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Charles Elbar

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# Etude mathématique d'équations de type Cahn-Hilliard dégénérées

**Thèse dirigée par** Benoît PERTHAME directeur  
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**Mots clés:** Cahn-Hilliard, limite incompressible, tension de surface, analyse asymptotique, flow gradient

**Keywords:** Cahn-Hilliard, incompressible limit, surface tension, asymptotic analysis, gradient flows



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*À mes parents, mon frère et Mohamed Amine Belabbes ...*





**ETUDE MATHÉMATIQUE D'ÉQUATIONS  
DE TYPE CAHN-HILLIARD DÉGÉNÉRÉES****Résumé**

Nous étudions des équations de type Cahn-Hilliard, équation qui fut introduite pour décrire la séparation de phases dans les systèmes multi-composants. Les résultats obtenus dans ce travail ont été motivés par des applications biologiques, notamment la formation de tissus et la croissance de tumeurs, ainsi que par des applications physiques, tels que les écoulements de fluides impliquant des phénomènes de tension de surface.

La première partie de cette thèse est une analyse de la relation entre l'équation de Cahn-Hilliard et les modèles Hele-Shaw, qui sont fréquemment utilisés pour modéliser l'écoulement des fluides ou l'évolution de tumeurs cancéreuses dans des espaces confinés. Nous examinons en particulier comment obtenir les modèles Hele-Shaw dans la limite dite *incompressible* de l'équation de Cahn-Hilliard. La deuxième partie de la thèse se concentre sur l'étude de l'équation de Cahn-Hilliard non-locale (et ses variations) et sa convergence vers l'équation locale. Cette équation, qui peut être dérivée rigoureusement à partir d'un système de particules en interaction, est obtenue en remplaçant le laplacien, qui est un terme local, par une approximation non-locale prenant en compte les interactions à longue distance entre les composants. Nous montrons que la solution de l'équation non-locale converge vers la solution de l'équation locale dans la limite d'interaction à courte distance. La troisième partie de la thèse se penche sur l'étude des modèles de fluides plus classiques, tels que les équations d'Euler et de Navier-Stokes, qui intègrent des phénomènes de tension de surface. Ces modèles sont utilisés pour décrire les écoulements de fluides ou les mouvements de cellules dans lesquels les forces interfaciales jouent un rôle important. La quatrième partie juxtapose la théorie cinétique, traditionnellement employée pour la représentation de phénomènes physiques à une échelle mésoscopique, avec l'équation de Cahn-Hilliard. Notre étude se concentre spécifiquement sur l'équation de Vlasov-Cahn-Hilliard, qui décrit les processus de transition de phase.

**Mots clés :** Cahn-Hilliard, limite incompressible, tension de surface, analyse asymptotique, flow gradient

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**Abstract**

We study Cahn-Hilliard type equations, an equation that was introduced to describe phase separation in multi-component systems. The results obtained in this work have been motivated by biological applications, such as tissue formation and tumor growth, as well as physical applications, such as fluid flows involving surface tension phenomena.

The first part of this thesis is an analysis of the relationship between the Cahn-Hilliard equation and the Hele-Shaw models, which are frequently used to model fluid flow or the evolution of cancerous tumors in confined spaces. In particular, we examine how to obtain Hele-Shaw models in the so-called *incompressible* limit of the Cahn-Hilliard equation. The second part of the thesis focuses on the study of the non-local Cahn-Hilliard equation (and its variations) and its convergence to the local equation. This equation, which can be derived rigorously from a system of interacting particles, is obtained by replacing the Laplacian, which is a local term, by a non-local approximation taking into account the long range interactions between the components. We show that the solution of the non-local equation converges to the solution of the local equation in the short range interaction limit. The third part of the thesis focuses on the study of more classical fluid models, such as the Euler and Navier-Stokes equations, which incorporate surface tension phenomena. These models are used to describe fluid flows or cell motions in which interfacial forces play an important role. The fourth part juxtaposes kinetic theory, traditionally used to represent physical phenomena on a mesoscopic scale, with the Cahn-Hilliard equation. Our study focuses specifically on the Vlasov-Cahn-Hilliard equation, which describes phase transition processes.

**Keywords:** Cahn-Hilliard, incompressible limit, surface tension, asymptotic analysis, gradient flows

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# List of publications

The following articles have been submitted or published and are included in this thesis.

1. C. Elbar, B. Perthame, J. Skrzeczkowski On the limit problem arising in the kinetic derivation of the Cahn-Hilliard equation Preprint 2023
2. C. Elbar, J. Skrzeczkowski On the inviscid limit connecting Brinkman's and Darcy's models of tissue growth with nonlinear pressure Preprint 2023
3. C. Elbar, B. Perthame, A. Poiatti, J. Skrzeczkowski Nonlocal Cahn-Hilliard equation with degenerate mobility: Incompressible limit and convergence to stationary states Archive for Rational Mechanics and Analysis 2024
4. C. Elbar, A. Poulain Analysis and numerical simulation of a generalized compressible Cahn-Hilliard-Navier-Stokes model with friction effects Preprint 2023
5. C. Elbar, P. Gwiazda, J. Skrzeczkowski, A. Świerczewska-Gwiazda From nonlocal Euler-Korteweg to local Cahn-Hilliard via the high-friction limit Preprint 2023
6. José A. Carrillo, C. Elbar, J. Skrzeczkowski Degenerate Cahn-Hilliard systems: From nonlocal to local Preprint 2023
7. C. Elbar, J. Skrzeczkowski Degenerate Cahn-Hilliard equation: From nonlocal to local Journal of Differential Equations 2023
8. C. Elbar, M. Mason, B. Perthame, J. Skrzeczkowski From Vlasov equation to degenerate nonlocal Cahn-Hilliard equation Communication in Mathematical Physics 2022
9. C. Elbar, B. Perthame, J. Skrzeczkowski Pressure jump and radial stationary solutions of the degenerate Cahn-Hilliard equation. Comptes rendus mécanique: The scientific legacy of Roland Glowinsky. 2022
10. C. Elbar, B. Perthame, A. Poulain Degenerate Cahn-Hilliard and incompressible limit of a Keller-Segel model Communication in Mathematical Sciences 2022



# Introduction

## The Cahn-Hilliard equation

The Cahn-Hilliard equation,

$$\partial_t u - \operatorname{div}(m(u)\nabla\mu) = 0, \quad \mu = -\delta\Delta u + F'(u), \quad \delta > 0,$$

is a fourth-order parabolic equation, generally studied on a bounded domain with Neumann Boundary conditions. This equation, was originally introduced by J.W. Cahn and J.E. Hilliard in 1958 [70] to explain the evolution of an isotropic system characterized by nonuniform composition or density, that is a system with a variation in one of these two intensive scalar properties.

To understand the genesis of this equation we have to go back to the phenomenon of interfacial energy in a system. A simple explanation is given in [134]: Interfacial energy characterizes the energy required to form an interface with a certain surface area. For instance consider a water molecule in the middle of others. The molecule is attracted on all sides by the polar nature of its neighbors. But what happens if we transport this molecule to the air-water interface? Where it was previously attracted equally in all directions, this is no longer the case. One side of the molecule is now in contact with air, which, as a gas, has far less attraction for the water molecule than water itself. Thus, there is a competition between a weak attractive force attempting to extract it from the liquid, and a strong attractive force seeking to draw it towards the bulk of water. This unequal attraction pushes the water molecule at the interface to attempt to return to the liquid. Only by doing substantial work can a water molecule be pulled to the junction between air and water. Each water molecule forced to the interface creates a small amount of new interfacial area. If we express the work required to move a molecule to the interface in terms of the new zone created, we obtain a measure of the interfacial energy of water when in contact with air.

To determine interfacial energies, some theoretical studies have imposed arbitrary constraints on the thickness of the interface. For example, Young and Becker assumed that two adjacent phases are homogeneous up to their shared interface. However, as Cahn and Hilliard pointed out, such assumptions may not hold in most cases, especially when the system's temperature and pressure are specified. In such situations, the interfacial thickness is no longer an independent variable, making computations of interfacial energy inaccurate. To give a satisfactory theory of the interfacial energy for a general nonuniform isotropic system, Cahn and Hilliard demonstrated that the free energy of such a system can be expressed as (at least approximated by a Taylor expansion):

$$E = N_V \int_V F(u) + \frac{\delta}{2} |\nabla u|^2 \, dV.$$

where the nonuniformity is given here by  $u$ , the mole fraction of one of the component and  $N_V$



is the number of molecules per unit volume. The previous equation can be understood as the sum of two contributions. One of these contributions represents the free energy that the volume would possess in a homogeneous solution, while the other is a "gradient energy" depending on the local composition. Once the free energy is derived, it is possible to compute the interfacial energy of a flat interface, which is the difference between the actual free energy of the system and that which it would have if the properties of the phases were continuous throughout the interface:

$$\sigma = N_V \int_V F(u) + \frac{\delta}{2} |\nabla u|^2 - u\mu_B - (1-u)\mu_A dV$$

where  $\mu_A$  and  $\mu_B$  are the chemical potentials per molecule of the species A and B.

With this expression, they were able to predict that the thickness of the interface is an increasing function of the temperature  $\theta$  and becomes infinite at a critical temperature  $\theta_c$ . Moreover, they computed that the interfacial energy behaves as  $(\theta - \theta_c)^{3/2}$ , in agreement with experiments. From the free energy expression, using the mass balance equation

$$\frac{\partial u}{\partial t} + \operatorname{div} j = 0,$$

and the generalized Fick's law

$$j = -m(u) \nabla \frac{\delta E}{\delta u}$$

we obtain the Cahn-Hilliard equation. The equation, originally introduced in the context of material science, has since been used to describe the evolution of nonuniform systems in different fields: biology, image inpainting, fluid mechanics and even to explain the origin of the irregular structure in Saturn's rings [320]. In general, it is used as the main tool to incorporate the surface tension effect in a model.

The results obtained in this work have been motivated by biological applications, such as tissue formation and tumor growth, as well as physical applications, such as fluid flows involving surface tension phenomena. The different models presented here, consists in using the Cahn-Hilliard equation, possibly coupled with other equations depending on the application.

## Contents of the thesis

We begin with a reminder of classical mathematical tools. Then, the study is centered around four distinct parts, each exploring various aspects of mathematical modeling and their connections to the Cahn-Hilliard equation. The chapters within each part are self-contained and can be read independently.

**Part 1: The incompressible limit of the Cahn-Hilliard equation** The first part deals with the link between the Cahn-Hilliard equation and Hele-Shaw models, commonly used in fluid dynamics and the growth of tumors within confined spaces. Specifically, we investigate how to derive some Hele-Shaw models in the context of the "incompressible" limit of the Cahn-Hilliard equation.

**Part 2: Nonlocal to local convergence of the Cahn-Hilliard Equation** Part two centers on the nonlocal Cahn-Hilliard equation and its variants. This equation replaces the local Laplacian term with a nonlocal approximation, accounting for long-range interactions between components. We prove that, as the range of interactions decreases, the solution of the nonlocal

equation converges to the one of its local counterpart.

**Part 3: Fluid Models with surface tension** The third part deals with classical fluid models, namely the Euler and Navier-Stokes equations, with additional surface tension effects. These models are used to describe fluid flow and cell motion, where interfacial forces play a crucial role.

**Part 4: Kinetic theory and the Cahn-Hilliard Equation** Part four juxtaposes kinetic theory, traditionally applied to describe mesoscopic-scale physical phenomena, with the Cahn-Hilliard equation. Our focus is on the Vlasov-Cahn-Hilliard equation, a model that characterizes phase transition processes.

At the end of the introduction, we discuss some current work and perspectives. In the appendix, we mention a recent work, concerning a porous medium system.

## Part 1: The incompressible limit of the Cahn-Hilliard equation

Consider two plates separated by a very small gap  $h$  as in Figure 1. In fact, an infinitesimally small gap. We assume that the plates are of size  $L \gg h$ .

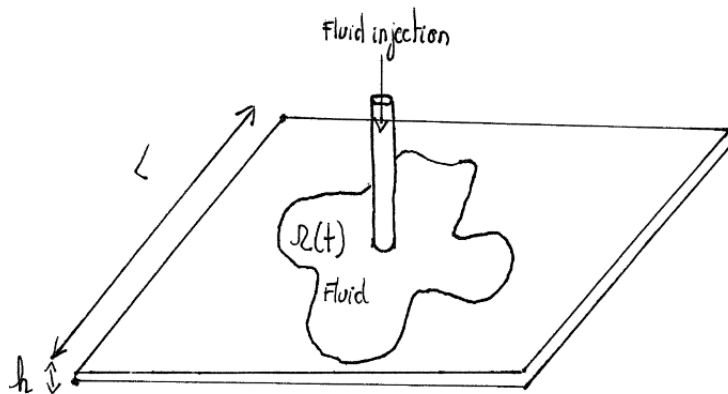


Figure 1 – Hele-Shaw flow

This system is called a Hele-Shaw cell and is often used to reproduce two dimensional fluid flows with low Reynolds number, generally around obstacles if an obstacle is inserted between the two plates. The derivation of the Hele-Shaw equations may be found in [8]. We outline here the fundamental idea behind the derivation. Consider an incompressible fluid with velocity  $u = (u_x, u_y, u_z)$  moving inside the Hele-Shaw cell. The fluid satisfies the Navier-Stokes equation

$$\rho \frac{\partial u}{\partial t} + \rho(u \cdot \nabla)u = -\nabla p + \nu \Delta u, \quad \text{div } u = 0, \quad u(x, y, 0) = u(x, y, h) = 0, \quad (0.0.1)$$

where  $\rho$  is the density of the fluid and  $\nabla p$  the pressure gradient. From the incompressibility relation  $\text{div } u = 0$  and assuming  $u_x \approx u_y$  we get the approximation

$$\frac{u_x}{L} + \frac{u_y}{L} + \frac{u_z}{h} = 0 \Rightarrow u_z \approx \frac{h}{L} u_{x,y} \ll u_{x,y} \text{ as } h \rightarrow 0$$

where  $u_{x,y}$  represents either  $u_x$  or  $u_y$ . Therefore we can neglect the  $z$  component of the velocity compared to the component in  $x$  and  $y$  and write  $u = (u_x, u_y, 0)$ . Also, since  $h$  is much smaller than  $L$

$$\frac{\partial^2 u_{x,y}}{\partial x^2} \ll \frac{\partial^2 u_{x,y}}{\partial z^2}.$$

Assuming that the fluid is of low Reynolds number we can neglect the first two terms of the Navier-Stokes equation (0.0.1) compared to the viscosity term  $\nu \Delta u$ :

$$\nu \frac{\partial^2 u_x}{\partial z^2} = \frac{\partial p}{\partial x}, \quad \nu \frac{\partial^2 u_y}{\partial z^2} = \frac{\partial p}{\partial y}, \quad 0 = \frac{\partial p}{\partial z}.$$

Therefore the pressure does not depend on  $z$ . Assuming that  $u_x(x, y, 0) = u_x(x, y, h) = u_y(x, y, 0) = u_y(x, y, h) = 0$  (Dirichlet conditions) yields

$$u_x = -\frac{1}{2\nu} \frac{\partial p}{\partial x} z(h-z), \quad u_y = -\frac{1}{2\nu} \frac{\partial p}{\partial y} z(h-z), \quad u_z = 0.$$

Noting  $u_{par} = (u_x, u_y)$ ,  $p_{par}(x, y) = p(x, y, z)$  the velocity and the pressure in the parallel plane ( $xOy$ ) we obtain

$$u_{par} = -\frac{1}{2\nu} \nabla p_{par} z(h-z).$$

Using the incompressibility relation one can also get the laplace equation

$$\Delta p_{par} = 0,$$

supplemented by the no-penetration boundary conditions on the side walls  $\nabla p_{par} \cdot \vec{n} = 0$  where  $\vec{n}$  is a unit vector perpendicular to the side wall. We note the important fact that *the equation of Hele-Shaw flows is the same to that of the flow of incompressible fluid through a porous medium with Darcy's law.* These models are referred to as 'free boundary models.' In this context, there exists a domain, often denoted as  $\Omega(t)$ , which defines the fluid region with a clear boundary. This boundary evolves over time according to a pressure law. Introduced to describe fluids, these models found application in the modeling, analysis and simulation of biological phenomena such as tumor growth. In the particular case of tumor growth, an additional term has to be taken into account for the proliferation of cells, resulting in a different formulation of the Hele-Shaw equation:  $\Delta p_{par} + G(p_{par}) = 0$ , where  $G$  is the growth term.

Using Hele-Shaw models to describe tumor growth Initiated by Greenspan [196], general mechanical models of tumor growth have been proposed and used internal pressure as the main effect that drives the movement and proliferation of cells. The prototypical example of a mechanical living tissue model is

$$\partial_t n = \text{div}(n \nabla p) + nG(p), \quad p = n^\gamma, \quad (0.0.2)$$

in which  $p$  is the pressure and  $n$  the cell density. In this kind of model, Darcy's law of movement is used to reflect the porous media formed by the extra-cellular matrix and the tendency of cells to move away from regions of high compression. Recall the sentence *the equation of Hele-Shaw flows is the same to that of the flow of incompressible fluid through a porous medium with Darcy's law.*

The previous equation models the flow of a fluid through a porous medium type with Darcy's law. However the fluid is not incompressible. Interestingly, Perthame *et al.* [289] have shown that the incompressible limit can be found sending  $\gamma$  to  $\infty$ . In that limit, solutions of Model (0.0.2) converge to a limit  $(n_\infty, p_\infty)$  solution of a free boundary limit problem of Hele-Shaw type for which the normal component of  $p_\infty$  gives the speed of the free boundary. In the limit, the solution to Equation (0.0.2) can be divided into two distinct regions:  $\Omega(t) = \{x, p_\infty(t, x) > 0\}$  where the pressure remains positive (representing the tumor), and the region outside where  $p_\infty = 0$ . Note also the important graph equation  $p_\infty(n_\infty - 1) = 0$  which implies the incompressibility of the fluid. Indeed, in the zone  $\Omega(t)$  we immediately get  $n_\infty = 1$  which corresponds to an incompressible tumor. The graph equation can be easily seen from the relation  $p_\gamma n_\gamma = (n_\gamma^\gamma)^{(\gamma+1)/\gamma}$  and sending gamma to  $\infty$ . The free boundary problem has also an additional equation specifying that the pressure satisfies

$$-\Delta p_\infty = G(p_\infty), \quad \text{in } \Omega(t), \quad \text{or equivalently } p_\infty(\Delta p_\infty + G(p_\infty)) = 0 \quad \text{almost everywhere in } \Omega. \quad (0.0.3)$$

These equations can be seen by taking  $n_\infty = 1$  in (0.0.2). When the source term is set as  $G(p) = 0$ , the classical Hele-Shaw system, which was derived earlier, is retrieved. The result established by Perthame, Quiros, and Vazquez stimulated the exploration of multiple variations of the general model (0.0.2), introducing different physical effects in mechanical models of tissue growth. As an example, the inclusion of viscosity effects has been incorporated into the model to represent cell friction [38, 50] using Stokes' or Brinkman's laws. A consistent aspect across these variations is the presence of a parameter  $\gamma$  within the pressure law, which is sent to infinity to achieve the incompressible limit of the model.

Surface tension A major assumption that was made in all the previous models in the absence of surface tension and interfacial energy. Surface tension is a concept associated with the internal cohesive forces between the molecules of a fluid: hydrogen bonds, van der Waals forces, metallic bonds, etc. Inside the fluid, molecules are attracted equally in all directions leading to a net force of zero; however molecules on the surface experience an attractive force that tends to pull them to the interior of the fluid: this is the origin of the interfacial (or surface) energy. This energy is equivalent to the work or energy required to remove the surface layer of molecules in a unit area. The value of the surface tension will vary greatly depending on the nature of the forces exerted between the atoms or molecules. In the case of solid tumor cells in a tissue, it reflects the cell-cell adhesion tendency between the cells and depends on the parameter  $\delta$  and the geometry of the tumor.

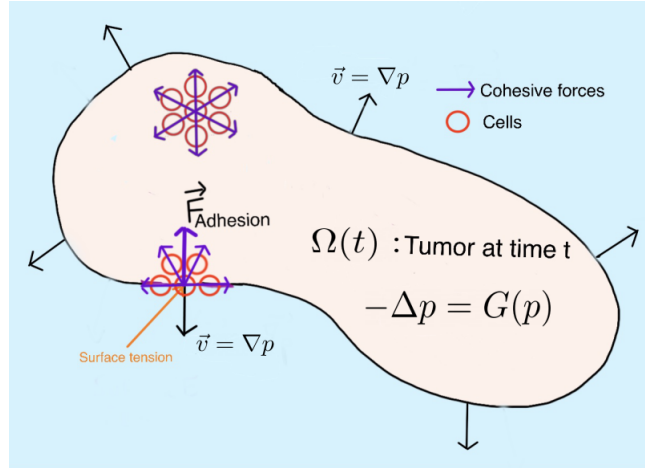


Figure 2 – Tumor evolving with pressure and surface tension

Figure 2 represents tumor (in white) denoted as  $\Omega(t)$  infiltrating a healthy tissue (in blue). The tumor cells are dividing through a proliferation (or source) term denoted as  $G(p)$ , resulting in the generation of a pressure gradient. This pressure gradient exerts a force on the surrounding tumor cells. It's noteworthy that within the tumor itself, the cohesive forces between the tumor cells do not matter, as the cells are attracted equally in all directions.

The pressure can be described using the Hele-Shaw equation as  $-\Delta p = G(p)$ . At the interface or boundary, the cohesive force is no longer zero. Instead, it gives rise to a force, named  $F_{adhesion}$ , which tends to draw the boundary cells into the tumor. Then, the boundary becomes a competition between growth and pulling, hence creating interfacial energy. One way to incorporate surface tension in the existing models is by considering the Cahn-Hilliard equation.

The Cahn-Hilliard equation and the Hele-Shaw models. Recall that Cahn and Hilliard have proved that the free energy of a volume  $V$  of an isotropic system of nonuniform composition or density is given by

$$E = N_V \int_V f_0(c) + \frac{\delta}{2} |\nabla c|^2 dV,$$

where  $N_V$  is the number of molecules per unit volume,  $\nabla c$  is the composition gradient, and  $f_0(c)$  is the free energy per molecule of a homogeneous system. The second component within the free energy  $E$  takes into account the fluctuations in composition or density, which is at the origin of the surface tension. The equation introduced by Perthame, Quirós, and Vazquez (0.0.2) corresponds to the situation where  $f_0(c) = \frac{n^{\gamma+1}}{\gamma+1}$  and  $\delta = 0$ , meaning there's no interfacial energy involved. An inherent question arises: what occurs when  $\delta$  is not equal to 0? Can we still send  $\gamma$  to  $\infty$  and find the incompressible limit of the model? The solution to this question remains open in the general context. Nevertheless, in specific cases it can be done. This is the case in the nonlocal Cahn-Hilliard equation and also in a 'smoothed' variant of the Cahn-Hilliard equation (which can be transformed into a Keller-Segel type equation), such as those discussed in the first and second chapters of this section. More precisely, in the first chapter we study the system

$$\begin{aligned} \partial_t n - \operatorname{div}(n \nabla \mu) &= n G(\mu), \\ -\sigma \Delta \mu + \mu &= \left( n - \frac{\sigma}{\delta} \mu \right)^\gamma - \delta \Delta n. \end{aligned} \tag{0.0.4}$$

The  $\sigma$  coefficient is typical of the Brinkman's law for porous medium and introduces viscosity effects in the equation. It also provides better regularity of the system. When  $\sigma \rightarrow 0$ , we expect to recover a solution of the Cahn-Hilliard equation

$$\begin{aligned}\partial_t n - \operatorname{div}(n \nabla \mu) &= nG(\mu), \\ \mu &= n^\gamma - \delta \Delta n.\end{aligned}\tag{0.0.5}$$

This assertion can be proved rigorously:

**Result 1** (Convergence of the relaxed system when  $\sigma \rightarrow 0$ . Cf. Theorem 1.2.1). *From the weak solutions  $(n_\sigma, \mu_\sigma)$  of the system (0.0.4) with parameter  $\sigma$  we can extract a subsequence converging to  $(n, \mu)$  weak solution of the system (0.0.5).*

We also study the incompressible limit  $\gamma \rightarrow \infty$  of the system (0.0.4) with  $\sigma > 0$  fixed. The case  $\sigma = 0$  is open. The major difficulty in the latter case is the lack of any estimates on  $\mu$  itself. Indeed, we are not able to control the vacuum zones  $n = 0$  that make the estimates degenerate. In the case  $\sigma > 0$  the main idea of the proof is to rewrite the system as a Keller-Segel system. More precisely, defining

$$w = n - \frac{\sigma}{\delta} \mu, \quad p = w^\gamma,\tag{0.0.6}$$

we can rewrite the system as

$$\partial_t n - \frac{\delta}{2\sigma} \Delta n^2 + \frac{\delta}{\sigma} \operatorname{div}(n \nabla w) = nG\left(\frac{\delta}{\sigma}(n - w)\right)\tag{0.0.7}$$

$$-\sigma \Delta w + \frac{\sigma}{\delta} w^\gamma + w = n.\tag{0.0.8}$$

In this formulation, we observe that  $\mu$ , the difficult term to control, does not appear anymore and we need to control mainly  $w$ , that satisfies an elliptic equation. Moreover using Alikakos iteration method we derive an  $L^\infty$  estimate on  $n$  which is crucial to find the incompressible limit of the system. Also, the solution can be proved to be unique in the case of no source term, controlling the  $H^{-1}$  between the difference of two solutions. Concerning the incompressible limit, we prove the following result

**Result 2** (Incompressible limit. Cf. Theorem 1.3.5). *Let  $(n_{\sigma,\gamma}, \mu_{\sigma,\gamma})$  be a weak solution to the system (0.0.4). Then, when  $\gamma \rightarrow \infty$ , after extraction of subsequences,  $(n_{\sigma,\gamma}, \mu_{\sigma,\gamma}, w_{\sigma,\gamma}) \rightarrow (n_{\sigma,\infty}, \mu_{\sigma,\infty}, w_{\sigma,\infty})$ . These functions satisfy in the weak sense*

$$\partial_t n_{\sigma,\infty} - \operatorname{div}(n_{\sigma,\infty} \nabla \mu_{\sigma,\infty}) = n_{\sigma,\infty} G(\mu_\infty).$$

and

$$\mu_{\sigma,\infty} = p_{\sigma,\infty} - \delta \Delta w_{\sigma,\infty}, \quad -\sigma \Delta w_{\sigma,\infty} + \frac{\sigma}{\delta} p_{\sigma,\infty} + w_{\sigma,\infty} = n_{\sigma,\infty}, \quad w_{\sigma,\infty} = n_{\sigma,\infty} - \frac{\sigma}{\delta} \mu_{\sigma,\infty}.$$

In this 'stiff pressure limit', the system has three unknowns  $n_{\sigma,\infty}, w_{\sigma,\infty}$  (or  $\mu_{\sigma,\infty}$ ) and  $p_{\sigma,\infty}$  and is completed with a type of incompressibility condition

$$p_{\sigma,\infty}(w_{\sigma,\infty} - 1) = 0.\tag{0.0.9}$$

The tumor is situated in the zone  $\Omega(t) := \{(t, x), p_\infty(t, x) > 0\}$  and we observe that inside  $\Omega(t)$ ,  $n_\infty = 1$ , and thus from the evolution equation  $-\Delta\mu = G(\mu)$ , which is a Hele-Shaw relation. The second chapter is dedicated to the study of the incompressible limit (*i.e.*  $\gamma \rightarrow \infty$ ) of the nonlocal Cahn-Hilliard equation, as well as its convergence to stationary states:

$$\begin{aligned}\partial_t u - \operatorname{div}(u \nabla \mu) &= u G(p), \\ \mu &= n^\gamma + B_\varepsilon(u).\end{aligned}$$

Here,  $B_\varepsilon$  denotes the nonlocal operator defined as

$$B_\varepsilon[u](x) = \frac{1}{\varepsilon^2}(u(x) - \omega_\varepsilon * u(x)) = \frac{1}{\varepsilon^2} \int_\Omega \omega_\varepsilon(y)(u(x) - u(x-y)) \, dy.$$

Because of the convolution term, there is no maximum principle and our approach relies on a new  $L^\infty$  estimate obtained by De Giorgi iterations. Also, since the diffusion term is given by  $n^\gamma + n^2$ , instead of the usual  $n^\gamma$  in the work of Perthame, Quiros, Vazquez, it seems not possible to use the classical Aronson-Benilan estimate, a natural estimate that provides an  $L^1$  bound on the laplacian of the pressure, hence provided compactness of the gradients. For this reason, we have to rely on an energy trick to achieve strong convergence of the gradients of the pressure.

**Result 3** (Incompressible limit. Cf. Theorem 2.1.5). *Let  $u_\gamma$  be a weak solution to the previous equation. Then up to a subsequence,  $u_\gamma \rightarrow u_\infty$  and  $p_\gamma = u_\gamma^\gamma \rightarrow p_\infty$  where  $u_\infty$  and  $p_\infty$  satisfy in*

$$\partial_t u_\infty - \operatorname{div}(u_\infty \nabla(p_\infty + B_\varepsilon(u_\infty))) = u_\infty G(p_\infty), \quad (0.0.10)$$

$$p_\infty \left( \Delta p_\infty + \frac{1}{2\varepsilon^2} \Delta u_\infty^2 - \frac{1}{\varepsilon^2} \operatorname{div}(u_\infty (\nabla \omega_\varepsilon * u_\infty)) + u_\infty G(p_\infty) \right) = 0, \quad (0.0.11)$$

$$0 \leq u_\infty \leq 1, \quad p_\infty \geq 0, \quad p_\infty(1 - u_\infty) = 0 \quad \text{almost everywhere in } \Omega \times (0, \infty), \quad (0.0.12)$$

We also prove the long-term convergence to a single constant stationary state of any weak solution using entropy methods, even when a source term is present. Indeed, and despite the source term, we get a uniform control in time of the energy. Our result implies in particular that the surface tension in the nonlocal (and even local) Cahn-Hilliard equation will not prevent the tumor from completely invading the domain.

**Result 4** (Long time behaviour. Cf. Theorem 2.1.6). *There are two cases:*

— for  $G(p) = p_H - p$ , we have

$$\|u(t) - p_H^{\frac{1}{\gamma}}\|_{L^q(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad \forall q \in [1, \gamma + 1). \quad (0.0.13)$$

— For  $G(p) = 0$  and  $\gamma \geq 1$ , we have an exponential decay towards the mean value: there exists a constant  $C$  such that

$$\|u(t) - \bar{u}_0\|_{L^q(\Omega)} \lesssim e^{-Ct}, \quad \forall q \in [1, \gamma + 1). \quad (0.0.14)$$

Although the incompressible limit is attainable, it is not clear what happens exactly at the boundary  $\partial\Omega(t)$ . Therefore we look for a deeper understanding of the tumor's boundary in this limit. To address this, the third chapter of this first part introduces an external confining  $V$

potential and we place ourselves in the radial setting:

$$\begin{aligned} \frac{\partial(rn)}{\partial t} - \frac{\partial}{\partial r} \left( rn \frac{\partial(\mu + V)}{\partial r} \right) &= 0, \\ \mu &= n^\gamma - \frac{\delta}{r} \frac{\partial}{\partial r} \left( r \frac{\partial n}{\partial r} \right). \end{aligned} \quad (0.0.15)$$

The equation is set on the domain  $(0, T) \times (0, R_b)$  for some  $R_b > 0$ . Then, we can compute the stationary states of the Cahn-Hilliard equation and prove their convergence towards a unique profile, completely determined, as the incompressible limit is reached. This profile showcases a pressure jump at the boundary, a hallmark of surface tension. The value of this jump is explicitly determined based on the applied external potential.

**Result 5** (Existence of solutions and long term asymptotic. Cf. Theorem 3.1.1). *There exists a global weak solution of 0.0.15. Moreover, up to a subsequence,  $\{r n(t + k, r)\}_k$  converge to a stationary solution  $r n_\infty(r) \geq 0$  where  $n_\infty \in C^1(\overline{[0, R_b]})$  satisfies  $m = \int_0^{R_b} r n_\infty(r) dr$  and*

$$r n_\infty \frac{\partial(\mu_\infty + V)}{\partial r} = 0, \quad \mu_\infty = n_\infty^\gamma - \frac{\delta}{r} \frac{\partial}{\partial r} \left( r \frac{\partial n_\infty}{\partial r} \right) \quad n'_\infty(0) = n'_\infty(R_b) = 0.$$

Our second result characterizes possible stationary states and shows we can distinguish an interval where  $n_\infty = 0$  and another where  $\mu_\infty + V$  is constant as expected from the first equation. We consider the confining potential  $V(r) = r^2$  for simplicity but the proof may be adapted to any increasing potential.

**Result 6** (Characterization of the stationary states. Cf. Theorem 3.1.2). *Let  $n_\infty$  be a solution as built in the previous theorem.*

(A) *Then,  $n_\infty$  is nonincreasing and it satisfies  $0 \leq n_\infty(R_b) < \frac{2m}{R_b^2}$ .*

(B) *Assume  $n_\infty(R_b) = 0$  and let  $R > 0$  be the smallest argument such that  $n_\infty(R) = 0$  and thus  $n_\infty > 0$  in  $[0, R)$ . Then, there is  $\lambda_\infty \in (0, R^2)$  such that*

$$\begin{cases} n_\infty^\gamma - \frac{\delta}{r} n'_\infty - \delta n''_\infty = R^2 - r^2 - \lambda_\infty & \text{in } (0, R), \\ n_\infty(R) = n'_\infty(R) = 0. \end{cases} \quad (0.0.16)$$

and, given  $R > 0$ , there is at most one couple  $(n, \lambda)$  solving (0.0.16).

(C) *Fix  $\delta \in (0, 1)$ . There exists  $R(m)$ , independent of  $\gamma$ , such that when  $R_b > R(m)$ , then  $n_\infty(R_b) = 0$ .*

Next, we focus on the incompressible limit of the solutions of (0.0.16), that is when  $\gamma_k \rightarrow \infty$ . We denote by  $n_k$  the steady state associated with  $\gamma_k$  and assume that  $R_b$  is large enough so that  $n_k(R_b) = 0$ .

**Result 7** (Incompressible limit of the stationary states. Cf. Theorem 3.1.3). *Let  $\{\gamma_k\}_{k \in \mathbb{N}}$  be any sequence such that  $\gamma_k \rightarrow \infty$ . Let  $\{n_k\}_{k \in \mathbb{N}}$  be a sequence of stationary states with the same mass  $m$  and with radius  $R_k$ , being the smallest argument such that  $n_k(r) = 0$ .*

*Then,  $n_k \rightarrow n_{inc}$  in  $C^1([0, R_b])$  and  $R_k \rightarrow R$ , where  $n_{inc}$  and  $R$  are uniquely defined and can be computed explicitly. Moreover, the sequence of pressures  $\{p_k := n_k^{\gamma_k}\}_{k \in \mathbb{N}}$  converges weakly to some pressure  $p_{inc}$  such that  $p_{inc}(n_{inc} - 1) = 0$  and  $p_{inc}$  has a jump at  $\partial\{n_{inc} = 1\}$*

$$[[p_{inc}]] \approx \sqrt[3]{6} R^{2/3} \delta^{1/3}, \quad \text{as } \delta \rightarrow 0.$$



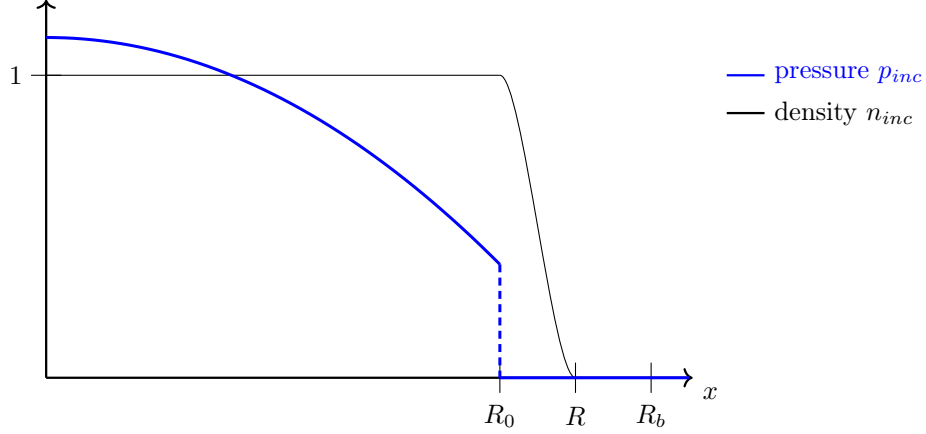


Figure 3 – Plot of the limiting profile  $n_{inc}$ , as  $\gamma_k \rightarrow \infty$ , for the potential  $V(r) = r^2$ . We can observe that the pressure has a discontinuity at  $R_0$  with  $(0, R_0) = \{n_{inc} = 1\} = \{p_{inc} > 0\}$ , while the density remains  $C^1$ .

## Part 2: Nonlocal to local convergence of the Cahn-Hilliard equation

The Cahn-Hilliard equation was derived in 1958 by Cahn and Hilliard. In 1997, Giacomini Lebowitz rigorously derived an equation called the nonlocal Cahn-Hilliard equation. Their derivation relies on a  $d$ -dimensional lattice gas evolving via Kawasaki exchange with respect to the Gibbs measure for a Hamiltonian which includes both short-range and long-range interactions. The two equations, nonlocal and local have many similarities, analytically and numerically, and it is often the case that the two may be used in applications, arguing that they are the same up to a Taylor expansion. Indeed, both of the equations can be written

$$\partial_t u - \operatorname{div}(m(u)\nabla\mu) = 0,$$

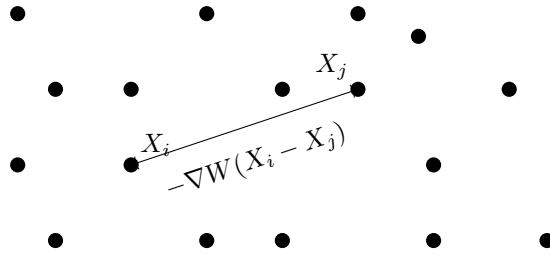
where  $\mu$  has to be chosen accordingly

$$\mu_{local} = -\Delta u + F'(u), \quad \mu_{nonlocal} = \int J_\varepsilon(x, y)(u(x) - u(y)) dy + F'(u).$$

When we write  $u(y) \approx u(x) + y \cdot \nabla u(x) + \frac{y^2}{2} D^2(u(x))$  and assume that the kernel is symmetric, with only non-zero second moments along the diagonal (yielding the Laplacian instead of the full Hessian), we observe that  $\mu_{nonlocal}$  serves as an approximation to  $\mu_{local}$ . The parameter  $\varepsilon$  in the kernel  $J_\varepsilon$  represents the range of the nonlocal interaction. Indeed, the kernel often takes the form of  $J_\varepsilon(|x - y|)$  and approximates a Dirac function as  $\varepsilon \rightarrow 0$ , containing all the information in the zone  $x \approx y$ . As  $\varepsilon \rightarrow 0$ , the nonlocal Cahn-Hilliard equation formally converges to the local Cahn-Hilliard equation. As previously mentioned, these two equations exhibit certain similarities, yet they also present distinct challenges. For instance, studying the nonlocal Cahn-Hilliard equation is in general simpler, as the equation is of second order. However the analysis on a bounded domain can become challenging due to the nonlocal operator, and it is often studied in the torus, as in the work of Giacomini-Lebowitz. Conversely, the local Cahn-Hilliard equation is often studied on a bounded domain with Neumann boundary conditions. Moreover, explicitly

computing the stationary states of the nonlocal Cahn-Hilliard equation proves to be a bit harder. Assuming  $\partial_t u = 0$  and  $m(u) > 0$ , we can then derive  $\mu = C$ . In one dimension, the resulting equation reduces to a simple second-order ODE for the local equation, and an equation evolving a convolution for the nonlocal one. The nonlocal Cahn-Hilliard equation offers the advantage of being easier to derive. We can rely on the work of Giacomini-Lebowitz, but when  $F'(u)$  is linear, there is a more straightforward derivation, from the particle level, see also [279, 36] for similar derivation in biological contexts.

Consider a system of  $N$  particles in interaction denoted  $\{X_i\}_{i=1,\dots,N}$ .



Suppose that these particles follow the law

$$\frac{dX_i}{dt} = \sum_{j \neq i} -\nabla W(X_i - X_j). \quad (0.0.17)$$

Let  $\rho^N$  be the empirical measure

$$\rho^N(t) := \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)},$$

then we see that the right-hand side of (0.0.17) can be rewritten as  $-\nabla W * \rho^N$ . As the law of motion is imposed on the velocity it makes sense to make a connection with the continuity equation, and one can prove that the empirical measure satisfies (in the measure sense)

$$\partial_t \rho^N - \operatorname{div}(\rho^N \nabla W * \rho^N) = 0.$$

Sending  $N \rightarrow +\infty$  and assuming that  $\rho^N$  converges to  $\rho$  we obtain

$$\partial_t \rho - \operatorname{div}(\rho \nabla W * \rho) = 0. \quad (0.0.18)$$

To consider volume exclusion effect, for instance the cells have some volume and cannot stand exactly on the top of each other one has to consider that the kernels are repulsive at the origin and we rename (up to changing the sign)

$$W \rightarrow -W + \nu \delta_0 \quad (0.0.19)$$

where  $\nu$  is the strength of the local repulsion. Equation (0.0.18) thus becomes

$$\partial_t \rho - \operatorname{div}(\rho \nabla(-W * \rho + \nu \rho)) = 0.$$

Then, assuming that  $W$  is of mass 1 and writing  $\nu \rho = \rho + (\nu - 1)\rho$ , we obtain a nonlocal Cahn-Hilliard equation. Note from (0.0.19) the competition between the long range attraction and repulsion at the origin. The convergence of the nonlocal to local Cahn-Hilliard equation states

that if these effects are scaled appropriately, we expect to obtain a fourth-order equation with surface tension effects.

More generally, consider the equation:

$$\begin{aligned} \partial_t u_\varepsilon &= \operatorname{div}(u_\varepsilon \nabla \mu), & \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \mu &= B_\varepsilon[u_\varepsilon] + F'(u_\varepsilon), & \text{in } (0, +\infty) \times \mathbb{T}^d. \end{aligned} \quad (0.0.20)$$

and its local counterpart:

$$\begin{aligned} \partial_t u &= \operatorname{div}(u \nabla \mu), & \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \mu &= -\Delta u + F'(u), & \text{in } (0, +\infty) \times \mathbb{T}^d. \end{aligned} \quad (0.0.21)$$

In the first chapter, we prove existence of weak solution and the nonlocal to local limit. More precisely we have the two following results:

**Result 8** (Existence of solutions for the nonlocal equation. Cf. Theorem 4.1.7). *There exists a global weak solution  $u_\varepsilon$  of the previous equation.*

**Result 9** (Convergence of nonlocal to local Cahn-Hilliard equation on the torus. Cf Theorem 4.1.8). *Let  $\{u_\varepsilon\}$  be a sequence of solutions of the degenerate nonlocal Cahn-Hilliard equation. Then, up to a subsequence,*

$$u_\varepsilon \rightarrow u \text{ in } L^2(0, T; H^1(\mathbb{T}^d))$$

where  $u$  is a weak solution of the local degenerate Cahn-Hilliard equation.

The proof of nonlocal to local relies on a new compactness argument in the spirit of Aubin-Lions for nonlocal operators where the compactness in space is achieved with a nonlocal Poincaré inequality proved by Ponce [294]. A similar result can be found for a system of two populations, and it has a direct biological application, which is the understanding of the cell-cell adhesion patterns and the so-called *Steinberg differential adhesion hypothesis*.

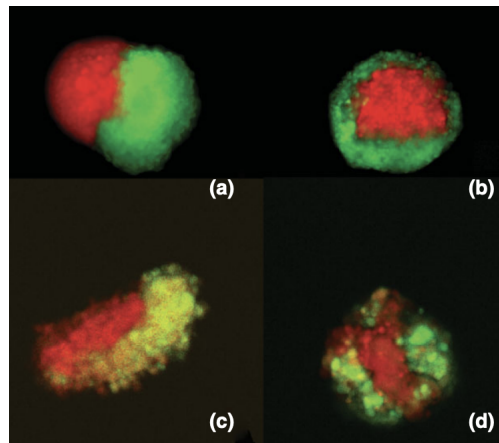
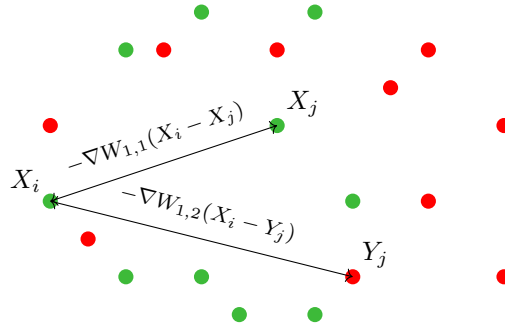


Figure 4 – Ramsey A. Foty, Malcolm S. Steinberg, *Differential adhesion in model systems*, Wiley Interdiscip Rev Dev Biol actions.

Consider two populations of cells. Let them evolve, and wait enough time to observe a stationary state. Then, one can observe 4 kind of different patterns: complete sorting, partial engulfment,

engulment and mixing corresponding to the 4 pictures above. The 4 patterns are a scale that describe how much the cells interact with each other: that is how the two populations interact and how individuals from a same population interact. These experiments lead Steinberg to the formulation of the differential adhesion hypothesis. As an analogy to immiscible fluids such as oil and water, Steinberg developed a thermodynamic model to explain this behavior by considering the different strengths of cell-cell adhesion bonds. In this context, when different cell populations are put together, they self-organize, with the objective of minimizing the overall energy within the tissue. Motivated by these ideas, Baker, Carrillo and Falco introduced in [165] a new local continuum model for cell-cell adhesion, aiming to replicate the diverse patterns observed in the DAH phenomenon. This model is based on a set of Cahn-Hilliard equations. The derivation is similar to that of the single equation.

Consider a system of  $N$  green particles denoted  $\{X_i\}_{i=1,\dots,N}$  and  $N$  red particles denoted  $\{Y_i\}_{i=1,\dots,N}$  that satisfy



$$\begin{aligned} X_i' &= -\frac{1}{N} \sum_{j \neq i} \nabla W_{1,1}(X_i - X_j) - \frac{1}{N} \sum_{j \neq i} \nabla W_{1,2}(X_i - Y_j) \\ Y_i' &= -\frac{1}{N} \sum_{j \neq i} \nabla W_{2,1}(Y_i - X_j) - \frac{1}{N} \sum_{j \neq i} \nabla W_{2,2}(Y_i - Y_j). \end{aligned}$$

To consider the repulsive effects at the origin (two cells cannot exactly stand on the top of each other) we rename the kernels  $W_{i,j} \rightarrow W_{i,j} + \nu \delta_0$ . Consider the empirical measures  $\rho^N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}$ ,  $\eta^N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{Y_i(t)}$  and send  $N \rightarrow \infty$ . In the limit we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nabla \cdot (\rho \nabla (W_{1,1} * \rho + W_{1,2} * \eta + \nu(\rho + \eta))), \\ \frac{\partial \eta}{\partial t} &= \nabla \cdot (\eta \nabla (W_{2,1} * \rho + W_{2,2} * \eta + \nu(\rho + \eta))). \end{aligned}$$

Assuming  $W_{i,j} = K_{i,j} \omega_\varepsilon$ ,  $W_{1,2} = W_{2,1}$  and doing some rescaling and renaming of parameters we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nabla \cdot (\rho \nabla (\kappa B_\varepsilon[\rho] + \alpha B_\varepsilon[\eta] - \gamma \rho - \beta \eta)), \\ \frac{\partial \eta}{\partial t} &= \nabla \cdot (\eta \nabla (\alpha B_\varepsilon[\rho] + B_\varepsilon[\eta] - \beta \rho - \eta)), \end{aligned} \tag{0.0.22}$$

with  $B_\varepsilon[u] = \frac{u - \omega_\varepsilon * u}{\varepsilon^2} \approx \Delta u$ .

Then, sending formally  $\varepsilon \rightarrow 0$  we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nabla \cdot (\rho \nabla (-\kappa \Delta \rho - \alpha \Delta \eta - \gamma \rho - \beta \eta)), \\ \frac{\partial \eta}{\partial t} &= \nabla \cdot (\eta \nabla (-\alpha \Delta \rho - \Delta \eta - \beta \rho - \eta)). \end{aligned} \tag{0.0.23}$$

In their paper, Baker, Carrillo and Falco studied the stationary states of the last system in dimension 1 and proved that depending of the parameters  $\kappa, \alpha, \gamma, \beta$ , we can observe four different shapes, which correspond to the four cell-cell adhesion hypothesis: complete sorting, partial engulfment, engulfment, and mixing. Three main questions remain

- Can we prove the existence of solutions to the nonlocal system and what is their regularity?
- Can we make the limit  $\varepsilon \rightarrow 0$  rigorous?
- Can we prove convergence to stationary states of the local problem?

The first two questions are solved in the second chapter of this part, where as the third one is still open at the moment. The existence result is based on the JKO scheme:

**Result 10** (Existence of weak solutions. Cf. Theorem 5.1.1). *When the matrix  $\begin{bmatrix} \kappa & \alpha \\ \alpha & 1 \end{bmatrix}$  is positive definite, there exists weak solutions of (0.0.22).*

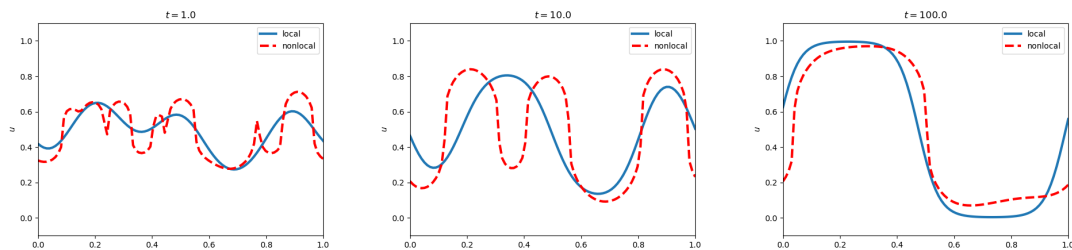
And for the convergence  $\varepsilon \rightarrow 0$ , the strategy is similar to the strategy for a single equation.

**Result 11** (Convergence nonlocal to local. Cf. Theorem 5.1.2). *Let  $(\rho_\varepsilon, \eta_\varepsilon)$  be weak solutions of (0.0.22). Then when  $\varepsilon \rightarrow 0$ ,*

$$\rho_\varepsilon \rightarrow \rho, \quad \eta_\varepsilon \rightarrow \eta, \quad \text{in } L^2(0, T; H^1(\mathbb{T}^d))$$

where  $\rho, \eta$  are weak solutions of (0.0.23).

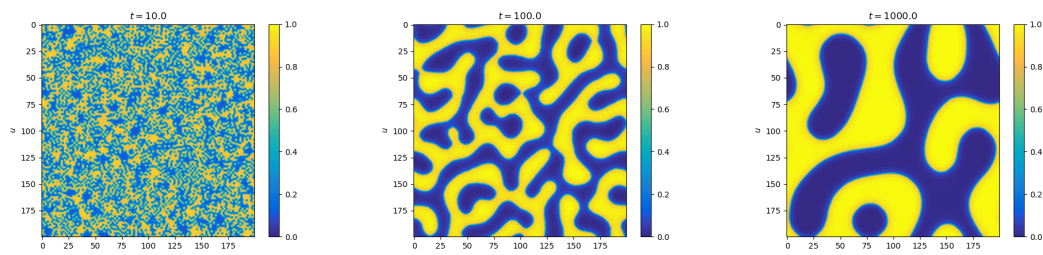
Choosing the right equation (nonlocal or local) to model a phenomenon is a natural question that arises. Since the equations are similar, as demonstrated by the theorem of nonlocal to local convergence, which one should be used to model a binary fluid, two types of interacting populations... And the answer is not entirely clear in the literature. Historically, the equation of local Cahn-Hilliard was derived for binary alloys and is still widely used in various fields such as tumor growth, image segmentation, and copolymer melts. The nonlocal equation, on the other hand, arrived later but has the advantage of being derived more simply. Moreover, it also highlights nonlocal phenomena, which is important in the study of collective movements, for example. It also allows for a nonlocality parameter epsilon that can be adjusted according to the application. Numerically, it is observed that the nonlocal equation tends to favor sharp interfaces, unlike the local equation, which always has a diffuse interface. Additionally, the regularity of the solutions of the nonlocal equation is also lower, which is easily noticeable in energy estimates. If we start with a less regular solution, for example, a constant state perturbed by random data, numerically it is observed that the solutions of the nonlocal equation take more time to become smooth. To provide a simple example, in the following numerical simulations we consider a Cahn-Hilliard equation (nonlocal and local) with constant mobility in dimension 1 and 2. The kernel employed in the nonlocal equation is of gaussian type.



nonlocal vs local  $t = 1$

nonlocal vs local  $t = 10$

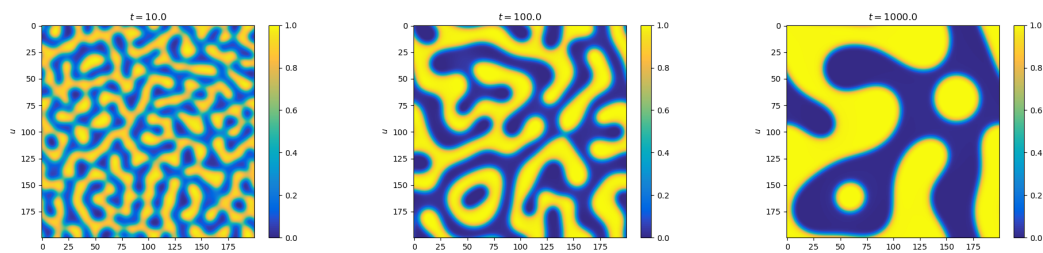
nonlocal vs local  $t = 100$



nonlocal  $t = 10$

nonlocal  $t = 100$

nonlocal  $t = 1000$



local  $t = 10$

local  $t = 100$

local  $t = 100$

Figure 6 – Comparison between the nonlocal and local Cahn-Hilliard equation in 1d and 2d

The pictures correspond to the numerical simulations of

$$\partial_t u = \Delta \mu, \quad \mu = F'(u) + \begin{cases} -\Delta u: & \text{local case} \\ \frac{u - \omega_\varepsilon * u}{\varepsilon^2}: & \text{nonlocal case} \end{cases}$$

where  $F(u) = u^2(1 - u^2)$  is a polynomial approximation of the double well potential and  $\omega_\varepsilon = \frac{1}{\varepsilon^d} \omega\left(\frac{x}{\varepsilon}\right)$  with  $\omega(x) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{|x|^2}{2}}$  a gaussian kernel. The numerical implementation was carried out using Python, and `numpy.fft`, `numpy.ifft` to efficiently calculate the Fourier and inverse Fourier transforms. The Fourier transform proves particularly useful for terms like  $\Delta^2 u$  and the convolution  $\Delta(u - \omega * u)$ . Then, an implicit Euler scheme is applied to the equation in Fourier variables.

### Part 3: Fluid Models with surface tension

The Cahn-Hilliard equation can be used to model a binary mixture in interaction with a diffuse interface. More precisely, consider the general Cahn-Hilliard equation

$$\partial_t c - \operatorname{div}(m(c)\nabla\mu) = 0, \quad \mu = F'(c) - \varepsilon\Delta c.$$

where  $c$  is an order parameter. During the evolution, an interface between the two components  $c = 0$ ,  $c = 1$  will form with size  $\sqrt{\varepsilon}$ .

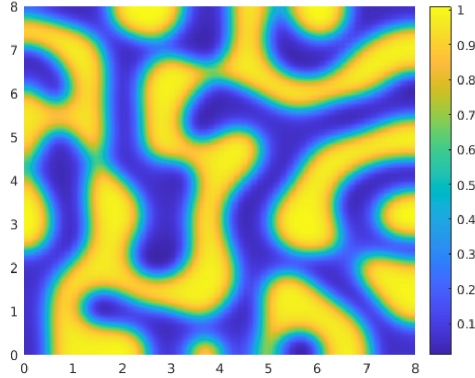


Figure 7 – Typical evolution of the Cahn-Hilliard equation. The interface is in green:  $c = 0.5$ .

A small argument in favor of this result is the form of the energy

$$E[c] = \int \varepsilon \frac{|\nabla c|^2}{2} + F(c).$$

As the energy is bounded we expect that  $|\nabla c|^2$  takes at most value like  $\frac{1}{\varepsilon}$ , and thus the transition zone from  $c = 0$  to  $c = 1$  is of size  $\sqrt{\varepsilon}$ . The idea of a diffuse interface model in a binary mixture is however not completely obvious in we refer to [22] for a review. The study of the interface between two fluids started with Young, Laplace, and Gauss in the early 1800s who imagined the interface as an infinitesimally thin surface. The physical quantities such as the density were assumed to be discontinuous across the interface. Boundary conditions were used to explain

physical phenomena like capillarity, which is the process where a liquid flows without the help of any external forces like gravity. For instance, when we impose the condition that the sum of all forces acting on an infinitesimal curved interface must equal zero, we obtain the Young-Laplace equation. This equation states that the pressure difference between the two sides of the interface (assuming each phase is in equilibrium) is equal to the product of the surface tension and the curvature.

