Rainbow subgraphs and properly colored subgraphs in colored graphs

Jie Hu

université PARIS-SACLAY

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Sous-graphes arc-en-ciel et sous-graphes correctement colorés

dans les graphes colorés

Thèse de doctorat de l'université Paris-Saclay

École doctorale n° 580, sciences et technologies
de l'information et de la communication (STIC)
Spécialité de doctorat : Informatique
Graduate School : Informatique et sciences du numérique
Référent : Faculté des sciences d'Orsay

Thèse préparée dans l'unité de recherche
Laboratoire Interdisciplinaire des Sciences du Numérique (Université Paris-Saclay, CNRS), sous la direction de **Hao LI**, Directeur de Recherche (HDR)

Thèse soutenue à Paris-Saclay, le 24 Mai 2022, par

Jie HU

Composition du Jury

Xueliang LI Professeur, Nankai University, Chine	Président
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Hao LI Directeur de Recherche (HDR), CNRS, Université Paris-Saclay	Directeur de thèse

THÈSE DE DOCTORAT

NNT : 2022UPASG045

université PARIS-SACLAY

ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Titre : Sous-graphes arc-en-ciel et sous-graphes correctement colorés dans les graphes colorés

Mots clés : Graphe à arêtes colorées; Sous-graphe arc-en-ciel; Sous-graphe correctement coloré; Système d'incompatibilité; Sous-graphe compatible.

Résumé : Dans cette thèse, nous étudions les sous-graphes arc-en-ciel et les sous-graphes correctement colorés dans les graphes à arêtes colorées, et les sous-graphes compatibles dans les graphes avec des systèmes d'incompatibilité, qui peuvent être considérés comme une généralisation des graphes à arêtes colorées. Par rapport aux graphes généraux, les graphes colorés contiennent plus d'informations et sont capables de modéliser des relations plus complexes dans les réseaux de communication, les sciences sociales, la biologie moléculaire, etc. Par conséquent, l'étude des structures dans les graphes aux arêtes colorées est importante à la fois pour la théorie des graphes et pour d'autres sujets connexes.

Nous étudions d'abord la condition de degré de couleur minimum forçant les triangles arc-en-ciel à sommets disjoints dans les graphes aux arêtes colorées. En 2013, Li s'est avéré être la meilleure condition de degré de couleur minimum possible pour l'existence d'un triangle arc-en-ciel. Motivés par cela, nous obtenons une condition de degré de couleur minimum précis garantissant l'existence de deux triangles arc-en-ciel à sommets disjoints et proposons une conjecture sur l'existence de k triangles arc-en-ciel à sommets disjoints.

Deuxièmement, nous considérons la relation entre l'ordre de l'arbre maximum correctement coloré dans le graphe à bords colorés et le degré de couleur minimum. On obtient que pour un graphe connexe G aux arêtes colorées, l'ordre du maximum d'arbre correctement coloré est au moins $\min\{|G|, 2\delta^{c}(G)\}$, ce qui généralise unrésultat de Cheng, Kano et Wang. De plus, la borne inférieure $2delta^{c}(G)$ dans notre résultat est la meilleure possible et nous caractérisons tous les graphes extrémaux.

Troisièmement, nous recherchons la condition de degré de couleur minimum garantissant l'existence de 2-facteurs correctement colorés dans les graphes aux bords colorés. Nous dérivons une condition de degré de couleur minimum asymptotique forçant chaque facteur 2 correctement coloré avec exactement t composants, ce qui généralise un résultat de Lo. Nous déterminons également la meilleure condition de degré de couleur minimum possible pour l'existence d'un facteur 2 correctement coloré dans un graphe bipartite à arêtes colorées.

Enfin, nous étudions les facteurs compatibles dans les graphes avec des systèmes d'incompatibilité. La notion de système d'incompatibilité a été introduite pour la première fois par Krivelevich, Lee et Sudakov, qui peut être considérée comme une mesure quantitative de la robustesse des propriétés du graphe. Récemment, il y a eu un intérêt croissant pour l'étude de la robustesse des propriétés des graphes, visant à renforcer les résultats classiques en théorie des graphes extrémaux et en combinatoire probabiliste. Nous étudions la version robuste du résultat d'Alon--Yuster par rapport au système d'incompatibilité.

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ÉCOLE DOCTORALE
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Title : Rainbow subgraphs and properly colored subgraphs in colored graphs

Keywords : Edge-colored graph; Rainbow subgraph; Properly colored subgraph; Incompatibility system; Compatible subgraph.

Abstract : In this thesis, we study rainbow subgraphs and properly colored subgraphs in edge-colored graphs, and compatible subgraphs in graphs with incompatibility systems, which can be viewed as a generalization of edge-colored graphs. Compared with general graphs, edge-colored graphs contain more information and are able to model more complicated relations in communication network, social science, molecular biology and so on. Hence, the study of structures in edge-colored graphs is significant to both graph theory and other related subjects.

We first study the minimum color degree condition forcing vertex-disjoint rainbow triangles in edge-colored graphs. In 2013, Li proved a best possible minimum color degree condition for the existence of a rainbow triangle. Motivated by this, we obtain a sharp minimum color degree condition guaranteeing the existence of two vertex-disjoint rainbow triangles and propose a conjecture about the existence of k vertex-disjoint rainbow triangles.

Secondly, we consider the relation between the order of maximum properly colored tree in edge-colored graph and the minimum color degree. We obtain that for an edge-colored connected graph G, the order of maximum properly colored tree is at least $\min\{|G|, 2\delta^{c}(G)\}$, which generalizes a result of Cheng, Kano and Wang. Moreover, the lower bound $2delta^{c}(G)$ in our result is best possible and we characterize all extremal graphs.

Thirdly, we research the minimum color degree condition guaranteeing the existence of properly colored 2-factors in edge-colored graphs. We derive an asymptotic minimum color degree condition forcing every properly colored 2-factor with exactly t components, which generalizes a result of Lo. We also determine the best possible minimum color degree condition for the existence of a properly colored 2-factor in an edge-colored bipartite graph.

Finally, we study compatible factors in graphs with incompatibility systems. The notion of incompatibility system was firstly introduced by Krivelevich, Lee and Sudakov, which can be viewed as a quantitative measure of robustness of graph properties. Recently, there has been an increasing interest in studying robustness of graph properties, aiming to strengthen classical results in extremal graph theory and probabilistic combinatorics. We study the robust version of Alon--Yuster's result with respect to the incompatibility system.

Acknowledgements

First of all, I would like to thank my supervisor Prof. Hao Li for offering the opportunity to study in France and work with him. In the past four years, Prof. Li not only taught me a lot in academic research, but also gave me a lot of help and support in daily life. Prof. Li taught me how to do research, including how to find problems, how to write academic papers, how to give presentations and so on. Whenever I had difficulties in my research problems, Prof. Li always guided me patiently and gave me insightful suggestions. During the COVID-19 outbreak, Prof. Li often called us to confirm our safety and offer help. I really appreciate all his help.

I would like to thank Prof. Xueliang Li, who is the president of the thesis defense jury. I would like to express my gratitude to Prof. Ingo Schiermeyer and Prof. Bing Wei, who are willing to take the time to review my thesis. I would also like to thank Prof. François Pirot, who accepted to be the member of the jury.

I would like to thank Prof. Yannis Manoussakis, who was a respected team leader. In our weekly seminars, he provided me with many good ideas and suggestions.

I would like to express my gratitude to my master's supervisor Prof. Guanghui Wang and his research team for the fruitful collaborations.

I would like to extend my appreciation to Shun-ichi Maezawa, Zengxian Tian, Tianjiao Dai, Qiancheng Ouyang, Wenling Zhou, Bin Wang and Lin Sun. Studying and living with them enriched my life in France. I would also like to thank my friendly colleagues in our lab for the good working atmosphere.

I would also like to appreciate China Scholarship Council for financial support in the past four years.

Last but not least, I owe my deepest gratitude to my parents, my younger sister and my fiance for their consistent support and encouragement.

Résumé

Dans cette thèse, nous étudions les sous-graphes arc-en-ciel et les sous-graphes correctement colorés dans les graphes à arêtes colorées, et les sous-graphes compatibles dans les graphes avec des systèmes d'incompatibilité, qui peuvent être considérés comme une généralisation des graphes à arêtes colorées. Par rapport aux graphes généraux, les graphes à arêtes colorées contiennent plus d'informations et sont capables de modéliser des relations plus complexes dans les réseaux de communication, les sciences sociales, la biologie moléculaire, etc. Par conséquent, l'étude des structures dans les graphes aux arêtes colorées est importante à la fois pour la théorie des graphes et pour d'autres sujets connexes.

Il y a eu beaucoup de recherches sur le sujet de la détermination de conditions suffisantes pour l'existence de sous-graphes correctement colorés et de sous-graphes arc-en-ciel dans les graphes à arêtes colorées. Les conditions suffisantes souvent considérées sont la condition de degré de couleur minimum, le degré de couleur monochromatique maximum, la somme des degrés de couleur minimum, la somme des degrés de couleur de tous les sommets, etc. Les sous-graphes qui ont reçu une attention considérable sont les appariements parfaits, les cycles courts, les cycles de Hamilton, les arbres couvrants et les facteurs. Pour le graphe à arêtes colorées de l'hôte G, il y a à la fois des études lorsque G est complet et des études lorsque G est un graphe relativement clairsemé. Pour la coloration des arêtes du graphe hôte G, il peut s'agir de n'importe quelle coloration des arêtes sans restriction, alors qu'il peut s'agir d'une coloration des arêtes appropriée ou d'une coloration des arêtes forte.

Dans le chapitre 2, nous étudions la condition de degré de couleur minimum forçant les triangles arc-en-ciel à sommets disjoints dans les graphes à arêtes colorées. Soit G un graphe à arêtes colorées d'ordre n. En 2013, Li [122] a montré que si $\delta^c(G) \geq (n+1)/2$, alors G contient un triangle arc-en-ciel et un arc-en-ciel $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ montre que la borne inférieure $(n+1)/2$ est la meilleure possible. Motivés par ce résultat, nous prouvons

que si $n \geq 20$ et $\delta^c(G) \geq (n+2)/2$, alors G contient deux triangles arc-en-ciel à sommets disjoints et la borne inférieure $(n+2)/2$ est aussi le meilleur possible. Un résultat classique de Dirac [59] affirme que pour tout entier positif n et k avec $n \geq 3k$, tout graphe d'ordre n de degré minimum au moins $(n+k)/2$ contient k triangles à sommets disjoints. En tant que version colorée par les arêtes, nous conjecturons que si $\delta^c(G) \geq (n+k)/2$, alors G contient k triangles arc-en-ciel à sommets disjoints. Si cette conjecture est vraie, alors la borne inférieure $(n+k)/2$ est la meilleure possible. Pour prouver la conjecture, nous montrons que pour tout entier $k \geq 2$, si $n \geq 16k-12$ et $\delta^c(G) \geq n/2+k-1$, alors G contient k triangles arc-en-ciel à sommets disjoints.

Dans le chapitre 3, nous considérons principalement la relation entre l'ordre de l'arbre maximum correctement coloré dans le graphe à arêtes colorées G et le degré de couleur minimum $\delta^c(G)$ de G. On obtient que pour un graphe connexe G aux arêtes colorées, l'ordre du maximum d'arbre correctement coloré est au moins $\min\{|G|, 2\delta^c(G)\}$, ce qui généralise un résultat de Cheng, Kano et Wang [39] sur la condition de degré de couleur minimum forçant des arbres couvrants correctement colorés. De plus, la borne inférieure $2\delta^c(G)$ dans notre résultat est la meilleure possible et nous caractérisons tous les graphes extrémaux G avec l'arbre maximum correctement coloré d'ordre $2\delta^c(G)$ $neq|G|$. En corollaire, nous obtenons une condition de degré de couleur minimum stricte pour l'existence d'une forêt couvrante correctement colorée avec au plus k composants. De plus, nous caractérisons tous les graphes extrémaux pour un résultat de Fujita et Magnant [74] sur la condition de degré de couleur minimum pour que les graphes à arêtes colorés soient correctement connectés.

Dans le chapitre 4, nous recherchons la condition de degré de couleur minimum garantissant l'existence de 2-facteurs correctement colorés dans les graphes à arêtes colorés. On en déduit que pour tout $\varepsilon > 0$ et entier positif $t \geq 1$, il existe un entier $N(\varepsilon, t)$ tel que tout graphe à arêtes colorées G avec $\delta^c(G) \geq (2/3+\varepsilon)|G|$ et $|G| > N(\varepsilon, t)$ contiennent tous les 2-facteurs correctement colorés avec exactement t composantes, ce qui généralise un résultat de Lo [134] sur la condition de degré de couleur minimum pour l'existence de cycles correctement colorés de n'importe quelle longueur. Un résultat classique de Moon et Moser [140] indique que tout graphe biparti équilibré G de degré minimum $\delta(G) \geq |G|/4$ est hamiltonien. Inspirés par ce résultat, nous étudions une version bipartite du résultat mentionné ci-dessus et prouvons que tout graphe bipartite équilibré aux arêtes colorées G avec $\delta^c(G) \geq |G|/3+1$ contient un graphe correctement

coloré 2-facteur, où la borne inférieure $|G|/3+1$ est la meilleure possible.

Dans le chapitre 5, nous étudions les facteurs compatibles dans les graphes avec des systèmes d'incompatibilité. La notion de système d'incompatibilité a d'abord été introduite par Krivelevich, Lee et Sudakov [116], qui peut être considérée comme une mesure quantitative de la robustesse des propriétés du graphe. Récemment, il y a eu un intérêt croissant pour l'étude de la robustesse des propriétés des graphes, visant à renforcer les résultats classiques en théorie des graphes extrémaux et en combinatoire probabiliste. Étant donné un graphe H, trouver une condition suffisante et extrême pour l'existence d'un facteur H est l'un des sujets les plus importants de la théorie des graphes extrémaux. Un résultat fondamental d'Alon et Yuster [8] montre que si le graphe hôte G à n sommets est de degré minimum $\delta(G) \geq \left(1-\frac{1}{\chi(H)}\right)+o(n)$, alors G contient un facteur H, où $\chi(H)$ est le nombre chromatique de H. Nous étudions la version robuste du résultat d'Alon–Yuster par rapport au système d'incompatibilité et prouvons que pour tout $\alpha>0$ et tout graphe H avec h sommets, il existe une constante $\mu>0$ tel que pour tout n suffisamment grand avec $n\in h\mathbb{N}$, si G est un graphe de n-vertex avec $\delta(G)\geq\left(1-\frac{1}{\chi(H)}+\alpha\right)n$ et $\mathcal{F}$ est un système d'incompatibilité μn-borné sur G, alors il existe un H compatible facteur dans G, où le terme d'erreur de αn ne peut pas être omis.

Nous concluons cette thèse au chapitre 6 et donnons quelques perspectives.

Mots clés: Graphe à arêtes colorées; Sous-graphe arc-en-ciel; Sous-graphe correctement coloré; Système d'incompatibilité; Sous-graphe compatible.

Abstract

In this thesis, we study rainbow subgraphs and properly colored subgraphs in edge-colored graphs, and compatible subgraphs in graphs with incompatibility systems, which can be viewed as a generalization of edge-colored graphs. Compared with general graphs, edge-colored graphs contain more information and are able to model more complicated relations in communication network, social science, molecular biology and so on. Hence, the study of structures in edge-colored graphs is significant to both graph theory and other related subjects.

There has been much research on the topic of determining sufficient conditions for the existence of properly colored subgraphs and rainbow subgraphs in edge-colored graphs. The sufficient conditions often considered are the minimum color degree condition, the maximum monochromatic color degree, the minimum color degree sum, the color degree sum of all vertices and so on. The subgraphs received considerable attention are perfect matchings, short cycles, Hamilton cycles, spanning trees and factors. For the host edge-colored graph G, there are both studies when G is complete and studies when G is a relatively sparse graph. For the edge-coloring of host graph G, it can be any edge-coloring with no restrictions, whereas, it can be a proper edge-coloring or a strong edge-coloring.

In Chapter 2, we study the minimum color degree condition forcing vertex-disjoint rainbow triangles in edge-colored graphs. Let G be an edge-colored graph of order n. In 2013, Li [122] showed that if $\delta^c(G) \geq (n+1)/2$, then G contains a rainbow triangle and a rainbow $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ shows that the lower bound $(n+1)/2$ is best possible. Motivated by this result, we prove that if $n \geq 20$ and $\delta^c(G) \geq (n+2)/2$, then G contains two vertex-disjoint rainbow triangles and the lower bound $(n+2)/2$ is also best possible. A classical result of Dirac [59] asserts that for all positive integers n and k with $n \geq 3k$, every graph of order n with minimum degree at least $(n+k)/2$ contains k vertex-disjoint triangles. As an edge-colored version, we conjecture that if $\delta^c(G) \geq (n+k)/2$, then

G contains k vertex-disjoint rainbow triangles. If this conjecture holds, then the lower bound $(n+k)/2$ is best possible. To provide evidence for the conjecture, we show that for any integer $k \geq 2$, if $n \geq 16k-12$ and $\delta^c(G) \geq n/2+k-1$, then G contains k vertex-disjoint rainbow triangles.

In Chapter 3, we mainly consider the relation between the order of maximum properly colored tree in edge-colored graph G and the minimum color degree $\delta^c(G)$ of G. We obtain that for an edge-colored connected graph G, the order of maximum properly colored tree is at least $\min\{|G|, 2\delta^c(G)\}$, which generalizes a result of Cheng, Kano and Wang [39] about the minimum color degree condition forcing properly colored spanning trees. Moreover, the lower bound $2\delta^c(G)$ in our result is best possible and we characterize all extremal graphs G with the maximum properly colored tree of order $2\delta^c(G) \neq |G|$. As a corollary, we obtain a tight minimum color degree condition for the existence of properly colored spanning forest with at most k components. Furthermore, we characterize all extremal graphs for a result of Fujita and Magnant [74] about the minimum color degree condition for edge-colored graphs to be properly connected.

In Chapter 4, we research the minimum color degree condition guaranteeing the existence of properly colored 2-factors in edge-colored graphs. We derive that for any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N(\varepsilon, t)$ such that every edge-colored graph G with $\delta^c(G) \geq (2/3+\varepsilon)|G|$ and $|G| > N(\varepsilon, t)$ contains every properly colored 2-factor with exactly t components, which generalizes a result of Lo [134] about the minimum color degree condition for the existence of properly colored cycles of any length. A classical result of Moon and Moser [140] states that every balanced bipartite graph G with minimum degree $\delta(G) \geq |G|/4$ is Hamiltonian. Inspired by this result, we study a bipartite version of the aforementioned result and prove that every edge-colored balanced bipartite graph G with $\delta^c(G) \geq |G|/3+1$ contains a properly colored 2-factor, where the lower bound $|G|/3+1$ is best possible.

In Chapter 5, we study compatible factors in graphs with incompatibility systems. The notion of incompatibility system was firstly introduced by Krivelevich, Lee and Sudakov [116], which can be viewed a quantitative measure of robustness of graph properties. Recently, there has been an increasing interest in studying robustness of graph properties, aiming to strengthen classical results in extremal graph theory and probabilistic combinatorics. Given a graph H, finding sufficient and extremal condition for the existence of an H-factor is one of the most important topic in extremal graph

theory. A fundamental result of Alon and Yuster [8] shows that if the host graph G with n vertices has minimum degree $\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) + o(n)$, then G contains an H-factor, where $\chi(H)$ is the chromatic number of H. We study the robust version of Alon–Yuster's result with respect to the incompatibility system and prove that for any $\alpha > 0$ and any graph H with h vertices, there exists a constant $\mu > 0$ such that for any sufficiently large n with $n \in h\mathbb{N}$, if G is an n-vertex graph with $\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \alpha\right) n$ and $\mathcal{F}$ is a μn-bounded incompatibility system over G, then there exists a compatible H-factor in G, where the error term of αn cannot be omitted.

We conclude this thesis in Chapter 6 and give some prospects.

Keywords: Edge-colored graph; Rainbow subgraph; Properly colored subgraph; Incompatibility system; Compatible subgraph.

Contents

Chapter 1

Introduction

A graph is one of the very basic combinatorial structures, and it is a way to formally represent a collection interconnected objects, such as a communication network. The paper written by Leonhard Euler on the Seven Bridges of Königsberg and published in 1736 is regarded as the first paper in the history of graph theory [19]. As it continues to develop, graph theory has many branches, including graph coloring, extremal graph theory, algebraic graph theory, topological graph theory, probabilistic graph theory and so on. Since graphs can be used to model many types of relations and processes in physical, biological, social and information systems, graph theory has wide applications in real-world systems. Meanwhile, it also has applications in other areas of mathematics, such as group theory and number theory.

In this thesis, we mainly study extremal problems in edge-colored graphs, which can be regarded as a generalization of graph coloring and extremal graph theory. For the convenience of the description of our research topic's background, we first give some terminology and notation in the first section of this chapter. Then we will describe background in detail, including motivations, known results and so on. In the last section, we will show contributions and outline of this thesis.

1.1 Terminology and notation

For terminology and notation not mentioned here, we will given at the beginning of the respective chapters or refer readers to [56].

1.1.1 Graphs and digraphs

Graph and digraph

A *graph* G is an ordered pair $(V(G), E(G))$ with a nonempty set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, where $E(G)$ is made up of some unordered pairs of (not necessarily distinct) vertices. A *digraph* or *directed graph* D is an ordered pair $(V(D), A(D))$ consisting of a nonempty set $V(D)$ of *vertices* and a set $A(D)$ of *arcs*, where $A(D)$ is made up of some ordered pairs of (not necessarily distinct) vertices.

Finite and simple graph (digraph)

A graph (digraph) is *finite* if both its vertex set and edge set (arc set) are finite. A graph G is *simple* if $E(G)$ is a collection of some distinct 2-subsets of $V(G)$. A digraph D is *simple* if $A(D)$ is a collection of some distinct ordered pairs of $V(D)$.

Note that unless otherwise stated, all graphs and digraphs considered in this thesis are finite and simple.

Order and size

The *order* of a graph G (digraph D) refers to the cardinality of $V(G)$ ($V(D)$), while the *size* refers to the cardinality of $E(G)$ ($A(D)$).

Adjacent and incident

Let G be a graph and $u, v \in V(G)$. We say that u, v are *adjacent* if the 2-set $\{u, v\} \in E(G)$. Let $e \in E(G)$ with $e = \{u, v\}$. Then we say that u, v are two *ends* of e and u, v are *incident* with e, respectively.

Degree

Let G be a graph and $u, v \in V(G)$. If u, v are adjacent, then u is called a *neighbor* of v and vice versa. For any vertex $v \in V(G)$, we use $N_G(v)$ to denote the set of all neighbors of v and call $N_G(v)$ the *neighborhood* of v. The cardinality of $N_G(v)$ is the *degree* of v, denoted by $d_G(v)$, i.e. $d_G(v) = |N_G(v)|$. Denote by $\delta(G)$ and $\Delta(G)$ the *minimum degree* and *maximum degree* of G, respectively.

Outdegree and indegree

Let D be a graph and $u, v \in V(D)$. If $(u, v) \in A(D)$, then we say that v is an *outneighbor* of u and u is an *inneighbor* of v. For any vertex $v \in V(D)$, let $N_D^+(v)$ and $N_D^-(v)$ be its *outneighborhood* and *inneighborhood*, i.e. the set of outneighbors and the set of inneighbors of v, respectively. Let $d_D^+(v) = |N_D^+(v)|$ and $d_D^-(v) = |N_D^-(v)|$ and call $d_D^+(v)$ and $d_D^-(v)$ *outdegree* and *indegree* of v, respectively. Denote by $\delta^+(D)$ and $\delta^-(D)$ the *minimum outdegree* and *minimum indegree*, respectively. Let $\delta(D) =$

$\min\{\delta^+(D), \delta^-(D)\}$ and call $\delta(D)$ the *minimum semi-degree* of D. Denote by $\Delta^+(D)$ and $\Delta^-(D)$ the *maximum outdegree* and *maximum indegree*, respectively.

Subgraph

Let G, H be two graphs. We say H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if H is a subgraph of G and H contains all the edges $uv \in E(G)$ with $u, v \in V(H)$, then H is an *induced* subgraph of G. If H is a subgraph of G and $V(H) = V(G)$, then H is a *spanning* subgraph of G. We say G is *F-free* if G does not contain F as a subgraph. Let H_1, H_2 be two subgraphs of G. If $V(H_1) \cap V(H_2) = \emptyset$, then we say H_1 and H_2 are *vertex-disjoint*. If $E(H_1) \cap E(H_2) = \emptyset$, then we say H_1 and H_2 are *edge-disjoint*. Let $V' \subset V(G)$ and $E' \subset E(G)$. We use $G - V'$ to denote the subgraph induced by $V(G) \setminus V'$ and use $G \setminus E'$ to denote the subgraph of G containing the same vertices as G but with all the elements of E' removed. In particular, if $V' = \{v\}$, then we write $G - v$ for simplicity. If $E' = \{e\}$, then $G \setminus \{e\}$ will be replaced with $G \setminus e$.

Decomposition

Let G be a graph and $H_1, \ldots, H_t$ be subgraphs of G. We say that $H_1, \ldots, H_t$ *decompose* G if $H_1, \ldots, H_t$ are edge-disjoint subgraphs of G covering all the edges of G.

Walk, trail, path and cycle

Let G be an n-vertex graph. A *walk* in G is defined as a sequence of alternating vertices and edges such as $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$, where each $e_i = \{v_{i-1}, v_i\}$. The *length* of this walk is k. A walk is considered to be *closed* if the starting vertex is the same as the ending vertex, that is $v_0 = v_k$. A walk is considered *open* otherwise. A *trail* is defined as a walk with no repeated edges. A *path* is defined as an open trail with no repeated vertices. A *cycle* is defined as a closed trail where no other vertices are repeated apart from the starting/ending vertex. We usually use P_k and C_k to denote a path of length of $k-1$ and a cycle of length k, respectively. A path (cycle) is called a *Hamilton path* (*Hamilton cycle*) if it visits each vertex of G exactly once. A cycle of length of 3 is called a *triangle*.

Directed path and directed cycle

A *directed path* in a digraph D is a sequence of alternating vertices (no repeated) and arcs such as $v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k$, where each $a_i = (v_{i-1}, v_i)$. A *directed cycle* is a sequence of alternating vertices (no repeated, except the starting/ending vertex) and arcs such as $v_0, a_1, v_1, a_2, v_2, \ldots, v_{k-1}, a_k, v_0$, where $a_i = (v_{i-1}, v_i)$ for each $1 \leq i \leq k-1$

and $a_k = (v_{k-1}, v_0)$. The *length* of a directed path (directed cycle) is the number of arcs appearing in the sequence.

Connected graph

Let G be a graph. We say G is *connected* if for every pair of vertices there is a path between the two vertices. Otherwise, G is *disconnected*. A *component* is a maximal connected subgraph of G. If for every pair of vertices in G there is a Hamilton path between the two vertices, then G is *Hamilton-connected*. If G has a Hamilton cycle, then G is a *Hamilton graph*.

Tree and forest

A *tree* is a connected graph that does not contain even a single cycle. A *rooted tree* is a tree in which one vertex has been designated the root. A *forest* is a disjoint collection of trees, i.e. each component of a forest is tree.

Complete graph and tournament

Let G be an n-vertex graph. We say G is a *complete graph* if any two vertices of it are adjacent, denoted by K_n. If we give every edge of G a direction, then we obtain a digraph and we call this digraph an *oriented graph* of G. A *tournament* refers to an oriented graph of complete graph.

Clique and independent set

Let G be a graph and K, I be two vertex subsets of G. We call K a *clique* of G if $G[K]$ is a complete graph. We call I an *independent set* if $G[I]$ has no edges.

Multipartite graph

Let k be a positive integer. A *k-partite graph* is a graph whose vertex set can be partitioned into k different independent sets, which are called *k parts* of the graph. When $k = 2$, these are the *bipartite graphs*. A k-partite graph is *balanced* if the k parts have same cardinality. A *complete* k-partite graph is a k-partite graph in which there is an edge between every pair of vertices from different independent sets. The *Turán graph*, denoted by $T(n, k)$, is a special case of complete k-partite graph in which each two independent sets differ in size by at most one vertex.

Regular graph and regular digraph

Let G be a graph and d be a positive integer. We say G is a *d-regular graph* if every vertex in G has degree d. A *matching* is a 1-regular graph. If G has a matching covering all vertices of G, then we call such a matching a *prefect matching*.

Let D be a digraph. We say D is a *d-regular digraph* if the outdegree and indegree

of every vertex in D both are d.

Factor

Let G be an n-vertex graph and k be an integer. A *k-factor* of G is a spanning subgraph where every vertex has degree k. In particular, a 1-factor of G is a perfect matching. A 2-factor of G is a collection of vertex-disjoint cycles covering all vertices of G. A connected 2-factor is a Hamilton cycle. Let H be an h-vertex graph. An *H-tiling* is a collection of vertex-disjoint copies of H in G. An *H-factor* is an H-tiling which covers all vertices of G. Note that $n \in h\mathbb{N}$ is a necessary condition for G containing an H-factor.

Power of a graph

Let G be a graph and k be an integer. The k-th power of G, denoted by G^k, is defined as the graph on the same vertex set whose edges join distinct vertices at distance at most k in G.

Vertex-coloring and edge-coloring

Let k be a positive integer. A *k-coloring* of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest value of k possible to obtain a k-coloring.

An *edge-coloring* c of a graph G is an assignment of colors to the edges of G. A *k-edge-coloring* of G is an edge-coloring using k colors overall, while a *local k-edge-coloring* of G is an edge-coloring using at most k colors at each vertex of G. An edge-coloring of G is *proper* if no two adjacent edges receive the same color. The *edge-chromatic number* of G, denoted by $\chi'(G)$, is the smallest value of k possible to obtain a proper k-edge-coloring. A *strong* edge-coloring is a proper edge-coloring where the edges at distance at most 2 receive distinct colors. An edge-coloring c of a graph is *(globally) g-bounded* if each color appears at most g times in the coloring, while c is *locally ℓ-bounded* if each color appears at most ℓ times at any given vertex (so c is a proper edge-coloring when $\ell = 1$).

1.1.2 Edge-colored graphs

Edge-colored graph

An *edge-colored graph* is a graph with an edge-coloring (not necessarily proper). Let G be an edge-colored graph. If the edge-coloring on G is g-bounded (locally ℓ-bounded),

then we usually call G a *g-bounded* (*locally ℓ-bounded*) edge-colored graph.

Monochromatic, properly colored, rainbow and properly connected

Given an edge-colored graph G, we say G is *monochromatic* if all edges of G have the same color, G is a *properly colored* graph if any two adjacent edges receive different colors, and G is a *rainbow* graph if all the edges receive pairwise different colors.

Properly connected and rainbow connected

Given an edge-colored graph G, we say G is *properly* (*rainbow*) *connected* if for any pair of vertices u and v, there exists a properly colored (rainbow) path between u and v. The *rainbow connection number* of a connected graph G, denoted $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected.

Color degree

For every vertex $v \in V(G)$, the *color degree* of v, denoted by $d^c_G(v)$, is the number of distinct colors appearing on the incident edges of v. The *minimum color degree* of G, denoted by $\delta^c(G)$, is the minimum $d^c_G(v)$ over all vertices $v \in V(G)$. By $\Delta^{mon}(v)$, we denote the maximum number of incident edges of v with the same color, and $\Delta^{mon}(G)$ is the maximum $\Delta^{mon}(v)$ over all vertices $v \in V(G)$. We call $\Delta^{mon}(G)$ the *maximum monochromatic color degree* of G. Note that if G is an edge-colored graph with $\Delta^{mon}(G) \leq \Delta$, then G is a locally Δ-bounded edge-colored graph, and vice versa. Let $\bar{\sigma}^c_2(G) = \min\{d^c_G(u) + d^c_G(v) : u, v \in V(G) \text{ and } uv \in E(G)\}$. We usually call $\bar{\sigma}^c_2(G)$ the *minimum color degree sum* of G.

1.1.3 Incompatibility systems

Incompatibility system

Let $G = (V, E)$ be a graph. An *incompatibility system* $\mathcal{F}$ over G is a family $\mathcal{F} = \{F_v\}_{v \in V}$ such that for every $v \in V$, F_v is a family of 2-subsets of edges incident with v, i.e. $F_v \subseteq \{\{e, e'\} \in \binom{E}{2} : e \cap e' = \{v\}\}$.

Compatible subgraph

For any two edges e and e', if there exists some vertex v such that $\{e, e'\} \in F_v$, then we say that e and e' are *incompatible* at v. Otherwise, they are *compatible*. A subgraph $H \subseteq G$ is *compatible* if all pairs of adjacent edges are compatible.

Δ-bounded incompatibility system

For a positive integer Δ, an incompatibility system $\mathcal{F}$ is *Δ-bounded* if for any vertex v and any edge e incident with v, there are at most Δ other edges incident with v that

are incompatible with e.

1.2 Backgrounds

Extremal graph theory lies at the intersection of extremal combinatorics and graph theory. The basic line of research in extremal graph theory is to determine sufficient and extremal condition for the existence of some desired structure. One of the most fundamental theorems is Turán's theorem [154], which states that every n-vertex graph does not contain K_{k+1} as a subgraph has at most as many edges as Turán graph $T(n,k)$. In 1946, Erdős and Stone [69] generalized Turán's theorem and bounded the number of edges in an H-free graph for a non-complete graph H. Another milestone in the study of extremal graph theory is Dirac's theorem [58], which asserts that every n-vertex graph all of whose vertices have degree at least $n/2$ has a Hamilton cycle. Note that the lower bound $n/2$ is best possible, as can be seen by the following two examples: first is a graph obtained by taking two vertex-disjoint complete graphs of order k and identifying one vertex from each of them, and second is the complete bipartite graph with parts of sizes k and $k-1$. Both graphs have $2k-1$ vertices and minimum degree $k-1$, but are not Hamiltonian. There are many other interesting branches of extremal graph theory, such as Ramsey theory [81].

There has also been much research on extremal problems in edge-colored graphs. An example is the canonical Ramsey theorem, proved by Erdős and Rado [65], a special case of which shows that any properly colored K_n produces a rainbow K_m, provided n is large relative to m. Ramsey's theorem is also a special case of the canonical Ramsey theorem and can be expressed in terms of edge-colored complete graphs. Ramsey's theorem states that there exists a least positive integer $R(r,s)$ for which every blue-red edge-colored complete graph on $R(r,s)$ vertices contains a blue clique on r vertices or a red clique on s vertices. There are also studies on the local variation of Ramsey problem, which is concerned only with the number of colors used on the edges incident to each vertex, instead of the total number of colors used. The local Ramsey problem asks for the smallest integer n such that any local k-edge-coloring of K_n contains a monochromatic subgraph isomorphic to a given graph H (see e.g. [83, 153]).

There is another interesting branch, called anti-Ramsey theory [75]. The anti-Ramsey problem is stated as follows: given a positive integer n and a graph H, the anti-Ramsey number $ar(K_n, H)$ is defined to be the minimum number of colors k such

that for any edge-coloring of K_n with exactly k colors, there exists a rainbow copy of H. The study of anti-Ramsey theory began with a paper by Erdős, Simonovits, and Sós [67] in 1975 (note that related ideas were studied even earlier in [66]). The anti-Ramsey number $ar(K_n, H)$ is closely related to the Turán number $ex(n, H)$, which is the maximum number of edges in a graph on n vertices with no subgraph isomorphic to H. The main result in [67] states that $ar(K_n, H) = \frac{n^2}{2}\left(1 - \frac{1}{\chi-1}\right)(1 + o(1))$, where $\chi = \min\{\chi(H \setminus e) : e \in E(H)\}$. Instead of forcing rainbow copies of a given graph H, one can consider forcing properly colored copies of H by using many colours, and study the threshold on the number of colors needed. This is thoroughly studied by Manoussakis, Spyratos, Tuza and Voigt in [137].

The local variation of anti-Ramsey problem was initially studied by Axenovich, Jiang and Tuza [9]. A k-edge-coloring of a complete graph K_n is called a *Gallai-coloring* if the edge-colored K_n contains no rainbow triangle. Ramsey-type problems in Gallai-colorings have also received considerable attention (see e.g. [84, 85, 132]). These works mainly focus on finding various monochromatic subgraphs in such colorings.

Moreover, the Turán-type problem has a generalization in edge-colored graphs, called rainbow Turán problem [53, 104]. For a fixed graph H, the rainbow Turán problem refers to determine the maximum number of edges in a properly colored graph on n vertices which does not contain a rainbow H. This maximum is the rainbow Turán number of H.

Recall that the rainbow connection number of a connected edge-colored graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. It was shown that computing the rainbow connection number of an arbitrary graph is NP-hard [32]. There has been much research on the bounds of $rc(G)$ with respect to the minimum degree, connectivity and so on [118, 130, 146, 147]. There are numerous other extremal problems in edge-colored graphs. For related surveys, we recommend [12, 75, 76, 100] and Chapter 16 of [13]. Whilst such problems are interesting in their own right, they also have applications in other areas.

Some classic problems can be transferred into extremal problems in edge-colored graphs. For example, finding directed cycles can be formulated as a special case of finding properly colored cycles. To see this, consider the following construction which was first introduced by Li [122] and also studied in [57]. Let D be an oriented graph of a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Define an edge-coloring τ of G by coloring the

edge v_iv_j with j for all arcs (v_i, v_j) in D. The resulting edge-colored graph, denoted by (D, τ), is called the *signature* of D (see Figure 1.1 for an example). Then the following two properties hold: (i) For every vertex $v \in V(G)$, $d^c_{(D,\tau)}(v) = d^+_D(v)$ if $d^-_D(v) = 0$, otherwise $d^c_{(D,\tau)}(v) = d^+_D(v) + 1$; (ii) A cycle in G is a directed cycle in D if and only if it is a properly colored cycle in (D, τ). Recall that the well-known Caccetta-Häggkvist conjecture [29] says that for all positive integers n, r with $n \geq r$, every digraph D of order n with $\delta^+(D) \geq \lceil n/r \rceil$ contains a directed cycle of length at most r. Hence, the study of Caccetta-Häggkvist conjecture in some sense can be transferred into the study of properly colored cycles in edge-colored graphs with minimum colore degree constraints.

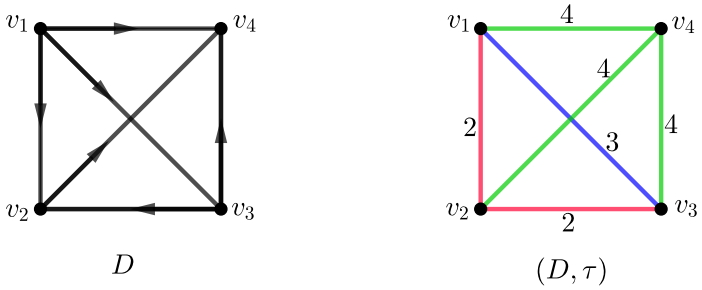


Figure 1.1: An illustration of the signature (D, τ) of an oriented graph D.

Besides a number of applications in graph theory and algorithms, some concepts and results in edge-colored graphs have also appeared in communication network [158], social science [42], biology [60, 61, 142] and so on. For example, edge-colored graphs can be used to model homogeneous faults in networks [158], study the order of chromosomes [60, 61] and DNA physical mapping [142]. In this thesis, we only focus on theoretical aspects and mainly study the existence of vertex-disjoint rainbow triangles, properly colored trees, properly-colored 2-factors and so on.

1.2.1 Edge-colored cycles

The study on the existence of properly colored or rainbow cycles in edge-colored graphs has a long history. In 1967, Gallai [78] characterized edge-colored complete graph K_n with no properly colored triangles (or rainbow triangles).

Theorem 1.1 (Gallai [78]). *For $n \geq 2$, if an edge-colored K_n has no properly colored triangles, then there exist at most two colors i and j and a partition of $V(G)$ into at least two parts such that for any two different parts, all edges between them are colored*

by the same color that is i or j.

In 1983, Grossman and Häggkvist [82] characterized edge-colored graphs with two colors but no properly colored cycles. Later, Yeo [160] extended the result to edge-colored graphs with any number of colors.

