativistic boundary fluid in Einstein spacetimes in the vanishing speed of light limit, supported by the conserved energy-momentum tensor $T_{\mu\nu}$. It is described in terms of the energy density ε , the heat currents Q^i and π^i , and the symmetric and traceless stress tensors Σ^{ij} and Ξ^{ij} [70, 71]. The associated momenta of the fluid dynamics in the sense of appendix B are as follows:

$$\Pi = \varepsilon, \quad \Pi^i = Q^i, \quad P^i = \pi^i, \quad \tilde{\Pi}^{ij} = -\Sigma^{ij}, \quad \Pi^{ij} = \frac{\varepsilon}{2}a^{ij} - \Xi^{ij}. \quad (3.8)$$

As opposed to the relativistic boundary fluid, however, the Carrollian fluid is not free, but sourced by the shear, the news and the Carrollian Cotton descendants. Put differently, its dynamical equations are (B.1), (B.2), (B.3) and (B.4) (at zero ξ^{ij}) with a non-vanishing right-hand side. These equations translate part of Einstein's, which furthermore impose¹⁵

$$Q^i = \frac{1}{8\pi G} * \chi^i, \quad \Sigma^{ij} = \frac{1}{8\pi G} * X^{ij}, \quad \Xi^{ij} = \frac{1}{8\pi G} * \Psi^{ij}. \quad (3.9)$$

Three of the Carrollian fluid data are thus tuned in terms of the boundary geometry through the Carrollian Cotton descendants displayed in eqs. (C.12), (C.14) and (C.15). Only two momenta remain independent ($\Pi = \varepsilon$ and $P^i = \pi^i$) and subject to two Carrollian-fluid evolution equations ((B.1) and (B.3) with zero ξ^{ij} and external force,¹⁶ often referred to as *flux-balance equations*) out of the four — the other two are automatically satisfied owing to the Cotton equations (B.1), (B.2), (B.3) and (B.4)

¹⁴Notice that they do not exactly coincide with the original shear and news defined in BMS gauge. They vanish in Robinson-Trautmann spacetimes expressed in the gauge at hand, which is their defining gauge, although these solutions are radiating.

¹⁵The presence of a non-vanishing energy flux $\Pi^i = Q^i$ betrays the breaking of local Carroll boost invariance (see appendix B, footnote 40) in the boundary Carrollian dynamics associated with Ricci-flat spacetimes. This breaking accounts for bulk gravitational radiation, which in the boundary-covariant gauge designed here does not originate solely in the news (3.7) but is also encoded in the Carrollian energy flux $\Pi^i = Q^i = \frac{1}{8\pi G} * \chi^i$ and the Carrollian stress $\tilde{\Pi}^{ij} = -\Sigma^{ij} = -\frac{1}{8\pi G} * X^{ij}$ obeying eq. (B.4) or equivalently (C.19). In Robinson-Trautman spacetimes e.g., the gravitational radiation is exclusively rooted in the latter Cotton descendants — see footnote 14 and ref. [23].

¹⁶We display for completeness these Carrollian equations, which coincide with eqs. (2.53) and (2.50) of ref. [72], once translated from our gauge into the BMS gauge:

$$\begin{aligned} \frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \hat{\mathcal{D}}_i \Pi^i &= \frac{1}{16\pi G} \left(\hat{\mathcal{D}}_i \hat{\mathcal{D}}_j \hat{\mathcal{N}}^{ij} + \mathcal{C}^{ij} \hat{\mathcal{D}}_i \hat{\mathcal{R}}_j + \frac{1}{2} \mathcal{C}_{ij} \frac{1}{\Omega} \hat{\mathcal{D}}_t \hat{\mathcal{N}}^{ij} \right), \\ \hat{\mathcal{D}}_j \Pi^{ij} + \frac{1}{\Omega} \hat{\mathcal{D}}_t P^i + 2 * \varpi * \Pi^i &= \frac{1}{16\pi G} \left[\mathcal{C}^{ij} \hat{\mathcal{D}}_j \hat{\mathcal{K}} + * \mathcal{C}^{ij} \hat{\mathcal{D}}_j \hat{\mathcal{L}} - 4 * \varpi * \mathcal{C}^{ij} \hat{\mathcal{R}}_j - \frac{1}{2} \hat{\mathcal{D}}^j \left(\hat{\mathcal{D}}_j \hat{\mathcal{D}}_k \mathcal{C}^{ik} - \hat{\mathcal{D}}^i \hat{\mathcal{D}}^k \mathcal{C}_{jk} \right) \right. \\ &\quad \left. + \mathcal{C}^{ij} \hat{\mathcal{D}}^k \hat{\mathcal{N}}_{jk} + \frac{1}{2} \hat{\mathcal{D}}^j \left(\mathcal{C}^{ik} \hat{\mathcal{N}}_{jk} \right) - \frac{1}{4} \hat{\mathcal{D}}^i \left(\mathcal{C}^{jk} \hat{\mathcal{N}}_{jk} \right) \right] \end{aligned}$$

with Π , Π^i , Π^{ij} , P^i as in eqs. (3.8) and (3.9).

with (C.16). These data are related to the Bondi mass and angular momentum aspects, $M(t, \mathbf{x})$ and $N^i(t, \mathbf{x})$:

$$8\pi G\varepsilon = 2M + \frac{1}{4}\mathcal{E}^{jk}\hat{\mathcal{N}}_{jk}, \tag{3.10}$$

$$8\pi G\pi^i = *\psi^i - N^i \tag{3.11}$$

with ψ^i given in (C.13).

Similarly to the expansion of Einstein spacetimes (in Fefferman-Graham or in the present gauge), fluid-related tensors appear *at every order* and not exclusively for $1/r$, as expression (3.6) might suggest.

Further degrees of freedom. Contrary to the asymptotically anti-de Sitter case, the above fluid data *are not* the only degrees of freedom besides the boundary geometry. An infinite number of Carrollian tensors are necessary to all orders in the radial expansion, as $E_{ij}(t, \mathbf{x})$ in (3.6) at order $1/r$, which obey Carrollian evolution — flux-balance — equations similar to those already displayed in footnote 16. These are dubbed “Chthonian” degrees of freedom.

We will not elaborate any further on the features of the expansion and the structure of the various evolution equations. The covariantization with respect to boundary Carroll diffeomorphisms and Weyl covariance is a powerful tool,¹⁷ rooted in the bulk general covariance. It can be supplemented with the boundary-fluid hydrodynamic-frame invariance at the expense of giving up radically the complete bulk gauge fixing. This requires a modified and *incomplete* Newman-Unti gauge, and has been performed for three bulk dimensions in refs. [57–59, 66].

Resumming the series expansion. In certain circumstances the series (3.6) can be resummed. As advertised in the introduction, this occurs when conditions are imposed on the boundary data, which enforce specific features for the bulk Weyl tensor:

1. the dynamic shear $\mathcal{E}_{ij}(t, \mathbf{x})$ should vanish, implying in particular the relation $M = 4\pi G\varepsilon$;
2. all non-Carrollian-fluid related degrees of freedom should be discarded, as e.g. $E_{ij}(t, \mathbf{x})$;
3. N^i in (3.11) should be set to zero, which amounts to demanding the Carrollian momentum $P^i = \pi^i$ be tuned with respect to a Carrollian Cotton descendant:¹⁸

$$\pi^i = \frac{1}{8\pi G} *\psi^i. \tag{3.12}$$

¹⁷The expression (3.10) matches with eq. (42) of ref. [73], reached through a completely different logical path. Similarly the Carroll Cotton scalar c given in (C.11) plays here the role of the dual mass aspect, captured in (53) of the quoted reference.

¹⁸Although eq. (3.12), which secretly tunes the bulk Weyl tensor, bares some resemblance with a self-duality condition, it isn't as the Ricci-flat spacetimes at hand are Lorentzian rather than Euclidean and this option is not available.

In the configuration reached with the above conditions, the remaining degrees of freedom are those describing the boundary Carrollian geometry (metric, field of observers and Ehresmann connection), and the Carrollian-fluid energy density i.e. the Bondi mass aspect. Expression (3.6) is now resummed into an exact Ricci-flat spacetime of algebraically special type:¹⁹

$$ds_{\text{res. Ricci-flat}}^2 = \mu \left[2dr + 2 \left(r\varphi_j - *\hat{\mathcal{D}}_j *\varpi \right) dx^j - \left(r\theta + \hat{\mathcal{K}} \right) \mu \right] + \rho^2 d\ell^2 + \frac{\mu^2}{\rho^2} [8\pi G \varepsilon r + *\varpi c] \quad (3.13)$$

with

$$\rho^2 = r^2 + *\varpi^2. \quad (3.14)$$

Ricci flatness is guaranteed by the Carrollian fluid equations, which are now genuine conservation equations without forcing term (B.1), (B.2), (B.3) and (B.4), where the momenta are (using (3.9) and (3.12))

$$\Pi = \varepsilon, \quad \Pi^i = \frac{1}{8\pi G} *\chi^i, \quad P^i = \frac{1}{8\pi G} *\psi^i, \quad \tilde{\Pi}^{ij} = -\frac{1}{8\pi G} *X^{ij}, \quad \Pi^{ij} = \frac{\varepsilon}{2} a^{ij} - \frac{1}{8\pi G} *\Psi^{ij}. \quad (3.15)$$

The same equations are identically obeyed by the Carrollian Cotton tensors (C.16) and the geometric shear is vanishing. We are therefore left with two independent equations, which are (B.1) and (B.3):

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \varepsilon + \frac{1}{8\pi G} \hat{\mathcal{D}}_i *\chi^i = 0, \quad (3.16)$$

$$\hat{\mathcal{D}}_j \varepsilon - \frac{1}{8\pi G} *\hat{\mathcal{D}}_j c = 0, \quad (3.17)$$

where c and χ^i are given in geometric terms in (C.11) and (C.12), and ε is proportional to the Bondi-mass aspect, as stressed in item 1 above. Equations (3.16) and (3.17) are those displayed in footnote 16 with vanishing right-hand side.

From the above eqs. (3.16) and (3.17) as well as eq. (C.17) one can foresee that the energy density ε and the Carrollian Cotton scalar c play dual roles. This will be formulated concretely in section 4 with reference to the boundary action of the Möbius group. Anticipating this argument, we introduce the following Carrollian complex scalar $\hat{\tau}(t, \mathbf{x})$ and vector $\hat{\chi}^j(t, \mathbf{x})$:

$$\hat{\tau} = -c + 8\pi i G \varepsilon, \quad (3.18)$$

$$\hat{\chi}^j = \chi^j - i *\chi^j. \quad (3.19)$$

The aforementioned equations are thus recast as²⁰

$$\begin{aligned} \frac{1}{\Omega} \hat{\mathcal{D}}_t \hat{\tau} &= \hat{\mathcal{D}}_j \hat{\chi}^j, & \hat{\mathcal{D}}_j \hat{\tau} \hat{\mathcal{D}}^j \hat{\tau} &= 0, \\ \hat{\mathcal{D}}_j \hat{\tau} \hat{\mathcal{D}}^j \hat{\tau} &= 8 \left(2 *\varpi *\chi_i + \frac{1}{\Omega} \hat{\mathcal{D}}_t \psi_i - \hat{\mathcal{D}}^j \Psi_{ji} \right) \left(2 *\varpi *\chi^i + \frac{1}{\Omega} \hat{\mathcal{D}}_t \psi^i - \hat{\mathcal{D}}_k \Psi^{ki} \right). \end{aligned} \quad (3.20)$$

¹⁹Details and examples are available in [23].

²⁰The first of eqs. (3.20) is flux-balance, driven exclusively by the Cotton vector $\hat{\chi}^j$ displayed in (3.19). The loss phenomenon concerns both the mass aspect ε and the “magnetic-mass aspect” c , as captured in eqs. (76) and (80) of [73] — see also appendix D of [75]. As opposed to ε , the time evolution (C.17) of the magnetic-mass aspect is not altered by \mathcal{C}_{ij} and \mathcal{N}_{ij} , in line with [76].

Acting with a second spatial derivative on (3.17) and using (A.19), we finally obtain

$$\hat{\mathcal{D}}^j \hat{\mathcal{D}}_j \hat{\tau} = 2i \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t * \varpi \hat{\tau} - \hat{\mathcal{A}} \hat{\tau} \right). \quad (3.21)$$

Let us mention for completeness that eqs. (3.16) and (3.17) coincide with eqs. (29.16) and (29.15) of [74].²¹ It is remarkable that complicated equations as the latter can actually be tamed into a simple fluid conservation supplemented with a kind of self-duality requirement. It would have been unthinkable to reach such a conclusion without the null boundary analysis performed here and the corresponding Carrollian geometric tools. The latter provide definitely the natural language for unravelling asymptotically flat spacetimes.

A last comment before closing this section concerns the algebraic-special nature of the metric (3.13). This is proven thanks to the Goldberg-Sachs theorem using the null, geodesic and, in the resummed instance, shear-free bulk congruence tangent to ∂_r . The latter is part of the canonical null tetrad parallelly transported along ∂_r (thanks to the affine nature of r) introduced in [23], which coincides with that of [74], eq. (29.13a), as well as with the original ref. [77]. In complex celestial-sphere coordinates ζ and $\bar{\zeta}$, see appendix C, the null tetrad reads:

$$\begin{cases} \mathbf{k} = \partial_r \\ \mathbf{l} = \frac{1}{2} \left(\frac{8\pi G \varepsilon r + * \varpi c}{\rho^2} - r\theta - \hat{\mathcal{K}} \right) \partial_r + \mathbf{v} \\ \mathbf{m} = \frac{P}{r - i * \varpi} \left(\hat{\partial}_{\bar{\zeta}} + (* \hat{\mathcal{D}}_{\bar{\zeta}} * \varpi - r \varphi_{\bar{\zeta}}) \partial_r \right) \end{cases} \quad (3.22)$$

with the usual relations $\mathbf{k} \cdot \mathbf{l} = -1$, $\mathbf{m} \cdot \bar{\mathbf{m}} = 1$ and $ds_{\text{res. Ricci-flat}}^2 = -2\mathbf{k}\mathbf{l} + 2\mathbf{m}\bar{\mathbf{m}}$. Generically, \mathbf{k} is a multiplicity-two principal null direction of the Weyl tensor, and using the tetrad at hand we find the following Weyl complex scalars:²²

$$\begin{cases} \Psi_0 = \Psi_1 = 0 \\ \Psi_2 = \frac{i\hat{\tau}}{2(r - i * \varpi)^3} \\ \Psi_3 = \frac{iP\chi_{\zeta}}{(r - i * \varpi)^2} + \mathcal{O}(1/(r - i * \varpi)^3) \\ \Psi_4 = \frac{iX_{\zeta}}{r - i * \varpi} + \mathcal{O}(1/(r - i * \varpi)^2). \end{cases} \quad (3.23)$$

Unsurprisingly, all Ψ s are spelled using the Carrollian descendants of the boundary Cotton tensor — as well as their derivatives in the higher-order terms.

3.2 Bulk versus boundary isometries

The geometries under consideration possess at least one Killing vector field. A natural question to address concerns the boundary manifestation of a bulk isometry. At the same time such an analysis provides the recipe for designing bulk isometries from a purely boundary perspective.

²¹For that purpose, the following identifications are necessary (in complex coordinates, as in appendix C): $b_{\zeta} = -L$, $*\varpi = -\Sigma$, $\hat{\tau} = 2(M + im)$, $\Omega = 1$, $t = u$, whereas their radial coordinate is $\tilde{r} = r - r_0$ with $r_0(t, \zeta, \bar{\zeta})$ the origin in the affine parameter of the geodesic congruence tangent to ∂_r .

²²Neither Ψ_3 nor Ψ_4 vanish in the instance of Petrov type D solutions, because \mathbf{l} is not a principal null direction. Another tetrad is reached with a Lorentz transformation suitably adjusted for \mathbf{l}' be a principal direction of multiplicity two whereas $\mathbf{k}' \propto \mathbf{k}$, and $\Psi'_3 = \Psi'_4 = 0$.

We will circumscribe our investigation to vector fields, which have no component along ∂_r , and whose other components depend only on t and \mathbf{x} . We could be more general without much effort assuming e.g. an expansion in inverse powers of r for the missing component and for the radial dependence of the others. However, this would unnecessarily sophisticate our presentation without shedding more light on our simple and robust conclusion: *the bulk isometries at hand are mapped onto boundary Carrollian diffeomorphisms generated by strong Killing vectors* (a summary on Carrollian isometries is available in appendix B).

It is convenient for the subsequent developments to adopt bulk Cartan frame and coframe aligned with the boundary (3.2), (A.2) and (3.5):

$$\begin{aligned} e_{\hat{t}} \equiv \mathbf{v} &= \frac{1}{\Omega} \partial_t, & e_{\hat{i}} \equiv \hat{\partial}_i &= \partial_i + \frac{b_i}{\Omega} \partial_t, & e_{\hat{r}} &\equiv \partial_r, \\ \theta^{\hat{t}} &\equiv -\mu = \Omega dt - b_i dx^i, & \theta^{\hat{i}} &\equiv dx^i, & \theta^{\hat{r}} &\equiv dr. \end{aligned} \quad (3.24)$$

The components for the bulk metric (3.13) read (in order to avoid cluttering, we keep the “hat” on the time indices only, where potential ambiguity exists):

$$\begin{aligned} g_{\hat{t}\hat{t}} &= \frac{1}{\rho^2} (8\pi G\epsilon r + *\varpi c) - r\theta - \hat{\mathcal{K}}, & g_{\hat{t}\hat{i}} &= *\hat{\mathcal{D}}_i *\varpi - r\varphi_i, & g_{\hat{t}\hat{r}} &= -1, \\ g_{r\hat{i}} &= 0, & g_{r\hat{r}} &= 0, & g_{\hat{i}\hat{j}} &= \rho^2 a_{ij}. \end{aligned} \quad (3.25)$$

Assuming a bulk vector of the form

$$\xi = \xi^{\hat{t}}(t, \mathbf{x}) \partial_{\hat{t}} + \xi^{\hat{k}}(t, \mathbf{x}) \partial_{\hat{k}} = \xi^{\hat{t}}(t, \mathbf{x}) \mathbf{v} + \xi^{\hat{k}}(t, \mathbf{x}) \hat{\partial}_k, \quad (3.26)$$

where $\xi^{\hat{t}} = \Omega \xi^t - \xi^k b_k$, we can determine the Lie derivative of the metric:

$$\begin{aligned} \mathcal{L}_\xi g_{r\hat{r}} &= 0, & \mathcal{L}_\xi g_{r\hat{i}} &= \mu, & \mathcal{L}_\xi g_{r\hat{i}} &= \nu_i, \\ \mathcal{L}_\xi g_{\hat{i}\hat{j}} &= 2\rho^2 \left(\hat{\nabla}_{(\hat{i}} \xi^{\hat{k}} a_{\hat{j})\hat{k}} + \xi^{\hat{t}} \hat{\gamma}_{\hat{i}\hat{j}} \right) - 2g_{\hat{i}(\hat{i}} \nu_{\hat{j})} + a_{\hat{i}\hat{j}} \xi \left(*\varpi^2 \right), \\ \mathcal{L}_\xi g_{\hat{t}\hat{i}} &= -g_{\hat{t}\hat{i}} \mu - g_{\hat{t}\hat{i}} \nu_i - r \left(\xi(\varphi_i) + \varphi_j \hat{\partial}_i \xi^j \right) + \xi \left(*\hat{\mathcal{D}}_i *\varpi \right) + \left(*\hat{\mathcal{D}}_j *\varpi \right) \hat{\partial}_i \xi^j + \rho^2 a_{ij} \frac{1}{\Omega} \partial_t \xi^j, \\ \mathcal{L}_\xi g_{\hat{t}\hat{t}} &= -2g_{\hat{t}\hat{t}} \mu + 2g_{\hat{t}\hat{i}} \frac{1}{\Omega} \partial_t \xi^i - \xi \left(\frac{1}{\rho^2} (8\pi G\epsilon r + *\varpi c) - r\theta - \hat{\mathcal{K}} \right) \end{aligned} \quad (3.27)$$

with $\mu(t, \mathbf{x})$ and $\nu_i(t, \mathbf{x})$ given in (B.10). Observe that everything is expressed in terms of boundary Carrollian geometric objects (see appendix A).

Since the Killing components are r -independent, the above Lie derivative vanishes if and only if the coefficients of every power of r do. The independent conditions we reach for this to occur are

$$\partial_t \xi^i = 0 \quad (3.28)$$

and (B.12), (B.13) and (B.17), which therefore map the bulk Killing field (3.26) onto a boundary Carrollian strong Killing vector (see appendix B). Some apparent extra conditions such as $\xi(*\varpi^2) = 0$ or $\xi(\varphi_i) + \varphi_j \hat{\partial}_i \xi^j = 0$ are the vanishing of ξ -Lie derivatives of some Carrollian tensors, which is guaranteed by the strong Killing requirement on ξ .

3.3 Towers of charges and dual charges

The Carrollian dynamics emerging on the boundary as a consequence of bulk Einstein’s equations, combined with the always available Carrollian conformal isometry group BMS_4 , enables us to define a variety of charges. These are not necessarily conserved, but even in that instance, their evolution properties are canonical and provide an alternative, tamed picture of the dynamics. Furthermore, they should ultimately pertain to those charges recently discovered and discussed from a bulk perspective [32–39], based as usual on the asymptotic symmetries — also and unsurprisingly BMS_4 , under appropriate fall-off conditions. Making the precise contact with those works would require a translation of our findings into the Newman-Penrose formalism [77] beyond what we have already observed in eqs. (3.23), namely $\Psi_2^0 = \frac{i}{2}\hat{\tau}$, $\Psi_3^0 = iP\chi_\zeta$ and $\Psi_4^0 = iX_\zeta\bar{\zeta}$. This would bring us far from our goal, and we will limit ourselves to pointing out that the ten Newman-Penrose conserved charges vanish here because the spacetimes are algebraically special. These charges would have been otherwise associated with the $s = 1$ “non-tilde” class introduced below, involving non-zero E_{ij} and N_i in the non-algebraic instance.

Ricci-flat metrics, either in the general form (3.6) or in its resummed version (3.13), exhibit two important features for the description of charges. Firstly, every order $1/r^{2s+1}$ reveals Carrollian dynamics of the type (B.1), (B.2), (B.3) and (B.4) with momenta $\Pi_{(s)}$, $\Pi_{(s)}^i$, $P_{(s)}^i$, $\tilde{\Pi}_{(s)}^{ij}$ and $\Pi_{(s)}^{ij}$, and possibly with right-hand sides — non-conservation. Every such set of momenta together with the Carrollian conformal Killings (C.20) lead to currents $\kappa_{(s)}$, $K_{(s)}^i$, $\tilde{\kappa}_{(s)}$, $\tilde{K}_{(s)}^i$ and charges $Q_{(s)T,Y}$ and $\tilde{Q}_{(s)T,Y}$, following (B.20) and (B.22). Their conservation or evolution encoded in (B.21) depends on $\mathcal{K}_{(s)}$, $\tilde{\mathcal{K}}_{(s)}$ in (B.23). The set associated with $s = 0$ corresponds to the fluid momenta (3.8) and its charges are *leading*; the sets with $s \geq 1$ reveal the *subleading charges*. Moreover, all these charges should be referred to as *electric* because their conservation, if valid, occurs on-shell.

Secondly, the Carrollian Cotton tensors obey conservation equations (B.1), (B.2), (B.3) and (B.4) with momenta (C.16), leading to two towers of Cotton charges $Q_{\text{Cot}T,Y}$ and $\tilde{Q}_{\text{Cot}T,Y}$, as discussed in appendix C. These charges are *magnetic* as the conservation of the Cotton is an identity valid off-shell.²³ Furthermore, the Carrollian Cotton tensors are not exclusive to $1/r^2$: each order $1/r^{2s+2}$ brings its share of off-shell Carrollian dynamics with momenta $\Pi_{\text{Cot}(s)}$, $\Pi_{\text{Cot}(s)}^i$, $P_{\text{Cot}(s)}^i$, $\tilde{\Pi}_{\text{Cot}(s)}^{ij}$, $\Pi_{\text{Cot}(s)}^{ij}$, currents $\kappa_{\text{Cot}(s)}$, $K_{\text{Cot}(s)}^i$, $\tilde{\kappa}_{\text{Cot}(s)}$, $\tilde{K}_{\text{Cot}(s)}^i$, and finally magnetic charges $Q_{\text{Cot}(s)T,Y}$ and $\tilde{Q}_{\text{Cot}(s)T,Y}$.

Incidentally, it should be noticed that due to the relationships amongst the fluid and the Cotton (eq. (3.9) in general plus eq. (3.12) in the resumable family), the electric and the magnetic towers have a non-empty intersection: $\tilde{Q}_{(s)T,Y}$ and $\tilde{Q}_{\text{Cot}(s)T,Y}$ generally coincide.

Let us for concreteness overview the situation in the resumable instance, eq. (3.13). Expanding the resummed factor $1/\rho^2$, we find the following results.

Electric towers. These have *sth* momenta $\Pi_{(s)}$, $\Pi_{(s)}^i$, $P_{(s)}^i$, $\tilde{\Pi}_{(s)}^{ij}$ and $\Pi_{(s)}^{ij}$ equal to (3.15) multiplied by $*\varpi^{2s}$. The same factor will multiply the leading Carrollian current ($s =$

²³We borrow here the phrasing *electric* and *magnetic* from refs. [76, 78].

0 i.e. κ , K^i , $\tilde{\kappa}$, \tilde{K}^i) and give the sth, $\kappa_{(s)}$, $K_{(s)}^i$, $\tilde{\kappa}_{(s)}$ and $\tilde{K}_{(s)}^i$, following (B.22). Using the Carroll-Bianchi identities (C.6), (C.7) and (C.8), we find the divergences (B.23), which contribute the time evolution of the charges computed as in (B.20), using (B.21):

$$\tilde{\mathcal{K}}_{(s)} = -s * \varpi^{2s-2} \left(* \varpi \hat{\mathcal{A}} \tilde{\kappa} + \frac{1}{3} \tilde{K}^i * \psi_i \right), \quad (3.29)$$

$$\mathcal{K}_{(s)} = -\frac{* \varpi^{2s}}{8\pi G} * \chi^i \left(\hat{\mathcal{D}}_i \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right) - s * \varpi^{2s-2} \left(* \varpi \hat{\mathcal{A}} \kappa + \frac{1}{3} K^i * \psi_i \right) \quad (3.30)$$

with

$$\begin{cases} \kappa = \frac{1}{8\pi G} \xi^i * \psi_i - \xi^{\hat{t}} \varepsilon \\ \tilde{\kappa} = \frac{1}{8\pi G} \xi^i * \chi_i \\ K^i = \frac{\varepsilon}{2} \xi^i - \frac{1}{8\pi G} \left(\xi^j * \Psi^i_j + \xi^{\hat{t}} * \chi^i \right) \\ \tilde{K}^i = -\frac{1}{8\pi G} \xi^j * X^i_j, \end{cases} \quad (3.31)$$

and the Killing components $\xi^{\hat{t}}$ and ξ^i read off in (C.20) following (B.6).

Regarding the charges and their evolution, only $\tilde{Q}_{(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \left(\tilde{\kappa} + b_j \tilde{K}^j \right) \equiv \tilde{Q}_{T,Y}$ are *always* conserved. These charges are purely geometric because they are integrals over \mathcal{S} ²⁴

$$\tilde{Q}_{T,Y} = -\frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} * \xi^i \left(\chi_i - b_j X^j_i \right), \quad (3.32)$$

which do not involve the energy density ε , as opposed to $Q_{(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \left(\kappa + b_j K^j \right) \equiv Q_{T,Y}$ spelled as

$$Q_{T,Y} = -\frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} \xi^{\hat{t}} \left(8\pi G \varepsilon + b_i * \chi^i \right) + \frac{1}{8\pi G} \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left(* \psi_i + 4\pi G \varepsilon b_i - b_j * \Psi^j_i \right). \quad (3.33)$$

The latter are conserved for strong Carrollian Killings. Other charges might also be conserved for specific Carrollian conformal Killings, or depending on the configuration.

Magnetic towers. The sth magnetic momenta $\Pi_{\text{Cot}(s)}$, $\Pi_{\text{Cot}(s)}^i$, $P_{\text{Cot}(s)}^i$, $\tilde{\Pi}_{\text{Cot}(s)}^{ij}$ and $\Pi_{\text{Cot}(s)}^{ij}$ are (C.16) multiplied by $* \varpi^{2s}$. As for the electric case, this latter factor will appear in all magnetic currents $\kappa_{\text{Cot}(s)}$, $K_{\text{Cot}(s)}^i$, $\tilde{\kappa}_{\text{Cot}(s)}$ and $\tilde{K}_{\text{Cot}(s)}^i$ built out of the leading $s = 0$:

$$\begin{cases} \kappa_{\text{Cot}} = \xi^i \psi_i - \xi^{\hat{t}} c \\ \tilde{\kappa}_{\text{Cot}} = \xi^i \chi_i \\ K_{\text{Cot}}^i = \frac{c}{2} \xi^i - \xi^j \Psi^i_j - \xi^{\hat{t}} \chi^i \\ \tilde{K}_{\text{Cot}}^i = -\xi^j X^i_j, \end{cases} \quad (3.34)$$

²⁴We use the property $V^i * W_i = - * V^i W_i$ — see (C.1).

Their divergences (B.23) read:

$$\tilde{\mathcal{K}}_{\text{Cot}(s)} = -s * \varpi^{2s-2} \left(* \varpi \hat{\mathcal{A}} \tilde{\kappa}_{\text{Cot}} + \frac{1}{3} \tilde{K}_{\text{Cot}}^i * \psi_i \right), \quad (3.35)$$

$$\mathcal{K}_{\text{Cot}(s)} = - * \varpi^{2s} \chi^i \left(\hat{\mathcal{D}}_i \xi^{\hat{i}} - 2 \xi^j \varpi_{ji} \right) - s * \varpi^{2s-2} \left(* \varpi \hat{\mathcal{A}} \kappa_{\text{Cot}} + \frac{1}{3} K_{\text{Cot}}^i * \psi_i \right). \quad (3.36)$$

These determine the evolution (B.21) of the charges (B.20), from which we learn that $\tilde{Q}_{\text{Cot}(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \left(\tilde{\kappa}_{\text{Cot}} + b_j \tilde{K}_{\text{Cot}}^j \right) \equiv \tilde{Q}_{\text{Cot}T,Y}$ are always conserved:

$$\tilde{Q}_{\text{Cot}T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left(\chi_i - b_j X^j_i \right). \quad (3.37)$$

For strong Carrollian Killing fields, $Q_{\text{Cot}(0)T,Y} = \int_{\mathcal{S}} d^2x \sqrt{a} \left(\kappa_{\text{Cot}} + b_j K_{\text{Cot}}^j \right) \equiv Q_{\text{Cot}T,Y}$ given by

$$Q_{\text{Cot}T,Y} = - \int_{\mathcal{S}} d^2x \sqrt{a} \xi^{\hat{i}} \left(c + b_i \chi^i \right) + \int_{\mathcal{S}} d^2x \sqrt{a} \xi^i \left(\psi_i + \frac{c}{2} b_i - b_j \Psi^j_i \right) \quad (3.38)$$

are also conserved off-shell, as other magnetic charges are in specific situations.

Several comments are in order here concerning the above sets of charges obtained for the resumable metrics (3.13). The tower of electric geometric charges $\tilde{Q}_{(s)T,Y}$, constructed upon multiplying the integrand of (3.32) by $*\varpi^{2s}$, coincides with its magnetic counterpart $\tilde{Q}_{\text{Cot}(s)T,Y}$ obtained likewise using (3.37). In $d = 2$, if ξ^i are the spatial components of a conformal Killing field, so are $*\xi^i$.²⁵ Hence the set of all ξ^i 's is identical to that of $*\xi^i$'s. The associated charges could be called “self-dual,” and in total three distinct towers emerge: the self-dual $\left\{ \tilde{Q}_{(s)T,Y} \right\} \equiv \left\{ \tilde{Q}_{\text{Cot}(s)T,Y} \right\}$, the electric $\left\{ Q_{(s)T,Y} \right\}$ and the magnetic $\left\{ Q_{\text{Cot}(s)T,Y} \right\}$ — the last two are reached by inserting $*\varpi^{2s}$ into the integrals (3.33) and (3.38). The $*\varpi^{2s}$ insertion pattern grants the subleading towers with the status of *multipolar moments* (see the original works [79–82] as well as [83] for a modern perspective). Making this statement precise would force us to deviate substantially from the analysis of the hidden Möbius group. This could fit more naturally in a comprehensive comparison of the present approach to subleading charges with the rich existing literature quoted earlier. Nonetheless, the pertinence of the proposition will be illustrated in the example of Kerr solution, at the very end of the forthcoming section 3.4.

Among the above towers of BMS₄ charges, always present but not always conserved, one finds those corresponding to the bulk isometries, whenever present. Indeed, as discussed in section 3.2, bulk Killings of the form (3.26) are associated with boundary strong Carrollian Killing vector fields. Combined as previously with the leading and subleading, electric and magnetic momenta, they generate two electric and two magnetic towers of charges: $\left\{ Q_{(s)}, \tilde{Q}_{(s)}, Q_{\text{Cot}(s)}, \tilde{Q}_{\text{Cot}(s)} \right\}$. The four leading charges $\left\{ Q_{(0)}, \tilde{Q}_{(0)}, Q_{\text{Cot}(0)}, \tilde{Q}_{\text{Cot}(0)} \right\}$ are always conserved, but part of them may be trivial or not independent. The subleading are neither necessarily conserved, nor always independent, and have the status of electric and magnetic multipole moments.

²⁵The proof of this statement is straightforward in complex coordinates, see footnote 50.

3.4 Time-independent solutions

Reconstruction from the boundary. In view of the forthcoming Ehlers-Geroch reduction, we will now assume the existence of a time-like Killing vector field ξ in Ricci-flat solutions of the resumable type (3.13). Such a vector could be generally of the form (3.26). In stationary spacetimes, the field ξ remains time-like in the asymptotic region. Then, it is possible to choose the field \mathbf{v} (3.2) of the modified Newman-Unti gauge such that $\mathbf{v} \equiv \xi$. Setting further $\Omega = 1$ brings the Killing to the simple form ∂_t (see e.g. [77] for a detailed description of the procedure). On the conformal boundary, the time-like Killing congruence thus coincides with the fibre of the Carrollian bundle. This feature is absent for spacetimes where a time-like Killing field exists but becomes space-like in the asymptotic region. Examples of this sort are captured by the Plebański-Demiański family (like the C-metric) [84] (see also [74, 85]), which is algebraically special of Petrov type D.²⁶ These include the black-hole acceleration parameter, which is responsible for the appearance of another Killing horizon, creating a new asymptotic region where the Killing vector fails to be time-like. Although interesting on its own right — of limited physical use, however — the inclusion of this parameter would render the presentation too convoluted, in particular because the action of the Ehlers group in this instance does not respect the algebraic feature of the spacetime. For the sake of clarity we will restrict our investigation to Killings of the form ∂_t , aligned with the fiber, i.e. to truly stationary spacetimes, which remain algebraically special under Ehlers transformations.

With the present choice, none of the Carrollian building blocks of $ds_{\text{res. Ricci-flat}}^2$ depends on t . As a consequence (see appendices A and C) $\theta = 0$ and $\varphi_i = \partial_i \ln \Omega$. The latter can be set to zero with a time-independent Weyl rescaling, which therefore amounts to setting $\Omega = 1$. This is an innocuous gauge fixing that will be assumed here because it allows to severely simplify the dynamics. Backed with time independence, Carrollian Weyl-covariant derivatives become ordinary Levi-Civita derivatives, and the only non-vanishing tensors are the following, in complex coordinates with $P = P(\zeta, \bar{\zeta})$ — see appendix C:

$$*\varpi = \frac{iP^2}{2} (\partial_\zeta b_{\bar{\zeta}} - \partial_{\bar{\zeta}} b_\zeta), \tag{3.39}$$

$$\hat{\mathcal{K}} = \hat{K} = K = \Delta \ln P, \tag{3.40}$$

$$c = (\Delta + 2K) *\varpi, \tag{3.41}$$

$$\chi_\zeta = \frac{i}{2} \partial_\zeta K, \quad \chi_{\bar{\zeta}} = -\frac{i}{2} \partial_{\bar{\zeta}} K, \tag{3.42}$$

$$\psi_\zeta = 3i \partial_\zeta *\varpi^2, \quad \psi_{\bar{\zeta}} = -3i \partial_{\bar{\zeta}} *\varpi^2, \tag{3.43}$$

$$\Psi_{\zeta\zeta} = \frac{1}{P^2} \partial_\zeta (P^2 \partial_\zeta *\varpi), \quad \Psi_{\bar{\zeta}\bar{\zeta}} = \frac{1}{P^2} \partial_{\bar{\zeta}} (P^2 \partial_{\bar{\zeta}} *\varpi), \tag{3.44}$$

where $\Delta f = 2P^2 \partial_{\bar{\zeta}} \partial_\zeta f$. To these one should add the energy density (i.e. the Bondi mass aspect) ε , as well as another scalar

$$\varpi = \frac{P^2}{2} (\partial_\zeta b_{\bar{\zeta}} + \partial_{\bar{\zeta}} b_\zeta), \tag{3.45}$$

²⁶Their Weyl components are given in eq. (3.23) — see also footnote 22.

which is $\frac{1}{2}\nabla_i b^i$ and should not be confused with the two-form $\varpi = \frac{1}{2}\varpi_{ij}dx^i \wedge dx^j$, i.e. the Hodge-dual of the scalar $*\varpi = -\frac{1}{2}\nabla_i *b^i$ displayed explicitly in (3.39). These two real twist scalars are adroitly combined into the *complex Carrollian twist*

$$\hat{\varpi} = *\varpi + i\varpi. \tag{3.46}$$

The equations of motion (3.16), (3.17) (or in the form (3.20), (3.21) with $\hat{\tau}$ defined in (3.18)) are recast as

$$\Delta K = 0, \tag{3.47}$$

$$\partial_\zeta \hat{\tau} = 0. \tag{3.48}$$

The first shows that the curvature is required to be a harmonic function i.e.

$$K(\zeta, \bar{\zeta}) = \frac{1}{2} \left(\hat{k}(\zeta) + \hat{k}(\bar{\zeta}) \right), \tag{3.49}$$

and although $\hat{k}(\zeta)$ is an arbitrary holomorphic function, the freedom is rather limited as K must also be the Laplacian of $\ln P$. Besides the constant-curvature cases, one solution has been exhibited thus far [74] (up to holomorphic coordinate transformations): $K = -3(\zeta + \bar{\zeta})$ realized with $P = (\zeta + \bar{\zeta})^{3/2}$. We will not specify any particular choice for the moment. For future use, we define the imaginary part of $\hat{k}(\zeta)$ as another harmonic function

$$K^*(\zeta, \bar{\zeta}) = \frac{1}{2i} \left(\hat{k}(\zeta) - \hat{k}(\bar{\zeta}) \right). \tag{3.50}$$

From eqs. (3.48) and (3.18), we infer that $-c$ is the real part of an arbitrary holomorphic function $\hat{\tau}(\zeta)$, whereas the imaginary part of the latter is $8\pi G\varepsilon$; both are harmonic functions. Given c and K , we can proceed with eq. (3.41) and find $*\varpi$, from which it is always possible to determine b_ζ and $b_{\bar{\zeta}}$.

Although the focus of the present work is not to solve Einstein’s equations, we will elaborate for illustrative purposes on the steps we’ve just described, without delving into fine questions like completeness or gauge redundancy of the solutions. Note in passing how remarkably the Carrollian boundary formalism is adapted to the framework of Ricci-flat spacetimes, allowing to convey often complicated expressions in a very elegant manner, and sorting naturally otherwise scattered classes of solutions (the ones we present can be found in various chapters of refs. [74, 85]). Several distinct instances appear, which require a separate treatment.

Non-constant K . This is the generic situation, although in practice the most obscure regarding the interpretation of the bulk geometries. As already mentioned, very few P s are expected to possess a non-constant harmonic curvature K , but assuming one has one, accompanied by its holomorphic function $\hat{k}(\zeta)$, and making a choice for the arbitrary holomorphic function $\hat{\tau}(\zeta)$, eq. (3.41) can be solved for $*\varpi$, which is expressed using (3.39) with Ehresmann connection

$$b_\zeta(\zeta, \bar{\zeta}) = \frac{i\hat{\tau}(\bar{\zeta})}{P^2(\zeta, \bar{\zeta})\partial_{\bar{\zeta}}\hat{k}(\bar{\zeta})}. \tag{3.51}$$

Constant K . This implies that $\hat{k}(\zeta)$ is also constant and the above solution is invalid.

The situation at hand is the most common, however, as it captures three standard instances: spherical, flat or hyperbolic foliations. We can parameterize the function P as follows:

$$P(\zeta, \bar{\zeta}) = A\zeta\bar{\zeta} + B\zeta + \bar{B}\bar{\zeta} + D \quad (3.52)$$

with A, D arbitrary real constants and B an arbitrary complex constant, leading to

$$K = 2(AD - B\bar{B}). \quad (3.53)$$

Several cases emerge, which must be treated separately.

$K \neq 0$. Here $c(\zeta, \bar{\zeta}) = -\frac{\hat{\tau}(\zeta) + \hat{\tau}(\bar{\zeta})}{2}$ is an arbitrary (possibly constant) harmonic function, and eq. (3.41) is solved with

$$*\varpi(\zeta, \bar{\zeta}) = \frac{c(\zeta, \bar{\zeta})}{2K} + i \left(\bar{f}(\bar{\zeta}) \partial_{\bar{\zeta}} \ln P(\zeta, \bar{\zeta}) - f(\zeta) \partial_{\zeta} \ln P(\zeta, \bar{\zeta}) + \frac{1}{2} \left(\partial_{\zeta} f(\zeta) - \partial_{\bar{\zeta}} \bar{f}(\bar{\zeta}) \right) \right) \quad (3.54)$$

with $f(\zeta)$ an arbitrary holomorphic function. It is reached with the following Ehresmann connection ($\hat{\tau}_0$ is a real constant):

$$b_{\zeta}(\zeta, \bar{\zeta}) = -\frac{\bar{\zeta}(\hat{\tau}_0 + i\hat{\tau}(\zeta))}{2K(B\zeta + D)P(\zeta, \bar{\zeta})} + \frac{\bar{f}(\bar{\zeta})}{P^2(\zeta, \bar{\zeta})}. \quad (3.55)$$

$K = 0$. This instance is obtained with $A = B = 0$ so that $P = D$. Now, given an arbitrary harmonic function $c(\zeta, \bar{\zeta}) = -\frac{\hat{\tau}(\zeta) + \hat{\tau}(\bar{\zeta})}{2}$ and an arbitrary holomorphic function $Z(\zeta)$, we find

$$*\varpi(\zeta, \bar{\zeta}) = \frac{i}{2} \left(Z(\zeta) - \bar{Z}(\bar{\zeta}) \right) - \frac{1}{4P^2} \left(\bar{\zeta} \int^{\zeta} dz \hat{\tau}(z) + \zeta \int^{\bar{\zeta}} d\bar{z} \hat{\tau}(\bar{z}) \right), \quad (3.56)$$

and ($\hat{\tau}_0$ is a real integration constant)

$$b_{\zeta}(\zeta, \bar{\zeta}) = \frac{1}{P^2} \int^{\bar{\zeta}} d\bar{z} \bar{Z}(\bar{z}) - \frac{\bar{\zeta}^2}{4P^4} \int^{\zeta} dz (\hat{\tau}_0 + i\hat{\tau}(z)). \quad (3.57)$$

The last two cases have in common the instance where $c = K = 0$, realized with vanishing $\hat{\tau}$ and constant P .

As already noticed, all solutions described in a unified fashion here can be found in the earlier quoted literature under distinct labels.²⁷ Discussing them would take us outside of our objectives. We will only emphasize a notorious subclass, which is the Kerr-Taub-NUT family. For the latter, the curvature K is constant (3.53) and realized e.g. with $B = 0$. Two distinct instances emerge: vanishing and non-vanishing K , respectively obtained with vanishing and non-vanishing A .

²⁷It should be stressed that part of the present solution space originates in gauge freedom. In particular, $c(\zeta, \bar{\zeta})$ being Weyl-covariant of weight 3 (see appendix C), it can always be reabsorbed by a boundary Weyl transformation, which is in turn neutralized by a bulk r -rescaling. Such a boundary transformation will bring Ω back with non-vanishing φ_i , which we have set to zero, and this is the reason we cannot here restrict to constant c and $\hat{\tau}$.

- For non-vanishing K , the holomorphic function $\hat{\tau}$ is

$$\hat{\tau} = 2i(M + iKn), \tag{3.58}$$

where M is the mass and n the nut charge, both constants. The holomorphic function $f(\zeta)$ reads

$$f(\zeta) = ia\zeta \tag{3.59}$$

with a the Kerr angular velocity. Using eqs. (3.54) and (3.55) with $\hat{\tau}_0 = 2M$ we find:

$$b_\zeta(\zeta, \bar{\zeta}) = -i\bar{\zeta} \left(\frac{a}{P^2} - \frac{n}{DP} \right) \tag{3.60}$$

and

$$* \varpi(\zeta, \bar{\zeta}) = n + a - \frac{2Da}{P}, \tag{3.61}$$

where $P = A\zeta\bar{\zeta} + D$ and $K = 2AD$.

- For $K = 0$ (i.e. $P = D$ constant), we use eqs. (3.56) and (3.57) with $\hat{\tau}_0 = 2M$,²⁸

$$\hat{\tau} = 2iM \tag{3.62}$$

and

$$Z = ia. \tag{3.63}$$

This leads to

$$b_\zeta(\zeta, \bar{\zeta}) = -i\frac{\bar{\zeta}a}{P^2} \tag{3.64}$$

and

$$* \varpi = -a. \tag{3.65}$$

Observe the absence of nut charge in the present case.²⁹

A remark on the rigidity theorem. The *rigidity theorem* asserts that under appropriate hypotheses, the isometry group of stationary asymptotically flat spacetimes contains $\mathbb{R} \times \text{U}(1)$. This theorem is best presented in refs. [86, 87], where the necessary assumptions are stated more accurately than in the original discussions (see e.g. [88]). Our framework does embrace stationary spacetimes. However, we have been agnostic regarding analyticity or regularity properties, which turn out to be fundamental for the applicability of the theorem at hand. Hence, we have no reason to foresee any additional $\text{U}(1)$ symmetry in all reconstructed solutions of the present section.

Aside from mathematical rigor, we can recast the conceivable disruption of the rigidity theorem from the boundary perspective, which has been our viewpoint. We have shown in section 3.2 that a bulk Killing field is mapped onto a Carrollian strong Killing on the

²⁸Both for vanishing and non-vanishing K , $\hat{\tau}_0$ has been tuned to ensure that M does not appear in b_ζ , displayed in (3.60) and (3.64). There is no principle behind this choice, it is simply in line with standard conventions for the Kerr-Taub-NUT family. As a consequence, ϖ defined in (3.45) vanishes.

²⁹Despite the absence of magnetic charges, the solution at hand belongs formally to the Taub-NUT family (see ref. [85], section 12.3.2).

boundary. The generator of the desired U(1) is of the form (3.26) with no time leg³⁰ i.e. $\xi^t = 0$, and no time dependence in ξ^i as imposed by (3.28):

$$\xi = - \left(\xi^\zeta b_\zeta + \xi^{\bar{\zeta}} b_{\bar{\zeta}} \right) \partial_t + \xi^\zeta \hat{\partial}_\zeta + \xi^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}}. \quad (3.66)$$

A strong Carrollian Killing field must obey eqs. (B.12), (B.13) and (B.17). Here (B.13) is identically satisfied, whereas (B.12) leads to

$$\partial_\zeta \xi^{\bar{\zeta}} = \partial_{\bar{\zeta}} \xi^\zeta = 0, \quad \partial_\zeta \frac{\xi^\zeta}{P^2} + \partial_{\bar{\zeta}} \frac{\xi^{\bar{\zeta}}}{P^2} = 0. \quad (3.67)$$

Finally (B.17) reads:

$$P^2 \partial_\zeta \left(\xi^\zeta b_\zeta + \xi^{\bar{\zeta}} b_{\bar{\zeta}} \right) + 2i \xi^{\bar{\zeta}} * \varpi = 0 \quad (3.68)$$

plus its complex conjugate.

For arbitrary $P(\zeta, \bar{\zeta})$, eqs. (3.67) have no solution, hence no extra Killing field is available. As mentioned earlier in the present section, the P s with harmonic curvature (required in (3.47)) are very restricted and probably lack the necessary analyticity properties, explaining why the rigidity theorem is not applicable. This indeed happens in the quoted example with $P = (\zeta + \bar{\zeta})^{3/2}$.

Alternatively, considering $P = A\zeta\bar{\zeta} + D$ with constant curvature $K = 2AD$, we find three more solutions to the equations (3.67):

$$\xi_1 = i \left(\zeta \partial_\zeta - \bar{\zeta} \partial_{\bar{\zeta}} \right), \quad (3.69)$$

$$\xi_2 = \frac{i}{2\sqrt{|AD|}} \left((D - A\zeta^2) \partial_\zeta - (D - A\bar{\zeta}^2) \partial_{\bar{\zeta}} \right), \quad (3.70)$$

$$\xi_3 = \frac{1}{2\sqrt{|AD|}} \left((D + A\zeta^2) \partial_\zeta + (D + A\bar{\zeta}^2) \partial_{\bar{\zeta}} \right), \quad (3.71)$$

closing in $\mathfrak{so}(3)$, \mathfrak{e}_2 and $\mathfrak{so}(2, 1)$ algebras³¹ for positive, zero or negative K . Using (3.60) and (3.61) one shows that for generic angular velocity a and nut charge n , only ξ_1 obeys the strong condition (3.68). This is then promoted to a bulk field generating the rotational U(1) isometry of the Kerr-Taub-NUT family. For vanishing a and n , all three Carrollian Killing fields are strong and the bulk Ricci flat solution is fully isotropic — Schwarzschild or A-class metric, see [74, 85].

Charge analysis. We would like to close the present section with a brief account on the charges of the Ricci-flat solutions under investigation. Gravitational charges disclose the identity of a background and, as we have proposed in section 3.3, boundary Carrollian geometry supplies alternative techniques for their determination and the study of their conservation. These techniques are still in an incipient stage though, because the contact with the standard methods still needs to be elaborated. Furthermore, non-radiating configurations, in particular stationary and algebraically special, offer a limited playground

³⁰We could keep non-vanishing ξ^t and perform a more thorough analysis. This would not alter the conclusions, which are meant here to illustrate possible boundary faults in the rigidity theorem.

³¹The Lie brackets of the ξ_s are $[\xi_1, \xi_2] = \xi_3$, $[\xi_3, \xi_1] = \xi_2$ and $[\xi_2, \xi_3] = \frac{K}{|K|} \xi_1$. For vanishing K , $\xi_2 = \frac{i}{2} (\partial_\zeta - \partial_{\bar{\zeta}})$ and $\xi_3 = \frac{1}{2} (\partial_\zeta + \partial_{\bar{\zeta}})$ are the translation commuting generators of \mathfrak{e}_2 .

in this programme. We would like nevertheless to summarize the situation, in view of the follow-up discussion on Möbius hidden-group action, section 4.2.

The simplest non-vanishing charge is the electric curvature defined in (C.9):³²

$$Q_{\text{ec}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} K. \quad (3.72)$$

Divided by the volume of \mathcal{S} , this is simply the average Gauss curvature. Note in passing that the charges defined here are extensive, hence the integrals may reveal convergence issues, in particular when \mathcal{S} is non-compact. Normalizing with $\text{Vol} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}$ is the simplest way to fix this divergence.³³ Alternatively, \mathcal{S} could be compactified — quotiented by a discrete isometry group. We will leave this discussion aside, as it would be better addressed within attempts to make sense of Ricci-flat black holes with non-compact horizons (see e.g. Ch. 9 of [85]).

The towers of charges introduced in section 3.3 are slightly simpler in the instance under consideration. Indeed, the Carrollian conformal Killings used in expressions (3.31) and (3.34) are (C.47) with

$$C(t, \zeta, \bar{\zeta}) = tP(\zeta, \bar{\zeta}) \quad (3.73)$$

(see (C.48)). Observe also that $\tilde{K}^i = \tilde{K}_{\text{Cot}}^i = 0$ so that $\tilde{\mathcal{K}}_{(s)} = \tilde{\mathcal{K}}_{\text{Cot}(s)} = 0$. Generically, $\mathcal{K}_{(s)}$ and $\mathcal{K}_{\text{Cot}(s)}$ are non-zero though, because the conformal Killing vectors are not necessarily strong and due to the time dependence, here encoded exclusively in their component $\xi^{\hat{t}}$. The corresponding charges are ultimately expressed as integrals of combinations of $\hat{k}, \hat{\bar{k}}, \hat{\tau}, \hat{\bar{\tau}}, *\varpi, \varpi$, and of their derivatives.³⁴

For concreteness, we will illustrate the above with the distinctive strong Carrollian conformal Killing field ∂_t , i.e. the generator of the Ehlers-Geroch bulk three-dimensional reduction. For this Killing field, the “tilde” Carrollian charges vanish. In example, for the leading charges ($s = 0$ in the coding of section 3.3), we find³⁵

$$Q_{\text{em}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (8\pi G\varepsilon + \varpi K), \quad Q_{\text{mm}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (-c + *\varpi K), \quad (3.74)$$

up to boundary terms with respect to (3.33) and (3.38) (and a factor $-8\pi G$ for the former), handily combined into

$$Q_{\text{m}} = Q_{\text{mm}} + iQ_{\text{em}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + \hat{\varpi} K). \quad (3.75)$$

The indices stand for magnetic and electric masses. These mass definitions carry some arbitrariness since, as a consequence of time independence, each of the terms in the integrals

³²Remember that here $\xi_{ij} = 0$, and the geometry is t -independent with vanishing $\theta, \varphi_i, \mathcal{A}, \hat{\mathcal{R}}_i$ as well as X_{ij} .

³³The integrals can be performed by setting $\zeta = Ze^{i\Phi}$, where $0 \leq \Phi < 2\pi$ and $Z = \sqrt{2} \tan \frac{\Theta}{2}$, $0 < \Theta < \pi$ for \mathbb{S}^2 ; $Z = \frac{R}{\sqrt{2}}$, $0 < R < +\infty$ for \mathbb{E}_2 ; $Z = \sqrt{2} \tanh \frac{\Psi}{2}$, $0 < \Psi < +\infty$ for \mathbb{H}_2 .

³⁴Although the components b_i of the Ehresmann connection enter the expression of the Carrollian charges (B.20), upon integration by parts, they are traded for $*\varpi$ or ϖ .

³⁵Using (3.31) and (3.34) with $\xi^{\hat{t}} = 1$ and $\xi^i = 0$, we find $\kappa = -\varepsilon$, $K^i = -\frac{1}{8\pi G} *\chi^i$, $\kappa_{\text{Cot}} = -c$ and $K_{\text{Cot}}^i = -\chi^i$.

provide a separate well-defined charge. We will turn back to this when discussing the action of the Möbius group, in section 4.2.

Following section 3.3, the subleading mass charges associated with the strong Carrollian conformal Killing field ∂_t are captured in

$$Q_{\text{m}(s)} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + \hat{\omega}K) * \varpi^{2s} \tag{3.76}$$

and define the higher- s *mass multipole moments*. In the instance of the $K = 1$ Kerr-Taub-NUT family displayed in eqs. (3.58), (3.59), (3.60), (3.61) with $A = 1/2$ and $D = 1$, we find:

$$Q_{\text{m}(s)} = 4\pi i (M + in) \left(\frac{(n+a)^{2s+1} - (n-a)^{2s+1}}{a(2s+1)} \right). \tag{3.77}$$

For this set of solutions, ξ_1 in (3.69) is a strong Carrollian Killing vector, which brings its own Carrollian rotational charges. Again the “tilde” (eqs. (3.32) and (3.37)) vanish whereas the “non-tilde” (see. (3.31) and (3.34)) are combined in the complex higher- s *angular-momentum multipole moments*

$$Q_{\text{r}(s)} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} 6\zeta\bar{\zeta} \left(\frac{n+iM}{P^2} (a-nP) \left(n+a - \frac{2a}{P} \right)^{2s} - \frac{2a}{P^2} \left(n+a - \frac{2a}{P} \right)^{2s+1} \right) \tag{3.78}$$

with $P = 1 + \frac{1}{2}\zeta\bar{\zeta}$, which are non-zero if one rotation parameter a or n is present. We find for example:

$$Q_{\text{r}(0)} = -8\pi [a(n+iM) + 3n(n-iM)]. \tag{3.79}$$

Expressions (3.77) and (3.79) are in line with the results obtained in refs. [79–82] (see also [89], where the electric part of $Q_{\text{r}(0)}$ is given) using standard methods circumscribed to bulk dynamics. They provide conserved moments since the divergences (3.30) and (3.36) vanish.

4 Ehlers transformations

4.1 Bulk reduction and Möbius action on the boundary

Our next and pivotal task is to unravel the action of the Ehlers group (2.11) on the boundary Carrollian observables, using the expression of the bulk Ricci-flat metric (3.6) assumed to possess a time-like Killing vector field. We will focus in the present work on the restricted class of resumable metrics (3.13), as exploited in section 3.4, i.e. equipped with a time-like Killing field $\xi = \partial_t$ and $\Omega = 1$.

In order to proceed, we are called to follow the steps for the Geroch reduction described in section 2, i.e. determine τ as defined in (2.9) for the metric (3.13) with λ and ω given in (2.1), (2.2) and (2.3). These should be expanded in inverse powers of r and thus deliver the boundary ingredients together with their transformations following (2.11). A remark should be made before hand. The Geroch reduction is followed by an oxidation, which defines the novel Ricci-flat solution. Nothing guarantees in this course that the oxidized

metric will assume again the form (3.13). Actually it doesn't and a redefinition of the radial coordinate is necessary to bring it back into the expected original gauge.

It is convenient for the present mission to adopt the Cartan frame defined in (3.24), leading to the bulk metric

$$\begin{aligned}
 g_{\hat{t}\hat{t}} &= \frac{1}{\rho^2} (8\pi G\epsilon r + *\varpi c) - K, & g_{\hat{t}i} &= *\partial_i *\varpi, & g_{\hat{t}r} &= -1, \\
 g_{ri} &= 0, & g_{rr} &= 0, & g_{ij} &= \rho^2 a_{ij}
 \end{aligned}
 \tag{4.1}$$

obtained using (3.25), assuming t -independence and $\Omega = 1$. In this expression $*\varpi$, K and c are given in eqs. (3.39), (3.40) and (3.41). The Killing form reads:

$$\xi = \left(K - \frac{1}{\rho^2} (8\pi G\epsilon r + *\varpi c) \right) \mu + *\partial_i *\varpi dx^i - dr,
 \tag{4.2}$$

with norm

$$\lambda = \frac{8\pi G\epsilon r + *\varpi c}{\rho^2} - K.
 \tag{4.3}$$

For the twist we use eq. (2.2), expressed as

$$w = -\star(\xi \wedge d\xi),
 \tag{4.4}$$

where “ \star ” stands for the four-dimensional Hodge duality. The latter one-form is exact on-shell and we find the following potential (eq. (2.3)):

$$\omega = \frac{8\pi G\epsilon *\varpi - cr}{\rho^2} + K^*.
 \tag{4.5}$$

On-shellness is implemented here through boundary dynamics as summarized in section 3.4, i.e. in eqs. (3.48), (3.49) and (3.50).

Inserting the above results into eqs. (2.9) and using (3.18), we find

$$\tau = \frac{\hat{\tau}}{r + i*\varpi} - i\hat{k}.
 \tag{4.6}$$

Likewise, we obtain the Geroch reduced and rescaled metric (2.8):

$$\tilde{h}_{AB} dx^A dx^B = - \left(dr - *\partial_k *\varpi dx^k \right)^2 + \lambda \rho^2 a_{ij} dx^i dx^j.
 \tag{4.7}$$

With this, τ given in (4.6) unsurprisingly solves the reduced Einstein's equations (2.10).

The premier Ehlers transformation rules are (2.11) and the invariance of \tilde{h}_{AB} . From these follows the rest of the construction, i.e. the transformation of h_{AB} and the oxidation toward g'_{AB} . In the present framework, we have to some extent locked the gauge, via the resummed bulk expression (3.13). Ehlers transformations are not designed a priori to maintain this form, and they are generally expected to require further coordinate transformations. It is rather remarkable that, to this end, a local (i.e. celestial-sphere dependent) shift in the radial coordinate suffices.

Using for convenience holomorphic and antiholomorphic coordinates as introduced in appendix C, expression (4.7) is recast as follows:

$$\tilde{h}_{AB}dx^A dx^B = - \left(dr - i\partial_\zeta * \varpi d\zeta + i\partial_{\bar{\zeta}} * \varpi d\bar{\zeta} \right)^2 + \frac{(\tau - \bar{\tau})(r + i * \varpi)(r - i * \varpi)}{iP^2} d\zeta d\bar{\zeta}. \quad (4.8)$$

Combining (2.11) with (4.6), we obtain the following *boundary* transformations:

$$\hat{\tau}' = - \frac{\hat{\tau}}{(\gamma\hat{k} + i\delta)^2}, \quad (4.9)$$

$$\hat{k}' = i \frac{\alpha\hat{k} + i\beta}{\gamma\hat{k} + i\delta}, \quad (4.10)$$

$$\hat{\omega}' = \hat{\omega} + \frac{\gamma\hat{\tau}}{\gamma\hat{k} + i\delta} \quad (4.11)$$

and

$$P' = \frac{P}{|\gamma\hat{k} + i\delta|}, \quad (4.12)$$

plus the radial shift³⁶

$$r' = r + \frac{i}{2} \left(\frac{\gamma\hat{\tau}}{\gamma\hat{k} + i\delta} - \frac{\gamma\hat{\tau}}{\gamma\hat{k} - i\delta} \right). \quad (4.13)$$

These transformation rules leave indeed (4.8) invariant. As advertised earlier, they are *local*, providing a direct transformation (4.12) of the boundary metric. The transformation of the energy density ε is obtained from (4.9) using (3.18):

$$8\pi G\varepsilon' = \frac{8\pi G\varepsilon \left((\gamma K^* + \delta)^2 - \gamma^2 K^2 \right) - 2c\gamma K (\gamma K^* + \delta)}{\left(\gamma^2 K^2 + (\gamma K^* + \delta)^2 \right)^2}. \quad (4.14)$$

The transformation of c is inferred similarly:

$$c' = \frac{c \left((\gamma K^* + \delta)^2 - \gamma^2 K^2 \right) + 16\pi G\varepsilon\gamma K (\gamma K^* + \delta)}{\left(\gamma^2 K^2 + (\gamma K^* + \delta)^2 \right)^2}. \quad (4.15)$$

All these rules are compatible with eqs. (3.40) and (3.41). Finally the transformations of the Carrollian Cotton tensors are reached using the above results combined with eqs. (3.42), (3.43) and (3.44).

The transformation of the Ehresmann connection is obtained directly from the expressions reached for the latter in (3.51), (3.55) and (3.57). To this end, observe that in the constant- \hat{k} instance, A , B , \bar{B} and D transform with a factor $1/|\gamma\hat{k} + i\delta|$ in order to comply

³⁶One could alternatively adopt a new radial coordinate defined as $\tilde{r} = r + \varpi$ that is invariant under Möbius transformations. This is actually mandatory in order to reach boundary $SL(2, \mathbb{R})$ -covariant tensors from the bulk, as we will discuss in section 4.2. It furthermore coincides with the radial coordinate of ref. [74] section 29 provided $r_0 = -\varpi$ (origin of the affine parameter along the geodesic congruence tangent to ∂_r — see footnote 21).

with (4.12). Similarly, $f(\zeta)$ and $Z(\zeta)$, introduced in (3.54) and (3.56), must be respectively invariant and transforming as

$$Z'(\zeta) = Z(\zeta) + i \frac{\gamma \hat{\tau}(\zeta)}{\gamma \hat{k} + i\delta}, \quad (4.16)$$

so that (4.11) be fulfilled.

Let us mention for completeness that once the Möbius transformation is performed on the boundary, the reconstruction of the new Ricci-flat solution is straightforward using the boundary-to-bulk formula (3.13), expressed with primed data — except for the unaltered boundary coordinates $\{t, \zeta, \bar{\zeta}\}$. This is equivalent to the oxidation procedure operated from three to four dimensions along the lines of eqs. (2.12), (2.13), (2.14) and (2.15) with

$$\eta' = -\mu' - \frac{1}{\lambda'} \left(dr' - i\partial_\zeta * \varpi' d\zeta + i\partial_{\bar{\zeta}} * \varpi' d\bar{\zeta} \right), \quad \mu' = -dt + b'_\zeta d\zeta + b'_{\bar{\zeta}} d\bar{\zeta}, \quad (4.17)$$

finally leading to (2.16), which assumes the form (3.13) primed. The new bulk Killing vector $\xi' = \lambda' \eta'$ is again ∂_t .

In the example of the Kerr-Taub-NUT family treated at the end of section 3.4, the specific choices of $P = \frac{1}{2}\zeta\bar{\zeta} + 1$, $K = 1$ and $K^* = 0$ (this was not explicitly demanded) are stable only under $\begin{pmatrix} \cos \chi & \sin \chi \\ -\sin \chi & \cos \chi \end{pmatrix} \in \text{SL}(2, \mathbb{R})$. For this transformation, using (4.9) we find $M' + in' = (M + in)e^{-2i\chi}$. Observe that (4.11) will switch on a non-zero ϖ though, as opposed to its original value in the family at hand (see footnote 28).

4.2 Charges and $\text{SL}(2, \mathbb{R})$ multiplets

Carrollian charges have been introduced in section 3.3 and further discussed for stationary and algebraic spacetimes in section 3.4. Two generic charges were found and displayed in (3.72) and (3.75). The former is purely geometric and stands for the integrated curvature of the celestial sphere; the latter carries genuine dynamic information captured in the electric and magnetic masses. It is legitimate to wonder how these quantities behave under Möbius transformations, and possibly tame them in $\text{SL}(2, \mathbb{R})$ multiplets. Although ideally this programme should be conducted for reductions along generic bulk Killing fields and no special algebraic structure — these would be non-resummable, i.e. of the form (3.6), and labelled by a possibly plethoric set of independent charges — we will pursue it here for illustrative purposes in the restricted framework at hand.

The curvature charge Q_{ec} in (3.72) is invariant under Ehlers' $\text{SL}(2, \mathbb{R})$, and this is inferred using the transformation laws (4.10) and (4.12). The mass charge Q_{m} , eq. (3.75), is not, but its transformation (see (4.9), (4.10) and (4.11)) suggests that it might belong to some $\text{SL}(2, \mathbb{R})$ multiplet or, more accurately, that it may be modified to this end — we have this freedom owing to time independence. Actually, a slight amendment to the charge Q_{m} , namely

$$Q'_{\text{m}} = \int_{\mathcal{S}} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2} (\hat{\tau} + 2\hat{\omega}K), \quad (4.18)$$

is $\text{SL}(2, \mathbb{R})$ -invariant. We can even go further and apply the following pattern to generate $\text{SL}(2, \mathbb{R})$ triplets. Suppose we identify a Carrollian two-form \mathbf{v} transforming under $\text{SL}(2, \mathbb{R})$

as

$$\mathbf{v} \rightarrow \mathbf{v}' = -\mathbf{v} \left(\gamma \hat{k} - i\delta \right)^2. \tag{4.19}$$

This allows to design an $SL(2, \mathbb{R})$ two-form triplet, i.e. a symmetric rank-two tensor, transforming as

$$\begin{pmatrix} \mathbf{v}'_3 & \mathbf{v}'_2 \\ \mathbf{v}'_2 & \mathbf{v}'_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \mathbf{v}_3 & \mathbf{v}_2 \\ \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \tag{4.20}$$

where

$$\mathbf{v}_1 = \mathbf{v}, \quad \mathbf{v}_2 = i\hat{k}\mathbf{v}, \quad \mathbf{v}_3 = -\hat{k}^2\mathbf{v}. \tag{4.21}$$

The same holds for the complex-conjugate triplet: $\bar{\mathbf{v}}_1 = \bar{\mathbf{v}}$, $\bar{\mathbf{v}}_2 = -i\hat{k}\bar{\mathbf{v}}$ and $\bar{\mathbf{v}}_3 = -\hat{k}^2\bar{\mathbf{v}}$. An $SL(2, \mathbb{R})$ triplet of charges is thus reached as

$$Q_I = \int_{\mathcal{S}} \mathbf{v}_I, \quad I = 1, 2, 3, \tag{4.22}$$

and $Q \equiv Q_1 Q_3 - Q_2^2$ is invariant under Möbius transformations.

The above strategy can be readily applied. Two-forms transforming as in (4.19) can be found, inspired by the structures of the charge (3.75) and of the Carrollian currents (3.31) and (3.34), given the expressions of the Carrollian twist (3.46), the Carrollian curvature (3.40), and the Carrollian Cotton tensors (3.41), (3.42), (3.43) and (3.44). We here exhibit two such Carrollian forms:

$$\mathbf{x} = -\frac{\hat{\tau}}{2(\hat{k} + \hat{k})} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}, \tag{4.23}$$

$$\mathbf{y} = -\left(\frac{P}{\hat{k} + \hat{k}} \right)^2 \partial_\zeta \hat{k} \partial_{\bar{\zeta}} \hat{\omega} \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}. \tag{4.24}$$

These lead along (4.22) to two triplets of charges, which do not carry more information than the original (3.72) and (3.75) though — in the constant- \hat{k} paradigm, which is in fact the most generic, these are K , M , n and possibly a , and the second triplet vanishes.

The last item in our Carrollian agenda is to setting the relationship amongst the charges introduced here using purely boundary methods and those computed directly by standard bulk techniques. This sort of question definitely deserves to be addressed in more general situations than ours, i.e. in the presence of a large set of non-trivial surface charges computed e.g. within covariant phase-space formalism [90]. Nonetheless some relevant observations can be made here, in relation with the original discussion on charges of ref. [2], in which the above two-forms (4.23) and (4.24) turn out to play a prominent role.

In ref. [2], an $SL(2, \mathbb{R})$ triplet of bulk two-forms, leading to surface charges upon integration on the celestial sphere of \mathcal{M} , is obtained by oxidizing the following two-form triplet of $\mathcal{S} \equiv \mathcal{M}/_{\text{orb}(\hat{\xi})}$ (eqs. (18) and (16) of the quoted reference):

$$\begin{aligned} V_1 &= \frac{1}{(\tau - \bar{\tau})^2} \star_{\text{h}}^3 (d\tau + d\bar{\tau}), \\ V_2 &= \frac{1}{(\tau - \bar{\tau})^2} \star_{\text{h}}^3 (\bar{\tau}d\tau + \tau d\bar{\tau}), \\ V_3 &= \frac{1}{(\tau - \bar{\tau})^2} \star_{\text{h}}^3 (\bar{\tau}^2 d\tau + \tau^2 d\bar{\tau}), \end{aligned} \tag{4.25}$$

where “ \star_h^3 ” stands for the three-dimensional Hodge-dual on \mathcal{S} equipped with \tilde{h}_{AB} displayed in (4.8). It is remarkable that the asymptotic limit of this two-form triplet coincides with those designed earlier from Carrollian boundary considerations. This statement is captured in the following result:

$$\lim_{\tilde{r} \rightarrow \infty} \begin{pmatrix} V_3 & V_2 \\ V_2 & V_1 \end{pmatrix} = \begin{pmatrix} -\hat{k}^2(\mathbf{x} + \mathbf{y}) - \hat{k}^2(\bar{\mathbf{x}} + \bar{\mathbf{y}}) & i\hat{k}(\mathbf{x} + \mathbf{y}) - i\hat{k}(\bar{\mathbf{x}} + \bar{\mathbf{y}}) \\ i\hat{k}(\mathbf{x} + \mathbf{y}) - i\hat{k}(\bar{\mathbf{x}} + \bar{\mathbf{y}}) & \mathbf{x} + \mathbf{y} + \bar{\mathbf{x}} + \bar{\mathbf{y}} \end{pmatrix}, \quad (4.26)$$

where $\tilde{r} = r + \varpi$ was introduced in footnote 36 as an $SL(2, \mathbb{R})$ -invariant radial coordinate, which must be used here in order to guarantee that the limit preserves the $SL(2, \mathbb{R})$ behaviour.

5 Conclusions

When a four-dimensional spacetime geometry is invariant under the action of a one-dimensional group of motions, a reduction can be performed and vacuum Einstein dynamics reveals a symmetry under Möbius transformations. Our main motivation was to exhibit this action from a holographic perspective, namely on the three-dimensional boundary of the Ricci-flat configuration at hand. We have successfully reached this goal for a class of *resummable* or *integrable* metrics, which are algebraic in Petrov’s classification and possess a time-like isometry. All of our findings can be extended to embody any Ricci-flat spacetime possessing an isometry at the expense of an augmented technical difficulty due to (i) the use of generic Killing vectors with Ehlers action ending outside the class of algebraically special, resummable metrics (3.13),³⁷ and (ii) the presence of an indefinitely increasing number of independent boundary observables transforming under $SL(2, \mathbb{R})$. The main features of the boundary $SL(2, \mathbb{R})$ action are however clearly captured by the simplest case treated here and we will now summarize them.

At the heart of the boundary Möbius transformations one finds the Carrollian Cotton tensors. The latter are a set of descendants of the original boundary pseudo-Riemannian Cotton, reached in the zero-speed-of-light limit. One finds in particular a scalar c , which is a dual-mass aspect, naturally combined with the Bondi mass aspect, another Carrollian scalar identified with the boundary Carrollian fluid energy density ε . The Möbius transformation hence mixes the geometric boundary variables i.e. those which determine the boundary itself with the dynamical variables like the boundary fluid (this is one of the infinite data, made redundant in the resummable situation studied here). Our analysis reveals that this duality transformation on the boundary is *algebraic* i.e. *local* for the metric, Ehresmann connection, field of observers, and for every other Carrollian boundary data. This is an important achievement summarized in eqs. (4.9), (4.10), (4.11) and (4.12), rooted in the decoupling of r close to the boundary.

An aside message this analysis conveys is the role of the Cotton tensor, which is manifestly dual to the energy/momentum. Before the advent of flat holography and Carrollian

³⁷See e.g. [91], where examples of space-like Killings are displayed with Ehlers groups connecting Petrov special to Petrov general Ricci-flat spacetimes (more recent works in a similar spirit are refs. [92, 93]), and [47] for a mathematical essay on the behaviour of the Weyl tensor under Ehlers’ Möbius group.

physics, the boundary Cotton tensor had been recognized in AdS/CFT as an unavoidable boundary trait carrying information on the bulk magnetic charges such as the nut [63, 64] (see also [94]). The Möbius transformation (4.9) for the $SO(2) \subset SL(2, \mathbb{R})$ subgroup was actually anticipated as a relationship on the conformal pseudo-Riemannian boundary of four-dimensional Einstein spacetimes [65, 95], in an attempt to relate electric and magnetic solutions to Einstein’s equations. Although such dual solutions exist irrespective of the cosmological constant, the relevant subgroup of Ehlers’ breaks down for $\Lambda \neq 0$ [46]. The bulk duality relationship fades in this case, but persists asymptotically and reveals on the conformal boundary. What we find here is a Λ -to-zero limit of this relationship.

Notwithstanding their role in boundary Ehlers duality manifestation, the Carrollian Cotton tensors obey off-shell conservation properties and generate towers of magnetic charges, some of them being conserved. This property is not exclusive to Ricci-flat spacetimes and Carrollian boundaries. Einstein bulk spacetimes and pseudo-Riemannian boundaries do provide a conserved Cotton tensor, which contracted with any boundary conformal Killing vector leads to a conserved current, hence a conserved charge. This powerful tool is undermined by the limited — if any — number of conformal isometries on arbitrary three-dimensional Riemannian spacetimes. The remarkable spin-off about Carrollian boundaries is the existence of an infinite-dimensional conformal group, which makes this method of charge determination a serious alternative to the more standard bulk asymptotic techniques. Following the Cotton pattern, electric towers of charges are constructed with the fluid dynamical data, which can only enjoy on-shell conservation — the same would hold in AdS boundaries with the aforementioned limitation. On both electric and magnetic sides, the towers of charges are multiplied ad nauseam, beyond their leading components.

Our present investigation on towers of charges designed from a boundary standpoint is radically novel and deserves a systematic extension. It has been here confined in the integrable case, where the infinite set of observables is redundant and shrinks to the elementary “leading” data — our tentative definition of subleading currents might have turned too naive, hadn’t it reproduced successfully the multipole moments. Moreover, our main goal being primarily on boundary Ehlers action, we have assumed a time-like bulk isometry, which further reduces this set. Besides, the chosen time-like Killing field was aligned with the fibre of the boundary Carrollian structure, which screens the black-hole acceleration parameter and avoids exploring head-on the uncharted subject of Carrollian reductions. The latter is the mathematical tool to be developed for unravelling the bulk-to-boundary relationship of hidden symmetries in Ricci-flat spacetimes. It could encompass bulk reductions along space-like isometries, which are interesting because they leave room for gravitational radiation,³⁸ probing the interplay between Ehlers Möbius group, time evolution and charge non-conservation. Last, we did not address the question of the charge algebra and its potential central extensions, or discussed other more general related physical aspects. All this calls for a thorough comparison to alternative approaches such as those of refs. [32–39] based on Newman-Penrose formalism — or to applications [83, 96–101].

³⁸The Petrov-algebraic spacetimes (3.13) accommodate axisymmetric time-dependent solutions of the Robinson-Trautman type, whose final state is the C-metric — see. [74] section 28.1.

In the chapter of charges, in spite of the various limitations just stated, we have successfully described the Ehlers Möbius action, and discussed the organisation of available charges in $SL(2, \mathbb{R})$ multiplets. This enabled us to recover from a Carrollian viewpoint the triplet of Komar charges inferred by Geroch in its original publication [2]. Again, this result should be considered as a first step toward a methodical $SL(2, \mathbb{R})$ taming of the above towers of electric/magnetic currents and charges in more general situations. These objects should include the boundary attributes of the bulk Weyl tensor, whose behaviour under Möbius transformations has been addressed in [47].

The importance of the boundary covariantization — Carroll and Weyl — is yet another feature we would like to stress, as it hasn't been sufficiently appreciated in the literature. This characteristic is absent from Bondi or Newman-Unti gauges, where the formalism might suggest that the relevant part of the conformal boundary is its two-dimensional spatial section — the celestial sphere. We heavily insist on the three-dimensional and Carrollian nature of the boundary, which is made manifest in the gauge we have been using. In ordinary AdS/CFT holography the Fefferman-Graham gauge is superior for this reason. One should likewise use a truly boundary-covariant gauge in flat holography and take advantage of it, as we modestly did for exhibiting the action of Ehlers' group, or for discussing the charges and their conservation. No boundary approach of this sort would have been possible in the more conventional gauges. Correspondingly, flat holography based on a purely celestial gauge is bound to be incomplete.

It is worth mentioning that Ehlers' $SL(2, \mathbb{R})$ group is the first and simplest example of a hidden symmetry. As pointed out in the introduction (see the references proposed there), more involved reductions reveal richer symmetries and the underlying dynamics is captured by elegant sigma models in various dimensions. Recasting this knowledge in a holographic fashion, we could possibly learn more, or at least differently, not only about hidden symmetries but also on flat holography. Carrollian reductions might again be the appropriate tool.

On a more speculative tone, our results suggest that a boundary analysis might reveal more general or unexpected duality properties. The paradigm of anti-de Sitter spacetimes, where the $SL(2, \mathbb{R})$ is broken in the bulk but restored on the boundary, calls for a systematic investigation that would complement the heuristic discussion of ref. [65], and possibly uncover novel instances of boundary duality symmetries, associated e.g. with an asymptotic Killing field rather than a plain reduction along Killing orbits. One could even be more audacious and entertain the idea of a “boundary” analysis for half-flat spaces (this is vaguely motivated by footnote 18), which have attracted some attention in relation with $w_{1+\infty}$ symmetry (see the original works [104–106] and [107–111] for a recent emanation). The main caveat foreseen here is the absence of Carrollian boundaries in Euclidean gravity, but this could be evaded in the ultra-hyperbolic instance ($2 + 2$ signature).

Acknowledgments

We would like to thank our colleagues G. Bossard, L. Ciambelli, A. Fiorucci, M. Godazgar, N. Lambert, C. Marteau, B. Oblak, G. Papadopoulos, A. Petkou, R. Ruzzi, A. Seraj,

K. Siampos, A. Stergiou and P. West for useful discussions and feedback. Nehal Mittal acknowledges the CPHT of the Ecole Polytechnique for hospitality during his M1 internship in 2021 and *DIM Quantip* for funding his PhD fellowship at the LKB. The work of D. Rivera-Betancour was funded by Becas Chile (ANID) Scholarship No. 72200301. The work of M. Vilatte was supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the *First Call for H.F.R.I. Research Projects to support Faculty members and Researchers and the procurement of high-cost research equipment grant* (MIS 1524, Project Number: 96048).

A Carrollian covariance in arbitrary dimension

Carroll structures on $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ were introduced in section 3.1 with emphasis on the covariance properties they enjoy when the time coordinate is aligned with the fiber of the structure. In the present appendix we will elaborate on this subject, treating in particular Carrollian covariant and Weyl-covariant derivatives.