In the middle of the nineteenth century, Poisson, Maxwell, and Gibbs recognized that the interface actually involved a smooth transition of physical quantities rather than abrupt changes. In particular, Gibbs introduced the concept of a dividing surface, denoting a surface of discontinuity, and introduced surface excess quantities to develop the equilibrium thermodynamics of interfaces. In the end of the nineteenth century, Rayleigh and Van der Waals expanded on these notions, describing the interface as having a non-zero thickness, or diffuseness, through thermodynamic gradient theories. In 1901 Korteweg further advanced these ideas by proposing a constitutive law for the capillary stress tensor based on density and its spatial gradients, and therefore on density variations. More precisely, he assumed the stress tensor to be of the form

$$\mathbf{T} = \mathbf{T}^{NS} + \mathbf{T}^C,$$

$$\mathbf{T}^{NS} = -p\mathbf{I} + 2\mu\mathbf{D} + \lambda\text{div}(v)\mathbf{I}, \quad \mathbf{T}^C = (\alpha|\nabla\rho|^2 + \beta\Delta\rho)\mathbf{I} + \gamma(\nabla\rho \otimes \nabla\rho) + \delta D^2\rho.$$

Here  $\mathbf{T}^{NS}$  is the usual stress tensor applied in Navier-Stokes model for compressible fluids, and  $\mathbf{T}^C$  is the introduced capillarity stress tensor. In particular,  $p$  is the pressure and  $\mathbf{D}$  is the rate-of-strain tensor.  $\alpha, \beta, \gamma, \delta$  are functions of the density  $\rho$ , representing long-range molecular effects. In fact, the previous formula of the capillarity stress tensor is mostly used in single component fluids and the density  $\rho$  is the variable that distinguishes the bulk fluids and the interface: it is an order parameter. For a binary fluid, the stress tensor can be formulated on another order parameter: the composition  $c$  which can be for instance the mass fraction of one component (Cahn and Hilliard 1958). This methodology was then expanded to simulate the phase separation of polymer blends and alloys (de Gennes, 1980). Additionally, in the 1970s, the diffuse interface approach merged with hydrodynamics, giving rise to a collection of conservation equations. Kawasaki, Siggia, Hohenberg and Halperin among others named this approach "model H" and it was later called the "diffuse interface method."

It is not completely clear whether or not the initial Korteweg model is consistent with the basic concepts of continuum thermodynamics. For instance, it was shown by Serrin in [309] that *"Unless certain conditions are satisfied, the only geometric phase boundaries that conform to Korteweg's theory are either spherical, cylindrical, or planar. In other words, disregarding the effect of gravity, the physical situations that can arise in phase transitions governed by Korteweg's theory are: spherical liquid bubbles in an ambient atmosphere, spherical vapor bubbles in an ambient liquid bath, planar interfaces between the phases, or circular cylindrical interfaces between the phases, which are likely unstable. This rather surprising conclusion can be understood as a consequence of the mechanical equations of equilibrium (Cauchy's equation) having three independent components, while the equilibrium of liquid-vapor phase transitions is determined by a single physical variable, the density."* Despite its uncertainty, Korteweg stresses have been successful in describing experiments. In [100], Chen and Meiburg show that Korteweg stresses can account for discrepancies observed between conventional Stokes flow simulations and experiments. They present an estimate of the vorticity and stream function fields induced by the Korteweg stresses, which shows these stresses result in the formation of a vortex ring structure near the tip of the concentration front. Through this mechanism, the propagation velocity of the concentration front is reduced, in agreement with the experimental observations.



If we assume that the Korteweg stresses represent well the interactions at the interface, a logical approach would be to formulate the corresponding Euler and Navier-Stokes equations. These equations are commonly referred to as the Euler-Korteweg and Navier-Stokes-Korteweg equations in the literature. The previous form of the Korteweg capillarity stress tensor is quite general and considerable simplifications are often made in the literature. For instance, in the first chapter, we only retain the Laplacian terms and consider

$$\nabla \cdot \mathbf{T}^C = \rho \nabla B_\eta[\rho]$$

where  $B_\eta$  is a nonlocal operator of first order approximating the Laplacian when  $\eta \rightarrow 0$ . Then we study the approximation of the (local) Cahn-Hilliard equation by the (nonlocal) Euler-Korteweg equation, particularly when the friction coefficient is large. More precisely, we consider

$$\begin{aligned} \partial_t \rho + \frac{1}{\varepsilon} \operatorname{div}(\rho \mathbf{u}) &= 0, \\ \partial_t(\rho \mathbf{u}) + \frac{1}{\varepsilon} \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) &= -\frac{1}{\varepsilon^2} \rho \mathbf{u} - \frac{1}{\varepsilon} \rho \nabla(F'(\rho) + B_\eta[\rho]). \end{aligned} \tag{0.0.24}$$

This equation models the long-time asymptotics of the motion of a compressible fluid with density  $\rho$ , velocity  $\mathbf{u}$  which is in fact a liquid-vapor mixture. The fluid experiences high friction (due to the term  $-\frac{1}{\varepsilon^2} \rho \mathbf{u}$ ) and additional capillary effects in the transition zone between liquid and vapor (due to the term  $-\frac{1}{\varepsilon} \rho \nabla(F'(\rho) + B_\eta[\rho])$ ).

When  $\varepsilon$  is very small, the friction is so big, that we mostly observe a phase separation phenomenon between the liquid and the vapor. When  $\varepsilon, \eta \rightarrow 0$  in some scaling to be determined, we prove the convergence of (0.0.24) to the local Cahn-Hilliard equation

$$\begin{aligned} \partial_t \rho &= \operatorname{div}(\rho \nabla \mu), \\ \mu &= -\Delta \rho + F'(\rho). \end{aligned}$$

which describes the dynamics of phase separation. The problem of deriving the Cahn-Hilliard equation from the Euler-Korteweg equation was studied only recently by [249] assuming two facts which are not guaranteed by the current theory: existence of weak dissipative solution of the Euler Korteweg equation for general initial data, and existence of classical solutions of the degenerate Cahn-Hilliard equation. For that reason, we propose to include a nonlocality in the Korteweg tensor and consider measure valued solutions, which allows to bypass the two previous assumptions to make the argument rigorous. Therefore, we start by proving the existence of measure-valued solutions, which are dissipative (this term will be defined in the related chapter), a tool particular useful to use relative-entropy method arguments.

**Result 12** (Existence of measure-valued solutions. Cf. Section 6.3). *There exists dissipative measure-valued solutions of (0.0.24).*

Then we prove the following result

**Result 13** (From Euler-Korteweg to Cahn-Hilliard. Cf. Theorem 6.1.4). *Let  $\rho_{\varepsilon, \eta}^0$  and  $\mathbf{u}_{\eta, \varepsilon}^0$  be a well prepared initial condition. Let  $(\rho_{\eta, \varepsilon}, \mathbf{u}_{\eta, \varepsilon})$  be a dissipative measure-valued solution of (0.0.24) with initial condition  $(\rho_{\varepsilon, \eta}^0, \mathbf{u}_{\varepsilon, \eta}^0)$ . Then, for each sequence  $\eta_k \rightarrow 0$ , there exists a subsequence  $\{\eta_k\}$  (not relabelled) and a sequence  $\{\varepsilon_k\}$  depending on  $\eta_k$  and the final time  $T$  such that  $\varepsilon_k \rightarrow 0$  and  $\rho_{\eta_k, \varepsilon_k} \rightarrow \rho$ , where  $\rho$  is a weak solution of the local Cahn-Hilliard equation.*

In the second chapter, we consider a generalized compressible Navier-Stokes-Cahn-Hilliard system

(a bit distinct from the Navier-Stokes-Korteweg equation). The tensor considered here is

$$\nabla \cdot \mathbf{T}^C = \gamma \operatorname{div} \left( \nabla c \otimes \nabla c - \frac{1}{2} |\nabla c|^2 \mathbf{1} \right).$$

and we study the system

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0, \\ \frac{\partial(\rho c)}{\partial t} + \operatorname{div}(\rho c \mathbf{v}) &= \operatorname{div}(b(c) \nabla \mu) + F_c, \\ \rho \mu &= -\gamma \Delta c + \rho \frac{\partial \psi_0}{\partial c}, \\ \frac{\partial(\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) &= - \left[ \nabla p + \gamma \operatorname{div} \left( \nabla c \otimes \nabla c - \frac{1}{2} |\nabla c|^2 \mathbf{1} \right) \right] + \operatorname{div}(\nu(c) (\nabla \mathbf{v} + \nabla \mathbf{v}^T)) \\ &\quad - \frac{2}{3} \nabla(\nu(c) (\operatorname{div}(\mathbf{v}))) - \kappa(\rho, c) \mathbf{v}, \end{aligned}$$

A rigorous derivation of the system is provided in the related article. It models the motion of a diphasic fluid composed of two immiscible components, *i.e.* the cells of the two different types, in a porous matrix and takes into account viscosity effects, surface tension, and friction on the rigid fibers constituting the medium.  $\rho$  is the total density of the mixture (*i.e.* the sum of the two partial densities),  $c$  is the relative mass fraction of one component (*e.g.* the cancer cells),  $\mathbf{v}$  is the mass averaged total velocity,  $\mu$  is called the chemical potential,  $p$  is the pressure. The coefficient  $\gamma$  is related to the surface tension and is equal to the square of the width of the diffuse interface existing between the two populations. The friction coefficient  $\kappa(\cdot)$  is a monotone increasing function of the density and takes into account the possible difference of friction strength between the two populations. We use this friction term to model possible adhesive effects on the extracellular matrix. The coefficient  $\nu(\cdot)$  represents the viscosity of the mixture and again possible differences of viscosities could be considered for the two populations. The function  $\psi_0$  represents the separation of the two components of the mixture and phenomenologically models the behavior of cells (*i.e.* cells tend to form aggregates of the same cell type). The function  $F_c(\cdot)$  accounts for the possible proliferation and death of cells. The non-negative function  $b(\cdot)$  models the mobility of cells and is assumed to be doubly degenerated to, again, correspond to the behavior of cells. The model aims to be as general as possible to study the possible mechanical effects playing a role in invasive growth of a tumor.

We focus on the analysis and numerical simulation of the model. Our system is derived rigorously and satisfies basic mechanics of fluids and the thermodynamics of particles. Under simplifying assumptions, we prove the existence of global weak solutions. We also propose a structure-preserving numerical scheme based on the scalar auxiliary variable method to simulate our system and present some numerical simulations validating the properties of the numerical scheme and illustrating the solutions of the model.

**Result 14** (Derivation, analysis and numerical simulations of the Cahn-Hilliard-Navier-Stokes model. Cf. Chapter 7). *The previous system can be rigorously derived in thermodynamic and fluid mechanics framework. There exists weak solutions and we can construct a structure preserving numerical scheme.*

## Part 4: Kinetic theory and the Cahn-Hilliard equation

Kinetic theory is a mathematical framework to describe the evolution of a gas or a plasma, and more generally a system of  $N$  identical point particles, where  $N$  is assumed to be large. Such a system considers three types of scales:

- *Microscopic scale.* The microscopic scale follows a Newtonian mechanical approach, treating each particle as a point in space. It uses Newton's laws of motion to determine particle acceleration, accounting for the interacting forces. Although this description offers remarkable precision, it becomes impractical for systems gases, where the number of molecules approaches Avogadro's number (approximately  $N \sim 10^{23}$ ). Moreover, even when dealing with a small number of particles, such as in the case of the 3-body problem, it can present significant challenges.
- *Macroscopic scale.* Imagine a cube positioned in space, small enough compared to the entire system yet capable of accommodating a large number of particles. Assuming that the molecules within the cube have reached a local equilibrium, this approach considers equations governing energy, momentum, and mass within the confined region. It yields equations of fluid mechanics, including the Euler and Navier-Stokes equations.
- *Mesoscopic scale.* Serving as a middle ground between the previous approaches, the mesoscopic scale considers molecular interactions without attributing specific effects to individual particles. Instead, it focuses on quantifying the number of molecules engaged in a particular activity, therefore providing a statistical perspective.

Kinetic theory follows this last approach. It represents a *statistical point of view*, which forgets about the individual trajectories and describes the distribution of particles in the phase space. This usually includes space and velocity, i.e. both macroscopic and microscopic variables.

The unknown is therefore the probability measure

$$f \equiv f(t, x, \xi), \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^d,$$

such that, for every infinitesimal volume  $dx d\xi$  around the point  $(x, \xi)$  in the phase space, the quantity  $f(t, x, \xi) dx d\xi$  is the number of particles which have position  $x$  and velocity  $\xi$  at fixed time  $t$ . For this reason,  $f$  is a nonnegative function and integrable in both space and velocity variables, but it is not directly observable. Nevertheless, at each point of the domain it provides all measurable macroscopic quantities, which can be expressed in terms of microscopic averages, for instance:

$$\rho(t, x) = \int_{\mathbb{R}^d} f(t, x, \xi) d\xi, \quad (\text{macroscopic density})$$

$$u(t, x) = \frac{1}{\rho} \int_{\mathbb{R}^d} \xi f(t, x, \xi) d\xi, \quad (\text{mean velocity})$$

$$T(t, x) = \frac{1}{\rho} \int_{\mathbb{R}^d} \frac{|\xi - u(t, x)|^2}{2} f(t, x, \xi) d\xi, \quad (\text{temperature})$$

We point out that such a statistical description makes sense only with a very large number of particles.

Applications to phase transitions. (For further details see [284]) Any substance with a fixed chemical composition, such as water can exist in different forms called states. Water can exist as a gas, a liquid, or a solid (ice). These states differ in many properties, such as heat capacity, density, and mechanical properties. Generally, a solid or liquid substance can have multiple

arrangements of atoms, molecules, or particles, each associated with distinct properties of the substance, forming different phases. Solid water, for instance, has several phases of ice. Through experiments, phase transitions or changes of state can be observed. For example, a substance transitions from a liquid to a solid state through solidification, while the application of pressure can modify molecular arrangements in a crystal, causing a transition from one crystalline phase to another. Phase transitions have been known for a long time and occur naturally (e.g., condensation of water droplets in clouds) or in our daily lives. They are also used in various systems and industrial processes. For instance, the evaporation of water in the steam generator of a nuclear power plant is a physical process that activates turbines in electric generators, and the melting and solidification of metals are crucial stages in metallurgical operations, among other examples. It is important to note that a phase transition is induced by external influences that modify an intensive thermodynamic variable of the system, such as temperature, pressure, magnetic field, or electric field.

Furthermore, phase transitions are observed to begin on a microscopic scale. For instance, small liquid droplets with radii smaller than one micron appear in the vapor phase before complete condensation into a liquid state. This phenomenon is known as nucleation.

The initial exploration of a microscopic approach to understanding phase transitions can be credited to Johannes Diderik van der Waals (1837-1923). In his doctoral thesis published in 1873, he introduced the "van der Waals equation of state," which is a theoretical model for the coexistence of liquid and gas phases using microscopic principles. Another approach, less computationally expensive than treating the fluid as a system of particles governed by mechanical laws, is the kinetic theory, which uses the mesoscopic scale mentioned above. During the early decades of the 20th century, the Swedish physicist David Enskog (1884-1947) extended the Boltzmann equation and, in collaboration with Vlasov, developed a model that addressed the problem of phase transitions.

Nevertheless, in many instances, these equations provide an abundance of details that may not always be necessary. This opens up the possibility of constructing simpler kinetic models that capture the essential features required to describe phase transition phenomena.

This is the idea of Noguchi and Takata, that consider in [317] a kinetic model to capture the dynamic of phase transition for the Van der Waals fluid. This model reads as follows:

$$\begin{aligned} \partial_t f + \xi \cdot \nabla_x f + F \cdot \nabla_\xi f &= A(\varrho)(\varrho M - f), \quad A(\varrho) > 0, \\ F &= F^1 + F^2, \quad F^{1,2} = -\nabla \Phi^{1,2}, \\ \Phi^1 &= \begin{cases} \varrho - \omega^L \star \varrho \\ \text{or } -\kappa \Delta \varrho \end{cases}, \quad \Phi^2 = -C_1 \log(1 - \varrho) + \frac{\varrho}{1 - \varrho} - C_2 \varrho, \quad C_1, C_2 \in \mathbb{R}, \end{aligned}$$

The first equation is a Vlasov equation with a BGK term. The force  $F$  is the gradient of a potential  $\Phi$  and it is divided into two parts:  $\Phi^1$  is either a nonlocal combination of short-range repulsion and long-range attraction, or a local laplacian (see Part 2 of this thesis for more details on nonlocal to local convergences when  $\omega^L$  approximates a dirac).  $\Phi^2$  is a short-range interaction potential. The explanation for this precise form of potential can be found in [317], but it follows by usual laws of thermodynamics and Van der Waals equation of state. The authors state that full details of intermolecular collision are not taken into account so that the collision term on the right-hand side plays just a thermal bath role and neither moment nor energy is conserved. However, despite this approximation, they show that the system exhibits the essential features of phase transition dynamics, both theoretically and numerically. A simple argument in favor of this model is that it has a connexion with the Cahn-Hilliard equation, which has been used intensively to describe the phase transition phenomena. The link with the Cahn-Hilliard equation

is obtained after introducing the diffusive scaling in time and space,

$$t \rightarrow \varepsilon^2 t, \quad x \rightarrow \varepsilon x,$$

and sending  $\varepsilon \rightarrow 0$ . Here  $\varepsilon$  is actually a physical parameter called the Knudsen number, which is the ratio between the mean free path of a particle and a typical length scale of the system. When  $\varepsilon$  is sent to 0, the limiting macroscopic density then solves a Cahn-Hilliard equation. Let's prove this statement formally. We start from the rescaled Vlasov equation (where we assumed  $A(\rho) = 1$ )

$$\varepsilon^2 \partial_t f_\varepsilon + \varepsilon \xi \cdot \nabla_x f_\varepsilon + \varepsilon F_\varepsilon \cdot \nabla_\xi f_\varepsilon = \varrho_\varepsilon M(\xi) - f_\varepsilon, \quad \varrho_\varepsilon = \int_{\mathbb{R}^d} f_\varepsilon(t, x, \xi) d\xi.$$

The potential is

$$F_\varepsilon = F_\varepsilon^1 + F_\varepsilon^2, \quad F^{1,2} = -\nabla \Phi^{1,2},$$

$$\Phi_\varepsilon^1 = \begin{cases} \varrho_\varepsilon - \omega^L \star \varrho_\varepsilon \\ \text{or } -\Delta \varrho_\varepsilon \end{cases}, \quad \Phi_\varepsilon^2 = -\log(1 - \varrho_\varepsilon) + \frac{\varrho_\varepsilon}{1 - \varrho_\varepsilon} - \varrho_\varepsilon.$$

Integrating the Vlasov equation with respect to  $\xi$ , we obtain the mass conservation equation

$$\partial_t \varrho_\varepsilon + \operatorname{div} J_\varepsilon = 0, \quad J_\varepsilon = \int_{\mathbb{R}^d} \frac{\xi}{\varepsilon} f_\varepsilon(t, x, \xi) d\xi.$$

Integrating the Vlasov equation against  $\xi$ , we obtain the flux equation

$$J_\varepsilon = \underbrace{-\varepsilon^2 \partial_t J_\varepsilon}_{\rightarrow 0} - \underbrace{\nabla \cdot \int_{\mathbb{R}^d} \xi \otimes \xi f_\varepsilon d\xi}_{\rightarrow -\nabla \varrho} + \underbrace{F_\varepsilon \varrho_\varepsilon}_{\rightarrow F \varrho}.$$

The second term of the right hand-side converges to  $-\nabla \varrho$  because we can decompose  $f_\varepsilon = f_\varepsilon - \varrho_\varepsilon M(\xi) + \varrho_\varepsilon M(\xi)$  and we know by  $f_\varepsilon \rightarrow \rho M(\xi)$  with rate  $\varepsilon$ .

Therefore, in the limit when  $\varepsilon \rightarrow 0$  we obtain the equation

$$\partial_t \varrho - \Delta \varrho + \operatorname{div}(\varrho F) = 0, \quad F = -\nabla(-\Delta \varrho + \Phi^2(\varrho)).$$

Noting that  $\Delta \varrho = \operatorname{div}(\varrho \nabla \log(\varrho))$  we obtain the Cahn-Hilliard equation with degenerate mobility

$$\partial_t \varrho - \operatorname{div}(\varrho \nabla \Phi) = 0, \quad \Phi = -\Delta \varrho + \log\left(\frac{\varrho}{1 - \varrho}\right) + \frac{\varrho}{1 - \varrho} - \varrho.$$

However, this formal approach is quite hard to make rigorous. And one of the main reason is that the Vlasov equation is not really well understood when the force depends directly on the macroscopic density  $\rho$  and its derivatives: recall that the Vlasov-Poisson system is studied with  $\Phi = \Delta^{-1} \rho$  and here we would like (retaining only the highest order)  $\Phi = \rho$  or even better  $\Phi = \Delta \rho$  so there is a loss of 2 or even 4 derivatives! In fact, the story may not be completely lost in the case  $\Phi = \rho$ , as there are some results of the existence of solutions in small time, but this requires a lot of assumptions on the initial data (Penrose stability criterion for instance), and even worse this time of existence may depend on our parameter  $\varepsilon$  that we intend to send to 0. A natural way to make this approach rigorous is to convolve the force with a smooth kernel and to pass to the limit. Then, we send the convolution kernels to a Dirac mass. A second problem is the logarithmic term  $\log(1 - \rho)$  and the term  $\frac{\rho}{1 - \rho}$  which are singular when  $\rho$  is close to 1. Handling

these terms would require some  $L^\infty$  bounds on  $\rho$ , but for this system it does not seem possible to obtain such bounds independently of  $\varepsilon$ . So we simply suppose that these terms are 0, but for applications, one can simply truncate these terms and add a convolution, the important property is that they should not grow more than  $\rho \log(\rho)$ .

Following this plan, the next chapter is dedicated to the convergence  $\varepsilon \rightarrow 0$  with a mollified force and with an interaction term set to 0 for simplicity.

**Result 15** (From Vlasov-Cahn-Hilliard to Cahn-Hilliard. Cf Theorem 8.1.1). *The solutions of the Vlasov-Cahn-Hilliard equation with a mollified force converge to*

$$\partial_t \rho - \Delta \rho + \operatorname{div}(\rho \nabla \Delta(\rho * \omega_\delta * \omega_\delta)) = 0.$$

Then it remains to send  $\delta \rightarrow 0$ , so that  $\omega_\delta \rightarrow \delta_0$ . This is achieved in the last chapter of this part.

**Result 16** (From mollified Cahn-Hilliard to Cahn-Hilliard. Cf Theorem 9.1.2). *When  $\delta \rightarrow 0$ , the solutions of the mollified Cahn-Hilliard equation converge to a solution of the Cahn-Hilliard equation.*

A similar result is found in the case of systems. The strategy is based on commutator estimates, but it turns out that the approach does not work for any type of kernel in any dimension. When the dimension is strictly greater than two, we introduce a novel condition that discriminates which kernels one can choose. This condition turns out to be pretty general as it covers for instance the Gaussian kernel and every compactly supported kernel.

## Perspectives

### Nonlocal equations

Recently, and in continuation of my work on the non-local to local limit, I got interested in different topics of nonlocal equations:

#### 1. Extension of results to more general aggregation diffusion systems:

- Examine the applicability of our results of the nonlocal to local limit to more complex aggregation-diffusion equations and systems such as [92], including considering various types of interaction kernels. As an example, I am working on nonlocal operators that could approximate systems  $\partial_t u - \operatorname{div}(A \nabla u)$  where  $A$  is a matrix and preserving some of its properties.

#### 2. Steady states, their stability, and asymptotic behaviour:

- When considering competing effects, in nonlocal and local equations, the study of the stationary states becomes nontrivial. Therefore we aim to work on the analysis of steady states in the case of competing effects for aggregation-diffusion equations and systems.
- Evaluate the convergence of solutions towards these steady states, including more general scenarios.

#### 3. Continuation of research projects:

- Explore laws other than the Brinkman law for the porous medium equations.
- Identify general conditions applicable to the convolution kernels mentioned in [153].

## The Vlasov-Cahn-Hilliard model

For the general system

$$\begin{aligned} \partial_t f + \xi \cdot \nabla_x f + F \cdot \nabla_\xi f &= A(\varrho)(\varrho M - f), \quad A(\varrho) > 0, \\ F &= F^1 + F^2, \quad F^{1,2} = -\nabla \Phi^{1,2}, \\ \Phi^1 &= \begin{cases} \varrho - \omega^L \star \varrho \\ \text{or } -\kappa \Delta \varrho \end{cases}, \quad \Phi^2 = -C_1 \log(1 - \varrho) + \frac{\varrho}{1 - \varrho} - C_2 \varrho, \quad C_1, C_2 \in \mathbb{R}, \end{aligned}$$

there remains a lack of established existence results and numerical simulations in the existing literature.

In my current work, I have succeeded in proving the existence of solutions for the system for analytical initial data. However, the question remains open when the initial data is in Sobolev spaces. It is widely recognized that similar equation, like the Vlasov-Benney equation, tends to pose challenges in such cases, unless, perhaps, certain criteria, as the Penrose stability criterion are met.

I am optimistic that the strategies employed to establish short-term existence of solutions in Sobolev spaces, where the initial condition satisfies a different Penrose criterion as in [208, 160], could work. Currently, I am actively working in this direction. Furthermore, I aim to propose numerical simulations of the model to validate its accuracy in describing phase transition processes. There are different possibilities for simulation: either use particle methods such as lattice Boltzmann methods, or hybrid methods as in [247] when the solution is close to an equilibrium.

### 1. Analysis and numerical simulation of the Vlasov-Cahn-Hilliard model

- Prove existence of the system for initial condition in Sobolev spaces.
- Design a numerical scheme at the kinetic level.

## Theoretical and numerical analysis of fluid models for biology

In the work on the Cahn-Hilliard-Navier-Stokes model cited above, we argue that, from the numerical simulations, our model seems capable to qualitatively represent patterns of invasive growth of tumors and could show the possible mechanical effects playing a role in the emergence of heterogeneous structures observed in tumor invasion of healthy tissue.

However, we emphasize that to achieve these latter goals, we have to be able to capture accurately the possible fine structures emerging during the numerical simulations. Thus, our numerical scheme will be improved to increase the spatial and temporal orders of accuracy. In a forthcoming work, we will develop a high-order finite element scheme for the compressible system we proposed in the present work. This numerical scheme will allow efficient simulations of compressible diphasic fluids and will be used to simulate relevant test cases with applications in fluid mechanics such as rising bubbles and Rayleigh-Taylor instabilities.

Objectives of this work:

### 1. Improve the actual numerical scheme

- Design an efficient discontinuous Galerkin finite element scheme which are well known to be effective to catch little structures [140].
- Use the scalar auxiliary variable (SAV) method [311] proposed in our previous paper to preserve energy decay.
- Benchmark the scheme to verify the ability to catch: Rayleigh-Taylor instabilities (difference of densities between the two fluids), viscous Saffmann-Taylor instabilities

(differences of viscosities between the two fluids), bubble rising test case (useful for aerosols). See for instance [170, 217, 303, 230]

## 2. Other applications of the model

- For instance application to the mathematical representation of tumor-on-chips [332] that can replicate in a very accurate manner the micro-environment of the tumor.

To implement a discontinuous galerkin method, we intend to use DUNE software [40], the DUNE-FEM module [129] and the DUNE-FEM-DG module [128].





# Mathematical tools

We present some of the key methods, concepts, and facts that are used in this thesis. This list is not exhaustive and for the reader familiar with these methodologies, we suggest skipping this chapter as we do not make any reference to it in other chapters, as they are mostly self-contained. We start by defining the space of Banach-valued functions and recall some useful inequalities.

## Spaces of Banach valued functions and classical facts

We are mostly concerned with evolution equations, meaning that some quantities is evolving in time at each point of a domain. The usual framework is to view the functions depending on time and space as function depending on time  $t$  with values in the functions of space in  $x$ . More precisely to  $u : (t, x) \rightarrow u(t, x)$  where  $(t, x) \in I \times \Omega$  we associate  $u : t \rightarrow u(t)$  where  $u(t) : x \rightarrow u(t, x)$ . Here  $I$  is an interval of  $\mathbb{R}$  and  $X$  is a Banach space. Now we provide some results that can be found in [54, 302, 63, 162].

**Definition 0.0.1** (Lebesgue measurability). Let  $X$  be a Banach space and  $I$  an interval of  $\mathbb{R}$ . We say that a function from  $I$  in  $X$  is Lebesgue measurable if

1. The inverse image under  $f$  of all open sets of  $X$  is a Borel set of  $I$ ,
2. We can change  $f$  on a subset of zero Lebesgue measure of  $I$ , so that  $f$  takes its values into a separable subspace of  $X$ .

With this definition we can generalize the concept of Lebesgue spaces to Bochner spaces.

**Definition 0.0.2** (Bochner spaces). Let  $I = (0, T)$ , the spaces  $L^p(I; X)$  with  $p \geq 1$  of Lebesgue measurable functions defined on  $I$  with values in  $X$  such that  $t \rightarrow \|f(t)\|_X^p$  is integrable on  $I$  is a Banach space with the norm

$$\|f\|_{L^p(I; X)} := \left( \int_I \|f(t)\|_X^p dt \right)^{1/p}.$$

They admit the following interpolation property

**Proposition 0.0.3.** *Let  $I$  be an interval of  $\mathbb{R}$  and  $\Omega$  be a open set of  $\mathbb{R}^d$ . Suppose  $f \in L^{p_1}(I; L^{q_1}(\Omega)) \cap L^{p_2}(I; L^{q_2}(\Omega))$  for some  $p_1, q_1, p_2, q_2 \in [1, +\infty]$ . Then  $f \in L^p(I; L^q(\Omega))$  for all  $p, q$  defined as*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

for all  $\theta \in (0, 1)$ . Moreover

$$\|f\|_{L^p(I; L^q(\Omega))} \leq \|f\|_{L^{p_1}(I; L^{q_1}(\Omega))}^\theta \|f\|_{L^{p_2}(I; L^{q_2}(\Omega))}^{1-\theta}.$$

As a corollary we obtain

**Proposition 0.0.4.** *Suppose that a sequence of functions converges strongly in  $L^{p_1}(I; L^{q_1}(\Omega))$  and is bounded in  $L^{p_2}(I; L^{q_2}(\Omega))$ . Then by interpolation we have strong convergence in every  $L^p(I; L^q(\Omega))$  where  $p, q$  are defined above.*

This corollary is fairly important as often we obtain strong compactness of a sequence in a low Lebesgue space, for instance  $L^1$  with the Frechet-Kolmogorov theorem (see below). Then if we prove for that the sequence is bounded in some  $L^p$  for some  $p > 1$ , we can upgrade this strong convergence to every  $L^q$  for  $q < p$ .

More generally, for  $X$  a Banach space and  $Y$  a locally convex space with  $X \subset Y$  we define the spaces

$$W^{1,p,q}(I; X, Y) := \left\{ u \in L^p(I; X); \frac{du}{dt} \in L^q(I; Y) \right\} \quad (0.0.25)$$

where the derivative  $\frac{du}{dt}$  has to be understood in the distributional sense *i.e.*

$$\frac{du}{dt}(\varphi) = - \int_I u \frac{d\varphi}{dt}, \quad \forall \varphi \in \mathcal{D}(I).$$

The inclusion  $\frac{du}{dt} \in L^q(I; Y)$  means that there exists a function  $v \in L^q(I; Y)$  such that

$$\frac{du}{dt}(\varphi) = \int_I v \varphi, \quad \forall \varphi \in \mathcal{D}(I).$$

As we have informations on the derivative in time, we expect some embedding into the space of continuous functions.

**Proposition 0.0.5.** *Let  $p, q \geq 1$  and let  $X \subset Y$  continuously. Then  $W^{1,p,q}(I; X, Y) \subset C(I; Y)$  continuously.*

It allows for instance to makes sense of the initial condition of an evolution problem. More precisely writing  $u(0) = u_0$  for a given initial condition has a meaning if we know some informations on  $\frac{du}{dt}$ . In the case  $X = V$  and  $Y = V'$  and  $H$  is a pivot Hilbert space (identified with its dual by the Riesz representation theorem) we can obtain a more precise result :

**Proposition 0.0.6.** *Let  $V \subset H \cong H' \subset V'$  and  $p' = \frac{p}{p-1}$  for some  $p \geq 1$ . then  $W^{1,p,p'}(I; V, V') \subset C(I; H)$  continuously and the following integration by part formula holds for any  $u, v \in W^{1,p,p'}(I; V, V')$  and  $0 \leq t_1 \leq t_2 \leq T$  :*

$$(u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H = \int_{t_1}^{t_2} \left\langle \frac{du}{dt}, v(t) \right\rangle_{V' \times V} + \left\langle u(t), \frac{dv}{dt} \right\rangle_{V' \times V} dt.$$

The previous formula is quite general and one can deduce for instance the classical Lions-Magenes lemma. An application of this result is the following : suppose we are given a function  $u$  solution of a PDE which is in  $L^2(0, T; H^1(\Omega))$  such that  $\partial_t u \in L^2(0, T; H^{-1}(\Omega))$ . Applying Proposition 0.0.5 only gives that  $u$  is  $C([0, T]; H^{-1}(\Omega))$  which is a weak continuity. Proposition 0.0.6 shows that  $u$  is in fact in  $C([0, T]; L^2(\Omega))$  where the continuity is understood for the strong topology of  $L^2$ . Also the integration by parts formula holds for instance for  $u, v = u$ .

## Functional analysis theorems

We start with the classical Sobolev embeddings that allow for the comparison of norms.

## Sobolev embeddings and the Gagliardo-Nirenberg inequality

Let  $\Omega \subset \mathbb{R}^d$  be a domain sufficiently smooth (not necessarily bounded). Let  $k, m \in \mathbb{N}$  and  $1 \leq p, q \leq +\infty$ . Then the following embeddings are continuous :

$$W^{m,p}(\Omega) \hookrightarrow \begin{cases} W^{k,q}(\Omega) & \text{if } m - \frac{d}{p} \geq k - \frac{d}{q}, q \geq p \text{ (unless } m - \frac{d}{p} = k, q = \infty), \\ C^{k+\mu}(\overline{\Omega}) & \text{if } 0 \leq \mu \leq m - k - \frac{d}{p} < 1 \text{ (unless } m - \frac{d}{p} \in \mathbb{N}). \end{cases}$$

Moreover, if  $\Omega$  is bounded the embeddings are compact for strict inequalities *i.e.* respectively  $m - \frac{d}{p} > k - \frac{d}{q}$  and  $m - \frac{d}{p} > k + \mu$ . For more details, we refer to [106]. It is also possible to interpolate these spaces, and we have the following interpolation inequality on the whole space :

**Theorem 0.0.7** (Gagliardo-Nirenberg inequality). *Let  $1 \leq q \leq +\infty$ . Let  $j, m \in \mathbb{N}$  such that  $j < m$ . Let  $1 \leq r \leq +\infty$ ,  $1 \leq p < +\infty$ , and  $\theta \in [0, 1]$  such that the relations*

$$\frac{1}{p} = \frac{j}{n} + \theta \left( \frac{1}{r} - \frac{m}{n} \right) + \frac{1-\theta}{q}, \quad \frac{j}{m} \leq \theta \leq 1$$

hold. Then,

$$\|D^j u\|_{L^p(\mathbb{R}^n)} \leq C \|D^m u\|_{L^r(\mathbb{R}^n)}^\theta \|u\|_{L^q(\mathbb{R}^n)}^{1-\theta}$$

for any  $u \in L^q(\mathbb{R}^n)$  such that  $D^m u \in L^r(\mathbb{R}^n)$ , with two exceptional cases :

1. If  $j = 0$ ,  $q = +\infty$ , and  $rm < n$ , then an additional assumption is needed : either  $u$  tends to 0 at infinity, or  $u \in L^s(\mathbb{R}^n)$  for some finite value of  $s$ .
2. If  $r > 1$  and  $m - j - \frac{n}{r} \in \mathbb{N}$ , then the additional assumption  $\frac{j}{m} \leq \theta < 1$  is needed.

In the case of bounded domains we have

**Theorem 0.0.8** (Gagliardo-Nirenberg on bounded domains). *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain sufficiently smooth. Let  $1 \leq q \leq \infty$ . Let  $j, m \in \mathbb{N}$  such that  $j < m$ . Let  $1 \leq r \leq +\infty$ ,  $1 \leq p < +\infty$  be real, and  $\theta \in [0, 1]$  such that the relations*

$$\frac{1}{p} = \frac{j}{n} + \theta \left( \frac{1}{r} - \frac{m}{n} \right) + \frac{1-\theta}{q}, \quad \frac{j}{m} \leq \theta \leq 1$$

hold. Then,

$$\|D^j u\|_{L^p(\Omega)} \leq C \|D^m u\|_{L^r(\Omega)}^\theta \|u\|_{L^q(\Omega)}^{1-\theta} + C \|u\|_{L^\sigma(\Omega)},$$

where  $u \in L^q(\Omega)$  such that  $D^m u \in L^r(\Omega)$  and  $\sigma$  is arbitrary, with one exceptional case :

1. if  $r > 1$  and  $m - j - \frac{n}{r}$  is a non-negative integer, then the additional assumption  $\frac{j}{m} \leq \theta < 1$  is needed.

## Convergence theorems

We first recall the definition of a uniformly integrable set.

**Definition 0.0.9** (Uniform integrability). Let  $M$  be a set of functions in  $L^1(\Omega; \mathbb{R}^d)$ . We say that  $M$  is uniformly integrable if and only if the two following conditions hold :

1. for any  $\eta > 0$  there exists a measurable set  $E \subset \Omega$  with  $\lambda(E) < +\infty$  such that

$$\sup_{u \in M} \int_{\Omega \setminus E} |u| dx < \eta;$$

2. for any  $\eta > 0$  there exists  $\delta > 0$  such that, for every measurable set  $E \subset \Omega$  with  $\lambda(E) < \delta$ ,

$$\sup_{u \in M} \int_E |u| dx < \eta.$$

Note that the first condition is only important when the domain  $\Omega$  is unbounded.

**Theorem 0.0.10** (Vitali convergence theorem). *Let  $\{u_k\}_{k \in \mathbb{N}} \subset L^1(\Omega)$  be a sequence converging a.e. to some  $u$ . Then the following assertions are equivalent :*

1.  $u \in L^p(\Omega)$  and  $u_k \rightarrow u$  in  $L^p(\Omega)$
2.  $\{|u_k|^p\}_{k \in \mathbb{N}}$  is uniformly integrable.

As a corollary we have the following theorem.

**Theorem 0.0.11** (Generalized Lebesgue dominated convergence theorem). *Let  $\{u_k\}_{k \in \mathbb{N}} \subset L^1(\Omega)$  be a sequence converging a.e. to some  $u$ . Let  $\{v_k\}_{k \in \mathbb{N}}$  be a sequence of non-negative Lebesgue measurable functions defined on  $\Omega$ . Suppose that :*

1.  $|u_k(x)| \leq |v_k(x)|$  for all  $k \in \mathbb{N}$  and for all a.e.  $x \in \Omega$ .
2.  $\{v_k(x)\}_{k \in \mathbb{N}}$  converges a.e. and in  $L^p(\Omega)$  to some  $v(x)$ .

Then  $u \in L^p(\Omega)$  and  $u_k \rightarrow u$  in  $L^p(\Omega)$ .

## Compactness toolbox for general PDEs

Compactness refers to the property of a set of functions that ensures that any sequence in the set contains a convergent subsequence. In general to obtain compactness we need to bound the sequence of functions in some norms. The easiest compactness result can be obtained by considering weak topologies :

**Theorem 0.0.12** (Banach-Alaoglu-Bourbaki). *In a Banach space with a separable predual, any bounded sequence contains a weakly\* convergent subsequence.*

For instance, if a sequence of functions is bounded in  $L^p(\Omega)$ , for  $1 < p \leq +\infty$  we can extract a subsequence that converges in the weak\* topology of  $L^p(\Omega)$ . When  $p < +\infty$  the weak\* and weak topology are equivalent. In the case  $p = 1$  the previous theorem cannot be applied. An easy counter example is the sequence of mollifiers that converge to a dirac. However we can still find compactness if we see the  $L^1(\Omega)$  functions as a sequence of measures and use Prokhorov's theorem. If we know that the functions are just a little bit better than  $L^1(\Omega)$ , we can still find weak compactness in  $L^1(\Omega)$ , this is the point of Dunford-Pettis theorem.

**Theorem 0.0.13** (Dunford-Pettis). *Let  $M \subset L^1(\Omega)$  be bounded. Then  $M$  is relatively weakly compact in  $L^1(\Omega)$  if and only if  $M$  is uniformly integrable.*

For instance, in kinetic equations, if we look at a sequence of functions  $\{u_k\}$ , we often obtain some bounds on  $\{u_k \log(u_k)\}$ . On a bounded domain it is sufficient to apply the previous theorem and find a weakly convergent subsequence in  $L^1$ . On the whole space, we need an additional assumption on the moments, for instance,  $\{|x|u_k\}$  is bounded in  $L^1$  to prove that the sequence is uniformly integrable.

In many cases, especially with nonlinear PDEs, weak convergence is not enough and we have to use strong compactness methods. A characterization for a subset of  $L^p(\mathbb{R}^d)$  to be (strongly) compact is provided by the Fréchet-Kolmogorov compactness theorem. Its proof is usually based on the Arzelà-Ascoli theorem, which answers the same question but in the space of continuous functions over a compact metric space.

**Theorem 0.0.14** (Arzelà-Ascoli). *Let  $(E, d_e)$  be a compact metric space,  $(F, d_f)$  a complete metric space and  $M$  be a subset of  $C(E, F)$ . The following properties are equivalent :*

1.  $M$  is relatively compact (i.e. its closure is compact) in  $(C(E, F), d_\infty)$  with  $d_\infty(u, v) := \sup_{x \in E} \{d_f(u(x), v(x))\}$ .
2.  $M$  is uniformly equicontinuous, i.e.

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that, if } d_e(x_1, x_2) < \delta, \text{ then } d_f(u(x_1), u(x_2)) < \varepsilon \quad \forall u \in M,$$

and  $M_x := \{u(x), u \in M\}$  is relatively compact in  $F$  for all  $x \in E$ .

**Theorem 0.0.15** (Fréchet–Kolmogorov theorem). *Let  $M$  be a bounded set in  $L^p(\Omega)$  with  $1 \leq p < +\infty$ . Then  $M$  is relatively compact if and only if the following properties hold :*

1. (Equicontinuity)

$$\lim_{|h| \rightarrow 0} \|\tau_h u - u\|_{L^p(\Omega)} = 0 \quad \text{uniformly in } u \in M,$$

i.e. for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\|\tau_h u - u\|_{L^p(\Omega)} < \varepsilon$  for every  $u \in M$  and for every  $h \in \Omega$  such that  $|h| < \delta$  ;

2. (Equitightness) For any  $\varepsilon > 0$  there exists a bounded and measurable subset  $E \subset \Omega$  such that  $\|u\|_{L^p(\Omega \setminus E)} < \varepsilon$  for every  $u \in M$ .

When  $\Omega$  is bounded, the equitightness condition is not required. To prove that a sequence satisfies the condition of the previous theorem, we often need some estimates on the derivatives. For instance this can be seen from the compact embeddings of the previous subsection. In fact, this result can be refined : if a nonlocal approximation of the derivatives is bounded, we have compactness.

**Theorem 0.0.16** (Bourgain-Brezis-Mironescu compactness result). *Let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^d$ . Let  $(\rho_k)$  be a smooth sequence of radial mollifiers that are radially non-increasing. Let  $\{u_k\}$  be a sequence of  $L^p(\Omega)$  functions for  $1 \leq p < +\infty$  satisfying the uniform estimate*

$$\int_{\Omega \times \Omega} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^p} \rho_k(|x - y|) dx dy \leq C_0.$$

Then there exists  $u \in L^p(\Omega)$  and a subsequence of  $\{u_k\}$  which converges in  $L^p(\Omega)$  to  $u$ . Moreover

1. If  $1 < p < +\infty$ ,  $u \in W^{1,p}(\Omega)$  and  $\|u\|_{W^{1,p}(\Omega)} \leq C(p, \Omega)C_0$ ,
2. If  $p = 1$ ,  $u \in BV(\Omega)$  and  $\|u\|_{BV(\Omega)} \leq C(\Omega)C_0$ .

One crucial assumption in the proof of this theorem is that the mollifiers are non-increasing. This assumption was later removed by Ponce in [294].

This theorem can be seen as a nonlocal version of the Rellich-Konchadrov theorem. As the Poincaré-Wirtinger inequality is a corollary of Rellich-Konchadrov we can expect a nonlocal Poincaré-Wirtinger inequality as a corollary of the previous result.

**Theorem 0.0.17** (Nonlocal Poincaré-Wirtinger). *There exists  $k_0$  such that for all  $u \in L^p(\Omega)$  and all  $k > k_0$*

$$\int_{\Omega} |u - (u)_\Omega|^p \leq C(p, \Omega) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_k(|x - y|) dx dy,$$

where  $(u)_\Omega = \frac{1}{|\Omega|} \int_{\Omega} u$  is the mass of  $u$ .

All the previous compactness methods mentioned above are mostly concerned with functions of one variable, the 'space variable'. When dealing with functions of space and time, the situation can become more tricky. Of course if we have for instance  $\partial_t u_k \in L^1(0, T; L^1(\Omega))$  and  $\nabla u_k \in L^1(0, T; L^1(\Omega))$  uniformly in  $k$ , we can apply the previous theorems to conclude compactness as here the time and space play the 'same' role. But in many cases, the PDE is of the form

$$\partial_t u = \operatorname{div}(A[u]),$$

where  $A[u]$  has at most range in some  $L^p$  spaces. Then we see that  $\partial_t u(t)$  takes values in the negative sobolev space  $W^{-1,p}(\Omega)$ . In those cases we can use the fundamental result of Aubin-Lions-Simon :

**Theorem 0.0.18** (Aubin-Lions-Simon). *Let  $X_0 \subset X \subset X_1$  be three Banach spaces. We assume that the embedding of  $X_0$  in  $X$  is compact and the embedding of  $X$  in  $X_1$  is continuous. For  $1 \leq p, q \leq +\infty$ , we have with the definition 0.0.25 :*

1. *If  $p < +\infty$ ,  $W^{1,p,q}((0, T); X_0, X_1)$  is compactly embedded in  $L^p(0, T; X)$ ,*
2. *If  $p = +\infty$  and if  $q > 1$ ,  $W^{1,p,q}((0, T); X_0, X_1)$  is compactly embedded in  $C([0, T], X)$ .*

This theorem shows that if we have some compactness in space, here represented by the compact embedding of  $X_0$  in  $X$ , and if we know that the derivative is in some  $L^p$  space in time, we can deduce compactness. In some situations, it may also happen that we know that a sequence is compact in space, however not necessarily bounded in a compact space, and we can prove a similar result that we state here in the case  $p = 2$ .

**Lemma 0.0.19.** *Let  $\{\varphi_\delta\}_\delta$  be the standard sequence of mollifiers. Suppose that  $\{f_\varepsilon\}$  is a sequence bounded in  $L^2((0, T) \times \mathbb{T}^d)$  such that*

1.  *$\partial_t f_\varepsilon = \nabla^k(J_\varepsilon)$ , where  $\nabla^k$  is any linear differential operator of order  $k \in \mathbb{N}$  and  $\{J_\varepsilon\}$  uniformly bounded in  $L^1((0, T) \times \mathbb{T}^d)$ ,*
2.  *$\{f_\varepsilon\}$  is compact in space in  $L^2((0, T) \times \mathbb{T}^d)$ , i.e.*

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{T}^d} |f_\varepsilon * \varphi_\delta(t, x) - f_\varepsilon(t, x)|^2 dx dt = 0. \quad (0.0.26)$$

*uniformly for all  $\varepsilon$ .*

*Then,  $\{f_\varepsilon\}$  is compact in time in  $L^2((0, T) \times \mathbb{T}^d)$ , i.e.*

$$\lim_{h \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t, x)|^2 dx dt \rightarrow 0 \text{ as } h \rightarrow 0 \quad (0.0.27)$$

*and so, it is compact in  $L^2((0, T) \times \mathbb{T}^d)$ .*

As the proof is quite interesting as it can be adapted in many situations, we detail it here. We recall that

$$\|\nabla^k \varphi_\delta\|_{L^p(\mathbb{T}^d)} \leq \frac{C}{\delta^{k+d-d/p}},$$

and for any function  $g \in L^p(\mathbb{T}^d)$ ,

$$\|g * \varphi_\delta\|_{L^p(\mathbb{T}^d)} \leq \|\varphi_\delta\|_{L^p(\mathbb{T}^d)} \|g\|_{L^1(\mathbb{T}^d)}.$$

*Démonstration.* Using the mollifiers with  $\delta = \delta(h)$  depending on  $h$  to be specified later in the way that  $\delta(h) \rightarrow 0$  as  $h \rightarrow 0$ , we first split

$$\begin{aligned} \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t, x)|^2 dx dt &\leq 4 \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t, x) - f_\varepsilon(t, \cdot) * \varphi_\delta(x)|^2 dx dt \\ &\quad + 4 \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t+h, \cdot) * \varphi_\delta(x)|^2 dx dt \\ &\quad + 4 \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, \cdot) * \varphi_\delta(x) - f_\varepsilon(t, \cdot) * \varphi_\delta(x)|^2 dx dt. \end{aligned}$$

When we apply limit  $\lim_{h \rightarrow 0} \limsup_{\varepsilon \rightarrow 0}$ , the first and second term vanish due to (0.0.26). It remains to study the third term which reads

$$\begin{aligned} \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, \cdot) * \varphi_\delta(x) - f_\varepsilon(t, \cdot) * \varphi_\delta(x)|^2 dx dt &= \\ &= \int_0^{T-h} \int_{\mathbb{T}^d} \left| \int_t^{t+h} \partial_t f_\varepsilon(s, \cdot) * \varphi_\delta(x) ds \right|^2 dx dt = \int_0^{T-h} \int_{\mathbb{T}^d} \left| \int_t^{t+h} J * \nabla^k \varphi_\delta(s, x) ds \right|^2 dx dt \\ &\leq Ch \int_0^{T-h} \int_{\mathbb{T}^d} \int_t^{t+h} |J * \nabla^k \varphi_\delta(s, x)|^2 ds dx dt, \end{aligned}$$

where we used Jensen's inequality. We perform the change of variables  $s \mapsto v = \frac{s-t}{h}$ , use Fubini's theorem, and obtain

$$h \int_0^{T-h} \int_{\mathbb{T}^d} \int_t^{t+h} |J * \nabla^k \varphi_\delta(s, x)|^2 ds dx dt = h^2 \int_0^1 \int_{\mathbb{T}^d} \int_0^{T-h} |J * \nabla^k \varphi_\delta(vh + t, x)|^2 dt dx dv.$$

Then we use the change of variables  $t \mapsto \tau = vh + t$  and obtain

$$\begin{aligned} h^2 \int_0^1 \int_{\mathbb{T}^d} \int_0^{T-h} |J * \nabla^k \varphi_\delta(vh + t, x)|^2 dt dx dv &= \\ &= h^2 \int_0^1 \int_{\mathbb{T}^d} \int_{vh}^{T+h(v-1)} |J_i * \nabla^k \varphi_\delta(\tau, x)|^2 d\tau dx dv \leq \frac{h^2}{\delta^{2k+d}} \|J_\varepsilon\|_{L^1_{t,x}}^2. \end{aligned}$$

Using the  $L^1((0, T) \times \mathbb{T}^d)$  bound on  $\{J_\varepsilon\}$  and choosing  $\delta$  such that  $\delta^{2k+d} = h$  we conclude that

$$\lim_{h \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t, x)|^2 dx dt \leq \theta(h).$$

Combined with the compactness in space (0.0.27) and the Fréchet-Kolmogorov theorem we obtain the compactness of  $\{f_\varepsilon\}$  in  $L^2((0, T) \times \mathbb{T}^d)$ .  $\square$

*Remark 0.0.20.* Compared with the usual version of the Fréchet-Kolmogorov theorem, one would expect that the condition for compactness in space should read

$$\lim_{y \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{T}^d} |f_\varepsilon(t, x+y) - f_\varepsilon(t, x)|^2 dx dt = 0. \quad (0.0.28)$$

However, by a careful inspection of the proof, (0.0.26) is sufficient and in fact, in the proof one



deduces (0.0.26) from (0.0.28).

When one deals with the convergence of a product of a sequence  $u_k v_k$  it is sufficient to prove that one of them converges weakly and the other one converges strongly in the dual space to prove that (at least in the distributional sense)  $u_k v_k \rightarrow uv$ . With the 'compensated compactness' approach in the space time setting is enough to require that one sequence is compact in time and the other is compact in space. Therefore, even if both sequence oscillates and do not converge strongly, as long as they oscillate in a compatible way we can conclude. We formulate this theorem in the very general setting from [280].

**Theorem 0.0.21** (Space-time compensated compactness). *Let  $1 \leq q \leq +\infty$ ,  $1 \leq p \leq d$  and  $\alpha < p'$ . Suppose that*

1.  $\{u_k\}$  is uniformly bounded in  $L^q(0, T; W^{1,p}(\Omega))$ ,
2.  $\{v_k\}$  is uniformly bounded in  $L^{q'}(0, T; L^{\alpha'}(\Omega))$ ,
3.  $\{\partial_t v_k\}$  is uniformly bounded in  $\mathcal{M}(0, T; H^{-m}(\Omega))$  for some  $m \in \mathbb{N}$ ,

and that they converge respectively weakly or weakly\* to  $u$  and  $v$ . Then  $u_k v_k \rightarrow uv$  in the distributional sense.

The following two sections are also compactness arguments. However as they are very specific to some situations (kinetic equations and measure valued solutions respectively), we separate them from the rest.

## Velocity averaging lemmas

The kinetic equations are generally of the form

$$\partial_t f + v \cdot \nabla_x f = S,$$

where  $f = f(t, x, v)$  is a function describing the probability of finding a particle at a time  $t$ , position  $x$  and velocity  $v$ . The transport operator being hyperbolic, we do not expect in general to gain any regularity or compactness for the solutions. However this property can be obtained for the moments of  $f$ , this is the so-called 'averaging lemma'. We state it in the  $L^2$  framework but more general cases [194, 286] have been proven.

**Theorem 0.0.22** (Averaging Lemma). *Let  $f \in L^2(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_v^d)$  be a solution of the free transport above. Then for all compactly supported test functions  $\varphi \in C_0^\infty(\mathbb{R}^d)$ ,*

$$\left\| \int_{\mathbb{R}^d} f \varphi(v) \, dv \right\|_{L^2(\mathbb{R}, H^{1/2}(\mathbb{R}^d))} \leq C (\|f\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_v^d)} + \|S\|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_v^d)}).$$

As a consequence we obtain compactness in space for the moments, since we are able to have informations on the  $\partial_x^{1/2}$  derivative. A version of the velocity averaging lemma that will be used in this manuscript is the following from [269, Lemma 4.2].

**Lemma 0.0.23.** *Assume that  $\{f^\varepsilon\}$  is bounded in  $L^2(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_v^d)$ ,  $\{f_0^\varepsilon\}$  and  $\{f_1^\varepsilon\}$  are bounded in  $L^1(\mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_v^d)$ . Moreover, suppose that*

$$\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = f_0^\varepsilon + \nabla_v \cdot f_1^\varepsilon.$$

Then, for all  $\varphi \in C_0^\infty(\mathbb{R}^d)$ ,

$$\left\| \int_{\mathbb{R}^d} (f^\varepsilon(t, x + y, v) - f^\varepsilon(t, x, v)) \varphi(v) \, dv \right\|_{L^1_{t,x}} \rightarrow 0,$$

when  $y \rightarrow 0$  uniformly in  $\varepsilon$ .

Therefore we can obtain compactness in space. It is not possible to obtain compactness in time here with the velocity averaging lemma as there is an  $\varepsilon$  factor in front of the time derivative.

## Young measures

We've seen in the previous sections that we need strong compactness to identify the limit on nonlinear terms. If  $u_k$  converges weakly to  $u$  and  $u_k^2$  converges weakly it is not true that its weak limit is  $u^2$ . A simple example is the sequence  $\sin(kx)$ , where the weak limits of  $u_k$  and  $u_k^2$  is respectively 0 and 1/2. However we can still describe the weak limit of  $u_k^2$  in terms of  $u$ , this is one of the purposes of the Young measures. The idea of Young measures is to embed the problem in a larger space and gain linearity. We write  $f(u_k(y)) = \langle f, \delta_{u_k(y)} \rangle$ , and if  $f \in C_0(\mathbb{R}^d)$ , using the duality  $(L^1(Q; C_0(\mathbb{R}^d)))^* = L_w^\infty(Q; \mathcal{M}(\mathbb{R}^d))$ , Banach-Alaoglu theorem and weak-star continuity of linear operators, we can pass to the limit. The same is true if  $\{f(u_k)\}$  is weakly compact in  $L^1(Q)$  and  $\{u_k\}$  does not grow too fast as the following theorem states :

**Theorem 0.0.24** (Fundamental Theorem of Young Measures). *Let  $Q \subset \mathbb{R}^d$  be a measurable set and let  $u_k : Q \rightarrow \mathbb{R}^d$  be measurable functions such that*

$$\sup_{k \in \mathbb{N}} \int_Q f(|u_k(y)|) \, dy < +\infty$$

for some continuous, nondecreasing function  $f : [0, +\infty) \rightarrow [0, +\infty)$  with  $\lim_{t \rightarrow +\infty} f(t) = +\infty$ . Then, there exists a subsequence (not relabeled) and a weakly star measurable family of probability measures  $\nu = \{\nu_y\}_{y \in Q}$  with the property that whenever the sequence  $\{\psi(y, u_k(y))\}_{k \in \mathbb{N}}$  is weakly compact in  $L^1(Q)$  for a Carathéodory function (measurable in the first and continuous in the second argument)  $\psi : Q \times \mathbb{R}^d \rightarrow \mathbb{R}$ , we have

$$\psi(y, u_k(y)) \rightharpoonup \int_{\mathbb{R}^n} \psi(y, \lambda) \, d\nu_y(\lambda) \quad \text{in } L^1(Q).$$

We say that the sequence  $\{u_k\}_{k \in \mathbb{N}}$  generates the sequence of Young measures  $\{\nu_y\}_{y \in Q}$ .