Theorem 1.2 (Grossman and Häggkvist [82], Yeo [160]). *Let G be an edge-colored graph with no properly colored cycles. Then there is a vertex $v \in V(G)$ such that no components of $G - v$ is joined to v with edges of more than one color.*

Long cycles in edge-colored complete graphs

In many classical problems about the existence of properly colored or rainbow cycles, the host graph is usually complete. For example, in 1976, Daykin [54] asked whether there exists a constant μ such that every edge-colored K_n with $\Delta^{mon}(K_n) \leq \mu n$ and $n \geq 3$ contains a properly colored Hamilton cycle. This question was answered independently by Bollobás and Erdős [21] with $\mu = 1/69$, and Chen and Daykin [35] with $\mu = 1/17$. In the same paper, Bollobás and Erdős proposed the following conjecture.

Conjecture 1.1 (Bollobás and Erdős [21]). *If $\Delta^{mon}(K_n) < \lfloor n/2 \rfloor$, then the edge-colored K_n contains a properly colored Hamilton cycle.*

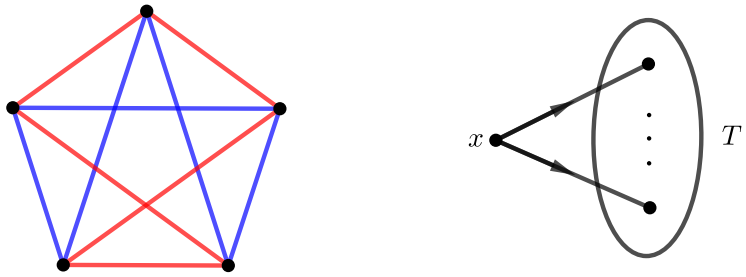
In 1979, Shearer [149] showed that $\Delta^{mon}(K_n) \leq n/7$ is sufficient. In 1997, Alon and Gutin [5] improved the bound to $(1 - 1/\sqrt{2} - o(1))n$. On the other hand, Li, Wang and Zhou [124] obtained that if $\Delta^{mon}(K_n) < \lfloor n/2 \rfloor$, then K_n contains a properly colored cycle of length at least $(n+2)/3 + 1$. In [135], Lo proved that Conjecture 1.1 is true asymptotically.

Theorem 1.3 (Lo [135]). *For any $\varepsilon > 0$, there exists an integer $N_0 = N_0(\varepsilon)$ such that every edge-colored K_n with $n \geq N_0$ and $\Delta^{mon}(K_n) \leq (1/2 - \varepsilon)n$ contains a properly colored Hamilton cycle.*

We now present three examples to show that the upper bound in Conjecture 1.1 is sharp. The first example was given by Bollobás and Erdős [21] for $n \equiv 1 \pmod 4$. The second and third examples were given by Lo [135] for even n.

Example 1.1 (Bollobás and Erdős [21]). *Let $n = 4k + 1$ and G be a $2k$-regular graph on n vertices. Hence the compliment $\bar{G}$ of G is also a $2k$-regular graph. Let edge-colored*

complete graph K_n be obtained by coloring all edges of G red and all edges of $\bar{G}$ blue. Note that $\Delta^{mon}(K_n) = 2k = \lfloor n/2 \rfloor$. However, the edge-colored K_n does not contain any properly colored Hamilton cycle C_n, since the edge-chromatic number of C_n is 3.



(i) $G \cup \bar{G}$ (ii) T_{2m}

Figure 1.2: (i) An illustration of Example 1.1 on 5 vertices, where G is the red 2-regular graph and $\bar{G}$ is the blue 2-regular graph. (ii) The tournament T_{2m} in Example 1.2, where T is the regular tournament on $2m-1$ vertices.

Example 1.2 (Lo [135]). *Let T_{2m} be a tournament on $2m$ vertices obtained from a regular tournament T on $2m-1$ vertices by adding a directed edge from a new vertex x to every $y \in V(T)$. Note that $\Delta^-(T_{2m}) = m$ and T_{2m} does not contain any directed Hamilton cycle. The signature (T_{2m}, τ) of T_{2m} is an edge-colored complete graph K_{2m} with $\Delta^{mon}(K_{2m}) = m$ and no properly colored Hamilton cycle.*

Example 1.3 (Lo [135]). *Let n be an even integer and $V(K_n) = \{x_1, \ldots, x_{n/2}, y_1, \ldots, y_{n/2}\}$. Set $X = \{x_1, \ldots, x_{n/2}\}$ and $Y = \{y_1, \ldots, y_{n/2}\}$. Let c be an edge-coloring of K_n such that*

(i) *$c(x_i y_j) = i$ for all $1 \le i, j \le n/2$;*

(ii) *$c(y_i y_j) = 1$ for all $1 \le i < j \le n/2$;*

(iii) *$K_n[X]$ is rainbow under c and does not contain any color in $\{1, \ldots, n/2\}$.*

Note that $\Delta^{mon}(K_n) = |Y| = n/2$. It can be proved that such an edge-colored K_n does not contain properly colored Hamilton cycles.

Short cycles in edge-colored complete graphs

For short cycles in edge-colored complete graphs, Erdős and Tuza [70] gave a minimum color degree condition for the existence of rainbow C_3, using a main idea of Theorem 1.1.

Theorem 1.4 (Erdős and Tuza [70]). *An edge-colored K_n admits a rainbow C_3 if the minimum color degree is at least* $\log_2 n + O(1)$.

Gyarfas and Simonyi [86] proved that each edge-colored complete graph K_n with $\Delta^{mon}(K_n) < 2n/5$ contains a rainbow C_3 and this bound is tight. As pointed out by Axenovich, Jiang and Tuza in [9], the lower bound in Theorem 1.4 is asymptotically best possible. In 2017, Fujita, Li and Zhang [73] proved the best possible bound.

Theorem 1.5 (Fujita, Li and Zhang [73]). *If the minimum color degree of K_n is larger than* $\log_2 n$, *then K_n contains a rainbow C_3.*

Axenovich, Jiang and Tuza [9] also proved that an edge-colored K_n admits a properly colored C_4 if the minimum color degree is at least 3 and the lower bound is tight. Furthermore, they gave a minimum color degree condition for the existence of a rainbow C_4.

Theorem 1.6 (Axenovich, Jiang and Tuza [9]). *Let n be a sufficiently large positive integer. Then every edge-colored K_n with minimum color degree at least $4n^{2/3}$ contains a rainbow C_4.*

In [74], Fujita and Magnant proposed the following conjecture in edge-colored complete graphs, which has also been received considerable attention.

Conjecture 1.2 (Fujita and Magnant [74]). *For an edge-colored complete graph K_n, if $\delta^c(K_n) \geq (n+1)/2$, then each vertex of K_n is contained in a properly colored cycle of length k, for all k with $3 \leq k \leq n$.*

In the same paper, they proved the following theorem to support their conjecture.

Theorem 1.7 (Fujita and Magnant [74]). *For an edge-colored complete graph K_n, if $\delta^c(K_n) \geq (n+1)/2$, then each vertex of K_n is contained in properly colored cycles of length 3 and 4. Moreover, if $n \geq 13$, then each vertex of K_n is contained in a properly colored cycle of length at least 5.*

In [125], Li, Broersma, Xu and Zhang proved that if $\delta^c(K_n) \geq (1/2+\varepsilon)n$, then each vertex of K_n is contained in a properly colored of length at least $\delta^c(G)$. By Theorem 1.3 and $\delta^c(K_n)+\Delta^{mon}(K_n) \leq n$, we know that if $\delta^c(K_n) \geq (1/2+\varepsilon)n$, then each vertex of K_n is contained in a properly colored Hamilton cycle. Very recently, Chen and Li [37] generalized Theorem 1.7 and proved the following result.

Theorem 1.8 (Chen and Li [37]). *For an edge-colored complete graph K_n, if $\delta^c(G) \geq (n+k)/2$, then every vertex of G is contained in at least k rainbow triangles.*

As we know, C_4 is the shortest possible cycle in a bipartite graph. Fujita, Li and Zhang [73] and Čada, Ozeki and Szabó [31] independently considered bipartite analogous of Theorem 1.1, i.e. characterized the structures of edge-colored complete bipartite graphs without properly colored C_4. As a corollary, Fujita, Li and Zhang [73] showed that an edge-colored complete bipartite graph contains a properly colore C_4 if the minimum color degree is at least 3. As mentioned above, the minimum color degree forcing a properly colored C_4 in an edge-colored complete graph is also 3. Motivated by these results, Li, Broersma, Yokota and Yoshimoto [126] recently gave a full characterization of edge-colored complete graphs without properly colored even cycles.

Existence of properly colored cycles and rainbow cycles in edge-colored graphs

There has also been much research on finding properly colored or rainbow cycles in relatively sparse graphs (instead of complete graphs, but still dense). It is well-known that if a graph has minimum degree at least two, then we are able to greedily find a cycle. However, establishing color degree conditions forcing properly colored or rainbow cycles in edge-colored graphs seems essentially more complicated. Indeed, Wang and Li [156] proved that for any positive integer i, there exists an edge-colored graph G_i with $\delta^c(G_i) \geq i$, but G_i contains no properly colored cycle. The graph G_i given in [156] is constructed inductively (see Figure 1.3) and G_1 has only one edge with a color. Recently, Fujita, Li and Zhang [73] obtained a tight minimum color degree condition guaranteeing the existence of a properly colored cycle.

Theorem 1.9 (Fujita, Li and Zhang [73]). *For all positive integers n and d with $d! \sum_{i=0}^{d} 1/i! \geq n+1$, every edge-colored graph G of order n with $\delta^c(G) \geq d$ contains a properly colored cycle.*

For the existence of rainbow cycles, Li and Wang [123] showed that every edge-colored graph G of order n with $\delta^c(G) \geq (n+1)/2$ contains a rainbow cycle. Furthermore, they [123] conjectured that if $\delta^c(G) \geq (n+1)/2$, then G contains a rainbow C_3. Later on, Li [122] proved this conjecture. A rainbow $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ shows that the bound $(n+1)/2$ is best possible.

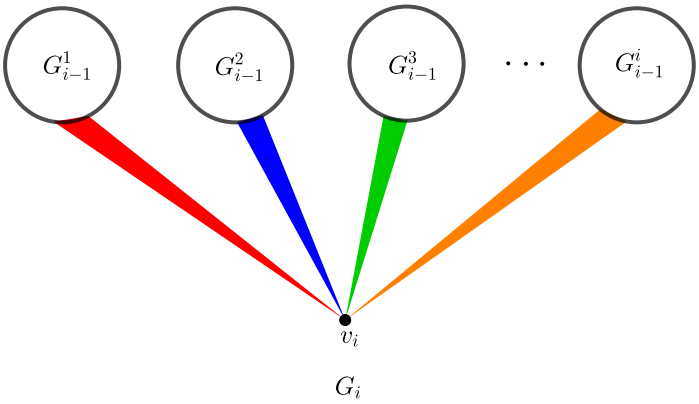


Figure 1.3: The construction of G_i. Note that each G^j_{i-1} $(1 \leq j \leq i)$ is a copy of G_{i-1} and all edges incident with v_i are colored with new colors.

Theorem 1.10 (Li [122]). *Let G be an n-vertex edge-colored graph. If $\delta^c(G) \geq (n+1)/2$, then G contains a rainbow C_3.*

Short cycles in edge-colored graphs

During the past decades, establishing sufficient conditions for the existence of properly colored or rainbow cycles of *certain lengths* has received considerable attention. In [121], Li, Ning, Xu and Zhang obtained a sufficient condition for the existence of rainbow C_3, which implies Theorem 1.10.

Theorem 1.11 (Li, Ning, Xu and Zhang [121]). *Let G be an n-vertex edge-colored graph. If $\Sigma_{v\in V(G)}d^c(v) \geq n(n+1)/2$, then G contains a rainbow C_3.*

In the same paper, they [121] also proved that if $e(G)+c(G) \geq n(n+1)/2$, then G contains a rainbow C_3, where $e(G)$ is the size of G and $c(G)$ is the number of colors appearing on $E(G)$. Later, Fujita, Ning, Xu and Zhang [77] characterized all the graphs which satisfy $e(G)+c(G) = n(n+1)/2-1$ but contain no rainbow triangles. Moreover, they [77] posed a problem: determine an integer valued function $f(k)$ as small as possible, such that if $e(G)+c(G) = n(n+1)/2+f(k)$ and n is sufficiently large, then G contains at least k rainbow C_3's. This problem was solved by Ehard and Mohr in [62]: there are at least k rainbow C_3's in an edge-colored graph G with $e(G)+c(G) = n(n+1)/2+k-1$.

Let $CN_G(v)$ be the set of colors assigned to the edges from v to $V(G)\setminus\{v\}$. Broersma, Li, Woeginger and Zhang [26] derived that if G is an edge-colored graph of order $n \geq 4$ such that $|CN(u)\cup CN(v)| \geq n-1$ for every pair of vertices u and v in $V(G)$,

then G contains a rainbow C_3 or a rainbow C_4. Li, Ning and Zhang [128] considered the minimum color degree sum $\bar{\sigma}_2^c(G)$ condition guaranteeing the existence of rainbow C_3 and proved that if $E(G) \neq \emptyset$ and $\bar{\sigma}_2^c(G) \geq n+1$, then G contains a rainbow C_3.

Very recently, Li, Ning, Shi and Zhang [129] studied the counting version of Theorem 1.10 and obtained the following result. Let $rt(G)$ be the number of rainbow triangles in an edge-colored graph G.

Theorem 1.12 (Li, Ning, Shi and Zhang [129]). *Let G be an edge-colored graph on n vertices. Then*

$$rt(G) \geq \delta^c(G)(2\delta^c(G) - n)n/6.$$

In particular, if $\delta^c(G) > cn$ for $c > 1/2$, then

$$rt(G) \geq c(2c-1)n^3/6.$$

In extremal graph theory, a classical result of Dirac [59] asserts that for all positive integers n and k with $n \geq 3k$, every graph of order n with minimum degree at least $(n+k)/2$ contains k vertex-disjoint triangles. It is natural to consider edge-colored version of this result, i.e. determine the minimum color degree condition which guarantees k vertex-disjoint rainbow triangles, where $k \geq 2$. Therefore the first open case is $k = 2$, and we almost resolve the case as follows.

Theorem 1.13 (Hu, Li and Yang [95]). *For all integers $n \geq 20$, every graph G of order n with $\delta^c(G) \geq (n+2)/2$ contains two vertex-disjoint rainbow triangles.*

Furthermore, we propose the following conjecture.

Conjecture 1.3 (Hu, Li and Yang [95]). *For all positive integers n and k with $n \geq 3k$, every edge-colored graph G of order n with $\delta^c(G) \geq (n+k)/2$ contains k vertex-disjoint rainbow triangles.*

By constructing Example 1.4, we know that if this conjecture holds, then the lower bound is best possible.

Example 1.4. *Let H be a properly colored graph where $V(H) = X_1 \cup X_2 \cup Y$ with $|X_1| = \lceil (n-k+1)/2 \rceil$, $|X_2| = \lfloor (n-k+1)/2 \rfloor$ and $|Y| = k-1$. $H[X_1 \cup X_2]$ is a balanced complete bipartite graph and $H[Y]$ is empty. In addition, each $y \in Y$ is complete to $X_1 \cup X_2$.*

For edge-colored C_4's, Xu, Magnant and Zhang [159] considered the $e(G)+c(G)$ and $\Sigma_{v\in V(G)}d^c(v)$ conditions for the existence of properly colored C_4, respectively, and characterized extremal graphs.

Theorem 1.14 (Xu, Magnant and Zhang [159]). *Let G be an edge-colored graph on n vertices. Then the following hold.*

(i) *If $e(G)+c(G)\geq n(n+1)/2+1$, then G contains a properly colored C_4.*

(ii) *If $n\geq 4$ and $\Sigma_{v\in V(G)}d^c(v)\geq n(n+1)/2+1$, then G contains a properly colored C_4.*

In [122], Li showed that every bipartite graph G with n vertices in each part and $\delta^c(G)>3n/5+1$ contains a rainbow C_4. More recently, we resolve this problem asymptotically by proving the following stronger result.

Theorem 1.15 (Ding, Hu, Wang and Yang [57]). *Let $G=(V_1,V_2;E)$ be an edge-colored bipartite graph with $|V_i|=n_i$, $\delta_i^c=\min_{v\in V_i}d^c(v)$ for $i\in\{1,2\}$. If $\delta_1^c\delta_2^c>n_1n_2/9+8n_1\sqrt{n_2}+8n_2\sqrt{n_1}$, then G contains a rainbow C_4. Therefore, if $\delta_1^c>n_2/3+24\sqrt{n_2}$ and $\delta_2^c>n_1/3+24\sqrt{n_1}$, then G contains a rainbow C_4. Moreover, this is asymptotically best possible by considering the signature of the $n/3$-blow-up of a directed C_6.*

As a corollary, we also obtained the following result about properly colored C_4 and rainbow C_4 in edge-colored graphs. The lower bounds are also asymptotically best possible by considering the signature of the $n/3$-blow-up of a directed C_3 (see Figure 1.4).

Theorem 1.16 (Ding, Hu, Wang and Yang [57]). *Let G be an edge-colored graph on n vertices. Then the following hold.*

(i) *If $\delta^c(G)>n/3+2\sqrt{n}+1$, then G contains a properly colored C_4.*

(ii) *If $\delta^c(G)>n/3+24\sqrt{n}$, then G contains a rainbow C_4.*

If the host graph G is an edge-colored triangle-free graph on n vertices, then $\delta^c(G)>n/3+1$ forces a rainbow C_4 by a result of Čada, Kaneko, Ryjáček and Yoshimoto [30]. We also asymptotically resolve this problem.

Theorem 1.17 (Ding, Hu, Wang and Yang [57]). *Let G be an edge-colored triangle-free graph on n vertices. If $\delta^c(G)>n/5+3\sqrt{n}$, then G contains a rainbow C_4, and*

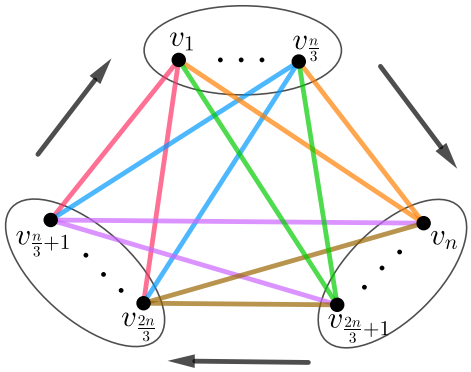


Figure 1.4: The signature of the $n/3$-blow-up of a directed C_3.

this is asymptotically best possible by considering the signature of the $n/5$-blow-up of a directed C_5.

It is worth mentioning that Czygrinow, Molla, Nagle and Oursler [51] recently proved an analogue of Theorem 1.10 for ℓ-cycles C_ℓ of fixed arbitrary length.

Theorem 1.18 (Czygrinow, Molla, Nagle and Oursler [51])**.** *For every integer $\ell \geq 3$, every edge-colored graph G on $n \geq n_0(\ell)$ many vertices satisfying $\delta^c(G) \geq (n+1)/2$ admits a rainbow ℓ-cycle C_ℓ.*

A rainbow $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ also establishes that Theorem 1.18 is best possible for all odd integers ℓ. For even integers $\ell \geq 4$, the authors earlier proved in [50] a stronger form of Theorem 1.18.

Theorem 1.19 (Czygrinow, Molla, Nagle and Oursler [50])**.** *There exists an absolute constant $\alpha > 0$ so that, for every even integer $\ell \geq 4$, every edge-colored graph G on $n \geq N_0(\ell)$ many vertices satisfying*

$$\delta^c(G) \geq \begin{cases} 1/3 - \alpha & \text{if } \ell \equiv 0 \mod 3 \\ (n+5)/3 & \text{if } \ell \not\equiv 0 \mod 3 \end{cases}$$

admits a rainbow ℓ-cycle C_ℓ.

Theorem 1.19 is best possible for $\ell \not\equiv 0 \mod 3$. The proof of Theorem 1.19 uses the stability method.

Long cycles in edge-colored graphs

For properly colored long cycles of desired length, Li and Wang [156] proved that every edge-colored graph G contains a properly colored path of length $2\delta^c(G)$ or a properly colored cycle of length at least $2\delta^c(G)/3$. Later on, Lo [133] improved their result by showing that G contains a properly colored path of length $2\delta^c(G)$ or a properly colored cycle of length at least $\delta^c(G)+1$. In the same paper, Lo [133] showed that every connected edge-colored graph G contains a properly colored Hamiltonian cycle or a properly colored path of length at least $6\delta^c(G)/5-1$.

Recall that Dirac's theorem states that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamilton cycle. It is natural to ask the minimum color degree condition for the existence of a properly colored Hamilton cycle in an edge-colored graph G of order n. Does $\delta^c(G) \geq n/2$ suffice? Unfortunately, the answer is no. Fujita and Magnant [74] showed that there exists an edge-colored K_{2m} with $\delta^c(K_{2m}) = m$ which has no properly colored Hamilton cycle. Since the longest path in a graph has length at most $n-1$, the result just mentioned above of Lo [133] implies that if $\delta^c(G) > 5n/6$, then G contains a properly colored Hamiltonian cycle. Furthermore, Lo [133] also proposed the following conjecture.

Conjecture 1.4 (Lo [133])**.** *Every connected edge-colored graph G contains a properly colored Hamilton cycle or a properly colored path of length $3\delta^c(G)/2$.*

If the conjecture is true, then every edge-colored graph G with $\delta^c(G) \geq 2|G|/3$ contains a properly colored Hamilton cycle. The bound on $\delta^c(G)$ is asymptotically best possible, i.e. there exists an edge-colored graph G on n vertices with $\delta^c(G) = \lceil 2n/3 \rceil - 1$, which does not contain a properly colored Hamilton cycle (see Figure 1.5). In [134], Lo proved that every edge-colored graph G on n vertices with $\delta^c(G) \geq (2/3+\varepsilon)n$ contains a properly colored cycle of length ℓ for all $3 \leq \ell \leq n$ provided $\varepsilon > 0$ and n is large enough. Using the absorbing technique and stability method, Lo [136] recently obtained the tight bound.

Theorem 1.20 (Lo [136])**.** *For sufficiently large n, every edge-colored graph G on n vertices with $\delta^c(G) \geq 2n/3$ contains a properly colored Hamilton cycle.*

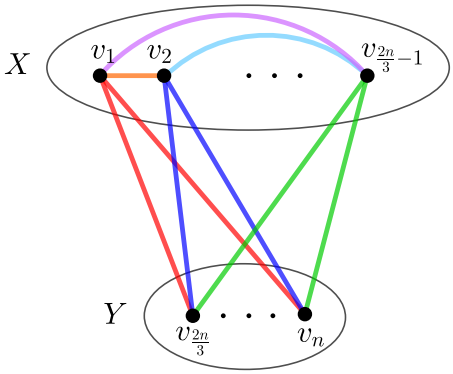


Figure 1.5: An example G shows the sharpness of the lower bound in Theorem 1.20, where $G[X]$ is a rainbow complete graph of order $2n/3-1$, $G[Y]$ is an independent set of size $n/3+1$ and edges between X and Y are colored with new colors. The minimum color degree of G is $2n/3-1$ but G has no properly colored Hamilton cycles since $|X|<2|Y|$.

1.2.2 Edge-colored trees and 2-factors

In addition to cycles, other types of subgraphs, especially spanning subgraphs, have also been well studied in edge-colored graphs, such as spanning trees and 2-factors.

Decompositions into rainbow spanning trees

As we know, there is a basic fact that the complete graph K_n can be decomposed into $n/2$ spanning trees. In 1996, Brualdi and Hollingsworth [27] proposed the following conjecture, which can be regarded as an edge-colored version of the above fact.

Conjecture 1.5 (Brualdi and Hollingsworth [27])**.** *For all $n>4$, every properly colored K_n with $n-1$ colors can be decomposed into $n/2$ rainbow spanning trees.*

Every properly colored K_n clearly contains a rainbow spanning tree, for example the star $K_{1,n-1}$ at any vertex. In the same paper, Brualdi and Hollingsworth [27] showed that there are at least two edge-disjoint rainbow spanning trees in every properly colored K_n with $n-1$ colors. Shortly afterwards, Krussel, Marshall and Verrall [119] proved that one can find at least three edge-disjoint rainbow spanning trees in such a K_n. Later, Horn [92] significantly improved on this by finding $\Omega(n)$ edge-disjoint rainbow spanning trees. Very recently, Montgomery, Pokrovskiy, and Sudakov [138] proved Conjecture 1.5 approximately by showing that one can guarantee $(1-o(1))n/2$ edge-disjoint rainbow spanning trees.

There are two generalizations of Conjecture 1.5. One requires the trees in the

decomposition to be isomorphic (see Conjecture 1.6). The other allows that the properly colored K_n can use any number of colors (see Conjecture 1.7).

Conjecture 1.6 (Constantine [44, 45]). *For all $n > 4$, every properly colored K_n with $n-1$ colors can be decomposed into $n/2$ isomorphic rainbow spanning trees.*

Conjecture 1.7 (Kaneko, Kano and Suzuki [99]). *For all $n > 4$, every properly colored K_n can be decomposed into $n/2$ rainbow spanning trees.*

For Conjecture 1.6, much less was known than for Conjecture 1.5. In [72], Fu and Lo showed that three isomorphic rainbow spanning trees can be guaranteed. In [143], Pokrovskiy and Sudakov derived that one can find $10^{-6}n$ edge-disjoint rainbow spanning trees all isomorphic to a *t-spider*, which is a tree obtained from a star by subdividing t of its edges once. Montgomery, Pokrovskiy, and Sudakov [138], and independently, Kim, Kühn, Kupavskii and Osthus [105], proved a weak asymptotic version of Conjecture 1.6 by showing that there are $(1-o(1))n/2$ edge-disjoint rainbow paths each of length $(1-o(1))n$.

More recently, Glock, Kuhn, Montgomery and Osthus[80] proved the following theorem, which implies Conjectures 1.5 and 1.6 for large n. This is the first general exact rainbow decomposition result for spanning subgraphs, where each subgraph in the decomposition has to use all the colors.

Theorem 1.21 (Glock, Kuhn, Montgomery and Osthus[80]). *For all sufficiently large n, there exists a tree T on n vertices such that for any properly colored K_n with $n-1$ colors, there exists a decomposition into rainbow subgraphs each isomorphic to T.*

For Conjecture 1.7, we first note that any proper edge-coloring is $n/2$-bounded, that is, every color appears on at most $n/2$ edges. Under the weaker assumption that the coloring is $n/2$-bounded, Akbari and Alipour [3] showed that one can guarantee two edge-disjoint rainbow spanning trees, and this was significantly improved by Carraher, Hartke, and Horn [33] who showed that $\Omega(n/\log n)$ such trees exist. For proper edge-colorings, a linear number of rainbow spanning trees was independently obtained by Pokrovskiy and Sudakov [143] and by Balogh, Liu and Montgomery [11], where in the former work, the trees are even isomorphic. Finally, the aforementioned result from [138] on Conjecture 1.5 also applies to proper edge-colorings, thus proving Conjecture 1.7 approximately.

Conditions forcing properly colored or rainbow spanning trees

In 2019, Borozan, Fernandez de La Vega, Manoussakis, Martinhon, Muthu, Pham and Saad [24] proved the following theorem. Hence, it is interesting to find sufficient conditions forcing properly colored spanning trees.

Theorem 1.22 (Borozan, Fernandez de La Vega, Manoussakis, Martinhon, Muthu, Pham and Saad [24])**.** *Finding a properly colored spanning tree in an edge-colored graph G is NP-complete.*

It is to see that "G is properly connected" is a necessary condition for "G has a properly colored spanning tree". Fujita and Magnant [74] derived a tight minimum color degree condition for edge-colored graphs to be properly connected.

Theorem 1.23 (Fujita and Magnant [74])**.** *Let G be an edge-colored connected graph. If $\delta^c(G) \geq |G|/2$, then G is properly connected.*

In [39], Cheng, Kano and Wang proved that under the same minimum color degree condition, there also exists a properly colored spanning tree and the lower bound is sharp (see Figure 1.6).

Theorem 1.24 (Cheng, Kano and Wang [39])**.** *Let G be an edge-colored connected graph. If $\delta^c(G) \geq |G|/2$, then G has a properly colored spanning tree.*

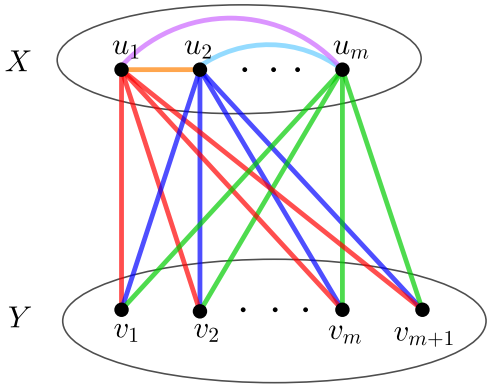


Figure 1.6: An example G shows the sharpness of lower bound in Theorem 1.24, where $G[X]$ is a rainbow complete graph of order m, $G[Y]$ is an independent set of size $m+1$ and edges between X and Y are colored with new colors. The minimum color degree of G is $m = (|G|-1)/2$ but G has no properly colored spanning tree since $|X| < |Y|$.

Moreover, they [39] derived a sufficient condition for the existence of rainbow spanning trees, where the lower bound of the minimum color degree is also sharp.

Theorem 1.25 (Cheng, Kano and Wang [39]). *Let G be an edge-colored connected graph. If $\delta^c(G) \geq |G|/2$ and for each color c, the set of edges colored with c forms a star, then G has a rainbow spanning tree.*

We consider maximum properly colored trees in edge-colored graphs and improve Theorems 1.24 and 1.25 as follows.

Theorem 1.26 (Hu, Li and Maezawa [93]). *Let G be an edge-colored connected graph.*

(i) *The order of maximum properly colored tree is at least* $\min\{|G|, 2\delta^c(G)\}$.

(ii) *Suppose that for each color c, the set of edges colored with c forms a star. Then the order of maximum rainbow tree is at least* $\min\{|G|, 2\delta^c(G)\}$.

The lower bounds in our result are best in the sense that we can not replace $2\delta^c(G)$ by $2\delta^c(G)+1$. In particular, we determine the extremal graph classes for Theorem 1.26 (i) and (ii).

Very recently, Kano and Tsugaki [103] showed that the minimum color degree condition given in Theorems 1.24 and 1.25 can be weakened for edge-colored bipartite graphs. Namely, they proved the following results.

Theorem 1.27 (Kano and Tsugaki [103]). *Let G be an edge-colored connected bipartite graph.*

(i) *If $\delta^c(G) \geq |G|/3 + 1$, then G has a properly colored spanning tree.*

(ii) *If $\delta^c(G) \geq |G|/3 + 1$ and for each color c, the set of edges colored with c forms a star, then G has a rainbow spanning tree.*

In 2020, Kano, Maezawa, Ota, Tsugaki and Yashima [101] obtained a minimum color degree sum condition for the existence of properly colored spanning tree and rainbow spanning tree, which generalized Theorems 1.24 and 1.25.

Theorem 1.28 (Kano, Maezawa, Ota, Tsugaki and Yashima [101]). *Let G be an edge-colored connected graph.*

(i) *If $\bar{\sigma}_2^c(G) \geq |G|$, then G has a properly colored spanning tree.*

(ii) *If $\bar{\sigma}_2^c(G) \geq |G|$ and for each color c, the set of edges colored with c forms a star, then G has a rainbow spanning tree.*

A Hamilton path is a special spanning tree. There are some results about properly colored Hamilton paths in edge-colored complete graphs. In 1998, Barr [15] proved that if an edge-colored K_n does not contain any monochromatic triangle, then there is a properly colored Hamilton path. Bang-Jensen, Gutin and Yeo [14] showed that an edge-colored K_n containing a properly colored 2-factor is also a sufficient condition.

Theorem 1.29 (Bang-Jensen, Gutin and Yeo [14]). *If an edge-colored K_n contains a properly colored 2-factor, then K_n contains a properly colored Hamilton path.*

This result was later improved by Feng, Giesen, Guo, Gutin, Jensen and Rafiey [71]. A graph G is defined to be a *1-path-cycle* if G is a vertex-disjoint union of at most one path and a number of cycles. Note that a spanning 1-path-cycle without any cycles is a Hamilton path, and a spanning 1-path-cycle without a path is a 2-factor.

Theorem 1.30 (Feng, Giesen, Guo, Gutin, Jensen and Rafiey [71]). *An edge-colored complete graph K_n contains a properly colored Hamilton path if and only if it contains a spanning properly colored 1-path-cycle.*

Properly colored 2-factors

Since every component of a 2-factor is a cycle and a connected 2-factor is a Hamilton cycle, the study on properly colored 2-factors could help to solve problems about properly colored cycles in some sense. Häggkvist [87] announced a non-trivial proof of the fact that every edge-colored complete graph K_n with $\Delta^{mon}(K_n) < \lfloor n/2 \rfloor$ contains a properly colored 2-factor. Note that for an edge-colored complete graph K_n, we have $\delta^c(K_n) + \Delta^{mon}(K_n) \leq n$. Hence, by the above fact, every edge-colored complete graph K_n with $\delta^c(K_n) > \lceil n/2 \rceil$ contains a properly colored 2-factor. For edge-colored graphs (not necessarily complete), Lo [134] proved the following theorem, where the lower bound is best possible by the construction in Figure 1.5.

Theorem 1.31 (Lo [134]). *Every edge-colored graph G with $\delta^c(G) \geq 2|G|/3$ contains a properly colored 2-factor.*

Resting on the shoulders of Lo's result [134] and the edge-colored version of absorption technique used in [134], we derive the following result.

Theorem 1.32 (Cheng, Hu and Wang [38]). *For any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N(\varepsilon, t)$ such that every edge-colored graph G with $\delta^c(G) \geq$*

$(2/3+\varepsilon)|G|$ *and* $|G| > N(\varepsilon, t)$ *contains every properly colored 2-factor with exactly* t *components.*

Moreover, we consider the bipartite version of Theorem 1.31 and obtain the following result, where the lower bound is tight.

Theorem 1.33 (Cheng, Hu and Wang [38])**.** *Every edge-colored balanced bipartite graph* G *with* $\delta^c(G) \geq |G|/3+1$ *contains a properly colored 2-factor.*

1.2.3 Compatible subgraphs

From the definitions, we know that graphs with incompatibility systems and compatible subgraphs generalize the concept of locally bounded edge-colored graphs and properly colored subgraphs, respectively. The definition of incompatibility system was firstly introduced by Krivelevich, Lee and Sudakov [116] when studying the robustness of Hamiltonicity of a *Dirac graph*, which is defined to be an n-vertex graph of minimum degree at least $n/2$.

What does "robustness" mean? Let G be a graph and $\mathcal{P}$ be a graph property. Many results in graph theory are of the form "under certain conditions, G has property $\mathcal{P}$". Once such a result is established, it is natural to ask how strongly does G possess $\mathcal{P}$? In other words, we want to determine the *robustness* of G with respect to $\mathcal{P}$. This has motivated recent trends in the study of robustness of graph properties, aiming to strengthen classical results in extremal graph theory and probabilistic combinatorics. What measures of robustness can one utilize? In this direction, several different measures of robustness have been explored in

(1) (sparse) random graphs $G(n,p)$ in the setting of so-called resilience (see e.g. [7, 10, 17, 55, 113, 114, 151]);

(2) edge-colored graphs (see e.g. [47, 48, 49, 68, 79, 105, 139, 161]);

(3) Maker-Breaker games (see e.g. [16, 22, 43, 112, 117]).

For further information on this area, the reader is referred to the comprehensive survey [150] by Sudakov.

In [116], the incompatibility system is defined as a new measure of robustness. Krivelevich, Lee and Sudakov [116] strengthened Dirac's theorem in the following way.

Theorem 1.34 (Krivelevich, Lee and Sudakov [116]). *There exists a constant μ such that the following holds for large enough n. For every n-vertex Dirac graph G and a μn-bounded incompatibility system $\mathcal{F}$ defined over G, there exists a compatible Hamilton cycle.*

Krivelevich, Lee and Sudakov [116] also suggested looking at how various extremal results can be strengthened using this notion. We consider the robustness of "a graph G contains some factor" with respect to the incompatibility system. Given a graph H, determining the sufficient and extrmal condition for the existence of an H-factor is one of the fundamental lines of research in extremal graph theory. A seminal result of Alon and Yuster [8] shows that if the host graph G with n vertices has minimum degree $\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) + o(n)$, then G contains an H-factor. We study the robust version of Alon–Yuster's result with respect to the incompatibility system. By adapting Szemerédi's regularity lemma and the absorption method, we prove the following result.

Theorem 1.35 (Hu, Li, Wang and Yang [94]). *For any $\alpha > 0$ and any graph H with h vertices, there exists a constant $\mu > 0$ such that for any sufficiently large n with $n \in h\mathbb{N}$, if G is an n-vertex graph with $\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \alpha\right) n$ and $\mathcal{F}$ is a μn-bounded incompatibility system over G, then there exists a compatible H-factor in G.*

By constructing examples, we show that the error term of αn in the lower bound of minimum degree cannot be omitted.

1.3 Contribution and outline of the thesis

In this section, we summarize main works and the organization of this dissertation.

(1) Motivated by Theorem 1.10 and a classical result of Dirac [59] about k vertex-disjoint triangles, we determine the minimum color degree condition which guarantees k vertex-disjoint rainbow triangles in an edge-colored graph, where $k \geq 2$.

Let G be an edge-colored graph of order n. We derive that if $n \geq 20$ and $\delta^c(G) \geq (n+2)/2$, then G contains two vertex-disjoint rainbow triangles. More generally, we conjecture that if $\delta^c(G) \geq (n+k)/2$, then G contains k vertex-disjoint rainbow triangles. If this conjecture holds, then the lower bound $(n+k)/2$ is tight. To provide more evidence for this conjecture, we prove the following result. For any integer $k \geq 2$, if $n \geq 16k - 12$ and $\delta^c(G) \geq n/2 + k - 1$, then G contains k

vertex-disjoint rainbow triangles. Moreover, we establish minimum color degree conditions for the existence of k edge-disjoint rainbow triangles, a properly colored cycle and k vertex-disjoint properly colored cycles, respectively. This work is illustrated precisely in Chapter 2.

[Corresponding paper: *Vertex-disjoint rainbow triangles in edge-colored graphs*, Discrete Math., 343 (2020) 112117, with H. Li and D. Yang.]

(2) We study the relation between the order of maximum properly colored tree in an edge-colored graph G and the minimum color degree $\delta^c(G)$ of G.

To be specific, we show that for an edge-colored connected graph G, the order of maximum properly colored tree in G is at least $\min\{|G|, 2\delta^c(G)\}$, which generalizes Theorem 1.24. Moreover, the lower bound $2\delta^c(G)$ in our result is sharp and we characterize all extremal graphs G with the maximum properly colored tree of order $2\delta^c(G)$ $(\neq |G|)$. We also prove that if for each color c, the set of edges colored with c forms a star, then the order of maximum rainbow tree in G is at least $\min\{|G|, 2\delta^c(G)\}$, which generalizes Theorem 1.25. As a corollary, we obtain a tight minimum color degree condition for the existence of properly colored spanning forest with at most k components. Furthermore, we characterize all extremal graphs for Theorem 1.23. This work is shown in Chapter 3.

[Corresponding paper: *Maximum properly colored trees in edge-colored graphs*, Journal of Combinatorial Optimization, (2021), https://doi.org/10.1007/s10878-021-00824-z, with H. Li and S. Maezawa.]

(3) Inspired by Theorem 1.31 and resting on the shoulder of it, we study properly colored 2-factors in edge-colored graphs and edge-colored balanced bipartite graphs.

We prove that for any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N(\varepsilon, t)$ such that every edge-colored graph G with $\delta^c(G) \geq (2/3 + \varepsilon)|G|$ and $|G| > N(\varepsilon, t)$ contains every properly colored 2-factor with exactly t components, which generalizes a result of Lo [134] about the minimum color degree condition for the existence of properly colored cycles of any length. We also obtain that every edge-colored balanced bipartite graph G with $\delta^c(G) \geq |G|/3 + 1$ contains a properly colored 2-factor and the lower bound is best possible. Furthermore, we show that for any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N'(\varepsilon, t)$ such that every edge-colored balanced bipartite graph G with $\delta^c(G) \geq (1/3+\varepsilon)|G|$

and $|G| > N'(\varepsilon, t)$ contains every properly colored even 2-factor with exactly t components. This work is discussed in Chapter 4.

[Corresponding paper: *Properly colored 2-factors in edge-colored graphs*, personal communication with Y. Cheng and G. Wang.]