The Carrollian transformations (eqs. (3.3) and (3.4)) are connection-like (non-covariant) for ∂_i and b_i , and density-like for ∂_t and Ω :

$$\partial'_j = J^{-1i}{}_j \left(\partial_i - \frac{j_i}{J} \partial_t \right), \quad b'_k = \left(b_i + \frac{\Omega}{J} j_i \right) J^{-1i}{}_k, \quad \partial'_t = \frac{1}{J} \partial_t, \quad \Omega' = \frac{\Omega}{J}. \quad (\text{A.1})$$

The vector fields dual to the forms dx^i are

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega} \partial_t \quad (\text{A.2})$$

and transform covariantly under (3.3) together with the metric (3.1), and the fields (3.2) and (3.5):

$$\mathbf{v}' = \mathbf{v}, \quad \boldsymbol{\mu}' = \boldsymbol{\mu}, \quad \hat{\partial}'_i = J^{-1j}{}_i \hat{\partial}_j, \quad a^{ij'} = J^i{}_k J^j{}_l a^{kl}. \quad (\text{A.3})$$

The vectors $\hat{\partial}_i$ and \mathbf{v} do not commute. They define the *Carrollian vorticity and acceleration*:

$$\left[\hat{\partial}_i, \hat{\partial}_j \right] = \frac{2}{\Omega} \varpi_{ij} \partial_t, \quad \left[\mathbf{v}, \hat{\partial}_i \right] = \frac{1}{\Omega} \varphi_i \partial_t, \quad \varpi_{ij} = \partial_{[i} b_{j]} + b_{[i} \varphi_{j]}, \quad \varphi_i = \frac{1}{\Omega} (\partial_t b_i + \partial_i \Omega), \quad (\text{A.4})$$

similarly appearing in

$$d\boldsymbol{\mu} = \varpi_{ij} dx^i \wedge dx^j + \varphi_i dx^i \wedge \boldsymbol{\mu}. \quad (\text{A.5})$$

A Carroll structure (strong definition) is also equipped with a metric-compatible and torsionless connection. Due to the degeneration of the metric, such a connection is not unique, but it can be chosen as the connection inherited from the parent relativistic space-time (see footnote 9),

$$\hat{\gamma}^i{}_{jk} = \frac{a^{il}}{2} \left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk} \right), \quad (\text{A.6})$$

obeying $\hat{\gamma}^k{}_{[ij]} = 0$, $\hat{\nabla}_i a_{jk} = 0$ and leading to the Levi-Civita-Carroll spatial covariant derivative $\hat{\nabla}_i$.³⁹

³⁹Details on the transformation rules can be found in the appendix A.2 of ref. [70].

The ordinary time-derivative operator $\frac{1}{\Omega}\partial_t$ acts covariantly on Carrollian tensors. However, it is not metric-compatible because a_{ij} depend on time and a temporal covariant derivative is defined requiring $\frac{1}{\Omega}\hat{D}'_t = \frac{1}{\Omega}\hat{D}_t$ and $\hat{D}_t a_{jk} = 0$. To this end, we introduce a temporal connection (a sort of extrinsic curvature of the spatial section \mathcal{S})

$$\hat{\gamma}_{ij} = \frac{1}{2\Omega}\partial_t a_{ij} = \xi_{ij} + \frac{1}{d}a_{ij}\theta, \quad \theta = \frac{1}{\Omega}\partial_t \ln \sqrt{a}, \quad (\text{A.7})$$

which is a symmetric Carrollian tensor splitting into the *geometric Carrollian shear (traceless)* and the *Carrollian expansion (trace)*. The action of \hat{D}_t on any tensor is obtained using Leibniz rule plus the action on scalars and vectors:

$$\frac{1}{\Omega}\hat{D}_t V^i = \frac{1}{\Omega}\partial_t V^i + \hat{\gamma}^i_j V^j, \quad \frac{1}{\Omega}\hat{D}_t V_i = \frac{1}{\Omega}\partial_t V_i - \hat{\gamma}_i^j V_j. \quad (\text{A.8})$$

The commutators of Carrollian covariant derivatives define Carrollian curvature tensors:

$$\begin{aligned} [\hat{\nabla}_k, \hat{\nabla}_l] V^i &= \left(\hat{\partial}_k \hat{\gamma}^i_{lj} - \hat{\partial}_l \hat{\gamma}^i_{kj} + \hat{\gamma}^i_{km} \hat{\gamma}^m_{lj} - \hat{\gamma}^i_{lm} \hat{\gamma}^m_{kj} \right) V^j + [\hat{\partial}_k, \hat{\partial}_l] V^i \\ &= \hat{r}^i_{jkl} V^j + \varpi_{kl} \frac{2}{\Omega} \partial_t V^i, \end{aligned} \quad (\text{A.9})$$

where \hat{r}^i_{jkl} is the *Riemann-Carroll* tensor. The Ricci-Carroll tensor and the Carroll scalar curvature are thus

$$\hat{r}_{ij} = \hat{r}^k_{ikj} \neq \hat{r}_{ji}, \quad \hat{r} = a^{ij} \hat{r}_{ij}. \quad (\text{A.10})$$

Similarly, space and time derivatives do not commute:

$$\left[\frac{1}{\Omega} \hat{D}_t, \hat{\nabla}_i \right] V^j = \varphi_i \left(\left(\frac{1}{\Omega} \hat{D}_t + \theta \right) V^j - \hat{\gamma}^j_k V^k \right) - \hat{\gamma}_i^k \hat{\nabla}_k V^j - d \hat{r}^j_{ik} V^k \quad (\text{A.11})$$

with

$$\hat{r}^j_{ik} = \frac{1}{d} \left(\theta \varphi_i \delta^j_k + \hat{\nabla}_i \hat{\gamma}^j_k - \frac{1}{\Omega} \partial_t \hat{\gamma}^j_{ik} \right), \quad \hat{r}^j_{jk} = \hat{r}_k = \frac{1}{d} \left(\hat{\nabla}_j \hat{\gamma}^j_k - \hat{\partial}_k \theta \right), \quad (\text{A.12})$$

further Carrollian curvature tensors.

The boundary geometry — be it pseudo-Riemannian or Carrollian — enjoy conformal properties. Weyl transformations are defined through their action on elementary geometric data

$$a_{ij} \rightarrow \frac{1}{\mathcal{B}^2} a_{ij}, \quad b_i \rightarrow \frac{1}{\mathcal{B}} b_i, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega \quad (\text{A.13})$$

with $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$ an arbitrary function. A Weyl-covariant derivative requires an appropriate connection built on φ_i and θ defined in (A.4) and (A.7), which transform as

$$\varphi_i \rightarrow \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \theta \rightarrow \mathcal{B} \theta - \frac{d}{\Omega} \partial_t \mathcal{B}. \quad (\text{A.14})$$

The Carrollian vorticity ϖ_{ij} (A.4) and the Carrollian shear ξ_{ij} (A.7) are Weyl-covariant of weight -1 .

The Weyl-Carroll space and time covariant derivatives are metric-compatible and torsionless. For a scalar function Φ and a vector V^l of weight w , we find:

$$\hat{\mathcal{D}}_j \Phi = \hat{\partial}_j \Phi + w \varphi_j \Phi, \quad (\text{A.15})$$

$$\hat{\mathcal{D}}_j V^l = \hat{\nabla}_j V^l + (w-1) \varphi_j V^l + \varphi^l V_j - \delta_j^l V^i \varphi_i. \quad (\text{A.16})$$

The weights are not altered by the spatial derivative and $\hat{\mathcal{D}}_j a_{kl} = 0$. One also defines

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi = \frac{1}{\Omega} \hat{D}_t \Phi + \frac{w}{d} \theta \Phi = \frac{1}{\Omega} \partial_t \Phi + \frac{w}{d} \theta \Phi, \quad (\text{A.17})$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t V^l = \frac{1}{\Omega} \hat{D}_t V^l + \frac{w-1}{d} \theta V^l = \frac{1}{\Omega} \partial_t V^l + \frac{w}{d} \theta V^l + \xi^l_i V^i, \quad (\text{A.18})$$

both are of weight $w + 1$. Furthermore $\hat{\mathcal{D}}_t a_{kl} = 0$, using Leibniz rule.

We finally obtain

$$\left[\hat{\mathcal{D}}_i, \hat{\mathcal{D}}_j \right] \Phi = \frac{2}{\Omega} \varpi_{ij} \hat{\mathcal{D}}_t \Phi + w \Omega_{ij} \Phi, \quad (\text{A.19})$$

$$\left[\hat{\mathcal{D}}_k, \hat{\mathcal{D}}_l \right] V^i = \left(\hat{\mathcal{R}}^i_{jkl} - 2\xi^i_j \varpi_{kl} \right) V^j + \varpi_{kl} \frac{2}{\Omega} \hat{\mathcal{D}}_t V^i + w \Omega_{kl} V^i, \quad (\text{A.20})$$

where

$$\begin{aligned} \hat{\mathcal{R}}^i_{jkl} &= \hat{r}^i_{jkl} - \delta_j^i \varphi_{kl} - a_{jk} \hat{\nabla}_l \varphi^i + a_{jl} \hat{\nabla}_k \varphi^i + \delta_k^i \hat{\nabla}_l \varphi_j - \delta_l^i \hat{\nabla}_k \varphi_j \\ &\quad + \varphi^i (\varphi_k a_{jl} - \varphi_l a_{jk}) - \left(\delta_k^i a_{jl} - \delta_l^i a_{jk} \right) \varphi_m \varphi^m + \left(\delta_k^i \varphi_l - \delta_l^i \varphi_k \right) \varphi_j, \end{aligned} \quad (\text{A.21})$$

$$\Omega_{ij} = \hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i - \frac{2}{d} \varpi_{ij} \theta \quad (\text{A.22})$$

are weight-0 Weyl-covariant tensors. Tracing them we obtain:

$$\hat{\mathcal{R}}_{ij} = \hat{\mathcal{R}}^k_{ikj}, \quad \hat{\mathcal{R}} = a^{ij} \hat{\mathcal{R}}_{ij} \quad (\text{A.23})$$

with

$$\hat{\mathcal{R}} = \hat{r} + (d-1) \left(2\hat{\nabla}_i \varphi^i - (d-2) \varphi_i \varphi^i \right), \quad (\text{A.24})$$

of weights zero and 2. The Weyl-covariant Carroll-Ricci tensor is not symmetric, $\hat{\mathcal{R}}_{[ij]} = -\frac{d}{2} \Omega_{ij}$, and a weight-1 curvature form also appears with

$$\left[\frac{1}{\Omega} \hat{\mathcal{D}}_t, \hat{\mathcal{D}}_i \right] \Phi = w \hat{\mathcal{R}}_i \Phi - \xi^j_i \hat{\mathcal{D}}_j \Phi, \quad (\text{A.25})$$

where

$$\hat{\mathcal{R}}_i = \frac{1}{\Omega} \partial_t \varphi_i - \frac{1}{d} \left(\hat{\partial}_i + \varphi_i \right) \theta. \quad (\text{A.26})$$

B Conformal Carrollian dynamics and charges

A complete account on the subject of dynamics and charges with the present conventions is available in refs. [70, 71]. We summarize here the necessary items, in particular regarding the Weyl-covariant side, which is relevant on the holographic boundaries.

The basics are encoded into four Carrollian momenta, replacing the relativistic energy-momentum tensor, which are obtained by varying some (effective) action with respect to a_{ij} , b_i and Ω (the fourth momentum is not necessarily obtained in this way — for details see [71]). These are the energy-stress tensor Π^{ij} , the energy flux Π^i , the energy density Π as well as the momentum P^i , of conformal weights $d + 3$, $d + 2$, $d + 1$ and $d + 2$. Extra momenta can also emerge as more degrees of freedom may be present. This phenomenon

occurs when studying the small- c limit of a relativistic energy-momentum tensor and the corresponding conservation equations. Keeping things rather minimal with only $\tilde{\Pi}^{ij}$ the equations read:⁴⁰

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \hat{\mathcal{D}}_i \Pi^i + \Pi^{ij} \xi_{ij} = 0, \tag{B.1}$$

$$\tilde{\Pi}^{ij} \xi_{ij} = 0, \tag{B.2}$$

$$\hat{\mathcal{D}}_i \Pi^i_j + 2\Pi^i \varpi_{ij} + \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \delta_j^i + \xi^i_j \right) P_i = 0, \tag{B.3}$$

$$\hat{\mathcal{D}}_i \tilde{\Pi}^i_j + \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \delta_j^i + \xi^i_j \right) \Pi_i = 0 \tag{B.4}$$

with

$$\tilde{\Pi}_i^i = 0, \quad \Pi_i^i = \Pi, \tag{B.5}$$

as a consequence of the assumed Weyl invariance.

Equations (B.1), (B.2), (B.3) and (B.4) are the Carrollian emanation of the relativistic conservation equation $\nabla_\mu T^{\mu\nu} = 0$. As for the relativistic instance, conformal isometries lead to conserved currents and conserved charges. Let ξ be a $d + 1$ -dimensional vector

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \left(\xi^t - \xi^i \frac{b_i}{\Omega} \right) \partial_t + \xi^i \left(\partial_i + \frac{b_i}{\Omega} \partial_t \right) = \xi^{\hat{t}} \frac{1}{\Omega} \partial_t + \xi^{\hat{i}} \hat{\partial}_i \tag{B.6}$$

restricted to $\xi^i = \xi^i(\mathbf{x})$, generator of a one-dimensional group of Carrollian diffeomorphisms on $\mathcal{M} = \mathbb{R} \times \mathcal{S}$. Its action on the elementary geometric data (3.1), (3.2) and (3.5) is as

⁴⁰Using the language of fluids, Π appears as the zero- c limit of the relativistic energy density, Π^i and P^i are the orders one and c^2 of the relativistic heat current, whereas $\tilde{\Pi}^{ij}$ and Π^{ij} are the orders $1/c^2$ and one of the relativistic stress. A non-vanishing Carrollian energy flux Π^i breaks local Carroll-boost invariance (see e.g. [102]) and makes its dual variable i.e. the Ehresmann connection $\mathbf{b} = b_i dx^i$ dynamical. This is neither a surprise nor a caveat. On the one hand, Carrollian dynamics, i.e. dynamics on geometries equipped with a degenerate metric, is often reached as a vanishing- c limit of relativistic dynamics and naturally breaks local Carroll boosts, even when the original relativistic theory is Lorentz-boost invariant. Indeed, invariance under local Lorentz boosts sets symmetry constraints on the components of the relativistic energy-momentum tensor, but not on their behaviour with respect to c^2 , leaving the possibility of persisting energy flux Π^i and “over-stress” $\tilde{\Pi}^{ij}$ related through eq. (B.4). A similar phenomenon occurs in Galilean theories, defined on spacetimes with a degenerate cometric, where the Galilean momentum is possibly responsible for the breaking of local Galilean-boost invariance. On the other hand, it is fortunate that this happens in the present instance (one of the very few known applications of Carrollian dynamics), when passing from the relativistic boundary of asymptotically anti-de Sitter spacetimes to the Carrollian boundary of their asymptotically flat relatives, as the Carrollian energy flux accounts for non-conservation properties resulting from bulk gravitational radiation, whereas the Ehresmann connection is part of the Ricci-flat solution space.

follows:⁴¹

$$\mathcal{L}_\xi a_{ij} = 2\hat{\nabla}_{(i}\xi^k a_{j)k} + 2\xi^{\hat{t}}\hat{\gamma}_{ij}, \quad (\text{B.7})$$

$$\mathcal{L}_\xi \mathbf{v} = \mu \mathbf{v}, \quad (\text{B.8})$$

$$\mathcal{L}_\xi \mu = -\mu \mu + \nu \quad (\text{B.9})$$

with $\mathbf{v} = \nu_i dx^i$ and

$$\begin{aligned} \mu(t, \mathbf{x}) &= -\left(\frac{1}{\Omega}\partial_t \xi^{\hat{t}} + \varphi_i \xi^i\right), \\ \nu_i(t, \mathbf{x}) &= -\left(\hat{\partial}_i - \varphi_i\right)\xi^{\hat{t}} + 2\xi^j \varpi_{ji}. \end{aligned} \quad (\text{B.10})$$

Due to the degeneration of the metric on \mathcal{M} , the variation of the field of observers \mathbf{v} is not identical to that of the clock form μ .

Isometries are generated by Killing fields of the Carrollian type (B.6), required to obey [30, 31, 41, 51]:

$$\mathcal{L}_\xi a_{ij} = 0, \quad \mathcal{L}_\xi \mathbf{v} = 0. \quad (\text{B.11})$$

i.e.

$$\hat{\nabla}_{(i}\xi^k a_{j)k} + \xi^{\hat{t}}\hat{\gamma}_{ij} = 0, \quad (\text{B.12})$$

$$\frac{1}{\Omega}\partial_t \xi^{\hat{t}} + \varphi_i \xi^i = 0. \quad (\text{B.13})$$

The clock form is not required to be invariant. Carrollian conformal Killing fields must satisfy

$$\mathcal{L}_\xi a_{ij} = \lambda a_{ij} \quad (\text{B.14})$$

with

$$\lambda(t, \mathbf{x}) = \frac{2}{d}\left(\hat{\nabla}_i \xi^i + \theta \xi^{\hat{t}}\right). \quad (\text{B.15})$$

This set of partial differential equations is insufficient for defining conformal Killing fields. One usually imposes to tune μ versus λ (see [30, 31, 51] for a detailed presentation) so that the scaling of the metric be twice that of the field of observers:

$$2\mu + \lambda = 2\left(\frac{1}{d}\hat{\mathcal{D}}_i \xi^i - \frac{1}{\Omega}\hat{\mathcal{D}}_t \xi^{\hat{t}}\right) = 0 \quad (\text{B.16})$$

(the conformal weight of $\xi^{\hat{t}}$ is -1 , that of ξ^i is zero). Again, the clock form is not involved. If one demands the latter be invariant under the action of a Killing field, or aligned with itself under the action of a conformal Killing, which in both cases amounts to setting

$$\nu_i \equiv -\hat{\mathcal{D}}_i \xi^{\hat{t}} + 2\xi^j \varpi_{ji} = 0 \quad (\text{B.17})$$

⁴¹The Lie derivative along $\xi = \xi^{\hat{t}}\frac{1}{\Omega}\partial_t + \xi^i\hat{\partial}_i$ of a general Carrollian tensor reads:

$$\begin{aligned} \mathcal{L}_\xi S_{i\dots}{}^{j\dots} &= \left(\xi^{\hat{t}}\frac{1}{\Omega}\partial_t + \xi^k\hat{\partial}_k\right) S_{i\dots}{}^{j\dots} + S_{k\dots}{}^{j\dots}\hat{\partial}_i \xi^k + \dots - S_{i\dots}{}^{k\dots}\hat{\partial}_k \xi^j - \dots \\ &= \left(\xi^{\hat{t}}\frac{1}{\Omega}\partial_t + \xi^k\hat{\nabla}_k\right) S_{i\dots}{}^{j\dots} + S_{k\dots}{}^{j\dots}\hat{\nabla}_i \xi^k + \dots - S_{i\dots}{}^{k\dots}\hat{\nabla}_k \xi^j - \dots \end{aligned}$$

(this is a conformal rewriting of ν_i given in (B.10)), then the corresponding (conformal) isometry generator will be referred to as *(conformal) strong Killing vector field*.⁴²

Transformations generated by ordinary Carrollian Killing fields leave invariant geometric markers that are built over the metric a_{ij} and Ω , such as the Carrollian expansion θ or the shear ξ_{ij} , encoded in $\hat{\gamma}_{ij}$, see (A.7). The Carrollian vorticity and acceleration given in (A.5) are not left invariant, however, since

$$\mathcal{L}_\xi d\mu = d\nu, \tag{B.18}$$

unless the Carrollian Killing field is strong (vanishing ν). Likewise, curvature invariance does also require the strong condition. This applies in particular to the Carrollian Cotton tensor discussed in $d = 2$ (see appendix C).

On a Carroll manifold a current has a scalar component κ as well as a Carrollian-vector set of components K^i . The divergence takes the form (see [40, 71, 103])

$$\mathcal{K} = \left(\frac{1}{\Omega} \partial_t + \theta \right) \kappa + \left(\hat{\nabla}_i + \varphi_i \right) K^i. \tag{B.19}$$

The charge associated with the current (κ, \mathbf{K}) is an integral at fixed t over the basis \mathcal{S}

$$Q_K = \int_{\mathcal{S}} d^d x \sqrt{a} \left(\kappa + b_i K^i \right), \tag{B.20}$$

and obeys the following time evolution:

$$\frac{dQ_K}{dt} = \int_{\mathcal{S}} d^d x \sqrt{a} \Omega \mathcal{K} - \int_{\partial \mathcal{S}} * \mathbf{K} \Omega. \tag{B.21}$$

The last term is of boundary type with $*\mathbf{K}$ the \mathcal{S} -Hodge dual of $K_i dx^i$. Generally, one can ignore it owing to adequate fall-off or boundary conditions on the fields.

Suppose that ξ is the generator (B.6) of a Carrollian diffeomorphism. It can be used to create two currents out⁴³ of Π^{ij} , $\tilde{\Pi}^{ij}$, Π^i , P^i and Π [40, 71]:

$$\begin{cases} \kappa = \xi^i P_i - \xi^t \Pi \\ \tilde{\kappa} = \xi^i \Pi_i \\ K^i = \xi^j \Pi_j^i - \xi^t \Pi^i \\ \tilde{K}^i = \xi^j \tilde{\Pi}_j^i, \end{cases} \tag{B.22}$$

If ξ is a (conformal) Carrollian Killing field, and assuming all momenta on-shell i.e. eqs. (B.1), (B.2), (B.3) and (B.4) (with (B.5) satisfied in the conformal instance), one finds the following Carrollian divergences (the conformal weights of κ and $\tilde{\kappa}$ are d , those of K^i and \tilde{K}^i , $d + 1$, and -1 for ν_i):

$$\begin{cases} \tilde{\mathcal{K}} = \frac{1}{\Omega} \hat{\mathcal{D}}_i \tilde{\kappa} + \hat{\mathcal{D}}_i \tilde{K}^i = 0 \\ \mathcal{K} = \frac{1}{\Omega} \hat{\mathcal{D}}_t \kappa + \hat{\mathcal{D}}_i K^i = \Pi^i \nu_i. \end{cases} \tag{B.23}$$

⁴²Carroll boosts, which are the archetype isometries of flat Carrollian spacetimes, *are not* generated by strong Killings [71].

⁴³We stress here that if more momenta were present, more currents would be available.

Two charges can be defined following (B.20): $Q_{\hat{K}}$ and Q_K . The former is conserved, whereas the latter isn't for generic isometries unless the field configuration has vanishing energy flux Π^i , i.e. if local Carroll-boost invariance is unbroken. The breaking of local Carroll-boost invariance hence appears as the trigger of non-conservation laws. This peculiarity was risen in [40, 71] and further illustrated with concrete field realizations in [103]. In four-dimensional Ricci-flat spacetimes, this boundary non-conservation is the consequence of bulk gravitational radiation, as mentioned previously in footnote 40. Observe nevertheless that irrespective of the energy flux Π^i , the (conformal) strong Killings introduced earlier do lead to full conservation properties as a consequence of (B.17).

C Three dimensions and the Carrollian Cotton tensor

Three-dimensional boundaries ($d = 2$) outline the framework of the Ehlers and Geroch investigation pursued in the main part of this article. Three dimensions have two remarkable properties. At the first place, if the geometric Carrollian shear ξ^{ij} defined in (A.7) vanishes, which occurs for the Carrollian boundaries of Ricci-flat spacetimes as a consequence of Einstein's equations (see section 3.1), the Carrollian conformal isometry group is infinite-dimensional: $\text{BMS}_4 \equiv \text{ccarr}(3) \equiv \mathfrak{so}(3,1) \times \text{supertranslations}$ [23, 41]. This potentially generates *infinite towers of charges*, possibly conserved.

Secondly, three-dimensional Carrollian spacetimes possess a *Carrollian Cotton tensor obeying conservation dynamics*. It appears as a set of Carrollian scalars, vectors and tensors emerging in the small- c expansion of the relativistic Cotton $C_{\mu\nu}$, which is symmetric, traceless, divergence-free and Weyl-covariant with weight 1. Reference [23] provides a complete account of the Carrollian descendants as they emerge from the pseudo-Riemannian Cotton tensor, in the absence of geometric Carrollian shear. Here we will circumscribe our exhibition to the basic output.

For $d = 2$, the \mathcal{S} -Hodge duality is induced by⁴⁴ $\eta_{ij} = \sqrt{a}\epsilon_{ij}$. This duality is involutive on Carrollian vectors as well as on two-index symmetric and traceless Carrollian tensors:

$$*V_i = \eta^l{}_i V_l, \quad *W_{ij} = \eta^l{}_i W_{lj}. \tag{C.1}$$

This fully antisymmetric form can be used to recast some of the expressions introduced in appendix A. The Carroll-Ricci tensor (A.10) is decomposed as

$$\hat{r}_{ij} = \hat{s}_{ij} + \hat{K}a_{ij} + \hat{A}\eta_{ij} \tag{C.2}$$

with

$$\hat{s}_{ij} = 2 * \varpi * \xi_{ij}, \quad \hat{K} = \frac{1}{2} a^{ij} \hat{r}_{ij} = \frac{1}{2} \hat{r}, \quad \hat{A} = \frac{1}{2} \eta^{ij} \hat{r}_{ij} = * \varpi \theta, \quad * \varpi = \frac{1}{2} \eta^{ij} \varpi_{ij}. \tag{C.3}$$

Similarly

$$\hat{\mathcal{R}}_{ij} = \hat{s}_{ij} + \hat{\mathcal{K}}a_{ij} + \hat{\mathcal{A}}\eta_{ij}, \tag{C.4}$$

⁴⁴We use here the conventions of ref. [23], namely $\epsilon_{12} = -1$, convenient when using complex coordinates $\{\zeta, \bar{\zeta}\}$. Notice that $\eta^{il}\eta_{jl} = \delta_j^i$ and $\eta^{ij}\eta_{ij} = 2$.

where we have introduced two weight-2 Weyl-covariant scalar Gauss-Carroll curvatures:

$$\hat{\mathcal{K}} = \frac{1}{2}a^{ij}\hat{\mathcal{R}}_{ij} = \hat{K} + \hat{\nabla}_k\varphi^k, \quad \hat{\mathcal{A}} = \frac{1}{2}\eta^{ij}\hat{\mathcal{R}}_{ij} = \hat{A} - *\varphi. \quad (\text{C.5})$$

These obey Carroll-Bianchi identities:

$$\frac{2}{\Omega}\hat{\mathcal{D}}_t*\varpi + \hat{\mathcal{A}} = 0, \quad (\text{C.6})$$

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t\hat{\mathcal{K}} - a^{ij}\hat{\mathcal{D}}_i\hat{\mathcal{R}}_j - \hat{\mathcal{D}}_i\hat{\mathcal{D}}_j\xi^{ij} = 0, \quad (\text{C.7})$$

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t\hat{\mathcal{A}} + \eta^{ij}\hat{\mathcal{D}}_i\hat{\mathcal{R}}_j = 0. \quad (\text{C.8})$$

Thanks to the identities (C.7) and (C.8), the couples $\{\hat{\mathcal{K}}, -\hat{\mathcal{R}}^i - \hat{\mathcal{D}}_j\xi^{ij}\}$ and $\{\hat{\mathcal{A}}, -*\hat{\mathcal{R}}^i\}$ allow to define *electric and magnetic curvature charges* as in eqs. (B.19) and (B.20):

$$Q_{\text{ec}} = \int_{\mathcal{S}} d^2x\sqrt{a} \left(\hat{\mathcal{K}} - b_i \left(\hat{\mathcal{R}}^i + \hat{\mathcal{D}}_j\xi^{ij} \right) \right), \quad Q_{\text{mc}} = \int_{\mathcal{S}} d^2x\sqrt{a} \left(\hat{\mathcal{A}} - b_i *\hat{\mathcal{R}}^i \right). \quad (\text{C.9})$$

Following (B.21), we find

$$\frac{dQ_{\text{ec}}}{dt} = \int_{\partial\mathcal{S}} * \left(\hat{\mathcal{R}} + \hat{\mathcal{D}} \cdot \xi \right) \Omega, \quad \frac{dQ_{\text{mc}}}{dt} = - \int_{\partial\mathcal{S}} \hat{\mathcal{R}} \Omega. \quad (\text{C.10})$$

Upon regular behaviour, the boundary terms vanish and the curvature charges are both conserved.

Besides the various curvature tensors, which are second derivatives of the metric and the Ehresmann connection, one defines third-derivative tensors, the descendants of the relativistic Cotton tensor. We will here limit our presentation to the instance $\xi_{ij} = 0$, which is the appropriate framework when solving Einstein's equations in the bulk. This reduces the number of tensors to five, a weight-3 scalar, two weight-2 forms and two weight-1 two-index symmetric and traceless tensors:

$$c = \left(\hat{\mathcal{D}}_l\hat{\mathcal{D}}^l + 2\hat{\mathcal{K}} \right) *\varpi, \quad (\text{C.11})$$

$$\chi_j = \frac{1}{2}\eta^l_j\hat{\mathcal{D}}_l\hat{\mathcal{K}} + \frac{1}{2}\hat{\mathcal{D}}_j\hat{\mathcal{A}} - 2*\varpi\hat{\mathcal{R}}_j, \quad (\text{C.12})$$

$$\psi_j = 3\eta^l_j\hat{\mathcal{D}}_l*\varpi^2, \quad (\text{C.13})$$

$$X_{ij} = \frac{1}{2}\eta^l_j\hat{\mathcal{D}}_l\hat{\mathcal{R}}_i + \frac{1}{2}\eta^l_i\hat{\mathcal{D}}_j\hat{\mathcal{R}}_l, \quad (\text{C.14})$$

$$\Psi_{ij} = \hat{\mathcal{D}}_i\hat{\mathcal{D}}_j*\varpi - \frac{1}{2}a_{ij}\hat{\mathcal{D}}_l\hat{\mathcal{D}}^l*\varpi - \eta_{ij}\frac{1}{\Omega}\hat{\mathcal{D}}_t*\varpi^2. \quad (\text{C.15})$$

As a consequence of the relativistic conservation of the Cotton tensor, its Carrollian descendants obey eqs. (B.1), (B.2),⁴⁵ (B.3) and (B.4) with

$$\Pi_{\text{Cot}} = c, \quad \Pi_{\text{Cot}}^i = \chi^i, \quad P_{\text{Cot}}^i = \psi^i, \quad \tilde{\Pi}_{\text{Cot}}^{ij} = -X^{ij}, \quad \Pi_{\text{Cot}}^{ij} = \frac{c}{2}a^{ij} - \Psi^{ij}, \quad (\text{C.16})$$

⁴⁵Equation (B.2), is trivially satisfied due to the vanishing of ξ_{ij} . If $\xi_{ij} \neq 0$, extra Cotton Carrollian descendants are available, and the conservation dynamics is encoded in more momenta and equations — in particular (B.2) is modified.

which read

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t c + \hat{\mathcal{D}}_i \chi^i = 0, \tag{C.17}$$

$$\frac{1}{2} \hat{\mathcal{D}}_j c + 2\chi^i \varpi_{ij} + \frac{1}{\Omega} \hat{\mathcal{D}}_t \psi_j - \hat{\mathcal{D}}_i \Psi^i_j = 0, \tag{C.18}$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \chi_j - \hat{\mathcal{D}}_i X^i_j = 0. \tag{C.19}$$

When the geometric Carrollian shear vanishes, the time dependence in the metric is factorized as $a_{ij}(t, \mathbf{x}) = e^{2\sigma(t, \mathbf{x})} \bar{a}_{ij}(\mathbf{x})$. One then shows [23, 41] that the Carrollian conformal isometry group is the semi-direct product of the conformal group of $\bar{a}_{ij}(\mathbf{x})$ with the infinite-dimensional supertranslation group. The former is generated by $Y^i(\mathbf{x})$, the latter by $T(\mathbf{x})$, and the Carrollian conformal Killing fields read:

$$\xi_{T,Y} = \left(T(\mathbf{x}) - Y^i(\mathbf{x}) \hat{\partial}_i C(t, \mathbf{x}) + \frac{1}{2} C(t, \mathbf{x}) \bar{\nabla}_i Y^i(\mathbf{x}) \right) \frac{e^{\sigma(t, \mathbf{x})}}{\Omega} \partial_t + Y^i(\mathbf{x}) \hat{\partial}_i \tag{C.20}$$

with

$$C(t, \mathbf{x}) \equiv \int^t d\tau e^{-\sigma(\tau, \mathbf{x})} \Omega(\tau, \mathbf{x}). \tag{C.21}$$

This result is valid in any dimension. At $d = 2$, $\bar{a}_{ij}(\mathbf{x})$ is conformally flat and $Y^i(\mathbf{x})$ generate $\mathfrak{so}(3, 1)$.⁴⁶

The conservation of the Carrollian Cotton momenta (C.16) makes it possible to define two infinite towers of Carrollian Cotton charges $Q_{\text{Cot}T,Y}$ and $\tilde{Q}_{\text{Cot}T,Y}$ following (B.20), based on the Carrollian Cotton currents κ_{Cot} , K_{Cot}^i , $\tilde{\kappa}_{\text{Cot}}$ and \tilde{K}_{Cot}^i (see (3.34)). According to (B.23), the latter are always conserved,⁴⁷ whereas the former are only if $\chi^i \nu_i = -\chi^i (\hat{\mathcal{D}}_i \xi^i - 2\xi^j \varpi_{ji}) = 0$. This occurs for special geometries ($\chi^i = 0$) or for the subset of strong Carrollian conformal Killing fields ($\nu_i = 0$).

In $d = 2$, it is convenient to use complex spatial coordinates ζ and $\bar{\zeta}$. With the permission of the authors of [23], we reproduce here the appendix of that reference, summarizing the useful formulas in this coordinate system. Using Carrollian diffeomorphisms (3.3), the metric (3.1) of the Carrollian geometry on the two-dimensional surface \mathcal{S} can be recast in conformally flat form,

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta} \tag{C.22}$$

with $P = P(t, \zeta, \bar{\zeta})$ a real function, under the necessary and sufficient condition that the Carrollian shear ξ_{ij} displayed in (A.7) vanishes. We will here assume that this holds and present a number of useful formulas for Carrollian and conformal Carrollian geometry. These geometries carry two further pieces of data: $\Omega(t, \zeta, \bar{\zeta})$ and

$$\mathbf{b} = b_\zeta(t, \zeta, \bar{\zeta}) d\zeta + b_{\bar{\zeta}}(t, \zeta, \bar{\zeta}) d\bar{\zeta} \tag{C.23}$$

⁴⁶The $\mathfrak{so}(3, 1)$ factor can also be promoted to superrotations (double Virasoro) if we give up the absolute regularity requirement.

⁴⁷The conformal Killing fields (C.20), (C.21) depend *explicitly* on time. Inside the charges they define, when conserved, this time dependence is confined, on-shell, in a boundary term, and hence drops — see concrete examples in [103].

with $b_{\bar{\zeta}}(t, \zeta, \bar{\zeta}) = \bar{b}_{\zeta}(t, \zeta, \bar{\zeta})$. Our choice of orientation is inherited from the one adopted for the relativistic boundary (see footnote 10) with $a_{\zeta\bar{\zeta}} = 1/P^2$ is⁴⁸

$$\eta_{\zeta\bar{\zeta}} = -\frac{i}{P^2}. \quad (\text{C.24})$$

The first-derivative Carrollian tensors are the acceleration (A.4), the expansion (A.7) and the scalar vorticity (A.4), (C.3):

$$\varphi_{\zeta} = \partial_t \frac{b_{\zeta}}{\Omega} + \hat{\partial}_{\zeta} \ln \Omega, \quad \varphi_{\bar{\zeta}} = \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \ln \Omega, \quad (\text{C.25})$$

$$\theta = -\frac{2}{\Omega} \partial_t \ln P, \quad * \varpi = \frac{i\Omega P^2}{2} \left(\hat{\partial}_{\zeta} \frac{b_{\bar{\zeta}}}{\Omega} - \hat{\partial}_{\bar{\zeta}} \frac{b_{\zeta}}{\Omega} \right) \quad (\text{C.26})$$

with

$$\hat{\partial}_{\zeta} = \partial_{\zeta} + \frac{b_{\zeta}}{\Omega} \partial_t, \quad \hat{\partial}_{\bar{\zeta}} = \partial_{\bar{\zeta}} + \frac{b_{\bar{\zeta}}}{\Omega} \partial_t. \quad (\text{C.27})$$

Curvature scalars and vector are second-derivative (see (C.3), (A.12)):⁴⁹

$$\hat{K} = P^2 \left(\hat{\partial}_{\bar{\zeta}} \hat{\partial}_{\zeta} + \hat{\partial}_{\zeta} \hat{\partial}_{\bar{\zeta}} \right) \ln P, \quad \hat{A} = iP^2 \left(\hat{\partial}_{\bar{\zeta}} \hat{\partial}_{\zeta} - \hat{\partial}_{\zeta} \hat{\partial}_{\bar{\zeta}} \right) \ln P, \quad (\text{C.28})$$

$$\hat{r}_{\zeta} = \frac{1}{2} \hat{\partial}_{\zeta} \left(\frac{1}{\Omega} \partial_t \ln P \right), \quad \hat{r}_{\bar{\zeta}} = \frac{1}{2} \hat{\partial}_{\bar{\zeta}} \left(\frac{1}{\Omega} \partial_t \ln P \right), \quad (\text{C.29})$$

and we also quote:

$$* \varphi = iP^2 \left(\hat{\partial}_{\zeta} \varphi_{\bar{\zeta}} - \hat{\partial}_{\bar{\zeta}} \varphi_{\zeta} \right), \quad (\text{C.30})$$

$$\hat{\nabla}_k \varphi^k = P^2 \left[\hat{\partial}_{\zeta} \partial_t \frac{b_{\bar{\zeta}}}{\Omega} + \hat{\partial}_{\bar{\zeta}} \partial_t \frac{b_{\zeta}}{\Omega} + \left(\hat{\partial}_{\zeta} \hat{\partial}_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}} \hat{\partial}_{\zeta} \right) \ln \Omega \right]. \quad (\text{C.31})$$

Regarding conformal Carrollian tensors we remind the weight-2 curvature scalars (C.5):

$$\hat{\mathcal{H}} = \hat{K} + \hat{\nabla}_k \varphi^k, \quad \hat{\mathcal{A}} = \hat{A} - * \varphi, \quad (\text{C.32})$$

and the weight-1 curvature one-form (A.26):

$$\hat{\mathcal{R}}_{\zeta} = \frac{1}{\Omega} \partial_t \varphi_{\zeta} - \frac{1}{2} \left(\hat{\partial}_{\zeta} + \varphi_{\zeta} \right) \theta, \quad \hat{\mathcal{R}}_{\bar{\zeta}} = \frac{1}{\Omega} \partial_t \varphi_{\bar{\zeta}} - \frac{1}{2} \left(\hat{\partial}_{\bar{\zeta}} + \varphi_{\bar{\zeta}} \right) \theta. \quad (\text{C.33})$$

The three-derivative Cotton descendants displayed in (C.11)–(C.15) are a scalar

$$c = \left(\hat{\mathcal{D}}_l \hat{\mathcal{D}}^l + 2\hat{\mathcal{H}} \right) * \varpi \quad (\text{C.34})$$

⁴⁸This amounts to setting $\sqrt{a} = i/P^2$ in coordinate frame and $\epsilon_{\zeta\bar{\zeta}} = -1$. The volume form reads $d^2x \sqrt{a} = \frac{d\zeta \wedge d\bar{\zeta}}{iP^2}$.

⁴⁹We also quote for completeness (useful e.g. in eq. (C.32)):

$$\hat{K} = K + P^2 \left[\partial_{\zeta} \frac{b_{\bar{\zeta}}}{\Omega} + \partial_{\bar{\zeta}} \frac{b_{\zeta}}{\Omega} + \partial_t \frac{b_{\zeta} b_{\bar{\zeta}}}{\Omega^2} + 2 \frac{b_{\bar{\zeta}}}{\Omega} \partial_{\zeta} + 2 \frac{b_{\zeta}}{\Omega} \partial_{\bar{\zeta}} + 2 \frac{b_{\zeta} b_{\bar{\zeta}}}{\Omega^2} \partial_t \right] \partial_t \ln P$$

with $K = 2P^2 \partial_{\bar{\zeta}} \partial_{\zeta} \ln P$ the ordinary Gaussian curvature of the two-dimensional metric (C.22).

of weight 3 ($*\varpi$ is of weight 1), two vectors

$$\chi_\zeta = \frac{i}{2} \hat{\mathcal{D}}_\zeta \hat{\mathcal{K}} + \frac{1}{2} \hat{\mathcal{D}}_\zeta \hat{\mathcal{A}} - 2 * \varpi \hat{\mathcal{R}}_\zeta, \quad \chi_{\bar{\zeta}} = -\frac{i}{2} \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{K}} + \frac{1}{2} \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{A}} - 2 * \varpi \hat{\mathcal{R}}_{\bar{\zeta}}, \quad (\text{C.35})$$

$$\psi_\zeta = 3i \hat{\mathcal{D}}_\zeta * \varpi^2, \quad \psi_{\bar{\zeta}} = -3i \hat{\mathcal{D}}_{\bar{\zeta}} * \varpi^2, \quad (\text{C.36})$$

of weight 2, and two symmetric and traceless tensors

$$X_{\zeta\zeta} = i \hat{\mathcal{D}}_\zeta \hat{\mathcal{R}}_\zeta, \quad X_{\bar{\zeta}\bar{\zeta}} = -i \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{R}}_{\bar{\zeta}}, \quad (\text{C.37})$$

$$\Psi_{\zeta\zeta} = \hat{\mathcal{D}}_\zeta \hat{\mathcal{D}}_\zeta * \varpi, \quad \Psi_{\bar{\zeta}\bar{\zeta}} = \hat{\mathcal{D}}_{\bar{\zeta}} \hat{\mathcal{D}}_{\bar{\zeta}} * \varpi, \quad (\text{C.38})$$

of weight 1. Notice that in holomorphic coordinates a symmetric and traceless tensor S_{ij} has only diagonal entries: $S_{\zeta\bar{\zeta}} = 0 = S_{\bar{\zeta}\zeta}$.

We also remind for convenience some expressions for the determination of Weyl-Carroll covariant derivatives. If Φ is a weight- w scalar function

$$\hat{\mathcal{D}}_\zeta \Phi = \hat{\partial}_\zeta \Phi + w \varphi_\zeta \Phi, \quad \hat{\mathcal{D}}_{\bar{\zeta}} \Phi = \hat{\partial}_{\bar{\zeta}} \Phi + w \varphi_{\bar{\zeta}} \Phi. \quad (\text{C.39})$$

For weight- w form components V_ζ and $V_{\bar{\zeta}}$ the Weyl-Carroll derivatives read:

$$\hat{\mathcal{D}}_\zeta V_\zeta = \hat{\nabla}_\zeta V_\zeta + (w+2) \varphi_\zeta V_\zeta, \quad \hat{\mathcal{D}}_{\bar{\zeta}} V_{\bar{\zeta}} = \hat{\nabla}_{\bar{\zeta}} V_{\bar{\zeta}} + (w+2) \varphi_{\bar{\zeta}} V_{\bar{\zeta}}, \quad (\text{C.40})$$

$$\hat{\mathcal{D}}_\zeta V_{\bar{\zeta}} = \hat{\nabla}_\zeta V_{\bar{\zeta}} + w \varphi_\zeta V_{\bar{\zeta}}, \quad \hat{\mathcal{D}}_{\bar{\zeta}} V_\zeta = \hat{\nabla}_{\bar{\zeta}} V_\zeta + w \varphi_{\bar{\zeta}} V_\zeta, \quad (\text{C.41})$$

while the Carrollian covariant derivatives are simply:

$$\hat{\nabla}_\zeta V_\zeta = \frac{1}{P^2} \hat{\partial}_\zeta (P^2 V_\zeta), \quad \hat{\nabla}_{\bar{\zeta}} V_{\bar{\zeta}} = \frac{1}{P^2} \hat{\partial}_{\bar{\zeta}} (P^2 V_{\bar{\zeta}}), \quad (\text{C.42})$$

$$\hat{\nabla}_\zeta V_{\bar{\zeta}} = \hat{\partial}_\zeta V_{\bar{\zeta}}, \quad \hat{\nabla}_{\bar{\zeta}} V_\zeta = \hat{\partial}_{\bar{\zeta}} V_\zeta. \quad (\text{C.43})$$

Finally,

$$\hat{\mathcal{D}}_k \hat{\mathcal{D}}^k \Phi = P^2 \left(\hat{\partial}_\zeta \hat{\partial}_{\bar{\zeta}} \Phi + \hat{\partial}_{\bar{\zeta}} \hat{\partial}_\zeta \Phi + w \Phi \left(\hat{\partial}_\zeta \varphi_{\bar{\zeta}} + \hat{\partial}_{\bar{\zeta}} \varphi_\zeta \right) + 2w \left(\varphi_\zeta \hat{\partial}_{\bar{\zeta}} \Phi + \varphi_{\bar{\zeta}} \hat{\partial}_\zeta \Phi + w \varphi_\zeta \varphi_{\bar{\zeta}} \Phi \right) \right). \quad (\text{C.44})$$

Using complex coordinates, we can recast the conformal Killing vectors of a shear-free Carrollian spacetime \mathcal{M} in three dimensions, given in eqs. (C.20) and (C.21). These are expressed in terms of an arbitrary real function $T(\zeta, \bar{\zeta})$, which encodes the *supertranslations*, and the conformal Killing vectors of flat space $d\ell^2 = 2d\zeta d\bar{\zeta}$. The latter are of the form $Y^\zeta(\zeta) \partial_\zeta + Y^{\bar{\zeta}}(\bar{\zeta}) \partial_{\bar{\zeta}}$, reached with any combination of $\ell_m + \bar{\ell}_m$ or $i(\ell_m - \bar{\ell}_m)$, where⁵⁰

$$\ell_m = -\zeta^{m+1} \partial_\zeta, \quad \bar{\ell}_m = -\bar{\zeta}^{m+1} \partial_{\bar{\zeta}}, \quad (\text{C.45})$$

obeying the Witt \oplus Witt algebra:

$$[\ell_m, \ell_n] = (m-n) \ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m-n) \bar{\ell}_{m+n}, \quad (\text{C.46})$$

⁵⁰Notice that combining (C.1) and (C.24), we find $*(\ell_m + \bar{\ell}_m) = -i(\ell_m - \bar{\ell}_m)$.

and referred to as *superrotations*. Usually one restricts to $\mathfrak{so}(3, 1)$, generated by $n = 0, \pm 1$. The conformal Killing fields of \mathcal{M} are thus

$$\xi_{T,Y} = \left(T - \left(Y^\zeta \hat{\partial}_\zeta + Y^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}} \right) C + \frac{C}{2} \left(\partial_\zeta Y^\zeta + \partial_{\bar{\zeta}} Y^{\bar{\zeta}} \right) \right) \frac{1}{P} \mathbf{v} + Y^\zeta \hat{\partial}_\zeta + Y^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}} \quad (\text{C.47})$$

with

$$C(t, \zeta, \bar{\zeta}) \equiv \int^t d\tau P(\tau, \zeta, \bar{\zeta}) \Omega(\tau, \zeta, \bar{\zeta}). \quad (\text{C.48})$$

The structure $\mathfrak{so}(3, 1) \oplus$ supertranslations — or $(\text{Witt} \oplus \text{Witt}) \oplus$ supertranslations — is recovered in

$$[\xi_{T,Y}, \xi_{T',Y'}] = \xi_{M_Y(T') - M_{Y'}(T), [Y, Y']} \quad (\text{C.49})$$

with

$$M_Y(f) = \left(Y^\zeta \hat{\partial}_\zeta + Y^{\bar{\zeta}} \hat{\partial}_{\bar{\zeta}} \right) f - \frac{f}{2} \left(\partial_\zeta Y^\zeta + \partial_{\bar{\zeta}} Y^{\bar{\zeta}} \right). \quad (\text{C.50})$$

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Revisiting the Carrollian scalar field

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(Received 20 July 2022; accepted 14 September 2022; published 11 October 2022)

We investigate the (conformally coupled) scalar field on a general Carrollian spacetime in arbitrary dimension. The analysis discloses electric and magnetic dynamics. For both, we provide the energy and the momenta of the field, accompanied by their conservation equations. We discuss the conservation and nonconservation properties resulting from the existence of conformal isometries and the associated charges. We illustrate those results for a scalar field propagating on the null boundary of four-dimensional Ricci-flat Robinson-Trautman spacetimes.

DOI: [10.1103/PhysRevD.106.085004](https://doi.org/10.1103/PhysRevD.106.085004)

The Carroll group was discovered in the seminal works of Lévy-Leblond [1] and Sen Gupta [2]. Its first application, due to Henneaux, appeared 14 years later [3] and it took almost 50 years for Carrollian physics to emerge as a full-blown research area, ranging from differential geometry to holographic duality. Carrollian physics is meant to embrace phenomena occurring on a Carrollian spacetime, such as hydrodynamics or, at a more fundamental level, field dynamics. The simplest field is a scalar and it has received some attention [4–11].

The aim of the present note is to present the dynamics of a (conformally coupled) scalar on a general Carrollian manifold, tame and illustrate scattered results, and unify two distinct and complementary approaches. The first relies on Carrollian structures and diffeomorphism invariance. The second consists in reaching Carrollian geometry and dynamics from a pseudo-Riemannian relative at vanishing speed of light. The set of features we address includes: (i) electric vs magnetic dynamics; (ii) action and equations of motion; (iii) energy, momentum, and their conservation; (iv) isometries and Noether's theorem. The basic technical tools are listed in the Appendix.

Carroll structures were introduced in [12–14] (see also [15–20]). They consist of a $(d + 1)$ -dimensional manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ equipped with a degenerate metric and a vector field, the kernel of the metric. For concreteness, we

will adopt coordinates (t, \mathbf{x}) and degenerate metrics of the form

$$d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i dx^j, \quad i, j \dots \in \{1, \dots, d\} \quad (1)$$

with a kernel generated by

$$\mathbf{v} = \frac{1}{\Omega} \partial_t, \quad (2)$$

which defines a field of observers. This coordinate system is adapted to the space/time splitting, which is in turn respected by Carrollian diffeomorphisms

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}). \quad (3)$$

The Carrollian manifold incorporates an Ehresmann connection, which is the background gauge field $\mathbf{b} = b_i dx^i$ appearing in the dual form of the field of observers (2), defined such as $\boldsymbol{\mu}(\mathbf{v}) = -1$:

$$\boldsymbol{\mu} = -\Omega dt + b_i dx^i, \quad (4)$$

the clock form (Ω and b_i depend on t and \mathbf{x}). The vector fields dual to the forms dx^i are

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega} \partial_t. \quad (5)$$

As shown in the Appendix, they transform covariantly under Carrollian diffeomorphisms (3). A Carroll structure (strong definition) is also equipped with a torsionless and metric-compatible connection. This is not unique, due to the degeneracy of the metric. We use here the connection inherited from the parent relativistic spacetime.

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A Carroll structure endowed with metric (1) and clock form (4) is naturally reached in the Carrollian limit ($c \rightarrow 0$) of a pseudo-Riemannian spacetime \mathcal{M} in Papapetrou-Randers gauge

$$ds^2 = -c^2(\Omega dt - b_i dx^i)^2 + a_{ij} dx^i dx^j, \quad (6)$$

where all functions are x dependent with $x \equiv (x^0 = ct, \mathbf{x})$. The connection we use on the Carrollian side is given in the Appendix, Eqs. (A4) and (A6). These are parts of the Levi-Civita connection attached to (6), and decomposed in powers of c .

The dynamics of scalar fields on an arbitrary Carrollian spacetime, limited to two-derivative kinetic terms encompasses two distinct situations dictated by Carrollian covariance. Their Lagrangian densities read

$$\mathcal{L}_e = \frac{1}{2} \left(\frac{1}{\Omega} \partial_t \Phi \right)^2 - V_e(\Phi), \quad (7)$$

$$\mathcal{L}_m = -\frac{1}{2} a^{ij} \hat{\partial}_i \Phi \hat{\partial}_j \Phi - V_m(\Phi), \quad (8)$$

and enter the Carrollian action $S_C = \int_{\mathcal{M}} dt d^d x \sqrt{a} \Omega \mathcal{L}$. The indices “e” and “m” stand for “electric” and “magnetic.” They refer to the origin of these actions in the parent relativistic theory [8,9]. Indeed starting from a relativistic scalar field on a Papapetrou-Randers background (6)

$$S = - \int_{\mathcal{M}} dt d^d x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) \right), \quad (9)$$

and assuming

$$V(\Phi) = \frac{1}{c^2} V_e(\Phi) + V_m(\Phi) + \mathcal{O}(c^2), \quad (10)$$

we find

$$S = \frac{1}{c^2} S_e + S_m + \mathcal{O}(c^2), \quad (11)$$

with S_e and S_m the Carrollian actions with Lagrangian densities (7) and (8). The existence of an expansion (10) for the original relativistic potential in powers of c^2 is a bona fide assumption, necessary to reach two actions invariant under Carrollian diffeomorphisms (3).¹

Due to the form of the metric (6), and to its subsequent behavior under Carrollian diffeomorphisms, the decomposition of any relativistic tensor as a (usually truncated) Laurent expansion, provides a Carrollian tensor

¹The actions associated with the $\mathcal{O}(c^2)$ terms are nondynamical as no kinetic term appears at this order. This will be illustrated in the subsequent analysis of a conformally coupled scalar, see Eq. (16).

for each term.² If we insist in reaching a single Carrollian tensor at vanishing c , then an appropriate rescaling by some power of c^2 is necessary—in order, e.g., to select one out of two options, if only two options are available as in the above scalar-field action (see [8], where this procedure is illustrated in Hamiltonian formalism and for flat spacetime).

An insightful scalar potential for a relativistic curved spacetime in $d + 1$ dimensions is the following:

$$V(\Phi) = \frac{d-1}{8d} R \Phi^2. \quad (12)$$

For a scalar field Φ of weight $w = \frac{d-1}{2}$, this is a conformal coupling. Indeed, the relativistic energy-momentum tensor for (9) with (12) has the form ($\nabla_\mu \Phi = \partial_\mu \Phi$)

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \Phi \nabla^\alpha \Phi \\ &\quad + \frac{d-1}{4d} (G_{\mu\nu} \Phi^2 + g_{\mu\nu} \square \Phi^2 - \nabla_\mu \nabla_\nu \Phi^2), \\ &= \mathcal{D}_\mu \Phi \mathcal{D}_\nu \Phi - \frac{1}{2} g_{\mu\nu} \mathcal{D}_\alpha \Phi \mathcal{D}^\alpha \Phi \\ &\quad + \frac{d-1}{4d} \left(\left(\mathcal{R}_{(\mu\nu)} - \frac{\mathcal{R}}{2} g_{\mu\nu} \right) \Phi^2 \right. \\ &\quad \left. + g_{\mu\nu} \mathcal{D}_\alpha \mathcal{D}^\alpha \Phi^2 - \mathcal{D}_{(\mu} \mathcal{D}_{\nu)} \Phi^2 \right), \end{aligned} \quad (13)$$

where³ $G_{\mu\nu}$ is the Einstein tensor, $\mathcal{R}_{\mu\nu}$ and \mathcal{R} the Weyl-covariant Ricci and scalar defined in the Appendix [Eqs. (A37) and (A38)], together with the Weyl-covariant derivative \mathcal{D}_μ . This energy-momentum tensor is traceless when Φ is on shell, and with a Weyl-covariant of weight $d-1$. The action is Weyl invariant (up to boundary terms⁴), whereas the equations of motion can be recast readily with Weyl-covariant attributes:

$$-\mathcal{D}_\mu \mathcal{D}^\mu \Phi + \frac{d-1}{4d} \mathcal{R} \Phi = 0. \quad (14)$$

As a consequence of diffeomorphism invariance, the energy-momentum tensor obeys a Weyl-covariant conservation equation, when the field Φ is on-shell:

$$\nabla_\mu T^{\mu\nu} = \mathcal{D}_\mu T^{\mu\nu} = 0. \quad (15)$$

²Phrased in more mathematical terms, the expansion in powers of c^2 , amounts to reducing the representations of the full diffeomorphism group, with respect to the Carrollian diffeomorphism subgroup.

³We thank Konstantinos Siampos for a useful discussion on this topic.

⁴Equation (14) is also simply $-\square \Phi + \frac{d-1}{4d} R \Phi = 0$.

The interest for studying relativistic conformally coupled scalar fields is originally found in inflationary models of cosmology.⁵ On the Carrollian side the motivation is entrenched in the attempts to generalize the gauge/gravity holographic correspondence for asymptotically flat spacetimes, where the boundary is null infinity, i.e., a Carrollian manifold par excellence.

Inserting inside (12) the Carrollian decomposition of R as displayed in the Appendix Eq. (A39), leads to

$$V(\Phi) = \frac{1}{c^2} V_e(\Phi) + V_m(\Phi) + c^2 V_{\text{nd}}(\Phi), \quad (16)$$

with

$$V_e(\Phi) = \frac{d-1}{8d} \left(\frac{2}{\Omega} \partial_t \theta + \frac{1+d}{d} \theta^2 + \xi_{ij} \xi^{ij} \right) \Phi^2, \quad (17)$$

$$V_m(\Phi) = \frac{d-1}{8d} (\hat{r} - 2\hat{\nabla}_i \varphi^i - 2\varphi^i \varphi_i) \Phi^2, \quad (18)$$

$$V_{\text{nd}}(\Phi) = \frac{d-1}{8d} \varpi_{ij} \varpi^{ij} \Phi^2. \quad (19)$$

In the last expression the index “nd” stands for “non-dynamical.” The reason is that when the expression (16) of the potential is used in the relativistic action (9), it produces the Carrollian electric and magnetic actions—with some boundary terms dropped here⁶

$$S_e = \int dt d^d x \sqrt{a\Omega} \left(\frac{1}{2} \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 - \frac{d-1}{8d} \xi_{ij} \xi^{ij} \Phi^2 \right), \quad (20)$$

$$S_m = \int dt d^d x \sqrt{a\Omega} \left(-\frac{1}{2} \hat{\mathcal{D}}_i \Phi \hat{\mathcal{D}}^i \Phi - \frac{d-1}{8d} \hat{\mathcal{R}} \Phi^2 \right), \quad (21)$$

as well as a third one $S_{\text{nd}} = -\int dt d^d x \sqrt{a\Omega} \frac{d-1}{8d} \varpi_{ij} \varpi^{ij} \Phi^2$, which has no kinetic term for Φ .⁷ The Carrollian equations of motion for the two nontrivial cases are as follows:

⁵See, e.g., [21] where more references are displayed.

⁶On the relativistic side we find: $\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{d-1}{8d} R \Phi^2 = \frac{1}{2} \mathcal{D}^\mu \Phi \mathcal{D}_\mu \Phi + \frac{d-1}{8d} \mathcal{R} \Phi^2 - \frac{d-1}{4\sqrt{-g}} \partial_\mu (\sqrt{-g} A^\mu \Phi^2)$.

⁷These results coincide with those obtained for $d=2$ in Ref. [7], where the authors proceed with a thorough investigation of the possible Weyl-compatible terms. The kinetic terms of the electric and magnetic actions, (20) and (21), can also be compared to the corresponding results of [8]. They also agree up to the magnetic constraint introduced in Ref. [8], which would read here $\Pi_m^i = 0$ [see (31)]. The latter guarantees the invariance of the action under local Carrollian boosts, which we have not required *a priori*—Carrollian invariance features here the covariance under Carrollian diffeomorphisms (3) of a theory defined on a Carrollian spacetime (1) and (2).

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi + \frac{d-1}{4d} \xi_{ij} \xi^{ij} \Phi = 0 \quad \text{electric}, \quad (22)$$

$$-\hat{\mathcal{D}}_i \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} \hat{\mathcal{R}} \Phi = 0 \quad \text{magnetic}, \quad (23)$$

where the detailed expressions for the derivatives and Carrollian tensors are available in the Appendix. These equations are Weyl-covariant of weight $w = \frac{d+1}{2}$.

Energy and momenta are part of the agenda when discussing field dynamics. These are conjugate variables to the geometric data, as is $T_{\mu\nu}$ in (13) for a relativistic theory, and inherit their conservation from the Carrollian diffeomorphism invariance. In Carrollian geometries there is no energy-momentum tensor, but instead an energy-stress tensor Π^{ij} , an energy flux Π^i and an energy density Π , defined as [4,22,23]

$$\Pi^{ij} = \frac{2}{\sqrt{a\Omega}} \frac{\delta S_C}{\delta a_{ij}}, \quad (24)$$

$$\Pi^i = \frac{1}{\sqrt{a\Omega}} \frac{\delta S_C}{\delta b_i}, \quad (25)$$

$$\Pi = -\frac{1}{\sqrt{a}} \left(\frac{\delta S_C}{\delta \Omega} + \frac{b_i}{\Omega} \frac{\delta S_C}{\delta b_i} \right), \quad (26)$$

with conformal weights $d+3$, $d+2$ and $d+1$. Requiring Weyl invariance for the action translates into

$$\Pi_i^i = \Pi, \quad (27)$$

which is valid on shell (as the tracelessness of the relativistic energy-momentum tensor).

A momentum P_i (weight d) is also defined but is not conjugate to a geometric variable. It enters the conservation equations that mirror the Carrollian diffeomorphism invariance. For Weyl-invariant dynamics these are [23]

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \hat{\mathcal{D}}_i \Pi^i + \Pi^{ij} \xi_{ij} = 0, \quad (28)$$

$$\hat{\mathcal{D}}_i \Pi^i_j + 2\Pi^i \varpi_{ij} + \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \delta^i_j + \xi^i_j \right) P_i = 0. \quad (29)$$

Conservation equations are satisfied when the field Φ is on shell, and this allows us to determine the momentum.

Using Eqs. (24)–(26), we obtain the following energy and momenta for the Carrollian electric and magnetic actions:

$$\begin{cases} \Pi_e^{ij} = \frac{a^{ij}}{2} \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 + \frac{d-1}{4d} \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t (\xi^{ij} \Phi^2) \right) - a^{ij} \left(\frac{1}{2} \xi_{lk} \xi^{lk} \Phi^2 + \frac{1}{\Omega} \hat{\mathcal{D}}_t \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi^2 \right) \\ \Pi_e^i = 0 \\ \Pi_e = \frac{1}{2} \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \right)^2 - \frac{d-1}{8d} \xi_{ij} \xi^{ij} \Phi^2 \end{cases}, \quad (30)$$

$$\begin{cases} \Pi_{nd}^{ij} = \frac{d-1}{4d} (2\varpi^{li} \varpi_l^j - \frac{a^{ij}}{2} \varpi_{lk} \varpi^{lk}) \Phi^2 \\ \Pi_m^i = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} (\hat{\mathcal{D}}^i \frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi^2 - \hat{\mathcal{D}}_j (\xi^{ij} \Phi^2)) \\ \Pi_m = \frac{1}{2} \hat{\mathcal{D}}_i \Phi \hat{\mathcal{D}}^i \Phi + \frac{d-1}{4d} \left(\frac{\hat{\mathcal{D}}^2}{2} \Phi^2 - \hat{\mathcal{D}}_i \hat{\mathcal{D}}^i \Phi^2 \right) \end{cases} \quad (31)$$

For the nondynamical action, which will turn useful in a short while, we find

$$\begin{cases} \Pi_{nd}^{ij} = \frac{d-1}{4d} (2\varpi^{li} \varpi_l^j - \frac{a^{ij}}{2} \varpi_{lk} \varpi^{lk}) \Phi^2 \\ \Pi_{nd}^i = \frac{d-1}{4d} \hat{\mathcal{D}}_j (\varpi^{ji} \Phi^2) \\ \Pi_{nd} = \frac{3(d-1)}{8d} \varpi_{ij} \varpi^{ij} \Phi^2 \end{cases}. \quad (32)$$

They all obey (27), and conservation equations (28) and (29) are satisfied with the electric momenta, assuming the field be on shell, i.e., obeying (22), and deliver the electric momentum:

$$P_e^i = \Pi_m^i. \quad (33)$$

In a similar fashion for the magnetic dynamics, and using the equation of motion (23), we obtain

$$P_m^i = \Pi_{nd}^i. \quad (34)$$

One might be puzzled at this stage by the interplay Eqs. (33) and (34) seem to entail amongst electric, magnetic, and nondynamics. There is no doubt that electric and magnetic Carrollian scalar dynamics resulting from \mathcal{L}_e and \mathcal{L}_m are distinct, and can be studied separately, on any Carrollian background. Likewise, the action $\mathcal{L}_{nd} = -V_{nd}$ is also Carrollian invariant with bona fide Carrollian momenta, but is nondynamical. What sets a deeper link between these dynamics, which is not visible when treating them directly in the Carrollian framework, is that they all emerge in the “small- c expansion” of a unique relativistic theory for the scalar field. This was one possible guideline for obtaining the Carrollian scalar theories. It can also be applied to the relativistic energy-momentum tensor, and will deliver in a similar expansion⁸ the Carrollian momenta:

⁸The wording “expansion” is an abuse because the result is exact here.

$$\begin{cases} T^{ij} = \frac{1}{c^2} \Pi_e^{ij} + \Pi_m^{ij} + c^2 \Pi_{nd}^{ij} \\ -\frac{c}{\Omega} T_0^i = \Pi_m^i + c^2 \Pi_{nd}^i \\ \frac{1}{\Omega^2} T_{00} = \frac{1}{c^2} \Pi_e + \Pi_m + c^2 \Pi_{nd} \end{cases}. \quad (35)$$

The relationship between relativistic and Carrollian dynamics can be thrust further. Following [23,24] we can expand the relativistic conservation of energy momentum (15) and recollect the Carrollian conservation equations for the electric, the magnetic and the nondynamical cases. In this process Eqs. (28) and (29) arise for each case at a different c order, and their momenta P_e^i and P_m^i are naturally determined in terms of the next-order energy fluxes. This explains the above results (33) and (34).

Conserved charges are fundamental ingredients for handling a dynamical system. They often appear as the consequence of symmetries. In a relativistic framework, if ξ is a Killing field of the spacetime \mathcal{M} , then the current defined as

$$I_\mu = \xi^\nu T_{\mu\nu} \quad (36)$$

has zero divergence and (\mathcal{S} is a d -dimensional spatial section of \mathcal{M} and $*I$ the \mathcal{M} -Hodge dual of $I = I_\mu dx^\mu$)

$$Q_I = \int_{\mathcal{S}} *I \quad (37)$$

is conserved. For Weyl-covariant dynamics this applies with conformal Killing fields.

In a Carrollian spacetime a current has a scalar component κ as well as a Carrollian-vector set of components K^i , and the divergence takes the form

$$\mathcal{K} = \left(\frac{1}{\Omega} \partial_t + \theta \right) \kappa + (\hat{\nabla}_i + \varphi_i) K^i. \quad (38)$$

This result can be inferred⁹ as from a relativistic computation, with a current I^μ such that

⁹See Ref. [23], where it is also shown how the current components are retrieved without reference to a relativistic ascendent.

$$-\frac{1}{c\Omega}I_0 = \kappa + \mathcal{O}(c^2), \quad I^k = K^k + \mathcal{O}(c^2), \quad (39)$$

leading in a Papapetrou-Randers background (6) to $\nabla_\mu I^\mu = \mathcal{K} + \mathcal{O}(c^2)$. Defining a charge associated with the current (κ, \mathbf{K}) as an integral at fixed t over the basis \mathcal{S} of the Carrollian structure

$$Q_K = \int_{\mathcal{S}} d^d x \sqrt{a} (\kappa + b_i K^i), \quad (40)$$

we obtain the following time evolution:

$$\frac{dQ_K}{dt} = \int_{\mathcal{S}} d^d x \sqrt{a} \Omega \mathcal{K} - \int_{\partial\mathcal{S}} \star \mathbf{K} \Omega, \quad (41)$$

where $\star \mathbf{K}$ is the \mathcal{S} -Hodge dual of $K_i dx^i$. For vanishing divergence \mathcal{K} , this is conserved if one can ignore the boundary term owing to adequate falloff or boundary conditions on the fields. Notice that if \mathcal{K} happens to be identical to the Carrollian divergence of some potential $(\phi, \boldsymbol{\phi})$, then a conserved charge is obtained with $\kappa - \phi$, $K^i - \phi^i$.

Suppose that ξ is the generator of a Carrollian diffeomorphism [see (A24) in the Appendix]. It can be used to create a current out of Π^{ij} , Π^i , Π , and P^i [4,23]:

$$\kappa = \xi^i P_i - \xi^i \Pi, \quad K^i = \xi^j \Pi_j^i - \xi^i \Pi^i. \quad (42)$$

For a Weyl-covariant system [Eq. (27)] with a conformal Killing vector [see the defining conditions in the Appendix, (A30) and (A31)], one obtains this:

$$\mathcal{K} = -\Pi^i ((\hat{\partial}_i - \varphi_i) \xi^i - 2\xi^j \varpi_{ji}). \quad (43)$$

As opposed to the relativistic situation, a conformal Killing field does not provide a conservation law in Weyl-invariant Carrollian dynamics, unless it satisfies (the conformal weight of ξ^i is -1 , that of ξ^i zero)

$$(\hat{\partial}_i - \varphi_i) \xi^i - 2\xi^j \varpi_{ji} \equiv \hat{\mathcal{D}}_i \xi^i - 2\xi^j \varpi_{ji} = 0. \quad (44)$$

This last condition amounts to further demanding the clock form (4) be invariant under the action of the conformal Killing [see Eq. (A27) in the Appendix]. In Carrollian dynamics, symmetry is generated by a subalgebra of the conformal isometry algebra.

Electric and magnetic Carrollian scalar fields with conformal coupling have different behavior regarding conservation. The former have vanishing energy flux [see (30)] and lead thus to conserved charges $Q_e = \int_{\mathcal{S}} d^d x \sqrt{a} (\kappa_e + b_i K_e^i)$ with

$$\kappa_e = \xi^i \Pi_{mi} - \xi^i \Pi_e, \quad K_e^i = \xi^j \Pi_{ej}^i, \quad (45)$$

where we have used (33). For the latter, $Q_m = \int_{\mathcal{S}} d^d x \sqrt{a} (\kappa_m + b_i K_m^i)$ with [see (31) and (34)],

$$\kappa_m = \xi^i \Pi_{ndi} - \xi^i \Pi_m, \quad K_m^i = \xi^j \Pi_{mj}^i - \xi^i \Pi_m^i \quad (46)$$

is not conserved since, according to (43),

$$\mathcal{K}_m = -\Pi_m^i ((\hat{\partial}_i - \varphi_i) \xi^i - 2\xi^j \varpi_{ji}). \quad (47)$$

Conservation is attainable for field configurations such that $\Pi_m^i = 0$, which translates local Carroll-boost invariance [17]. Following (31), this happens, e.g., when $\hat{\mathcal{D}}_i \Phi = 0$ in backgrounds with vanishing geometric Carrollian shear [$\xi_{ij} = 0$, defined in the Appendix, Eq. (A6)], which is possibly compatible with the magnetic dynamics (23).

Conformal Killing fields on general Carrollian spacetimes are obtained upon solving a set of complicated partial differential equations and this is not an easy task. It is remarkable that when the Carrollian shear [see (A6) in the Appendix] ξ^{ij} vanishes, the conformal Killing fields are known [19]. Zero shear implies that the time dependence in the metric is factorized: $a_{ij}(t, \mathbf{x}) = \mathcal{B}^{-2}(t, \mathbf{x}) \tilde{a}_{ij}(\mathbf{x})$. This drives the conformal algebra of the Carrollian structure to the standard infinite-dimensional semidirect sum of the conformal algebra of $\tilde{a}_{ij}(\mathbf{x})$ with supertranslations. For conformally flat $\tilde{a}_{ij}(\mathbf{x})$, the latter coincides with $\text{ccarr}(d+1)$. One recovers in particular BMS_{d+2} in $d = 1$ and 2 —possibly in higher dimensions.¹⁰

The Carrollian spacetimes emerging as null boundaries of asymptotically locally flat solutions to Einstein equations turn out to satisfy the vanishing-shear condition.¹¹ That makes this class of Carrollian structures particularly appealing and the forthcoming example will illustrate their properties regarding the propagation of a conformally coupled scalar field.

Robinson-Trautman spacetimes are four-dimensional, time-dependent Ricci-flat solutions of algebraically special Petrov type. They describe configurations emitting gravitational radiation and settling down in the far future into a Schwarzschild black hole.¹² Their null boundary is a Carrollian manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$, where \mathcal{S} is equipped with a conformally flat $d = 2$ metric:

¹⁰The standard conformal Carrollian algebra $\text{ccarr}(d+1)$ is also referred to as “level-2” $\text{ccarr}_2(d+1)$. More general level- N algebras $\text{ccarr}_N(d+1)$ emerge in the presence of a dynamical exponent $z = 2/N$ —see footnote 21. For $d > 2$ the BMS algebra is finite-dimensional, whereas $\text{ccarr}_N(d+1)$ is not. Infinite-dimensional extensions of the BMS_{d+2} require adjustments in the fall-off behaviors and have been considered in the literature (see, e.g., [25] for a recent account and further reading suggestions).

¹¹See, e.g., [26], Eq. (3.40) at vanishing Λ (see also [27]). One should not confuse the shear of the boundary Carrollian manifold, with the Bondi shear which is another boundary Carrollian tensor, nonvanishing in general and carrying information about the bulk gravitational radiation.

¹²The original solution is available in [28,29]. Robinson-Trautman spacetimes have been discussed in the framework of AdS/CFT in Refs. [30–33], and further in flat holography in Refs. [27,34].

$$d\ell^2 = \frac{2}{P^2} d\zeta d\bar{\zeta}. \quad (48)$$

Here $P = P(t, \zeta, \bar{\zeta})$ obeys a fourth-order partial-differential equation known as Robinson-Trautman's equation, which also involves the Bondi mass aspect $M(t)$.¹³ The field of observers and the clock form are ($\Omega = 1$, $b_i = 0$)

$$\mathbf{v} = \partial_t, \quad \boldsymbol{\mu} = -dt. \quad (49)$$

Hence, one can compute the basic geometric data¹⁴:

$$\begin{aligned} \theta &= -2\partial_t \ln P, & \varphi_i &= 0, & \varpi_{ij} &= 0, \\ \xi_{ij} &= 0, & \hat{\mathcal{R}} &= 4P^2 \partial_{\bar{\zeta}} \partial_{\zeta} \ln P. \end{aligned} \quad (50)$$

Although the Robinson-Trautman solutions have no isometries, they have asymptotic symmetries, and these are actually reflected in the conformal isometries of the Carrollian boundary. Following [19], we find that the conformal Killing fields of \mathcal{M} are expressed in terms of an arbitrary real function $T(\zeta, \bar{\zeta})$, which encodes the supertranslations and the conformal Killing vectors $\mathbf{Y} = Y^\zeta \partial_\zeta + Y^{\bar{\zeta}} \partial_{\bar{\zeta}}$ of $d\tilde{\ell}^2 = 2d\zeta d\bar{\zeta}$, which is flat space. The latter generate $\mathfrak{so}(3, 1)$ —or even a double copy of Witt algebras referred to as “superrotations,” if we are ready to give up invertibility. We find that \mathbf{Y} is any combination of $\ell_m + \bar{\ell}_m$ or $i(\ell_m - \bar{\ell}_m)$ with

$$\ell_m = -\zeta^{m+1} \partial_\zeta, \quad \bar{\ell}_m = -\bar{\zeta}^{m+1} \partial_{\bar{\zeta}}, \quad (51)$$

obeying Witt \oplus Witt:

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n}, \quad [\bar{\ell}_m, \bar{\ell}_n] = (m-n)\bar{\ell}_{m+n}. \quad (52)$$

In this representation, $\mathfrak{so}(3, 1)$ is generated by $n = 0, \pm 1$. The conformal Killing fields of \mathcal{M} are [see (A24) in the Appendix]¹⁵

$$\xi_{T,Y} = (T - M_Y(C)) \frac{1}{P} \partial_t + Y^i \partial_i, \quad (53)$$

where

¹³We will not specifically use the Robinson-Trautman equation [displayed in the aforementioned literature—footnote 12—e.g., Ref. [33], Eq. (2.35)] in our subsequent analysis, which is thus valid for arbitrary $P(t, \zeta, \bar{\zeta})$.

¹⁴Conventions: $\sqrt{a} = i/P^2$ and $\epsilon_{\zeta\bar{\zeta}} = 1$.

¹⁵The existence of conformal Killing fields for the Carrollian structure at hand is remarkable. Actually, the relativistic ascendent of this structure $ds^2 = -c^2 dt^2 + \frac{2}{P^2} d\zeta d\bar{\zeta}$, appearing as the conformal timelike boundary of AdS (anti-de Sitter)-Robinson-Trautman spacetimes, has generically no conformal Killings. In particular, it is not conformally flat because it has a nonzero Cotton tensor, see [30–33].

$$C(t, \zeta, \bar{\zeta}) = \int^t d\tau P(\tau, \zeta, \bar{\zeta}), \quad (54)$$

and M_Y is an operator acting on scalar functions $f(t, \zeta, \bar{\zeta})$ as

$$M_Y(f) = Y^k \partial_k f - \frac{f}{2} \partial_k Y^k. \quad (55)$$

The structure $\mathfrak{so}(3, 1) \ltimes$ supertranslations—or (Witt \oplus Witt) \ltimes supertranslations—is recovered in

$$[\xi_{T,Y}, \xi_{T',Y'}] = \xi_{M_Y(T') - M_{Y'}(T), [Y, Y']}. \quad (56)$$

We are now ready to discuss the dynamics of a conformally coupled scalar field and its conserved charges. The ultimate motivation for this study is flat holography and the possible usefulness of the Carrollian dynamics for describing modes that propagate all the way inside the bulk towards the null boundary of asymptotically flat spacetimes. The electric equation of motion (22) reads as follows in the three-dimensional Carrollian spacetime under consideration:

$$\partial_t \frac{1}{P} \partial_t \frac{\Phi}{\sqrt{P}} = 0. \quad (57)$$

Its general solution is given in terms of two arbitrary functions $f(\zeta, \bar{\zeta})$ and $g(\zeta, \bar{\zeta})$:

$$\Phi = \sqrt{P}(Cf + g). \quad (58)$$

With this, we can compute the energy density Π_e and the electric momentum P_e^i as in (33), using (30) and (31), and combine them into the scalar component of the current (45) associated with the conformal Killing fields (53):

$$\kappa_{eT,Y} = P^2 \left[Y^i \left(\frac{1}{4} \partial_i (fg) - f \partial_i g \right) - \frac{Tf^2}{2} - \frac{1}{4} \partial_i (Y^i C f^2) \right]. \quad (59)$$

This leads to the charges

$$\begin{aligned} Q_{eT,Y} &= -i \int_{\mathcal{S}} d\zeta \wedge d\bar{\zeta} \left(Y^i \left(\frac{1}{4} \partial_i (fg) - f \partial_i g \right) - \frac{Tf^2}{2} \right) \\ &\quad - \frac{1}{4} \int_{\partial\mathcal{S}} \star Y C f^2 P^2. \end{aligned} \quad (60)$$

On shell, the time dependence is exclusively encoded in the last term through P (and C). This is a flux at infinity, and thus vanishes upon appropriate falloff behavior of the field f . Hence, the charges are indeed conserved.

The infinite number of conserved charges, awkward at first glance, translates the separation of time and space imposed by Carrollian symmetry. The field equation (57) contains no spatial derivative, hence every locus $(\zeta, \bar{\zeta})$

provides a decoupled degree of freedom. This often happens in Carrollian field theory (as, e.g., in the magnetic conformally stationary scalar field—see below), although the general equations at hand (22) and (23) contain actually both time and space derivatives— $\hat{\mathcal{D}}_t$ and $\hat{\mathcal{D}}_i$ contain both—making the advertised decoupling less transparent.

The magnetic equation (23) is

$$4\partial_\zeta\partial_{\bar{\zeta}}\Phi = \Phi\partial_\zeta\partial_{\bar{\zeta}}\ln P. \quad (61)$$

According to (47), magnetic charges are conserved with those conformal Killing fields obeying the extra condition (44), which leads to

$$T = SP + M_Y(C), \quad (62)$$

where S is a function of time only. Since P and C are time dependent while T is not, Eq. (62) restricts severely the allowed subset of \mathcal{S} -conformal Killings Y , which may even turn empty. Assuming this set is not empty, due to the vanishing of the magnetic momentum P_m^i [see (34) with (32)—here $\varpi_{ij} = 0$], Eq. (46) leads to a single conserved charge based on $\kappa_{mS} = -S\Pi_m$ with Π_m given in (31): $Q_{mS} = -S \int_{\mathcal{S}} \frac{d\zeta d\bar{\zeta}}{P^2} \Pi_m$. This charge is nothing but the total energy, but it turns out to vanish here. Indeed, on shell, Π_m reads [Eqs. (23) and (31)] irrespective of the dimension and of the geometric background:

$$\Pi_m = \frac{1}{2d} \hat{\mathcal{D}}_i(\Phi \hat{\mathcal{D}}^i \Phi). \quad (63)$$

In the case under consideration ($b_i = 0$ and $\varphi_i = 0$), $\Pi_m = \frac{P^2}{4} [\partial_\zeta(\Phi \partial_{\bar{\zeta}} \Phi) + \partial_{\bar{\zeta}}(\Phi \partial_\zeta \Phi)]$, which is a divergence. Hence Q_{mS} receives only an \mathcal{S} -boundary contribution, vanishing under appropriate falloff or boundary conditions.¹⁶

¹⁶This property of vanishing scalar-field conserved magnetic charges is actually valid more generally, in any dimension d , and for a Carrollian background structure with $b_i = 0$. Indeed, this implies $\varpi_{ij} = 0$, leading therefore to $Q_m = - \int_{\mathcal{S}} d^d x \sqrt{a} \xi^i \Pi_m$. For Killing fields obeying the extra condition (44), using (63) we find that the on-shell integral is again a boundary term.

¹⁷Notice in passing that the general solution of (61) is in this case $\Phi(t, \zeta, \bar{\zeta}) = f(t, \zeta) + \bar{f}(t, \bar{\zeta})$, where $f(t, \zeta)$ is arbitrary.

¹⁸With $\Phi = \sqrt{P}g(\zeta, \bar{\zeta})$, Eq. (61) reads $4P\partial_\zeta\partial_{\bar{\zeta}}g + 2(\partial_\zeta P\partial_{\bar{\zeta}}g + \partial_{\bar{\zeta}} P\partial_\zeta g) + g\partial_\zeta\partial_{\bar{\zeta}}P = 0$ (also valid if P is traded for C).

It is worth stressing that Eq. (62) is extremely constraining. For instance, if the function $P(t, \zeta, \bar{\zeta})$ obeys the Robinson-Trautman equation, it can awkwardly entangle time and space dependence (see, e.g., [29]), leaving little room for finding $T(\zeta, \bar{\zeta})$ and $Y^i(\zeta, \bar{\zeta})$ that satisfy (62). In the simplest possible instance, which is flat space ($P = 1$ and $C = t$),¹⁷ the two special conformal transformations of the $\mathfrak{so}(3, 1)$ are excluded ($\partial_i Y^i = C_0$ constant), and only constant time translations are allowed ($T = T_0$ constant, and $S(t) = T_0 + C_0 t/2$); this is a five-dimensional subgroup of the infinite-dimensional BMS_4 .