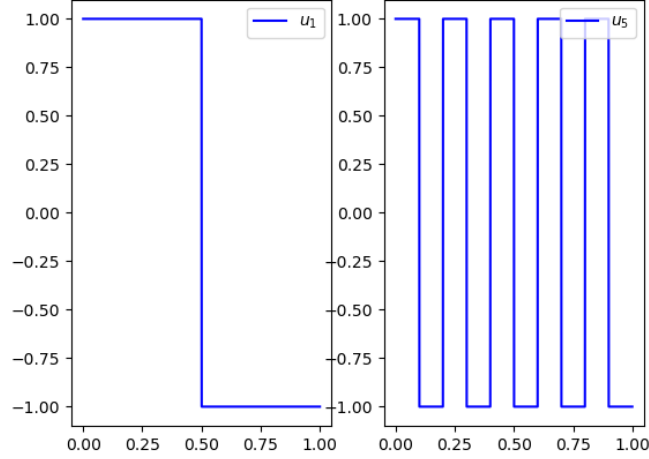
**Example 0.0.25** ([297]). *Consider the oscillating sequence for  $x \in (0, 1)$*

$$u_k(x) := \begin{cases} 1 & \text{if } kx - [kx] \in [0, 1/2), \\ -1 & \text{if } kx - [kx] \in [1/2, 1). \end{cases}$$

Then  $u_k \rightharpoonup 1/2 * 1 + 1/2 * (-1) = 0$  and the associated family of Young measures is

$$\nu_x = \frac{1}{2} \delta_1 + \frac{1}{2} \delta_{-1}.$$

$\{\nu_x\}_x$  reflects the asymptotic distribution of values in the sequence  $u_k$  which is in this case -1 and 1 with equal probability 1/2.

FIGURE 8 – Plot of  $u_1$  and  $u_5$ .

Note that in the above theorem, we require that the sequence  $\{\psi(y, u_k(y))\}$  is weakly compact in  $L^1(Q)$ . This prevents the concentration effect to appear (think about the family of standard mollifiers). When we do not have weak compactness, we use the following proposition which follows from the Banach-Alaoglu theorem. We formulate it with a distinguishment between time and space variables (that is,  $Q = (0, T) \times \Omega$ ,  $y = (t, x)$  with  $t \in (0, T)$  and  $x \in \Omega$ ) as usually in applications one has better integrability in time which results in better characterization of the resulting measure. The following proposition is a consequence of the Banach-Alaoglu theorem and the Radon-Nikodym theorem, see [55].

**Proposition 0.0.26.** *Let  $f$  be a continuous function and a sequence  $\{f(t, x, u_k(t, x))\}_{k \in \mathbb{N}}$  be bounded in  $L^p(0, T; L^1(\Omega))$  with  $p \geq 1$ . Let  $\{\nu_{t,x}\}_{t,x}$  be the Young measure generated by  $\{u_k\}_k$ . Then there exists a measure  $m^f$  such that (up to a subsequence not relabelled)*

$$f(t, x, u_k(t, x)) - \langle \nu_{t,x}, f \rangle \xrightarrow{*} m^f \quad \text{in } L^p(0, T; \mathcal{M}(\Omega)) \text{ if } p > 1,$$

$$f(t, x, u_k(t, x)) - \langle \nu_{t,x}, f \rangle \xrightarrow{*} m^f \quad \text{in } \mathcal{M}((0, T) \times \Omega) \text{ if } p = 1.$$

Moreover, if  $p > 1$ , the measure  $m^f$  is absolutely continuous with respect to time : for a.e.  $t \in (0, T)$ , there exists measure  $m^f(t, \cdot)$  such that

$$\int_{(0,T) \times \Omega} \psi(t, x) dm^f(t, x) = \int_0^T \int_{\Omega} \psi(t, x) m^f(t, dx) dt.$$

Let us remark that by the fundamental theorem, we have  $m^f = 0$  when the sequence  $\{f(u_k)\}_{k \in \mathbb{N}}$  is weakly compact in  $L^1((0, T) \times \Omega)$ . We use the notation :

$$\bar{f} = \langle f(\lambda), \nu_{t,x} \rangle + m^f \tag{0.0.29}$$

to represent weak limit of  $f(t, x, u_k(t, x))$ . We also have the following result which allows comparing two concentration measures  $m^{f_1}$  and  $m^{f_2}$  for two different nonlinearities  $f_1, f_2$ . For the

proof, we refer to [167, Lemma 2.1].

**Proposition 0.0.27.** *Let  $\{\nu_{t,x}\}_{(t,x) \in (0,T) \times \Omega}$  be a Young measure generated by a sequence  $\{u_k\}_{k \in \mathbb{N}}$ . If two continuous functions  $f_1 : (0,T) \times \Omega \rightarrow \mathbb{R}^d$  and  $f_2 : (0,T) \times \Omega \rightarrow \mathbb{R}^+$  satisfy  $|f_1(z)| \leq f_2(z)$  for every  $z$ , and if  $\{f_2(u_k)\}$  is uniformly bounded in  $L^1((0,T) \times \Omega)$ , then we have*

$$|m^{f_1}(A)| \leq m^{f_2}(A),$$

for any borel set  $A \subset (0,T) \times \Omega$ .

Here,  $|\mu|$  is the total variation measure defined as  $|\mu|(A) = \mu^+(A) + \mu^-(A)$  where  $\mu^+$ ,  $\mu^-$  are positive and negative parts of  $\mu$ .

Let us conclude with a few comments about the measure  $m^f$  which captures concentration effects. One can describe it more precisely. The first attempts to do so by some generalizations of the Young measures were initiated by DiPerna and Majda in the case of the incompressible Euler equations [139]. Then, Alibert and Bouchitté extended the result to a more general class of nonlinearities in [15]. They proved that there exists a subsequence (not relabeled) as well as a parametrized probability measure  $\nu \in L_w^\infty(Q; \mathcal{P}(\mathbb{R}^n))$  (which is identical with the "classical" Young measure), a non-negative measure  $m \in \mathcal{M}^+(Q)$ , and a parametrized probability measure  $\nu^\infty \in L_w^\infty(Q, m; \mathcal{P}(\mathbb{S}^{n-1}))$  such that for any Carathéodory function  $f$  such that  $f(x, z)/(1 + |z|)$  is bounded and uniformly continuous with respect to  $z$ ,

$$f(y, u_k(y)) \xrightarrow{*} \int_{\mathbb{R}^d} f(y, \lambda) d\nu_y(\lambda) + \int_{\mathbb{S}^{n-1}} f^\infty(y, \beta) d\nu_y^\infty(\beta) m(y)$$

weakly\* in the sense of measures. Here,

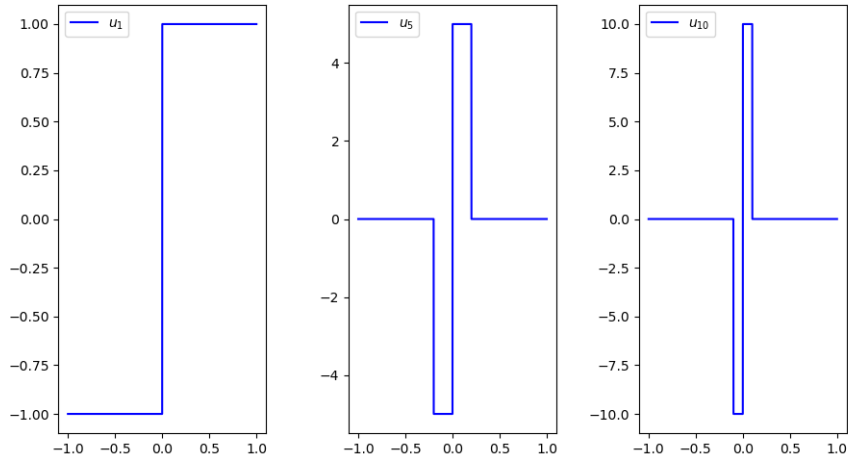
$$f^\infty(y, \beta) := \lim_{s \rightarrow \infty} \frac{f(y, t\beta)}{t}.$$

Their result was also extended to the case when  $f$  has different growth with respect to different variables, see for instance [207].

**Example 0.0.28** ([297]). *Consider the concentrating sequence at 0 for  $x \in (-1, 1)$   $u_k(x) := k(\mathbf{1}_{(0,1/k)} - \mathbf{1}_{(-1/k,0)})$ . Then we have*

$$\nu_x = \delta_0, \quad m(y) = 2\delta_0, \quad \nu_0^\infty = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}.$$

*We get the information that the sequence concentrates around 0 in the direction 1 and -1 with the same probability.*

FIGURE 9 – Plot of  $u_1, u_5, u_{10}$ .

## Gradient flows and the JKO scheme

Gradient flows : The Cahn-Hilliard equation is a gradient flow. Let's try to understand this statement. First of all, a gradient flow system in its more general form can be written as

$$x'(t) = -\nabla F(x(t)).$$

Here  $x : t \rightarrow X$  where  $X$  is a linear space and  $F : X \rightarrow \mathbb{R}$ . Therefore  $x$  evolves in the direction where  $F$  decreases the most. Here  $F$  is a functional and we have to specify for which metric we consider its gradient.

**Example 0.0.29.** *The heat equation*

$$\partial_t u = \Delta u$$

is a gradient flow for the energy  $F(u) = \frac{1}{2} \int |\nabla u|^2$  for the metric defined by the  $L^2$  scalar product. It is also a gradient flow of the energy  $F(u) = \frac{1}{2} \int u^2$  for the metric defined by the  $H^{-1}$  scalar product.

This setting is actually enough to understand that the Cahn-Hilliard equation

$$\partial_t u = \Delta \mu, \quad \mu = -\Delta u + W'(u)$$

is an  $H^{-1}$  gradient flow of the energy functional

$$F(u) = \int \frac{1}{2} |\nabla u|^2 + W(u). \quad (0.0.30)$$

More tricky to understand is the Cahn-Hilliard equation with degenerate mobility :

$$\partial_t u = \operatorname{div}(u \nabla \mu), \quad \mu = -\Delta u + W'(u). \quad (0.0.31)$$

We define the second Wasserstein distance as

$$W_2^2(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\Omega \times \Omega} d(x, y)^2 d\pi(x, y), \quad \mu, \nu \in \mathcal{P}_2(\Omega),$$

where  $\mathcal{P}_2(\Omega)$  is the space of probability measures with finite second moments, and  $\Pi(\mu, \nu)$  is the set of transport plans with first marginals  $\mu$  and second marginals  $\nu$ . More precisely

$$\Pi(\mu, \nu) := \left\{ \pi \in \mathcal{P}(\Omega \times \Omega), \int \int \psi(x) + \phi(y) d\pi = \int \psi d\mu + \int \phi d\nu \right\}.$$

With this definition in mind, it can be justified with optimal transport theory, and more precisely the Benamou-Brenier theorem below and Otto calculus that we can make sense of the object  $\nabla_{W_2} F$  and

$$\nabla_{W_2} F(u) = -\operatorname{div} \left( u \nabla \frac{\delta F}{\delta u} \right), \quad (0.0.32)$$

where  $\frac{\delta F}{\delta u}$  is the gradient of  $F$  in the  $L^2$  metric.

**Theorem 0.0.30** (Benamou-Brenier).

$$W_2^2(\mu, \nu) = \inf \left\{ \int_0^1 \int |\nabla \phi_s|^2 d\mu_s ds, \mu_0 = \mu, \mu_1 = \nu, \frac{d\mu_s}{ds} = -\operatorname{div}(\mu_s \nabla \phi_s) \right\}.$$

Let us try to understand Equation (0.0.32). The idea of Otto is to use the Benamou-Brenier theorem to define a metric structure  $\langle \cdot, \cdot \rangle_u$  on each tangent space  $T_\mu \mathcal{P}$  tangent to  $\mathcal{P}(\Omega)$  at  $\mu$ . This metric should satisfy (as in the case of the geodesic formula for Riemannian geometry)

$$W_2^2(\mu, \nu) = \inf \left\{ \int_0^1 \left\| \frac{d\mu_s}{ds} \right\|_{\mu_s}^2 ds, \mu_0 = \mu, \mu_1 = \nu \right\}.$$

A natural guess is to take for two tangent vectors at  $\mu : \frac{d\mu_{t_1}}{dt}$  and  $\frac{d\mu_{t_2}}{dt}$  the scalar product

$$\left\langle \frac{d\mu_{t_1}}{dt}, \frac{d\mu_{t_2}}{dt} \right\rangle_\mu = \int \nabla \phi_1 \cdot \nabla \phi_2 d\mu \quad (0.0.33)$$

where  $\phi_1$  and  $\phi_2$  solve

$$-\operatorname{div}(\mu \nabla \phi_i) = \frac{d\mu_{t_i}}{dt}, \quad i = 1, 2.$$

Then for a functional  $\mathcal{F} : \mathcal{P}(\Omega) \rightarrow \mathbb{R}$  and  $(\mu_t)_t$  a path we have

$$\frac{d}{dt} \mathcal{F}(\mu_t) = \left\langle \nabla_{\mu_t} \mathcal{F}, \frac{d\mu_t}{dt} \right\rangle_{\mu_t}.$$

Using the fact that

$$\frac{d}{dt} \mathcal{F}(\mu_t) = \left\langle \frac{\delta \mathcal{F}}{\delta \mu_t}, \frac{d\mu_t}{dt} \right\rangle_{L^2}$$

and formula (0.0.33), we deduce formula (0.0.32). Note that the notation  $\nabla_{W_2}$  in (0.0.32) is a renaming of  $\nabla_{\mu_t}$  to emphasize the metric structure of the Wasserstein distance. This shows that the Cahn-Hilliard equation with degenerate mobility (0.0.31) is a gradient flow for the Wasserstein metric of the functional (0.0.30).

Existence theory for gradient flows : From the theory of gradient flows, we can deduce existence of solutions to some PDE. For instance in the case of an ODE

$$x'(t) = -\nabla F(x(t)),$$

we can construct a sequence of points

$$x_{k+1} = \operatorname{argmin}_x \left\{ F(x) + \frac{|x - x_k|^2}{2\tau} \right\},$$

provided we have some assumptions of  $F$  like lower semi-continuity. By definition of  $x_{k+1}$  as a minimum we obtain

$$\nabla F(x_{k+1}) + \frac{x_{k+1} - x_k}{\tau} = 0$$

which is a discretization of the ODE. We can define a linear interpolation  $x_\tau$  of the sequence  $x_k$  and the goal is to send  $\tau \rightarrow 0$ . For Wasserstein gradient flows, the idea is similar and the scheme, first introduced in [234] and called the JKO scheme, becomes

$$x_{k+1} = \operatorname{argmin}_x \left\{ F(x) + \frac{\mathcal{W}_2^2(x, x_k)}{2\tau} \right\}.$$

In the case of a PDE, to send  $\tau \rightarrow 0$  we require compactness in time and space. We recall that by definition,  $x_{k+1}$  is bounded since it is a minimum and therefore we can expect  $F(x_\tau) \leq C$ ,  $\mathcal{W}_2^2(x_\tau, x_s) \leq C|\tau - s|$ . The first bound usually gives informations on compactness in space and the second one compactness in time. Of course it may happen that another functional gives informations on the equation, this is exactly the case of the Cahn-Hilliard equation with degenerate mobility : it is a gradient flow in the Wasserstein metric of the energy, but it is mainly the entropy that provides estimates. It is important to note that this information is not lost in the JKO scheme and one can use the so-called 'flow interchange lemma' which states that *the dissipation of one functional along the gradient flow of another functional equals the dissipation of the second functional along the gradient flow of the first one.*

Première partie

The incompressible limit of the  
Cahn-Hilliard equation





# Chapitre 1

## Degenerate Cahn-Hilliard and incompressible limit of a Keller-Segel model

### Abstract

The Keller-Segel model is a well-known system representing chemotaxis in living organisms. We study the convergence of a generalized nonlinear variant of the Keller-Segel to the degenerate Cahn-Hilliard system. This analysis is made possible from the observation that the Keller-Segel system is equivalent to a relaxed version of the Cahn-Hilliard system. Furthermore, this latter equivalent system has an interesting application in the modelling of living tissues. Indeed, compressible and incompressible porous medium type equations are widely used to describe the mechanical properties of living tissues. The relaxed degenerate Cahn-Hilliard system, can be viewed as a compressible living tissue model for which the movement is driven by Darcy's law and takes into account the effects of the viscosity as well as surface tension at the surface of the tissue. We study the convergence of the Keller-Segel system to the Cahn-Hilliard equation and some of the analytical properties of the model such as the incompressible limit of our model. Our analysis relies on a priori estimates, compactness properties, and on the equivalence between the Keller-Segel system and the relaxed degenerate Cahn-Hilliard system.

### 1.1 Introduction

When describing living tissue dynamics, the most relevant Hele-Shaw free boundary models come with a surface tension term, [161, 177]. In the regime of a smooth free boundary, these problems are well established and obtained as the sharp-interface limits of Cahn-Hilliard equations [17]. However, it is an open question to prove such a derivation in the global regime of weak solutions. Our goals are to establish the incompressible limit departing from the Relaxed Cahn-Hilliard system (RCH in short), and to make the link with a specific Generalized Keller-Segel (GKS)

type system, namely

$$\partial_t n - \frac{\delta}{2\sigma} \Delta n^2 + \frac{\delta}{\sigma} \operatorname{div}(n \nabla w) = n G \left( \frac{\delta}{\sigma} (n - w) \right), \quad \text{in } (0, +\infty) \times \Omega, \quad (1.1.1)$$

$$-\sigma \Delta w + \frac{\sigma}{\delta} w^\gamma + w = n, \quad \text{in } (0, +\infty) \times \Omega, \quad (1.1.2)$$

where  $\Omega \subset \mathbb{R}^d$  is a smooth open bounded domain. We are going to prove that as  $\sigma$  vanishes, its limit is the Degenerate Cahn-Hilliard (DCH) model

$$\partial_t n - \operatorname{div}(n \nabla \mu) = n G(\mu), \quad \text{in } (0, +\infty) \times \Omega, \quad (1.1.3)$$

$$\mu = n^\gamma - \delta \Delta n, \quad \text{in } (0, +\infty) \times \Omega, \quad (1.1.4)$$

where  $\mu$  is formed of a repulsion potential  $n^\gamma$  and a surface tension potential  $-\delta \Delta n$  (see further explanations below). In this work,  $n^\gamma$  actually stands for  $\max(0, n)^\gamma$  but since the solutions are nonnegative, we keep the notation  $n^\gamma$  for sake of clarity. The growth (or source) term  $G(\mu)$  takes into account death and birth of cells.

It is crucial for the analysis of the convergence of the GKS system to the DCH equation to remark that the nonlinear system (1.1.1)–(1.1.2) is equivalent to a relaxed version of the Cahn-Hilliard model (see [287]). Indeed, defining

$$\mu = \frac{\delta}{\sigma} (n - w), \quad w = n - \frac{\sigma}{\delta} \mu, \quad p = w^\gamma, \quad \mu = p - \delta \Delta w, \quad (1.1.5)$$

System (1.1.1)–(1.1.2) can be rewritten in the form

$$\partial_t n - \operatorname{div}(n \nabla \mu) = n G(\mu), \quad \text{in } (0, +\infty) \times \Omega, \quad (1.1.6)$$

$$-\sigma \Delta \mu + \mu = \left( n - \frac{\sigma}{\delta} \mu \right)^\gamma - \delta \Delta n, \quad \text{in } (0, +\infty) \times \Omega. \quad (1.1.7)$$

Although they are equivalent, Systems (1.1.1)–(1.1.2) and (1.1.6)–(1.1.7) carry different informations. For instance, since we consider non-negative densities  $n \geq 0$ , we also have

$$w \geq 0, \quad \mu \leq \frac{\delta}{\sigma} n.$$

The Keller-Segel formulation of the system also provides higher regularity of the solutions. In the following, we combine analytical methods better adapted to each of these two equivalent formulations in order to pass to the incompressible limit  $\gamma \rightarrow \infty$  and obtain

$$\partial_t n_{\sigma, \infty} - \operatorname{div}(n_{\sigma, \infty} \nabla \mu_{\sigma, \infty}) = n_{\sigma, \infty} G(\mu_{\sigma, \infty}), \quad (1.1.8)$$

$$\begin{cases} \mu_{\sigma, \infty} = p_{\sigma, \infty} - \delta \Delta w_{\sigma, \infty}, \\ -\sigma \Delta w_{\sigma, \infty} + \frac{\sigma}{\delta} p_{\sigma, \infty} + w_{\sigma, \infty} = n_{\sigma, \infty}, \quad w_{\sigma, \infty} = n_{\sigma, \infty} - \frac{\sigma}{\delta} \mu_{\sigma, \infty}. \end{cases} \quad (1.1.9)$$

In this 'stiff pressure limit', the system (1.1.8)–(1.1.9) has three unknowns  $n_{\sigma, \infty}$ ,  $w_{\sigma, \infty}$  (or  $\mu_{\sigma, \infty}$ ) and  $p_{\sigma, \infty}$  and is completed with a type of incompressibility condition

$$p_{\sigma, \infty} (w_{\sigma, \infty} - 1) = 0. \quad (1.1.10)$$

For these systems, we use the zero mass and energy flux boundary conditions

$$n \frac{\partial \mu}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad (0, \infty) \times \partial \Omega, \quad (1.1.11)$$

where  $\nu$  is the outward normal vector to the boundary of  $\Omega$ , and the initial condition

$$n(0, \cdot) = n^0 \in H^1(\Omega) \cap L^\infty(\Omega), \quad \text{and} \quad 0 \leq n^0 \leq 1 \quad \text{a.e.} \quad (1.1.12)$$

We also need assumptions on the pressure-dependent proliferation rate of the cells

$$G \in C(\mathbb{R}; \mathbb{R}), \quad \sup_{\mu \in \mathbb{R}} (1 + |\mu|) |G(\mu)| < +\infty. \quad (1.1.13)$$

For instance we can suppose that there exists  $\mu_H$  such that  $G(\mu) = 0$  for  $|\mu| > \mu_H$ . In that case, the value  $\mu_H$  can be viewed as the *homeostatic pressure* which is the lowest level of pressure that prevents cell multiplication due to contact-inhibition. For the pressure law exponent, we assume

$$\gamma > 1. \quad (1.1.14)$$

Using assumptions (1.1.13) the total mass of the system is bounded. More precisely, we find a constant  $C_T$  such that for all  $t \in (0, T)$  and for all  $\sigma, \gamma$ , it holds for all  $T > 0$

$$\frac{1}{|\Omega|} \int_{\Omega} n_{\sigma, \gamma}(t, \cdot) \leq C_T, \quad \frac{1}{|\Omega|} \int_{\Omega} w_{\sigma, \gamma}(t, \cdot) \leq C_T. \quad (1.1.15)$$

This allows us to use the Poincaré-Wirtinger inequality.

System (1.1.6)–(1.1.7) comes with the energy and entropy respectively defined by

$$\mathcal{E}[n, \mu] = \int_{\Omega} \frac{(n - \frac{\sigma}{\delta} \mu)^{\gamma+1}}{\gamma+1} + \frac{\delta}{2} \left| \nabla \left( n - \frac{\sigma}{\delta} \mu \right) \right|^2 + \frac{\sigma}{\delta} \frac{|\mu|^2}{2},$$

$$\Phi[n] = \int_{\Omega} n \log n.$$

They formally satisfy, as already used in [287], the relations

$$\frac{d}{dt} \mathcal{E}[n, \mu] = - \int_{\Omega} n |\nabla \mu|^2 + \int_{\Omega} n \mu G(\mu),$$

$$\begin{aligned} \frac{d}{dt} \Phi[n] = & - \int_{\Omega} \delta \left| \Delta \left( n - \frac{\sigma}{\delta} \mu \right) \right|^2 + \frac{\sigma}{\delta} |\nabla \mu|^2 \\ & + \gamma \left( n - \frac{\sigma}{\delta} \mu \right)^{\gamma-1} \left| \nabla \left( n - \frac{\sigma}{\delta} \mu \right) \right|^2 + \int_{\Omega} n G(\mu) (\log(n) + 1). \end{aligned}$$

From (1.1.5), they also provide informations on  $w$ . The purpose of this work is to establish estimates which allow us to study the convergence of the nonlinear GKS model to the DCH model, and to analyze the incompressible limit  $\gamma \rightarrow \infty$ .

**Modelling of living tissues and assumptions** Our study lies among other models which represent biological phenomena that we present here.

System (1.1.6)–(1.1.7) models the movement and proliferation of a population of cells constituting a biological tissue and driven by the combined effects of the pressure, the surface tension occurring

at the surface of the tissue, as well as its viscosity. As in the context of the modelling of diphasic fluid, in System (1.1.6)–(1.1.7),  $n$  is the order parameter, *i.e.* the relative cell density  $n = n_1/(n_1 + n_2)$ . The unknown  $\mu$  is a quantity related to the effective pressure and is also used to relax the fourth order Cahn-Hilliard type equation in a system of two-second order equations. Following the Cahn-Hilliard terminology or Mechanobiology, we refer to  $\mu$  indistinctly as the chemical potential or as the effective pressure. The equation for the effective pressure (*i.e.* Equation (1.1.7)) contains the effects of both the pressure, through the term  $(n - \frac{\sigma}{\delta}\mu)^\gamma$  with  $\gamma > 1$  that controls the stiffness of the pressure law, and surface tension by  $-\delta\Delta n$ , where  $\sqrt{\delta}$  is the width of the interface in which partial mixing of the two components  $n_1, n_2$  occurs. Equation (1.1.7) also contains a diffusive term  $-\sigma\Delta\mu$  used to relax the Cahn-Hilliard system (1.1.6)–(1.1.7) as in [287]. This relaxation term can also be interpreted as the effect of viscosity.

Modelling tissue growth and understanding the dynamics of cells have been the center of many research pieces in the past decade. Initiated by Greenspan [126], general mechanical models of tumor growth [67, 177, 264] have been proposed and used the internal pressure as the main effect that drives the movement and proliferation of cells. The prototypical example of a mechanical living tissue model is

$$\partial_t n = \operatorname{div}(n\nabla p) + nG(p), \quad p = P_\gamma(n) := \frac{\gamma}{\gamma-1}n^{\gamma-1}, \quad (1.1.16)$$

in which  $p$  is the pressure and  $n$  the cell density. In this kind of model, Darcy's law of movement is used to reflect the porous media formed by the extra-cellular matrix and the tendency of cells to move away from regions of high compression. The dependency on the pressure of the growth function has also been used to model the sensitivity of tissue proliferation to compression. The behavior and stability of monotone traveling waves has been studied in [114]. Interestingly, Perthame *et al.* [289] have shown that in the incompressible limit (*i.e.*  $\gamma \rightarrow \infty$ ), solutions of Model (1.1.16) converge to a limit  $(n_\infty, p_\infty)$  solution of a free boundary limit problem of Hele-Shaw type for which the speed of the free boundary is given by the normal component of  $p_\infty$ . In this limit, the solution of Equation (1.1.16) organizes into 2 regions :  $\Omega(t)$  in which the pressure is positive (corresponding to the tissue) and outside this zone where  $p = 0$ . Furthermore, the free Boundary problem is supplemented by a complementary equation that indicates that the pressure satisfies

$$-\Delta p_\infty = G(p_\infty), \quad \text{in } \Omega(t), \quad \text{or similarly } p_\infty(\Delta p_\infty + G(p_\infty)) = 0 \quad \text{a.e. in } \Omega. \quad (1.1.17)$$

However, the crucial role of the cell-cell adhesion at the surface of the tissue is not retrieved at the limit. Indeed, as pointed by Lowengrub *et al.* [264], the velocity of the free surface should depend on its local curvature denoted by  $\kappa$ .

Thus, multiple variants of the general model (1.1.16) have been suggested to consider other physical effects in mechanical models of tissue growth. The addition of the effect of viscosity in the model has been made to represent the friction between cells [38, 50] through the use of Stokes' or Brinkman's law (see [18] for a rigorous derivation of Brinkman's law in inhomogeneous materials). However, as pointed out by Perthame and Vauchelet [291], Brinkman's law leads to a simpler version of the model and, therefore, is a preferential choice for its mathematical analysis. Adding viscosity through the use of Brinkman's law leads to the model

$$\begin{cases} \partial_t n = \operatorname{div}(n\nabla\mu) + nG(p), & \text{in } (0, +\infty) \times \Omega, \\ -\sigma\Delta\mu + \mu = p, & \text{in } (0, +\infty) \times \Omega. \end{cases} \quad (1.1.18)$$

The incompressible limit of this system also yields the complementary relation (see [291])

$$p_\infty(p_\infty - \mu_\infty - \sigma G(p_\infty)) = 0, \quad \text{a.e. in } \Omega.$$

In the incompressible limit, notable changes compared to the system with Darcy's law are found. First the previous complementary relation is different compared to Equation (1.1.17), and the pressure  $p_\infty$  in the limit is discontinuous, *i.e.* there is a jump of the pressure located at the surface of  $\Omega(t)$ . However, the pressure jump is related to the potential  $\mu$  and not to the local curvature of the free boundary  $\partial\Omega(t)$ . The authors already indicated that a possible explanation to this is that the previous model does not include the effect of surface tension. Indeed, classical solutions for the Hele-Shaw problem with the addition of the effect of surface tension leads to the free boundary problem (see *e.g.* [161])

$$\begin{cases} -\Delta\mu = 0 & \text{in } \Omega \setminus \partial\Omega(t), \\ \mu = \sigma\kappa & \text{on } \partial\Omega(t). \end{cases} \quad (1.1.19)$$

This correct Hele-Shaw limit has been formally obtained as the sharp-interface asymptotic model of the Cahn-Hilliard equation [17].

Cahn-Hilliard type models are widely used nowadays to represent living tissues and in particular tumors [328, 176]. Being of fourth-order, the Cahn-Hilliard equation is often rewritten in a system of two second-order equations, *i.e.*

$$\partial_t n = \operatorname{div}(n\nabla(\psi'(n) - \delta\Delta n)) \rightarrow \begin{cases} \partial_t n & = \operatorname{div}(n\nabla\mu), \\ \mu & = -\delta\Delta n + \psi'(n), \end{cases} \quad (1.1.20)$$

where  $n$  is the concentration of a phase and  $\mu$  is called the chemical potential in material sciences but is often used as an effective pressure for living tissues (see [97, 108, 11, 98]). Also, the interaction potential  $\psi(n)$  contained in this effective pressure term comprises the effects of attraction and repulsion between cells. The physically relevant form of this potential is a double-well logarithmic potential and is often approximated by a smooth polynomial function. However, recent studies show that for the modelling of living tissues and for the particular application where only one of the component of the mixture experiences attractive and repulsive forces, a single-well logarithmic potential is more relevant [67]. In our particular application, the potential  $\psi'(n)$  is only related to the modelling of pressure and, therefore, the final model includes only repulsive forces.

We emphasize that multiple variants of the Cahn-Hilliard model have been studied to represent living tissues and tumors including passive transport by Darcy's Law [187, 186, 179] and with the effect of viscosity [144, 145]. The existence of weak solutions for degenerate mobility has been first studied by Elliott and Garcke in [157]. We also refer to [147] for the existence of weak solutions for a multicomponent mixture with cross-diffusion effects. The case of a smooth potential and mobility was treated by Dai and Du in [113]. In the definition of their weak solutions, the authors did not identify the potential  $\mu$  except in the zone where the density does not vanishes. Indeed this difficulty comes from the energy which provides the bound  $\|n^{1/2}\nabla\mu\|_{L^2(\Omega_T)} \leq C$ . The term  $\operatorname{div}(n\nabla\mu)$  in the weak solutions is then treated as  $\operatorname{div}(n^{1/2}\zeta)$  where  $\zeta$  is an  $L^2$  vector field that can be identified in the zone  $n \neq 0$ . For our system, we identify the potential  $\mu$  by considering a weaker type of solutions.

**Contents** We present and analyze the convergence of the nonlinear GKS model to the DCH model (corresponding to the study of the vanishing viscosity limit, *i.e.*  $\sigma \rightarrow 0$ ). We also study

rigorously the incompressible limit, *i.e.*  $\gamma \rightarrow \infty$  for this new living tissue model.

Section 1.2 contains the proof that the nonlinear GKS system (1.1.1)–(1.1.2) converges to the DCH model (1.1.3)–(1.1.4). This analysis is made possible from the fact that the nonlinear GKS model is equivalent to the RCH model and from the use of standard compactness properties. To the best of our knowledge, this is the first time that a Keller-Segel system is interpreted as a Cahn-Hilliard system. The study of the incompressible limit is the purpose of Section 1.3. Since uniform bounds in  $\gamma$  are required to use standard compactness properties and pass to the limit, we provide new estimates relying on results obtained for the non-linear GKS model. Section 1.4 contains a proof of the existence of weak solution for the system (1.1.1)–(1.1.2) and can be read first for the reader's convenience. The proof follows the lines of the works [157, 287]. Indeed, using a regularization of the mobility, we apply a Galerkin approximation to show the existence of weak solutions to the regularized model. The main novelty in the proof of this latter result is the need of strong convergence for the chemical potential which is required to pass to the limit for the source term. Then, a priori estimates on the regularized model give us sufficient control to use standard compactness results and pass to the limit of the regularization, hence obtaining the existence of weak solutions to the RCH-DKS model. Uniqueness of the weak solution is also shown in the case where there is no proliferation term.

Therefore, our analysis relies on a combination of results obtained for the Cahn-Hilliard model as well as the non-linear GKS limit. The addition of the regularization term in the Cahn-Hilliard model opens a new angle of attack to solve new problems and to find the correct Hele-Shaw limit for a simple living tissue model.

## 1.2 From the GKS to the DCH system ( $\sigma \rightarrow 0$ )

We consider weak solutions of System (1.1.1)–(1.1.2) as built in Section 1.4 and prove that weak solutions of the DCH equation can be obtained as the limit of the GKS system. In the definition of the weak solutions, there is not enough regularity on  $\mu$  to treat the term  $-\operatorname{div}(n\nabla\mu)$  as  $\int n\nabla\mu\nabla\chi$  where  $\chi$  is a test function. We need to rely on another integration by parts based on the definition of  $\mu$  and Equation (1.2.13).

Weak solutions of the DCH equation (1.1.3)–(1.1.4) satisfy both forms

$$\int_0^T \langle \chi, \partial_t n \rangle = \int_{\Omega_T} \left[ \frac{\gamma}{\gamma+1} n^{\gamma+1} - \delta n \Delta n - \delta \frac{|\nabla n|^2}{2} \right] \Delta \chi + \delta \sum_{i=1}^d \partial_i n \nabla n \cdot \nabla \partial_i \chi + n G_\infty \chi, \quad (1.2.1)$$

$$\int_0^T \langle \chi, \partial_t n \rangle = \int_{\Omega_T} \frac{\gamma}{\gamma+1} n^{\gamma+1} \Delta \chi - \delta \Delta n (\nabla n \cdot \nabla \chi + n \Delta \chi) + n G_\infty \chi, \quad (1.2.2)$$

for all  $\chi \in L^\infty(0, T, W^{2,\infty}(\Omega))$  with  $\nabla \chi \cdot \nu = 0$ , and

$$\int_0^T \langle \chi, \partial_t n \rangle = - \int_{\Omega_T} n \partial_t \chi - \int_{\Omega} \chi(t=0) n^0, \quad (1.2.3)$$

when, additionally,  $\chi \in C^1([0, T] \times \Omega)$  and  $\chi(T) = 0$ . The source term can be identified as  $G_\infty = G(\mu)$  in dimension  $d \leq 4$ . The weak formulation (1.2.1) comes from Equation (1.2.13). The forms (1.2.1) and (1.2.2) are equivalent with the formula

$$\frac{1}{2} \nabla (|\nabla f|^2) = \operatorname{div}(\nabla f \otimes \nabla f) - \Delta f \nabla f.$$

We have the following convergence theorem.

**Theorem 1.2.1** (Convergence of the GKS system to the DCH eq.). *Assume (1.1.12)–(1.1.14). The weak solutions  $(n_\sigma, \mu_\sigma)$  of the RCH-DKS model converge to  $(n, \mu)$  weak solution of (1.1.3)–(1.1.4) in the sense defined by (1.2.1)–(1.2.3). They satisfy the regularity estimates  $n \in L^\infty(0, T, H^1(\Omega)) \cap L^2(0, T, H^2(\Omega))$ ,  $\partial_t n \in (L^4(0, T, W^{1,s}(\Omega)))'$  where  $s$  is defined in Proposition 1.2.2,  $\mu \in L^2(0, T, L^{\frac{\gamma+1}{\gamma}}(\Omega)) \cap L^{\frac{\gamma+1}{\gamma}}(0, T, L^{\frac{d(\gamma+1)}{(d-2)\gamma}}(\Omega))$ . In dimension  $d \leq 4$ ,  $\mu$  belongs to  $L^2(\Omega)$  and for all  $\chi \in L^\infty(0, T, W^{2,\infty}(\Omega))$  with  $\nabla \chi \cdot \nu = 0$ , Equation (1.2.1) can be written as*

$$\int_0^T \langle \chi, \partial_t n \rangle = \int_{\Omega_T} \mu (\nabla n \cdot \nabla \chi + n \Delta \chi) + n G(\mu) \chi. \quad (1.2.4)$$

with  $\mu = n^\gamma - \delta \Delta n$  a.e. in  $\Omega$ .

The rest of this section is devoted to prove this theorem.

### 1.2.1 A priori estimates

Since the two systems (1.1.1)–(1.1.2) and (1.1.6)–(1.1.7) are equivalent, all estimates proved for a system apply to the other. Based on the construction in Section 1.4, the energy and entropy structure of the RCH model provide us with the following bounds. Assuming (1.1.12)–(1.1.14), there is a constant  $C$  independent of  $\gamma, \sigma$ , such that, for all  $T \geq 0$ ,

$$\frac{\sigma}{\delta} \int_{\Omega_T} |\nabla \mu|^2 \leq C, \quad \frac{\sigma}{\delta} \int_{\Omega} |\mu(t)|^2 \leq C \quad \forall t \in (0, T), \quad (1.2.5)$$

$$\delta \int_{\Omega} \left| \nabla \left( n(t) - \frac{\sigma}{\delta} \mu(t) \right) \right|^2 \leq C, \quad \forall t \in (0, T), \quad (1.2.6)$$

$$\delta \int_{\Omega_T} \left| \Delta \left( n - \frac{\sigma}{\delta} \mu \right) \right|^2 \leq C, \quad (1.2.7)$$

$$\int_{\Omega_T} n |\nabla \mu|^2 \leq C, \quad (1.2.8)$$

$$\gamma \int_{\Omega_T} \left( n - \frac{\sigma}{\delta} \mu \right)^{\gamma-1} \left| \nabla \left( n - \frac{\sigma}{\delta} \mu \right) \right|^2 \leq C, \quad (1.2.9)$$

$$\int_{\Omega} \frac{(n(t) - \frac{\sigma}{\delta} \mu(t))^{\gamma+1}}{\gamma+1} \leq C, \quad \forall t \in (0, T). \quad (1.2.10)$$

Next, we deduce further a priori estimates which are enough to pass to the limit as  $\sigma$  vanishes.

**Proposition 1.2.2** (A priori bounds). *Assuming (1.1.12)–(1.1.14), there is a constant  $C(T)$  independent of  $\gamma$  and  $\sigma$  such that the a priori estimates hold*

$$\|n\|_{L^2(0, T, H^1(\Omega))} \leq C, \quad \|\partial_t n\|_{(L^4(0, T, W^{1,s}(\Omega)))'} \leq C,$$

with  $s = 2$  for  $d = 1$ ,  $s > 2$  for  $d = 2$  and  $s = \frac{4d}{d+2}$  otherwise. Furthermore, we have

$$\|w\|_{L^\infty(0, T, L^{\gamma+1}(\Omega))} \leq C, \quad \frac{1}{\gamma} \|w\|_{L^{\gamma+1}(0, T, L^{\frac{d(\gamma+1)}{d-2}}(\Omega))}^{\gamma+1} \leq C, \quad \frac{1}{\gamma} \|\mu\|_{L^{\frac{\gamma+1}{\gamma}}(0, T, L^q(\Omega))}^{\frac{\gamma+1}{\gamma}} \leq C,$$

with  $q = \min(2, \frac{d(\gamma+1)}{(d-2)\gamma})$ .



*Démonstration.* From Inequalities (1.2.5)–(1.2.6), we deduce that  $\nabla n \in L^2(0, T, L^2(\Omega))$ . Thus, using the Poincaré-Wirtinger inequality, we obtain that  $n \in L^2(0, T, H^1(\Omega))$  uniformly in  $\gamma, \sigma$ . Next, we recall that in dimension  $d$ ,  $H^1 \hookrightarrow L^r$  where  $r = \frac{2d}{d-2}$  for  $d > 2$ ,  $1 \leq r < \infty$  for  $d = 2$ , and  $1 \leq r \leq \infty$  for  $d = 1$ . Thus,  $n^{1/2} \in L^4(0, T, L^{2r}(\Omega))$ . Therefore, for all  $\varphi \in L^4(0, T, W^{1,s}(\Omega))$ , we can compute

$$\begin{aligned} \left| \int_{\Omega} \partial_t n \varphi \right| &= \left| \int_{\Omega} n^{1/2} n^{1/2} \nabla \mu \cdot \nabla \varphi + \int_{\Omega} n G(\mu) \varphi \right| \\ &\leq \|n^{1/2}\|_{L^{2r}(\Omega)} \|n^{1/2} \nabla \mu\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^s} + C \|n\|_{L^r(\Omega)} \|\varphi\|_{L^{s/2}(\Omega)}, \end{aligned}$$

because  $\frac{2r}{r-1} = \frac{4d}{d+2} := s$  for  $d > 2$ . For  $d \leq 2$ , we can consider any  $r > 1$ . Assumption (1.1.13) is used to control the source term in  $L^\infty$ . Integrating in time, using the triangle inequality on the integrals, Inequality (1.2.8) and Hölder's inequality, we obtain the second estimate,

$$\left| \int_{\Omega_T} \partial_t n \varphi \right| \leq C \|\varphi\|_{L^4(0, T, W^{1,s}(\Omega))}.$$

The first Lebesgue estimate for  $w$  is just an application of Inequality (1.2.10). For the second Lebesgue estimate for  $w$  we write Inequality (1.2.9) as

$$\frac{4\gamma}{(\gamma+1)^2} \int_{\Omega_T} |\nabla w^{\frac{\gamma+1}{2}}|^2 \leq C.$$

With Sobolev's embedding and the Poincaré-Wirtinger inequality, we obtain

$$\frac{1}{\gamma} \|w^{\frac{\gamma+1}{2}}\|_{L^2(0, T, L^{\frac{2d}{d-2}}(\Omega))}^2 \leq C.$$

This yields the second estimate. In addition, this estimate allows us to bound  $w^\gamma$  uniformly in  $L^{\frac{\gamma+1}{\gamma}}(0, T, L^{\frac{d(\gamma+1)}{(d-2)\gamma}}(\Omega))$ . Finally, we recall the last formula for  $\mu$  in (1.1.5),  $\mu = w^\gamma - \Delta w$ . Using the previous bound on  $w^\gamma$  and the bound (1.2.7), we obtain the estimate on  $\mu$ .

This ends the proof of Proposition 1.2.2.  $\square$

## 1.2.2 Convergence $\sigma \rightarrow 0$

To study the limit  $\sigma \rightarrow 0$ , we write the equivalent RCH and GKS systems as

$$\begin{aligned} \partial_t n_\sigma - \operatorname{div}(n_\sigma \nabla \mu_\sigma) &= n_\sigma G(\mu_\sigma), \quad \text{in } (0, +\infty) \times \Omega, \\ \mu_\sigma &= w_\sigma^\gamma - \delta \Delta w_\sigma, \quad w_\sigma = n_\sigma - \frac{\sigma}{\delta} \mu_\sigma \quad \text{in } (0, +\infty) \times \Omega. \end{aligned}$$

The main difficulty is the convergence of the term  $n_\sigma \nabla \mu_\sigma$  in the first equation. We know that  $n_\sigma \rightarrow n$  strongly in  $L^2(\Omega_T)$  thanks to the Lions-Aubin lemma and Proposition 1.2.2. In fact, the strong convergence is obtained in  $L^{<\infty}(0, T, L^2(\Omega))$  (see below). No useful information can be obtained on  $\nabla \mu_\sigma$  and for this reason, after an integration by parts, it remains to prove the weak convergence in  $L^{>1}(0, T, L^2(\Omega))$  of  $\mu_\sigma$ . The estimate on  $\mu_\sigma$  given in Proposition 1.2.2 is not enough in high dimension (i.e.  $d > 4$ ) and we need to consider weak solutions given by (1.2.1)–(1.2.2) in the limit. Strong convergence of  $\mu_\sigma$  is also needed in order to identify the source term. Finally, with the definition of  $w_\sigma$ , notice that Inequalities (1.2.5) show that  $w_\sigma$  is a compact

perturbation of  $n_\sigma$  (see Proposition 1.2.4) which has much higher regularity than  $n_\sigma$  due to the entropy/energy structure of our system. Therefore the limit in the terms  $\text{div}(n_\sigma \nabla \mu_\sigma)$  and  $n_\sigma G(\mu_\sigma)$  is treated by writing  $n_\sigma = w_\sigma +$  'compact perturbation'.

**Remark 1.2.3.** In the following proof and until the end, we consider convolutions which we integrate on a bounded domain to prove compactness properties. This is just a formal writing because the functions we consider are only defined on  $\Omega$ . To make it rigorous, instead of considering a function  $f$  we need to consider  $f\zeta$  where  $\zeta$  is a truncature function. Then, the compactness has to be obtained in domains  $\Omega_\varepsilon$  for every  $\varepsilon$  small enough where  $\Omega_\varepsilon \subset \Omega$  and every points in  $\Omega_\varepsilon$  is at distance  $\varepsilon$  of the boundary of  $\Omega$ . From now on, we ignore this technical issue in the proof below.

**Proposition 1.2.4.** *With the assumptions of Theorem 1.2.1 we have  $\nabla w_\sigma, \nabla n_\sigma \rightarrow \nabla n$  strongly in  $L^2(\Omega_T)$  when  $\sigma \rightarrow 0$ .*

*Démonstration.* Inequalities (1.2.5) show that  $\sigma \mu_\sigma, \sigma \nabla \mu_\sigma \rightarrow 0$  in  $L^2(\Omega_T)$ . In addition, Proposition 1.2.2 yields the weak convergence of  $\nabla w_\sigma, \nabla n_\sigma \rightharpoonup \nabla n$ . It remains to prove strong convergence. We rely on arguments similar to those of Appendix 1.5.1 using the Fréchet-Kolmogorov theorem.

We know that  $\nabla w_\sigma$  is bounded uniformly in  $L^\infty(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$  using Inequalities (1.2.6)-(1.2.7) and elliptic regularity. Then, by interpolation we have an  $L^p(\Omega_T)$  bound for some  $p > 2$ . It remains to prove strong convergence in  $L^1$ .

We already have compactness in space thanks to the  $H^1$  bound. With the notations of Appendix 1.5.1, it only remains to prove compactness in time, that means

$$\int_0^{T-h} \int_\Omega |\nabla w_\sigma(t+h, \cdot) * \varphi_\varepsilon(x) - \nabla w_\sigma(t, \cdot) * \varphi_\varepsilon(x)| dx dt \leq \theta(h), \quad \theta(h) \rightarrow 0 \text{ when } h \rightarrow 0, \quad (1.2.11)$$

uniformly in  $\sigma$ .

From (1.1.5) we can write

$$\begin{aligned} \int_0^{T-h} \int_\Omega |\nabla w_\sigma(t+h, \cdot) * \varphi_\varepsilon(x) - \nabla w_\sigma(t, \cdot) * \varphi_\varepsilon(x)| dx dt &\leq \int_0^{T-h} \int_\Omega |(n_\sigma(t+h, \cdot) - n_\sigma(t, \cdot)) * \nabla \varphi_\varepsilon(x)| dx dt \\ &\quad + \sigma \int_0^{T-h} \int_\Omega |(\nabla \mu_\sigma(t+h, \cdot) - \nabla \mu_\sigma(t, \cdot)) * \varphi_\varepsilon(x)| dx dt. \end{aligned}$$

The term  $\sigma \nabla \mu_\sigma$  converges strongly to 0 in  $L^2(\Omega_T)$ . This provides equicontinuity of the last term. The first term is dealt in the same way as in Appendix 1.5.1. This yields the strong convergence of  $\nabla w_\sigma$ . Using once again the previous decomposition and  $\sigma \nabla \mu_\sigma \rightarrow 0$  show that  $\nabla n_\sigma$  converges strongly.  $\square$

Now, we can prove Theorem 1.2.1.

*Démonstration. Step 1 : Strong compactness for  $w_\sigma, n_\sigma, \nabla w_\sigma, \nabla n_\sigma$ .* This step is a consequence of Proposition 1.2.2 and Proposition 1.2.4.

*Step 2 : Convergence of  $n \nabla \mu$ .* Now, most of the convergences come from properties of the weak limit and the bounds provided at the beginning of this section. For the case of any dimension, we write  $n_\sigma \nabla \mu_\sigma$  as

$$n_\sigma \nabla \mu_\sigma = w_\sigma \nabla \mu_\sigma + \frac{\sigma}{\delta} \mu_\sigma \nabla \mu_\sigma.$$

The last term converges strongly to 0 as  $\sigma \rightarrow 0$ . Indeed, from Inequalities (1.2.5) together with the Sobolev injection and the Poincaré-Wirtinger inequality, we know that

$$\|\sigma^{1/2}\mu_\sigma\|_{L^r(\Omega_T)} \leq C,$$

for some  $r > 2$  and  $C$  independent of  $\sigma$ . Using also that  $\mu$  is bounded in  $L^1(\Omega_T)$  and interpolation we find

$$\|\sigma^{1/2}\mu\|_{L^2(\Omega_T)} \leq C\sigma^{\frac{1-\theta}{2}}, \quad \theta = \frac{r}{2(r-1)} < 1. \quad (1.2.12)$$

With Inequalities (1.2.5)-(1.2.12) we find  $\sigma\mu_\sigma\nabla\mu_\sigma = (\sigma^{1/2}\mu_\sigma)(\sigma^{1/2}\nabla\mu_\sigma) \rightarrow 0$  strongly. We now treat the first term on the right hand side. We recall that  $\mu_\sigma = w_\sigma^\gamma - \delta\Delta w_\sigma$  and therefore we write  $w_\sigma\nabla\mu_\sigma$  as

$$w_\sigma\nabla\mu_\sigma = \frac{\gamma}{\gamma+1}\nabla w_\sigma^{\gamma+1} - \delta\left[\nabla(w_\sigma\Delta w_\sigma) + \nabla\frac{|\nabla w_\sigma|^2}{2} - \sum_{i=1}^d\partial_i(\partial_i w_\sigma\nabla w_\sigma)\right]. \quad (1.2.13)$$

We recall the strong convergence of  $w_\sigma$  and  $\nabla w_\sigma$  in  $L^2(\Omega_T)$  to  $n, \nabla n$  and we have the weak convergence of  $\Delta w_\sigma$  to  $\Delta n$  thanks to Inequality (1.2.7). Finally we obtain the convergence in the distributional sense of the bracket in the right hand side. For the first term on the right hand side, we use Proposition 1.2.2. This provides that  $w_\sigma^{\gamma+1}$  is bounded uniformly  $L^1(0, T, L^p(\Omega)) \cap L^\infty(0, T, L^1(\Omega))$  for some  $p > 1$  and therefore by interpolation in some  $L^q(0, T, L^r(\Omega))$  for some  $1 < q < \infty$  and  $r = \frac{1}{1-1/q+1/(qp)}$ . Using the strong convergence of  $w_\sigma$  and the Lebesgue dominated convergence theorem allows us to identify  $w^{\gamma+1}$  in the limit. Using integration by parts we obtain the weak formulations (1.2.1)-(1.2.2).

The regularity of  $n, \nabla n, \Delta n$  in the limit comes from the uniform bounds on  $w_\sigma, \nabla w_\sigma, \Delta w_\sigma$  and the convergence of  $\sigma\mu_\sigma$  to 0.

In dimension  $d \leq 4$  we have better regularity for  $\mu$  and we can find another weak formulation. This is achieved in the following step.

*Step 3 : The weak formulation (1.2.4) in dimension  $d \leq 4$ .* In the case  $d \leq 4$  we have

$$\|\mu_\sigma\|_{L^{\frac{\gamma+1}{\gamma}}(0, T, L^2(\Omega_T))} \leq C, \quad \|\nabla w_\sigma\|_{L^\infty(0, T, L^2(\Omega))} \leq C,$$

thanks to Proposition 1.2.2 and Inequality (1.2.6). We write

$$\mu_\sigma\nabla n_\sigma = \mu_\sigma\nabla w_\sigma + \frac{\sigma}{\delta}\mu_\sigma\nabla\mu_\sigma.$$

We know from above that the second term of the right hand side strongly converges to 0. For the first term of the right hand side we know that  $\nabla w_\sigma \rightarrow \nabla n$  strongly in  $L^2(\Omega_T)$ . With the previous bound on  $\nabla w_\sigma$  the convergence actually holds in every  $L^p(0, T, L^2(\Omega))$  for  $p < \infty$ . Using the weak convergence of  $\mu_\sigma$  we find the weak convergence of the product.

The weak convergence of  $n_\sigma\mu_\sigma$  is similar. Therefore, with integration by parts of the term  $\text{div}(n_\sigma\nabla\mu_\sigma)$  against a test function  $\chi$  we find the result.

*Step 4 : Identification of the source term when  $d \leq 4$ .* The last difficulty in the proof is to identify the source term (i.e  $nG(\mu)$ ). Indeed, we do not know that  $\mu_\sigma$  converges a.e. (up to a subsequence). However, we know that  $n_\sigma$  converges a.e to  $n$  and  $G(\mu_\sigma)$  is bounded. It remains to prove that  $\mu_\sigma$  converges a.e. in the zone  $\{n > 0\}$ .

For this reason we search for estimates on  $n_\sigma\mu_\sigma$  to prove its convergence almost everywhere. Then, the convergence of  $n_\sigma\mu_\sigma$  and  $n_\sigma$  yields the convergence a.e. of  $\mu_\sigma$  in the zone  $\{n > 0\}$ .

Now, we write  $n_\sigma \mu_\sigma = w_\sigma \mu_\sigma + \frac{\sigma}{\delta} \mu_\sigma^2$  and compute

$$\nabla(n_\sigma \mu_\sigma) = n_\sigma^{1/2} n_\sigma^{1/2} \nabla \mu_\sigma + \mu_\sigma \nabla w_\sigma + \frac{3\sigma}{2\delta} \nabla(\mu_\sigma^2).$$

As previously, the last term is bounded uniformly in  $L^1(\Omega_T)$  (and even converges to 0). The first term is also bounded in  $L^1(\Omega_T)$  with Proposition 1.2.2 and Inequality (1.2.8). For the second term we use that  $\mu_\sigma$  is bounded uniformly in  $L^{\frac{\gamma+1}{\gamma}}(0, T, L^{\frac{d(\gamma+1)}{(d-2)\gamma}}(\Omega))$ . In dimension  $d \leq 4$ , Inequality (1.2.6) provides a bound on the second term. Therefore we have compactness in space. The proof of the time compactness uses arguments and lemmas reported in Appendix 1.5.1. Since  $\nabla(w\mu)$  is bounded, it only remains to show

$$\int_0^{T-h} \int_\Omega |w_\sigma(t+h, \cdot) \mu_\sigma(t+h, \cdot) * \varphi_\varepsilon(x) - w_\sigma(t, \cdot) \mu_\sigma(t, \cdot) * \varphi_\varepsilon(x)| dx dt \xrightarrow{h \rightarrow 0} 0, \quad \text{uniformly in } \sigma,$$

where the smooth functions  $\varphi_\varepsilon$  are defined in the Appendix 1.5.1.