(4) The notion of incompatibility system was firstly introduced in [116] as a quantitative measure of robustness of graph properties. Recently, there has been an increasing interest in studying robustness of graph properties, aiming to strengthen classical results in extremal graph theory and probabilistic combinatorics. We study the robust version of Alon–Yuster's result [8] about the existence of H-factor with respect to the incompatibility system.

We prove that for any $\alpha > 0$ and any graph H with h vertices, there exists a constant $\mu > 0$ such that for any sufficiently large n with $n \in h\mathbb{N}$, if G is an n-vertex graph with $\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \alpha\right) n$ and $\mathcal{F}$ is a μn-bounded incompatibility system over G, then there exists a compatible H-factor in G, where the error term of αn cannot be omitted. This part of work is stated in Chapter 5.

[Corresponding paper: *Compatible H-factors in dense graphs*, manuscript, with H. Li, Y. Wang and D. Yang.]

Remark. Some of my works are not included in this thesis (see [34, 36, 52, 57, 90, 96, 97, 98]), which have been (or will be) contained in theses of my collaborators or my master's thesis. We will introduce these works briefly in the last chapter Appendix.

Chapter 2

Vertex-disjoint rainbow triangles in edge-colored graphs

2.1 Introduction

It has been a long and rich history to study the existence of cycles in graphs, digraphs and edge-colored graphs. For graphs, there is a trivial fact that every graph with minimum degree at least two contains a cycle. In 1963, Corrádi and Hajnal [46] generalized this fact as follows.

Theorem 2.1. (Theorem 1 of [46]) *For any positive integer k, every graph of order at least $3k$ and minimum degree at least $2k$ contains k vertex-disjoint cycles.*

Compared to graphs without edge-colorings, establishing color degree conditions forcing properly colored or rainbow cycles in edge-colored graphs seems essentially more complicated. Indeed, Wang and Li (Proposition 2.2 of [156]) proved that for any positive integer l, there exists an edge-colored graph G with $\delta^c(G) \geq l$, but G contains no properly colored cycle. Recently, Fujita, Li and Zhang [73] obtained a tight minimum color degree condition as follows.

Theorem 2.2. (Theorem 2 of [73]) *For all positive integers n and d with $d! \sum_{i=0}^{d} 1/i! \geq n+1$, every edge-colored graph G of order n with $\delta^c(G) \geq d$ contains a properly colored cycle.*

As an analogy of Theorem 2.1, it seems more challenging to determine sufficient conditions for the existence of vertex-disjoint properly colored cycles. Compared to

Theorem 2.2, we first give an alternative tight color degree condition for a properly colored cycle, which will be of independent interest. To see this, we say that a star $K_{1,s}$ is *nontrivial* when $s > 1$. Given an edge-colored graph G and a vertex $v \in V(G)$, let $m(v)$ be the number of maximal monochromatic nontrivial stars centred at v. For any positive integer k, let $f(k)$ be the smallest integer such that every digraph of minimum outdegree at least $f(k)$ contains k vertex-disjoint directed cycles. There has been plenty of work around $f(k)$. Alon (Theorem 1.1 of [4]) proved that $f(k) \leq 64k$. Recently, Bucić (Theorem 4.1 of [28]) improved on this bound to show $f(k) \leq 18k$.

Theorem 2.3. *For every positive integer k and every edge-colored graph G, we have the following.*

(i) *If $d^c(v) \geq 2 + m(v)$ for all vertices $v \in V(G)$, with at most one exception v_0 satisfying $d^c(v_0) = 1 + m(v_0)$, then G contains a properly colored cycle.*

(ii) *If $d^c(v) \geq 2f(k)+m(v)$ for each vertex $v \in V(G)$, then G contains k vertex-disjoint properly colored cycles.*

It is worth noting that the color degree condition given in Theorem 2.3 (i) is tight by the following construction.

Example 2.1. *Let G_1 be an edge-colored K_2 with color c_0. For all $i \geq 1$, let G_{i+1} be obtained from $2i$ vertex-disjoint copies $\{H_1, H_2, \ldots, H_{2i}\}$ of G_i and a new edge x_iy_i with color c_i as follows. First, join x_i completely to each H_j with $1 \leq j \leq i$, and y_i completely to each H_j with $i + 1 \leq j \leq 2i$. Then color all the edges between x_i (or y_i) and H_j with one color c_i^j. Note that all the colors c_i and c_i^j are different.*

By the above construction, we can easily observe that in each G_{i+1}, $d^c(x_i) = d^c(y_i) = i + 1$, $m(x_i) = m(y_i) = i$, $d^c(v) \geq 2 + m(v)$ for each $v \in V(G_{i+1}) \setminus \{x_i, y_i\}$ and G_{i+1} contains no properly colored cycle.

Let K_n^c be an edge-colored complete graph of order n. Li, Broersma and Zhang [127] initiated the study of vertex-disjoint properly colored short cycles in K_n^c and proposed the following conjecture.

Conjecture 2.1. (Conjecture 5 of [127]) *If $\Delta^{mon}(K_n^c) \leq n - 3k + 1$, then K_n^c contains k vertex-disjoint properly colored cycles of length at most 4.*

In the same paper, they confirmed this conjecture for $k = 1, 2$ (Observation 3 and Theorem 6 of [127]). Moreover, they constructed some examples to show the tightness of the bound in Conjecture 2.1.

Closely related questions concern rainbow cycles. For instance, Li and Wang (Theorem 2.1 of [123]) showed that every edge-colored graph G of order n with $\delta^c(G) \geq (n+1)/2$ contains a rainbow cycle. Furthermore, they conjectured that if $\delta^c(G) \geq (n+1)/2$, then G contains a rainbow triangle. Later on, Li (Theorem 6 of [122]) proved this conjecture. In particular, Li, Ning, Xu and Zhang [121] generalized this result as follows.

Theorem 2.4. (Theorem 2 of [121]) *Every edge-colored graph G of order n with $\sum_{v\in V(G)} d^c(v) \geq n(n+1)/2$ contains a rainbow triangle.*

To the end, it is worth mentioning that Czygrinow, Molla, Nagle and Oursler (Theorem 1.2 of [51]) recently confirmed that the minimum color degree condition $\delta^c(G) \geq (n+1)/2$ for a graph G also guarantees a rainbow cycle C_ℓ for any given odd integer ℓ, where the order of G is essentially large with respect to ℓ, and the bound is also tight.

Inspired by Conjecture 2.1 and the result of Li [122], we study the minimum color degree condition which guarantees k vertex-disjoint rainbow triangles, where $k \geq 2$. Therefore the first open case is $k = 2$, and we almost resolve the case as follows.

Theorem 2.5. *For all integers $n \geq 20$, every graph G of order n with $\delta^c(G) \geq (n+2)/2$ contains two vertex-disjoint rainbow triangles.*

It has been observed from a result of Hajnal and Szemerédi [88] that for all positive integers n and k with $n \geq 3k$, every graph of order n with minimum degree at least $(n+k)/2$ contains k vertex-disjoint triangles. As an edge-colored version, we propose the following conjecture.

Conjecture 2.2. *For all positive integers n and k with $n \geq 3k$, every edge-colored graph G of order n with $\delta^c(G) \geq (n+k)/2$ contains k vertex-disjoint rainbow triangles.*

If Conjecture 2.2 holds, then the lower bound is tight by the following example.

Example 2.2. *Let H be a properly colored graph where $V(H) = X_1 \cup X_2 \cup Y$ with $|X_1| = \lceil (n-k+1)/2 \rceil$, $|X_2| = \lfloor (n-k+1)/2 \rfloor$ and $|Y| = k-1$. $H[X_1 \cup X_2]$ is a balanced complete bipartite graph and $H[Y]$ is empty. In addition, each $y \in Y$ is complete to $X_1 \cup X_2$.*

Hence $\delta^c(H) = \lfloor (n+k-1)/2 \rfloor$. Since every rainbow triangle in H must contain at least one vertex from Y and $|Y| = k-1$, H contains at most $k-1$ vertex-disjoint rainbow triangles.

We say that two rainbow triangles uvw and xyz have *totally different colors* if $\{c(uv), c(vw),\ c(wu)\} \cap \{c(xy), c(yz), c(zx)\} = \emptyset$. Note that when n and k have the same parity, $\delta^c(G) \geq (n+k)/2$ cannot guarantee k vertex-disjoint rainbow triangles such that any two of them have totally different colors. To see this, we obtain the following construction H^* by modifying H.

Example 2.3. *Let $V(H^*) = X_1 \cup X_2 \cup Y$ and $|X_1| = |X_2| = (n-k)/2$, $|Y| = k$, where n and k have the same parity. Moreover, $H^*[X_1 \cup X_2]$ is a balanced complete bipartite graph and $H^*[Y]$ is a clique. Also, each $y \in Y$ is complete to $X_1 \cup X_2$. Arbitrarily take two vertices y_1, y_2 from Y and color all the edges in $H^*[X_1 \cup \{y_1\}]$ and $H^*[X_2 \cup \{y_2\}]$ with one color c_0. Next we properly color the remaining edges of H^* with other colors.*

Hence $\delta^c(H^*) = (n+k)/2$ and among any collection of k vertex-disjoint rainbow triangles in H^*, the two rainbow triangles containing y_1 and y_2, respectively, always have a common color c_0.

By induction on k, one can easily observe the following fact.

Fact 2.1. *If $\delta^c(G) \geq (n+3k)/2$, then G contains k vertex-disjoint rainbow triangles.*

To provide more evidence for Conjecture 2.2, we obtain the following extension of Theorem 2.5.

Theorem 2.6. *For all integers n, k with $k \geq 2$ and $n \geq 16k-12$, every edge-colored graph G of order n with $\delta^c(G) \geq n/2+k-1$ contains k vertex-disjoint rainbow triangles.*

By similar arguments in the proof of Theorem 2.6, we obtain the following result.

Proposition 2.1. *For all positive integers n, k with $n \geq k+4$, every edge-colored graph G of order n with $\delta^c(G) \geq (n+k)/2$ contains k edge-disjoint rainbow triangles.*

2.2 Preliminaries

Recall that G is an edge-colored graph of order n. Let $\delta = \delta^c(G)$ and $\Delta = \Delta^{mon}(G)$. By deleting edges in G, we may assume that G is *edge-minimal*, i.e., any additional edge deletion would lead to a decrease in $d^c(v)$ for some vertex v in G. Therefore, every

monochromatic subgraph in G is a disjoint union of stars. For every vertex $v \in V(G)$, let $N_G(v)$ be the set of neighbors of v in G. We say $xy \in E(G)$ is a *good edge* for v if vxy is a rainbow triangle, and denote by $g(v)$ the number of good edges for v.

Lemma 2.1. *Let v be a vertex of G with $\Delta^{mon}(v) = \Delta$. Then $g(v) \geq (2\delta - n)(\delta + \Delta - 1)/2$.*

Proof. Let A be the maximal subset of $N_G(v)$ such that $c(vx) = c(vy)$ for any two vertices $x, y \in A$. Hence $|A| = \Delta$. Let B be a maximal subset of $N_G(v) \setminus A$ such that $c(vx) \neq c(vy)$ for any two vertices $x, y \in B$. So we have $|B| \geq \delta - 1$ and $|V(G) \setminus (A \cup B \cup \{v\})| \leq n - \delta - \Delta$.

Now we define a directed graph D on $A \cup B$ such that $\overrightarrow{xy}$ is an arc in D if and only if $xy \in E(G)$ and $c(xy) \neq c(vx)$. Hence for any two vertices $x, y \in A$, there is either a 2-cycle or no arcs between x and y. Denote by e_a the number of 2-cycles in $D[A]$. Moreover, if $\overleftrightarrow{xy}$ is a 2-cycle with at least one end-vertex in B, then xy is a good edge for v. Let e_0 be the number of such 2-cycles in D. Obviously, $g(v) \geq e_0$. In the remaining proof, we obtain a lower bound of e_0.

For every $x \in A \cup B$, write $d_D^+(x)$ and $d_D^-(x)$ for its outdegree and indegree in D, respectively. We have

$$d_D^+(x) \geq d^c(x) - 1 - |V(G) \setminus (A \cup B \cup \{v\})| \geq 2\delta + \Delta - n - 1.$$

Therefore,

$$\sum_{x \in A \cup B} d_D^+(x) \geq (\Delta + |B|)(2\delta + \Delta - n - 1).$$

Next we consider $\sum_{x \in A \cup B} d_D^-(x)$. If an arc $\overrightarrow{xy}$ is not contained in a 2-cycle, then we have $c(xy) = c(vy)$. Hence for every vertex $x \in A \cup B$, there are at most $\Delta^{mon}(x) - 1 \leq \Delta - 1$ arcs $\overrightarrow{zx}$ which are not contained in 2-cycles. Moreover, the arcs in $D[A]$ and the arcs from B to A are all contained in 2-cycles. Thus, by double counting, we have

$$\sum_{x \in A \cup B} d_D^-(x) \leq |B|(\Delta - 1) + 2(e_a + e_0).$$

Since $\sum_{x \in A \cup B} d_D^-(x) = \sum_{x \in A \cup B} d_D^+(x)$ and $2e_a \leq \Delta(\Delta - 1)$, we conclude that

$$e_0 \geq (2\delta - n)(|B| + \Delta)/2 \geq (2\delta - n)(\delta + \Delta - 1)/2.$$

□

The following lemma is due to Li, Ning, Xu and Zhang [121].

Lemma 2.2. (Theorem 3 of [121]) *Let G be an edge-colored graph of order n. If $d^c(v) \geq n/2$ for every vertex $v \in V(G)$ and G contains no rainbow triangles, then n is even and G is the complete bipartite graph $K_{n/2,n/2}$, unless $G = K_4 - e$ or K_4 when $n = 4$.*

The rest of this chapter is organized as follows. In Section 3, we first prove Theorem 2.3. Next we proceed to prove Theorem 2.6 while the proof of Theorem 2.5 is included in that of Theorem 2.6 as an initial step. The proof of Proposition 2.1 is presented at the end of this section. Section 4 will propose some related problems.

2.3 Main proofs

Proof of Theorem 2.3. Let G be given in the statement of (i) (or (ii)) and suppose G contains no properly colored cycles (or k vertex-disjoint properly colored cycles). We still assume that G is edge-minimal. Therefore, every monochromatic subgraph of G is a union of vertex-disjoint stars. Now we define a directed graph D on G in the following way: for every edge xy in G, we have an arc from x to y in D if there is no $z \in V(G)\backslash\{x, y\}$ with $c(xy) = c(xz)$. Note that since there is no monochromatic path of length 3 in G, for every edge xy, there is an arc or a directed 2-cycle corresponding to it in D. We can easily observe that every directed cycle of length at least 3 in D is actually a properly colored cycle in G. Moreover, for each vertex $v \in V(D)$, we have $d^+(v) = d^c(v) - m(v)$. By assumption, we can greedily find a directed cycle of length at least 3 if each vertex v has $d^+(v) \geq 2$, even though there exists one exception v_0 with $d^+(v) = d^c(v) - m(v) = 1$.

We proceed the proof of (ii) by defining an orientation D of G in the same way as above. It follows that $d^+(v) = d^c(v) - m(v) \geq 2f(k)$ for all vertices $v \in V(D)$. By the observation that every directed cycle of length at least 3 in D is indeed a properly colored cycle in G, it remains to prove that D contains k vertex-disjoint directed cycles of length at least 3.

Claim 2.1. *There exists a subdigraph D' of D such that $d^+_{D'}(v) \geq f(k)$ for each $v \in V(D)$ and D' contains no 2-cycle.*

Proof. Let H be a spanning subgraph of G with $E(H) = \{xy \in E(G) \mid \overleftrightarrow{xy} \in D\}$. Assume that H is connected, otherwise we can consider each component separately. We pair up the vertices of odd degree in H and join each pair with an extra edge to form a multigraph H'. Hence, H' has an Eulerian circuit C, that is, a walk that begins and ends at the same vertex, and contains each edge in H' exactly once. We can see C as a directed cycle, and split it into a family of directed walks $\{W_1, W_2, \ldots, W_l\}$ by deleting the extra edges added to H to form H'. Hence each walk W_i can be treated as a subdigraph of D. Now we obtain D' from D by deleting all the arcs in W_i for each $i \in [l]$. It follows that D' contains no 2-cycle and $d^+_{D'}(v) \geq d^+_D(v) - \lceil d_H(v)/2 \rceil \geq f(k)$ for each $v \in V(D)$. □

By the definition of $f(k)$ and Claim 2.1, one can easily obtain k vertex-disjoint properly colored cycles of length at least 3. This completes the proof of Theorem 2.3. □

Proof of Theorem 2.6. As an initial step, we first prove Theorem 2.5. Let v be a vertex of G with $\Delta^{mon}(v) = \Delta$. We assume that $\Delta \geq 2$, otherwise G is a properly colored graph with $\delta(G) = \delta^c(G) \geq (n+2)/2$. It follows that one can find two vertex-disjoint triangles by a result of Hajnal and Szemerédi [88]. Since $\delta^c(G-v) \geq (|G-v|+1)/2$, there is a rainbow triangle, say xyz, in $G - v$. For all good edges uw for v, we assume that $\{u, w\} \cap \{x, y, z\} \neq \emptyset$, otherwise we can find two vertex-disjoint rainbow triangles xyz and uvw in G. So there are at least $g(v)/3$ good edges for v which are incident with one vertex, say x, from $\{x, y, z\}$. Let $N^*(v, x)$ be the set of vertices u such that uvx is a rainbow triangle. By Lemma 2.1, we have $|N^*(v, x)| \geq g(v)/3 \geq 4$. Let $G' = G - \{v, x\}$, then $\delta^c(G') \geq |G'|/2$. If there exists a rainbow triangle, say $x'y'z'$ in G', then we can find a vertex $u \in N^*(v, x) \setminus \{x', y', z'\}$ such that uvx and $x'y'z'$ are two vertex-disjoint rainbow triangles in G. Therefore we assume that there is no rainbow triangle in G'. By Lemma 2.2, we know that G' is a properly colored balanced complete bipartite graph. Since $\delta^c(G) \geq (n+2)/2$, we have $vu, xu \in E(G)$ and $d^c_G(u) = (n+2)/2$ for every vertex $u \in V(G')$. Thus, we can easily find two vertex-disjoint rainbow triangles containing x and v, respectively. This completes the proof of Theorem 2.5.

Now we proceed the proof of Theorem 2.6 by induction on k. The basic case $k = 2$ is easily derived from Theorem 2.5. Let $k \geq 3$ and assume that Theorem 2.6 is true

for $k-1$. Similar to the proof of Theorem 2.5, let v be a vertex of G with maximum monochromatic degree. Since $\delta^c(G-v) \geq n/2+k-2 > (n-1)/2+(k-1)-1$ and $n-1 \geq 16(k-1)-12$, there are $k-1$ vertex-disjoint rainbow triangles in $G-v$ by induction hypothesis. Let S be the set of vertices of such $k-1$ rainbow triangles. For any good edge uw for v, we may assume that $\{u,w\} \cap S \neq \emptyset$, otherwise one can find k vertex-disjoint rainbow triangles in G. By averaging arguments, there is a vertex $x \in S$ which is incident with at least $g(v)/(3k-3)$ good edges for v. We also use $N^*(v,x)$ to denote the set of vertices u such that uvx is a rainbow triangle. Since $\delta^c(G-\{v,x\}) \geq (n-2)/2+(k-1)-1$ and $n-2 \geq 16(k-1)-12$, there are $k-1$ vertex-disjoint rainbow triangles in $G-\{v,x\}$ by induction hypothesis. Since $|N^*(v,x)| \geq g(v)/(3k-3)$ and by Lemma 2.1, we have $|N^*(v,x)| \geq 3k-2$. It follows that there is a vertex $u \in N^*(v,x) \setminus S$ such that uvx together with the $k-1$ vertex-disjoint rainbow triangles in $G-\{v,x\}$ forms k vertex-disjoint rainbow triangles in G. □

Proof of Proposition 2.1. We prove Proposition 2.1 by induction on k. The case $k=1$ follows from the result of Li [122]. Let $k \geq 2$ and suppose Proposition 2.1 holds for $k-1$. Let v be a vertex of G with maximum monochromatic degree. Since $\delta^c(G-v) \geq (n+k)/2-1 = [(n-1)+(k-1)]/2$ and $n-1 \geq (k-1)+4$, $G-v$ contains $k-1$ edge-disjoint rainbow triangles by induction hypothesis. Since $g(v) \geq k(k+2)/2 \geq 3(k-1)+1$, there is a good edge uw for v such that uvw and the $k-1$ edge-disjoint rainbow triangles in $G-v$ form k edge-disjoint rainbow triangles in G. □

2.4 Concluding Remarks

In this chapter, we mainly consider vertex-disjoint rainbow triangles in edge-colored graphs. Theorem 2.3 concerns vertex-disjoint properly colored cycles. Based on Theorems 2.2 and 2.3, it will be interesting to determine the minimum color degree condition for k vertex-disjoint properly colored cycles. For digraphs, Henning and Yeo (Conjecture 1 of [91]) conjectured that every digraph of minimum out-degree at least 4, contains two vertex-disjoint directed cycles of different lengths. Later on, Lichiardopol confirmed it (Theorem 1.1 of [131]) and proposed the following conjecture.

Conjecture 2.3. (Conjecture 3.1 of [131]) *For every integer $k \geq 2$, there exists an integer $g(k)$ such that any digraph of minimum out-degree at least $g(k)$ contains k*

vertex-disjoint cycles of different lengths.

Recently, Bensmail, Harutyunyan, Le, Li and Lichiardopol [18] proved Conjecture 6.1 for tournaments, regular digraphs and digraphs of small order. Moreover, for undirected graphs, they proved that every graph with minimum degree at least $(k^2+5k-2)/2$ contains k vertex-disjoint cycles of different lengths (Theorem 12 of [18]), where the degree bound is best possible. In particular, using the same arguments in the proof of Theorem 2.3, we can easily obtain the following result.

Proposition 2.2. *If Conjecture 6.1 holds for each positive integer k, then every edge-colored graph G with $d^c(v) \geq 2g(k) + m(v)$ for all vertices $v \in V(G)$ contains k vertex-disjoint properly colored cycles of different lengths.*

It will also be interesting to establish the minimum color degree condition for k vertex-disjoint properly colored cycles of different lengths, and we will return to this topic in the near future.

Chapter 3

Maximum properly colored trees in edge-colored graphs

3.1 Introduction

For an edge $e \in E(G)$, we use $c(e)$ to denote the color of e. For a color c, let E_c be the set of all edges colored with c. For a vertex $v \in V(G)$, we use $N_G(v)$ to denote the set of all neighbors of v in G and $N_G^c(v)$ is a maximum subset of $N_G(v)$ such that $c(vu) \neq c(vw)$ for any pair $u, w \in N_G^c(v)$.

The problem of finding properly colored or rainbow (spanning) subgraphs in edge-colored graphs, such as cycles, matchings or paths, has been investigated widely, see [40, 73, 100, 123, 155]. In particular, there are many results in edge-colored complete graphs, see [5, 14, 21, 71, 135]. For general edge-colored graphs, Li and Wang [156] proved that G contains a properly colored path of length $2\delta^c(G)$ or a properly colored cycle of length at least $2\delta^c(G)/3$. Later, Lo [133] improved $2\delta^c(G)/3$ to $\delta^c(G)+1$ and conjectured that every edge-colored connected graph G contains a properly colored Hamilton cycle or a properly colored path of length $\lfloor 3\delta^c(G)/2 \rfloor$. If this conjecture is true, then the bound is sharp. In [136], Lo proved the following theorem, which implies the above conjecture is true for $\delta^c(G) \geq (1/2+\varepsilon)|G|$ and $|G|$ large.

Theorem 3.1 (Lo [136]). *For $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that every edge-colored graph G on $|G| \geq n_0$ vertices with $\delta^c(G) \geq (1/2+\varepsilon)|G|$ contains a properly colored cycle of length at least* $\min\{\lfloor 3\delta^c(G)/2 \rfloor, |G|\}$.

Another type of properly colored subgraph received much attention is trees, see

[27, 92, 102, 152]. Borozan et al. [24] proved that the problem of determining whether an edge-colored graph has a properly colored spanning tree is NP-complete. Recently, Cheng, Kano and Wang [39] gave a minimum color degree condition forcing properly colored spanning trees.

Theorem 3.2 (Cheng, Kano and Wang [39])**.** *Let G be an edge-colored connected graph. If $\delta^c(G) \geq |G|/2$, then G has a properly colored spanning tree.*

In the process of proving Theorem 3.2, they also obtained the following theorem.

Theorem 3.3 (Cheng, Kano and Wang [39])**.** *Let G be an edge-colored connected graph. Suppose that E_c forms a star for each color c. If $\delta^c(G) \geq |G|/2$, then G has a rainbow spanning tree.*

The lower bounds in Theorems 3.2 and 3.3 are both tight. In this chapter, we improve Theorems 3.2 and 3.3 as follows.

Theorem 3.4. *Let G be an edge-colored connected graph. Then the order of maximum properly colored tree is at least* $\min\{|G|, 2\delta^c(G)\}$.

Theorem 3.5. *Let G be an edge-colored connected graph. Suppose that E_c forms a star for each color c. Then the order of maximum rainbow tree is at least* $\min\{|G|, 2\delta^c(G)\}$.

The lower bounds in Theorems 3.4 and 3.5 are best in the sense that we can not replace $2\delta^c(G)$ by $2\delta^c(G) + 1$ as shown in Theorems 3.6 and 3.7. In particular, we determine the extremal graph classes for Theorems 3.4 and 3.5 in Section 3.2. The characterization and proof about extremal graphs are our main work and the most important part in this chapter.

By Theorem 3.4, we obtain Corollary 3.1, where the k components can be a properly colored tree with order $|G| - k + 1$ and $k - 1$ independent vertices. The lower bound in Corollary 3.1 is sharp.

Corollary 3.1. *Let k be a positive integer and G be an edge-colored connected graph. If $\delta^c(G) \geq (|G| - k + 1)/2$, then G has a properly colored spanning forest with at most k components.*

3.2 Extremal graphs of Theorems 3.4 and 3.5

In this section, we characterize extremal graphs of Theorems 3.4 and 3.5. We describe six graphs $G^1_{m,k}$, G^2_k, $G^3_{m,k}$, $G^4_{m,k}$, $G^5_{m,k}$ and $G^6_{m,k}$ as follows, where $G^3_{m,k}$ and $G^5_{m,k}$ are

extremal graphs for Theorem 3.4 and the others are extremal graphs for Theorems 3.4 and 3.5. Note that $G^5_{m,k}$ also shows the sharpness of Corollary 3.1.

- Let $m \geq 3$ be an odd positive integer and $c_1, c_2, \ldots, c_m$ be distinct colors. Let K_m be an edge-colored complete graph with $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $c(u_iu_j) = c_i$ for all $1 \leq i \leq m$ and $i+1 \leq j \leq i+(m-1)/2$, where the indices are taken modulo m. Let $k \geq 2$ be a positive integer and $v_1, v_2, \ldots, v_k$ be distinct vertices not contained in $V(K_m)$. For each $1 \leq j \leq k$, let U_j be an arbitrary subset of $V(K_m)$ with size at least $(m+1)/2$. For each $1 \leq i \leq m$ and $1 \leq j \leq k$, if $u_i \in U_j$, then we add an edge u_iv_j and color it with c_i. Let $G^1_{m,k}$ be the resulting graph. Then $\delta^c(G^1_{m,k}) = (m+1)/2$ and E_{c_i} forms a star for each color c_i $(1 \leq i \leq m)$. Since any properly colored tree T in $G^1_{m,k}$ is rainbow and $G^1_{m,k}$ has m colors, $|T| \leq m+1$.

- Let $k \geq 3$ be a positive integer. Let $u_1, u_2, v_1, v_2, \ldots, v_k$ be distinct vertices. Let c_1, c_1', c_2 be distinct colors. We add an edge u_1u_2 colored with c_1. Let V_1 and V_1' be disjoint subsets of $\{v_1, v_2, \ldots, v_k\}$ such that $V_1' \neq \emptyset$ and $V_1 \cup V_1' = \{v_1, v_2, \ldots, v_k\}$. For each $1 \leq i \leq k$, if $v_i \in V_1$, then we add an edge u_1v_i colored with c_1, otherwise we add an edge u_1v_i colored with c_1'. Moreover, for each $1 \leq i \leq k$, we add an edge u_2v_i colored with c_2. Let G^2_k be the resulting graph. Then $\delta^c(G^2_k) = 2$ and every monochromatic subgraph in G^2_k is a star. Since any properly colored tree T in G^2_k is rainbow and G^2_k has three colors, $|T| \leq 4$.

- Let m and k be positive integers. Let K_{m+1} be a properly colored complete graph and $V(K_{m+1}) = \{u, u_1, u_2, \ldots, u_m\}$. Let $v, v_1, v_2, \ldots, v_{m+k}$ be distinct vertices not contained in $V(K_{m+1})$. Let $c, c_1, c_2, \ldots, c_m$ be distinct colors such that for each $1 \leq i \leq m$, c_i does not appear on edges incident with u_i in K_{m+1} and c does not appear on edges incident with u in K_{m+1}. For each $1 \leq i \leq m$ and $1 \leq j \leq m+k$, we add edges u_iv_j and u_iv colored with c_i and add edges uv and v_jv colored with c. Let $G^3_{m,k}$ be the resulting graph. Then $\delta^{col}(G^3_{m,k}) = m+1$. Let T be a maximum properly colored tree in $G^3_{m,k}$. For edges in $E(G^3_{m,k}) \setminus E(K_{m+1})$, only $m+1$ of them colored with $c, c_1, c_2, \ldots, c_m$, respectively, can be contained in T. Adding a properly colored spanning path in K_{m+1}, we have $|T| = 2m+2$.

- Let m and k be positive integers. Let $G^4_{m,k}$ be the graph obtained from $G^3_{m,k}$ by replacing the properly colored complete graph K_{m+1} with a rainbow complete

graph K_{m+1} which does not contain the colors $c, c_1, c_2, \ldots, c_m$. Then $\delta^c(G^4_{m,k}) = m+1$ and every monochromatic subgraph in $G^4_{m,k}$ is a star. By the same argument used in $G^3_{m,k}$, we obtain that the order of a maximum rainbow tree in $G^4_{m,k}$ is $2m+2$.

- Let m and k be positive integers. Let K_m be a properly colored complete graph and $V(K_m) = \{u_1, u_2, \ldots, u_m\}$. Let $v_1, v_2, \ldots, v_{m+k}$ be distinct vertices not contained in $V(K_m)$. Let $c_1, c_2, \ldots, c_m$ be distinct colors such that for each $1 \leq i \leq m$, c_i does not appear on edges incident with u_i in K_m. Based on the properly colored K_m and the independent set $\{v_1, v_2, \ldots, v_{m+k}\}$, we construct an edge-colored graph $G^5_{m,k}$ by adding a new edge u_iv_j and coloring it with c_i for all $1 \leq i \leq m$ and $1 \leq j \leq m+k$. Hence $\delta^{col}(G^5_{m,k}) = m$. For any properly colored tree T in $G^5_{m,k}$, the number of edges between K_m and $\{v_1, v_2, \ldots, v_{m+k}\}$ contained in T is at most m. Hence, $|T| \leq 2m$.

- Let m and k be positive integers. Let $G^6_{m,k}$ be the graph obtained from $G^5_{m,k}$ by replacing the properly colored complete graph K_m with a rainbow complete graph K_m which does not contain the colors $c_1, c_2, \ldots, c_m$. Then $\delta^c(G^6_{m,k}) = m$ and every monochromatic subgraph in $G^6_{m,k}$ is a star. By the same argument used in $G^5_{m,k}$, the order of a rainbow tree in $G^6_{m,k}$ is at most $2m$.

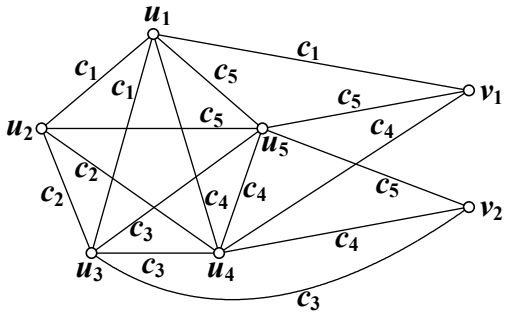
Let $\mathcal{G}^1 = \{G^1_{m,k} \mid m \geq 3 \text{ is an odd integer and } k \geq 2 \text{ is a positive integer}\}$, $\mathcal{G}^2 = \{G^2_k \mid k \geq 3 \text{ is a positive integer}\}$ and $\mathcal{G}^i = \{G^i_{m,k} \mid m, k \in \mathbb{Z}^+\}$ for each $3 \leq i \leq 6$.

Theorem 3.6. *Let G be an edge-colored connected graph. If G has no properly colored tree with order at least $\min\{|G|, 2\delta^c(G)+1\}$, then $G \in \mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^3 \cup \mathcal{G}^5$.*

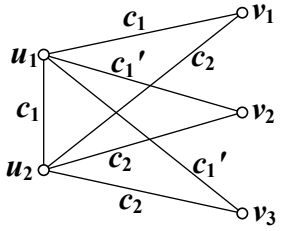
Theorem 3.7. *Let G be an edge-colored connected graph. Suppose that E_c forms a star for each color c. If G has no rainbow tree with order at least $\min\{|G|, 2\delta^c(G)+1\}$, then $G \in \mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^4 \cup \mathcal{G}^6$.*

Moreover, we give an application of Thereom 3.6. An edge-colored graph G is *properly connected* if there exists a properly colored path connecting u and v for any $u, v \in V(G)$. Fujita and Magnant [74] showed a minimum color degree condition for edge-colored graphs to be properly connected.

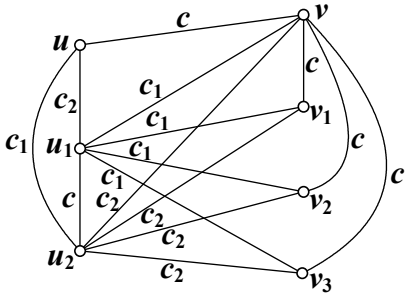
Theorem 3.8 (Fujita and Magnant [74])**.** *Let G be an edge-colored graph with order at least three. If $\delta^c(G) \geq |G|/2$, then G is properly connected.*



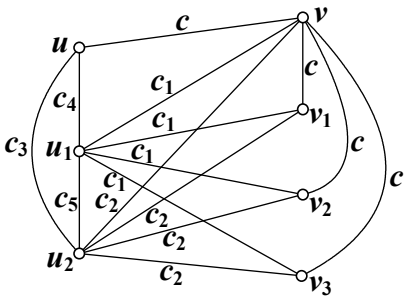
(1) $G^1_{5,2}$



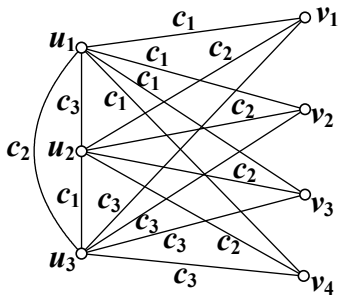
(2) G^2_3



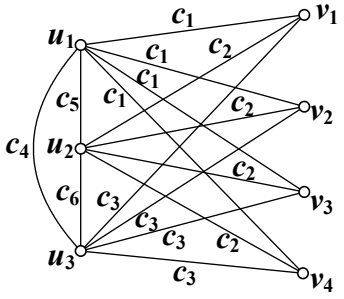
(3) $G^3_{2,1}$



(4) $G^4_{2,1}$



(5) $G^5_{3,1}$



(6) $G^6_{3,1}$

Figure 3.1: Examples of $G^1_{m,k}$, G^2_k, $G^3_{m,k}$, $G^4_{m,k}$, $G^5_{m,k}$, and $G^6_{m,k}$, respectively.

By Theorem 3.6, we characterize all extremal graphs of Theorem 3.8 when $|G|$ is odd. If G has a properly colored spanning tree, then G is properly connected. Hence by Theorems 3.4 and 3.6, we only need to consider graphs G in $\mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^3 \cup \mathcal{G}^5$ with $\delta^c(G) = (|G| - 1)/2$.

Theorem 3.9. *Let G be an edge-colored graph with $\delta^c(G) = (|G| - 1)/2$. If G is not properly connected, then G is in $\{G^1_{m,2} \mid m \geq 3$ is an odd integer$\} \cup \{G^3_{1,1}, G^5_{1,1}\}$.*

Proof. For a given odd integer $m \geq 3$, there is no properly colored path connecting the two vertices v_1 and v_2 in $V(G^1_{m,2})$, which are not contained in $V(K_m)$. Hence graphs in $\{G^1_{m,2} \mid m \geq 3$ is an odd integer$\}$ are not properly connected. It is easy to see that $G^3_{1,1}$ and $G^5_{1,1}$ are not properly connected. Note that graphs in $\{G^1_{m,k} \mid m \geq 3$ is an odd integer and $k \geq 3$ is a positive integer$\} \cup \{G^2_k \mid k \geq 4$ is a positive integer$\} \cup \{G^3_{m,k} \mid m \in \mathbb{Z}^+, k \geq 2$ is a positive integer$\} \cup \{G^5_{m,k} \mid m \in \mathbb{Z}^+, k \geq 2$ is a positive integer$\}$ do not satisfy the minimum color degree condition in Theorem 3.9.

Hence we have to show that any graph in $\{G^2_3\} \cup \{G^3_{m,1} \mid m \geq 2$ is a positive integer$\} \cup \{G^5_{m,1} \mid m \geq 2$ is a positive integer$\}$ is properly connected. It is easy to check that G^2_3 is properly connected. Let m be a fixed integer with $m \geq 2$ and $G = G^3_{m,1} - v_{m+1}$. Since $\delta^c(G) \geq |G|/2$, G is properly connected by Theorem 3.8. Now, we show that there is a properly colored path in $G^3_{m,1}$ from v_{m+1} to any vertex $w \in V(G)$. If $w = v$ or $w = u_i$ for any $1 \leq i \leq m$, then wv_{m+1} is a properly colored path. If $w = u$, then $v_{m+1}u_1u$ is a properly colored path. If $w = v_i$ for any $1 \leq i \leq m$, then $v_{m+1}u_2u_1v_i$ is a properly colored path. Thus, $G^3_{m,1}$ is properly connected. By the similar argument, we can obtain that $G^5_{m,1}$ is properly connected. □

The rest of this chapter is organized as follows. Since the proofs of Theorems 3.4 and 3.6 are based on Theorems 3.5 and 3.7, we first prove Theorems 3.5 and 3.7 in Section 3. Next, we deal with Theorems 3.4 and 3.6 in Section 4.

3.3 Proofs of Theorems 3.5 and 3.7

Before proceeding with the proofs of Theorems 3.5 and 3.7, we give some lemmas and claims. Let G be an edge-colored connected graph of order n such that E_c forms a star for each color c. Suppose that G has no rainbow spanning tree. Let T be a maximum rainbow tree in G. Note that $V(G) \setminus V(T) \neq \emptyset$. Firstly, we use the following useful lemma, which is implicit in the proof of Theorem 7 in [152].

Lemma 3.1 (Suzuki [152]). *Let G be an edge-colored connected graph and let T be a maximum rainbow tree in G. If T is not a spanning tree, then for some $1 \leq r \leq |G|-2$, there exists r colors $c_1, \ldots, c_r$ in T such that*

- *the number of components of $G[V(T)] - (E_{c_1} \cup \ldots \cup E_{c_r})$ is $r+1$ and*
- *the edges between $V(T)$ and $V(G) \setminus V(T)$ are contained in $E_{c_1} \cup \ldots \cup E_{c_r}$.*

We briefly explain how to derive Lemma 3.1 from [152]. In the proof of Theorem 7 in [152], Suzuki assumes that T is a maximum rainbow tree in G but not a spanning tree. Let E_0 be the set of edges of G between $V(T)$ and $V(G) \setminus V(T)$. Suzuki shows that there is an edge set $E_p \subseteq E(G[V(T)])$ such that

(i) T has every color appearing in $E_0 \cup E_p$ and

(ii) there is no edge between any two components of $T - (E_0 \cup E_p)$ in G.

Let r be the number of colors in $E_0 \cup E_p$. Since T is a rainbow tree, it follows from (i) that the number of components of $T - (E_0 \cup E_p)$ is $r+1$. By (ii), the number of components of $G[V(T)] - (E_0 \cup E_p)$ is equal to the number of components of $T - (E_0 \cup E_p)$ and so it is also equal to $r+1$. Hence we obtain Lemma 3.1.