When Π_m^i given in (31) vanishes, the magnetic charges are all conserved, as inferred by Eq. (47). This occurs in particular [the Carrollian geometric shear vanishes here, see (50)] for conformally stationary scalars obeying $\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi \equiv \sqrt{P} \partial_t \frac{\Phi}{\sqrt{P}} = 0$, thus of the form $\Phi = \sqrt{P}g(\zeta, \bar{\zeta})$, where $g(\zeta, \bar{\zeta})$ is further determined by solving the magnetic equation of motion (61). The latter¹⁸ may not be solvable in a general Robinson-Trautman background $P(t, \zeta, \bar{\zeta})$ under the present ansatz. If it is, the conserved magnetic charges are found using Eqs. (46) and (53). On shell, these lead to

$$\kappa_{mTY} = -\xi^i \Pi_m = \frac{P^2}{2} (M_Y(C) - T) (\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g), \quad (64)$$

which are integrated as in (40):

$$Q_{mT} = \frac{i}{2} \int_{\mathcal{S}} d\zeta \wedge d\bar{\zeta} T (\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g) - \frac{1}{4} \int_{\partial\mathcal{S}} \star X P^2 \quad (65)$$

with

$$\begin{cases} X^\zeta = C(Y^\zeta(\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g) + Y^{\bar{\zeta}}(3(\partial_{\bar{\zeta}} g)^2 - g \partial_{\bar{\zeta}}^2 g)) - \frac{1}{2} Y^{\bar{\zeta}} g^2 \partial_{\bar{\zeta}}^2 C \\ X^{\bar{\zeta}} = C(Y^{\bar{\zeta}}(\partial_\zeta g \partial_{\bar{\zeta}} g - g \partial_\zeta \partial_{\bar{\zeta}} g) + Y^\zeta(3(\partial_\zeta g)^2 - g \partial_\zeta^2 g)) - \frac{1}{2} Y^\zeta g^2 \partial_\zeta^2 C \end{cases}. \quad (66)$$

As in the electric case [see Eq. (60)], the time dependence is confined into a boundary term, which ultimately drops, taking with it all the dependence on the $\mathfrak{so}(3, 1)$ vectors Y . For a conformally stationary scalar field in Robinson-Trautman background, the magnetic charges are nonzero and conserved on shell without restriction on the Carrollian conformal Killing vector ξ (the energy flux vanishes), but they only depend on its supertranslation component $T(\zeta, \bar{\zeta})$.

Concluding, we would like to summarize our results. The present framework is set by a general Carrollian spacetime and the systems under investigation are general covariant with respect to Carrollian diffeomorphisms.

The Carrollian scalar field dynamics is either electric or magnetic. The same holds for a conformally coupled scalar, and the two options are rather different. The electric is “timelike,” whereas the magnetic looks “spacelike,” and they couple to distinct pieces of the Carrollian curvature. We have determined the energy-stress tensor, the energy flux, the energy density, and the momentum in both situations, and shown that Carrollian conformal isometries imply conservation laws in the electric instance but not in the magnetic. The physical reason behind this cleavage is rather easy to understand. A Carrollian (conformal) isometry translates the invariance of the metric and the field of observers, but not that of its dual clock form. Time (supported by the field of observers) and space (associated with the clock form) directions behave differently and this ultimately reveals in the conservation properties of electric versus magnetic dynamics. A similar phenomenon is expected to occur in Newton-Cartan manifolds, where a scalar field will also have electric and magnetic dynamics.¹⁹ Isometries will guarantee conservation laws for the latter, as opposed to the former, because the clock form is invariant under the action of a Killing vector, while the field of observers is not.

The above findings have been illustrated in the case of the null boundary of Robinson-Trautman asymptotically locally flat spacetimes, which are Carrollian with vanishing geometric shear and vorticity. The electric conformally coupled scalar field has been worked out thoroughly, accompanied with its infinite tower of conserved charges. For the magnetic dynamics, we have found that all charges associated with the subalgebra of the conformal Carrollian algebra satisfying the extra conservation condition [Eq. (44)] vanish—i.e., amount to purely boundary terms. Nonvanishing conserved magnetic charges appear for field configurations with $\Pi_m^i = 0$, and this happens, e.g., for conformally stationary fields.

From our general discussion one should probably retain the contrast between the infinite tower of conformal Killing fields available in most Carrollian structures and the often lesser conserved Carrollian charges. In this picture one should not underestimate the role of the nonconserved ones, usually infinite in number. When the Carrollian structures are null boundaries of asymptotically flat spacetimes, the presence of nonconserved charges betrays, among others, gravitational radiation.

Even though we have focused our analysis on conformally coupled scalar fields, ordinary scalars share these properties—with Killings instead of conformal Killings.

¹⁹The magnetic and electric scalar-field actions are, respectively, $S_m = - \int_{\mathcal{M}} dt d^d x \sqrt{a} \Omega (\frac{1}{2} a^{ij} \partial_i \Phi \partial_j \Phi + V_m(\Phi))$ and $S_e = \int_{\mathcal{M}} dt d^d x \sqrt{a} \Omega ((\frac{1}{\Omega} \frac{\hat{D}\Phi}{dt})^2 - V_e(\Phi))$ in torsionless Newton-Cartan geometries with degenerate cometric a^{ij} , clock form Ωdt , field of observers $\frac{1}{\Omega}(\partial_t + w^j \partial_j)$, and metric-compatible time derivative $\frac{1}{\Omega} \frac{\hat{D}\Phi}{dt} = \frac{1}{\Omega} \partial_t \Phi + \frac{w^j}{\Omega} \partial_j \Phi$.

The motivation behind conformal couplings lies in the role these may play in flat holography—for scalar or more general fields. This calls for a better understanding of the classical dynamics, and above all of the quantum properties. The conservation of charges, the associated algebras and the distinction of electric versus magnetic representatives, should ultimately be translated into bulk language. Our example of the null three-dimensional boundary of Robinson-Trautman Ricci-flat spacetimes is meant to illustrate this bridge, although discussed here in a primitive fashion, revealing a generically trivial magnetic conservation as opposed to an infinite set of electric conserved charges. How this reflects flat-holographic properties remains in limbo.

We would like to express our gratitude to our colleagues Marios Petropoulos and Konstantinos Siampos for carefully reading the draft of this manuscript, for many useful discussions and for suggestions of improvements. We also thank Jelle Hartong and Gerben Oling for interesting exchanges on Carrollian covariance and Carroll-boost invariance. The work of D.R.-B. was funded by Becas Chile (ANID) Scholarship No. 72200301. The work of M.V. was supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the *First Call for H.F.R.I. Research Projects to support Faculty members and Researchers and the procurement of high-cost research equipment grant* (MIS 1524, Project No. 96048).

APPENDIX: CARROLLIAN MANIFOLDS

Under Carrollian diffeomorphisms (3) with Jacobian

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J_j^i(\mathbf{x}) = \frac{\partial x'^i}{\partial x^j}, \quad (\text{A1})$$

the transformations are noncovariant (connectionlike) for ∂_i and b_i , and densitylike for ∂_t and Ω :

$$\begin{aligned} \partial'_j &= J^{-1i}{}_j \left(\partial_i - \frac{j_i}{J} \partial_t \right), & b'_k &= \left(b_i + \frac{\Omega}{J} j_i \right) J^{-1i}{}_k, \\ \partial'_t &= \frac{1}{J} \partial_t, & \Omega' &= \frac{\Omega}{J}. \end{aligned} \quad (\text{A2})$$

They are covariant for the other objects:

$$\mathbf{v}' = \mathbf{v}, \quad \boldsymbol{\mu}' = \boldsymbol{\mu}, \quad \hat{\partial}'_i = J^{-1j}{}_i \hat{\partial}_j, \quad a^{ij'} = J^i{}_k J^j{}_l a^{kl}. \quad (\text{A3})$$

Carrollian tensors depend on time t and space \mathbf{x} , carry indices i, j, \dots lowered and raised with a_{ij} and its inverse a^{ij} , and transform covariantly under Carrollian diffeomorphisms (3) with J_i^j and $J^{-1j}{}_i$ defined in (A1). A Levi-Civita-Carroll spatial covariant derivative $\hat{\nabla}_i$ is defined with connection coefficients

$$\hat{\gamma}_{jk}^i = \frac{a^{il}}{2} (\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk}), \quad (\text{A4})$$

which emerge naturally in the vanishing- c limit of a Levi-Civita connection in the Papapetrou-Randers coordinates (6). This connection is torsionless and metric compatible²⁰: $\hat{\gamma}_{[ij]}^k = 0$, $\hat{\nabla}_i a_{jk} = 0$. The vectors $\hat{\partial}_i$ do not commute and define the Carrollian vorticity and acceleration:

$$\begin{aligned} [\hat{\partial}_i, \hat{\partial}_j] &= \frac{2}{\Omega} \varpi_{ij} \partial_t, & \varpi_{ij} &= \partial_{[i} b_{j]} + b_{[i} \varphi_{j]}, \\ \varphi_i &= \frac{1}{\Omega} (\partial_t b_i + \partial_i \Omega). \end{aligned} \quad (\text{A5})$$

The usual time-derivative operator $\frac{1}{\Omega} \partial_t$ acts covariantly on Carrollian tensors, but it is not metric compatible because a_{ij} depend on time. A temporal covariant derivative is defined by requiring $\frac{1}{\Omega} \hat{D}'_t = \frac{1}{\Omega} \hat{D}_t$ and $\hat{D}_t a_{jk} = 0$, and is also inherited from the Papapetrou-Randers Levi-Civita connection. To this end, we introduce a temporal connection

$$\hat{\gamma}_{ij} = \frac{1}{2\Omega} \partial_t a_{ij} = \xi_{ij} + \frac{1}{d} a_{ij} \theta, \quad \theta = \frac{1}{\Omega} \partial_t \ln \sqrt{a}, \quad (\text{A6})$$

which is a symmetric Carrollian tensor split into the Carrollian shear (traceless) and the Carrollian expansion (trace). The action of \hat{D}_t on scalars is ∂_t whereas on vectors or forms it is defined as

$$\frac{1}{\Omega} \hat{D}_t V^i = \frac{1}{\Omega} \partial_t V^i + \hat{\gamma}^i_j V^j, \quad \frac{1}{\Omega} \hat{D}_t V_i = \frac{1}{\Omega} \partial_t V_i - \hat{\gamma}_i^j V_j. \quad (\text{A7})$$

Generalization to any tensor uses the Leibniz rule.

The commutators of Carrollian covariant derivatives define Carrollian curvature tensors. We keep it minimal here with

$$\begin{aligned} [\hat{\nabla}_k, \hat{\nabla}_l] V^i &= (\hat{\partial}_k \hat{\gamma}_{lj}^i - \hat{\partial}_l \hat{\gamma}_{kj}^i + \hat{\gamma}_{km}^i \hat{\gamma}_{lj}^m - \hat{\gamma}_{lm}^i \hat{\gamma}_{kj}^m) V^j + [\hat{\partial}_k, \hat{\partial}_l] V^i \\ &= \hat{r}^i_{jkl} V^j + \varpi_{kl} \frac{2}{\Omega} \partial_t V^i. \end{aligned} \quad (\text{A8})$$

In this expression \hat{r}^i_{jkl} should be called the ‘‘Riemann-Carroll’’ tensor. The Ricci-Carroll tensor and the Carroll scalar curvature are thus

$$\hat{r}_{ij} = \hat{r}^k_{ikj} \neq \hat{r}_{ji}, \quad \hat{r} = a^{ij} \hat{r}_{ij}. \quad (\text{A9})$$

²⁰Details on the transformation rules can be found in Appendix A.2 of Ref. [24], together with further useful properties.

Weyl covariance under Weyl transformations

$$a_{ij} \rightarrow \frac{1}{\mathcal{B}^2} a_{ij}, \quad b_i \rightarrow \frac{1}{\mathcal{B}} b_i, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega, \quad (\text{A10})$$

with $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$ an arbitrary function, requires a Weyl-Carroll connection built on φ_i and θ defined in (A5) and (A6), which transform as

$$\varphi_i \rightarrow \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \quad \theta \rightarrow \mathcal{B} \theta - \frac{d}{\Omega} \partial_t \mathcal{B}. \quad (\text{A11})$$

The Carrollian vorticity ϖ_{ij} (A5) and the Carrollian shear ξ_{ij} (A6) are Weyl covariant of weight -1 .

The Weyl-Carroll space and time covariant derivatives are torsionless and metric compatible. For a weight- w scalar function Φ and a vector with weight- w components V^l , the action is

$$\hat{\mathcal{D}}_j \Phi = \hat{\partial}_j \Phi + w \varphi_j \Phi, \quad (\text{A12})$$

$$\hat{\mathcal{D}}_j V^l = \hat{\nabla}_j V^l + (w-1) \varphi_j V^l + \varphi^l V_j - \delta_j^l V^i \varphi_i. \quad (\text{A13})$$

The Weyl-Carroll spatial derivative does not alter the weight, and one checks that $\hat{\mathcal{D}}_j a_{kl} = 0$. Regarding time, one defines

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi = \frac{1}{\Omega} \hat{D}_t \Phi + \frac{w}{d} \theta \Phi = \frac{1}{\Omega} \partial_t \Phi + \frac{w}{d} \theta \Phi, \quad (\text{A14})$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t V^l = \frac{1}{\Omega} \hat{D}_t V^l + \frac{w-1}{d} \theta V^l = \frac{1}{\Omega} \partial_t V^l + \frac{w}{d} \theta V^l + \xi^l_i V^i, \quad (\text{A15})$$

and both are of weight $w+1$. Similarly for any tensor by Leibniz rule and in particular we find $\hat{\mathcal{D}}_t a_{kl} = 0$.

We now close this paragraph with the Weyl-Carroll curvature tensors, appearing in the commutation of Weyl-Carroll covariant derivatives. We find

$$[\hat{\mathcal{D}}_i, \hat{\mathcal{D}}_j] \Phi = \frac{2}{\Omega} \varpi_{ij} \hat{\mathcal{D}}_t \Phi + w \Omega_{ij} \Phi, \quad (\text{A16})$$

$$[\hat{\mathcal{D}}_k, \hat{\mathcal{D}}_l] V^i = (\hat{\mathcal{R}}^i_{jkl} - 2\xi_j^i \varpi_{kl}) V^j + \varpi_{kl} \frac{2}{\Omega} \hat{\mathcal{D}}_t V^i + w \Omega_{kl} V^i, \quad (\text{A17})$$

where we have introduced the following Carrollian, weight-0 Weyl-covariant tensors:

$$\begin{aligned} \hat{\mathcal{R}}^i_{jkl} &= \hat{r}^i_{jkl} - \delta^i_j \varphi_{kl} - a_{jk} \hat{\nabla}_l \varphi^i + a_{jl} \hat{\nabla}_k \varphi^i \\ &\quad + \delta^i_k \hat{\nabla}_l \varphi_j - \delta^i_l \hat{\nabla}_k \varphi_j + \varphi^i (\varphi_k a_{jl} - \varphi_l a_{jk}) \\ &\quad - (\delta^i_k a_{jl} - \delta^i_l a_{jk}) \varphi_m \varphi^m + (\delta^i_l \varphi_k - \delta^i_k \varphi_l) \varphi_j, \end{aligned} \quad (\text{A18})$$

$$\Omega_{ij} = \hat{\partial}_i \varphi_j - \hat{\partial}_j \varphi_i - \frac{2}{d} \varpi_{ij} \theta. \quad (\text{A19})$$

Additionally, we define traces as

$$\hat{\mathcal{R}}_{ij} = \hat{\mathcal{R}}^k{}_{ikj}, \hat{\mathcal{R}} = a^{ij} \hat{\mathcal{R}}_{ij} \quad (\text{A20})$$

with

$$\hat{\mathcal{R}} = \hat{r} + (d-1)(2\hat{\nabla}_i \varphi^i - (d-2)\varphi_i \varphi^i). \quad (\text{A21})$$

Observe that the Weyl-covariant Carroll-Ricci tensor is not symmetric: $\hat{\mathcal{R}}_{[ij]} = -\frac{d}{2}\Omega_{ij}$. Finally, we recall that

$$\left[\frac{1}{\Omega} \hat{\mathcal{D}}_t, \hat{\mathcal{D}}_i \right] \Phi = w \hat{\mathcal{R}}_i \Phi - \xi^j \hat{\mathcal{D}}_j \Phi, \quad (\text{A22})$$

where

$$\hat{\mathcal{R}}_i = \frac{1}{\Omega} \partial_t \varphi_i - \frac{1}{d} (\hat{\partial}_i + \varphi_i) \theta \quad (\text{A23})$$

are the components of a Weyl-covariant weight-1 Carrollian curvature one-form.

Isometries and conformal isometries are associated with Killing and conformal Killing fields. Carrollian diffeomorphisms (3) are generated by vector fields

$$\begin{aligned} \xi &= \xi^t \partial_t + \xi^i \partial_i = \left(\xi^t - \xi^i \frac{b_i}{\Omega} \right) \partial_t + \xi^i \left(\partial_i + \frac{b_i}{\Omega} \partial_t \right) \\ &= \xi^{\hat{t}} \frac{1}{\Omega} \partial_t + \xi^{\hat{i}} \hat{\partial}_i \end{aligned} \quad (\text{A24})$$

restricted to $\xi^i = \xi^i(\mathbf{x})$. Their action operates with the Lie derivative, and for the geometric data one finds

$$\mathcal{L}_\xi a_{ij} = 2\hat{\nabla}_{(i} \xi^k a_{j)k} + 2\xi^{\hat{t}} \hat{\gamma}_{ij}, \quad (\text{A25})$$

$$\mathcal{L}_\xi \mathbf{v} = \mu \mathbf{v}, \quad (\text{A26})$$

$$\mathcal{L}_\xi \boldsymbol{\mu} = -\mu \boldsymbol{\mu} - ((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^{\hat{j}} \varpi_{ji}) dx^i, \quad (\text{A27})$$

with

$$\mu(t, \mathbf{x}) = -\left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^{\hat{i}} \right). \quad (\text{A28})$$

The significant observation is here that due to the degeneration of the metric on \mathcal{M} , the variation of the clock form μ is not identical to that of the field of observers v . For further use, we also introduce the trace of (A25) divided by d :

$$\lambda(t, \mathbf{x}) = \frac{2}{d} (\hat{\nabla}_i \xi^i + \theta \xi^{\hat{t}}). \quad (\text{A29})$$

Carrollian isometries are Carrollian diffeomorphisms generated by Killing fields, obeying $\mathcal{L}_\xi a_{ij} = 0$ and $\mathcal{L}_\xi v = 0$. In the strong Carroll structure, this requirement is completed with the invariance of the connection. For conformal Carrollian isometries one demands

$$\mathcal{L}_\xi a_{ij} = \lambda a_{ij}. \quad (\text{A30})$$

This set of partial differential equations is insufficient for defining conformal Killing vectors and one usually imposes to tune μ versus λ so that the scaling of the metric be twice that of the field of observers²¹:

$$2\mu + \lambda = 0. \quad (\text{A31})$$

The projective structure associated with some Carroll connection should also be preserved.

For a pseudo-Riemannian manifold \mathcal{M} in $d+1$ dimensions with metric $g_{\mu\nu}$ (weight-2), a Weyl-covariant derivative \mathcal{D}_μ maintains the weight w of a Weyl-covariant tensor. The corresponding connection uses a (weight-1) vector u^μ of norm $-c^2$, as well as its expansion Θ and acceleration a^μ :

$$A = \frac{1}{c^2} \left(a - \frac{\Theta}{d} u \right). \quad (\text{A32})$$

The Weyl covariant derivative is metric compatible with

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu) f = w f F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (\text{A33})$$

where the action on a weight- w scalar f is

$$\mathcal{D}_\lambda f = \nabla_\lambda f + w A_\lambda f. \quad (\text{A34})$$

The action of \mathcal{D}_λ on a weight- w form v_μ is

$$\mathcal{D}_\lambda v_\mu = \nabla_\lambda v_\mu + (w+1) A_\lambda v_\mu + A_\mu v_\lambda - g_{\mu\lambda} A^\rho v_\rho, \quad (\text{A35})$$

and we obtain

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu) v^\rho = \mathcal{R}^\rho{}_{\sigma\mu\nu} v^\sigma + w v^\rho F_{\mu\nu}. \quad (\text{A36})$$

The Weyl-covariant Ricci (weight 0) and scalar (weight 2) curvatures read

²¹One usually considers in the literature $2\mu + z\lambda = 0$, where z is minus the conformal weight of Ω , referred to as the dynamical exponent. Here, due to the relationship of the considered Carrollian spacetimes with relativistic parents, the weight of Ω is -1 .

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + (d-1)(\nabla_\nu A_\mu + A_\mu A_\nu - g_{\mu\nu} A_\lambda A^\lambda) + g_{\mu\nu} \nabla_\lambda A^\lambda - F_{\mu\nu}, \quad (\text{A37})$$

$$\mathcal{R} = R + 2d\nabla_\lambda A^\lambda - d(d-1)A_\lambda A^\lambda. \quad (\text{A38})$$

Observe that $\mathcal{R}_{\mu\nu}$ is not symmetric.

If the metric is of the Papapetrou-Randers form (6), the dependence with respect to the velocity of light c is explicit. Thus every relativistic tensor, i.e., a tensor with respect to the full diffeomorphism group, can be reduced with respect to the Carrollian subgroup (3), and exhibits a finite number of Carrollian tensors. We find

$$R = \frac{1}{c^2} \left(\frac{2}{\Omega} \partial_t \theta + \frac{1+d}{d} \theta^2 + \xi_{ij} \xi^{ij} \right) + \hat{r} - 2\hat{\nabla}_i \varphi^i - 2\varphi^i \varphi_i + c^2 \varpi_{ij} \varpi^{ij}, \quad (\text{A39})$$

$$\mathcal{R} = \frac{1}{c^2} \xi_{ij} \xi^{ij} + \hat{\mathcal{R}} + c^2 \varpi_{ij} \varpi^{ij}. \quad (\text{A40})$$

Actually, the Carroll and Weyl-Carroll connections introduced earlier are also obtained from the ordinary Levy-Civita and the Weyl connections of the pseudo-Riemannian spacetime at hand, in the form of an exact, truncated Laurent expansion.

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Relativistic fluids, hydrodynamic frames and their Galilean versus Carrollian avatars

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ABSTRACT: We comprehensively study Galilean and Carrollian hydrodynamics on arbitrary backgrounds, in the presence of a matter/charge conserved current. For this purpose, we follow two distinct and complementary paths. The first is based on local invariance, be it Galilean or Carrollian diffeomorphism invariance, possibly accompanied by Weyl invariance. The second consists in analyzing the relativistic fluid equations at large or small speed of light, after choosing an adapted gauge, Arnowitt-Deser-Misner-Zermelo for the former and Papapetrou-Randers for the latter. Unsurprisingly, the results agree, but the second approach is superior as it effortlessly captures more elaborate situations with multiple degrees of freedom. It furthermore allows to investigate the fate of hydrodynamic-frame invariance in the two limits at hand, and conclude that its breaking (in the Galilean) or its preservation (in the Carrollian) are fragile consequences of the behaviour of transport attributes at large or small c . Both methods do also agree on the doom of Noetherian currents generated in the relativistic theory by isometries: conserved currents are not always guaranteed in Newton-Cartan or Carroll spacetimes as a consequence of Galilean or Carrollian isometries. Comparison of Galilean and Carrollian fluid equations exhibits a striking but often superficial resemblance, which we comment in relation to black-hole horizon dynamics, awkwardly akin to Navier-Stokes equations. This congruity is authentic in one instance though and turns out then to describe Aristotelian dynamics, which is the last item in our agenda.

KEYWORDS: Classical Theories of Gravity, Holography and Hydrodynamics, Space-Time Symmetries

ARXIV EPRINT: [2205.09142](https://arxiv.org/abs/2205.09142)

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1 Introduction

Fluid dynamics is 19th century physics par excellence. It has been thoroughly investigated, expanded and applied in various areas, but continues to raise questions and challenges, which are sometimes conceptual. In relativistic fluids for example, the issue of hydrodynamic-frame invariance is rather subtle. It reflects the freedom to choose arbitrarily the velocity of the fluid, and is rooted in the impossibility to distinguish the mass flow from the energy flow in

a relativistic system. This invariance was made popular by Landau and Lifshitz in their formulation of dissipative relativistic hydrodynamics without heat current [1], as opposed to the first formalism for relativistic fluids due to Eckart [2]. The freedom to set freely the velocity field has drawbacks recognized long ago, when implemented in the linearized (or, more generally, truncated) constitutive relations, which accompany the fluid equations of motion. A comprehensive presentation of the subject can be found in [3–6], where the main difficulties, namely causality, stability and shock structure, are discussed in length.¹ More recent progress has been reported in [9–19], showing that the debate is still ongoing. It is worth stressing that the above phenomenological descriptions of out-of-equilibrium phenomena are enshrined by relativistic kinetic theory. In particular, the various quoted formalisms — Eckart, Landau-Lifshitz or others — arise using the relativistic Boltzmann’s equation. Further reading on this subject is proposed in [20–23].

Hydrodynamic-frame invariance has also emerged in a slightly more formal context. The asymptotic isometries of the gravitational field in general relativity² are related to the symmetries of a fictitious fluid defined on the conformal boundary.³ When the gravitational field is asymptotically anti-de Sitter, the boundary is time-like and the dual fluid is relativistic. A local transformation of the fluid velocity amounts to a diffeomorphism on the gravitational side. For an asymptotically flat gravitational field, the boundary is null and the associated fluid is Carrollian [28]. Does the hydrodynamic-frame invariance survive in that case and does it share the above relationship with the asymptotic isometries? Similarly, and irrespective of any bearing to gravity, why is hydrodynamic-frame invariance lost in ordinary Galilean fluids, where the velocity and the mass density are measurable quantities?

The purpose of the present work is to elaborate on properties of Galilean and Carrollian fluids, in the spirit and as a follow up of ref. [29]. This includes the discussion of hydrodynamic-frame invariance, the addition of a conserved current and its associated chemical potential, the interpretation of Galilean and Carrollian fluid equations as conservation laws stemming out of appropriate diffeomorphism invariances, and the dearth of conservation properties ensuing isometries.

The emergence of Carrollian physics goes back to the works of Lévy-Leblond [30] and Sen Gupta [31]. The Carroll group is an ultra-relativistic contraction of Poincaré group. It is dual to the better-known non-relativistic contraction, the Galilean group.⁴ Carrollian symmetry has triggered interest in several directions. On the mathematical side, new geometric structures were discovered *dubbed Carrollian manifolds* [32–46], following patterns similar to those leading to the Galilean duals i.e. the Newton-Cartan geometries. From a more physical perspective, the connection of Carrollian symmetry with asymptotic isometries of Ricci-flat gravitational backgrounds and in particular its role in the growing subject of flat holography have attracted outmost attention [47–56].

¹Modern textbooks on fluid mechanics are e.g. [7, 8].

²See e.g. the lecture notes [24] for a recent review and further references on this subject.

³This fluid is often referred to as “dual” or “holographic” — see refs. [25–27] for the precise symmetry interplay and a complete bibliography of fluid/gravity holographic correspondence.

⁴On a purely semantic vein, the given names “relativistic, ultra-relativistic, non-relativistic” are all unfortunate, as pointed out with brio by Jean-Marc Lévy-Leblond. Although it is probably too late to give up the first, one should try to replace the others by *Carrollian* and *Galilean*. Incidentally, Niels Obers and Stefan Vandoren rightfully insist on the ultra-local nature of the Carrollian limit.

Ordinary, Galilean fluid dynamics is the non-relativistic limit of relativistic hydrodynamics. It was originally circumscribed to three-dimensional Euclidean space with absolute time i.e. to the strict Newtonian framework with full Galilean isometry. Efforts have been made to evade this restriction [57–66] and finally reach the general equations describing a non-relativistic viscous fluid moving on a space endowed with a spatial, time-dependent metric, and covariant under Galilean diffeomorphisms such as $t' = t'(t)$ and $\mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$ [29, 67].⁵

The more exotic Carrollian fluids are “flowing” on Carrollian manifolds and their equations of motion are invariant under Carrollian coordinate transformations, $t' = t'(t, \mathbf{x})$ and $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$. Although particle motion is forbidden due to the shrinking of the light cone, and despite the absence of a microscopic analysis based on thermodynamics or kinetic theory, the dynamics for a continuous medium seems to make formally sense, involving an “inverse velocity,” energy density, pressure etc. The first instance where Carrollian fluids were quoted is ref. [70]. There, it was realized that contrary to a forty-year lore, Einstein dynamics on black-hole horizons and the associated membrane paradigm were not related to the Navier-Stokes equations, but instead to their Carrollian duals. This observation was further discussed in [71], and aspects of Carrollian hydrodynamics were analyzed in [72–77]. It is fair to repeat that this sort of fluids lack of microscopic settlement and laboratory applications. Nevertheless, their dynamical equations emerge in various instances where null hypersurfaces are at work, and this justifies a thorough investigation.

The hydrodynamic equations for Galilean or Carrollian fluids were obtained in [29] as limits of the fully covariant relativistic equations on general pseudo-Riemannian manifolds. For the Galilean case, the suitable form of the metric was Zermelo, whereas Papapetrou-Randers was better adapted to the Carrollian limit. These metrics are indeed form-invariant under Galilean or Carrollian diffeomorphisms respectively (see [78] for further properties). Our study is performed here in the presence of a conserved current, which contributes the dynamics, and fosters the attainment of the Galilean continuity equation. The infinite or vanishing velocity limits are accompanied with some assumptions on the behaviour of the physical quantities such as energy density, heat current or stress tensor, including important sub-leading terms. We show that the resilience or the failure of the relativistic hydrodynamic-frame invariance in the non-relativistic or ultra-relativistic limits are closely tied to those behaviours. In a nutshell our conclusions can be summarized as follows. For the Carrollian case, the behaviours at vanishing velocity of light are suggested by the experience acquired with holographic fluids, and turn out to be compatible with hydrodynamic-frame invariance. This is no longer true in the Galilean limit, where the rules are dictated by non-relativistic physics and disrupt this invariance, unless one concedes to give up matter conservation and at the cost of altering the Navier-Stokes equations.

The relativistic hydrodynamic equations, namely the vanishing of the covariant energy-momentum tensor divergence, translate the invariance of some effective action with respect to general diffeomorphisms $t' = t'(t, \mathbf{x})$ and $\mathbf{x}' = \mathbf{x}'(t, \mathbf{x})$. Similarly, Galilean or Carrollian equations can be reached upon imposing the corresponding diffeomorphism invariance on the effective actions. The energy-momentum tensor is in these cases traded for other generalized

⁵See also [68, 69] for a discussion on symmetries.

momenta adapted to the local symmetries at hand. Our analysis, performed along the lines of [79], reveals subtleties and slightly differ in comparison with the large- c or small- c limits of the relativistic theory. This betrays that when considering these limits, as opposed to working with action principles directly in Newton-Cartan or Carrollian spacetimes, more information is stored in the equations, and more constraints emerge due to the larger original local invariance. Specifically, Galilean mass conservation (continuity equation) is built in (as shown in [29]) irrespective of extra matter current conservation. This confirms that the most striving and economical approach for reaching the dynamical equations is indeed the one based on the limiting procedure, originally used in [29], which can be even extended at wish for incorporating naturally more degrees of freedom, which would require more conjugate variables in the Galilean or Carrollian action principles.

In order to deliver a comprehensive picture of the web of dynamics emanating upon contractions of the plain relativistic group and their associate spacetimes, we briefly venture out and explore the realm of Aristotelian geometries. Introduced by Penrose in [80], they became suddenly popular because they do not possess any boost invariance (see e.g. [74, 76, 81–83]). The absence of boosts⁶ features that both time and space are absolute in these geometries. Hence motion and light cone are trivialized, and the notion of fluid becomes even more questionable than in the Carrollian framework. Nonetheless, dynamics can be defined from invariance principles — no limit involving the speed of light exists that would connect Aristotelian spacetimes to relativistic theories — and is worth investigating as it appears to stand at the intersection of Galilean and Carrollian physics.

When discussing dynamics in general, and fluid dynamics in particular, part of the duty is to exhibit conserved quantities. These are generally the consequence of symmetries, but this concept should be scrutinized on a case-by-case basis. In a relativistic theory, any (conformal) Killing field provides a conserved current upon contraction with a (traceless) conserved energy-momentum tensor. We prove that this is no longer systematically true for Galilean or Carrollian hydrodynamics, and for instance, boosts present in flat space are not spared — this would be circumvented in Aristotelian spacetimes if boosts were available. Hence Noetherian currents in Newton-Cartan or Carroll manifolds arise for a restricted subset of isometries. This is an important spin-off of our study, that compromises former attempts to describe hydrodynamics in flat Newton-Cartan or Carroll spacetimes on the ground of Noetherian conservation laws.

An executive outline of the present work is as follows. We remind the basics on relativistic fluids in the presence of a conserved matter current with emphasis on hydrodynamic-frame invariance. This analysis is further expanded into two distinct frames, the Zermelo and Papapetrou-Randers, appropriate for the subsequent investigation about Galilean and Carrollian fluids. Galilean fluids are first studied from the conservation perspective mirroring the Galilean-diffeomorphism invariance of Newton-Cartan spacetimes with emphasis on the effect of isometries, when present in the background; next as the infinite- c limit of the relativistic hydrodynamics in Zermelo frame. Hydrodynamic-frame invariance is revisited and we move next to the Galilean massless case. The current analysis is repeated along

⁶The associated group is the *static group*, introduced in [84].

cognate lines for Carrollian fluid dynamics with a paragraph specifically devoted to the possible multiplication of degrees of freedom in the zero- c limit. Finally we describe the case of Aristotelian fluids in a short section before concluding.

The subject of hydrodynamic-frame invariance and Carrollian fluids has been lately in the spotlight. Some debatable statements have been promoted in the literature, and our views are not always in line with those of other authors. Wherever necessary, we stress it and provide the adequate elements to make the comparison clear and avoid confusion. Our approach is meant to be a constructive criticism, and we intentionally supply a wealth of technical details, some appended in four sections, to back-up our conclusions, sometimes at the expense of considerably increased length.

2 Relativistic hydrodynamics

2.1 Basic concepts and general equations

Energy–momentum and matter conservation

Fluid mechanics is the description of irreversible off-equilibrium thermodynamics under the assumption that the wave lengths of dynamical phenomena are large compared to typical kinetic scales. It is thus legitimate to assume local thermal equilibrium and use the laws of thermostatics (recalled in appendix A), although the definitions of temperature, chemical potential, entropy etc. are possibly questionable, or at least ambiguous.

Without external forces and springs or sinks of matter, the basic requirements are covariant energy-momentum and matter (rather than mass) macroscopic conservation, encoded in the following $d + 2$ equations:⁷

$$\nabla_\mu T^{\mu\nu} = 0, \tag{2.1}$$

$$\nabla_\mu J^\mu = 0, \tag{2.2}$$

where we assume the spacetime, of dimension $d + 1$, be equipped with a metric $g_{\mu\nu}$. The energy-momentum tensor and the matter current can be decomposed along a vector congruence u^μ playing the role of velocity field normalized as $u^\mu u_\mu = -c^2$:

$$T^{\mu\nu} = (\varepsilon + p) \frac{u^\mu u^\nu}{c^2} + p g^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\nu}{c^2} + \frac{u^\nu q^\mu}{c^2}, \tag{2.3}$$

$$J^\mu = \varrho_0 u^\mu + j^\mu. \tag{2.4}$$

The viscous stress tensor $\tau^{\mu\nu}$ and the heat current q^μ are purely transverse:

$$u^\mu q_\mu = 0, \quad u^\mu \tau_{\mu\nu} = 0, \quad u^\mu T_{\mu\nu} = -q_\nu - \varepsilon u_\nu, \quad \varepsilon = \frac{1}{c^2} T_{\mu\nu} u^\mu u^\nu. \tag{2.5}$$

They are expressed in terms of u^i and their spatial components q_i and τ_{ij} with $i = 1, 2, \dots, d$. Similarly is the imperfect particle current from its components j_i , since

$$u^\mu j_\mu = 0, \quad \varrho_0 = -\frac{1}{c^2} u^\mu J_\mu. \tag{2.6}$$

In the above expressions — see also appendix A

⁷Matter conservation could be multiple, or even absent as e.g. in a gas of photons, although no principle forbids the existence of conserved currents in fluids made of massless carriers — see appendix A.

- ε and ϱ_0 are the energy and the matter per unit of proper volume, as measured by an observer moving at velocity u^μ (comoving);
- ε and ϱ_0 are related and so are $T^{\mu\nu}$ and J^μ ;
- p is the local-equilibrium thermodynamic pressure obeying an equation of state of the form $p = p(T, \mu_0)$, where T and μ_0 are the local temperature and chemical potential;
- the quantities j_i , q_i and τ_{ij} capture the physical properties of the out of equilibrium state, and are usually expressed as expansions in temperature, chemical potential and velocity derivatives: *the constitutive relations in their hydrodynamic expansion*.

It is worth recalling that from the perspective of an effective action⁸ $S = \frac{1}{c} \int d^{d+1}x \sqrt{-g} \mathcal{L}$, the energy-momentum tensor is defined as

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}, \tag{2.7}$$

whereas for the matter current, a U(1) gauge field with components B_μ is needed:

$$J^\mu = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta B_\mu}. \tag{2.8}$$

On the one hand, invariance under diffeomorphisms generated by arbitrary vector fields $\xi = \xi^\mu(t, \mathbf{x})\partial_\mu$ as

$$\delta_\xi g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu}, \tag{2.9}$$

where

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \xi^\rho + g_{\nu\rho} \partial_\mu \xi^\rho = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{2.10}$$

implies the conservation equation (2.1). On the other hand, the matter-conservation equation (2.2) is a consequence of invariance under

$$\delta_\Lambda B_\mu = -\partial_\mu \Lambda \tag{2.11}$$

with $\Lambda = \Lambda(t, \mathbf{x})$.

It is important to stress at this early stage that we do not assume any isometry, neither here, nor in the subsequent limiting geometries. The energy-momentum tensor and the current should not be confused with any sort of Noetherian currents, and their conservation is a direct consequence of local invariances. This should be opposed to other approaches presented in the quoted literature.

Isometries, conformal isometries and extra conservations

If $\xi = \xi^\mu \partial_\mu$ is a Killing field it obeys

$$\mathcal{L}_\xi g_{\mu\nu} = 0. \tag{2.12}$$

⁸As usual $\frac{1}{c} d^{d+1}x = dt \wedge dx^1 \wedge \dots \wedge dx^d$.

Hence, due to (2.1), the current defined as

$$I_\mu = \xi^\nu T_{\mu\nu} \tag{2.13}$$

is divergence-free

$$\nabla_\mu I^\mu = 0. \tag{2.14}$$

Using Stokes and Gauss theorems,

$$\int_{\mathcal{W}} d^{d+1}x \sqrt{-g} \nabla_\mu I^\mu = \oint_{\partial\mathcal{W}} *I, \tag{2.15}$$

where \mathcal{W} is a domain inside \mathcal{M} and $*I$ is the \mathcal{M} -Hodge dual of $I = I_\mu dx^\mu$ ($\epsilon_{01\dots d} = 1$), we infer that

$$Q_I = \frac{1}{c} \int_{\Sigma_d} *I \tag{2.16}$$

is conserved. Here, Σ_d is an arbitrary space-like hypersurface of \mathcal{M} , and “conserved” means that the value of Q_I is independent of the choice of Σ_d .⁹

When the energy-momentum is trace-free

$$T^\mu{}_\mu = 0, \tag{2.17}$$

a conformal isometry suffices for producing a conservation, along the lines described above. The conformal Killing satisfies

$$\mathcal{L}_\xi g_{\mu\nu} = \frac{2}{d+1} \nabla_\rho \xi^\rho g_{\mu\nu}. \tag{2.18}$$

Entropy current and entropy equation

The variational definitions of the energy-momentum tensor and the matter current as conjugate momenta to some elementary background fields are elegant and general as we will see in the forthcoming sections, but not indispensable. The physics of the fluid relies in fact on the decomposition of these momenta into observable quantities, expressed themselves as derivative expansions, and this requires more information than the knowledge of local symmetries. To this, one should add that the entropy current S^μ is yet another physical object, which has no variational definition of the sort (2.7) or (2.8). It has actually no microscopic definition such as an expectation value of some observable, but is built order by order in the derivative expansion,¹⁰ requiring among others that its divergence (or, alternatively for theories which incorporate memory effects, the integrated divergence) be non-negative. The generic form of the entropy current is

$$S^\mu = \frac{1}{T} (pu^\mu - T^{\mu\nu} u_\nu - \mu_0 J^\mu) + R^\mu = \Sigma^\mu + R^\mu, \tag{2.19}$$

⁹If Σ_d belongs to a family of hypersurfaces defined as $\tau(t, \mathbf{x}) = \text{const.}$, the conservation is expressed as $\frac{dQ_I}{d\tau} = 0$. This needs not be so though, and the wording “conservation” is to some extent reductive, as no reference to time is needed. Care should also be taken with the behaviour of the fields at spatial infinity and if Σ_d has itself a boundary.

¹⁰The first order is often referred to as *classical irreversible thermodynamics*, the second *extended irreversible thermodynamics*, etc.

where Σ^μ is a kind of universal piece of the current, and R^μ depends on the specific off-equilibrium thermodynamic theory. Using (2.5), (2.6), (A.3) and (A.5), Σ^μ is recast as follows:

$$\Sigma^\mu = \sigma u^\mu - \frac{\mu_0}{T} j^\mu + \frac{1}{T} q^\mu = \frac{\sigma}{\varrho_0} J^\mu - \frac{w}{T\varrho_0} \left(j^\mu - \frac{\varrho_0}{w} q^\mu \right) \quad (2.20)$$

with σ the entropy, w the relativistic enthalpy (A.3) and μ_0 the relativistic chemical potential.

It is convenient, both for the relativistic dynamics and for its Galilean or Carrollian limits, to consider the longitudinal and transverse projections of (2.1), possibly combined with (2.2) and the thermodynamic laws (A.5), (A.6) and (A.8) in order to trade the energy for the entropy. For the longitudinal projection, we find:¹¹

$$-u_\nu \nabla_\mu T^{\mu\nu} = u(\varepsilon) + \left(w + \frac{\tau}{d} \right) \Theta + \tau^{\mu\nu} \sigma_{\mu\nu} + \nabla_\nu q^\nu + \frac{a_\nu q^\nu}{c^2}, \quad (2.21)$$

$$= T \nabla_\nu \Sigma^\nu + \frac{\tau}{d} \Theta + \tau^{\mu\nu} \sigma_{\mu\nu} + T j^\nu \partial_\nu \frac{\mu_0}{T} + q^\nu \left(\frac{\partial_\nu T}{T} + \frac{a_\nu}{c^2} \right), \quad (2.22)$$

where $\tau = \tau^{\mu\nu} g_{\mu\nu}$ is the *relativistic non-equilibrium pressure* and $u(f)$ stands for $u^\mu \partial_\mu f$. We have also introduced the following kinematical tensors:¹²

$$a_\mu = u^\nu \nabla_\nu u_\mu, \quad \Theta = \nabla_\mu u^\mu, \quad (2.23)$$

$$\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)} + \frac{1}{c^2} u_{(\mu} a_{\nu)} - \frac{1}{d} \Theta h_{\mu\nu}, \quad (2.24)$$

$$\omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]} + \frac{1}{c^2} u_{[\mu} a_{\nu]}, \quad (2.25)$$

which are the acceleration, the expansion, the shear and the vorticity of the velocity field, with $h_{\mu\nu}$ and $U_{\mu\nu}$ the projectors onto the space transverse and longitudinal to the velocity field:

$$h_{\mu\nu} = \frac{u_\mu u_\nu}{c^2} + g_{\mu\nu}, \quad U_{\mu\nu} = -\frac{u_\mu u_\nu}{c^2}. \quad (2.26)$$

Relativistic hydrodynamic-frame invariance

The absence of sharp distinction between energy and mass flow in relativistic theories brings some redundancy in the above fluid data such as q^μ and j^μ . In his seminal theory of relativistic fluids [2], Eckart shed this redundancy by making the choice $j^\mu = 0$, whereas Landau and Lifshitz [1] required instead $q^\mu = 0$. It is more generally admitted that one has the freedom to redefine

$$T(x) \rightarrow T'(x), \quad \mu_0(x) \rightarrow \mu'_0(x), \quad u^\mu(x) \rightarrow u'^\mu(x), \quad (2.27)$$

provided we modify accordingly $\varepsilon(x)$, $p(x)$, $\varrho_0(x)$, $q^\mu(x)$, $\tau^{\mu\nu}(x)$ and $j^\mu(x)$ so that the energy-momentum tensor, the conserved current and the entropy current remain unaltered. This is the gauge symmetry, associated with local Lorentz transformations, known as

¹¹This is the generalization of eq. (127,5) in [1] for general gravitational backgrounds and in an arbitrary hydrodynamic frame.

¹²Our conventions for (anti-) symmetrization are $A_{(\mu\nu)} = \frac{1}{2} (A_{\mu\nu} + A_{\nu\mu})$ and $A_{[\mu\nu]} = \frac{1}{2} (A_{\mu\nu} - A_{\nu\mu})$.

hydrodynamic-frame invariance. It translates, that the velocity field has no first-principle definition in relativistic hydrodynamics.¹³

It should be stressed here that irrespective of the aforementioned local Lorentz freedom, dealing with out-of-equilibrium phenomena brings its share of indetermination. Once the global equilibrium is abandoned, thermodynamic functions become local and their very existence relies on local thermodynamic equilibrium. They are furthermore supposed to be slow-varying for hydrodynamics to make sense. Nonetheless, even within these assumptions, for non-perfect fluids, neither $p(x)$, $\varepsilon(x)$ and $\varrho_0(x)$ appearing in the fluid equations, nor $T(x)$ and $\mu_0(x)$ entering the constitutive relations need a priori to be identified with the corresponding local-equilibrium thermodynamic quantities. We have made that choice here, and we will not further discuss this side of hydrodynamic-frame invariance, but rather focus on the kinematical aspects. More information can be found in the already quoted literature, and in particular in [10–19].

Suppose we perform a local Lorentz transformation on the fluid, i.e. a transformation on the velocity vector field

$$\mathbf{u} \rightarrow \mathbf{u} + \delta\mathbf{u}, \quad \delta\mathbf{u} \cdot \mathbf{u} = 0. \quad (2.28)$$

One can transform accordingly the various pieces that appear in the decomposition of $T_{\mu\nu}$ (2.3) so that $\delta T_{\mu\nu} = 0$.¹⁴

$$\delta\varepsilon = -\frac{2}{c^2}q^\mu\delta u_\mu, \quad (2.29)$$

$$\delta q_\nu = \frac{u_\nu}{c^2}q^\mu\delta u_\mu - w\delta u_\nu - \tau_{\nu\mu}\delta u^\mu, \quad (2.30)$$

$$\begin{aligned} \delta(p h_{\mu\nu} + \tau_{\mu\nu}) &= \frac{p}{c^2}(u_\mu\delta u_\nu + u_\nu\delta u_\mu) + \frac{1}{c^2}(u_\mu\tau_{\nu\rho} + u_\nu\tau_{\mu\rho})\delta u^\rho \\ &\quad - \frac{1}{c^2}(\delta u^\mu q^\nu + \delta u^\nu q^\mu). \end{aligned} \quad (2.31)$$

Similarly, in the presence of a matter current, one requires $\delta J_\mu = 0$, and using (2.4) one obtains:

$$\delta j_\mu = -u_\mu\delta\varrho_0 - \varrho_0\delta u_\mu. \quad (2.32)$$

The interplay between the energy-momentum tensor and the matter current is ensured by the thermodynamic laws and the entropy current. The latter must also be hydrodynamic-frame-invariant, as are the thermodynamic relations, in particular the Gibbs-Duhem equation

¹³In fact, the two extreme options embraced by Eckart and Landau-Lifshitz may not be achievable in all systems. The former demands a time-like conserved current J^μ and selects a velocity field aligned with it. The latter requires that the energy-momentum tensor has a time-like eigenvector with positive eigenvalue and aligns the velocity with this eigenvector. In both cases, the defining assumption is reasonable, and the output unique, leaving no residual hydrodynamic-frame invariance. The equivalence amongst these frames relies on a deep interplay between the energy-momentum and the conserved current, when it exists. This interplay reveals in thermodynamics and specifically in the entropy current. However, although equivalent, the two frames are not equally suited for the non-relativistic limit, as we will see in section 3.2 (see also [8]).

¹⁴We cannot disentangle at this stage the transverse components p and $\tau_{\mu\nu}$, as their separation relies on thermodynamics (p is the equilibrium pressure, $\tau_{\mu\nu}$ stands for the non-equilibrium stress and its trace is the non-equilibrium pressure).

and the equation of state. These requirements provide the tools for computing $\delta\sigma$, δp , δT , and $\delta\mu_0$, in terms of the above variations already expressed with the basic data δu_μ and $\delta\rho_0$. We will not delve in analyzing the effects induced on thermodynamic observables by the local Lorentz transformations (for that see e.g. [10–19, 85]), but will conclude this discussion with some remarks. At the first place, as already stressed, there is no microscopic definition of the entropy current. Hydrodynamic-frame invariance serves therefore as a prerequisite for completing expressions like (2.20), order by order in the derivative expansion. Secondly, q_μ , $\tau_{\mu\nu}$ and j_μ are given in the form of constitutive relations, which are asymptotic series, terminated at finite order. This blurs the fluid-frame invariance, and is at the heart of the caveats mentioned in the introduction (stability and causality). Privileged hydrodynamic frames unavoidably emerge, depending on the physical situation under consideration. Last but not least, even when the previous items are under control (as in two dimensions [25]), the question of global issues remains, in the lines discussed in refs. [25–27]: like any gauge symmetry, hydrodynamic-frame invariance suffers from possible global breaking.

A remark on perfect fluids

For a perfect fluid, the heat current, the viscous stress tensor and the transverse part of the matter current vanish. From an intrinsic view, given an energy-momentum tensor and a matter current, these requirements are met if the matter current is an eigenvector of the energy-momentum tensor, and if the transverse part of the energy-momentum tensor with respect to this eigenvector is proportional to the projector orthogonal to the current. Given the normalisation of the velocity, here aligned with the current, this provides unambiguously ρ_0 , p and ε (see (2.3), (2.4), (2.5) and (2.6)), and guarantees at the same time the absence of j^ν , q^ν and $\tau^{\mu\nu}$. The entropy current (2.19) is now given by (2.20) and is proportional to the velocity. From (2.22) we infer it has zero divergence.

Formally, one could perform local hydrodynamic-frame transformations. However, following the rules (2.28), (2.29), (2.30), (2.31) and (2.32), no transformation exists, which preserves the perfect forms of the matter current and of the energy-momentum tensor. They all generate non-perfect components, which should in this case be considered spurious because they do not reflect any genuine microscopic interaction.¹⁵ In the absence of dissipative phenomena and heat transport, energy is carried by matter and there is no ambiguity in defining a physical fluid velocity, in line with the presentation of ref. [1].

Weyl-invariant fluids

A physical system such as a fluid can be invariant under Weyl transformations. Those act on the background metric and fluid velocity as

$$ds^2 \rightarrow \mathcal{B}^{-2} ds^2, \quad u^\mu \rightarrow \mathcal{B} u^\mu, \tag{2.33}$$

and more general tensors are Weyl-covariant if they rescale with some power of \mathcal{B} (weight w) — *not to be confused with the relativistic enthalpy w* . A Weyl-covariant derivative \mathcal{D}_μ

¹⁵Although unphysical, this formal freedom is important from a holographic perspective, and was discussed extensively in [25–27].

maintains the canonical Weyl transformation of a Weyl-covariant tensor, and calls for a Weyl connection one-form:¹⁶

$$A = \frac{1}{c^2} \left(a - \frac{\Theta}{d} u \right). \quad (2.34)$$

The Weyl covariant derivative is metric-compatible:

$$\mathcal{D}_\rho g_{\mu\nu} = 0, \quad (2.35)$$

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu) f = w f F_{\mu\nu}, \quad (2.36)$$

where the action on a weight- w scalar f is

$$\mathcal{D}_\lambda f = \nabla_\lambda f + w A_\lambda f, \quad (2.37)$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2.38)$$

is the Weyl curvature (weight-0). For concreteness, the action of \mathcal{D}_λ on a weight- w form v_μ and a weight- w tensor $t_{\mu\nu}$ is

$$\mathcal{D}_\lambda v_\mu = \nabla_\lambda v_\mu + (w+1)A_\lambda v_\mu + A_\mu v_\lambda - g_{\mu\lambda} A^\rho v_\rho, \quad (2.39)$$

$$\mathcal{D}_\lambda t_{\mu\nu} = \nabla_\lambda t_{\mu\nu} + (w+2)A_\lambda t_{\mu\nu} + A_\mu t_{\lambda\nu} + A_\nu t_{\mu\lambda} - g_{\mu\lambda} A^\rho t_{\rho\nu} - g_{\nu\lambda} A^\rho t_{\mu\rho}. \quad (2.40)$$

Commuting the Weyl-covariant derivatives acting on vectors, as usual one defines the Weyl-covariant Riemann tensor

$$(\mathcal{D}_\mu \mathcal{D}_\nu - \mathcal{D}_\nu \mathcal{D}_\mu) v^\rho = \mathcal{R}^\rho_{\sigma\mu\nu} v^\sigma + w v^\rho F_{\mu\nu} \quad (2.41)$$

(v^ρ are the components of a weight- w vector) and the usual subsequent quantities. The Weyl-covariant Ricci (weight 0) and scalar (weight 2) curvatures read:

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + (d-1) \left(\nabla_\nu A_\mu + A_\mu A_\nu - g_{\mu\nu} A_\lambda A^\lambda \right) + g_{\mu\nu} \nabla_\lambda A^\lambda - F_{\mu\nu}, \quad (2.42)$$

$$\mathcal{R} = R + 2d \nabla_\lambda A^\lambda - d(d-1) A_\lambda A^\lambda. \quad (2.43)$$

The fluid dynamics captured by (2.1) and (2.2) is Weyl-invariant under the necessary and sufficient condition that the energy-momentum tensor and the matter current are Weyl-covariant and such that

$$\nabla_\mu T^{\mu\nu} = \mathcal{D}_\mu T^{\mu\nu}, \quad \nabla_\mu J^\mu = \mathcal{D}_\mu J^\mu. \quad (2.44)$$

This demands the conformal weights of $T_{\mu\nu}$ and J_μ be $d-1$, and $T_{\mu\nu}$ be traceless. The required weight for the energy-momentum tensor is the translation of Weyl invariance for the underlying action, as one infers from eq. (2.7); this Weyl invariance also imposes the

¹⁶The explicit form of A is obtained by demanding $\mathcal{D}_\mu u^\mu = 0$ and $u^\lambda \mathcal{D}_\lambda u_\mu = 0$. The Weyl connection is not unique. Any weight- w conformal vector V is associated with a bona fide Weyl connection $A_V = \frac{1}{\|V\|^2} \left(\frac{(2-w)\Theta_V}{d+1-w} V - a_V \right)$ with Θ_V and a_V , the expansion and acceleration of V . An example of alternative Weyl connection will be presented in section 3.4.

weight	observables
$d + 1$	ε, p
d	q_μ, ϱ_0
$d - 1$	$\tau_{\mu\nu}, j_\mu$

Table 1. Conformal weights.

absence of trace.¹⁷ In the decomposition (2.3) the latter condition reads $-\varepsilon + dp + \tau^\mu{}_\mu = 0$, usually split into the conformal global-equilibrium equation of state plus a condition on the piece associated with dynamical irreversible phenomena:

$$\varepsilon = dp, \quad \tau^\mu{}_\mu = 0. \tag{2.45}$$

Furthermore we learn from eq. (2.8) that the gauge field B_μ conjugate to J^μ is weight-zero to comply with the expected weight for J_μ . We have summarized the weights of the various physical quantities in the table 1.

2.2 Fluids in Zermelo coordinates

Zermelo frame

In a pseudo-Riemannian manifold \mathcal{M} of $d + 1$ dimensions, one can always assume the Arnowitt-Deser-Misner form of the metric

$$ds^2 = -\Omega^2 c^2 dt^2 + a_{ij} (dx^i - w^i dt) (dx^j - w^j dt) \tag{2.46}$$

with a_{ij} , w^i and Ω functions of $x = (ct, \mathbf{x}) = \{x^\mu, \mu = 0, 1, \dots, d\}$ and \mathbf{x} stands for $\{x^1, \dots, x^d\}$. These coordinates are well-suited for the implementation of the Galilean limit [29]. Indeed, *Galilean diffeomorphisms*

$$t' = t'(t), \quad \mathbf{x}' = \mathbf{x}'(t, \mathbf{x}) \tag{2.47}$$

have Jacobian

$$J(t) = \frac{\partial t'}{\partial t}, \quad j^i(t, \mathbf{x}) = \frac{\partial x'^i}{\partial t}, \quad J_j^i(t, \mathbf{x}) = \frac{\partial x'^i}{\partial x^j}, \tag{2.48}$$

and the transformation of the metric components is nicely reduced:

$$a'_{ij} = a_{kl} J^{-1k}{}_i J^{-1l}{}_j, \quad w'^k = \frac{1}{J} (J^k{}_i w^i + j^k), \quad \Omega' = \frac{\Omega}{J}. \tag{2.49}$$

This fits with the Newton-Cartan structure emerging at infinite c .

We call *Zermelo metrics* (see [78]) the restricted class of (2.46) for which Ω depends on t only. This class is stable under Galilean diffeomorphisms because J in (2.49) does not depend on spatial coordinates. The corresponding Newton-Cartan geometries reached in the

¹⁷We recall that $\delta S = \int d^{d+1}x \sqrt{-g} (\frac{1}{2} T^{\mu\nu} \delta g_{\mu\nu} + J^\mu \delta B_\mu)$. Hence for an infinitesimal Weyl rescaling (i.e. \mathfrak{B} close to the identity), $\delta_{\mathfrak{B}} S = - \int d^{d+1}x \sqrt{-g} \ln \mathfrak{B} T_\mu{}^\mu$.

Galilean limit are torsion-free (see [42]) and they feature an absolute, invariant Newtonian time $\int dt \Omega(t) = \int dt' \Omega'(t')$.

The above reduction with respect to the Galilean diffeomorphisms (2.48), can be completed as follows. Any tensor component with an upper time index transforms as a Galilean density, and thus is a scalar upon multiplication by Ω . Similarly the components with lower spatial indices transform as Galilean tensors. As an example, the transformation of the $d + 1$ vector components u^μ under a Galilean diffeomorphism leads to¹⁸

$$u'^0 = J u^0, \quad u'_i = u_k J^{-1k}_i. \quad (2.50)$$

A relativistic fluid moving in (2.46) is described by the components of its velocity \mathbf{u} , normalized as $\|\mathbf{u}\|^2 = -c^2$:

$$u^\mu = \frac{dx^\mu}{d\tau} \Rightarrow \quad u^0 = \gamma c, \quad u^i = \gamma v^i, \quad (2.51)$$

where the Lorentz factor γ is, in the Zermelo frame¹⁹

$$\gamma = \frac{dt}{d\tau} = \frac{1}{\Omega \sqrt{1 - \left(\frac{\mathbf{v}-\mathbf{w}}{c\Omega}\right)^2}}. \quad (2.52)$$

Under a Galilean diffeomorphism (2.48), the transformation of u^μ (see footnote 18) induces the expected transformation on v^i :

$$v'^k = \frac{1}{J} \left(J^k_i v^i + j^k \right). \quad (2.53)$$

Orthogonality conditions (2.5) and (2.6) imply that the fundamental data for the non-perfect matter current, the heat current and the stress tensor are j_i , q_j and τ_{kl} . Other components are e.g.

$$j^0 = \frac{(v^i - w^i) j_i}{c\Omega^2}, \quad q^0 = \frac{(v^i - w^i) q_i}{c\Omega^2}, \quad \tau^{00} = \frac{(v^k - w^k)(v^l - w^l) \tau_{kl}}{c^2 \Omega^4}, \quad \tau^0_j = \frac{(v^k - w^k) \tau_{kj}}{c\Omega^2}, \quad (2.54)$$

which transform as tensors under Galilean diffeomorphisms.

Hydrodynamic-frame transformations and invariants

The fluid velocity is parameterized in (2.51) with d components v^i . We can thus formulate the relativistic hydrodynamic-frame transformations in terms of arbitrary $\delta v^i(x)$. In the Zermelo frame, we obtain:

$$\delta\gamma = \frac{\gamma^3}{c^2} \delta v^i (v_i - w_i), \quad (2.55)$$

hence

$$\delta\mathbf{u} = \gamma \delta v^i \left(\partial_i + \frac{\gamma^2}{c^2} (v_i - w_i) (\partial_t + v^k \partial_k) \right). \quad (2.56)$$

¹⁸When the indices are inverted, the transformations are of the connection type: $u'^i = J^i_k u^k + J^i u^0$, $u'_0 = \frac{1}{J} (u_0 - u_j J^{-1j}_k J^k)$. For those, the tensorial structure is restored at the infinite- c limit, where indices are lowered and raised with a_{ij} and its inverse.

¹⁹Expressions as \mathbf{v}^2 stand for $a_{ij} v^i v^j$, not to be confused with $\|\mathbf{u}\|^2 = g_{\mu\nu} u^\mu u^\nu$.

Using eqs. (2.29), (2.30) and (2.31) together with (2.54) and (2.56) we find:²⁰

$$\delta\varepsilon = -2\frac{\gamma}{c^2}\delta v^i q_i, \quad (2.57)$$

$$\delta q_i = \gamma\delta v^k \left(\frac{\gamma}{c^2} (v_i - w_i) q_k - w h_{ki} - \tau_{ki} \right), \quad (2.58)$$

$$\begin{aligned} \delta(ph_{ij} + \tau_{ij}) &= \frac{\gamma^2}{c^2}\delta v^k ((v_i - w_i)(\tau_{jk} + ph_{jk}) + (v_j - w_j)(\tau_{ik} + ph_{ik})) \\ &\quad - \frac{\gamma}{c^2}\delta v^k (q_i h_{jk} + q_j h_{ik}), \end{aligned} \quad (2.59)$$

where (see (2.26))

$$h_{ik} = a_{ik} + \frac{\gamma^2}{c^2} (v_i - w_i)(v_k - w_k). \quad (2.60)$$

When a matter current is available, the above is completed with (2.32), which gives

$$\delta\rho_0 = -\frac{\gamma}{c^2}\delta v^i j_i \quad (2.61)$$

and

$$\delta j_i = \delta v^k \left(\frac{\gamma^2}{c^2} j_k (v_i - w_i) - \gamma\rho_0 h_{ki} \right). \quad (2.62)$$

The transformations at hand translate the invariance of the energy-momentum tensor and current components. The latter define therefore *invariants*, which are simply the energy density, the heat current, the stress tensor, the matter density and the matter non-perfect current in a privileged frame, that we will call “at rest” or “proper,”²¹ borrowing the standard expressions of special relativity:

$$T^{00} = \frac{\varepsilon_r}{\Omega^2}, \quad T^0_i = \frac{1}{c\Omega} q_{ri}, \quad T_{ij} = p_r a_{ij} + \tau_{rij} \quad (2.63)$$

with trace

$$T_\mu{}^\mu = -\varepsilon_r + dp_r + a^{ij}\tau_{rij}, \quad (2.64)$$

and

$$J^0 = \frac{c}{\Omega} \rho_{0r}, \quad J_i = j_{ri}. \quad (2.65)$$

We find explicitly

$$\varepsilon_r = \varepsilon\gamma^2\Omega^2 + \frac{2}{c^2}\gamma q_i (v^i - w^i) + (ph_{ij} + \tau_{ij}) \frac{(v^i - w^i)(v^j - w^j)}{c^2\Omega^2}, \quad (2.66)$$

$$\begin{aligned} q_{ri} &= \varepsilon\gamma^2\Omega (v_i - w_i) + \gamma\Omega q_j \left(\delta_i^j + \frac{(v^j - w^j)(v_i - w_i)}{c^2\Omega^2} \right) \\ &\quad + (ph_{ij} + \tau_{ij}) \frac{v^j - w^j}{\Omega}, \end{aligned} \quad (2.67)$$

$$\begin{aligned} p_r a_{ij} + \tau_{rij} &= \frac{\varepsilon\gamma^2}{c^2} (v_i - w_i)(v_j - w_j) + \frac{\gamma}{c^2} (q_i (v_j - w_j) + q_j (v_i - w_i)) \\ &\quad + ph_{ij} + \tau_{ij}, \end{aligned} \quad (2.68)$$

²⁰Notice that $q_\mu \delta u^\mu = \gamma \delta v^i q_i$.

²¹We call this frame “fiducial” in section 3.2 and show it is associated with an observer at velocity $u_z = e_i$ given in (3.3).

and

$$\varrho_{0r} = \varrho_0 \Omega \gamma + j_i \frac{v^i - w^i}{c^2 \Omega}, \quad (2.69)$$

$$j_{ri} = j_i + \varrho_0 \gamma (v^i - w^i). \quad (2.70)$$

It should be stressed that the above quantities are hydrodynamic-frame invariant but also covariant under Galilean diffeomorphisms. This latter property will be useful when considering the Galilean limit.

Killings and conserved currents

Consider a Killing field on \mathcal{M} satisfying (2.12)

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \xi^{\hat{t}} e_{\hat{t}} + \xi^{\hat{i}} e_{\hat{i}}, \quad (2.71)$$

where we have introduced a somewhat more convenient frame and coframe

$$e_{\hat{t}} = \frac{1}{\Omega} (\partial_t + w^j \partial_j), \quad e_{\hat{i}} = \partial_i, \quad \theta^{\hat{t}} = \Omega dt, \quad \theta^{\hat{i}} = dx^i - w^i dt, \quad (2.72)$$

so that the metric (2.46) reads:

$$ds^2 = -c^2 (\theta^{\hat{t}})^2 + a_{ij} \theta^{\hat{i}} \theta^{\hat{j}}. \quad (2.73)$$

Hence

$$\xi^{\hat{t}} = \xi^t \Omega, \quad \xi^{\hat{i}} = \xi^i - \xi^t w^i, \quad \xi_{\hat{t}} = -c^2 \xi^{\hat{t}}, \quad \xi_{\hat{i}} = a_{ij} \xi^{\hat{j}} = \xi_i. \quad (2.74)$$

With these data, the components of the conserved current (2.13) are²²

$$I^0 = \frac{c}{\Omega} \iota_{0r}, \quad I_i = i_{ri}, \quad (2.75)$$

where

$$\iota_{0r} = \frac{1}{c^2} \xi^{\hat{i}} q_{ri} - \xi^{\hat{t}} \varepsilon_r, \quad (2.76)$$

$$i_{ri} = \xi^{\hat{j}} (p_r a_{ij} + \tau_{rij}) - \xi^{\hat{t}} q_{ri}. \quad (2.77)$$

The associated conserved charge is obtained using (2.16):

$$\begin{aligned} Q_I &= \int_{\Sigma_d} \sqrt{a} \iota_{0r} (dx^1 - w^1 dt) \wedge \dots \wedge (dx^d - w^d dt) \\ &\quad - \int_{\Sigma_d} \sqrt{a} \sum_{i=1}^d (dx^1 - w^1 dt) \wedge \dots \wedge a^{ij} i_{rj} \Omega dt \wedge \dots \wedge (dx^d - w^d dt), \end{aligned} \quad (2.78)$$

where $a^{ij} i_{rj} \Omega dt$ is the i th factor in the exterior product of the last term.

²²We use the standard decomposition $I^\mu = \iota_0 u^\mu + i^\mu$ with $u^\mu i_\mu = 0$ and $\iota_0 = -u^\mu I_\mu$, and introduce ι_{0r} as a proper or fiducial density, following the footsteps of the energy-momentum tensor and the matter current, eqs. (2.63) and (2.65).

2.3 Fluids in Papapetrou-Randers coordinates

Papapetrou–Randers frame

An alternative frame for a $d + 1$ -dimensional pseudo-Riemannian manifold \mathcal{M} is defined as

$$ds^2 = -c^2 \left(\Omega dt - b_i dx^i \right)^2 + a_{ij} dx^i dx^j, \quad (2.79)$$

where all functions are x -dependent — again $x \equiv (x^0 = ct, \mathbf{x})$. It is known as *Papapetrou-Randers*, and this form is stable under *Carrollian diffeomorphisms*

$$t' = t'(t, \mathbf{x}) \quad \text{and} \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \quad (2.80)$$

with Jacobian

$$J(t, \mathbf{x}) = \frac{\partial t'}{\partial t}, \quad j_i(t, \mathbf{x}) = \frac{\partial t'}{\partial x^i}, \quad J_j^i(\mathbf{x}) = \frac{\partial x'^i}{\partial x^j}. \quad (2.81)$$

Under Carrollian diffeomorphisms, Ω and the metric transform as in (2.49) i.e.

$$\Omega' = \frac{\Omega}{J}, \quad a'^{ij} = J_k^i J_l^j a^{kl}, \quad (2.82)$$

while b^i obeys a connection transformation

$$b'_k = \left(b_i + \frac{\Omega}{J} j_i \right) J^{-1}{}^i_k. \quad (2.83)$$

The Papapetrou-Randers frame realizes a reduction with respect to the Carrollian diffeomorphisms (2.80). Any tensor component with a lower time index transforms as a Carrollian density and provides a scalar upon division by Ω ; the components with upper spatial indices transform as Carrollian tensors. The $d + 1$ vector components u^μ transform under a Carrollian diffeomorphism as

$$u'_0 = \frac{u_0}{J}, \quad u'^i = u^k J_k^i. \quad (2.84)$$

One can again express the components of a velocity field normalized to $-c^2$ as $u^0 = \gamma c$ and $u^i = \gamma v^i$. It is furthermore convenient to parameterize v^i as

$$v^i = \frac{c^2 \Omega \beta^i}{1 + c^2 \beta^j b_j} \Leftrightarrow \beta^i = \frac{v^i}{c^2 \Omega \left(1 - \frac{v^j b_j}{\Omega} \right)}, \quad (2.85)$$

because of future use in the Carrollian limit, and due to the simple Carrollian transformation property this definition leads to²³

$$\beta'^i = J_j^i \beta^j. \quad (2.86)$$

Now the Lorentz factor reads:

$$\gamma = \frac{1 + c^2 \boldsymbol{\beta} \cdot \mathbf{b}}{\Omega \sqrt{1 - c^2 \boldsymbol{\beta}^2}}. \quad (2.87)$$

In Papapetrou-Randers frame (2.79), the fundamental hydrodynamic variables are naturally chosen as j^i , q^j and τ^{kl} . Using the transversality conditions (2.5) we find:

$$j_0 = -c \Omega \beta_i j^i, \quad q_0 = -c \Omega \beta_i q^i, \quad \tau_{00} = c^2 \Omega^2 \beta_k \beta_l \tau^{kl}, \quad \tau^i_0 = -c \Omega \beta_k \tau^{ik}. \quad (2.88)$$

These are all Carrollian tensors (or densities).

²³This is easily proven by observing that $\beta_i + b_i = -\frac{\Omega u_i}{c u_0}$. We define as usual $b^i = a^{ij} b_j$, $\beta_i = a_{ij} \beta^j$, $v_i = a_{ij} v^j$, $\mathbf{b}^2 = b_i b^i$, $\boldsymbol{\beta}^2 = \beta_i \beta^i$ and $\mathbf{b} \cdot \boldsymbol{\beta} = b_i \beta^i$.

Hydrodynamic-frame transformations and invariants

Following the same pattern as for the Zermelo frame, we investigate the hydrodynamic-frame transformations, i.e. local Lorentz transformations captured here in the d components $\delta\beta^i(x)$. We obtain

$$\delta u^0 = c\delta\gamma, \quad \delta u^i = c^2 \frac{h^{ik}\delta\beta_k}{\sqrt{1-c^2\boldsymbol{\beta}^2}} \quad (2.89)$$

with

$$\delta\gamma = c^2\gamma\delta\beta^i \left(\frac{b_i}{1+c^2\boldsymbol{\beta}\cdot\mathbf{b}} + \frac{\beta_i}{1-c^2\boldsymbol{\beta}^2} \right), \quad (2.90)$$

and

$$h^{ik} = a^{ik} + \frac{c^2\beta^i\beta^k}{1-c^2\boldsymbol{\beta}^2}. \quad (2.91)$$

Using the general transformation rules (2.29), (2.30) and (2.31) together with (2.88) and (2.89) we find (w is the relativistic enthalpy (A.3)):²⁴

$$\delta\varepsilon = -2 \frac{q^i\delta\beta_i}{\sqrt{1-c^2\boldsymbol{\beta}^2}}, \quad (2.92)$$

$$\delta q^i = \frac{c^2\delta\beta_k}{\sqrt{1-c^2\boldsymbol{\beta}^2}} \left(\frac{q^k\beta^i}{\sqrt{1-c^2\boldsymbol{\beta}^2}} - w h^{ki} - \tau^{ki} \right), \quad (2.93)$$

$$\begin{aligned} \delta(p h^{ij} + \tau^{ij}) &= \frac{c^2\delta\beta_k}{1-c^2\boldsymbol{\beta}^2} \left(\beta^i (p h^{jk} + \tau^{jk}) + \beta^j (p h^{ik} + \tau^{ik}) \right) \\ &\quad - \frac{\delta\beta_k}{\sqrt{1-c^2\boldsymbol{\beta}^2}} (q^i h^{jk} + q^j h^{ik}). \end{aligned} \quad (2.94)$$

Similarly, using eq. (2.32), we obtain:

$$\delta\rho_0 = - \frac{j^i\delta\beta_i}{\sqrt{1-c^2\boldsymbol{\beta}^2}}, \quad (2.95)$$

and

$$\delta j^i = \frac{c^2\delta\beta_k}{\sqrt{1-c^2\boldsymbol{\beta}^2}} \left(\frac{j^k\beta^i}{\sqrt{1-c^2\boldsymbol{\beta}^2}} - \rho_0 h^{ki} \right). \quad (2.96)$$

The energy-momentum tensor is by definition invariant under hydrodynamic-frame transformations. This invariant can be nicely tamed in three canonical objects, which are the energy density ε_r , the heat current q_r^i and the stress tensor τ_r^{ij} , in the fluid proper hydrodynamic frame:²⁵

$$T_{00} = \varepsilon_r \Omega^2, \quad T_0^i = -\frac{\Omega}{c} q_r^i, \quad T^{ij} = p_r a^{ij} + \tau_r^{ij} \quad (2.97)$$

with trace

$$T_\mu{}^\mu = -\varepsilon_r + dp_r + a_{ij}\tau_r^{ij}, \quad (2.98)$$

²⁴Notice in passing $q_\mu\delta u^\mu = c^2 \frac{q^i\delta\beta_i}{\sqrt{1-c^2\boldsymbol{\beta}^2}}$.

²⁵In section 4.2 this frame is referred to as “fiducial” and is associated with an observer moving at velocity u_{PR} given in (4.72).

and

$$J_0 = -c\Omega\varrho_{0r}, \quad J^i = j_r^i. \quad (2.99)$$

It is easy to relate these *invariants* to the fluid data in an arbitrary frame encoded in β^i . We find

$$\varepsilon_r = \frac{\varepsilon}{1 - c^2\beta^2} + \frac{2\beta_i q^i}{\sqrt{1 - c^2\beta^2}} + c^2\beta_i\beta_j (ph^{ij} + \tau^{ij}), \quad (2.100)$$

$$q_r^i = \frac{c^2\varepsilon\beta^i}{1 - c^2\beta^2} + \frac{q^j}{\sqrt{1 - c^2\beta^2}} (\delta_j^i + c^2\beta^i\beta_j) + c^2\beta_j (ph^{ij} + \tau^{ij}), \quad (2.101)$$

$$p_r a^{ij} + \tau_r^{ij} = \frac{c^2\varepsilon\beta^i\beta^j}{1 - c^2\beta^2} + \frac{\beta^i q^j + \beta^j q^i}{\sqrt{1 - c^2\beta^2}} + ph^{ij} + \tau^{ij}, \quad (2.102)$$

and similarly

$$\varrho_{0r} = \frac{\varrho_0}{\sqrt{1 - c^2\beta^2}} + \beta_i j^i, \quad (2.103)$$

$$j_r^i = j^i + \frac{c^2\varrho_0\beta^i}{\sqrt{1 - c^2\beta^2}}. \quad (2.104)$$

Killings and conserved currents

Consider a Killing field of the generic form (2.71), satisfying (2.12) on \mathcal{M} in Papapetrou-Randers coordinates, where the convenient frame and coframe are now²⁶

$$e_{\hat{t}} = \frac{1}{\Omega}\partial_t, \quad e_i = \partial_i + \frac{b_i}{\Omega}\partial_t, \quad \theta^{\hat{t}} = \Omega dt - b_i dx^i, \quad \theta^{\hat{i}} = dx^i, \quad (2.105)$$

so that the metric (2.79) becomes (2.73). The Killing components are

$$\xi^{\hat{t}} = \xi^t\Omega - \xi^i b_i, \quad \xi^{\hat{i}} = \xi^i, \quad \xi_{\hat{t}} = -c^2\xi^{\hat{t}}, \quad \xi_i = a_{ij}\xi^{\hat{j}} = \xi_i + \xi_{\hat{t}}b_i, \quad (2.106)$$

and those of the conserved current (2.13), following the familiar decomposition procedure in a proper frame:

$$I_0 = -c\Omega\iota_{0r}, \quad I^i = i_r^i, \quad (2.107)$$

where

$$\iota_{0r} = \frac{1}{c^2}\xi_i q_r^i - \xi^{\hat{t}}\varepsilon_r, \quad (2.108)$$

$$i_r^i = \xi_j (p_r a^{ij} + \tau_r^{ij}) - \xi^{\hat{t}}q_r^i. \quad (2.109)$$

Using (2.16), one can express the conserved charge in the Papapetrou-Randers frame as follows:

$$Q_I = \int_{\Sigma_d} \sqrt{a} \iota_{0r} dx^1 \wedge \dots \wedge dx^d - \int_{\Sigma_d} \sqrt{a} \sum_{i=1}^d dx^1 \wedge \dots \wedge i_r^i (\Omega dt - b_j dx^j) \wedge \dots \wedge dx^d, \quad (2.110)$$

where in the exterior product of the second term, $i_r^i (\Omega dt - b_j dx^j)$ is the i th factor.

²⁶Later on e_i will be alternatively displayed as $\hat{\partial}_i$.

3 Galilean fluid dynamics

3.1 Newton-Cartan manifolds and general Galilean covariance

Newton–Cartan in a nutshell

The natural geometric framework for describing non-relativistic fluids is torsionless Newton-Cartan — see e.g. [37–44] for a comprehensive presentation and further reading suggestions. Newton-Cartan manifolds are members of a wide web including Bargmann spaces or Leibnizian structures (see [34, 39, 40]). We will here consider a manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ equipped with coordinates (t, \mathbf{x}) and a degenerate cometric²⁷

$$\partial_a^2 = a^{ij} \partial_i \partial_j, \quad i, j \dots \in \{1, \dots, d\}, \quad (3.1)$$

as well as a *clock form*

$$\theta^{\hat{t}} = \Omega dt. \quad (3.2)$$

The dual vector of the latter, referred to as a *field of observers*, is

$$e_{\hat{t}} = \frac{1}{\Omega} (\partial_t + w^j \partial_j). \quad (3.3)$$

Here, a^{ij} and w^i are general functions of (t, \mathbf{x}) whereas $\Omega = \Omega(t)$. This last feature makes the clock form $\theta^{\hat{t}}$ in (3.2) *exact* and this qualifies for the torsionless nature of the Newton-Cartan manifold. As pointed out in section 2.2, this guarantees the existence of an absolute time $\int dt \Omega(t) = \int dt' \Omega'(t')$, invariant under Galilean diffeomorphisms (2.47) — the only allowed now. For completeness, one should emphasize that even in general, torsionfull Newton-Cartan spacetimes the time interval is invariant but depends on the location \mathbf{x} of the clock: $\int dt \Omega(t, \mathbf{x})$. One still — abusively — call it absolute as a way to stress that the differences are due to the location of the clock and not directly to its motion, if any. Motion affects directly the measurements of spatial distances.

The submanifold \mathcal{S} plays the role of d -dimensional Newtonian space, endowed with a positive-definite metric, inverse of a^{ij}

$$d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i dx^j, \quad (3.4)$$

and observed from a frame with respect to which the locally inertial frame has velocity $\mathbf{w} = w^i \partial_i$ (see footnote 39). A moving particle or a fluid cell will have velocity $\mathbf{v} = v^i \partial_i$ with $v^i = \frac{dx^i}{dt}$. Under Galilean diffeomorphisms (2.47) with Jacobian (2.48), the transformation rules are as in (2.49), (2.53), and

$$\partial'_t = \frac{1}{J} (\partial_t - j^k J^{-1k} \partial_k), \quad (3.5)$$

$$\partial'_j = J^{-1j} \partial_j. \quad (3.6)$$

The clock form and the field of observers remain invariant:

$$\theta^{\hat{t}'} = \theta^{\hat{t}}, \quad e_{\hat{t}'} = e_{\hat{t}}. \quad (3.7)$$

²⁷We systematically omit the tensor product symbol \otimes in the metric and in the cometric.