We know using (1.1.5) that

$$w_\sigma \mu_\sigma = w_\sigma^{\gamma+1} - \delta w_\sigma \Delta w_\sigma = w_\sigma^{\gamma+1} - \frac{\delta}{2} \Delta w_\sigma^2 + \delta |\nabla w_\sigma|^2.$$

Since  $w_\sigma$  converges strongly in  $L^2(\Omega_T)$  and is bounded in  $L^\infty(0, T, L^{\gamma+1}(\Omega)) \cap L^{\gamma+1}((0, T), L^{\frac{d(\gamma+1)}{d-2}}(\Omega))$  thanks to Inequality (1.2.10) and Proposition 1.2.2, we conclude by interpolation the strong convergence of  $w_\sigma$  in  $L^{\gamma+1}(\Omega_T)$ . Therefore the first term is equicontinuous by the converse of the Fréchet-Kolmogorov theorem.

It only remains to estimate

$$\begin{aligned} & \int_0^{T-h} \int_\Omega |[w_\sigma(t+h, \cdot) \Delta w_\sigma(t+h, \cdot) - w_\sigma(t, \cdot) \Delta w_\sigma(t, \cdot)] * \varphi_\varepsilon(x)| dx dt \\ & \leq \int_0^{T-h} \int_\Omega \left| \int_\Omega (|\nabla w_\sigma(t, y)|^2 - |\nabla w_\sigma(t+h, y)|^2) \varphi_\varepsilon(x-y) dy dx dt \right. \\ & \quad \left. + \int_0^{T-h} \int_\Omega \left| \int_\Omega (w_\sigma^2(t+h, y) - w_\sigma^2(t, y)) \Delta \varphi_\varepsilon(x-y) dy dx dt \right|. \end{aligned}$$

With the strong convergence of  $\nabla w_\sigma$  and  $w_\sigma^2$  we find the equicontinuity of the second and third term in the decomposition. This yields the equicontinuity with respect to time of  $w\mu$ .

Therefore thanks to the Fréchet-Kolmogorov theorem we have strong convergence in some  $L^p$  and the convergence a.e. up to a subsequence of  $\mu$  in the zone  $n > 0$ . The Lebesgue dominated convergence theorem then allows us to identify the source term in the definition of the weak solutions.  $\square$

### 1.3 Incompressible limit $\gamma \rightarrow +\infty$

We now fix  $\delta, \sigma > 0$  and study the incompressible limit  $\gamma \rightarrow \infty$  for the RCH-DKS system. These two constants are the main link between RCH and DKS models. We recall that in the case  $\delta = 0$ , the incompressible limit is given in [291]. In the case  $\sigma = 0$ , the regularity provided by the DCH model is not sufficient to pass to the limit. From now on, we keep  $\delta, \sigma > 0$  fixed and we may

consider them equal to 1 in some computations. We summarize the main bounds proved in this section in the following proposition

**Proposition 1.3.1.** *For all  $T > 0$ , there exists a constant  $C(T)$  independent of  $\gamma$  such that the weak solutions built in Section 1.4 satisfy*

$$\|\mu\|_{L^2(0,T,H^1(\Omega))} \leq C, \quad \|n\|_{L^2(0,T,H^1(\Omega))} \leq C, \quad (1.3.1)$$

$$\|n\|_{L^\infty(\Omega_T)} \leq C, \quad (1.3.2)$$

$$\begin{aligned} \|w\|_{L^\infty(0,T,H^2(\Omega))} \leq C, \quad \|w^\gamma\|_{L^\infty(\Omega_T)} \leq C, \quad \|\nabla w^\gamma\|_{L^\infty(0,T,L^1(\Omega))} \leq C, \\ \|\Delta w\|_{L^\infty(\Omega_T)} \leq C, \end{aligned} \quad (1.3.3)$$

$$\begin{aligned} \|\partial_t n\|_{L^2(0,T,H^{-1}(\Omega))} \leq C, \quad \|\partial_t w\|_{L^2(0,T,H^1(\Omega))} \leq C, \quad \|\partial_t \mu\|_{L^2(0,T,H^{-1}(\Omega))} \leq C, \\ \|\partial_t w^\gamma\|_{L^2(0,T,H^{-1}(\Omega))} \leq C. \end{aligned} \quad (1.3.4)$$

The weak solutions in the case of no source term, i.e  $G = 0$ , are unique.

### 1.3.1 Uniform a priori estimates in Proposition 1.3.1

*Proof of Proposition 1.3.1.* We start with the first two estimates (1.3.1) of Proposition 1.3.1. The first inequality is a consequence of Inequalities (1.2.5). The second inequality has been proven in the previous section.

*$L^\infty$  bound for  $n$ .* We now establish Inequality (1.3.2) under the assumption (1.1.12) on the initial condition. The proof requires a variant of Gagliardo-Nirenberg inequality, namely there exists  $C > 0$  such that for every  $0 < \varepsilon < 1/2$  and every  $v \in H^1(\Omega)$ ,

$$\|v\|_{L^2(\Omega)}^2 \leq \varepsilon \|\nabla v\|_{L^2(\Omega)}^2 + \frac{C}{\varepsilon^{d/2}} \|v\|_{L^1(\Omega)}^2. \quad (1.3.5)$$

This inequality is an application of the classical Gagliardo-Nirenberg and Young inequalities. We refer the reader to [106], Equation (9.3.8).

First, to begin the proof, we wish to stress a few comments. The proof of this proposition relies on the use of the Alikakos iteration method [16]. We prove it for smooth solutions of the equation but the method can be applied for weak solutions, since the bounds only depend on the a priori estimates already found.

We start by choosing  $\sigma = \delta = 1$  to simplify the notation. We notice however that the  $L^\infty$  bound is not uniform in  $\sigma$  and therefore this result does not apply to Subsection 1.2.2. In fact, with the same proof, it is possible to show that the  $L^\infty$  bound varies as  $1/\sigma$ . We multiply Equation (1.1.1) by  $n^{2^k-1}$ , integrate over the domain, and after integration by parts that uses the boundary conditions (1.1.11), we obtain

$$\frac{1}{2^k} \frac{d}{dt} \int_{\Omega} n^{2^k} + (2^k - 1) \int_{\Omega} n^{2^k-1} |\nabla n|^2 = (2^k - 1) \int_{\Omega} n^{2^k-1} \nabla w \cdot \nabla n + \int_{\Omega} n^{2^k} G(\mu). \quad (1.3.6)$$

Then, multiplying Equation (1.1.2) by  $n^{2^k}$ , integrating over  $\Omega$ , and after integration by parts, we have from the non-negativity of both  $n$  and  $w$  (see Proposition 1.4.5)

$$\int_{\Omega} n^{2^k-1} \nabla w \cdot \nabla n = \frac{1}{2^k} \int_{\Omega} -\Delta w n^{2^k} \leq \frac{1}{2^k} \int_{\Omega} n^{2^k+1}. \quad (1.3.7)$$

Therefore, rearranging the second term on the left-hand side of Equation (1.3.6), passing it to the right-hand side, using Inequality (1.3.7), and, finally, using  $n^{2^k} \leq n^{2^k+1} + 1$  for the second

term of the right hand side, we obtain

$$\frac{1}{2^k} \frac{d}{dt} \int_{\Omega} n^{2^k} \leq -\frac{4(2^k - 1)}{(2^k + 1)^2} \int_{\Omega} |\nabla n^{\frac{2^k+1}{2}}|^2 + \left( \frac{2^k - 1}{2^k} + C \right) \int_{\Omega} n^{2^k+1} + C.$$

This means exactly that for some  $C > 0$ ,

$$\frac{d}{dt} \|n\|_{L^{2^k}(\Omega)}^{2^k} \leq -\frac{4(2^k - 1)2^k}{(2^k + 1)^2} \|\nabla n^{\frac{2^k+1}{2}}\|_{L^2(\Omega)}^2 + C2^k \|n\|_{L^{2^k+1}}^{2^k+1} + C2^k.$$

Applying Equation (1.3.5) with  $v = n^{\frac{2^k+1}{2}}$ , we obtain for any  $\varepsilon > 0$

$$\frac{d}{dt} \|n\|_{L^{2^k}(\Omega)}^{2^k} \leq \left( C2^k - \frac{4(2^k - 1)2^k}{\varepsilon(2^k + 1)^2} \right) \|n\|_{L^{2^k+1}}^{2^k+1} + \frac{C}{\varepsilon^{(d+2)/2}} \frac{4(2^k - 1)2^k}{(2^k + 1)^2} \|n^{\frac{2^k+1}{2}}\|_{L^1(\Omega)}^2 + C2^k.$$

Choosing  $\varepsilon = C2^{-k}$  in order to let the first term of the right-hand side to be non-positive, leads to

$$\frac{d}{dt} \|n\|_{L^{2^k}(\Omega)}^{2^k} \leq C2^{(d+2)k/2} \|n^{\frac{2^k+1}{2}}\|_{L^1(\Omega)}^2 + C2^k.$$

Moreover, using Riesz-Thorin interpolation theorem, we have

$$\frac{d}{dt} \|n\|_{L^{2^k}(\Omega)}^{2^k} \leq C2^{(d+2)k/2} \|n\|_{L^{2^k}(\Omega)}^2 \|n\|_{L^{2^{k-1}}(\Omega)}^{2^k-1} + C2^k.$$

Denoting  $m_k = \sup_{t \in (0, T)} \|n(t)\|_{L^{2^k}(\Omega)}$  and after integrating in time, we obtain

$$m_k \leq \left( C2^{\frac{(d+2)k}{2}} m_{k-1}^{2^k-1} m_k^2 + C2^k \right)^{1/2^k}.$$

Following [106] (Lemma 9.3.1, p.213) the sequence  $m_k$  can be dominated by  $m'_k$  which satisfies

$$m'_k = (C2^{\frac{(d+2)k}{2}})^{1/2^k} m_{k-1}^{1-1/2^k} m_k^{1/2^{k-1}} \quad \text{with } C \text{ large enough and } m'_0 \geq 1.,$$

i.e.,

$$m'_k = (C2^{\frac{(d+2)k}{2}})^{\frac{1}{2^{k-2}}} m_{k-1}^{\frac{2^k-1}{2^{k-2}}}.$$

Letting  $k \rightarrow \infty$  and by induction we find  $\|n\|_{L^\infty(\Omega_T)} = m_\infty \leq m'_\infty \leq C$ . We refer to [106] for the details. This yields the result. This in turn provides also higher regularity for  $w$  thanks to Equation (1.1.2) and one can find estimates (1.3.3).

*Proof of (1.3.3).* The first estimate is a consequence of Equation (1.1.2) together with elliptic regularity.

For the second estimate we multiply Equation (1.1.2) by  $w^{\gamma(r-1)}$ . After an integration by parts and using the nonnegativity of  $n, w$  we find for every  $t$ ,

$$\|w^\gamma(t)\|_{L^r(\Omega)}^r \leq \|n(t)w(t)^{\gamma(r-1)}\|_{L^r(\Omega)}.$$

With Hölder inequality, we obtain

$$\|w^\gamma(t)\|_{L^r(\Omega)} \leq \|n(t)\|_{L^r(\Omega)}.$$

Letting  $r \rightarrow \infty$  thanks to Inequality (1.3.2) and taking the supremum in time yields the result. We refer the reader to Theorem 2.14 in [9] for further details. The third inequality is just the result of the two previous inequalities as well as Equation (1.1.2). To get the fourth inequality we differentiate in space Equation (1.1.2) and multiply by  $\text{sgn}(\partial_x w^\gamma)$ . Since  $w$  is nonnegative,  $\text{sgn}(\partial_x w^\gamma) = \text{sgn}(\partial_x w)$ . Using integration by parts on the first term yields the result. The computations can be made rigorous with the derivative of a convex approximation of the absolute value.

**Remark 1.3.2.** With Inequality (1.3.2) and Equation (1.1.2) it is possible to find that  $w \in L^2(0, T, H^3(\Omega))$  and therefore  $w^\gamma \in L^2(0, T, H^1(\Omega))$  thanks to Equation (1.1.2). However, the bound is not uniform in  $\gamma$  and we cannot gain compactness.

**Remark 1.3.3.** When  $G = 0$ , Inequalities (1.3.3) provides uniqueness of the weak solutions, see Appendix 1.5.2.

*Time compactness of  $n$ .* Compactness in time for  $n$  follows using once again the Lions-Aubin lemma with the  $(L^4(0, T, W^{1,s}(\Omega)))'$  bound on  $\partial_t n$  from proposition 1.2.2. One can prove compactness in time for  $w$  using the Fréchet-Kolmogorov theorem. However, with the previous regularity results, we can get the better bounds (1.3.4) on the time derivative of  $n$ ,  $w$ ,  $p = w^\gamma$  and  $\mu$ .

*Proof of (1.3.4).* Since we have found that  $n \in L^\infty(\Omega_T)$ , we have for any test function  $\phi \in L^2(0, T; H^1(\Omega))$ ,

$$\left| \int_{\Omega_T} \partial_t n \phi \right| \leq \|n\|_{L^\infty(\Omega_T)}^{1/2} \int_{\Omega_T} |n^{1/2} \nabla \mu \nabla \phi| \leq C \|\nabla \phi\|_{L^2(\Omega_T)},$$

where we have used (1.3.2) and (1.2.8). Hence, we have  $\partial_t n \in L^2(0, T, H^{-1}(\Omega))$  uniformly. To find compactness of  $\partial_t \mu$  it is enough to find compactness for  $\partial_t w$  thanks to Equation (1.1.5). Computing the time derivative of Equation (1.1.2), multiplying by a test function  $\phi \in L^2(0, T; H^1(\Omega))$ , using the notation  $z = \partial_t w$ , and integrating in space and time, we have

$$\int_{\Omega_T} \sigma \nabla z \cdot \nabla \phi + \frac{\sigma}{\delta} \gamma w^{\gamma-1} z \phi + z \phi = \int_{\Omega_T} \partial_t n \phi.$$

Choosing  $\phi = z$ , and from Young's Inequality, we have

$$\int_{\Omega_T} \left( \sigma - \frac{1}{2\kappa} \right) |\nabla z|^2 + \left( \frac{\sigma}{\delta} \gamma w^{\gamma-1} + 1 - \frac{1}{2\kappa} \right) |z|^2 \leq 2\kappa \|\partial_t n\|_{L^2(0, T; H^{-1}(\Omega))}^2.$$

Therefore, choosing  $\kappa$  large enough, we obtain that  $\partial_t w \in L^2(0, T, H^1(\Omega))$ . Using Equation (1.1.2) provides the compactness for  $\partial_t w^\gamma$ . This achieves the proof of (1.3.4) and Proposition 1.3.1.  $\square$

Finally, one can show that the regularity of the solutions provides continuity with respect to time

**Proposition 1.3.4.** *Assume (1.1.12)–(1.1.13), the functions  $n$ ,  $w$  and  $w^\gamma$  belong to  $C(0, T, L^p(\Omega))$  for every  $1 \leq p < \infty$ .*

### 1.3.2 Convergence $\gamma \rightarrow +\infty$

With all the ingredients of the previous subsections we find

**Theorem 1.3.5** (Incompressible limit). *Assume (1.1.12)–(1.1.13) and let  $(n_{\sigma, \gamma}, \mu_{\sigma, \gamma})$  be a weak solution to the RDCH model (1.1.6)–(1.1.7). Then, when  $\gamma \rightarrow \infty$ , after extraction of subsequences,  $(n_{\sigma, \gamma}, \mu_{\sigma, \gamma}, w_{\sigma, \gamma}) \rightarrow (n_{\sigma, \infty}, \mu_{\sigma, \infty}, w_{\sigma, \infty})$  with the regularity  $n_{\sigma, \infty} \in L^2(0, T, H^1(\Omega)) \cap$*

$L^\infty(\Omega_T)$ ,  $\partial_t n_{\sigma,\infty} \in L^2(0, T, H^{-1}(\Omega))$ ,  $w_{\sigma,\infty} \in C(0, T, L^p(\Omega)) \cap L^\infty(0, T, W^{2,p}(\Omega))$  and  $\mu_{\sigma,\infty} \in C(0, T, L^p(\Omega)) \cap L^2(0, T, H^1(\Omega))$  for every  $1 \leq p < \infty$ . These functions satisfy in the weak sense

$$\partial_t n_{\sigma,\infty} - \operatorname{div}(n_{\sigma,\infty} \nabla \mu_{\sigma,\infty}) = n_{\sigma,\infty} G(\mu_\infty).$$

and

$$\mu_{\sigma,\infty} = p_{\sigma,\infty} - \delta \Delta w_{\sigma,\infty}, \quad -\sigma \Delta w_{\sigma,\infty} + \frac{\sigma}{\delta} p_{\sigma,\infty} + w_{\sigma,\infty} = n_{\sigma,\infty}, \quad w_{\sigma,\infty} = n_{\sigma,\infty} - \frac{\sigma}{\delta} \mu_{\sigma,\infty}.$$

where  $p_{\sigma,\infty}$  is the strong  $L^p(\Omega_T)$ -limit of  $w^\gamma$  and belongs to  $L^\infty(0, T, L^\infty(\Omega)) \cap C(0, T, L^p(\Omega))$ .

*Démonstration.* For the first term on the left hand side, the weak convergence of  $\partial_t n_{\sigma,\gamma}$  given by (1.3.4) is enough. For the second term of the left hand side, we use (1.3.1)–(1.3.2) to prove the weak convergence of  $n_{\sigma,\gamma} \nabla \mu_{\sigma,\gamma}$ . Then, to identify its limit, we use the strong convergence of  $n_{\sigma,\gamma}$  from (1.3.1)–(1.3.4) and the weak convergence of  $\nabla \mu_{\sigma,\gamma}$  given by (1.3.1). For the term on the right hand side, we use the strong convergence of  $n_{\sigma,\gamma}$  and  $\mu_{\sigma,\gamma}$ . Finally for the equation on  $\mu_{\sigma,\gamma}$  and  $w_{\sigma,\gamma}$  weak convergence is enough. To prove strong convergence of  $p_{\sigma,\gamma} = w_{\sigma,\gamma}^\gamma$  we use the bounds on  $\partial_t p_{\sigma,\gamma}$ ,  $\nabla p_{\sigma,\gamma}$  from (1.3.3)–(1.3.4). An application of the Fréchet-Kolmogorov theorem as in Appendix 1.5.1 yields the strong convergence. The continuity with respect to time follows from the regularity of the solutions.  $\square$

Moreover, we find two propositions on  $w_{\sigma,\infty}$ , the first one provides an  $L^\infty$ -bound on  $w_{\sigma,\infty}$ , and the second one gives some information about the behaviour of the potential in the zones where  $w_{\sigma,\infty} \neq 1$ .

**Proposition 1.3.6** ( $L^\infty$ -bound for  $w_{\sigma,\infty}$ ). *For the limit solution  $w_{\sigma,\infty}$  defined in Theorem 1.3.5, we have, with  $w_{\sigma,\gamma} = n_{\sigma,\gamma} - \frac{\sigma}{\delta} \mu_{\sigma,\gamma}$ ,*

$$\|w_{\sigma,\infty}\|_{L^\infty(\Omega_T)} \leq 1,$$

*Démonstration.* In the case  $\sigma > 0$ , this estimate is trivial with the  $L^\infty$  bound on  $w^\gamma$  from Proposition 1.3.1. However, we also provide a proof that works in the case  $\sigma = 0$ , i.e., when  $w_{0,\gamma} = n_{0,\gamma}$ .

We start by using Inequality (1.2.10), we have

$$\|w_{\sigma,\gamma}(t)\|_{L^{\gamma+1}(\Omega)} \leq (C(\gamma+1))^{1/(\gamma+1)} \leq C^{1/(\gamma+1)}.$$

By interpolation, and with  $\frac{1}{q} = \theta + \frac{1-\theta}{\gamma+1}$  where  $q \in (1, \gamma+1)$ , we have

$$\|w_{\sigma,\gamma}(t)\|_{L^q(\Omega)} \leq \|w_{\sigma,\gamma}(t)\|_{L^1(\Omega)}^\theta \|w_{\sigma,\gamma}(t)\|_{L^{\gamma+1}}^{1-\theta}.$$

From the Cauchy-Schwarz inequality for the  $L^1$  norm and from the previous inequality for the  $L^{\gamma+1}$ -norm, we easily find two constants  $C, \tilde{C}$  such that

$$\|w_{\sigma,\gamma}(t)\|_{L^q(\Omega)} \leq \tilde{C}^\theta C^{(1-\theta)/(\gamma+1)}.$$

Since we know that  $w_\gamma \rightharpoonup w_\infty$ , and by lower semi-continuity of the norm as well as the fact that  $\theta \rightarrow 1/q$  when  $\gamma \rightarrow \infty$ , we have

$$\|w_{\sigma,\infty}\|_{L^\infty(0,T,L^q(\Omega))} \leq \liminf_{\gamma \rightarrow \infty} \|w_{\sigma,\gamma}\|_{L^\infty(0,T,L^q(\Omega))} \leq \tilde{C}^{1/q},$$



for any  $q \in (1, \infty)$ . Therefore, letting  $q \rightarrow \infty$ , we obtain

$$\|w_{\sigma,\infty}\|_{L^\infty(\Omega_T)} \leq \liminf_{q \rightarrow \infty} \|w_{\sigma,\infty}\|_{L^\infty(0,T,L^q)(\Omega)} \leq 1.$$

□

Compared to previous results on incompressible limits for living tissue models (see *e.g.* [289]), we have a slightly different relation linking the pressure and the density. We have the following proposition.

**Proposition 1.3.7** (Relation between  $p_\infty$  and  $w_\infty$ ). *The following relation holds for the limits of  $p_{\sigma,\gamma} = w_{\sigma,\gamma}^\gamma$  and  $w_{\sigma,\gamma}$ ,*

$$p_{\sigma,\infty}(w_{\sigma,\infty} - 1) = 0, \quad \text{a.e. in } \Omega_T.$$

*Démonstration.* The inequality  $p_{\sigma,\infty}(w_{\sigma,\infty} - 1) \leq 0$  is found in a straightforward manner using Proposition 1.3.6, and the fact that  $w_{\sigma,\gamma} \geq 0 \implies w_{\sigma,\gamma}^\gamma \geq 0 \implies p_{\sigma,\infty} \geq 0$ .

It remains to show that  $p_{\sigma,\infty}(w_{\sigma,\infty} - 1) \geq 0$ . We borrow the argument of [254]. For  $\nu > 0$ , there exists  $\gamma_0$  such that for  $\gamma \geq \gamma_0$ ,

$$w_{\sigma,\gamma}^{\gamma+1} \geq w_{\sigma,\gamma}^\gamma - \nu.$$

Then, from the convergence of  $w_{\sigma,\gamma}$  and  $w_{\sigma,\gamma}^\gamma$  we know that  $w_{\sigma,\gamma}^\gamma w_{\sigma,\gamma}$  converges to  $p_{\sigma,\infty} w_{\sigma,\infty}$ . Passing to the limit we get

$$p_{\sigma,\infty} w_{\sigma,\infty} \geq p_{\sigma,\infty} - \nu,$$

for every  $\nu > 0$ . Letting  $\nu \rightarrow 0$  yields the result.

□

**Remark 1.3.8.** From this result we find that in the zone  $w_\infty \neq 1$ , we get  $p_\infty = 0$ , and thus  $\mu_\infty = -\delta\Delta w_\infty$  which can be interpreted as a term representing surface tension.

When the relaxation parameter satisfy  $\sigma = 0$ , we expect to have  $w = n$ . Therefore, in the zone  $\Omega_+(t) = \{x, p_\infty(t, x) > 0\}$  we obtain that the density stays constant  $n_\infty = 1$ . With Equation (1.1.6) this means that formally

$$\begin{cases} -\Delta\mu = G(\mu) & \text{in } \Omega_+, \\ \mu = -\delta\Delta n & \text{on } \partial\Omega_+. \end{cases}$$

## 1.4 Existence of weak solutions

The proof of existence of weak solutions for system (1.1.6)–(1.1.7) follows the standard method for the DCH (see *e.g.* [157, 287, 113]). We start by regularizing the model : using a small positive parameter  $\varepsilon$  we define a positive approximation of the degenerate mobility  $b(n) := n$ . Existence of a solution to the regularized system is found using standard methods for nonlinear parabolic equations. Then, we derive a priori estimates on the regularized problem that allow us to pass to the limit  $\varepsilon \rightarrow 0$  and, hence, show the existence of weak solution for System (1.1.6)–(1.1.7).

### 1.4.1 Regularized mobility

We consider a small parameter  $0 < \varepsilon < 1$  and define the regularized mobility

$$B_\varepsilon(n) = \begin{cases} \frac{1}{\varepsilon} & \text{for } n \geq \frac{1}{\varepsilon}, \\ \varepsilon & \text{for } n \leq \varepsilon, \\ n & \text{otherwise.} \end{cases} \quad (1.4.1)$$

Thereby, we write the regularized analogous of System (1.1.6)–(1.1.7)

$$\partial_t n_\varepsilon - \operatorname{div}(B_\varepsilon(n_\varepsilon) \nabla \mu_\varepsilon) = n_\varepsilon G(\mu_\varepsilon), \quad \text{in } (0, +\infty) \times \Omega, \quad (1.4.2)$$

$$-\sigma \Delta \mu_\varepsilon + \mu_\varepsilon = \left(n_\varepsilon - \frac{\sigma}{\delta} \mu_\varepsilon\right)^\gamma - \delta \Delta n_\varepsilon, \quad \text{in } (0, +\infty) \times \Omega, \quad (1.4.3)$$

supplemented with the zero-flux boundary conditions

$$\frac{\partial w_\varepsilon}{\partial \nu} = n_\varepsilon \frac{\partial \mu_\varepsilon}{\partial \nu} = 0 \quad \text{on } (0, \infty) \times \partial\Omega, \quad (1.4.4)$$

where  $w_\varepsilon$  is defined by (1.1.5) and with the initial conditions (1.1.12).

We have the following existence theorem

**Theorem 1.4.1** (Weak solutions for the regularized system). *There exists a pair of functions  $(n_\varepsilon, \mu_\varepsilon)$  such that for all  $T \geq 0$ ,*

$$\begin{aligned} n_\varepsilon &\in L^2(0, T, H^1(\Omega)), & \partial_t n_\varepsilon &\in L^2(0, T, H^{-1}(\Omega)), \\ \mu_\varepsilon &\in L^2(0, T, H^1(\Omega)), & \partial_t \mu_\varepsilon &\in L^2(0, T, H^{-1}(\Omega)), \\ w_\varepsilon &\in L^2(0, T, H^2(\Omega)) \cap L^\infty(0, T, H^1(\Omega)), & w_\varepsilon^\gamma &\in L^2(0, T, L^2(\Omega)) \quad \partial_t w_\varepsilon \in L^2(0, T, H^1(\Omega)). \end{aligned}$$

*These functions satisfy the regularized Cahn-Hilliard model (1.4.2)–(1.4.4) in the following weak sense : for all test functions  $\chi \in L^2(0, T, H^1(\Omega))$ , it holds*

$$\int_0^T \langle \chi, \partial_t n_\varepsilon \rangle = - \int_{\Omega_T} B_\varepsilon(n_\varepsilon) \nabla \mu_\varepsilon \nabla \chi + \int_{\Omega_T} n_\varepsilon G(\mu_\varepsilon) \chi, \quad (1.4.5)$$

$$\sigma \int_{\Omega_T} \nabla \mu_\varepsilon \nabla \chi + \int_{\Omega_T} \mu_\varepsilon \chi = \delta \int_{\Omega_T} \nabla n_\varepsilon \nabla \chi + \int_{\Omega_T} \left(n_\varepsilon - \frac{\sigma}{\delta} \mu_\varepsilon\right)^\gamma \chi, \quad (1.4.6)$$

$$-\sigma \Delta w_\varepsilon + \frac{\sigma}{\delta} w_\varepsilon^\gamma + w_\varepsilon = n_\varepsilon \quad \text{a.e.}, \quad (1.4.7)$$

$$\mu_\varepsilon = w_\varepsilon^\gamma - \delta \Delta w_\varepsilon \quad \text{a.e.} \quad (1.4.8)$$

*Démonstration. Step 1. Galerkin approximation.* We consider  $\{\phi_i\}_{i \in \mathbb{N}}$  the eigenfunctions of the Laplace operator with zero Neumann boundary conditions.

$$-\Delta \phi_i = \lambda_i \phi_i \in \Omega \quad \text{with} \quad \nabla \phi_i \cdot \nu = 0 \quad \text{on } \partial\Omega, \quad (1.4.9)$$

which form an orthogonal basis of both  $H^1(\Omega)$  and  $L^2(\Omega)$  and we normalize them such that  $(\phi_i, \phi_j)_{L^2(\Omega)} = \delta_{ij}$ . Furthermore we assume without loss of generality that  $\lambda_1 = 0$ .

We consider the following discrete approximation of System (1.4.2)-(1.4.3)

$$n^N(t, x) = \sum_{i=1}^N c_i^N(t) \phi_i(x), \quad \mu^N(t, x) = \sum_{i=1}^N d_i^N(t) \phi_i(x), \quad (1.4.10)$$

$$\int_{\Omega} \partial_t n^N \phi_j = - \int_{\Omega} B_{\varepsilon}(n^N) \nabla \mu^N \nabla \phi_j + \int_{\Omega} n^N G(\mu^N) \phi_j, \quad \text{for } j = 1, \dots, N, \quad (1.4.11)$$

$$\int_{\Omega} \mu^N \phi_j = \delta \int_{\Omega} \nabla \left( n^N - \frac{\sigma}{\delta} \mu^N \right) \nabla \phi_j + \int_{\Omega} \left( n^N - \frac{\sigma}{\delta} \mu^N \right)^{\gamma} \phi_j, \quad \text{for } j = 1, \dots, N, \quad (1.4.12)$$

$$n^N(0, x) = \sum_{i=1}^N (n_0, \phi_i)_{L^2(\Omega)} \phi_i. \quad (1.4.13)$$

where the coefficients  $c_j^N, d_j^N$  for  $j = 1, \dots, N$  are determined by

$$\partial_t c_j^N = - \int_{\Omega} B_{\varepsilon} \left( \sum_{i=1}^N c_i^N \phi_i \right) \nabla \mu^N \nabla \phi_j + c_j^N \int_{\Omega} G(\mu^N), \quad (1.4.14)$$

$$d_j^N (1 + \sigma \lambda_j) = \delta \lambda_j c_j^N + \int_{\Omega} \left( \sum_{i=1}^N (c_i^N - \frac{\sigma}{\delta} d_i^N) \phi_i \right)^{\gamma} \phi_j, \quad (1.4.15)$$

$$c_j^N(0) = (n_0, \phi_j)_{L^2(\Omega)}. \quad (1.4.16)$$

Since the right hand side of Equation (1.4.14) depends continuously on the coefficients  $c_j^N$ , standard results on ODE systems gives the existence and uniqueness of a local solution to the initial value problem (1.4.14)–(1.4.16).

*Step 2. Inequalities and convergences.* Multiplying Equation (1.4.11) by  $d_j(t)$  and summing over  $j$  leads to

$$\int_{\Omega} \partial_t n^N \mu^N = - \int_{\Omega} B_{\varepsilon}(n^N) |\nabla \mu^N|^2 + \int_{\Omega} n^N G(\mu^N) \mu^N.$$

Rearranging the left-hand side, we obtain

$$\int_{\Omega} \partial_t n^N \mu^N = \int_{\Omega} \partial_t \left( n^N - \frac{\sigma}{\delta} \mu^N \right) \mu^N + \frac{1}{2} \frac{\sigma}{\delta} \frac{d}{dt} \int_{\Omega} |\mu^N|^2.$$

Using in Equation (1.4.12),  $\phi_j = \frac{d}{dt} (c_j^N - \frac{\sigma}{\delta} d_j^N) \phi_j$  and summing over  $j$ , we have

$$\int_{\Omega} \partial_t \left( n^N - \frac{\sigma}{\delta} \mu^N \right) \mu^N = \frac{\delta}{2} \frac{d}{dt} \int_{\Omega} \left| \nabla \left( n^N - \frac{\sigma}{\delta} \mu^N \right) \right|^2 + \frac{d}{dt} \int_{\Omega} \frac{\left( n^N - \frac{\sigma}{\delta} \mu^N \right)^{\gamma+1}}{\gamma+1}.$$

Altogether, we obtain the discrete energy dissipation

$$\frac{d}{dt} E(t) + \int_{\Omega} B_{\varepsilon}(n^N) |\nabla \mu^N|^2 = \int_{\Omega} n^N G(\mu^N) \mu^N, \quad (1.4.17)$$

in which the energy is defined by

$$E(t) = \frac{\delta}{2} \int_{\Omega} \left| \nabla \left( n^N - \frac{\sigma}{\delta} \mu^N \right) \right|^2 + \int_{\Omega} \frac{\left( n^N - \frac{\sigma}{\delta} \mu^N \right)^{\gamma+1}}{\gamma+1} + \frac{1}{2} \frac{\sigma}{\delta} \int_{\Omega} |\mu^N|^2.$$

Taking  $j = 1$  in (1.4.11) leads to  $\partial_t \int_{\Omega} n^N = \int_{\Omega} \phi_1 \int_{\Omega} n^N G(\mu^N) \phi_1$  where  $\phi_1$  is constant, and with the assumptions on  $G$  (see (1.1.13)), together with Gronwall's inequality we find

$$\left| \int_{\Omega} n^N \right| \leq C, \quad (1.4.18)$$

where  $C$  is a positive constant independent of  $N$ . Using this inequality in (1.4.17) and the assumptions on the source term  $G$ , we have

$$\frac{d}{dt} E(t) + \int_{\Omega} B_{\varepsilon}(n^N) |\nabla \mu^N|^2 \leq C. \quad (1.4.19)$$

Altogether, we find the following inequalities

$$\frac{\delta}{2} \int_{\Omega} \left| \nabla \left( n^N - \frac{\sigma}{\delta} \mu^N \right) \right|^2 \leq C, \quad (1.4.20)$$

$$\frac{\sigma}{2\delta} \int_{\Omega} |\mu^N|^2 \leq C, \quad (1.4.21)$$

$$\int_{\Omega_T} B_{\varepsilon}(n^N) |\nabla \mu^N|^2 \leq C, \quad (1.4.22)$$

$$\int_{\Omega} \frac{\left( n^N - \frac{\sigma}{\delta} \mu^N \right)^{\gamma+1}}{\gamma+1} \leq C. \quad (1.4.23)$$

Using the definition (1.4.1) of  $B_{\varepsilon}(n)$  and Inequality (1.4.22) we find a control on  $|\nabla \mu^N|^2$ . Combined with Inequality (1.4.20), we have

$$\int_{\Omega_T} |\nabla \mu^N|^2 \leq C, \quad \int_{\Omega_T} |\nabla n^N|^2 \leq C. \quad (1.4.24)$$

Using (1.4.18), (1.4.24) and the Poincaré-Wirtinger inequality, we obtain the following convergence as  $N \rightarrow +\infty$

$$n^N \rightharpoonup n_{\varepsilon} \quad \text{weakly in } L^2(0, T, H^1(\Omega)). \quad (1.4.25)$$

From (1.4.20)-(1.4.21), we find that the coefficients  $c_j^N, d_j^N$  are bounded and a global solution to (1.4.14)-(1.4.16) exists. Choosing  $j = 1$  in (1.4.12) gives

$$\int_{\Omega} \mu^N = \int_{\Omega} \left( n^N - \frac{\sigma}{\delta} \mu^N \right)^{\gamma},$$

which is bounded thanks to (1.4.23) and the Hölder inequality. Therefore, combining (1.4.24) and the Poincaré-Wirtinger inequality, we obtain

$$\mu^N \rightharpoonup \mu_{\varepsilon} \quad \text{weakly in } L^2(0, T, H^1(\Omega)). \quad (1.4.26)$$

Then, denoting by  $\Pi_N$  the projection operator from  $L^2(\Omega)$  to  $\text{span}\{\phi_1, \dots, \phi_N\}$ , using Equa-

tion (1.4.11), we have for every test functions  $\phi \in L^2(0, T, H^1(\Omega))$ ,

$$\begin{aligned} \left| \int_{\Omega_T} \partial_t n^N \phi \right| &= \left| \int_{\Omega_T} \partial_t n^N \Pi_N \phi \right| \\ &= \left| \int_{\Omega_T} B_\varepsilon(n^N) \nabla \mu^N \nabla \Pi_N \phi + \int_{\Omega_T} n^N G(\mu^N) \Pi_N \phi \right| \\ &\leq C \left( \int_{\Omega_T} B_\varepsilon(n^N) |\nabla \mu^N|^2 \right)^{1/2} \left( \int_{\Omega_T} |\nabla \Pi_N \phi|^2 \right)^{1/2} + C \left( \int_{\Omega_T} |\Pi_N \phi|^2 \right)^{1/2} \\ &\leq C \|\phi\|_{L^2(0, T, H^1(\Omega))}, \end{aligned}$$

where the last inequality is obtain from (1.4.22). Therefore, from the previous result, we can extract a subsequence such that

$$\partial_t n^N \rightharpoonup \partial_t n_\varepsilon \text{ weakly in } L^2(0, T, H^{-1}(\Omega)). \quad (1.4.27)$$

Hence, combining the weak convergences (1.4.25) and (1.4.27), and using the Lions-Aubin lemma we obtain the strong convergence

$$n^N \rightarrow n_\varepsilon \text{ strongly in } L^2(0, T, L^2(\Omega)). \quad (1.4.28)$$

*Step 3. Strong compactness for  $\mu^N$ .* To identify the limit in the source term, we need to find strong compactness for the potential  $\mu^N$ . As we have strong compactness for  $n^N$ , it is enough to find strong compactness for  $w^N$ .

Using the notation  $w^N = n^N - \frac{\sigma}{\delta} \mu^N$ , we change Equation (1.4.12) in

$$\sigma \int_{\Omega} \nabla w^N \nabla \phi_j + \frac{\sigma}{\delta} \int_{\Omega} (w^N)^\gamma \phi_j + \int_{\Omega} w^N \phi_j = \int_{\Omega} n^N \phi_j,$$

with  $w^N(t, x) = \sum_{i=1}^N q_i^N(t) \phi_i(x)$ . The coefficients  $q_j^N$  for  $j = 1, \dots, N$  are determined by the equation

$$(\sigma \lambda_j + 1) q_j^N + \frac{\sigma}{\delta} \int_{\Omega} \left( \sum_{i=1}^N q_i^N(t) \phi_i(x) \right)^\gamma \phi_j = c_j^N. \quad (1.4.29)$$

Thus, denoting  $z = \partial_t w$ , and computing the time derivative of the previous equation, we obtain

$$(\sigma \lambda_j + 1) \frac{d}{dt} q_j^N + \frac{\sigma}{\delta} \frac{d}{dt} \int_{\Omega} \left( \sum_{i=1}^N q_i^N(t) \phi_i(x) \right)^\gamma \phi_j = \frac{d}{dt} c_j^N.$$

We use the notation  $\partial_t w^N = z^N$ , Multiplying the previous equation by  $\phi_j(\partial_t w^N)$ , summing over the  $j$  and integrating over the domain lead to

$$\sigma \int_{\Omega} |\nabla z^N|^2 + \int_{\Omega} |z^N|^2 + \frac{\gamma \sigma}{\delta} \int_{\Omega} (w^N)^{\gamma-1} |z^N|^2 = \int_{\Omega} \partial_t n^N z^N.$$

Integrating the previous equation in time, and from the use of Young's inequality on the right hand side, we have

$$\int_{\Omega_T} \sigma |\nabla z^N|^2 + |z^N|^2 \leq C \|\partial_t n^N\|_{L^2(0, T, H^{-1}(\Omega))}^2 + \frac{\sigma}{2} \|z^N\|_{L^2(0, T, H^1(\Omega))}^2$$

Then, assuming  $1 - \frac{\sigma}{2} > 0$ , we find  $\|z\|_{L^2(0,T,H^1(\Omega))}^2 \leq C$ .

This previous result allows us to pass to the limit in the source term using the Lions-Aubin lemma. We refer to [287] to pass to the limit in other terms of the equation and detail the differences.

Lastly, it remains to prove the regularity of  $w$ ,  $w^\gamma$ . To do so, we use Equation (1.4.29) and successively multiply it by  $-\phi_j \Delta w^N$ , integrate it in space, use integration by parts, and integrate with respect to time, to arrive to

$$\sigma \int_{\Omega_T} |\Delta w^N|^2 + \int_{\Omega} |\nabla w^N|^2 + \frac{\sigma}{\delta} \gamma (w^N)^{\gamma-1} |\nabla w^N|^2 = - \int_{\Omega_T} n^N \Delta w^N.$$

Since  $n^N$  is bounded  $L^2(0, T, H^1(\Omega))$ , we can use Young inequality and elliptic regularity to find that  $w^N$  is bounded in  $L^2(0, T, H^2(\Omega))$ . Inequality (1.4.20) provides the  $L^\infty(0, T, H^1(\Omega))$  bound. Finally the regularity of  $(w^N)^\gamma$  comes from Equation (1.1.2).

This concludes the proof.  $\square$

### 1.4.2 A priori estimates

To show the existence of weak solutions of the non-regularized model (1.1.6)–(1.1.7), the idea is to pass to the limit  $\varepsilon \rightarrow 0$  in System (1.4.2)–(1.4.3). A priori estimates derived from the regularized model help us to find the required compactness and pass to the limit in the model. The computations of the a priori estimates follow closely the paper [287] where the case of a single-well potential was considered. The addition of the source term  $G$ , which is not present in [287], induces the need of new computations. The second main difference is that we consider a potential term  $w^\gamma$  in (1.1.7) instead of a smooth and bounded potential as it was done before. We start by defining the entropy of the system

$$\Phi_\varepsilon[n] = \int_{\Omega} \phi_\varepsilon(n(x)) dx, \quad \phi_\varepsilon''(x) = \frac{1}{B_\varepsilon(n)}, \quad \phi_\varepsilon(0) = \phi_\varepsilon'(0) = 0,$$

and we recall the energy

$$\mathcal{E}_\varepsilon[n] = \int_{\Omega} \frac{(n - \frac{\sigma}{\delta} \mu)^{\gamma+1}}{\gamma+1} + \frac{\delta}{2} \left| \nabla \left( n - \frac{\sigma}{\delta} \mu \right) \right|^2 + \frac{\sigma}{\delta} \frac{|\mu|^2}{2}.$$

In comparison with the previous subsection, when  $\varepsilon \rightarrow 0$  we lose the uniform  $L^2(\Omega_T)$  bound on  $\nabla \mu$  which was obtained with the regularized mobility. This bound is actually recovered with the estimates provided by the entropy that could not be used in the Galerkin scheme. Also, when  $\varepsilon \rightarrow 0$ , the time derivative of the density lies in a larger space than  $L^2(0, T, H^{-1}(\Omega))$ , which is  $(L^4(0, T, W^{1,s}(\Omega)))'$  where  $s > 2$ . This prevents us from using the previous computations and we lose the bound on  $\partial_t w$  and  $\partial_t \mu$ . To recover their strong convergence and thus to identify the source term, we need to rely on the Fréchet-Kolmogorov theorem. This last difficulty is dealt with the following proposition

**Proposition 1.4.2** (Compactness for  $w_\varepsilon$  and  $\mu_\varepsilon$ ). *The sequences  $(w_\varepsilon)_\varepsilon$  and  $(\mu_\varepsilon)_\varepsilon$  converge strongly in  $L^1(\Omega_T)$ .*

*Démonstration.* We use the Fréchet-Kolmogorov theorem as in Appendix 1.5.1. Indeed, we have the compactness in space for  $w$  and Equation (1.1.2) provides

$$-\sigma \Delta [w(t+h) - w(t)] + \frac{\sigma}{\delta} [w^\gamma(t+h) - w^\gamma(t)] + w(t+h) - w(t) = n(t+h) - n(t).$$

Multiplying formally by  $\text{sgn}(w(t+h) - w(t))$  (rigorously by  $\phi'(w(t+h) - w(t))$  where  $\phi$  is a convex approximation of the absolute value), integrating in space and using Appendix 1.5.1, we obtain the compactness for  $w$ . Since  $\mu = \frac{\delta}{\sigma}(n - w)$  we have compactness for  $\mu$ .  $\square$

**Remark 1.4.3.** The strong convergence actually holds in higher Lebesgue spaces since  $(w_\varepsilon)_\varepsilon$  is bounded uniformly in  $L^2(0, T, H^2(\Omega)) \cap L^\infty(0, T, L^{\gamma+1}(\Omega))$  and  $(\mu_\varepsilon)_\varepsilon$  in  $L^2(0, T, H^1(\Omega))$ .

### 1.4.3 Limit $\varepsilon \rightarrow 0$

From the previous estimates on the regularized model, we are now in position to prove the existence of global weak solutions for System (1.1.6)–(1.1.7).

**Theorem 1.4.4** (Existence of weak solutions). *Assume an initial condition satisfying (1.1.12). Then, for  $\sigma$  small enough, there exists a global weak solutions  $(n, \mu)$  of Equations (1.1.6)–(1.1.7) such that*

$$\begin{aligned} n &\in L^2(0, T, H^1(\Omega)), \quad \partial_t n \in (L^4(0, T, W^{1,s}(\Omega)))', \\ \mu &\in L^2(0, T, H^1(\Omega)), \\ w &\in L^2(0, T, H^2(\Omega)) \cap L^\infty(0, T, H^1(\Omega)), \quad w^\gamma \in L^2(0, T, L^2(\Omega)), \\ n &\geq 0, \quad w \geq 0, \quad \text{a.e. in } \Omega_T, \end{aligned}$$

where  $s$  is defined in proposition 1.2.2. Moreover, as  $\varepsilon \rightarrow 0$ , the inequalities provided by the energy and entropy hold true.

*Démonstration.* The estimates on  $\partial_t n$  has been proved in Section 2. The proof of this theorem is a straightforward adaptation of Theorem 5 in [287] using the computation of Section 1.4.2, therefore, we do not repeat the proof arguments here. The nonnegativities of  $n$  and  $w$  are a consequence of Proposition 1.4.5.  $\square$

The regularity of the solutions is higher than it is expected in Theorem 1.4.4. We refer to Section 1.3 for the proof of this result.

**Proposition 1.4.5** (Nonnegativity of  $n$  and  $w$ ). *The solution  $(n, w)$  of System (1.1.1)–(1.1.2) defined by Theorem 1.4.4 is nonnegative, i.e.*

$$n(x, t), w(x, t) \geq 0, \quad \text{a.e. in } \Omega.$$

*Démonstration.* The proof of the nonnegativity of the density  $n$  follows the same argument than [287] that uses the boundedness of the entropy (uniform in  $\varepsilon$ ). Hence, we do not repeat the proof here. Now, we recall that the repulsive potential from (1.1.2) is actually defined by assumption in the introduction as  $w^\gamma = \max(0, w^\gamma)$ . Then, since  $n \geq 0$ , Equation (1.1.2) implies that for every function  $\eta \geq 0$ , we have

$$\int_{\Omega} \nabla w \cdot \nabla \eta + \int_{\Omega} \max(0, w^\gamma) \eta + \int_{\Omega} w \eta \geq 0,$$

where we supposed  $\sigma = \delta = 1$  for the sake of clarity. Using the previous inequality, and choosing  $\eta = -w_-$  (where  $w_-$  represents the negative part of  $w$ ), we obtain

$$\int_{\Omega} |\nabla w_-|^2 + \int_{\Omega} w_-^2 \leq 0.$$

This achieves the proof of the nonnegativity of  $w$ .  $\square$

**Remark 1.4.6.** A main difference with [287] is that we can not find an upper bound using an entropy argument. Indeed, the result in [287] relies on the singularity of the potential at  $n = 1$ . Furthermore, Dai and Du [113] notice that with a smooth potential at  $n = 1$ , one cannot prevent the solution from exceeding this threshold. This is due to the Gibbs-Thomson effect, where the mean curvature of the interface plays a role in the concentration of the phases. However, from a different argument relying on the Alikakos iteration method [16], see Inequality (1.3.2), we are able to show an  $L^\infty$ -bound for System (1.1.1)–(1.1.2) and therefore better regularity for the functions of our system.

## 1.5 Appendix

### 1.5.1 Compactness with the Fréchet-Kolmogorov theorem

We provide another method to prove strong compactness in  $\gamma$  for  $n$  only using the fact that  $n\nabla\mu$  and  $\nabla n$  are integrable. This proposition can be applied to prove compactness in  $\sigma$  and  $\varepsilon$  with the parameters defined respectively in Section 2 and 4.

**Proposition 1.5.1** (Strong compactness for  $n$ ). *The sequence  $(n_{\sigma,\gamma})_\gamma$  is compact in  $L^2(0, T, L^p(\Omega))$  for  $1 \leq p < 2d/(d-2)$  if  $d > 2$  and  $1 \leq p < \infty$  else.*

The proof of this proposition uses a sequence  $(\varphi_\delta)_{\delta>0} \in C_c(\mathbb{R}^d)$  of standard mollifiers with mass 1 such that

$$\|\nabla^k \varphi_\delta\|_{L^1(\Omega)} \leq \frac{C}{\delta^k},$$

for any function  $g \in L^p(\Omega)$ ,

$$\|g * \varphi_\delta\|_{L^p(\Omega)} \leq \|\varphi_\delta\|_{L^1(\Omega)} \|g\|_{L^p(\Omega)},$$

and when  $g \in W^{1,p}(\Omega)$  it holds

$$\|g * \varphi_\delta - g\|_{L^p} \leq \delta \|\nabla g\|_{L^p(\Omega)}.$$

*Proof of proposition 1.5.1.* Since  $\nabla n$  is bounded in  $L^1(\Omega_T)$  we only need to prove the time compactness

$$\lim_{|h| \rightarrow 0} \int_0^{T-h} \int_\Omega |n_{\sigma,\gamma}(t+h, x) - n_{\sigma,\gamma}(t, x)| dx dt = 0 \quad \text{uniformly in } \gamma.$$

Using the mollifiers with  $\delta$  depending on  $h$  to be specified later on, we first notice that

$$\begin{aligned} \int_0^{T-h} \int_\Omega |n(t+h, x) - n(t, x)| dx dt &\leq \int_0^{T-h} \int_\Omega |n(t, x) - n(t, \cdot) * \varphi_\delta(x)| dx dt \\ &+ \int_0^{T-h} \int_\Omega |n(t+h, x) - n(t+h, \cdot) * \varphi_\delta(x)| dx dt \\ &+ \int_0^{T-h} \int_\Omega |n(t+h, \cdot) * \varphi_\delta(x) - n(t, \cdot) * \varphi_\delta(x)| dx dt. \end{aligned}$$



For the first and second term, the computations are the same, hence, we only present it for the first term. Using the properties of the mollifiers, we have

$$\int_0^{T-h} \int_{\Omega} |n(t, x) - n(t, \cdot) * \varphi_{\delta}(x)| dx dt \leq \delta \int_0^T \|\nabla n(t)\|_{L^1(\Omega)} dt \leq C\delta.$$

To control  $\|\nabla n\|_{L^1(\Omega)}$  we used the Cauchy-Schwarz inequality and Proposition 1.2.2. The third term can be written as

$$\int_0^{T-h} \int_{\Omega} |n(t+h, \cdot) * \varphi_{\delta}(x) - n(t, \cdot) * \varphi_{\delta}(x)| dx dt = \int_0^{T-h} \int_{\Omega} \left| \int_t^{t+h} \partial_s n(s, x) * \varphi_{\delta}(x) ds \right| dx dt.$$

And using Equation (1.1.6) yields

$$\begin{aligned} & \int_0^{T-h} \int_{\Omega} \left| \int_t^{t+h} \partial_s n(s, \cdot) * \varphi_{\delta}(x) ds \right| dx dt \\ &= \int_0^{T-h} \int_{\Omega} \left| \int_t^{t+h} [\operatorname{div}(n\nabla\mu) + nG(\mu)](s, \cdot) * \varphi_{\delta}(x) ds \right| dx dt \\ &\leq \int_0^{T-h} \int_{\Omega} \left| \int_t^{t+h} \sum_{i=1}^d n \partial_{x_i} \mu(s, \cdot) * \partial_{x_i} \varphi_{\delta}(x) ds \right| dx dt \\ &+ \int_0^{T-h} \int_{\Omega} \left| \int_t^{t+h} nG(\mu)(s, \cdot) * \varphi_{\delta}(x) ds \right| dx dt. \end{aligned}$$

The first term is bounded by

$$\int_0^{T-h} \int_{\Omega} \left| \int_t^{t+h} \sum_{i=1}^d n \partial_{x_i} \mu(s, \cdot) * \partial_{x_i} \varphi_{\delta}(x) ds \right| dx dt \leq C \int_0^{T-h} \int_t^{t+h} \|n\nabla\mu(s)\|_{L^1(\Omega)} \|\nabla\varphi_{\delta}\|_{L^1(\Omega)}.$$

Writing  $n\nabla\mu = n^{1/2}n^{1/2}\nabla\mu$ , using the Cauchy-Schwarz inequality, Inequality (1.2.8), the first term is bounded by  $Ch/\delta$ .

The second term is bounded by  $Ch$  using assumptions on  $G$ . Choosing  $\delta = h^{1/2}$  gives compactness in  $L^1(\Omega_T)$ .

Finally using Proposition 1.2.2 and interpolation, we get the result.  $\square$

## 1.5.2 Uniqueness with no source term

We consider Equations (1.1.1)-(1.1.2) with  $G = 0$ . Therefore we have conservation of the mass. To simplify the notations we suppose  $\sigma = \delta = 1$ . We retain the regularity of the solutions from Section 1.3. Writing  $n = n_2 - n_1$  and  $w = w_2 - w_1$  the difference of two solutions we consider  $\varphi(t)$  such that

$$\begin{aligned} -\Delta\varphi(t) &= n \quad \text{in } \Omega, \\ \nabla\varphi(t) \cdot \nu &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Multiplying Equation (1.1.1) by  $\varphi$ , integrating in space and using integrations by parts we obtain

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \phi|^2 - \int (n_2^2 - n_1^2) \Delta \varphi - \int n \nabla w_2 \cdot \nabla \varphi - \int n_1 \nabla w \cdot \nabla \varphi = 0.$$

We denote by  $I_1, I_2, I_3$  the last three terms. Using the definition of  $\varphi$ , we have

$$I_1 = - \int (n_2^2 - n_1^2) \Delta \varphi = \int (n_2^2 - n_1^2) (n_2 - n_1),$$

which is nonnegative since  $x \mapsto x^2$  is increasing for  $x \geq 0$ . Now, we treat  $I_2$ . Using integration by parts and  $n = -\Delta \varphi$  one can show that

$$\begin{aligned} I_2 &= - \sum_{i,j} \int \partial_i \varphi \partial_{ij} w_2 \partial_j \varphi - \sum_{i,j} \int \partial_i \varphi \partial_j w_2 \partial_{ij} \varphi \\ &= J_1 + J_2. \end{aligned}$$

Using integrations by parts on the second term we find

$$\begin{aligned} J_2 &= \sum_{i,j} \int \partial_j \frac{|\partial_i \varphi|^2}{2} \partial_j w_2 + \sum_{i,j} \int |\partial_i \varphi|^2 \partial_{jj} w_2 \\ &= \frac{1}{2} \int |\nabla \varphi|^2 \Delta w_2. \end{aligned}$$

We finally obtain

$$I_2 \leq C \int |D^2 w_2| |\nabla \varphi|^2.$$

From Inequality (1.3.3) together with the Calderón–Zygmund lemma and using  $\nabla \varphi \in L^\infty$  we find

$$\begin{aligned} I_2 &\leq Cp \left( \int |\nabla \varphi|^{2p/(p-1)} \right)^{(p-1)/p} \\ &\leq Cp \|\nabla \varphi\|_{L^\infty}^{2/p} \left( \int |\nabla \varphi|^2 \right)^{(p-1)/p}, \end{aligned}$$

for every  $1 \leq p < \infty$ . For  $I_3$  we recall that  $n_1$  is bounded in  $L^\infty$ . Using the Cauchy-Schwarz inequality we only need to show that  $\|\nabla w\|_{L^2} \leq C \|\nabla \varphi\|_{L^2}$ .

Equation (1.1.2) for the difference of the two solutions can be written as

$$-\Delta w + w_2^\gamma - w_1^\gamma + w = -\Delta \varphi.$$

Now, we multiply by  $w$ , use integration by parts and the fact that  $x \mapsto x^\gamma$  is increasing for  $x \geq 0$  to get

$$\int |\nabla w|^2 + w^2 \leq \int |\nabla \varphi \cdot \nabla w|.$$

Applying Young's inequality yields the result. Therefore

$$I_3 \leq C \|\nabla \varphi\|_{L^2}^2.$$

Combining the previous results we obtain

$$\frac{d}{dt}\eta(t) \leq Cp \max(\eta(t)^{1-1/p}, \eta(t)),$$

for every  $1 \leq p < \infty$  where  $\eta(t) = \int |\nabla\varphi(t)|^2$ .

Following [48, 41], for  $t < 1/C$ , the solution is bounded by

$$\eta(t) \leq (Ct)^p.$$

For  $t < 1/(2C)$  we then find

$$\eta(t) \leq 2^{-p}.$$

Letting  $p \rightarrow \infty$  we find uniqueness on  $[0, \frac{1}{2C})$  and this procedure can be iterated on the whole interval of existence.