Let $W_1, W_2, \ldots, W_{r+1}$ be the $r+1$ components of $G[V(T)] - (E_{c_1} \cup \ldots \cup E_{c_r})$. For any monochromatic star which is isomorphic to K_2, we choose an arbitrary endpoint of it as a center. For any $v \in V(G)$ and a color c, if v is a center of a monochromatic star whose edges are colored by c, then we say that c *belongs* to v. For any $v \in V(G)$, let

$$C(v) = \{c \in \{c_1, c_2, \ldots, c_r\} : c \text{ belongs to } v\}.$$

And let

$$S = \{v \in V(G) : C(v) \neq \emptyset\}.$$

Since E_c forms a star for each color c and $c_1, c_2, \ldots, c_r$ appear in T, we have the following claim.

Claim 3.1. *The set S is contained in $V(T)$.*

Let $1 \leq m \leq r+1$ be an integer such that $W_i \cap S \neq \emptyset$ for $1 \leq i \leq m$, and $W_i \cap S = \emptyset$ for $m+1 \leq i \leq r+1$. Then $m \leq |S| \leq r$. Note that there is no edge between $V(G) \setminus V(T)$ and W_j for $m+1 \leq j \leq r+1$, and there is no edge between W_i and W_j for $m+1 \leq i < j \leq r+1$.

For each $1 \leq i \leq m$, choose a vertex x_i from $W_i \cap S$ arbitrarily. For any $x, y \in V(G)$ with $xy \in E(G)$, $c(xy)$ belongs to either x or y. In particular, for any distinct $1 \leq i, j \leq m$ with $x_ix_j \in E(G)$, $col(x_ix_j) \in C(x_i)$ or $col(x_ix_j) \in C(x_j)$. For each $1 \leq i \leq m$, let

$$\beta_i = |(S \setminus \{x_1, x_2, \ldots, x_m\}) \cap W_i|.$$

Note that $\sum_{1 \leq i \leq m} |C(x_i)| \leq r - \sum_{1 \leq i \leq m} \beta_i$.

Claim 3.2. *For each $1 \leq i \leq m$,*

$$d^c_G(x_i) \leq |C(x_i)| + |W_i| - 1 + |\{c(x_ix_j) : j \neq i \text{ and } c(x_ix_j) \notin C(x_i)\}| + \sum_{1 \leq j \leq m, j \neq i} \beta_j.$$

Proof. It is easy to see that

$$|\{c(x_iy) : y \in V(W_i) \text{ and } c(x_iy) \notin C(x_i)\}| \leq |W_i| - 1.$$

By the definition of $W_1, W_2, \ldots, W_{r+1}$ and S, we obtain

$$\begin{aligned} &|\{c(x_iy) : y \in V(T) \setminus V(W_i) \text{ and } c(x_iy) \notin C(x_i)\}| \\ &\leq |\{c(x_ix_j) : j \neq i \text{ and } c(x_ix_j) \notin C(x_i)\}| + \sum_{1 \leq j \leq m, j \neq i} \beta_j. \end{aligned}$$

By Lemma 3.1 and Claim 3.1, for $x \in V(G) \setminus V(T)$ with $xx_i \in E(G)$, we have $c(x_ix) \in C(x_i)$. We can obtain Claim 3.2 by the above inequalities. □

For convenience, let $\beta = \sum_{1 \leq i \leq m} \beta_i$.

Claim 3.3. *We have $\sum_{1 \leq i \leq m} d^c_G(x_i) \leq \sum_{1 \leq i \leq m} (|C(x_i)| + |W_i|) + (m-1)\beta + m(m-3)/2$.*

Proof. It is easy to see that

$$\sum_{1 \leq i \leq m} \Big(\sum_{1 \leq j \leq m, j \neq i} \beta_j \Big) = (m-1)\beta.$$

Now, we show that

$$\sum_{1 \leq i \leq m} |\{c(x_ix_j) : j \neq i \text{ and } c(x_ix_j) \notin C(x_i)\}| \leq m(m-1)/2. \tag{3.1}$$

We give an orientation D to $G[\{x_1, x_2, \ldots, x_m\}]$ such that for any distinct $1 \leq i, j \leq m$, x_ix_j is oriented from x_i to x_j if $c(x_ix_j) \in C(x_i)$. Then $\sum_{1 \leq i \leq m} |\{c(x_ix_j) : j \neq$

i and $c(x_ix_j) \notin C(x_i)\}| = \sum_{1\le i\le m} d_D^-(x_i) \le m(m-1)/2$, where $d_D^-(x_i)$ is the in-degree of x_i in D. By the above equality, inequality, and Claim 3.2, we obtain the desired inequality. □

Claim 3.4. *We have* $|T| \ge m + 2\beta + 1$.

Proof. Since $r \ge m + \beta$ and $|W_i| \ge |\{x_i\}| + \beta_i = 1 + \beta_i$ for $1 \le i \le m$, we obtain

$$\begin{aligned} |T| &= \sum_{1\le i\le m} |W_i| + \sum_{m+1\le j\le r+1} |W_j| \\ &\ge \big(m + \beta\big) + (r - m + 1) \qquad (3.2) \\ &= \beta + r + 1 \\ &\ge m + 2\beta + 1. \qquad (3.3) \end{aligned}$$

□

Claim 3.5. *We have* $m\delta^c(G) \le |T| + (m-2)\beta + m(m-1)/2 - 1$.

Proof. By Claim 3.3, we obtain

$$\begin{aligned} m\delta^c(G) &\le \sum_{1\le i\le m} d_G^c(x_i) \\ &\le \sum_{1\le i\le m} (|C(x_i)| + |W_i|) + (m-1)\beta + m(m-3)/2. \end{aligned}$$

Note that

$$\sum_{1\le i\le m} |C(x_i)| \le r - \beta \qquad (3.4)$$

and

$$\begin{aligned} \sum_{1\le i\le m} |W_i| &= |T| - \sum_{m+1\le i\le r+1} |W_i| \\ &\le |T| - r + m - 1. \qquad (3.5) \end{aligned}$$

Hence we obtain

$$\begin{aligned} m\delta^c(G) &\le r - \beta + |T| - r + m - 1 + (m-1)\beta + m(m-3)/2 \\ &= |T| + (m-2)\beta + m(m-1)/2 - 1. \end{aligned}$$

□

Proof of Theorem 3.5. Suppose that $|T| \leq 2\delta^c(G) - 1$. By Claim 3.5, we obtain

$$2\delta^c(G)(m-2) \leq (m + 2\beta + 1)(m-2) - 2.$$

On the other hand, by Claim 3.4, we obtain

$$2\delta^c(G) \geq m + 2\beta + 2.$$

This implies that $m = 1$.

Choose $w_1 \in V(W_1)$ and $w_2 \in V(W_2)$ such that $w_1 \in S$ and $w_1w_2 \in E(T)$. Note that $w_2 \notin S$. Since $m = 1$, we obtain

$$\begin{aligned} d^c_G(w_1) + d^c_G(w_2) &\leq (|W_1| - 1 + |C(w_1)|) + (|W_2| - 1 + r - |C(w_1)| + 1) \\ &\leq |W_1| + |W_2| + r - 1 \\ &\leq |T| \\ &\leq 2\delta^c(G) - 1. \end{aligned}$$

This implies that either $d^c_G(w_1)$ or $d^c_G(w_2)$ is less than $\delta^c(G)-1$, which is a contradiction.

□

Proof of Theorem 3.7. Suppose that $|T| = 2\delta^c(G) \neq |G|$. By Claim 3.5, we have

$$2\delta^c(G)(m-2) \leq (m + 2\beta + 1)(m-2). \tag{3.6}$$

Meanwhile, by Claim 3.4, we obtain

$$2\delta^c(G) \geq m + 2\beta + 1. \tag{3.7}$$

We divide the proof of Theorem 3.7 into two cases according to the value of m.

Case 1. $m \geq 2$.

In this case, we will show that $G \in \mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^4$.

Claim 3.6. *If $m \geq 3$, then all inequalities of Claims 3.2-3.5 are equalities. If $m = 2$, then all inequalities of Claims 3.2, 3.3 and 3.5 are equalities.*

Proof. If $m \geq 3$, then (3.6) and (3.7) imply that all inequalities of Claims 3.2-3.5 are equalities. If $m = 2$, then $2\delta^c(G) \leq |T|$ by Claim 3.5. Since $|T| = 2\delta^c(G)$, the inequality in Claim 3.5 is an equality. This implies that all inequalities of Claims 3.2, 3.3 and 3.5 are equalities. □

Note that when $m \geq 2$, $d^c_G(x_i) = \delta^c(G)$ for each $1 \leq i \leq m$ by Claims 3.5 and 3.6.

Claim 3.7. *The graph $G[\{x_1, x_2, \ldots, x_m\}]$ is a clique.*

Proof. By Claim 3.6 and (3.1), we obtain $|E(G[\{x_1, x_2, \ldots, x_m\}])| = m(m-1)/2$. This implies that $G[\{x_1, x_2, \ldots, x_m\}]$ is a clique and so the claim holds. □

Claim 3.8. *For each $1 \leq i \leq m$,*

(i) *x_i is adjacent to all vertices in $(V(W_i) \cup S) \setminus \{x_1, \ldots, x_m\}$, and the colors of these incident edges are pairwise distinct and are not contained in $C(x_i)$, and*

(ii) *if $|V(W_i) \cap S| \geq 2$, then for each vertex $x \in V(W_i) \cap S$, the number of colors in $C(x)$ is one and $c(xx_j) \in C(x)$ (for any $j \neq i$).*

Proof. **(i)** Since the right side of the inequality of Claim 3.2 contains $\sum_{1 \leq j \leq m, j \neq i} \beta_j$ and $|W_i| - 1$, by Claim 3.6, x_i is adjacent to all vertices in $(V(W_i) \cup S) \setminus \{x_1, x_2, \ldots, x_m\}$ and the colors of these incident edges are pairwise distinct. Since the right side of the inequality of Claim 3.2 also contains $|C(x_i)|$, the set of the colors between x_i and $(V(W_i) \cup S) \setminus \{x_1, x_2, \ldots, x_m\}$ and $C(x_i)$ are disjoint.

(ii) By Claim 3.6 and (3.4), for each $x \in V(W_i) \cap S \setminus \{x_i\}$, the number of colors in $C(x)$ is one. Since we can take an arbitrary vertex from $V(W_i) \cap S$ as x_i, then for each $x \in V(W_i) \cap S$, the number of colors in $C(x)$ is one. By the same reason and Claim 3.8 (i), for each x_j with $j \neq i$, $xx_j \in E(G)$ and $c(xx_j) \in C(x)$.

□

Claim 3.9. *The following two statements hold.*

(i) $|W_{m+1}| = \cdots = |W_{r+1}| = 1$.

(ii) *If $m \geq 3$, then $V(W_1) \cup \ldots \cup V(W_m) = S$. If $m = 2$, then either $V(W_1) \subseteq S$ or $V(W_2) \subseteq S$.*

(iii) *$V(W_{m+1}) \cup \ldots \cup V(W_{r+1})$ is an independent set in G.*

Proof. **(i) (ii)** When $m \geq 3$, by Claim 3.6 and (3.2), we have

$$\sum_{m+1 \leq j \leq r+1} |W_j| = r - m + 1 \quad \text{and} \quad \sum_{1 \leq i \leq m} |W_i| = m + \beta.$$

Hence $|W_{m+1}| = \cdots = |W_{r+1}| = 1$ and $V(W_1) \cup \ldots \cup V(W_m) = S$.

When $m = 2$, by Claim 3.6 and (3.5), we have

$$\sum_{m+1 \leq j \leq r+1} |W_j| = r - m + 1.$$

Hence $|W_{m+1}| = \cdots = |W_{r+1}| = 1$. Now we show Claim 3.9 (ii) for the case $m = 2$. Suppose that there exist two vertices $w_1 \in V(W_1) \setminus S$ and $w_2 \in V(W_2) \setminus S$. Since $m = 2$, we obtain

$$\begin{aligned} d^c_G(w_1) + d^c_G(w_2) &\leq |W_1| - 1 + |N_G(w_1) \cap (S \setminus V(W_1))| + |W_2| - 1 + |N_G(w_2) \cap (S \setminus V(W_2))| \\ &\leq |W_1| + |W_2| - 2 + r \\ &\leq |T| - 1 \\ &= 2\delta^c(G) - 1. \end{aligned}$$

This implies that either $d^c_G(w_1)$ or $d^c_G(w_2)$ is less than $\delta^c(G)-1$, which is a contradiction. Hence either $V(W_1) \subseteq S$ or $V(W_2) \subseteq S$.

(iii) By Lemma 3.1 and Claim 3.9 (i), any two vertices in $V(W_{m+1}) \cup \ldots \cup V(W_{r+1})$ are not adjacent in G and so Claim 3.9 (iii) holds. □

Next, we consider $m \geq 3$ and $m = 2$, respectively.

Subcase 1.1. *$m \geq 3$.*

We will show that $G \in \mathcal{G}^1$ in this subcase.

Claim 3.10. *The following statements hold.*

(i) *For any $1 \leq i < j \leq r$, c_i and c_j belong to distinct vertices in S.*

(ii) *For each $1 \leq i \leq m$, $|\{c(x_ix_j) : j \neq i \text{ and } c(x_ix_j) \notin C(x_i)\}| = (m-1)/2$.*

(iii) *For each $1 \leq i \leq m$, $V(W_i) = \{x_i\}$.*

Proof. **(i)** By Claim 3.6 and (3.3), we have $r = m + \beta$. Hence c_i and c_j belong to distinct vertices in S for any $1 \leq i < j \leq r$.

(ii) By Claims 3.9 (ii) and 3.10 (i), for $1 \leq i \leq m$, $|W_i| = |W_i \cap S| = 1 + \beta_i$ and $|C(x_i)| = 1$. This together with Claims 3.2 and 3.6 implies that for any $1 \leq i \leq m$,

$$\delta^c(G) = d^c_G(x_i) = 1 + \beta + |\{c(x_ix_j) : j \neq i \text{ and } c(x_ix_j) \notin C(x_i)\}|.$$

Hence we obtain

$$|\{c(x_ix_j) : j \neq i \text{ and } c(x_ix_j) \notin C(x_i)\}| = |\{c(x_{i'}x_j) : j \neq i' \text{ and } c(x_{i'}x_j) \notin C(x_{i'})\}|$$

for $1 \leq i < i' \leq m$. Let D be the digraph as defined in the proof of Claim 3.3. Then $d^-_D(x_i) = d^-_D(x_{i'})$ for any $1 \leq i < i' \leq m$. By Claim 3.7, $|E(G[\{x_1, x_2, \ldots, x_m\}])| = m(m-1)/2$. Hence $d^-_D(x_i) = (m-1)/2$ for each $1 \leq i \leq m$, i.e., $|\{c(x_ix_j) : j \neq i \text{ and } c(x_ix_j) \notin C(x_i)\}| = (m-1)/2$.

(iii) Suppose that there exists a vertex $x'_i \in W_i \setminus \{x_i\} \subseteq S$ for some $1 \leq i \leq m$. By Claim 3.8 (ii), $|\{c(x'_ix_j) : j \neq i \text{ and } c(x'_ix_j) \notin C(x'_i)\}| = 0$. Since x_i and x'_i are symmetric in W_i, we can replace x_i with x'_i when choosing x_i from $W_i \cap S$ for each $1 \leq i \leq m$. If we do so, then we obtain a contradiction to Claim 3.10 (ii) since $m \geq 3$. Hence $V(W_i) = \{x_i\}$ for each $1 \leq i \leq m$. □

By Claim 3.10 (i) (iii), we have $r = m$ and $S = \{x_1, x_2, \ldots, x_m\}$. By Claim 3.10 (i) (ii), $d^c_G(x_i) = \delta^c(G) = (m+1)/2$ which implies that m is an odd integer.

Claim 3.11. *The set $V(G) \setminus V(T)$ is an independent set of G.*

Proof. Suppose that the set $V(G) \setminus V(T)$ is not an independent set of G. Since G is connected, there exist vertices x and y in $V(G) \setminus V(T)$ such that $xy \in E(G)$ and x is adjacent to some vertices in $V(T)$. Let x_i be a vertex in $V(T)$ adjacent to x. Since each monochromatic subgraph in G is a star, it follows from Lemma 3.1 that $c(xy) \neq c(xx_i)$ and $c(xy) \notin \{c_1, c_2, \ldots, c_r\}$. Since $G[\{x_1, x_2, \ldots, x_m\}]$ corresponds to the edge-colored graph K_m in $G^1_{m,k}$, $G[\{x_1, x_2, \ldots, x_m\}]$ has a rainbow spanning tree T' not containing the color $c(xx_i)$. Hence $T' + xy + xx_i$ is a rainbow tree with order $m + 2 = 2\delta^c(G) + 1 = |T| + 1$, which contradicts to the maximality of T. □

Let $k = |G| - m$. By Claim 3.10 (ii), $G[\{x_1, x_2, \ldots, x_m\}]$ corresponds to the edge-

colored graph K_m in $G^1_{m,k}$. Since the vertex in $V(W_{r+1})$ is not adjacent to any vertices in $V(G) \setminus V(T)$, it follows from Claim 3.11 that $V(W_{r+1}) \cup (V(G) \cup V(T))$ corresponds to $\{v_1, v_2, \ldots, v_k\}$ in $G^1_{m,k}$. Hence $G = G^1_{m,k} \in \mathcal{G}^1$.

Subcase 1.2. $m = 2$.

Recall that we have $V(W_1) \subseteq S$ or $V(W_2) \subseteq S$ by Claim 3.9 (ii). We will show that if $V(W_1) \subseteq S$ and $V(W_2) \subseteq S$, then $G \in \mathcal{G}^2$. Otherwise, $G \in \mathcal{G}^4$.

Claim 3.12. *If* $|V(W_i) \cap S| \geq 2$ *for some* $i \in \{1, 2\}$*, then* $|V(W_{3-i}) \cap S| = 1$.

Proof. Suppose that $|V(W_{3-i}) \cap S| \geq 2$. Note that we can choose any vertex in $V(W_{3-i}) \cap S$ as x_{3-i}. Let $y \in V(W_{3-i}) \cap S \setminus \{x_{3-i}\}$. By Claim 3.8 (i), we have $x_i y \in E(G)$ and $col(x_i y)$ does not belong to x_i. Now we choose y as x_{3-i}. Since $|V(W_i) \cap S| \geq 2$, we have $c(x_i y)$ belongs to x_i by Claim 3.8 (ii), a contradiction. □

Claim 3.13. *If* $V(W_i) \subseteq S$ *for some* $i \in \{1, 2\}$*, then* $V(W_i) = \{x_i\}$.

Proof. Suppose that $V(W_i) \subseteq S$ and $V(W_i) \setminus \{x_i\} \neq \emptyset$. By Claim 3.12, $\beta_{3-i} = |(V(W_{3-i}) \cap S) \setminus \{x_{3-i}\}| = 0$. By Claims 3.2 and 3.8 (ii), we obtain

$$d^c_G(x_i) = |C(x_i)| + |W_i| - 1 = |W_i|,$$
$$d^c_G(x_{3-i}) = |C(x_{3-i})| + |W_{3-i}| - 1 + |W_i| \geq |W_i| + 1.$$

However $d^c_G(x_i) = \delta^c(G)$ for each $1 \leq i \leq 2$, a contradiction. □

By Claim 3.9 (ii), we may assume that $V(W_1) \subseteq S$. By Claim 3.13, $V(W_1) = \{x_1\}$. Since $m = 2$, W_3 exists and $|W_3| = 1$ by Claim 3.9 (i). Let w_3 be the unique vertex in $V(W_3)$. Note that $\delta^c(G) \leq d^c_G(w_3) \leq |S|$.

Claim 3.14. *If* $V(W_2) \subseteq S$*, then* $|C(x_i)| = 2$*,* $|C(x_{3-i})| = 1$*, and* $c(x_i x_{3-1}) \in C(x_i)$ *for some* $i \in \{1, 2\}$.

Proof. By Claim 3.13, $V(W_i) = \{x_i\}$ for each $1 \leq i \leq 2$. We may assume that $c(x_1 x_2)$ belongs to x_1. Since $S = \{x_1, x_2\}$, $\delta^c(G) \leq d^c_G(w_3) \leq 2$. It is easy to see that $2\delta^c(G) = |T| \geq 3$. Hence $\delta^c(G) = 2$. Since $d^c_G(x_i) = \delta^c(G) = 2$ for each $1 \leq i \leq 2$, we obtain that $|C(x_1)| = 2$ and $|C(x_2)| = 1$. □

We show the construction of $G[V(T)]$ when the case $V(W_2) \subseteq S$. Suppose that $V(W_2) \subseteq S$. Then $\delta^c(G) = 2$ and so $|T| = 4$. This together with Claims 3.9 (i) and 3.13 implies that $V(T) = V(W_1) \cup V(W_2) \cup V(W_3) \cup V(W_4)$. Since $\delta^c(G) = 2$, every vertex in $V(W_3) \cup V(W_4)$ is adjacent to both x_1 and x_2. This together with Claim 3.9 (iii) show that the vertices in $V(W_3) \cup V(W_4)$ correspond to two of $v_1, v_2, \ldots, v_k$ in G_k^2, where $k = |V(G) \setminus (V(W_1) \cup V(W_2))|$. We may assume that $|C(x_1)| = 2$, $|C(x_2)| = 1$, and $col(x_1x_2) \in C(x_1)$. Then x_1 and x_2 correspond to u_1 and u_2 in G_k^2, respectively.

Next, we consider the construction of $G[V(T)]$ when the case $V(W_2) \subsetneq S$.

Claim 3.15. *If $V(W_2) \subsetneq S$, then the following statements hold.*

(i) *For any $1 \leq i < j \leq r$, c_i and c_j belong to distinct vertices in S.*

(ii) $|V(W_2) \setminus S| = 1$.

(iii) *W_2 is a rainbow clique not containing $c_1, c_2, \ldots, c_r$.*

(iv) *For any $x \in V(W_2) \cap S$, $c(xx_1) \in C(x)$.*

(v) *Every vertex in $V(W_3) \cup \ldots \cup V(W_{r+1})$ is adjacent to any vertices in S.*

Proof. **(i)** Suppose that there exists a vertex in S such that at least two colors in $\{c_1, c_2, \ldots, c_r\}$ belong to the vertex. Then $r \geq |S|+1$ and so $|V(W_3) \cup \ldots \cup V(W_{r+1})| = r - 1 \geq |S|$ by Claim 3.9 (i). Since $V(W_2) \subsetneq S$, $|V(W_1) \cup V(W_2)| \geq |S| + 1$. Hence $|T| \geq 2|S| + 1 > 2\delta^c(G)$, a contradiction.

(ii) Suppose that $|V(W_2) \setminus S| \geq 2$. Since $V(W_1) = \{x_1\}$, it follows from Claim 3.15 (i) that $|W_2| \geq (r-1) + 2 = r + 1$. Since $d_G^c(w_3) \leq |S| = r$, $d_G^c(x_i) = \delta^c(G) \leq r$ for each $1 \leq i \leq 2$. By Claims 3.2 and 3.6, we obtain

$$d_G^c(x_2) \geq |C(x_2)| + |W_2| - 1 = |W_2| \geq r + 1,$$

a contradiction.

(iii) Let w_2 be the unique vertex in $V(W_2) \setminus S$. Since we can take an arbitrary vertex from $V(W_2) \cap S$ as x_2, it follows from Claim 3.8 (i) that W_2 does not contain $c_1, c_2, \ldots, c_r$ and $G[V(W_2) \cap S]$ is a rainbow clique. By Claims 3.8 (i) and 3.15 (i) (ii), $\delta^c(G) = d_G^c(x_2) = |W_2| - 1 + 1 = r$. If w_2 is incident with two edges colored with same color, then $d_G^c(w_2) = |W_2| - 2 + 1 = r - 1$, a contradiction. Recall that each

monochromatic subgraph in G is a star. Hence W_2 is a rainbow clique not containing $c_1, c_2, \ldots, c_r$.

(iv) If $|V(W_2) \cap S| \geq 2$, then the claim holds by Claim 3.8 (ii). Hence we may assume $V(W_2) \cap S = \{x_2\}$. Then $\delta^c(G) = d^c_G(x_i) = 2$ for each $1 \leq i \leq 2$. Suppose that $c(x_1x_2) \in C(x_1)$. By Claim 3.15 (iii) and $c(x_1x_2) \in C(x_1)$, $d^c_G(x_2) = |c(x_1x_2)| + |C(x_2)| + |W_2| - 1 \geq 3$, a contradiction.

(v) By Claims 3.9 (i) and 3.15 (i) (ii), $2\delta^c(G) = |T| = |V(W_1) \cup V(W_2)| + |V(W_3) \cup \ldots \cup V(W_{r+1})| = (r+1) + (r-1) = 2r$ and so $\delta^c(G) = r$ which implies that every vertex in $V(W_3) \cup \ldots \cup V(W_{r+1})$ must be adjacent to any vertices in S. □

We show the construction of $G[V(T)]$ when $V(W_2) \subsetneq S$. Suppose that $V(W_2) \subsetneq S$. By Claims 3.9 (i) and 3.15 (i) (ii), $r = 1 + (|W_2| - 1) = |W_2|$ and $|V(W_3) \cup \ldots \cup V(W_{r+1})| = r - 1$. Hence by Claims 3.9 (iii) and 3.15, $G[V(W_2)]$, the vertex in $V(W_2) \setminus S$, the vertex in $V(W_1)$, and the vertices in $V(W_3) \cup \ldots \cup V(W_{r+1})$ correspond to the rainbow K_r, u, v, and $r-1$ of $v_1, v_2, \ldots, v_{r+k-1}$ in $G^4_{r-1,k}$, respectively, where $k = |V(G) \setminus V(T)|$.

Next, we consider the construction of $G[V(G) \setminus V(T)]$.

Claim 3.16. *The set $V(G) \setminus V(T)$ is an independent set of G.*

Proof. Suppose that the set $V(G) \setminus V(T)$ is not an independent set of G. Since G is connected, there exists a vertex x and y in $V(G) \setminus V(T)$ such that $xy \in E(G)$ and x is adjacent to a vertex z in $V(T)$. Note that $z \in S$. Since each monochromatic subgraph in G is a star, it follows from Lemma 3.1 that $c(xy) \neq c(xz)$.

Suppose that $V(W_2) \subseteq S$. Then $z = x_1$ or $z = x_2$. Recall that we assume that $c(x_1x_2)$ belongs to x_1, $|C(x_1)| = 2$ and $|C(x_2)| = 1$ when $V(W_2) \subseteq S$. Since the two colors in $C(x_1)$ are contained in T by Lemma 3.1, there exists a vertex y_1 in $V(W_3) \cup V(W_4)$ such that $c(x_1y_1) \neq c(x_1x_2)$. If $z = x_2$, then $x_1x_2 + xz + xy + x_1y_1$ is a rainbow tree with order five. Since T is a maximum rainbow tree and $|T| = 2\delta^{col}(G) = 4$, we get a contradiction. If $z = x_1$ and $c(xz) \neq c(x_1x_2)$, then $x_1x_2 + xz + xy + x_2y_1$ is a rainbow tree with order five, a contradiction. If $z = x_1$ and $c(xz) = c(x_1x_2)$, then $xz + xy + x_1y_1 + x_2y_1$ is a rainbow tree with order five, a contradiction.

Suppose that $V(W_2) \subsetneq S$. By Claim 3.15 (iii) (iv), for any $1 \leq i \leq r$, $G[V(W_1) \cup V(W_2)]$ has a rainbow spanning tree T' such that c_i is the only color of $c_1, c_2, \ldots, c_r$ contained in T'. Since $r \geq 2$, we can choose a color different from $c(xz)$ as c_i. By

choosing $r-2$ independent edges between $V(W_1) \cup V(W_2)$ and $V(W_3) \cup \ldots \cup V(W_r)$ colored with colors in $\{c_1, c_2, \ldots, c_r\} \setminus \{col(xz), c_i\}$, we can construct a rainbow tree containing $V(T') \cup V(W_3) \cup \ldots V(W_r) \cup \{x, y\}$. The order of the rainbow tree is $|T| - |W_{r+1}| + |\{x, y\}| = |T| + 1$. This contradicts to the maximality of T. □

Since $\delta^c(G) = r$, it follows from Claim 3.16 that every vertex in $V(G) \setminus V(T)$ is adjacent to any vertices in S. By Claim 3.16 and the explanation after the proof of Claim 3.14, if $V(W_2) \subseteq S$, then $G = G_l^2 \in \mathcal{G}^2$ with $l = |V(G) \setminus (V(W_1) \cup V(W_2))| \geq 3$. By Claim 3.16 and the explanation after the proof of Claim 3.15, if $V(W_2) \subsetneq S$, then $G = G_{r-1,k}^4 \in \mathcal{G}^4$ with $k = |V(G) \setminus V(T)|$.

Case 2. $m = 1$.

We will show that $G \in \mathcal{G}^6$.

Claim 3.17. *We have $V(W_1) = S$.*

Proof. Suppose that there exists a vertex u in $V(W_1) \setminus S$. Let v be a vertex in $V(W_2)$. Since $m = 1$, we obtain

$$\begin{aligned} d_G^c(u) + d_G^c(v) &\leq |W_1| - 1 + |W_2| - 1 + r \\ &\leq |W_1| + |W_2| + r - 2 \\ &\leq |T| - 1 \\ &= 2\delta^c(G) - 1. \end{aligned}$$

This implies that either $d_G^c(u)$ or $d_G^c(v)$ is less than $\delta^c(G) - 1$, a contradiction. □

Claim 3.18. *The following statements hold.*

(i) $|W_2| = \cdots = |W_{r+1}| = 1$.

(ii) *Each vertex in $V(T) \setminus V(W_1)$ is adjacent to all vertices of W_1.*

(iii) *For each $x \in V(T)$, $d_G^c(x) = \delta^{col}(G)$.*

(iv) *For any $1 \leq i < j \leq r$, c_i and c_j belong to distinct vertices in S.*

(v) *W_1 is a rainbow clique not containing $c_1, c_2, \ldots, c_r$.*

Proof. Since G is connected, for each $2 \leq i \leq r+1$, there exists a vertex $w \in V(W_i)$ adjacent to some vertices of W_1. We choose an arbitrary vertex $w \in V(W_2) \cup \ldots \cup V(W_{r+1})$ which has neighbors in W_1. We may assume that $w \in V(W_i)$, where $2 \leq i \leq r+1$. We choose an arbitrary vertex $w_1 \in V(W_1) \cap N_G(w)$. Then

$$\begin{aligned} d^c_G(w_1) + d^c_G(w) &\leq (|W_1| - 1 + |C(w_1)|) + (|W_i| - 1 + r - |C(w_1)| + 1) \\ &\leq |W_1| + |W_i| + r - 1 \\ &\leq |T| \\ &= 2\delta^c(G). \end{aligned}$$

Since $d^c_G(w_1) + d^c_G(w) \geq 2\delta^c(G)$, the above inequalities are equalities and so we obtain the following equalities:

$$d^c_G(w_1) = |W_1| - 1 + |C(w_1)|, \tag{3.8}$$

$$d^c_G(w) = |W_i| - 1 + r - |C(w_1)| + 1, \tag{3.9}$$

$$|T| = |W_1| + |W_i| + r - 1. \tag{3.10}$$

(i) By (3.10), we have $|W_2| + |W_3| + \cdots + |W_{r+1}| - |W_i| = r - 1$ and it implies that $|W_j| = 1$ for each $2 \leq j \neq i \leq r+1$. Since i is arbitrarily chosen from $\{2, \ldots, r+1\}$, we have $|W_2| = \cdots = |W_{r+1}| = 1$.

(ii) Since $d^c_G(w) = |W_i| - 1 + r - |C(w_1)| + 1$, it follows from Claim 3.17 that w is adjacent to all vertices of W_1. Since we choose an arbitrary i from $\{2, \ldots, r+1\}$, Claim 3.18 (ii) holds.

(iii) By Claim 3.18 (ii), we can choose any pair of vertices from $V(W_1)$ and $V(T) \setminus V(W_1)$ as w_1 and w, respectively. Since $d^c_G(w_1) + d^c_G(w) = 2\delta^c(G)$, we obtain $d^c_G(w_1) = d^c_G(w) = \delta^c(G)$. Hence, Claim 3.18 (iii) holds.

(iv) When $|S| = 1$, we obtain that $|W_1| = 1$ and $W_1 = \{x_1\}$. Then for each $w \in V(T) \setminus V(W_1)$, $d^c_G(w) = 1$. By Claim 3.18 (iii), $d^c_G(x_1) = 1$. Hence $r = 1$, c_1 belongs to x_1 and so Claim 3.18 (iv) holds when $|S| = 1$.

Suppose that $|S| \geq 2$ and there exists a vertex x in S such that two different colors c_i and c_j belong to x. By Claim 3.18 (ii), each vertex in W_1 is adjacent to all vertices of $V(T) \setminus V(W_1)$. Let $w_1 \in V(W_1) \setminus \{x\}$ and $w \in V(T) \setminus V(W_1)$. Then $d^c_G(w) \leq r - |C(x)| + 1 - |C(w_1)| + 1 \leq r - |C(w_1)|$, which contradicts to (3.9).

(v) By Claim 3.18 (ii) and (3.8), we have $d^c_G(w_1) = |W_1| - 1 + |C(w_1)|$ for each $w_1 \in V(W_1)$. Hence w_1 is adjacent to all vertices in $V(W_1) \setminus \{w_1\}$, and colors of these incident edges are distinct and not contained in $C(w_1)$. Since every monochromatic subgraph is a star, W_1 is a rainbow clique not containing $c_1, c_2, \ldots, c_r$. □

By Claim 3.18, $\delta^c(G) = |S| = |W_1| = |W_2| + \ldots + |W_{r+1}| = r$. Note that $G[V(W_1)]$ and the vertices in $V(W_2) \cup \ldots \cup V(W_{r+1})$ correspond to the rainbow K_r and r of $v_1, v_2, \ldots, v_{r+k}$ in $G^6_{r,k}$, where $k = |V(G) \setminus V(T)|$.

Claim 3.19. *The set $V(G) \setminus V(T)$ is an independent set of G.*

Proof. Suppose that $V(G) \setminus V(T)$ is not an independent set. Then there exist two vertices x and y in $V(G) \setminus V(T)$ such that $xy \in E(G)$ and x is adjacent to a vertex $z \in V(T)$. Since $V(W_1) = S$, we have $z \in V(W_1)$. Since E_c forms a star for each color c, it follows from Lemma 3.1 that $c(xy) \neq c(xz)$. By choosing $r - 1$ independent edges between $V(W_1)$ and $V(W_2) \cup \ldots \cup V(W_r)$ not colored with $c(xz)$, we can construct a rainbow tree containing $V(W_1) \cup V(W_2) \cup \ldots \cup V(W_r) \cup \{x, y\}$. The order of the rainbow tree is $|T| + 1$, which contradicts to the maximality of T. □

Since $\delta^c(G) = r$, it follows from Claim 3.19 that every vertex in $V(G) \setminus V(T)$ is adjacent to every vertex in W_1. Then $G = G^6_{r,k} \in \mathcal{G}^6$, where $k = |V(G) \setminus V(T)|$. □

3.4 Proofs of Theorems 3.4 and 3.6

Proof of Theorem 3.4. Suppose the contrary and let G be a counterexample to Theorem 3.4 such that

(i) G is edge-minimal, i.e., any additional edge deletion would lead to a decrease in $d^c_G(v)$ for some vertex v in G;

(ii) subject to **(i)**, the number of colors used in G is maximum among all edge-minimal counterexamples of order $|G|$.

We claim that there is no monochromatic P_4 in G. Suppose that there is a monochromatic P_4, say $uvwx$, in G. We delete the edge vw from G. If $G - vw$ is disconnected, then we can find a properly colored tree with order $2\delta^c(G)$ which consists of v, w, $N^c_G(v) \setminus \{u, w\}$ and $N^c_G(w) \setminus \{x, v\}$, a contradiction. Hence $G - vw$ is connected. Moreover, the deletion of vw does not lead to any decrease in color degree,

which contradicts to the edge-minimality of G. Thus there is no monochromatic P_4 in G and every monochromatic subgraph is a disjoint union of stars.

Next, we show that every monochromatic subgraph in G is a star. Suppose that there are at least two disjoint stars colored with b. We recolor one of them, say S, with a new color b' and denote the new graph by G'. Now, by the condition (ii), G' is not a counterexample and there is a maximum properly colored tree T with order at least $\min\{|G|, 2\delta^c(G)\}$. There must exist an edge in T colored with b'. We recolor S with b. Since there is no edge colored with b incident with vertices of S in G', we obtain a maximum properly colored tree with order at least $\min\{|G|, 2\delta^c(G)\}$ in G, a contradiction.

Therefore, every monochromatic subgraph in G is a star, i.e., E_c forms a star for each color c. By Theorem 3.5, the order of maximum rainbow tree in G is at least $\min\{|G|, 2\delta^c(G)\}$, a contradiction. □

Proof of Theorem 3.6. Let G be an edge-colored connected graph satisfying the condition of Theorem 3.6, i.e., the order of maximum properly colored tree in G is $2\delta^c(G) \neq |G|$, but $G \notin \mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^3 \cup \mathcal{G}^5$. We still assume that

(i) G is edge-minimal;

(ii) subject to **(i)**, the number of colors used in G is maximum among all edge-minimal counterexamples of order $|G|$.

Firstly, we also show that there is no monochromatic P_4 in G. Suppose that there is a monochromatic P_4, say $uvwx$, in G. We delete the edge vw from G.

Suppose that $G - vw$ is disconnected. Denote the two connected components of $G - vw$ by G_1 and G_2. Assume $\{u, v\} \subseteq V(G_1)$. Let A_v and A_w be maximum subsets of $N^c_{G_1}(v)$ and $N^c_{G_2}(w)$, respectively, such that $u \in A_v$ and $x \in A_w$. If $|A_v| \geq \delta^c(G)+1$, then $G[(A_v \setminus \{u\}) \cup (A_w \setminus \{x\}) \cup \{v, w\}]$ has a properly colored spanning tree with order $2\delta^c(G) + 1$, a contradiction. Hence $|A_v| = \delta^c(G)$. For every vertex $y \in A_v \setminus \{u\}$ and its neighbor $z \in (V(G_1) \setminus (A_v \cup \{v\})) \cup \{u\}$, we have $c(yz) = c(yv)$, otherwise $G[(A_v \setminus \{u\}) \cup (A_w \setminus \{x\}) \cup \{v, w, z\}]$ has a properly colored spanning tree with order $2\delta^c(G) + 1$, a contradiction. Hence $d^c_G(y) \leq |(A_v \setminus \{y, u\}) \cup \{v\}| = \delta^c(G) - 1$, a contradiction.

Thus $G - vw$ is connected. Similar to the proof of Theorem 3.4, we can obtain that every monochromatic subgraph in G is a disjoint union of stars.

We also claim that every monochromatic subgraph in G is a star. Suppose that there are at least two disjoint stars colored with b_1. We recolor one of them, say S_1, with a new color b_1' and denote the new graph by G'. The order of maximum properly colored tree in G' is also $2\delta^c(G) \neq |G|$. By the condition (ii), G' is not a counterexample and $G' \in \mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^3 \cup \mathcal{G}^5$. Since any two monochromatic stars of a graph in $\mathcal{G}^1 \cup \mathcal{G}^2$ have at least one common vertex, $G' \notin \mathcal{G}^1 \cup \mathcal{G}^2$ and $G' \in \mathcal{G}^3 \cup \mathcal{G}^5$.

Suppose $G' = G^3_{m_3,k_3} \in \mathcal{G}^3$ for some positive integers m_3 and k_3. If S_1 is in the properly colored K_{m_3+1} of G', then S_1 has only one edge. If S_1 is not completely in the properly colored K_{m_3+1}, then S_1 is not isomorphic to K_2, and its center is v or in $K_{m_3+1} \setminus \{u\}$ (u and v are followed $G^3_{m,k}$ in Section 2). Since every monochromatic subgraph in G is a disjoint union of stars, in above two cases, we can recolor S_1 with b_1 and obtain that G also belongs to $\mathcal{G}^3$, a contradiction.