Galilean tensors carry only spatial indices $i, j, \dots \in \{1, \dots, d\}$, which are lowered and raised with a_{ij} and a^{ij} . They transform covariantly under Galilean diffeomorphisms (2.47) with Jacobian J_i^j and J^{-1j}_i defined in (2.48).²⁸ Tensors depend generically on time t and space \mathbf{x} . Connections can be defined on the geometries at hand, which lead to space and time derivatives, covariant with respect to Galilean diffeomorphisms. These are not uniquely defined (see the literature quoted above) as are torsion-free and metric-compatible connections in Riemannian geometries. We will here make the specific choice, which naturally emerges when the present geometry is reached as an infinite- c limit of a pseudo-Riemannian manifold in the Zermelo frame (2.46) (see appendix A.1 of ref. [29]). This choice makes a sharp separation between space materialized in \mathcal{S} and time. Our spatial connection is

$$\gamma_{jk}^i = \frac{a^{il}}{2} (\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}). \quad (3.8)$$

The associated covariant derivative is spelled $\hat{\nabla}_i$, as opposed to ∇_i , the spatial component of the Levi-Civita covariant derivative ∇_μ defined on the ascendent pseudo-Riemannian spacetime.²⁹ This connection is torsionless

$$\hat{t}^k_{ij} = 2\gamma^k_{[ij]} = 0, \quad (3.9)$$

and metric-compatible

$$\hat{\nabla}_i a_{jk} = 0. \quad (3.10)$$

Its Riemann, Ricci and scalar curvature tensors are defined as usual d -dimensional Levi-Civita curvature tensors would be on \mathcal{S} , except that they are t -dependent:

$$[\hat{\nabla}_k, \hat{\nabla}_l] V^i = (\partial_k \gamma^i_{lj} - \partial_l \gamma^i_{kj} + \gamma^i_{km} \gamma^m_{lj} - \gamma^i_{lm} \gamma^m_{kj}) V^j = \hat{r}^i_{jkl} V^j. \quad (3.11)$$

It is worth stressing that Galilean tensors can be constructed from an object which is not a vector but rather transforming like a connection,

$$A'^k = \frac{1}{J} (J_i^k A^i + j^k). \quad (3.12)$$

Indeed

$$\frac{1}{\Omega} \hat{\nabla}^{(k} A^{l)} - \frac{1}{2\Omega} \partial_t a^{kl} = -\frac{1}{2\Omega} (\mathcal{L}_{\mathbf{A}} a^{kl} + \partial_t a^{kl}) \quad (3.13)$$

($\mathcal{L}_{\mathbf{A}}$ is the Lie derivative along $\mathbf{A} = A^i \partial_i$) and

$$\frac{1}{\Omega} \hat{\nabla}_{(k} A_{l)} + \frac{1}{2\Omega} \partial_t a_{kl} = \frac{1}{2\Omega} (\mathcal{L}_{\mathbf{A}} a_{kl} + \partial_t a_{kl}) \quad (3.14)$$

have tensorial transformation rules, and their trace is a scalar.³⁰ We can apply this to \mathbf{w} or \mathbf{v} (see (2.49) and (2.53)) and define

$$\hat{\gamma}^w_{ij} = \frac{1}{\Omega} \left(\hat{\nabla}_{(i} w_{j)} + \frac{1}{2} \partial_t a_{ij} \right), \quad \hat{\gamma}^v_{ij} = \frac{1}{\Omega} \left(\hat{\nabla}_{(i} v_{j)} + \frac{1}{2} \partial_t a_{ij} \right), \quad (3.15)$$

²⁸For a vector e.g. the transformation is $V'^k = J_i^k V^i$.

²⁹In [29] the hat was not used in the Galilean covariant derivative, and this might have caused confusion.

³⁰Observe that neither $\frac{1}{\Omega} \partial_t$ nor $\frac{1}{\Omega} \mathcal{L}_{\mathbf{A}}$ acting on Galilean tensors give separately tensors because of (3.5) and A^i transforming as (3.12).

where the upper indices refer to the vectors \mathbf{w} and \mathbf{v} , corresponding to the geometry and fluid respectively. The former is purely geometrical (and emerges in the large- c expansion of the relativistic-spacetime Levi-Civita connection in Zermelo frame³¹); the latter is associated with a fluid of velocity v^i . They coincide for a fluid at rest in the locally inertial frame, i.e. for $v^i = w^i$. From these tensors, one defines their traceless relatives and the traces: the *geometric Galilean shear*

$$\xi^w_{ij} = \frac{1}{\Omega} \left(\hat{\nabla}_{(i} w_{j)} + \frac{1}{2} \partial_t a_{ij} \right) - \frac{1}{d} a_{ij} \theta^w, \quad (3.16)$$

and the *geometric Galilean expansion*

$$\theta^w = \frac{1}{\Omega} \left(\partial_t \ln \sqrt{a} + \hat{\nabla}_i w^i \right), \quad (3.17)$$

as well as the *fluid Galilean shear*

$$\xi^v_{ij} = \frac{1}{\Omega} \left(\hat{\nabla}_{(i} v_{j)} + \frac{1}{2} \partial_t a_{ij} \right) - \frac{1}{d} a_{ij} \theta^v, \quad (3.18)$$

and the *fluid Galilean expansion*

$$\theta^v = \frac{1}{\Omega} \left(\partial_t \ln \sqrt{a} + \hat{\nabla}_i v^i \right). \quad (3.19)$$

One similarly defines a time, metric-compatible covariant derivative (again emerging in the Galilean expansion of the spacetime Levi-Civita covariant derivative in the time direction of a Zermelo frame). For a scalar function Φ it is simply

$$\frac{1}{\Omega} \frac{\hat{D}\Phi}{dt} = e_i(\Phi) = \frac{1}{\Omega} \partial_t \Phi + \frac{w^j}{\Omega} \partial_j \Phi, \quad (3.20)$$

whereas for vectors one finds

$$\begin{aligned} \frac{1}{\Omega} \frac{\hat{D}V^i}{dt} &= \frac{1}{\Omega} \partial_t V^i + \frac{w^j}{\Omega} \partial_j V^i - V^j \partial_j \frac{w^i}{\Omega} + \hat{\gamma}^{wi}_j V^j \\ &= \frac{1}{\Omega} \left(\partial_t V^i + \mathcal{L}_{\mathbf{w}} V^i \right) + \hat{\gamma}^{wi}_j V^j. \end{aligned} \quad (3.21)$$

More generally, the Leibniz rule leads to

$$\frac{1}{\Omega} \frac{\hat{D}K^{i\dots}_{j\dots}}{dt} = \frac{1}{\Omega} \left(\partial_t K^{i\dots}_{j\dots} + \mathcal{L}_{\mathbf{w}} K^{i\dots}_{j\dots} \right) + \hat{\gamma}^{wi}_k K^{k\dots}_{j\dots} + \dots - \hat{\gamma}^{wk}_j K^{i\dots}_{k\dots} - \dots, \quad (3.22)$$

and as a consequence

$$\frac{1}{\Omega} \frac{\hat{D}a^{ij}}{dt} = \frac{1}{\Omega} \frac{\hat{D}a_{ij}}{dt} = 0. \quad (3.23)$$

³¹As a general comment, valid both in the present section on Galilean dynamics as well as in the next on Carrollian, the c -dependence of our relativistic metrics is always explicit and in line with the Galilean (or Carrollian) reduction. Hence, every term in the power expansions is Galilean-covariant (or Carrollian-covariant).

In the presence of a fluid one can also introduce the more physical *material derivative*

$$\frac{d}{dt} = \partial_t + v^i \hat{\nabla}_i, \quad (3.24)$$

which produces a scalar density (or a scalar upon division by Ω) when acting on a scalar function. When acting on arbitrary tensors, it should be supplemented with appropriate \mathbf{w} and/or \mathbf{v} terms in order to maintain the tensorial transformation properties. Several options exist and we here quote the most physical (see [29]):³²

$$\frac{1}{\Omega} \frac{DV^i}{dt} = \frac{1}{\Omega} \frac{dV^i}{dt} - \frac{1}{\Omega} V^j \hat{\nabla}_j w^i, \quad \frac{1}{\Omega} \frac{DV_i}{dt} = \frac{1}{\Omega} \frac{dV_i}{dt} + \frac{1}{\Omega} V_j \hat{\nabla}_i w^j, \quad (3.25)$$

resulting in genuine tensors under Galilean diffeomorphisms. As opposed to (3.22), this time-covariant derivative is not metric compatible:

$$\frac{1}{\Omega} \frac{Da_{ij}}{dt} = 2\hat{\gamma}^w_{ij}. \quad (3.26)$$

Space and time Galilean covariant derivatives do not commute. They define a Galilean tensor, rooted in the Riemann tensor of the ascendent relativistic spacetime at finite velocity of light. We find

$$\left[\frac{1}{\Omega} \frac{\hat{D}}{dt}, \hat{\nabla}_i \right] \Phi = -\hat{\gamma}^{wk}_i \partial_k \Phi, \quad (3.27)$$

$$\left[\frac{1}{\Omega} \frac{\hat{D}}{dt}, \hat{\nabla}_i \right] V^j = -\hat{\gamma}^{wk}_i \hat{\nabla}_k V^j + \hat{\gamma}^j_{ik} V^k, \quad (3.28)$$

and similarly for higher-rank Galilean tensors, where

$$\hat{\gamma}^j_{ik} = \frac{1}{\Omega} \left(\partial_t \gamma^j_{ik} + \hat{\nabla}_i \hat{\nabla}_k w^j - \hat{\nabla}_i \hat{\gamma}^{wj}_k + w^l \hat{\gamma}^j_{kli} \right). \quad (3.29)$$

Galilean diffeomorphisms and conservation equations

Without referring specifically to a fluid, one may consider an effective action describing the dynamics of a system defined on the geometry $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ discussed previously. This effective action is thus a functional of a^{ij} , Ω and w^i : $S = \int dt d^d x \sqrt{a} \Omega \mathcal{L}$. The standard relativistic energy-momentum tensor (2.7) is now traded for the following *Galilean momenta*, namely the *energy-stress tensor*, the *momentum* and the *energy density*:

$$\Pi_{ij} = -\frac{2}{\sqrt{a} \Omega} \frac{\delta S}{\delta a^{ij}}, \quad (3.30)$$

$$P_i = -\frac{1}{\sqrt{a} \Omega} \frac{\delta S}{\delta w^i}, \quad (3.31)$$

$$\Pi = -\frac{1}{\sqrt{a} \Omega} \left(\Omega \frac{\delta S}{\delta \Omega} - \frac{w^i}{\Omega} \frac{\delta S}{\delta w^i} \right), \quad (3.32)$$

³²For a detailed and general presentation of Galilean affine connections see [39, 40].

which can likewise be combined as $\frac{\delta S}{\delta \Omega} = -\sqrt{a} \left(\Pi + \frac{w^i}{\Omega} P_i \right)$. These momenta are summarized in the following variation — at least for the gravitational sector:

$$\delta S = - \int dt \Omega \int d^d x \sqrt{a} \left(\frac{1}{2} \Pi_{ij} \delta a^{ij} + P_i \delta \frac{w^i}{\Omega} + \left(\Pi + \frac{w^i}{\Omega} P_i \right) \delta \ln \Omega \right). \quad (3.33)$$

The above momenta obey conservation equations as a consequence of the assumed invariance of the action under Galilean diffeomorphisms, which simultaneously guarantees their Galilean-covariant transformation rules. Galilean diffeomorphisms (2.47) are generated by vector fields on \mathcal{M} whose time component depends only on t :

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \xi^{\hat{t}} e_{\hat{t}} + \xi^{\hat{i}} \partial_i \quad (3.34)$$

(this is the same expression as (2.71)), where $\xi^{\hat{t}}(t) = \xi^t(t)\Omega(t)$ is a Galilean scalar, and $\xi^{\hat{i}}(t, \mathbf{x}) = \xi^i - \xi^t w^i$ are the components of a Galilean vector. The variation under diffeomorphisms is implemented through the Lie derivative (the minus sign is conventional):

$$- \delta_{\xi} a^{ij} = \mathcal{L}_{\xi} a^{ij} = -2 \left(\hat{\nabla}^{(i} \xi^{\hat{j})} + \hat{\gamma}^{w_{ij}} \xi^{\hat{t}} + \frac{1}{\Omega} w^{(i} a^{j)k} \partial_k \xi^{\hat{t}} \right), \quad (3.35)$$

where the last term drops for Galilean diffeomorphisms. Furthermore

$$\mathcal{L}_{\xi} e_{\hat{t}} = -\frac{1}{\Omega} \left(\partial_t \xi^{\hat{t}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{t}} \right) e_{\hat{t}} - \frac{1}{\Omega} \left(\partial_t \xi^{\hat{i}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{i}} \right) \partial_i, \quad (3.36)$$

from which, using (3.3), we infer

$$- \delta_{\xi} \Omega = \mathcal{L}_{\xi} \Omega = \partial_t \xi^{\hat{t}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{t}}, \quad \delta_{\xi} w^i = -\mathcal{L}_{\xi} w^i = \partial_t \xi^{\hat{i}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{i}}. \quad (3.37)$$

Notice also the action on the clock form:

$$\mathcal{L}_{\xi} \theta^{\hat{t}} = \frac{1}{\Omega} \left(\partial_t \xi^{\hat{t}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{t}} \right) \theta^{\hat{t}} = \frac{1}{\Omega} \frac{\hat{D} \xi^{\hat{t}}}{dt} \theta^{\hat{t}} = \mu \theta^{\hat{t}}, \quad (3.38)$$

where we introduced

$$\mu(t, \mathbf{x}) = \frac{1}{\Omega} \frac{\hat{D} \xi^{\hat{t}}}{dt} \quad (3.39)$$

not to be confused with the chemical potential introduced in thermodynamics.

We can now determine the variation of the action under Galilean diffeomorphisms:

$$\begin{aligned} \delta_{\xi} S = & \int dt d^d x \sqrt{a} \Omega \left\{ -\xi^{\hat{t}} \left[\frac{1}{\Omega} \frac{\hat{D} \Pi}{dt} + \theta^w \Pi + \Pi_{ij} \hat{\gamma}^{w_{ij}} \right] \right. \\ & \left. + \xi^{\hat{i}} \left[\frac{1}{\Omega} \frac{\hat{D} P_i}{dt} + \theta^w P_i + P_j \hat{\gamma}^{w_j}{}_i + \hat{\nabla}^j \Pi_{ij} \right] \right\} \\ & + \int dt d^d x \left\{ \partial_t \left(\sqrt{a} \left(\Pi \xi^{\hat{t}} - P_j \xi^{\hat{j}} \right) \right) \right. \\ & \left. + \partial_i \left(\sqrt{a} w^i \left(\Pi \xi^{\hat{t}} - P_j \xi^{\hat{j}} \right) - \sqrt{a} \Omega \Pi^i{}_j \xi^{\hat{j}} \right) \right\}. \quad (3.40) \end{aligned}$$

Requiring that $\delta_\xi S$ vanishes and ignoring the boundary terms (last two lines in eq. (3.40)), we reach two equations. The momentum equation is the simplest because ξ^j being functions of both t and \mathbf{x} , their factor must vanish:

$$\boxed{\left(\frac{1}{\Omega} \hat{D} + \theta^w\right) P_i + P_j \hat{\gamma}^{wj} + \hat{\nabla}^j \Pi_{ij} = 0.} \quad (3.41)$$

The energy equation is more subtle because ξ^t depends on t only. As a consequence it is enough to require that its factor be the Galilean divergence of a vector:

$$\boxed{\left(\frac{1}{\Omega} \hat{D} + \theta^w\right) \Pi + \Pi_{ij} \hat{\gamma}^{wij} = -\hat{\nabla}_i \Pi^i,} \quad (3.42)$$

where Π^i is undetermined a priori. Indeed, $\sqrt{a} \Omega \xi^t \hat{\nabla}_i \Pi^i = \partial_i (\sqrt{a} \Omega \xi^t \Pi^i)$, which leads to a boundary term and vanishes inside the integral. One can interpret Π^i as the *energy current* (also *energy flux*).

Gauge invariance and matter conservation

Besides Galilean covariance, the action might also be invariant under a local U(1) symmetry, parameterized by $\Lambda(t, \mathbf{x})$ and acting on the components of a gauge field $B = B(t, \mathbf{x})dt + B_i(t, \mathbf{x})dx^i$ as

$$\delta_\Lambda B_i = -\partial_i \Lambda, \quad \delta_\Lambda B = -\partial_t \Lambda. \quad (3.43)$$

The conjugate momenta are now the *matter density* and the *matter current*:

$$\varrho = -\frac{1}{\sqrt{a}} \frac{\delta S}{\delta B}, \quad (3.44)$$

$$N^i = \frac{1}{\Omega \sqrt{a}} \left(w^i \frac{\delta S}{\delta B} - \frac{\delta S}{\delta B_i} \right) \quad (3.45)$$

with $\frac{\delta S}{\delta B_i} = -\sqrt{a} (\Omega N^i + \varrho w^i)$, and

$$\delta S = - \int dt d^d x \sqrt{a} \left(\varrho \delta B + (\Omega N^i + \varrho w^i) \delta B_i \right) \quad (3.46)$$

for the matter sector. The gauge variation of the action reads:

$$\begin{aligned} \delta_\Lambda S &= \int dt d^d x \sqrt{a} \left(\varrho \partial_t \Lambda + (\Omega N^i + \varrho w^i) \partial_i \Lambda \right) \\ &= - \int dt d^d x \sqrt{a} \Omega \Lambda \left(\frac{1}{\Omega} \hat{D} \varrho + \theta^w \varrho + \hat{\nabla}_i N^i \right) \\ &\quad + \int dt d^d x \left\{ \partial_t (\sqrt{a} \Lambda \varrho) + \partial_i (\sqrt{a} \Lambda (\Omega N^i + \varrho w^i)) \right\}. \end{aligned} \quad (3.47)$$

Invariance of S leads to the Galilean continuity equation:

$$\boxed{\left(\frac{1}{\Omega} \hat{D} + \theta^w\right) \varrho + \hat{\nabla}_i N^i = 0.} \quad (3.48)$$

The continuity equation can be alternatively presented in an integral form, using Stokes and Gauss theorems:³³

$$\begin{aligned} & \int_{\mathcal{W}} dt d^d x \Omega \sqrt{a} \left(\left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w \right) \varrho + \hat{\nabla}_i N^i \right) = \\ & \oint_{\partial \mathcal{W}} \sqrt{a} \varrho \left(dx^1 - w^1 dt \right) \wedge \dots \wedge \left(dx^d - w^d dt \right) \\ & - \oint_{\partial \mathcal{W}} \sqrt{a} \sum_{i=1}^d \left(dx^1 - w^1 dt \right) \wedge \dots \wedge N^i \theta^i \wedge \dots \wedge \left(dx^d - w^d dt \right), \end{aligned} \quad (3.49)$$

where $\mathcal{W} \subset \mathcal{M} = \mathbb{R} \times \mathcal{S}$ and $N^i \theta^i$ is the i th factor in the exterior product of the last term (θ^i is the clock form given in (3.2)). From this we obtain a conserved charge — under the usual assumptions for the behaviour of the fields — expressed as an integral over an arbitrary hypersurface Σ_d of $\mathcal{M} = \mathbb{R} \times \mathcal{S}$. It coincides with the relativistic Zermelo result captured e.g. in (2.78).

Although not compulsory, it is convenient to chose $\Sigma_d \equiv \mathcal{S}$ i.e. a constant- t hypersurface. We then find

$$Q_N = \int_{\mathcal{S}} d^d x \sqrt{a} \varrho, \quad (3.50)$$

which fits the usual definition of charge in Galilean physics. In this case, the conservation of Q_N is often phrased as independence with respect to ordinary time t , although it is actually a stronger statement, even in Newton-Cartan spacetimes, where there is a privileged time direction (for the relativistic case, see the comment in footnote 9). Time-independence appears explicitly if one trades \mathcal{S} in the integral (3.50) with $\mathcal{V} \subset \mathcal{S}$. Assuming for simplicity that the boundary $\partial \mathcal{V}$ of that domain does not depend on t and using (3.48), the time evolution of the matter/charge content of \mathcal{V} is

$$\frac{1}{\Omega} \frac{d}{dt} \int_{\mathcal{V}} d^d x \sqrt{a} \varrho = - \int_{\mathcal{V}} d^d x \partial_i \left(\sqrt{a} \left(N^i + \varrho \frac{w^i}{\Omega} \right) \right) = - \int_{\partial \mathcal{V}} \star \left(\mathbf{N} + \varrho \frac{\mathbf{w}}{\Omega} \right), \quad (3.51)$$

where \star stands for the d -dimensional \mathcal{S} -Hodge dual based on \sqrt{a} and on the antisymmetric symbol $\epsilon_{i_1 \dots i_d}$ with $\epsilon_{1 \dots d} = 1$. If the integral is performed over the entire \mathcal{S} it vanishes (assuming a reasonable asymptotic behaviour), and Q_N in (3.50) is conserved.

Equation (3.48) and its variables are a priori independent of the energy-momentum equations (3.41), (3.42) and their variables. As we will see, thermodynamics sets a relationship among the momentum P_i and the current N_i .

Isometries, conservation and non-conservation laws

In (pseudo-)Riemannian geometry, isometries are diffeomorphisms generated by vectors leaving the metric invariant, i.e. requiring (2.12).³⁴ Newton-Cartan spacetimes may also have

³³Stokes theorem is valid irrespective of the metric. Gauss' requires a dual exterior derivative d^\dagger , which can be introduced consistently despite the cometric being degenerate. We will not elaborate on this matter here. For the Carrollian case, this was discussed for $d = 1$ in ref. [26].

³⁴For historical reasons, some authors use the name “Killing fields” for generators of isometries in a (pseudo-)Riemannian manifold exclusively. We take the freedom here to call every isometry generator a Killing field, be it for weak or strong, Newton-Cartan or Carrollian manifolds.

isometries. Due to the degenerate cometric, however, their definition and the determination of the Killing fields are more subtle. This subject has been abundantly discussed in the literature. We will summarize the features important for our purpose regarding the Galilean fluid dynamics.

The Killing fields are of the Galilean type (3.34) and are required to obey

$$\mathcal{L}_\xi a^{ij} = 0, \quad \mathcal{L}_\xi \theta^{\hat{t}} = 0, \tag{3.52}$$

since the fundamental geometric data are the cometric (3.1) and the clock form (3.2). Using expressions (3.35) and (3.38) for Galilean diffeomorphisms ($\xi^{\hat{t}}$ is only t -dependent), we obtain the Galilean Killing equations:

$$\hat{\nabla}^{(i} \xi^{\hat{j})} + \hat{\gamma}^{wij} \xi^{\hat{t}} = 0, \quad \frac{1}{\Omega} \frac{\hat{D} \xi^{\hat{t}}}{dt} = 0. \tag{3.53}$$

These equations generally admit an infinite number of solutions. The reason is that they refer to the *weak definition* of Newton-Cartan spacetimes [36] in terms of the cometric and the clock form, and express exclusively the invariance of these data. A *strong definition* exists and requires additionally a symmetric affine connection, which is metric-compatible and parallel-transport the clock form.³⁵ Isometries are thus restricted to comply with the strong definition, and leave the affine connection invariant. This reduces the set of generators to a finite number [87].

Notice also that the fundamental geometric piece of data for a Newton-Cartan spacetime is genuinely the clock form $\theta^{\hat{t}}$ rather than the field of observers $e_{\hat{t}}$. The latter is not required to have a vanishing Lie derivative along Killing vectors, and using (3.36) and (3.53) we indeed find

$$\mathcal{L}_\xi e_{\hat{t}} = -\frac{1}{\Omega} \left(\partial_t \xi^{\hat{t}} + \mathcal{L}_w \xi^{\hat{t}} \right) \partial_i, \tag{3.54}$$

for a generic Killing field ξ .

Consider as an example the Newton-Cartan manifold with $a^{ij} = \delta^{ij}$, $\Omega = 1$ and w^i constant. This is our familiar $\mathbb{R} \times \mathbb{E}_3$ spacetime, which is flat for the connection introduced earlier. Equations (3.53) possess an infinite number of solutions:

$$\xi = \left(\Omega_i^j(t) x^i + Z^j(t) \right) \partial_j + T \partial_t \tag{3.55}$$

with T a constant and $\Omega_{ij} = \Omega_i^k \delta_{kj}$ antisymmetric. Imposing the invariance of the affine connection, one recovers [87] the $(d+2)(d+1)/2$ -dimensional Galilean algebra $\mathfrak{gal}(d+1)$ with constant Ω_{ij} generating the $\mathfrak{so}(d)$ rotations, $Z^j(t) = V^j t + X^j$ for the Galilean boosts and spatial translations, and T for the time translations. We find in particular that

$$\mathcal{L}_\xi e_{\hat{t}} = - \left(V^i + w^k \Omega_k^i \right) \partial_i \neq 0, \tag{3.56}$$

showing among others that the boosts produce a displacement in the field of observers. This is expected because w^i describes the constant velocity of the original inertial frame, which

³⁵It turns out that the connection we have introduced in this section obeys these properties.

is shifted during a Galilean boost, and nicely illustrates why it would have been unnatural to impose the invariance of $e_{\hat{t}}$ under the action of an isometry.

Assuming the existence of an isometry, we can now address the conservation law that would take the Galilean form (3.48) with a Galilean scalar κ and a Galilean vector K^i determined from the Killing components $\xi^{\hat{t}}$ and $\xi^{\hat{i}}$, and from the Galilean momenta, i.e. the energy density Π , the momentum P_i and the energy-stress tensor Π_{ij} defined in eqs. (3.30), (3.31), (3.32), as well as the energy flux Π_i , and satisfying the conservation equations (3.41) and (3.42). The Galilean scalar

$$\mathcal{K} = \left(\frac{1}{\Omega} \hat{D} + \theta^w \right) \kappa + \hat{\nabla}_i K^i \tag{3.57}$$

would then vanish on-shell. The components of the Galilean current κ and K^i are read off in the boundary terms of $\delta_{\xi} S$ given in (3.40) and set on-shell:³⁶

$$\kappa = \xi^{\hat{i}} P_i - \xi^{\hat{t}} \Pi, \tag{3.58}$$

$$K_i = \xi^{\hat{j}} \Pi_{ij} - \xi^{\hat{t}} \Pi_i. \tag{3.59}$$

Using the conservation equations (3.41) and (3.42) we obtain the following result:

$$\mathcal{K} = -\frac{\Pi \hat{D} \xi^{\hat{t}}}{\Omega} + \frac{P_i}{\Omega} \left(\partial_t \xi^{\hat{i}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{i}} \right) + \Pi_{ij} \left(\hat{\nabla}^i \xi^{\hat{j}} + \hat{\gamma}^{w_{ij}} \xi^{\hat{t}} \right) \tag{3.60}$$

$$= \frac{P_i}{\Omega} \left(\partial_t \xi^{\hat{i}} + \mathcal{L}_{\mathbf{w}} \xi^{\hat{i}} \right). \tag{3.61}$$

We have used the Killing equations (3.53) to reach (3.61) from (3.60), which shows that *in Newton-Cartan spacetimes, a Killing field fails to systematically support an on-shell conservation law for Galilean dynamics*. This could have been anticipated, actually. Indeed, P_i is conjugate to w^i/Ω (see (3.31)) and w^i transforms under diffeomorphisms according to (3.37), *even when this diffeomorphism is generated by a Killing vector field*. This is precisely what eq. (3.61) conveys. Still, a conservation law exists for those Killing fields, which happen to satisfy $\mathcal{L}_{\xi} e_{\hat{t}} \equiv [\xi, e_{\hat{t}}] = 0$ — using Jacobi identity, one checks that the commutator of two such Killings leaves also $e_{\hat{t}}$ invariant. As we mentioned earlier, ordinary Galilean boosts in $\mathbb{R} \times \mathbb{E}_3$ do not, eq. (3.56). In other instances with multiple degrees of freedom arising from a Laurent expansion in powers of c^2 (see section 3.3), several currents coexist in the presence of a Killing, and some can be conserved due to the accidental — as opposed to a priori demanded — absence of a P_i -like component.

The above result sounds iconoclastic, in view of the robustness and generality of Noether's theorems. In ref. [42], for instance, the existence of an isometry-related conserved current in torsional Newton-Cartan spacetimes has not been demonstrated, not even questioned — it was just assumed to be true. However, *it is not ipso facto*, for the simple reason that isometries are less restraining in Newton-Cartan spacetimes, so that only a

³⁶They are in fact inherited from the relativistic-current components i.e. as a large- c expansion of (2.76) and (2.77), and the precise computation is performed at the end of section 3.3, eq. (3.129).

restricted class of Galilean Killing fields might indeed be eligible for conservation laws — boosts in flat spacetime aren't.

One may naturally contemplate that ξ being a Killing field, the right-hand side of eq. (3.61) is associated with a *boundary term*. If this happens, the boundary term contributes the current components κ and K^i , leading to an effective κ' and K'^i , truly conserved. However, this does not seem to be the rule, not even in flat space, where $\mathcal{K} = P_i (V^i + w^k \Omega_k^i)$, obtained using eqs. (3.55) and (3.56).³⁷

The same conclusion will be reached for Carrollian spacetimes in section 4.1, and further discussed from the large- c perspective in the upcoming paragraphs, as well as in appendix D.

Weyl invariance, conformal isometries, conservation and non-conservation laws

Fluids involving massless excitations have observables with remarkable scaling properties. We can introduce Weyl transformations acting as follows on the fundamental geometric data of a Newton-Cartan geometry:

$$a^{ij} \rightarrow \mathcal{B}^2 a^{ij}, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega, \quad w^i \rightarrow w^i, \quad w_i \rightarrow \frac{1}{\mathcal{B}^2} w_i. \quad (3.62)$$

Since Ω is a function of t only, the second of (3.62) imposes $\mathcal{B} = \mathcal{B}(t)$. Weyl-invariance requirement of an effective action S leads to following weights for the Galilean momenta in (3.30), (3.31), (3.32): the energy-stress tensor Π_{ij} has weight $d - 1$, the momentum P_i , d , and the energy density Π , $d + 1$. The energy flux Π_i introduced in (3.42) has also weight d . Furthermore, using (3.33) with $\delta_{\mathcal{B}} S = 0$ implies that

$$\Pi_i{}^i = \Pi. \quad (3.63)$$

On the matter sector, the gauge fields B and B_i are weight-zero, whereas ϱ is weight- d and N_i , $d - 1$.

Weyl covariance can be implemented with the appropriate Galilean-Weyl covariant derivatives for both time and space. These will be introduced latter in section 3.4, our aim being here to foster on the case of conformal Killing fields and their possible role in supporting conservation laws within Weyl-invariant Galilean dynamics.

Following [34, 36] a conformal isometry is generated by a vector field ξ satisfying

$$\mathcal{L}_\xi a^{ij} = \lambda a^{ij}, \quad (3.64)$$

where

$$\lambda(t, \mathbf{x}) = \frac{2}{d} \left(\hat{\nabla}_i \xi^i + \theta^w \xi^t \right). \quad (3.65)$$

³⁷For the conservation to occur in flat space, P_i should be a *potential flow* (also called *irrotational*, see [1] section 9) i.e. obey $P_i = (\partial_t + w^j \partial_j) \phi_i + \partial_i \phi$ for some set of functions $\phi(t, \mathbf{x})$ and $\phi_i(t, \mathbf{x})$ — scalar and vector potentials. Then $\mathcal{K} = (\partial_t + w^j \partial_j) \phi_i W^i + \partial_i \phi W^i$ with $W^i = V^i + w^k \Omega_k^i$ and the conservation works out ((3.57) vanishes) with current components $\kappa - \phi_i W^i$ and $K^i - \phi W^i$. Notice that in contrast to the present framework, the conservation is generally valid for free-particle motion on Newton-Cartan flat spacetimes, as in that case the momentum is a total derivative (typically \dot{x}^i). Appendix C summarizes this instance and provides the necessary details for making the statement sound.

This set of partial differential equations is not sufficient for defining conformal Killing vectors — again a consequence of the cometric degeneration. Besides, imposing a requirement similar to (3.64) on the clock form $\theta^{\hat{t}}$ does not help since according to (3.38), the Lie derivative of $\theta^{\hat{t}}$ is already proportional to $\theta^{\hat{t}}$ with factor (3.39). What is rather natural is to tune μ versus λ so that the scaling of the metric be twice that of the clock form:³⁸

$$2\mu + \lambda = 0. \tag{3.66}$$

This is a consistent Weyl-covariant condition, leading to a reasonable set of conformal Killing fields. Using the strong versus the weak definition of Newton-Cartan structures further reduces the freedom for conformal isometries, and opens the Pandora box for investigating conformal Galilean algebras in flat Newton-Cartan spacetimes: $\mathfrak{cga}(d+1)$ and their multiple variations. Discussing these in detail is beyond our scope and ample information can be found in the quoted references.

Assuming Weyl invariance i.e. (3.63), and the presence of a conformal Killing field, the conservation equations (3.41) and (3.42) can be exploited for computing \mathcal{K} defined in (3.57), (3.58) and (3.59):

$$\mathcal{K} = -\Pi \left(\frac{\lambda}{2} + \mu \right) + \frac{P_i}{\Omega} \left(\partial_t \xi^i + \mathcal{L}_{\mathbf{w}} \xi^i \right). \tag{3.67}$$

The extra condition (3.66) defining the conformal Killing vectors of Newton-Cartan spacetimes emerges naturally in the quest of conserved Galilean currents and gives $\mathcal{K} = \frac{P_i}{\Omega} \left(\partial_t \xi^i + \mathcal{L}_{\mathbf{w}} \xi^i \right) \neq 0$. As for the ordinary Killings, *a conformal Killing vector does not guarantee a conservation law for Weyl-invariant Galilean dynamics.*

3.2 Galilean hydrodynamics as a non-relativistic limit

Philosophy and large- c behaviour

Equations (3.41), (3.42) and (3.48) summarize the conservation properties of Galilean dynamics on a general, curved and time-dependent space \mathcal{S} , spatial section of a torsionless Newton-Cartan spacetime $\mathbb{R} \times \mathcal{S}$. They are general-Galilean-covariant and are a consequence of this invariance, as much as the relativistic equations (2.1) and (2.2) reflect general Riemannian covariance.

Fluid dynamics is more. Its description requires expressing the momenta in terms of the velocity field \mathbf{v} , the heat current \mathbf{Q} , the stress tensor $\mathbf{\Sigma}$, together with local-equilibrium thermodynamic variables such as e , p , h , ϱ , μ , T and s , obeying further thermodynamic laws (see appendix A), and ultimately entering the constitutive relations. A systematic approach to this programme is based on the large- c expansion of relativistic hydrodynamics in Zermelo frame, which is the natural pseudo-Riemannian ascendent of the present Galilean framework. This method was applied successfully in [29], where the Eckart frame was

³⁸More generally, one considers $2\mu + z\lambda = 0$, where z is the *dynamical exponent* i.e. minus the conformal weight of Ω . Here, due to the close relationship of our Newton-Cartan spacetimes with relativistic ascendents, the weight of Ω is inherited from the latter and $z = 1$. One also defines the level $N = 2/z$, which appears in the conformal algebras emerging in flat Newton-Cartan spacetimes.

implicitly assumed on the relativistic side, subsequently ignoring the role of the current and the chemical potential. We will generalize it here in order to address the issue of the non-relativistic hydrodynamic-frame invariance, which relies intimately on the behaviour of the various observables.

The dependence with respect to the velocity of light is explicit in the relativistic geometric data (see the metric (2.46)). This is our choice for making the bridge with Newton-Cartan straightforward. As a consequence, the behaviour of kinematical observables is also known — see (2.23), (2.24), (2.25), (2.51) and (2.52). We find in particular

$$u^0 = \frac{c}{\Omega} + \mathcal{O}(1/c), \quad u_i = \frac{v_i - w_i}{\Omega} + \mathcal{O}(1/c^2), \quad (3.68)$$

and

$$\sigma_{ij} = \xi^v_{ij} + \mathcal{O}(1/c^2), \quad (3.69)$$

$$\Theta = \theta^v + \mathcal{O}(1/c^2), \quad (3.70)$$

$$\omega_{ij} = \frac{1}{\Omega} \left(\partial_{[i}(v - w)_{j]} \right) + \mathcal{O}(1/c^2). \quad (3.71)$$

The non-relativistic limit of thermodynamic variables is standard and is recalled in appendix A. The precise relation of the Galilean density ϱ to the proper density ϱ_0 deserves however a comment. Indeed, the density measured by an observer is the projection of the current (2.4) onto the observer’s velocity. For instance, an observer at rest with respect to the fluid, i.e. running with velocity u , finds

$$\varrho_0 = -\frac{1}{c^2} J_\mu u^\mu. \quad (3.72)$$

The density ϱ is the one measured by a fiducial observer of Zermelo frame. Since ϱ is the “non-relativistic density,” this fiducial observer should have some “absolute” status. In Minkowski spacetime we would have simply taken the inertial observer at rest in the inertial frame at hand. In Zermelo it is natural to consider an observer with velocity $u_Z \equiv e_t$ as given in (3.3) because

$$u_Z^\mu \nabla_\mu u_Z^\nu = 0, \quad (3.73)$$

and thus this observer indeed defines a locally inertial frame.³⁹ Hence we obtain

$$\varrho = -\frac{1}{c^2} J_\mu u_Z^\mu = \frac{\Omega}{c} J^0 = \Omega \gamma \varrho_0 + j_i \frac{v^i - w^i}{c^2 \Omega}, \quad (3.74)$$

which naturally coincides with the hydrodynamic-frame invariant ϱ_{0r} introduced in (2.69). This expression agrees with ref. [29] only in the *Eckart frame*, i.e. when $j_i = 0$.

The behaviour of ϱ_0 in terms of ϱ at large c depends on the behaviour of j_i and this brings us to the heart of the discussion of the non-relativistic limit in hydrodynamics:

³⁹This is precisely why it was stated earlier that a frame with velocity $\mathbf{w} = w^i \partial_i$ was inertial in the Newton-Cartan geometry (3.3), (3.4). Notice that the property (3.73) holds because Ω is a function of time only, which is the emanation of the torsionless nature of the limiting Newton-Cartan structure, and enables to define in turn an absolute Newtonian time in this Galilean limit. This framework, where no isometry is assumed, is the closest we can go to the standard classical Newtonian physics on $\mathbb{R} \times \mathbf{E}_d$ with clock form dt .

how do the stress tensor τ_{ij} , the heat current q_i and the non-perfect current j_i behave at large c ? There is no absolute answer to these questions, it depends on the microscopic properties of the system (the interactions in particular). These properties are encapsulated in the transport coefficients appearing in the constitutive relations for τ_{ij} , q_i and j_i as derivative expansions. For instance, at first-derivative order $\tau_{(1)ij} = -2\eta\sigma_{ij} - \zeta h_{ij}\Theta$, $q_{(1)i} = -\kappa h_i^\nu \left(\partial_\nu T + \frac{T}{c^2} a_\nu \right)$ and $j_{(1)i} = -h_i^\nu \sigma_T T \partial_\nu \frac{\mu_0}{T}$ with a typical relationship among the transport coefficients: $\kappa = \frac{\sigma_T \omega^2}{T \varrho_0^2}$ (see e.g. ref. [10], where σ_T is referred to as the charge conductivity). Under the reasonable assumption that η , ζ and κ (the shear and bulk viscosities, and the heat conductivity) are of dominant order 1, σ_T is of order $1/c^4$ (due to ω^2). Since μ_0 is of order c^2 (rest-mass contribution in (A.4)) we conclude that τ_{ij} and q_i are of order 1, whereas j_i is of order $1/c^2$. For reasons that will become clear in the discussion on hydrodynamic-frame invariance, we would like to be slightly more general at this stage, and assume the following behaviour:

$$\tau_{ij} = -\Sigma_{ij} + \mathcal{O}(1/c^2), \tag{3.75}$$

$$q_l = c^2 r_l + k_l + \mathcal{O}(1/c^2), \tag{3.76}$$

$$j_l = n_l + \frac{m_l}{c^2} + \mathcal{O}(1/c^4). \tag{3.77}$$

Following the above discussion, r_l and n_l are expected to vanish for ordinary non-relativistic fluids. Their presence will disclose some interesting properties though.

Before we proceed with the Galilean fluid equations a comment should be made. Considering several distinct orders in the expansion of the relativistic data amounts to allow for a multiplication of the degrees of freedom in the Galilean limit (see the end of section 3.3). This is accompanied with further dynamical (or possibly constraint) equations. Sometimes, these degrees of freedom and their dynamics can emerge separately, by performing an appropriate c^2 rescaling in the relativistic data before taking the limit. This is how the electric versus magnetic options occur, as e.g. in refs. [34, 101].

Inserting the expression (3.77) in (3.74), we find:⁴⁰

$$\varrho_0 = \varrho - \frac{1}{c^2} \left(\frac{\varrho}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 + \frac{\mathbf{n} \cdot (\mathbf{v} - \mathbf{w})}{\Omega} \right) + \mathcal{O}(1/c^4). \tag{3.78}$$

The latter can in turn be used inside (A.2) leading to

$$\varepsilon = c^2 \varrho + \varrho \left(e - \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \right) - \frac{\mathbf{n} \cdot (\mathbf{v} - \mathbf{w})}{\Omega} + \mathcal{O}(1/c^2), \tag{3.79}$$

where the first term is the rest energy, the second is the internal energy corrected by the kinetic energy with respect to the local inertial frame, and the third is a contribution originating from the leading term in the matter current (3.77). We already foresee that this amounts to the presence of a spring or a sink that create or consume matter, and this

⁴⁰Although $\lim_{c \rightarrow \infty} \Omega \gamma = 1$, we must keep terms of order $1/c^2$ because of the rest mass contributions.

will be confirmed when reaching the conservation equation. This situation is usually not considered except when discussing diffusion or similar phenomena (see for instance [86], chapter VI).

A last comment concerns the role of (A.2), or its large- c emanation (3.79). This equation of thermodynamic nature, makes the bridge between energy-momentum and matter conservations. As we will see soon, it establishes a relationship amongst P_i and N_i respectively defined in (3.32) and (3.45) as the fluid momentum and the matter current.

Notice that if several charges and associated conserved currents are present, only one will have a privileged relationship with the energy-momentum, being therefore affiliated with mass conservation — and entering the thermodynamic relation (A.2), as mentioned in appendix A.

Galilean momenta

The Galilean momenta defined earlier in (3.30), (3.31), (3.32), (3.44) and (3.45) will now appear in the large- c expansion of the relativistic energy-momentum and current components (2.63) and (2.65) in Zermelo frame, as explicit expressions in terms of the various hydrodynamic and thermodynamic observables. This includes the energy current Π^i introduced in eq. (3.42) — and not a priori defined as a variation of the effective action with respect to some conjugate variable. As we have already emphasized, the thermodynamic laws set relationships amongst the energy-momentum and the matter.

Using eqs. (2.66), (2.67), (2.68), (2.69), (2.70) and (3.74), we obtain:

$$\varrho_{0r} = \varrho, \tag{3.80}$$

$$j_{ri} = N_i + \frac{1}{c^2} p_i + \mathcal{O}(1/c^4), \tag{3.81}$$

where we introduced the leading and subleading matter currents

$$N^i = \varrho \frac{v^i - w^i}{\Omega} + n^i, \tag{3.82}$$

$$p_i = m_i - \frac{\mathbf{n} \cdot (\mathbf{v} - \mathbf{w})(v_i - w_i)}{\Omega^2}. \tag{3.83}$$

The subleading terms must be kept because they are multiplied in the expansions by the rest-mass term and contribute the equations. Anticipating the next steps, we set

$$P^i = \varrho \frac{v^i - w^i}{\Omega} + r^i. \tag{3.84}$$

We recognize in P_i (defined generically in (3.31) — indices raised with a^{ij}) a slight extension of the usual fluid momentum, while the matter current N^i (introduced in (3.45)) is related to the former as

$$N^i = P^i + n^i - r^i. \tag{3.85}$$

The more standard equality $N^i = P^i$ occurs when $n^i = r^i$.

Similarly we find for the energy-momentum

$$\varepsilon_{\mathbf{r}} = c^2 \varrho + \Pi + \mathcal{O}(1/c^2), \quad (3.86)$$

$$q_{\mathbf{r}i} = c^2 P_i + \Pi_i + p_i + \mathcal{O}(1/c^2), \quad (3.87)$$

$$p_{\mathbf{r}} a_{ij} + \tau_{\mathbf{r}ij} = \Pi_{ij} + \mathcal{O}(1/c^2) \quad (3.88)$$

with⁴¹

$$\Pi_{ij} = \varrho \frac{(v_i - w_i)(v_j - w_j)}{\Omega^2} + p a_{ij} - \Sigma_{ij} + 2 \frac{(v_{(i} - w_{(i)} r_{j)})}{\Omega}, \quad (3.89)$$

$$\Pi = \varrho \left(e + \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \right) + \frac{(2\mathbf{r} - \mathbf{n}) \cdot (\mathbf{v} - \mathbf{w})}{\Omega}, \quad (3.90)$$

$$\begin{aligned} \Pi_i = \varrho \frac{v_i - w_i}{\Omega} \left(h + \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \right) - \frac{v^j - w^j}{\Omega} \Sigma_j^i \\ + \frac{r_i}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 + \frac{\mathbf{r} \cdot (\mathbf{v} - \mathbf{w})(v_i - w_i)}{\Omega^2} + k_i - m_i, \end{aligned} \quad (3.91)$$

the explicit expressions for (3.30) and (3.32). Observe that we have split the order-1 contribution in (3.87) as $\Pi_i + p_i$. This is not arbitrary because p_i , defined in (3.83), occurs also as subleading term in the matter current (3.81), and these will annihilate in the final equations, which will turn exactly as (3.41), (3.42) and (3.48). We recover in the above formulas the fluid energy-stress tensor, energy density and energy current, as defined in ref. [1] §§ 15 and 49, generalized though in a covariant fashion for arbitrary torsionless Newton-Cartan geometries. They all receive exotic contributions from the n_i and r_i , absent in standard Galilean fluids. The combination

$$Q_l = k_l - m_l, \quad (3.92)$$

inside the energy current, appears as the Galilean heat current. It receives contributions from both the relativistic heat current q_i and the relativistic non-perfect matter current j_i (see (3.76) and (3.77)). This is exactly how it should: in Landau-Lifshitz frame $q_i = 0$ and the Galilean heat current originates exclusively from the relativistic non-perfect matter current, whereas in Eckart frame where $j_i = 0$ it is the other way around. A more complete discussion on hydrodynamic frames will be brought off in a short while. As anticipated in footnote 13, we can however notice that for fluids without conserved current, although sensible, the Landau-Lifshitz hydrodynamic frame is not suited for the Galilean limit — it leads to $Q_i = 0$ and Eckart frame is always preferred [8].

⁴¹The energy current Π_i defined in (3.91) differs slightly from the expression (3.44) of [29], even at vanishing \mathbf{r} . In that reference, neither were the momenta defined as variations of an effective action, nor was the hydrodynamic-frame invariance a guide.

Hydrodynamic equations

The fluid equations (2.1) and (2.2) translate the vanishing of

$$c\Omega\nabla_\mu T^{\mu 0} = c^2 \left(\mathcal{E} + \hat{\nabla}_i (r^i - n^i) \right) + \hat{\nabla}_i p^i + \mathcal{E} + \mathcal{O} \left(\frac{1}{c^2} \right), \quad (3.93)$$

$$\nabla_\mu T^\mu_i = \mathcal{M}_i + \mathcal{O} \left(\frac{1}{c^2} \right), \quad (3.94)$$

$$\nabla_\mu J^\mu = \mathcal{C} + \frac{1}{c^2} \hat{\nabla}_i p^i + \mathcal{O} \left(\frac{1}{c^4} \right), \quad (3.95)$$

with

$$\mathcal{E} = \frac{1}{\Omega} \frac{\hat{D}\Pi}{dt} + \theta^w \Pi + \Pi_{ij} \hat{\gamma}^{wj} + \hat{\nabla}_i \Pi^i, \quad (3.96)$$

$$\mathcal{M}_i = \frac{1}{\Omega} \frac{\hat{D}P_i}{dt} + \theta^w P_i + P_j \hat{\gamma}^{wj}_i + \hat{\nabla}^j \Pi_{ij}, \quad (3.97)$$

$$\mathcal{C} = \frac{1}{\Omega} \frac{\hat{D}\varrho}{dt} + \theta^w \varrho + \hat{\nabla}_i N^i. \quad (3.98)$$

Two fluid equations emerge from (3.93) because of the presence of order-1 and order- c^2 terms that should both be zero in the infinite- c limit. The latter should be taken after we combine (3.95) with (3.94) as

$$c\Omega\nabla_\mu T^{\mu 0} = c^2 \hat{\nabla}_i (r^i - n^i) + \mathcal{E} + \mathcal{O}(1/c^2), \quad (3.99)$$

leading to

$$\hat{\nabla}_i (r^i - n^i) = 0, \quad (3.100)$$

and

$$\boxed{\mathcal{E} = 0}, \quad (3.101)$$

which is the fluid energy equation. Using (3.85), eq. (3.100) is recast as

$$\boxed{\hat{\nabla}_i (N^i - P^i) = 0}, \quad (3.102)$$

which is a constraint equation for the divergences of the matter current and the fluid momentum. Finally, (3.94) provides the fluid momentum equation

$$\boxed{\mathcal{M}_i = 0}, \quad (3.103)$$

whereas (3.95) exhibits the continuity equation, which thanks to (3.102) also reads:⁴²

$$\boxed{\mathcal{C} = 0}, \quad (3.104)$$

possibly recast in several forms

$$\mathcal{C} = \frac{1}{\Omega} \frac{\hat{D}\varrho}{dt} + \theta^w \varrho + \hat{\nabla}_i P^i = \frac{1}{\Omega} \frac{d\varrho}{dt} + \theta^v \varrho + \hat{\nabla}_i r^i = \frac{1}{\Omega} \frac{d\varrho}{dt} + \theta^v \varrho + \hat{\nabla}_i n^i = 0. \quad (3.105)$$

⁴²This is the typical equation describing phenomena, where several fluid components are present but are not separately conserved. Examples are diffusion or superfluid dynamics (e.g. [86], chapters VI and XVI, eqs. (58,3) or (139,3)).

We can now summarize our findings. The Galilean fluid equations on a general background are by essence fully covariant under Galilean diffeomorphisms (alternative views about this statement are available in ref. [29]) and extend the standard hydrodynamic equations on flat Euclidean space with absolute time. The momentum equation (3.103) coincides with (3.41), whereas the energy equation (3.101) is (3.42). Similarly, the continuity equation (3.105) is identified with (3.48), provided the matter current and the fluid momentum have equal divergence (3.102). This requirement is subsequent to the relationship (3.79), which finally relates the energy-momentum equations with the matter equation.

Strictly speaking, eq. (3.105) is *not* a conservation law. Integrated inside a static domain \mathcal{V} , the density ϱ varies in time, not only because of the expansion or the contraction of \mathcal{V} (term $\varrho\theta^v$), but also due to the flux of \mathbf{n} through $\partial\mathcal{V}$. Using (3.51) we find:

$$\frac{1}{\Omega} \frac{d}{dt} \int_{\mathcal{V}} d^d x \sqrt{a} \varrho + \int_{\partial\mathcal{V}} \varrho \frac{\star\mathbf{v}}{\Omega} = - \int_{\partial\mathcal{V}} \star\mathbf{n}. \quad (3.106)$$

Allegedly, \mathbf{n} appears as the flux of matter brought about by a sink or a spring. Furthermore, eq. (3.100) transcribes that matter loss or gain goes along with heat loss or gain. In usual Galilean hydrodynamics r^i is required to vanish, which then forbids such spring or of matter. At the same time, in those cases $n^i = 0$, and the fluid momentum and matter current are identical: $P_i = N_i$. The systems under investigation here are more general and possess a remarkable property, which is broken in ordinary non-relativistic fluids: Galilean hydrodynamic frame invariance.

The fate of hydrodynamic-frame invariance

The relativistic fluid equations are invariant under arbitrary unimodular transformations of the velocity field \mathbf{u} , captured in⁴³ $v^k \rightarrow v^k + \delta v^k(t, \mathbf{x})$, provided they are accompanied with the transformations of all other dynamical quantities, as described in eqs. (2.57), (2.58), (2.59), (2.61) and (2.62). Does this survive in the Galilean limit?

The intuitive answer to this question is no. The velocity field is a physical and observable quantity and only variations by constant values in directions associated with isometries of the underlying Galilean spacetime, if any, might leave the equations invariant. The fluid density ϱ is also physical and has furthermore a microscopic definition in terms of an observable expectation value. It is hard to imagine how one could maintain the continuity equation invariant without altering the density. Although this might still be considered as an abstract field redefinition,⁴⁴ it would be at the expense of giving up the physical meaning of the various quantities at hand.

The above intuitive answer seems to contradict the mathematical structure of the equations describing the dynamics. Indeed, on the one hand the operators entering equations (3.96), (3.97) and (3.98) are velocity-independent; on the other hand, the momenta ϱ ,

⁴³The infinitesimal local Lorentz transformations are parameterized with Lorentz boost and rotation generators, $V^i(t, \mathbf{x})$ and $\Omega^{ij}(t, \mathbf{x})$ — antisymmetric, as follows: $\delta v^i = V^i - V^j \frac{(v_j - w_j)(v^i - w^i)}{c^2 \Omega^2} + \Omega^{ij} (v_j - w_j)$. In the Galilean limit the general local velocity transformation is thus $\delta v^i = V^i + \Omega^{ij} (v_j - w_j)$ — Galilean boosts and rotations.

⁴⁴This point of view, slightly different from ours, is adopted to some extent in [63, 64]. We acknowledge a rich exchange with P. Kovtun on that matter.

N_i , P_i , Π , Π_i and Π_{ij} , appear as coefficients in the expansion of the hydrodynamic-frame-invariant relativistic momenta (3.80), (3.81), (3.86), (3.87) and (3.88). It is however too naive to infer that the Galilean momenta are automatically invariant. They are obtained assuming a behaviour of the stress, and of the heat and matter currents with respect to c^2 (eqs. (3.75), (3.76), (3.77), (3.78), (3.79)), and this behaviour may or may not be stable under velocity transformations.

The precise answer calls for a thorough examination of the transformations (2.57), (2.58), (2.59), (2.61) and (2.62) in the infinite- c limit, and of their effect on the non-relativistic quantities introduced through (3.75), (3.76), (3.77), (3.78) and (3.79). We find the following transformations under the action of general local Galilean boosts and rotations:⁴⁵

$$a_{ij}\delta p - \delta\Sigma_{ij} = -\frac{2}{\Omega}r_{(i}\delta v_{j)}, \tag{3.107}$$

$$\delta r_i = -\frac{\varrho}{\Omega}\delta v_i, \tag{3.108}$$

$$\delta k_i = \frac{\delta v_i (\mathbf{v} - \mathbf{w}) \cdot \mathbf{n}}{\Omega} + \frac{\delta v^j}{\Omega} \left(\frac{v_i - w_i}{\Omega} r_j - \varrho \frac{(v_i - w_i)(v_j - w_j)}{\Omega^2} - \varrho h a_{ij} + \Sigma_{ij} \right), \tag{3.109}$$

$$\delta Q_i = \frac{\delta v^j}{\Omega} \left((r_j - n_j) \frac{v_i - w_i}{\Omega} - \varrho h a_{ij} + \Sigma_{ij} \right), \tag{3.110}$$

$$\varrho \delta e = (n_i - 2r_i) \frac{\delta v^i}{\Omega}, \tag{3.111}$$

$$\delta \varrho = 0, \tag{3.112}$$

$$\delta n_i = -\frac{\varrho}{\Omega}\delta v_i, \tag{3.113}$$

$$\delta m_i = \frac{\delta v^j}{\Omega} \left(n_j - \varrho \frac{v_j - w_j}{\Omega} \right) \frac{v_i - w_i}{\Omega} + \frac{\delta v_i (\mathbf{v} - \mathbf{w}) \cdot \mathbf{n}}{\Omega}, \tag{3.114}$$

and thus

$$\delta (n^i - r^i) = 0. \tag{3.115}$$

In turn, this action translates into the invariance of the fundamental momenta, i.e. the fluid energy density, the fluid energy current, the fluid energy-stress tensor, the fluid momentum, the matter density and the matter current:

$$\delta \Pi = 0, \quad \delta \Pi^i = 0, \quad \delta \Pi^{ij} = 0, \quad \delta P^i = 0, \quad \delta \varrho = 0, \quad \delta N^i = 0. \tag{3.116}$$

These imply that *the Galilean fluid equations are invariant under an arbitrary local redefinition of the fluid velocity field $v^i(t, \mathbf{x})$.*

This result is important but should be reckoned with great care. On the one hand, the non-relativistic density ϱ is not sensitive to the velocity field \mathbf{v} (in contrast with the relativistic result (2.61)). On the other hand, the standard momentum $\varrho \frac{v^i - w^i}{\Omega}$ does depend on the velocity. The actual momentum emerging here, P^i displayed in eq. (3.84), is therefore invariant thanks to n^i . Hence, the non-conservation of matter discussed previously saves

⁴⁵Notice the following useful formulas: $\delta\theta^v = \frac{1}{\Omega}\hat{\nabla}_i\delta v^i$ and $\delta\xi_{ij}^v = \frac{1}{\Omega}(\hat{\nabla}_{(i}\delta v_{j)} - \frac{1}{d}a_{ij}\hat{\nabla}_k\delta v^k)$, whereas $\delta\theta^w = \delta\xi_{ij}^w = 0$.

hydrodynamic-frame invariance. Put differently, this invariance is disabled upon demanding a genuine conservation i.e. $n^i = 0$: *truly conserved, ordinary non-relativistic fluids are not hydrodynamic-frame-invariant*. For these fluids the behaviours of the relativistic heat and matter currents are $q_l = k_l + \mathcal{O}(1/c^2)$ and $j_l = \frac{m_l}{c^2} + \mathcal{O}(1/c^4)$. Following our earlier discussion these behaviours are physical, but are at the same time unstable under velocity transformations.⁴⁶

Although negative, our last conclusion makes clear the origin of the physically expected breaking of velocity invariance in non-relativistic fluids obeying an authentic conservation equation. In this respect, it is worth quoting a substantial literature on trials to explore the symmetries of Navier-Stokes equations (i.e. first-derivative truncated Galilean fluid equations) in their compressible or incompressible form. A nice overview is given in [69], from which it comes out that most of these extra symmetries are probably accidental, and bound to the specific truncation or system (compressible/incompressible), rather than emanating from the original relativistic hydrodynamic-frame invariance.

Heat and entropy equations

As we have emphasized in section 2.1, the investigation of hydrodynamic-frame transformations should be completed with the analysis of the entropy current, which is also invariant in the relativistic theory. We will not pursue this endeavour any further, which provides ultimately the transformations δs , δp , δT , and $\delta \mu$. Instead, we will combine the above results in order to reach the non-relativistic entropy equation.

As a first step we can perform the usual combination $\mathcal{E} - \frac{v^i - w^i}{\Omega} \mathcal{M}_i$, which leads to the heat equation. It takes the following form:

$$\begin{aligned} & \frac{1}{\Omega} \frac{d}{dt} \left(e \varrho + \frac{(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{r} - \mathbf{n})}{\Omega} \right) + \left(h \varrho + \frac{(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{r} - \mathbf{n})}{\Omega} - \frac{\Sigma}{d} \right) \theta^v - \xi^v_{ij} \Sigma^{ij} \\ & + \hat{\nabla}_i \left(Q^i - \frac{v^i - w^i}{\Omega} \frac{(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{r} - \mathbf{n})}{\Omega} \right) + \frac{r^i}{\Omega} \frac{D}{dt} \frac{v_i - w_i}{\Omega} - \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \hat{\nabla}_j r^j = 0. \end{aligned} \tag{3.117}$$

This equation can be alternatively established within the relativistic framework, by considering $-u_\nu \nabla_\mu T^{\mu\nu}$ as in (2.21), and its subsequent Galilean limit. Notice that due to the explicit appearance of the velocity field (u_ν or v^i), the equation at hand is hydrodynamic-frame invariant only on-shell.

⁴⁶The requirements $r^i = 0$ or $n^i = 0$ are not compatible with the transformations (3.108) or (3.113). Observe however that a choice, stable under hydrodynamic-frame transformations, is $n_i = r_i$, thanks to (3.115). With this, $P^i = N^i$ and we are the closest possible to ordinary non-relativistic fluids, without genuine conservation though.

Assuming the thermodynamic properties⁴⁷ of section A and using in particular eq. (A.15) together with the continuity equation (3.105), eq. (3.117) is recast as follows:

$$\begin{aligned}
 \frac{\varrho T}{\Omega} \frac{ds}{dt} + \frac{1}{\Omega} \frac{d}{dt} \left(\frac{(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{r} - \mathbf{n})}{\Omega} \right) = \\
 \left(\frac{\Sigma}{d} - \frac{(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{r} - \mathbf{n})}{\Omega} \right) \theta^v + \xi^v_{ij} \Sigma^{ij} \\
 - \hat{\nabla}_i \left(Q^i - \frac{v^i - w^i}{\Omega} \frac{(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{r} - \mathbf{n})}{\Omega} - r^i \left(h + \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \right) \right) \\
 - r^i \left(\frac{1}{\Omega} \frac{D}{dt} \frac{v_i - w_i}{\Omega} + \hat{\nabla}_i \left(h + \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \right) \right), \tag{3.118}
 \end{aligned}$$

which can also be reached from the relativistic equation (2.22) in the infinite- c limit. This equation is intricate and can be somehow simplified by setting $\mathbf{n} = \mathbf{r}$, which does not spoil the Galilean hydrodynamic-frame invariance (again realized on-shell):

$$\begin{aligned}
 \frac{\varrho T}{\Omega} \frac{ds}{dt} = \frac{\Sigma}{d} \theta^v + \xi^v_{ij} \Sigma^{ij} - \hat{\nabla}_i \left(Q^i - r^i \left(h + \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \right) \right) \\
 - r^i \left(\frac{1}{\Omega} \frac{D}{dt} \frac{v_i - w_i}{\Omega} + \hat{\nabla}_i \left(h + \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 \right) \right). \tag{3.119}
 \end{aligned}$$

The latter resembles the entropy equations found when studying diffusion phenomena or dissipative processes in superfluids (see eqs. (58,6) and (140,4) in [86]). The combination $h + \frac{1}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2$ materializes a sort of effective chemical potential for the current \mathbf{r} . For vanishing \mathbf{r} , (3.119) is the standard non-relativistic entropy equation in arbitrary Galilean backgrounds. We will not pursue this discussion any longer.

3.3 A comment on Galilean conservation versus non-relativistic limit

The hidden local $U(1)$

We have so far pursued two distinct approaches. The first (section 3.1) relies on the requirement of Galilean general covariance for a system defined on a (torsionless) Newton-Cartan spacetime — possibly but not necessarily — obtained as an infinite- c limit of a pseudo-Riemannian geometry in Zermelo frame. The second (section 3.2) amounts to taking the $c \rightarrow \infty$ limit after the general-covariance conservation has been imposed on the relativistic system.

It is legitimate to wonder whether the two approaches are equivalent, in other words, whether the conservation requirement and the $c \rightarrow \infty$ limit commute.

In order to present a clean answer to this question, we must consider the simplest possible situation. Aiming at this, we focus on the energy and momentum only (no matter) and make no reference to their expressions in terms of fluid variables such as density, thermodynamic quantities, velocity etc. Starting with a Newton-Cartan set up, we define

⁴⁷This might turn naive due to the extra underlying degrees of freedom carried by the *effective* creation/destruction currents \mathbf{n} and \mathbf{r} . Investigating this issue is not in our agenda here.

the energy-stress Π_{ij} , the momentum P_i and the energy density Π as in (3.30), (3.31) and (3.32). General Galilean covariance translates into two equations, (3.41) and (3.42), which reveal a novel, a priori undetermined vector Π_i , interpreted as the energy current.

Alternatively, one may work in a relativistic spacetime equipped with Zermelo coordinates, and a full-fledged and conserved energy-momentum tensor $T_{\mu\nu}$ with the following large- c expansion:

$$\begin{cases} \Omega^2 T^{00} = \varepsilon_r = \Pi + \mathcal{O}(1/c^2) \\ c\Omega T^0_i = q_{ri} = c^2 P_i + \Pi_i + \mathcal{O}(1/c^2) \\ T^i_{ij} = p_r a_{ij} + \tau_{rij} = \Pi_{ij} + \mathcal{O}(1/c^2). \end{cases} \quad (3.120)$$

This determines the energy current Π_i as a subleading term with respect to the momentum P_i in the expansion of the relativistic heat current. Furthermore, the expansion of the relativistic equations is now

$$\begin{cases} c\Omega \nabla_\mu T^{\mu 0} = c^2 \hat{\nabla}_j P^j + \mathcal{E} + \mathcal{O}(1/c^2) = 0 \\ \nabla_\mu T^\mu_i = \mathcal{M}_i + \mathcal{O}(1/c^2) = 0. \end{cases} \quad (3.121)$$

with \mathcal{E} and \mathcal{M}_i given in eqs. (3.96) and (3.97). We recover, as in the first way, the Galilean conservation equations (3.41) and (3.42), *supplemented now by an extra constraint on the current P_i* :

$$\hat{\nabla}_j P^j = 0. \quad (3.122)$$

The punch line of the current discussion is that taking the $c \rightarrow \infty$ limit followed by the general-covariance requirement is less restrictive than following the pattern in the reverse order. With this latter order, not only the energy current is provided explicitly but the fluid current obeys an extra constraint equation. The reason for this is simple. When the infinite- c limit is the last step, the system secretly remembers the full diffeomorphism invariance present at the first step, which contracts during the limit into the Galilean covariance accompanied with a central extension [88]. This extra hidden local U(1) invariance accounts for the supplementary equation (3.122). When the Galilean diffeomorphism invariance is the second step, the \mathbf{x} -independence of ξ^t leaves Π_i undetermined, and no further equation is found.

All this shows how the full — as opposed to (3.122) — continuity equation (3.105)

$$\frac{1}{\Omega} \hat{D}\varrho + \theta^w \varrho + \hat{\nabla}_i P^i = 0, \quad (3.123)$$

can emerge provided ε_r contains an extra $c^2 \varrho$ term, *without* introducing a relativistic conserved current with a local U(1) symmetry, as in [29].⁴⁸ When such an explicit U(1) current is present, as in sections 3.1 and 3.2, it is promptly identified with the hidden one through (3.102), originating in the deep relationship between energy and mass, eq. (A.2).

⁴⁸With this method, however, the relativistic hydrodynamic frame is locked to Eckart's since the Galilean heat current Q^i in (3.92) receives only the contribution k^i .

More general abstract equations

The large- c behaviours (3.80), (3.81), (3.86), (3.87) and (3.88) (or a simpler version of the latter without ϱ (3.120)) are motivated by the physics standing behind these momenta, captured in the behaviours of transport coefficients and materialized in (3.75), (3.76) and (3.77). As an echo to the comments made when setting the latter, one could be more abstract and consider order- c^2 terms in the stress tensor τ_{ij} as we have already introduced in the heat and matter currents, and possibly further powers in those (r_l and n_l were already beyond normalcy, but introduced as a mean of restoring hydrodynamic-frame invariance — or demonstrating its breaking in genuine non-relativistic fluids). The net effect of this sort of options is to bring the energy-momentum tensor in the form

$$\begin{cases} \Omega^2 T^{00} = \varepsilon_r = c^2 \varrho + \Pi + \mathcal{O}(1/c^2) \\ c\Omega T^0_i = q_{ri} = c^4 \tilde{P}_i + c^2 P_i + \Pi_i + \mathcal{O}(1/c^2) \\ T_{ij} = p_r a_{ij} + \tau_{rij} = c^2 \tilde{\Pi}_{ij} + \Pi_{ij} + \mathcal{O}(1/c^2), \end{cases} \quad (3.124)$$

and produce at infinite c a genuine hierarchy of equations, which are replicas of those we have already met:

$$\begin{cases} \left(\frac{1}{\Omega} \hat{D} + \theta^w \right) \Pi + \Pi_{ij} \hat{\gamma}^{wij} + \hat{\nabla}_i \Pi^i = 0 \\ \left(\frac{1}{\Omega} \hat{D} + \theta^w \right) \varrho + \tilde{\Pi}_{ij} \hat{\gamma}^{wij} + \hat{\nabla}_i P^i = 0 \\ \hat{\nabla}_j \tilde{P}^j = 0 \\ \left(\frac{1}{\Omega} \hat{D} + \theta^w \right) P_i + P_j \hat{\gamma}^{wj}_i + \hat{\nabla}^j \Pi_{ij} = 0 \\ \left(\frac{1}{\Omega} \hat{D} + \theta^w \right) \tilde{P}_i + \tilde{P}_j \hat{\gamma}^{wj}_i + \hat{\nabla}^j \tilde{\Pi}_{ij} = 0. \end{cases} \quad (3.125)$$

This sort of situation is the archetype of multiplication of degrees of freedom, mentioned earlier. It comes naturally thanks to the existence of a parameter c , which makes it possible to organize a Laurent expansion. It is more artificial to interpret this system as conservation equations portraying local symmetries, because this would require introducing further variables, conjugate to the new momenta, such as \tilde{a}^{ij} , \tilde{w}^i , $\tilde{\Omega}$ etc. We will not elaborate on that, but keep the structure in mind for comparison with the forthcoming analysis of section 4.3 about Carrollian fluids. For the latter, no physical intuition can possibly serve as a guide — basic thermodynamics is even missing. Only a blind $1/c^2$ expansion applies, as suggested by the only known application field of Carrollian fluids, which is flat holography [25–28]. Then the hierarchy obtained is dual to (3.125), and this plainly justifies our present excursion from standard, physical non-relativistic fluids.

Multiplication of degrees of freedom occurs also in the matter sector. One could indeed abstractly assume that some matter current behaves like

$$\frac{\Omega}{c} I^0 = \iota_{0r} = c^2 \tilde{\kappa} + \kappa + \mathcal{O}(1/c^2), \quad I_k = i_{rk} = c^4 \tilde{K}_k + c^2 \tilde{K}_k + K_k + \mathcal{O}(1/c^2). \quad (3.126)$$

Using these expansions in the relativistic divergence of the matter current J^μ in Zermelo background we find:

$$\nabla_\mu I^\mu = c^4 \tilde{\mathcal{K}} + c^2 \tilde{\mathcal{K}} + \mathcal{K} + \mathcal{O}(1/c^2) \quad (3.127)$$

with

$$\begin{cases} \tilde{\mathcal{K}} = \hat{\nabla}_j \tilde{K}^j \\ \tilde{\mathcal{K}} = \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w\right) \tilde{\kappa} + \hat{\nabla}_j \tilde{K}^j \\ \mathcal{K} = \left(\frac{1}{\Omega} \frac{\hat{D}}{dt} + \theta^w\right) \kappa + \hat{\nabla}_j K^j, \end{cases} \quad (3.128)$$

which must vanish if $\nabla_\mu I^\mu = 0$.

As an aside application of the latter results, we can insert the behaviour (3.124) inside the components (2.76) and (2.77) of a relativistic conserved current resulting from the combination of the energy-momentum tensor with a Killing field. We find thus

$$\begin{cases} \kappa = \xi^i P_i - \xi^t \Pi \\ \tilde{\kappa} = \xi^i \tilde{P}_i - \xi^t \tilde{\rho} \\ K_i = \xi^j \Pi_{ij} - \xi^t \Pi_i \\ \tilde{K}_i = \xi^j \tilde{\Pi}_{ij} - \xi^t \tilde{P}_i \\ \tilde{\tilde{K}}_i = -\xi^t \tilde{P}_i, \end{cases} \quad (3.129)$$

where κ and K_i are precisely as anticipated in (3.58) and (3.59). On-shell i.e. assuming (3.125), and using (3.53) we find for (3.128)

$$\begin{cases} \tilde{\mathcal{K}} = 0 \\ \tilde{\mathcal{K}} = \frac{\tilde{P}_i}{\Omega} \left(\partial_t \xi^i + \mathcal{L}_{\mathbf{w}} \xi^i\right) \\ \mathcal{K} = \frac{P_i}{\Omega} \left(\partial_t \xi^i + \mathcal{L}_{\mathbf{w}} \xi^i\right), \end{cases} \quad (3.130)$$

in agreement with the result (3.61) for the last two. The first vanishes, and this shows that even though the existence of a Killing field does not guarantee the conservation of a Galilean current, such a conservation can occur if the appropriate vector vanishes. Ordinary Galilean fluids as those studied in section 3.2 have $\tilde{P}_i = \tilde{\Pi}_{ij} = 0$ in (3.124) so that $\tilde{K}^j = 0$. Hence two currents survive, one with $\mathcal{K} \neq 0$ (non-conserved) and another, which is conserved ($\tilde{\mathcal{K}} = 0$) but already known. Indeed, it is the very same that appears inside the ordinary continuity equation, and no extra conservation arises as a consequence of isometries.

One might be legitimately skeptical about the validity of the above conclusion on non-conservation: how can the bona fide law $\nabla_\mu I^\mu = 0$ of a relativistic current $I^\mu = \xi_\nu T^{\mu\nu}$ based on a Killing field ξ of a pseudo-Riemannian spacetime, break down suddenly in the infinite- c limit? The answer is captured by the very definition of a Galilean Killing, which ultimately leaves non-vanishing terms in the divergence. The precise way this comes about is exposed in appendix D.1.

3.4 Massless carriers and Weyl properties

Generic hydrodynamic equations

Although massless particles are ultra-relativistic, a macroscopic collection of them forming a fluid can be compatible with Galilean symmetries. The latter appear as a phenomenological

emanation and are agnostic on their microscopic origin. Such a system can also have a conserved current and a chemical potential, which are not related to mass, but to some charge. Some examples of this sort are mentioned in appendix A (see ref. [89]) together with their basic thermodynamic properties.

The difference with respect to the previous section (section 3.2) is the Galilean limit. Here, instead of (3.78), we can simply consider

$$\varrho_0 = \varrho + \mathcal{O}(1/c^2), \quad \varepsilon = \varrho e + \mathcal{O}(1/c^2), \quad (3.131)$$

where e is the energy per charge unit and ϱ is the charge volume density — as opposed to proper volume — and $\varepsilon = \varrho e$ the non-relativistic energy density (see also appendix A). Again, our goal is to find the fundamental variables as well as the dynamical equations, and probe the behaviour of the latter under Galilean hydrodynamic-frame transformations.

Why do we expect this sort of system be potentially Galilean hydrodynamic-frame invariant? A gas of photons, perhaps under some isotropy and homogeneity requirement regarding interactions, gives no handle for measuring a global velocity. In other words, we do not expect v^i to enter the hydrodynamic equations. This reasoning is not a proof, but a guideline to pursue here. For that, we will adopt again the behaviours (3.75), (3.76) and (3.77) for the stress tensor, the heat and the charge currents, even though we are aware that on physics grounds r_i and n_i are bound to vanish. For the charge current, the subleading term m_l turns out to be irrelevant here because of the absence of rest mass.

Equations (2.66), (2.67), (2.68), (2.69) and (2.70), now give:

$$\varrho_{0r} = \varrho, \quad j_{ri} = N_i + \mathcal{O}(1/c^2) \quad (3.132)$$

with Galilean charge current

$$N^i = \varrho \frac{v^i - w^i}{\Omega} + n^i. \quad (3.133)$$

From the energy-momentum tensor one obtains

$$\varepsilon_r = \Pi + \mathcal{O}(1/c^2), \quad (3.134)$$

$$q_{ri} = c^2 P_i + \Pi_i + \mathcal{O}(1/c^2), \quad (3.135)$$

$$p_r a_{ij} + \tau_{rij} = \Pi_{ij} + \mathcal{O}(1/c^2), \quad (3.136)$$

which coincides with (3.120), where

$$\Pi_{ij} = p a_{ij} - \Sigma_{ij} + 2 \frac{(v_{(i} - w_{(i}) r_{j)})}{\Omega}, \quad (3.137)$$

$$P^i = r^i, \quad (3.138)$$

$$\Pi = \varepsilon + \frac{2 \mathbf{r} \cdot (\mathbf{v} - \mathbf{w})}{\Omega}, \quad (3.139)$$

$$\Pi_i = ((\varepsilon + p) a_{ij} - \Sigma_{ij}) \frac{v^j - w^j}{\Omega} + \frac{r_i}{2} \left(\frac{\mathbf{v} - \mathbf{w}}{\Omega} \right)^2 + \frac{\mathbf{r} \cdot (\mathbf{v} - \mathbf{w}) (v_i - w_i)}{\Omega^2} + k_i, \quad (3.140)$$

are the explicit expressions for (3.30), (3.31) and (3.32), as well as for the energy current Π_i , which will appear in the energy equation (3.42). The Galilean heat current receives now a single contribution as

$$Q_l = k_l. \quad (3.141)$$

For the fluid under consideration, the structure of the conservation equations is as follows (actually as in (3.121) because the energy-momentum (3.134), (3.135), (3.136) is as in (3.120)):

$$c\Omega\nabla_{\mu}T^{\mu 0} = c^2\hat{\nabla}_jr^j + \mathcal{E} + \mathcal{O}(1/c^2), \quad (3.142)$$

$$\nabla_{\mu}T^{\mu}_i = \mathcal{M}_i + \mathcal{O}(1/c^2), \quad (3.143)$$

$$\nabla_{\mu}J^{\mu} = \mathcal{C} + \mathcal{O}(1/c^2), \quad (3.144)$$

with \mathcal{E} , \mathcal{M}_i and \mathcal{C} as in (3.96), (3.97), (3.98). At infinite c the hydrodynamic equations are again (3.101), (3.103), (3.104), and we recover eqs. (3.41), (3.42) and (3.48), as expected, plus the extra equation (same as (3.122))

$$\hat{\nabla}_jr^j = 0, \quad (3.145)$$

which is absent when the unphysical vector r^j originating from the c^2 term of the relativistic heat current vanishes. The difference with respect to the massive case studied in the previous section dwells in the expression of the momenta (energy-stress tensor, fluid current, fluid energy density and fluid energy current — the charge current is the same as the matter current before).

We can now combine the above results in order to reach the heat and next the entropy equations. Equivalently these are obtained as infinite- c limits of eqs. (2.21) and (2.22). We find for the former

$$\begin{aligned} \mathcal{E} - \frac{v_i - w_i}{\Omega} \mathcal{M}_i &= \frac{1}{\Omega} \frac{d}{dt} \left(\varepsilon + \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \left(\varepsilon + p + \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} - \frac{\Sigma}{d} \right) \theta^v - \xi^v_{ij} \Sigma^{ij} \\ &+ \hat{\nabla}_i \left(Q^i - \frac{v^i - w^i}{\Omega} \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \frac{r^i}{\Omega} \frac{D}{dt} \frac{v_i - w_i}{\Omega} = 0. \end{aligned} \quad (3.146)$$

For the entropy equations there are two options. If no conserved charge current exists, the equation (3.104) is immaterial, the chemical potential vanishes and (A.19) gives $d\varepsilon = Td\sigma$, which can be substituted in (3.146). This happens e.g. for a gas of photons. If a conserved charge current is available then ε can be traded for ϱe , $\varepsilon + p$ for ϱh , σ for ϱs , and using (A.18), (A.19) and (3.104) one obtains

$$\frac{1}{\Omega} \frac{d\varrho e}{dt} = \frac{\varrho T}{\Omega} \frac{ds}{dt} - \varrho h \theta^v - h \hat{\nabla}_i n^i, \quad (3.147)$$

which can be inserted back in (3.146):

$$\begin{aligned} \frac{\varrho T}{\Omega} \frac{ds}{dt} + \frac{1}{\Omega} \frac{d}{dt} \left(\frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) &+ \left(\frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} - \frac{\Sigma}{d} \right) \theta^v - \xi^v_{ij} \Sigma^{ij} - h \hat{\nabla}_i n^i \\ &+ \hat{\nabla}_i \left(Q^i - \frac{v^i - w^i}{\Omega} \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \frac{r^i}{\Omega} \frac{D}{dt} \frac{v_i - w_i}{\Omega} = 0. \end{aligned} \quad (3.148)$$

Given the above Galilean hydrodynamical equations, one may reconsider their behaviour under velocity local transformations. The absence of rest mass for the carriers modifies

the scalings with respect to the speed of light, and possibly the invariance properties. Bringing together the transformations (2.57), (2.58), (2.59), (2.61) and (2.62), and the scalings (3.75), (3.76), (3.77) and (3.131), we find in the infinite- c limit that $\delta\rho$ and δn_i are still as in (3.112) and (3.113), while

$$a_{ij}\delta p - \delta\Sigma_{ij} = -\frac{2}{\Omega}r_{(i}\delta v_{j)}, \quad (3.149)$$

$$\delta r_i = 0, \quad (3.150)$$

$$\begin{aligned} \delta k_i &= \delta Q_i \\ &= \frac{\delta v^j}{\Omega} \left(\frac{v_i - w_i}{\Omega} r_j - \rho h a_{ij} + \Sigma_{ij} \right), \end{aligned} \quad (3.151)$$

$$\begin{aligned} \delta\epsilon &= \rho\delta e \\ &= -2r_i \frac{\delta v^i}{\Omega}. \end{aligned} \quad (3.152)$$

These transformations ensure the invariance of the Galilean momenta as in eqs. (3.116), which thus implies that *for fluids consisting of massless particles, the Galilean fluid equations established above are invariant under arbitrary hydrodynamic frame transformations.*⁴⁹

One should stress that, thanks to (3.150), the hydrodynamic-frame invariance holds even when $r_i = 0$, which is the physically interesting situation, following our previous discussion on the behaviour of the relativistic heat current. The momentum equation obtained from (3.97) greatly simplifies in this case:

$$\mathcal{M}_i = \partial_i p - \nabla_j \Sigma_i^j = 0. \quad (3.153)$$

Nonetheless, due to (3.113), hydrodynamic-frame invariance does not resist when n_i is required to vanish in the charge current, which is necessary for the continuity equation (3.98), (3.104) to be a genuine conservation. This caveat, which opposes again hydrodynamic-frame invariance to conservation, is evaded precisely for fluids without conserved charge, as are photon gases, which are therefore truly hydrodynamic-frame-invariant in the Galilean regime with entropy equation (for the physical situation where $r_i = 0$)

$$\frac{T}{\Omega} \frac{d\sigma}{dt} + \left(\epsilon + p - \frac{\Sigma}{d} \right) \theta^v - \xi^v_{ij} \Sigma^{ij} + \hat{\nabla}_i Q^i = 0. \quad (3.154)$$

Weyl invariance

Galilean groups can accommodate conformal extensions. The subject has generated an abundant literature, part of which is already quoted here [33, 34, 38, 52, 53] — more can be found in those references. The analysis of conformal symmetry in non-relativistic fluid dynamics has been in the agenda of many groups. No real guiding principle has been followed though, the search has been usually blind and the output often looks accidental.⁵⁰ From the fluid perspective on non-isometric and non-conformal-isometric backgrounds, conformal

⁴⁹Equations (3.146) and (3.148) are hydrodynamic-frame-invariant only on-shell.