## Chapitre 2

# Nonlocal Cahn-Hilliard equation with degenerate mobility : Incompressible limit and convergence to stationary states

### Abstract

The link between compressible models of tissue growth and the Hele-Shaw free boundary problem of fluid mechanics has recently attracted a lot of attention. In most of these models, only repulsive forces and advection terms are taken into account. In order to take into account long range interactions, we include a surface tension effect by adding a nonlocal term which leads to the degenerate nonlocal Cahn-Hilliard equation, and study the incompressible limit of the system. The degeneracy and the source term are the main difficulties. Our approach relies on a new  $L^\infty$  estimate obtained by De Giorgi iterations and on a uniform control of the energy despite the source term. We also prove the long-term convergence to a single constant stationary state of any weak solution using entropy methods, even when a source term is present. Our result shows that the surface tension in the nonlocal (and even local) Cahn-Hilliard equation will not prevent the tumor from completely invading the domain.

### 2.1 Introduction

Nonlocal parabolic equations are commonly used to describe living tissues because cells experience two types of forces : repulsive and attractive. The repulsion arises at high volume fraction because, locally, cells occupy a non-vanishing volume, while cell adhesion and chemotaxis create attraction at long range, i.e., low densities. These effects, as well as surface tension effects, can be considered by using the Cahn-Hilliard equation (see, e.g., [277] for a review on these models). Our work is dedicated to the analysis of the nonlocal Cahn-Hilliard equation for long range interactions with a repulsive potential. More precisely we are interested in two results : the incompressible limit connecting mechanistic and free boundary descriptions of the tissue and the long-time asymptotics of equations. Concerning the first result, the main difficulty is that we lose any maximum principle and we have to rely on different arguments to obtain the same results concerning the incompressible limit. Concerning the convergence to the stationary state,

we prove that it converges to a nonnegative constant which shows that the surface tension effect (modeled by the Cahn-Hilliard equation) is not strong enough to prevent the tumor from invading the entire domain.

### 2.1.1 Mathematical setting

Our settings is as follows : we let  $\Omega$  be the  $d = 1, 2, 3$  dimensional flat torus in  $\mathbb{R}^d$ , which is particularly useful when treating nonlocal terms and we consider the equation

$$\begin{aligned} \partial_t u - \operatorname{div}(u \nabla \mu) &= uG(p) \quad \text{in } \Omega \times (0, T), \\ \mu &= p + B_\varepsilon(u) \quad \text{in } \Omega \times (0, T), \end{aligned} \quad (2.1.1)$$

with the initial condition  $u(0) = u_0 \geq 0$  in  $\Omega$  and  $u(x, t) \geq 0$  represents the cell density. Here,  $B_\varepsilon$  denotes the nonlocal operator defined as

$$B_\varepsilon[u](x) = \frac{1}{\varepsilon^2}(u(x) - \omega_\varepsilon * u(x)) = \frac{1}{\varepsilon^2} \int_{\Omega} \omega_\varepsilon(y)(u(x) - u(x-y)) \, dy \quad (2.1.2)$$

for fixed  $\varepsilon$  small enough (in order to be able to use [155, Lemma C.1] and Lemma 2.5.3) and  $\omega_\varepsilon$  is a usual mollification kernel  $\omega_\varepsilon(x) = \frac{1}{\varepsilon^d} \omega(\frac{x}{\varepsilon})$  with  $\omega$  compactly supported in the unit ball of  $\mathbb{R}^d$  satisfying

$$\int_{\mathbb{R}^d} \omega(y) \, dy = 1, \quad \omega(\cdot) \text{ is radial.} \quad (2.1.3)$$

The pressure and source term are defined, for  $u \geq 0$ , as

$$p = u^\gamma, \quad \gamma \geq 1, \quad G(p) = p_H - p, \quad (2.1.4)$$

with  $p_H > 0$  a constant called the homeostatic pressure, which is the threshold where cells begin to die, assuming that pressure produces an inhibitory effect on cell proliferation.

We comment the different terms appearing in the equation. First,  $u$  is the density of tumor cells and can be thought of as being between 0 and 1. However, this fact is not easy to prove since the maximum principle does not hold here. Using a De Giorgi iteration technique, we can prove however that the bound holds with a small perturbation terms which vanishes as  $\gamma \rightarrow \infty$  (see Lemma 2.3.1). From the Cahn-Hilliard terminology, we refer to  $\mu$  as the chemical potential, which is composed by two terms : one is the pressure  $p = u^\gamma$  and the other is  $B_\varepsilon$ , the approximation of the Laplace operator, which takes into account surface tension effects, see for instance [155]. Concerning the initial condition, we distinguish two sets of assumptions.

*Assumption 2.1.1* (Initial condition). We assume :

(A)  $0 \leq u_0 \leq p_H^{\frac{1}{\gamma}}$  for almost any  $x \in \Omega$ .

Note that the same assumption has already been considered, e.g., in [116] and that it implies  $u_0 \in L^q(\Omega)$ , for any  $q \geq 1$ , since  $\Omega$  is of finite Lebesgue measure. For the single Section 2.3, we need additionally :

*Assumption 2.1.2* (Additional assumption for Section 2.3). We assume :

(B) There is  $\gamma_0 > 0$  and  $C = C(\gamma_0) > 0$  such that  $\|\Delta(u_0^{1+\gamma})\|_{L^1(\Omega)} + \|\nabla u_0\|_{L^2(\Omega)} + \|\Delta u_0\|_{L^1(\Omega)} \leq C, \quad \forall \gamma \geq \gamma_0$ .

System (2.1.1) is associated with the energy  $\mathcal{E}$  and entropy  $\Phi$ , respectively defined by

$$\mathcal{E}(u) := \frac{1}{4\varepsilon^2} \int_{\Omega} \int_{\Omega} \omega_{\varepsilon}(y) |u(x) - u(x-y)|^2 dy dx + \int_{\Omega} \frac{u^{\gamma+1}}{\gamma+1} dx \geq 0, \quad (2.1.5)$$

$$\Phi(u) := \int_{\Omega} \left( \frac{u}{p_H^{\frac{1}{\gamma}}} \log \left( \frac{u}{p_H^{\frac{1}{\gamma}}} \right) - \frac{u}{p_H^{\frac{1}{\gamma}}} + 1 \right) dx \geq 0. \quad (2.1.6)$$

They formally satisfy the identities

$$\frac{d}{dt} \mathcal{E}(u) + \int_{\Omega} u |\nabla \mu|^2 = \int_{\Omega} u \mu G(p), \quad (2.1.7)$$

$$\begin{aligned} \frac{d}{dt} \Phi(u) + \frac{1}{2\varepsilon^2 p_H^{\frac{1}{\gamma}}} \int_{\Omega} \int_{\Omega} \omega_{\varepsilon}(y) |\nabla u(x) - \nabla u(x-y)|^2 dx dy \\ + \frac{1}{p_H^{\frac{1}{\gamma}}} \int_{\Omega} \frac{4\gamma}{(\gamma+1)^2} |\nabla |u|^{\frac{\gamma+1}{2}}|^2 dx - \frac{1}{p_H^{\frac{1}{\gamma}}} \int_{\Omega} u \log \left( \frac{u}{p_H^{\frac{1}{\gamma}}} \right) G(p) dx = 0. \end{aligned} \quad (2.1.8)$$

Moreover they provide us with direct a priori estimates, provided we can control the integral related to the source term in (2.1.7), which may change sign. Here, we assume that we have existence of solutions with regularity typical of the Cahn-Hilliard equation. We do not include the proof, since most of the a priori estimates are derived in Section 2.2. For a rigorous proof of existence by means of an approximating scheme, we refer for instance to [155].

**Lemma 2.1.3.** *Let  $u_0$  satisfy assumption (2.1.1). Then, for any  $T > 0$ , there exist constants  $C_0(T, \mathcal{E}(u_0), \Phi(u_0), \varepsilon)$  and  $C_1(T, \mathcal{E}(u_0), \Phi(u_0), \varepsilon, \gamma)$  and a global weak solution  $u$  such that,*

$$u \geq 0 \quad \text{a.e. in } \Omega \times (0, \infty), \quad (2.1.9)$$

$$u \in C([0, \infty); (W^{1,r}(\Omega))') \cap C_{weak}([0, \infty); L^{\gamma+1}(\Omega)), \quad r = \frac{(\gamma+1)(2\gamma+1)}{\gamma^2}, \quad (2.1.10)$$

$$\bar{u}(t) := \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx \leq p_H^{\frac{1}{\gamma}} \quad \forall t \geq 0, \quad (2.1.11)$$

$$\|u\|_{L^2(0,T;H^1(\Omega))} + \frac{1}{\gamma+1} \|u\|_{L^{\infty}(0,T;L^{\gamma+1}(\Omega))}^{\gamma+1} + \|u\|_{L^{2\gamma+1}(\Omega \times (0,T))} \leq C_0, \quad (2.1.12)$$

$$\|\partial_t u\|_{L^2(0,T;(W^{1,r}(\Omega))')} + \|\partial_t u\|_{L^{q'}(0,T;(W^{1,q}(\Omega))')} \leq C_1, \quad (2.1.13)$$

where  $\frac{1}{q} + \frac{1}{q'} = 1$ ,  $q = \frac{2(2\gamma+1)}{\gamma}$ . Moreover, for any  $v \in W^{1,r}(\Omega)$ ,  $u$  satisfies

$$\begin{aligned} \langle \partial_t u, v \rangle_{(W^{1,r}(\Omega))', W^{1,r}(\Omega)} + \int_{\Omega} u \nabla p \cdot \nabla v dx + \frac{1}{\varepsilon^2} \int_{\Omega} u \nabla u \cdot \nabla v dx - \frac{1}{\varepsilon^2} \int_{\Omega} u (\nabla \omega_{\varepsilon} * u) \cdot \nabla v dx \\ = \int_{\Omega} u G(p) v dx, \quad \text{for almost every } t > 0, \end{aligned} \quad (2.1.14)$$

with  $u(0) = u_0$  almost everywhere in  $\Omega$ . Here  $\langle \cdot, \cdot \rangle_{(W^{1,r}(\Omega))', W^{1,r}(\Omega)}$  denotes the duality product between  $(W^{1,r}(\Omega))'$  and  $W^{1,r}(\Omega)$ .

**Remark 2.1.4.** The first continuity result follows from the bounds on  $\partial_t u$  in (2.1.13). Moreover,

being  $u \in L^\infty(0, T; L^{\gamma+1}(\Omega))$ , we also deduce that  $u \in C_{weak}([0, T]; L^{\gamma+1}(\Omega))$ , see [54, Lemma II.5.9]. This clearly also implies that  $\bar{u} = \frac{\int_\Omega u \, dx}{|\Omega|} \in C([0, T])$ .

### 2.1.2 The main results

Our first result establishes the incompressible limit  $\gamma \rightarrow \infty$  of the system (2.1.1) which links two descriptions of the tumor growth : mechanistic and free-boundary. The main mathematical novelty here is the nonlocality which makes it difficult to establish the uniform  $L^\infty$  bound on  $u_\gamma$ . To overcome this problem, we apply the De Giorgi iterations, see Lemma 2.3.1, in the spirit of [182, 293].

**Theorem 2.1.5** (Incompressible limit). *Let  $u_\gamma$  be a weak solution to (2.1.1) as defined in Lemma 2.1.3 and initial datum satisfying Assumptions 2.1.1-2.1.2. Then, as  $\gamma \rightarrow \infty$ , we have for all  $T > 0$ , up to a (not relabeled) subsequence*

$$\begin{aligned} u_\gamma &\overset{*}{\rightharpoonup} u_\infty \text{ in } L^\infty(\Omega \times (0, \infty)), \\ u_\gamma &\rightarrow u_\infty \text{ in } L^q(\Omega \times (0, T)) \quad \forall q \in [2, +\infty), \\ u_\gamma &\rightharpoonup u_\infty \text{ in } L^2(0, T; H^1(\Omega)), \\ \partial_t u_\gamma &\rightharpoonup \partial_t u_\infty \text{ in } L^2(0, T; (H^1(\Omega))'), \\ p_\gamma &\rightharpoonup p_\infty \text{ in } L^2(0, T; H^1(\Omega)), \\ p_\gamma &\rightarrow p_\infty \text{ in } L^r(\Omega \times (0, T)), \quad \forall r \in [2, 3), \end{aligned}$$

where  $u_\infty$  and  $p_\infty$  satisfy in  $\mathcal{D}'(\Omega \times [0, \infty))$

$$\partial_t u_\infty - \operatorname{div}(u_\infty \nabla(p_\infty + B_\varepsilon(u_\infty))) = u_\infty G(p_\infty), \quad (2.1.15)$$

$$p_\infty \left( \Delta p_\infty + \frac{1}{2\varepsilon^2} \Delta u_\infty^2 - \frac{1}{\varepsilon^2} \operatorname{div}(u_\infty (\nabla \omega_\varepsilon * u_\infty)) + u_\infty G(p_\infty) \right) = 0, \quad (2.1.16)$$

$$0 \leq u_\infty \leq 1, \quad p_\infty \geq 0, \quad p_\infty(1 - u_\infty) = 0 \quad \text{almost everywhere in } \Omega \times (0, \infty), \quad (2.1.17)$$

with  $u_\infty(0) = u_0$ . Furthermore it holds  $\langle \partial_t u_\infty, p_\infty \rangle_{(H^1(\Omega))', H^1(\Omega)} = 0$  for almost any  $t \in (0, \infty)$ .

This theorem entails that in the limit  $\gamma \rightarrow \infty$  we can consider the measurable set  $\Omega(t) := \{x \in \Omega : p_\infty(t) > 0\}$ , where  $u_\infty = 1$  by the graph relation (2.1.17) so that it can be interpreted as the ‘tumor zone’. Note that it must hold

$$-\Delta p_\infty - \frac{1}{2\varepsilon^2} \Delta u_\infty^2 = -\frac{1}{\varepsilon^2} \operatorname{div}(u_\infty (\nabla \omega_\varepsilon * u_\infty)) + u_\infty G(p_\infty) \quad \text{in } \mathcal{I}nt(\Omega(t))$$

which yields a Hele-Shaw type equation.

Our second result is concerned with the convergence to stationary states. We distinguish two cases : when  $G(p) = p_H - p$  and when  $G(p) = 0$ . The main novelty lies in the first case which is not conservative and its proof requires a careful analysis of the entropy. We prove that as  $t \rightarrow \infty$ , the solution converges to the constant  $p_H^{\frac{1}{2}} > 0$ , which shows that the surface tension is not strong enough to prevent the tumor from invading the entire domain. We have the

**Theorem 2.1.6** (Long time behaviour). *Let  $u$  be a solution to (2.1.1) with fixed  $\gamma \geq 1$  in the sense of Lemma 2.1.3 and initial datum satisfying Assumption 2.1.1. Then, if  $u_0 \not\equiv 0$ , there are two cases :*

— for  $G(p) = p_H - p$ , we have

$$\|u(t) - p_H^{\frac{1}{\gamma}}\|_{L^q(\Omega)} \rightarrow 0 \text{ as } t \rightarrow \infty, \quad \forall q \in [1, \gamma + 1). \quad (2.1.18)$$

— For  $G(p) = 0$  and  $\gamma \geq 1$ , we have an exponential decay towards the mean value : there exists a constant  $C = C(\Omega, \gamma, q, \Phi(u_0), \bar{u}_0)$  such that

$$\|u(t) - \bar{u}_0\|_{L^q(\Omega)} \lesssim e^{-Ct}, \quad \forall q \in [1, \gamma + 1). \quad (2.1.19)$$

There are two possibilities to prevent steady states to be constant. The first one is to consider different potentials than just repulsive ones like  $u^\gamma$ . The second possibility is to include an external force, which can be taken into account either by a generic force that acts directly on the cells like it was done in [154]. One can also include the effects of nutrients and impose that the tumor cells die in the regions where there are no nutrients.

**Remark 2.1.7.** In the case of nonzero source term  $G$ , Theorem 2.1.6 implies that the constant solution  $u \equiv 0$  is unstable, whereas  $p_H^{\frac{1}{\gamma}}$  is the only asymptotically stable equilibrium and *any* weak solution  $u$  (in the sense of Lemma 2.1.3) departing from *any* nonzero initial datum converges to  $p_H^{\frac{1}{\gamma}}$ .

### 2.1.3 Literature review

**Incompressible limit for tumor model.** The incompressible limit connects two models of tumor growth : the mechanistic one proposed in [289] and the free-boundary one analysed in [275]. Since [289], many substantial contributions were made, allowing nutrients [119], advection effects [120, 239, 237], additional structuring variable [116], congesting flows [211], concerning two species [127, 206] or including additional surface tension via Cahn-Hilliard equation [151, 154].

One of the major difficulties for establishing the incompressible limit is proving strong compactness of the gradient of the pressure. The main tool is the celebrated Aronson-Benilan estimate [28, 112]. The estimate has been recently readressed in [49] but the generalization available there are not applicable for the pressure  $p = u^\gamma + \frac{1}{\varepsilon^2}u$  as in our case cf. [49, Theorem 4.1]. Another direct technique was developed in [260] which is based on deducing strong convergence from a sort of energy equality. This is the strategy we follow in our proof.

**Nonlocal Cahn-Hilliard equation.** The nonlocal Cahn-Hilliard is an approximation of the Cahn-Hilliard equation proposed to model dynamics of phase separation [70]. While originally introduced in the context of material science, it is currently widely applied also in biology [265, 12, 287]. The nonlocal equation was obtained for the first time by Giacomin and Lebowitz as the limit of interacting particle systems [190, 189]. Their work can be considered as the first derivation of the Cahn-Hilliard equation up to a delicate passage to the limit from the nonlocal equation to the local one. The latter problem was in fact addressed only recently, first for the case of the constant mobility [123, 121, 273, 122] and finally for the case of degenerate mobility [155, 81]. Another derivation as a hydrodynamic limit of the Vlasov-type equation was proposed recently in [150], following the work of Takata and Noguchi [317].

**Entropy dissipation methods and asymptotic analysis.** For establishing convergence to stationary states we use methods based on the entropy dissipation. In the simplest scenario, it



can be applied to the PDEs equipped with the entropy  $\Phi$  which decreases with some dissipation  $\mathcal{D}\Phi(t) \geq 0$  :

$$\partial_t \Phi(t) + \mathcal{D}\Phi(t) \leq 0.$$

Then, one tries to prove that the dissipation is bounded from below by the entropy  $|\Phi(t)|^\alpha \leq \mathcal{D}\Phi(t)$  so that  $\Phi(t) \rightarrow 0$  with an explicit convergence rate (exponential if  $\alpha = 1$  and polynomial if  $\alpha > 1$ ). Finally, by virtue of Csiszár-Kullback inequality, one deduces convergence in  $L^1$ . The last step requires conservation of mass which is not available if  $G(p) \neq 0$  in (2.1.1).

The approach presented above was successfully applied to study classical problems, including heat equation, porous media equation, fast diffusion equation or  $p$ -Laplace equation (see [78] and references therein). In particular, we refer to Section 2.4.2 where we present the method in detail for the Cahn-Hilliard equation without the source term. Finally, we point out that the entropy dissipation method is also applied to several problems of current interest, including Landau equation [74], reaction-cross-diffusion systems [115] and reaction-diffusion systems with nonlinear diffusion [172]. Another method to obtain convergence to equilibrium, applied in the context of the Cahn-Hilliard equation, is via the Łojasiewicz-Simon inequality [105, 7].

## 2.2 Basic a priori estimates

The energy/entropy structure usually provides a priori estimates on the solutions. However, in the case of a source term which may change sign, we first need to prove that we can control their dissipation. Before tackling the problem of energy/entropy we first show a basic estimate which ensures the control of the mass of the system, uniformly in time. This estimate is useful to obtain a first  $L_t^\infty L_x^1$  bound on the solution. Our proof of these estimates is somehow formal but can be carried out rigorously with an approximation scheme as, e.g., in [155]. These estimates are also fundamental to prove the existence Lemma 2.1.3.

### 2.2.1 Control of the mass

We recall that the total mass of the system is defined in (2.1.11) and we prove the corresponding bound.

**Proposition 2.2.1** (Mass control). *For all  $t \geq 0$  we have  $\bar{u}(t) \leq p_H^{\frac{1}{\gamma}}$ .*

*Démonstration.* Integrating equation (2.1.1), we obtain

$$\frac{d}{dt} \bar{u} = \frac{1}{|\Omega|} \int_{\Omega} (u p_H - u^{\gamma+1}) dx. \quad (2.2.1)$$

By the Hölder inequality,

$$\int_{\Omega} u dx \leq \left( \int_{\Omega} u^{\gamma+1} dx \right)^{\frac{1}{\gamma+1}} |\Omega|^{\frac{\gamma}{\gamma+1}}$$

we deduce

$$\frac{d}{dt} \bar{u} \leq \frac{p_H}{|\Omega|} \int_{\Omega} u dx - \left( \int_{\Omega} u dx \right)^{\gamma+1} \frac{1}{|\Omega|^{\gamma+1}} = p_H \bar{u} - \bar{u}^{1+\gamma},$$

so that, with the Assumption 2.1.1 on the initial condition, we can conclude (2.1.11).  $\square$

### 2.2.2 Energy and entropy estimates

We recall that the energy, the entropy as well as their dissipation have been defined in (2.1.5)–(2.1.8). We prove that, for a fixed time horizon  $T$ , we have the following inequalities.

**Proposition 2.2.2** (Control of the energy and entropy dissipation). *The inequalities hold*

$$\begin{aligned} & \sup_{t \geq 0} \Phi(u(t)) + \frac{1}{2\varepsilon^2 p_H^{\frac{1}{\gamma}}} \int_0^\infty \int_\Omega \int_\Omega \omega_\varepsilon(y) |\nabla u(x) - \nabla u(x-y)|^2 dx dy dt \\ & + \frac{1}{p_H^{\frac{1}{\gamma}}} \frac{4\gamma}{(\gamma+1)^2} \int_0^\infty \int_\Omega |\nabla |u|^{\frac{\gamma+1}{2}}|^2 dx dt + \int_0^\infty \int_\Omega u \log \left( \frac{u}{p_H^{\frac{1}{\gamma}}} \right) (p - p_H) dx dt \leq \Phi(u_0), \end{aligned} \quad (2.2.2)$$

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}(u) + \frac{1}{2\varepsilon^2} \int_\Omega \omega_\varepsilon(y) (u(x) - u(x-y)) (u^{\gamma+1}(x) - u^{\gamma+1}(x-y)) dx + \int_\Omega u |\nabla \mu|^2 dx \\ & + \frac{1}{2} \int_\Omega u^{2\gamma+1} dx \leq C(\mathcal{E}(u) + 1), \end{aligned} \quad (2.2.3)$$

and thus there exists  $C(\mathcal{E}(u_0), T, \varepsilon) > 0$  such that

$$\sup_{t \in [0, T]} \mathcal{E}(u(t)) + \frac{1}{2} \int_0^T \int_\Omega u^{2\gamma+1} dx dt \leq C(\mathcal{E}(u_0), T, \varepsilon).$$

**Remark 2.2.3.** The above estimate in the energy  $\mathcal{E}$  depends exponentially on the final time  $T$ . We improve this result to a global one in Proposition 2.2.5.

*Démonstration. Control of the entropy.* Note that  $\Phi(u) \geq 0$  by the inequality  $x \log \left( \frac{x}{y} \right) - x + y \geq (\sqrt{x} - \sqrt{y})^2$  for  $x \geq 0$  and  $y > 0$  ( $x = \frac{u}{p_H^{\frac{1}{\gamma}}}$ ,  $y = 1$  here). Then we have

$$- \int_\Omega u \log \left( \frac{u}{p_H^{\frac{1}{\gamma}}} \right) G(p) dx = - \int_\Omega u \log \left( \frac{u}{p_H^{\frac{1}{\gamma}}} \right) (p_H - p) dx \geq 0,$$

since  $u \geq 0$  and  $x \mapsto \log x$  is increasing for  $x \geq 0$ . Therefore, all the terms in the dissipation of entropy in (2.1.8) are nonnegative so that we can integrate in time and obtain (2.2.2), which clearly implies the control of the entropy *independently* of  $T$ .

*Energy control.* Turning to the energy  $\mathcal{E}$ , departing from (2.1.7), we observe that

$$\int_\Omega u G(p) \mu dx = \int_\Omega u (p_H - p) p dx + \int_\Omega u B_\varepsilon(u) (p_H - p) dx.$$

The first term can be written as

$$\begin{aligned} \int_\Omega u (p_H - p) p dx &= - \int_\Omega u p^2 dx + \int_\Omega u p_H p dx \\ &\leq - \int_\Omega u p^2 dx + \frac{1}{2} \int_\Omega u p^2 dx + \frac{1}{2} p_H^2 \int_\Omega u dx \leq - \frac{1}{2} \int_\Omega u p^2 dx + C, \end{aligned}$$

where we used the mass control (2.1.11). For the second term we have, by symmetry of  $\omega$ ,

$$\begin{aligned} \int_{\Omega} u B_{\varepsilon}(u)(p_H - p) \, dx &= \frac{p_H}{2\varepsilon^2} \int_{\Omega} \int_{\Omega} \omega_{\varepsilon}(y) |u(x) - u(x-y)|^2 \, dx \, dy \\ &\quad - \frac{1}{2\varepsilon^2} \int_{\Omega} \int_{\Omega} \omega_{\varepsilon}(y) (u(x) - u(x-y))(u^{\gamma+1}(x) - u^{\gamma+1}(x-y)) \, dx \, dy. \end{aligned}$$

All together, these inequalities give immediately (2.2.3) and by the Gronwall lemma the energy control.  $\square$

**Remark 2.2.4.** In the limit  $\gamma \rightarrow \infty$  it holds that  $u_{\infty} \leq 1$ . This follows from the bound obtained with the energy in Proposition 2.2.2 since

$$\|u_{\infty}\|_{L^{\infty}} = \lim_{\gamma \rightarrow \infty} \|u_{\infty}\|_{L^{\gamma}} \quad \text{and} \quad \|u\|_{L^{\gamma+1}} \leq C^{\frac{1}{\gamma+1}} (\gamma+1)^{\frac{1}{\gamma+1}} \rightarrow 1$$

because of the weak convergence of  $u^{\gamma}$ . We refer for instance to [151, 322]. However in the next section we obtain a better control on  $u$  by the De Giorgi iteration method.

Now, we improve the local in time estimate on  $\mathcal{E}$  to a global one. Since Proposition 2.2.2 gives the uniform control  $\Phi(u(t)) \leq \Phi(u_0)$  for any  $t \geq 0$ , our aim is to control in a uniform way the energy  $\mathcal{E}$  as well.

**Proposition 2.2.5** (Uniform in time estimates for the energy). *There exists a constant independent of time and  $\gamma$  such that*

$$\mathcal{E}(t) \leq C, \quad \forall t \geq 0. \quad (2.2.4)$$

*Démonstration.* We estimate separately the two terms defining the energy in (2.1.5).

Firstly, from the entropy estimate (2.2.2), we immediately infer that, for a constant  $C > 0$  independent of time and for any sufficiently small  $\varepsilon$ ,

$$\begin{aligned} \int_t^{t+1} \frac{1}{4\varepsilon^2} \int_{\Omega} \int_{\Omega} \omega_{\varepsilon}(y) |u(x) - u(x-y)|^2 \, dx \, dy \, ds &\leq C(\Omega) \int_t^{t+1} \|u\|_{H^1(\Omega)}^2 \, ds \\ &\leq C \int_t^{t+1} \frac{1}{4\varepsilon^2} \int_{\Omega} \int_{\Omega} \omega_{\varepsilon}(y) |\nabla u(x) - \nabla u(x-y)|^2 \, dx \, dy \, ds + C \int_t^{t+1} \|u\|_{L^1(\Omega)}^2 \, ds \quad (2.2.5) \\ &\leq C + Cp_H^{\frac{2}{\gamma}}, \quad \forall t \geq 0. \end{aligned}$$

where the second inequality follows from Equations (2.5.4) and (2.1.11).

Secondly, we control the remaining part of the energy  $\mathcal{E}$ , the one related to  $\frac{1}{\gamma+1} \int_{\Omega} u^{\gamma+1} \, dx$ . To this aim, we integrate Equation (2.2.1) in time over  $[t, t+1]$  and get

$$\bar{u}(t+1) - \bar{u}(t) = \frac{1}{|\Omega|} \int_t^{t+1} \int_{\Omega} (up_H - u^{\gamma+1}) \, dx \, ds,$$

so that, rearranging the terms, we control the second term of the energy as

$$\frac{1}{|\Omega|} \int_t^{t+1} \int_{\Omega} u^{\gamma+1} \, dx \, ds \leq \frac{1}{|\Omega|} \int_t^{t+1} \int_{\Omega} up_H \, dx \, ds + \bar{u}(t) = (p_H + 1) \int_t^{t+1} \bar{u}(s) \, ds \leq C$$

because  $\bar{u}(\cdot) \leq p_H^{\frac{1}{\gamma}}$  for any  $t \geq 0$  thanks to (2.1.11). This, together with (2.2.5), implies that

$$\int_t^{t+1} \mathcal{E}(s) ds \leq C, \quad (2.2.6)$$

with  $C > 0$  independent of  $t \geq 0$  and  $\gamma$ .

We now conclude the energy estimate. By Proposition 2.2.2, we have

$$\frac{d}{dt} \mathcal{E}(u) \leq C(\mathcal{E}(u) + 1).$$

Using the Gronwall lemma, we obtain for all  $t \geq 0$  and all  $0 \leq s \leq 1$ ,

$$\mathcal{E}(u(t+1)) \leq C \mathcal{E}(u(t+s)) + C.$$

Integrating in  $s$  and using the bound (2.2.6), we conclude the proof of Proposition 2.2.5.  $\square$

### 2.2.3 A control on $\partial_t u$

Here we prove the estimate on time derivative, which appears also in Lemma 2.1.3 and is used in the proof of Theorem 2.1.6.

**Proposition 2.2.6.** *There exists  $C = C(\gamma, T) > 0$  such that the bounds hold*

$$\|\partial_t u\|_{L^{q'}(0,T;(W^{1,q}(\Omega))')} \leq C, \quad q = \frac{2(2\gamma+1)}{\gamma}, \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad (2.2.7)$$

$$\|\partial_t u\|_{L^2(0,T;(W^{1,r}(\Omega))')} \leq C, \quad r = \frac{(\gamma+1)(2\gamma+1)}{\gamma^2}. \quad (2.2.8)$$

*Démonstration.* For  $\gamma \geq 1$  fixed, any  $T > 0$ , any  $\varphi \in L^q(0, T; W^{1,q}(\Omega))$ , we have

$$\begin{aligned} & \left| \int_0^T \langle \partial_t u, \varphi \rangle_{(W^{1,q}(\Omega))', W^{1,q}(\Omega)} dt \right| \\ & \leq \left| \int_0^T \int_{\Omega} \gamma u^\gamma \nabla u \cdot \nabla \varphi dx dt \right| + \left| \int_0^T \frac{1}{\varepsilon^2} \int_{\Omega} u \nabla u \cdot \nabla \varphi dx dt \right| \\ & \quad + \left| \int_0^T \frac{1}{\varepsilon^2} \int_{\Omega} u (\nabla \omega_\varepsilon * u) \cdot \nabla \varphi dx dt \right| + \left| \int_0^T \int_{\Omega} u (p_H - u^\gamma) \varphi dx dt \right| \\ & \leq \gamma^{\frac{1}{2}} \|u\|_{L^{2\gamma+1}(\Omega \times (0,T))}^{\frac{\gamma+1}{2}} \|\gamma^{\frac{1}{2}} u^{\frac{\gamma-1}{2}} \nabla u\|_{L^2(\Omega \times (0,T))} \|\nabla \varphi\|_{L^q(\Omega \times (0,T))} \\ & \quad + C(\varepsilon) \|u\|_{L^{2\gamma+1}(\Omega \times (0,T))} \|\nabla u\|_{L^2(\Omega \times (0,T))} \|\nabla \varphi\|_{L^{\frac{2(2\gamma+1)}{2\gamma-1}}(\Omega \times (0,T))} \\ & \quad + C(\varepsilon) \|u\|_{L^2(\Omega \times (0,T))} \|\nabla \omega_\varepsilon * u\|_{L^\infty(\Omega \times (0,T))} \|\nabla \varphi\|_{L^2(\Omega \times (0,T))} \\ & \quad + C \|u\|_{L^{2\gamma+1}(\Omega \times (0,T))} \|p_H - u^\gamma\|_{L^2(\Omega \times (0,T))} \|\varphi\|_{L^{\frac{2(2\gamma+1)}{2\gamma-1}}(\Omega \times (0,T))} \\ & \leq C(\varepsilon, T, \gamma) \|\varphi\|_{L^q(0,T;W^{1,q}(\Omega))}. \end{aligned}$$

More precisely, to estimate the first and second terms, we used the Hölder inequality with exponents  $\frac{2(2\gamma+1)}{\gamma+1}$ , 2,  $q$  and  $2\gamma+1$ , 2 and  $\frac{2(2\gamma+1)}{2\gamma-1}$ , respectively. Then,  $\|u\|_{L^{\frac{\gamma+1}{2(2\gamma+1)}}(\Omega \times (0,T))}$  is bounded due to (2.2.3),  $\|\gamma^{\frac{1}{2}} u^{\frac{\gamma-1}{2}} \nabla u\|_{L^2(\Omega \times (0,T))}$  is estimated by (2.2.2) while the bound on  $\nabla u$  follows from (2.2.2) and nonlocal Poincaré inequality (2.5.4). The fourth term is bounded in the same spirit. Concerning the third one, we simply estimate

$$\|\nabla \omega_\varepsilon * u\|_{L^\infty(\Omega \times (0,T))} \leq \|\nabla \omega_\varepsilon\|_{L^\infty(\Omega)} \|u\|_{L^\infty(0,T;L^1(\Omega))} \leq C(\varepsilon)$$

and use the estimate on the total mass (2.1.11). The final conclusion follows from the inequalities  $q > 2$  and  $q \geq \frac{2(2\gamma+1)}{2\gamma-1}$  for any  $\gamma \geq 1$ .

Concerning (2.2.8), let  $\varphi \in L^2(0,T;W^{1,r}(\Omega))$ , with  $r = \frac{(\gamma+1)(2\gamma+1)}{\gamma^2}$ . Then,

$$\begin{aligned} & \left| \int_0^T \langle \partial_t u, \varphi \rangle_{(W^{1,r}(\Omega))', W^{1,r}(\Omega)} dt \right| \\ & \leq \left| \int_0^T \int_\Omega u \nabla \mu \cdot \nabla \varphi dx dt \right| + \left| \int_0^T \int_\Omega u (p_H - u^\gamma) \varphi dx dt \right| \\ & \leq \|u^{\frac{1}{2}}\|_{L^\infty(0,T;L^{2(\gamma+1)})} \|u^{\frac{1}{2}} \nabla \mu\|_{L^2(\Omega \times (0,T))} \|\nabla \varphi\|_{L^2(0,T;L^{\frac{2(\gamma+1)}{\gamma}}(\Omega))} \\ & \quad + p_H \|u\|_{L^\infty(0,T;L^{\gamma+1}(\Omega))} \|\varphi\|_{L^1(0,T;L^{\frac{\gamma+1}{\gamma}}(\Omega))} \\ & \quad + \|u\|_{L^\infty(0,T;L^{\gamma+1}(\Omega))} \|u\|_{L^{2\gamma+1}(\Omega \times (0,T))} \|\varphi\|_{L^{\frac{2\gamma+1}{2\gamma}}(0,T;L^r(\Omega))} \\ & \leq C \|\varphi\|_{L^2(0,T;W^{1,r}(\Omega))}, \end{aligned}$$

exploiting Proposition 2.2.2, recalling that  $r \geq \frac{2(\gamma+1)}{\gamma} > \frac{\gamma+1}{\gamma}$  and  $\frac{2\gamma+1}{2\gamma} \leq 2$ . This concludes the proof.  $\square$

### 2.3 Incompressible limit : proof of Theorem 2.1.5

We see the incompressible limit of (2.1.1) as the limit  $\gamma \rightarrow \infty$  and the resulting problem turns out to be a free boundary problem of Hele-Shaw type. Concerning the techniques adopted here, we first show that, for any fixed  $T > 0$ ,  $u_\gamma$  is bounded in  $L^\infty(\Omega \times (0,T))$  by a quantity which converges to 1 as  $\gamma \rightarrow \infty$  (see (2.3.1)). Due to the presence of the convolution term, which makes the equation nonlocal, we cannot apply any classical maximum principle, so that we need to resort to De Giorgi's iterations, exploiting the fact that the equation is a second order differential equation.

With uniform estimates in  $L^\infty$  at hand, we may apply standard energy estimates to gain sufficient regularity on the pressure  $p_\gamma$ , which we bounded in  $L^3(\Omega \times (0,T)) \cap L^2(0,T;H^1(\Omega))$  uniformly in  $\gamma$ . Then, one can obtain uniform controls in  $L^\infty(0,T;L^1(\Omega))$  for  $\partial_t u_\gamma$  and in  $L^1(\Omega \times (0,T))$  for  $\partial_t p_\gamma$ , so as to deduce the strong convergence of  $u_\gamma$  and  $p_\gamma$  in  $L^2(\Omega \times (0,T))$  to some  $u_\infty$  and  $p_\infty$ . With the help of these bounds, we are able to pass to the limit as  $\gamma \rightarrow \infty$  in Equation (2.1.1) and obtain Equations (2.1.15) and (2.1.17) for the limit concentration  $u_\infty$ .

In order to obtain more information on  $u_\infty, p_\infty$ , like complementarity conditions (2.1.16), we need a stronger convergence for  $\nabla p_\gamma$ . The standard technique uses some control on  $\Delta p$  thanks to the Aronson-Bénilan inequality (see, e.g., [120]) which do not apply here due to the higher-order term. In particular, the (formal) CH equation can be written as

$$\partial_t u_\gamma - \frac{\gamma}{\gamma+1} \Delta u_\gamma^{\gamma+1} - \frac{1}{2\varepsilon^2} \Delta u_\gamma^2 + \frac{1}{\varepsilon^2} \operatorname{div}(u_\gamma (\nabla \omega_\varepsilon * u)_\gamma) = u_\gamma G(p_\gamma),$$

and the extra term  $\Delta u_\gamma^2$ , independent of  $\gamma$ , appearing, and this prevents us from obtaining the Aronson-Bénilan inequality. Taking inspiration from [116, 260], in the second part of the present section, we instead show the strong  $L^2(\Omega \times (0, T))$  convergence of

$$\nabla \left( \frac{\gamma}{\gamma+1} u_\gamma^{\gamma+1} + \frac{1}{2\varepsilon^2} u_\gamma^2 \right),$$

which is shown to be enough to guarantee the validity of the condition (2.1.16).

### 2.3.1 An $L^\infty$ bound on $u_\gamma$

**Lemma 2.3.1.** *Assume  $0 \leq u_0 \leq p_H^{\frac{1}{\gamma}}$ . For any  $T > 0$  there exists  $\bar{\gamma}(T, \varepsilon) > 2$ , explicitly computed as a function of  $T$ , such that*

$$0 \leq u_\gamma \leq p_H^{\frac{1}{\gamma}} + \frac{2}{\sqrt[3]{\gamma}}, \quad \text{a.e. on } \Omega \times (0, T), \quad \forall \gamma \geq \bar{\gamma}(T, \varepsilon). \quad (2.3.1)$$

**Remark 2.3.2.** Notice that the bound (2.3.1) is useless to control the pressure, since  $\left(p_H^{\frac{1}{\gamma}} + \frac{2}{\sqrt[3]{\gamma}}\right)^\gamma \rightarrow +\infty$  as  $\gamma \rightarrow +\infty$ .

*Démonstration.* To simplify notations, we set  $\xi := \frac{1}{\sqrt[3]{\gamma}}$ . The iterative scheme is as follows. Let us consider the sequence

$$k_n = p_H^{\frac{1}{\gamma}} + 2\xi - \frac{\xi}{2^n}, \quad n \geq 0,$$

and note that  $p_H^{\frac{1}{\gamma}} + \xi \leq k_n < p_H^{\frac{1}{\gamma}} + 2\xi$ . Now we define the sequences

$$A_n(t) := \{x \in \Omega : u_\gamma \geq k_n\}, \quad y_n := \int_0^T \int_{A_n(s)} dx ds.$$

By testing the equation against  $u_{n,\gamma} := (u_\gamma - k_n)^+$ , we immediately infer that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(u_\gamma - k_n)^+\|_{L^2(\Omega)}^2 + \int_\Omega \gamma u_\gamma^\gamma |\nabla (u_\gamma - k_n)^+|^2 dx + \int_\Omega u_\gamma \nabla B_\varepsilon(u_\gamma) \cdot \nabla (u_\gamma - k_n)^+ dx \\ = \int_\Omega u_\gamma G(p_\gamma) (u_\gamma - k_n)^+ dx. \end{aligned}$$

By the definition of  $B_\varepsilon$ , we get

$$\int_\Omega u_\gamma \nabla B_\varepsilon(u_\gamma) \cdot \nabla (u_\gamma - k_n)^+ dx = \frac{1}{\varepsilon^2} \int_\Omega u_\gamma |\nabla u_{n,\gamma}|^2 dx - \frac{1}{\varepsilon^2} \int_\Omega u_\gamma (\nabla \omega_\varepsilon * u_\gamma) \cdot \nabla u_{n,\gamma} dx.$$

The first term is nonnegative. For the second we use that  $\gamma > 2$ ,  $\|u_\gamma\|_{L^\infty(0,\infty;L^1(\Omega))} \leq p_H^{\frac{1}{\gamma}}|\Omega|$  thanks to (2.1.11) as well as  $u_\gamma \geq k_n \geq p_H^{1/\gamma} + \xi$  on  $A_n$  to obtain

$$\begin{aligned} & \frac{1}{\varepsilon^2} \int_{\Omega} u_\gamma (\nabla \omega_\varepsilon * u_\gamma) \cdot \nabla u_{n,\gamma} \, dx \leq \frac{1}{\varepsilon^4} \frac{1}{2\gamma} \int_{A_n} u_\gamma^{2-\gamma} |\nabla \omega_\varepsilon * u_\gamma|^2 \, dx + \frac{\gamma}{2} \int_{\Omega} u_\gamma^\gamma |\nabla u_{n,\gamma}|^2 \, dx \\ & \leq \frac{1}{\varepsilon^4} \frac{1}{2\gamma \left(p_H^{\frac{1}{\gamma}} + \xi\right)^{\gamma-2}} \|\nabla \omega_\varepsilon * u_\gamma\|_{L^\infty(\Omega)}^2 \int_{A_n} \, dx + \frac{\gamma}{2} \int_{\Omega} u_\gamma^\gamma |\nabla u_{n,\gamma}|^2 \, dx \\ & \leq \frac{1}{\varepsilon^4} \frac{1}{2\gamma \left(p_H^{\frac{1}{\gamma}} + \xi\right)^{\gamma-2}} \|\nabla \omega_\varepsilon\|_{L^\infty(\Omega)}^2 \|u_\gamma\|_{L^1(\Omega)}^2 \int_{A_n} \, dx + \frac{\gamma}{2} \int_{\Omega} u_\gamma^\gamma |\nabla u_{n,\gamma}|^2 \, dx \\ & \leq \frac{C_1}{\gamma \left(p_H^{\frac{1}{\gamma}} + \xi\right)^{\gamma-2}} \int_{A_n} \, dx + \frac{\gamma}{2} \int_{\Omega} u_\gamma^\gamma |\nabla u_{n,\gamma}|^2 \, dx, \end{aligned}$$

where  $C_1 > 0$  is a constant that depends on  $\varepsilon$  but not on  $T$  and  $\gamma$ . Moreover, since on  $A_n$  we have  $u_\gamma \geq p_H^{1/\gamma}$  and thus  $p_\gamma \geq p_H$ , we immediately infer that

$$\int_{\Omega} u_\gamma G(p_\gamma) (u_\gamma - k_n)^+ \, dx \leq 0.$$

We can then sum up the results above to obtain

$$\frac{1}{2} \frac{d}{dt} \|(u_\gamma - k_n)^+\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \int_{\Omega} u_\gamma^\gamma |\nabla (u_\gamma - k_n)^+|^2 \, dx \leq \frac{C_1}{\gamma \left(p_H^{\frac{1}{\gamma}} + \xi\right)^{\gamma-2}} \int_{A_n} \, dx,$$

which also implies, since on  $A_n(t)$  we have  $u_\gamma^\gamma \geq \left(p_H^{\frac{1}{\gamma}} + \xi\right)^\gamma$ ,

$$\frac{1}{2} \frac{d}{dt} \|(u_\gamma - k_n)^+\|_{L^2(\Omega)}^2 + \frac{\gamma \left(p_H^{\frac{1}{\gamma}} + \xi\right)^\gamma}{2} \int_{\Omega} |\nabla (u_\gamma - k_n)^+|^2 \, dx \leq \frac{C_1}{\gamma \left(p_H^{\frac{1}{\gamma}} + \xi\right)^{\gamma-2}} \int_{A_n} \, dx.$$

It is now clear that,

$$\sup_{t \in [0, T]} \|u_{n,\gamma}(t)\|_{L^2(\Omega)}^2 \leq \frac{2C_1}{\gamma \left(p_H^{\frac{1}{\gamma}} + \xi\right)^{\gamma-2}} y_n =: Z_n, \quad (2.3.2)$$

$$\gamma \left(p_H^{\frac{1}{\gamma}} + \xi\right)^\gamma \int_0^T \int_{\Omega} |\nabla (u_\gamma - k_n)^+(s)|^2 \, dx \, ds \leq Z_n. \quad (2.3.3)$$

where we used, by the assumptions on the initial conditions,  $\|u_{n,\gamma}(0)\|_{L^2(\Omega)} = 0$ . Now for any  $t$

and for almost any  $x \in A_{n+1}(t)$ , we get

$$u_{n,\gamma}(x,t) = \underbrace{u_\gamma(x,t) - \left[ p_H^{\frac{1}{\gamma}} + 2\xi - \frac{\xi}{2^{n+1}} \right]}_{u_{n+1,\gamma}(x,t) \geq 0} + \xi \left[ \frac{1}{2^n} - \frac{1}{2^{n+1}} \right] \geq \frac{\xi}{2^{n+1}}.$$

Then we have

$$\int_0^T \int_\Omega |u_{n,\gamma}|^3 dx ds \geq \int_0^T \int_{A_{n+1}(s)} |u_{n,\gamma}|^3 dx ds \geq \left( \frac{\xi}{2^{n+1}} \right)^3 \int_0^T \int_{A_{n+1}(s)} dx ds = \left( \frac{\xi}{2^{n+1}} \right)^3 y_{n+1}.$$

Then we have

$$\left( \frac{\xi}{2^{n+1}} \right)^3 y_{n+1} \leq \left( \int_0^T \int_\Omega |u_{n,\gamma}|^{\frac{10}{3}} dx ds \right)^{\frac{9}{10}} \left( \int_0^T \int_{A_n(s)} dx ds \right)^{\frac{1}{10}}. \quad (2.3.4)$$

For the sake of clarity we now present the argument in the case  $d = 3$ , but it can be easily adapted to any dimension  $d \geq 1$ . We recall that by a variant of the three-dimensional Sobolev-Gagliardo-Nirenberg inequality (see, e.g., [63, Ch.9]) we get

$$\|v - \bar{v}\|_{L^{\frac{10}{3}}(\Omega)} \leq C_G \|v\|_{L^2(\Omega)}^{\frac{2}{5}} \|\nabla v\|_{L^2(\Omega)}^{\frac{3}{5}} \quad \forall v \in H^1(\Omega),$$

with  $C_G > 0$  depending only on  $\Omega$ . Therefore,

$$\begin{aligned} \int_0^T \int_\Omega |u_{n,\gamma}|^{\frac{10}{3}} dx ds &\leq 2^{\frac{7}{3}} \int_0^T \int_\Omega |u_{n,\gamma} - \bar{u}_{n,\gamma}|^{\frac{10}{3}} dx ds + 2^{\frac{7}{3}} \int_0^T \int_\Omega |\bar{u}_{n,\gamma}|^{\frac{10}{3}} dx ds \\ &\leq C_G 2^{\frac{7}{3}} \int_0^T \|\nabla u_{n,\gamma}\|_{L^2(\Omega)}^2 \|u_{n,\gamma}\|_{L^2(\Omega)}^{\frac{4}{3}} ds + \frac{2^{\frac{7}{3}}}{|\Omega|^{\frac{7}{3}}} \int_0^T \|u_{n,\gamma}\|_{L^1(\Omega)}^{\frac{10}{3}} \\ &\leq C_G 2^{\frac{7}{3}} \int_0^T \|\nabla u_{n,\gamma}\|_{L^2(\Omega)}^2 \|u_{n,\gamma}\|_{L^2(\Omega)}^{\frac{4}{3}} ds + \frac{2^{\frac{7}{3}}}{|\Omega|^{\frac{2}{3}}} \int_0^T \|u_{n,\gamma}\|_{L^2(\Omega)}^{\frac{10}{3}}, \end{aligned}$$

so that by (2.3.2) and (2.3.3) we immediately infer

$$\begin{aligned} &\int_0^T \int_\Omega |u_{n,\gamma}|^{\frac{10}{3}} dx ds \\ &\leq \frac{C_G}{\gamma \left( p_H^{\frac{1}{\gamma}} + \xi \right)^\gamma} \sup_{t \in [0, T]} \|u_{n,\gamma}(t)\|_{L^2(\Omega)}^{\frac{4}{3}} 2^{\frac{7}{3}} \gamma \left( p_H^{\frac{1}{\gamma}} + \xi \right)^\gamma \int_0^T \|\nabla u_{n,\gamma}(s)\|_{L^2(\Omega)}^2 ds \\ &\quad + \sup_{t \in [0, T]} \|u_{n,\gamma}(t)\|_{L^2(\Omega)}^{\frac{10}{3}} \frac{T 2^{\frac{7}{3}}}{|\Omega|^{\frac{2}{3}}} \\ &\leq \frac{2^{\frac{7}{3}} C_G}{\gamma \left( p_H^{\frac{1}{\gamma}} + \xi \right)^\gamma} Z_n^{\frac{5}{3}} + \frac{T 2^{\frac{7}{3}}}{|\Omega|^{\frac{2}{3}}} Z_n^{\frac{5}{3}} \leq \frac{T 2^{\frac{10}{3}}}{|\Omega|^{\frac{2}{3}}} Z_n^{\frac{5}{3}} \leq \frac{C_2 T}{\gamma^{\frac{5}{3}} \left( p_H^{\frac{1}{\gamma}} + \xi \right)^{\frac{5}{3} \gamma - \frac{10}{3}}} y_n^{\frac{5}{3}}, \end{aligned}$$



with  $C_2 = C_2(\varepsilon) > 0$ . Note that we have assumed  $\gamma$  sufficiently large, say  $\gamma \geq \gamma_0(T) > 2$  so that

$$\gamma \left( p_H^{\frac{1}{\gamma}} + \xi \right)^\gamma \geq \frac{C_G |\Omega|^{\frac{2}{3}}}{T}. \quad (2.3.5)$$

Coming back to (2.3.4), we get

$$\left( \frac{\xi}{2^{n+1}} \right)^3 y_{n+1} \leq \frac{C_2^{\frac{9}{10}} T^{\frac{9}{10}}}{\gamma^{\frac{3}{2}} \left( p_H^{\frac{1}{\gamma}} + \xi \right)^{\frac{3}{2}\gamma-3}} y_n^{\frac{8}{5}},$$

i.e., recalling the definition of  $\xi$ ,

$$y_{n+1} \leq \frac{2^{3n+3} C_2^{\frac{9}{10}} T^{\frac{9}{10}}}{\gamma^{\frac{1}{2}} \left( p_H^{\frac{1}{\gamma}} + \xi \right)^{\frac{3}{2}\gamma-3}} y_n^{\frac{8}{5}}.$$

Due to Lemma 2.5.1 with  $b = 2^3 > 1$ ,  $C = \frac{2^3 C_2^{\frac{9}{10}} T^{\frac{9}{10}}}{\gamma^{\frac{1}{2}} \left( p_H^{\frac{1}{\gamma}} + \xi \right)^{\frac{3}{2}\gamma-3}} > 0$ ,  $\varepsilon = \frac{3}{5}$ , we get that  $y_n \rightarrow 0$  if

$$y_0 \leq C^{-\frac{5}{3}} b^{-\frac{25}{9}} \iff y_0 \leq 2^{-\frac{25}{3}} \frac{\gamma^{\frac{5}{6}} \left( p_H^{\frac{1}{\gamma}} + \xi \right)^{\frac{5}{2}\gamma-5}}{2^5 C_2^{\frac{3}{2}} T^{\frac{3}{2}}}. \quad (2.3.6)$$

As we have  $y_0 \leq T|\Omega|$ , it is enough to ask for  $\gamma$  sufficiently large, say  $\gamma \geq \gamma_1(T, \varepsilon) \geq \gamma_0(T)$  such that

$$T^{\frac{5}{2}} |\Omega| \leq 2^{-\frac{25}{3}} \frac{\gamma^{\frac{5}{6}} \left( p_H^{\frac{1}{\gamma}} + \xi \right)^{\frac{5}{2}\gamma-5}}{2^5 C_2^{\frac{3}{2}}},$$

i.e.,

$$y_0 \leq T|\Omega| \leq 2^{-\frac{25}{3}} \frac{\gamma^{\frac{5}{6}} \left( p_H^{\frac{1}{\gamma}} + \xi \right)^{\frac{5}{2}\gamma-5}}{2^5 C_2^{\frac{3}{2}} T^{\frac{3}{2}}}.$$

This way  $y_n \rightarrow 0$  as long as  $\bar{\gamma}(T, \varepsilon) \geq \gamma_1(T, \varepsilon)$  and any  $\gamma \geq \bar{\gamma}(T, \varepsilon)$ .  $\square$

### 2.3.2 Higher-order regularity results, uniformly in $\gamma$ .

**Lemma 2.3.3.** *For any  $T > 0$  there exists  $C = C(T, \varepsilon) > 0$  such that*

$$\|p_\gamma\|_{L^2(0,T;H^1(\Omega))} + \|p_\gamma\|_{L^3(\Omega \times (0,T))} \leq C, \quad \forall \gamma > 1, \quad (2.3.7)$$

$$\|\partial_t u_\gamma\|_{L^2(0,T;(H^1(\Omega))')} \leq C, \quad \forall \gamma \geq \bar{\gamma}(T, \varepsilon) \quad (\text{see (2.3.1)}), \quad (2.3.8)$$

$$\|\partial_t u_\gamma\|_{L^\infty(0,T;L^1(\Omega))} + \|\partial_t p_\gamma\|_{L^1(\Omega) \times (0,T)} \leq C, \quad \forall \gamma \geq 1. \quad (2.3.9)$$

*Démonstration.* The arguments of the proof are often written formally for simplicity, but can be easily made rigorous in a suitable approximating scheme. Note that from Proposition 2.2.2 we

are able to deduce that

$$p_\gamma = u_\gamma^\gamma \in L^{2+\frac{1}{\gamma}}(\Omega \times (0, T)) \hookrightarrow L^2(\Omega \times (0, T)) \quad (2.3.10)$$

uniformly in  $\gamma$ . Thus, to prove the  $H^1$  bound in (2.3.7), we only need to find an estimate for the gradient of  $p_\gamma$ . Let us consider  $\int_\Omega u_\gamma^\gamma dx$  and compute its time derivative : from (2.1.1) we infer

$$\begin{aligned} \frac{d}{dt} \int_\Omega u_\gamma^\gamma dx &= \int_\Omega \gamma u_\gamma^{\gamma-1} \partial_t u_\gamma dx \\ &= -\gamma \int_\Omega u_\gamma \nabla(u_\gamma^{\gamma-1}) \nabla(u_\gamma^\gamma) dx - \gamma \int_\Omega u_\gamma \nabla(u_\gamma^{\gamma-1}) \cdot \nabla B_\varepsilon(u_\gamma) dx + \gamma \int_\Omega u_\gamma (p_H - p_\gamma) u_\gamma^{\gamma-1} dx \\ &= -\gamma^2(\gamma-1) \int_\Omega u_\gamma^{2\gamma-2} |\nabla u_\gamma|^2 dx - \gamma \int_\Omega u_\gamma \nabla(u_\gamma^{\gamma-1}) \cdot \nabla B_\varepsilon(u_\gamma) dx + \gamma \int_\Omega u_\gamma (p_H - p_\gamma) u_\gamma^{\gamma-1} dx \\ &= -(\gamma-1) \int_\Omega |\nabla(u_\gamma^\gamma)|^2 dx - \gamma(\gamma-1) \int_\Omega u_\gamma^{\gamma-1} \nabla u_\gamma \cdot \nabla B_\varepsilon(u_\gamma) dx + \gamma \int_\Omega u_\gamma (p_H - p_\gamma) u_\gamma^{\gamma-1} dx. \end{aligned} \quad (2.3.11)$$

Due to (2.1.2), we have

$$\begin{aligned} &\gamma(\gamma-1) \int_\Omega u_\gamma^{\gamma-1} \nabla u_\gamma \cdot \nabla B_\varepsilon(u_\gamma) dx \\ &= \frac{1}{\varepsilon^2} \gamma(\gamma-1) \int_\Omega u_\gamma^{\gamma-1} |\nabla u_\gamma|^2 dx - \frac{1}{\varepsilon^2} \gamma(\gamma-1) \int_\Omega u_\gamma^{\gamma-1} \nabla u_\gamma \cdot \nabla(\omega_\varepsilon * u_\gamma) dx \end{aligned}$$

By the Young inequality, recalling that  $\|u_\gamma\|_{L^\infty(0, T; L^2(\Omega))} \leq C$  (due to the uniform bound on the energy and the non-local Poincaré inequality (2.5.3), see Lemma 2.1.3), we get

$$\begin{aligned} &\frac{1}{\varepsilon^2} \gamma(\gamma-1) \int_\Omega u_\gamma^{\gamma-1} \nabla u_\gamma \cdot \nabla(\omega_\varepsilon * u_\gamma) dx \\ &\leq \frac{\gamma^2(\gamma-1)}{2} \int_\Omega u_\gamma^{2\gamma-2} |\nabla u_\gamma|^2 dx + \frac{1}{\varepsilon^4} \frac{\gamma-1}{2} \int_\Omega |\nabla \omega_\varepsilon * u_\gamma|^2 dx \\ &\leq \frac{\gamma^2(\gamma-1)}{2} \int_\Omega u_\gamma^{2\gamma-2} |\nabla u_\gamma|^2 dx + \frac{1}{\varepsilon^4} \frac{\gamma-1}{2} \|\nabla \omega_\varepsilon\|_{L^1(\Omega)}^2 \|u_\gamma\|_{L^2(\Omega)}^2 \\ &\leq \frac{(\gamma-1)}{2} \int_\Omega |\nabla(u_\gamma^\gamma)|^2 dx + \frac{C(T, \varepsilon)(\gamma-1)}{2}. \end{aligned}$$

The last term in (2.3.11) can be controlled by

$$\int_\Omega u_\gamma (p_H - p_\gamma) \gamma u_\gamma^{\gamma-1} dx \leq \int_\Omega (p_H - p_\gamma) \gamma u_\gamma^\gamma dx \leq \int_{p_\gamma \leq p_H} (p_H - p_\gamma) \gamma u_\gamma^\gamma dx \leq \gamma |\Omega| p_H^2.$$

Therefore, from (2.3.11) we get

$$\frac{1}{\gamma-1} \frac{d}{dt} \int_\Omega u_\gamma^\gamma dx + \frac{1}{2} \int_\Omega |\nabla(u_\gamma^\gamma)|^2 dx \leq \frac{C}{2} + C \frac{\gamma}{\gamma-1} \leq C(T, \varepsilon), \quad (2.3.12)$$

showing that  $\nabla u_\gamma^\gamma \in L^2(\Omega \times (0, T))$  uniformly in  $\gamma$  so that  $u_\gamma^\gamma$  is bounded uniformly in  $L^2(0, T; H^1(\Omega))$ .