Suppose $G' = G^5_{m_5,k_5} \in \mathcal{G}^5$ for some positive integers m_5 and k_5. If S_1 is in the properly colored K_{m_5} of G', then S_1 has only one edge. If the edges of S_1 are between the properly colored K_{m_5} and the independent set $\overline{K_{m_5+k_5}}$ in G', then S_1 is not isomorphic to K_2 and its center is in K_{m_5}. Similarly, for both cases, we can recolor S_1 with b_1 and obtain that G also belongs to $\mathcal{G}^5$, a contradiction.

Therefore, every monochromatic subgraph in G is a star. By Theorem 3.7, $G \in \mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^4 \cup \mathcal{G}^6 \subseteq \mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^3 \cup \mathcal{G}^5$, a contradiction. □

3.5 Concluding Remarks

Recall that for an edge-colored connected graph G, Cheng, Kano and Wang [39] proved that if $\delta^c(G) \geq |G|/2$, then G has a properly colored spanning tree. Later, Kano, Maezawa, Ota, Tsugaki and Yashima [101] derived that if $\bar{\sigma}^c_2(G) \geq |G|$, then G has a properly colored spanning tree. It is also interesting to study other types of sufficient condition forcing a properly colored spanning tree, such as $\Sigma_{v \in V(G)} d^c(v)$ and $e(G) + c(G)$.

Chapter 4

Properly colored 2-factors in edge-colored graphs

4.1 Introduction

Recall that a graph G is said to be a 1-*path-cycle* if G is a vertex-disjoint union of at most one path and a number of cycles. In particular, 1-*edge-cycle* is a vertex-disjoint union of at most one edge and a number of cycles.

There is a basic fact that every graph G with $\delta(G) \geq 2$ contains a cycle. However, for any positive integer k, there exists an edge-colored graph G such that $\delta^c(G) \geq k$, but G contains no properly colored cycles [156]. Grossman and Häggkvist [82], and Yeo [160], independently, gave the following structural property: If G is an edge-colored graph containing no properly colored cycle, then there is a vertex $v \in V(G)$ such that no component of $G - v$ is joint to v with edges of more than one color. An interesting problem arises: Determine the least function $f(n)$ such that G contains a properly colored cycle if $\delta^c(G) \geq f(n)$, where G is an edge-colored graph on n vertices. There are many results [74, 121, 122, 133, 136, 157] to this problem for properly colored cycles of given lengths. Recently, Fujita, Li and Zhang [73] proved that if $\delta^c(G) \geq D$ such that $D! \sum_{i=0}^{D} \frac{1}{i!} \geq n + 1$, then G contains a properly colored cycle, where the lower bound D is tight.

A classical theorem of Dirac [58] states that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamiltonian cycle. A natural generalization of Dirac's theorem is to determine the minimum color degree threshold for the existence of a properly colored Hamiltonian cycle. Many researchers investigate this problem in

edge-colored complete graphs K_n^c. Let $\Delta^{mon}(K_n^c) = \max \Delta(H)$ over all monochromatic subgraphs H in K_n^c. Note that $\Delta^{mon}(K_n^c)+\delta^c(K_n^c) \leq n$. Bollobás and Erdős [21] proved that if $\Delta^{mon}(K_n^c) \leq n/69$, then K_n^c contains a properly colored Hamiltonian cycle. They further conjectured that $\Delta^{mon}(K_n^c) < \lfloor n/2 \rfloor$ suffices. Their result was subsequently improved by Chen and Daykin [35], Shearer [149] and Alon and Gutin [5]. Moreover, Alon and Gutin [5] showed that for every $\varepsilon > 0$, there exists an $m_0 = m_0(\varepsilon)$ so that for every $m > m_0$, every $K_{m,m}^c$ satisfying $\Delta(K_{m,m}^c) \leq (1-1/\sqrt{2}-\varepsilon)m$ contains a properly colored cycle of every even length between 4 and $2m$. In 2016, Lo [135] proved that the conjecture of Bollobás and Erdős is asymptotically true.

In [134], Lo generalized Dirac's theorem to edge-colored graphs and proved that every edge-colored graph G with $\delta^c(G) \geq 2|G|/3$ contains a properly colored 2-factor, which is best possible. Moreover, by establishing an edge-colored version of the absorbing technique, he proved the following result.

Theorem 4.1. [134] *For any $\varepsilon > 0$, there exists an integer n_0 such that every edge-colored graph G with $\delta^c(G) \geq (2/3+\varepsilon)|G|$ and $|G| \geq n_0$ contains a properly colored cycle of length l for all $3 \leq l \leq |G|$.*

Resting on the shoulders of Lo's result and method, we generalize the result above and prove the following theorem.

Theorem 4.2. *For any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N(\varepsilon, t)$ such that every edge-colored graph G with $\delta^c(G) \geq (2/3+\varepsilon)|G|$ and $|G| > N(\varepsilon, t)$ contains every properly colored 2-factor with exactly t components.*

It can be seen that Theorem 4.2 implies Theorem 4.1. In fact, let $t = 2$ in Theorem 4.2 and we get that G contains every properly colored cycle of length $3 \leq l \leq |G| - 3$ when $|G| \geq N(\varepsilon, 2)$. Let $t = 1$ and we get that G contains a properly colored Hamiltonian cycle when $|G| \geq N(\varepsilon, 1)$. Now let G_i $(i = 1, 2)$ be the subgraph of G by deleting exactly i vertices of G. We have $\delta^c(G_i) \geq (2/3+\varepsilon/2)|G_i|$ when $|G| > 4/\varepsilon$ for $i = 1, 2$. Suppose that $|G| > N(\varepsilon/2, 1) + 2$, then $|G_i| > N(\varepsilon/2, 1)$ and G_i contains a properly colored Hamiltonian cycle for $i = 1, 2$, i.e., G contains every properly colored cycle of length $l \in \{|G|-2, |G|-1\}$.

As we know, there are some interesting results in bipartite graphs for the existence of 2-factors or Hamiltonian cycles, e.g. Moon and Moser's Theorem [140].

Theorem 4.3. [140] *Every balanced bipartite graph G with minimum degree $\delta(G) > |G|/4$ is Hamiltonian.*

Basing on Lo's results and methods in [134], we consider analogues of these results for edge-colored balanced bipartite graphs. Firstly, we give a sharp color degree condition for the existence of 2-factors in edge-colored balanced bipartite graphs.

Theorem 4.4. *Every edge-colored balanced bipartite graph G with $\delta^c(G) \geq |G|/3 + 1$ contains a properly colored 2-factor.*

Furthermore, we give an asymptotic bound for the existence of arbitrary properly colored 2-factors in edge-colored balanced bipartite graphs.

Theorem 4.5. *For any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N'(\varepsilon, t)$ such that every edge-colored balanced bipartite graph G with $\delta^c(G) \geq (1/3 + \varepsilon)|G|$ and $|G| > N'(\varepsilon, t)$ contains every properly colored even 2-factor with exactly t components.*

By the same argument as above, we have the following corollary.

Corollary 4.1. *For any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N'(\varepsilon, t)$ such that every edge-colored balanced bipartite graph G with $\delta^c(G) \geq (1/3 + \varepsilon)|G|$ and $|G| > N'(\varepsilon, t)$ contains a properly colored cycle of length l for all even $4 \leq l \leq |G|$.*

We now construct a graph to show that the bound in Theorem 4.4 is tight. Assume that $G = (X, Y)$ is a balanced bipartite graph with $|X| = |Y| = n = 3k$. Let $X = A_0 \cup B_0$ and $Y = A_1 \cup B_1$. Moreover, let $|A_i| = 2k - 1$, $|B_i| = k + 1$ and $A_i \cap B_i = \emptyset$ for $i \in \{0, 1\}$. Let $G[A_0 \cup A_1]$ be a rainbow complete bipartite graph. For every vertex $v \in A_i$ $(i \in \{0, 1\})$, add an edge between v and u for each $u \in B_{(i+1) \bmod 2}$ and color these edges with a new color. Further, choose exactly one vertex $x \in B_0$ and one vertex $y \in B_1$. Then add an edge between x and v for each $v \in B_1 \backslash \{y\}$ and color these edges with a new color. Moreover, add an edge between y and v for each $v \in B_0 \backslash \{x\}$ and color these edges with a new color. By our construction, $d^c(v) = 2k$ for all $v \in V(G)$ and so $\delta^c(G) = |G|/3$. (See Figure 4.1.) However, there is no properly colored 2-factor in G. If not, without loss of generality, suppose that $F = \{C_1, C_2, \ldots, C_t\}$ is a properly colored 2-factor in G and $B_0 \subseteq V(C_1) \cup V(C_2) \cup \ldots \cup V(C_l)$ where $V(C_i) \cap B_0 \neq \emptyset$ for each $i \in \{1, 2, \ldots, l\}$ and $l \leq t$. For each $v \in V(G)$, let $C_v \in F$ be the cycle that contains v and $N_{C_v}(v)$ be the two neighbors of v on C_v. Then we have

$$2|B_0| = |\cup_{v \in B_0} N_{C_v}(v)| \leq |A_1 \cup \{y, u\}|, \tag{4.1}$$

where $u \in B_1 \backslash \{y\}$. Since $2|B_0| = 2k+2$ and $|A_1 \cup \{y, u\}| = 2k+1$, (4.1) is a contradiction.

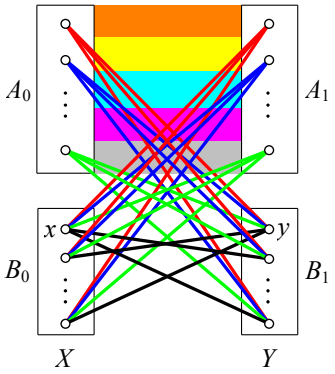


Figure 4.1: An example G to show the sharpness of bound in Theorem 4.4.

The absorption technique was introduced by Rödl, Ruciński and Szemerédi [145] which was used to tackle Hamiltonian problems in hypergraphs. In [134], Lo established the "edge-colored" version of absorption technique. In the proof of Theorem 4.2 and Theorem 4.5, we generalize this technique and obtain "many" absorbing sets to deal with arbitrary properly colored 2-factors. Moreover, in the proofs of Theorem 4.5, we rebuild up some lemmas for the "bipartite" case, which is also of independent interest. In the proof of Theorem 4.4, we want to prove that the edge-colored graph G contains a properly colored 2-factor, and we choose a counterexample with order $|G|$ under the color degree condition using the maximum number of colors. So G can not be properly colored, otherwise we are done since G contains a 2-factor. Hence there exist two adjacent edges e_1 and e_2 having the same color, by changing the color of e_1 to a new color, then we get a new edge-colored graph G' containing a properly colored 2-factor C containing the edge e_1. Hence G contains a properly colored 1-path cycle $C - e_1$. This method is introduced in [41] for the existence of rainbow matchings and also works in various edge-colored type problems. It can avoid some tedious case analysis. Since the proof of Theorem 4.2 is the same as (easier than) the proof of Theorem 4.5 based on the results in [134], we only give the latter's proof.

4.2 Terminology and notation

Unless stated otherwise, G will be assumed to be an edge-colored graph with edge-coloring c. Let $c(G)$ be the size of the set $\{c(e) \mid e \in E(G)\}$. For $v \in V(G)$, we denote the neighborhood of v in G by $N_G(v)$. Let $N_G^c(v)$ be a maximal subset of $N_G(v)$

such that $c(vy_1) \neq c(vy_2)$ for all distinct $y_1, y_2 \in N^c_G(v)$. Let $CN_G(v)$ be the set of colors assigned to edges incident to v. If the graph G is understood, then we omit the subscript. Let $U \subseteq V(G)$. We write $G[U]$ for the edge-colored subgraph of G induced by U. We also write $G \setminus U$ for the edge-colored subgraph obtained from G by deleting all vertices in U. For edge-disjoint edge-colored graphs G and H, we denote by $G + H$ the union of G and H.

In this chapter, every cycle and every path are assumed to have directions. Let H be a union of vertex-disjoint cycles with directions. For each $y \in V(H)$, let C_y be the cycle in H that contains y. For every vertex y in a cycle or a path, denote by y_+ and y_- the successor and ancestor of y, respectively. Write y_{--} for $(y_-)_-$ and y_{++} for $(y_+)_+$. For distinct vertices y, z in a cycle C, define yC^+z and yC^-z to be the paths $yy_+ \dots z_-z$ and $yy_- \dots z_+z$ in C, respectively. For distinct vertices v_i, v_j in a path P, define v_iPv_j to be the subpath of P between v_i and v_j, including v_i and v_j.

In all figures of this chapter, if there is a "$\times$" on an edge, then it means that we would delete the edge.

4.3 Proof of Theorem 4.4

Suppose the contrary and let G be a counterexample with $c(G)$ is maximum among all counterexamples of order $|G|$.

Claim 4.1. *G contains a properly colored spanning 1-path-cycle.*

Proof. Since $\delta(G) \geq \delta^c(G) \geq |G|/3 + 1$, by the theorem of Moon and Moser [140], G is not properly colored, otherwise there is a properly colored Hamiltonian cycle in G. So there exist two adjacent edges $e_1, e_2 \in E(G)$ with $c(e_1) = c(e_2)$. If we recolor e_1 with a color that does not appear in G and denote the new coloring of G by c', then G contains a properly colored 2-factor, following by the maximality of $c(G)$. Deleting e_1 from the 2-factor, we will get a properly colored spanning 1-path-cycle of G. □

We choose a properly colored spanning 1-path-cycle H from G in which the path is longest among all properly colored spanning 1-path-cycles of G. Denote the path by $P = z_1z_2 \cdots z_l$ and the direction is from z_1 to z_l. Since G is bipartite, we have $l \geq 4$ and l is even. Note that if $z_1z_l \in E(G)$, then $c(z_1z_l) = c(z_1z_2)$ or $c(z_lz_{l-1})$, otherwise $H + z_1z_l$ is a properly colored 2-factor of G, a contradiction. Without loss of generality, we assume that $c(z_1z_l) = c(z_1z_2)$.

Let $N_1 = \{y \in N_G(z_1) : c(z_1y) \neq c(z_1z_2)\} \cap V(P)$ and $N_l = \{y \in N_G(z_l) : c(z_ly) \neq c(z_lz_{l-1})\} \cap V(P)$. Let N_i^c be a maximal subset of N_i such that $c(z_iy_1) \neq c(z_iy_2)$ for all distinct $y_1, y_2 \in N_i^c$, where $i \in \{1, l\}$. If there exists $v \in N_G(z_1)\backslash V(P)$ and $c(z_1v) \neq c(z_1z_2)$, then orient C_v into a directed cycle such that $c(vv_+) \neq c(z_1v)$. Thus $H - C_v - P + v_-C_v^-vP$ contradicts the maximality of l. Therefore, $|N_1^c| \geq \delta^c(G) - 1 > \delta^c(G) - 2$ and $|N_l^c| \geq \delta^c(G) - 1$.

For $i \in \{1, l\}$, define

$$R_i = \{y : y_- \in N_i^c \text{ and } c(z_iy_-) = c(yy_-)\},$$
$$S_i = \{y : y_+ \in N_i^c \text{ and } c(z_iy_+) \neq c(y_+y_{++})\}.$$

Hence, $|R_i| + |S_i| = |N_i^c|$.

Set $T_1 = (R_1 \cup \{z_3\}) \cap S_1$ and $T_l = R_l \cap S_l$. Note that $z_3 \notin R_1$. This is because $y_- \in N_1^c$ for every $y \in R_1$ and $z_{3-} = z_2 \notin N_1^c$. Let

$$R_1' = (R_1 \cup \{z_3\})\backslash T_1, \quad S_1' = S_1\backslash T_1, \quad W_1 = \{z_1\} \cup R_1' \cup S_1' \cup T_1, \quad U_1 = W_1\backslash T_1,$$
$$R_l' = R_l\backslash T_l, \quad S_l' = S_l\backslash T_l, \quad W_l = \{z_l\} \cup R_l' \cup S_l' \cup T_l, \quad U_l = W_l\backslash T_l.$$

Then we have $|R_1'| + |S_1'| + 2|T_1| = |R_1 \cup \{z_3\}| + |S_1| = |N_1^c| + 1 \geq \delta^c(G) - 1$ and $|R_l'| + |S_l'| + 2|T_l| = |R_l| + |S_l| = |N_l^c| \geq \delta^c(G) - 1$. Note that W_1 and W_l are contained in different parts of the bipartite graph G.

For an edge $xy \in E(G[W_1 \cup W_l])$, we say that xy is *incompatible* with y

- if $y = z_1$ and $c(xy) = c(z_1z_2)$;
- if $y = z_l$ and $c(xy) = c(z_lz_{l-1})$;
- if $y \in R_1' \cup R_l'$ and $c(xy) = c(yy_+)$;
- if $y \in S_1' \cup S_l'$ and $c(xy) = c(yy_-)$.

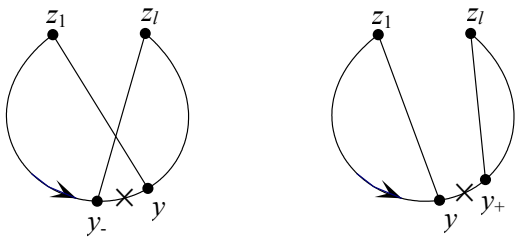
We define a digraph D on $W_1 \cup W_l$. There is a directed edge from $x \in W_p$ to $y \in W_q$ if $xy \notin E(G)$ or xy is incompatible with y, where $\{p, q\} = \{1, l\}$. So the in-degree of a vertex $v \in U_1 \cup U_l$ reflects numbers of monochromatic incident edges of v and vertices that are not adjacent to v in $G[U_1 \cup U_l]$. By Claim 4.2 and Claim 4.3, we can get that there is a vertex in $U_1 \cup U_l$ with a large in-degree.

Claim 4.2. *If $xy \in E(G[U_1 \cup U_l])$, then xy is incompatible with x or y.*

Proof. Suppose that there exists an edge $xy \in E(G[U_1 \cup U_l])$ satisfying xy is not incompatible with x and y. Since G is bipartite, we may assume that $x \in U_1$ and $y \in U_l$. We will show that there exists a properly colored 2-factor in $G[V(P)]$ or a properly colored spanning 1-edge-cycle in G. It is a contradiction as G is a counterexample to Theorem 4.4 and G contains no properly colored spanning 1-edge-cycle (see Claim 4.4).

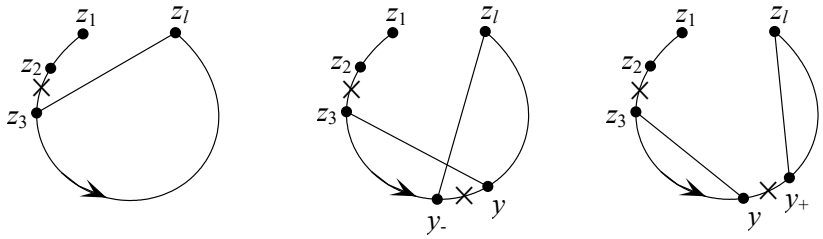
Case 1. Let $x = z_1$ or $y = z_l$ and $x \neq z_3$. If $x = z_1$ and $y = z_l$, then $c(xy) = c(z_1z_2)$ or $c(xy) = c(z_lz_{l-1})$, a contradiction. This is because xy is not incompatible with x and y. If $x = z_1$ and $y \in R'_l$, then $y_- \in N^c_l$ and $c(z_ly_-) \neq c(y_-y_{--})$. Since $c(z_1z_2) \neq c(z_1y) \neq c(yy_+)$, $z_1Py_-z_lPyz_1$ is a properly colored cycle spanning $V(P)$ (see Figure 4.2(a)). If $x = z_1$ and $y \in S'_l$, then $y_+ \in N^c_l$ and $c(z_ly_+) \neq c(y_+y_{++})$. Since $c(z_1z_2) \neq c(z_1y) \neq c(yy_-)$, $z_1Pyz_1 + z_lPy_+z_l$ is a properly colored 2-factor in $G[V(P)]$ (see Figure 4.2(b)).

If $y = z_l$ and $x \in R'_1 \cup S'_1 \backslash \{z_3\}$, then the proof is similar to the above. So we omit the detail.



(a) $x=z_1$, $y\in R_l'$ (b) $x=z_1$, $y\in S_l'$

Figure 4.2



(a) $x=z_3$, $y=z_l$ (b) $x=z_3$, $y\in R_l'$ (c) $x=z_3$, $y\in S_l'$

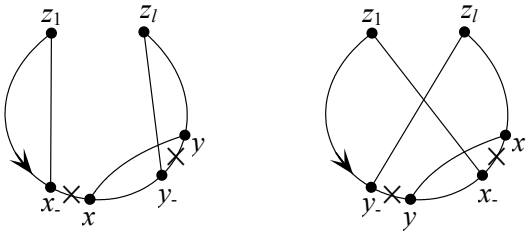
Figure 4.3

Case 2. Let $x = z_3 \in R'_1$. If $y = z_l$, then $c(z_3z_4) \neq c(z_3z_l) \neq c(z_lz_{l-1})$. So $H - P + z_3Pz_lz_3 + z_1z_2$ is a properly colored spanning 1-edge-cycle in G (see Figure 4.3(a)). If $y \in R'_l$, then $y_- \in N^c_l$ and $c(z_ly_-) \neq c(y_-y_{--})$. Since $c(z_3z_4) \neq c(z_3y) \neq c(yy_+)$, $C = z_3Py_-z_lPyz_3$ is a properly colored cycle and $H - P + C + z_1z_2$ is a properly colored spanning 1-edge-cycle in G (see Figure 4.3(b)). If $y \in S'_l$, then $y_+ \in N^c_l$ and

$c(z_l y_+) \neq c(y_+ y_{++})$. Since $c(z_3 z_4) \neq c(z_3 y) \neq c(y y_-)$, $C_1 = z_3 P y z_3$ and $C_2 = z_l P y_+ z_l$ are properly colored cycles. Then $H - P + C_1 + C_2 + z_1 z_2$ is a properly colored spanning 1-edge-cycle in G (see Figure 4.3(c)).

Case 3. Let $x \in R'_1 \cup S'_1 \backslash \{z_3\}$ and $y \in R'_l \cup S'_l$.

Suppose that $x \in R'_1 \backslash \{z_3\}$ and $y \in R'_l$, then $c(x x_+) \neq c(xy) \neq c(y y_+)$. In particular, $x_+ \neq y$ and $x \neq y_+$. If $x_+ < y$, then let $C_1 = z_1 P x_- z_1$, $C_2 = x P y_- z_l P y x$ and $H - P + C_1 + C_2$ is a properly colored 2-factor in G (see Figure 4.4(a)). If $x > y_+$, then let $C = z_1 P y_- z_l P x y P x_- z_1$ and $H - P + C$ is a properly colored 2-factor in G (see Figure 4.4(b)).



(a) $x_+ < y$ (b) $y_+ < x$

Figure 4.4

Suppose that $x \in R'_1 \backslash \{z_3\}$ and $y \in S'_l$, then $x \neq y_+$; otherwise, let $C_1 = z_1 P y z_1$, $C_2 = z_l P x z_l$ and $H - P + C_1 + C_2$ is a properly colored 2-factor in G. If $x > y_+$, then let $C = z_1 P y x P z_l y_+ P x_- z_1$ and $H - P + C$ is a properly colored 2-factor in G. If $x_+ = y$, then $c(xy) = c(x x_+)$, a contradiction. If $x_+ < y$, then let $C_1 = z_1 P x_- z_1$, $C_2 = x P y x$, $C_3 = y_+ P z_l y_+$ and $H - P + C_1 + C_2 + C_3$ is a properly colored 2-factor in G.

Suppose that $x \in S'_1$ and $y \in S'_l$, then by reversing the orientation of P, we are back in the case that $x \in R'_1 \backslash \{z_3\}$ and $y \in R'_l$. So we are done.

Suppose that $x \in S'_1$ and $y \in R'_l$, if $x_+ < y$, then let $C = z_1 P x y P z_l y_- P x_+ z_1$ and $H - P + C$ is a properly colored 2-factor in G. If $x_+ = y$, then let $C = z_1 P x z_l P y z_1$ $H - P + C$ is a properly colored 2-factor in G. If $x = y_+$, then $c(xy) = c(y y_+)$, a contradiction. If $x > y_+$, then let $C_1 = z_1 P y_- z_l P x_+ z_1$, $C_2 = y P x y$ and $H - P + C_1 + C_2$ is a properly colored 2-factor in G. □

Claim 4.3. *If xy is an edge with $x \in T_p$ and $y \in U_q$, where $\{p, q\} = \{1, l\}$, then xy is incompatible with y.*

Proof. Suppose not, we will find a properly colored 2-factor or a properly colored

spanning 1-edge-cycle, which is a contradiction.

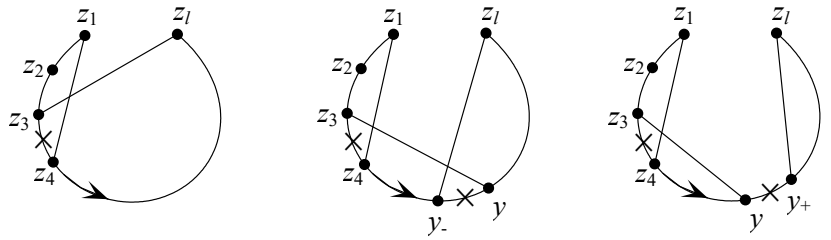
Case 1. $x \in T_1$ and $y \in U_l$.

If $x = z_3 \in T_1$, then we have $z_3 \in S_1$. So $z_4 \in N_1$ and $c(z_1z_4) \neq c(z_4z_{4+})$. Since P is properly colored, $c(z_3y) \neq c(z_2z_3)$ or $c(z_3y) \neq c(z_3z_4)$.

- $y = z_l$. If $c(z_3z_l) \neq c(z_2z_3)$, then $c(z_3z_l) = c(z_lz_{l-1})$, i.e., z_3z_l is incompatible with z_l. Otherwise, $z_1Pz_3z_lPz_4z_1$ is a properly colored cycle spanning $V(P)$ (see Figure 4.5(a)). If $c(z_3z_l) \neq c(z_3z_4)$, then $c(z_3z_l) = c(z_lz_{l-1})$, i.e., z_3z_l is incompatible with z_l. Otherwise, let $C = z_3Pz_lz_3$ and then $H - P + C + z_1z_2$ is a properly colored spanning 1-edge-cycle in G (see Figure 4.3(a)).

- $y \in R'_l$. If $c(z_3y) \neq c(z_2z_3)$, then $c(z_3y) = c(yy_+)$, i.e., z_3y is incompatible with y. Otherwise, $z_1Pz_3yPz_ly_-z_4z_1$ is a properly colored cycle spanning $V(P)$ (see Figure 4.5(b)). If $c(z_3y) \neq c(z_3z_4)$, then $c(z_3y) = c(yy_+)$, i.e., z_3y is incompatible with y. Otherwise, $C = z_3Py_-z_lPyz_3$ is a properly colored cycle and $H - P + C + z_1z_2$ is a properly colored spanning 1-edge-cycle in G (see Figure 4.3(b)).

- $y \in S'_l$. If $c(z_3y) \neq c(z_2z_3)$, then $c(z_3y) = c(yy_-)$, i.e., z_3y is incompatible with y. Otherwise, $z_1Pz_3yPz_4z_1 + z_lPy_+z_l$ is a properly colored 2-factor in $G[V(P)]$ (see Figure 4.5(c)). If $c(z_3y) \neq c(z_3z_4)$, then $c(z_3y) = c(yy_-)$, i.e., z_3y is incompatible with y. Otherwise, $C_1 = z_3Pyz_3$ and $C_2 = z_lPy_+z_l$ are properly colored cycles. Then $H - P + C_1 + C_2 + z_1z_2$ is a properly colored spanning 1-edge-cycle in G (see Figure 4.3(c)).



(*a*) $x=z_3$, $y=z_l$ (*b*) $x=z_3$, $y\in R_l'$ (*c*) $x=z_3$, $y\in S_l'$

Figure 4.5

If $x \in T_1 \backslash z_3$, then $x \in R_1 \cap S_1$, i.e., x has properties of vertices in R_1 and vertices in S_1.

- $y = z_l$. Since $x \in R_1$, we have $c(xz_l) = c(xx_+)$ or $c(xz_l) = c(z_lz_{l-1})$. Otherwise, $z_1Px_-z_1 + z_lPxz_l$ is a properly colored 2-factor in $G[V(P)]$. Moreover, since

$x \in S_1$, we have $c(xz_l) = c(xx_-)$ or $c(xz_l) = c(z_l z_{l-1})$. Otherwise, $z_1 x_+ P z_l x P z_1$ is a properly colored cycle spanning $V(P)$. Thus, $c(xz_l) = c(z_l z_{l-1})$ which means that xz_l is incompatible with z_l.

- $y \in R'_l$. Firstly, we consider $x_+ \neq y$ and $x \neq y_+$. If $x_+ = y$, then $z_1 P x z_l P y z_1$ is a properly colored cycle spanning $V(P)$. If $x = y_+$, then $C = z_1 P y_- z_l P x_+ z_1$ is a properly colored cycle and $H - P + C + xy$ is a properly colored spanning 1-edge-cycle in G.

  Now we consider $y > x_+$. Since $x \in R_1$, referring to the proof of Claim 4.2 Case 3, we have $c(xy) = c(xx_+)$ or $c(xy) = c(yy_+)$. Meanwhile, since $x \in S_1$, we have $c(xy) = c(xx_-)$ or $c(xy) = c(yy_+)$. Thus, $c(xy) = c(yy_+)$ which means that xy is incompatible with y.

  The last subcase is $y < x_-$. Since $x \in R_1$, we have $c(xy) = c(xx_+)$ or $c(xy) = c(yy_+)$. Meanwhile, since $x \in S_1$, we have $c(xy) = c(xx_-)$ or $c(xy) = c(yy_+)$. Thus, $c(xy) = c(yy_+)$ which means that xy is incompatible with y.

- $y \in S'_l$. Similar to the proof of the case $y \in R'_l$, we can obtain $c(xy) = c(yy_-)$ which means that xy is incompatible with y.

Case 2. $x \in U_1$ and $y \in T_l$. The proof is similar to $x \in T_1 \backslash \{z_3\}$ and $y \in U_l$ in Case 1. □

Now we consider the digraph D and prove that there is a vertex in D with a large in-degree. By Claim 4.2, the base graph of $D[U_1 \cup U_l]$ is complete. Moreover, the directed edge from x to y exists for all $x \in T_p$ and $y \in U_q$ by Claim 4.3, where $\{p, q\} = \{1, l\}$.

Since $\sum_{v \in U_1 \cup U_l} d^-_{D[U_1 \cup U_l]}(v) \geq |U_1||U_l|$, without loss of generality, we suppose that $\sum_{v \in U_l} d^-_{D[U_1 \cup U_l]}(v) \geq |U_1||U_l|/2$. So $\sum_{v \in U_l} d^-_D(v) \geq |U_1||U_l|/2 + |T_1||U_l|$. By an averaging argument, there exists $v \in U_l$ with $d^-_D(v) \geq |U_1|/2 + |T_1| = (|U_1| + 2|T_1|)/2 \geq (|\{z_1\}| + |R'_1| + |S'_1| + 2|T_1|)/2 \geq \delta^c(G)/2$. Recall that $d^-_D(v)$ reflects the numbers of monochromatic incident edges of v and vertices that are not adjacent to v. Hence $n \geq d^-_D(v) + (\delta^c(G) - 1)$, then we have $\delta^c(G) \leq |G|/3 + 2/3$, contradicting with $\delta^c(G) \geq |G|/3 + 1$. Combining with Claim 4.4, this completes the proof of Theorem 4.4.

Claim 4.4. *There is no properly colored spanning 1-edge-cycle in G.*

Proof. Suppose that G contains a properly colored spanning 1-edge-cycle H. We will show that $\delta^c(G) < |G|/3 + 1$.

Let the edge be z_1z_2. Recall that $N^c(z)$ is a maximal subset of neighbors of z in G such that $c(zy_1) \neq c(zy_2)$ for all distinct $y_1, y_2 \in N^c(z)$. Let $N_1^c = \{y \in N^c(z_1) : c(z_1y) \neq c(z_1z_2)\}$ and $N_2^c = \{y \in N^c(z_2) : c(z_2y) \neq c(z_1z_2)\}$. So $|N_i^c| \geq \delta^c(G) - 1$ for $i \in \{1, 2\}$. And we define

$$
\begin{aligned}
&R_i = \{y : y_- \in N_i^c \text{ and } c(z_iy_-) = c(yy_-)\},\\
&S_i = \{y : y_+ \in N_i^c \text{ and } c(z_iy_+) \neq c(y_+y_{++})\},\\
&W_i = R_i \cup S_i \cup \{z_i\}, \quad T_i = R_i \cap S_i, \quad U_i = W_i \backslash T_i,\\
&R_i' = R_i \backslash T_i, \quad S_i' = S_i \backslash T_i.
\end{aligned}
$$

Hence, $|R_i| + |S_i| = |N_i^c|$. Moreover, $|R_i'| + |S_i'| + 2|T_i| = |N_i^c| \geq \delta^c(G) - 1$.

For an edge $xy \in E(G[W_1 \cup W_2])$, we say that xy is *incompatible* with y

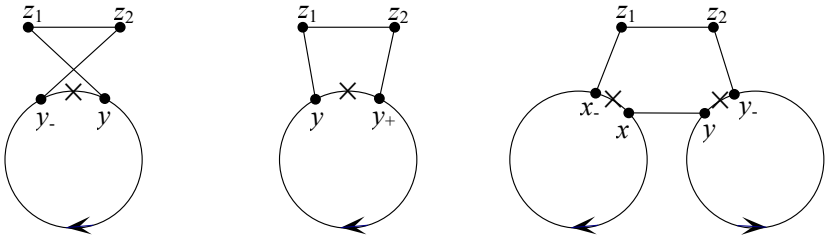
- if $y = z_1$ and $c(xy) = c(z_1z_2)$;
- if $y = z_2$ and $c(xy) = c(z_1z_2)$;
- if $y \in R_1' \cup R_2'$ and $c(xy) = c(yy_+)$;
- if $y \in S_1' \cup S_2'$ and $c(xy) = c(yy_-)$.

Claim 4.4.1 *If $xy \in E(G[U_1 \cup U_2])$, then xy is incompatible with x or y.*

Proof. Suppose to the contrary that there exists an edge $xy \in E(G[U_1 \cup U_2])$ such that xy is not incompatible with x and y. We will show that there exists a properly colored 2-factor in G. It is a contradiction as G is a counterexample to Theorem 4.4.

Case 1. Let $x = z_1$ or $y = z_2$. If $xy = z_1z_2$, then $c(xy) = c(z_1z_2)$, i.e., xy is incompatible with x and y, a contradiction. If $x = z_1$ and $y \in R_2'$, then $y_- \in N_2^c$ and $c(z_2y_-) \neq c(y_-y_{--})$. Since $c(z_1y) \neq c(z_1z_2)$ and $c(z_1y) \neq c(yy_+)$, $C' = z_1yC_y^+y_-z_2z_1$ is a properly colored cycle and $H - z_1z_2 - C_y + C'$ is a properly colored 2-factor in G (see Figure 4.6(a)). If $x = z_1$ and $y \in S_2'$, then $y_+ \in N_2^c$ and $c(z_2y_+) \neq c(y_+y_{++})$. Since $c(z_1y) \neq c(z_1z_2)$ and $c(z_1y) \neq c(yy_-)$, $C' = z_1yC_y^-y_+z_2z_1$ is a properly colored cycle and $H - z_1z_2 - C_y + C'$ is a properly colored 2-factor in G (see Figure 4.6(b)). For the case $y = z_2$ and $x \in R_1' \cup S_1'$, the proof is similar.

Case 2. Let $x \in R_1' \cup S_1'$ and $y \in R_2' \cup S_2'$.



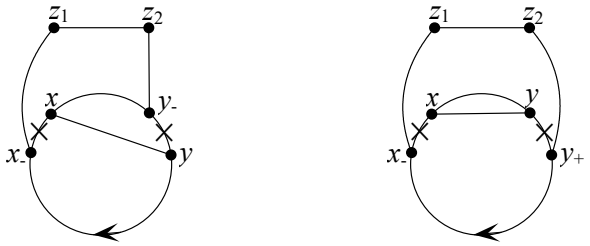
(*a*) $x=z_1$, $y\in R_2'$ (*b*) $x=z_1$, $y\in S_2'$ (*c*) $C_x \neq C_y$, $x\in R_1'$, $y\in R_2'$

Figure 4.6

Assume that $C_x \neq C_y$. If $x \in R_1'$ and $y \in R_2'$, then $x_- \in N_1^c$, $y_- \in N_2^c$, $c(z_1x_-) \neq c(x_-x_{--})$ and $c(z_2y_-) \neq c(y_-y_{--})$. Since $c(xx_+) \neq c(xy) \neq c(yy_+)$, $C' = z_1x_-C_x^-xyC_y^+y_-z_2z_1$ is a properly colored cycle and $H - z_1z_2 - C_x - C_y + C'$ is a properly colored 2-factor in G (see Figure 4.6(c)). If $x \in S_1'$ or $y \in S_2'$, then we reverse the orientation of C_x or C_y. And we are back in the case when $x \in R_1'$ and $y \in R_2'$.

Next, assume that $C = C_x = C_y$. If $x \in R_1'$ and $y \in R_2'$, then we have $C' = z_1x_-C^-yxC^+y_-z_2z_1$ is a properly colored cycle and $H - z_1z_2 - C + C'$ is a properly colored 2-factor in G (see Figure 4.7(a)). Note that $x_+ \neq y$ and $y_+ \neq x$ since $c(xx_+) \neq c(xy) \neq c(yy_+)$.

If $x \in R_1'$ and $y \in S_2'$, then $y_+ \neq x$; otherwise $C' = z_1x_-C^-xy_2y_1$ is a properly colored cycle and $H - z_1z_2 - C + C'$ is a properly colored 2-factor in G. Since $c(xx_+) \neq c(xy) \neq c(yy_-)$, $C' = z_1x_-C^-y_+z_2z_1$ and $C'' = xC^+yx$ are properly colored cycles and $H - z_1z_2 - C + C' + C''$ is a properly colored 2-factor in G (see Figure 4.7(b)).



(*a*) $C_x=C_y$, $x\in R_1'$, $y\in R_2'$ (*b*) $C_x=C_y$, $x\in R_1'$, $y\in S_2'$

Figure 4.7

If $x \in S_1'$ and $y \in R_2'$, then reverse the orientation of C and we are back to the case when $x \in R_1$ and $y \in S_2$. If $x \in S_1'$ and $y \in S_2'$, then reverse the orientation of C and we are back to the case when $x \in R_1'$ and $y \in R_2'$.

This completes the proof of Claim 4.1. □

Claim 4.4.2 *Suppose that* $xy \in E(G)$, *if* $x \in T_p$ *and* $y \in U_q$, *where* $\{p, q\} = \{1, 2\}$,

then xy is incompatible with y.

Since the proof of Claim 4.4.2 is similar to Claims 4.3 and 4.4.1, we omit it.

Let $F = G[W_1 \cup W_2]$. Define F' to be the directed graph on $V(F)$ such that there is a directed edge from x to y if $xy \notin E(F)$ or xy is incompatible with y. By Claim 4.1, the base graph of $F'[U_1 \cup U_2]$ is complete. Moreover, the directed edge from x to y exists for all $x \in T_p$ and $y \in U_q$ by Claim 4.2, where $\{p, q\} = \{1, 2\}$. Since $\sum_{v \in U_1 \cup U_2} d^-_{F'[U_1 \cup U_2]}(v) \geq |U_1||U_2|$, without loss of generality, we suppose that $\sum_{v \in U_1} d^-_{F'[U_1 \cup U_2]}(v) \geq |U_1||U_2|/2$. So $\sum_{v \in U_1} d^-_{F'}(v) \geq |U_1||U_2|/2 + |T_2||U_1|$. By an averaging argument, there exists $v \in U_1$ with $d^-_{F'}(v) \geq |U_2|/2 + |T_2| = (|U_2| + 2|T_2|)/2 = (|\{z_2\}| + |R'_2| + |S'_2| + 2|T_2|)/2 \geq \delta^c(G)/2$. Since $n \geq \delta^c(G)/2 + (\delta^c(G) - 1)$, we have $\delta^c(G) \leq |G|/3 + 2/3$ as required, contradicting with $\delta^c(G) \geq |G|/3 + 1$. This completes the proof of Claim 4.4. □

4.4 Proof of Theorem 4.5

Given an edge x_1x_2, we say that a path P is an *absorbing path* for x_1x_2 if the following conditions hold:

(i) $P = z_1z_2z_3z_4$ is a properly colored path of length 3;

(ii) $V(P) \cap \{x_1, x_2\} = \emptyset$;

(iii) $z_1z_2x_1x_2z_3z_4$ is a properly colored path.