⁵⁰Reference [69] makes better contact with the relativistic fluid equations, and provides a nice and critical overview of the field.

symmetry is rather meant to be Weyl symmetry, which is expected to portray the dynamics when no massive excitations are present. This instance was touched upon in section 3.1, when presenting the basic features of Newton-Cartan geometry, and will be considered in the present section from the perspective of the large- c limit in Zermelo backgrounds.

The fundamental quantities of the Zermelo geometry (2.46) behave as follows under a Weyl transformation:

$$a_{ij} \rightarrow \frac{1}{\mathcal{B}^2} a_{ij}, \quad w^i \rightarrow w^i, \quad w_i \rightarrow \frac{1}{\mathcal{B}^2} w_i, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega, \quad (3.155)$$

and since Ω depends on time only, the last of (3.155) imposes $\mathcal{B} = \mathcal{B}(t)$. The velocity components u^μ have weight 1. This gathers the following for the ordinary spatial fluid velocity:

$$v^i \rightarrow v^i, \quad v_i \rightarrow \frac{1}{\mathcal{B}^2} v_i. \quad (3.156)$$

With this at hand, one can wonder what is the Galilean Weyl-covariant derivative, acting on Galilean Weyl-covariant tensors. From a purely mathematical perspective taming the connections on Newton-Cartan (or Carrollian) geometries is a thriving subject (see e.g. [38–40]). We will here answer modestly this question by examining the infinite- c limit of the connection (2.34) and the corresponding Weyl-covariant derivative used in the relativistic case. As we will also witness later in the Carrollian side, this splits into time and space Weyl derivatives, associated with time and space Weyl connections, inherited from the limit of A given in (2.34). Owing to the fact that

$$\lim_{c \rightarrow \infty} \Omega c A^0 = -\frac{\theta^v}{d}, \quad \lim_{c \rightarrow \infty} A_i = 0, \quad (3.157)$$

there is no spatial Weyl connection in the Galilean limit. The ordinary Galilean spatial covariant derivative $\hat{\nabla}_i$ used here as the usual d -dimensional metric-compatible and torsionless covariant derivative with connection coefficients (3.8) (possibly time-dependent since generally $a_{ij} = a_{ij}(t, \mathbf{x})$) is thus Weyl-covariant on its own right. This is not a surprise since a Weyl rescaling with $\mathcal{B}(t)$ leaves the Christoffel symbols (3.8) unaltered.

The Galilean time covariant derivative $\frac{D}{dt}$ given in (3.25) is not Weyl-covariant, though. It can be promoted to a Weyl-covariant Galilean time derivative \mathcal{D}_t thanks to θ^v , which transforms indeed as a connection:

$$\theta^v \rightarrow \mathcal{B} \theta^v - \frac{d}{\Omega} \partial_t \mathcal{B}. \quad (3.158)$$

Consequently, if $S^{ij\dots}_{kl\dots}$ are the components of a weight- w Galilean tensor, then

$$\frac{1}{\Omega} \mathcal{D}_t S^{ij\dots}_{kl\dots} = \left(\frac{1}{\Omega} \frac{D}{dt} + \frac{w}{d} \theta^v \right) S^{ij\dots}_{kl\dots} \quad (3.159)$$

are the components of Galilean tensor with weight $w + 1$. Observe that the components of the Galilean shear given in (3.18) is of weight -1 :

$$\xi^v_{ij} \rightarrow \frac{1}{\mathcal{B}} \xi^v_{ij}. \quad (3.160)$$

The Weyl transformation (3.158) holds equally for θ^w defined in (3.17), and so does in fact (3.160) for ξ_{ij}^w defined in (3.16). One can therefore introduce an alternative Galilean Weyl-covariant time derivative, defined in purely geometrical terms:⁵¹

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Phi = \left(\frac{1}{\Omega} \frac{\hat{\mathcal{D}}}{dt} + \frac{w}{d} \theta^w \right) \Phi, \quad (3.161)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t V_i = \left(\frac{1}{\Omega} \frac{\hat{\mathcal{D}}}{dt} + \frac{w+1}{d} \theta^w \right) V_i, \quad (3.162)$$

for weight- w Galilean scalars or forms, extendable by the Leibniz rule. For convenience, both \mathcal{D}_t and $\hat{\mathcal{D}}_t$ will be used in the following and should not be confused.

Equipped with the above tools and imposing Weyl invariance (3.63), the fundamental equations (3.41), (3.42) and (3.48) are recast as follows:⁵²

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t P_i + P_j \xi^{wj}_i + \hat{\nabla}^j \Pi_{ij} = 0, \quad (3.163)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \Pi_{ij} \xi^{wij} + \hat{\nabla}_i \Pi^i = 0, \quad (3.164)$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \varrho + \hat{\nabla}_i N^i = 0. \quad (3.165)$$

They are Weyl-covariant of weights $d+1$, $d+2$ and $d+1$.

When dealing with Galilean fluids, the Galilean momenta ϱ , N_i , P_i , Π , Π_i and Π_{ij} emerge in the large- c expansion of ϱ_{r0} , j_{ri} , ε_r , q_{ri} and $p_r a_{ij} + \tau_{rij}$ (see eqs. (3.132), (3.134), (3.135), (3.136)). The weights inherited in this limiting procedure (the relativistic weights are available in table 1) are in agreement with those previously defined through the effective-action definition of the momenta. These momenta are expressed in terms of the Galilean velocity v^i together with the usual list of variables emanating from the relativistic stress, heat current and charge/matter current.

From the expressions (3.137), (3.138), (3.139), (3.140), we infer that the forms r_i , k_i (and thus Q_i) have weight d , while n_i and the Galilean stress Σ_{ij} have weight $d-1$. The Weyl condition⁵³ (3.63) now reads $\varepsilon = dp - \Sigma$. The stress being considered as a correction to perfect fluids, absent at global thermodynamic equilibrium, this condition splits into the conformal equation of state

$$\varepsilon = dp, \quad (3.166)$$

accompanied with the Weyl-invariance requirement

$$\Sigma \equiv \Sigma_{ij} a^{ij} = 0. \quad (3.167)$$

Other thermodynamic observables like e , T , μ or h have all weight 1, and s is weight zero.

⁵¹This sort of Weyl-covariant derivative is insensitive to the fluid velocity and is thus better suited for discussing hydrodynamic-frame invariance. Its relativistic ascendent is a Weyl connection A^Z constructed, as explained generally in footnote 16, with the vector field $uz = e_i$ defined in (3.3) (and used in section 3.2), which has norm $-c^2$ in the Zermelo background (2.46). This connection exists irrespective of the fluid velocity: $A^Z = \frac{\theta^w}{d} \Omega dt$.

⁵²We saw in section 3.1 that Π_{ij} has weight $d-1$, P_i and Π_i weight d , and Π weight $d+1$; similarly ϱ is weight- d and N_i weight $d-1$.

⁵³Notice in passing that the Weyl-invariance requirement (3.63) determined from the effective action, is also the large- c expression of the relativistic condition $T_\mu{}^\mu = 0$ discussed at the end of section 2.1, obtained using (2.64) with (3.134), (3.135) and (3.136).

It is important to stress that the above analysis is consistent because we have been systematically referring to fluids with microscopic massless degrees of freedom, and have thus used eqs. (3.132), (3.133), (3.134), (3.135), (3.136), (3.137), (3.138), (3.139), (3.140). Had one considered fluids with massive carriers, conflicts would have appeared in the conformal weights, as e.g. in eq. (3.102) setting a relationship among N^i and P^i , which in a Weyl-covariant system are expected to have different weights ($d + 1$ and $d + 2$).

Hydrodynamic equations (3.146) and (3.154) are recast as

$$\begin{aligned} & \frac{1}{\Omega} \mathcal{D}_t \left(\varepsilon + \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \frac{r^i}{\Omega} \mathcal{D}_t \frac{v_i - w_i}{\Omega} - \xi^v{}_{ij} \Sigma^{ij} \\ & + \hat{\nabla}_i \left(Q^i - \frac{v^i - w^i}{\Omega} \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) = 0, \end{aligned} \tag{3.168}$$

$$\begin{aligned} & \frac{\rho T}{\Omega} \mathcal{D}_t s + \frac{1}{\Omega} \mathcal{D}_t \left(\frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) + \frac{r^i}{\Omega} \mathcal{D}_t \frac{v_i - w_i}{\Omega} - \xi^v{}_{ij} \Sigma^{ij} \\ & - h \hat{\nabla}_i n^i + \hat{\nabla}_i \left(Q^i - \frac{v^i - w^i}{\Omega} \frac{(\mathbf{v} - \mathbf{w}) \cdot \mathbf{r}}{\Omega} \right) = 0. \end{aligned} \tag{3.169}$$

For more conventional conformal fluids with $r^i = 0$ and no conserved charge we find

$$\boxed{\frac{1}{\Omega} \mathcal{D}_t \varepsilon - \xi^v{}_{ij} \Sigma^{ij} + \hat{\nabla}_i Q^i = \frac{T}{\Omega} \mathcal{D}_t \sigma - \xi^v{}_{ij} \Sigma^{ij} + \hat{\nabla}_i Q^i = 0,} \tag{3.170}$$

which are Weyl-covariant of weight $d + 2$. The Euler (transverse) equation (3.153) remains unchanged and can be expressed in terms of the energy thanks to (3.166), or further using (A.18):

$$\boxed{\frac{1}{d} \partial_i \varepsilon - \hat{\nabla}^j \Sigma_{ij} = \frac{1}{(d+1)} \partial_i (T\sigma) - \hat{\nabla}^j \Sigma_{ij} = 0.} \tag{3.171}$$

It is Weyl-covariant of weight $d + 1$.

Besides expressing hydrodynamic equations for fluids based on massless microscopic constituents in a Galilean general-covariant fashion, our present analysis exhibits one class of physical fluids, where hydrodynamic-frame invariance survives the Galilean limit: the conformal fluids without any conserved charge. This is in contrast to the more general Galilean hydrodynamics studied in section 3.2, where hydrodynamic-frame invariance was only emerging in exotic fluids, where matter conservation was not fulfilled. Furthermore, under the physical assumption that the heat current remains of order 1 in the infinite- c limit ($r_i = 0$ in (3.76)), the energy density is insensitive to the choice of fluid velocity (see (3.152)).

Conformal isometries and conservation laws

There is not much we can add on conservation laws that has not yet been processed. Summarizing, the large- c expansion of the energy-momentum tensor (3.134), (3.135), (3.136), combined with the components (2.76), (2.77) in the presence of a conformal Killing field ξ , produces

$$\iota_{0r} = \kappa + \mathcal{O}(1/c^2), \quad \iota_{rk} = K_k + \mathcal{O}(1/c^2) \tag{3.172}$$

with κ and K_k as in (3.58) and (3.59), defining a Galilean current. Using the Weyl-invariance condition (3.63), the equations (3.163), (3.164), and the Galilean Killing equations (3.64), (3.66), the Galilean divergence (3.57) of this current turns out to satisfy (3.67). As usual, conservation demands a symmetry wider than a conformal isometry, for which $\partial_t \xi^i + \mathcal{L}_w \xi^i = 0$.

4 Carrollian fluid dynamics

4.1 Carroll structures and general Carrollian covariance

Carrollian manifolds

Carroll structures are alternatives to Newton-Cartan spacetimes, introduced in [34–36]. They consist of a $d + 1$ -dimensional manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ endowed with a degenerate metric and a vector field, which is the kernel of the metric. These manifolds are described in terms of fibre bundles⁵⁴ with one-dimensional fibre and a d -dimensional base \mathcal{S} thought of as space, the fibre being time. The Carroll group [30, 31] emerges as the isometry group of flat Carrollian structures, but our framework is here more general with no assumption about isometries, but Carrollian diffeomorphisms instead; the Carrollian transformations are realized locally, in the tangent space. These diffeomorphisms have the virtue of preserving the time/space separation, as opposed to general diffeomorphisms.

For concreteness \mathcal{M} will be equipped with coordinates (t, \mathbf{x}) and we will restrict to degenerate metrics of the form

$$d\ell^2 = a_{ij}(t, \mathbf{x}) dx^i dx^j, \quad i, j \dots \in \{1, \dots, d\} \tag{4.1}$$

with kernel generated by

$$e_{\hat{t}} = \frac{1}{\Omega} \partial_t, \tag{4.2}$$

which defines a *field of observers*. This coordinate system is adapted to the fiber/base splitting, which is in turn respected by Carrollian diffeomorphisms (2.80). It is also naturally reached in the Carrollian limit of a pseudo-Riemannian spacetime in Papapetrou-Randers gauge (2.79). The Carrollian structure naturally incorporates an *Ehresmann connection*, which is a background gauge field $\mathbf{b} = b_i dx^i$, appearing in the dual form of the kernel generator (4.2):

$$\theta^{\hat{t}} = \Omega dt - b_i dx^i, \tag{4.3}$$

the *clock form*. The scale factor Ω and the gauge components b_i depend on t and \mathbf{x} .

⁵⁴Carrollian structures were defined as “ambient structures” in refs. [39, 40]. Notice that we use equally the wording “manifolds,” “spacetimes” and “structures.” Mathematically the latter is more precise for it embraces the various attributes. Depending on these attributes, it can be even refined into *weak* or *strong* structure as we will see in the following.

Under Carrollian diffeomorphisms (2.80) (the Jacobian is defined in (2.81)), the transformation rules of the various geometric objects are as in (2.82), (2.83) as well as

$$\partial'_t = \frac{1}{J} \partial_t, \tag{4.4}$$

$$\partial'_j = J^{-1i} \left(\partial_i - \frac{J_i}{J} \partial_t \right), \tag{4.5}$$

$$\theta^{\hat{t}'} = \theta^{\hat{t}}, \tag{4.6}$$

$$\hat{\partial}'_i = J^{-1j} \hat{\partial}_j, \tag{4.7}$$

where

$$\hat{\partial}_i = \partial_i + \frac{b_i}{\Omega} \partial_t, \tag{4.8}$$

are the vector fields dual to the forms dx^i , also spelled e_i in (2.105).

For Carrollian manifolds it is customary to say that space is absolute, whereas time isn't, as opposed to their dual relatives, the Newton-Cartan spacetimes. This is again somehow abusive, except when a_{ij} depends on space only, and it mostly refers to the form of the Jacobian (2.81). It is rooted to the properties of the primitive Carrollian manifold obtained as the $c \rightarrow 0$ limit of Minkowski spacetime.

Carrollian tensors depend generically on time t and space \mathbf{x} . They carry indices $i, j, \dots \in \{1, \dots, d\}$, which are lowered and raised with a_{ij} and its inverse spatial cometric a^{ij} , and transform covariantly under Carrollian diffeomorphisms (2.80) with Jacobian J^j_i and J^{-1i}_j defined in (2.81). Following [29], we introduce a Levi-Civita-Carroll connection with coefficients

$$\hat{\gamma}^i_{jk} = \frac{a^{il}}{2} \left(\hat{\partial}_j a_{lk} + \hat{\partial}_k a_{lj} - \hat{\partial}_l a_{jk} \right). \tag{4.9}$$

This connection is not unique (see the already quoted literature [36, 40, 41]), but emerges naturally in the vanishing- c limit of a Levi-Civita connection in the Papapetrou-Randers coordinates (2.79). It defines a spatial Carrollian covariant derivative $\hat{\nabla}_i$ with tensorial transformation properties under Carrollian diffeomorphisms (details on the transformation properties can be found in the appendix A.2 of ref. [29]).⁵⁵

The Levi-Civita-Carroll connection is torsionless and metric-compatible:

$$\hat{t}^k_{ij} = 2\hat{\gamma}^k_{[ij]} = 0, \quad \hat{\nabla}_i a_{jk} = 0. \tag{4.10}$$

The vectors $\hat{\partial}_i$ do not commute and define the Carrollian vorticity:

$$[\hat{\partial}_i, \hat{\partial}_j] = \frac{2}{\Omega} \varpi_{ij} \partial_t, \quad \varpi_{ij} = \partial_{[i} b_{j]} + b_{[i} \varphi_{j]} \tag{4.11}$$

with

$$\varphi_i = \frac{1}{\Omega} (\partial_t b_i + \partial_i \Omega), \tag{4.12}$$

⁵⁵Important remark: many symbols are common to the Galilean and Carrollian sides investigated in the present paper. The context should leave no room for confusion.

the Carrollian acceleration. Notice that

$$d\theta^{\hat{t}} = \varphi_i dx^i \wedge \theta^{\hat{t}} - \varpi_{ij} dx^i \wedge dx^j, \quad (4.13)$$

so that vanishing Carrollian acceleration and vorticity are necessary and sufficient conditions for $\theta^{\hat{t}}$ be closed and define a family of hypersurfaces inside $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ as $\tau(t, \mathbf{x}) = \text{const.}$, where locally $\theta^{\hat{t}} = d\tau$.

The ordinary time derivative operator $\frac{1}{\Omega} \partial_t$ acts covariantly on Carrollian tensors. However, it is not metric-compatible because a_{ij} depends on time. Hence one defines a new Carrollian temporal covariant derivative by requiring covariance, i.e. $\frac{1}{\Omega'} \hat{D}'_t = \frac{1}{\Omega} \hat{D}_t$, and

$$\hat{D}_t a_{jk} = 0. \quad (4.14)$$

This is achieved by introducing a temporal Carrollian connection

$$\hat{\gamma}_{ij} = \frac{1}{2\Omega} \partial_t a_{ij} = \xi_{ij} + \frac{1}{d} a_{ij} \theta, \quad (4.15)$$

which is a genuine symmetric Carrollian tensor split into a traceless part, the Carrollian shear, and the trace, which is the Carrollian expansion:

$$\theta = \frac{1}{\Omega} \partial_t \ln \sqrt{a}. \quad (4.16)$$

The action of \hat{D}_t on scalars is ∂_t

$$\hat{D}_t \Phi = \partial_t \Phi, \quad (4.17)$$

whereas on vectors or forms it is defined as

$$\frac{1}{\Omega} \hat{D}_t V^i = \frac{1}{\Omega} \partial_t V^i + \hat{\gamma}^i_j V^j, \quad \frac{1}{\Omega} \hat{D}_t V_i = \frac{1}{\Omega} \partial_t V_i - \hat{\gamma}_i^j V_j. \quad (4.18)$$

Leibniz rule generalizes the latter to any tensor and allows to demonstrate the property (4.14).

The commutators of Carrollian covariant spatial derivatives define further Carrollian tensors (Φ and V^i are a Carrollian scalar and a Carrollian vector):⁵⁶

$$[\hat{\nabla}_i, \hat{\nabla}_j] \Phi = \varpi_{ij} \frac{2}{\Omega} \partial_t \Phi, \quad (4.19)$$

$$[\hat{\nabla}_k, \hat{\nabla}_l] V^i = \left(\hat{\partial}_k \hat{\gamma}_{lj}^i - \hat{\partial}_l \hat{\gamma}_{kj}^i + \hat{\gamma}_{km}^i \hat{\gamma}_{lj}^m - \hat{\gamma}_{lm}^i \hat{\gamma}_{kj}^m \right) V^j + [\hat{\partial}_k, \hat{\partial}_l] V^i \quad (4.20)$$

$$= \hat{R}^i_{jkl} V^j + \varpi_{kl} \frac{2}{\Omega} \hat{D}_t V^i. \quad (4.21)$$

Similarly, time and space derivatives do not commute:

$$\left[\frac{1}{\Omega} \hat{D}_t, \hat{\nabla}_i \right] V^j = \varphi_i \left(\left(\frac{1}{\Omega} \hat{D}_t + \theta \right) V^j - \hat{\gamma}^j_k V^k \right) - \hat{\gamma}_i^k \hat{\nabla}_k V^j - d\hat{\gamma}^j_{ik} V^k \quad (4.22)$$

⁵⁶In [29] an alternative tensor was defined as $\hat{r}^i_{jkl} = \hat{R}^i_{jkl} + 2\hat{\gamma}^i_j \varpi_{kl}$ with $[\hat{\nabla}_k, \hat{\nabla}_l] V^i = \hat{r}^i_{jkl} V^j + \varpi_{kl} \frac{2}{\Omega} \partial_t V^i$.

with⁵⁷

$$\hat{r}^j_{ik} = \frac{1}{d} \left(\theta \varphi_i \delta_k^j + \hat{\nabla}_i \hat{\gamma}^j_k - \frac{1}{\Omega} \partial_t \hat{\gamma}^j_{ik} \right) \quad (4.23)$$

and

$$\hat{r}^j_{jk} = \hat{r}_k = \frac{1}{d} \left(\hat{\nabla}_j \hat{\gamma}^j_k - \hat{\partial}_k \theta \right), \quad (4.24)$$

further Carrollian curvature tensors.

Carrollian dynamics and Carrollian diffeomorphisms

Consider now a dynamical system on a Carrollian manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ described with an (effective) action $S = \int dt d^d x \sqrt{a} \Omega \mathcal{L}$, functional of a_{ij} , Ω and b_i . The associated *Carrollian momenta*, which replace the corresponding relativistic energy-momentum tensor (2.7) are now (see [79, 90])⁵⁸

$$\Pi^{ij} = \frac{2}{\sqrt{a} \Omega} \frac{\delta S}{\delta a_{ij}}, \quad (4.25)$$

$$\Pi^i = \frac{1}{\sqrt{a} \Omega} \frac{\delta S}{\delta b_i}, \quad (4.26)$$

$$\Pi = -\frac{1}{\sqrt{a}} \left(\frac{\delta S}{\delta \Omega} + \frac{b_i}{\Omega} \frac{\delta S}{\delta b_i} \right), \quad (4.27)$$

with $\frac{\delta S}{\delta \Omega} = -\sqrt{a} (\Pi + b_i \Pi^i)$. These are the *energy-stress tensor*, the *energy current* and the *energy density*.

Diffeomorphisms are generated by vector fields as in (2.71)

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \left(\xi^t - \xi^i \frac{b_i}{\Omega} \right) \partial_t + \xi^i \left(\partial_i + \frac{b_i}{\Omega} \partial_t \right) = \xi^{\hat{t}} \frac{1}{\Omega} \partial_t + \xi^i \hat{\partial}_i. \quad (4.28)$$

Carrollian diffeomorphisms (2.80) are restricted to $\xi^i = \xi^i(\mathbf{x})$. As usual, the variation under diffeomorphisms is implemented through the Lie derivative and we find the following:

$$-\delta_\xi a_{ij} = \mathcal{L}_\xi a_{ij} = 2 \hat{\nabla}_{(i} \xi^k a_{j)k} + 2 \xi^{\hat{t}} \hat{\gamma}_{ij} - 2 b_{(i} a_{j)k} \frac{1}{\Omega} \partial_t \xi^k, \quad (4.29)$$

where the last term drops for Carrollian diffeomorphisms. Furthermore

$$\mathcal{L}_\xi e_{\hat{t}} = - \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i \right) e_{\hat{t}} = \mu e_{\hat{t}}, \quad (4.30)$$

and (the form $\theta^{\hat{t}}$ is defined in (4.3))

$$\mathcal{L}_\xi \theta^{\hat{t}} = \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i \right) \theta^{\hat{t}} + \left((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2 \xi^j \varpi_{ji} \right) dx^i. \quad (4.31)$$

From the latter we infer

$$-\delta_\xi \ln \Omega = \frac{1}{\Omega} \mathcal{L}_\xi \Omega = \frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i, \quad -\delta_\xi b_i = \mathcal{L}_\xi b_i = b_i \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_j \xi^j \right) - \left((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2 \xi^j \varpi_{ji} \right). \quad (4.32)$$

⁵⁷Notice that $\frac{1}{\Omega} \partial_t \hat{\gamma}^j_{ik} = (\hat{\nabla}_i + \varphi_i) \hat{\gamma}^j_k + (\hat{\nabla}_k + \varphi_k) \hat{\gamma}^j_i - (\hat{\nabla}^j + \varphi^j) \hat{\gamma}_{ik}$ is a Carrollian tensor, even though $\hat{\gamma}^j_{ik}$ is not.

⁵⁸The fluid energy density Π was spelled e_e in [29].

We can now move to the variation of the action under Carrollian diffeomorphisms:

$$\delta_\xi S = \int dt d^d x \sqrt{a} \Omega \left(\frac{1}{2} \Pi^{ij} \delta_\xi a_{ij} + \Pi^i \delta_\xi b_i - \frac{1}{\Omega} (\Pi + b_i \Pi^i) \delta_\xi \Omega \right). \quad (4.33)$$

Using (4.29) and (4.32) with $\xi^i = \xi^i(\mathbf{x})$, we obtain (indices are here lowered with a_{ij})

$$\begin{aligned} \delta_\xi S = & \int dt d^d x \sqrt{a} \Omega \left\{ -\xi^{\hat{t}} \left[\left(\frac{1}{\Omega} \partial_t + \theta \right) \Pi + (\hat{\nabla}_i + 2\varphi_i) \Pi^i + \Pi^{ij} \hat{\gamma}_{ij} \right] \right. \\ & + \xi^i \left[(\hat{\nabla}_j + \varphi_j) \Pi^j_i + 2\Pi^j \varpi_{ji} + \Pi \varphi_i \right] \\ & + \int dt d^d x \left\{ \partial_t \left[\sqrt{a} \left(\xi^{\hat{t}} (\Pi + b_i \Pi^i) - \xi^j b_i \Pi^i_j \right) \right] \right. \\ & \left. \left. + \partial_i \left[\sqrt{a} \Omega \left(\xi^{\hat{t}} \Pi^i - \xi^j \Pi^i_j \right) \right] \right\} \right\}. \quad (4.34) \end{aligned}$$

Ignoring the boundary terms (last two lines of (4.34)), $\delta_\xi S = 0$ implies that the Carrollian momenta defined previously in (4.25), (4.26), (4.27) are Carrollian-covariant, and leads to two equations. The energy equation is simple because $\xi^{\hat{t}}$ depends on t and \mathbf{x} :

$$\boxed{\left(\frac{1}{\Omega} \partial_t + \theta \right) \Pi + (\hat{\nabla}_i + 2\varphi_i) \Pi^i + \Pi^{ij} \hat{\gamma}_{ij} = 0.} \quad (4.35)$$

The momentum equation calls for a careful treatment. Indeed, ξ^i is a function of \mathbf{x} only, hence the factor in brackets in the second line of (4.34) needs not vanish, but rather

$$\boxed{(\hat{\nabla}_j + \varphi_j) \Pi^j_i + 2\Pi^j \varpi_{ji} + \Pi \varphi_i = - \left(\frac{1}{\Omega} \partial_t + \theta \right) P_i} \quad (4.36)$$

because $\sqrt{a} \Omega \xi^i \left(\frac{1}{\Omega} \partial_t + \theta \right) P_i = \partial_t (\sqrt{a} \xi^i P_i)$, which is a boundary term and vanishes inside the integral.

The new vector P_i , which we will refer to as *momentum*, is not defined directly through a variation of the action with respect to some conjugate variable. It is however inescapable, and this can be verified whenever a microscopic action is available in terms of fundamental fields leading to full-fledged equations of motion. In this instance, eqs. (4.35) and (4.36) must be obeyed on-shell, and this procedure determines the momentum P_i (for a Carrollian scalar field see [103]).

A plethora of comments and comparison to the existing literature is appropriate at this point. The Carrollian equations at hand, which are ultimately the Carrollian fluid equations, have been the source of confusion or misinterpretation, and unfortunately this is not fading.

As a first and minor remark, the term in the right-hand side of (4.36) was missing in [79, 90].⁵⁹ More importantly, it has been claimed that both vectors, the energy current Π_i and the momentum P_i , should vanish [74, 77]. Systems where this happens are not

⁵⁹More precisely, eq. (4.36) here is in disagreement with eq. (2.30) in [79] resulting from the invariance of the action under Carrollian diffeomorphisms. It agrees however with eq. (4.4) in the same reference, reached via a vanishing- c limiting procedure (see also our eq. (4.81)).

excluded, but there is no principle that demands a priori such a property — as no reason exist for the Galilean energy current and momentum to vanish.⁶⁰

Another feature of the above equations (4.35) and (4.36) is their superficial resemblance to eqs. (3.42) and (3.41). This is superficial because when coming to the genuine fluid equations, the various momenta are expressed in terms of kinematical and physical parameters, which are different in the two instances (e.g. there is no “velocity” in Carrollian dynamics — see next section for details on the Carrollian fluids). But even at the superficial level, the resemblance is alleviated by the symmetries, which are undoubtedly distinct: Carrollian versus Galilean. Nonetheless, confusion has settled for good in the literature around the membrane paradigm, which was originally carefully stated [91], but has drifted in time giving an overwhelming importance to Navier-Stokes equations, which are Galilean par excellence, for the description of phenomena occurring in the vicinity of black-hole horizons, which call for Carrollian physics (ref. [92] is a good example of abuse). Recently, efforts were made to clarify this issue [70, 71].

A U(1) local invariance and conservation law

If the action on the Carrollian manifold is further invariant under a local U(1) associated with a gauge field $B = B(t, \mathbf{x})dt + B_i(t, \mathbf{x})dx^i$ as in (3.43), then a further conservation is available. This sort of conservation is not as useful as it was in the Galilean framework. Indeed, the thermodynamic law (A.2) that sets the relationship between a conserved charge and the energy (see also the discussion in section 3.3) is invalidated here by the vanishing- c limit, and plays no subsequent role in the fluid dynamics.

The conjugate momenta are again the *charge density* and the *charge current*:

$$\varrho = \frac{1}{\sqrt{a}} \left(\frac{\delta S}{\delta B} - \frac{b_i}{\Omega} \frac{\delta S}{\delta B_i} \right), \tag{4.37}$$

$$N^i = \frac{1}{\Omega \sqrt{a}} \frac{\delta S}{\delta B_i} \tag{4.38}$$

with $\frac{\delta S}{\delta B} = \sqrt{a} (\varrho + b_i N^i)$. The gauge variation of the action is here:

$$\delta_\Lambda S = \int dt d^d x \sqrt{a} \left((\varrho + b_i N^i) \delta_\Lambda B + \Omega N^i \delta_\Lambda B_i \right) \tag{4.39}$$

$$\begin{aligned} &= - \int dt d^d x \sqrt{a} \left((\varrho + b_i N^i) \partial_t \Lambda + \Omega N^i \partial_i \Lambda \right) \\ &= \int dt d^d x \sqrt{a} \Omega \Lambda \left(\frac{1}{\Omega} \partial_t \varrho + \theta \varrho + (\hat{\nabla}_i + \varphi_i) N^i \right) \\ &\quad - \int dt d^d x \left\{ \partial_t \left(\sqrt{a} \Lambda (\varrho + b_i N^i) \right) + \partial_i \left(\sqrt{a} \Lambda \Omega N^i \right) \right\}. \end{aligned} \tag{4.40}$$

Invariance of S leads to a Carrollian continuity equation:

$$\boxed{\left(\frac{1}{\Omega} \partial_t + \theta \right) \varrho + (\hat{\nabla}_i + \varphi_i) N^i = 0.} \tag{4.41}$$

⁶⁰As an aside comment, in flat as in AdS holography, the momentum P_i of the boundary fluid is mapped onto the bulk angular-momentum aspect [28], and this is not expected to vanish. From a different perspective, as already mentioned, microscopic systems such as the Carrollian scalar field exhibit non-vanishing Π_i and P_i [79, 103].

Using Stokes and Gauss theorems (see also footnote 33) and the Carrollian continuity equation (4.41) we find

$$\int_{\mathcal{W}} dt d^d x \Omega \sqrt{a} \left(\left(\frac{1}{\Omega} \partial_t + \theta \right) \varrho + \left(\hat{\nabla}_i + \varphi_i \right) N^i \right) = \oint_{\partial \mathcal{W}} \sqrt{a} \varrho dx^1 \wedge \dots \wedge dx^d - \oint_{\partial \mathcal{W}} \sqrt{a} \sum_{i=1}^d dx^1 \wedge \dots \wedge N^i \theta^{\hat{i}} \wedge \dots \wedge dx^d, \tag{4.42}$$

where $\mathcal{W} \subset \mathcal{M} = \mathbb{R} \times \mathcal{S}$ and $N^i \theta^{\hat{i}}$ ($\theta^{\hat{i}}$ given in (4.3)) is the i th factor in the exterior product of the last term. Assuming a good behaviour for the fields, a conserved charge exists and can be expressed as an integral over an arbitrary space-like hypersurface Σ_d of $\mathcal{M} = \mathbb{R} \times \mathcal{S}$. This conserved charge is identical to the relativistic Papapetrou-Randers result obtained e.g. in (2.110). As for the Galilean instance, it is suitable to chose $\Sigma_d \equiv \mathcal{S}$ i.e. a constant- t hypersurface, and the charge then reads:⁶¹

$$Q_N = \int_{\mathcal{S}} d^d x \sqrt{a} \left(\varrho + b_i N^i \right). \tag{4.43}$$

Following the Galilean steps, time-independence reveals by replacing \mathcal{S} in (4.43) with $\mathcal{V} \subset \mathcal{S}$, where the boundary $\partial \mathcal{V}$ does not depend on t , for convenience. Using (4.41), the time evolution of the matter/charge content of \mathcal{V} is the following:

$$\frac{d}{dt} \int_{\mathcal{V}} d^d x \sqrt{a} \left(\varrho + b_i N^i \right) = - \int_{\mathcal{V}} d^d x \partial_i \left(\sqrt{a} \Omega N^i \right) = - \int_{\partial \mathcal{V}} \Omega \star \mathbf{N}. \tag{4.44}$$

If \mathcal{V} is extended to the whole \mathcal{S} the time dependence fades and we find that Q_N is conserved.

Isometries, conservation and non-conservation

Isometries of Carrollian spacetimes are generated by Killing fields of the Carrollian type (4.28), required to obey

$$\mathcal{L}_{\xi} a_{ij} = 0, \quad \mathcal{L}_{\xi} e_{\hat{i}} = 0, \tag{4.45}$$

because the metric (4.1) and the field of observers (4.2) are the fundamental geometric data in the spacetimes at hand (see refs. [34–36, 45]). For Carrollian diffeomorphisms ($\xi^i \equiv \xi^{\hat{i}}$ is only \mathbf{x} -dependent), eqs. (4.29) and (4.30) lead to

$$\boxed{\hat{\nabla}_{(i} \xi^k a_{j)k} + \xi^{\hat{i}} \hat{\gamma}_{ij} = 0, \quad \frac{1}{\Omega} \partial_t \xi^{\hat{i}} + \varphi_i \xi^i = 0.} \tag{4.46}$$

⁶¹It should be noticed that the presence of b_i apparently breaks the manifest covariance, since according to (2.83) the form of the integrand is respected only by coordinate transformations such that $t' = t'(t)$ i.e. a subset of Carrollian diffeomorphisms. This is actually innocuous, and merely translates a feature of the hypersurface chosen for computing the charge, which is otherwise an absolute constant, as emphasized in footnote 9 for the relativistic case, and further discussed in section 3.1 within the Galilean framework. If the clock form is closed (see eq. (4.13)), locally $\theta^{\hat{i}} = d\tau$ and one may alternatively choose the integration hypersurface Σ_{τ} as $\tau(t, \mathbf{x})$ kept constant. In this instance, we obtain $Q_N = \int_{\Sigma_{\tau}} d^d x \sqrt{a} \varrho$. Nevertheless, all choices of space-like hypersurface Σ_d lead to the same charge.

These equations refer to the invariance of a *weak Carroll structure* [35] and possess an infinite set of solutions. As for the Newton-Cartan case discussed in section 3.1, *strong Carroll structures* are further equipped with a torsionless metric-compatible connection, which is also required to be invariant under Carrollian isometries. This constricts the solution space of (4.46). Observe however that one does not demand the Ehresmann be invariant, hence for a Carrollian Killing field ξ , using (4.46) inside (4.31) we obtain:

$$\mathcal{L}_\xi \theta^{\hat{t}} = \left((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right) dx^i. \tag{4.47}$$

The case of a Carroll spacetime with $a^{ij} = \delta^{ij}$, $\Omega = 1$ and constant b_i (standard flat Carroll spacetime with our connection) provides a nice illustration of the above. Equations (4.46) possess an infinite number of solutions:

$$\xi = \left(\Omega_i^j x^i + X^j \right) \partial_j + f(\mathbf{x}) \partial_t \tag{4.48}$$

with constant and antisymmetric $\Omega_{ij} = \Omega_i^k \delta_{kj}$ generating the rotations in $\mathfrak{so}(d)$, constant X^j for the space translations, and an arbitrary function of space $f(\mathbf{x})$. The latter is only linear if the connection of the strong Carroll structure is required to remain invariant under ξ : $f = T - B_i x^i$, T generating time translations and B_i being the Carroll boosts. The total number of solutions is now $(d+2)(d+1)/2$, which is the dimension of the Carroll algebra $\mathfrak{cart}(d+1)$. Besides, we find that

$$\mathcal{L}_\xi \theta^{\hat{t}} = - \left(B_i + \Omega_i^j b_j \right) dx^i \neq 0, \tag{4.49}$$

exhibiting a constant shift in the Ehresmann connection.

We can now handle the conservation law that would take the Carrollian form (4.41) with a Carrollian scalar κ and a Carrollian vector K^i determined from the components $\xi^{\hat{t}}$ and $\xi^{\hat{i}}$ of a Carrollian Killing, and from the Carrollian momenta, i.e. the energy density Π , the energy current Π^i and the energy-stress tensor Π^{ij} defined in eqs. (4.25), (4.26), (4.27), as well as the momentum P^i , and satisfying the conservation equations (4.36) and (4.35). If such a conservation exists, the Carrollian scalar

$$\mathcal{K} = \left(\frac{1}{\Omega} \partial_t + \theta \right) \kappa + \left(\hat{\nabla}_i + \varphi_i \right) K^i \tag{4.50}$$

shall vanish on-shell. In fine, κ and K^i are disclosed in the on-shell boundary terms of $\delta_\xi S$ (see (4.34)) — or likewise, inherited from the relativistic-current components (2.107), (2.108), (2.109):⁶²

$$\kappa = \xi^i P_i - \xi^{\hat{t}} \Pi, \tag{4.51}$$

$$K^i = \xi^j \Pi_j^i - \xi^{\hat{t}} \Pi^i. \tag{4.52}$$

⁶²We anticipate here section 4.3, where eqs. (4.133) are obtained as a small- c expansions of (2.108) and (2.109), leading to (4.134), which includes (4.51) and (4.52).

The scalar \mathcal{K} can be determined using the conservation equations (4.36) and (4.35):

$$\mathcal{K} = -\Pi \left(\frac{1}{\Omega} \partial_i \xi^{\hat{t}} + \varphi_i \xi^{\hat{t}} \right) - \Pi^i \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right) + \Pi^i_j \left(\hat{\nabla}_i \xi^j + \xi^{\hat{t}} \hat{\gamma}_i^j \right) \quad (4.53)$$

$$= -\Pi^i \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right). \quad (4.54)$$

We have obtained (4.54) from (4.53) thanks to the Killing equations (4.46). This latter result shows that *in Carroll structures, a Killing field does not guarantee an on-shell conservation law for Carrollian dynamics.*⁶³

The above result is expected as in the Galilean case, where a similar non-conservation was proven: the energy current Π^i is conjugate to b_i (4.26) and b_i does transform under diffeomorphisms (see (4.32)), *even when this diffeomorphism is an isometry*. Nonetheless, eq. (4.54) infers that a conservation law exists for the subalgebra of Killing vectors such that $\mathcal{L}_\xi \theta^{\hat{t}} = 0$,⁶⁴ in agreement with general Noether's theorem, for which isometry seems insufficient and a stronger symmetry required. Incidentally, one cannot exclude that the right-hand side of (4.54) originates from a boundary term (as for the corresponding Galilean equation (3.61)). Conservation would then occur — put differently (4.50) would vanish with effective Carrollian current κ' and $K^{i'}$, amended by the boundary-term contributions.

As we have emphasized slightly above, *ordinary Carrollian boosts in flat Carroll structures* ($a_{ij} = \delta_{ij}$, $\Omega = 1$, b_i constants) *do not satisfy the extra condition* $\mathcal{L}_\xi \theta^{\hat{t}} = 0$ (see (4.49)), *and thus no conservation law is necessarily associated with them*. This property was disregarded in refs. [74, 76, 77], where the authors took for granted that such a conservation should exist in the primitive sense i.e. with $\mathcal{K} = 0$ in (4.54).⁶⁵ This assumption led to the conclusion that Π^i should always vanish, as we have already pointed out. A similar reasoning in the Galilean case (see (3.61)) amounts to stating that the fluid momentum P_i ought to vanish. This is a notoriously degenerate state, where either the fluid is absent, or motion is absent — global equilibrium is reached. Having no intuition for Carrollian fluids, we leave open a teensy possibility for a state with vanishing energy flux to make sense. Such a state is by no means a consequence of any spacetime symmetry though.⁶⁶ This feature will be recast at the end of section 4.3, where besides the Carrollian current (4.51), (4.52), more currents of the same sort appear, which may or may not be conserved as a consequence of the vanishing of a vector, be it the energy flux Π^i met here, or another vector emerging in the small- c expansion of the relativistic heat current.

⁶³Credit should be given to the authors of [79] for observing this phenomenon in a quite general framework.

⁶⁴The Jacobi identity is used to show that the commutator of two ξ s obeying $\mathcal{L}_\xi \theta^{\hat{t}} = 0$, satisfies the same condition.

⁶⁵Notice that conservation might be valid with non-vanishing \mathcal{K} , if the latter is a generalized divergence. In flat space we find $\mathcal{K} = \Pi^i (B_i + \Omega_i^j b_j)$ using eqs. (4.48) and (4.49). Under the assumption of *potential flow* (we borrow the Galilean language) $\Pi_i = \hat{\partial}_i \phi + \partial_i \phi_i$ with $\phi(t, \mathbf{x})$ and $\phi_i(t, \mathbf{x})$ the potential functions so that $\mathcal{K} = \hat{\partial}_i \phi U^i + \partial_i \phi_i U^i$ with $U^i = B^i + \Omega^{ij} b_j$. In this very specific instance, there is a conserved Carrollian current associated with boosts and rotations, and components $\kappa - \phi_i U^i$ and $K^i - \phi U^i$ making (4.50) vanish.

⁶⁶Situations of this sort are not forbidden but are not demanded a priori. They may occur outside the realm of fluids. For instance, a Carrollian scalar field has always vanishing energy flux in its electric edition, and can be set to zero in magnetic some configurations [77, 79, 103]. However, this flux is generically non-zero in the magnetic version, although configurations do exist for which it vanishes.

Weyl invariance, conformal isometries, conservation and non-conservation

As for the Newton-Cartan spacetimes, Weyl transformations can be investigated in Carrollian manifolds. They act on their basic geometric data as

$$a_{ij} \rightarrow \frac{1}{\mathcal{B}^2} a_{ij}, \quad \Omega \rightarrow \frac{1}{\mathcal{B}} \Omega, \quad b_i \rightarrow \frac{1}{\mathcal{B}} b_i, \quad (4.55)$$

where $\mathcal{B} = \mathcal{B}(t, \mathbf{x})$ is an arbitrary function. The Carrollian momenta Π^{ij} , Π^i and Π defined in (4.25), (4.26) and (4.27) inherit conformal weights $d+3$, $d+2$ and $d+1$ when the effective action is presumed Weyl-invariant. The momentum P^i appearing in (4.36) has also weight $d+2$, and in the matter sector, assuming the gauge field B and B_i be weight-zero, we conclude from (4.37) and (4.38) that the density ϱ and the matter current N^i have weights d and $d+1$.

Requiring Weyl invariance for the effective action $\delta_{\mathcal{B}} S = 0$, expression (4.33) implies that

$$\Pi_i{}^i = \Pi. \quad (4.56)$$

In order to implement elegantly Weyl covariance, the appropriate covariant derivatives will be introduced in section 4.2 for time and space, dubbed Weyl-Carroll. For the moment, we wish to circumscribe our discussion and adapt to Carroll the pattern discussed in section 3.1 for Newton-Cartan spacetimes, Weyl-invariant dynamics and conformal isometries. This will lead to the same conclusion as above: *a conformal Killing field does not always provide a conservation law in Weyl-invariant Carrollian dynamics.*

Following [33–36, 45] a conformal isometry is generated by a vector field ξ satisfying

$$\mathcal{L}_{\xi} a_{ij} = \lambda a_{ij}, \quad (4.57)$$

where

$$\lambda(t, \mathbf{x}) = \frac{2}{d} \left(\hat{\nabla}_i \xi^i + \theta \xi^t \right). \quad (4.58)$$

The extra condition imposed for reaching an operational definition of conformal Killing vectors is again (3.66) i.e. $2\mu + \lambda = 0$ with (see (4.30))

$$\mu(t, \mathbf{x}) = - \left(\frac{1}{\Omega} \partial_t \xi^t + \varphi_i \xi^i \right). \quad (4.59)$$

A dynamical exponent z , here equal to 1, can also be defined (see footnote 38), and the distinction of weak versus strong Carroll structures supplements the discussion on the web of conformal Killing fields.

Assuming the existence of a conformal isometry, the conservation equations (4.35) and (4.36) can be used for computing the Carrollian scalar \mathcal{K} (4.50) with (4.51), (4.52) and (4.56):

$$\mathcal{K} = \Pi \left(\frac{\lambda}{2} + \mu \right) - \Pi^i \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^t - 2\xi^j \varpi_{ji} \right). \quad (4.60)$$

The defining equation (3.66) for conformal Killing vectors on Carrollian spacetimes expectedly arises in (4.60), but is insufficient to ensure $\mathcal{K} = 0$. As anticipated, *a plain*

conformal Killing field does not generically provide a conservation law in Weyl-invariant Carrollian dynamics.

The same conclusion has been reached for Galilean spacetimes in section 3.1. It will be further investigated from the small- c viewpoint in the following paragraphs, and finally in appendix D.2.

As an example, let us again consider the standard flat Carroll spacetime ($a^{ij} = \delta^{ij}$, $\Omega = 1$ and constant b_i). Equations (4.57) and (3.66) possess an infinite number of solutions, which for a strong Carroll structure read [35, 36, 45]:

$$\xi = Y^j(\mathbf{x})\partial_j + \left(T(\mathbf{x}) + \frac{t}{d}\partial_i Y^i\right)\partial_t \tag{4.61}$$

with $T(\mathbf{x})$ an arbitrary function generating the *supertranslations* and $Y^i(\mathbf{x})\partial_j$ being conformal Killing fields of Euclidean d -dimensional space, generating $\mathfrak{so}(d+1, 1)$. This is the conformal Carroll algebra $\mathfrak{ccarr}(d+1) \equiv \mathfrak{so}(d+1, 1) \ltimes \text{supertranslations}$, also known as BMS_{d+2} (for Bondi-van der Burg-Metzner-Sachs) [35, 36].⁶⁷ We also find how the clock form behaves:

$$\mathcal{L}_\xi \theta^t = \left(\partial_i (T - Y^j b_j) + \frac{b_i}{d}\partial_j Y^j + \frac{t}{d}\partial_i \partial_j Y^j\right) dx^i. \tag{4.62}$$

The associated current is not conserved since \mathcal{K} in (4.60) does not generically vanish,⁶⁸ unless $\partial_j Y^j = C_0$ and $T = T_0 + Y^j b_j - \frac{C_0}{d} b_i x^i$, thus linear in x^i (C_0 and T_0 are constants). This excludes the d special conformal transformations of $\mathfrak{so}(d+1, 1)$ and leaves the supertranslations with the time translation as unique freedom, leading to a symmetry subgroup of finite dimension $\frac{d^2}{2} + \frac{d}{2} + 2$.

One should stress again that generally, $\Pi^i \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^t - 2\xi^j \varpi_{ji} \right)$ with ξ a conformal Killing field of a Carrollian manifold \mathcal{M} , might be a boundary term possibly leading to a conserved Carrollian current. This needs however to be appreciated case-by-case and not premised.

4.2 Carrollian hydrodynamics from relativistic fluids — I

The small- c expansion

Following the pattern introduced in [29], we will now study the vanishing- c limit of relativistic hydrodynamics on a pseudo-Riemannian manifold, in Papapetrou-Randers coordinates. As for the Galilean case, all c -dependence in the geometry is explicit. In particular, the fluid velocity is parameterized with β^i introduced in (2.85), and for small c we obtain:

$$v^i = c^2 \Omega \beta^i + \mathcal{O}(c^4). \tag{4.63}$$

⁶⁷In the presence of a dynamical exponent defined via $2\mu + z\lambda = 0$ (in this case the Carrollian structure is not inherited from the Carrollian limit of a pseudo-Riemannian spacetime), the algebra exhibits a level $N = 2/z$: $\mathfrak{ccarr}_N(d+1)$. Strictly speaking $\mathfrak{ccarr}(d+1) \equiv \mathfrak{ccarr}_2(d+1)$ is BMS_{d+2} for $d = 1, 2$ only, because for higher d the BMS algebra is finite-dimensional, whereas $\mathfrak{ccarr}_N(d+1)$ is not. Infinite-dimensional extensions of the BMS algebra have been nevertheless presented in the literature (see e.g. [99] for a recent account and further references).

⁶⁸For general Carrollian fluids it has no a priori reason to vanish, and no Noether current exists. Nonetheless, one finds explicit dynamics where these currents are enforced (see e.g. [98] for the scalar electrodynamics), and where the full conformal Carroll group is realized in terms of the associated conserved charges.

The fluid velocity vanishes at zero c — this is not a surprise — but a kinematical parameter with dimensions of an inverse velocity is bound to remain in as a Carrollian-fluid variable. The full fluid congruence then reads:

$$u_0 = -c\Omega + \mathcal{O}(c^3), \quad u^i = c^2\beta^i + \mathcal{O}(c^4), \quad (4.64)$$

whereas the expansion and the shear behave as

$$\Theta = \frac{1}{\Omega} \partial_t \ln \sqrt{a} + \mathcal{O}(c^2) = \theta + \mathcal{O}(c^2), \quad (4.65)$$

$$\sigma^{ij} = -\frac{1}{\Omega} \left(\frac{1}{2} \partial_t a^{ij} + \frac{1}{d} a^{ij} \partial_t \ln \sqrt{a} \right) + \mathcal{O}(c^2) = \xi^{ij} + \mathcal{O}(c^2) \quad (4.66)$$

with ξ^{ij} and θ defined for a Carrollian manifold in (4.15) and (4.16).

In order to go on with the fluid equations, we should handle the behaviour of the energy-momentum tensor at small c . This includes ε , p , q^i and τ^{ij} . The matter current j^i , if present, plays no role since no relationship amongst ε and ϱ survives in the Carrollian limit. In the absence of thermodynamic or transport hints for the behaviour of these quantities, we will consider an ansatz motivated by the Carrollian fluids emerging in flat holography [28]. The simplest is

$$\varepsilon = \eta + \mathcal{O}(c^2), \quad (4.67)$$

$$p = \varpi + \mathcal{O}(c^2), \quad (4.68)$$

$$q^i = Q^i + c^2 \pi^i + \mathcal{O}(c^4), \quad (4.69)$$

$$\tau^{ij} = -\Xi^{ij} + \mathcal{O}(c^2). \quad (4.70)$$

This follows the pattern of the Galilean counterpart (3.75) and (3.76), except that the energy is now of order 1, as for the case of a massless-carrier Galilean fluid. In section 4.3 we will consider a Laurent expansion with order- $1/c^2$ terms, as required in flat-holography fluids.

Although a conserved current is not essential in the discussion, we present it for completeness with the following ansatz

$$\varrho_0 = \chi + \mathcal{O}(c^2), \quad j^k = n^k + \mathcal{O}(c^2). \quad (4.71)$$

We recall that ϱ_0 is the proper density i.e. the density measured by an observer with velocity u^μ . We could consider a fiducial observer, who would play here the role of u_Z in Zermelo frame:

$$u_{\text{PR}} = \frac{1}{\Omega} \partial_t. \quad (4.72)$$

This observer is not geodesic (unless ∂_t it is a Killing field), but this is of secondary importance since inertial frames play no role in the Carrollian limit. For this observer, the fluid density is $-\frac{1}{c^2} J_\mu u_{\text{PR}}^\mu = \frac{c}{\Omega} J_0$, which coincides with ϱ_{0r} given in (2.103).

Carrollian momenta and hydrodynamic equations

With the data (4.67), (4.68), (4.69), (4.70), the invariant pieces of the relativistic energy-momentum tensor defined in (2.97), (2.100), (2.101) and (2.102) read:

$$q_r^i = \Pi^i + c^2 P^i + \mathcal{O}(c^4), \quad (4.73)$$

$$\varepsilon_r = \Pi + \mathcal{O}(c^2), \quad (4.74)$$

$$p_r a^{ij} + \tau_r^{ij} = \Pi^{ij} + \mathcal{O}(c^2) \quad (4.75)$$

with

$$\Pi = \eta + 2\beta_i Q^i, \quad \Pi^i = Q^i, \quad \Pi^{ij} = Q^i \beta^j + \beta^i Q^j + \varpi a^{ij} - \Xi^{ij}, \quad (4.76)$$

and

$$P^i = \pi^i + \beta^i \left(\eta + \varpi + \beta_k Q^k \right) - \beta_k \Xi^{ki} + \frac{\beta^2}{2} Q^i. \quad (4.77)$$

Equations (2.1) with the energy-momentum tensor at hand demand the following expressions

$$\frac{c}{\Omega} \nabla_\mu T^\mu_0 = \mathfrak{E} + \mathcal{O}(c^2), \quad (4.78)$$

$$\nabla_\mu T^{\mu i} = \frac{1}{c^2} \left\{ \left(\frac{1}{\Omega} \hat{D}_t + \theta \right) \Pi^i + \Pi^j \hat{\gamma}_j^i \right\} + \mathfrak{G}^i + \mathcal{O}(c^2) \quad (4.79)$$

be zero with

$$\mathfrak{E} = - \left(\frac{1}{\Omega} \hat{D}_t + \theta \right) \Pi - \left(\hat{\nabla}_i + 2\varphi_i \right) \Pi^i - \Pi^{ij} \hat{\gamma}_{ij}, \quad (4.80)$$

$$\mathfrak{G}_j = \left(\hat{\nabla}_i + \varphi_i \right) \Pi^i_j + 2\Pi^i \varpi_{ij} + \Pi \varphi_j + \left(\frac{1}{\Omega} \hat{D}_t + \theta \right) P_j + P^i \hat{\gamma}_{ij}. \quad (4.81)$$

Hence, we recover the Carrollian momenta conservation equations (4.35) and (4.36) (notice the use of the Carrollian time covariant derivative (4.18)) *augmented with an extra equation on the energy current*

$$\left(\frac{1}{\Omega} \hat{D}_t + \theta \right) \Pi^i + \Pi^j \hat{\gamma}_j^i = 0. \quad (4.82)$$

The whole Carrollian scheme of the present section, and eq. (4.82) in particular, resonate with the discussion made on the Galilean side, section 3.3. This equation is indeed absent when working directly in the framework of a Carrollian manifold with Carrollian diffeomorphisms, as in section 4.1. It is in fact a boundary term that could not be retrieved by a variational principle — as the Carrollian momentum P^i appeared to be necessary but undetermined. Getting the Carrollian dynamics as a zero- c limit of relativistic hydrodynamics is richer.

On the one hand, equation (4.82) emerges as a vestige of the original full-diffeomorphism relativistic invariance,⁶⁹ and is the dual of the Galilean constraint equation (3.122) —

⁶⁹Precise statements on this reminiscence of full diffeomorphisms are illusive. In particular, no central extension can accompany the Carrollian contraction. According to ref. [84], the prerequisite for this to occur is the existence of an absolute time i.e. Galilean or Aristotelian frameworks.

remember that time and space play dual roles in the two limits considered here, interchanging, among others, momentum and energy current. On the other hand, the momentum P^i is no longer undetermined, and stands for the subleading term of the relativistic heat current (4.73), expressed explicitly in terms of the kinematical and “thermodynamic-transport” observables — β^i and η , ϖ , Q^i , π^i , Ξ^{ij} in (4.77).

This last observation calls for a comparison with the existing literature. At the first place it should be accepted that the very concept of spacetime energy-momentum tensor is loose in Carrollian (and Galilean) physics. This was clearly emphasized in the early work [79], where the role of Carrollian momenta as a necessary replacement to the energy-momentum tensor was proposed. It is stated again here, and one should moreover avoid the latent confusion amongst momenta (or energy-momentum tensor, if any) as a response of the system to geometry disturbances, and Noetherian currents, which are generically absent as no isometries have been assumed. Next, it is clear that nothing requires the vanishing of neither the energy flux Π^i , nor the momentum P^i , irrespective of the approach — Carrollian conservation or zero- c limit. This has been undermined in [74, 76, 77]. Finally, the momenta are expressed in terms of an “inverse velocity” β^i and not a velocity v^i , since no velocity is compatible with Carrollian physics due to the shrinking of the light cone. It should be added that no matter density enters the momenta because no relationship exists any longer between energy and mass. Equation (A.2) is obsolete in the Carrollian limit, and it is fair to admit that Carrollian thermodynamics remains in limbo — as pointed out in appendix A. This is sometimes overlooked.

From a more abstract viewpoint, a relation between energy and some other charge (possibly, but not necessarily the mass) might appear only if a conservation exists that defines this charge, independently of the conservation involving the energy. Such a conservation may or may not be present for the relativistic fluids, and is emergent in the Galilean limit. For the Carrollian case, if a U(1) conservation law of the type (2.2) is available in the ascendent theory, we find, after inserting (4.71) inside (2.103) and (2.104):

$$\varrho_{0r} = \varrho + \mathcal{O}(c^2), \tag{4.83}$$

$$j_r^i = N^i + \mathcal{O}(c^2), \tag{4.84}$$

with

$$\varrho = \chi + \beta_i n^i, \quad N^i = n^i, \tag{4.85}$$

the matter Carrollian momenta explicitly determined in terms of β^i . We can now compute the divergence of (2.4) in the Papapetrou-Randers background (2.79). The result is

$$\nabla_\mu J^\mu = \mathcal{F} + \mathcal{O}(c^2) \tag{4.86}$$

with

$$\mathcal{F} = \left(\frac{1}{\Omega} \partial_t + \theta \right) \varrho + \left(\hat{\nabla}_j + \varphi_j \right) N^j, \tag{4.87}$$

and demanding the conservation, we recover the Carrollian continuity equation (4.41).

Hydrodynamic-frame invariance

We discussed hydrodynamic-frame invariance in the framework of relativistic fluids, where it states that the relativistic fluid equations remain invariant under arbitrary unimodular transformations of the velocity field u , performed together with transformations of the energy density, pressure, heat current and stress tensor — see e.g. (2.92), (2.93) and (2.94).⁷⁰ The question we want now to answer is again whether this invariance survives the Carrollian limit.

From the experience we have acquired in the Galilean section, answering requires a careful analysis, and the output depends on several options. Assumptions are made about the small- c behaviour of the various observables (see (4.67), (4.68), (4.69), and (4.70)) and this behaviour may not be stable under Carrollian hydrodynamic-frame transformations. As opposed to the Galilean situation, there is no physical intuition that can argue in favour or against. There are however concrete results from flat holography [26, 27] suggesting that Carrollian hydrodynamic-frame invariance should hold as a local boundary symmetry, translating in a bulk diffeomorphism transformation.

As for the Galilean case, the operators entering (4.80), (4.81) and (4.82) are velocity-independent, and the momenta P_i , Π , Π_i and Π_{ij} appear as coefficients in the expansion of the hydrodynamic-frame-invariant relativistic momenta (4.73), (4.74) and (4.75). In order to conclude about hydrodynamic-frame invariance, we must investigate the stability of the scaling properties encoded in (4.67), (4.68), (4.69) and (4.70), using the transformation rules set in the Papapetrou-Randers frame (2.92), (2.93) and (2.94). We find the following:

$$\delta\eta = -2\delta\beta_i Q^i, \tag{4.88}$$

$$\delta Q^i = 0, \tag{4.89}$$

$$\delta\pi^i = \delta\beta_j \left(\Xi^{ij} - (\eta + \varpi) a^{ij} + \beta^i Q^j \right), \tag{4.90}$$

$$\delta \left(\Xi^{ij} - \varpi a^{ij} \right) = \delta\beta_k \left(Q^i a^{jk} + Q^j a^{ik} \right). \tag{4.91}$$

Remarkably, under the frame transformations at hand, the Carrollian densities and fluxes defined in (4.76) and (4.77) are invariant:

$$\delta\Pi = 0, \quad \delta\Pi^i = 0, \quad \delta\Pi^{ij} = 0, \quad \delta P^i = 0. \tag{4.92}$$

Contrary to the Galilean massive case (section 3.2), but similarly to the Galilean massless case (section 3.4), the energy-momentum-tensor dynamics and the current dynamics are decoupled here. This decoupling holds in particular for hydrodynamic-frame invariance, and we are invited to iterate the above course for matter dynamics. For matter, the transformation rules in Papapetrou-Randers frame are (2.95) and (2.96), whereas the invariant relativistic momenta (4.83), (4.84) should be used in conjunction with the small- c

⁷⁰In the Papapetrou-Randers frame, the local unimodular transformations (2.28) are captured by $\beta^i \rightarrow \beta^i + \delta\beta^i(t, \mathbf{x})$ (see eqs. (2.51)), (2.85), (2.87)) parameterized as $\delta\beta^i = B^i - c^2 B^j \beta_j \beta^i + \Omega^{ij} \beta_j$. Infinitesimal Lorentz boosts are associated with $B^i(t, \mathbf{x})$, while infinitesimal rotations go along with the antisymmetric $\Omega^{ij}(t, \mathbf{x})$. In the Carrollian limit, the general transformation, which captures Carrollian boosts and rotations, reads: $\delta\beta^i = B^i + \Omega^{ij} \beta_j$.

behaviour (4.71). The output is now

$$\delta\chi = -\delta\beta_i n^i, \tag{4.93}$$

$$\delta n^i = 0. \tag{4.94}$$

Using (4.85), one shows that

$$\delta\rho = 0, \quad \delta N^i = 0. \tag{4.95}$$

This result demonstrates the invariance of (4.87). In conclusion, *matter dynamics is hydrodynamic-frame-invariant*, establishing thereby that *Carrollian fluid dynamics is hydrodynamic-frame invariant*.

Weyl-invariant Carrollian fluids

Carrollian fluids are fundamental ingredients of flat holography, where they appear in their Weyl-invariant version [28]. On a Papapetrou-Randers frame, Weyl transformations generated by $\mathcal{B}(t, \mathbf{x})$ act as they do on the fundamental data of a Carrollian spacetime, eqs. (4.55). The Weyl-invariance condition (4.56) is here reached as the zero- c limit of the relativistic Weyl-invariance condition $T^\mu{}_\mu = 0$ discussed at the end of section 2.1, using (2.98), (4.73), (4.74) and (4.75). For Carrollian fluids, the various Carrollian momenta (4.76), (4.77), (4.85) are expressed in terms of fluid variables such as the inverse velocity β^i , the energy density η , the pressure ϖ as well as Q_i , π_i and Ξ_{ij} . Their conformal weights are⁷¹ $1, d+1, d+1, d, d$ and $d-1$. Similarly the weights of χ and n^i are d and $d+1$. Condition (4.56) reads $\eta = d\varpi - \Xi^i{}_i$, which is split as usual:

$$\eta = d\varpi, \quad \Xi^i{}_i. \tag{4.96}$$

The geometric tools necessary for handling Carrollian Weyl covariance were introduced in appendix A.2 of [29] and we summarize them here. We define *Weyl-Carroll* covariant time and space derivatives using θ and φ_i defined in (4.16) and (4.12), which transform as connections

$$\theta \rightarrow \mathcal{B}\theta - \frac{d}{\Omega}\partial_t\mathcal{B}, \quad \varphi_i \rightarrow \varphi_i - \hat{\partial}_i \ln \mathcal{B}, \tag{4.97}$$

as opposed to the Carrollian shear ξ_{ij} (4.15) and Carrollian vorticity ϖ_{ij} (4.11), which are Weyl-covariant of weight -1 . The action of the Carrollian Weyl-covariant time derivative on a weight- w function Φ is

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t\Phi = \frac{1}{\Omega}\hat{D}_t\Phi + \frac{w}{d}\theta\Phi = \frac{1}{\Omega}\partial_t\Phi + \frac{w}{d}\theta\Phi, \tag{4.98}$$

and this is a scalar of weight $w+1$. On a weight- w vector, the action is

$$\frac{1}{\Omega}\hat{\mathcal{D}}_t V^l = \frac{1}{\Omega}\hat{D}_t V^l + \frac{w-1}{d}\theta V^l = \frac{1}{\Omega}\partial_t V^l + \frac{w}{d}\theta V^l + \xi^l{}_i V^i. \tag{4.99}$$

⁷¹We mentioned in section 4.1 that Π^{ij} , Π^i , Π^i and Π have conformal weights $d+3$, $d+2$, $d+2$ and $d+1$, whereas the density ρ and the matter current N^i have weights d and $d+1$.

These are the components of a Carrollian vector of weight $w + 1$. Similarly for any tensor by Leibniz rule and in particular we find:

$$\hat{\mathcal{D}}_t a_{kl} = 0. \tag{4.100}$$

For a weight- w scalar function Φ , we introduce the space Weyl-covariant Carrollian derivative

$$\hat{\mathcal{D}}_j \Phi = \hat{\partial}_j \Phi + w \varphi_j \Phi, \tag{4.101}$$

which has the same conformal weight. Similarly, for a vector with weight- w components V^l :

$$\hat{\mathcal{D}}_j V^l = \hat{\nabla}_j V^l + (w - 1) \varphi_j V^l + \varphi^l V_j - \delta_j^l V^i \varphi_i. \tag{4.102}$$

The Weyl-Carroll spatial derivative does not modify the weight of the tensor it acts on. The action on any other tensor is obtained using the Leibniz rule, as in example for rank-two tensors:

$$\hat{\mathcal{D}}_j t_{kl} = \hat{\nabla}_j t_{kl} + (w + 2) \varphi_j t_{kl} + \varphi_k t_{jl} + \varphi_l t_{kj} - a_{jl} t_{ki} \varphi^i - a_{jk} t_{il} \varphi^i. \tag{4.103}$$

Moreover, it is metric-compatible:

$$\hat{\mathcal{D}}_j a_{kl} = 0. \tag{4.104}$$

Time and space Weyl-Carroll covariant derivatives do not commute. Their commutators allow to define further geometric tensors such as the Weyl-Carroll curvature (spatial and mixed space-time), which do also emerge in the small- c expansion of the relativistic Weyl curvature tensors introduced in (2.41), (2.42), (2.43), and evaluated in a Papapetrou-Randers background. More information is available in the already quoted reference [29].

With these derivatives, Carrollian equations (4.35), (4.36) and equation (4.82) read for a conformal fluid:

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi + \hat{\mathcal{D}}_i \Pi^i + \Pi^{ij} \xi_{ij} = 0, \tag{4.105}$$

$$\hat{\mathcal{D}}_i \Pi^i_j + 2 \Pi^i \varpi_{ij} + \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \delta_j^i + \xi^i_j \right) P_i = 0, \tag{4.106}$$

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi_j + \Pi_i \xi^i_j = 0. \tag{4.107}$$

These equations are Weyl-covariant of weights $d+2$, $d+1$ and $d+1$ (P_i is weight- d). They are also manifestly hydrodynamic-frame invariant. Similarly, for the matter sector, (4.41) reads:

$$\frac{1}{\Omega} \hat{\mathcal{D}}_t \varrho + \hat{\mathcal{D}}_j N^j = 0, \tag{4.108}$$

and is Weyl-covariant of weight $d + 1$ and hydrodynamic-frame invariant.

4.3 Carrollian hydrodynamics from relativistic fluids — II

More degrees of freedom

The behaviours (4.67), (4.68), (4.69) and (4.70) of energy, pressure, heat current and stress tensor are required for matching the zero- c limit of hydrodynamic equations with the Carrollian momenta conservation, while preserving hydrodynamic-frame invariance. At the end of section 3.3, we contemplated exotic Galilean situations involving more degrees of freedom and obeying extended systems of fluid equations reached at infinite c , such as (3.125). Although of limited use in the non-relativistic framework, this sort of extensions play a pivotal role when studying Carrollian fluids in flat holography, were more divergent terms appear to be needed [28]. It is worth elaborating in this direction in the spirit of [29], and consider in particular⁷²

$$\varepsilon = \frac{\zeta}{c^2} + \eta + \mathcal{O}(c^2), \tag{4.109}$$

$$p = \frac{\phi}{c^2} + \varpi + \mathcal{O}(c^2), \tag{4.110}$$

$$q^i = \frac{\psi^i}{c^2} + Q^i + c^2 \pi^i + \mathcal{O}(c^4), \tag{4.111}$$

$$\tau^{ij} = -\frac{\Sigma^{ij}}{c^2} - \Xi^{ij} + \mathcal{O}(c^2). \tag{4.112}$$

We can at the same time extend the matter sector with

$$\varrho_0 = \frac{\omega}{c^2} + \chi + \mathcal{O}(c^2), \quad j^k = \frac{m^i}{c^2} + n^k + \mathcal{O}(c^2). \tag{4.113}$$

With the new scalings, the expansion of the energy-momentum tensor components (2.100), (2.101), (2.102) is now

$$\begin{cases} \varepsilon_r = \frac{\tilde{\Pi}}{c^2} + \Pi + \mathcal{O}(c^2), \\ q_r^i = \frac{\tilde{\Pi}^i}{c^2} + \Pi^i + c^2 P^i + \mathcal{O}(c^4), \\ p_r a^{ij} + \tau_r^{ij} = \frac{\tilde{\Pi}^{ij}}{c^2} + \Pi^{ij} + \mathcal{O}(c^2). \end{cases} \tag{4.114}$$

The Carrollian momenta are displayed in (B.1). Similarly, the matter current (eqs. (2.103) and (2.104)) exhibits the following:

$$\varrho_{0r} = \frac{\tilde{\varrho}}{c^2} + \varrho + \mathcal{O}(c^2), \quad j_r^k = \frac{\tilde{N}^k}{c^2} + N^k + \mathcal{O}(c^2) \tag{4.115}$$

with $\tilde{\varrho}$, ϱ , \tilde{N}^k and N^k given in (B.2).

Using the above expansions in the relativistic divergence of the energy-momentum tensor on Papapetrou-Randers background (2.79) one obtains

$$\frac{c}{\Omega} \nabla_\mu T^\mu_0 = \frac{\mathcal{F}}{c^2} + \mathcal{G} + \mathcal{O}(c^2), \tag{4.116}$$

$$\nabla_\mu T^{\mu i} = \frac{\mathcal{X}^i}{c^4} + \frac{\mathcal{H}^i}{c^2} + \mathcal{G}^i + \mathcal{O}(c^2), \tag{4.117}$$

⁷²In holographic Carrollian fluids, one keeps terms up to order $1/c^4$. The pattern is the same.

while the divergence of the matter current reads:

$$\nabla_\mu J^\mu = \frac{\mathcal{N}}{c^2} + \mathcal{F} + \mathcal{O}(c^2). \quad (4.118)$$

In these expressions, \mathcal{E} , \mathcal{G}_j and \mathcal{F} are given in (4.80), (4.81) and (4.87), whereas the new expressions are

$$\mathcal{F} = -\left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\tilde{\Pi} - \left(\hat{\nabla}_i + 2\varphi_i\right)\tilde{\Pi}^i - \tilde{\Pi}^{ij}\hat{\gamma}_{ij}, \quad (4.119)$$

$$\mathcal{H}_j = \left(\hat{\nabla}_i + \varphi_i\right)\tilde{\Pi}^i_j + 2\tilde{\Pi}^i\varpi_{ij} + \tilde{\Pi}\varphi_j + \left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\Pi_j + \Pi^i\hat{\gamma}_{ij}, \quad (4.120)$$

$$\mathcal{X}_j = \left(\frac{1}{\Omega}\hat{D}_t + \theta\right)\tilde{\Pi}_j + \tilde{\Pi}^i\hat{\gamma}_{ij}, \quad (4.121)$$

and

$$\mathcal{N} = \left(\frac{1}{\Omega}\partial_t + \theta\right)\tilde{\varrho} + \left(\hat{\nabla}_j + \varphi_j\right)\tilde{N}^j. \quad (4.122)$$

At zero c , the Carrollian energy and momenta equations are thus $\mathcal{E} = \mathcal{F} = \mathcal{G}_j = \mathcal{H}_j = \mathcal{X}_j = 0$, and similarly $\mathcal{J} = \mathcal{N} = 0$ describe the matter sector. All these equations are invariant under hydrodynamic-frame transformations because the differential operators are geometric and thus invariant, and because the momenta $\tilde{\Pi}$, Π , $\tilde{\Pi}^i$, Π^i , P^i , $\tilde{\Pi}^{ij}$, Π^{ij} , $\tilde{\varrho}$, ϱ , \tilde{N}^i , N^i also are, as shown in appendix B.

Weyl-invariant Carrollian fluids

For the system under investigation, the use of an effective action is not convenient, as it would require a complete set of variables conjugate to the momenta Π , Π^i , Π^{ij} and $\tilde{\Pi}$, $\tilde{\Pi}^i$, $\tilde{\Pi}^{ij}$, which is bigger than a_{ij} , b_i and Ω . Weyl invariance is here easier to impose as a zero- c limit of $T^\mu{}_\mu = 0$ discussed at the end of section 2.1, using (2.98) with (4.114):

$$T^\mu{}_\mu = \frac{1}{c^2}\left(\tilde{\Pi}_i{}^i - \tilde{\Pi}\right) + \Pi_i{}^i - \Pi + \mathcal{O}(c^2) = 0, \quad (4.123)$$

leading to

$$\tilde{\Pi}_i{}^i = \tilde{\Pi}, \quad \Pi_i{}^i = \Pi. \quad (4.124)$$

These conditions can be recast in terms of Carrollian-fluid observables using the explicit expressions of the momenta (B.1). Splitting them again à la (2.45), into global-equilibrium equations of state plus conditions for the dynamical irreversible components, we find the following:

$$\zeta = d\phi, \quad \Sigma^i{}_i = 0, \quad \eta = d\varpi, \quad \Xi^i{}_i = \beta_i\beta_j\Sigma^{ij}. \quad (4.125)$$

The conformal weights of $\tilde{\Pi}$, $\tilde{\Pi}^i$, $\tilde{\Pi}^{ij}$ match those of Π , Π^i , Π^{ij} ; those of the extra variables ζ , ϕ , ψ_i and Σ_{ij} are $d+1$, $d+1$, d and $d-1$, while for ω and m^i we find d and $d+1$.

Finally, using (4.124), the equations $\mathcal{E} = \mathcal{F} = \mathcal{G}_j = \mathcal{H}_j = \mathcal{X}_j = 0$ and $\mathcal{J} = \mathcal{N} = 0$ become Weyl-covariant with

$$\mathcal{F} = \frac{1}{\Omega}\hat{D}_t\varrho + \hat{D}_jN^j, \quad (4.126)$$

$$\mathcal{N} = \frac{1}{\Omega}\hat{D}_t\tilde{\varrho} + \hat{D}_j\tilde{N}^j, \quad (4.127)$$

and

$$\mathfrak{E} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \Pi - \hat{\mathcal{D}}_i \Pi^i - \Pi^{ij} \xi_{ij}, \quad (4.128)$$

$$\mathfrak{F} = -\frac{1}{\Omega} \hat{\mathcal{D}}_t \tilde{\Pi} - \hat{\mathcal{D}}_i \tilde{\Pi}^i - \tilde{\Pi}^{ij} \xi_{ij}, \quad (4.129)$$

$$\mathfrak{G}_j = \hat{\mathcal{D}}_i \Pi^i_j + 2\Pi^i \varpi_{ij} + \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \delta_j^i + \xi^i_j \right) P_i, \quad (4.130)$$

$$\mathfrak{H}_j = \hat{\mathcal{D}}_i \tilde{\Pi}^i_j + 2\tilde{\Pi}^i \varpi_{ij} + \left(\frac{1}{\Omega} \hat{\mathcal{D}}_t \delta_j^i + \xi^i_j \right) \Pi_i, \quad (4.131)$$

$$\mathfrak{X}_j = \frac{1}{\Omega} \hat{\mathcal{D}}_t \tilde{\Pi}_j + \tilde{\Pi}_i \xi^i_j. \quad (4.132)$$

These equations are the seed for flat holography, see [25–28]. They are covariant under Carrollian diffeomorphisms, covariant under Weyl rescalings and invariant under hydrodynamic-frame transformations, which are local Carroll transformations (Carroll boosts and rotations).

Isometries and conformal isometries

As a final application of the above results on multiplication of degrees of freedom, we can insert the behaviour (4.114) inside the components (2.107), (2.108) and (2.109) of a relativistic conserved current resulting from the combination of the energy-momentum tensor with a Killing or a conformal Killing field. We obtain

$$-\frac{1}{c\Omega} I_0 = \iota_{0r} = \frac{\tilde{\kappa}}{c^4} + \frac{\tilde{\kappa}}{c^2} + \kappa + \mathcal{O}(c^2), \quad I^k = i_r^k = \frac{\tilde{K}^k}{c^2} + K^k + \mathcal{O}(c^2) \quad (4.133)$$

with (remember that $\xi^i \equiv \xi^{\hat{t}}$ is a function of \mathbf{x} only for Carrollian diffeomorphisms)

$$\begin{cases} \kappa = \xi^i P_i - \xi^{\hat{t}} \Pi \\ \tilde{\kappa} = \xi^i \Pi_i - \xi^{\hat{t}} \tilde{\Pi} \\ \tilde{\tilde{\kappa}} = \xi^i \tilde{\tilde{\Pi}}_i \\ K^i = \xi^j \Pi_j^i - \xi^{\hat{t}} \Pi^i \\ \tilde{K}^i = \xi^j \tilde{\Pi}_j^i - \xi^{\hat{t}} \tilde{\Pi}^i, \end{cases} \quad (4.134)$$

where κ and K_i are precisely as anticipated in (4.51) and (4.52), and described in footnote 62. Inserting (4.133) in the relativistic divergence of the matter current I^μ in Papapetrou-Randers background we recover a multiplication of (4.50) in the form

$$\nabla_\mu I^\mu = \frac{\tilde{\tilde{\mathcal{K}}}}{c^4} + \frac{\tilde{\mathcal{K}}}{c^2} + \mathcal{K} + \mathcal{O}(c^2) \quad (4.135)$$

with

$$\begin{cases} \tilde{\tilde{\mathcal{K}}} = \left(\frac{1}{\Omega} \partial_t + \theta \right) \tilde{\tilde{\kappa}} = 0 \\ \tilde{\mathcal{K}} = \left(\frac{1}{\Omega} \partial_t + \theta \right) \tilde{\kappa} + \left(\hat{\nabla}_i + \varphi_i \right) \tilde{K}^i = -\tilde{\Pi}^i \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right) \\ \mathcal{K} = \left(\frac{1}{\Omega} \partial_t + \theta \right) \kappa + \left(\hat{\nabla}_i + \varphi_i \right) K^i = -\Pi^i \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right). \end{cases} \quad (4.136)$$

The last equalities are obtained owing to the Carrollian Killing (4.46) (or conformal Killing (3.66), (4.57), (4.58), (4.59)) conditions and the equations of motion $\mathcal{E} = \mathcal{F} = \mathcal{G}_j = \mathcal{H}_j = \mathcal{X}_j = 0$ (see eqs. (4.128), (4.129), (4.130), (4.131), (4.132) — and (4.124) for the Weyl-covariant situation). One of the three currents is conserved as a consequence of the (conformal) isometry, whereas the other two are not.

The multiplication of degrees of freedom induced by the behaviour of the energy-momentum tensor (4.114), triggers a multiplication of currents in the presence of an isometry (or a conformal isometry if the dynamics is Weyl-invariant) — here three, but possibly more if more terms are present in (4.114). These currents are generically non-conserved, as we already observed in section 4.1 for a single current (eqs. (4.54) or (4.60)), unless the symmetry is stronger than a Carrollian (conformal) isometry i.e. if $\mathcal{L}_\xi \theta^{\hat{t}} = \left((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right) dx^i = 0$. Alternatively each of those currents may or may not be conserved, irrespective of the others, when an appropriate vector in the expansion of q_r^i vanishes, be it Π^i or $\tilde{\Pi}^i$, and this explains why $\tilde{\mathcal{K}} = 0$ in the above paradigm, whereas $\tilde{\mathcal{K}} \neq 0$ and $\mathcal{K} \neq 0$.

As we emphasized at the end of section 3.3 in the Galilean framework, i.e. for infinite c , it is puzzling that the conservation law $\nabla_\mu I^\mu = 0$ of a relativistic current $I^\mu = \xi_\nu T^{\mu\nu}$ produced by a Killing vector ξ of a pseudo-Riemannian spacetime, similarly fails when the limit c -to-zero is taken. This is once again due to the nature of the Carrollian Killing fields; details are available in appendix D.2.