We now need a similar estimate to show the  $L^3$  bound in (2.3.7). We have

$$\begin{aligned}
\frac{1}{2\gamma} \frac{d}{dt} \int_{\Omega} u_{\gamma}^{2\gamma} dx &= \int_{\Omega} u_{\gamma}^{2\gamma-1} \partial_t u_{\gamma} dx \\
&= - \int_{\Omega} u_{\gamma} \nabla(u_{\gamma}^{2\gamma-1}) \nabla(u_{\gamma}^{\gamma}) dx - \int_{\Omega} u_{\gamma} \nabla(u_{\gamma}^{2\gamma-1}) \cdot \nabla B_{\varepsilon}(u_{\gamma}) dx + \int_{\Omega} u_{\gamma} (p_H - p) u_{\gamma}^{2\gamma-1} dx \\
&= -\gamma(2\gamma-1) \int_{\Omega} u_{\gamma}^{3\gamma-2} |\nabla u_{\gamma}|^2 dx - \frac{1}{\varepsilon^2} (2\gamma-1) \int_{\Omega} u_{\gamma}^{2\gamma-1} |\nabla u_{\gamma}|^2 dx \\
&\quad + \frac{1}{\varepsilon^2} (2\gamma-1) \int_{\Omega} u_{\gamma}^{2\gamma-1} \nabla u_{\gamma} \cdot (\nabla \omega_{\varepsilon} * u_{\gamma}) dx + \int_{\Omega} (p_H - p_{\gamma}) u_{\gamma}^{2\gamma} dx. \tag{2.3.13}
\end{aligned}$$

Notice now that, by Young's inequality for convolutions and after integration by parts,

$$\begin{aligned}
\frac{1}{\varepsilon^2} (2\gamma-1) \int_{\Omega} u_{\gamma}^{2\gamma-1} \nabla u_{\gamma} \cdot (\nabla \omega_{\varepsilon} * u_{\gamma}) dx \\
&= \frac{1}{\varepsilon^2} \frac{2\gamma-1}{2\gamma} \int_{\Omega} \nabla u_{\gamma}^{2\gamma} \cdot (\nabla \omega_{\varepsilon} * u_{\gamma}) dx = \frac{1}{\varepsilon^2} \frac{1-2\gamma}{2\gamma} \int_{\Omega} (\Delta \omega_{\varepsilon} * u_{\gamma}) u_{\gamma}^{2\gamma} dx \\
&\leq \frac{1}{\varepsilon^2} \frac{2\gamma-1}{2\gamma} \|\Delta \omega_{\varepsilon}\|_{L^{\infty}(\Omega)} \|u_{\gamma}\|_{L^1(\Omega)} \int_{\Omega} p_{\gamma}^2 dx \leq C(\varepsilon) \frac{2\gamma-1}{2\gamma} \int_{\Omega} p_{\gamma}^2 dx.
\end{aligned}$$

We thus get, integrating (2.3.13) over  $(0, T)$  and using the  $L^2$  bound on  $p_{\gamma}$  in (2.3.10),

$$\begin{aligned}
\frac{1}{2\gamma} \int_{\Omega} p_{\gamma}^2(T) dx + \gamma(2\gamma-1) \int_0^T \int_{\Omega} u_{\gamma}^{3\gamma-2} |\nabla u_{\gamma}|^2 dx ds + \frac{2\gamma-1}{\varepsilon^2} \int_0^T \int_{\Omega} u_{\gamma}^{2\gamma-1} |\nabla u_{\gamma}|^2 dx ds \\
+ \int_0^T \int_{\Omega} p_{\gamma}^{3\gamma} dx ds \leq C(T, \varepsilon) \left( \frac{2\gamma-1}{2\gamma} + 1 \right) \leq C(T, \varepsilon),
\end{aligned}$$

where we used that  $\frac{2\gamma-1}{2\gamma} \rightarrow 1$  as  $\gamma \rightarrow \infty$ . From this we deduce the uniform  $L^3$  estimate in (2.3.7).

To prove (2.3.8), we compute for any  $\varphi \in L^2(0, T; H^1(\Omega))$  and  $\gamma \geq \bar{\gamma}$ , with  $\bar{\gamma}(T)$  as in (2.3.1)

$$\begin{aligned}
\left| \int_0^T \langle \partial_t u_{\gamma}, \varphi \rangle dt \right| &\leq \left| \int_0^T \int_{\Omega} u_{\gamma} \nabla p_{\gamma} \cdot \nabla \varphi dx dt \right| + \left| \int_0^T \frac{1}{\varepsilon^2} \int_{\Omega} u_{\gamma} \nabla u_{\gamma} \cdot \nabla \varphi dx dt \right| \\
&\quad + \left| \int_0^T \frac{1}{\varepsilon^2} \int_{\Omega} u_{\gamma} (\nabla \omega_{\varepsilon} * u_{\gamma}) \cdot \nabla \varphi dx dt \right| + \left| \int_0^T \int_{\Omega} u_{\gamma} (p_H - u_{\gamma}^{\gamma}) \varphi dx dt \right|.
\end{aligned}$$

Note that by Young's inequality for convolutions we get

$$\begin{aligned}
\|\nabla \omega_{\varepsilon} * u_{\gamma}\|_{L^2(\Omega \times (0, T))}^2 &\leq \int_0^T \|\nabla \omega_{\varepsilon} * u_{\gamma}\|_{L^{\infty}(\Omega)}^2 |\Omega| dt \leq |\Omega| \|\nabla \omega_{\varepsilon}\|_{L^{\infty}(\Omega)}^2 \int_0^T \|u_{\gamma}\|_{L^1(\Omega)}^2 dt \\
&\leq |\Omega| \|\nabla \omega_{\varepsilon}\|_{L^{\infty}(\Omega)}^2 \int_0^T \left( \int_{\Omega} p_H^{\frac{1}{2}} dx \right)^2 dt \leq |\Omega|^3 \|\nabla \omega_{\varepsilon}\|_{L^{\infty}(\Omega)}^2 p_H^{\frac{2}{3}} T \leq C(\varepsilon, T),
\end{aligned}$$

by (2.1.11). Therefore, by Lemma 2.1.3, (2.3.1) and (2.3.7)

$$\begin{aligned} \left| \int_0^T \langle \partial_t u_\gamma, \varphi \rangle dt \right| &\leq \|u_\gamma\|_{L^\infty(\Omega \times (0, T))} \|\nabla p_\gamma\|_{L^2(0, T; L^2(\Omega))} \|\nabla \varphi\|_{L^2(0, T; L^2(\Omega))} \\ &\quad + C(\varepsilon) \|u_\gamma\|_{L^\infty(\Omega \times (0, T))} \|\nabla u_\gamma\|_{L^2(\Omega \times (0, T))} \|\nabla \varphi\|_{L^2(\Omega \times (0, T))} \\ &\quad + C(\varepsilon) \|u_\gamma\|_{L^\infty(\Omega \times (0, T))} \|(\nabla \omega_\varepsilon * u)_\gamma\|_{L^2(\Omega \times (0, T))} \|\nabla \varphi\|_{L^2(\Omega \times (0, T))} \\ &\quad + C \|u_\gamma\|_{L^\infty(\Omega \times (0, T))} \|p_H - u_\gamma^\gamma\|_{L^2(\Omega \times (0, T))} \|\varphi\|_{L^2(\Omega \times (0, T))} \\ &\leq C(\varepsilon, T) \|\varphi\|_{L^2(0, T; H^1(\Omega))}, \end{aligned}$$

Therefore, for any  $\gamma \geq \bar{\gamma}$ , we infer that  $\|\partial_t u_\gamma\|_{L^2(0, T; (H^1(\Omega))')} \leq C(T)$ , thus showing (2.3.8).

It remains to prove (2.3.9). First note that, clearly,  $\partial_t u_\gamma$  and  $\partial_t p_\gamma$  share the same sign since  $u_\gamma \geq 0$  almost everywhere in  $\Omega \times (0, T)$ . Then we differentiate in time (2.1.1) and get

$$\begin{aligned} \partial_{tt} u_\gamma - \frac{\gamma}{\gamma+1} \Delta \partial_t (u_\gamma^{\gamma+1}) - \frac{1}{2\varepsilon^2} \Delta \partial_t (u_\gamma^2) + \frac{1}{\varepsilon^2} \operatorname{div}(\partial_t u_\gamma (\nabla \omega_\varepsilon * u_\gamma)) + \frac{1}{\varepsilon^2} \operatorname{div}(u_\gamma (\nabla \omega_\varepsilon * \partial_t u_\gamma)) \\ = \partial_t u_\gamma (p_H - p_\gamma) - u_\gamma \partial_t p_\gamma. \end{aligned}$$

We test it against  $\operatorname{sign}(\partial_t u_\gamma)$  and use Kato's inequality to obtain

$$\begin{aligned} \partial_t |\partial_t u_\gamma| &\leq \frac{\gamma}{\gamma+1} \Delta (|\partial_t (u_\gamma^{\gamma+1})|) + \frac{1}{2\varepsilon^2} \Delta (|\partial_t (u_\gamma^2)|) + \frac{1}{\varepsilon^2} \operatorname{div}(|\partial_t u_\gamma| (-\nabla \omega_\varepsilon * u_\gamma)) \\ &\quad - \frac{1}{\varepsilon^2} \operatorname{div}(u_\gamma (\nabla \omega_\varepsilon * \partial_t u_\gamma)) \operatorname{sign}(\partial_t u_\gamma) + |\partial_t u_\gamma| p_H - p_\gamma |\partial_t u_\gamma| - u_\gamma |\partial_t p_\gamma|. \end{aligned}$$

Now we rearrange the terms and integrate in space, deducing

$$\begin{aligned} \frac{d}{dt} \int_\Omega |\partial_t u_\gamma| dx + \int_\Omega u_\gamma |\partial_t p_\gamma| dx + \int_\Omega p_\gamma |\partial_t u_\gamma| dx \\ \leq -\frac{1}{\varepsilon^2} \int_\Omega \operatorname{div}(u_\gamma (\nabla \omega_\varepsilon * \partial_t u_\gamma)) \operatorname{sign}(\partial_t u_\gamma) dx + p_H \int_\Omega |\partial_t u_\gamma| dx. \end{aligned}$$

Then we have, by Young's inequality for convolutions,

$$\begin{aligned} -\frac{1}{\varepsilon^2} \int_\Omega \operatorname{div}(u_\gamma (\nabla \omega_\varepsilon * \partial_t u_\gamma)) \operatorname{sign}(\partial_t u_\gamma) dx \\ = -\frac{1}{\varepsilon^2} \int_\Omega \nabla u_\gamma \cdot (\nabla \omega_\varepsilon * \partial_t u_\gamma) \operatorname{sign}(\partial_t u_\gamma) dx - \frac{1}{\varepsilon^2} \int_\Omega u_\gamma (\Delta \omega_\varepsilon * \partial_t u_\gamma) \operatorname{sign}(\partial_t u_\gamma) dx \\ \leq \frac{1}{\varepsilon^2} \|\nabla u_\gamma\|_{L^1(\Omega)} \|\nabla \omega_\varepsilon * \partial_t u_\gamma\|_{L^\infty(\Omega)} + \frac{1}{\varepsilon^2} \|u_\gamma\|_{L^1(\Omega)} \|\Delta \omega_\varepsilon * \partial_t u_\gamma\|_{L^\infty(\Omega)} \\ \leq C(\varepsilon) (1 + \|\nabla u_\gamma\|_{L^1(\Omega)}) \int_\Omega |\partial_t u_\gamma| dx, \end{aligned}$$

since

$$\|\nabla \omega_\varepsilon * \partial_t u_\gamma\|_{L^\infty(\Omega)} \leq \|\nabla \omega_\varepsilon\|_{L^\infty(\Omega)} \int_\Omega |\partial_t u_\gamma| dx \leq C(\varepsilon) \int_\Omega |\partial_t u_\gamma| dx,$$

and the same for  $\|\Delta\omega_\varepsilon * \partial_t u_\gamma\|_{L^\infty(\Omega)}$ . Therefore we end up with

$$\frac{d}{dt} \int_{\Omega} |\partial_t u_\gamma| dx + \int_{\Omega} u_\gamma |\partial_t p_\gamma| dx + \int_{\Omega} p_\gamma |\partial_t u_\gamma| dx \leq C(\varepsilon)(1 + \|\nabla u_\gamma\|_{L^1(\Omega)}) \int_{\Omega} |\partial_t u_\gamma| dx. \quad (2.3.14)$$

To apply the Gronwall inequality, we need to estimate  $\|\partial_t u_\gamma(0)\|_{L^1(\Omega)}$ . From (2.1.1) we have

$$\begin{aligned} \|\partial_t u_\gamma(0)\|_{L^1(\Omega)} &\leq C(\varepsilon) \left( \frac{\gamma}{\gamma+1} \|\Delta(u_0^{\gamma+1})\|_{L^1(\Omega)} + \|\Delta(u_0^2)\|_{L^1(\Omega)} + \right. \\ &\quad \left. + \|\operatorname{div}(u_0(\nabla\omega_\varepsilon * u_0))\|_{L^1(\Omega)} + \|u_0(p_H - p_0)\|_{L^1(\Omega)} \right). \end{aligned}$$

Due to Assumption 2.1.2, the first term is bounded. Concerning next terms, we have by Assumption 2.1.1 and 2.1.2

$$\|\Delta(u_0^2)\|_{L^1(\Omega)} \leq 2\|\nabla u_0\|_{L^2(\Omega)}^2 + 2\|u_0\Delta u_0\|_{L^1(\Omega)} \leq 2\|\nabla u_0\|_{L^2(\Omega)}^2 + 2p_H^{\frac{1}{2}}\|\Delta u_0\|_{L^1(\Omega)} \leq C,$$

$$\begin{aligned} \|\operatorname{div}(u_0(\nabla\omega_\varepsilon * u_0))\|_{L^1(\Omega)} &\leq \|\nabla u_0 \cdot (\nabla\omega_\varepsilon * u_0)\|_{L^1(\Omega)} + \|u_0(\Delta\omega_\varepsilon * u_0)\|_{L^1(\Omega)} \\ &\leq \|\nabla u_0\|_{L^1(\Omega)} \|\nabla\omega_\varepsilon\|_{L^\infty(\Omega)} \|u_0\|_{L^1(\Omega)} + p_H^{\frac{1}{2}} |\Omega| \|\Delta\omega_\varepsilon\|_{L^\infty(\Omega)} \|u_0\|_{L^1(\Omega)} \leq C(\varepsilon). \end{aligned}$$

It follows that  $\|\partial_t u_\gamma(0)\|_{L^1(\Omega)} \leq C(\varepsilon)$ . Since  $u_\gamma \in L^2(0; T; H^1(\Omega))$  uniformly in  $\gamma$  (Lemma 2.1.3), we may apply the Gronwall Lemma in (2.3.14) and obtain

$$\|\partial_t u_\gamma\|_{L^\infty(0, T; L^1(\Omega))} + \|u_\gamma \partial_t p_\gamma\|_{L^1(0, T; L^1(\Omega))} \leq C(T, \varepsilon), \quad (2.3.15)$$

with  $C = C(T, \varepsilon) > 0$  independent of  $\gamma$ . To conclude the argument we notice that

$$\int_{\Omega} |\partial_t p_\gamma| dx \leq \int_{u_\gamma \leq \frac{1}{2}} |\partial_t p_\gamma| dx + 2 \int_{u_\gamma > \frac{1}{2}} u_\gamma |\partial_t p_\gamma| dx \leq \frac{\gamma}{2\gamma-1} \int_{\Omega} |\partial_t u_\gamma| dx + 2 \int_{\Omega} u_\gamma |\partial_t p_\gamma| dx,$$

so that, being  $\frac{\gamma}{2\gamma-1} \leq 1$  for any  $\gamma \geq 1$ , from (2.3.15) we deduce the second estimate in (2.3.9) and conclude the proof of Lemma 2.3.3.  $\square$

### 2.3.3 The limit $\gamma \rightarrow \infty$

We complete the convergence as  $\gamma \rightarrow \infty$  distinguishing two steps.

**Step 1. Consequences of Lemmas 2.1.3, 2.3.1 and 2.3.3.** Exploiting those lemmas, we can obtain the following convergences, up to subsequences, which are deduced by standard compactness arguments : for any  $T > 0$ , as  $\gamma \rightarrow \infty$ ,

$$u_\gamma \rightarrow u_\infty \text{ almost everywhere in } \Omega \times (0, T), \quad (2.3.16)$$

$$u_\gamma \overset{*}{\rightharpoonup} u_\infty \text{ in } L^\infty(\Omega \times (0, T)), \quad (2.3.17)$$

$$u_\gamma \rightarrow u_\infty \text{ in } L^p(\Omega \times (0, T)) \quad \forall p \in [1, +\infty), \quad (2.3.18)$$

$$u_\gamma \rightharpoonup u_\infty \text{ in } L^2(0, T; H^1(\Omega)), \quad (2.3.19)$$

$$\partial_t u_\gamma \rightharpoonup \partial_t u_\infty \text{ in } L^2(0, T; (H^1(\Omega))'), \quad (2.3.20)$$

$$p_\gamma \rightharpoonup p_\infty \text{ in } L^2(0, T; H^1(\Omega)), \quad L^3((0, T) \times \Omega). \quad (2.3.21)$$

Moreover, by the Aubin-Lions-Simon Lemma,

$$p_\gamma \rightarrow p_\infty \text{ in } L^2(0, T; L^2(\Omega)), \quad (2.3.22)$$

which, thanks to (2.3.7) can be improved by interpolation, to

$$p_\gamma \rightarrow p_\infty \text{ in } L^q(0, T; L^q(\Omega)), \quad \forall q \in [2, 3). \quad (2.3.23)$$

In order to obtain the complementarity condition, we study the function  $v_\gamma := u_\gamma^{\gamma+1} = u_\gamma p_\gamma$ . First we have

$$\|v_\gamma\|_{L^2(\Omega \times (0, T))} \leq \|u_\gamma\|_{L^\infty(\Omega \times (0, T))} \|p_\gamma\|_{L^2(\Omega \times (0, T))} \leq C(T, \varepsilon),$$

and

$$\|\nabla v_\gamma\|_{L^2(\Omega \times (0, T))} = \frac{\gamma+1}{\gamma} \|u_\gamma \nabla p_\gamma\|_{L^2(\Omega \times (0, T))} \leq \frac{\gamma+1}{\gamma} \|u_\gamma\|_{L^\infty(\Omega \times (0, T))} \|\nabla p_\gamma\|_{L^2(\Omega \times (0, T))} \leq C(T, \varepsilon),$$

since  $\frac{\gamma+1}{\gamma} \rightarrow 1$  as  $\gamma \rightarrow \infty$ . Therefore, up to subsequences, we have

$$v_\gamma = u_\gamma p_\gamma \rightharpoonup v_\infty \quad \text{in } L^2(0, T; H^1(\Omega)),$$

for some  $v_\infty \in L^2(0, T; H^1(\Omega))$ . To identify  $v_\infty$ , we observe that

$$\begin{aligned} \|u_\gamma p_\gamma - u_\infty p_\infty\|_{L^2(\Omega \times (0, T))} &\leq \|u_\gamma(p_\gamma - p_\infty)\|_{L^2(\Omega \times (0, T))} + \|p_\infty(u_\gamma - u_\infty)\|_{L^2(\Omega \times (0, T))} \\ &\leq \|u_\gamma\|_{L^\infty(\Omega \times (0, T))} \|p_\gamma - p_\infty\|_{L^2(\Omega \times (0, T))} + \|u_\gamma - u_\infty\|_{L^6(\Omega \times (0, T))} \|p_\infty\|_{L^3(\Omega \times (0, T))} \rightarrow 0 \end{aligned}$$

as  $\gamma \rightarrow \infty$ , thanks to the above results, in particular (2.3.1), (2.3.7), (2.3.18) and (2.3.22). From this we clearly identify  $v_\infty = u_\infty p_\infty$  and obtain

$$v_\gamma = u_\gamma p_\gamma \rightharpoonup v_\infty = u_\infty p_\infty \text{ in } L^2(0, T; H^1(\Omega)). \quad (2.3.24)$$

$$v_\gamma \rightarrow p_\infty u_\infty \text{ in } L^2(\Omega \times (0, T)). \quad (2.3.25)$$

We are now able to pass to the limit in  $\gamma$  to obtain (2.1.15) and

$$p_\infty(1 - u_\infty) = 0, \quad 0 \leq u_\infty \leq 1, \quad p_\infty \geq 0. \quad (2.3.26)$$

Indeed, we can argue as in [254] to see that, for any  $\epsilon > 0$  there exists  $\gamma_0 = \gamma_0(\epsilon)$  such that for any  $y \geq 0$  and  $\gamma \geq \gamma_0$

$$y^{\gamma+1} \geq y^\gamma - \epsilon.$$

Applying this to  $y = u_\gamma$ , since we have that, up to subsequences,  $u^\gamma = p_\gamma \rightarrow p_\infty$  and  $v_\gamma = u_\gamma^{\gamma+1} \rightarrow p_\infty u_\infty$  almost everywhere in  $\Omega \times (0, T)$ , we can pass to the limit and obtain,

$$u_\infty p_\infty \geq p_\infty - \epsilon, \quad \forall \epsilon > 0,$$

which implies that  $p_\infty u_\infty \geq p_\infty$ . Since by (2.3.1) and (2.3.16) we get  $0 \leq u_\infty \leq 1$  almost everywhere in  $\Omega \times (0, T)$ , we get  $p_\infty \geq p_\infty u_\infty \geq p_\infty$ , i.e., (2.3.26), since it holds for any  $T > 0$ . In order to pass to Step 2, we introduce the quantities

$$\tilde{v}_\gamma := \frac{\gamma}{\gamma+1} v_\gamma + \frac{1}{2\varepsilon^2} u_\gamma^2, \quad \tilde{v}_\infty = v_\infty + \frac{1}{2\varepsilon^2} u_\infty^2, \quad (2.3.27)$$

and study  $\nabla \tilde{v}_\gamma$  which is essential to obtain the complementarity condition (2.1.16).

## Step 2. Strong convergence of $\nabla \tilde{v}_\gamma$ .

**Lemma 2.3.4.** *Let  $\tilde{v}_\gamma, \tilde{v}_\infty$  be as in (2.3.27). Then, for any  $T > 0$ ,*

$$\tilde{v}_\gamma \rightarrow \tilde{v}_\infty \quad \text{in } L^2(0, T; H^1(\Omega)) \quad \text{as } \gamma \rightarrow \infty.$$

*Démonstration.* Using (2.3.18) and (2.3.25) we obtain  $\tilde{v}_\gamma \rightarrow \tilde{v}_\infty$  in  $L^2(\Omega \times (0, T))$  so it is sufficient to prove  $\nabla \tilde{v}_\gamma \rightarrow \nabla \tilde{v}_\infty$  in  $L^2(\Omega \times (0, T))$ . Of course, by (2.3.16)-(2.3.21) and (2.3.24), we have weak convergence

$$\nabla \tilde{v}_\gamma \rightharpoonup \nabla \tilde{v}_\infty \text{ in } L^2(0, T; L^2(\Omega)), \quad \gamma \rightarrow \infty. \quad (2.3.28)$$

Let us first observe that (2.1.1) can be rewritten highlighting the presence of  $\tilde{v}_\gamma$  :

$$\partial_t u_\gamma - \Delta \tilde{v}_\gamma = u_\gamma G(p_\gamma) - \frac{1}{\varepsilon^2} \operatorname{div}(u_\gamma (\nabla \omega_\varepsilon * u_\gamma)). \quad (2.3.29)$$

We multiply (2.3.29) by  $\tilde{v}_\gamma - \tilde{v}_\infty$  and integrate over  $\Omega_T := \Omega \times (0, T)$ . Since

$$\int_\Omega \partial_t u_\gamma v_\gamma \, dx = \frac{1}{\gamma + 2} \frac{d}{dt} \int_\Omega u_\gamma^{\gamma+2} \, dx.$$

we obtain

$$\begin{aligned} & \frac{\gamma}{(\gamma + 2)(\gamma + 1)} \int_\Omega u_\gamma^{\gamma+2}(T) \, dx + \int_{\Omega_T} \nabla \tilde{v}_\gamma \cdot \nabla (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, ds \\ &= \frac{\gamma}{(\gamma + 2)(\gamma + 1)} \int_\Omega u_\gamma^{\gamma+2}(0) \, dx - \frac{1}{2\varepsilon^2} \int_{\Omega_T} \partial_t u_\gamma (u_\gamma^2 - u_\infty^2) \, dx \, dt + \int_{\Omega_T} u_\gamma G(p_\gamma) (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, dt \\ &+ \frac{1}{\varepsilon^2} \int_{\Omega_T} u_\gamma (\nabla \omega_\varepsilon * u_\gamma) \cdot \nabla (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, dt + \int_0^T \langle \partial_t u_\gamma, v_\infty \rangle \, dt. \end{aligned} \quad (2.3.30)$$

The plan is to estimate  $\limsup_{\gamma \rightarrow \infty} \int_{\Omega_T} |\nabla \tilde{v}_\gamma - \nabla \tilde{v}_\infty|^2 \, dx \, dt$  from (2.3.30). First, we rewrite the term  $\frac{1}{2\varepsilon^2} \int_{\Omega_T} \partial_t u_\gamma (u_\gamma^2 - u_\infty^2) \, dx \, dt$ . We have

$$\begin{aligned} & \frac{1}{2\varepsilon^2} \int_\Omega \partial_t u_\gamma (u_\gamma^2 - u_\infty^2) \, dx = \frac{1}{2\varepsilon^2} \langle \partial_t u_\gamma - \partial_t u_\infty, u_\gamma^2 - u_\infty^2 \rangle + \frac{1}{2\varepsilon^2} \langle \partial_t u_\infty, u_\gamma^2 - u_\infty^2 \rangle \\ &= \frac{1}{4\varepsilon^2} \frac{d}{dt} \int_\Omega (u_\gamma - u_\infty)^2 (u_\gamma + u_\infty) \, dx - \frac{1}{4\varepsilon^2} \langle \partial_t (u_\gamma + u_\infty), (u_\gamma - u_\infty)^2 \rangle + \frac{1}{2\varepsilon^2} \langle \partial_t u_\infty, u_\gamma^2 - u_\infty^2 \rangle \\ &= \frac{1}{4\varepsilon^2} \frac{d}{dt} \int_\Omega (u_\gamma - u_\infty)^2 (u_\gamma + u_\infty) \, dx - \frac{1}{12\varepsilon^2} \frac{d}{dt} \int_\Omega (u_\gamma - u_\infty)^3 \, dx - \frac{1}{2\varepsilon^2} \langle \partial_t u_\infty, (u_\gamma - u_\infty)^2 \rangle \\ &+ \frac{1}{2\varepsilon^2} \langle \partial_t u_\infty, u_\gamma^2 - u_\infty^2 \rangle, \end{aligned}$$

so that, integrating over  $[0, T]$  and recalling that  $u_\infty(0) \equiv u_\gamma(0) \equiv u_0$ , we get

$$\begin{aligned} & -\frac{1}{2\varepsilon^2} \int_{\Omega_T} \partial_t u_\gamma (u_\gamma^2 - u_\infty^2) \, dx \, dt = -\frac{1}{4\varepsilon^2} \int_{\Omega} (u_\gamma(T) - u_\infty(T))^2 (u_\gamma(T) + u_\infty(T)) \, dx \\ & + \frac{1}{12\varepsilon^2} \int_{\Omega} (u_\gamma(T) - u_\infty(T))^3 \, dx + \frac{1}{2\varepsilon^2} \int_0^T \langle \partial_t u_\infty, (u_\gamma - u_\infty)^2 \rangle \, dt - \frac{1}{2\varepsilon^2} \int_0^T \langle \partial_t u_\infty, u_\gamma^2 - u_\infty^2 \rangle \, dt. \end{aligned} \quad (2.3.31)$$

Note that

$$\begin{aligned} & \frac{1}{4\varepsilon^2} \int_{\Omega} (u_\gamma(T) - u_\infty(T))^2 (u_\gamma(T) + u_\infty(T)) \, dx - \frac{1}{12\varepsilon^2} \int_{\Omega} (u_\gamma(T) - u_\infty(T))^3 \, dx \\ & = \frac{1}{4\varepsilon^2} \int_{\Omega} (u_\gamma(T) - u_\infty(T))^2 \left( u_\gamma(T) + u_\infty(T) - \frac{1}{3}u_\gamma(T) + \frac{1}{3}u_\infty(T) \right) \, dx \geq 0. \end{aligned} \quad (2.3.32)$$

Therefore, taking into account (2.3.31), (2.3.32) and  $u_\gamma \geq 0$  we obtain from (2.3.30)

$$\begin{aligned} & \int_{\Omega_T} \nabla \tilde{v}_\gamma \cdot \nabla (\tilde{v} - \tilde{v}_\infty) \, dx \, ds \leq \frac{\gamma}{(\gamma+2)(\gamma+1)} \int_{\Omega} u_\gamma^{\gamma+2}(0) \, dx + \frac{1}{2\varepsilon^2} \int_0^T \langle \partial_t u_\infty, (u_\gamma - u_\infty)^2 \rangle \, dt \\ & - \int_0^T \langle \partial_t u_\infty, \frac{1}{2\varepsilon^2} u_\gamma^2 - \frac{1}{2\varepsilon^2} u_\infty^2 \rangle \, dt + \int_{\Omega_T} u_\gamma G(p_\gamma) (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, dt \\ & + \frac{1}{\varepsilon^2} \int_{\Omega_T} u_\gamma (\nabla \omega_\varepsilon * u_\gamma) \cdot \nabla (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, dt + \int_0^T \langle \partial_t u_\gamma, v_\infty \rangle \, dt. \end{aligned} \quad (2.3.33)$$

Observe that

$$\int_{\Omega_T} \nabla \tilde{v}_\gamma \cdot \nabla (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, ds = \int_{\Omega_T} |\nabla (\tilde{v}_\gamma - \tilde{v}_\infty)|^2 \, dx \, ds + \int_{\Omega_T} \nabla \tilde{v}_\infty \cdot \nabla (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, ds,$$

but by the weak convergence in (2.3.28) we have

$$\int_{\Omega_T} \nabla \tilde{v}_\infty \cdot \nabla (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, ds \rightarrow 0,$$

since  $\nabla \tilde{v}_\infty \in L^2(0, T; L^2(\Omega))$ . Thus we deduce that

$$\limsup_{\gamma \rightarrow \infty} \int_{\Omega_T} |\nabla (\tilde{v}_\gamma - \tilde{v}_\infty)|^2 \, dx \, ds = \limsup_{\gamma \rightarrow \infty} \int_{\Omega_T} \nabla \tilde{v}_\gamma \cdot \nabla (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, ds.$$

Then, recalling also that  $\partial_t u_\gamma \rightharpoonup \partial_t u_\infty$  in  $L^2(0, T; (H^1(\Omega))')$  and that  $v_\infty \in L^2(0, T; H^1(\Omega))$ , we deduce from (2.3.33)

$$\begin{aligned} & \limsup_{\gamma \rightarrow \infty} \int_{\Omega_T} |\nabla (\tilde{v}_\gamma - \tilde{v}_\infty)|^2 \, dx \, ds \leq \limsup_{\gamma \rightarrow \infty} \left( \frac{\gamma}{(\gamma+2)(\gamma+1)} \int_{\Omega} u_\gamma^{\gamma+2}(0) \, dx \right. \\ & + \frac{1}{2\varepsilon^2} \int_0^T \langle \partial_t u_\infty, (u_\gamma - u_\infty)^2 \rangle \, dt - \frac{1}{2\varepsilon^2} \int_0^T \langle \partial_t u_\infty, u_\gamma^2 - u_\infty^2 \rangle \, dt + \int_{\Omega_T} u_\gamma G(p_\gamma) (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, dt \\ & \left. + \frac{1}{\varepsilon^2} \int_{\Omega_T} u_\gamma \nabla \omega_\varepsilon * u_\gamma \cdot \nabla (\tilde{v}_\gamma - \tilde{v}_\infty) \, dx \, dt \right) + \int_0^T \langle \partial_t u_\infty, v_\infty \rangle \, dt. \end{aligned} \quad (2.3.34)$$



The plan is to prove that all the terms on the (RHS) of (2.3.34) converge to 0. First, we have that  $0 \leq u_\gamma(0) = u_0 \leq p_H^{\frac{1}{\gamma}}$  for any  $\gamma > 0$  by assumption, so that in the end we deduce  $u_\gamma^{\gamma+1}(0) \leq p_H^{\frac{\gamma+2}{\gamma}} \leq C$  and thus, as  $\gamma \rightarrow \infty$ ,

$$\frac{\gamma}{(\gamma+2)(\gamma+1)} \int_{\Omega} u_\gamma^{\gamma+2}(0) dx \rightarrow 0.$$

Then, using (2.3.18) and the fact that  $\partial_t u_\infty \in L^2(0, T; (H^1(\Omega))')$ , we immediately deduce

$$\frac{1}{2\varepsilon^2} \int_0^T \langle \partial_t u_\infty, (u_\gamma - u_\infty)^2 \rangle dt - \frac{1}{2\varepsilon^2} \int_0^T \langle \partial_t u_\infty, u_\gamma^2 - u_\infty^2 \rangle dt \rightarrow 0 \quad \text{as } \gamma \rightarrow \infty.$$

Concerning the term  $\int_{\Omega_T} u_\gamma G(p_\gamma)(\tilde{v}_\gamma - \tilde{v}_\infty) dx dt$ , we simply use the fact that  $\{u_\gamma\}$  is bounded in  $L^\infty(\Omega \times (0, T))$  (cf. (2.3.1)),  $\{G(p_\gamma)\}$  is bounded in  $L^2(\Omega \times (0, T))$  (cf. (2.3.7)) and  $\tilde{v}_\gamma \rightarrow \tilde{v}_\infty$  strongly in  $L^2(\Omega \times (0, T))$ . Similarly,  $\int_{\Omega_T} u_\gamma \nabla \omega_\varepsilon * u_\gamma \cdot \nabla(\tilde{v}_\gamma - \tilde{v}_\infty) dx dt \rightarrow 0$  because of weak convergence (2.3.28) and strong convergence  $u_\gamma \nabla \omega_\varepsilon * u_\gamma \rightarrow u_\infty \nabla \omega_\varepsilon * u_\infty$  in  $L^2(\Omega \times (0, T))$  (which follows by (2.3.18) and simple properties of convolutions).

We are left with the analysis of the last term, i.e.,  $\int_0^T \langle \partial_t u_\infty, v_\infty \rangle dt$ . Our aim is to show that this term vanishes, exploiting Theorem 2.5.2. We introduce the following indicator function on  $\mathbb{R}$  :

$$I_S(s) := \begin{cases} 0 & \text{if } s \leq 1, \\ +\infty & \text{if } s > 1, \end{cases} \quad (2.3.35)$$

and define  $S = (-\infty, 1]$ , which is a closed, convex and nonempty set, so that  $I_S : \mathbb{R} \rightarrow (-\infty, +\infty]$  is proper, convex and lower semicontinuous (see, e.g., [283, Appendix 1]) and it holds

$$\partial I_S(x) = \{y \in \mathbb{R} : y \cdot (x - s) \geq 0, \quad \forall s \leq 1\}.$$

We see that we have

- $u_\infty \in L^2(0, T; H^1(\Omega))$  and  $\partial_t u_\infty \in L^2(0, T; (H^1(\Omega))')$ ;
- for almost any  $(x, t) \in \Omega_T$ ,

$$v_\infty(x, t) \in \partial I_S(u_\infty(x, t)),$$

by (2.3.26) and being  $v_\infty = p_\infty$ . Indeed,

$$v_\infty(x, t)(u_\infty(x, t) - s) \geq 0, \quad \forall s \leq 1,$$

since, when  $u_\infty(x, t) = 1$ , being  $v_\infty(x, t) \geq 0$ , the inequality is always verified for any  $s \leq 1$ , whereas, when  $u_\infty(x, t) < 1$ , it holds  $v_\infty(x, t) = 0$  and thus the inequality is verified for any  $s \leq 1$  as well.

- $v_\infty \in L^2(0, T; H^1(\Omega))$ .

Therefore, all the assumptions are verified and we can apply Theorem 2.5.2 with  $h = I_S$ ,  $f = u_\infty$ ,  $g = v_\infty$ , to infer, after an integration over  $[0, T]$ ,

$$0 = \int_{\Omega} I_S(u_\infty(x, T)) dx - \int_{\Omega} I_S(u_\infty(x, 0)) dx = \int_0^T \langle \partial_t u_\infty, v_\infty \rangle dt, \quad (2.3.36)$$

by the definition of  $I_S$ . Indeed, it holds  $\int_{\Omega} I_S(u_\infty(x, \cdot)) dx \equiv 0$  on  $[0, T]$  for any  $T > 0$ . To see

this, first notice that, being  $u_\infty \leq 1$  almost everywhere in  $\Omega \times [0, \infty)$ , we deduce that, for almost any  $t \in [0, \infty)$ ,

$$u_\infty(x, t) \leq 1 \quad \text{for almost any } x \in \Omega.$$

Therefore,  $\int_\Omega I_S(u_\infty(x, \cdot)) dx = 0$ , for almost any  $t \in [0, T]$  and for any  $T > 0$ . Recall now by Theorem 2.5.2 that  $\int_\Omega I_S(u_\infty(x, \cdot)) dx \in AC([0, T])$  for any  $T > 0$ , which ensures that  $\int_\Omega I_S(u_\infty(x, t)) dx \equiv 0$  for any  $t \in [0, T]$  and for any  $T > 0$ .

Having studied all the terms in the right-hand side of (2.3.34), in the end we conclude that

$$\limsup_{\gamma \rightarrow \infty} \int_{\Omega_T} |\nabla(\tilde{v}_\gamma - \tilde{v}_\infty)|^2 dx ds \leq 0,$$

implying that

$$\lim_{\gamma \rightarrow \infty} \int_{\Omega_T} |\nabla(\tilde{v}_\gamma - \tilde{v}_\infty)|^2 dx ds = 0.$$

□

**Step 3. The complementarity condition.** Now that we have all the necessary convergences, let us consider the following equation in distributional sense (this equation comes from multiplying (2.1.1) by  $v_\gamma = u_\gamma^{\gamma+1}$ ) :

$$\frac{1}{\gamma+2} \partial_t u_\gamma^{\gamma+2} - v_\gamma \Delta \tilde{v}_\gamma + \frac{1}{\varepsilon^2} v_\gamma \operatorname{div}(u_\gamma (\nabla \omega_\varepsilon * u_\gamma)) = u_\gamma G(p_\gamma) v_\gamma.$$

Thanks to the results of Step 2. and Lemma 2.3.4, we can then pass the limit as  $\gamma \rightarrow \infty$  and obtain the complementarity condition :

$$v_\infty (\Delta \tilde{v}_\infty - \frac{1}{\varepsilon^2} \operatorname{div}(u_\infty (\nabla \omega * u_\infty)) + u_\infty G(p_\infty)) = 0 \quad \text{in } \mathcal{D}'(\Omega \times (0, T)), \quad (2.3.37)$$

for any  $T > 0$ , so that in the end, recalling  $v_\infty = p_\infty$  and  $\tilde{v}_\infty = p_\infty + \frac{1}{2\varepsilon^2} u_\infty^2$ ,

$$p_\infty (\Delta p_\infty + \frac{1}{2\varepsilon^2} \Delta u_\infty^2 - \frac{1}{\varepsilon^2} \operatorname{div}(u_\infty (\nabla \omega_\varepsilon * u_\infty)) + u_\infty G(p_\infty)) = 0 \quad \text{in } \mathcal{D}'(\Omega \times (0, \infty)), \quad (2.3.38)$$

which is the complementarity condition (2.1.16). In conclusion, since  $v_\infty = p_\infty$ , we can repeat the argument leading to (2.3.36), to infer that, for any  $t \in [0, \infty)$ ,

$$\int_0^t \langle \partial_t u_\infty, p_\infty \rangle ds = 0, \quad \langle \partial_t u_\infty(t), p_\infty(t) \rangle = 0,$$

thus concluding the proof of Theorem 2.1.5.

## 2.4 Convergence to equilibria : proof of Theorem 2.1.6

Here, we prove Theorem 2.1.6. Numerical simulations illustrating the result in dimension 1 with a source term are depicted in Figure 2.1.

### 2.4.1 Case $G(p) = p_H - p$

*Proof of Theorem 2.1.6.* We now divide the proof of Theorem 2.1.6 in different steps.

**Step 1 : characterization of possible limits.** We fix the initial datum  $u_0 \neq 0$  as in the statement of Theorem 2.1.6. We show the following key result :

**Lemma 2.4.1.** *From any divergent sequence  $\{t_n\}_n \subset \mathbb{R}^+$  we can extract a subsequence (not relabeled) such that, as  $n \rightarrow \infty$ ,  $u_n(t) := u(t + t_n)$  converges to the same limit  $u_*$  strongly in  $L^p_{t,x}$  for all  $1 \leq p < 2\gamma + 1$ . Moreover, either  $u_* \equiv 0$  and  $\Phi(u(t)) \rightarrow |\Omega|$  as  $t \rightarrow \infty$ , otherwise  $u_* \equiv p_H^{1/\gamma}$  and  $\Phi(u(t)) \rightarrow 0$  as  $t \rightarrow \infty$ .*

We fix  $T > 0$ , we also consider  $n$  large enough such that  $t_n > T$ . Observe that  $u_n$  solves the problem

$$\begin{cases} \partial_t u_n - \operatorname{div}(u_n \nabla \mu_n) = u_n G(p_n) & \text{in } \Omega \times (-T, T), \\ \mu_n = p_n + B_\varepsilon(u_n), & p_n = u_n^\gamma. \end{cases} \quad (2.4.1)$$

By Propositions 2.2.2 and 2.2.5, we have the following uniform-in- $n$  bounds :

$$\mathcal{E}(u_n(t)) \leq C \quad \forall t \in [-T, T], \quad (2.4.2)$$

$$\Phi(u_n(t)) \leq \Phi(u_0) \quad \forall t \in (-T, T), \quad \Phi(u_n(t)) \text{ non-increasing.} \quad (2.4.3)$$

From this and Propositions 2.2.2 and 2.2.6, we deduce the following uniform bounds, for any  $T > 0$ ,

$$\begin{aligned} & \|u_n\|_{L^\infty(-T, T; L^{\gamma+1}(\Omega))} + \|u_n\|_{L^2(-T, T; H^1(\Omega))} + \|u_n\|_{L^{2\gamma+1}(\Omega \times (-T, T))} \\ & + \|\partial_t u_n\|_{L^{q'}(-T, T; (W^{1,q}(\Omega))')} + \|\partial_t u_n\|_{L^2(-T, T; (W^{1,r}(\Omega))')} + \|\nabla u_n^{\frac{\gamma+1}{2}}\|_{L^2(-T, T; L^2(\Omega))} \leq C(T), \end{aligned} \quad (2.4.4)$$

with  $q$  and  $r$  as in Proposition 2.2.6 which implies, by standard arguments, the following convergences (up to subsequences) as  $n \rightarrow \infty$  to the same function  $u_* \geq 0$

$$\begin{aligned} u_n & \rightharpoonup u_* & \text{in } L^2(-T, T; H^1(\Omega)) & \text{ and } L^{2\gamma+1}(\Omega \times (-T, T)) \\ \partial_t u_n & \rightharpoonup \partial_t u_* & \text{in } L^{q'}(-T, T; (W^{1,q}(\Omega))') & \text{ and in } L^2(-T, T; (W^{1,r}(\Omega))'), \\ u_n & \rightarrow u_* & \text{in } L^p(\Omega \times (0, T)), & \quad p \in [1, 2\gamma + 1), \quad \text{and almost everywhere,} \\ \nabla u_n^{\frac{\gamma+1}{2}} & \rightharpoonup \nabla u_*^{\frac{\gamma+1}{2}} & \text{in } L^2(\Omega \times (0, T)). \end{aligned}$$

We want to characterize  $u_*$ . Notice that from (2.2.2), we have for all  $T > 0$ ,

$$\begin{aligned} & \int_{-T}^T \int_{\Omega} \int_{\Omega} \omega_\varepsilon(y) |\nabla u_n(x) - \nabla u_n(x-y)|^2 dx dy ds \\ & = \int_{t_n-T}^{t_n+T} \int_{\Omega} \int_{\Omega} \omega_\varepsilon(y) |\nabla u(x) - \nabla u(x-y)|^2 dx dy ds \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

by integrability on  $(0, \infty)$ . By weak-lower semicontinuity, in the limit

$$\int_{-T}^T \int_{\Omega} \int_{\Omega} \omega_\varepsilon(y) |\nabla u_*(x) - \nabla u_*(x-y)|^2 dx dy ds = 0$$

so that  $u_*(t)$  is constant in space for a.e.  $t \in [0, T]$ . In fact,  $u_*(t)$  is constant in space for all  $t \in [0, T]$  because  $u_* \in C_{weak}([-T, T]; L^{\gamma+1}(\Omega))$  (see Remark 2.1.4) so that for all  $\varphi \in C_c^\infty(\Omega)$ , the function  $t \mapsto \int_\Omega u_*(t, x) \operatorname{div} \varphi(x) dx$  is continuous. Similarly, from (2.2.2) and the Fatou lemma,

$$u_* \log \left( \frac{u_*}{p_H^{\frac{1}{\gamma}}} \right) (u_*^\gamma - p_H) = 0, \text{ for a.e. } t \in (-T, T),$$

so that either  $u_*(t) = 0$  or  $u_*(t) = p_H^{\frac{1}{\gamma}}$ . Since  $u_* \in C_{weak}([-T, T]; L^{\gamma+1}(\Omega))$ , the average  $\bar{u}_*(\cdot) \in C([0, T])$  so that  $u_* \equiv \bar{u}_*$  can attain only one of the values for all times.

Because  $\Phi(u(t))$  is non-increasing in time, it has a limit as  $t \rightarrow \infty$ , say  $\Phi_*$ . Then clearly  $\Phi_* = \lim_{n \rightarrow \infty} \Phi(u(t+t_n)) = \Phi(u_*)$ , and thus either  $\Phi_* = \Phi(0) = |\Omega|$  if  $u_* = 0$  or  $\Phi_* = \Phi(p_H^{\frac{1}{\gamma}}) = 0$  if  $u_* = p_H^{\frac{1}{\gamma}}$ . Clearly this also implies that, given another sequence of times  $\{t_m\}_m$ , we can repeat the same argument and extract a (non relabeled) subsequence  $\{u(t_m)\}_m$  converging to the same constant  $u_*$ . This concludes the proof of Lemma 2.4.1.

**Step 2. Stability of the equilibria.** We complete Lemma 2.4.1 with the following

**Lemma 2.4.2.** *Under notation of Lemma 2.4.1, if  $u_* \equiv 0$  then  $u_0 = 0$  almost everywhere in  $\Omega$ .*

*Démonstration.* Since  $\Phi(u(t))$  is nonincreasing in time and, by Lemma 2.4.1,  $\Phi(u(t)) \rightarrow |\Omega|$ , we obtain

$$|\Omega| = \lim_{t \rightarrow \infty} \Phi(u(t)) \leq \Phi(u_0) \leq |\Omega|.$$

Thus we infer  $\Phi(u_0) = |\Omega|$ . Being the entropy function  $g : x \mapsto \frac{x}{p_H^{\frac{1}{\gamma}}} \log \left( \frac{x}{p_H^{\frac{1}{\gamma}}} \right) - \frac{x}{p_H^{\frac{1}{\gamma}}} + 1$  decreasing for  $x \in [0, p_H^{\frac{1}{\gamma}})$ , and since  $u_0 \leq p_H^{\frac{1}{\gamma}}$ , it follows that  $u_0 = 0$  almost everywhere in  $\Omega$ .  $\square$

**Step 3. Existence of the  $L^q(\Omega)$ -limit as  $t \rightarrow \infty$ .**

**Lemma 2.4.3.** *Assume that  $u_0 \not\equiv 0$ . Then it holds*

$$\lim_{t \rightarrow \infty} u(t) = p_H^{\frac{1}{\gamma}} \quad \text{in } L^q(\Omega) \quad \forall q \in [1, \gamma + 1).$$

*Démonstration.* First note that, the pointwise values  $u(t)$  as an  $L^{\gamma+1}(\Omega)$ -function makes sense thanks to the weak continuity obtained in Lemma 2.1.3. Now, we consider the decomposition of  $\Phi$  :

$$\Phi(u) = \frac{1}{p_H^{\frac{1}{\gamma}}} \left( \int_\Omega u \log \left( \frac{u}{\bar{u}} \right) dx + \int_\Omega u \log \left( \frac{\bar{u}}{p_H^{\frac{1}{\gamma}}} \right) dx + \int_\Omega (p_H^{\frac{1}{\gamma}} - u) dx \right) \quad (2.4.5)$$

and we study the limits of terms appearing in (2.4.5). Since  $u_0 \not\equiv 0$ , from Lemma 2.4.1 we obtain  $\lim_{t \rightarrow \infty} \Phi(u(t)) = \Phi(p_H^{\frac{1}{\gamma}}) = 0$ .

By Proposition 2.2.1, for any sequence  $\{t_n\}_n$  there exists a (nonrelabeled) subsequence such that, for some  $\kappa$ ,

$$\bar{u}(t_n) \rightarrow \kappa \in [0, p_H^{\frac{1}{\gamma}}]. \quad (2.4.6)$$

We prove that  $\kappa = p_H^{\frac{1}{\gamma}}$ . Indeed, the function  $g : x \mapsto \frac{x}{p_H^{\frac{1}{\gamma}}} \log \left( \frac{x}{p_H^{\frac{1}{\gamma}}} \right) - \frac{x}{p_H^{\frac{1}{\gamma}}} + 1$  is convex and continuous. As  $\Phi(u(t)) = \int_{\Omega} g(u(t)) dx$ , by Jensen's inequality we get

$$0 \leq g(\bar{u}(t)) \leq \frac{1}{|\Omega|} \Phi(u(t)) \rightarrow 0 \text{ as } t \rightarrow \infty$$

so that  $g(\kappa) = 0$  and the claim follows. It follows that  $\bar{u}(t) \rightarrow p_H^{\frac{1}{\gamma}}$  as  $t \rightarrow \infty$ .

Hence, passing to the limit in (2.4.5)

$$\int_{\Omega} u(t) \log \left( \frac{u(t)}{\bar{u}(t)} \right) dx \rightarrow 0 \text{ when } k \rightarrow \infty. \quad (2.4.7)$$

From Lemma 2.5.4 and (2.4.7), together with the fact that  $\bar{u}(t) \rightarrow p_H^{\frac{1}{\gamma}}$  as  $n \rightarrow \infty$ , we then deduce

$$\|u(t) - p_H^{\frac{1}{\gamma}}\|_{L^1(\Omega)} \rightarrow 0 \text{ when } k \rightarrow \infty.$$

Furthermore, by the bound in  $L_t^\infty L_x^{\gamma+1}$  given by the control of the energy  $\mathcal{E}$  in Proposition 2.2.2, we can deduce the convergence (2.1.18) by interpolation. The proof of Theorem 2.1.6 in the case of a nonzero source term  $G$  is thus concluded.  $\square$

### 2.4.2 Case $G(p) = 0$

When there is no source term, the solution converges to the mean value. The argument is a simple consequence of the logarithmic Sobolev inequality and the Csiszár–Kullback–Pinsker inequality. We refer to [276] for other systems where it is applied. We consider the relative entropy between the solution  $u$  and a stationary state  $u_*$ , defined as

$$\Phi(u|u_*) = \int_{\Omega} \left( u \log \left( \frac{u}{u_*} \right) - u + u_* \right) dx.$$

In our case  $u_* = \bar{u}_0$  a.e. in  $\Omega$ . Notice that, by the conservation of mass, we have

$$\Phi(u|\bar{u}) = \int_{\Omega} u \log \left( \frac{u}{\bar{u}} \right) dx.$$

Since  $\bar{u}(\cdot) \equiv \bar{u}_0$ , we see that  $\Phi(u|\bar{u})$  satisfies the identity (or at least the inequality for weak solutions) : for almost any  $t \geq 0$ ,

$$\frac{d\Phi(u|\bar{u})}{dt} + \frac{1}{2\varepsilon^2} \int_{\Omega} \int_{\Omega} \omega_\varepsilon(y) |\nabla u(x) - \nabla u(x-y)|^2 dx dy + \int_{\Omega} \frac{4\gamma}{(\gamma+1)^2} \left| \nabla |u|^{\frac{\gamma+1}{2}} \right|^2 dx = 0.$$

The generalized logarithmic Sobolev inequality [276, Section 3.1] provides us the exponential decay of the relative entropy. Indeed we have

**Lemma 2.4.4.** *For any  $m \geq (d-2)_+/d$ , there exists  $C(\Omega, m)$  such that*

$$\int_{\Omega} |\nabla (u^{\frac{m}{2}})|^2 dx \geq C(\Omega, m) \bar{u}^{m-1} \int_{\Omega} u \log \left( \frac{u}{\bar{u}} \right) dx.$$

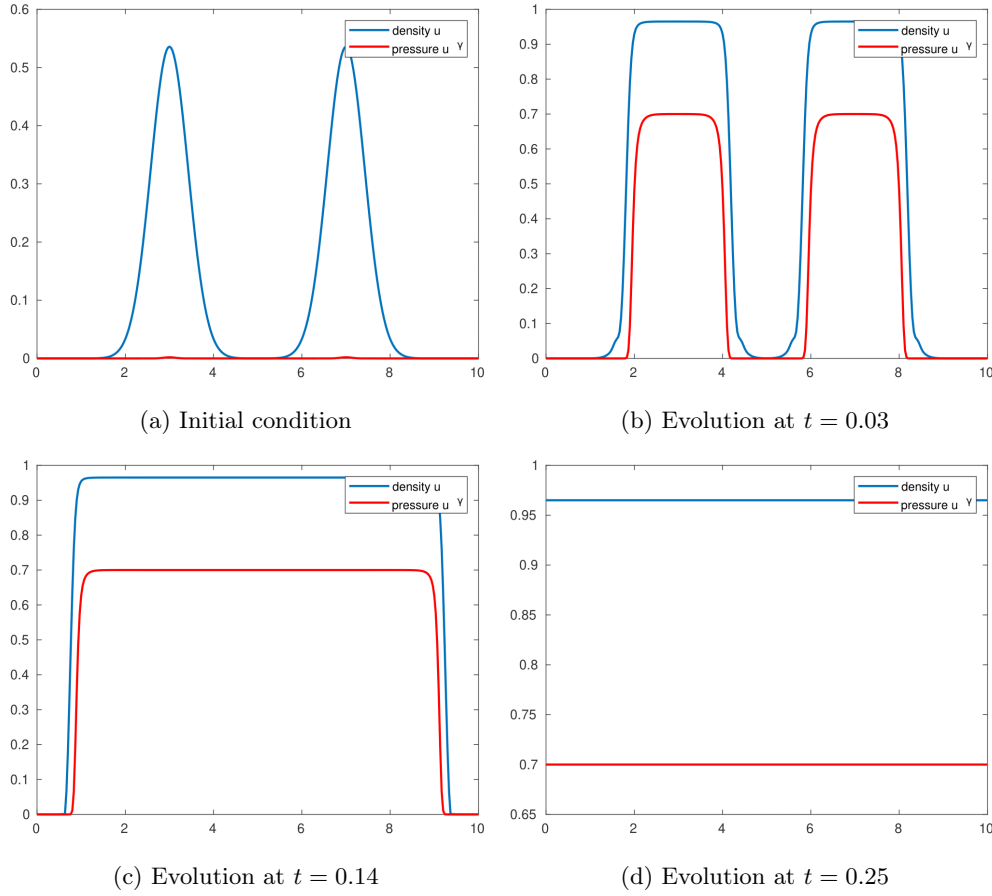


FIGURE 2.1 – We provide here 1D numerical simulations illustrating this theoretical result. We assume  $\gamma = 10$  and  $p_H = 0.7$ . We thus have  $p_H^{1/\gamma} \approx 0.96$ . The initial condition is taken as a double gaussian as in Figure (a).