Next we define an absorbing path for two disjoint edges. Given two vertex-disjoint edges x_1x_2 and y_1y_2 with $x_1, y_1 \in X$ and $x_2, y_2 \in Y$, we say that a path P is an *absorbing path* for $(x_1, x_2; y_1, y_2)$ if the following conditions hold:

(i) $P = z_1z_2z_3z_4$ is a properly colored path of length 3;

(ii) $V(P) \cap \{x_1, x_2, y_1, y_2\} = \emptyset$;

(iii) both $z_1z_2x_1x_2$ and $y_1y_2z_3z_4$ are properly colored paths of length 3.

Note that the ordering of $(x_1, x_2; y_1, y_2)$ is important. Given an edge x_1x_2, let $\mathcal{L}(x_1, x_2)$ be the set of absorbing paths for x_1x_2. Similarly, given two vertex-disjoint edges x_1x_2 and y_1y_2 with $x_1, y_1 \in X$ and $x_2, y_2 \in Y$, let $\mathcal{L}(x_1, x_2; y_1, y_2)$ be the set of absorbing paths for $(x_1, x_2; y_1, y_2)$.

Lemma 4.1. *Let $\varepsilon > 0$. Suppose that $G = (X, Y)$ is an edge-colored bipartite graph with $|X| = |Y| = n$ and $\delta^c = \delta^c(G) \geq (1/2 + \varepsilon)n$. Then $|\mathcal{L}(x_1, x_2)| \geq \varepsilon n^4/8$ and $|\mathcal{L}(x_1, x_2; y_1, y_2)| \geq \varepsilon n^4/8$ for any distinct vertices x_1, x_2, y_1, y_2 with $x_1, y_1 \in X$, $x_2, y_2 \in$*

Y and $x_1x_2, y_1y_2 \in E(G)$ when n is sufficiently large.

Proof. Firstly, for an edge x_1x_2 with $x_1 \in X$ and $x_2 \in Y$, choose a vertex $z_2 \in N(x_1)$ such that $c(x_1z_2) \neq c(x_1x_2)$. Next, pick another vertex $z_3 \in N(z_2) \cap N(x_2)$ such that $c(x_2z_3) \neq c(x_1x_2)$ and $c(x_1z_2) \neq c(z_2z_3)$. The total number of such z_2, z_3 is at least $(\delta^c - 1)(2\varepsilon n - 1)$. Now, fixing z_2 and z_3, choose $z_1 \in N(z_2)$ such that $c(z_1z_2) \neq c(z_2z_3)$ and $c(z_1z_2) \neq c(x_1z_2)$. Choose another vertex $z_4 \in N(z_3)$ such that $c(z_3z_4) \neq c(z_2z_3)$ and $c(z_3z_4) \neq c(x_2z_3)$. Note that the total number of such z_1, z_4 is at least $(\delta^c-2)(\delta^c-2)$ and hence we derive that there exist at least

$$(\delta^c - 1)(2\varepsilon n - 1)(\delta^c - 2)(\delta^c - 2) \geq n^4\varepsilon(1+\varepsilon)^3/8 \geq \varepsilon n^4/8$$

absorbing paths for (x_1, x_2) when n is sufficiently large.

For two vertex-disjoint edges x_1x_2 and y_1y_2 with $x_1, y_1 \in X$ and $x_2, y_2 \in Y$, choose a vertex $z_2 \in N(x_1)\backslash\{y_2\}$ such that $c(x_1z_2) \neq c(x_1x_2)$. Pick another vertex $z_3 \in N(z_2) \cap N(y_2)\backslash\{x_1\}$ such that $c(y_2z_3) \neq c(y_1y_2)$. The total number of such z_2, z_3 is at least $(\delta^c - 2)(2\varepsilon n - 2)$. Now fix z_2 and z_3. Choose $z_1 \in N(z_2)\backslash\{y_1\}$ such that $c(z_1z_2) \neq c(z_2z_3), c(z_1z_2) \neq c(x_1z_2)$. Choose another vertex $z_4 \in N(z_3)\backslash\{x_2\}$ such that $c(z_3z_4) \neq c(z_2z_3), c(z_3z_4) \neq c(y_2z_3)$. Note that the total number of such z_1, z_4 is at least $(\delta^c - 3)(\delta^c - 3)$ and hence we derive that there exist at least

$$(\delta^c - 2)(2\varepsilon n - 2)(\delta^c - 3)(\delta^c - 3) \geq n^4\varepsilon(1+\varepsilon)^3/8 \geq \varepsilon n^4/8$$

absorbing paths for $(x_1, x_2; y_1, y_2)$ when n is sufficiently large. □

We will need the following Chernoff bound for the binomial distribution (see, e.g. [6]).

Proposition 4.1. *Suppose that X has the binomial distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-a^2\mathbb{E}X/3}$.*

Lemma 4.2. *Let $0 < \gamma < 1$ and t be a positive integer. Then there exist integers n_0 and n_1 such that whenever $n \geq n_1 \geq n_0$ the following holds. Let $G = (X, Y)$ be an edge-colored bipartite graph with $|X| = |Y| = n$. Suppose that $|\mathcal{L}(x_1, x_2)| \geq \gamma n^4$ and $|\mathcal{L}(x_1, x_2; y_1, y_2)| \geq \gamma n^4$ for any distinct vertices x_1, x_2, y_1, y_2 with $x_1, y_1 \in X$, $x_2, y_2 \in Y$ and $x_1x_2, y_1y_2 \in E(G)$. Then there exist t disjoint families $\mathcal{F}_i'$ $(1 \leq i \leq t)$*

of vertex-disjoint properly colored paths each of length 3*, which satisfy the properties*

$$
\begin{aligned}
&|\bigcup_{i=1}^{t}\mathcal{F}_i'| \le 2^{-6}\gamma n,\\
&|\mathcal{L}(x_1,x_2)\cap\mathcal{F}_i'| \ge \frac{1}{t}2^{-10}\gamma^2 n,\\
&|\mathcal{L}(x_1,x_2;y_1,y_2)\cap\mathcal{F}_i'| \ge \frac{1}{t}2^{-10}\gamma^2 n
\end{aligned}
$$

for any $1 \le i \le t$ *and any distinct vertices* x_1, x_2, y_1, y_2 *with* $x_1, y_1 \in X$*,* $x_2, y_2 \in Y$ *and* $x_1x_2, y_1y_2 \in E(G)$*.*

Proof. Choose $n_0 \in \mathbb{N}$ large enough so that

$$2\exp(-\gamma n_0/(3\times 2^7)) + 2(n_0^2+n_0^4)\exp(-\gamma^2 n_0/(3\times 2^9)) \le 1/6. \tag{4.2}$$

Recall that we consider every path as a directed path. So a path $z_1z_2z_3z_4$ is identified as a 4-tuple (z_1,z_2,z_3,z_4). Choose a family $\mathcal{F}$ of 4-tuples in $V(G)$ by selecting each of the $n!/(n-4)!$ possible 4-tuples independently and randomly with probability

$$p = 2^{-7}\gamma\frac{(n-4)!}{(n-1)!} \ge 2^{-7}\gamma n^{-3}.$$

Note that we now have

$$
\begin{aligned}
&\mathbb{E}|\mathcal{F}| = p\frac{n!}{(n-4)!} = 2^{-7}\gamma n,\\
&\mathbb{E}|\mathcal{L}(x_1,x_2)\cap\mathcal{F}| = p|\mathcal{L}(x_1,x_2)| \ge 2^{-7}\gamma^2 n,\\
&\mathbb{E}|\mathcal{L}(x_1,x_2;y_1,y_2)\cap\mathcal{F}| = p|\mathcal{L}(x_1,x_2;y_1,y_2)| \ge 2^{-7}\gamma^2 n
\end{aligned}
$$

for any distinct vertices x_1, x_2, y_1, y_2 with $x_1, y_1 \in X$, $x_2, y_2 \in Y$ and $x_1x_2, y_1y_2 \in E(G)$. Therefore by Proposition 4.1 and (4.2), choose $a = 1$ and $a = \frac{1}{2}$ independently and we get that with probability at least $5/6$, the family $\mathcal{F}$ satisfies the following properties

$$|\mathcal{F}| \le 2\mathbb{E}|\mathcal{F}| = 2^{-6}\gamma n, \tag{4.3}$$

$$|\mathcal{L}(x_1,x_2)\cap\mathcal{F}| \ge 2^{-1}\mathbb{E}|\mathcal{L}(x_1,x_2)\cap\mathcal{F}| \ge 2^{-8}\gamma^2 n, \tag{4.4}$$

$$|\mathcal{L}(x_1,x_2;y_1,y_2)\cap\mathcal{F}| \ge 2^{-1}\mathbb{E}|\mathcal{L}(x_1,x_2;y_1,y_2)\cap\mathcal{F}| \ge 2^{-8}\gamma^2 n. \tag{4.5}$$

for any distinct vertices x_1, x_2, y_1, y_2 with $x_1, y_1 \in X$, $x_2, y_2 \in Y$ and $x_1x_2, y_1y_2 \in E(G)$.

We say that two 4-tuples (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) are *intersecting* if $a_i = b_j$ for some $1 \leq i, j \leq 4$. We can bound the expected number of pairs of 4-tuples in $\mathcal{F}$ that are intersecting by

$$\frac{n!}{(n-4)!} \times 4^2 \times \frac{(n-1)!}{(n-4)!} \times p^2 = 2^{-10}\gamma^2 n.$$

Therefore, by using Markov's inequality, we derive that with probability at least $1/2$, $\mathcal{F}$ contains at most $2^{-9}\gamma^2 n$ intersecting pairs of 4-tuples. Hence, with positive probability the family $\mathcal{F}$ satisfies (4.3), (4.4), (4.5) and contains at most $2^{-9}\gamma^2 n$ intersecting pairs of 4-tuples. By removing one 4-tuple in each intersecting pair in $\mathcal{F}$ and those 4-tuples that are not absorbing paths, we get a subfamily $\mathcal{F}'$ consisting of pairwise disjoint 4-tuples, which satisfies

$$|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}'| \geq 2^{-8}\gamma^2 n - 2^{-9}\gamma^2 n = 2^{-9}\gamma^2 n$$

and also $|L(x_1, x_2) \cap \mathcal{F}'| \geq 2^{-9}\gamma^2 n$ for any distinct vertices x_1, x_2, y_1, y_2 with $x_1, y_1 \in X$, $x_2, y_2 \in Y$ and $x_1x_2, y_1y_2 \in E(G)$. Since each 4-tuple in $\mathcal{F}'$ is an absorbing path, $\mathcal{F}'$ is a set of vertex disjoint properly colored path of length 3.

When $t = 1$, let $\mathcal{F}'_1 = \mathcal{F}'$ and Lemma 4.2 is true by the construction of $\mathcal{F}'$. Assume that $n \geq n_0$ and $t \geq 2$. We partition $\mathcal{F}'$ into t subfamilies $\mathcal{F}'_1, ..., \mathcal{F}'_t$ uniformly at random and we allow that $\mathcal{F}'_j = \emptyset$ for some j. For any $1 \leq i \leq t$ and any distinct vertices x_1, x_2 with $x_1 \in X$, $x_2 \in Y$, let $|\mathcal{L}(x_1, x_2) \cap \mathcal{F}'| = L$ and for any $0 \leq L_0 \leq L$, we have

$$\mathbb{P}(|\mathcal{L}(x_1, x_2) \cap \mathcal{F}'_i| = L_0) = \frac{\binom{L}{L_0}(t-1)^{L-L_0}}{t^L} = \binom{L}{L_0}\left(1 - \frac{1}{t}\right)^{L-L_0}\left(\frac{1}{t}\right)^{L_0}.$$

Thus $|\mathcal{L}(x_1, x_2) \cap \mathcal{F}'_i|$ is a binomial distribution with parameter $(L, \frac{1}{t})$. We obtain that

$$\mathbb{E}|\mathcal{L}(x_1, x_2) \cap \mathcal{F}'_i| = \frac{1}{t}|\mathcal{L}(x_1, x_2) \cap \mathcal{F}'| \geq \frac{1}{t}2^{-9}\gamma^2 n. \tag{4.6}$$

Similarly, $|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}'_i|$ is a binomial distribution and we obtain that

$$\mathbb{E}|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}'_i| = \frac{1}{t}|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}'| \geq \frac{1}{t}2^{-9}\gamma^2 n. \tag{4.7}$$

Choose n_1 large enough such that $n_1 \geq n_0$ and

$$2t(n_1^2 + n_1^4)\exp(-\gamma^2 n_1/(3 \times 2^{11} \times t)) < 1. \tag{4.8}$$

Therefore, let $a = \frac{1}{2}$ and using the Chernoff bound we obtain that when $n \geq n_1$ with positive probability there exists a partition of $\mathcal{F}'$ into $\mathcal{F}'_1, ..., \mathcal{F}'_t$ such that

$$|\mathcal{L}(x_1, x_2) \cap \mathcal{F}'_i| \geq 2^{-1}\mathbb{E}|\mathcal{L}(x_1, x_2) \cap \mathcal{F}'_i| \geq \frac{1}{t}2^{-10}\gamma^2 n, \tag{4.9}$$

$$|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}'_i| \geq 2^{-1}\mathbb{E}|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}'_i| \geq \frac{1}{t}2^{-10}\gamma^2 n. \tag{4.10}$$

for any $1 \leq i \leq t$ and any distinct vertices x_1, x_2, y_1, y_2 with $x_1, y_1 \in X$, $x_2, y_2 \in Y$ and $x_1x_2, y_1y_2 \in E(G)$. Thus we finish the proof. □

The following connecting lemma is just an analogue of Lemma 4.6 in [134] and the proof is almost same. Since the proof is not very complicated, here we omit it.

Lemma 4.3. *Suppose that $G = (V_1, V_2)$ is an edge-colored bipartite graph with $|V_1| = |V_2| = n$ and $\delta^c(G) \geq 2n/3 + 2$. Let x_1x_2 and y_1y_2 be two edges in G with $x_2 \neq y_2$, $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$. Then there exists an edge z_1z_2 with $z_1, z_2 \in V(G)\backslash\{x_1, x_2, y_1, y_2\}$ and $z_1 \in V_1, z_2 \in V_2$ such that both $x_1x_2z_1z_2$ and $z_1z_2y_1y_2$ are properly colored paths. In particular, if x_1, x_2, y_1, y_2 are distinct, then $x_1x_2z_1z_2y_1y_2$ is a properly colored path.*

Let $K = K(t) = 10^{10}t$ be a positive integer which is relying on t. Thus we have $K(1) = 10^{10}$, $K(t) > 2^9$ and $K(t) > 9t$.

Lemma 4.4. (Absorbing Cycles Lemma) *Let $\varepsilon > 0$ and t be a positive integer. There exists an integer n_2 such that whenever $n \geq n_2$ the following holds. Suppose that $G = (X, Y)$ is an edge-colored bipartite graph with $|X| = |Y| = n$ and $\delta^c(G) \geq (2/3 + \varepsilon)n$. Then there exist t vertex-disjoint properly colored cycles C_i $(1 \leq i \leq t)$ such that the following holds: (i) $|\bigcup_{i=1}^t V(C_i)| \leq \frac{6\varepsilon n}{K}$; (ii) For any $1 \leq a \leq t$ and for all $k \leq \frac{4\varepsilon^2 n}{tK^2}$, if $P_1, ..., P_k$ are vertex-disjoint properly colored paths in $G\backslash\bigcup_{i=1}^t V(C_i)$ with odd lengths, then there exists a properly colored cycle with vertex set $V(C_a) \cup (\bigcup_{1\leq j\leq k} V(P_j))$.*

Proof. Let $\gamma = \frac{2^6\varepsilon}{K}$ and choose $n_2 \in N$ large enough so that Lemma 4.1 and Lemma 4.2 hold. By Lemma 4.1, $|\mathcal{L}(x_1, x_2)| \geq \gamma n^4$ and $|\mathcal{L}(x_1, x_2; y_1, y_2)| \geq \gamma n^4$ for any distinct vertices x_1, x_2, y_1, y_2 with $x_1, y_1 \in X$, $x_2, y_2 \in Y$ and $x_1x_2, y_1y_2 \in E(G)$. Let $\mathcal{F}'_i$

$(1 \le i \le t)$ be the families of properly colored paths obtained by Lemma 4.2. Therefore, $|\bigcup_{i=1}^{t} \mathcal{F}_i'| \le 2^{-6}\gamma n = \varepsilon n/K$ and we obtain

$$|\mathcal{L}(x_1, x_2) \cap \mathcal{F}_i'| \ge \frac{1}{t} 2^{-10}\gamma^2 n = \frac{4\varepsilon^2 n}{tK^2}, \tag{4.11}$$

$$|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}_i'| \ge \frac{1}{t} 2^{-10}\gamma^2 n = \frac{4\varepsilon^2 n}{tK^2}. \tag{4.12}$$

for any $1 \le i \le t$ and any distinct vertices x_1, x_2, y_1, y_2 with $x_1, y_1 \in X$, $x_2, y_2 \in Y$ and $x_1x_2, y_1y_2 \in E(G)$. We now construct t cycles C_i $(1 \le i \le t)$ satisfying the desired property. Suppose that we have already constructed cycles $C_1, ..., C_{l-1}$ with $|V(C_i)| \le 6|\mathcal{F}_i'|$ $(1 \le i \le l-1)$, now we construct the cycle C_l. Let $P_1', ..., P_{|\mathcal{F}_l'|}'$ be the properly colored paths in $\mathcal{F}_l'$. For each $1 \le j \le |\mathcal{F}_l'|$, we are going to find edge $z_1^j z_2^j$ such that $\{z_1^j, z_2^j\} \cap V(\mathcal{F}_l') = \emptyset$, $P_j' z_1^j z_2^j P_{j+1}'$ is a properly colored path, where we take $P_{|\mathcal{F}_l'|+1}' = P_1'$, and $\{z_1^j, z_2^j\} \cap \{z_1^{j'}, z_2^{j'}\} = \emptyset$ for all $j \ne j'$. Assume that we have already found edges $z_1^1 z_2^1, ..., z_1^{j-1} z_2^{j-1}$ for some $1 \le j \le |\mathcal{F}_l'|$. Let $P_j' = v_1v_2v_3v_4$ and $P_{j+1}' = v_1'v_2'v_3'v_4'$ where $v_1, v_1' \in X$. Let

$$W_l^j = \left(\bigcup_{i=1}^{l-1} V(C_i) \cup \bigcup_{i=l}^{t} V(\mathcal{F}_i') \cup \bigcup_{1 \le j' < j} \{z_1^{j'}, z_2^{j'}\} \right) \backslash \{v_3, v_4, v_1', v_2'\}.$$

Note that

$$|W_l^j| \le \sum_{i=1}^{l-1} 6|\mathcal{F}_i'| + \sum_{i=l+1}^{t} 4|\mathcal{F}_i'| + 4|\mathcal{F}_l'| + 2(j-1) - 4 < 6\sum_{i=1}^{t} |\mathcal{F}_i'| \le \frac{6\varepsilon n}{K}.$$

Define $G_l^j = G[V(G) \backslash W_l^j]$, then $\delta^c(G_l^j) \ge 2n/3 + 2$. By Lemma 4.3, we obtain an edge $z_1^j z_2^j$ in G_l^j such that $v_3v_4z_1^jz_2^jv_1'v_2'$ is a properly colored path in G_l^j. This implies that $P_j' z_1^j z_2^j P_{j+1}'$ is a properly colored path in G. Therefore, there exist vertex-disjoint edges $z_1^1 z_2^1, ..., z_1^{|\mathcal{F}_l'|} z_2^{|\mathcal{F}_l'|}$ as desired. Let C_l be the properly colored cycle obtained by concatenating $P_1', z_1^1 z_2^1, P_2', z_1^2 z_2^2, ..., P_{|\mathcal{F}_l'|}', z_1^{|\mathcal{F}_l'|} z_2^{|\mathcal{F}_l'|}$. Note that $|C_l| = 6|\mathcal{F}_l'|$. Finally we can obtain t cycles C_i $(1 \le i \le t)$ such that $\sum_{i=1}^{t} |V(C_i)| = 6\sum_{i=1}^{t} |\mathcal{F}_i'| \le \frac{6\varepsilon n}{K}$.

Suppose that $\mathcal{P}$ is a family of at most $\frac{4\varepsilon^2 n}{tK^2}$ vertex-disjoint properly colored paths with odd length in $V(G) \backslash \bigcup_{i=1}^{t} V(C_i)$. For fixed $1 \le a \le t$, we assign each $P_0 \in \mathcal{P}$ to a path $Q \in \mathcal{F}_a'$ such that $Q \in \mathcal{L}(x_1, x_2)$ if $P_0 = x_1x_2$ and $Q \in \mathcal{L}(x_1, x_2; x_{l-1}, x_l)$ if $P_0 = x_1x_2...x_l$ with $l \ge 4$. Moreover, no two paths in $\mathcal{P}$ are assigned to the same $Q \in \mathcal{F}_a'$. Note that such an assignment exists by equations (4.11) and (4.12). Thus,

we can obtain a new cycle C' with $V(C') = V(C_a) \cup V(\mathcal{P})$ by replacing the paths in $\mathcal{F}'$. □

Lemma 4.5. *For each integer $k \geq 6$, every edge-colored balanced bipartite graph G on $2n$ vertices with $\delta^c(G) \geq 2n/3 + k$ contains a properly colored 2-factor in which every cycle has length at least $k/2$.*

Based on some skills that we developed in the proof of Theorem 4.4, the proof of Lemma 4.5 is similar to the proof of Lemma 3.1 in [134] and we do not need to add new skills. For convenience of the reader, we put the proof of Lemma 4.5 in the last section (Appendix).

Lemma 4.6. *For any $\varepsilon > 0$, there exists an integer $N_0(\varepsilon)$ such that every edge-colored balanced bipartite graph G with $\delta^c(G) \geq (1/3 + \varepsilon)|G|$ and $|G| \geq N_0(\varepsilon)$ contains a properly colored Hamiltonian cycle.*

Proof. Let $t = 1$ in Lemma 4.4. Choose $n_0 \in \mathbb{N}$ large enough so that $\varepsilon(1-\varepsilon/3)n_0/3 \geq 6$ and $4\varepsilon^3 n_0/10^{20} \geq 12$. Let $N_0 = 2n_0$. Suppose that $G = (X, Y)$ is an edge-colored bipartite graph with $|X| = |Y| = n \geq n_0$ and $\delta^c(G) \geq (2/3 + \varepsilon)n$.

Let C be the properly colored cycle given by Lemma 4.4 when $t = 1$, so $|C| \leq 2\varepsilon n/3$. Let G'' be the subgraph of G by removing all the vertices of C. We have

$$\delta^c(G'') > \delta^c(G) - |C| \geq (2 + \varepsilon)n/3 \geq (2 + \varepsilon)|G''|/6.$$

Note that $\varepsilon|G''|/6 \geq \varepsilon(1 - \varepsilon/3)n/3 \geq 6$. By Lemma 4.5, G'' can be covered by at most $12\varepsilon^{-1}$ vertex-disjoint properly colored odd paths. Suppose these vertex-disjoint odd paths are $P_1, ..., P_k$ such that $k \leq 12\varepsilon^{-1} \leq 4\varepsilon^2 n/10^{20}$. By the property of C, we can get a properly colored cycle C' with $V(C') = V(C) \cup \bigcup_{1 \leq i \leq k} V(P_i)$. Note that $|C'| = |C| + \sum_{1 \leq i \leq k} |P_i| = 2n$. So C' is the properly colored Hamiltonian cycle. □

For any $\varepsilon > 0$ and positive integer $t \geq 2$, suppose that $G = (X, Y)$ is an edge-colored bipartite graph with $|X| = |Y| = n$ and $\delta^c(G) \geq (2/3 + \varepsilon)n$. A *large 2-factor* with t components is a 2-factor in G such that each cycle of it has size larger than $\frac{6\varepsilon n}{K(t)}$.

Before proving Theorem 4.5, we firstly give the following lemma.

Lemma 4.7. *For any $\varepsilon > 0$ and positive integer $t \geq 2$, suppose that $G = (X, Y)$ is an edge-colored bipartite graph with $|X| = |Y| = n$ and $\delta^c(G) \geq (2/3 + \varepsilon)n$, then there*

exists an integer $N(\varepsilon, t)$ such that G contains every large properly colored even 2-factor with exactly t components when $n \geq N(\varepsilon, t)$.

Proof. Choose $N \in \mathbb{N}$ large enough so that $\varepsilon(1-\varepsilon/3)N/3 \geq 6$ and $4\varepsilon^3 N \geq 12tK^2$. Let $n \geq N$. Suppose F is a large even 2-factor with t cycles $C'_1, ..., C'_t$ such that $|V(C'_i)| > \frac{6\varepsilon n}{K(t)}$ for any $1 \leq i \leq t$. Let $C_1, ..., C_t$ be the t cycles that we obtain in Lemma 4.4 and hence we have $|\bigcup_{i=1}^{t} V(C_i)| \leq 2\varepsilon n/3$. Let G'' be the subgraph of G by removing all the vertices of C_i $(1 \leq i \leq t)$. We have

$$\delta^c(G'') > \delta^c(G) - |\bigcup_{i=1}^{t} V(C_i)| \geq (2+\varepsilon)n/3 \geq (2+\varepsilon)|G''|/6.$$

Note that $\varepsilon|G''|/6 \geq \varepsilon(1-\varepsilon/3)n/3 \geq 6$. By Lemma 4.5, G'' can be covered by at most $12\varepsilon^{-1}$ vertex-disjoint properly colored odd paths. Suppose these vertex-disjoint odd paths are $P_1, ..., P_k$ such that $k \leq 12\varepsilon^{-1} \leq \frac{4\varepsilon^2 n}{tK^2}$. Now we construct families of odd paths inductively from $P_1, ..., P_k$. Suppose the subscript s_1 satisfies that $|P_1| + ... + |P_{s_1-1}| < |C'_1| - |C_1|$ and $|P_1| + ... + |P_{s_1}| \geq |C'_1| - |C_1|$, we partition P_{s_1} into two subpaths P'_{s_1} and P''_{s_1} such that $V(P_{s_1}) = V(P'_{s_1}) \cup V(P''_{s_1})$ and $|P_1| + ... + |P_{s_1-1}| + |P'_{s_1}| = |C'_1| - |C_1|$. Let the family $W_1 = \{P_1, ..., P_{s_1-1}, P'_{s_1}\}$. Suppose that we have already constructed $W_1, ..., W_{l-1}$ and the remaining set of odd paths is $\{P''_{s_{l-1}}, P_{s_{l-1}+1}, ..., P_k\}$ where $P''_{s_{l-1}}$ is an odd subpath of $P_{s_{l-1}}$ or empty. Update $P_{s_{l-1}}$ and let $P_{s_{l-1}} := P''_{s_{l-1}}$ and take the subscript s_l such that $|P_{s_{l-1}}| + |P_{s_{l-1}+1}| + ... + |P_{s_l-1}| < |C'_l| - |C_l|$ and $|P_{s_{l-1}}| + |P_{s_{l-1}+1}| + ... + |P_{s_l}| \geq |C'_l| - |C_l|$. We partition the path P_{s_l} into two subpaths P'_{s_l} and P''_{s_l} such that $|P_{s_{l-1}}| + |P_{s_{l-1}+1}| + ... + |P_{s_l-1}| + |P'_{s_l}| = |C'_l| - |C_l|$. Let the family $W_l = \{P_{s_{l-1}}, ..., P_{s_l-1}, P'_{s_l}\}$. By induction, we finally construct t families of odd paths $W_1, ..., W_t$ such that $|W_i| \leq k \leq 12\varepsilon^{-1} \leq \frac{4\varepsilon^2 n}{tK^2}$ for all $1 \leq i \leq t$. By Lemma 4.4, there exist t cycles $X_1, ..., X_t$ such that

$$|X_i| = |V(C_i) \cup (\bigcup_{P \in W_i} V(P))| = |C'_i|.$$

Thus the proof is finished. □

Proof of Theorem 4.5.

Suppose that $G = (X, Y)$ is an edge-colored bipartite graph with $|X| = |Y| = n$. When $t = 1$, by Lemma 4.6, we can easily obtain Theorem 4.5. Let $N'(\varepsilon, 1) = N_0(\varepsilon)$, here $N_0(\varepsilon)$ is defined in Lemma 4.6. When $t \geq 2$, we prove Theorem 4.5 by induction

on t.

Suppose that for every $2 \leq j \leq t-1$ and any $\varepsilon > 0$, there exists an integer $N'(\varepsilon, j)$ satisfies the condition we need. We now consider the case when the 2-factor contains exactly t even cycles. Let $n' > N(\varepsilon, t)$ where $N(\varepsilon, t)$ is defined in Lemma 4.7. Suppose $n \geq n'$, then by Lemma 4.7, G contains every large properly colored 2-factor with exactly t components. Let $F = \{C'_1, ..., C'_t\}$ be a 2-factor that is not large. Without loss of generality, suppose that for $1 \leq i \leq t'$, $|C'_i| \leq \frac{6\varepsilon n}{K(t)}$ and for $t'+1 \leq i \leq t$, $|C'_i| > \frac{6\varepsilon n}{K(t)}$. We first show that G contains t' vertex-disjoint properly colored cycles $C_1, ..., C_{t'}$ such that $|C_i| = |C'_i|$ for $1 \leq i \leq t'$. By $\delta^c(G) \geq (\frac{2}{3} + \varepsilon)n$, we can greedily construct a properly colored path of length $\frac{2n}{3}$. Since $\sum_{i=1}^{t'} |C'_i| \leq \frac{6t\varepsilon n}{K(t)} < \frac{2\varepsilon n}{3}$, we can construct t' vertex-disjoint properly colored paths $P_1, ..., P_{t'}$ with $|P_i| = |C'_i| - 2$ for $1 \leq i \leq t'$. Let $end(P_i)$ denote the set of points which are incident to the first edge or the last edge in path P_i $(1 \leq i \leq t')$ and $G'_1 = G - \{\bigcup_{i=1}^{t'} V(P_i) \backslash end(P_1)\}$, since $\delta^c_{G'_1} \geq \frac{2n}{3} + \frac{\varepsilon n}{3}$ and Lemma 4.3, we can find an edge z_1, z'_1 not in $\bigcup_{i=1}^{t'} V(P_i)$ such that $z_1 P_1 z'_1$ is a properly colored cycle. Similarly, let $G'_j = G - \{\bigcup_{i=1}^{t'} V(P_i) \cup \bigcup_{i=1}^{j-1} \{z_i, z'_i\} \backslash end(P_j)\}$ and we can find z_j, z'_j such that $z_j P_j z'_j$ is a properly colored cycle.

Let $G'' = G - \bigcup_{i=1}^{t'} V(C_i)$. Then we get $\delta^c(G'') \geq \frac{2n}{3} + \frac{\varepsilon n}{3} \geq \frac{|G''|}{3} + \frac{\varepsilon |G''|}{6}$ since $|G''| \leq 2n$. Choose n'' large enough such that $2n'' > \max_{1 \leq j \leq t-1} N'(\frac{\varepsilon}{6}, j)$. Choose n' such that $n' > n''/(1 - \frac{\varepsilon}{3})$. Thus $|G''| > 2(1 - \frac{\varepsilon}{3})n > 2n'' > N'(\frac{\varepsilon}{6}, t - t')$ when $n > n'$. Therefore, by induction, G'' contains every properly colored 2-factors with exactly $t - t'$ cycles. Thus G contains a properly colored 2-factor which is isomorphic to F when $n > n'$. □

4.5 Concluding Remarks

In 1984, El-Zahár [63] proposed the following famous conjecture:

Conjecture 4.1. [63] *Let t be a positive integer, and let G be a graph of order $n = \sum_{i=1}^{t} n_i$, where $n_i \geq 3$ for $1 \leq i \leq t$. If $\delta(G) \geq \sum_{i=1}^{t} \lceil n_i/2 \rceil$, then G can be partitioned into t cycles of lengths $n_1, n_2, \ldots, n_t$, respectively.*

El-Zahár proved the case $t = 2$. Aigner and Brandt [2] proved that if G is a graph of order $n = \sum_{i=1}^{t} n_i$ $(n_i \geq 3)$ with $\delta(G) \geq (2n-1)/3$, then G contains t vertex disjoint cycles of lengths $n_1, n_2, \ldots, n_t$, respectively. Abbasi, in his Ph.D. Thesis [1], settled El-Zahár's Conjecture for sufficiently large graphs by using the regularity lemma. Theorem

4.2 and Theorem 4.5 can be regarded as asymptotic edge-colored version results for El-Zahár's Conjecture in some sense. So it is very interesting to get a sharp bound for the minimum color degree in edge-colored graphs.

4.6 Appendix

Proof of Lemma 4.5.

Suppose the contrary and let G be a counterexample with $c(G)$ is maximum among all counterexamples. Since $\delta(G) \geq \delta^c(G)$, by the theorem of Moon and Moser [140], G is not properly colored, otherwise there is a properly colored Hamiltonian cycle in G, a contradiction.

Just like the proof of Theorem 4.4, G contains properly colored spanning 1-path-cycles with every cycle has length at least $k/2$. We also choose a properly colored spanning 1-path-cycle H in G with the path is longest among all properly colored spanning 1-path-cycles. Denote the path by $P = z_1 z_2 \cdots z_l$. Note that if $z_1 z_l \in E(G)$, then $c(z_1 z_l) = c(z_1 z_2)$ or $c(z_l z_{l-1})$, otherwise $H + z_1 z_l$ is a properly colored 2-factor of G, a contradiction.

Let $N_1 = \{y \in N_G(z_1) : c(z_1 y) \neq c(z_1 z_2)\} \cap V(P)$ and $N_l = \{y \in N_G(z_l) : c(z_l y) \neq c(z_l z_{l-1})\} \cap V(P)$. Let N_i^c be a maximal subset of N_i such that $c(z_i y_1) \neq c(z_i y_2)$ for all distinct $y_1, y_2 \in N_i^c$, where $i \in \{1, l\}$. Similar to the proof of Theorem 4.4, we have $|N_1^c| \geq \delta^c(G) - 1 > \delta^c(G) - 2$, $|N_l^c| \geq \delta^c(G) - 1$, $\{z_1, z_2\} \cap N_1^c = \emptyset$ and $\{z_{l-1}, z_l\} \cap N_l^c = \emptyset$. Therefore, $l \geq \max\{|N_1^c|, |N_l^c|\} + 3 \geq 2n/3 + k + 2$. Let $N_q' = N_q^c \backslash (\{z_1, z_2, \cdots, z_{\lfloor k/2 \rfloor}\} \cup \{z_l, z_{l-1}, \cdots, z_{l-\lceil k/2 \rceil + 1}\})$ for $q \in \{1, l\}$. Since $k \geq 6$, $|N_1'| \geq 2n/3 + 1$ and $|N_l'| \geq 2n/3 + 1$.

For $q \in \{1, l\}$, define

$$
\begin{aligned}
R_q &= \{y : y_- \in N_q' \text{ and } c(z_q y_-) = c(y y_-)\},\\
S_q &= \{y : y_+ \in N_q' \text{ and } c(z_q y_+) \neq c(y_+ y_{++})\},\\
W_q &= R_q \cup S_q \cup \{z_q\},\\
T_q &= R_q \cap S_q,\\
U_q &= W_q \backslash T_q.
\end{aligned}
$$

Hence, $|R_q| + |S_q| = |N'_q|$ and $|W_q| + |T_q| = |N'_q| + 1$. Moreover,

$$2|W_q| = |W_q| + |T_q| + |U_q| = |U_q| + |N'_q| + 1.$$

Therefore, $|W_q| \geq n/3 + |U_q|/2$.

Now define an auxiliary directed bipartite graph D as follows. The vertex classes of D are W_1 and W_l. For $\{q, q'\} = \{1, l\}$, there is a directed edge in D from $x \in W_q$ to $y \in W_{q'}$ if and only if $xy \in E(G)$ and one of the following statements holds:

(i) $x \in R_q \backslash T_q$ and $c(xy) \neq c(xx_+)$.

(ii) $x \in S_q \backslash T_q$ and $c(xy) \neq c(xx_-)$.

(iii) $x \in T_q$.

(iv) $x = z_1$ and $c(xy) \neq c(z_1 z_2)$.

(v) $x = z_l$ and $c(xy) \neq c(z_l z_{l-1})$.

Claim 4.5. *There exists a vertex $u \in U_1 \cup U_l$ such that u is in at least $(k-1)$ directed 2-cycles in D.*

Proof. For $x \in \{z_1\} \cup R_1 \backslash T_1$,

$$\begin{aligned} d_D^+(x) &= |\{y \in N^c(x) : c(xy) \neq c(xx_+)\} \cap W_l| \\ &\geq |\{y \in N^c(x) : c(xy) \neq c(xx_+)\}| + |W_l| - n \\ &\geq (2n/3 + k - 1) + (n/3 + |U_l|/2) - n \\ &= k - 1 + |U_l|/2. \end{aligned}$$

A similar statement holds for $x \in U_1 \backslash R_1$, in which case $d_D^+(x) = |\{y \in N^c(x) : c(xy) \neq c(xx_-)\} \cap W_l|$. In summary, for all $x \in U_1$,

$$d_D^+(x) \geq k - 1 + |U_l|/2.$$

Hence, there are at least $|U_1|(k-1+|U_l|/2)$ directed edges from U_1 to W_l, and similarly there are at least $|U_l|(k-1+|U_1|/2)$ directed edges from U_l to W_1. Note that if $\overrightarrow{xy}$ is a directed edge in D with $x \in U_1$ and $y \in T_l$, then $xy \in E(G)$ and so $\{x, y\}$ forms a directed 2-cycle in D by (iii). Hence, if $\overrightarrow{xy}$ is a directed edge in D not contained in a directed 2-cycle with $x \in U_1$ and $y \in W_l$, then $y \in U_l$. A similar statement holds for $\overrightarrow{xy}$ with $x \in U_l$ and $y \in W_1$. Therefore, at most $|U_1||U_l|$ directed edges from $U_1 \cup U_l$ are

not contained in directed 2-cycles in D. The number of directed edges $\overrightarrow{xy}$ in D such that $x \in U_1 \cup U_l$ and $\overrightarrow{xy}$ is contained in a directed 2-cycle is at least

$$|U_1|(k-1+|U_l|/2)+|U_l|(k-1+|U_1|/2)-|U_1||U_l| = (k-1)(|U_1|+|U_l|).$$

By an averaging argument, there exists a vertex $u \in U_1 \cup U_l$ that is in at least $(k-1)$ directed 2-cycles in D.

□

Assume that $u \in U_1$. There are at least $(k-1)$ vertices $w \in W_l$ such that uw is a directed 2-cycle in D. Pick one such $w \in W_l$ such that the subpath uPw in G has length at least $(k-1)/2$. Similarly, if $u \in U_l$, then there is a vertex $w \in W_1$ such that uw is a directed 2-cycle in D and the subpath uPw in G has length at least $(k-1)/2$. Without loss of generality, we assume that $u \in W_1$ and $w \in W_l$.

If $u \in T_1 = R_1 \cap S_1$, then we can assume that $u \in R_1$ with $c(uw) \neq c(uu_+)$, or $u \in S_1$ with $c(uw) \neq c(uu_-)$, as P is a properly colored path. Together with (i), (ii), (iv) and the definitions of R_1, S_1, we may assume that at least one of the following statements holds:

(a_1) $u \in R_1$, $c(uw) \neq c(uu_+)$ and $c(z_1z_2) \neq c(z_1u_-) \neq c(u_-u_{--})$.

(a_2) $u \in S_1$, $c(uw) \neq c(uu-)$ and $c(z_1z_2) \neq c(z_1u_+) \neq c(u_+u_{++})$.

(a_3) $u = z_1$ and $c(z_1w) \neq c(z_1z_2)$.

By symmetry, at least one of the following statements holds for w:

(b_1) $w \in R_l$, $c(uw) \neq c(ww_+)$ and $c(z_lz_{l-1}) \neq c(z_lw_-) \neq c(w_-w_{--})$.