5 Aristotelian dynamics

Aristotelian spacetimes

Aristotelian spacetimes were introduced in [80]. They are part of a rich web of geometric structures, which incorporate Newton-Cartan and Carroll, among others. Their defining feature is a manifold $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ of dimension $d + 1$, endowed with a degenerate metric and a degenerate cometric. The first implies that there is a symmetric rank-two tensor acting on tangent-space elements

$$d\ell^2 = \ell_{\mu\nu}(t, \mathbf{x}) dx^\mu dx^\nu, \quad \mu, \nu \dots \in \{0, 1, \dots, d\} \tag{5.1}$$

with one-dimensional kernel, the field of observers

$$\mathbf{v} = v^\mu(t, \mathbf{x}) \partial_\mu : \quad \ell_{\mu\nu} v^\mu = 0. \tag{5.2}$$

The second translates into the existence of symmetric rank-two tensor acting on cotangent-space vectors:

$$\partial_m^2 = m^{\mu\nu}(t, \mathbf{x}) \partial_\mu \partial_\nu, \tag{5.3}$$

where

$$\boldsymbol{\mu} = \mu_\nu(t, \mathbf{x}) dx^\nu : \quad m^{\mu\nu} \mu_\nu = 0, \tag{5.4}$$

defines a clock form (one should say “anti-clock” because of the sign), such that

$$-v^\mu \mu_\nu + m^{\mu\lambda} \ell_{\lambda\nu} = \delta_\nu^\mu, \quad v^\nu \mu_\nu = -1, \quad m^{\mu\nu} \ell_{\mu\nu} = d. \tag{5.5}$$

The topological structure $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ provides here a genuine, vertical and horizontal foliation, as opposed to Carrollian manifolds, where it generally supports a fiber bundle over a d -dimensional basis, and to Newton-Cartan where the fibration is defined over a line. In other words, Aristotelian spacetimes lie in the intersection of Carrollian and Newton-Cartan geometries with trivial Ehresmann connection for the Carrollian and trivial field of observers for the Newton-Cartan.⁷³

The vector \mathbf{v} and the covector μ are intrinsic geometric objects. Aristotelian transformations are meant to leave them invariant and respect the associated foliation. This forbids boosts, which is a characteristic feature of Aristotelian spacetimes translating into the absolute nature of time *and* space. In practice it allows to adopt a convenient choice.⁷⁴

$$d\ell^2 = a_{ij} dx^i dx^j, \quad \partial_m^2 = a^{ij} \partial_i \partial_j \tag{5.6}$$

with $a^{ik} a_{kj} = \delta_j^i$ and

$$\mathbf{v} = \frac{1}{\Omega} \partial_t, \quad \mu = -\Omega dt, \tag{5.7}$$

where a_{ij} and Ω are functions of t and \mathbf{x} . Aristotelian diffeomorphisms act as

$$t' = t'(t), \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x}) \tag{5.8}$$

with Jacobian

$$J(t) = \frac{\partial t'}{\partial t}, \quad J_j^i(\mathbf{x}) = \frac{\partial x'^i}{\partial x^j}. \tag{5.9}$$

The transformations of the metric and kernel components are thus

$$a'_{ij} = a_{kl} J^{-1k}_i J^{-1l}_j, \quad \Omega' = \frac{\Omega}{J}. \tag{5.10}$$

Aristotelian manifolds can be equipped with covariant derivatives, which allow to obtain genuine tensors under Aristotelian diffeomorphisms (5.8). As for the various spacetimes met earlier, we will focus here on the simplest torsionless and metric-compatible space and time connections, which naturally appear in the conservation equations.

The spatial connection is the same as the one introduced for the Galilean manifolds in (3.8):

$$\gamma_{jk}^i = \frac{a^{il}}{2} (\partial_j a_{lk} + \partial_k a_{lj} - \partial_l a_{jk}). \tag{5.11}$$

The associated covariant derivative is spelled ∇_i since it needs not be distinguished from the ordinary Levi-Civita derivative on a Riemannian d -dimensional manifold — it just depends on time here. It is torsionless, because γ_{jk}^i are symmetric and also obeys $\nabla_i a_{jk} = 0$.⁷⁵ Its

⁷³Our definition of Aristotelian manifolds is that of [80], also used in [74, 76]. In refs. [39, 40] “Aristotelian” is somewhat less restrictive. It is meant to be a Leibnizian structure and a Fröbenius-integrable distribution associated with the absolute clock form ($\mu \wedge d\mu = 0$ obvious for (5.7)). The necessary extra ingredient to match the more conventional picture we use is the field of observers \mathbf{v} (5.2) (or (5.7) in the concrete realization), as described in Props. A.5 and A.6 of [40].

⁷⁴This is not the most general choice, as (3.1)–(3.2) and (4.1)–(4.2) were not the most general either for Newton-Cartan and Carrollian manifolds.

⁷⁵For a more general discussion on compatible connections, see Props. A.17 and A.18 of [40].

Riemann, Ricci and scalar curvature tensors are defined as usual d -dimensional Levi-Civita curvature tensors would be, except that they are t -dependent.

The time derivative operator $\frac{1}{\Omega}\partial_t$ is also promoted to an Aristotelian temporal metric-compatible covariant derivative

$$D_t a_{jk} = 0, \tag{5.12}$$

using

$$\gamma_{ij} = \frac{1}{2\Omega}\partial_t a_{ij} = \xi_{ij} + \frac{1}{d}a_{ij}\theta. \tag{5.13}$$

We have introduced the familiar by now traceless shear tensor ξ_{ij} and expansion scalar $\theta = \frac{1}{\Omega}\partial_t \ln \sqrt{a}$, which can be completed with the acceleration form

$$\varphi_i = \partial_i \ln \Omega. \tag{5.14}$$

The action of D_t is as in the Carrollian case, eqs. (4.17) and (4.18): $D_t \Phi = \partial_t \Phi$ and

$$\frac{1}{\Omega}D_t V^i = \frac{1}{\Omega}\partial_t V^i + \gamma^i_j V^j, \quad \frac{1}{\Omega}D_t V_i = \frac{1}{\Omega}\partial_t V_i - \gamma_i^j V_j \tag{5.15}$$

generalized using the Leibniz rule.

Time and space derivatives do not commute:

$$\left[\frac{1}{\Omega}D_t, \nabla_i \right] V^j = \varphi_i \left(\left(\frac{1}{\Omega}D_t + \theta \right) V^j - \gamma^j_k V^k \right) - \gamma_i^k \nabla_k V^j - dr^j_{ik} V^k \tag{5.16}$$

with⁷⁶

$$r^j_{ik} = \frac{1}{d} \left(\theta \varphi_i \delta^j_k + \nabla_i \gamma^j_k - \frac{1}{\Omega} \partial_t \gamma^j_{ik} \right) \tag{5.17}$$

another Aristotelian curvature tensor, similar to the one introduced for our Carrollian connection.

Dynamics from diffeomorphism invariance

Aristotelian fluids were introduced in [74], and mentioned as “self-dual” in [29]. They received further attention [76] amid a soaring interest for non-boost-invariant dynamics [81–83]. It is abusive though to call this “hydrodynamics” for two reasons. The first is shared with the Carrollian instance: a fluid is meant to assimilate a phenomenological description of a many-body system under local-equilibrium and slow-variation assumptions, and those assumptions are non controllable in the Carrollian or Aristotelian framework due the absence of a thermodynamic or kinetic theory — the elementary particle motion is forbidden in both cases. Additionally, as opposed to Galilean or Carrollian manifolds, Aristotelian cannot be obtained naturally as limits of pseudo-Riemannian spacetimes. Thus, no guide for the would-be Aristotelian fluid equations exists that could be based on a limiting procedure, and this is the second reason. The only safe way to reach a set of equations covariant under Aristotelian diffeomorphisms (5.8) is to require this diffeomorphism invariance at the level of an effective action, and study its conservation consequences for the conjugate

⁷⁶Again $\frac{1}{\Omega}\partial_t \gamma^j_{ik} = (\nabla_i + \varphi_i)\gamma^j_k + (\nabla_k + \varphi_k)\gamma^j_i - (\nabla^j + \varphi^j)\gamma_{ik}$ is an Aristotelian tensor, even though γ^j_{ik} is not.

Aristotelian momenta. The latter remain however agnostic on a decomposition in terms of kinematical (as v_i for Galilean, β^i for Carrollian) and kinetic (energy, pressure, heat and stress) variables.

Starting again with an action $S = \int dt d^d x \sqrt{a} \Omega \mathcal{L}$ on $\mathcal{M} = \mathbb{R} \times \mathcal{S}$, now functional of a_{ij} and Ω , the Aristotelian momenta conjugate to those variables are

$$\Pi^{ij} = \frac{2}{\sqrt{a} \Omega} \frac{\delta S}{\delta a_{ij}}, \quad (5.18)$$

$$\Pi = -\frac{1}{\sqrt{a}} \frac{\delta S}{\delta \Omega}. \quad (5.19)$$

These are the *energy-stress tensor* and the *energy density*, and with those the variation of the action in the gravitational sector is

$$\delta S = - \int dt \Omega \int d^d x \sqrt{a} \left(\frac{1}{2} \Pi_{ij} \delta a^{ij} + \Pi \delta \ln \Omega \right). \quad (5.20)$$

Aristotelian diffeomorphisms (5.8) are generated by vector fields

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \xi^{\hat{t}} \frac{1}{\Omega} \partial_t + \xi^i \partial_i, \quad (5.21)$$

where $\xi^t = \xi^t(t)$ and $\xi^i = \xi^i(\mathbf{x})$. The variation under diffeomorphisms is implemented as usual with

$$- \delta_{\xi} a_{ij} = \mathcal{L}_{\xi} a_{ij} = 2 \nabla_{(i} \xi^k a_{j)k} + 2 \xi^{\hat{t}} \gamma_{ij}. \quad (5.22)$$

Now

$$\mathcal{L}_{\xi} \mathbf{v} = \mu \mathbf{v}, \quad \mathcal{L}_{\xi} \mu = -\mu \mu \quad (5.23)$$

with

$$\mu(t, \mathbf{x}) = - \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i \right). \quad (5.24)$$

Thus

$$- \delta_{\xi} \ln \Omega = \frac{1}{\Omega} \mathcal{L}_{\xi} \Omega = -\mu. \quad (5.25)$$

Using (5.22) and (5.25) in (5.20) with $\xi^i = \xi^i(\mathbf{x})$, we obtain (indices are here lowered with a_{ij})

$$\begin{aligned} \delta_{\xi} S = & \int dt d^d x \sqrt{a} \Omega \left\{ -\xi^{\hat{t}} \left[\left(\frac{1}{\Omega} \partial_t + \theta \right) \Pi + \Pi^{ij} \gamma_{ij} \right] + \xi^i \left[(\nabla_j + \varphi_j) \Pi^j_i + \Pi \varphi_i \right] \right\} \\ & + \int dt d^d x \left\{ \partial_t \left[\sqrt{a} \xi^{\hat{t}} \Pi \right] - \partial_i \left[\sqrt{a} \Omega \xi^j \Pi^i_j \right] \right\}. \end{aligned} \quad (5.26)$$

The boundary terms (last line of (5.26)) are ignored and $\delta_{\xi} S = 0$ implies that the momenta defined earlier in (5.18), (5.19) are Aristotelian-covariant. It leads to two equations: the energy equation

$$\boxed{\left(\frac{1}{\Omega} \partial_t + \theta \right) \Pi + \Pi^{ij} \gamma_{ij} = - (\nabla_i + 2\varphi_i) \Pi^i}, \quad (5.27)$$

and the momentum equation

$$\boxed{(\nabla_j + \varphi_j) \Pi^j_i + \Pi \varphi_i = - \left(\frac{1}{\Omega} \partial_t + \theta \right) P_i.} \quad (5.28)$$

In these equations the *energy current* Π^i and the *momentum* P^i are undetermined. They arise because ξ^t depends on time and ξ^i on space, exclusively. As a consequence, $\sqrt{a} \Omega \xi^t (\nabla_i + 2\varphi_i) \Pi^i = \partial_i (\sqrt{a} \Omega \xi^t \Pi^i)$ and $\sqrt{a} \Omega \xi^i \left(\frac{1}{\Omega} \partial_t + \theta \right) P_i = \partial_t (\sqrt{a} \xi^i P_i)$, which are boundary terms and vanish inside the integral.

Aristotelian dynamics stands at the intersection of Galilean and Carrollian. This is expected from first principles and can be verified by comparing eqs. (5.27) and (5.28) with (3.42)⁷⁷ and (3.41) in the Galilean at $w^i = 0$, as well as (4.35) and (4.36) in the Carrollian $b_i = 0$ instances. However, contrary to these, Aristotelian dynamics cannot be reached as a limit of relativistic hydrodynamics. No kinematical parameter such as v_i or β^i can be foreseen and no decomposition of the momenta à la (3.84), (3.89), (3.90), (3.91) or (4.76), (4.77). Finally, hydrodynamic-frame invariance cannot be argued or disputed.

The presence of a matter/charge sector can be treated as in the previous families of dynamics. We assume the existence of a gauge field $B = B(t, \mathbf{x})dt + B_i(t, \mathbf{x})dx^i$ associated with a local U(1) transformation as in (3.43), and the conjugate momenta

$$\varrho = \frac{1}{\sqrt{a}} \frac{\delta S}{\delta B}, \quad (5.29)$$

$$N^i = \frac{1}{\Omega \sqrt{a}} \frac{\delta S}{\delta B_i} \quad (5.30)$$

are the *charge density* and the *charge current*. The gauge variation

$$\begin{aligned} \delta_\Lambda S &= \int dt d^d x \sqrt{a} \left(\varrho \delta_\Lambda B + \Omega N^i \delta_\Lambda B_i \right) \\ &= \int dt d^d x \left\{ \sqrt{a} \Omega \Lambda \left(\frac{1}{\Omega} \partial_t \varrho + \theta \varrho + (\nabla_i + \varphi_i) N^i \right) - \partial_t (\sqrt{a} \Lambda \varrho) - \partial_i (\sqrt{a} \Lambda \Omega N^i) \right\} \end{aligned} \quad (5.31)$$

is assumed to vanish, and this leads to

$$\boxed{\left(\frac{1}{\Omega} \partial_t + \theta \right) \varrho + (\nabla_i + \varphi_i) N^i = 0.} \quad (5.32)$$

In integrated form, thanks to Stokes and Gauss theorems, this conservation reads ($\mathcal{V} \subset \mathcal{S}$, a constant-time section, and $\partial \mathcal{V}$ does not depend on time):

$$\frac{d}{dt} \int_{\mathcal{V}} d^d x \sqrt{a} \varrho = - \int_{\mathcal{V}} d^d x \partial_i (\sqrt{a} \Omega N^i) = - \int_{\partial \mathcal{V}} \Omega \star \mathbf{N} \quad (5.33)$$

⁷⁷Notice that a term of the type $2\varphi_i \Pi^i$ would also have been present in this Galilean energy equation if we had considered a torsionfull Newton-Cartan spacetime, with $\Omega = \Omega(t, \mathbf{x})$, as we assume in the present Aristotelian case.

with the conserved charge defined as an integral over the entire space \mathcal{S} :

$$Q_N = \int_{\mathcal{S}} d^d x \sqrt{a} \varrho. \quad (5.34)$$

As for the Galilean or the Carrollian situations discussed in sections 3.1 and 4.1, one can chose any space-like hypersurface Σ_d for the determination of the charge:

$$Q_N = \int_{\Sigma_d} \sqrt{a} \varrho dx^1 \wedge \dots \wedge dx^d + \int_{\Sigma_d} \sqrt{a} \sum_{i=1}^d dx^1 \wedge \dots \wedge N^i \mu \wedge \dots \wedge dx^d, \quad (5.35)$$

with the clock form μ given in (5.7), and $N^i \mu$ at the i th position in the last exterior product.

Weyl invariance

Weyl invariance can also be present in Aristotelian dynamics. The reasoning is familiar, starting from $\delta_{\mathcal{B}} S = 0$, we are lead to the condition

$$\Pi_i^i = \Pi. \quad (5.36)$$

The geometric tools for handling Weyl covariance are similar to those introduced in the previous sections. The connections are θ and φ_i (see eqs. (5.13) and (5.14)) with

$$\theta \rightarrow \mathcal{B}\theta - \frac{d}{\Omega} \partial_t \mathcal{B}, \quad \varphi_i \rightarrow \varphi_i - \partial_i \ln \mathcal{B}. \quad (5.37)$$

The action of the Aristotelian Weyl-covariant time derivative on any tensor increases its weight by one unit. On a weight- w function Φ it is

$$\frac{1}{\Omega} \mathcal{D}_t \Phi = \frac{1}{\Omega} D_t \Phi + \frac{w}{d} \theta \Phi = \frac{1}{\Omega} \partial_t \Phi + \frac{w}{d} \theta \Phi, \quad (5.38)$$

while on a weight- w vector, we find using (5.15)

$$\frac{1}{\Omega} \mathcal{D}_t V^l = \frac{1}{\Omega} D_t V^l + \frac{w-1}{d} \theta V^l = \frac{1}{\Omega} \partial_t V^l + \frac{w}{d} \theta V^l + \xi_i^l V^i. \quad (5.39)$$

Similarly for any tensor by Leibniz rule and in particular we find $\mathcal{D}_t a_{kl} = 0$.

A spatial Aristotelian-Weyl-covariant derivative can also be introduced, and does not alter the conformal weight. For a weight- w scalar function Φ it acts as

$$\mathcal{D}_j \Phi = \partial_j \Phi + w \varphi_j \Phi; \quad (5.40)$$

for a vector we find

$$\mathcal{D}_j V^l = \nabla_j V^l + (w-1) \varphi_j V^l + \varphi^l V_j - \delta_j^l V^i \varphi_i. \quad (5.41)$$

The action on any other tensor is obtained using the Leibniz rule, and in particular $\mathcal{D}_j a_{kl} = 0$.

Time and space Aristotelian-Weyl covariant derivatives do not commute and curvature tensors follow, which resemble those already quoted for the Carrollian or Galilean cases. We

will not elaborate on this subject here, in particular because all these connections can be generalized and studied in a more abstract geometric framework, as for example in [39, 40].

With the above tools, Aristotelian equations (5.27) and (5.28) read, under the assumption of conformal dynamics (5.36):

$$\frac{1}{\Omega} \mathcal{D}_t \Pi + \mathcal{D}_i \Pi^i + \Pi^{ij} \xi_{ij} = 0, \quad (5.42)$$

$$\mathcal{D}_i \Pi^i_j + \left(\frac{1}{\Omega} \mathcal{D}_t \delta_j^i + \xi^i_j \right) P_i = 0. \quad (5.43)$$

These equations are Weyl-covariant of weights $d+2$ and $d+1$ (Π and Π^i_j have weight $d+1$, whereas Π_i and P_i are weight- d). Similarly, for the matter sector, (5.32) reads:

$$\frac{1}{\Omega} \mathcal{D}_t \varrho + \mathcal{D}_j N^j = 0, \quad (5.44)$$

and is Weyl-covariant of weight $d+1$.

Isometries, conformal isometries and conservation laws

Isometries of Aristotelian spacetimes are generated by vectors fields (5.21) required to satisfy

$$\mathcal{L}_\xi a_{ij} = 0, \quad \mathcal{L}_\xi \mu = 0. \quad (5.45)$$

The latter lead to a set of Killing equations for $\xi^{\hat{t}}(t)$ and $\xi^i(\mathbf{x})$:

$$\boxed{\nabla_{(i} \xi^k a_{j)k} + \xi^{\hat{t}} \gamma_{ij} = 0, \quad \frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i = 0.} \quad (5.46)$$

For conformal isometries, the generators must satisfy

$$\mathcal{L}_\xi a_{ij} = \lambda a_{ij} \quad (5.47)$$

where λ is obtained by tracing the latter:

$$\lambda(t, \mathbf{x}) = \frac{2}{d} \left(\nabla_i \xi^i + \theta \xi^{\hat{t}} \right). \quad (5.48)$$

As for the previous Galilean and Carrollian cases, the requirement (5.47) must be supplemented with the usual extra condition (3.66) $2\mu + \lambda = 0$, in order to reach a well-defined set of conformal generators — $\mu(t, \mathbf{x})$ is displayed in (5.24).

In the presence of isometries or conformal isometries, we can define an Aristotelian current κ and K^i (as ϱ and N^i in (5.29), (5.30)), following the previous definitions in Newton-Cartan ((3.58) and (3.59)) or Carrollian spacetimes ((4.51) and (4.52)) — see also the boundary terms in $\delta_\xi S$ (5.26) evaluated on-shell:

$$\kappa = \xi^i P_i - \xi^{\hat{t}} \Pi, \quad K^i = \xi^j \Pi_j^i - \xi^{\hat{t}} \Pi^i. \quad (5.49)$$

Going on-shell and using the Killing equations, or the conformal Killing equations when the system is Weyl-invariant i.e. when (5.36) is satisfied, we find:

$$\mathfrak{K} = \left(\frac{1}{\Omega} \partial_t + \theta \right) \kappa + (\nabla_i + \varphi_i) K^i = 0. \quad (5.50)$$

Consequently, *on Aristotelian structures, a (conformal) Killing field always supports an on-shell conservation law for (Weyl-invariant) Aristotelian dynamics.*

6 Conclusions

Hydrodynamic equations for Galilean and Carrollian fluids on arbitrary backgrounds were displayed in ref. [29]. The present work augments these achievements and points towards an extensive set of features: (i) the fluid momenta conjugate to the geometric variables that define the spacetime and their role in the dynamics; (ii) the inclusion of a matter conserved current and its potential interplay with the energy and momentum through thermodynamics; (iii) the hydrodynamic-frame invariance; (iv) the consequence of isometries in terms of conservation. For this purpose, two distinct and complementary approaches have been pursued.

The first relies on local symmetries, primarily diffeomorphism invariance, but also local Weyl and U(1) gauge invariance. The energy and momenta are defined through geometry and are conserved as a consequence of diffeomorphism invariance. They are different for the relativistic, Galilean or Carrollian theories, and it should be stressed that *only the relativistic theory has an energy-momentum tensor* $T_{\mu\nu}$ with zero general-covariant divergence. In Galilean dynamics there is an energy-stress tensor Π_{ij} , a momentum P_i and an energy density Π , obeying a set of conservation equations, necessarily involving an extra variable, the energy flux Π_i , which is not conjugate to any available geometric piece of data. Similarly, Carrollian dynamics goes through with an energy-stress tensor Π^{ij} , an energy flux Π^i and an energy density Π , whereas the momentum P^i comes now aside.

The second approach consists in working within an appropriate coordinate system of the relativistic theory, and reach the Galilean or Carrollian dynamics in the infinite or zero speed-of-light limit. This method does not allow to retrieve Aristotelian dynamics, which is neither a large- c nor a small- c limit, but offers the possibility to capture more degrees of freedom by keeping overleading terms in the Laurent expansions. It is also better suited for unravelling the subtleties in the contraction of the relativistic group of invariance, as the persistence in the limit of a supplementary equation, absent in the direct approach. This is for instance the continuity equation in the Galilean framework or a similar conservation equation in the Carrollian, which betray that the relativistic diffeomorphism group valid before the limit is bigger than the actual Galilean or Carrollian groups. Finally, only when considering a limit from a relativistic system, can we express the various momenta in terms of the relevant kinematical and thermodynamic variables. This is how the velocity appears in the Galilean limit, whereas the inverse velocity arises as the relevant observable for Carrollian fluids — particularly important for the latter, where the we have no handle or hint from thermodynamics. The kinematical observables play a pivotal role in the further analysis of hydrodynamic-frame invariance.

The energy-momentum variables entering the dynamical equations are variations of some effective action with respect to geometric data. These *are not* any sort of Noetherian currents, which have no reason to exist since no isometry is assumed. Furthermore, even when isometries are present, Noetherian conservations appear only in relativistic dynamics. In Galilean and Carrollian systems, conserved currents cannot be designed on the basis of an isometry, unless the latter obeys an extra condition. This unexpected result should be viewed as one of our main accomplishments. It jeopardizes Noetherian descriptions

of Galilean or Carrollian hydrodynamics on isometric backgrounds, and calls for extra attention when studying charges in the framework of flat holography.

Focusing specifically on Galilean hydrodynamics, a few words should be devoted to the extra conserved matter current introduced in the relativistic theory. As shown in [29], this current is not necessary for recovering the continuity equation in the Galilean limit. Additionally, its associated matter density defines the rest contribution to the energy density, and their relationship is dictated by thermodynamics. This is how the chemical potential, absent in [29] enters the dynamics. Most importantly, in the limiting process, this current contributes to the Galilean heat current, as much as the relativistic heat current does, confirming that both Eckart and Landau-Lifshitz hydrodynamic-frame choices are viable. This brings us to a central theme of the present work: the hydrodynamic-frame invariance in the Galilean limit.

The fluid velocity in the relativistic system is merely a book-keeping device. The effect of modifying it by local Lorentz transformations propagates on the various observables (heat current, stress tensor, thermodynamic variables) so as to keep the energy-momentum tensor, the matter current and the entropy current invariant. When considering a limit on the speed of light, the expansion of the observables matters on the possible preservation of the hydrodynamic-frame invariance. Using the standard behaviour dictated by the physical transport coefficients, this invariance does not survive in the Navier-Stokes equations. This is not a surprise because the velocity and the matter density of a Galilean fluid are measurable observables, protected by a symmetry supported by the conservation of mass — itself a consequence of a central extension of the Galilean group emerging in a Poincaré-group contraction. Adopting an alternative behaviour, with overleading terms in the heat current saves the hydrodynamic-frame invariance at the expense of altering the continuity equation. The resulting dynamics is reminiscent of diffusion processes or superfluids, where indeed all species are not simultaneously conserved.

Fluids based on massless energy carriers are revealed as an exception to this Galilean scheme. For these fluids, with or without a charge conservation, the behaviour of the heat current, possibly but not necessarily with exotic terms, is compatible with Galilean hydrodynamic-frame invariance. It should be emphasized that this holds irrespective of Weyl invariance, which is not even a priori assumed. The deep reason of this feature is rather the absence of a rest term in the energy density, behaving like c^2 — and thereupon, the inexistence of mass conservation, usually in conflict with hydrodynamic-frame invariance.

Although to some extent dual to Galilean, Carrollian fluids exhibit different features. As opposed to the Galilean case, we have no inkling on what thermodynamics for these systems is, due to the absence of motion and thus of kinetic theory. For the same reasons, the energy density cannot be decomposed into rest plus kinetic contributions. Therefore, if a conserved charge exists, it decouples from the hydrodynamic equations. More importantly, no transport theory is available that would serve as a guide for the behaviour of the physical observables (heat current and the stress tensor) when the speed of light gets extinct. Instead, Carrollian fluids have been recognized as holographic duals of asymptotically flat spacetimes — their unique successful application so far — and this gives a handle on the terms to keep in the small- c expansion. With those, hydrodynamic-frame invariance is maintained in the Carrollian limit, reflecting holographically residual bulk diffeomorphisms.

Aristotelian dynamics was the last subject treated in this context. As we stressed earlier, only the geometric method is applicable, based on Aristotelian diffeomorphisms. The set of equations reached in this way appears as the intersection of Galilean (with torsion though) and Carrollian dynamics. The reduction of the light cone onto points, the degenerate nature of motion and the impossibility to bridge this theory with the relativistic theory through a limit,⁷⁸ leaves little room for a more in-depth discussion in terms of fluids.

All this summarizes our achievements, their relationships and the general context. This activity, mostly based on classical physics and differential geometry, is part of a palette of timely developments. The subject of hydrodynamic-frame invariance has been treated here in a kinematic fashion, ignoring the entropy current and the constitutive relations. This latter facet has come back in the forefront [11–19] (see also [63, 64]), and further investigation of these physical features is certainly desirable in light of our more formal results. In particular, certain aspects of the photon fluid deserve some attention, even at an anecdotic level [93] with the notorious Planck-Ott paradox: *does a moving body appear cool?* According to Israël about this paradox [5] “*it is not yet quite dead,*” and one indeed finds articles where it is still debated [94]. On a less frivolous tone, relativistic and Galilean hydrodynamics can be studied using Boltzmann equation, and this should also apply to Carrollian fluids. More ambitiously, one may even try to root Boltzmann equation inside field theory, which in turn would require mastering Galilean or Carrollian fields on general curved spacetimes. At the classical level, some results are available [63–65, 79, 95–98, 100–103], but the quantum theory remains elusive — see e.g. [104].

Besides the caveats plaguing hydrodynamic-frame invariance in relation with causality or stability and rooted in the constitutive relations, this local gauge symmetry does also disclose global issues. Local velocity transformations may leave the system with global distinctnesses. To handle them, one should be able to design charges associated with the energy and momenta, the matter current or the entropy current, possibly sensitive to global properties. For relativistic or Carrollian fluids, those charges can be handled holographically via a gravity dual, asymptotically anti-de Sitter or flat. They actually obey algebras, which depend on the chosen hydrodynamic frame [25–27]. Extensions and refinements to this analysis would be expedient.

Acknowledgments

We thank our colleagues G. Barnich, X. Bekaert, G. Bossard, A. Campoleoni, L. Ciambelli, L. Donnay, A. Fiorucci, L. Freidel, P. Gauduchon, M. Geiller, J. Gomis, E. Gourgoulhon, D. Grumiller, M. Henneaux, Y. Herfray, P. Horvathy, A. Kleinschmidt, C. Kopper, P. Kovtun, C. Lorcé, C. Markou, C. Marteau, G. Montambaux, N. Obers, B. Oblak, G. Oling, J. Rossell, R. Ruzziconi, S. Vandoren, M. Vilatte and C. Zwickel for rich scientific discussions. We are also grateful to the *IP2I de l’Université de Lyon*, the *Université libre de Bruxelles* and the *Université de Mons*, where presentations were made by us on these topics, triggering fecund exchanges. Tasos Petkou and Marios Petropoulos have also often visited the *Ecole*

⁷⁸Notice that Aristotelian spacetimes are related to the infinite-mass limit introduced by Henri Bacry and Jean-Marc Lévy-Leblond in [84], as a mean to reach the static group by contraction of the Galilean group.

Polytechnique and the *Aristotelian University of Thessaloniki*, where parts of the present work were carried on. Finally, many thanks to the *TU Wien* and to the organizers of the *Carroll Workshop* held in Vienna in February 2022, where our results were previewed and refined, taking advantage of the inimitable emulation created during this event. The work of A. C. Petkou was supported by the Hellenic Foundation for Research and Innovation (H.F.R.I.) under the *First Call for H.F.R.I. Research Projects to support Faculty members and Researchers and the procurement of high-cost research equipment grant* (MIS 1524, Project Number: 96048). The work of D. Rivera-Betancour was funded by Becas Chile (ANID) Scholarship No. 72200301.

A A primer on thermodynamics

Relativistic thermodynamics

We remind here the usual observables of global-equilibrium thermodynamics. These are supposed to make sense also in local-equilibrium thermodynamics, as for fluids where the absence of short wave-length modes is assumed. In this case they depend on time and space and refer to measurements performed by an observer comoving with respect to the fluid. Matter conservation is generically (but not necessarily) akin to the existence of massive carriers in conserved number.

- The temperature T .
- The mass density ϱ_0 per unit proper volume.
- The entropy per unit proper volume σ , and the entropy per unit mass s (specific entropy):

$$\sigma = s\varrho_0. \tag{A.1}$$

- The relativistic internal energy density per unit proper volume ε , which contains the rest mass, and the specific energy per unit mass e :

$$\varepsilon = (e + c^2)\varrho_0. \tag{A.2}$$

- The pressure p and the relativistic enthalpy w per unit proper volume:

$$w = \varepsilon + p. \tag{A.3}$$

- The relativistic chemical potential per unit mass (specific chemical potential) μ_0 . This contains the rest-mass contribution, as opposed to μ :

$$\mu_0 = \mu + c^2. \tag{A.4}$$

These quantities obey the grand-potential equation, sometimes referred to as the Gibbs-Duhem equation:

$$w = T\sigma + \mu_0\varrho_0 \Leftrightarrow p = Ts\varrho_0 + \mu\varrho_0 - e\varrho_0. \tag{A.5}$$

The energy density is a functional of two thermodynamic variables: $\varepsilon = \varepsilon(\sigma, \varrho_0)$. The first law of thermodynamics reads:

$$d\varepsilon = Td\sigma + \mu_0 d\varrho_0 \Leftrightarrow de = Tds - pd\left(\frac{1}{\varrho_0}\right). \quad (\text{A.6})$$

The Gibbs-Duhem equation allows to exhibit the dependence of the enthalpy per unit proper volume $w = w(\sigma, p, \varrho_0)$

$$dw = Td\sigma + dp + \mu_0 d\varrho_0, \quad (\text{A.7})$$

whereas a double Legendre transformation on ε infers the dependence of the grand potential $p = p(T, \mu_0)$

$$dp = \sigma dT + \varrho_0 d\mu_0 = s\varrho_0 dT + \varrho_0 d\mu. \quad (\text{A.8})$$

We would like to mention the situations where no massive degrees of freedom are present in the microscopic theory.⁷⁹ A gas of photons is the prime example but other instances exist in condensed matter, in particular when fermions are involved, as in graphene (see e.g. [89]). In the latter case, as opposed to the gas of photons, there is a conserved quantity. So ϱ_0 is non-vanishing, but it *is not* a mass density; $\varepsilon = e\varrho_0$ and $\mu_0 = \mu$, without rest-mass contribution. These systems can be conformally invariant, and in that case the dependence $p = p(T, \mu)$ is

$$p = T^D f(\mu/T) \quad (\text{A.9})$$

in $D = d + 1$ spacetime dimensions.⁸⁰

Coming back to a system with massless carriers and no conserved charge, as for the gas of photons, the above thermodynamic relationships simplify by setting $\mu = 0$ and dropping the rest-mass terms. Specific quantities are no longer significant in this instance. Fluid dynamics of such systems does not involve any conserved current.⁸¹ The basic laws are summarized as follows:

$$\left\{ \begin{array}{l} w = \mathcal{T}\sigma \\ d w = T d\sigma + dp \\ d\varepsilon = T d\sigma \\ dp = \sigma dT, \end{array} \right. \quad (\text{A.10})$$

and when the system is furthermore conformal, $p \propto T^D$.

Several conserved charges might exist simultaneously in a thermodynamic system. They would each be associated with a density and a chemical potential. Only one, if any, would however enter the energy density (A.2).

⁷⁹This happens effectively in the usual ultra-relativistic limit, meant to be relevant microscopically at high temperature or high pressure.

⁸⁰The precise bearing between conformal invariance, absence of mass and existence of conserved currents is subtle and tight to the microscopic theory.

⁸¹This instance was discussed in the precise framework of relativistic fluid dynamics in [86] section 134, footnote 1 and exercise 2. The general thermodynamic aspects are presented in greater detail in [105] section 60.

Non-relativistic limit

The thermodynamic variables introduced earlier in the relativistic theory such as $\varrho_0, \varepsilon, \mu$ etc. are referring to a comoving observer. Measurements performed by another observer, be this an inertial observer in special relativity or some fiducial observer in a general gravitational background, are more relevant for the Galilean framework, but are not equal and should be spelled *stricto sensu* with some distinctive index. Their differences, however, are of order $1/c^2$ and vanish in the infinite- c limit. In order to avoid inflation in notation, we will keep the same symbols, e, T, p, s, μ , with the exception of ϱ_0 , which becomes ϱ for the fiducial observer. The $1/c^2$ corrections amongst ϱ_0 and ϱ (see (3.74)) play no role in thermodynamics, but are indispensable in recovering Navier-Stokes equations as the Galilean limit of the relativistic hydrodynamic equations.

In non-relativistic thermodynamics, it is customary to introduce the specific volume (not to be confused with the velocity)

$$v = \frac{1}{\varrho}, \tag{A.11}$$

as well as specific enthalpy $h = h(s, p)$ as⁸²

$$h = e + pv, \tag{A.12}$$

which also enters in

$$\mu = h - Ts. \tag{A.13}$$

Using these definitions and the various relativistic laws mentioned above, we find the standard expressions:

$$dh = Tds + vdp, \tag{A.14}$$

$$d(e\varrho) = \varrho Tds + h d\varrho, \tag{A.15}$$

$$de = Tds - p dv, \tag{A.16}$$

$$d\mu = -s dT + v dp. \tag{A.17}$$

Before closing this section, let us quote that Galilean thermodynamics can accommodate fluids with massless energy carriers, as long as the macroscopic velocity is small compared to c — although at the microscopic level the dynamics is ultra-relativistic. Again, a conserved current may or may not exist. In case such a current is available, ϱ is the charge density with⁸³ $\varepsilon = e\varrho$ the internal energy density and μ the chemical potential. The basic relationships are now

$$w = p + \varepsilon = T\sigma + \mu\varrho \Leftrightarrow p = Ts\varrho + \mu\varrho - e\varrho, \tag{A.18}$$

and

$$\left\{ \begin{array}{l} dw = Td\sigma + dp + \mu d\varrho \\ d\varepsilon = Td\sigma + \mu d\varrho \\ dp = \sigma dT + \varrho d\mu. \end{array} \right. \tag{A.19}$$

⁸²This is spelled w_{nr} in footnote 1, section 134 of [86].

⁸³Notice the distinction: $\varepsilon = \lim_{c \rightarrow \infty} \varepsilon = e\varrho$, $\sigma = \lim_{c \rightarrow \infty} \sigma = s\varrho$. In order to avoid multiplication of symbols, we keep $w = h\varrho$, p and μ both for the relativistic quantities and for their Galilean limits.

Equations (A.11), (A.12), (A.13), (A.14), (A.15), (A.16), (A.17) remain also valid, together with (A.9) in case of conformal invariance.

If no conserved charge is present, the chemical potential vanishes (as does $d\mu$) and the relevant equations are expressed with w , ε and σ rather than $h\rho$, $e\rho$ or $s\rho$.

Carrollian thermodynamics

Carrollian thermodynamics is poorly understood. In most parts of this work dealing with the fluid equations, we have kept the energy density ε and the pressure p unaltered in the limit of vanishing velocity of light. Neither have we introduced any temperature, nor discussed an entropy equation, and when a conserved current was assumed (as eluded in [85]), no relationship was established or set among energy and conserved-charge densities. This is minimalistic by default. Indeed, the shrinking of the light cone and the absence of particle motion or signal propagation, raise fundamental questions regarding the origin — and even the definition — of energy, pressure, entropy, temperature and thermalization processes. Even the kinematic parameter of the fluid is an inverse velocity, which could point towards the dynamics of instantonic space-filling branes, as mentioned in [29]. Obviously, this sort of objects are tachyonic — like those introduced later in [77] — and we feel uneasy advocating any sort of kinetic theory for setting up thermodynamic laws and deviations from equilibrium.

B Carrollian momenta and hydrodynamic-frame invariance

In section 4.3 we obtained the Carrollian fluid equations under c^2 -scaling assumptions involving more degrees of freedom than the standard ones treated in section 4.2: (4.109), (4.110), (4.111) and (4.112) for the relativistic energy ε , pressure p , heat current q^i and stress tensor τ^{ij} , and similarly for the matter sector with the matter density ϱ_0 and the non-perfect current j^i in (4.113). These equations involve the Carrollian momenta $\tilde{\Pi}$, Π , $\tilde{\Pi}^i$, Π^i , P^i , $\tilde{\Pi}^{ij}$, Π^{ij} , $\tilde{\varrho}$, ϱ , \tilde{N}^i , N^i , which have the following expressions in terms of the observables entering the already quoted c -expansions:

$$\left\{ \begin{array}{l} \tilde{\Pi}^i = \psi^i \\ \Pi^i = Q^i - \beta_j (\Sigma^{ij} - \phi a^{ij}) + \beta^i (\zeta + \beta_j \psi^j) + \frac{\beta^2}{2} \psi^i \\ P^i = \pi^i - \beta_j (\Xi^{ij} - \varpi a^{ij}) + \beta^i (\eta + \beta_j Q^j) + \frac{\beta^2}{2} (Q^i + \frac{3\beta^2}{4} \psi^i) + \beta^i \beta^2 (\zeta + \phi + \frac{1}{2} \beta_j \psi^j) \\ \tilde{\Pi} = \zeta + 2\beta_i \psi^i \\ \Pi = \eta + 2\beta_i Q^i - \beta_i \beta_j (\Sigma^{ij} - \phi a^{ij}) + \beta^2 (\zeta + \beta_i \psi^i) \\ \tilde{\Pi}^{ij} = \psi^i \beta^j + \beta^i \psi^j + \phi a^{ij} - \Sigma^{ij} \\ \Pi^{ij} = Q^i \beta^j + \beta^i Q^j + \varpi a^{ij} - \Xi^{ij} + \beta^i \beta^j (\zeta + \phi) + \frac{\beta^2}{2} (\psi^i \beta^j + \beta^i \psi^j), \end{array} \right. \quad (\text{B.1})$$

and

$$\begin{cases} \tilde{N}^j = m^j \\ N^j = n^j + \beta^j \omega \\ \tilde{\varrho} = \omega + \beta_k m^k \\ \varrho = \chi + \beta_k n^k + \frac{\beta^2}{2} \omega. \end{cases} \quad (\text{B.2})$$

The aim of the present appendix is to show that these are hydrodynamic-frame-invariant.

Although hydrodynamic-frame invariance is built-in for the relativistic momenta (2.100), (2.101), (2.102), (2.103) and (2.104), it is not guaranteed to persist in the vanishing- c limit because it can be incompatible with the presumed small- c behaviour of the physical observables. This happens in the Galilean (infinite- c) limit, as we have witnessed in section 3.2 for the ordinary i.e. with $n^i = 0$ non-relativistic fluids, because $\delta n^i \propto \delta v^i$ (see eq. (3.113)). Here it turns out to hold and in order to prove that we use the relativistic transformation rules (2.92), (2.93), (2.94), (2.95) and (2.96) in the Papapetrou-Randers frame (2.79). Using (4.109), (4.110), (4.111), (4.112) and (4.113), and expanding we find

$$\begin{cases} \delta m^j = 0 \\ \delta n^j = -\delta \beta^j \omega + \delta \beta_k m^k \beta^j \\ \delta \omega = -\delta \beta_k m^k \\ \delta \chi = -\delta \beta_k \left(n^k + \frac{\beta^2}{2} m^k \right), \end{cases} \quad (\text{B.3})$$

and

$$\begin{cases} \delta \psi^i = 0 \\ \delta Q^i = \delta \beta_j (\Sigma^{ij} - \phi a^{ij}) - \delta \beta^i \zeta + \delta \beta_j \psi^j \beta^i \\ \delta \pi^i = \delta \beta_j (\Xi^{ij} - (\eta + \varpi) a^{ij}) + \frac{\beta^2}{2} \delta \beta_j (\Sigma^{ij} - (\zeta + \phi) a^{ij}) + \delta \beta_j \beta^i (\beta^j (\zeta + \phi) + Q^j) \\ \quad + \beta^2 \beta^i \delta \beta^j \psi^j \\ \delta \zeta = -2\delta \beta_i \psi^i \\ \delta \eta = -2\delta \beta_i \left(Q^i + \frac{\beta^2}{2} \psi^i \right) \\ \delta \Sigma^{ij} - \delta \phi a^{ij} = \psi^i \delta \beta^j + \psi^j \delta \beta^i \\ \delta \Xi^{ij} - \delta \varpi a^{ij} = Q^i \delta \beta^j + Q^j \delta \beta^i + \delta \beta_k (\Sigma^{ki} \beta^j + \Sigma^{kj} \beta^i) + \frac{\beta^2}{2} (\psi^i \delta \beta^j + \psi^j \delta \beta^i) \\ \quad + \delta \beta_k \beta^k (\psi^i \beta^j + \psi^j \beta^i) + \beta^i \beta^j \delta \phi. \end{cases} \quad (\text{B.4})$$

It is straightforward to show that the variations of all momenta (B.1) and (B.2) vanish.

C Free motion

Our results on the failure of conservation laws associated with some Galilean or Carrollian (conformal) Killing vector fields are generic and rooted to the nature of the underlying

geometries. The same phenomenon occurs when studying free-particle motion in Newton-Cartan spacetimes, or instantonic branes on Carrollian structures (see [29] for motivations on the latter paradigm). For concreteness we will illustrate here the former case.

The stage is set with an action

$$S[\mathbf{x}] = \int_{\mathcal{C}} dt \Omega(t) \mathcal{L}(t, \mathbf{x}, \mathbf{v}), \quad (\text{C.1})$$

where $L = \Omega \mathcal{L}$ is the Lagrangian — as opposed to the Lagrangian density. The generalized Lagrange momenta are

$$p_i = \frac{\partial \mathcal{L}}{\partial v^i} \quad (\text{C.2})$$

and the energy $E = \Omega \mathcal{E}$ with

$$\mathcal{E} = \frac{p_i v^i}{\Omega} - \mathcal{L}. \quad (\text{C.3})$$

The equations of motion are Euler-Lagrange

$$\frac{1}{\Omega} \dot{p}_i - \frac{\partial \mathcal{L}}{\partial x^i} = 0. \quad (\text{C.4})$$

The dot stands for the total derivative along the trajectory, which can act also as $\partial_t + v^i \partial_i$ on *any* tensor, and should not be confused with d/dt defined in (3.24) unless they act on scalars (cf. ordinary vs. covariant spatial derivative).

Consider Galilean diffeomorphisms generated by

$$\xi = \xi^t \partial_t + \xi^i \partial_i, \quad (\text{C.5})$$

where $\xi^t = \xi^t(t)$ and

$$\xi^{\hat{t}} = \xi^t \Omega, \quad \xi^{\hat{i}} = \xi^i - \xi^t w^i, \quad \xi_{\hat{t}} = -c^2 \xi^{\hat{t}}, \quad \xi_i = a_{ij} \xi^{\hat{j}} = \xi_i. \quad (\text{C.6})$$

Their effect on the dynamical variables is

$$\begin{cases} t \rightarrow t + \xi^t \\ x^i \rightarrow x^i + \xi^i \\ v^i \rightarrow v^i + \partial_t \xi^i + v^j \partial_j \xi^i - v^i \partial_t \xi^t. \end{cases} \quad (\text{C.7})$$

On the one hand, the invariance of the action is characterized as follows:

$$\delta S = 0 \Leftrightarrow \Omega \delta \mathcal{L} + \mathcal{L} \partial_t \xi^{\hat{t}} = \frac{d\phi}{dt}, \quad (\text{C.8})$$

where $\phi = \phi(t, \mathbf{x})$ is an arbitrary function, that needs not be zero. On the other hand, one can determine the on-shell variation of the Lagrangian density:

$$\delta \mathcal{L} = -\frac{\mathcal{L}}{\Omega} \partial_t \xi^{\hat{t}} + \frac{1}{\Omega} \frac{d}{dt} (p_i \xi^i - \mathcal{E} \xi^{\hat{t}}). \quad (\text{C.9})$$

The simplest of Noether's theorems states that

$$\delta S = 0 \Leftrightarrow p_i \xi^i - \mathcal{E} \xi^{\hat{t}} - \phi = \text{constant of motion}. \quad (\text{C.10})$$

Suppose now that the motion is free on a Newton-Cartan spacetime featured by a^{ij} , w^i and Ω . The Lagrangian density is

$$\mathcal{L} = \frac{1}{2\Omega^2} a_{ij} (v^i - w^i) (v^j - w^j) \quad (\text{C.11})$$

with

$$p_i = \frac{1}{\Omega} (v_i - w_i). \quad (\text{C.12})$$

Euler-Lagrange equations read:

$$\left(\frac{1}{\Omega} \hat{\mathcal{D}} + p^j \hat{\nabla}_j \right) p_i + p_j \hat{\gamma}^{wj}_i = 0. \quad (\text{C.13})$$

As an aside remark, the latter equation is the infinite- c limit of the spatial component of the geodesic equation $u^\mu \nabla_\mu u_i = 0$, in a Zermelo background. The time component $u^\mu \nabla_\mu u^0 = 0$ leads to

$$\frac{1}{2\Omega} \frac{dp_i p^i}{dt} + p_i p_j \hat{\gamma}^{wij} = 0, \quad (\text{C.14})$$

which is the energy equation, obtained by contracting (C.13) with p^i .

We can now compute the generic variation of (C.11) under Galilean diffeomorphisms acting as (C.7). We obtain the following:

$$\delta\mathcal{L} = p_i p_j \left(\hat{\nabla}^{(i} \xi^{j)} + \xi^{\hat{t}} \hat{\gamma}^{wij} \right) - p_i p^i \frac{1}{\Omega} \frac{\hat{\mathcal{D}} \xi^{\hat{t}}}{dt} + p_i \left(\frac{1}{\Omega} \frac{\hat{\mathcal{D}} \xi^i}{dt} - \hat{\gamma}^{wi}_j \xi^{\hat{j}} \right). \quad (\text{C.15})$$

If ξ is a Killing field it satisfies (3.53), the first two terms drop and

$$\delta\mathcal{L} = p_i \left(\frac{1}{\Omega} \frac{\hat{\mathcal{D}} \xi^i}{dt} - \hat{\gamma}^{wi}_j \xi^{\hat{j}} \right) \quad (\text{C.16})$$

does not vanish, exactly as in the Galilean fluid dynamics in the presence of an isometry. This betrays the break down of conservation, unless the right-hand side of eq. (C.16) happens to be of the form (C.9), in which case Noether's theorem applies in its version (C.10).

As already emphasized repeatedly, this pattern works the same way in all situations we have met, involving Galilean or Carrollian dynamics. In flat spacetimes (either Galilean or Carrollian) boosts belong invariably to the class of isometries with non-vanishing Lagrangian variation (see (3.56) and (4.49)). There is not much we could extract from this in fluid dynamics (except for the case of flat-space potential flows, see footnotes 37 and 65), but for Galilean free-particle motion on flat spacetime ($a_{ij} = \delta_{ij}$, $\Omega = 1$, w^i constants) the situation is simpler. We find indeed:

$$\delta\mathcal{L} = (\dot{x}^i - w^i) (V_i + w^k \Omega_{ki}) = \frac{d}{dt} (x^i V_i - w^i V_i t + w^k x^i \Omega_{ki}). \quad (\text{C.17})$$

In this particular case, (C.10) applies and gives the general constant of motion as (see also (3.55))

$$V_i (\dot{x}^i t - x^i) - \frac{T}{2} (\dot{\mathbf{x}}^2 - \mathbf{w}^2) + X_i (\dot{x}^i - w^i) + \Omega_{ij} x^i \dot{x}^j. \quad (\text{C.18})$$

The boosts V^i do not generate any useful first integral (the initial position x_0^i), as opposed to time translation T , space translations X^i and rotations Ω_{ij} , which lead to energy, momentum and angular momentum conservations.

D From conservation to potential non-conservation

D.1 Galilean law from infinite speed of light

Our starting point is a pseudo-Riemannian spacetime in Zermelo frame (2.46)

$$ds^2 = -\Omega^2 c^2 dt^2 + a_{ij} (dx^i - w^i dt) (dx^j - w^j dt) \quad (\text{D.1})$$

with an energy-momentum tensor $T^{\mu\nu}$ obeying $\nabla_\mu T^{\mu\nu} = 0$, and a vector field

$$\xi = \xi^t \partial_t + \xi^i \partial_i = \xi^{\hat{t}} e_{\hat{t}} + \xi^{\hat{i}} e_{\hat{i}}, \quad (\text{D.2})$$

where the frame and coframe are defined as in (2.72), and

$$\xi^{\hat{t}} = \xi^t \Omega, \quad \xi^{\hat{i}} = \xi^i - \xi^t w^i, \quad \xi_{\hat{t}} = -c^2 \xi^{\hat{t}}, \quad \xi_{\hat{i}} = a_{ij} \xi^{\hat{j}} = \xi_i. \quad (\text{D.3})$$

We define a current as in (2.13), $I_\mu = T_{\mu\nu} \xi^\nu$, and compute its on-shell divergence, using eqs. (2.63):

$$\begin{aligned} \nabla_\mu I^\mu &= -\frac{1}{2} T_{\mu\nu} \mathcal{L}_\xi g^{\mu\nu} = -\frac{\varepsilon_r}{\Omega} \frac{\hat{D}\xi^{\hat{t}}}{dt} + (p_r a_{ij} + \tau_{rij}) (\hat{\nabla}^i \xi^{\hat{j}} + \xi^{\hat{t}} \hat{\gamma}^{wij}) \\ &\quad + \frac{1}{c^2} q_{ri} \left(\frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi}{}_j \xi^{\hat{j}} - c^2 a^{ij} \partial_j \xi^{\hat{t}} \right). \end{aligned} \quad (\text{D.4})$$

This result is relativistic, expressed with Galilean derivatives though. It vanishes iff

$$\begin{cases} \frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{t}}}{dt} = 0 \\ \hat{\nabla}^{(i} \xi^{\hat{j})} + \xi^{\hat{t}} \hat{\gamma}^{wij} = 0 \\ \frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi}{}_j \xi^{\hat{j}} - c^2 a^{ij} \partial_j \xi^{\hat{t}} = 0, \end{cases} \quad (\text{D.5})$$

which are simply the conditions for ξ be a Killing field of the pseudo-Riemannian manifold.

We would like now to consider the infinite- c limit of (D.4). At the first place, we must provide the behaviour of ε_r , q_{ri} and $p_r a_{ij} + \tau_{rij}$ for large c . This is typically of the form (3.120)⁸⁴

$$\begin{cases} \varepsilon_r = \Pi + \mathcal{O}(1/c^2) \\ q_{ri} = c^2 P_i + \Pi_i + \mathcal{O}(1/c^2) \\ p_r a_{ij} + \tau_{rij} = \Pi_{ij} + \mathcal{O}(1/c^2), \end{cases} \quad (\text{D.6})$$

and (D.4) becomes:

$$\nabla_\mu I^\mu = -\frac{\Pi}{\Omega} \frac{\hat{D}\xi^{\hat{t}}}{dt} + \Pi_{ij} (\hat{\nabla}^i \xi^{\hat{j}} + \xi^{\hat{t}} \hat{\gamma}^{wij}) + \left(P_i + \frac{\Pi_i}{c^2} \right) \left(\frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi}{}_j \xi^{\hat{j}} - c^2 a^{ij} \partial_j \xi^{\hat{t}} \right) + \mathcal{O}(1/c^2). \quad (\text{D.7})$$

⁸⁴More general behaviours have appeared in (3.86), (3.87), (3.88), or in (3.124). These choices wouldn't change our present argument though.

For this expression to remain finite at infinite c , we must impose that⁸⁵

$$\partial_j \xi^{\hat{t}} = 0, \tag{D.8}$$

which is the requirement that ξ generates a Galilean diffeomorphism. Conservation holds in the limit if expression (D.7) vanishes, which is again a threefold condition:

$$\frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{t}}}{dt} = 0, \tag{D.9}$$

$$\hat{\nabla}^{(i}\xi^{\hat{j})} + \xi^{\hat{t}}\hat{\gamma}^{wij} = 0, \tag{D.10}$$

$$\frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi}{}_j \xi^{\hat{j}} = 0. \tag{D.11}$$

Equations (D.9) and (D.10) are nothing but (3.53) i.e. the definition of a Galilean Killing field. Equation (D.11) is an extra condition, absent for generic Galilean isometries. The latter *do not guarantee the existence of a conserved Galilean current*. The break down of the conservation is read off in

$$\lim_{c \rightarrow \infty} \nabla_\mu I^\mu = P_i \left(\frac{1}{\Omega} \frac{\hat{D}\xi^{\hat{i}}}{dt} - \hat{\gamma}^{wi}{}_j \xi^{\hat{j}} \right) = \frac{P_i}{\Omega} \left(\partial_t \xi^{\hat{i}} + \mathcal{L}_w \xi^{\hat{i}} \right), \tag{D.12}$$

which agrees with (3.61) or (3.130). As stressed in section 3.1, the failure might be only apparent, if the term $\frac{P_i}{\Omega} \left(\partial_t \xi^{\hat{i}} + \mathcal{L}_w \xi^{\hat{i}} \right)$ turns out to be a boundary term, that would then contribute the Galilean current.

D.2 Carrollian law from zero speed of light

We will here consider pseudo-Riemannian spacetime in Papapetrou-Randers frame (2.79)

$$ds^2 = -c^2 \left(\Omega dt - b_i dx^i \right)^2 + a_{ij} dx^i dx^j. \tag{D.13}$$

We assume a conserved energy-momentum tensor $T^{\mu\nu}$ and a vector field as in (D.2) with

$$\xi^{\hat{t}} = \xi^t \Omega - \xi^i b_i, \quad \xi^{\hat{i}} = \xi^i, \quad \xi_{\hat{t}} = -c^2 \xi^{\hat{t}}, \quad \xi_{\hat{i}} = a_{ij} \xi^{\hat{j}} = \xi_i + \xi_{\hat{t}} b_i. \tag{D.14}$$

The frame and coframe are defined in (2.105).

We now compute the on-shell divergence of the current (2.13) $I_\mu = T_{\mu\nu} \xi^\nu$, using eqs. (2.97):

$$\begin{aligned} \nabla_\mu I^\mu &= \frac{1}{2} T^{\mu\nu} \mathcal{L}_\xi g_{\mu\nu} = -\varepsilon_r \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^{\hat{i}} \right) + \left(p_r a^{ij} + \tau_r^{ij} \right) \left(\hat{\nabla}_i \xi_{\hat{j}} + \xi^{\hat{t}} \hat{\gamma}_{ij} \right) \\ &\quad - q_r^i \left(\left(\hat{\partial}_i - \varphi_i \right) \xi^{\hat{t}} - 2\xi^{\hat{j}} \varpi_{ji} - \frac{1}{c^2 \Omega} a_{ij} \partial_t \xi^{\hat{j}} \right). \end{aligned} \tag{D.15}$$

⁸⁵Following the discussion on section 3.3, one may refine the limiting procedure for the Killing fields, and reach the Galilean diffeomorphisms as $\xi^{\hat{t}}(t, \mathbf{x}) = \xi_G^{\hat{t}}(t) + \frac{1}{c^2} \nu(t, \mathbf{x}) + \mathcal{O}\left(\frac{1}{c^4}\right)$. This would alter eq. (D.11) as $\frac{1}{\Omega} \frac{\hat{D}\xi_{\hat{t}}}{dt} - \hat{\gamma}^{wj}{}_i \xi_{\hat{j}} - \partial_t \nu = 0$. Similarly the arbitrary function $\nu(t, \mathbf{x})$ would also appear in the large- c expansions of eqs. (2.76) and (2.77), altering the Galilean currents (3.129). Ultimately, this would have no incidence on our conclusions about the interplay between Galilean isometries and conservation. It may nevertheless provide a complementary view on the large- c contraction of the relativistic diffeomorphisms, possibly in line with the approach followed in ref. [106], where a further duality relationship has been established among leading Galilean and subleading Carrollian contributions (see footnote 87), and vice-versa.

Although expressed with Carrollian derivatives, this is relativistic and vanishes iff

$$\begin{cases} \frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i = 0 \\ \hat{\nabla}_i(\xi_j) + \xi^{\hat{t}} \hat{\gamma}_{ij} = 0 \\ (\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} - \frac{1}{c^2 \Omega} a_{ij} \partial_t \xi^j = 0. \end{cases} \quad (\text{D.16})$$

These conditions define a Killing field ξ on a pseudo-Riemannian manifold.

We would like now to consider the zero- c limit of (D.15). We must provide the behaviour of ε_r , q_r^i and $p_r a^{ij} + \tau_r^{ij}$ for small c , which is typically of the form (4.73), (4.74), (4.75)⁸⁶

$$\begin{cases} \varepsilon_r = \Pi + \mathcal{O}(c^2) \\ q_r^i = \Pi^i + c^2 P^i + \mathcal{O}(c^4) \\ p_r a^{ij} + \tau_r^{ij} = \Pi^{ij} + \mathcal{O}(c^2). \end{cases} \quad (\text{D.17})$$

Equation (D.15) reads now:

$$\begin{aligned} \nabla_\mu I^\mu &= -\Pi \left(\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i \right) + \Pi^{ij} \left(\hat{\nabla}_i \xi_j + \xi^{\hat{t}} \hat{\gamma}_{ij} \right) \\ &\quad - \left(\Pi^i + c^2 P^i \right) \left((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} - \frac{1}{c^2 \Omega} a_{ij} \partial_t \xi^j \right) + \mathcal{O}(c^2). \end{aligned} \quad (\text{D.18})$$

Finiteness at zero c , demands⁸⁷

$$\partial_t \xi^i = 0, \quad (\text{D.19})$$

hence ξ generates a Carrollian diffeomorphism. Conservation holds in the limit if expression (D.18) vanishes. This occurs if

$$\frac{1}{\Omega} \partial_t \xi^{\hat{t}} + \varphi_i \xi^i = 0, \quad (\text{D.20})$$

$$\hat{\nabla}_{(i} \xi_{j)} + \xi^{\hat{t}} \hat{\gamma}_{ij} = 0, \quad (\text{D.21})$$

$$(\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} = 0. \quad (\text{D.22})$$

Equations (D.20) and (D.21) are as in (4.46) i.e. the definition of a Carrollian Killing field. Equation (D.22) is an extra condition, absent for generic Carrollian isometries, which therefore *do not guarantee the existence of a conserved Carrollian current*. The disruption to the conservation is measured as

$$\lim_{c \rightarrow 0} \nabla_\mu I^\mu = -\Pi^i \left((\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi^j \varpi_{ji} \right), \quad (\text{D.23})$$

in agreement with (4.54) or (4.136).

⁸⁶More general behaviours have appeared in (4.114). The latter wouldn't change our present conclusions though.

⁸⁷Mirroring footnote 85, an option is to set $\xi^i(t, \mathbf{x}) = \xi_C^i(\mathbf{x}) + c^2 \nu^i(t, \mathbf{x}) + \mathcal{O}(c^4)$. With this, eq. (D.22) becomes $(\hat{\partial}_i - \varphi_i) \xi^{\hat{t}} - 2\xi_C^j \varpi_{ji} - \frac{1}{\Omega} a_{ij} \partial_t \nu^j = 0$, and further work would be necessary on eqs. (2.108), (2.109) and (4.134), that would not alter our final conclusions, but could shed light on the small- c contraction of general diffeomorphisms.

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Renormalized holographic entanglement entropy for quadratic curvature gravity

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(Received 20 May 2021; accepted 31 August 2021; published 1 October 2021)

We derive a covariant expression for the renormalized holographic entanglement entropy for conformal field theories (CFTs) dual to quadratic curvature gravity in arbitrary dimensions. This expression is written as the sum of the bare entanglement entropy functional obtained using standard conical defect techniques, and a counterterm defined at the boundary of the extremal surface of the functional. The latter corresponds to the cod-2 self-replicating part of the extrinsic counterterms when evaluated on the replica orbifold. This renormalization method isolates the universal terms of the holographic entanglement entropy functional. We use it to compute the standard C -function candidate for CFTs of arbitrary dimension, and the type-B anomaly coefficient c for four-dimensional CFTs.

DOI: [10.1103/PhysRevD.104.086003](https://doi.org/10.1103/PhysRevD.104.086003)

I. INTRODUCTION

Higher-curvature gravity theories have attracted considerable attention in the literature for various reasons. This is mainly due to their better ultraviolet (UV) completion properties [1–5], what provides a promising scenario for deciphering quantum gravity properties. Indeed, the low energy effective action of string theory is characterized by the presence of a series of higher-derivative terms added on top of the Einstein-Hilbert action.

Quadratic curvature gravity (QCG) is one of the simplest examples of a higher-curvature gravitational theory. Its action is given by

$$I_{\text{QCG}} = \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^{d+1}x \sqrt{|\mathcal{G}|} (R - 2\Lambda_0 + \alpha R^2 + \beta R^\mu_\nu R^\nu_\mu + \gamma GB), \quad (1)$$

where the *bare* cosmological constant is $\Lambda_0 = -\frac{d(d-1)}{2L_0^2}$, in terms of the radius of anti-de Sitter (AdS) space L_0 . Here, the scalar density GB stands for the Gauss-Bonnet term, which reads $GB = R^2 - 4R^\mu_\nu R^\nu_\mu + R^{\mu\nu} R_{\mu\nu}^{\kappa\lambda}$. The equation of motion (EOM) of the theory (1) is fourth order in differential terms of the metric $\mathcal{G}_{\mu\nu}$, when arbitrary couplings (α, β, γ) are considered. However, if $\alpha = \beta = 0$, i.e., the Gauss-Bonnet term is the only modification considered, then the EOM becomes second order [6], as the theory goes back to the Lovelock class [7,8].

A larger family is that of generalized quasitopological gravities that have sparked a lot of interest in recent years [9–14]. Its main feature is the fact that the EOM is reduced to second order when evaluated in a static and spherically symmetric ansatz.

In $d + 1 = 4$ dimensions, of particular importance is the point of the parametric space of QCG where $\alpha = -1/(2\Lambda_0)$ and $\beta = 3/(2\Lambda_0)$. In this dimension, γ plays no role as the Gauss-Bonnet term is not dynamical. It corresponds to critical gravity, introduced in Ref. [15], a theory which becomes trivial when evaluated on Einstein spacetimes [16]. Interestingly enough, for relaxed AdS asymptotic conditions, this theory constitutes the gravitational dual of three-dimensional logarithmic conformal field theory (CFT).

The holographic principle has a concrete realization in the form of the AdS/CFT correspondence introduced in Refs. [17–19]. In this context, higher-curvature gravity

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theories have been the subject of extensive study as toy models due to the appealing properties of their dual field theories. This class of theories probes a broader range of CFTs than standard Einstein gravity. Indeed, the type-A and type-B anomalies of their four-dimensional holographic counterparts do not coincide, unlike in the Einstein case [20,21]. Furthermore, the couplings of higher curvature theories modify the ratio of shear viscosity η to entropy density s of their hydrodynamic duals, with respect to the value provided by Einstein gravity of $\eta/s = 1/(4\pi)$ [22–24], as can be seen in five-dimensional Einstein-Gauss-Bonnet theory [25]. This indicates that the Kovtun-Son-Starinets bound is not universal in the sense that its value has explicit dependence on the higher curvature couplings of the gravity theory. However, as causality considerations further constrain said couplings [26], it is possible to obtain modified Kovtun-Son-Starinets bounds applicable to classes of higher curvature theories [27–29]. This is another example of how the presence of higher curvature terms can break degeneracies in holographic quantities, helping to pinpoint the holographic dictionary more precisely.

A similar nontrivial behavior is present in the universal terms of the entanglement entropy (EE) for CFTs dual to higher-curvature gravity theories [30–35]. EE appears as a useful tool to explore aspects of quantum entanglement in strongly coupled field theories. In a quantum model described by the density matrix ρ , the EE between a subsystem A and its complement A^c is given by the von Neumann entropy of the reduced matrix $\rho_A = \text{tr}_{A^c} \rho$,

$$S_A = -\text{tr}_A(\rho_A \log \rho_A). \quad (2)$$

As many observables in field theory, EE is a UV divergent measure. Its main characteristic is the fact that the leading divergence always scales as the cod-2 area of the entangling surface ∂A that separates the subsystem A with its complement, independently of its shape.¹ This is due to the UV degrees of freedom which are localized at the vicinity of the entangling surface [36–38].

Of great interest are the universal features associated to the renormalization group (RG) flow of the quantum field theory (QFT), in the expansion of the EE near ∂A . They correspond to scheme-independent terms that probe the number of effective degrees of freedom of the respective theory. In odd dimensions, it is the finite part of the EE that can be related to the F function that is monotonic along the RG flow [39–41]. In even dimensions, one can identify the type-A and type-B anomaly coefficients of the corresponding field theory from the logarithmic term of the EE expansion. As the type-A anomaly monotonically decreases when going from ultraviolet to infrared (IR), it is a good C -function candidate [42–44].

¹This assumes local field theories and continuity of the fields and their derivatives across the entangling surface.

In principle, the calculation of the EE in QFT is rather involved. Using the real-time formalism or the replica trick in the path-integral quantization demands analytic techniques that can be applied only for particular shapes of A , in low enough dimensions [45].

Major progress was made in this direction by considering the AdS/CFT duality. Ryu and Takayanagi (RT) conjectured that EE is given by a generalization of the Bekenstein-Hawking formula for cod-2 surfaces which are not the fixed point of a continuous isometry [46,47]. More specifically, the EE of a spatial subregion A is given by the area of a minimal homologous cod-2 surface Σ , i.e.,

$$S_A = \frac{\text{Area}(\Sigma)}{4G_N}. \quad (3)$$

The surface is embedded in the bulk solution of the dual gravity theory and anchored at ∂A .

The proof behind the conjectured RT formula (3), came a few years later by Lewkowycz and Maldacena [48]. The central idea is the relation between holographic entanglement entropy (HEE) and Euclidean gravity action, in the saddle-point approximation of AdS/CFT. This is achieved by constructing bulk solutions from a set of boundary conditions that are not $U(1)$ invariant. As a consequence, one may extend the replica symmetry, needed for the calculation of the EE, to the bulk. In particular, the HEE is given by the limit

$$S_{\text{EE}} = -\lim_{\vartheta \rightarrow 1} \partial_{\vartheta} I_E[\mathcal{M}^{(\vartheta)}], \quad (4)$$

where $I_E[\mathcal{M}^{(\vartheta)}]$ denotes the Euclidean gravity action evaluated on a conically singular manifold. The orbifold $\mathcal{M}^{(\vartheta)}$ is constructed as the bulk gravity dual of the replica CFT obtained through the standard replica trick in [48–50]. As the replica symmetry is discrete, $\mathcal{M}^{(\vartheta)}$ is a squashed cone [having no $U(1)$ isometry], and it has an angular deficit of $2\pi(1 - \vartheta)$, such that $\vartheta = \frac{1}{m}$ is related to the replica index m .

As pointed out in Ref. [51], in the case of HEE in Einstein-AdS gravity, $\frac{(1-\vartheta)}{4G_N}$ can be identified with the tension of a cosmic brane coupled to the ambient geometry through the Nambu-Goto action. Thus, in the tensionless limit ($\vartheta \rightarrow 1$), the area of the cosmic brane anchored at the boundary gives the HEE, trivially recovering the RT prescription [46]. However, Eq. (4) also holds for generic gravity theories, such that upon evaluating the action on the $\mathcal{M}^{(\vartheta)}$ orbifold, the correct entropy functional emerges naturally. The most notable contributions in this direction have been given by Dong [52], Camps [53], and Miao [54]. In the case of QCG, the evaluation of the arbitrary quadratic curvature invariants on squashed cones was firstly done by Fursaev, Patrushev, and Solodukhin in Ref. [50] using distributional geometry techniques. In the present work we

exploit the fact that, as the HEE functional depends only on the gravity action under consideration, its renormalization is inherited from the renormalization of the bulk gravity action [55].

On previous papers, based on a renormalization scheme defined by the addition of extrinsic counterterms, we showed that for odd-dimensional CFTs dual to Einstein-AdS gravity, the finite part of the HEE is the sum of the Euler characteristic of the RT surface, and a curvature term on that surface [56].² The coefficient of the Euler characteristic is matched to the F term, a quantity that has been shown to be monotonic along RG flows [30,31,39,40,49,58]. In particular, in three-dimensional CFTs, the F quantity is robust under continuous deformations of the entangling region [59]. In the case of even-dimensional CFTs, the C -function candidate is identified as the coefficient of the log part of the EE. As shown in Ref. [60], this universal contribution can be isolated when the HEE is written in terms of a covariant functional which is the usual area term plus a cod-3 extrinsic boundary counterterm. The latter term cancels power-law divergences in the HEE formula.

In this paper, we turn our interest to the case of QCG. There, we will explicitly show that the same form of extrinsic counterterms (only with a different coupling constant) is able to isolate the universal part of the HEE for CFTs dual to this type of higher curvature gravity theories. This can be achieved by considering particular shapes of entangling regions (i.e., spheres and cylinders), where the C -function candidates and other holographic quantities like the type-B anomaly coefficient c can be directly obtained [20,21,61].

II. COUNTERTERMS OF A DIFFERENT SORT: KOUNTERTERMS

In general relativity, it is straightforward to prove that second derivatives in the normal coordinate appear in the Lagrangian. One may think of an analogous system in classical mechanics: a Lagrangian with a linear dependence on the acceleration \dot{q} , of the type $L(q, \dot{q}, \ddot{q}) = \dot{q}h(q) + \ell(q, \dot{q})$, where q is the generalized coordinate. An arbitrary variation of this action gives rise to a field equation which is still of second order in the time derivative, while producing a term that contains both δq and $\delta \dot{q}$ at the boundary. The addition of a total derivative to L leads to a surface term where only δq appears, what is a feature of first-order Lagrangians. The above reasoning can be mimicked in gravity, what implies that the bulk action has to be augmented by the Gibbons-Hawking term at the boundary, in order to ensure a well-defined action principle for a Dirichlet boundary condition on the boundary metric h_{ab} [62]. The introduction of a cosmological constant does not modify the surface terms of the theory, but it does

change the asymptotic behavior of the metric, such that the canonical momentum is no longer finite at the boundary. In the early days of the AdS/CFT correspondence, the renormalization of AdS gravity was achieved by the addition of intrinsic counterterms, such that the Dirichlet boundary condition was not spoiled [63–65]. It is then that, by a proper rescaling of the metric h_{ab} , holographic correlation functions are obtained as variations with respect to the source at the conformal boundary $g_{(0)ab}$.

However, it was later pointed out by Papadimitriou and Skenderis [66] that a Dirichlet condition on the boundary metric h_{ab} does not make sense in asymptotically AdS (AAdS) spaces. As a matter of fact, the asymptotic expansion of this field $h_{ab} = \frac{g_{(0)ab}}{z^2} + \dots$ reveals an infinite conformal factor near the boundary. As a consequence, the only way to have a well-posed variational principle in AdS gravity is fixing instead the metric $g_{(0)ab}$ at the conformal boundary. This argument implies that the addition of counterterms is required not only for canceling divergent terms in the variation of the action, but also for the consistency of the variational problem on $g_{(0)ab}$.

Along the same line, we stress the fact that there is a blissful accident in AdS gravity: the leading order in the asymptotic expansion of the extrinsic curvature is (up to a numerical factor) the same as the one in h_{ab} . Indeed, in the Fefferman-Graham frame, $K_{ab} = \frac{1}{\ell} \frac{g_{(0)ab}}{z^2} + \dots$, what has been recently emphasized in Ref. [67]. This simple observation means that one can express also variations of K_{ab} in terms of variations of $g_{(0)ab}$. This also implies that one may consider surface terms which depend on the extrinsic curvature and act as counterterms, in the sense that they cancel divergent contributions in the AdS gravity action. So, even though they are plain incompatible with a Dirichlet condition for the full boundary metric h_{ab} , they can still reproduce the correct holographic stress tensor varying with respect to $g_{(0)ab}$.

The above reasoning, which opens the possibility to look for an alternative sort of counterterms, is justified by the lack of a closed expression for the series in arbitrary dimensions. More than 20 years ago, evidence was provided on the fact that topological terms were able to regulate the variation of the AdS gravity action in even dimensions [68,69], though based on the study of particular solutions. As for the Euclidean action, the addition of the Euler term at the boundary of $d = 2n$ dimensions renders it finite in AAdS solutions if the coupling is adequately chosen [70]. However, it was not clear what this prescription to renormalize AdS gravity had to do with holographic renormalization and the addition of standard counterterms.

A first step towards the understanding of this issue was given in Ref. [70], where topological terms in the bulk are equivalently written as the corresponding Chern form B_{2n-1} at the boundary,

²In Ref. [57], the authors arrive at the same conclusion using standard holographic renormalization techniques.

$$B_{2n-1} = -2n \int_0^1 ds \sqrt{|h|} \delta_{b_1 \dots b_{2n-1}}^{a_1 \dots a_{2n-1}} K_{a_1}^{b_1} \left(\frac{1}{2} \mathcal{R}_{a_2 a_3}^{b_2 b_3} - s^2 K_{a_2}^{b_2} K_{a_3}^{b_3} \right) \\ \times \dots \times \left(\frac{1}{2} \mathcal{R}_{a_{2n-2} a_{2n-1}}^{b_{2n-2} b_{2n-1}} - s^2 K_{a_{2n-2}}^{b_{2n-2}} K_{a_{2n-1}}^{b_{2n-1}} \right). \quad (5)$$

For the first time, counterterms which depend on K_{ab} were proposed to deal with the renormalization of AdS gravity. Here, $h_{ab} = g_{ab}(z, x)/z^2$ is the induced metric at constant Poincaré coordinate z , \mathcal{R}_{cd}^{ab} is the intrinsic Riemann curvature tensor, K_{ab} is the extrinsic curvature and $\delta_{b_1 \dots b_{2n-1}}^{a_1 \dots a_{2n-1}}$ is the generalized Kronecker delta. In this notation, the indices of the generalized Kronecker delta are contracted with those of the rest of the tensors in the integrand.

$$B_{2n} = -2n \int_0^1 ds \int_0^s dt \sqrt{|h|} \delta_{b_1 \dots b_{2n-1}}^{a_1 \dots a_{2n-1}} K_{a_1}^{b_1} \left(\frac{1}{2} \mathcal{R}_{a_2 a_3}^{b_2 b_3} - s^2 K_{a_2}^{b_2} K_{a_3}^{b_3} + \frac{t^2}{L_{\text{eff}}^2} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} \right) \times \dots \\ \dots \times \left(\frac{1}{2} \mathcal{R}_{a_{2n-2} a_{2n-1}}^{b_{2n-2} b_{2n-1}} - s^2 K_{a_{2n-2}}^{b_{2n-2}} K_{a_{2n-1}}^{b_{2n-1}} + \frac{t^2}{L_{\text{eff}}^2} \delta_{a_{2n-2}}^{b_{2n-2}} \delta_{a_{2n-1}}^{b_{2n-1}} \right), \quad (6)$$

where L_{eff} is the effective AdS radius of the theory.

One may think that the renormalization procedure described above, dubbed *Kounterterms*, may lead to a variational principle which is at odds with the holographic description of AdS gravity in terms of the boundary source $g_{(0)ab}$, as it seems to require a different boundary condition on the extrinsic curvature. But the analysis portrayed above gives a firmer ground to the addition of Kounterterms to the gravitational action: the total action is consistent with a holographic description, as its variation is both finite and given in terms of $\delta g_{(0)ab}$.

This simple reasoning suggests the resummation of the counterterm series as an expression in terms of the extrinsic curvature. As a matter of fact, an asymptotic expansion of the term B_d reproduces the counterterms, once the Gibbons-Hawking term is correctly isolated. This was first sketched in Ref. [72] and analyzed in greater detail in a recent paper [73].³ In addition, earlier works in the

A similar structure at the boundary of $d+1 = 2n+1$ dimensions was far more difficult to obtain. In particular, due to the fact that there is no equivalent form in the bulk for such boundary term. The extensive use of field-theory tools in the context of anomalies (Chern-Simons and transgression forms, homotopy operator, etc.) allows us to make a concrete proposal for that case. The resulting term, in essence, shares common properties with the Chern form, as it is a given polynomial of the extrinsic and intrinsic curvatures [71], but it did not exist in the mathematical literature before. Its specific form is given by the following expression:

mathematical literature [74,75], indicate that the Chern form is fundamental in defining the renormalized volume of an Einstein space.

The Kounterterm method has been used to deal with the construction of conserved quantities and the thermodynamic description of black holes in Einstein-Gauss-Bonnet AdS and, in general, Lovelock AdS gravity. Furthermore, it has also linked the concept of conformal mass to the addition of boundary terms in Einstein-Hilbert [76] and higher-curvature gravity [77,78]. Evidence has been given that Kounterterms can provide finite conserved charges in QCG, as well [79–81].

In the present paper, we extensively use its properties to deal with the problem of renormalization of HEE in QCG.

III. RENORMALIZED EE FROM THE REPLICATOR ORBIFOLD

In this section, we compute the renormalized HEE in CFTs dual to QCG. In order to determine the EE functional from the bulk gravity Lagrangian we use the results by Fursaev, Patrushev, and Solodukhin for the evaluation of quadratic curvature invariants in conically singular manifolds [50]. The renormalization of HEE is then inherited from the renormalization of the bulk action by the addition of cod-1 Kounterterms and derived from a set of cod-2 relations with respect to the bulk and boundary dimensions. In particular, it is a remarkable property of the Kounterterm B_d —when evaluated on orbifolds—that the singular part gives rise to the same structure in two dimensions lower (B_{d-2}). This self-replicating feature of B_d induces cod-3 Kounterterms on $\partial\Sigma$ which renormalize the cod-2 Fursaev,

³In Ref. [73], it was shown that, in Einstein-AdS gravity, the Kounterterms are the resummation of the counterterms for asymptotically conformally flat manifolds in arbitrary dimensions. For a generic AAdS space, there is a mismatch between counterterms and Kounterterms, consisting on terms which are the dimensional continuation of conformal invariants at the boundary. At the lowest order, this difference is a Weyl-squared term, which is identically vanishing for a conformally flat boundary. However, it may be the case that this condition is relaxed by taking $\text{Weyl}^2 = 0$ instead, which would be the analog of demanding local flatness ($Rie = 0$) vs a vanishing Kretschmann scalar for a given spacetime.

Patrushev, and Solodukhin functional. Based on this construction, we determine the universal terms of the HEE for this theory.

A. Curvature invariants on the cone

Now, we consider the expressions for the quadratic curvature invariants evaluated on squashed cones [without U(1) isometry], as given in Refs. [50,82]. Following this procedure, we have

$$\int_{\mathcal{M}^{(\theta)}} d^{d+1}x \sqrt{|\mathcal{G}|} R^2 = \int_{\mathcal{M}^{(\theta)} \setminus \Sigma} d^{d+1}x \sqrt{|\mathcal{G}|} R^2 + 8\pi(1-\vartheta) \int_{\Sigma} d^{d-1}y \sqrt{|\sigma|} (\tilde{R} + 2R^{(i)(i)} - R^{(i)(j)(i)(j)} - 2\mathcal{K}^{(i)\bar{\mu}}_{[\bar{\mu}} \mathcal{K}^{(i)\bar{\nu}}_{\bar{\nu}]}) + \dots, \quad (7)$$

$$\int_{\mathcal{M}^{(\theta)}} d^{d+1}x \sqrt{|\mathcal{G}|} R^\mu_\nu R^\nu_\mu = \int_{\mathcal{M}^{(\theta)} \setminus \Sigma} d^{d+1}x \sqrt{|\mathcal{G}|} R^\mu_\nu R^\nu_\mu + 4\pi(1-\vartheta) \int_{\Sigma} d^{d-1}y \sqrt{|\sigma|} \left(R^{(i)(i)} - \frac{1}{2} \mathcal{K}^{(i)}_{\bar{\mu}} \bar{\mu} \mathcal{K}^{(i)}_{\bar{\nu}} \bar{\nu} \right) + \dots, \quad (8)$$

$$\int_{\mathcal{M}^{(\theta)}} d^{d+1}x \sqrt{|\mathcal{G}|} R^{\mu\nu}_{\kappa\lambda} R^{\kappa\lambda}_{\mu\nu} = \int_{\mathcal{M}^{(\theta)} \setminus \Sigma} d^{d+1}x \sqrt{|\mathcal{G}|} R^{\mu\nu}_{\kappa\lambda} R^{\kappa\lambda}_{\mu\nu} + 8\pi(1-\vartheta) \int_{\Sigma} d^{d-1}y \sqrt{|\sigma|} (R^{(i)(j)(i)(j)} - \mathcal{K}^{(i)}_{\bar{\mu}} \bar{\nu} \mathcal{K}^{(i)}_{\bar{\nu}} \bar{\mu}) + \dots, \quad (9)$$

where the ellipsis denotes terms of higher order in $(1-\vartheta)$, Σ is the fixed-point set of the replica symmetry (i.e., the cod-2 surface at the apex of the cone), i, j are indices normal to Σ , $\bar{\mu}, \bar{\nu}$ are indices in the world volume of Σ , $\mathcal{K}^{(i)}_{\bar{\mu}\bar{\nu}}$ is the extrinsic curvature of the cod-2 foliation along the i direction, and summation is implied on repeated i, j indices.