In our case  $m = \gamma + 1$  and  $\bar{u} \equiv \bar{u}_0$  is bounded from below by a positive constant since we consider an initial condition  $u_0 \not\equiv 0$ . Therefore, by the Gronwall Lemma we conclude that the entropy  $\Phi$  experiences an exponential decay as  $t \rightarrow \infty$  :

$$\frac{d\Phi(u|\bar{u})}{dt} + C(\Omega, \gamma)\bar{u}_0^\gamma \Phi(u|\bar{u}) \leq 0$$

Therefore, we have, for some  $C = C(\Omega, \gamma, \Phi(u_0), \bar{u}_0)$ ,

$$\Phi(u|\bar{u}) \leq C e^{-Ct}, \quad \forall t \geq 0.$$

To prove that this implies the exponential decay of the solution we use the Csiszár–Kullback–Pinsker inequality of Lemma 2.5.4 : there exists  $C(\Omega)$  such that

$$\Phi(u|\bar{u}) \geq C(\Omega) \|u - \bar{u}_0\|_{L^1(\Omega)}^2.$$

The exponential decay in  $L^1(\Omega)$  of  $u$  towards  $\bar{u}_0$  then easily follows. By the  $L^\infty(0, T; L^{\gamma+1}(\Omega))$  bound given by the control of the energy  $\mathcal{E}$  (which is the same as in the case with a source term  $G$  given in Proposition 2.2.2), we can in conclusion deduce the exponential convergence (2.1.19) by interpolation. This ends the proof of Theorem 2.1.6.

### 2.4.3 Longtime behavior of the local Cahn-Hilliard equation

The nonlocal Cahn-Hilliard equation can be also seen as an approximation of the local Cahn-Hilliard equation :

$$\begin{cases} \partial_t u - \operatorname{div}(u \nabla \mu) = uG(p) & \text{in } \Omega \times (0, T), \\ \mu = p - \Delta u, \quad p = u^\gamma, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (2.4.8)$$

This follows, at least formally, with a Taylor expansion, using the symmetry of the kernel  $\omega_\varepsilon$  in the operator  $B_\varepsilon$  of (2.1.1), and for a rigorous proof we refer, e.g., to [155]. Therefore, one may wonder whether the previous results obtained for the nonlocal Cahn-Hilliard equation also hold for the local one. It turns out that for the convergence to the stationary states, the result is the same. Indeed, one mainly uses arguments based on the entropy, so that the nonlocal term does not play a role : concerning the entropy  $\Phi$ , we can consider again (2.1.6) and formally get for (2.4.8)

$$\frac{d}{dt} \Phi(u) + \frac{1}{p_H^{\frac{1}{\gamma}}} \int_{\Omega} |\Delta u|^2 dx + \frac{1}{p_H^{\frac{1}{\gamma}}} \int_{\Omega} \frac{4\gamma}{(\gamma+1)^2} |\nabla |u|^{\frac{\gamma+1}{2}}|^2 dx - \int_{\Omega} u \log \left( \frac{u}{p_H^{\frac{1}{\gamma}}} \right) G(p) dx = 0, \quad (2.4.9)$$

which is very similar to the result in Proposition 2.2.2. Concerning the energy, we set

$$\mathcal{E}_l(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} \frac{u^{1+\gamma}}{1+\gamma} dx$$

and thus

$$\frac{d}{dt} \mathcal{E}_l(u) + \int_{\Omega} u |\nabla \mu|^2 dx = \int_{\Omega} uG(p)\mu dx.$$

Now we observe that, integrating by parts,

$$\begin{aligned} \int_{\Omega} uG(p)\mu dx &= \int_{\Omega} u(p_H - p)p dx - \int_{\Omega} u(p_H - p)\Delta u dx \\ &= \int_{\Omega} u(p_H - p)p dx + \int_{\Omega} |\nabla u|^2 (p_H - p) dx - \int_{\Omega} u \nabla p \cdot \nabla u dx \\ &= \int_{\Omega} u(p_H - p)p dx + \int_{\Omega} |\nabla u|^2 (p_H - p) dx - \gamma \int_{\Omega} p |\nabla u|^2 dx. \end{aligned}$$

Moreover, notice that, since the relation (2.1.11) still holds with the same proof,

$$p_H \int_{\Omega} up dx \leq \frac{1}{2} \int_{\Omega} up^2 dx + \frac{p_H^2}{2} \int_{\Omega} u dx \leq \frac{1}{2} \int_{\Omega} up^2 dx + C.$$

Then we can rewrite the energy inequality as

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}_l(u) + \int_{\Omega} u |\nabla \mu|^2 dx + \frac{1}{2} \int_{\Omega} p^2 u dx + \int_{\Omega} |\nabla u|^2 p dx + \gamma \int_{\Omega} p |\nabla u|^2 dx \\ & \leq \int_{\Omega} |\nabla u|^2 dx + C \leq C(\mathcal{E}_l(u) + 1), \end{aligned}$$

which is again very similar to the one obtained in Proposition 2.2.2 for the nonlocal case. Therefore, with these estimates we can basically perform again all the arguments of Section 2.4 and obtain again the same result as in Theorem 2.1.6. Note that in the local case, differently from the nonlocal one, we can also repeat the same arguments in the case of a smooth bounded domain  $\Omega \subset \mathbb{R}^d$  with homogeneous Neumann boundary conditions  $u \nabla \mu \cdot \mathbf{n} = 0$  and  $\nabla u \cdot \mathbf{n} = 0$  on  $\partial\Omega \times (0, \infty)$ , where  $\mathbf{n}$  is the outward unit normal.

**Remark 2.4.5.** The incompressible limit,  $\gamma \rightarrow \infty$ , is very different and remains an open question in the local Cahn-Hilliard case. Indeed, obtaining an equation for the pressure  $p$  from which to deduce a uniform-in- $\gamma$   $L^2(0, T; H^1(\Omega))$ -control on  $p$  seems still out of reach. Therefore no analogous of Theorem 2.1.5 can be stated in this local case.

## 2.5 Appendix

### 2.5.1 Technical tools

Several tools have been used to carry out some proofs. First, we present a lemma about geometric convergence of numerical sequences, whose proof can be easily obtained by induction (see, e.g., [246, Ch.2, Lemma 5.6]) :

**Lemma 2.5.1.** *Let  $\{y_n\}_{n \in \mathbb{N} \cup \{0\}} \subset \mathbb{R}^+$  satisfy the recursive inequality*

$$y_{n+1} \leq C b^n y_n^{1+\epsilon}, \quad \forall n \geq 0, \quad \text{and} \quad y_0 \leq \theta := C^{-\frac{1}{\epsilon}} b^{-\frac{1}{\epsilon^2}}, \quad (2.5.1)$$

for some  $C > 0$ ,  $b > 1$  and  $\epsilon > 0$ . Then,  $y_n \rightarrow 0$  for  $n \rightarrow \infty$  with geometric rate

$$y_n \leq \theta b^{-\frac{n}{\epsilon}}, \quad \forall n \geq 0. \quad (2.5.2)$$

Next, we state a theorem concerning the absolute continuity of some integrals of convex functions in  $\mathbb{R}$ , whose proof can be found, e.g. in [209, p.101] :

**Theorem 2.5.2.** *Let  $T > 0$  and let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a convex and lower semicontinuous function. Assume that*

- $f \in L^2(0, T; H^1(\Omega))$  and  $\partial_t f \in L^2(0, T; (H^1(\Omega))')$ ,
- $g(x, t) \in \partial h(x, t)$  for almost every  $(x, t) \in \Omega \times (0, T)$ ,
- $g \in L^2(0, T; H^1(\Omega))$ .

Then, the function  $t \mapsto \int_{\Omega} h(f(x, t))$  is absolutely continuous on  $[0, T]$  and,

$$\frac{d}{dt} \int_{\Omega} h(f) dx = \langle \partial_t f, g \rangle \quad \text{for almost any } t \in (0, T).$$

We then propose a control on the  $H^1(\Omega)$ -norm related to the use of  $\omega_\epsilon$ .



**Lemma 2.5.3.** *There exists  $\varepsilon_0 > 0$  and a constant  $C$  such that for  $\varepsilon \in (0, \varepsilon_0)$  and all  $f \in L^2(\Omega)$  we have*

$$\|f - \bar{f}\|_{L^2(\Omega)}^2 \leq \frac{C}{2\varepsilon^2} \int_{\Omega} \int_{\Omega} \omega_{\varepsilon}(y) |f(x) - f(x-y)|^2 dx dy, \quad (2.5.3)$$

where  $\bar{f}$  is the average of  $f$  over  $\Omega$ . Similarly, for all  $\alpha$ , there exists  $\varepsilon_0(\alpha) > 0$  and constant  $C(\alpha)$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  and all  $f \in H^1(\Omega)$  we have

$$\|f\|_{H^1(\Omega)}^2 \leq \frac{\alpha}{2\varepsilon^2} \int_{\Omega} \int_{\Omega} \omega_{\varepsilon}(y) |\nabla f(x) - \nabla f(x-y)|^2 dx dy + C(\alpha) \|f\|_{L^1(\Omega)}^2. \quad (2.5.4)$$

*Démonstration.* The proof is identical to the one in [155, Lemma C.3] by substituting the norm  $\|\cdot\|_{L^2(\mathbb{T}^d)}$  with the norm  $\|\cdot\|_{L^1(\mathbb{T}^d)}$ . Indeed, with the notation of the proof of that Lemma, also  $n\|g_n\|_{L^1(\mathbb{T}^d)} < 1$  implies that the limit function  $g = 0$ , exactly as in the case  $L^2(\mathbb{T}^d)$ .  $\square$

In conclusion, we recall the Csiszár–Kullback–Pinsker inequality (see, e.g., [52]), which is essential to study the asymptotic behavior of weak solutions

**Lemma 2.5.4.** *For any non-negative  $u \in L^1(\Omega)$*

$$4|\Omega|\bar{u} \int_{\Omega} u \log\left(\frac{u}{\bar{u}}\right) dx \geq \|u - \bar{u}\|_{L^1(\Omega)}^2.$$

## Chapitre 3

# Pressure jump and radial stationary solutions of the degenerate Cahn-Hilliard equation.

### Abstract

The Cahn-Hilliard equation with degenerate mobility is used in several areas including the modeling of living tissues. We are interested in quantifying the pressure jump at the interface in the case of incompressible flows. To do so, we include an external force and consider stationary radial solutions. This allows us to compute the pressure jump in the small dispersion regime. We also characterize compactly supported stationary solutions in the incompressible case, prove the incompressible limit and prove convergence of the parabolic problems to stationary states.

### 3.1 Introduction

The degenerate Cahn-Hilliard equation is now commonly used in tumor growth modeling and takes into account surface tensions at the interface between different types of cells, leading to a jump of pressure. In order to compute this jump, we propose to set the problem in a spherically symmetric domain with a boundary determined by the radius  $R_b$ , and to include an external force. Therefore we consider, in two dimensions for simplicity, the equation

$$\frac{\partial(rn)}{\partial t} - \frac{\partial}{\partial r} \left( rn \frac{\partial(\mu + V)}{\partial r} \right) = 0, \quad \text{in } (0, +\infty) \times I_{R_b}, \quad (3.1.1)$$

$$\mu = n^\gamma - \frac{\delta}{r} \frac{\partial}{\partial r} \left( r \frac{\partial n}{\partial r} \right), \quad \text{in } (0, +\infty) \times I_{R_b}, \quad (3.1.2)$$

where  $I_{R_b} = (0, R_b)$  is the line segment of length  $R_b$ . Equations (3.1.1)–(3.1.2) are equipped with Neumann boundary conditions

$$\frac{\partial n}{\partial r} \Big|_{r=0} = \frac{\partial n}{\partial r} \Big|_{r=R_b} = n \frac{\partial(\mu + V)}{\partial r} \Big|_{r=0} = n \frac{\partial(\mu + V)}{\partial r} \Big|_{r=R_b} = 0, \quad (3.1.3)$$

and with an initial condition satisfying

$$n_0 \in H^1(I_{R_b}), \quad n_0 \geq 0. \quad (3.1.4)$$

We only consider nonnegative solutions and thus the term  $n^\gamma$  is well defined and the (normalized by a factor  $\frac{\pi}{2}$ ) total mass is

$$m := \int_0^{R_b} r n_0(r) dr = \int_0^{R_b} r n(t, r) dr. \quad (3.1.5)$$

Finally, the confining potential  $V(r)$  is of class  $C^1$ .

### 3.1.1 Main results

Our first result concerns the existence of solutions of (3.1.1)-(3.1.2), their regularity, and asymptotic behaviour.

**Theorem 3.1.1** (Existence of solutions and long term asymptotic). *There exists a global weak solution of (3.1.1)-(3.1.4) in the sense of Definition 3.2.1 and it satisfies estimates as in Remark 3.2.2. Moreover, up to a subsequence,  $\{r n(t+k, r)\}_k$  converges locally in time uniformly in space to a stationary solution  $r n_\infty(r) \geq 0$  where  $n_\infty \in C^1(\overline{I_{R_b}})$  satisfies  $m = \int_0^{R_b} r n_\infty(r) dr$  and*

$$r n_\infty \frac{\partial(\mu_\infty + V)}{\partial r} = 0, \quad \mu_\infty = n_\infty^\gamma - \frac{\delta}{r} \frac{\partial}{\partial r} \left( r \frac{\partial n_\infty}{\partial r} \right) \quad n'_\infty(0) = n'_\infty(R_b) = 0. \quad (3.1.6)$$

Our second result characterizes possible stationary states and shows we can distinguish an interval where  $n_\infty = 0$  and another where  $\mu_\infty + V$  is constant as expected from the first equation in (3.1.6). From now on, we consider the confining potential  $V(r) = r^2$  for simplicity. The proof may be adapted to any increasing potential.

**Theorem 3.1.2** (Characterization of the stationary states). *Let  $n_\infty \in C^1([0, R_b])$ ,  $n_\infty \geq 0$ , be a solution of (3.1.6) as built in Theorem 3.1.6.*

(A) *Then,  $n_\infty$  is nonincreasing and it satisfies  $0 \leq n_\infty(R_b) < \frac{2m}{R_b^2}$ .*

(B) *Assume  $n_\infty(R_b) = 0$  and let  $R > 0$  be the smallest argument such that  $n_\infty(R) = 0$  and thus  $n_\infty > 0$  in  $[0, R)$ . Then, there is  $\lambda_\infty \in (0, R^2)$  such that*

$$\begin{cases} n_\infty^\gamma - \frac{\delta}{r} n'_\infty - \delta n''_\infty = R^2 - r^2 - \lambda_\infty & \text{in } (0, R), \\ n_\infty(R) = n'_\infty(R) = 0. \end{cases} \quad (3.1.7)$$

and, given  $R > 0$ , there is at most one couple  $(n, \lambda)$  solving (3.1.7).

(C) *Fix  $\delta \in (0, 1)$ . There exists  $R(m)$ , independent of  $\gamma$ , such that when  $R_b > R(m)$ , then  $n_\infty(R_b) = 0$ .*

Next, we focus on the incompressible limit of the solutions of (3.1.7), that is when  $\gamma_k \rightarrow \infty$ . We denote by  $n_k$  the steady state associated with  $\gamma_k$  and assume that  $R_b$  is large enough so that  $n_k(R_b) = 0$ .

**Theorem 3.1.3** (Incompressible limit of the stationary states). *Let  $\{\gamma_k\}_{k \in \mathbb{N}}$  be any sequence such that  $\gamma_k \rightarrow \infty$ . Let  $\{n_k\}_{k \in \mathbb{N}}$  be a sequence of stationary states with the same mass  $m$  and with radius  $R_k$ , being the smallest argument such that  $n_k(r) = 0$ .*

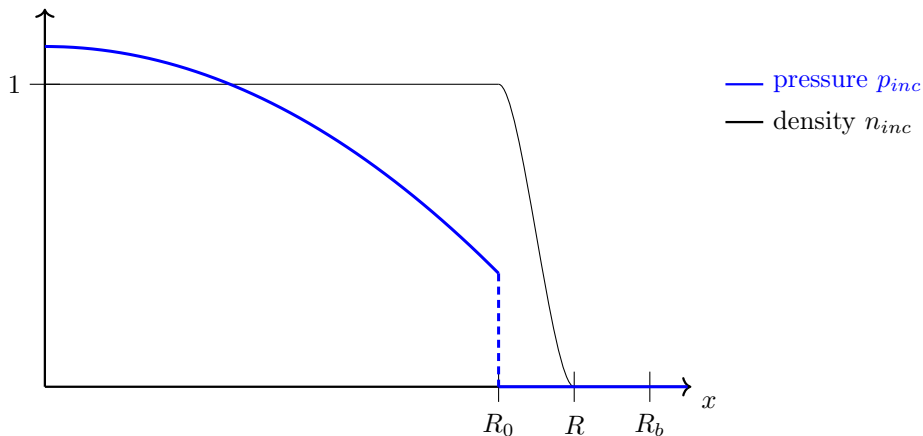


FIGURE 3.1 – Plot of the limiting profile  $n_{inc}$ , as  $\gamma_k \rightarrow \infty$ , for the potential  $V(r) = r^2$ . We can observe that the pressure has a discontinuity at  $R_0$  with  $(0, R_0) = \{n_{inc} = 1\} = \{p_{inc} > 0\}$ , while the density remains  $C^1$ .

Then,  $n_k \rightarrow n_{inc}$  in  $C^1([0, R_b])$  and  $R_k \rightarrow R$ , where  $n_{inc}$  and  $R$  are uniquely defined in Proposition 3.4.1. Moreover, the sequence of pressures  $\{p_k := n_k^{\gamma_k}\}_{k \in \mathbb{N}}$  converges weakly to some pressure  $p_{inc}$  such that  $p_{inc}(n_{inc} - 1) = 0$  and  $p_{inc}$  has a jump at  $\partial\{n_{inc} = 1\}$

$$\llbracket p_{inc} \rrbracket \approx \sqrt[3]{6} R^{2/3} \delta^{1/3}, \quad \text{as } \delta \rightarrow 0.$$

The profile  $n_{inc}$  obtained for the incompressible limit of stationary states is depicted in Figure 3.1. The density is equal to 1 on a certain interval  $(0, R_0)$  where the pressure is positive. Then, the pressure vanishes and the density decreases to 0 on a small interval  $(R_0, R)$ . At the boundary point  $R_0$  the pressure undergoes a jump, which depends on the surface tension coefficient  $\delta$  and on the shape of the confinement potential  $V$ . More precisely, for a general potential  $V(r)$ , this jump is determined by

$$\llbracket p_{inc} \rrbracket \approx \frac{\sqrt[3]{12}}{2} \delta^{1/3} (V'(R))^{2/3} \quad \text{as } \delta \rightarrow 0, \quad (3.1.8)$$

where  $R$  is the smallest value where  $n(R) = 0$  and we have the estimate  $R^2 - R_0^2 \approx \frac{2\sqrt[3]{12}\delta^{1/3}R}{\sqrt[3]{V'(R)}}$ .

We point out that the limiting profile (including parameters  $R_0$  and  $R$ ) is uniquely determined in terms of mass  $m$ ,  $\delta$  and  $V$  cf. Proposition 3.4.1.

In the above statements, the main novelty concerns the incompressible limit  $\gamma \rightarrow \infty$  for the stationary states. A previous work in this direction [151] made use of viscosity relaxation, which provided additional estimates implying compactness. In our case, assuming the radial symmetry of the problem, we are able to characterize the incompressible limit of the sequence of compactly supported stationary solutions. While our setting is restrictive, it allows performing many computations explicitly. In particular, we find how the pressure jump depends on  $V$  and  $\delta$ , cf. (3.1.8).

**Open question.** We prove that the stationary states are compactly supported or at least zero on the boundary if the domain is large enough. It is logical to ask whether the solutions of the parabolic equation are compactly supported for a large domain and a strong confining potential.

This question is still open. However, a work in this direction [99] has proved that in dimension 1, one could expect the solutions of the Cahn-Hilliard equation without confining potential to propagate with finite speed. By adding this potential, we can expect to have a better result, and compactly supported solutions, with time-independent support.

**Contents.** The above theorems are proved in the following sections. Section 3.2, is devoted to prove Theorem 3.1.1. In Section 3.3 we prove Theorem 3.1.2 and in Section 3.4 we give the proof of our main new result, namely Theorem 3.1.3. Numerical simulations of the model with a source term and no confining potential are presented in Section 3.5. The appendix contains the computation of the pressure jump for a general confining potential.

**Notations.** For a function  $n(x, t)$  we associate a function in radial coordinates that is still denoted by  $n(r, t)$ . For  $1 \leq p, s \leq +\infty$  or  $s = -1$  and  $\Omega$  a domain,  $L^p(\Omega), H^s(\Omega)$  denote the usual Lebesgue and Sobolev spaces. When  $s = -1$ ,  $H^{-1}(\Omega)$  is the topological dual of  $H_0^1(\Omega)$ . Here  $H^s(\Omega) = W^{s,2}(\Omega)$  in the usual notation. We also consider the Bochner spaces  $L^p(0, T; H^s(\Omega))$  associated with the norm

$$\|f\|_{L^p(0,T;H^s(\Omega))} = \left( \int_0^T \|f\|_{H^s(\Omega)}^p \right)^{1/p}.$$

The partial derivative with respect to the radial variable is written as  $\partial_r u(r) = \frac{\partial u}{\partial r}(r) = u'(r)$ . Finally,  $C$  denotes a generic constant which appears in inequalities and whose value can change from one line to another. This constant can depend on various parameters unless specified otherwise.

### 3.1.2 Literature review and biological relevancy of the system

**Tissue growth models and Hele-Shaw limits.** Development of tissue growth models is presently a major line of research in mathematical biology. Nowadays, number of models are available [67, 177, 264] with the common feature that they use the tissue internal pressure as the main driver of both the cell movement and proliferation. The simplest example of a mechanical model of living tissue is the *compressible* equation

$$\partial_t n = \operatorname{div}(n \nabla p) + n G(p), \quad p = P_\gamma(n) := n^\gamma, \quad (3.1.9)$$

in which  $p(t, x) = P(n(t, x))$ , with  $P$  a law of state, is the pressure and  $n$  the density of cell number. Here, the cell velocity is given via Darcy's law which captures the effect of cells moving away from regions of high compression. Dependence on growth function pressure has also been used to model the sensitivity of tissue proliferation to compression (contact inhibition, [68]).

An important problem is to understand the so-called incompressible limit (*i.e.*  $\gamma \rightarrow \infty$ ) of this model. Perthame *et al.* [289] have shown that in this limit, solutions of (3.1.9) converge to a limit solution  $(n_\infty, p_\infty)$  of a Hele-Shaw-type free boundary limit problem for which the speed of the free boundary is given by the normal component of  $\nabla p_\infty$ , see also other approaches in [238, 261]. In this limit, the solution of (3.1.9) is organized into 2 regions :  $\Omega(t)$  in which the pressure is positive (corresponding to the tissue) and outside of this zone where  $p = 0$ . Furthermore, the free boundary problem is supplemented by a complementary equation that indicates that the

pressure satisfies

$$-\Delta p_\infty = G(p_\infty), \quad \text{in } \Omega(t), \quad \text{or similarly } p_\infty(\Delta p_\infty + G(p_\infty)) = 0 \quad \text{a.e. in } \Omega. \quad (3.1.10)$$

In this model, the pressure stays continuous in space, with jumps in time, and is equal to 0 at the interface. This is because only repulsive forces were taken into account. Hence, the crucial role of the cell-cell adhesion and thus the pressure jump at the surface of the tissue is not retrieved at the limit. Additionally, as pointed out by Lowengrub *et al.* [264], the velocity of the free surface should depend on its geometry and more precisely on the local curvature denoted by  $\kappa$ .

This motivated considering variants of the general model (3.1.9), where other physical effects of mechanical models of tissue growth are introduced. One of them is the addition of the effect of viscosity in the model, which has been made to represent the friction between cells [38, 50] through the use of Stokes' or Brinkman's law. Moreover, as pointed out by Perthame and Vauchelet [291], Brinkman's law leads to a simpler version of the model and, therefore, is a preferential choice for its mathematical analysis. Adding viscosity through the use of Brinkman's law leads to the model

$$\begin{cases} \partial_t n = \operatorname{div}(n \nabla \mu) + nG(p), & \text{in } (0, +\infty) \times \Omega, \\ -\sigma \Delta \mu + \mu = p, & \text{in } (0, +\infty) \times \Omega. \end{cases} \quad (3.1.11)$$

The incompressible limit of this system also yields the complementary relation (see [291])

$$p_\infty(p_\infty - \mu_\infty - \sigma G(p_\infty)) = 0, \quad \text{a.e. in } \Omega.$$

In the incompressible limit, notable changes compared to the system with Darcy's law are found. First, the previous complementary relation is different compared to Equation (3.1.10), and the pressure  $p_\infty$  in the limit is discontinuous, *i.e.* there is a jump of the pressure located at the surface of  $\Omega(t)$ . However, the pressure jump is related to the potential  $\mu$  and not to the local curvature of the free boundary  $\partial\Omega(t)$ . The authors already indicated that a possible explanation for this is that the previous model does not include the effect of surface tension.

**Surface tension and pressure jump.** Surface tension is a concept associated with the internal cohesive forces between the molecules of a fluid : hydrogen bonds, van der Waals forces, metallic bonds, etc. Inside the fluid, molecules are attracted equally in all directions leading to a net force of zero ; however molecules on the surface experience an attractive force that tends to pull them to the interior of the fluid : this is the origin of the surface energy. This energy is equivalent to the work or energy required to remove the surface layer of molecules in a unit area. The value of the surface tension will vary greatly depending on the nature of the forces exerted between the atoms or molecules. In the case of solid tumor cells in a tissue, it reflects the cell-cell adhesion tendency between the cells and depends on the parameter  $\delta$  and the geometry of the tumor.

In the previous definition, the surface tension is associated with a single body that has an interface with the vacuum. When one considers two bodies, the surface energy of each body is modified by the presence of the other and we speak of interfacial tension. The latter depends on the surface tension of each of the two compounds, as well as the interaction energy between the two compounds. In the system considered above, it is then possible to imagine that the vacuum in which the tumor grows is in fact another body that has an internal pressure of the form  $V(r)$  which increases with respect to  $r$  so that the tumor is stopped at some point and we can consider the stationary states.

For such a tumor to be in equilibrium, it is necessary that the interior is overpressured relative to the exterior by an amount. This amount is called the pressure jump and is computed explicitly

in our case.

Surface tension effects can be introduced in the Hele-Shaw model as follows (see *e.g.* [161])

$$\begin{cases} -\Delta\mu = 0 & \text{in } \Omega \setminus \partial\Omega(t), \\ \mu = \sigma\kappa & \text{on } \partial\Omega(t). \end{cases} \quad (3.1.12)$$

where  $\sigma$  is a positive constant, called a surface tension and  $\kappa$  is a mean curvature of  $\partial\Omega(t)$ . This correct Hele-Shaw limit has been formally obtained as the sharp-interface asymptotic model of the Cahn-Hilliard equation [17]; see also [101] for a convergence result in a weak varifold formulation. This suggests that the Cahn-Hilliard equation is an appropriate model to capture surface tension effects.

**The Cahn Hilliard equation.** Cahn-Hilliard type models for tissue growth have been developed based on the theory of mixtures in mechanics, see [176, 97, 307] and the references therein. Nowadays, they are widely used, in particular for tumor growth, and analysed, [187, 11, 186, 179, 144, 145, 287]. Originally introduced in the context of materials sciences [70], they are currently applied in numerous fields, including complex fluids, polymer science, and mathematical biology. For the overview of mathematical theory, we refer to [277].

Usually, in mechanical models, the Cahn-Hilliard equation takes the form

$$\partial_t\varphi = \operatorname{div}(b(\varphi)\nabla(\psi'(\varphi) - \delta\Delta\varphi)) \iff \begin{cases} \partial_t\varphi & = \operatorname{div}(b(\varphi)\nabla\mu), \\ \mu & = -\delta\Delta\varphi + \psi'(\varphi), \end{cases} \quad (3.1.13)$$

where  $\varphi$  represents the relative density of cells  $\varphi = n_1/(n_1 + n_2)$ ,  $b$  is the mobility,  $\psi$  is the potential while  $\mu$  is the quantity of chemical potential, which is a quantity related to the effective pressure. From the point of view of mathematical biology, the most relevant case is  $b(\varphi) = \varphi(1 - \varphi)$ , which is referred to as degenerate mobility.

In our context, (3.1.1) models the motion of a population of cells constituting a biological tissue in the form of a continuity equation. It takes into account pressure, the surface tension occurring at the surface of the tissue and its viscosity. More precisely, the equation for  $\mu$  (*i.e.* equation (3.1.2)) includes the effects of both the pressure, through the term  $n^\gamma$  with  $\gamma > 1$  that controls the stiffness of the pressure law, and surface tension by  $-\delta\Delta n$ , where  $\sqrt{\delta}$  is the width of the interface in which partial mixing of the two components  $n_1, n_2$  occurs.

A similar Cahn-Hilliard problem, without radial symmetry assumption, has previously been considered in [151], but including a relaxation (viscosity) term and a proliferation source term in place of the confinement potential. In the incompressible limit, the authors obtain a jump in pressure at the interface at all times for the relaxed system. The aim here is to justify a rigorous limit without viscosity relaxation and mostly to compute the pressure jump by analyzing the stationary states of a system with confining potential.

## 3.2 Existence, regularity, and long term behavior

The existence of weak solutions for the Cahn-Hilliard equation with degenerate mobility usually follows the method from [157]. The idea is to apply a Galerkin scheme with a non-degenerate regularized mobility, *i.e.*, calling  $b(n)$  the mobility, then one considers an approximation  $b_\varepsilon(n) \geq \varepsilon$ . Then, using standard compactness methods one can prove the existence of weak solutions for the initial system. However, the uniqueness of the weak solutions is still an open question.

In the case of a radially symmetric solution, the resulting system has only one dimension in space, and it is possible to apply a fixed point theorem, see [331], to obtain better regularity results. Since the solutions have radial symmetry, the equation is singular at  $r = 0$ . Therefore, the first step is to consider the system with  $r + \varepsilon$  instead of  $r$  and a regularized mobility. The existence of solutions for a similar regularized system has been achieved in [331] based on a result of [330] and we do not repeat the arguments here. Then, we can pass to the limit  $\varepsilon \rightarrow 0$ . Finally, the nonnegativity of the limiting solution is achieved with the bounds provided by the entropy.

We finally point out that since we are also interested in the convergence to the stationary states, one needs to carefully verify that the bounds do not depend on time.

**Definition 3.2.1** (Weak solutions). We say that  $n(t, r)$  is a global weak solution of the equation (3.1.1)-(3.1.2) provided that

- $n$  is nonnegative,
- $rn$  is continuous in  $[0, \infty) \times \overline{I_{R_b}}$ ,  $\sqrt{r}n \in L^\infty((0, \infty) \times I_{R_b})$  and  $r \partial_t n \in L^2((0, \infty); H^{-1}(I_{R_b}))$ ,
- $\sqrt{rn} \partial_r \mu \in L^2((0, \infty) \times \overline{I_{R_b}} \setminus \{rn = 0\})$  and  $\mu$  is defined in (3.1.2),
- for every test function  $\varphi \in L^2((0, \infty); H^1(I_{R_b})) \cap C_c^1([0, \infty) \times I_{R_b})$

$$\int_0^T r \langle \partial_t n, \varphi \rangle_{H^{-1}, H^1} dt + \int_0^T \int_0^{R_b} \mathbf{1}_{rn>0} r n \partial_r (\mu + V) \partial_r \varphi dr dt = 0,$$

and

$$\int_0^T r \langle \partial_t n, \varphi \rangle_{H^{-1}, H^1} dt = - \int_0^T \int_0^{R_b} rn \partial_t \varphi dr dt - \int_0^{R_b} \varphi(0, r) n_0(r) dr,$$

- $n'(t, R_b) = 0$  for a.e.  $t \in (0, T)$ .

**Remark 3.2.2** (Energy, entropy properties of weak solutions). In fact, we construct solutions satisfying additionally mass, energy, and entropy relations as follows : for a.e.  $\tau \in [0, T]$

$$\int_0^{R_b} r n(\tau, r) dr = \int_0^{R_b} r n_0(r) dr, \quad (3.2.1)$$

$$\mathcal{E}[n(\tau, \cdot)] + \int_0^\tau \int_0^{R_b} \mathbf{1}_{rn>0} rn |\partial_r (\mu + V)|^2 dr dt \leq \mathcal{E}[n_0], \quad (3.2.2)$$

$$\Phi[n(\tau, \cdot)] + \int_0^\tau \int_0^{R_b} \left( \gamma rn^{\gamma-1} |\partial_r n|^2 + \delta r |\partial_{rr} n|^2 + \delta \frac{|\partial_r n|^2}{r} + r \partial_r n \partial_r V \right) dr dt \leq \Phi[n_0], \quad (3.2.3)$$

where energy and entropy are defined as follows :

$$\mathcal{E}[n] = \int_0^{R_b} r \left( \frac{n^{\gamma+1}}{\gamma+1} + \frac{\delta}{2} |\partial_r n|^2 + nV \right) dr, \quad \Phi[n] = \int_0^{R_b} r \phi(n) dr,$$

and  $\phi(n) = n(\log(n) - 1) + 1$ . Equations (3.2.1)–(3.2.3) provide the basic a priori estimates. Moreover, we construct Hölder continuous solutions; there is a constant  $C$ , such that for all  $r, r_1, r_2 \in [0, R_b]$ ,  $t, t_1, t_2 \in [0, \infty)$

$$|r_1 n(t, r_1) - r_2 n(t, r_2)| \leq C|r_1 - r_2|^{1/2}, \quad (3.2.4)$$

$$|r(n(t_2, r) - n(t_1, r))| \leq C|t_2 - t_1|^{1/8}. \quad (3.2.5)$$



### 3.2.1 Regularized system

We consider the existence of a regularized system, which reads :

$$\partial_t(r + \varepsilon)n_\varepsilon - \partial_r((r + \varepsilon)B_\varepsilon(n_\varepsilon)\partial_r(\mu_\varepsilon + V)) = 0, \quad \text{in } (0, +\infty) \times I_{R_b}, \quad (3.2.6)$$

$$\mu_\varepsilon = n_\varepsilon^\gamma - \frac{\delta}{r + \varepsilon}\partial_r((r + \varepsilon)\partial_r n_\varepsilon), \quad \text{in } (0, +\infty) \times I_{R_b}, \quad (3.2.7)$$

where

$$B_\varepsilon(n) = \begin{cases} \varepsilon & \text{for } n \leq \varepsilon, \\ n & \text{otherwise.} \end{cases} \quad (3.2.8)$$

We impose Neumann boundary conditions

$$\frac{\partial n_\varepsilon}{\partial r} \Big|_{r=0} = \frac{\partial n_\varepsilon}{\partial r} \Big|_{r=R_b} = B_\varepsilon(n_\varepsilon) \frac{\partial(\mu_\varepsilon + V)}{\partial r} \Big|_{r=0} = B_\varepsilon(n_\varepsilon) \frac{\partial(\mu_\varepsilon + V)}{\partial r} \Big|_{r=R_b} = 0. \quad (3.2.9)$$

We admit the following theorem of existence, for a result for a similar system we refer to [331],

**Theorem 3.2.3.** *For  $\varepsilon > 0$  and  $T > 0$ , Problem (3.2.6)-(3.2.7) with boundary conditions (3.2.9) and smooth initial condition admits a unique strong solution  $n_\varepsilon$ .*

**Remark 3.2.4.** Note that the assumption on the initial condition is stronger than the one asked in (3.1.4). This means that for the regularized system we need to consider a smooth approximation of the initial condition, for instance  $n_\varepsilon^0 = n^0 * \omega_\varepsilon$  with  $\omega$  a smooth kernel that we send to a dirac mass when  $\varepsilon \rightarrow 0$ .

Next, we prove some conservation properties for the system (3.2.6)-(3.2.7), see for instance [151, 287].

**Lemma 3.2.5** (Conservation of mass, energy and entropy). *We define  $\phi_\varepsilon$  such that  $\phi_\varepsilon''(n) = \frac{1}{B_\varepsilon(n)}$  and  $\phi_\varepsilon(1) = \phi_\varepsilon'(1) = 0$ , and*

$$\mathcal{E}_\varepsilon[n] := \int_0^{R_b} (r + \varepsilon) \left( \frac{n^{\gamma+1}}{\gamma+1} + \frac{\delta}{2} |\partial_r n|^2 + nV \right) dr,$$

$$\Phi_\varepsilon[n] := \int_0^{R_b} (r + \varepsilon) \phi_\varepsilon(n) dr.$$

Then, we have

$$\frac{d}{dt} \int_0^{R_b} (r + \varepsilon) n_\varepsilon(t, r) dr = 0, \quad (3.2.10)$$

$$\frac{d}{dt} \mathcal{E}_\varepsilon[n_\varepsilon] + \int_0^{R_b} (r + \varepsilon) B_\varepsilon(n_\varepsilon) |\partial_r(\mu_\varepsilon + V)|^2 dr = 0, \quad (3.2.11)$$

$$\frac{d}{dt} \Phi_\varepsilon[n_\varepsilon] + \int_0^{R_b} \left( \gamma(r + \varepsilon) n_\varepsilon^{\gamma-1} |\partial_r n_\varepsilon|^2 + \delta(r + \varepsilon) |\partial_{rr} n_\varepsilon|^2 + \delta \frac{|\partial_r n_\varepsilon|^2}{r + \varepsilon} + (r + \varepsilon) \partial_r n_\varepsilon \partial_r V \right) dr = 0. \quad (3.2.12)$$

**Remark 3.2.6.** The function  $\phi_\varepsilon$  is given by an explicit formula

$$\phi_\varepsilon(x) = \begin{cases} x(\log(\varepsilon) - 1) + 1 + x^2/(2\varepsilon) - \varepsilon/2 & \text{for } x \leq \varepsilon \\ x(\log(x) - 1) + 1 & \text{for } \varepsilon < x. \end{cases} \quad (3.2.13)$$

With  $\varepsilon < 1$ , it enjoys three properties :

1.  $\phi_\varepsilon(x) \rightarrow \phi(x) := x(\log(x) - 1) + 1$  for  $x \geq 0$  as  $\varepsilon \rightarrow 0$ ,
2.  $\phi_\varepsilon(x) \geq 0$  for all  $x \in \mathbb{R}$ ,
3.  $\phi'_\varepsilon(x) \leq 0$  for  $x \leq \varepsilon$ ,
4.  $\phi_\varepsilon(x) \leq \phi(x) + 1 - \varepsilon/2$  for  $x \geq 0$ .

The first one is trivial. To see the second one, we observe that the function  $x \mapsto x(\log(x) - 1) + 1$  is nonnegative, which implies  $\phi_\varepsilon(x) \geq 0$  for  $x \geq \varepsilon$ . Then, for  $x \leq \varepsilon$  we discover

$$\phi'_\varepsilon(x) = \log(\varepsilon) - 1 + \frac{x}{\varepsilon} \leq 0. \quad (3.2.14)$$

As  $\phi_\varepsilon(\varepsilon) \geq 0$ , this implies  $\phi_\varepsilon(x) \geq 0$  for all  $x \in \mathbb{R}$ . Then, (3.2.14) also implies the third property while the fourth follows by estimating  $\phi_\varepsilon(x) \leq \phi_\varepsilon(0)$  for  $x \leq \varepsilon$ .

*Proof of Lemma 3.2.5.* Mass conservation (3.2.10) follows from integrating (3.2.6) in space and using the boundary conditions (3.2.9).

To see (3.2.11), we multiply (3.2.6) by  $\mu_\varepsilon + V$ , integrate in space and use boundary conditions to obtain :

$$\int_0^{R_b} (r + \varepsilon) \partial_t n_\varepsilon (\mu_\varepsilon + V) dr + \int_0^{R_b} (r + \varepsilon) B_\varepsilon(n_\varepsilon) |\partial_r(\mu_\varepsilon + V)|^2 dr = 0.$$

Using (3.2.7) and integrating by parts, we obtain

$$\int_0^{R_b} (r + \varepsilon) \partial_t n_\varepsilon (\mu_\varepsilon + V) dr = \frac{d}{dt} \mathcal{E}_\varepsilon[n_\varepsilon],$$

which concludes the proof of (3.2.11).

To see (3.2.12), we multiply (3.2.6) by  $\phi'_\varepsilon(n_\varepsilon)$  and integrate in space to obtain

$$\frac{d}{dt} \Phi_\varepsilon[n_\varepsilon] + \int_0^{R_b} (r + \varepsilon) n'_\varepsilon(r) \partial_r(\mu_\varepsilon + V) dr = 0.$$

In view of (3.2.12), it is sufficient to prove

$$\int_0^{R_b} (r + \varepsilon) n'_\varepsilon(r) \partial_r \mu_\varepsilon dr = \gamma \int_0^{R_b} (r + \varepsilon) |n'_\varepsilon(r)|^2 n_\varepsilon^{\gamma-1} dr + \delta \int_0^{R_b} \frac{|n'_\varepsilon|^2}{r + \varepsilon} + (r + \varepsilon) |n''_\varepsilon|^2 dr.$$

We have

$$\int_0^{R_b} (r + \varepsilon) n'_\varepsilon(r) \partial_r \mu_\varepsilon dr = \gamma \int_0^{R_b} (r + \varepsilon) |n'_\varepsilon(r)|^2 n_\varepsilon^{\gamma-1} dr - \delta \int_0^{R_b} (r + \varepsilon) n'_\varepsilon(r) \partial_r \left( \frac{1}{r + \varepsilon} \partial_r((r + \varepsilon) \partial_r n_\varepsilon) \right) dr.$$

In the second part, we can integrate by parts (using Neumann's boundary conditions)

$$\begin{aligned}
& - \int_0^{R_b} (r + \varepsilon) n'_\varepsilon(r) \partial_r \left( \frac{1}{r + \varepsilon} \partial_r((r + \varepsilon) \partial_r n_\varepsilon) \right) dr \\
&= \int_0^{R_b} n'_\varepsilon(r) \frac{1}{r + \varepsilon} \partial_r((r + \varepsilon) \partial_r n_\varepsilon) dr + \int_0^{R_b} (r + \varepsilon) n''_\varepsilon(r) \frac{1}{r + \varepsilon} \partial_r((r + \varepsilon) \partial_r n_\varepsilon) dr \\
&= \int_0^{R_b} \frac{|n'_\varepsilon|^2}{r + \varepsilon} + (r + \varepsilon) |n''_\varepsilon|^2 dr + 2 \int_0^{R_b} n'_\varepsilon(r) n''_\varepsilon(r) dr.
\end{aligned}$$

The last term vanishes thanks to boundary conditions :

$$2 \int_0^{R_b} n'_\varepsilon(r) n''_\varepsilon(r) dr = \int_0^{R_b} \partial_r |n'_\varepsilon(r)|^2 dr = 0$$

and this concludes the proof.  $\square$

From Lemma 3.2.5, we may deduce uniform bounds (in  $\varepsilon$ ) for the solutions  $n_\varepsilon$  as follows

**Proposition 3.2.7.** *Let  $T > 0$ . The following sequences are uniformly bounded with respect to  $\varepsilon > 0$  :*

- (A1)  $\{\sqrt{r + \varepsilon} \partial_r n_\varepsilon\}$  in  $L^\infty((0, \infty); L^2(I_{R_b}))$ ,
- (A2)  $\{\sqrt{r + \varepsilon} n_\varepsilon\}$  in  $L^\infty((0, \infty) \times I_{R_b})$ ,
- (A3)  $\{\sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \partial_r(\mu_\varepsilon + V)\}$  in  $L^2((0, \infty) \times I_{R_b})$ ,
- (A4)  $\{\sqrt{r + \varepsilon} \partial_{rr} n_\varepsilon\}$  and  $\left\{ \frac{\partial_r n_\varepsilon}{\sqrt{r + \varepsilon}} \right\}$  in  $L^2((0, T) \times I_{R_b})$ ,
- (A5)  $\{\Phi_\varepsilon(n_\varepsilon)\}$  in  $L^\infty(0, T)$ ,
- (A6)  $\{(r + \varepsilon) \partial_t n_\varepsilon\}$  in  $L^2((0, \infty); H^{-1}(I_{R_b}))$ ,

where the estimates (A1)–(A3) and (A6) depend only on the initial energy  $\mathcal{E}(n_0)$ . Moreover, there is a constant  $C$ , independent of  $\varepsilon$ , such that for all  $r, r_1, r_2 \in [0, R_b]$ ,  $t, t_1, t_2 \in [0, \infty)$

$$|(r_1 + \varepsilon) n_\varepsilon(r_1, t) - (r_2 + \varepsilon) n_\varepsilon(r_2, t)| \leq C |r_1 - r_2|^{1/2}, \quad (3.2.15)$$

$$|(r + \varepsilon)(n_\varepsilon(t_2, r) - n_\varepsilon(t_1, r))| \leq C |t_2 - t_1|^{1/8}. \quad (3.2.16)$$

In fact, the constant  $C$  depends only on initial energy  $\mathcal{E}(n_0)$ .

*Proof of Proposition 3.2.7.* We divide the reasoning into a few steps.

*Step 1 : Estimates (A1)–(A2).* First, from (3.2.11) we deduce (A1). For estimate (A2) we adapt

the method from [331]. For any  $\rho \in (0, R_b)$ ,

$$\begin{aligned}
& \frac{R_b^2 + 2\varepsilon R_b}{2} n_\varepsilon(t, \rho) - \int_0^{R_b} (z + \varepsilon) n_\varepsilon(t, z) dz = \int_0^{R_b} (z + \varepsilon) [n(t, \rho) - n(t, z)] dz \\
&= \int_0^{R_b} \int_z^\rho (z + \varepsilon) \partial_r n_\varepsilon(t, r) dr dz \\
&= \int_0^\rho \int_z^\rho (z + \varepsilon) \partial_r n_\varepsilon(t, r) dr dz + \int_\rho^{R_b} \int_z^\rho (z + \varepsilon) \partial_r n_\varepsilon(t, r) dr dz \\
&= \int_0^\rho \int_0^r (z + \varepsilon) \partial_r n_\varepsilon(t, r) dz dr + \int_\rho^{R_b} \int_r^{R_b} (z + \varepsilon) \partial_r n_\varepsilon(t, r) dz dr \\
&= \int_0^\rho \left( \frac{r^2}{2} + \varepsilon r \right) \partial_r n_\varepsilon(t, r) dr + \int_\rho^{R_b} \left[ \frac{1}{2}(R_b^2 - r^2) + \varepsilon(R_b - r) \right] \partial_r n_\varepsilon(t, r) dr \\
&\leq R_b \int_0^\rho (r + \varepsilon) |\partial_r n_\varepsilon(t, r)| dr + 2R_b^2 \int_\rho^{R_b} |\partial_r n_\varepsilon(t, r)| dr.
\end{aligned}$$

Multiplying the previous inequality by  $2(\rho + \varepsilon)^{1/2}$  yields

$$\begin{aligned}
& \left| (R_b^2 + 2\varepsilon R_b)(\rho + \varepsilon)^{1/2} n_\varepsilon(t, \rho) - 2(\rho + \varepsilon)^{1/2} \int_0^{R_b} (z + \varepsilon) n_\varepsilon(t, z) dz \right| \leq \\
& \leq 2(\rho + \varepsilon)^{1/2} R_b \int_0^\rho (r + \varepsilon) |\partial_r n_\varepsilon(t, r)| dr + 4R_b^2 \int_\rho^{R_b} (r + \varepsilon)^{1/2} |\partial_r n_\varepsilon(t, r)| dr \\
& \leq C(R_b) \left( \int_0^{R_b} (r + \varepsilon) |\partial_r n_\varepsilon(t, r)|^2 dr \right)^{1/2}.
\end{aligned}$$

Thanks to the conservation of mass (3.2.10), we obtain (A2).

*Step 2 : Estimates (A3)–(A5).* The bound (A3) follows from the conservation of energy (3.2.11). To see (A4) and (A5), we want to use the conservation of entropy (3.2.12), but this has to be done carefully, as the term  $(r + \varepsilon) \partial_r n_\varepsilon \partial_r V$  can be negative. Therefore, we fix  $T > 0$ , consider  $\phi_\varepsilon$  as in Remark 3.2.6 and integrate (3.2.12) on  $(0, T)$  to deduce

$$\Phi_\varepsilon(n_\varepsilon(T, \cdot)) + \delta \int_0^T \int_0^{R_b} (r + \varepsilon) |\partial_{rr} n_\varepsilon|^2 + \frac{|\partial_r n_\varepsilon|^2}{r + \varepsilon} dr dt \leq \Phi_\varepsilon(n_0) + \int_0^T \int_0^{R_b} (r + \varepsilon) \partial_r n_\varepsilon \partial_r V dr dt.$$

The last term can be easily bounded (using estimates (A1)–(A2)) by a constant depending on  $T$ . The conclusion follows from  $\Phi_\varepsilon(n_\varepsilon(T, \cdot)) \geq 0$  and  $\Phi_\varepsilon(n_0)$  can be bounded in terms of  $\Phi(n_0)$ , cf. (4) in Remark 3.2.6.

*Step 3 : Estimate (A6).* Let  $\chi \in L^2(0, T; H^1(I_{R_b}))$ . We multiply (3.2.6) by  $\chi$  and integrate with respect to  $r$  between 0 and  $R_b$ . Using an integration by parts and Neumann boundary conditions, we obtain

$$\begin{aligned}
\int_0^\infty \int_0^{R_b} (r + \varepsilon) \partial_t n_\varepsilon \chi \, dr \, dt &= - \int_0^\infty \int_0^{R_b} (r + \varepsilon) B_\varepsilon(n_\varepsilon) \partial_r(\mu_\varepsilon + V) \partial_r \chi \, dr \, dt \\
&= - \int_0^\infty \int_0^{R_b} \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \partial_r(\mu_\varepsilon + V) \partial_r \chi \, dr \, dt \\
&\leq \| \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \|_\infty \| \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \partial_r(\mu_\varepsilon + V) \|_2 \| \partial_r \chi \|_2.
\end{aligned}$$

where the norms are taken over  $(0, \infty) \times I_{R_b}$ . The conclusion follows.

*Step 4 : Hölder estimate in space (3.2.15).* By differentiation

$$(r_2 + \varepsilon) n(t, r_2) - (r_1 + \varepsilon) n(t, r_1) = \int_{r_1}^{r_2} (r + \varepsilon) \partial_r n(t, r) \, dr + \int_{r_1}^{r_2} n(t, r) \, dr.$$

For the first term, we have

$$\int_{r_1}^{r_2} (r + \varepsilon) \partial_r n_\varepsilon(t, r) \, dr \leq \left( \int_{r_1}^{r_2} (r + \varepsilon) \, dr \right)^{1/2} \| \sqrt{r + \varepsilon} \partial_r n_\varepsilon(t, \cdot) \|_2 \leq C |r_1 - r_2|^{1/2}$$

due to (A1). For the second term, we compute, using (A2),

$$\begin{aligned}
\int_{r_1}^{r_2} n_\varepsilon(t, r) \, dr &= \int_{r_1}^{r_2} n_\varepsilon(t, r) \frac{\sqrt{r + \varepsilon}}{\sqrt{r + \varepsilon}} \, dr \leq \| \sqrt{r + \varepsilon} n_\varepsilon \|_\infty \int_{r_1}^{r_2} \frac{1}{\sqrt{r + \varepsilon}} \, dr \leq \\
&\leq C | \sqrt{r_2 + \varepsilon} - \sqrt{r_1 + \varepsilon} | \leq C |r_1 - r_2|^{1/2}.
\end{aligned}$$

*Step 5 : Hölder estimate in time (3.2.16).* The idea is to deduce the regularity in time from the regularity in space. We extend the function  $r \mapsto n_\varepsilon r$  for  $r < 0$  with a constant to preserve continuity. We consider  $\eta_\nu$  to be a usual one-dimensional mollifier in the spatial variable  $r$  where  $\nu$  will be chosen later in terms of  $|t_2 - t_1|$ . Mollifying (3.2.6) with  $\eta_\nu$  and integrating in time (from  $t_1$  to  $t_2$ ) we obtain

$$((r + \varepsilon) n_\varepsilon) * \eta_\nu(t_2, r) - ((r + \varepsilon) n_\varepsilon) * \eta_\nu(t_1, r) = \int_{t_1}^{t_2} \partial_r \eta_\nu * ((r + \varepsilon) B_\varepsilon(n_\varepsilon) \partial_r(\mu_\varepsilon + V)) \, dt. \quad (3.2.17)$$

First, we estimate (RHS). We notice that Young's convolutional inequality and Hölder's inequality are implying for fixed  $t \in [t_1, t_2]$

$$\begin{aligned}
\| \partial_r \eta_\nu * ((r + \varepsilon) B_\varepsilon(n_\varepsilon) \partial_r(\mu_\varepsilon + V)) \|_\infty &\leq \\
&\leq \| \partial_r \eta_\nu \|_2 \| \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \|_\infty \| \sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \partial_r(\mu_\varepsilon + V) \|_2.
\end{aligned}$$

By the definition of a mollifier,

$$\| \partial_r \eta_\nu \|_2 = \frac{1}{\nu^2} \left| \int_{\mathbb{R}} (\eta')^2 \left( \frac{r}{\nu} \right) \, dr \right|^{1/2} \leq \frac{C}{\nu^{3/2}}.$$

Therefore, applying (A2), (A3) and Hölder's inequality in time, we deduce

$$\int_{t_1}^{t_2} \|\partial_r \eta_\nu * ((r + \varepsilon)B_\varepsilon(n_\varepsilon) \partial_r(\mu_\varepsilon + V))\|_\infty dt \leq C \frac{|t_2 - t_1|^{1/2}}{\nu^{3/2}}. \quad (3.2.18)$$

To conclude the proof, we need to estimate (3.2.17) using (3.2.15) from Step 4, we get

$$\begin{aligned} & |((r + \varepsilon)n_\varepsilon) * \eta_\nu(t_1, r) - ((r + \varepsilon)n_\varepsilon)(t_1, r)| \leq \\ & \leq \int_{\mathbb{R}} |(r + y + \varepsilon)n_\varepsilon(t_1, r + y) - (r + \varepsilon)n_\varepsilon(t_1, r)| \eta_\nu(y) dy \\ & \leq C \int_{\mathbb{R}} |y|^{1/2} \eta_\nu(y) dy \leq C \nu^{1/2}, \end{aligned} \quad (3.2.19)$$

where we used that on the support of  $\eta_\nu$  we have  $|y| \leq \nu$ . Exactly the same estimate holds if we replace  $t_1$  with  $t_2$ . Combining (3.2.17), (3.2.18) and (3.2.19) we obtain

$$|(r + \varepsilon)n_\varepsilon(t_2, r) - (r + \varepsilon)n_\varepsilon(t_1, r)| \leq C \frac{|t_2 - t_1|^{1/2}}{\nu^{3/2}} + C \nu^{1/2}.$$

We choose  $\nu = |t_2 - t_1|^{1/4}$  and this concludes the proof.  $\square$

**Remark 3.2.8.** In the above proof, Step 1 shows more generally that when  $n(t, r) : [0, \infty) \times [0, R_b] \rightarrow \mathbb{R}$  satisfies  $\int_0^{R_b} r n(t, r) dr = m$  and  $\sqrt{r} \partial_r n \in L^\infty(0, T; L^2(0, R_b))$  then

$$|\sqrt{r} n(t, r)| \leq C (R_b, m, \|\sqrt{r} \partial_r n\|_{L^\infty(0, T; L^2(0, R_b))}).$$

### 3.2.2 Proof of Theorem 3.1.1 (existence part)

We are concerned with the first part of Theorem 3.1.1 *i.e.* the convergence  $\varepsilon \rightarrow 0$  of the approximation scheme.

*Proof of Theorem 3.1.1 (existence).* The proof is divided into several steps.