(b_2) $w \in S_l$, $c(uw) \neq c(ww-)$ and $c(z_lz_{l-1}) \neq c(z_lw_+) \neq c(w_+w_{++})$.

(b_3) $w = z_l$ and $c(z_lu) \neq c(z_lz_{l-1})$.

Claim 4.6. *There exists a properly colored 2-factor H' in $G[V(P)]$ such that each cycle has length at least $k/2$.*

Proof. We need to prove Claim 4.6 for each combination of (a_i) and (b_j) for $1 \leq i, j \leq 3$. We say that $u < w$ if $u = z_i$ and $w = z_j$ with $i < j$.

Case 1. (a_3) and (b_3) hold. Since $u = z_1$ and $w = z_l$, (a_3) and (b_3) imply that $c(z_1z_2) \neq c(z_1z_l) \neq c(z_lz_{l-1})$, a contradiction.

Case 2. (a_1) and (b_2) hold. If $u < w$, then set $H' = z_1Pu_-z_1 + uPwu + w_+Pz_lw_+$ which is a properly colored 2-factor in $G[V(P)]$ (see Figure 4.8(a)). It suffices to show that each of $z_1Pu_-z_1$, $uPwu$ and $w_+Pz_lw_+$ is a cycle of length at least $k/2$. Since $u \in$

$R_1 = \{y : y_- \in N_1' \text{ and } c(z_1y_-) = c(yy_-)\}$, we have $u_- \in N_1' \subseteq N_1^c \backslash \{z_1, z_2, \cdots, z_{\lfloor k/2 \rfloor}\}$ and $z_1Pu_-z_1$ is a cycle of length at least $k/2$. Similarly, $w_+Pz_lw_+$ is a cycle of length at least $k/2$. Since u and w are chosen such that the subpath uPw has length at least $(k-1)/2$, we have $uPwu$ has length at least $(k-1)/2 + 1 > k/2$.

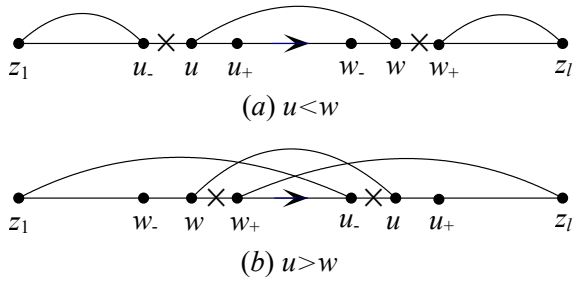


Figure 4.8

If $u > w$, then set $H' = z_1u_-Pw_+z_lPuwPz_1$ which is a properly colored cycle with vertex set $V(P)$ (see Figure 4.8(b)).

We would like to point out that other combinations of (a_i) and (b_j) for $1 \leq i, j \leq 3$ are proved by similar arguments used in Case 2. So in the following cases, the proofs are written briefly.

Case 3. (a_1) and (b_1) hold. If $u < w$, then set $H' = z_1Pu_-z_1 + uPw_-z_lPwu$ (see Figure 4.9(a)). If $u > w$, then set $H' = z_1Pw_-z_lPuwPu_-z_1$ (see Figure 4.9(b)).

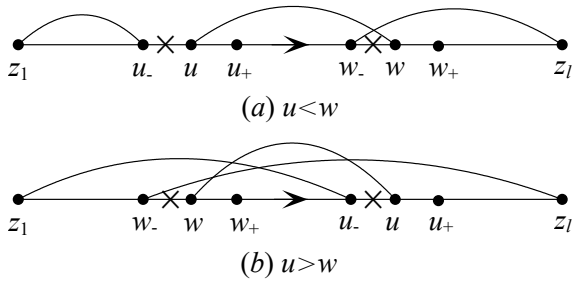


Figure 4.9

Case 4. (a_1) and (b_3) hold. We set $H' = z_1Pu_-z_1 + uPz_lu$ (see Figure 4.10).

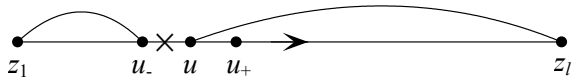


Figure 4.10

Case 5. (a_2) and (b_1) hold. If $u < w$, then set $H' = z_1PuwPz_1w_-Pu_+z_1$ (see Figure 4.11(a)). If $u > w$, then set $H' = wPuw + z_1Pw_-z_lPu_+z_1$ (see Figure 4.11(b)).

Case 6. (a_2) and (b_3) hold. We set $H' = z_1Puz_1Pu_+z_1$ (see Figure 4.12).

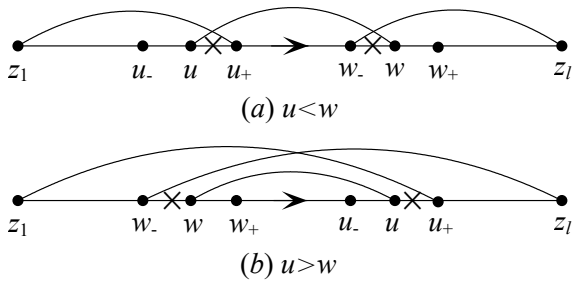


Figure 4.11

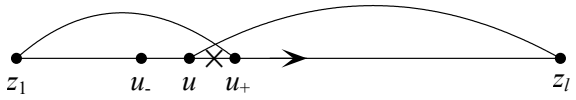


Figure 4.12

Case 7. (a_i) and (b_j) hold for $(i,j) \in \{(3,2),(2,2),(3,1)\}$. Reversing the orientation of P, we get a properly colored path $P' = z'_1 z'_2 \dots z'_l$. So $z'_i = z_{l-i+1}$. Moreover, set $u' = w$ and $w' = u$. Then we get that the case when (a_3) and (b_2) ((a_2) and (b_2), (a_3) and (b_1)) hold with respect to P corresponds to the case when (a_1) and (b_3) ((a_1) and (b_1), (a_2) and (b_3)) hold with respect to P', i.e., we are in Case 4 (Case 2, Case 6). □

Therefore, $H - P + H'$ is a properly colored 2-factor in G such that each cycle has length at least $k/2$, contradicting that G is a counterexample to Lemma 4.5. This completes the proof of Lemma 4.5.

Chapter 5

Compatible H-factors in dense graphs

5.1 Introduction

All the graphs considered here are finite, undirected and simple. Let H be an h-vertex graph and G be an n-vertex graph. An *H-tiling* is a collection of vertex-disjoint copies of H in G. An *H-factor* is an H-tiling which covers all vertices of G. Note that $n \in h\mathbb{N}$ is a necessary condition for G containing an H-factor.

5.1.1 Perfect graph tilings

One of the most fundamental research topics in extremal graph theory is to determine sufficient conditions forcing spanning structures, such as perfect matchings, Hamilton cycles, H-factors, etc. Textbook results of Hall and Tutte give a sufficient condition for the existence of a perfect matching (see e.g. [56]). A classical theorem of Dirac [58] states that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq \frac{n}{2}$ contains a Hamilton cycle. For H-factors, when $H = K_k$, a seminal result of Hajnal and Szemerédi [88] shows that any n-vertex graph G with $n \in k\mathbb{N}$ and $\delta(G) \geq \left(1 - \frac{1}{k}\right) n$ contains a K_k-factor and the bound is best possible. For general H, Alon and Yuster [8] first gave an asymptotic result by showing that if $\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n + o(n)$, then G contains an H-factor, where $\chi(H)$ is the chromatic number of H. Later, it was proved in [109] that the $o(n)$ term can be replaced with a constant $C = C(H)$. Further, Kühn and Osthus [120] managed to characterise, up to an additive constant, the minimum degree which forces an H-factor in a host graph G.

5.1.2 Locally ℓ-bounded

Finding spanning subgraphs in edge-colored graphs with certain constraints has also been widely and well studied. An edge-coloring c of a graph is *(globally) g-bounded* if each color appears at most g times in the coloring, while c is *locally ℓ-bounded* if each color appears at most ℓ times at any given vertex (so c is a proper edge-coloring when $\ell = 1$). A fundamental line of research is to find *rainbow subgraphs* (all edges have distinct colors) in g-bounded edge-colorings of graphs and *properly colored subgraphs* (any adjacent edges receive different colors) in locally ℓ-bounded edge-colorings of graphs. For perfect matchings, Erdős and Spencer [68] proved that any $(n-1)/16$-bounded edge-coloring of $K_{n,n}$ contains a rainbow perfect matching. Recently, Coulson and Perarnau [48] considered the sparse version and obtained that there exists $\mu > 0$ such that any μn-bounded edge-coloring of a Dirac bipartite graph contains a rainbow perfect matching. For Hamilton cycles, a conjecture of Bollobás and Erdős [21] states that every locally $\left(\lfloor \frac{n}{2} \rfloor - 1\right)$-bounded edge-colored K_n contains a properly colored Hamilton cycle. There is a series of partial results toward this conjecture (see e.g. [5, 35, 149]). In [135], Lo proved that the conjecture of Bollobás and Erdős is true asymptotically. For H-factors, Coulson, Keevash, Perarnau and Yepremyan [47] showed that there is a constant μ such that any μn-bounded edge-coloring of G with $\delta(G) \geq (\delta_H + o(1))n$ contains a rainbow H-factor, where δ_H is the minimum degree threshold for the existence of an H-factor.

In this chapter, we consider a more general setting of so called *incompatibility systems*, which was first proposed in [116].

Definition 5.1 (Incompatibility system)**.** *Let $G = (V, E)$ be a graph.*

- *An* incompatibility system *$\mathcal{F}$ over G is a family $\mathcal{F} = \{F_v\}_{v \in V}$ such that for every $v \in V$, F_v is a family of 2-subsets of edges incident with v, i.e. $F_v \subseteq \{\{e, e'\} \in \binom{E}{2} : e \cap e' = \{v\}\}$.*

- *For any two edges e and e', if there exists some vertex v such that $\{e, e'\} \in F_v$, then we say that e and e' are* incompatible *at v. Otherwise, they are* compatible*. A subgraph $H \subseteq G$ is* compatible *if all pairs of adjacent edges are compatible.*

- *For a positive integer Δ, an incompatibility system $\mathcal{F}$ is* Δ-bounded *if for any vertex v and any edge e incident with v, there are at most Δ other edges incident with v that are incompatible with e.*

Locally ℓ-bounded edge-colorings can be viewed as transitive $(\ell-1)$-bounded incompatibility systems, and compatible subgraphs generalize the concept of properly colored subgraphs. Note that there is also a similar generalization of g-bounded edge-colorings, called *systems of conflicts* (see [48]). Definition 5.1 was firstly introduced in [116] as a measure of *robustness* and motivated by two concepts in graph theory. First, it generalizes *transition systems* introduced by Kotzig [111] in 1968. In our terminology, a transition system is simply a 1-bounded incompatibility system. Kotzig's work was motivated by a problem of Nash–Williams on cycle covering of Eulerian graphs (see, e.g. Section 8.7 in [23]).

In [116], Krivelevich, Lee and Sudakov studied the robustness of Hamiltonicity of *Dirac graphs* with respect to the incompatibility system and derived Theorem 5.1, where a Dirac graph is defined as an n-vertex graph with minimum degree at least $n/2$.

Theorem 5.1 ([116]). *There exists a constant μ such that the following holds for large enough n. For every n-vertex Dirac graph G and a μn-bounded incompatibility system $\mathcal{F}$ defined over G, there exists a compatible Hamilton cycle.*

Their main tool is based on Pósa's rotation-extension technique. Theorem 5.1 settled in a very strong form, a conjecture of Häggkvist from 1988 (see Conjecture 8.40 in [23]). They further studied compatible Hamilton cycles in random graphs in [115] and proved that there is a constant $\mu > 0$ such that if $p \gg \frac{\log n}{n}$, then w.h.p. $G = G(n,p)$ contains a compatible Hamilton cycle for every μnp-bounded incompatibility system defined over G, which strengthen the result about Hamilton cycles in $G(n,p)$ without restrictions of incompatibility systems.

The concept of incompatibility systems appears to provide a new and interesting take on robustness of graph properties. Krivelevich, Lee and Sudakov [116] also suggested looking at how various extremal results can be strengthened using this notion. It is natural to consider other compatible spanning structures. We study robust version of the above-mentioned Alon–Yuster result [8] about H-factors in incompatibility system, and obtain the following result. Note that throughout this chapter, we always assume $\chi(H) \geq 2$, since $\chi(H) = 1$ is a trivial case. For convenience, given constants μ, δ and $n \in \mathbb{N}$, an (n, δ, μ)-*incompatibility system* $(G, \mathcal{F})$ consists of an n-vertex graph G with $\delta(G) \geq \delta n$ and a μn-bounded incompatibility system $\mathcal{F}$ over G.

Theorem 5.2 (Main Theorem). *Let $h, r \in \mathbb{N}$ and H be any h-vertex graph with $\chi(H) = r$. For any $\alpha > 0$, there exists a constant $\mu > 0$ such that for any sufficiently large n*

with $n \in h\mathbb{N}$, every $(n, 1-\frac{1}{r}+\alpha, \mu)$-incompatibility system $(G, \mathcal{F})$ contains a compatible H-factor. In particular, the error term of α in the minimum degree condition cannot be omitted.

We adapt the absorption method to prove Theorem 5.2, so our main tasks are to build an absorbing set (Lemma 5.1) and find a compatible H-tiling covering almost all vertices of G (Lemma 5.2). The rest of this chapter is organised as follows. In the next section, we set up some basic notation and show why the error term of αn in Theorem 5.2 cannot be omitted. In Section 3, we state Lemmas 5.1 and 5.2 and give the proof of Theorem 5.2. Sections 4 and 5 are devoted to proving Lemmas 5.1 and 5.2, respectively.

5.2 Notation and preliminaries

For a graph G, we use $e(G)$ to denote the number of edges of G. Let $r, s \geq 2$. We use $K_r(n_1, n_2, \ldots, n_r)$ to denote the complete r-partite graph with each part of size n_i. In particular, when $n_i = s$ for any $i \in [r]$, we use K_r^s instead of $K_r(s, s, \ldots, s)$. We also use K_1^s to denote an independent set with size s. For a graph $G = (V, E)$ and k pairwise disjoint vertex subsets $U_1, \ldots, U_k \subseteq V$, we use $G[U_1, \ldots, U_k]$ to denote the r-partite subgraph induced by $U_1, \ldots, U_r$. In the proofs, if we choose $a \ll b$, then this means that for any $b > 0$, there exists $a_0 > 0$ such that for any $a < a_0$ the subsequent statement holds. Hierarchies of other lengths are defined similarly.

5.2.1 A barrier for a perfect K_r^s-tiling

Given $r \geq 3, s \geq 1$, we now show the minimum degree condition is in some sense tight when $H = K_r^s$ by constructing for $0 < \mu \leq \frac{1}{2r}$ an n-vertex graph G and an incompatibility system $\mathcal{F}$ over G such that $\delta(G) \geq \left(1 - \frac{1}{r} + \frac{\mu}{2}\right) n$ and $\mathcal{F}$ is a μn-bounded, but there is no compatible K_r^s-factor. Let $n = k \cdot |K_r^s| = krs$ and G_0 be a complete r-partite graph with r parts $V_1, \ldots, V_r$, where $|V_1| = ks + 1$, $|V_2| = ks - 1$ and $|V_i| = ks$ for each $i \in \{3, \ldots, r\}$. Inside each part V_i of G_0, we add a bipartite spanning subgraph with minimum degree at least $\frac{\mu n}{2} + 1$ and maximum degree at most μn, whose two parts differ in size by at most 1. Denote the resulting graph by G. Hence, each $G[V_i]$ is a triangle-free graph with $\delta(G[V_i]) \geq \frac{\mu n}{2} + 1$ and $\Delta(G[V_i]) \leq \mu n$, and $\delta(G) \geq \left(1 - \frac{1}{r} + \frac{\mu}{2}\right) n$. Now we define an incompatibility system $\mathcal{F}$ over G. For any

two different parts V_i, V_j of G, let v_i be any vertex in V_i and v_{j_1}, v_{j_2} be any two different vertices in V_j. If $v_{j_1}v_{j_2}$ is an edge in $G[V_j]$, then let $v_iv_{j_1}$ and $v_iv_{j_2}$ be incompatible at v_i. Since $\Delta(G[V_j]) \leq \mu n$, $\mathcal{F}$ is μn-bounded.

Now it remains to verify that there is no compatible K_r^s-factor. Before that we define the *index vector* $\mathbf{v}_F = (x_1, \ldots, x_r)$ for any subgraph $F \subseteq G$, where $x_i = |V(F) \cap V_i|$. We claim that the index vector of every compatible K_r^s in G is $(s, \ldots, s)$. Assume that there is a compatible copy of K_r^s in G, denoted by K, whose index vector is not $(s, \ldots, s)$. Then one of coordinates of its index vector is large than s. Without loss of generality, suppose that the first coordinate is $s + \ell$, where $\ell \geq 1$. Then K has an edge uv in $G[V_1]$. This is because any $s + 1$ vertices of a K_r^s induce at least one edge. Since $G[V_1]$ is triangle-free, it can contain at most two parts of K. Since $r \geq 3$, K has a vertex w in some V_i $(i \neq 1)$ such that wu and wv are compatible, which contradicts the definition of $\mathcal{F}$. Hence, the assumption is false and the index vector of every compatible copy of K_r^s in G is $(s, \ldots, s)$. Since $V_1, \ldots, V_r$ are not balanced, we cannot find k vertex-disjoint compatible copies of K_r^s, i.e. there is no compatible K_r^s-factor.

5.2.2 Proof strategy and main tools

Our proof uses the the absorption method, pioneered by the work of Rödl, Ruciński and Szemerédi [144] on perfect matchings in hypergraphs. In recent years, the method has become an extremely important tool for studying the existence of spanning structures in graphs, digraphs and hypergraphs

Before its appearance the only known systematic way is the blow-up lemma due to Komlós, Sárközy and Szemerédi [107], which has successfully been used in the proof the Alon–Yuster Theorem and various classical results. However, in our setting of incompatibility systems, there has been no extension of the blow-up lemma so far. Instead, we utilize the absorption framework and develop a different counting argument for embedding compatible graphs under certain regularity conditions. A key step in the absorption method for H-factor problem is to show that for (almost) every set of $h := |V(H)|$ vertices, the host graph G contains polynomially many absorbers (to be defined shortly). Here, a much weaker counting strategy is used in this chapter, that is, we aim to show that *for (almost) every set of h vertices, the host graph G contains $\Omega(n)$ vertex-disjoint absorbers.*

In what follows we sketch our proofs and introduce some key concepts and reduce the proofs to two main lemmas: The general idea of absorption is to split the problem of finding an perfect tilings into two subproblems. The first major task is to define and find an absorbing structure in the host graph which can 'absorb' left-over vertices. We will use the following notion of an absorber.

Definition 5.2 (Absorber)**.** *Let H be an h-vertex graph and G be an n-vertex graph. For any h-set $S \subseteq V(G)$ and integer t, we say that a set $A_S \subseteq V(G) \setminus S$ is an* (H,t)-absorber *for S if $|A_S| \leq h^2 t$ and both $G[A_S]$ and $G[A_S \cup S]$ contain compatible H-factors.*

Definition 5.3 (Absorbing set)**.** *Let H be an h-vertex graph, G be an n-vertex graph and ξ be a constant. A set $A \subseteq V(G)$ is called a* ξ-absorbing set *if for any set $R \subseteq V(G) \setminus A$ with $|R| \leq \xi n$ and $|A \cup R| \in h\mathbb{N}$, the graph $G[A \cup R]$ contains a compatible H-factor.*

Lemma 5.1 (Absorbing lemma)**.** *Let $h, r \in \mathbb{N}$ and H be an h-vertex graph with $\chi(H) = r \geq 2$. For any $\alpha, \sigma > 0$, there exist $\mu, \xi > 0$ such that for any sufficiently large n, if $(G, \mathcal{F})$ is an $\left(n, 1 - \frac{1}{r} + \alpha, \mu\right)$-incompatibility system, then G contains a ξ-absorbing set A of size at most σn.*

The second major task in absorption arguments for tilings is to find a tiling that covers most of the vertices, leaving just a small linear number of vertices uncovered. We will sometimes call such a tiling an *almost perfect tiling* or an *almost cover*.

Lemma 5.2 (Almost cover)**.** *Let $h, r \in \mathbb{N}$ and H be an h-vertex graph with $\chi(H) = r \geq 2$. For any $\alpha, \tau > 0$, there exists $\mu > 0$ such that for any sufficiently large n, if $(G, \mathcal{F})$ is an $\left(n, 1 - \frac{1}{r} + \alpha, \mu\right)$-incompatibility system, then there exists a compatible H-tiling covering all but at most τn vertices of G.*

Proof of Theorem 5.2. Given $\alpha > 0$, we choose

$$\frac{1}{n} \ll \mu \ll \tau \ll \xi \ll \sigma \ll \alpha.$$

Applying Lemma 5.1 with $\sigma \leq \frac{\alpha}{2}$, G contains a ξ-absorbing set A of size at most σn. Since $\delta(G - A) \geq \left(1 - \frac{1}{r} + \frac{\alpha}{2}\right) n$, we can apply Lemma 5.2 on $G - A$, and obtain a compatible H-tiling $\mathcal{H}_1$ covering all but at most τn vertices in $G - A$. Denote the set

of uncovered vertices in $G - A$ by R. Since $\tau \ll \xi$, $G[A \cup R]$ contains a compatible H-factor $\mathcal{H}_2$. Thus, $\mathcal{H}_1 \cup \mathcal{H}_2$ is a compatible H-factor of G. □

5.3 Almost compatible H-factor

The main tool in the proof of Lemma 5.2 is Szemerédi's Regularity Lemma. To state it, we first give some definitions and lemmas.

5.3.1 Regularity

Definition 5.4 (Regular pair)**.** *Given a graph G and disjoint vertex subsets $X, Y \subseteq V(G)$, the* density *of the pair (X, Y) is defined as*

$$d(X,Y) := \frac{e(X,Y)}{|X||Y|},$$

where $e(X,Y) := e(G[X,Y])$. For $\varepsilon > 0$, the pair (X,Y) is ε-regular *if for any $A \subseteq X, B \subseteq Y$ with $|A| \geq \varepsilon|X|, |B| \geq \varepsilon|Y|$, we have*

$$|d(A,B) - d(X,Y)| < \varepsilon.$$

Additionally, if $d(X,Y) \geq d$ for some $d > 0$, then we say that (X,Y) is (ε, d)-regular.

Definition 5.5 (Regular partition)**.** *For a graph $G = (V,E)$ and $\varepsilon, d > 0$, a partition $V = V_0 \cup V_1 \cup \ldots \cup V_k$ is* (ε, d)-regular*, if*

- $|V_0| \leq \varepsilon|V|$*;*
- $|V_1| = |V_2| = \ldots = |V_k| \leq \lceil \varepsilon|V| \rceil$*;*
- *all but at most εk^2 pairs (V_i, V_j) with $1 \leq i < j \leq k$ are (ε, d)-regular.*

Moreover, we usually call $V_0, V_1, \ldots, V_k$ clusters *and call the cluster V_0* exceptional set*.*

Fact 5.1. *Let (X,Y) be an (ε, d)-regular pair, and $B \subseteq Y$ with $|B| \geq \varepsilon|Y|$. Then all but $\varepsilon|X|$ vertices in X have degree at least $(d - \varepsilon)|B|$ in B.*

Fact 5.2 (Slicing lemma, [110])**.** *Let (X,Y) be an (ε, d)-regular pair, and for some $\eta > \varepsilon$, let $X' \subseteq X, Y' \subseteq Y$ with $|X'| \geq \eta|X|, |Y'| \geq \eta|Y|$. Then (X', Y') is an ε'-regular pair with $\varepsilon' = \max\{\varepsilon/\eta, 2\varepsilon\}$, and for its density d' we have $|d' - d| < \varepsilon$.*

Lemma 5.3 (Degree form of the Regularity Lemma, [110]). *For every $\varepsilon > 0$, there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in (0, 1]$ is any real number, then there is an (ε, d)-regular partition $V = V_0 \cup V_1 \cup \ldots \cup V_k$ with $|V_i| = m$ for each $i \in [k]$, and a spanning subgraph $G' \subseteq G$ with the following properties:*

- $1/\varepsilon \leq k \leq M$,
- $d_{G'}(v) > d_G(v) - (d + \varepsilon)|V|$ *for all* $v \in V$,
- $e(G'[V_i]) = 0$ *for all* $i \geq 1$,
- *all pairs* (V_i, V_j) $(1 \leq i < j \leq k)$ *are* ε*-regular in* G' *with density* 0 *or at least* d.

Definition 5.6 (Reduced graph). *Given an arbitrary graph $G = (V, E)$, a partition $V = V_1 \cup \ldots \cup V_k$, and two parameters $\varepsilon, d > 0$, the* reduced graph $R = R(\varepsilon, d)$ *of G is defined as follows:*

- $V(R) = [k]$,
- $ij \in E(R)$ *if and only if* (V_i, V_j) *is* (ε, d)*-regular.*

As remarked in [110], a typical application of degree form of the Regularity Lemma begins with a graph $G = (V, E)$ and appropriate parameters $\varepsilon, d > 0$, and then obtains an (ε, d)-regular partition $V = V_0 \cup V_1 \cup \ldots \cup V_k$ and a subgraph G' with above-mentioned properties. Then we usually drop the exceptional set V_0 to get a *pure graph* $G'' = G' - V_0$ of G and study the properties of reduced graph $R = R(\varepsilon, d)$ of G''. By Lemma 5.3,

$$\delta(R) \geq \frac{\delta(G) - (d + \varepsilon)|V| - |V_0|}{m} \geq \frac{\delta(G) - (d + 2\varepsilon)|V|}{m}.$$

In particular, if $\delta(G) \geq c|V|$, then $\delta(R) \geq (c - d - 2\varepsilon)k$.

5.3.2 Embedding compatible subgraphs

We will use the following lemma to prove Lemma 5.2.

Lemma 5.4. *For any $h, d, \eta > 0$ and $r \geq 1$, there exist $\mu, \varepsilon^*, c = c(r, d, \eta) > 0$ such that for any sufficiently large n, if $(G, \mathcal{F})$ is an $\left(n, 1 - \frac{1}{r} + \alpha, \mu\right)$-incompatibility system and $U_1, \ldots, U_r \subseteq V(G)$ are pairwise (ε^*, d)-regular with $|U_i| \geq \eta n$ for any $i \in [r]$, then there exist at least cn^{rh} compatible copies of K_r^h in $G[U_1, \ldots, U_r]$.*

Proof of Lemma 5.2. Given α and τ, we choose

$$\frac{1}{n} \ll \mu \ll \varepsilon \ll \varepsilon^* \ll \rho \ll \alpha, \tau.$$

We first apply Lemma 5.3 on G with $d = \rho$, and obtain an (ε, ρ)-regular partition $V(G) = V_0 \cup V_1 \cup \ldots \cup V_k$ for some $1/\varepsilon \le k \le M$ and $|V_i| = m \ge \frac{(1-\varepsilon)n}{k}$ for each $i \in [k]$. Let $R = R(\varepsilon, \rho)$ be the reduced graph for this partition. Since $\varepsilon \ll \rho \ll \alpha$ (we actually need $\alpha/2 \ge \rho + 2\varepsilon$), we have $\delta(R) \ge \left(1 - \frac{1}{r} + \alpha - \rho - 2\varepsilon\right) k \ge \left(1 - \frac{1}{r} + \frac{\alpha}{2}\right) k$. By the Hajnal–Szemerédi theorem [88], R has a K_r-tiling $\mathcal{K} = \{K_r^{(1)}, K_r^{(2)} \ldots, K_r^{(\lfloor k/r \rfloor)}\}$ covering all but at most $r-1$ vertices.

Recall that $\chi(H) = r$. Suppose that the sizes of r color classes of H are $h_1, \ldots, h_r$ with $h = h_1 + \ldots + h_r$. Since $H \subseteq K_{h_1,\ldots,h_r}$ and a balanced complete r-partite graph K_r^h contains r vertex-disjoint copies of $K_{h_1,\ldots,h_r}$, it suffices to find a compatible K_r^h-tiling of G with at most τn vertices uncovered. Suppose that the corresponding clusters of $K_r^{(1)}$ are $V_1, \ldots, V_r$. By Definition 5.6, $V_1, \ldots, V_r$ are pairwise (ε, ρ)-regular. Iteratively applying Lemma 5.4 on $G[V_1, \ldots, V_r]$ with $d = \rho/2$ and $\eta = \frac{(1-\varepsilon)\tau}{2k}$ and the fact $\varepsilon \ll \varepsilon^*$, we can find a compatible K_r^h-tiling that leaves at most $\eta n \le \tau m/2$ vertices uncovered in each V_i for $i \in [r]$. For $K_r^{(2)} \ldots, K_r^{(\lfloor k/r \rfloor)}$, we do the same operation as $K_r^{(1)}$. Finally, we can find a compatible H-tiling with at most

$$|V_0| + (r-1)m + (k-r+1)\tau m/2 \le r\varepsilon n + km\tau/2 \le \tau n$$

vertices uncovered, where the last inequality follows since $2r\varepsilon \le \tau$. □

In the rest of this section, we will focus on the proof of Lemma 5.4. Before that, we give a definition.

Definition 5.7 (Good pair)**.** *Given a vertex v and a balanced complete i-partite graph K_i^h for any $i \ge 1$, we say that (v, K_i^h) is a* good pair *if*

- *K_i^h is compatible;*
- *v is adjacent to all vertices of K_i^h;*
- *all edges from v to K_i^h are mutually compatible at v.*

Moreover, the vertex v is called the center *of good pair (v, K_i^h).*

Proof of Lemma 5.4. We prove this lemma by induction on r. Given $d, \eta > 0$, we choose

$$\frac{1}{n} \ll \mu \ll \varepsilon_r \ll \varepsilon_{r-1} \ll \cdots \ll \varepsilon_2 \ll \varepsilon_1 \ll d, \eta.$$

The base case $r = 1$ is trivial since we can find $\binom{\eta n}{h}$ h-subsets in U_1.

Suppose that Lemma 5.4 is true for $r-1$. Now we consider the case r, that is, $U_1, \ldots, U_r$ are pairwise (ε_r, d)-regular and $|U_i| \geq \eta n$ for any $i \in [r]$. We want to find $c(r, d, \eta)n^{rh}$ compatible copies of K_r^h in $G[U_1, \ldots, U_r]$ for some $c(r, d, \eta)$.

We choose a subset $U_r' \subseteq U_r$ such that any vertex in U_r' has degree at least $(d-\varepsilon_r)|U_i|$ in U_i for every $i \in [r-1]$. By Fact 5.1, there are at least $(1-\varepsilon_r)|U_r|$ vertices of U_r with degree at least $(d-\varepsilon_r)|U_i|$ in U_i for any fixed $i \in [r-1]$. Hence, $|U_r'| \geq (1-(r-1)\varepsilon_r)|U_r|$.

Let v be an arbitrary vertex in U_r'. Denote the neighborhood of v in U_i by U_i' for every $i \in [r-1]$. Then $|U_i'| \geq (d-\varepsilon_r)|U_i| \geq (d-\varepsilon_r)\eta n \geq d\eta n/2$. By Fact 5.2, for any $1 \leq i < j \leq r-1$, U_i' and U_j' is an (ε', d')-regular pair with $\varepsilon' = \max\{\varepsilon_r/(d-\varepsilon_r), 2\varepsilon_r\}$ and $|d'-d| < \varepsilon_r$, which is also an $(\varepsilon_{r-1}, \frac{d}{2})$-regular pair by the fact that $\varepsilon_r \ll \varepsilon_{r-1}$ and $d' > d - \varepsilon_r \geq \frac{d}{2}$. By the induction hypothesis, there are at least $c(r-1, \frac{d}{2}, \frac{d}{2}\eta)n^{(r-1)h}$ compatible copies of K_{r-1}^h in $G[U_1', \ldots, U_{r-1}']$. Among these compatible copies of K_{r-1}^h, at least $(c(r-1, \frac{d}{2}, \frac{d}{2}\eta) - \mu)n^{(r-1)h}$ of them can form good pairs with v. This is because there are at most $\mu n^{(r-1)h}$ compatible copies of K_{r-1}^h, each of which contains two vertices v_1, v_2 such that vv_1 and vv_2 are incompatible at v. Indeed, since $(G, \mathcal{F})$ is an $\big(n, 1-\frac{1}{r}+\alpha, \mu\big)$-system, there are at most μn^2 incompatible pairs $\{vv_1, vv_2\}$ and for each such pair $\{vv_1, vv_2\}$, there are at most $n^{(r-1)h-2}$ copies of K_{r-1}^h containing v_1 and v_2.

Since $|U_r'| \geq (1-(r-1)\varepsilon_r)|U_r|$, there are at least

$$\begin{aligned}
&(1-(r-1)\varepsilon_r)|U_r| \cdot (c(r-1, d/2, d\eta/2) - \mu)n^{(r-1)h} \\
\geq &(1-(r-1)\varepsilon_r)(c(r-1, d/2, d\eta/2) - \mu)\eta n^{(r-1)h+1} \\
= &c_1 n^{(r-1)h+1}
\end{aligned}$$

good pairs (v, K_{r-1}^h) between U_r and $G[U_1, \ldots, U_{r-1}]$ (with centers v in U_r), where $c_1 = (1-(r-1)\varepsilon_r)(c(r-1, \frac{d}{2}, \frac{d}{2}\eta) - \mu)\eta$.

We claim that if a compatible copy K of K_{r-1}^h is contained in $c_1 n/2$ good pairs (with a set X of centers in U_r), then there are at least $c_2 n^h$ compatible copies of K_r^h in $G[U_1, \ldots, U_r]$ containing K, where c_2 will be determined later. To build such a copy

of K_r^h, we need to choose h proper centers $v_1, \ldots, v_h$ from X:

- there are at least $c_1 n/2 - 2\mu n \cdot e(K_{r-1}^h)$ choices for v_1 such that $\{v_1\} \cup V(K)$ induces a compatible copy of $K_r(1, h, \ldots, h)$ in $G[U_1, \ldots, U_r]$. Fix one of the choices as v_1;
- for $i \geq 2$, there are at least $c_1 n/2 - (i-1) - 2\mu n \cdot e(K_{r-1}^h) - \mu n \cdot e(K_{i-1,(r-1)h}) \geq (c_1/2 - \mu r^2 h^2)n$ choices for v_i such that $\{v_1, \ldots, v_i\} \cup V(K)$ induces a compatible copy of $K_r(i, h, \ldots, h)$ in $G[U_1, \ldots, U_r]$.

Hence, there are at least $c_2 n^h$ compatible copies of K_r^h in $G[U_1, \ldots, U_r]$ containing K, where $c_2 = \frac{1}{h!}(c_1/2 - \mu r^2 h^2)^h$.

From above, it suffices to find $\Omega(n^{(r-1)h})$ such copies of compatible K_{r-1}^h contained in $c_1 n/2$ good pairs with centers in U_r. We define $\mathcal{X}_1 := \{K \mid K$ is a compatible copy of K_{r-1}^h which is contained in at least $c_1 n/2$ good pairs with centers in $U_r\}$ and $\mathcal{X}_2$ be the family of remaining compatible copies of K_{r-1}^h in $G[U_1, \ldots, U_{r-1}]$. Since all copies of K_{r-1}^h in $\mathcal{X}_2$ totally occupy at most $n^{(r-1)h} \cdot c_1 n/2 = c_1 n^{(r-1)h+1}/2$ good pairs, the number of remaining good pairs is at least $c_1 n^{(r-1)h+1}/2$. Hence $|\mathcal{X}_1| \geq c_1 n^{(r-1)h}/2$. Thus, there are at least $c_1 c_2 n^{rh}/2$ compatible copies of K_h^r in $G[U_1, \ldots, U_r]$. The proof is completed by letting $c = c(r, d, \eta) = c_1 c_2/2$ and $\varepsilon^* = \varepsilon_r$. □

5.4 Absorbing lemma

To derive Lemma 5.1, it suffices to prove the following two lemmas.

Lemma 5.5. *Let H be an h-vertex graph and $\chi(H) = r \geq 2$. For any $\alpha, \delta, \gamma > 0$ with $\delta \leq \alpha/2$, there exist $\mu, \xi > 0$ such that for any sufficiently large n, if $(G, \mathcal{F})$ is an $\big(n, 1 - \frac{1}{r} + \alpha, \mu\big)$-incompatibility system, then $V(G)$ can be partitioned into B and U such that $|B| \leq \delta n$ and $G[U]$ contains a ξ-absorbing set A_U of size at most γn.*

Lemma 5.6. *Let H be an h-vertex graph and $\chi(H) = r \geq 2$. For any $\alpha > 0$, there exist $\mu, \varrho > 0$ such that for any sufficiently large n, if $(G, \mathcal{F})$ is an $\big(n, 1 - \frac{1}{r} + \alpha, \mu\big)$-incompatibility system, then every vertex v in G is contained in ϱn compatible copies of H which contain v and are otherwise vertex-disjoint.*

Proof of Lemma 5.1. Given $\alpha, \sigma > 0$, we choose

$$\frac{1}{n} \ll \mu \ll \xi \ll \gamma, \delta \ll \varrho \ll \alpha, \sigma.$$

By Lemma 5.5, $V(G)$ can be partitioned into B and U such that $|B| \leq \delta n$ and $G[U]$ contains a ξ-absorbing set A_U of size at most γn. Since $\delta(G - A_U) \geq (1 - \frac{1}{r} + \frac{\alpha}{2})n$, by applying Lemma 5.6 on $G - A_U$, we obtain that every vertex $v \in B$ is contained in ϱn compatible copies of H in $G - A_U$ which contain v and are otherwise vertex-disjoint. We greedily choose $|B|$ vertex-disjoint compatible copies of H each containing exactly one vertex of B, denoted by $H_1, \ldots, H_{|B|}$. This can be done because $|B| \cdot h \leq \delta n h \leq \varrho n$. The proof of Lemma 5.1 is completed by letting $A := A_U \cup H_1 \cup \ldots \cup H_{|B|}$. □

5.4.1 Building an absorbing set

Lemma 5.7 (Lemma 2.2 in [141], Lemma 3.2 in [89]). *Let $t \in \mathbb{N}$, H be an h-vertex graph and $\gamma > 0$. Then there exists $\xi = \xi(t, h, \gamma) > 0$ such that the following holds. If $(G, \mathcal{F})$ is an incompatibility system with $|V(G)| = n$ such that for every $S \in \binom{V(G)}{h}$ there is a family of at least γn vertex-disjoint (H, t)-absorbers, then G contains a ξ-absorbing set of size at most γn.*

Definition 5.8 (Closed). *Let $m, t \in \mathbb{N}$, H be an h-vertex graph and G be an n-vertex graph. We say that two vertices $u, v \in V(G)$ are* (H, m, t)-reachable *if for any set W of m vertices, there is a set $S \subseteq V(G) \setminus W$ of size at most $ht - 1$ such that both $G[S \cup \{u\}]$ and $G[S \cup \{v\}]$ have compatible H-factors, where we call such S an* H-connector *for u, v. Moreover, $V(G)$ is* (H, m, t)-closed *if every two vertices u, v in $V(G)$ are (H, m, t)-reachable.*

Definition 5.9 (Inner-closed). *For a vertex subset $U \subseteq V(G)$, two vertices $u, v \in U$ are* (H, m, t)-inner reachable *if they are (H, m, t)-reachable and their H-connectors are contained in U. Moreover, U is* (H, m, t)-inner closed *if every two vertices u, v in U are (H, m, t)-inner reachable.*

Lemma 5.8 (Lemma 3.9 in [89]). *Given $h, t \in \mathbb{N}$ with $h \geq 3$ and $\beta > 0$, the following holds for any h-vertex graph H and sufficiently large $n \in \mathbb{N}$. If $(G, \mathcal{F})$ is an incompatibility system with $|V(G)| = n$ such that $V(G)$ is $(H, \beta n, t)$-closed, then every $S \in \binom{V(G)}{h}$ has at least $\frac{\beta}{h^3 t} n$ vertex-disjoint (H, t)-absorbers.*

Note that the original versions of Lemmas 5.7 and 5.8 deal with absorbing sets and absorbers without restrictions of incompatibility systems. By going through their proofs, the two lemmas also valid for compatible version.