We also consider the evaluation of the Ricci scalar and the GB term on the squashed cone, following the results of Ref. [50]. We therefore have

$$\int_{\mathcal{M}^{(\theta)}} d^{d+1}x \sqrt{|\mathcal{G}|} R = \int_{\mathcal{M}^{(\theta)} \setminus \Sigma} d^{d+1}x \sqrt{|\mathcal{G}|} R + 4\pi(1-\vartheta) \int_{\Sigma} d^{d-1}y \sqrt{|\sigma|}, \quad (10)$$

$$\int_{\mathcal{M}^{(\theta)}} d^{d+1}x \sqrt{|\mathcal{G}|} \text{GB} = \int_{\mathcal{M}^{(\theta)} \setminus \Sigma} d^{d+1}x \sqrt{|\mathcal{G}|} \text{GB} + 8\pi(1-\vartheta) \int_{\Sigma} d^{d-1}y \sqrt{|\sigma|} \tilde{R}. \quad (11)$$

B. Extrinsic counterterms on the cone

We begin our analysis of Kounterterms evaluated on squashed cones by considering the topological origin of the Chern form. When \mathcal{M} is even dimensional, the existence of extrinsic counterterms B_{2n-1} is guaranteed by the Euler theorem

$$\int_{\mathcal{M}} d^{2n}x \mathcal{E}_{2n} = (4\pi)^n n! \chi(\mathcal{M}) + \int_{\partial\mathcal{M}} d^{2n-1}X B_{2n-1}, \quad (12)$$

which unveils the dynamic equivalence to the topological term in the bulk

$$\mathcal{E}_{2n} = \frac{\sqrt{|\mathcal{G}|}}{2^n} \delta^{\nu_1 \dots \nu_{2n}}_{\mu_1 \dots \mu_{2n}} R^{\mu_1 \mu_2}_{\nu_1 \nu_2} \dots R^{\mu_{2n-1} \mu_{2n}}_{\nu_{2n-1} \nu_{2n}}. \quad (13)$$

The above relation also identifies B_{2n-1} with the corresponding Chern form, which appears as the correction to the Euler characteristic in a manifold with boundary.

For the purposes of the current study, we highlight the fact that the equivalence between bulk and boundary terms

can be generalized to manifolds with conical deficits. Indeed, any Euler term \mathcal{E}_{2n} self-replicates in cod-2, that is,

$$\int_{\mathcal{M}^{(\theta)}} d^{2n}x \mathcal{E}_{2n} = \int_{\mathcal{M}^{(\theta)} \setminus \Sigma} d^{2n}x \mathcal{E}_{2n} + 4\pi n(1-\vartheta) \int_{\Sigma} d^{2n-2}y \mathcal{E}_{2n-2} \quad (14)$$

when constructed on an orbifold $M^{(\theta)}$ [60,83]. This fact implies that the self-replication property extends to the Chern form, as well [56,84]

$$\int_{\partial\mathcal{M}^{(\theta)}} d^{2n-1}X B_{2n-1} = \int_{\partial\mathcal{M}^{(\theta)} \setminus \partial\Sigma} d^{2n-1}X B_{2n-1} + 4\pi n(1-\vartheta) \int_{\partial\Sigma} d^{2n-3}Y B_{2n-3}. \quad (15)$$

Unlike the Chern form, Kounterterms B_{2n} for odd bulk dimensions are not associated to the Euler density. Therefore, the previous analysis for orbifolds cannot be repeated verbatim. However, even though B_{2n-1} and B_{2n}

are different geometrical objects, it can be shown that, in presence of squashed conical singularities,

$$\int_{\partial\mathcal{M}^{(\vartheta)}} d^d X B_d = \int_{\partial\mathcal{M}^{(\vartheta)} \setminus \partial\Sigma} d^d X B_d + 4\pi \left[\frac{d+1}{2} \right] (1-\vartheta) \int_{\partial\Sigma} d^{d-2} Y B_{d-2}, \quad (16)$$

for either odd or even boundary dimension d [60].

The extrinsic counterterms B_{d-2} (living in cod-3 respect to the bulk space) cancel the divergences coming from the set of points anchoring the surface Σ to the conformal boundary. It has been shown that they correctly isolate the universal terms of the entanglement and Rényi entropies for CFTs which are dual to Einstein gravity [56,60,82,84]. In that situation, the relation between EE and the notion of renormalized area was made manifest.

In what follows, we seek to extend the renormalization scheme for cod-2 surfaces to one of the simplest examples of a higher-derivative gravity, such as QCG.

C. The Kounterterm-renormalized QCG action

We start by considering the generic renormalized QCG action [80]

$$I_{\text{QCG}}^{\text{ren}} = I_{\text{QCG}} + \frac{c_d}{16\pi G_N} \int_{\partial\mathcal{M}} B_d, \quad (17)$$

where B_d is the boundary Kounterterm defined in Eq. (16).

By solving the usual characteristic equation to find the vacuum of the theory, we obtain the following relation for the effective cosmological constant and its corresponding effective AdS radius

$$-\frac{1}{2\Lambda_{\text{eff}}} + \frac{\Lambda_0}{2\Lambda_{\text{eff}}^2} = \frac{(d-3)}{(d-1)^2} [(d+1)\alpha + \beta] + \frac{(d-3)(d-2)}{d(d-1)} \gamma, \quad (18)$$

where $\Lambda_{\text{eff}} = -\frac{d(d-1)}{2L_{\text{eff}}^2}$. Also, the coupling c_d of the boundary Kounterterm B_d is fixed by requiring the action of the pure AdS solution (vacuum) to be finite, thus finding

$$c_d = \begin{cases} a_d \frac{(-1)^{\frac{d+1}{2}} L_{\text{eff}}^{d-1}}{\left(\frac{d+1}{2}\right)!(d-1)!} & \text{if } d \text{ odd} \\ a_d \frac{(-1)^{\frac{d}{2}} L_{\text{eff}}^{d-2}}{2^{d-3} d \left(\left(\frac{d}{2}-1\right)!\right)^2} & \text{if } d \text{ even} \end{cases}, \quad (19)$$

where the auxiliary function a_d reads

$$a_d = 1 - \frac{2d}{L_{\text{eff}}^2} \left[(d+1)\alpha + \beta + \frac{(d-2)(d-1)}{d} \gamma \right]. \quad (20)$$

Based on the preceding relations, we evaluate the QCG action augmented with the boundary term (17) on the squashed cone, obtaining

$$I_{\text{QCG}}^{\text{ren}}[M^{(\vartheta)}] = I_{\text{QCG}}^{\text{ren}}[M^{(\vartheta)} \setminus \partial\Sigma] + (1-\vartheta) S_{\text{QCG}}^{\text{Univ}} + \dots \quad (21)$$

When considering the Lewkowycz-Maldacena prescription (4), applied to the renormalized action (17), $S_{\text{QCG}}^{\text{Univ}}$ corresponds to the universal part of the HEE for CFTs dual to QCG, which is finite for odd- d and log-divergent for even- d . Explicitly, the $S_{\text{QCG}}^{\text{Univ}}$ functional is given by

$$S_{\text{QCG}}^{\text{Univ}} = S_{\text{QCG}} + S_{\text{KT}}, \quad (22)$$

where

$$S_{\text{QCG}} = \frac{1}{4G_N} \left\{ \text{Area}[\Sigma] + \int_{\Sigma} d^{d-1} y \sqrt{|\sigma|} \left[2\alpha(\tilde{R} + 2R^{(i)(i)} - R^{(i)(j)(i)(j)} - 2\mathcal{K}^{(i)\bar{\mu}}_{[\bar{\mu}} \mathcal{K}^{(i)\bar{\nu}}_{\bar{\nu}]}) + \beta \left(R^{(i)(i)} - \frac{1}{2} \mathcal{K}^{(i)\bar{\mu}}_{\bar{\mu}} \mathcal{K}^{(i)\bar{\nu}}_{\bar{\nu}} \right) + 2\gamma \tilde{R} \right] \right\}, \quad (23)$$

and the corresponding curvature terms are defined after Eqs. (7)–(9). Alternatively, S_{QCG} can be derived either following the prescription provided by Dong in Ref. [52] or the one by Camps in Ref. [53].⁴ For gravity theories with curvature terms of cubic order or higher, these procedures give different HEE functionals, which is referred to as the *splitting problem* [54,86–88].

On the other hand, the Kounterterm obtains the form

$$S_{\text{KT}} = \frac{c_d}{4G_N} \left[\frac{(d+1)}{2} \right] \int_{\partial\Sigma} B_{d-2}. \quad (24)$$

For this expression, the cod-2 boundary term reads

⁴Furthermore, in Ref. [85], S_{QCG} was computed using field redefinition in the gravity theory.

$$B_{d-2} = \begin{cases} - (d-1) \int_0^1 ds \sqrt{|\tilde{\sigma}|} \delta_{\bar{b}_1 \dots \bar{b}_{d-2}}^{\bar{a}_1 \dots \bar{a}_{d-2}} k_{\bar{a}_1}^{\bar{b}_1} \left(\frac{1}{2} \tilde{\mathcal{R}}_{\bar{a}_2 \bar{a}_3}^{\bar{b}_2 \bar{b}_3} - s^2 k_{\bar{a}_2}^{\bar{b}_2} k_{\bar{a}_3}^{\bar{b}_3} \right) \times \dots \\ \dots \times \left(\frac{1}{2} \tilde{\mathcal{R}}_{\bar{a}_{d-3} \bar{a}_{d-2}}^{\bar{b}_{d-3} \bar{b}_{d-2}} - s^2 k_{\bar{a}_{d-3}}^{\bar{b}_{d-3}} k_{\bar{a}_{d-2}}^{\bar{b}_{d-2}} \right), & \text{odd } d \\ - (d-2) \int_0^1 ds \int_0^s dt \sqrt{|\tilde{\sigma}|} \delta_{\bar{b}_1 \dots \bar{b}_{d-3}}^{\bar{a}_1 \dots \bar{a}_{d-3}} k_{\bar{a}_1}^{\bar{b}_1} \left(\frac{1}{2} \tilde{\mathcal{R}}_{\bar{a}_2 \bar{a}_3}^{\bar{b}_2 \bar{b}_3} - s^2 k_{\bar{a}_2}^{\bar{b}_2} k_{\bar{a}_3}^{\bar{b}_3} + \frac{t^2}{L_{\text{eff}}^2} \delta_{\bar{a}_2}^{\bar{b}_2} \delta_{\bar{a}_3}^{\bar{b}_3} \right) \times \dots \\ \dots \times \left(\frac{1}{2} \tilde{\mathcal{R}}_{\bar{a}_{d-4} \bar{a}_{d-3}}^{\bar{b}_{d-4} \bar{b}_{d-3}} - s^2 k_{\bar{a}_{d-4}}^{\bar{b}_{d-4}} k_{\bar{a}_{d-3}}^{\bar{b}_{d-3}} + \frac{t^2}{L_{\text{eff}}^2} \delta_{\bar{a}_{d-4}}^{\bar{b}_{d-4}} \delta_{\bar{a}_{d-3}}^{\bar{b}_{d-3}} \right), & \text{even } d \end{cases}, \quad (25)$$

where the corresponding terms are defined after Eq. (5). Having the renormalized EE functional at hand, we evaluate it on certain configurations, i.e., sphere and cylinder, whose universal terms encode significant information for the corresponding CFT.

IV. HEE FOR SPHERES IN VACUUM CFT

The bulk dual to the vacuum state of a d -dimensional CFT is pure AdS $_{d+1}$, whose metric in Poincaré coordinates is given by

$$ds^2 = \mathcal{G}_{\mu\nu} dx^\mu dx^\nu = \frac{L_{\text{eff}}^2}{z^2} (dt^2 + dz^2 + dr^2 + r^2 d\Omega_{d-2}^2), \quad (26)$$

where Ω_{d-2}^2 represents the angular directions of an \mathbb{S}^{d-2} sphere. For ball-shaped entangling regions of radius R in the CFT, the bulk extremal surface is given by the spherical hemisphere of the same radius [89], whose embedding is described by

$$\Sigma: \{t = \text{const.}; r^2 + z^2 = R^2\}. \quad (27)$$

For the following analysis, it is convenient to foliate pure AdS with warped spherical hemispheres. In order to make the extremal surface explicit, the change of coordinates $r = X \sin U$, $z = X \cos U$ is performed. After this change, metric (26) reads

$$ds^2 = \frac{L_{\text{eff}}^2}{X^2 \cos^2 U} (dt^2 + dX^2 + X^2 dU^2 + X^2 \sin^2 U d\Omega_{d-2}^2). \quad (28)$$

In this metric, the hemispheres are the constant (t, X) cod-2 hypersurfaces and the extremal one is located at $X = R$. Also, the nonzero components of the normal vectors to the hypersurfaces read

$$n_X^{(X)} = n_t^{(t)} = \frac{L_{\text{eff}}}{X \cos U}. \quad (29)$$

Therefore, the nonzero components of the projected Riemann and Ricci tensors along these directions read

$$R^{(X)(X)} = R^{\mu\nu} n_\mu^{(X)} n_\nu^{(X)} = -\frac{d}{L_{\text{eff}}^2},$$

$$R^{(t)(t)} = R^{\mu\nu} n_\mu^{(t)} n_\nu^{(t)} = -\frac{d}{L_{\text{eff}}^2}, \quad (30)$$

$$R^{(X)(t)(X)(t)} = R^{\mu\nu\kappa\lambda} n_\mu^{(X)} n_\nu^{(t)} n_\kappa^{(X)} n_\lambda^{(t)} = -\frac{1}{L_{\text{eff}}^2}. \quad (31)$$

Regarding the extrinsic curvatures, since the foliation defines a sphere, they identically vanish, i.e.,

$$\mathcal{K}^{(X)\bar{\mu}}_{\bar{\nu}} = \mathcal{K}^{(t)\bar{\mu}}_{\bar{\nu}} = 0. \quad (32)$$

In Poincaré coordinates, the induced metric reads

$$ds_\sigma^2 = \sigma_{\bar{\mu}\bar{\nu}} dy^{\bar{\mu}} dy^{\bar{\nu}} = \frac{L_{\text{eff}}^2}{z^2} \left[\frac{R^2 dz^2}{R^2 - z^2} + (R^2 - z^2) d\Omega_{d-2}^2 \right], \quad (33)$$

which admits a Fefferman-Graham (FG)-like expansion

$$r ds_\sigma^2 = \frac{L_{\text{eff}}^2}{z^2} \left[1 + \frac{z^2}{R^2} + \frac{z^4}{R^4} + \mathcal{O}(z^6) \right] dz^2 + \tilde{\sigma}_{\bar{a}\bar{b}} dY^{\bar{a}} dY^{\bar{b}}, \quad (34)$$

$$\tilde{\sigma}_{\bar{a}\bar{b}} = \frac{R^2 L_{\text{eff}}^2}{z^2} \left(1 - \frac{z^2}{R^2} \right) \Omega_{\bar{a}\bar{b}}, \quad (35)$$

where Ω_{ij} is the metric of the $(d-2)$ -dimensional sphere. The induced metric $\tilde{\sigma}_{ij}$ is fixed at the regulator $z = \epsilon$, i.e., when $\epsilon \rightarrow 0$.

Under the previous considerations, the quantities present in Eq. (22) read

$$\text{Area}[\Sigma] = \text{Vol}(\mathbb{S}^{d-2})R \int_e^{z_{\max}} dz (R^2 - z^2)^{\frac{d-3}{2}} \left(\frac{L_{\text{eff}}}{z}\right)^{d-1}, \quad (36)$$

$$\tilde{R} = -\frac{(d-1)(d-2)}{L_{\text{eff}}^2}, \quad (37)$$

$$R^{(i)(i)} = -\frac{2d}{L_{\text{eff}}^2}, \quad (38)$$

$$R^{(i)(j)(i)(j)} = -\frac{2}{L_{\text{eff}}^2}, \quad (39)$$

$$\mathcal{K}^{(i)\bar{\mu}}\mathcal{K}^{(i)\bar{\nu}} = \mathcal{K}^{(i)\bar{\mu}}\mathcal{K}^{(i)\bar{\nu}} = 0. \quad (40)$$

The cod-2 extrinsic curvatures vanish because the geometry after the foliations describes a sphere.

Therefore, considering all the terms that appear in Eq. (23), we have that the bare HEE S_{QCG} is given by

$$S_{\text{QCG}} = \frac{1}{4G_N} \left(\text{Vol}(\mathbb{S}^{d-2})R \int_e^{z_{\max}} dz (R^2 - z^2)^{\frac{d-3}{2}} \left(\frac{L_{\text{eff}}}{z}\right)^{d-1} + \int_{\Sigma} d^{d-1}\sqrt{|h|} \left\{ -\frac{2\alpha}{L_{\text{eff}}^2} [(d-1)(d-2) + 4d - 2] - \frac{2d\beta}{L_{\text{eff}}^2} - \gamma \frac{(d-1)(d-2)}{L_{\text{eff}}^2} \right\} \right), \quad (41)$$

which can be rearranged, using the definition of a_d in Eq. (20), to

$$S_{\text{QCG}} = \frac{a_d \text{Vol}(\mathbb{S}^{d-2})}{4G_N} \int_e^{z_{\max}} dz R (R^2 - z^2)^{\frac{d-3}{2}} \left(\frac{L_{\text{eff}}}{z}\right)^{d-1}, \\ = a_d \frac{\text{Area}[\Sigma]}{4G_N}. \quad (42)$$

Also, the HEE Kounterterm S_{KT} of Eq. (24) can be written as

$$S_{\text{KT}} = a_d \frac{c_d^{\text{EH}}}{4G_N} \left[\frac{(d+1)}{2} \right] \int_{\partial\Sigma} B_{d-2}, \quad (43)$$

where $c_d^{\text{EH}} = \frac{c_d}{a_d}$ is the Kounterterm coupling for Einstein-AdS gravity. Thus, we have that

$$S_{\text{QCG}}^{\text{Univ}} = \frac{a_d}{4G_N} \left[\text{Area}[\Sigma] + c_d^{\text{EH}} \left[\frac{(d+1)}{2} \right] \int_{\partial\Sigma} B_{d-2} \right], \\ S_{\text{QCG}}^{\text{Univ}} = \frac{a_d}{4G_N} \text{Area}_{\text{Univ}}[\Sigma], \quad (44)$$

such that the universal part of the HEE for ball-shaped entangling regions becomes proportional to the universal part of the area of the minimal surface Σ .

Finally, using our results of Refs. [56,60], $\text{Area}_{\text{Univ}}$ is given by⁵

$$\text{Area}_{\text{Univ}} = \begin{cases} (-1)^{\frac{d-1}{2}} \frac{2^{d-1} \pi^{\frac{d-1}{2}} L_{\text{eff}}^{d-1}}{(d-1)!} & \text{if } d \text{ odd} \\ (-1)^{\frac{d}{2}-1} \frac{2\pi^{\frac{d}{2}-1} L_{\text{eff}}^{d-1}}{(\frac{d}{2}-1)!} \log\left(\frac{2R}{\epsilon}\right) & \text{if } d \text{ even,} \end{cases} \quad (45)$$

where a_d was defined in Eq. (20). The explicit cancellation of the IR divergences in the area functional and the identification of the universal term are given in Appendix B.

A. C-function candidates in CFTs dual to QCG

In order to characterize the properties of CFTs, such as their central charges, it is useful to compute the C -function candidates, which are quantities conjectured to decrease along RG flows [39,40,42–44,58]. In the context of AdS/CFT, these quantities can be computed by holographic methods. For instance, in the case of ball-shaped entangling regions, we can read out the C -function candidates for both odd- and even-dimensional CFTs directly from the expression for $S_{\text{QCG}}^{\text{Univ}}$ [30,31,49]. In particular, we have

$$S_{\text{QCG}}^{\text{Univ}} = \begin{cases} (-1)^{\frac{d-1}{2}} F & \text{if } d \text{ odd} \\ (-1)^{\frac{d}{2}-1} 4A \log\left(\frac{2R}{\epsilon}\right) & \text{if } d \text{ even} \end{cases}, \quad (46)$$

where

$$F = a_d \frac{2^d \pi^{\frac{d-1}{2}} L_{\text{eff}}^{d-1}}{8G_N (d-1)!} \quad (47)$$

is the F quantity, defined in terms of the partition function of the CFT evaluated on a sphere as $F = (-1)^{\frac{d-1}{2}} \log(Z_{\text{CFT}}[\mathbb{S}^d])$, and

⁵Note the choice of $2R$ as the characteristic scale inside the logarithm of the universal term. This choice allows us to absorb the finite term as part of the logarithmically divergent term.

$$A = a_d \frac{\pi^{\frac{d-1}{2}} L_{\text{eff}}^{d-1}}{8G_N (\frac{d}{2} - 1)!}, \quad (48)$$

is the type-A anomaly coefficient. They correspond to quantities that are conjectured to be monotonic along RG flows for odd- and even-dimensional CFTs, respectively [30,31].⁶ Therefore, the C -function candidate for CFTs dual to QCG is proportional to the one of Einstein-AdS gravity, but multiplied by an overall coefficient that depends on the parameters of the theory a_d . An identical behavior of the universal terms has been found in other higher-curvature theories of gravity such as Einstein cubic gravity, quasitopological gravity, and Lovelock theories, at least, at the perturbative level [91].

The explanation behind this proportionality becomes clear when Eqs. (44) and (45) are considered. Note that the computation of $S_{\text{QCG}}^{\text{Univ}}$ is simplified, for the case of ball-shaped entangling regions, as the resulting functional is proportional to the renormalized area $\text{Area}_{\text{Univ}}$ of the minimal surface in the bulk. This is due to the fact that for spheres, the bare entropy functional becomes proportional to the area of Σ what is the RT functional. Therefore, the C -function candidates of both QCG and Einstein-AdS gravity correspond to the universal terms of the area of Σ , given by $\text{Area}_{\text{Univ}}$, up to a factor that depends on the QCG couplings.

The fact that the entropy has to be proportional to the area in the spherical case is universal, and can be inferred directly from the Casini-Huerta-Myers map [41]. In particular, due to the conformal symmetry of the CFT, the EE of the ball-shaped subregion can be mapped to the thermal entropy of the CFT at a certain temperature that depends on the replica index. This entropy can be computed, using AdS/CFT, as the Wald entropy of a hyperbolic black hole of constant curvature, which is trivially proportional to the area of the black hole horizon.

Note also, that in the expression for the log universal term of Eq. (46), one can consider the radius of the sphere R as the characteristic size scale. In which case the logarithmic term can be written as $\log \frac{R}{\epsilon} + \log 2$. This extra $\log 2$ appears in even d . Because of the robustness of the term in different dimensions, it is suggestive to consider it as coming from a topological term. Indeed, it can be written as $\log(\chi[\partial\Sigma])$, where $\chi[\partial\Sigma]$ is the Euler characteristic of the entangling surface in the CFT.

⁶Although the F and A theorems have been proven for arbitrary (unitary) CFTs only in three and four dimensions, respectively, the monotonicity of the a^* charge of which they are particular cases is conjectured to apply for arbitrary dimension. Furthermore, the holographic version of the corresponding C theorem has been proven for CFTs dual to Einstein gravity in arbitrary dimension. For the case of CFTs dual to higher curvature gravities, the case of Einsteinian cubic gravity was studied in [90]. Therefore, even though the C theorem has not been proven in the generic case, the corresponding C functions for CFTs dual to QCG are expected to be monotonic along RG flows.

In the following section, we consider cylinder-shaped entangling regions, from which it is possible to compute the type-B anomaly coefficient in four-dimensional CFTs [35,89].

V. HEE FOR A CYLINDER IN VACUUM CFT

In order to characterize the type-B anomaly of a CFT, it is useful to consider the log part of the HEE for a cylindrical entangling region. For instance, in the case of AdS₅/CFT₄, this universal term is related to c (the type-B anomaly coefficient) according to

$$S_{\text{EE}}^{\text{Univ}} = -\frac{cH}{2l} \log \frac{l}{\epsilon}, \quad (49)$$

where l is the radius of the cylinder, H is its length along the axis and ϵ is the usual UV cutoff in the CFT [35,89]. When computing the EE holographically, by comparing the obtained result with the previous expression, it is possible to identify the c coefficient in terms of the bulk gravity quantities.

We start by considering the metric of pure AdS₅ written as

$$ds^2 = \mathcal{G}_{\mu\nu} dx^\mu dx^\nu = \frac{L_{\text{eff}}^2}{z^2} (dt^2 + dz^2 + dx_3^2 + dr^2 + r^2 d\theta^2), \quad (50)$$

where θ represents the angular direction of an \mathbb{S}^1 sphere. For cylindrical entangling regions of radius l in the CFT, with their axis extending infinitely along the x_3 direction, the bulk extremal surface, in the near-boundary region, is described by the embedding

$$\Sigma: \left\{ t = \text{const}; r = l \left[1 - \frac{z^2}{4l^2} + \mathcal{O}(z^4) \right] \right\}. \quad (51)$$

The normal vectors to the hypersurface read

$$\begin{aligned} n_\mu^{(X)} &= \frac{L_{\text{eff}}}{\sqrt{4l^2 + z^2}} \left(0, 1, 0, \frac{2l}{z}, 0 \right), \\ n_\mu^{(t)} &= \left(\frac{L_{\text{eff}}}{z}, 0, 0, 0, 0 \right). \end{aligned} \quad (52)$$

In this case, the projected Riemann and Ricci tensors read

$$R^{(t)(t)} = R^{tt} n_t^{(t)} n_t^{(t)} = -\frac{4}{L_{\text{eff}}^2}, \quad (53)$$

$$R^{(X)(X)} = R^{rr} n_r^{(X)} n_r^{(X)} + R^{zz} n_z^{(X)} n_z^{(X)} = -\frac{4}{L_{\text{eff}}^2}, \quad (54)$$

$$\begin{aligned} R^{(X)(t)(X)(t)} &= R^{ztzt} n_z^{(X)} n_t^{(t)} n_z^{(X)} n_t^{(t)} + R^{rtrt} n_r^{(X)} n_t^{(t)} n_r^{(X)} n_t^{(t)} \\ &= -\frac{1}{L_{\text{eff}}^2}. \end{aligned} \quad (55)$$

The extrinsic curvature along the temporal axis vanishes, i.e., $\mathcal{K}_v^{\bar{t}} = 0$. However, the foliation in the z coordinates gives a nonzero extrinsic curvature whose components read

$$\mathcal{K}_z^{z(X)} = -\frac{z^3}{8l^3 L_{\text{eff}}} + \mathcal{O}(z^5), \quad (56)$$

$$\mathcal{K}_{x_3}^{x_3(X)} = -\frac{z}{2l L_{\text{eff}}} + \frac{z^3}{16l^3 L_{\text{eff}}} + \mathcal{O}(z^5), \quad (57)$$

$$\mathcal{K}_\theta^{\theta(X)} = \frac{z}{2l L_{\text{eff}}} + \frac{3z^3}{16l^3 L_{\text{eff}}} + \mathcal{O}(z^5). \quad (58)$$

The induced metric in the cod-2 manifold Σ reads

$$ds_\sigma^2 = \frac{L_{\text{eff}}^2}{z^2} \left\{ \left[1 + \frac{z^2}{4l^2} + \mathcal{O}(z^4) \right] dz^2 + dx_3^2 + l^2 \left[1 - \frac{z^2}{4l^2} + \mathcal{O}(z^4) \right]^2 d\theta^2 \right\}. \quad (59)$$

As in the spherical entangling region case, this expression admits a FG expansion as well.

Based on these considerations, the geometric quantities appearing in the EE functional in Eq. (23) are given by

$$\text{Area}[\Sigma] = \int_0^{2\pi} d\theta \int_0^H dx_3 \int_\epsilon^{z_{\text{max}}} dz \sqrt{|\bar{h}|}, \quad (60)$$

$$\bar{R} = -\frac{6}{L_{\text{eff}}^2} - \frac{z^2}{2l^2 L_{\text{eff}}^2} + \mathcal{O}(z^4), \quad (61)$$

$$R^{(i)(i)} = -\frac{8}{L_{\text{eff}}^2}, \quad (62)$$

$$R^{(i)(j)(i)(j)} = -\frac{2}{L_{\text{eff}}^2}, \quad (63)$$

$$\mathcal{K}_a^{(i)a} \mathcal{K}_b^{(i)b} = \mathcal{O}(z^6), \quad (64)$$

$$\mathcal{K}_b^{(i)a} \mathcal{K}_a^{(i)b} = \frac{z^2}{2l^2 L_{\text{eff}}^2} + \mathcal{O}(z^4). \quad (65)$$

In even-dimensional CFTs, the finite part of the EE is nonuniversal, and, therefore, upon evaluating the integral in the Poincaré coordinate of the area functional, the upper limit (at z_{max}) can be neglected. For the lower limit we expand the metric determinant, finding

$$\sqrt{|\sigma|} = \frac{l L_{\text{eff}}^3}{z^3} - \frac{L_{\text{eff}}^3}{8l z} + \mathcal{O}(z). \quad (66)$$

Now, plugging all these results into the functional, we obtain

$$S_{\text{QCG}} = \frac{\pi H L_{\text{eff}}^3}{4l G_N} \left(a_4 \frac{l^2}{\epsilon^2} - \frac{b_4}{4} \log \frac{l}{\epsilon} \right) + \mathcal{O}(1), \quad (67)$$

where we define the coefficient

$$b_4 = 1 - \frac{4}{L_{\text{eff}}^2} (10\alpha + 2\beta + \gamma). \quad (68)$$

This factor differs from a_4 defined in Eq. (20).

Finally, we check that the boundary term cancels the power law term in Eq. (67). The induced metric at the boundary $z = \epsilon$ reads

$$\sqrt{\bar{\sigma}} = \frac{l L_{\text{eff}}^2}{\epsilon^2} - \frac{L_{\text{eff}}^2}{4l} + \mathcal{O}(\epsilon), \quad (69)$$

yielding

$$S_{\text{KT}} = -\frac{a_4 \pi H L_{\text{eff}}^3}{4G_N \epsilon^2}. \quad (70)$$

As we can see, the power law divergence in Eq. (67) is indeed canceled by the Kounterterm.

Thus, up to a nonuniversal finite part, one has that for the cylinder entangling region in $d = 4$, the universal part of the HEE is given by

$$S_{\text{QCG}}^{\text{Univ}} = -b_4 \frac{\pi H L_{\text{eff}}^3}{16l G_N} \log \frac{l}{\epsilon}. \quad (71)$$

Finally, comparing this expression with that of Eq. (49), we have that

$$c = b_4 \frac{\pi L_{\text{eff}}^3}{8G_N}. \quad (72)$$

In the QCG case, it is evident from our results that the A and c central charges are different. However, for Einstein-AdS ($b_4 = a_4 = 1$), they coincide.

A. Extremal surface for the cylinder

In Refs. [35,89], it was shown that the hypersurface (51) extremizes the HEE functional for Gauss-Bonnet theory. In QCG, higher-order terms appear in the entanglement entropy. However, the same embedding function yields the extremal surface at order $\mathcal{O}(z^3)$. In order to see this, consider an arbitrary surface parametrized with $r = r(z)$. In $d = 4$, the HEE functional for QCG reads

$$S_{\text{QCG}} = \frac{\pi H L_{\text{eff}}}{4G_N} \int_{\epsilon}^{z_{\text{max}}} dz \frac{1}{z^3 r (r^2 + 1)^{5/2}} \{ 2L_{\text{eff}}^2 r^2 (r^2 + 1)^3 + 16\alpha r (r^2 + 1) [z(z + 2rr')r'' - (r^2 + 1)(r(5 + 3r^2) - zr')] - \beta [(r(r^3 + r' + zr'')) + z(r^2 + 1)]^2 + 16r^2 (r^2 + 1)^3 - 4\gamma r (r^2 + 1) [2z(2rr' + z)r'' + 2(3r - 2zr')(r^2 + 1)] \}. \quad (73)$$

This expression is obtained once the terms given by

$$\begin{aligned} \text{Area}[\Sigma] &= \int_0^{2\pi} d\theta \int_0^H du \int_{\epsilon}^{z_{\text{max}}} dz \sqrt{|\sigma|} \\ &= 2\pi H L_{\text{eff}}^3 \int_{\epsilon}^{z_{\text{max}}} dz \frac{r\sqrt{1+r^2}}{z^3}, \end{aligned} \quad (74)$$

$$\tilde{R} = -\frac{2z(2rr' + z)r'' + 2(3r - 2zr')(r^2 + 1)}{L_{\text{eff}}^2 r (r^2 + 1)^2}, \quad (75)$$

$$R^{(i)(i)} = -\frac{8}{L_{\text{eff}}^2}, \quad (76)$$

$$R^{(i)(j)(i)(j)} = -\frac{2}{L_{\text{eff}}^2}, \quad (77)$$

$$\mathcal{K}^{(i)\bar{\mu}} \mathcal{K}^{(i)\bar{\nu}} = \frac{[(rr' + z)(r^2 + 1) + zr'r'']^2}{L_{\text{eff}}^2 r^2 (r^2 + 1)^3}, \quad (78)$$

$$\mathcal{K}^{(i)\bar{\mu}} \mathcal{K}^{(i)\bar{\nu}} = \frac{[(z + rr')^2 + r^2 r'^2] (1 + r^2)^2 + r^2 (r' + r^3 - zr'')}{L_{\text{eff}}^2 r^2 (1 + r^2)^3}, \quad (79)$$

are plugged into Eq. (23). The resulting functional (73) constitutes a Lagrangian $\mathcal{L} = \mathcal{L}(z, r, r', r'')$ that contains second-order derivatives of the dynamical function $r(z)$. Because of this, the Euler-Lagrange equation needed to find the extremal surface reads

$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dz} \frac{\partial \mathcal{L}}{\partial r'} + \frac{d^2}{dz^2} \frac{\partial \mathcal{L}}{\partial r''} = 0. \quad (80)$$

From this expression, an EOM containing fourth-derivative terms in the function $r(z)$ is found. However, the ansatz

$$r = l \left[1 - \frac{z^2}{4l^2} + \mathcal{O}(z^4) \right], \quad (81)$$

from Ref. [35] is verified to satisfy the EOM up to order $\mathcal{O}(z^3)$. Because of this, the extremal surface for QCG coincides with that for Gauss-Bonnet gravity in this perturbative regime. This is an expected result due to the universality of the second term in the asymptotic expansion of the embedding function, which is linked to the universality of the $g_{(2)ij}$ coefficient in the FG expansion in terms of the Schouten tensor of $g_{(0)}$, as discussed in Ref. [92].

The case for the cylindrical entangling region in $d = 4$ is interesting as it isolates the contribution from the type-B anomaly in the universal part. The same should be the case for higher-dimensional cylinders, as the coefficient obtained should represent a linear combination of the couplings of different conformal invariants.⁷ However, the embedding function is not known in the higher-dimensional case as it would require knowledge of the subleading terms in the expansion of Eq. (81), which are not universal. Furthermore, the Kounterterm renormalization procedure has limitations regarding the types of entangling surfaces that it can accommodate for dual bulk manifolds of dimension greater than five. In particular, it requires the dimensional continuation of cod-2 conformal invariants at the entangling surface to vanish. For example, to the next-to-leading order, the method only works for surfaces such that

$$\mathcal{W}_{\bar{a}\bar{b}}^{\bar{a}\bar{b}} - \hat{k}_{\bar{a}\bar{b}}^{(\bar{i})} \hat{k}_{(\bar{i})}^{\bar{a}\bar{b}} = 0, \quad (82)$$

as shown in Ref. [60]. Here, $\mathcal{W}_{\bar{a}\bar{b}}^{\bar{a}\bar{b}}$ is the Weyl tensor of the conformal boundary projected along the entangling surface directions and $\hat{k}_{\bar{a}\bar{b}}^{(\bar{i})}$ is the traceless extrinsic curvature of $\partial\Sigma$ along the orthogonal directions. Equation (82) is trivially satisfied for spheres, but not for cylinders or arbitrary shapes.⁸

VI. DISCUSSION

The results for HEE for CFTs dual to QCG presented here come as the natural blend between the Kounterterm method applied to this gravity theory [79,80] and a remarkable feature of the boundary term B_d when evaluated in spacetimes with a conical defect, Eq. (16) [60]. In this respect, we have recovered the universal part of the HEE found in the literature regarding the computation of the C -function candidates [91]. This function captures essential properties of CFTs, which are given by the type-A anomaly coefficient in the case of even d and by the generalized F quantity (or a^* charge) for odd d [31,49,94].

⁷Although little is known about conformal invariants beyond eight dimensions, one may think this computation would provide information on the part of these invariants which is polynomial in the Weyl tensor.

⁸In bulk dimensions up to five, the Kounterterm procedure works for arbitrary entangling regions. Furthermore, as proven in [93], the procedure correctly renormalizes actions for gravity theories of arbitrary order in the Riemannian curvature, and therefore, it is expected to work for renormalizing HEE for CFTs dual to said theories as well.

The above calculation requires a ball-shaped entangling region in the CFT, where for the case of pure AdS (dual to the vacuum of the CFT), the embedding of the extremal surface (27) for the QCG HEE functional (23) is explicitly given. Then, the FG-like expansion of all the terms involved in the functional can be obtained. In both even and odd boundary dimension d , it can be seen that the C -function candidate derived is proportional to the one for Einstein-AdS gravity, but with an overall coupling-dependent factor a_d , whose form is given in Eq. (20).

We have also obtained the type-B anomaly coefficient c in the case of four-dimensional CFTs. In order to perform this computation, we have considered a cylindrical entangling region in the CFT, and the near-boundary expansion (up to cubic order in the Poincaré coordinate) of the embedding for the minimal surface. In this situation, we have derived the corresponding Euler-Lagrange equation for the embedding function $r(z)$, by taking variations of the entropy functional of Eq. (73). We have verified that the same embedding function considered for Einstein-AdS gravity is also a solution of the extremization equation in the QCG case, in $d = 4$, and up to cubic order. The condition for the minimal surface obtained in this way contains higher-order derivative terms in the dynamical variable (akin to the *acceleration*). Thus, extra boundary conditions are required beyond setting the border of the surface to coincide with the entangling region. When the surface is extremal (i.e., the intersection of the surface with the conformal boundary is orthogonal), the boundary problem is completely fixed. This is the case of the cylinder in $d = 4$, due to the fact that the entropy functional becomes proportional to the area up to cubic order. In the result for the c coefficient we also find agreement with the literature [91]. It is evident from the expression obtained in Eq. (72), that the type-B anomaly coefficient can be written as the one for Einstein-AdS gravity, but multiplied with a factor b_4 , given in Eq. (68), which incorporates the information on the couplings of QCG theory. The fact that b_4 is different from the a_4 of Eq. (20) allows for different central charges in four-dimensional CFTs dual to QCG, unlike the Einstein-AdS case.

All in all, for both even- and odd-dimensional CFT cases, the Kounterterm procedure allows us to isolate the universal part of the HEE of the dual gravity theory. For bulk dimensions lower than 6, the Kounterterm procedure works on entangling regions of arbitrary shape. Also, despite its limitations on the type of entangling regions that can be renormalized in higher dimensions (as discussed in the previous section), the Kounterterms prescription is the only method available so far for renormalizing HEE in higher curvature gravity theories.⁹

⁹The alternative renormalization procedure of [55], based on holographic renormalization [95], was only applied for Einstein and Einstein-Gauss-Bonnet gravity theories.

In isolating the universal part of the HEE, we have been able to express it as a covariant functional which is given by the standard HEE functional plus an extrinsic counterterm in cod-3 (24). In the particular case of spherical entangling regions in pure AdS (vacuum CFTs), the renormalized HEE functional becomes proportional to the renormalized area (44), which is logarithmically divergent for even d and finite for odd d .

For a cylindrical entangling region in $d > 4$, the renormalized entropy functional is no longer proportional to the renormalized cod-2 volume. However, in $d = 4$, the expressions coincide—up to the normalizable order—albeit with a different proportionality constant than for the spherical case.

We point out that the Kounterterm procedure is a nonperturbative method, in the sense that nowhere it is assumed that the couplings of the quadratic terms are small. In other words, the prescription does not rely on the linearization of the EOM such that the theory behaves like Einstein gravity with a modified Newton's constant. In particular, Kounterterms give a consistent renormalization prescription even at degenerate points, such as at the single-vacuum point of Einstein-Gauss-Bonnet theory (Chern-Simons AdS gravity in five dimensions). Thus, the obtained form of the holographic central charges is robust, and would hold in degenerate points of the parametric space as well (in the case of Lovelock, see Ref. [78]). Although we have not developed this point any further in this work, it would certainly be interesting to explore it in the near future.

ACKNOWLEDGMENTS

We thank Andrés Argandoña for useful comments. The work of G. A. and R. O. was funded in part by FONDECYT Grants No. 3190314 *Holographic Complexity from Anti-de Sitter gravity* and No. 1170765 *Boundary dynamics in anti-de Sitter gravity and gauge/gravity duality*. The work of J. M. is funded by the Agencia Nacional de Investigación y Desarrollo (ANID) Scholarship No. 21190234 and by Pontificia Universidad Católica de Valparaíso. J. M. is also grateful to the QMAP faculty for their hospitality. The work of I. J. A. is funded by ANID, REC Convocatoria Nacional Subvención a Instalación en la Academia Convocatoria Año 2020, Folio PAI77200097. The work of D. R. B. is funded by Becas Chile (ANID) Scholarship No. 72200301.

APPENDIX A: NOTATION AND CONVENTIONS

Here, we present the conventions used throughout the paper in the Table I. In the first column, we provide a list of objects defined on the different manifolds presented in the first line.

TABLE I. Notation and conventions.

	\mathcal{M}	$\partial\mathcal{M}$	Σ	$\partial\Sigma$
Indices	$\mu, \nu, \kappa, \lambda$	a, b, c, d	$\bar{\mu}, \bar{\nu}, \bar{\kappa}, \bar{\lambda}$	$\bar{a}, \bar{b}, \bar{c}, \bar{d}$
Coordinates	x^μ	X^a	$y^{\bar{\mu}}$	$Y^{\bar{a}}$
Metric	$\mathcal{G}_{\mu\nu}$	h_{ab}	$\sigma_{\bar{\mu}\bar{\nu}}$	$\tilde{\sigma}_{\bar{a}\bar{b}}$
Riemann tensor	$R_{\mu\nu}^{\kappa\lambda}$	\mathcal{R}_{ab}^{cd}	$\tilde{R}_{\bar{\mu}\bar{\nu}}^{\bar{\kappa}\bar{\lambda}}$	$\tilde{\mathcal{R}}_{\bar{a}\bar{b}}^{\bar{c}\bar{d}}$
Extrinsic curvature		K_{ab}	$\mathcal{K}_{\bar{\mu}\bar{\nu}}$	$k_{\bar{a}\bar{b}}$

APPENDIX B: DIVERGENCE CANCELLATION FOR SPHERICAL ENTANGLING REGIONS

In this section, we present explicit computations and show the cancellation of divergences in the Kounterterms scheme for spherical entangling regions.

1. Three-dimensional case

For three dimensions, the universal part of the EE (22) reduces to

$$S_{\text{QCG}}^{\text{Univ}} = \frac{\pi R}{2G_N} \int_\epsilon^{z_{\text{max}}} \frac{dz}{z^2} (L_{\text{eff}}^2 - 24\alpha - 6\beta) + \frac{c_3}{2G_N} \int_{\partial\Sigma} B_1, \quad (\text{B1})$$

where the auxiliary function c_3 and the boundary term B_1 , defined in Eqs. (19) and (25), respectively, read

$$c_3 = \frac{1}{4} (L_{\text{eff}}^2 - 24\alpha - 6\beta), \quad (\text{B2})$$

$$B_1 = -2\sqrt{\tilde{\sigma}} \text{tr} k \, d\Omega_1 = -\frac{2}{\epsilon} R [1 + \mathcal{O}(\epsilon^2)] d\Omega_1. \quad (\text{B3})$$

On the other hand, the determinant of the metric $\tilde{\sigma}$ is given by

$$\sqrt{\tilde{\sigma}} = \frac{RL_{\text{eff}}}{\epsilon} \sqrt{1 - \frac{\epsilon^2}{R^2}} = L_{\text{eff}} \left[\frac{R}{\epsilon} - \frac{\epsilon}{2R} - \frac{\epsilon^3}{R^3} + \mathcal{O}(\epsilon^4) \right]. \quad (\text{B4})$$

Also, $\text{tr} k$ is the trace of the extrinsic curvature $k_{\bar{a}\bar{b}} = -\frac{1}{2\sqrt{\tilde{\sigma}_{zz}}} \partial_z \tilde{\sigma}_{\bar{a}\bar{b}}$ of the FG-like expansion (34), which reads

$$k_{\bar{a}\bar{b}} = L_{\text{eff}} \left[\frac{R^2}{\epsilon^2} - \frac{1}{2} - \frac{\epsilon^2}{8R^2} + \mathcal{O}(\epsilon^4) \right] \Omega_{\bar{a}\bar{b}}. \quad (\text{B5})$$

Since the inverse metric reads,

$$\tilde{\sigma}^{\bar{a}\bar{b}} = \frac{1}{L_{\text{eff}}^2} \left[\frac{\epsilon^2}{R^2} + \mathcal{O}(\epsilon^4) \right] \Omega^{\bar{a}\bar{b}}, \quad (\text{B6})$$

then, the expansion of the trace yields

$$\text{tr} k = \tilde{\sigma}^{\bar{a}\bar{b}} k_{\bar{a}\bar{b}} = \frac{1}{L_{\text{eff}}} \left[1 + \frac{\epsilon^2}{2R^2} + \mathcal{O}(\epsilon^4) \right]. \quad (\text{B7})$$

In consequence, according to Eq. (24) and up to leading order, the Kounterterm in this case reads

$$S_{\text{KT}} = -\frac{\pi R}{2G_N \epsilon} [L_{\text{eff}}^2 - 24\alpha - 6\beta] + \mathcal{O}(\epsilon). \quad (\text{B8})$$

Thus, upon performing the integral in Eq. (B1) on the extremal surface, along the Poincar coordinate from $z = \epsilon$ to $z = R$, one gets

$$S_{\text{QCG}}^{\text{Univ}} = S_{\text{finite}} + (L_{\text{eff}}^2 - 24\alpha - 6\beta) \left[\frac{\pi R}{2G_N \epsilon} - \frac{\pi R}{2G_N} \right] + \mathcal{O}(\epsilon), \quad (\text{B9})$$

where $S_{\text{finite}} = -\frac{\pi}{2G} (L_{\text{eff}}^2 - 24\alpha - 6\beta)$ is the universal finite part. Thus, it becomes manifest that upon taking the $\epsilon \rightarrow 0$ limit, $S_{\text{QCG}}^{\text{Univ}}$ recovers the universal finite part and thus the HEE is renormalized correctly.

2. Four-dimensional case

In the four-dimensional case, from Eq. (25) it can be seen that the corresponding boundary term is given by

$$B_2 = -2L_{\text{eff}} \frac{\sqrt{R^2 - \epsilon^2}}{\epsilon^2}, \quad (\text{B10})$$

and hence, the Kounterterm expanded around $\epsilon = 0$ reads

$$S_{\text{KT}} = -\frac{a_4 \pi L_{\text{eff}}^3}{2G_N} \left(\frac{R^2}{\epsilon^2} - \frac{1}{2} \right). \quad (\text{B11})$$

On the other hand, the bare EE is given by

$$S_{\text{QCG}} = \frac{a_4 \pi}{G_N} \int_\epsilon^{z_{\text{max}}} dz R (R^2 - z^2)^{\frac{1}{2}} \left(\frac{L_{\text{eff}}}{z} \right)^3, \quad (\text{B12})$$

$$= \frac{a_4 \pi L_{\text{eff}}^3}{G_N} \left[\coth^{-1} \left(\frac{R}{\sqrt{R^2 - z_{\text{max}}^2}} \right) - \frac{R}{z_{\text{max}}^2} \sqrt{R^2 - z_{\text{max}}^2} \right] + \frac{a_4 \pi L_{\text{eff}}^3}{2G_N} \left[\frac{R^2}{\epsilon^2} - \log \frac{R}{\epsilon} - \frac{1}{2} (1 + \log 4) \right]. \quad (\text{B13})$$

Since the first line of (B13) vanishes in the limit $z_{\text{max}} \rightarrow R$, then, after adding the Kounterterm, the renormalized EE reads

$$S_{\text{QCG}}^{\text{Univ}} = -\frac{a_4 \pi L_{\text{eff}}^3}{2G_N} \log \frac{2R}{\epsilon}. \quad (\text{B14})$$

Notice that the Kounterterm isolates the logarithmic divergence, whose coefficient is universal and related to the

type-A charge of the CFT. A part of the finite term is canceled and there is a piece left that is reabsorbed in the logarithmic divergence.

3. Bare HEE in the arbitrary-dimensional case

For arbitrary dimensions, evaluating the quantities present in the EE expression using the metric (26) yields

$$S_{\text{QCG}} = \frac{1}{4G_N} \left(\text{Vol}(\mathbb{S}^{d-2}) R \int_{\epsilon}^{z_{\text{max}}} dz (R^2 - z^2)^{\frac{d-3}{2}} \left(\frac{L_{\text{eff}}}{z} \right)^{d-1} + \int_{\Sigma} d^{d-1} \sqrt{|h|} \left\{ -\frac{2\alpha}{L_{\text{eff}}^2} [(d-1)(d-2) + 4d-2] - \frac{2d\beta}{L_{\text{eff}}^2} - \gamma \frac{(d-1)(d-2)}{L_{\text{eff}}^2} \right\} \right), \quad (\text{B15})$$

which can be rearranged, using definition of a_d in Eq. (20), to

$$S_{\text{QCG}} = \frac{a_d \text{Vol}(\mathbb{S}^{d-2})}{4G_N} \int_{\epsilon}^{z_{\text{max}}} dz R (R^2 - z^2)^{\frac{d-3}{2}} \left(\frac{L_{\text{eff}}}{z} \right)^{d-1}. \quad (\text{B16})$$

After computing the integral and expanding around $\epsilon = 0$, the EE yields

$$S_{\text{QCG}} = S_{\text{QCG}}^{\text{Univ}} + \frac{a_d \text{Vol}(\mathbb{S}^{d-2}) L_{\text{eff}}^{d-1}}{4G_N} \left[\frac{1}{d-2} \frac{R^{d-2}}{\epsilon^{d-2}} - \frac{d-3}{2(d-4)} \frac{R^{d-4}}{\epsilon^{d-4}} + \mathcal{O}(\epsilon^{-(d-6)}) \right]. \quad (\text{B17})$$

In the previous expression, the universal term is given by

$$S_{\text{QCG}}^{\text{Univ}} = \begin{cases} (-1)^{\frac{d-1}{2}} \frac{a_d (4\pi)^{\frac{d-1}{2}} (\frac{d-1}{2})! L_{\text{eff}}^{d-1}}{4G(d-1)!} & \text{if } d \text{ odd} \\ (-1)^{\frac{d}{2}-1} \frac{a_d \pi^{\frac{d}{2}-1} L_{\text{eff}}^{d-1}}{2G_N (\frac{d}{2}-1)!} \log \frac{2R}{\epsilon} & \text{if } d \text{ even} \end{cases}, \quad (\text{B18})$$

where z_{max} is set to the radius R and the volume of the sphere is given by

$$\text{Vol}(\mathbb{S}^{d-2}) = \frac{2^d \pi^{\frac{d-1}{2}} (\frac{d}{2})!}{d(d-2)}. \quad (\text{B19})$$

4. Cancellation of divergences for odd dimensions

Let us now focus on the Kounterterm. In the odd-dimensional case, the boundary form can be read, as before, from Eq. (25). Since we are considering a spherical entangling region, the extrinsic curvature and the Riemann tensor read, respectively,

$$k_{\bar{b}}^{\bar{a}} = \frac{1}{L_{\text{eff}}} \frac{R}{\sqrt{R^2 - \epsilon^2}} \delta_{\bar{b}}^{\bar{a}}, \quad \tilde{\mathcal{R}}_{\bar{b}_1 \bar{b}_2}^{\bar{a}_1 \bar{a}_2} = \frac{1}{L_{\text{eff}}^2} \frac{\epsilon^2}{R^2 - \epsilon^2} \delta_{\bar{b}_1 \bar{b}_2}^{\bar{a}_1 \bar{a}_2}. \quad (\text{B20})$$

Plugging these values into the boundary form, we find

$$B_{d-2} = -(d-1) \int_0^1 \sqrt{|\tilde{\sigma}|} \delta_{\bar{b}_1 \dots \bar{b}_{d-2}}^{\bar{a}_1 \dots \bar{a}_{d-2}} \delta_{\bar{a}_1}^{\bar{b}_1} \frac{1}{L_{\text{eff}}} \frac{R}{\sqrt{R^2 - \epsilon^2}} \left(\frac{1}{2L_{\text{eff}}^2} \frac{\epsilon^2}{R^2 - \epsilon^2} \delta_{\bar{a}_2 \bar{a}_3}^{\bar{b}_2 \bar{b}_3} - s^2 \frac{1}{L_{\text{eff}}^2} \frac{R^2}{R^2 - \epsilon^2} \delta_{\bar{a}_2}^{\bar{b}_2} \delta_{\bar{a}_3}^{\bar{b}_3} \right) \times \dots \\ \dots \times \left(\frac{1}{2L_{\text{eff}}^2} \frac{\epsilon^2}{R^2 - \epsilon^2} \delta_{\bar{a}_{d-3} \bar{a}_{d-2}}^{\bar{b}_{d-3} \bar{b}_{d-2}} - s^2 \frac{1}{L_{\text{eff}}^2} \frac{R^2}{R^2 - \epsilon^2} \delta_{\bar{a}_{d-3}}^{\bar{b}_{d-3}} \delta_{\bar{a}_{d-2}}^{\bar{b}_{d-2}} \right). \quad (\text{B21})$$

In this expression, we make use the relations

$$\delta_{\bar{b}_1 \dots \bar{b}_{d-3}}^{\bar{a}_1 \dots \bar{a}_{d-3}} \delta_{\bar{a}_2 \bar{a}_3}^{\bar{b}_2 \bar{b}_3} = 2 \delta_{\bar{b}_1 \dots \bar{b}_{d-3}}^{\bar{a}_1 \dots \bar{a}_{d-3}} \delta_{\bar{a}_2}^{\bar{b}_2} \delta_{\bar{a}_3}^{\bar{b}_3}, \quad \delta_{\bar{b}_1 \dots \bar{b}_{d-2}}^{\bar{a}_1 \dots \bar{a}_{d-2}} \delta_{\bar{a}_{d-2}}^{\bar{b}_{d-2}} = \delta_{\bar{b}_1 \dots \bar{b}_{d-3}}^{\bar{a}_1 \dots \bar{a}_{d-3}}, \quad \delta_{\bar{b}_1 \dots \bar{b}_{d-3}}^{\bar{a}_1 \dots \bar{a}_{d-3}} \delta_{\bar{a}_1}^{\bar{b}_1} \dots \delta_{\bar{a}_{d-3}}^{\bar{b}_{d-3}} = (d-2)!. \quad (\text{B22})$$

Besides, writing explicitly the determinant $\sqrt{\tilde{\sigma}}$ given in Eq. (B4), the boundary form reads

$$B_{d-2} = -(d-1)! \int_0^1 ds \left[\frac{R^2}{\epsilon^2} \left(1 - \frac{\epsilon^2}{R^2} \right) \right]^{\frac{d-2}{2}} \frac{R}{\sqrt{R^2 - \epsilon^2}} \left(\frac{\epsilon^2}{R^2 - \epsilon^2} - s^2 \frac{R^2}{R^2 - \epsilon^2} \right)^{\frac{d-3}{2}}. \quad (\text{B23})$$

This expression can be expanded around $\epsilon = 0$ as

$$B_{d-2} = -(d-1)! \int_0^1 ds \left[(-1)^{\frac{d+1}{2}} s^{d-3} \frac{R^{d-2}}{\epsilon^{d-2}} - (-1)^{\frac{d+1}{2}} s^{d-5} \frac{d-3}{2} \frac{R^{d-4}}{\epsilon^{d-4}} + \mathcal{O}(\epsilon^{-(d-6)}) \right], \quad (\text{B24})$$

which after the integration reads

$$B_{d-2} = -(d-1)! \left[(-1)^{\frac{d+1}{2}} \frac{1}{d-2} \frac{R^{d-2}}{\epsilon^{d-2}} - (-1)^{\frac{d+1}{2}} \frac{d-3}{2(d-4)} \frac{R^{d-4}}{\epsilon^{d-4}} + \mathcal{O}(\epsilon^{-(d-6)}) \right]. \quad (\text{B25})$$

Therefore the Kounterterm S_{KT} , after using the definition c_d in the odd-dimensional case from Eq. (20), becomes

$$S_{\text{KT}} = -\frac{a_d \text{Vol}(\mathbb{S}^{d-2}) L_{\text{eff}}^{d-1}}{4G_N} \left[\frac{1}{d-2} \frac{R^{d-2}}{\epsilon^{d-2}} - \frac{d-3}{2(d-4)} \frac{R^{d-4}}{\epsilon^{d-4}} + \mathcal{O}(\epsilon^{-(d-6)}) \right]. \quad (\text{B26})$$

Therefore, adding the previous expression to the bare EE found in Eq. (B17), the renormalized EE reads

$$S_{\text{QCG}}^{\text{Univ}} = S_{\text{finite}}. \quad (\text{B27})$$

5. Cancellation of divergences in even dimensions

Proceeding as in the odd-dimensional case, the extrinsic curvature and the Riemann tensor, given in Eq. (B20), are used along with the identities

$$\delta_{\bar{b}_1 \dots \bar{b}_{d-3}}^{\bar{a}_1 \dots \bar{a}_{d-3}} \delta_{\bar{b}_2 \bar{b}_3}^{\bar{a}_2 \bar{a}_3} = 2 \delta_{\bar{a}_1 \dots \bar{a}_{d-3}}^{\bar{b}_1 \dots \bar{b}_{d-3}} \delta_{\bar{a}_2}^{\bar{b}_2} \delta_{\bar{a}_3}^{\bar{b}_3}, \quad \delta_{\bar{b}_1 \dots \bar{b}_{d-3}}^{\bar{a}_1 \dots \bar{a}_{d-3}} \delta_{\bar{a}_{d-3}}^{\bar{b}_{d-3}} = 2 \delta_{\bar{b}_1 \dots \bar{b}_{d-4}}^{\bar{a}_1 \dots \bar{a}_{d-4}}, \quad \delta_{\bar{b}_1 \dots \bar{b}_{d-4}}^{\bar{a}_1 \dots \bar{a}_{d-4}} \delta_{\bar{a}_1}^{\bar{b}_1} \dots \delta_{\bar{a}_{d-4}}^{\bar{b}_{d-4}} = \frac{(d-2)!}{2}. \quad (\text{B28})$$

After implementing all these relations and substituting $\sqrt{|\tilde{\sigma}|}$ the boundary form reads

$$B_{d-2} = -(d-2)^2 (d-3)! L_{\text{eff}} \int_0^1 ds \int_0^s dt b(t, s, \epsilon), \quad (\text{B29})$$

where the introduced function $b(t, s)$ reads

$$b(t, s, \epsilon) = \frac{R^{d-2}}{\epsilon^{d-2}} \left(1 - \frac{\epsilon^2}{R^2} \right)^{d-2} \frac{R}{\sqrt{R^2 - \epsilon^2}} \left(\frac{\epsilon^2}{R^2 - \epsilon^2} - s^2 \frac{R^2}{R^2 - \epsilon^2} + t^2 \right)^{\frac{d-2}{2}}. \quad (\text{B30})$$

If the function $b(t, s, \epsilon)$ is expanded around $\epsilon = 0$, then

$$b(t, s, \epsilon) = (t^2 - s^2)^{\frac{d-2}{2}} \frac{R^{d-2}}{\epsilon^{d-2}} + \frac{1}{2} (t^2 - s^2)^{\frac{d-3}{2}} [d-4 + s^2 - (d-3)t^2] \frac{R^{d-4}}{\epsilon^{d-4}} + \mathcal{O}(\epsilon^{-(d-6)}). \quad (\text{B31})$$

In this expression, the convergence condition demands that the order $\mathcal{O}(\epsilon^{-(d-2)})$ term appears for $d > 2$. Likewise, the order $\mathcal{O}(\epsilon^{-(d-4)})$ term appears when $d > 4$ and successively. Now, computing the integrals

$$\int_0^1 ds \int_0^s dt (t^2 - s^2)^{\frac{d-2}{2}} = \frac{(-1)^{\frac{d}{2}} 2^{d-2} (\frac{d}{2}-1)!^2}{(d-2)^2 (d-2)!}, \quad (\text{B32})$$

$$\int_0^1 ds \int_0^s dt (t^2 - s^2)^{\frac{d}{2}-3} [d-4 + s^2 - (d-3)t^2] = -\frac{(-1)^{\frac{d}{2}} 2^{d-2} (d-3) (\frac{d}{2}-1)!^2}{(d-2)(d-4)(d-2)!}, \quad (\text{B33})$$

the boundary term reduces to

$$B_{d-2} = (-1)^{\frac{d}{2}+1} 2^{d-2} L_{\text{eff}} \left[\left(\frac{d}{2} - 1 \right)! \right]^2 \left[\frac{1}{d-2} \frac{R^{d-2}}{\epsilon^{d-2}} - \frac{d-3}{2(d-4)} \frac{R^{d-4}}{\epsilon^{d-4}} + \mathcal{O}(\epsilon^{-(d-6)}) \right] + \frac{(d-2)! L_{\text{eff}}}{2} H_{\frac{d}{2}-1}. \quad (\text{B34})$$

Notice that, in this expression, the last term is finite and it is written in terms of the $(\frac{d}{2}-1)$ th harmonic number $H_{\frac{d}{2}-1} = \sum_{i=0}^{\frac{d}{2}-1} \frac{1}{i}$.

Once the boundary term is computed, the Kounterterm S_{KT} is derived easily from Eq. (24) for the even-dimensional case, obtaining

$$S_{\text{KT}} = -\frac{a_d \text{Vol}(\mathbb{S}^{d-2}) L_{\text{eff}}^{d-1}}{4G_N} \left[\frac{1}{d-2} \frac{R^{d-2}}{\epsilon^{d-2}} - \frac{d-3}{2(d-4)} \frac{R^{d-4}}{\epsilon^{d-4}} + \mathcal{O}(\epsilon^{-(d-6)}) \right] + \frac{(d-2)!}{2} H_{\frac{d}{2}-1}. \quad (\text{B35})$$

The structure of power-law divergences is the same as in the odd-dimensional case. However, in even-dimensional CFTs, the bare EE (B17) differs in a finite term and in a log term, whose coefficient is the universal part of the EE. Following the procedure in Ref. [60], the log term is successfully isolated in arbitrary even d , reading

$$S_{\text{QCG}}^{\text{Univ}} = (-1)^{\frac{d}{2}-1} \frac{a_d \pi^{\frac{d}{2}-1} L_{\text{eff}}^{d-1}}{2G_N (\frac{d}{2}-1)!} \log \frac{2R}{\epsilon}. \quad (\text{B36})$$

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Shape dependence of renormalized holographic entanglement entropy

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ABSTRACT: We study the holographic entanglement entropy of deformed entangling regions in three-dimensional CFTs dual to Einstein-AdS gravity, using a renormalization scheme based on the addition of extrinsic counterterms. In this prescription, when even-dimensional manifolds are considered, the universal contribution to the entanglement entropy is identified as the renormalized volume of the Ryu-Takayanagi hypersurface, which is written as the sum of a topological and a curvature term. It is shown that the change in the renormalized entanglement entropy due to the deformation of the entangling surface is encoded purely in the curvature contribution. In turn, as the topological part is given by the Euler characteristic of the Ryu-Takayanagi surface, it remains shape independent. Exploiting the covariant character of the extrinsic counterterms, we apply the renormalization scheme for the case of deformed entangling regions in AdS₄/CFT₃, recovering the results found in the literature. Finally, we provide a derivation of the relation between renormalized entanglement entropy and Willmore energy. The presence of a lower bound of the latter makes manifest the relation between the AdS curvature of the Ryu-Takayanagi surface and the strong subadditivity property.

KEYWORDS: AdS-CFT Correspondence, Gauge-gravity correspondence

ARXIV EPRINT: [2002.06111](https://arxiv.org/abs/2002.06111)

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1 Introduction

Entanglement Entropy (EE) has attracted great interest in recent literature, as it appears in areas of theoretical physics as diverse as quantum information, condensed matter and quantum gravity. It also unveils unexpected links between these fields (see refs. [1–7] for reviews on the subject).

In the context of gauge/gravity duality, the Ryu-Takayanagi (RT) formula [8] relates the EE of a entangling subregion in a Conformal Field Theory (CFT) with the area of a codimension-2 hypersurface immersed in Einstein-anti-de Sitter (AdS) spacetime. This relation was proven in ref. [9].

This idea has inspired extensive work in the subject, full of appealing relations and conjectures. Some concrete examples include the emergence of spacetime from the first law of entanglement entropy [10] and the proposed solution of the firewall paradox [11]. In the case of non trivial topologies, the entropy of de Sitter space was recently interpreted as the holographic entanglement entropy between two disconnected conformal boundaries [12].

In the CFT side, in ref. [3] it was shown that EE is obtained at the limit of Rényi entropy when the replica parameter m tends to the unity. The introduction of the replica trick led Lewkowycz and Maldacena [9, 13] to consider a squashed-cone $(d+1)$ -dimensional replica orbifold $\mathcal{M}_{d+1}^{(\alpha)}$. That is, in the bulk gravity side, a conically singular manifold without U(1) symmetry. Here, α is a conical angular variable such that the cone has an

angular deficit given by $2\pi(1 - \alpha)$, and related to the replica parameter by $\alpha = \frac{1}{m}$. Based on these considerations, the EE is defined as

$$S_{\text{EE}} = - \lim_{\alpha \rightarrow 1} \partial_\alpha I_{\text{E}} \left[\mathcal{M}_{d+1}^{(\alpha)} \right], \quad (1.1)$$

where $I_{\text{E}} \left[\mathcal{M}_{d+1}^{(\alpha)} \right]$ is the Euclidean action evaluated on the orbifold $\mathcal{M}_{d+1}^{(\alpha)}$. By definition, Rényi entropy considers an integer replica parameter. The fact that it is related to the aperture of the cone allows for non-integer values, such that the limit (1.1) is well defined. In the particular case of I_{E} being the Einstein-Hilbert (EH) action, the limit reproduces the RT formula for the EE.

Let A be a smooth entangling region on a time slice of a d -dimensional CFT, the general form of the EE is given by the expansion [14, 15]

$$S_{\text{EE}}(A) = c_{d-2} \frac{l^{d-2}}{\delta^{d-2}} + c_{d-4} \frac{l^{d-4}}{\delta^{d-4}} + \dots + \begin{cases} c_2 \frac{l^2}{\delta^2} + s_{\text{univ}}(A) \log \frac{l}{\delta} + c_0 & \text{for even } d, \\ c_1 \frac{l}{\delta} + (-1)^{(d-1)/2} s_{\text{univ}}(A) & \text{for odd } d. \end{cases} \quad (1.2)$$

In this expression, $\{c_i\}$ are scheme-dependent coefficients. Thus, they are not physically observable. In turn, δ and l are the energy cut-off of the theory and a characteristic scale of the entangling region, respectively.

In even dimensions, s_{univ} is a linear combination of local integrals on the entangling surface, whose coefficients corresponds to the conformal anomaly of the theory (see refs. [3, 16, 17] for examples). If the entangling surface is spherical, the only contribution to s_{univ} comes from the type A -anomaly. On the other hand, if it is cylindrical, the surviving contributions come from the B -type ones [6]. For odd-dimensional CFTs, the lack of logarithmic term reflects the absence of conformal anomaly. Nevertheless, the finite part is physically relevant. It is also shown to be highly non-local, as opposed to the even-dimensional case. Interestingly, when computed for ball-shaped entangling regions, s_{univ} is equivalent to the free energy $F_{\mathbb{S}^d}$ of a CFT placed on \mathbb{S}^d background [18, 19]. The sign is introduced in order to maintain positivity regardless the dimension of the CFT [20].

In the particular case of CFT_3 on \mathbb{S}^3 , the free energy is a monotonic function of the energy under Renormalization Group (RG) flows [21–24]. For this reason, it is considered as an F -function,¹ a measure of the number of degrees of freedom of the effective field theory at a certain energy scale [28, 29]. This fact makes the universal term s_{univ} , evaluated at a circular entangling surface, a valuable probe of the F -theorem.

For arbitrary entangling regions, physical information of the field theory can be extracted from its shape. Studies on smooth entangling regions with symmetry can be found in refs. [30–32]. In the case of non-smooth entangling regions, the expansion (1.2) is modified and new universal contributions to EE emerge [33–38].

The shape dependence of EE is also studied perturbatively around maximally symmetric entangling regions in refs. [39–48]. More specifically, in refs. [30, 39], it is shown

¹It has been proposed that the F -theorem holds also to higher dimensions, but no definite proof has been provided [25–27].

that the EE of a spherical entangling surface with deformations \mathbb{S}_ϵ^1 in a CFT adopts the expansion

$$S_{\text{EE}}^{\text{ren}}(\mathbb{S}_\epsilon^1) = S_{\text{EE}}^{\text{ren},(0)}(\mathbb{S}^1) + \epsilon^2 S_{\text{EE}}^{\text{ren},(2)}(\mathbb{S}^1) + \mathcal{O}(\epsilon^3). \quad (1.3)$$

Here, ϵ is a small deformation parameter and $S_{\text{EE}}^{\text{ren},(0)}$ is the renormalized EE of the unperturbed sphere. The linear term in ϵ vanishes as the sphere is a minimum of the universal term amongst all shapes. The subleading term in expansion (1.3) is proportional to the coefficient C_T of the two-point function of the stress tensor

$$S_{\text{EE}}^{\text{ren},(2)}(\mathbb{S}^1) \propto C_T, \quad (1.4)$$

where

$$\langle T_{ij}(x)T_{kl}(0) \rangle = \frac{C_T}{x^{2d}} \left[I_{i(k}I_{l)j} - \frac{\delta_{ij}\delta_{kl}}{d} \right], \quad (1.5)$$

and $I_{ij} = \delta_{ij} - 2\frac{x_i x_j}{x^2}$. In the case of three-dimensional CFTs dual to Einstein gravity, the coefficient is given by $C_T = \frac{3L^2}{\pi^3 G}$.

These holographic results for deformed entangling surfaces were extended to arbitrary dimensions in refs. [30, 39]. They were later supported by field theory computations [43].

The connection between the renormalized EE and renormalized volume of dual RT surface [49], provides a novel geometric interpretation on the origin of the shape-dependent terms. When a bulk AdS_4 spacetime is considered, the renormalized area of the RT surface is associated to the Willmore energy of a closed manifold immersed on \mathbb{R}^3 [32, 50]. A similar connection between the Willmore energy and the renormalized volume have been provided earlier in mathematical literature [51].

The Willmore energy is a geometrical quantity that measures the deviation of a closed surface from sphericity in \mathbb{R}^3 [52–54]. It has appeared in different fields of study, even beyond mathematics and physics. Applications of it can be found in biology, in order to study elastic properties of cell membranes (along with its generalization, the Helfrich energy [55]). It also applies to computer graphics [56] and mesh processing [57]. In the context of holographic EE, we propose it as a useful probe of the shape deformations of an entangling region. As it is a functional invariant under conformal transformations of the ambient metric, it induces a conformal structure. This will play an important role on our analysis.

Motivated by the results outlined above, in this paper, we study the shape dependence of the EE and its connection to Willmore energy. This paper is organized as follows.

In section 2, we review Kounterterms renormalization scheme in connection to the renormalized EE for a spherical entangling region developed in refs. [49, 58, 59]. This quantity is found to be proportional to the renormalized volume of the RT surface. Following this idea, we provide additional examples and the corresponding interpretation of the results.

In section 3, we focus on an entangling region which is a deformed disk in CFT_3 to compute holographically the renormalized EE following the scheme described in section 2. We obtain a formula that reads

$$S_{\text{EE}}^{\text{ren}}(A) = -\frac{\pi L^2}{2G_N} \chi(\Sigma_{\text{RT}}) + \frac{L^2}{8G_N} \int_{\Sigma_{\text{RT}}} d^2x \sqrt{\gamma} \mathcal{F}, \quad (1.6)$$

where L is the AdS radius. We show that the information on the shape deformation is controlled by the trace of the AdS curvature \mathcal{F} of the RT surface. The first term is a topological contribution, given by the Euler characteristic $\chi(\Sigma_{\text{RT}})$, being non-local in the same way as free energy of the corresponding theory on \mathbb{S}^3 .

In section 4, we derive the relation between renormalized EE and Willmore energy \mathcal{W} of the doubled minimal surface Σ_{RT} , given by

$$S_{\text{EE}}^{\text{ren}}(A) = -\frac{L^2}{8G_N} \mathcal{W}(2\Sigma_{\text{RT}}), \tag{1.7}$$

when the RT surface is embedded in AdS_4 bulk. This relation allows to map the strong subadditivity property of EE to a constraint on the AdS curvature. We show that the validity of the renormalized area formula holds for non-minimal surfaces, as well.

2 Renormalization of entanglement entropy from extrinsic counterterms

In this section, we review the cancellation of divergences that arise in the Einstein-Hilbert action when evaluated in asymptotically AdS (AAdS) spacetimes. We apply the extrinsic counterterms scheme, worked out in refs. [60–63]. This produces a finite Euclidean action in order to obtain a renormalized entanglement entropy $S_{\text{EE}}^{\text{ren}}$ by means of the relation (1.1). In that respect, Kounterterms is a prescription alternative to standard holographic renormalization developed in refs. [64–70].

Renormalized holographic EE has been computed for CFTs dual to Einstein-Hilbert gravity in an arbitrary dimension [49, 58, 59, 71]. In these works, the universal contribution to EE is successfully extracted, removing all scheme-dependent quantities. In odd-dimensional CFTs, for spherical entangling surface, the renormalized EE corresponds to the free energy of a CFT residing on \mathbb{S}^d . In the case of even-dimensional CFTs, the only nonvanishing term is the logarithmic divergence, whose coefficient is the Weyl anomaly of the theory.

In what follows, we will restrict ourselves to odd d -dimensional CFTs, which correspond to even-dimensional dual gravity theories on an AAdS $(d+1)$ -dimensional spacetime. The metric of this class of spacetimes is written in the Fefferman-Graham (FG) gauge as

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{1}{z^2} \left(L^2 dz^2 + g_{ab}(z, x) dx^a dx^b \right), \tag{2.1}$$

where z is the holographic radial coordinate. The singularity at $z = 0$, where the conformal boundary is located, induces a conformal structure at asymptotic infinity. The conformal boundary is endowed with a metric $g_{ab}(z, x)$ which accepts an expansion of the form

$$g_{ab}(z, x) = g_{ab}^{(0)}(x) + z^2 g_{ab}^{(2)}(x) + \dots + z^d g_{ab}^{(d)}(x) + z^d h_{ab}^{(d)}(x) \log(z^2) + \dots \tag{2.2}$$

In the Kounterterms method, for even-dimensional manifolds \mathcal{M}_{2n} with $2n = d+1$, the renormalized Einstein-AdS action $I_{\text{E}}^{\text{ren}}$ is achieved through the addition of the corresponding n -th Chern form B_{2n-1} , as

$$I_{\text{E}}^{\text{ren}}[\mathcal{M}_{2n}] = \frac{1}{16\pi G_N} \int_{\mathcal{M}_{2n}} d^{2n}x \sqrt{|G|} (R - 2\Lambda) + \frac{c_{2n}}{16\pi G_N} \int_{\partial\mathcal{M}_{2n}} B_{2n-1}, \tag{2.3}$$

where the coefficient c_{2n} is defined as

$$c_{2n} = (-1)^n \frac{L^{2n-2}}{n\Gamma(2n-1)}, \quad (2.4)$$

and the n -th Chern form reads

$$\begin{aligned} B_{2n-1} = & -2n \int_0^1 dt \sqrt{h} \delta_{a_1 \dots a_{2n-1}}^{b_1 \dots b_{2n-1}} K_{b_1}^{a_1} \left(\frac{1}{2} \hat{\mathcal{R}}_{b_2 b_3}^{a_2 a_3} - t^2 K_{b_2}^{a_2} K_{b_3}^{a_3} \right) \times \dots \\ & \dots \times \left(\frac{1}{2} \hat{\mathcal{R}}_{b_{2n-2} b_{2n-1}}^{a_{2n-2} a_{2n-1}} - t^2 K_{b_{2n-2}}^{a_{2n-2}} K_{b_{2n-1}}^{a_{2n-1}} \right). \end{aligned} \quad (2.5)$$

Here, $h_{ab} = g_{ab}(z, x)/z^2$ is the induced metric at a constant radius, $\hat{\mathcal{R}}_{cd}^{ab}$ is the intrinsic Riemann curvature tensor, K_b^a the extrinsic curvature and $\delta_{b_1 \dots b_{2n-1}}^{a_1 \dots a_{2n-1}}$ is the generalized Kronecker delta.

Note that the Euler theorem for manifolds with a boundary takes the form,

$$\int_{\mathcal{M}_{2n}} d^{2n}x \sqrt{|G|} \mathcal{E}_{2n} = (4\pi)^n \Gamma(n+1) \chi[\mathcal{M}_{2n}] + \int_{\partial\mathcal{M}_{2n}} B_{2n-1}, \quad (2.6)$$

expressing the equivalence of B_{2n-1} with the topological term

$$\mathcal{E}_{2n} = \frac{1}{2^n} \delta_{\mu_1 \dots \mu_{2n}}^{\nu_1 \dots \nu_{2n}} R_{\nu_1 \nu_2}^{\mu_1 \mu_2} \dots R_{\nu_{2n-1} \nu_{2n}}^{\mu_{2n-1} \mu_{2n}}, \quad (2.7)$$

up to the Euler characteristic of the manifold $\chi[\mathcal{M}_{2n}]$. Using this result, we can rewrite expression (2.3) exclusively in terms of bulk quantities as

$$I_E^{\text{ren}} = \frac{1}{16\pi G_N} \int_{\mathcal{M}_{2n}} d^{2n}x \sqrt{|G|} (R - 2\Lambda + c_{2n} \mathcal{E}_{2n}) - \frac{(-1)^n \pi^{(2n-1)/2} L^{2n-2}}{4G_N \Gamma[(2n-1)/2]} \chi[\mathcal{M}_{2n}]. \quad (2.8)$$

In ref. [49], it was shown that the quantity inside the integral in the above formula can be rewritten in terms of a polynomial of the tensor

$$F_{\nu_1 \nu_2}^{\mu_1 \mu_2} = R_{\nu_1 \nu_2}^{\mu_1 \mu_2} + \frac{1}{L^2} \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2}, \quad (2.9)$$

known as *AdS curvature*. In doing so, the action adopts the form

$$I_E^{\text{ren}} = \frac{1}{16\pi G_N} \int_{\mathcal{M}_{2n}} d^{2n}x \sqrt{|G|} L^{2n-2} P_{2n}(F) - \frac{(-1)^n \pi^{(2n-1)/2} L^{2n-2}}{4G_N \Gamma[(2n-1)/2]} \chi[\mathcal{M}_{2n}], \quad (2.10)$$

where the polynomial of the AdS curvature introduced reads

$$P_{2n}(F) = \frac{1}{2^n n \Gamma(2n-1)} \sum_{k=1}^n \frac{(-1)^k [2(n-k)]! 2^{(n-k)}}{L^{2(n-k)}} \binom{n}{k} \delta_{\mu_1 \dots \mu_{2k}}^{\nu_1 \dots \nu_{2k}} F_{\nu_1 \nu_2}^{\mu_1 \mu_2} \dots F_{\nu_{2k-1} \nu_{2k}}^{\mu_{2k-1} \mu_{2k}}. \quad (2.11)$$

The AdS curvature, of particular convenience in AdS gravity, measures the deviation of the space with respect to global AdS. Notice that the renormalized action consists on the addition of two terms: a topological one, given by the Euler characteristic of the manifold,

and another one, characterized by the AdS curvature. This decomposition has been earlier found on the mathematical literature [51] in connection to the concept of renormalized volume. It reflects the equivalence between this quantity and renormalized EE, up to a proportionality constant that depends on the dimension of the manifold [49]. This result will be of central importance afterwards for the renormalized EE of deformed entangling surfaces, as information on the deformation is entirely contained in the polynomial $P_{2n-2}(\mathcal{F})$.

Once we have the renormalized form of the Euclidean action I_E^{ren} , we evaluate it on the conically singular manifold $\mathcal{M}_{2n}^{(\alpha)}$ in order to use eq. (1.1). Properties of curvature invariants defined on squashed cone manifolds like $\mathcal{M}_{2n}^{(\alpha)}$ have been developed in refs. [72–76]. For our purposes, we recall the relations

$$R^{(\alpha)} = R + 4\pi(1 - \alpha)\delta_\Sigma, \quad (2.12)$$

$$R^{(\alpha)\mu\nu}_{\rho\sigma} = R^{\mu\nu}_{\rho\sigma} + 2\pi(1 - \alpha)N^{\mu\nu}_{\rho\sigma}\delta_\Sigma \quad (2.13)$$

where $N^{\mu\nu}_{\rho\sigma} = n^{(i)\mu}n^{(i)}_{\rho}n^{(j)\nu}n^{(j)}_{\sigma} - n^{(i)\mu}n^{(i)}_{\sigma}n^{(j)\nu}n^{(j)}_{\rho}$ is a linear combination of the i -th normal vector to the surface Σ , $n^{(i)\mu}$. Here $R^{(\alpha)}$ and $R^{(\alpha)\mu\nu}_{\rho\sigma}$ denote the Ricci scalar and Riemann tensor evaluated at the orbifold, respectively. The unindexed tensors indicate the regular part of the corresponding bulk tensor and δ_Σ is a $(2n - 2)$ -dimensional Dirac delta localized at the conical singularity. As a consequence,

$$\int_{\mathcal{M}_{2n}^{(\alpha)}} d^{2n}x\sqrt{G}\delta_\Sigma = \int_{\Sigma} d^{2n-2}y\sqrt{\gamma}, \quad (2.14)$$

where Σ is the codimension-2 locus of the conical singularity and γ the induced metric on the Σ hypersurface. We assigned the coordinate y^a to parametrize the worldvolume of Σ . In ref. [49], it was shown that the Einstein-AdS action evaluated on the orbifold consists on the sum of a regular part and a term localized at the conical defect. The explicit form is

$$\begin{aligned} I_E^{\text{ren}}[\mathcal{M}_{2n}^{(\alpha)}] &= \frac{L^{2n-2}}{16\pi G_N} \int_{\mathcal{M}_{2n}^{(\alpha)} \setminus \Sigma} d^{2n}x\sqrt{|G|}P_{2n}(F) - \frac{(-1)^n \pi^{(2n-1)/2} L^{2n-2}}{4G_N \Gamma[(2n-1)/2]} \chi[\mathcal{M}_{2n}^{(\alpha)} \setminus \Sigma] \\ &+ \frac{1-\alpha}{4G_N} \text{Vol}^{\text{ren}}[\Sigma] + \mathcal{O}[(1-\alpha)^2], \end{aligned} \quad (2.15)$$

where $\mathcal{M}_{2n}^{(\alpha)} \setminus \Sigma$ is identified as the regular manifold \mathcal{M}_{2n} given by the $\alpha \rightarrow 1$ limit, and

$$\text{Vol}^{\text{ren}}[\Sigma] = -\frac{L^{2n-2}}{2(2n-3)} \int_{\Sigma} d^{2n-2}y\sqrt{\gamma}P_{2n-2}(\mathcal{F}) - \frac{(-1)^n \pi^{(2n-1)/2} L^{2n-2}}{4G_N \Gamma[(2n-1)/2]} \chi[\Sigma], \quad (2.16)$$

is the renormalized volume of the codimension-2 manifold. The factor proportional to the angular deficit

$$T = \frac{1-\alpha}{4G}, \quad (2.17)$$

can be regarded as the cosmic brane tension of the Nambu-Goto action, in the interpretation given by Dong [13].