To obtain Lemma 5.5, by Lemmas 5.7 and 5.8, we only need to prove that $V(G)$ can be partitioned into B and U such that $|B| \leq \delta n$ and U is $(H, \beta n, t)$-inner closed for some β and t. Before that, we give a couple of lemmas which will be used in the proof of Lemma 5.5.

Lemma 5.9. *Let H be an h-vertex graph and $\chi(H) = r \geq 2$. For any $d, \eta > 0$, there exist $\mu, \varepsilon, \beta' > 0$ such that the following holds for any sufficiently large n. If $(G, \mathcal{F})$ is an incompatibility system such that $|V(G)| = n$ and $\mathcal{F}$ is μn-bounded, $U_1, \ldots, U_r \subseteq V(G)$ are pairwise (ε, d)-regular with $|U_i| \geq \eta n$ and $u, v \notin \cup_{i=1}^{r} U_i$ are both adjacent to all vertices in $\cup_{i=2}^{r} U_i$, then u, v are $(K_r^h, \beta' n, 1)$-reachable.*

Lemma 5.10. *Let H be an h-vertex graph and $\chi(H) = r \geq 2$. For any $\alpha, \delta > 0$, there exist $\mu, \beta', \gamma' > 0$ such that for any sufficiently large n, if $(G, \mathcal{F})$ is an $\left(n, 1 - \frac{1}{r} + \alpha, \mu\right)$-incompatibility system, then $V(G)$ can be partitioned into B and U such that $|B| \leq \delta n$ and for every vertex $u \in U$, u is $(K_r^h, \beta' n, 1)$-inner reachable to at least $\gamma' n$ vertices in U.*

Fact 5.3. *Let G be an n-vertex graph and $v_1, \ldots, v_k$ be k distinct vertices in G. Then*

$$|\cap_{i=1}^{k} N(v_i)| \geq \Sigma_{i=1}^{k} |N(v_i)| - (k-1)n.$$

Lemma 5.11. *Let $t_1, t_2 \in \mathbb{N}, \beta_1, \beta_2 > 0$ and H be an h-vertex graph. The following holds for any sufficiently large n. Let G be an n-vertex graph, $u, v \in V(G)$ and $Z \subseteq V(G)$ with $|Z| = cn$ for some positive constant c. If u is $(H, \beta_1 n, t_1)$-reachable to every vertex in Z, while v is $(H, \beta_2 n, t_2)$-reachable to every vertex in Z, then u, v are $(H, \beta_3 n, t_1 + t_2)$-reachable, where $\beta_3 n = \min\{cn - 1, \beta_1 n/2, \beta_2 n/2\}$.*

Proof. Let W be an arbitrary subset of G with $|W| \leq \beta_3 n$. Since $|W| \leq \min\{cn - 1, \beta_1 n/2\}$, we can find a vertex $u_1 \in Z \setminus W$ and an H-connector S_1 for u, u_1 in $V(G) \setminus (W \cup \{u, u_1, v\})$ with $|S_1| \leq ht_1 - 1$. Meanwhile, since $|W| \leq \beta_2 n/2$, we can also find an H-connector S_2 for u_1, v in $V(G) \setminus (W \cup \{u, u_1, v\} \cup S_1)$ with $|S_2| \leq ht_2 - 1$. Hence, u, v are $(H, \beta_3 n, t_1 + t_2)$-reachable. □

By a similar proof, we have the following lemma.

Lemma 5.12. *Let $t_1, t_2 \in \mathbb{N}, \beta_1, \beta_2 > 0$ and H be an h-vertex graph. The following holds for any sufficiently large n. Let G be an n-vertex graph and $u_1, u_2, v \in V(G)$. If u_i*

is $(H, \beta_i n, t_i)$-reachable to v for each $i \in [2]$, then u_1, u_2 are $(H, \beta_3 n, t_1 + t_2)$-reachable, where $\beta_3 = \min\{\beta_1/2, \beta_2/2\}$.

Proof of Lemma 5.5. Given $\alpha, \delta, \gamma > 0$, we choose

$$\frac{1}{n} \ll \mu, \beta', \gamma' \ll \varepsilon \ll d \ll \alpha, \delta, \gamma.$$

By Lemma 5.10, $V(G)$ can be partitioned into B and U such that $|B| \leq \delta n$ and for every vertex $u \in U$, u is $(K_r^h, \beta' n, 1)$-inner reachable to at least $\gamma' n$ vertices in U. We will show that U is $(K_r^h, \beta n, t)$-inner closed for some β and t. Taking this for granted, U is also $(H, \beta n, t)$-inner closed. By Lemma 5.8, every $S \in \binom{U}{h}$ has at least $\frac{\beta(1-\delta)}{h^3 t} n$ vertex-disjoint (H, t)-absorbers. By Lemma 5.7, $G[U]$ contains a ξ-absorbing set A_U of size at most $\frac{\beta(1-\delta)}{h^3 t} n \leq \gamma n$ (here we need $\frac{\beta(1-\delta)}{h^3 t} \leq \gamma$). It remains to show that for any two vertices $x, y \in U$, x, y are $(K_r^h, \beta n, t)$-inner reachable.

Let $X, Y \subseteq U$ be the sets of vertices that can be $(K_r^h, \beta' n, 1)$-inner reachable to x, y respectively. Then $|X|, |Y| \geq \gamma' n$. If $|X \cap Y| \geq \gamma' n/2$, then by Lemma 5.11, x, y are $(K_r^h, \beta_1 n, 2)$-inner reachable, where $\beta_1 n = \min\{\gamma' n/2 - 1, \beta' n/2\}$. Hence, we assume that $|X \cap Y| < \gamma' n/2$. Let $X_1 := X \setminus Y$ and $Y_1 := Y \setminus X$. Then $|X_1|, |Y_1| \geq \gamma' n/2$.

Apply Lemma 5.3 on $G[U]$ to refine X_1, Y_1 and $U \setminus (X_1 \cup Y_1)$ and obtain an (ε, d)-regular partition $U = V_0 \cup V_1 \cup \ldots \cup V_k$ for some $1/\varepsilon \leq k \leq M$ with $|V_i| = m$ for each $i \in [k]$. Then we arbitrarily choose two clusters (except V_0) from the partitions of X_1, Y_1 respectively. Without loss of generality, denote the two chosen clusters by V_1, V_2 respectively. Since $\delta(G[U]) \geq (1 - 1/r + \alpha)n - |B| \geq (1 - 1/r + \alpha/2)n$, the minimum degree of reduced graph is at least $(1 - 1/r + \alpha/3)k$. For each $i \in [2]$, denote the corresponding vertex of V_i in the reduced graph by v_i. By Fact 5.3, v_1 and v_2 have at least $(1 - 2/r + 2\alpha/3)k$ common neighbors in the reduced graph, and arbitrarily choose one of them, denoted by z. Let Z be the corresponding cluster of z. We can greedily construct two copies of K_{r+1}, denoted by $K_{r+1}^{(1)}$ and $K_{r+1}^{(2)}$ respectively, in the reduced graph such that $v_i, z \in V(K_{r+1}^{(i)})$ and z is the unique common vertex of $K_{r+1}^{(1)}$ and $K_{r+1}^{(2)}$. By Fact 5.3, this can be done as long as we choose ε small enough such that

$$r(1 - 1/r + \alpha/3)k - (r-1)k = \alpha r k/3 \geq r + 1.$$

Suppose that the corresponding clusters of $K_{r+1}^{(1)}$ are $V_1, Z, V_3, \ldots, V_{r+1}$. Then any two of them are (ε, d)-regular. We choose an arbitrary vertex w_1 from V_1 which has

degree at least $(d-\varepsilon)|V_j|$ in each V_j ($j \in \{3,4,\ldots,r+1\}$). Denote the neighborhood of w_1 in V_j by V_j'. Then $|V_j'| \geq (d-\varepsilon)|V_j|$ and Z has at least $(1-\varepsilon(r-1))|Z|$ vertices with degree at least $(d-\varepsilon)|V_j'|$ in V_j' for each $j \in \{3,4,\ldots,r+1\}$. Denote the set of these vertices by Z', and then $|Z'| \geq (1-\varepsilon(r-1))|Z| \geq (1-\varepsilon(r-1))(1-\varepsilon)n/k$. Choose an arbitrary vertex v from Z' and let V_j'' be the neighborhood of v in V_j' for each $j \in \{3,4,\ldots,r+1\}$. Then $|V_j''| \geq (d-\varepsilon)|V_j'| \geq (d-\varepsilon)^2|V_j| \geq (d-\varepsilon)^2(1-\varepsilon)n/k$. Note that w_1, v are both adjacent to all vertices in V_j'' for each $j \in \{3,4,\ldots,r+1\}$. Let $\eta = \min\{1-\varepsilon(r-1),(d-\varepsilon)^2\}$. Then $Z' \setminus \{v\}, V_3'', \ldots, V_{r+1}''$ are pairwise (ε', d')-regular, where $\varepsilon' = \max\{\varepsilon/\eta, 2\varepsilon\}$ and $|d'-d| < \varepsilon$. By Lemma 5.9, w_1, v are $(K_r^h, \beta' n, 1)$-inner reachable. By the arbitrariness of v, w_1 is $(K_r^h, \beta' n, 1)$-inner reachable to at least $(1-\varepsilon(r-1))|Z|$ vertices of Z. For $K_{r+1}^{(2)}$, we do the similar operation. Choose a large-degree vertex w_2 from V_2 and obtain that w_2 is also $(K_r^h, \beta' n, 1)$-inner reachable to at least $(1-\varepsilon(r-1))|Z|$ vertices of Z. Hence, w_1 and w_2 are both $(K_r^h, \beta' n, 1)$-inner reachable to at least $(1-2\varepsilon(r-1))|Z| \geq \frac{(1-2\varepsilon(r-1))(1-\varepsilon)}{k} n$ vertices of Z.

By Lemma 5.11, w_1, w_2 are $(K_r^h, \beta'' n, 2)$-inner reachable, where $\beta'' n = \min\{(1-2\varepsilon(r-1))(1-\varepsilon)n/k - 1, \beta' n/2\}$. Recall that x, w_1 are $(K_r^h, \beta' n, 1)$-inner reachable and y, w_2 are $(K_r^h, \beta' n, 1)$-inner reachable. By Lemma 5.12, x and y are $(K_r^h, \beta_2 n, 4)$-inner reachable, where $\beta_2 = \min\{\beta'/4, \beta''/4\}$. Thus, for any two vertices $x, y \in U$, x, y are $(H, \beta n, 4)$-inner reachable, where $\beta = \min\{\beta_1, \beta_2\}$. □

5.4.2 Proofs of Lemmas 5.9 and 5.10

Proof of Lemma 5.9. Given $d, \eta > 0$, we choose

$$1/n \ll \mu, \beta' \ll \varepsilon \ll d, \eta.$$

By Lemma 5.4, there exist at least cn^{rh} compatible copies of K_r^h in $G[U_1,\ldots,U_r]$. Since a copy of $K_r(h-1,h,\ldots,h)$ is contained in at most n copies of K_r^h, there are at least cn^{rh-1} compatible copies of $K_r(h-1,h,\ldots,h)$ in $G[U_1,\ldots,U_r]$ with $h-1$ vertices in U_1. Now, we find a family $\mathcal{K}$ of compatible copies of $K_r(h-1,h,\ldots,h)$, each of which can form compatible copies of K_r^h with u and v respectively.

We claim that the number of compatible copies of $K_r(h-1,h,\ldots,h)$ such that either there is an edge v_1v_2 in the $K_r(h-1,h,\ldots,h)$ with v_1v_2, uv_i incompatible at v_i for some $i \in [2]$ or there are two vertices u_1, u_2 in the $K_r(h-1,h,\ldots,h)$ with

uu_1, uu_2 incompatible at u is at most $2\mu n^{rh-1}$. This is because the number of edges v_1v_2 with v_1v_2, uv_i incompatible at v_i for some $i \in [2]$ is at most μn^2, while the number of 2-subsets $\{u_1, u_2\}$ with uu_1, uu_2 incompatible at u is at most μn^2.

Hence, $|\mathcal{K}| \geq (c - 4\mu)n^{rh-1}$. Moreover, deleting $\beta' n$ vertices destroys at most $\beta' n^{rh-1} < |\mathcal{K}|$ compatible copies of $K_r(h-1, h, \ldots, h)$ (here we need $\beta' n^{rh-1} < |\mathcal{K}|$). Thus, u and v are $(K_r^h, \beta' n, 1)$-reachable. □

Proof of Lemma 5.10. Given $\alpha, \delta > 0$, we choose

$$1/n \ll \mu, \beta', \gamma' \ll \varepsilon \ll \rho \ll \alpha, \delta.$$

Apply Lemma 5.3 on G and obtain an (ε, ρ)-regular partition $V(G) = V_0 \cup V_1 \cup \ldots \cup V_k$ for some $1/\varepsilon \leq k \leq M$ with $|V_i| = m \geq \frac{(1-\varepsilon)n}{k}$ for each $i \in [k]$. Let $R = R(\varepsilon, \rho)$ be the reduced graph for this partition. Then $\delta(R) \geq (1 - \frac{1}{r} + \frac{\alpha}{2})k$. By the Hajnal–Szemerédi theorem [88], taking $\ell = \lfloor k/r \rfloor$, R has a K_r-tiling $\mathcal{K} = \{K_r^{(1)}, K_r^{(2)} \ldots, K_r^{(\ell)}\}$ covering all but at most $r-1$ vertices. Suppose that the corresponding clusters of $\mathcal{K}$ are $V_1, \ldots, V_{r\ell}$ and the corresponding clusters of uncovered vertices of R are $V_{r\ell+1}, \ldots, V_k$.

Suppose that the corresponding clusters of $K_r^{(1)}$ are $V_1, \ldots, V_r$. Then $V_1, \ldots, V_r$ are pairwise (ε, ρ)-regular. For any $i \in [r]$, V_i has at least $(1 - (r-1)\varepsilon)|V_i|$ vertices with degree at least $(\rho - \varepsilon)|V_j|$ in V_j for any $j \in [r] \setminus \{i\}$. Denote the set of these vertices by V_i' and let $\eta_1 = 1 - (r-1)\varepsilon$. Then $|V_i'| \geq \eta_1 |V_i|$. Let u be an arbitrary vertex in V_1' and $V_1'' := V_1' \setminus \{u\}$. Then u has at least $(\rho - \varepsilon)|V_i| - |V_i \setminus V_i'| \geq (\rho - r\varepsilon)|V_i|$ neighbors in V_i'. Denote the neighborhood of u in V_i' by V_i'' for any $i \in \{2, \ldots, r\}$ and let $\eta_2 = \rho - r\varepsilon$. Then $|V_i''| \geq \eta_2 |V_i|$. Thus, V_1'' has at least $|V_1''| - (r-1)\varepsilon|V_1|$ vertices with degree at least $(\rho - \varepsilon)|V_i''|$ in V_i'' for any $i \in \{2, \ldots, r\}$. Denote the set of these vertices by V_1^0. Then $|V_1^0| \geq |V_1''| - (r-1)\varepsilon|V_1| \geq (\eta_1 - (r-1)\varepsilon)|V_1|$. Let v be an arbitrary vertex in V_1^0 and denote the neighborhood of v in V_i'' by V_i^0 for any $i \in \{2, \ldots, r\}$. Then $|V_i^0| \geq (\rho - \varepsilon)|V_i''| \geq (\rho - \varepsilon)\eta_2 |V_i|$. Note that u, v are both adjacent to all vertices in $\cup_{i=2}^r V_i^0$. Let $\eta_3 = \min\{\eta_1 - (r-1)\varepsilon, (\rho - \varepsilon)\eta_2\}$. We have $V_1^0 \setminus \{v\}, V_2^0, \ldots, V_r^0$ are pairwise (ε', ρ')-regular, where $\varepsilon' = \max\{\varepsilon/\eta_3, 2\varepsilon\}$ and $|\rho - \rho'| < \varepsilon$, and $|V_i^0| \geq \eta_3 |V_i| \geq \frac{(1-\varepsilon)\eta_3}{k} n$.

Now we can apply Lemma 5.9 with $d = \rho'$ and $\eta = \frac{(1-\varepsilon)\eta_3}{k}$, and obtain that u, v are $(K_r^h, \beta' n, 1)$-reachable. Since v can be any vertex in V_1^0, there are at least

$$|V_1^0| \geq (\eta_1 - (r-1)\varepsilon)(1-\varepsilon)n/k \geq \gamma' n$$

(here we need $\frac{(1-\varepsilon)(\eta_1-(r-1)\varepsilon)}{k} \geq \gamma'$) vertices that are $(K_r^h, \beta' n, 1)$-reachable to u. By the arbitrariness of $u \in V_1'$, every vertex in V_1' can be $(K_r^h, \beta' n, 1)$-reachable to $\gamma' n$ vertices. Similarly, for every V_i $(1 \leq i \leq r\ell)$, there exists a subset $V_i' \subseteq V_i$ with $|V_i'| \geq (1-(r-1)\varepsilon)|V_i|$ such that every vertex in V_i' is $(K_r^h, \beta' n, 1)$-reachable to $\gamma' n$ vertices.

Let $U = \cup_{i=1}^{r\ell} V_i'$ and $B = V(G) \setminus U$. Then

$$\begin{aligned}|B| \leq & |V_0| + \Sigma_{i=1}^{r\ell}(r-1)\varepsilon|V_i| + \Sigma_{i=r\ell+1}^{k}|V_i| \\ \leq & \varepsilon n + (r-1)\varepsilon mk + m(r-1) \\ \leq & \delta n.\end{aligned}$$

(here we need $2r\varepsilon \leq \delta$) Moreover, every vertex $u \in U$ is $(K_r^h, \beta' n, 1)$-inner reachable to $\gamma' n$ vertices in U. □

5.4.3 Covering a vertex with many copies of H

In this section, we shall focus on proving Lemma 5.6.

Proof of Lemma 5.6. Given $\alpha > 0$, we choose

$$\frac{1}{n} \ll \mu, \varrho \ll \varepsilon \ll d \ll \alpha.$$

Apply Lemma 5.3 on G and obtain an (ε, d)-regular partition $V(G) = V_0 \cup V_1 \cup \ldots \cup V_k$ for some $1/\varepsilon \leq k \leq M$ and $|V_i| = m$ for each $i \in [k]$. Let $R = R(\varepsilon, d)$ be the reduced graph for this partition. Then $\delta(R) \geq \left(1 - \frac{1}{r} + \frac{\alpha}{2}\right) k$.

Choose an arbitrary vertex v from $V(G)$. Let $N^{\alpha/2} = \{V_i : i \in [k] \text{ and } |N_G(v) \cap V_i| \geq \alpha m/2\}$. By the double counting, we have

$$(1 - 1/r + \alpha)n - |V_0| \leq d_{G-V_0}(v) \leq |N^{\alpha/2}| \cdot m + (k - |N^{\alpha/2}|) \cdot \alpha m/2,$$

that is, $|N^{\alpha/2}| \geq \left(1 - \frac{1}{r} + \frac{\alpha}{4}\right) k$ (here we need $\varepsilon \leq \alpha/4$). Let $\ell = \left(1 - \frac{1}{r} + \frac{\alpha}{4}\right) k$. We choose arbitrary ℓ clusters from $N^{\alpha/2}$. Suppose that the corresponding vertices in R of these ℓ clusters are $x_1, \ldots, x_\ell$. Since $\delta(R) \geq \left(1 - \frac{1}{r} + \frac{\alpha}{2}\right) k$, we have $\delta(R[x_1, \ldots, x_\ell]) \geq \left(1 - \frac{2}{r} + \frac{3\alpha}{4}\right) k$. We can greedily find a copy of K_r in $R[x_1, \ldots, x_\ell]$. By Fact 5.3, this can be done as long as we choose ε small enough such that $(r-1)\left(1 - \frac{2}{r} + \frac{3\alpha}{4}\right) k - (r-2)\ell \geq 1$. Suppose that the corresponding clusters of this copy of K_r are $V_1, \ldots, V_r$. Let

$V_i' = V_i \cap N_G(v)$ for each $i \in [r]$. Then $|V_i'| \geq \frac{\alpha m}{2} \geq \frac{\alpha(1-\varepsilon)}{2k}n$. By Lemma 5.2, $V_1', \ldots, V_r'$ are (ε', d')-regular with $\varepsilon' = \max\{2\varepsilon/\alpha, 2\varepsilon\}$ and $|d' - d| < \varepsilon$. By Lemma 5.4 on $V_1', \ldots, V_r'$ with $d = d', \eta = \frac{\alpha(1-\varepsilon)}{2k}$, there are at least cn^{rh-1} compatible copies of $K_r(h-1, h, \ldots, h)$ in $G[V_1', \ldots, V_r']$.

Now we construct compatible copies of K_r^h based on v and these compatible copies of $K_r(h-1, h, \ldots, h)$. Similar to the proof of Lemma 5.9, there are at least $cn^{rh-1} - 2\mu n^{rh-1}$ compatible copies of $K_r(h-1, h, \ldots, h)$ that can form compatible copies of K_r^h with v. Then we greedily find compatible copies of K_r^h which contain v and are otherwise vertex-disjoint: We first choose an arbitrary compatible $K_r(h-1, h, \ldots, h)$ that can form a compatible K_r^h with v, then delete the $rh-1$ vertices of it, which can destroy at most $(rh-1)n^{rh-2}$ compatible copies of $K_r(h-1, h, \ldots, h)$. We repeat this operation. Finally, we can find ϱn compatible copies of K_r^h which contain v and are otherwise vertex-disjoint as long as we choose μ and ϱ small enough such that

$$cn^{rh-1} - 2\mu n^{rh-1} > (\varrho n - 1)(rh-1)n^{rh-2}.$$

Thus, there are at least ϱn compatible copies of H as desired. □

5.5 Concluding Remarks

In this chapter, we study the robust version of Alon–Yuster's result [8] about the existence of H-factor with respect to the incompatibility system. As we know, in extremal graph theory, characterizing the best possible bound on a minimum degree forcing an H-factor has attracted significant attention until it was finally settled by Kühn and Osthus [120]. They proved the following result, which improves Alon–Yuster's result [8].

Theorem 5.3 ([120]). *Let $\delta(H, n)$ denote the smallest integer k such that every graph G whose order n is divisible by $|H|$ and with $\delta(G) \geq k$ contains an H-factor. Then*

$$\delta(H, n) = \left(1 - \frac{1}{\chi^*(H)}\right) n + O(1).$$

The value of $\chi^*(H)$ depends on the relative sizes of the colour classes in the optimal colorings of H and satisfies $\chi(H) - 1 < \chi^*(H) \leq \chi(H)$.

We would like to improve our result by studying the robust version of Theorem 5.3

with respect to the incompatibility system in the near future.

Chapter 6

Conclusions and prospects

6.1 Vertex-disjoint properly colored cycles

In Chapter 2, we mainly consider vertex-disjoint rainbow triangles in edge-colored graphs. We derive a tight minimum color degree condition for the existence of two vertex-disjoint rainbow triangles (Theorem 2.5). In general, we conjecture that for an edge-colored graph G with n vertices, if the minimum color degree of G is at least $(n+k)/2$, then G contains k vertex-disjoint rainbow triangles (Conjecture 2.2). By constructing examples, we show that if the conjecture is true, then the lower bound $(n+k)/2$ is best possible. Although we provide some evidence for this conjecture, it is still open now. We will try to prove it by establishing some new methods.

Theorem 2.3 concerns vertex-disjoint properly colored cycles. Based on Theorems 2.2 and 2.3, it will be interesting to determine the minimum color degree condition for k vertex-disjoint properly colored cycles. For digraphs, Henning and Yeo (Conjecture 1 of [91]) conjectured that every digraph of minimum out-degree at least 4, contains two vertex-disjoint directed cycles of different lengths. Later on, Lichiardopol confirmed it (Theorem 1.1 of [131]) and proposed the following conjecture.

Conjecture 6.1. (Conjecture 3.1 of [131]) *For every integer $k \geq 2$, there exists an integer $g(k)$ such that any digraph of minimum out-degree at least $g(k)$ contains k vertex-disjoint cycles of different lengths.*

Recently, Bensmail, Harutyunyan, Le, Li and Lichiardopol [18] proved Conjecture 6.1 for tournaments, regular digraphs and digraphs of small order. Moreover, for undirected graphs, they proved that every graph with minimum degree at least $(k^2+5k-2)/2$

contains k vertex-disjoint cycles of different lengths (Theorem 12 of [18]), where the degree bound is best possible. In particular, using the same arguments in the proof of Theorem 2.3, we can easily obtain the following result.

Proposition 6.1. *If Conjecture 6.1 holds for each positive integer k, then every edge-colored graph G with $d^c(v) \geq 2g(k) + m(v)$ for all vertices $v \in V(G)$ contains k vertex-disjoint properly colored cycles of different lengths.*

It will also be interesting to establish the minimum color degree condition for k vertex-disjoint properly colored cycles of different lengths, and we will return to this topic in the near future.

6.2 Properly colored spanning trees

In Chapter 3, we mainly study the relation between the order of maximum properly colored tree in an edge-colored graph G and the minimum color degree $\delta^c(G)$ of G. To be specific, we show that for an edge-colored connected graph G, the order of maximum properly colored tree in G is at least $\min\{|G|, 2\delta^c(G)\}$ (Theorem 3.4), which generalizes Theorem 1.24. Moreover, the lower bound $2\delta^c(G)$ in our result is sharp and we characterize all extremal graphs G with the maximum properly colored tree of order $2\delta^c(G)$ $(\neq |G|)$. We also prove that if for each color c, the set of edges colored with c forms a star, then the order of maximum rainbow tree in G is at least $\min\{|G|, 2\delta^c(G)\}$ (Theorem 3.5), which generalizes Theorem 1.25. As a corollary, we obtain a tight minimum color degree condition for the existence of properly colored spanning forest with at most k components (Corollary 3.1). Furthermore, we characterize all extremal graphs for Theorem 1.23.

Recall that for an edge-colored connected graph G, Cheng, Kano and Wang [39] proved that if $\delta^c(G) \geq |G|/2$, then G has a properly colored spanning tree. Later, Kano, Maezawa, Ota, Tsugaki and Yashima [101] derived that if $\bar{\sigma}_2^c(G) \geq |G|$, then G has a properly colored spanning tree. It is also interesting to study other types of sufficient condition guaranteeing the existence of a properly colored spanning tree, such as $\Sigma_{v \in V(G)} d^c(v)$ and $e(G) + c(G)$.

An edge-colored tree T with fixed root r is *weakly proper* if every path in T connecting the root r and any leaf is a properly colored path. In 2019, Borozan, Fernandez de La Vega, Manoussakis, Martinhon, Muthu, Pham and Saad [24] introduced the concept

of weakly proper tree and proved the following theorem.

Theorem 6.1 ([24]). *For a given vertex r in an edge-colored graph G, the problem of determining whether G has a weakly proper spanning tree with root r is NP-complete.*

It is interesting to find sufficient conditions forcing a weakly proper spanning tree with root r for any vertex $r \in V(G)$. Although we have obtained some sufficient conditions in [96], there is still room for improvement.

6.3 Properly colored 2-factors

In Chapter 4, we study properly colored 2-factors in edge-colored graphs and edge-colored balanced bipartite graphs. We prove that for any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N(\varepsilon, t)$ such that every edge-colored graph G with $\delta^c(G) \geq (2/3 + \varepsilon)|G|$ and $|G| > N(\varepsilon, t)$ contains every properly colored 2-factor with exactly t components (Theorem 4.2), which generalizes Theorem 4.1. We also obtain that every edge-colored balanced bipartite graph G with $\delta^c(G) \geq |G|/3 + 1$ contains a properly colored 2-factor and the lower bound is best possible (Theorem 4.4). Furthermore, we show that for any $\varepsilon > 0$ and positive integer $t \geq 1$, there exists an integer $N'(\varepsilon, t)$ such that every edge-colored balanced bipartite graph G with $\delta^c(G) \geq (1/3 + \varepsilon)|G|$ and $|G| > N'(\varepsilon, t)$ contains every properly colored even 2-factor with exactly t components (Theorem 4.5).

In 1984, El-Zahár [63] proposed the following famous conjecture:

Conjecture 6.2 ([63]). *Let t be a positive integer, and let G be a graph of order $n = \sum_{i=1}^{t} n_i$, where $n_i \geq 3$ for $1 \leq i \leq t$. If $\delta(G) \geq \sum_{i=1}^{t} \lceil n_i/2 \rceil$, then G can be partitioned into t cycles of lengths $n_1, n_2, \ldots, n_t$, respectively.*

El-Zahár proved the case $t = 2$. Aigner and Brandt [2] proved that if G is a graph of order $n = \sum_{i=1}^{t} n_i$ ($n_i \geq 3$) with $\delta(G) \geq (2n-1)/3$, then G contains t vertex disjoint cycles of lengths $n_1, n_2, \ldots, n_t$, respectively. Abbasi, in his Ph.D. Thesis [1], settled El-Zahár's Conjecture for sufficiently large graphs by using the regularity lemma. Theorem 4.2 and Theorem 4.5 can be regarded as asymptotic edge-colored version results for El-Zahár's Conjecture in some sense. So it is very interesting to get a sharp bound for the minimum color degree in edge-colored graphs.

6.4 Compatible spanning subgraphs

In Chapter 5, we study the robust version of Alon–Yuster's result [8] about the existence of H-factor with respect to the incompatibility system. We prove that for any $\alpha > 0$ and any graph H with h vertices, there exists a constant $\mu > 0$ such that for any sufficiently large n with $n \in h\mathbb{N}$, if G is an n-vertex graph with $\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \alpha\right) n$ and $\mathcal{F}$ is a μn-bounded incompatibility system over G, then there exists a compatible H-factor in G, where the error term of αn cannot be omitted.

As we know, in extremal graph theory, characterizing the best possible bound on a minimum degree forcing an H-factor has attracted significant attention until it was finally settled by Kühn and Osthus [120]. They proved the following result, which improves Alon–Yuster's result [8].

Theorem 6.2 ([120])**.** *Let $\delta(H, n)$ denote the smallest integer k such that every graph G whose order n is divisible by $|H|$ and with $\delta(G) \geq k$ contains an H-factor. Then*

$$\delta(H, n) = \left(1 - \frac{1}{\chi^*(H)}\right) n + O(1).$$

The value of $\chi^*(H)$ depends on the relative sizes of the colour classes in the optimal colorings of H and satisfies $\chi(H) - 1 < \chi^*(H) \leq \chi(H)$.

We would like to improve our result by studying the robust version of Theorem 6.2 with respect to the incompatibility system.

The notion of incompatibility system was firstly introduced in [116] as a quantitative measure of robustness of graph properties. Krivelevich, Lee and Sudakov [116] studied the robustness of Hamiltonicity of a Dirac graph with respect to the incompatibility system, which strengthens Dirac's theorem. They also suggested looking at how various extremal results can be strengthened using this notion. Besides H-factors, there are many results about other spanning subgraphs in extremal graph theory:

The Pósa–Seymour's conjecture [64, 148] states that any graph G on n vertices with $\delta(G) \geq kn/(k+1)$ contains the k-th power of a Hamilton cycle. Komlós, Sárközy and Szemerédi [108] proved this conjecture for sufficiently large graphs. A decade ago, Böttcher, Schacht and Taraz [25] proved the so-called bandwidth theorem, which gives a condition on the minimum degree of a graph G on n vertices that ensures G contains every k-chromatic graph on n vertices of bounded degree and of bandwidth $o(n)$. There are also results about universality for bounded degree spanning trees.

Bollobás [20] conjectured in the 1970s that graphs on n vertices with minimum degree at least $n/2 + o(n)$ would contain every n-vertex tree of bounded maximum degree as a subgraph. Komlós, Sárközy and Szemerédi [106] proved this conjecture in 1995, introducing a prototype version of what is now known as the blow-up lemma. It is interesting to study robust version of these results with respect to the incompatibility system.

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Publications and manuscripts

(1) **Jie Hu**, Qiancheng Ouyang, Guanghui Wang, Local antimagic orientations of d-degenerate graphs, *Discrete Applied Mathematics*, 257 (2019) 206-215.

(2) **Jie Hu**, Guanghui Wang, Jianliang Wu, Donglei Yang, Xiaowei Yu, Adjacent vertex distinguishing total coloring of planar graphs with maximum degree 9, *Discrete Mathematics*, 342 (2019) 1392-1402.

(3) Ming Chen, **Jie Hu**, Xiaowei Yu, Shan Zhou, List strong edge coloring of planar graphs with maximum degree 4, *Discrete Mathematics*, 342 (2019) 1471-1480.

(4) Yulin Chang, **Jie Hu**, Guanghui Wang, Xiaowei Yu, Adjacent vertex distinguishing total coloring of planar graphs with maximum degree 8, *Discrete Mathematics*, 343 (2020) 112014.

(5)* **Jie Hu**, Hao Li, Donglei Yang, Vertex-disjoint rainbow triangles in edge-colored graphs, *Discrete Mathematics*, 343 (2020) 112117.

(6)* **Jie Hu**, Hao Li, Shun-ichi Maezawa, Maximum properly colored trees in edge-colored graphs, *Journal of Combinatorial Optimization*, (2021) https://doi.org/10.1007/s10878-021-00824-z.

(7) John Haslegrave, **Jie Hu**, Jaehoon Kim, Hong Liu, Bingyu Luan, Guanghui Wang, Crux and long cycles in graphs, arXiv:2107.02061.

(8) Laihao Ding, **Jie Hu**, Guanghui Wang, Donglei Yang, Properly colored short cycles in edge-colored graphs, *European Journal of Combinatorics*, 100 (2022) 103436.

(9) Tianjiao Dai, **Jie Hu**, Hao Li, Shun-ichi Maezawa, On DP-coloring of outerplanar graphs, submitted.

(10)* Properly colored 2-factors in edge-colored graphs, personal communication with Yangyang Cheng and Guanghui Wang.

(11)* **Jie Hu**, Hao Li, Yue Wang, Donglei Yang, Compatible H-factors in dense graphs, manuscript.

(12) **Jie Hu**, Shun-ichi Maezawa, Yannis Manoussakis, Qiancheng Ouyang, On sufficient conditions for weakly proper spanning trees in edge-colored graphs, manuscript.

Note that the papers marked with * form the main content of this thesis. Other papers are introduced briefly in Appendix.

Appendix

Some of my works are not included in this thesis (see [34, 36, 52, 57, 90, 96, 97, 98]), which have been (or will be) contained in theses of my collaborators or my master's thesis. I introduce these works briefly here.

(1) We study local antimagic orientations of d-degenerate graphs in [97]. A *k-antimagic labeling* of a digraph D with n vertices and m arcs is an injection from the set of arcs of D to the integers $\{1, \ldots, m+k\}$ such that all n vertex-sums are pairwise distinct, where the vertex-sum of a vertex v is the sum of labels of all arcs entering v minus the sum of labels of all arcs leaving v. An orientation D of a graph G is called *k-antimagic* if D has a k-antimagic labeling. Hefetz, Mütze, and Schwartz [On antimagic directed graphs, J. Graph Theory 64 (3) (2010) 219-232] conjectured that every connected graph admits an antimagic orientation, where "antimagic" is short for "0-antimagic". In [97], we consider local k-antimagic orientations of graphs. An orientation D of a graph G is called *local k-antimagic* if there is an injective edge labeling from $E(G)$ to $\{1, \ldots, |E(G)| + k\}$ such that any two adjacent vertices of D have different vertex-sums. We prove that every d-degenerate graph admits a local $(d+2)$-antimagic orientation.

(2) We consider adjacent vertex distinguishing total colorings of planar graphs with maximum degree 9 in [98]. Let k be a positive integer. An *adjacent vertex distinguishing* (for short, AVD) total-k-coloring of a graph G is a proper total-k-coloring of G such that any two adjacent vertices have different color sets, where the color set of a vertex v contains the color of v and the colors of its incident edges. It was conjectured that any graph with maximum degree Δ has an AVD total-$(\Delta+3)$-coloring. The conjecture was confirmed for any graph with maximum degree at most 4 and any planar graph with maximum degree at least 10. In [98], we verify the conjecture for all planar graphs with maximum degree at least 9. Moreover, we prove that any planar graph with maximum degree at least 10 has an AVD total-$(\Delta+2)$-coloring and the bound $\Delta+2$ is sharp.

(3) We research list strong edge-colorings of planar graphs with maximum degree 4 in [36]. A *strong edge-coloring* of a graph is a proper edge-coloring in which every color class is an induced matching. The *strong chromatic index* of a graph G is the minimum number of colors to construct such a coloring. In an analogous way, we can define the list version of strong chromatic index. In [36], we prove that if G is a planar graph with maximum degree at most four, then the list strong chromatic index of G is at most 19.

(4) We study adjacent vertex distinguishing total colorings of planar graphs with maximum degree 8 in [34]. We obtain that any planar graph G with maximum degree at least 8 has an adjacent vertex distinguishing total-$(\Delta(G)+3)$-coloring by using Alon's Combinatorial Nullstellensatz and discharging method, which improves the result in [98].

(5) We introduce a notion of the *crux* of a graph G in [90], measuring the order of a smallest dense subgraph in G. This simple-looking notion leads to some generalisations of known results about cycles, offering an interesting paradigm of 'replacing average degree by crux'. In particular, we prove that every graph contains a cycle of length linear in its crux.

Long proved that every subgraph of a hypercube Q^m (resp. discrete torus C_3^m) with average degree d contains a path of length $2^{d/2}$ (resp. $2^{d/4}$), and conjectured that there should be a path of length 2^d-1 (resp. $3^{d/2}-1$). As a corollary of our result, together with isoperimetric inequalities, we close these exponential gaps giving asymptotically optimal bounds on long paths in hypercubes, discrete tori, and more generally Hamming graphs.

We also consider random subgraphs of C_4-free graphs and hypercubes, proving near optimal lower bounds on the lengths of long cycles.

(6) We consider properly colored short cycles in edge-colored graphs in [57]. Properly colored cycles in edge-colored graphs are closely related to directed cycles in oriented graphs. As an analogy of the well-known Caccetta-Häggkvist conjecture, we study the existence of properly colored cycles of bounded length in an edge-colored graph. We first prove that for all integers s and t with $t \geq s \geq 2$, every edge-colored graph G with no properly colored $K_{s,t}$ contains a spanning subgraph H which admits an orientation D such that every directed cycle in D

is a properly colored cycle in G. Using this result, we show that for $r \geq 4$, if the Caccetta-Häggkvist conjecture holds , then every edge-colored graph of order n with minimum color degree at least $n/r + 2\sqrt{n} + 1$ contains a properly colored cycle of length at most r. In addition, we also obtain an asymptotically tight total color degree condition which ensures a properly colored (or rainbow) $K_{s,t}$.

(7) We research DP-colorings of outerplanar graphs in [52]. While solving a question on list coloring of planar graphs, Dvořák and Postle introduced the new notion which is called a DP-coloring. A *DP-coloring* of a graph G reduces the problem of finding a coloring of G from a given list L to the problem of finding an independent set of size $|G|$ in an auxiliary graph $\mathcal{M}_L$-cover with vertex set $\{(v, c) : v \in V(G)$ and $c \in L(v)\}$. The DP-coloring is a generalization of a list coloring. Hutchinson [On list-coloring outerplanar graphs, J. Graph Theory 59 (2008) 59-74] showed that

- if a 2-connected bipartite outerplanar graph G with a list of colors $L(v)$ for each vertex v such that $|L(v)| \geq \min\{d_G(v), 4\}$, then G is L-colorable;
- if a maximal outerplanar graph G with at least four vertices having a list of colors $L(v)$ for each vertex v such that $|L(v)| \geq \min\{d_G(v), 5\}$, then G is L-colorable.

In [52], we study whether the bounds of Hutchinson's results hold for the DP-coloring or not and we obtain that the first one is not sufficient for a DP-coloring while the second one is sufficient.

(8) We devote to finding sufficient conditions for the existence of weakly proper spanning trees in edge-colored graphs in [96]. An edge-colored tree T with fixed root r is weakly proper if every path in T, from the root r to any leaf, is a properly colored one. Borozan et al. [Maximum colored trees in edge-colored graphs, European J. Combin., 80 (2019), 296-310] proved that for a given vertex r in an edge-colored graph G, the problem of determining whether G has a weakly proper spanning tree with root r is NP-complete. In [96], we give some sufficient conditions for an edge-colored graph G to have a weakly proper spanning tree with root r for any vertex $r \in V(G)$.