It is important to stress that the expression given for the renormalized volume in eq. (2.16) is generic and not restricted to minimal hypersurfaces. In particular, for CFTs which are dual to Einstein-AdS gravity, when Σ is minimal, corresponds to the RT surface. When the limit $\alpha \rightarrow 1$ is taken in the renormalized action (2.15), the only surviving term (1.1) is the one coming from the Nambu-Goto action

$$S_{\text{EE}}^{\text{ren}}(A) = \frac{\text{Vol}^{\text{ren}}(\Sigma_{\text{RT}})}{4G_N}. \tag{2.18}$$

On the l.h.s., A is a spatial entangling region in CFT_d while on the r.h.s. appearing the renormalized volume of the homologous RT surface Σ_{RT} . Therefore, the computation of the renormalized entanglement entropy depends on AdS curvature and the Euler characteristic of the codimension-2 surface, attending to expression (2.16).

This calculation can be equivalently be interpreted as the renormalized volume of a tensionless codimension-2 brane Σ embedded in a $2n$ -dimensional AAdS Einstein spacetime, for a minimal surface Σ [13].

For a spherical entangling surface, the polynomial $P_{2n-2}(\mathcal{F})$ vanishes identically. The contribution to the holographic EE is coming uniquely from the topology of the RT surface, which is an hemisphere. Because the Euler characteristic is $\chi[\Sigma_{\text{RT}}] = 1$, the finite part of the EE of a ball-shaped surface takes the form

$$S_{\text{EE}}^{\text{ren}} = \frac{(-1)^{(d-1)/2} \pi^{d/2} L^{d-1}}{4G_N \Gamma(d/2)}, \tag{2.19}$$

where we have re-expressed the result in terms of the odd-dimensional d of the CFT. Notice that this result is in agreement with the universal part of the EE [6]. As shown by Casini, Huerta and Myers in ref. [18], $S_{\text{EE}}^{\text{ren}}$ is equivalent to the free energy of a CFT_d on a spherical background \mathbb{S}^d . This relation is of relevance for RG flows, as $F_{\mathbb{S}^d}$ is a monotonic function in $d = 3$. Once the general picture has been discussed, we will illustrate explicitly the duality $\text{AdS}_4/\text{CFT}_3$ in this context by particular examples.

2.1 Entanglement entropy in $\text{AdS}_4/\text{CFT}_3$ in the global coordinate patch

The use of the extrinsic counterterms in the renormalization of holographic EE has been applied for spatial entangling regions embedded on a flat background, in the Poincaré-AdS patch, in the context of gauge/gravity duality [58]. In particular, in what follows, we study the EE of a polar cap-like entangling region immersed on an Einstein Static Universe background (ESU), i.e., $\mathbb{R} \times \mathbb{S}^2$ to account for properties of a CFT_3 . In this case, the dual bulk geometry is given by global AdS_4 spacetime, whose line element reads

$$ds^2 = - \left(1 + \frac{r^2}{L^2}\right) dt^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{2.20}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric of \mathbb{S}^2 .

In the RT picture, the minimality condition for Σ , in order to be homologous to the circular entangling surface at the boundary, amounts to the vanishing of the trace of the extrinsic curvature $K_{ab}^{(i)}$ along whichever normal direction to Σ . Here, the label index

$i = \{1, 2\}$ represents these directions. Indeed the equations of motion of the surface can be derived from the Nambu-Goto action. That, for the case of the Einstein-AdS gravity, results in the condition [77, 78]

$$K^{(i)} = 0. \tag{2.21}$$

Considering that the two-dimensional orthogonal space is spanned along $i = t, r$, the induced metric γ of Σ , is given by

$$ds_\gamma^2 = - \left(1 + \frac{r^2}{L^2} \right) dt^2 + \left(r^2 + \frac{L^2 r'^2}{L^2 + r^2} \right) d\theta^2 + r^2 \sin^2 \theta d\phi^2 \tag{2.22}$$

where we have parametrized the geometry with the embedding function $r = r(\theta)$ and $r' = \partial_\theta r(\theta)$.

Solving the second order differential equation that results from eq. (2.21), we find that the RT surface is characterized by [79–81] the function

$$r^2(\theta) = \frac{L^2 \cos^2 \theta_0}{\cos^2 \theta \sin^2 \theta_0 - \sin^2 \theta \cos^2 \theta_0}. \tag{2.23}$$

For this embedding, the polynomial $P_2(\mathcal{F})$ in eq. (2.16) vanishes identically, as it is a constant-curvature subspace. The only nonvanishing part is the topological one, for which the universal part of the EE takes the form

$$S_{\text{EE}}^{\text{ren}} = -\frac{\pi L^2}{2G_N}. \tag{2.24}$$

Thus, even though this time the spherical entangling surface is immersed in the curved background of ESU metric, eq. (2.24) matches the one obtained for the flat case [58].

3 Renormalized entanglement entropy of a deformed disk

In this section, we calculate the finite contribution to the EE of a spatial entangling region for a CFT₃ on the ground state. To this end, we consider a deformed disk whose dual geometry corresponds to global AdS₄. The universal part of the holographic EE for such region was first obtained in refs. [30, 39] for a general class of gravity theories. Such result was later confirmed from field theory computations in ref. [43].

We shall study the deformation in two coordinate systems: polar coordinates (following [30]) and spherical coordinates (in order to make contact with refs. [39] and [43]). Using the Kounterterms, we make contact with the renormalized volume of the RT surface (2.16), which contains both local (curvature) and global (topological) terms [59]. Our analysis below allows us to track the origin of the shape-dependent contributions to the curvature part in eq. (2.16).

3.1 Deformed disk in polar coordinates

Consider the Poincaré-AdS₄ spacetime, written in polar coordinates as

$$ds^2 = \frac{L^2}{z^2} (-dt^2 + dz^2 + d\rho^2 + \rho^2 d\phi^2). \tag{3.1}$$

We define the embedding function of the RT surface Σ_{RT} by $\rho(z, \phi)$, where ρ and ϕ are the radial and the angular coordinate at the boundary, respectively. The deformation breaks the azimuthal symmetry of Σ_{RT} . Hence, the simplification used in the section 2.1 is not applicable. In this case, the codimension-2 induced metric reads

$$ds_\gamma^2 = \frac{L^2}{z^2} \left[(1 + \rho'^2) dz^2 + (\rho^2 + \dot{\rho}^2) d\phi^2 + 2\rho'\dot{\rho} dzd\phi \right], \quad (3.2)$$

where $\rho' = \partial_z \rho(z, \phi)$ and $\dot{\rho} = \partial_\phi \rho(z, \phi)$. It is indeed easy to find the equations of motion of the RT surface following eq. (2.21). If we consider the binormal directions as $i = t, r$, we find that

$$K^{(r)}{}_z{}^z + K^{(r)}{}_\theta{}^\theta = 0, \quad (3.3)$$

provided that the temporal foliation is constant, what implies into $K^{(t)}{}_z{}^z = K^{(t)}{}_\theta{}^\theta = 0$. This leads to the equations of motion

$$\frac{\rho(1 + \rho'^2)}{mz^2} - \partial_z \left(\frac{\rho^2 \rho'}{mz^2} \right) - \frac{1}{z^2} \partial_\phi \left(\frac{\dot{\rho}}{m} \right) = 0, \quad (3.4)$$

where we have introduced an auxiliary function

$$m = m(z, \phi) = \sqrt{\rho^2(1 + \rho'^2) + \dot{\rho}^2}. \quad (3.5)$$

In absence of deformations, the embedding function (3.4) is parametrized by a hemisphere of unit radius, $\rho^2 = 1 - z^2$. The shape can be deformed as linear perturbations around the unitary circle of the form $\rho(\phi) = [1 + \epsilon f(\phi)]$, where ϵ is the deformation parameter [30]. Altogether, we assume that its embedding in AdS₄ geometry is given by the ansatz

$$\rho(z, \phi) = \sqrt{1 - z^2} [1 + \epsilon f(z, \phi)], \quad (3.6)$$

for the separation of variables $f(z, \phi) = R(z)\Phi(\phi)$. The corresponding functions satisfy the conditions $R(0) = 1$ and $\Phi(\phi) = \Phi(\phi + 2\pi)$ at the conformal boundary. This is a consequence of the homologous constraint on the RT surface, as it is anchored to the conformal boundary $z = 0$. An additional condition comes from the fact that the maximum reach of the embedding does not change when the RT surface is deformed, what leads to $R(1) = 0$ [30, 82] (see figure 1).

Solving eq. (3.4) for $R(z)$ and $\Phi(\phi)$, we obtain

$$\rho(z, \phi) = \sqrt{1 - z^2} \left[1 + \epsilon \sum_\ell \left(\frac{1 - z}{1 + z} \right)^{\ell/2} \frac{1 + \ell z}{1 - z^2} (a_\ell \cos(\ell\phi) + b_\ell \sin(\ell\phi)) \right], \quad (3.7)$$

where ℓ is the degree of the harmonic function and labels the deformation with respect to the circle.

Once we have obtained the embedding function (3.7), we are able to compute the renormalized EE for the perturbed circle by using eq. (2.18). For Einstein gravity in four dimensions this reads

$$S_{\text{EE}}^{\text{ren}}(\mathbb{S}_\epsilon^1) = -\frac{\pi L^2}{2G_N} \chi(\Sigma_{\text{RT}}) + \frac{L^2}{8G_N} \int_{\Sigma_{\text{RT}}} d^2x \sqrt{\gamma} \mathcal{F}, \quad (3.8)$$

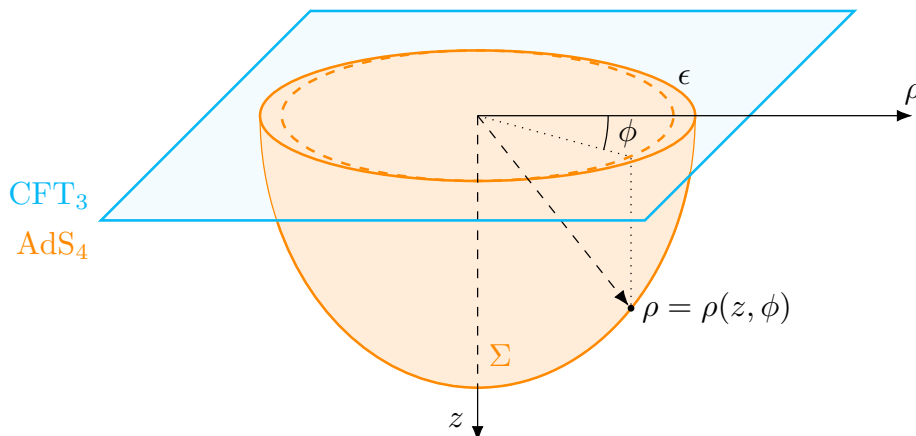


Figure 1. Time slice of minimal co-dimension two surface Σ with an elliptical deformation ϵ ($\ell = 2$).

where \mathcal{F} is the trace of the AdS curvature tensor defined in (2.11). Replacing the embedding function (3.7) into eq. (3.8), we obtain

$$S_{\text{EE}}^{\text{ren}}(\mathbb{S}_\epsilon^1) = -\frac{\pi L^2}{2G_N} \left[1 + \epsilon^2 \sum_\ell \frac{\ell(\ell^2 - 1)}{4} (a_\ell^2 + b_\ell^2) + \mathcal{O}(\epsilon^4) \right], \quad (3.9)$$

what is in agreement with the holographic computation for an arbitrary perturbation of a circle performed in ref. [30].

3.2 Deformed disk in spherical coordinates

Consider now the Poincaré-AdS spacetime written in spherical coordinates as

$$ds^2 = \frac{L^2}{r^2 \cos^2 \theta} (-dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2). \quad (3.10)$$

Polar and spherical coordinates are mapped into each other by the transformation

$$r = \sqrt{\rho^2 + z^2}, \quad \theta = \arctan \frac{\rho}{z}. \quad (3.11)$$

In this coordinate system, the embedding function of the minimal surface Σ_{RT} is defined by $r = r(\theta, \phi)$, such that the induced metric is

$$ds_\gamma^2 = \frac{L^2}{r^2 \cos^2 \theta} \left[(1 + r'^2) d\theta^2 + (1 + \dot{r}^2) d\phi^2 + 2r' \dot{r} d\theta d\phi \right], \quad (3.12)$$

where we denoted $r' = \partial_\theta r(\theta, \phi)$ and $\dot{r} = \partial_\phi r(\theta, \phi)$. The minimality condition (2.21) leads to the equation for $r(\theta, \phi)$,

$$\frac{1}{mr^3 \cos^2 \theta} (r'^2 \sin^2 \theta + \dot{r}^2) + \partial_\theta \left(\frac{r' \tan^2 \theta}{r^2 m} \right) + \frac{1}{\cos^2 \theta} \partial_\phi \left(\frac{\dot{r}}{r^2 m} \right) = 0, \quad (3.13)$$

with the corresponding function

$$m = m(\theta, \phi) = \sin \theta \sqrt{1 + \frac{r'^2 + \dot{r}^2}{r^2}}. \quad (3.14)$$

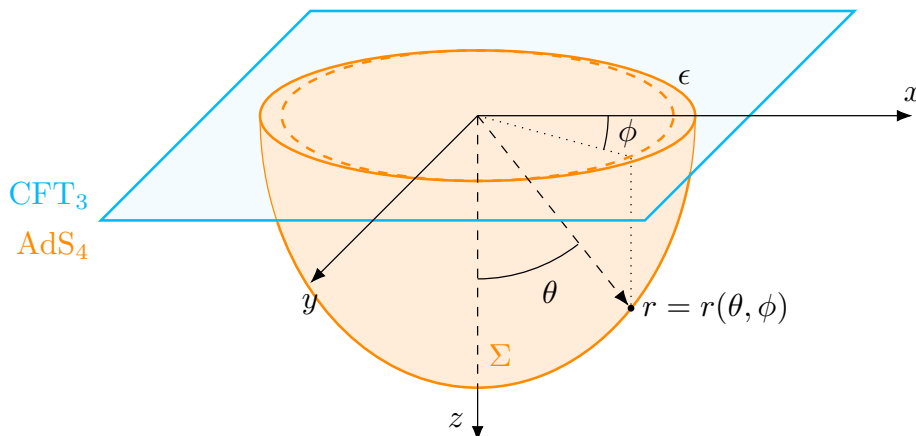


Figure 2. Time slice of the extremal co-dimension two surface Σ with an elliptical deformation ϵ ($\ell = 2$).

From eq. (3.13), in the undeformed case, the parametrization of the embedding function of the RT surface is given by the unit hemisphere, $r^2 = 1$. In a similar fashion as in the previous parametrization, we consider the linear perturbation of the entangling region as [39]

$$r(\theta, \phi) = 1 + \epsilon f(\theta, \phi). \quad (3.15)$$

For a choice $f(\theta, \phi) = \Theta(\theta)\Phi(\phi)$, the boundary conditions correspond to a periodic function Φ with period 2π and $\Theta \rightarrow 1$ at the conformal boundary, i.e., $\Theta(\frac{\pi}{2}) = 1$. Here, the maximal reach of the RT surface implies $\Theta(0) = 0$ (see figure 2). Thus, eq. (3.13) for the ansatz (3.15) leads to a solution of the form

$$r(\theta, \phi) = 1 + \epsilon \sum_{\ell} \tan^{\ell} \frac{\theta}{2} (1 + \ell \cos \theta) [a_{\ell} \cos(\ell\phi) + b_{\ell} \sin(\ell\phi)]. \quad (3.16)$$

For this embedding function, the nonvanishing AdS curvature component reads

$$\mathcal{F}_{\theta\phi}^{\theta\phi} = -\epsilon^2 \sum_{\ell} \frac{\ell^2(\ell^2 - 1)}{\pi L^2} (a_{\ell} + b_{\ell}) \tan\left(\frac{\theta}{2}\right)^{2\ell} \cot^4(\theta) + \mathcal{O}(\epsilon^3). \quad (3.17)$$

Introducing this expression into eq. (3.8), yields

$$S_{\text{EE}}^{\text{ren}}(\mathbb{S}_{\epsilon}^1) = -\frac{\pi L^2}{2G_N} \left[1 + \epsilon^2 \sum_{\ell} \frac{\ell(\ell^2 - 1)}{4} (a_{\ell}^2 + b_{\ell}^2) + \mathcal{O}(\epsilon^4) \right], \quad (3.18)$$

what matches exactly the result (3.9) of the previous subsection, and, in turn, agrees with the formula for $d = 3$ in ref. [39].

3.3 Interpretation of the results

A quick analysis of the results above leads to the fact that the $\mathcal{O}(\epsilon^2)$ contribution is coming only from the curvature part in formula (3.8). Indeed, the information on the deformation

of the entangling region is only contained in the polynomial $\mathcal{P}_2(\mathcal{F})$. As it shall be discussed below, this behavior can be explained once the equivalence between the renormalized EE and the renormalized volume of the RT surface (2.16), is taken into account.

Continuous perturbations on the hemisphere do not modify its topology, leaving intact the Euler characteristic in eq. (2.16). As a consequence, its shape dependence is encoded only on the local properties of the manifold, which are reflected in the polynomial in the curvature (\mathcal{F} term).

The term of the renormalized EE that is quadratic in the perturbation carries information on *entanglement susceptibility*, associated to the change of shape of the entangling region [43, 83–86]. This quantity contains universal information due to the coefficient C_T of the two-point correlation function of the energy-momentum tensor in a ground state of the CFT_3 . Indeed, the subleading term of the formula (3.9), can equivalently be written as

$$S_{\text{EE}}^{\text{ren},(2)}(\mathbb{S}_\epsilon^1) = \frac{\pi^4 C_T}{24} \sum_\ell \ell(\ell^2 - 1) (a_\ell^2 + b_\ell^2). \quad (3.19)$$

This expression² makes manifest the analogy between the entanglement susceptibility and C_T . A posteriori, one can say that the AdS curvature of a deformed entangling region is a geometrical probe of C_T .

The leading-order contribution is a shape-independent constant that corresponds to the universal part of the EE of a circular entangling surface. This term is a topological number which is identified as the free energy of a CFT_3 in a \mathbb{S}^3 background, using gauge/gravity duality [18]. As mentioned in the Introduction, the latter quantity provides a realization of the F -theorem. The matching to a notion of EE in terms of the Euler characteristic provides firmer ground to a connection between the topology and the effective number of degrees of freedom of the field theory.

4 Renormalized volume and Willmore energy

4.1 Minimal and non-minimal surfaces

The connection between quantum information theoretic measures and geometry can be extended beyond EE. Dong in ref. [13] showed that a similar area formula is valid for the calculation of the modular entropy. In this case, the codimension-2 hypersurface Σ_T is not minimal, but its location is determined by the minimization of the Nambu-Goto action of a cosmic brane with tension T (2.17).

The prescription used in the present work for the cancellation of divergences in the holographic EE of entangling surfaces is linked to the volume renormalization given in the mathematical literature [51]. As shown in ref. [59], isolating the finite contribution of the modular entropy amounts to the renormalization of the volume of Σ_T

$$\tilde{S}_m^{\text{ren}} = \frac{\text{Vol}^{\text{ren}}(\Sigma_T)}{4G_N}. \quad (4.1)$$

²In ref. [39], the proportionality constant differs by a factor π . This corresponds to a different normalization for the spherical harmonics, leading to an overall factor $1/\sqrt{\pi}$ for each one of them.

Interestingly enough, both quantities, EE and modular entropy, are described by the same geometrical object, the renormalized volume of a codimension-2 hypersurface Σ . In 4D Einstein-AdS gravity, the corresponding renormalized volume of Σ is given by (2.16) and reads

$$\text{Vol}^{\text{ren}}(\Sigma) = -2\pi L^2 \chi(\Sigma) + \frac{L^2}{4} \int_{\Sigma} d^2 y \sqrt{\gamma} \delta_{ab}^{cd} \left(\mathcal{R}_{cd}^{ab} + \frac{1}{L^2} \delta_{cd}^{ab} \right). \quad (4.2)$$

This expression matches the renormalized area expression given in ref. [87]. Notice that this formula holds whether Σ is minimal or not. A physical example of minimal surface is a soap film that spans between two wires. As there is no pressure difference between the sides, the membrane has zero mean curvature. In turn, soap bubbles are non-minimal, due to the difference of pressure at the interface [88, 89]. In the latter case, they are constant mean curvature surfaces, and modelled by Helfrich energy [90].

For extremal surfaces, minimality condition amounts to the vanishing of the trace of the extrinsic curvature of the surface Σ (2.21). For this reason, it is useful to rewrite eq. (4.2) in terms of $K_{ab}^{(i)}$ using the Gauss-Codazzi relation, for codimension-2 surfaces

$$R_{cd}^{ab} = \mathcal{R}_{cd}^{ab} - K^{(i)c}_a K^{(i)b}_d + K^{(i)a}_d K^{(i)b}_c. \quad (4.3)$$

Taking into account the antisymmetry of Kronecker delta, we find that

$$\text{Vol}^{\text{ren}}(\Sigma) = -2\pi L^2 \chi(\Sigma) + \frac{L^2}{4} \int_{\Sigma} d^2 y \sqrt{\gamma} \delta_{ab}^{cd} \left[R_{cd}^{ab} + 2K^{(i)a}_c K^{(i)b}_d + \frac{1}{L^2} \delta_{cd}^{ab} \right]. \quad (4.4)$$

In addition, the Weyl tensor for Einstein-AdS spaces can be written as

$$W_{\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{\alpha\beta} + \frac{1}{L^2} \delta_{\mu\nu}^{\alpha\beta}, \quad (4.5)$$

what allows us to express the renormalized volume of Σ as

$$\text{Vol}^{\text{ren}}(\Sigma) = -2\pi L^2 \chi(\Sigma) + \frac{L^2}{4} \int_{\Sigma} d^2 y \sqrt{\gamma} \delta_{ab}^{cd} \left[W_{cd}^{ab} + 2K^{(i)a}_c K^{(i)b}_d \right]. \quad (4.6)$$

In turn, the extrinsic curvature in eq. (4.6) can be decomposed into its trace $K^{(i)}$ and a traceless part $P_{ab}^{(i)}$ as

$$K^{(i)}_{ab} = P_{ab}^{(i)} + \frac{1}{2} \gamma_{ab} K^{(i)}. \quad (4.7)$$

We can also replace the trace by the mean curvature $H^{(i)}$, which expresses a linear combination of the eigenvalues of $K^{(i)}_{ab}$, that is, $H^{(i)} = K^{(i)}/2$. Armed with these tools, we deduce that the renormalized volume in eq. (4.2) can be equivalently written as

$$\text{Vol}^{\text{ren}}(\Sigma) = -2\pi L^2 \chi(\Sigma) + \frac{L^2}{2} \int_{\Sigma} d^2 y \sqrt{\gamma} \left[W_{ab}^{ab} + 2H^{(i)2} - P_b^{(i)a} P_a^{(i)b} \right]. \quad (4.8)$$

When a minimal two-dimensional surface Σ_{min} is considered, the above relation reduces to

$$\text{Vol}^{\text{ren}}(\Sigma_{\text{min}}) = -2\pi L^2 \chi(\Sigma_{\text{min}}) + \frac{L^2}{2} \int_{\Sigma_{\text{min}}} d^2 y \sqrt{\gamma} \left[W_{ab}^{ab} - P_b^{(i)a} P_a^{(i)b} \right]. \quad (4.9)$$

The last two equations are in agreement with the result of Alexakis and Mazzeo in ref. [51] for the renormalized area of submanifolds. These equivalent expressions allow us to tell between the two prescriptions in eqs. (2.18) and (4.1): the RT surface satisfies the minimality condition while the cosmic brane used in modular entropy not. Renormalized area relations in eqs. (4.8) and (4.9) can be further simplified when considering entangling regions for a vacuum CFT. Since its gravity dual is global AdS₄ spacetime, which is conformally flat, the bulk Weyl tensor vanishes identically.

4.2 Renormalized area and Willmore energy

The relation between renormalized EE and the renormalized area of the RT surface, has two key ingredients. On one hand, the topology of the minimal surface, expressed by the Euler characteristic, which captures global properties of Σ_{RT} . On the other hand, the local properties of Σ_{RT} are dictated by the AdS curvature term inside the integral in eq. (2.9).

According to the analysis in section 3, the deformation in the shape of a disk entangling region is encoded only at the curvature part of the renormalized EE, leaving the topological contribution unchanged.

A functional with similar properties, called the Willmore energy, has been introduced in mathematical literature [52–54]. It is defined for a smooth, closed and orientable surface X embedded in \mathbb{R}^3 and adopts the form,

$$\mathcal{W}(X) = \int_X H^2 dS, \tag{4.10}$$

where H is the mean curvature of X and dS is the area element of the 2D metric. Furthermore, it acquires a minimal value when evaluated on spherical surfaces

$$\mathcal{W}(X) \geq 4\pi. \tag{4.11}$$

Therefore, it measures the deviation of X from sphericity. It was conjectured by Willmore that a new bound arises when the genus of X changes from zero to one. According to this, the functional for the $g = 1$ submanifold has a bound

$$\mathcal{W}(X_{g=1}) \geq 2\pi^2. \tag{4.12}$$

The conjecture was proved recently in ref. [91]. These properties are fundamental for the analysis below, where we establish the connection between Willmore energy and quantum information theoretic measures, through the concept of renormalized volume of the entangling surface.

Willmore energy is defined for a closed surface. Therefore, the first obstacle is the fact formula (4.2) involves an open two-dimensional surface anchored to the boundary of an AAdS space.

In order to overcome this problem, we generalize the field doubling method proposed in ref. [32], for AAdS manifolds. In this case, we consider the embedding of the codimension-2 surface Σ and its reflection with respect to the $z = 0$ plane, Σ' . The intersection of Σ and Σ' is the entangling curve ∂A , at the conformal boundary, such that $\partial\Sigma = \partial\Sigma' = \partial A$.

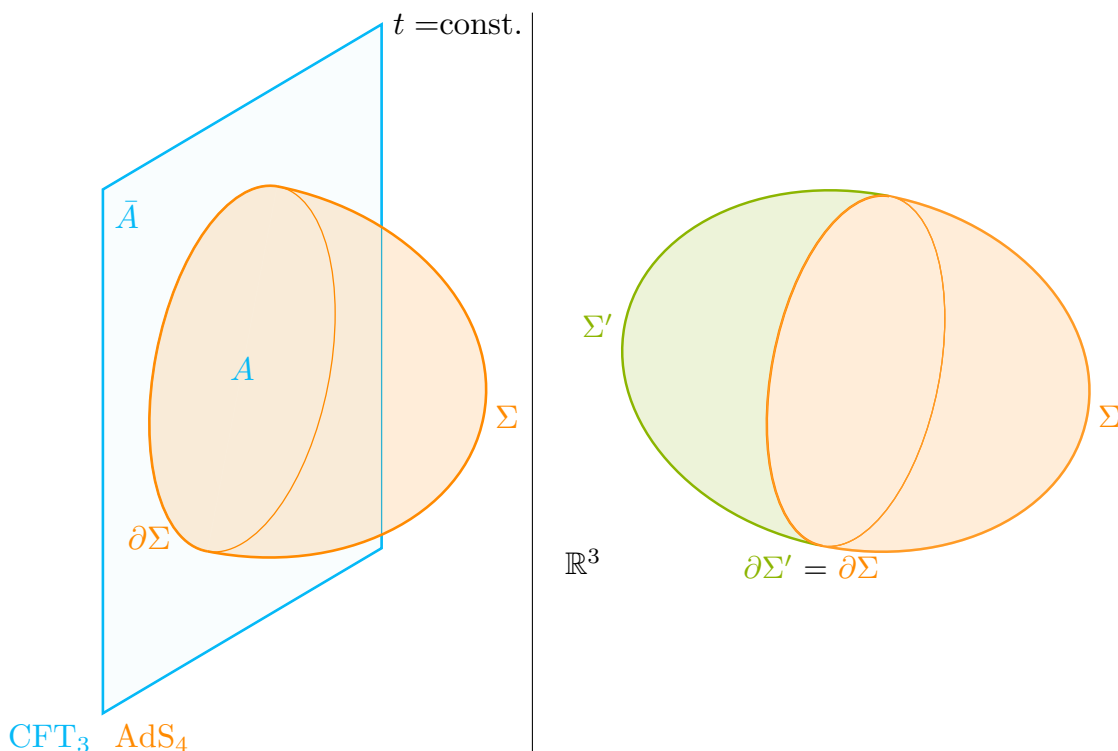


Figure 3. The doubling of the minimal surface Σ is achieved by gluing a copy Σ' so that they are cobordant.

Continuity conditions at the interface situated at $z = 0$, require the two surfaces to be immersed in a regular spacetime. In this case, its union produces a closed two-dimensional surface $2\Sigma = \Sigma \cup \Sigma'$, which is embedded into the smooth spacetime $\tilde{G}_{\mu\nu}$. A pictorial representation of the method is shown in figure 3.

With this geometrical setup in mind, we examine the rescaling properties of the renormalized volume of Σ (4.8) under generic Weyl transformations of the ambient metric $G_{\mu\nu} = e^{2\varphi}\tilde{G}_{\mu\nu}$. Notice that, when $\varphi = -\log(z/L)$, one recovers the eq. (2.1). The Euler characteristic is a topological invariant and does not change under metric rescalings. Thus, we focus on the quantities which appear under the integral symbol in eq. (4.8).

By definition of the bulk line element in terms of the codimension-2 metric γ_{ab}

$$G^{\mu\nu} = \left(n^{(i)\mu} n^{(i)\nu} + e_a^\mu e_b^\nu \gamma^{ab} \right), \tag{4.13}$$

where $n^{(i)\mu}$ are the corresponding normal vectors and e_a^μ are the frame vectors, we have that

$$n^{(i)}_\mu = e^\varphi \tilde{n}^{(i)}_\mu, \quad \gamma_{ab} = e^{2\varphi} \tilde{\gamma}_{ab}. \tag{4.14}$$

Here, the quantities with tilde indicate an embedding with respect to the regular metric $\tilde{G}_{\mu\nu}$.

On the other hand, the extrinsic curvature of Σ transforms as

$$K^{(i)}_{ab} = e^\varphi \left(\tilde{K}^{(i)}_{ab} + \tilde{\gamma}_{ab} \tilde{n}^{(i)} \cdot \partial\varphi \right), \quad (4.15)$$

where we have omitted the indices in the contraction $\tilde{n}^{(i)c} \partial_c \varphi$. This expression allows us to write down the trace of $K^{(i)}_{ab}$ as

$$K^{(i)} = \gamma^{ab} K^{(i)}_{ab} = e^{-\varphi} \left(\tilde{K}^{(i)} + 2\tilde{n}^{(i)} \cdot \partial\varphi \right), \quad (4.16)$$

and its traceless part as

$$P^{(i)}_{ab} = e^{-\varphi} \tilde{P}^{(i)}_{ab}. \quad (4.17)$$

A Weyl transformation of the area element is given by $d^2y\sqrt{\gamma} = d^2y\sqrt{\tilde{\gamma}}e^{2\phi}$. Then, it is straightforward to show that the following object is Weyl invariant

$$\int_{\Sigma} d^2y\sqrt{\gamma} P^{(i)a}_b P^{(i)b}_a = \int_{\Sigma} d^2y\sqrt{\tilde{\gamma}} \tilde{P}^{(i)a}_b \tilde{P}^{(i)b}_a. \quad (4.18)$$

In turn, the square of the trace of the extrinsic curvature is not Weyl invariant

$$\int_{\Sigma} d^2y\sqrt{\gamma} K^{(i)2} = \int_{\Sigma} d^2y\sqrt{\tilde{\gamma}} \left[\tilde{K}^{(i)2} + 4\tilde{K}^{(i)} \left(\tilde{n}^{(i)} \cdot \partial\varphi \right) + 4 \left(\tilde{n}^{(i)} \cdot \partial\varphi \right)^2 \right]. \quad (4.19)$$

Altogether, the bulk Weyl tensor satisfies $W^a_{bcd} = \tilde{W}^a_{bcd}$, what implies the relation

$$W^{ab}_{cd} = e^{-2\varphi} \tilde{G}^{bmn} \tilde{W}^a_{mcd} = e^{-2\varphi} \tilde{W}^{ab}_{cd}. \quad (4.20)$$

In doing so, the integral of the double subtrace of the Weyl tensor on the area element is proved to be invariant

$$\int_{\Sigma} d^2y\sqrt{\gamma} W^{ab}_{ab} = \int_{\Sigma} d^2y\sqrt{\tilde{\gamma}} \tilde{W}^{ab}_{ab}. \quad (4.21)$$

Therefore, the renormalized volume (4.2), when expressed in terms of the smooth metric, reads

$$\begin{aligned} \text{Vol}^{\text{ren}}(\Sigma) &= \frac{L^2}{2} \int_{\Sigma} d^2y\sqrt{\tilde{\gamma}} \left[\tilde{W}^{ab}_{ab} + 2\tilde{H}^{(i)2} - \tilde{P}^{(i)a}_b \tilde{P}^{(i)b}_a + 2\tilde{K}^{(i)} \left(\tilde{n}^{(i)} \cdot \partial\varphi \right) + 2 \left(\tilde{n}^{(i)} \cdot \partial\varphi \right)^2 \right] - \\ &\quad - 2\pi L^2 \chi(\Sigma). \end{aligned} \quad (4.22)$$

This formula adopts a more compact form by taking Gauss-Codazzi eq. (4.3) and the relations between bulk and codimension-2 curvature tensors [71]

$$\begin{aligned} \tilde{P}^{(i)a}_b \tilde{P}^{(i)b}_a &= \tilde{K}^{(i)a}_b \tilde{K}^{(i)b}_a - 2\tilde{H}^{(i)2}, \\ \tilde{K}^{(i)a}_b \tilde{K}^{(i)b}_a &= \tilde{R} + \tilde{R}_{abcd} n^{(i)a} n^{(i)c} n^{(j)b} n^{(j)d} - 2\tilde{R}_{ab} n^{(i)a} n^{(i)b} - \tilde{\mathcal{R}} + 4\tilde{H}^{(i)2}, \\ \tilde{W}^{ab}_{ab} &= \tilde{R}^{ab}_{ab} - 2\tilde{S}^a_a, \\ \tilde{R}^{ab}_{ab} &= \tilde{R} + \tilde{R}_{abcd} n^{(i)a} n^{(i)c} n^{(j)b} n^{(j)d} - 2\tilde{R}_{ab} n^{(i)a} n^{(i)b}, \end{aligned}$$

where \tilde{S}_b^μ is the Schouten tensor of $\tilde{G}_{\mu\nu}$. Combining these expressions, we find that

$$\tilde{P}_b^{(i)a} \tilde{P}_a^{(i)b} = \tilde{W}_{ab}^{ab} + 2\tilde{S}_a^a - \mathcal{R} + 2\tilde{H}^{(i)2}, \quad (4.23)$$

what leads to

$$\text{Vol}^{\text{ren}}(\Sigma) = -2\pi L^2 \chi(\Sigma) + L^2 \int_{\Sigma} d^2y \sqrt{\tilde{\gamma}} \left[\frac{1}{2} \mathcal{R} + \tilde{K}^{(i)} \left(\tilde{n}^{(i)} \cdot \partial\varphi \right) + \left(\tilde{n}^{(i)} \cdot \partial\varphi \right)^2 - \tilde{S}_a^a \right]. \quad (4.24)$$

Until now, we have treated the two-dimensional sheet Σ as an open surface anchored to the $z = 0$ plane. As discussed above, Σ is also half of the closed surface 2Σ , we have

$$\text{Vol}^{\text{ren}}(\Sigma) = -2\pi L^2 \chi(\Sigma) + \frac{L^2}{2} \int_{2\Sigma} d^2y \sqrt{\tilde{\gamma}} \left[\frac{1}{2} \mathcal{R} + \tilde{K}^{(i)} \left(\tilde{n}^{(i)} \cdot \partial\varphi \right) + \left(\tilde{n}^{(i)} \cdot \partial\varphi \right)^2 - \tilde{S}_a^a \right]. \quad (4.25)$$

For the compact manifold 2Σ , the Euler theorem in two dimensions states that

$$\int_{2\Sigma} d^2y \sqrt{\tilde{\gamma}} \mathcal{R} = 4\pi \chi(2\Sigma), \quad (4.26)$$

as for the Euler characteristic, the relation

$$\chi(2\Sigma) = 2\chi(\Sigma), \quad (4.27)$$

holds without loss of generality. With all the above equations, we have

$$\text{Vol}^{\text{ren}}(\Sigma) = \frac{L^2}{2} \int_{2\Sigma} d^2y \sqrt{\tilde{\gamma}} \left[\tilde{K}^{(i)} \left(\tilde{n}^{(i)} \cdot \partial\varphi \right) + \left(\tilde{n}^{(i)} \cdot \partial\varphi \right)^2 - \tilde{S}_a^a \right], \quad (4.28)$$

what is equivalent to the renormalized area formula in eq. (4.2). As a matter of fact, it is more general as it is valid for both minimal and non-minimal surfaces embedded in an AAdS₄ spacetime. Any constraint on the shape of Σ can be readily implemented as a relation between the different terms in eq. (4.28).

For instance, minimality condition leads to

$$K^{(i)} = 0 \Rightarrow \tilde{K}^{(i)} = -2\tilde{n}^{(i)} \cdot \partial\varphi, \quad (4.29)$$

when written in terms of the smooth metric $\tilde{G}_{\mu\nu}$. As a consequence, when a minimal surface Σ_{min} is considered, the renormalized volume reduces to

$$\text{Vol}^{\text{ren}}(\Sigma_{\text{min}}) = -\frac{L^2}{2} \int_{2\Sigma_{\text{min}}} d^2y \sqrt{\tilde{\gamma}} \left[\tilde{H}^{(i)2} + \tilde{S}_a^a \right]. \quad (4.30)$$

In particular, when Σ_{min} is a spatial subregion of an AdS₄ spacetime, which corresponds to the conditions $\tilde{H}^{(t)} = 0$ and $\tilde{S}_a^a = 0$ ($\tilde{G}_{\mu\nu}$ is a locally flat space), the last equation reads

$$\text{Vol}^{\text{ren}}(\Sigma_{\text{min}}) = -\frac{L^2}{2} \int_{2\Sigma_{\text{m}}} d^2y \sqrt{\tilde{\gamma}} \tilde{H}^2 = -\frac{L^2}{2} \mathcal{W}(2\Sigma_{\text{min}}). \quad (4.31)$$

Therefore, it is made explicit the connection between the renormalized area and the Willmore energy. The equivalence of these two geometric concepts lead to interesting inequalities about the AdS curvature of the minimal surface.

Indeed, when the closed surface $2\Sigma_{\min}$ belongs to the topological class of the sphere ($g = 0$), the combination of eqs. (4.2), (4.31) and (4.27) leads to the inequality

$$\int_{\Sigma_{\min}} d^2y \sqrt{\gamma} \mathcal{F} \leq 0. \tag{4.32}$$

For a toroidal closed surface ($g = 1$), the inequality in eq. (4.12) gives

$$\int_{\Sigma_{\min}} d^2y \sqrt{\gamma} \mathcal{F} \leq -2\pi^2. \tag{4.33}$$

In this geometry, the bound is saturated by the Clifford torus [92]. Notice that in both cases the integral of the trace of the AdS curvature is non-positive.

From a different starting point, Alexakis and Mazzeo arrived at the same type of inequalities in ref. [51]. Note that our derivation of the Willmore energy from renormalized volume relies on the existence of an AdS bulk. Thus, the bounds (4.32) and (4.33) cannot be extended to other backgrounds. A generalization of these results to generic bulk and boundary geometries can be seen in ref. [87].

5 Holographic entanglement entropy and Willmore energy

The connection between renormalized area and Willmore energy provide us insight on surfaces immersed in a higher-dimensional manifold. In particular, the dependence of EE on the geometry becomes manifest when taking a minimal surface Σ_{\min} .

For RT surfaces, the renormalized area of Σ_{RT} is equivalent to the renormalized EE of the subregion A , and the following formula holds

$$S_{EE}^{\text{ren}}(A) = -\frac{L^2}{8G_N} \mathcal{W}(2\Sigma_{RT}). \tag{5.1}$$

One can map the universal part of EE of an entangling region for a vacuum state of the CFT_3 to the Willmore energy of a closed geometry constructed by gluing two copies of the RT surface.

Inequalities (4.11) and (4.12) set a bound to renormalized EE (5.1). For a doubled RT surface which correspond to $g = 0$ [32], we get

$$S_{EE}^{\text{ren}}(A) \leq -\frac{\pi L^2}{2G_N}, \tag{5.2}$$

that means the finite part of the EE is maximized for a circular surface among all the possible shapes within the same topological class. The same bound was obtained in ref. [51].

In a similar fashion, when the closed surface $2\Sigma_{RT}$ is of genus $g = 1$, the finite term of the EE satisfies

$$S_{EE}^{\text{ren}}(A) \leq -\frac{\pi^2 L^2}{4G_N}. \tag{5.3}$$

Therefore, the sphere results as the global maximum of the EE between surfaces of genus up to one. Astaneh, Gibbons and Solodukhin arrived at the same conclusion by extending their study to higher dimensional surfaces of genus larger than one in ref. [92].

In section 3 we showed that a measure of the deformation of an entangling surface is given by the trace of the AdS curvature, subjected to the inequalities (4.32) and (4.33). This quantity is a holographic geometric probe of entanglement susceptibility in the dual CFT. Interestingly enough, the susceptibility is negative as a consequence of the strong subadditivity property of EE [43, 84, 86]. Hence, strong subadditivity imposes a restriction on the curvature of the RT surface side which reads

$$\int_{\Sigma_{\text{RT}}} d^2y \sqrt{\gamma} \left(\mathcal{R} + \frac{2}{L^2} \right) \leq 0. \tag{5.4}$$

An analogous constraint on the spacetime curvature was derived in ref. [85] in the context of covariant EE in AdS₃/CFT₂.

5.1 *F*-theorem and Willmore energy

The universal term s_{univ} of EE of a disk-like entangling region is a relevant quantity, as it is identified with the free energy $F_{\mathbb{S}^3}$ of a CFT₃ on a spherical background \mathbb{S}^3 [18]. In addition, $F_{\mathbb{S}^3}$ has been proven to be a *F*-function along the RG flows in $d = 3$ [21, 22], what reflects the degrees of freedom of the theory.

Consider the EE of a spatial subregion A for a 3-dim CFT

$$S_{\text{EE}}(A) = \frac{\text{Area}(\partial A)}{\delta} - s_{\text{univ}}(A), \tag{5.5}$$

where δ is the regulator in eq. (1.2). By an adequate manipulation of eqs. (5.1) and (5.5), it is straightforward to show that

$$s_{\text{univ}}(A) = \frac{L^2}{8G_N} \mathcal{W}(2\Sigma_{\text{RT}}), \tag{5.6}$$

whereas combining eqs. (5.2) and (5.5), one can arrive at the inequality

$$s_{\text{univ}}(A) \geq \frac{\pi L^2}{2G_N} = F_{\mathbb{S}^3}. \tag{5.7}$$

One can assume that the Casini-Huerta-Myers (CHM) map linking the finite term in the EE and the number of degrees of freedom of the theory is valid for any shape of the entangling surface. On the other hand, the Wilsonian picture of the RG flows indicates that the microscopic degrees of freedom depend on the energy of the theory. In this picture, the energy acquire global characteristics as it affects the degrees of freedom independently on their position and local properties on the manifold. Therefore, $s_{\text{univ}}(A)$ should depend on global features of the entangling surface, namely the topological contribution of the renormalized EE in eq. (5.1).

Furthermore, for surfaces within the same topological class only the ones having the maximum renormalized area are adequate probes of the degrees of freedom of the theory.

This can be seen as coming from their maximum capacity of information storage. Due to the fact that the circle is the global maximum of the area among all 2D geometries of different genus — as shown in eqs. (5.2) and (5.3) — it represents a strong candidate to a proper measure of the degrees of freedom.

In addition, surfaces of maximal renormalized area with $g > 0$, are entangling regions that cannot fully cover a spatial slice of the CFT in their maximum extension. Physically, that means that they cannot account for all the degrees of freedom of the theory. In turn, entangling regions of $g = 0$ can potentially cover the full manifold in its totality. Indeed, the fact that the disk has the maximum renormalized area in the $g = 0$ topological class, implies that is able to encode all the information in the theory.³

6 Discussion

In the present paper, we have studied the shape dependence of entanglement entropy in 3-dim CFTs which are holographically dual to Einstein-AdS gravity. The finite part of the entanglement entropy is expressed as the renormalized volume of the RT surface (2.18) for CFTs in odd dimensions. It consists on two contributions: a topological part, proportional to the Euler characteristic of Σ (shape independent); and a curvature term, which encodes the information of the deformation of the entangling region with respect to a constant-curvature condition.

We have presented explicit computations on entangling regions with deformations for 3-dimensional CFTs, along the line of refs. [49, 58, 59]. We match the results found in the literature given in refs. [30, 39]. Our analysis shows that the number of degrees of freedom of the field theory is given by the topological part. In turn, the quadratic term in the deformation is coming from the integral of the AdS curvature. This means, that the AdS curvature of the RT surface carries information on the coefficients of the correlation function of the dual CFT₃. Future directions of this work considers its extension to higher dimensions and to higher-curvature gravity theories.

We have also shown that Willmore energy arises as a special case of renormalized volume formula of a two-dimensional surface. Indeed, expression (4.2) is general, as there is no distinction between minimal and non-minimal surfaces. Demanding a minimal surface in a constant time slice of global AdS₄ bulk spacetime, makes eq. (4.2) equivalent to the Willmore functional. The latter provides a lower bound, saturated by a circular entangling surface. This also shows that renormalized EE of a disk-like entangling region is maximal among all the shapes with the same perimeter. This is in consonance with the observations made in ref. [30], which points out that the universal contribution s_{univ} of the entanglement entropy is minimized by a circular entangling surface. At the same time, we know that s_{univ} matches the free energy of a CFT₃ on a spherical \mathbb{S}^3 background due to the Casini-Huerta-Myers relation [18]. What we learnt here is that, as prescribed by eq. (5.6), s_{univ} can be equivalently seen as the Willmore energy of \mathbb{S}^2 .

³We thank I.J. Araya for comments on this point.

Acknowledgments

We would like to thank I.J. Araya, C. Corral, F. Díaz for useful discussions and comments and P. Bueno for introducing us the concept of Willmore energy. The work of RO and GA was funded in part by FONDECYT grants No. 1170765 *Boundary dynamics in anti-de Sitter gravity and gauge/gravity duality* and No. 3190314 *Holographic Complexity from Anti-de Sitter gravity*. The work of JM is funded by the Comisión Nacional de Investigación Científica y Tecnológica (CONICYT) scholarship No. 21190234 and by Pontificia Universidad Católica de Valparaíso. The work of DRB is funded by Becas Chile (CONICYT) scholarship No. 72200301.

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Topological invariants and the definition of energy in quadratic gravity theory

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(Received 31 January 2020; accepted 10 March 2020; published 20 March 2020)

We provide a simple method to compute the energy in higher curvature gravity in asymptotically AdS spacetimes in even dimensions. It follows from the combined use of topological terms added to the gravity action, and the Wald charges derived from the augmented action functional. No additional boundary terms are needed. As a consistency check, we show that the formula for the conserved quantities derived in this way yields the correct result for the mass of asymptotically AdS black holes.

DOI: [10.1103/PhysRevD.101.064046](https://doi.org/10.1103/PhysRevD.101.064046)

I. INTRODUCTION

Higher-curvature corrections to Einstein-Hilbert action are ubiquitous in effective field theory when gravity is involved. In fact, a sensible theory of quantum gravity is generically expected to yield such corrections, and string theory, as an antonomastic example of this, does predict them at next-to-leading order [1–3]. Higher-curvature corrections, on the other hand, are interesting by their own right: they have been studied for a long time and in many different contexts, including mathematical aspects of general relativity (GR) [4–6], cosmology [7], black holes [8], massive gravity [9], supergravity [10], and quantum gravity [11].

Regarding quantum gravity, it is well known that the introduction of higher-order terms in the gravity action suffices to render the theory renormalizable [11] but at the

price of introducing ghosts [9]. This is generically the case, with a few notable exceptions [3]. Therefore, the absence of ghosts and of other potential pathologies, such as causality issues, can be used as a criterion to select the sensible higher-order theory or ultraviolet completion to work with [3,12,13].

In recent years, the interest in *healthy* higher-order corrections to Einstein theory has been renewed. In particular, there have been very interesting works studying higher-curvature models in anti-de Sitter (AdS) space. Such is the case of the so-called critical gravity (CG) theories [14,15], which provide ghost-free models of gravity in asymptotically AdS spacetimes in $D \geq 4$ dimensions. In $D = 3$ dimensions, higher-curvature terms were also considered as a toy model of a consistent gravity theory [16]. Other recent works explore, for instance, the connection between higher-curvature conformally invariant theories and Einstein gravity in (A)dS in $D = 4$ [17]. Other models in $D \geq 4$ dimensions studied recently, such as the so-called (generalized) quasitopological gravity [18–23] and Einsteinian gravity [24,25], are also very interesting and provide a new perspective on this old topic.

The computation of conserved charges in higher-curvature theories in both asymptotically flat and asymptotically AdS spaces is an important problem that has been addressed by many authors in the last 20 years; notably by

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Deser and Tekin, who found in Refs. [26,27] the higher-derivative generalization of the Abbot-Deser method for GR [28]. Other papers discussing conserved charges in related contexts are [29–31], and of course others using the covariant formalism of Ref. [32]; see references therein.

Recently, we proposed in [33] a novel definition of gravitational energy for an arbitrary theory of gravity including quadratic-curvature corrections to Einstein equations. We focused on the theory in $D = 4$ dimensions and in the presence of a negative cosmological constant. Unlike some other methods considered in the literature, the method proposed in [33] is intrinsically nonlinear and permits one to deal with the boundary terms [34,35] in a systematic way. It relies on the idea of adding to the gravity action topological invariant terms, which suffice to regularize the Noether charges and render the variational problem well posed. More precisely, the method amounts to adding to the action the bulk part of a topological invariant. This is an idea that has been previously considered in [36,37] in the case of second-order theories, such as Einstein-AdS gravity. Here, we show how this method can be generalized to generic quadratic-curvature theories in any even dimension.

The paper is organized as follows: In Sec. II, we will introduce the action of generic quadratic-curvature gravity in $D = 2n$ dimensions, including topological invariants that will eventually serve to regularize the action. In Sec. III, we provide the definition of gravitational energy and we use it to compute the mass of black holes in asymptotically AdS $_D$ space. Section IV contains our conclusions.

II. QUADRATIC-CURVATURE GRAVITY IN $D = 2n$ DIMENSIONS

The most general gravity action which adds up quadratic corrections in the curvature to GR in even dimensions is given by the expression

$$I = \int_M d^{2n}x \sqrt{-g} \left(\frac{R - 2\Lambda_0}{\kappa} + \alpha R^\mu_\nu R^\nu_\mu + \beta R^2 + \gamma GB \right) + \alpha_{2n} \int_M d^{2n}x \mathcal{E}_{2n}. \quad (1)$$

Here, $GB \equiv \sqrt{-g}(R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} - 4R^{\mu\nu}R_{\mu\nu} + R^2)$ is the Gauss-Bonnet term, which in $D > 4$ is a dynamical term. The last term in the integral (1) is, by contrast, the $2n$ -dimensional topological Euler density, which reads

$$\mathcal{E}_{2n} = \frac{\sqrt{-g}}{2^n} \delta_{[\nu_1 \dots \nu_{2n}] }^{\mu_1 \dots \mu_{2n}} R^{\nu_1 \nu_2 \dots \nu_{2n-1} \nu_{2n}}. \quad (2)$$

This contribution accommodates the maximal number of curvatures in $2n$ dimension, such that it does not contribute to the field equations. In $D = 4$, this invariant is the same as the Gauss-Bonnet term.

We shall use the conventions in Ref. [27], where $\kappa = 2\text{Vol}(S^{2n-2})G_{2n}$ in terms of the D -dimensional Newton's constant G_D . In addition, in order to compare with existing literature, we are considering the inclusion of a bare cosmological constant $\Lambda_0 = -(2n-1)(2n-2)/2\ell^2$ in terms of the original AdS radius ℓ . In our notation, $\delta_{[\nu_1 \dots \nu_p]}^{\mu_1 \dots \mu_p} = \det[\delta_{\nu_i}^{\mu_i} \dots \delta_{\nu_p}^{\mu_p}]$ is a totally antisymmetric product of p Kronecker deltas.

Varying the action with respect to the metric, one obtains the equations of motion (EOM), which correspond to Einstein tensor plus fourth-order contributions coming from curvature-squared terms; namely

$$0 = \frac{1}{\kappa} G_{\mu\nu} + 2\beta R \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) + (2\beta + \alpha)(g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) R - \gamma H_{\mu\nu} + \alpha \square G_{\mu\nu} + 2\alpha \left(R_{\mu\sigma\nu\rho} - \frac{1}{4} g_{\mu\nu} R_{\sigma\rho} \right) R^{\sigma\rho}, \quad (3)$$

where the part that is linear in the curvature, namely,

$$G_\nu^\mu = R_\nu^\mu - \frac{1}{2} R \delta_\nu^\mu + \Lambda_0 \delta_\nu^\mu, \quad (4)$$

is the usual field equation of GR, and the quadratic part

$$H_\nu^\mu = \frac{1}{8} \delta_{[\nu_1 \nu_2 \nu_3 \nu_4]}^{\mu_1 \mu_2 \mu_3 \mu_4} R^{\nu_1 \nu_2} R^{\nu_3 \nu_4}, \quad (5)$$

corresponds to the Lanczos tensor [4–6].

Vacuum states of the theory correspond to maximally symmetric spaces, which satisfy the constant curvature condition

$$R_{\alpha\beta}^{\mu\nu} = -\frac{1}{\ell_{\text{eff}}^2} \delta_{[\alpha\beta]}^{[\mu\nu]}. \quad (6)$$

Here, ℓ_{eff} is the effective (A)dS radius. From the equation of motion one can readily obtain an expression for the effective (A)dS radius in terms of the couplings of the theory. Using the standard relation between (A)dS length and the corresponding cosmological constant $\Lambda_{\text{eff}} = -(2n-1)(2n-2)/2\ell_{\text{eff}}^2$, one arrives at the equation

$$-\frac{1}{2\kappa\Lambda_{\text{eff}}} + \frac{\Lambda_0}{2\kappa\Lambda_{\text{eff}}^2} = \frac{(2n-4)}{(2n-2)^2} (2n\beta + \alpha) + \gamma \frac{(2n-4)(2n-3)}{(2n-2)(2n-1)}. \quad (7)$$

Note that $\Lambda_{\text{eff}} = \Lambda_0$ in four dimensions. This is no longer the case in $D > 4$, as higher curvature terms modify the effective cosmological constant.

The surface term of the theory arises from integrating by parts the variation of the gravitational action in order to

construct the EOMs. Without loss of generality, one can always split such contribution in the pieces that contain δg and $\delta\Gamma$, respectively. They can be easily derived employing Wald construction of Noether conserved quantities in gravity theories [34]. Then, the surface term is cast in the form

$$\Theta^\alpha(\delta g, \delta\Gamma) = 2E_{\mu\nu}^{\alpha\beta} g^{\nu\lambda} \delta\Gamma_{\beta\lambda}^\mu + 2\nabla^\mu E_{\mu\nu}^{\alpha\beta} (g^{-1}\delta g)_{\beta}^{\nu}, \quad (8)$$

where the tensor $E_{\mu\nu}^{\alpha\beta} = \delta\mathcal{L}/\delta R_{\alpha\beta}^{\mu\nu}$ is the derivative of the Lagrangian density with respect to the Riemann tensor which, for QCG theory, yields

$$E_{\mu\nu}^{\alpha\beta} = \frac{1}{2\kappa} \delta_{[\mu\nu]}^{[\alpha\beta]} + \frac{1}{2} \alpha R_{[\mu}^{\alpha} \delta_{\nu]}^{\beta]} + \beta R \delta_{[\mu\nu]}^{[\alpha\beta]} + \frac{1}{2} \gamma \delta_{[\mu\nu\nu_3\nu_4]}^{[\alpha\beta\mu_3\mu_4]} R_{\mu_3\mu_4}^{\nu_3\nu_4} + \frac{n\alpha_{2n}}{2^n} \delta_{[\mu\nu\nu_3\cdots\nu_{2n}]}^{[\alpha\beta\mu_3\cdots\mu_{2n}]} R_{\mu_3\mu_4}^{\nu_3\nu_4} \cdots R_{\mu_{2n-1}\mu_{2n}}^{\nu_{2n-1}\nu_{2n}}. \quad (9)$$

In what follows, we construct an energy definition for QCG in even dimensions, as the natural generalization of the procedure shown in Ref. [33].

III. ENERGY DEFINITION

In the literature, there is a plethora of different approaches to deal with the general issue of defining energy for a given gravity theory [38]. Here, we shall apply Wald formalism [34,35], as the charges derived with such formalism for the case of quadratic theories take a relatively simple form: For a Lagrangian which is a function of the metric and the Riemann curvature $\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\alpha\beta})$, the conserved quantity associated to any Killing vector ξ^μ is expressed as a surface integral in the codimension-2 surface Σ

$$Q_W^\alpha[\xi] = 2 \int_\Sigma dS_\beta (E_{\mu\nu}^{\alpha\beta} \nabla^\mu \xi^\nu + 2\nabla^\mu E_{\mu\nu}^{\alpha\beta} \xi^\nu). \quad (10)$$

The use of Wald's derivation of conserved quantities from the gravitational bulk Lagrangian does not guarantee by itself that the value of the energy is correct. This can be seen in GR, where Wald charge coincides with the Komar formula. So, even in the asymptotically flat case, the black hole mass computed from (10) does not coincide with the one obtained by the Hamiltonian method. The situation worsens for AdS gravity, as the behavior of solutions turns Komar integral divergent at the spatial infinity. These facts imply that the charges (10) need to be corrected by adding suitable boundary terms to the original Lagrangian. Prescribing appropriate boundary conditions makes possible the integration of the conserved quantities, although getting closed expressions for the charges is not always guaranteed. In this regard, the covariant formalism [32], which provides a robust method to compute conserved charges in which the boundary terms are constructed systematically, is particularly useful. As we will see, we will obtain a result in agreement with that method.

The addition of topological terms to Einstein-Hilbert action in GR with a negative cosmological constant provides a remarkably simple method to circumvent the drawbacks in the procedure described above. In four dimensions, the consistent coupling of the Gauss-Bonnet and Pontryagin terms allows us to express the gravitational charges as the electric/magnetic part of the Weyl tensor [39,40]. The formulas, which give rise to the correct energy of different asymptotically AdS solutions, turn the discussions on background-subtraction methods and the inclusion of extra boundary terms in the Lagrangian superfluous. In a previous work [33], we have shown that the addition of the Gauss-Bonnet invariant to quadratic-curvature gravity (QCG) action in $D = 4$ also acts as a regulator of the Noether charges in both Einstein and non-Einstein sectors of the theory. Here, we extend this result to an arbitrary even-dimensional QCG theory. A single topological term added to the action suffices to render the conserved charges finite. The information about the background of the corresponding sector of the theory is encoded in the asymptotic value of the curvature, and gets reflected in the coupling constant of the Euler topological term. In other words, even in the higher-derivative theory it happens that the addition of topological invariants to the action is sufficient to regularize the conserved charges, making the job of the otherwise needed boundary terms.

In QCG, due to Bianchi identity, the second part of the integrand in Eq. (10) does not feature any term containing γ . Altogether, the rest of $\nabla^\mu E_{\mu\nu}^{\alpha\beta}$ vanishes identically for Einstein spaces,

$$R_{\mu\nu} = -\frac{(2n-1)g_{\mu\nu}}{\ell_{\text{eff}}^2}. \quad (11)$$

The gravity theory under analysis here admits a number of analytic solutions with different asymptotic behavior, such as asymptotically AdS, Lifshitz, etc. These different asymptotic behaviors represent different sectors of the theory. For the purpose of the present discussion, we will consider solutions which are continuously connected to a global AdS spacetime, which defines the background configuration. We then fix the coupling of the Euler density in terms of the parameters α , β , and γ by the following criterion: The total surface term must vanish identically for the vacuum state (6) corresponding to the class of solutions we are interested in. This basic assumption implies a well-posed variational principle, at least for global AdS spacetime. In doing so, the coupling constant of the Euler density reads

$$\alpha_{2n} = \frac{(-1)^n \ell_{\text{eff}}^{2n-2}}{n(2n-2)! \kappa} \left[1 + \frac{4\kappa \Lambda_{\text{eff}}}{2n-2} \left(2n\beta + \alpha + \gamma \frac{(2n-3)(2n-2)}{2n-1} \right) \right]. \quad (12)$$

The quartic relation for the effective AdS radius in (7) has a single root provided the derivative of it with respect to ℓ_{eff}^2 is different from zero. Indeed, this degeneracy condition poses an obstruction for the linearization of EOM when the two maximally symmetric vacua of the theory coincide. This is similar to what happens, e.g., in odd dimensions with the Chern-Simons gravity theories [41]. The construction of conserved quantities is sensitive to this issue, as linearized charges cannot be obtained for the degenerate case. In relation to this, it is worth emphasizing that the method of adding the Euler term (12) is not affected by this consideration, as it does not rely on linear perturbations of the geometry around a given background. Once we have suitably identified the vacuum solution, we can ask whether the massive deviations from the maximally symmetric solution are such that the total surface integral keeps being finite. This can be checked in concrete examples.

The full charge is obtained as the surface integral

$$Q^\alpha[\xi] = \int_\Sigma dS_\beta (q_{(1)}^{\alpha\beta} + q_{(2)}^{\alpha\beta}), \quad (13)$$

where $q_{(1)}^{\alpha\beta}$ and $q_{(2)}^{\alpha\beta}$ are the prepotentials

$$q_{(1)}^{\alpha\beta} = \nabla^\mu \xi^\nu \left(\frac{1}{\kappa} \delta_{[\mu\nu]}^{[\alpha\beta]} + \alpha R_{[\mu}^{\alpha} \delta_{\nu]}^{\beta]} + 2\beta R \delta_{[\mu\nu]}^{[\alpha\beta]} + \gamma \delta_{[\mu\nu\nu_3\nu_4]}^{[\alpha\beta\mu_3\mu_4]} R_{\mu_3\mu_4}^{\nu_3\nu_4} + \frac{n\alpha_{2n}}{2^{n-1}} \delta_{[\mu\nu\cdots\nu_{2n}] }^{[\alpha\beta\cdots\mu_{2n}]} R_{\mu_3\mu_4}^{\nu_3\nu_4} \cdots R_{\mu_{2n-1}\mu_{2n}}^{\nu_{2n-1}\nu_{2n}} \right),$$

and

$$q_{(2)}^{\alpha\beta} = 2\nabla^\mu (\alpha R_{[\mu}^{\alpha} \delta_{\nu]}^{\beta]} + 2\beta R \delta_{[\mu\nu]}^{[\alpha\beta]}) \xi^\nu. \quad (14)$$

Now, we can consider a static black hole with standard AdS asymptotics, whose metric is given by

$$ds^2 = -f^2(r) dt^2 + \frac{1}{f^2(r)} dr^2 + r^2 d\Omega_{2n-2}^2, \quad (15)$$

where $d\Omega_{2n-2}^2$ is the metric on the unit $(2n-2)$ -sphere. This discussion can be straightforwardly generalized to topological black holes with planar or hyperbolic horizons. The nonvanishing components of the Riemann curvature for this static configuration are

$$\begin{aligned} R_{tr}^{tr} &= -\frac{1}{2}(f^2)'' , \\ R_{tm}^{tm} &= R_{rm}^{rm} = -\frac{1}{2r}(f^2)' \delta_m^n , \\ R_{kl}^{mn} &= \frac{1-f^2}{r^2} \delta_{[kl]}^{[mn]} , \end{aligned} \quad (16)$$

while the nonvanishing components of the Ricci tensor are

$$\begin{aligned} R_t^t &= R_r^r = -\frac{1}{2r} [r(f^2)'' + 2(n-1)(f^2)'] , \\ R_m^m &= -\frac{1}{r^2} [r(f^2)' - (2n-3)(1-f^2)] \delta_m^n . \end{aligned} \quad (17)$$

Here, the prime stands for derivatives with respect to r . In the case of Boulware-Deser black holes [8] of Einstein-Gauss-Bonnet gravity ($\alpha = \beta = 0$), which is the working example in QCG, the metric function $f^2(r)$ takes the asymptotic form

$$f^2(r) \simeq \frac{r^2}{\ell_{\text{eff}}^2} + 1 - \left(\frac{r_0}{r}\right)^{2n-3} + \cdots, \quad (18)$$

with ℓ_{eff}^2 being a root of the polynomial equation

$$\gamma(2n-3)(2n-4) \frac{1}{\ell_{\text{eff}}^4} - \frac{1}{\kappa \ell_{\text{eff}}^2} + \frac{1}{\kappa \ell^2} = 0. \quad (19)$$

The ellipsis in (18) stands for higher powers of $1/r$. In a more general case, in which the couplings α and β are nonzero and generic, a similar falloff is expected, with the difference being the value of the AdS radius, which turns out to be given by Eq. (7). For special relations between α , β , and Λ_0 , a weakened version of the asymptotic condition (18) might be possible. This has been extensively studied in the literature. However, for generic values of the parameters α and β , (18) is the expected behavior, as the quadratic terms represent ultraviolet corrections. A more general case could eventually involve two different metric functions corresponding to g_{tt} and g_{rr}^{-1} , each one with the asymptotic behavior that respects the AdS asymptotics as given by Eq. (18), what we discuss at the end of this section.

For static massive objects, the energy formula takes the form

$$\begin{aligned} E &= Q^t[\partial_t] \\ &= \int_\Sigma dS_r \nabla^r \xi^t \left[\alpha (R_t^t + R_r^r) + \left(\frac{1}{\kappa} + 2\beta R\right) \delta_{[rt]}^{[rt]} \right. \\ &\quad \left. + \gamma \delta_{[rtm_1m_2]}^{[rtm_1m_2]} R_{p_1p_2}^{m_1m_2} + \frac{n\alpha_{2n}}{2^{n-1}} \delta_{[rtm_1\cdots p_{2n}]}^{[rtm_1\cdots p_{2n}]} R_{p_1p_2}^{m_1m_2} \cdots R_{p_{2n-3}p_{2n-2}}^{m_{2n-3}m_{2n-2}} \right]. \end{aligned} \quad (20)$$

Given the form of the line element for the black hole (15), explicit evaluation of the above energy also requires the derivative of the timelike Killing vector, $\nabla^r \xi^t = (f^2)'/2$, and the codimension-2 surface element dS_r . If the local coordinates on Σ are denoted by y^m and the corresponding line element of Σ is

$$r^2 d\Omega_{2n-2}^2 = r^2 \sigma_{mn}(y) dy^m dy^n, \quad (21)$$

where σ_{mn} is the metric of the unit sphere, then we find along the radial normal $dS_r = r^{2n-2} d^{2n-2}y \sqrt{\sigma}$. In turn, the

curvature expression in the square bracket can be recast in terms of the metric function $f^2(r)$ and its derivatives. The result is

$$\begin{aligned}
 E = \int_{\Sigma} d^{2n-2}y \sqrt{\sigma} \frac{1}{2} (f^2)' & \left\{ \frac{1}{\kappa} - \frac{\alpha}{r} (r(f^2)'' + 2(n-1)(f^2)') \right. \\
 & - \frac{2\beta}{r^2} [r^2(f^2)'' + 4(n-1)r(f^2)'] \\
 & - 2(n-1)(2n-3)(1-f^2)] \\
 & + 4\gamma(n-1)(2n-3) \frac{1-f^2}{r^2} \\
 & \left. + n\alpha_{2n}(2n-2)! \left(\frac{1-f^2}{r^2} \right)^{n-1} \right\}. \quad (22)
 \end{aligned}$$

It is important to notice here that the Ricci tensor tends to a constant value, namely $R_{\nu}^{\mu} \simeq -(2n-1)\delta_{\nu}^{\mu}/\ell_{\text{eff}}^2$.

Replacing the asymptotic value of the metric function and its derivatives and performing the integral on the angular variables $\int_{\Sigma} d^{2n-2}y \sqrt{\sigma} = \text{Vol}(S^{2n-2})$, the expression for the energy takes the form

$$\begin{aligned}
 E = \text{Vol}(S^{2n-2}) \lim_{r \rightarrow \infty} r^{2n-2} f f' & \left[\frac{1}{\kappa} - (\alpha + 2\beta)(f^2)'' \right. \\
 & - 2(n-1)(\alpha + 4\beta) \frac{(f^2)'}{r} \\
 & + 4(n-1)(2n-3) \frac{1-f^2}{r^2} (\beta + \gamma) \\
 & \left. + n\alpha_{2n}(2n-2)! \left(\frac{1-f^2}{r^2} \right)^{n-1} \right], \quad (23)
 \end{aligned}$$

where the last term of this expression can be expanded as

$$\left(\frac{1-f^2}{r^2} \right)^{n-1} = \frac{(-1)^{n-1}}{\ell_{\text{eff}}^{2n-2}} + (n-1) \frac{(-1)^{n-2} r_0^{2n-3}}{\ell_{\text{eff}}^{2n-4} r^{2n-1}} + \dots \quad (24)$$

The remaining terms in the sum can be neglected as they vanish in the limit $r \rightarrow \infty$, in which the surface integral is evaluated. Keeping all the coupling constants arbitrary, we obtain the divergent expression

$$\begin{aligned}
 E = \text{Vol}(S^{2n-2}) \lim_{r \rightarrow \infty} & \left[C \left(\frac{r^{2n-1}}{\ell_{\text{eff}}^2} + \frac{(2n-3)r_0^{2n-3}}{2} \right) \right. \\
 & + \left(2(2n-3)\gamma + n\alpha_{2n}(2n-2)! \frac{(-1)^{n-2}}{2\ell_{\text{eff}}^{2n-4}} \right) \\
 & \left. \times \frac{(2n-2)r_0^{2n-3}}{\ell_{\text{eff}}^2} \right], \quad (25)
 \end{aligned}$$

where we introduced the constant

$$\begin{aligned}
 C = \frac{1}{\kappa} - \frac{2}{\ell_{\text{eff}}^2} & ((2n-1)(\alpha + 2n\beta) + (2n-2)(2n-3)\gamma) \\
 & + n\alpha_{2n}(2n-2)! \frac{(-1)^{n-1}}{\ell_{\text{eff}}^{2n-2}}.
 \end{aligned}$$

Now it is clear that fixing the value of α_{2n} as in Eq. (12), which produces $C = 0$ and regularizes the r^{2n-1} term, makes the gravitational energy finite.

Finally, writing γ in terms of α and β [as given in Eq. (7)], the energy is

$$E = \left(-1 + \frac{8\Lambda_{\text{eff}}\kappa}{(2n-2)^2} (2n\beta + \alpha) + \frac{2\Lambda_0}{\Lambda_{\text{eff}}} \right) \frac{(2n-2)r_0^{2n-3}}{4G_{2n}}, \quad (26)$$

which is the correct result for a black hole mass in QCG theory with $\Lambda_0 = 0$, cf. [27], in our case generalized to include the bare cosmological constant Λ_0 .

In the case when the metric component g_{tt} in the ansatz is replaced by a more general function $-N^2(r)f^2(r)$ that respects the AdS asymptotics, the resulting formula for the energy still yields a finite result. Here, in order to compare with known results in the literature, we consider the case $N = 1$.

The above discussion shows the effect of the Euler topological invariant added to the QCG action, which is similar to the one in GR and Einstein-Gauss-Bonnet gravity actions. Namely, without such term, the gravitational energy is both divergent and yields an incorrect value for the finite part in the mass computation. The addition of properly fixed Euler coupling α_{2n} resolves both issues at the same time.

IV. CONCLUSIONS

In this paper, we derived a general formula for the conserved charges for asymptotically AdS black holes in quadratic curvature gravity in $2n$ dimensions. The addition of topological terms to the gravity action acts as a regulator at the level of the surface term, and yields finite Wald charges in AdS without need of extra boundary terms. It remains an open question to know whether the method of adding Euler characteristics can also be implemented in order to compute conserved charges in sectors with other asymptotic behaviors, such Lifshitz black holes, AdS-waves, or spacetimes with weakened asymptotics in AdS.

In the past, the use of topological terms to regularize the conserved quantities and the Euclidean action of a given gravity theory has not only provided a powerful, yet simple computational tool to understand physical properties of black holes, but it has also unveiled remarkable features of the gravitational action in critical points of the parameter space in four dimensions such as critical gravity [14]. Indeed, the addition of the Gauss-Bonnet term to the critical gravity action makes manifest the fact that the energy, the entropy, and the action are zero for Einstein spaces [42,43].

Implications of this procedure in the computation of holographic correlation functions have been explored in Ref. [44]. We expect that the addition of Euler terms will make more clear certain properties of critical theories in higher dimensions too.

ACKNOWLEDGMENTS

This work was funded in part by the NSF through Grant No. PHY-1214302, Chilean FONDECYT Grants

No. 1170765 and No. 1190533, and the Grant VRIEA-PUCV No. 123.764/2019. D. R. B. is supported by Becas Chile (CONICYT) Ph.D. scholarship No. 72200301. G. G. is partially supported by CONICET through Grant No. PIP 1109 (2017).

Note added.—While finishing our paper, the work [45] appeared, in which the higher-dimensional extension of [33] is also worked out.

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Titre : Aspects de la physique carrollienne

Mots clés : la gravité, holographie, théorie des champs

Résumé : Nous étudions divers aspects et applications gravitationnelles de la dynamique carrollienne. Les systèmes carrolliens se manifestent lorsque la vitesse de la lumière s'annule. Nous construisons des équations dynamiques générales carrolliennes et galiléennes valables pour des géométries carrolliennes/newtoniennes arbitraires, courbes et dépendantes du temps, l'accent étant mis sur l'hydrodynamique. La présence d'un courant $U(1)$ est aussi prise en compte. Cette démarche suit deux approches : basée sur l'invariance de l'action sous Carroll/Galilée et Weyl, ou par la prise d'une limite de grande/petite vitesse de la lumière dans le tenseur énergie-impulsion relativiste et le courant $U(1)$. La dynamique est régie par la conservation d'un ensemble de moments qui résultent soit de la variation de l'action par rapport aux différentes composantes de la géométrie carrollienne/newtonienne, soit apparaissent à différents ordres dans le développement en c du tenseur énergie-impulsion. Ces deux approches concordent, mais il est montré que la procédure de limite est plus riche en raison de la possibilité d'embrasser des situations avec des degrés de liberté supplémentaires. En fait, c'est cette liberté qui nous permet de déterminer dans quelles conditions l'invariance hydrodynamique est préservée lorsqu'on prend la limite de grande/petite vitesse de la lumière. Nous montrons que dans la limite galiléenne standard l'invariance hydrodynamique est perdue, mais récupérée en ajoutant deux degrés de liberté supplémentaires dans le développement en c du courant de chaleur et des courants $U(1)$. Dans le cas carrollien, l'invariance sous changement de repère hydrodynamique survit lorsque le comportement du tenseur énergie-impulsion est inspiré des fluides carrolliens holographiques. Nous présentons enfin l'analyse des courants associés aux isométries carrolliennes/galiléennes. Dans le cas carrollien/galiléen, ces courants ne sont pas toujours conservés et des conditions supplémentaires doivent être imposées.

La dérivation présentée pour la dynamique carrollienne transcende les fluides. Nous étudions le champ scalaire conforme sur une géométrie carrollienne générale et analysons les extensions carrolliennes de la théorie de Chern-Simons gravitationnelle à trois dimensions. Dans cette analyse, on découvre des dynamiques électriques et magnétiques codées à différents ordres en puissances de la vitesse de la lumière de l'action relativiste parente. Deux actions supplémentaires apparaissent, nommées paramagnétique et paraélectrique, respectivement. Dans l'esprit de la dualité jauge/gravité plate, nous étudions la dynamique des espace-temps asymptotiquement plats d'un point de vue carrollien. Nous montrons que les espace-temps à Ricci nul sont exprimés dans une jauge covariante vis-à-vis du bord nul. Cette jauge est une extension de la jauge de Newman-Unti, valable pour constante cosmologique finie ou nulle. Le cas plat correspond à une limite carrollienne au bord. L'espace de solutions à Ricci nul résultant est constitué d'un ensemble infini de données carrolliennes. On y trouve la géométrie conforme carrollienne, les moments en nombre fini de la théorie du bord et un nombre infini de tenseurs arbitraires, obtenus en développant le tenseur énergie-impulsion relativiste d'origine en série de Laurent. Tous obéissent aux équations de bilan de flux carrolliennes. Pour les solutions de type Petrov algébrique, cette structure carrollienne au bord permet de déterminer les charges gravitationnelles usuelles. Nous retrouvons le développement multipolaire de la masse et du moment angulaire pour la famille de Kerr-Taub-NUT. Nous étudions enfin comment le groupe d'Ehlers de type Möbius agit sur les données du bord nul. Pour les espace-temps stationnaires, ce groupe se manifeste comme une transformation locale des observables carrolliennes du bord nul. Pour la solution de Kerr-Taub-NUT par exemple, la transformation de la masse/nut est une rotation de l'énergie/Cotton.

Title : Aspects of Carrollian physics

Keywords : Gravity, Holography, Field Theory

Abstract : The purpose of this thesis is to study aspects of Carrollian dynamics and its application to gravity with zero cosmological constant. Carrollian systems arise as the vanishing speed of light (k) limit of Lorentzian theories. Here, general Carrollian and Galilean dynamical equations valid for arbitrary curved and time dependent Carrollian/Newton-Cartan geometries are constructed, with focus on fluid mechanics. In both cases the presence of a $U(1)$ current is considered. The latter is done in two approaches : Carrollian/Galilean and Weyl invariance of the action, and by taking a large/small- k limit of the relativistic energy-momentum tensor and the $U(1)$ current. In both cases, the dynamic is given by the conservation of a set of momenta that arise either as the variation of the action with respect to the different pieces of the Carrollian/Newton-Cartan geometry or appear at different orders in the k -expansion of the energy-momentum tensor. Although these two approaches agree, the limiting procedure is shown to be richer due to the possibility of capturing more general situations with extra degrees of freedom. In fact, it is this freedom that allows us to find under which conditions hydrodynamic-frame invariance is preserved when taking the large/small- k limit. We show that, although in the standard Galilean limit hydrodynamic-frame invariance is lost, it is recovered by adding two extra degrees of freedom in the large- k expansion of the heat current and $U(1)$ currents. In the Carrollian case, hydrodynamic-frame invariance survives when the behavior of the energy-momentum tensor is guided by holographic Carrollian fluid results. We also present the analysis of the currents generated by Carrollian/Galilean isometries. In the Carrollian/Galilean instances, these currents are not guaranteed to be conserved and additional conditions must be imposed.

The presented derivation for Carrollian dynamics is not valid only for fluids. The investigation of the scalar field on a general Carrollian spacetime is also pre-

ented, as well as the analysis of three dimensional Carrollian gravitational Chern-Simons extensions. In this analysis one finds electric and magnetic dynamics that are encoded at different orders in powers of the speed of light of the relativistic action. We furthermore unravel two more Carrollian Chern-Simons actions, dubbed paramagnetic and paraelectric, respectively.

In relation to a possible flat version of the gauge/gravity duality, we also study some aspects of Ricci-flat dynamics from a Carrollian perspective. We show that Ricci-flat spacetimes can be expressed in a gauge covariant with respect to the null boundary. This gauge is an extension of the Newman-Unti gauge which is valid for asymptotically anti-de Sitter and flat spacetimes. The flat instance is reached as the vanishing cosmological constant limit of the anti-de Sitter case, which corresponds to a Carrollian limit at the boundary. Therefore, the resulting Ricci-flat solution space is reconstructed in terms of an infinite set of boundary Carrollian data. These are composed by the Carrollian conformal geometry, a finite set of momenta of the theory hosted at the boundary, and an infinite number of arbitrary tensors, obtained by expanding the original energy-momentum tensor in Laurent series, which obey Carrollian flux balance equations. We take advantage of the latter to define gravitational charges by using Carrollian boundary techniques and restricting the spacetime to the algebraically special Petrov type. With this construction we recover the mass and angular momentum multipolar expansion for the Kerr-Taub-NUT family. We also learn how the hidden Ehlers Möbius group acts on the boundary data at null infinity. We find that, for stationary spacetimes, this group is manifested as a local transformation for the Carrollian geometry and the boundary Carrollian observables. We reproduce the mass/nut rotation as the energy/Cotton rotation for the Kerr-Taub-NUT solution.