*Step 1 : Compactness.* By the estimates in Proposition 3.2.7, the Banach-Alaoglu and Arzela-Ascoli theorems, we can extract a subsequence such that, for some  $\xi \in L^2((0, T) \times I_{R_b})$ ,

- (C1)  $(r + \varepsilon) n_\varepsilon \rightarrow r n$  uniformly in  $C([0, T] \times I_{R_b})$ ,
- (C2)  $(r + \varepsilon) \partial_t n_\varepsilon \rightharpoonup r \partial_t n$  in  $L^2(0, T; H^{-1}(I_{R_b}))$ ,
- (C3)  $\sqrt{(r + \varepsilon) B_\varepsilon(n_\varepsilon)} \partial_r(\mu_\varepsilon + V) \rightharpoonup \xi$  in  $L^2((0, T) \times I_{R_b})$ ,
- (C4)  $\sqrt{r + \varepsilon} \partial_{rr} n_\varepsilon \rightharpoonup \sqrt{r} \partial_{rr} n$  and  $\frac{\partial_r n_\varepsilon}{\sqrt{r + \varepsilon}} \rightharpoonup \frac{\partial_r n}{\sqrt{r}}$  in  $L^2((0, T) \times I_{R_b})$ .

*Step 2 : Nonnegativity of  $n$ .* The plan is to obtain a contradiction with the uniform estimate of the entropy. For  $\alpha > 0$ , we define the sets

$$\begin{aligned} V_{\alpha, \varepsilon} &= \{(t, r) \in (0, T) \times I_{R_b} : n_\varepsilon(t, r) \leq -\alpha, r \geq \alpha\}, \\ V_{\alpha, 0} &= \{(t, r) \in (0, T) \times I_{R_b} : n(t, r) \leq -\alpha, r \geq \alpha\}. \end{aligned}$$

By Remark 3.2.6 (nonnegativity of  $\phi_\varepsilon$ ) and (A5) in Lemma 3.2.7 there is a constant such that

$$\int_{V_{\alpha, \varepsilon}} (r + \varepsilon) \phi_\varepsilon(n_\varepsilon) dr dt \leq \int_{(0, T) \times I_{R_b}} (r + \varepsilon) \phi_\varepsilon(n_\varepsilon) dr dt \leq C(T).$$

For  $n_\varepsilon \leq -\alpha$ , we have  $0 \leq \phi_\varepsilon(-\alpha) \leq \phi_\varepsilon(n_\varepsilon)$  ((3) in Remark 3.2.6) so that

$$(-\alpha(\log(\varepsilon) - 1) + 1 + \alpha^2/(2\varepsilon) - \varepsilon/2) \int_{V_{\alpha,\varepsilon}} (r + \varepsilon) dr dt \leq C(T).$$

Sending  $\varepsilon \rightarrow 0$  and using uniform convergence of  $n_\varepsilon \rightarrow n$  for  $r \geq \alpha > 0$  we discover that

$$\int_{V_{\alpha,0}} r dr dt = \lim_{\varepsilon \rightarrow 0} \int_{V_{\alpha,\varepsilon}} (r + \varepsilon) dr dt = 0$$

using, from measure theory, that on a measure space  $(X, \mu)$  if  $f_n, f : X \rightarrow \mathbb{R}$  and  $f_n \rightarrow f$  in  $L^1(X, \mu)$  then for  $\alpha \in \mathbb{R}$  we have  $\int_{f_n < \alpha} d\mu \rightarrow \int_{f < \alpha} d\mu$  as  $n \rightarrow \infty$ . This means that  $V_{\alpha,0}$  is a null set for each  $\alpha > 0$ , concluding the proof.

*Step 3 : Identification of the limit  $(r + \varepsilon)B_\varepsilon(n_\varepsilon)\partial_r(\mu_\varepsilon + V)$ .* The last difficulty is to pass to the limit in  $\int_0^T \int_0^{R_b} (r + \varepsilon)B_\varepsilon(n_\varepsilon)\partial_r(\mu_\varepsilon + V)\partial_r\varphi dr dt$ . Indeed, since the mobility is degenerate it is not clear that we can identify the derivative of the potential  $\partial_r\mu$  in the limit. However, due to the uniform convergence of  $(r + \varepsilon)n_\varepsilon$  and the nonnegativity of  $n$  we can conclude. By (C3) and the uniform convergence of  $\sqrt{(r + \varepsilon)B_\varepsilon(n_\varepsilon)}$ , we have

$$(r + \varepsilon)B_\varepsilon(n_\varepsilon)\partial_r(\mu_\varepsilon + V) \rightharpoonup \sqrt{rn}\xi = \begin{cases} \sqrt{rn}\xi & \text{when } rn > 0 \\ 0 & \text{when } rn = 0 \end{cases} \text{ in } L^2((0, T) \times I_{R_b}). \quad (3.2.20)$$

We first claim that

$$\xi(t, r) = \sqrt{rn}\partial_r(\mu + V) \text{ when } rn > 0. \quad (3.2.21)$$

We introduce the family of open sets

$$\{(t, r) : rn(t, r) > 0\} = \cup_{\nu > 0} P_\nu, \quad P_\nu = \{(t, r) : rn(r, t) > \nu, r > \nu\},$$

so that it is sufficient to identify the limit  $\xi$  in  $P_\nu$  for fixed  $\nu > 0$ .

Because of the uniform convergence we know that for every  $\varepsilon < \varepsilon(\nu)$  for  $\varepsilon(\nu)$  small enough,

$$(r + \varepsilon)B_\varepsilon(n_\varepsilon(r, t)) \geq \frac{\nu}{2}, \quad (r, t) \in P_\nu.$$

Therefore, the estimate (A3) implies

$$\|\partial_r(\mu_\varepsilon + V)\|_{L^2(P_\nu)} \leq \frac{C}{\nu^{1/2}}.$$

As  $\partial_r V$  is uniformly bounded, we deduce that

$$\|\partial_r\mu_\varepsilon\|_{L^2(P_\nu)} \leq \frac{C}{\nu^{1/2}}.$$

By definition of  $\mu_\varepsilon$

$$\partial_r\mu_\varepsilon = \gamma n_\varepsilon^{\gamma-1} \partial_r n_\varepsilon - \delta \partial_r \left( \frac{1}{r + \varepsilon} \partial_r((r + \varepsilon)\partial_r n_\varepsilon) \right).$$

For the first term of (RHS), we use the strong convergence (C1) that yields a uniform convergence

of  $n_\varepsilon$  in the zone  $\{(r, t) : r > \nu\}$ . Then, because  $P_\nu \subset \{(r, t) : r > \nu\}$  we obtain

$$\gamma n_\varepsilon^{\gamma-1} \rightarrow \gamma n^{\gamma-1} \quad \text{uniformly in } L^\infty(P_\nu).$$

Combined with the weak convergence provided by estimate (A1) in  $P_\nu$  we obtain that up to a subsequence,

$$\gamma n_\varepsilon^{\gamma-1} \partial_r n_\varepsilon \rightharpoonup \gamma n^{\gamma-1} \partial_r n \quad \text{weakly in } L^2(P_\nu).$$

Then we combine the  $L^2(P_\nu)$  bound on  $\gamma n_\varepsilon^{\gamma-1} \partial_r n_\varepsilon$  with the  $L^2(P_\nu)$  estimate of  $\partial_r \mu_\varepsilon$ . We obtain an  $L^2(P_\nu)$  bound on the second term on the right-hand side. Together with estimates (A1)-(A2)-(A3)-(A4) we obtain the weak convergence up to a subsequence

$$\partial_r \left( \frac{1}{r+\varepsilon} \partial_r ((r+\varepsilon) \partial_r n_\varepsilon) \right) \rightharpoonup \partial_r \left( \frac{1}{r} \partial_r (r \partial_r n) \right) \quad \text{weakly in } L^2(P_\nu).$$

Finally, we obtain

$$\partial_r \mu_\varepsilon \rightharpoonup \partial_r \mu = \gamma n^{\gamma-1} \partial_r n - \delta \partial_r \left( \frac{1}{r} \partial_r (r \partial_r n) \right) \quad \text{weakly in } L^2(P_\nu). \quad (3.2.22)$$

Using uniform convergence, we conclude the proof of (3.2.21). Finally, (3.2.20) and (3.2.21) implies

$$\int_0^T \int_0^{R_b} (r+\varepsilon) B_\varepsilon(n_\varepsilon) \partial_r (\mu_\varepsilon + V) \partial_r \varphi \, dr \, dt \rightarrow \int_{rn>0} r n \partial_r (\mu + V) \partial_r \varphi \, dr \, dt.$$

*Step 4 : existence of a weak solution.* Steps 1-3 show that  $n$  satisfies the condition of Definition 3.2.1.

*Step 5 : Properties (3.2.1)–(3.2.5) from Remark 3.2.2.* First, properties (3.2.1), (3.2.4) and (3.2.5) follow from uniform convergence (C1) and estimates (3.2.15)–(3.2.16) for  $r n_\varepsilon$ . To see (3.2.2), we notice that weak lower semicontinuity of  $L^2$  norm implies

$$\mathcal{E}[n(\tau, \cdot)] + \int_0^\tau \int_0^{R_b} |\xi(t, r)|^2 \, dr \, dt \leq \mathcal{E}[n_0].$$

By (3.2.21), the integral on the (LHS) can be estimated from below by

$$\mathcal{E}[n(\tau, \cdot)] + \int_0^\tau \int_0^{R_b} \mathbf{1}_{rn>0} r n |\partial_r (\mu + V)|^2 \, dr \, dt \leq \mathcal{E}[n_0].$$

To see (3.2.3), it is sufficient to prove

$$\Phi(n(t, \cdot)) \leq \liminf_{\varepsilon \rightarrow 0} \Phi_\varepsilon(n_\varepsilon(t, \cdot)).$$

Let  $\delta > 0$ . By nonnegativity of  $\phi_\varepsilon$  we estimate

$$\liminf_{\varepsilon \rightarrow 0} \Phi_\varepsilon(n_\varepsilon(t, \cdot)) \geq \liminf_{\varepsilon \rightarrow 0} \int_{r \geq \delta} \phi_\varepsilon(n_\varepsilon)(r + \varepsilon) \, dr = \int_{r \geq \delta} \phi(n) r \, dr,$$

because on the set  $\{r \geq \delta\}$ , we have uniform convergence  $n_\varepsilon \rightarrow n$ . As  $\phi(n) \geq 0$ , we can send



$\delta \rightarrow 0$  by monotone convergence and conclude the proof.

*Step 6 : Neumann boundary condition  $n'(t, R_b) = 0$ .* First, if  $\varphi, \phi \in C^1[a, b] \cap H^2(a, b)$  we have (via approximation)

$$\int_a^b \varphi'(r) \phi'(r) + \varphi''(r) \phi(r) dr = \varphi'(b) \phi(b) - \varphi'(a) \phi(a). \quad (3.2.23)$$

Let  $\phi$  be a smooth function with  $\phi(R_0) = 0$  and  $\phi(R_b) = 1$  for some  $R_0 \in (0, R_b)$ . We know from estimates (C1)-(C4) that  $n \in L^2(0, T; H^2(R_0, R_b))$ . Let  $t$  be such that  $r \mapsto n(t, r) \in H^2(R_0, R_b)$ . Applying (3.2.23) with  $\varphi(r) = n_\varepsilon(t, r)$ , we deduce, thanks to the Neumann boundary condition  $n'_\varepsilon(t, R_b) = 0$ , that

$$\int_{R_0}^{R_b} n'_\varepsilon(t, r) \phi'(r) + n''_\varepsilon(t, r) \phi(r) dr = 0.$$

Multiplying by a smooth test function  $\eta(t)$ , we have

$$\int_0^T \int_{R_0}^{R_b} \eta(t) (n'_\varepsilon(t, r) \phi'(r) + n''_\varepsilon(t, r) \phi(r)) dr dt = 0.$$

Passing to the weak limit  $\varepsilon \rightarrow 0$  and using that  $\eta$  is arbitrary we conclude

$$\int_{R_0}^{R_b} n'(t, r) \phi'(r) + n''(t, r) \phi(r) dr = 0, \quad \text{for a.e. } t > 0.$$

As  $n(t, \cdot) \in H^2(R_0, R_b)$ , we can apply (3.2.23) again and deduce

$$\int_{R_0}^{R_b} n'(t, r) \phi'(r) + n''(t, r) \phi(r) dr = n'(t, R_b),$$

which finally proves  $n'(t, R_b) = 0$ . □

### 3.2.3 Proof of Theorem 3.1.1 (Long term asymptotics)

With global solutions at hand, we can study the long term behaviour. For that purpose, we fix  $k, T$ ,  $k \geq T$  and define  $n_k(t, x) = n(t + k, x)$ ,  $\mu_k(t, x) = \mu(t + k, x)$ . Consider the solution  $n$  in the interval  $(-T + k, T + k)$ , it satisfies

$$\int_{-T+k}^{T+k} r \langle \partial_t n, \varphi \rangle_{H^{-1}, H^1} dt + \int_{-T+k}^{T+k} \int_0^{R_b} \mathbf{1}_{rn>0} r n \partial_r (\mu + V) \partial_r \varphi dr dt = 0,$$

and a change of variables yields

$$\int_{-T}^T r \langle \partial_t n_k, \varphi \rangle_{H^{-1}, H^1} dt + \int_{-T}^T \int_0^{R_b} \mathbf{1}_{rn_k>0} r n_k \partial_r (\mu_k + V) \partial_r \varphi dr dt = 0. \quad (3.2.24)$$

We also recall the Neumann boundary condition  $n'_k(t, R_b) = 0$  and the conservation of mass  $\int_0^{R_b} r n_k dr = \int_0^{R_b} r n_0 dr$ . We want to pass to the limit  $k \rightarrow \infty$  in this equation and prove the

**Proposition 3.2.9.** *Let  $(n, \mu)$  be a weak solution of (3.1.1)-(3.1.2). Then, we can extract a subsequence, still denoted by the index  $k$ , of  $(n_k, \mu_k)$  such that  $\sqrt{r} n_k \rightarrow \sqrt{r} n_\infty$  strongly in*

$L^\infty((-T, T) \times I_{R_b})$  and  $\sqrt{rn_k} \partial_r(\mu_k + V) \rightharpoonup \sqrt{rn} \partial_r(\mu_\infty + V)$  weakly in  $L^2((-T, T) \times I_{R_b} \setminus \{rn = 0\})$ . We have  $n_\infty \in C^1(\mathbb{R} \times \bar{B}_{R_b})$  and the relations

$$rn_\infty \partial_r(\mu_\infty + V) = 0, \quad \mu_\infty = n_\infty^\gamma - \frac{\delta}{r} \partial_r(r \partial_r n_\infty), \quad (3.2.25)$$

with the Neumann boundary conditions

$$\frac{\partial n_\infty}{\partial r} \Big|_{r=0} = \frac{\partial n_\infty}{\partial r} \Big|_{r=R_b} = 0.$$

The mass  $\int_0^{R_b} rn_\infty(t) dr$  is constant and equal to the initial mass  $\int_0^{R_b} rn_0 dr$ .

This proposition implies the assertions of Theorem 3.1.1.

*Démonstration. Step 1 : Bounds coming from the energy.* We claim that the following uniform estimates (with respect to  $k$ ) are true :

- (B1)  $\{\sqrt{r} \partial_r n_k\}$  in  $L^\infty((-T, T); L^2(I_{R_b}))$ ,
- (B2)  $\{\sqrt{r} n_k\}$  in  $L^\infty((-T, T) \times I_{R_b})$ ,
- (B3)  $\{r \partial_t n_k\}$  in  $L^2((-T, T); H^{-1}(I_{R_b}))$ ,
- (B4)  $|r_2 n_k(t_2, r_2) - r_1 n_k(t_1, r_1)| \leq C(|t_2 - t_1|^{1/8} + |r_2 - r_1|^{1/2})$ ,
- (B5)  $L_k(T) := \int_{-T}^T \int_0^{R_b} \mathbb{1}_{rn_k > 0} rn_k |\partial_r(\mu_k + V)|^2 \xrightarrow{k \rightarrow +\infty} 0$ .

The energy decay estimate (3.2.2) and assumption  $\mathcal{E}[n_0] < \infty$  imply that  $\mathcal{E}[n_k(t)]$  remains bounded with respect to  $k$  for all  $k > T$ . Therefore, (B1) follows directly from (3.2.2) and then (B2) follows from Remark 3.2.8. As  $rn_k(t, r)$  is obtained as the pointwise limit of  $(r + \varepsilon) n_\varepsilon(t + k, r)$ , estimates (B3) and (B4) follow directly from passing to the limit  $\varepsilon \rightarrow 0$  in (A6) and from (3.2.15)–(3.2.16) in Proposition 3.2.7. Finally, to see (B5), we note that

$$\int_0^\infty \int_0^{R_b} \mathbb{1}_{rn > 0} rn |\partial_r(\mu + V)|^2 dr dt \leq \mathcal{E}(n_0),$$

so by change of variables we obtain

$$L_k(T) \leq \int_{k-T}^\infty \int_0^{R_b} \mathbb{1}_{rn > 0} rn |\partial_r(\mu + V)|^2 dr dt \xrightarrow{k \rightarrow +\infty} 0.$$

*Step 2 : Bounds coming from the entropy.* We prove now uniform estimates

- (C1)  $\{\sqrt{r} \partial_{rr} n_k\}$  in  $L^2((-T, T); L^2(I_{R_b}))$ ,
- (C2)  $\{\frac{\partial_r n_k}{\sqrt{r}}\}$  in  $L^2((-T, T); L^2(I_{R_b}))$ .

To this end, we integrate the entropy relation (3.2.3) between  $k - T$  and  $k + T$  and perform a change of variables to obtain

$$\begin{aligned} \int_{-T}^T \int_0^{R_b} \left( \gamma rn_k^{\gamma-1} |\partial_r n_k|^2 + \delta r |\partial_{rr} n_k|^2 + \delta \frac{|\partial_r n_k|^2}{r} \right) dr dt &\leq \\ &\leq \Phi[n_k(-T, \cdot)] - \Phi[n_k(T, \cdot)] + \int_{-T}^T \int_0^{R_b} r \partial_r n_k \partial_r V dr dt. \end{aligned}$$

We need to bound the right-hand side. Concerning the entropy term, we recall the inequality  $\log n \leq n - 1$  valid for  $n > 0$  so that, by bound (B2),

$$\begin{aligned} \Phi(n_k(T, \cdot)) &= \int_0^{R_b} r(n_k(T, r)(\log n_k(T, r) - 1) + 1) dr \leq \\ &\leq \int_0^{R_b} r((n_k(T, r))^2 + n_k(T, r)) dr \leq C \|\sqrt{r} n_k\|_\infty \leq C. \end{aligned}$$

The same estimate is satisfied by  $\Phi(n_k(T))$ . Concerning  $\int_{-T}^T \int_0^{R_b} r \partial_r n_k \partial_r V dr dt$ , we estimate it using (B1) and uniform bound on  $\partial_r V$ . Therefore,

$$\int_{-T}^T \int_0^{R_b} \left( \gamma r n_k^{\gamma-1} |\partial_r n_k|^2 + \delta r |\partial_{rr} n_k|^2 + \delta \frac{|\partial_r n_k|^2}{r} \right) dr dt \leq C(T, \mathcal{E}(n_0)).$$

*Step 3 : Convergence in equation (3.2.24).* Reasoning as in the proof of Theorem 3.1.1 we obtain in the limit  $k \rightarrow \infty$

$$\int_{-T}^T r \langle \partial_t n_\infty, \varphi \rangle_{H^{-1}, H^1} dt + \int_{-T}^T \int_0^{R_b} \mathbf{1}_{rn_\infty > 0} r n_\infty \partial_r (\mu_\infty + V) \partial_r \varphi dr dt = 0.$$

We can show even better, namely that  $\partial_t n_\infty = 0$ . Indeed, from the Cauchy-Schwarz inequality we obtain that for every test function  $\chi$  compactly supported in  $(-T, T) \times (0, R_b)$ ,

$$\begin{aligned} \left| \int_{-T}^T \int_0^{R_b} r \partial_t n_k \chi \right| &= \left| \int_{-T}^T \int_0^{R_b} \mathbf{1}_{rn_k > 0} r n_k \partial_r (\mu_k + V) \partial_r \chi \right| \\ &\leq C(T, R_b) \|\partial_r \chi\|_{L^\infty} \int_{-T}^T \int_0^{R_b} \mathbf{1}_{rn_k > 0} r n_k |\partial_r (\mu_k + V)|^2 \xrightarrow{k \rightarrow \infty} 0. \end{aligned}$$

where we used (B2) and (B5). This means that in the limit,  $n_\infty$  does not depend on the time variable  $t$ . Then, in the limit, we obtain that, for every test function  $\chi$ ,

$$\int_{-T}^T \int_0^{R_b} \mathbf{1}_{rn_\infty > 0} r n_\infty \partial_r (\mu_\infty + V) \partial_r \chi = 0.$$

*Step 4 :  $n'_\infty$  is uniformly continuous and  $n_\infty$  satisfies Neumann boundary condition  $n'_\infty(0) = 0$ .* We recall that  $n_\infty$  does not depend on time. Moreover, the estimate (C1) implies that  $n'_\infty$  is continuous on  $(0, R_b]$ . Furthermore, from the estimates (C1)-(C2), we obtain the absolute continuity in space of the derivative of  $n_\infty$ . Indeed, for every  $r_1, r_2 \in (0, R_b)$  we obtain

$$\begin{aligned} (\partial_r n_\infty(r_2))^2 - (\partial_r n_\infty(r_1))^2 &= 2 \int_{r_1}^{r_2} \partial_r n_\infty(r) \partial_{rr} n_\infty(r) dr \\ &= 2 \int_{r_1}^{r_2} \frac{\partial_r n_\infty(r)}{\sqrt{r}} \sqrt{r} \partial_{rr} n_\infty(r) dr \\ &\leq 2 \left( \int_{r_1}^{r_2} \frac{|\partial_r n_\infty(r)|^2}{r} dr \right)^{1/2} \left( \int_{r_1}^{r_2} r |\partial_{rr} n_\infty(r)|^2 dr \right)^{1/2}. \end{aligned}$$

From this, we deduce that  $\partial_r n_\infty$  is bounded so that by the Sobolev embedding,  $n_\infty$  is continuous

and

$$n_\infty(r_2) - n_\infty(r_1) = \int_{r_1}^{r_2} \partial_r n_\infty(r) dr.$$

Next, we discover that  $(0, R_b] \ni r \mapsto (\partial_r n_\infty(r))^2$  is uniformly continuous, so that by Lemma 3.2.10 below,  $n'_\infty(r)$  is uniformly continuous on  $(0, R_b]$ . Therefore, there is the unique extension of  $r \mapsto n'_\infty(r)$  to  $[0, R_b]$  which is uniformly continuous. Furthermore, in view of

$$\int_0^{R_b} \frac{|\partial_r n_\infty|^2}{r} dr \leq C,$$

this extension has to be obtained by setting  $n'_\infty(0) = 0$ .

It remains to prove that  $n_\infty$  is differentiable (in the classical sense) at  $r = 0$  and  $n'_\infty(0) = 0$ . To this end, we write

$$\left| \frac{n_\infty(r) - n_\infty(0)}{r} \right| \leq \frac{1}{r} \int_0^r |\partial_r n_\infty(u)| du \leq \sup_{u \in (0, r]} |\partial_r n_\infty(u)| \rightarrow 0$$

as  $r \rightarrow 0$  by uniform continuity which, again, implies that  $n'_\infty(0)$  exists and  $n'_\infty(0) = 0$ .

*Step 5 : Neumann boundary condition  $n'_\infty(R_b) = 0$ .* The proof is similar to Step 6 in Section 3.2.2. For a fixed  $k \in \mathbb{N}$ , there is a set of times  $\mathcal{N}_k \subset (0, T)$  of full measure such that, when  $t \in \mathcal{N}_k$ , we have  $n'_k(t, R_b) = 0$  and  $n_k(t, \cdot) \in H^2(R_0, R_b)$ . Let  $\mathcal{N} = \bigcap_{k \in \mathbb{N}} \mathcal{N}_k$ , which is again the set of full measure. For  $t \in \mathcal{N}$  and  $\phi$  as in Step 6 in Section 3.2.2, we have

$$\int_{R_0}^{R_b} n'_k(t, r) \phi'(r) + n''_k(t, r) \phi(r) dr = 0.$$

We multiply by a smooth test function  $\eta(t)$  and pass to the weak limit  $k \rightarrow \infty$  to deduce

$$\int_0^T \eta(t) dt \int_{R_0}^{R_b} (n'_\infty(r) \phi'(r) + n''_\infty(r) \phi(r)) dr = 0.$$

As  $n_\infty \in H^2(R_0, R_b)$  we deduce  $n'_\infty(R_b) = 0$ . □

**Lemma 3.2.10.** *Let  $f : (a, b) \rightarrow \mathbb{R}$  be a continuous function such that  $f^2$  is uniformly continuous. Then  $|f|$  and  $f$  are also uniformly continuous.*

*Démonstration.* First, we observe that  $|f|$  is uniformly continuous as a composition of a  $\frac{1}{2}$ -Hölder continuous function and a uniformly continuous one. Therefore,

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in (a, b) \quad |x - y| \leq \delta \implies ||f(x)| - |f(y)|| \leq \varepsilon. \quad (3.2.26)$$

Fix  $\varepsilon > 0$  and choose  $\delta > 0$  such that (3.2.26) holds with  $\varepsilon/2$ . Let  $x, y \in (a, b)$  be such that  $|x - y| \leq \delta$ . If  $f(x), f(y)$  have the same sign we are done. Otherwise, by continuity, there exists  $z$  between  $x$  and  $y$  such that  $f(z) = 0$ . As  $|x - z|, |y - z| \leq \delta$ , we can apply (3.2.26) again to deduce

$$|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \square$$

### 3.3 Properties of the stationary states

The stationary solution built previously has compact support for  $R_b$  large enough. This is the main content of Theorem 3.1.2 which we prove here. We still use, to simplify notations, the potential  $V(r) = r^2$ . We postpone to Appendix 3.6.1 the case of a more general potential  $V(r)$ .

#### 3.3.1 Proof of Theorem 3.1.2 (A)

We recall that, from Theorem 3.1.6,  $n_\infty \geq 0$  is  $C^1$ ,  $n'_\infty(R_b) = 0$ ,  $n'_\infty(0) = 0$ .

*Proof of Theorem 3.1.2 (A).* To prove that  $n_\infty$  is non-increasing, the main idea is to show that it cannot have a local maximum except at the point  $r = 0$ .

To do so, by contradiction, we assume there is local maximum at  $R_2 \in (0, R_b]$ . This implies that  $n'_\infty(R_2) = 0$ ,  $n''_\infty(R_2) \leq 0$ . Also by  $C^1$  regularity, in a neighborhood of  $R_2$  the equation hold

$$n_\infty^\gamma(r) - \frac{\delta}{r}n'_\infty(r) - \delta n''_\infty(r) = C - r^2,$$

for some constant  $C$ . This equation implies that the local maximum is strict.

Also, still by  $C^1$  regularity, in this neighborhood of  $R_2$  there is a point  $0 < R_1 < R_2$  such that  $0 < n_\infty(R_1) < n_\infty(R_2)$  and  $n'_\infty(R_1) > 0$ . Evaluating the equation at the points  $R_1$  and  $R_2$ , and eliminating the constant  $C$ , we obtain

$$\delta n''_\infty(R_1) = R_1^2 - R_2^2 + n_\infty^\gamma(R_1) - n_\infty^\gamma(R_2) - \frac{\delta}{R_1}n'_\infty(R_1) + \delta n''_\infty(R_2) < 0.$$

Therefore  $n_\infty$  is strictly concave at  $R_1$ . Consequently,  $R_1$  can be continued to smaller values,  $n_\infty(R_1)$  staying concave increasing (and thus  $n'_\infty$  larger and larger as  $R_1$  decreases) until either  $R_1 = 0$  or  $n_\infty(R_1) = 0$ . In both cases we get a contradiction with the condition  $n'_\infty(R_1) = 0$  which holds at 0 and at values where  $n_\infty(R_1) = 0$ .

Consequently, the only possible local maximum is at 0 and  $n_\infty$  is non-increasing.

The upper bound on  $n_\infty(R_b)$  is just to say that  $n_\infty(r) \geq n_\infty(R_b)$  on the full interval. □

#### 3.3.2 Proof of Theorem 3.1.2 (B)

We now consider a stationary state such that  $n_\infty(R_b) = 0$ . Theorem 3.1.2 (A) asserts that there is  $R \in [0, R_b]$  such that  $n_\infty(r) = 0$  on  $[R, R_b]$  and  $n_\infty$  is positive on  $[0, R]$ . Hence, on  $[0, R]$ , the relation (3.2.25) shows that there exists a constant, that we write  $R^2 - \lambda$ , such that  $n_\infty$  solves

$$\begin{cases} n^\gamma(r) - \frac{\delta}{r}n'(r) - \delta n''(r) = R^2 - r^2 - \lambda, & 0 \leq r \leq R, \\ n(R) = 0. \end{cases} \quad (3.3.1)$$

Because it is  $C^1$ , the stationary solution also satisfies  $n'_\infty(R) = 0$  (and this is also true for  $R = R_b$  as stated in Theorem 3.1.1). We prove that there exists only one value  $\lambda$  such that the solution of Equation (3.3.1) also satisfies the condition  $n'(R) = 0$ .

Firstly, we exclude some values of  $\lambda$ . Here, we use the notation  $n^\gamma$  for  $\max(0, n)^\gamma$ .

**Lemma 3.3.1.** *Being given  $\lambda \in \mathbb{R}$ , let  $n$  be the solution of Equation (3.3.1) Then, we have*

- when  $\lambda \geq R^2$ ,  $n(r) \leq 0 \quad \forall r \in [0, R]$  and  $n'(R) > 0$ ,
- when  $\lambda \leq 0$ ,  $n(r) \geq 0 \quad \forall r \in [0, R]$  and  $n'(R) < 0$ .

*Démonstration.* For  $\lambda \geq R^2$ ,  $n \leq 0$  is a consequence of the maximum principle and it follows immediately that  $n'(R) \geq 0$ . If we had  $n'(R) = 0$ , the equation gives  $n''(R) = \frac{\lambda}{\delta} > 0$  which is in contradiction with the fact that  $n$  is nonpositive in a small left neighborhood of  $R$ .

For  $\lambda \leq 0$ ,  $n \geq 0$  is a consequence of the maximum principle and it follows that  $n'(R) \leq 0$ . To exclude the possibility that  $n'(R) = 0$ , we suppose by contradiction that  $n'(R) = 0$ . Since we also have  $n(R) = 0$ , we find that  $n''(R) = \frac{\lambda}{\delta}$ . As before, for  $\lambda < 0$ , we find contradiction. For  $\lambda = 0$ , we have  $n''(R) = 0$ . Differentiating the equation, we find

$$\gamma n^{\gamma-1}(r) n'(r) - \delta n^{(3)}(r) - \delta \frac{n''(r)}{r} + \delta \frac{n'(r)}{r^2} = -2r,$$

and thus  $n^{(3)}(R) = 2R/\delta > 0$ . As  $n(R) = n'(R) = n''(R) = 0$ , it follows that in a small neighbourhood of  $R$ ,  $n$  has to be negative raising a contradiction. The lemma is proved.  $\square$

Secondly, from Lemma 3.3.1, we may conclude that there is at least one value  $\lambda \in (0, R^2)$  such that the Neumann condition is satisfied. This value is unique

**Lemma 3.3.2.** *There exists only one  $\lambda \in (0, R^2)$  such that the solution of (3.3.1) satisfies  $n'(R) = 0$ .*

*Démonstration.* Suppose there are two solutions  $n_1, n_2$  of (3.3.1) with  $0 < \lambda_1 < \lambda_2 < R^2$  such that  $n_i(R) = n'_i(R) = 0$  for  $i = 1, 2$ . From (3.3.1), we find  $n''_i(R) = \frac{\lambda_i}{\delta}$ . Therefore  $0 < n''_1(R) < n''_2(R)$  and we conclude by a Taylor expansion that  $n'_2$  is smaller than  $n'_1$  in a small left neighborhood of  $R$  which contradicts that  $n$  decreases with  $\lambda$ . This proves Lemma 3.3.2.  $\square$

*Proof of Theorem 3.1.2 (B).* Clearly,  $n_\infty$  is a solution to the problem (3.3.1) with some  $\lambda$ . By Lemma 3.3.1, we know that  $\lambda \in (0, R^2)$  and then Lemma 3.3.2 yields the unique value of  $\lambda$ .

For the second assertion, if there are two solutions  $(n_1, \lambda_1), (n_2, \lambda_2)$  of (3.1.7), Lemma 3.3.2 applies and we obtain that  $\lambda_1 = \lambda_2$ . The conclusion follows from uniqueness of solutions of the elliptic PDE (3.3.1).  $\square$

### 3.3.3 Proof of Theorem 3.1.2 (C)

Consider a solution  $n_\infty$  of (3.1.6) with  $a := n_\infty(R_b) > 0$ . From Theorem 3.1.2 (A), we know that  $n_\infty$  is  $C^1$  and  $n'_\infty \leq 0$  so that  $n_\infty > 0$ . Therefore, from the equation for  $n_\infty$ ,  $rn_\infty(r)$  is  $C^2$ , and (3.1.6) boils down to

$$\begin{cases} n_\infty^\gamma - \frac{\delta}{r} n'_\infty - \delta n''_\infty = R_b^2 - r^2 - \lambda & \text{in } (0, R_b), \\ n_\infty(R_b) = a > 0, n'_\infty(R_b) = 0, \\ m = \int_0^{R_b} r n_\infty(r) dr, \end{cases} \quad (3.3.2)$$

where  $\lambda$  is some constant. Our goal is to prove that if  $R_b$  is sufficiently large with respect to  $m$ , there is no such solution of (3.3.2). Therefore we now assume that  $R_b > 2$ .

A useful formula in the sequel is, because of radial symmetry and after integration between 0 and  $r$ ,

$$\int_0^r \bar{r} n_\infty^\gamma(\bar{r}) d\bar{r} - \delta r \partial_r n_\infty = (R_b^2 - \lambda) \frac{r^2}{2} - \frac{r^4}{4}. \quad (3.3.3)$$

Another useful general observation is that we may assume

$$n_\infty(R_b)^\gamma \leq R_b^2.$$

Otherwise, by Theorem 3.1.2 (A), we have  $m \geq \frac{R_b^{2+2/\gamma}}{2}$  which proves the result.

Firstly, we provide lower and upper bounds on admissible values of the constant  $\lambda$

$$-R_b^2 \leq -n_\infty(R_b)^\gamma \leq \lambda \leq \frac{R_b^2}{2}. \quad (3.3.4)$$

The first inequality is the above restriction on  $n_\infty(R_b)^\gamma$ . The second inequality is valid because  $n_\infty''(R_b) \geq 0$  since  $n_\infty$  is decreasing and  $n_\infty'(R_b) = 0$ . The third inequality is just (3.3.3) at  $r = R_b$ .

Secondly, we provide a control of  $n_\infty(0)$ . To do so, using (3.3.3),  $\partial_r n_\infty \leq 0$  and the above upper bound on  $\lambda$ , we estimate  $|\partial_r n_\infty|$  from above as

$$\delta |\partial_r n_\infty| \leq R_b^2 r.$$

This gives

$$n_\infty(r) \geq n_\infty(0) - \frac{R_b^2}{2\delta} r^2$$

and, with  $\alpha > 0$  such that  $\alpha^2 = \frac{\delta}{2R_b^2} \leq 1$ ,

$$m \geq \int_0^{\alpha R_b} r n_\infty(r) dr \geq \frac{\alpha^2 R_b^2}{2} \left( n_\infty(0) - \frac{R_b^2}{2\delta} \frac{\alpha^2 R_b^2}{2} \right) = \frac{\alpha^2 R_b^2}{2} \left( n_\infty(0) - \frac{R_b^2}{8} \right).$$

As a conclusion of this step, we may assume

$$n_\infty(0) \leq \frac{R_b^2}{4},$$

otherwise  $m \geq \frac{\alpha^2 R_b^2}{2} \frac{R_b^2}{8} = \delta \frac{R_b^2}{32}$  and the result is proved again.

Thirdly, we prove that with this control from above of  $n_\infty(0)$ , the derivative  $|\partial_r n_\infty|$  is large, thus again there is a control on the mass since  $n_\infty$  is decreasing. To do so, we use again (3.3.3) and the third inequality in (3.3.4). This gives

$$\delta r |\partial_r n_\infty| \geq -n_\infty(0)^\gamma \frac{r^2}{2} + \frac{R_b^2}{2} \frac{r^2}{2} - \frac{r^4}{4} \geq \frac{r^2}{2} \left( -\frac{R_b^2}{4} + \frac{R_b^2}{2} - \frac{r^2}{2} \right) = \frac{r^2}{4} \left( \frac{R_b^2}{2} - r^2 \right)$$

where we have used the smallness assumption on  $n_\infty(0)$  and  $\gamma \geq 1$ . On the range  $r \in (0, \frac{R_b}{2})$ , we control

$$\delta |\partial_r n_\infty| \geq \frac{r}{4} \frac{R_b^2}{4}, \quad \text{thus} \quad n_\infty(r) \geq \frac{R_b^2}{32\delta} \left( \frac{R_b^2}{4} - r^2 \right),$$

and thus

$$m \geq \frac{R_b^2}{32\delta} \int_0^{R_b/2} \left( \frac{R_b^2}{4} - r^2 \right) dr \geq \frac{R_b^5}{4 \cdot 128 \delta}.$$

Again we have the desired control and Theorem 3.1.2 (C) is proved.

### 3.4 Proof of Theorem 3.1.3

To study the incompressible limit of stationary states of the Cahn-Hilliard equation, the difficulty comes from the singularity of the pressure. However it is possible to fully characterize them, and calculate the pressure jump at the tumor boundary. We begin with establishing the existence and uniqueness for the solution  $n_{inc}$  of the limiting equation. Then, we show that all limits of  $n_k$ 's are determined by this profile  $n_{inc}$ .

#### 3.4.1 Preliminary steps

If a sequence  $\gamma_k \rightarrow \infty$  of stationary states  $n_k$  converges to  $n_{inc}$  and the sequence of pressures  $p_k = n_k^{\gamma_k}$  converges to  $p_{inc}$ . Then we expect that  $p_{inc}(n_{inc} - 1) = 0$ . Therefore, there should be a 'tumor zone' where  $n_{inc} = 1$  and the pressure vanishes outside. This leads us to study the following problem in the zone  $(R_0, R)$  where  $p_{inc} = 0$  :

$$\begin{cases} -\frac{\delta}{r}u'_c - \delta u''_c = R^2 - r^2 - \lambda_c & \text{in } (R_0, R), \\ u_c(R) = u'_c(R) = 0, & u_c(R_0) = 1, u'_c(R_0) = 0, \\ \int_0^R r n_{inc}(r) dr = m, \end{cases} \quad (3.4.1)$$

where  $n_{inc}$  is the extension of  $u_c$  by 1 on  $[0, R_0]$ .

In a later subsection, we prove the convergence of the stationary states  $n_k$  to this limiting profile.

Notice that System (3.4.1) has three free parameters  $(R, R_0, \lambda_c)$  and three constraints (2 additional boundary conditions and mass  $m$ ). The following proposition gives the existence of a solution.

**Proposition 3.4.1** (Unique limiting profile). *Let  $m > 72\delta^{1/2}$ . There exist uniquely determined  $R > 0$ ,  $\lambda_c \in (0, R^2)$  and  $R_0 \in (0, R)$  such that Equation (3.4.1) has a solution. Furthermore,*

$$R_0 = \sqrt{R^2 - 2\lambda_c} \quad \text{and} \quad \lambda_c \approx \sqrt[3]{6} R^{2/3} \delta^{1/3} \quad \text{for small } \delta > 0. \quad (3.4.2)$$

We postpone the proof of this proposition to the next subsection. Its proof uses an explicit solution obtained by the following problem. Find a couple  $(\lambda_u, u)$  such that

$$\begin{cases} -\frac{\delta}{r}u' - \delta u'' = R^2 - r^2 - \lambda_u & \text{in } (0, R) \\ u(R) = u'(R) = 0. \end{cases} \quad (3.4.3)$$

**Proposition 3.4.2** (Lower bound profile). *Let  $\lambda_u \in [0, R^2]$ , then the solution  $u$  of (3.4.3) satisfies*

(A) *the explicit formula for  $u$*

$$u(r) = \frac{R^2}{4\delta} (R^2 - 2\lambda_u) \ln\left(\frac{r}{R}\right) + \frac{(r^2 - R^2)^2}{16\delta} + \frac{R^2 - 2\lambda_u}{8\delta} (R^2 - r^2),$$



$$u'(r) = \frac{(R^2 - r^2)(R^2 - r^2 - 2\lambda_u)}{4\delta r},$$

(B) the function  $u(r)$  is decreasing and positive for  $r$  such that  $0 < R^2 - r^2 < 2\lambda_u$ ,

(C) for a solution  $n_\infty$  of (3.1.7) as in Theorem 3.1.2, if  $\lambda_\infty \geq \lambda_u$  then  $n_\infty(r) \geq u(r)$  for  $r \in (0, R]$ .

*Proof of Proposition 3.4.2.* To prove (A), we first compute  $u'(r)$  and  $u''(r)$  :

$$u'(r) = \frac{R^2}{4\delta}(R^2 - 2\lambda_u) \frac{1}{r} + \frac{(r^2 - R^2)r}{4\delta} - \frac{R^2 - 2\lambda_u}{4\delta} r = \frac{(R^2 - r^2)(R^2 - r^2 - 2\lambda_u)}{4\delta r},$$

$$u''(r) = \frac{-4r(R^2 - r^2) + 4\lambda_u r}{4\delta r} - \frac{(R^2 - r^2)(R^2 - r^2 - 2\lambda_u)}{4\delta r^2} = -\frac{1}{\delta}(R^2 - r^2 - \lambda_u) - \frac{u'(r)}{r}.$$

Therefore, we obtain the desired equation (3.4.3).

The statement (B) is an immediate consequence of the formula for  $u'(r)$ .

Finally, we prove (C). We introduce  $h(r) = n_\infty(r) - u(r)$  and we have to prove that  $h(r) \geq 0$ . From the equations we get

$$n^\gamma - \frac{\delta}{r}h' - \delta h'' = \lambda_u - \lambda \text{ in } (0, R].$$

So, thanks to our assumptions and letting  $g'(r) = rh'(r)$ , we have

$$h''(r) + \frac{h'(r)}{r} \geq 0, \quad g''(r) \geq 0.$$

Integrating this from  $r$  to  $R$  and using the boundary conditions, we obtain

$$g'(r) \leq 0 \implies rh'(r) \leq 0 \implies h'(r) \leq 0.$$

Integrating this once again and using boundary conditions, we discover  $h(r) \geq 0$  as desired.  $\square$

### 3.4.2 Proof of Proposition 3.4.1.

The explicit solution built in Proposition 3.4.2 allows us to characterize the parameters  $\lambda_c$  and  $R_0$ . Indeed, we are looking for  $\lambda_c$  and  $R_0$  such that

$$u'_c(R_0) = \frac{(R^2 - R_0^2)(R^2 - R_0^2 - 2\lambda_c)}{4\delta R_0} = 0, \quad (3.4.4)$$

$$u_c(R_0) = \frac{R^2}{4\delta}(R^2 - 2\lambda_c) \ln\left(\frac{R_0}{R}\right) + \frac{(R_0^2 - R^2)^2}{16\delta} + \frac{R^2 - 2\lambda_c}{8\delta}(R^2 - R_0^2) = 1. \quad (3.4.5)$$

**Lemma 3.4.3** (Solving for  $R_0$  and  $\lambda_c$ ). *Let  $R > 0$ . Then (3.4.4)-(3.4.5) has a unique solution if and only if  $16\delta < R^4$ . Moreover, the solution is given by*

$$R_0 = \sqrt{R^2 - 2\lambda_c}, \quad \lambda_c = \frac{R^2 x_c}{2}, \quad (3.4.6)$$

where  $x_c \in (0, 1)$  is the unique solution of

$$(1 - x_c) \ln(1 - x_c) + \frac{1}{2}x_c^2 + (1 - x_c)x_c = \frac{8\delta}{R^4}. \quad (3.4.7)$$

*Démonstration.* We split the reasoning into several steps.

*Step 1 : Equation for  $R_0$ .* Because  $R_0 = R$  cannot fit (3.4.5), from (3.4.4) we immediately deduce the formula for  $R_0$  in (3.4.6).

*Step 2 : Equation for  $\lambda_c$ .* We plug the formula for  $R_0$  into (3.4.5) to deduce

$$\frac{R^2}{4\delta}(R^2 - 2\lambda_c) \ln\left(\frac{\sqrt{R^2 - 2\lambda_c}}{R}\right) + \frac{4\lambda_c^2}{16\delta} + \frac{R^2 - 2\lambda_c}{8\delta}2\lambda_c = 1.$$

Using properties of logarithm and simple algebra, we have

$$\frac{R^2}{8\delta}(R^2 - 2\lambda_c) \ln\left(1 - \frac{2\lambda_c}{R^2}\right) + \frac{\lambda_c^2}{4\delta} + \frac{R^2 - 2\lambda_c}{4\delta}\lambda_c = 1.$$

Introducing the auxiliary variable  $x_c = \frac{2\lambda_c}{R^2}$  and after multiplication by  $\frac{8\delta}{R^4}$ , this equation is equivalent to Equation (3.4.7).

*Step 3 : Existence and uniqueness of  $x_c$  and  $\lambda_c$ .* We prove that if  $16\delta < R^4$ , equation (3.4.7) has a unique solution. To this end, we define

$$f(x) := (1 - x) \ln(1 - x) - \frac{1}{2}x^2 + x, \quad f(0) = 0, \quad f(1) = \frac{1}{2}. \quad (3.4.8)$$

Then, we compute

$$f'(x) = -\ln(1 - x) - x, \quad f''(x) = \frac{1}{1 - x} - 1. \quad (3.4.9)$$

Since  $f'(0) = 0$  and  $f''(x) > 0$  for  $x \in (0, 1)$ , it follows that  $f'(x) > 0$  so that  $f(x)$  is increasing. It follows that  $f$  is one-to-one from  $(0, 1)$  into  $(0, \frac{1}{2})$ . Therefore, when  $16\delta < R^4$ , there exists a unique  $x_c \in (0, 1)$  such that  $f(x_c) = \frac{8\delta}{R^4}$ .  $\square$

**Lemma 3.4.4** (Estimates for  $x_c$ ). *Let  $x_c$  be a solution to (3.4.7) and  $16\delta < R^4$ . Then, we have*

$$x_c \approx 2\sqrt[3]{6}\delta^{1/3}R^{-4/3}, \quad \lambda_c \approx \sqrt[3]{6}\delta^{1/3}R^{2/3} \quad (\text{as } \delta \rightarrow 0). \quad (3.4.10)$$

*More precisely, we have*

$$x_c \leq 2\sqrt[3]{6}\delta^{1/3}R^{-4/3}. \quad (3.4.11)$$

*Moreover, if  $6^4 8\delta < R^4$ , we have*

$$x_c \geq 2\sqrt[3]{5}\delta^{1/3}R^{-4/3}. \quad (3.4.12)$$

*Démonstration.* For  $\delta$  small, Equation (3.4.7) shows that  $x_c$  is small. More precisely, using (3.4.8), (3.4.9), we obtain  $f(0) = f'(0) = f''(0) = 0$  and  $f^{(3)}(0) = 1$  since  $f^{(3)}(x) = \frac{1}{(1-x)^2}$ . Hence, by the Taylor expansion, for small  $x$ ,  $f(x) \approx \frac{x^3}{6}$ . Plugging this approximation into (3.4.7), we obtain Estimate (3.4.10).

Next, we observe that

$$f^{(k)}(x) = \frac{(k-2)!}{(1-x)^{k-1}}, \quad f^{(k)}(0) = (k-2)!.$$

In particular, the Taylor expansion around  $x = 0$  gives

$$f(x) = \sum_{k \geq 3} \frac{(k-2)!}{k!} x^k = \sum_{k \geq 3} \frac{1}{k(k-1)} x^k.$$

Therefore,  $f(x)$  is controlled by

$$\frac{x^3}{6} \leq f(x) \leq \frac{x^3}{6} \sum_{k \geq 0} x^k = \frac{x^3}{6(1-x)}. \quad (3.4.13)$$

The control (3.4.11) follows from the lower bound.

Finally, using this, we can find  $\delta$  such that  $2\sqrt[3]{6}\delta^{1/3}R^{-4/3} \leq \frac{1}{6}$ , namely  $6^4 8\delta \leq R^4$ . Then, we have  $x_c \leq \frac{1}{6}$  so that  $1 - x_c \geq \frac{5}{6}$  and then the estimate (3.4.13) gives us

$$\frac{8\delta}{R^4} = f(x_c) \leq \frac{x_c^3}{6(1-x_c)} \leq \frac{x_c^3}{5}$$

so that

$$\frac{8\delta}{R^4} \leq \frac{x_c^3}{5} \iff \frac{40\delta}{R^4} \leq x_c^3 \iff 2\sqrt[3]{5}\delta^{1/3}R^{-4/3} \leq x_c.$$

□

**Lemma 3.4.5.** *Let  $u_c$  and  $n_{inc}$  be as in Equation (3.4.1), then the total mass of  $n_{inc}$  satisfies*

$$\mathcal{M}(n_{inc}) := \int_0^R r n_{inc}(r) dr = \frac{R^6 x_c^3(R)}{96 \delta}.$$

Moreover, the map  $R \mapsto \frac{R^6 x_c^3(R)}{96 \delta}$  is increasing if  $R^4 x_c^3(R) > 32 \delta$ .

*Démonstration.* Because  $n_{inc}$  is a  $C^1$  function, integrating by parts, we find

$$\mathcal{M}(n_{inc}) = \int_0^R r n_{inc}(r) dr = -\frac{1}{2} \int_0^R r^2 n'_{inc}(r) dr = -\frac{1}{2} \int_{R_0}^R r^2 u'_c(r) dr.$$

Inserting the formula for  $u'_c(r)$  stated in Proposition 3.4.2, we deduce that

$$\mathcal{M}(n_{inc}) = -\frac{1}{8\delta} \int_{R_0}^R r (R^2 - r^2)(R^2 - r^2 - 2\lambda_c) dr.$$

With the notations  $\lambda_c := R^2 x_c/2$  and  $R_0 = R\sqrt{1-x_c}$ , we obtain

$$\mathcal{M}(n_{inc}) = -\frac{1}{8\delta} \int_{R\sqrt{1-x_c}}^R r (R^2 - r^2)(R^2 - r^2 - R^2 x_c) dr.$$

We change variables  $\tau = R^2 - r^2$  to get the desired formula

$$\mathcal{M}(n_{inc}) = -\frac{1}{16\delta} \int_0^{R^2 x_c} \tau(\tau - R^2 x_c) d\tau = \frac{R^6 x_c^3}{96\delta}.$$

For the second assertion, it is sufficient to prove that the map  $R \mapsto R^6 x_c^3(R)$  is strictly increasing. Note that  $x_c(R)$  is given implicitly via equation (3.4.7). Differentiating it with respect to  $R$ , we discover that

$$\frac{dx_c}{dR}(x_c + \log(1 - x_c)) = \frac{32\delta}{R^5} \implies \frac{dx_c}{dR} = \frac{32\delta}{R^5(x_c + \log(1 - x_c))}.$$

Then, we study the derivative of  $R^6 x_c^3(R)$ ,

$$\frac{d(R^6 x_c^3(R))}{dR} = 6R^5 x_c^3 + 3R^6 x_c^2 \frac{dx_c}{dR} = 6R^5 x_c^3 + \frac{96\delta R x_c^2}{(x_c + \log(1 - x_c))}.$$

Using a simple Taylor estimate, we have  $\frac{1}{x + \log(1-x)} \geq \frac{2}{x^2}$  and we conclude that  $R^6 x_c^3(R)$  is increasing since

$$\frac{d(R^6 x_c^3(R))}{dR} \geq 6R^5 x_c^3 - 192\delta R = 6R(R^4 x_c^3 - 32\delta).$$

□

*Proof of Proposition 3.4.1.* First, we notice that if  $(R_0, R, \lambda)$  satisfy conditions of the Proposition 3.4.1, then  $R_0, \lambda_c$  are given by (3.4.6) (Lemma 3.4.3) and  $\frac{R^6 x_c^3}{96\delta} = m$  (Lemma 3.4.5). Then, by the control  $6^2 2\delta^{1/2} \leq m$  as well as an upper bound on  $x_c$ , cf. (3.4.11), we deduce

$$6^2 2\delta^{1/2} \leq m = \frac{R^6 x_c^3}{96\delta} \leq \frac{R^2}{2} \implies 6^4 4^2 \delta \leq R^4.$$

This means that we can apply the lower bound (3.4.12) to deduce

$$R^4 x_c^3 \geq 40\delta^{1/3}.$$

It follows that the necessary condition for existence of  $(R_0, R, \lambda)$  is  $6^4 4^2 \delta \leq R^4$  which implies  $R^4 x_c^3 \geq 40\delta^{1/3}$ . Therefore, by Lemma 3.4.5, the map  $R \mapsto R^6 x_c^3(R)$  is invertible and we can find uniquely determined  $R$  such that

$$m = \frac{R^6 x_c^3(R)}{96\delta}.$$

With such a value  $R$  (because  $16\delta < R^4$ ), we can find unique  $R_0$  and  $\lambda_c$  solving (3.4.4)-(3.4.5) so that the formula for the mass is satisfied and the conclusion follows. □

### 3.4.3 Proof of Theorem 3.1.3

The solutions of Theorem 3.1.2 satisfy, with  $\lambda_k \in (0, R^2)$ ,  $R_k > 0$ ,

$$\begin{cases} n_k^{\gamma_k} - \delta n_k'' - \frac{\delta}{r} n_k' = R_k^2 - r^2 - \lambda_k & \text{in } (0, R_k), & n_k = 0 & \text{in } (R_k, R_b), \\ n_k(R_k) = n_k'(R_k) = 0, & & n_k'(0) = 0. & \end{cases}$$

Thanks to the maximum principle, the sequence  $\{n_k^{\gamma_k}\}_k$  is bounded in  $L^\infty(I_{R_b})$ . Moreover, multiplying this equation by  $n_k''$  and integrating by parts, we obtain

$$\int_0^{R_k} \left( \delta (n_k'')^2 + \gamma n_k^{\gamma_k-1} (n_k')^2 + \delta \frac{(n_k')^2}{2r^2} \right) dr = \int_0^{R_k} n_k'' (R_k^2 - r^2 - \lambda_k) dr.$$

Since  $0 \leq R_k, \lambda_k \leq R_b$ , the right-hand side is bounded by  $\frac{\delta}{2} \int_0^{R_k} (n_k'')^2 + C(\delta, R_b)$ . Thus,  $\{n_k''\}_k$  is uniformly bounded in  $L^2(0, R_b)$ . Therefore, up to a subsequence, as  $k \rightarrow \infty$

$$n_k^{\gamma_k} \rightharpoonup p_{inc} \geq 0 \quad \text{weakly}^* \text{ in } L^\infty(I_{R_b}), \quad n_k \rightarrow n_{inc} \leq 1 \quad \text{in } C^1(\overline{I_{R_b}}).$$

We also have the algebraic relation  $p_{inc}(n_{inc} - 1) = 0$ . The inequality  $p_{inc}(n_{inc} - 1) \leq 0$  is straightforward using  $p_{inc} \geq 0$  and  $n_{inc} \leq 1$ . It remains to show that  $p_{inc}(n_{inc} - 1) \geq 0$ . For  $\nu > 0$ , there exists  $\gamma_0$  such that for  $\gamma_k \geq \gamma_0$

$$n_k^{\gamma_k+1} \geq n_k^{\gamma_k} - \nu$$

because the function  $x \mapsto x^\gamma(x-1)$  is nonpositive on  $[0, 1]$  and attains its minimum  $-\left(\frac{\gamma}{\gamma+1}\right)^\gamma \frac{1}{\gamma+1} \rightarrow 0$  as  $\gamma \rightarrow \infty$ . Then, from the strong convergence of  $n_k$  and the weak convergence of  $n_k^{\gamma_k}$  we know that  $n_k^{\gamma_k} n_k$  converges weakly to  $p_{inc} n_{inc}$ . Passing to the limit, we obtain

$$p_{inc} n_{inc} \geq p_{inc} - \nu,$$

for every  $\nu > 0$ . Letting  $\nu \rightarrow 0$  yields the result.

Since  $\{\lambda_k\}_k$  and  $\{R_k\}_k$  are also bounded subsequences, we can extract converging subsequences to  $\lambda$  and  $R$  respectively. Thanks to the  $C^1$  convergence we know that  $n$  satisfies the boundary condition  $n_{inc}(R) = n'_{inc}(R) = 0$  and  $n_{inc}$  is radially decreasing as the uniform limit of radially decreasing functions. Finally, we can pass to the limit in the equation of mass conservation and obtain  $\int_0^R r n_{inc} dr = m$ . To sum up, in the limit we obtain a  $C^1$ , nonincreasing function  $n_{inc}$  satisfying

$$\begin{cases} p_{inc} - \frac{\delta}{r} n'_{inc} - \delta n''_{inc} = R^2 - r^2 - \lambda & \text{in } (0, R), \\ n_{inc}(R) = n'_{inc}(R) = n'_{inc}(0) = 0, \\ \int_0^R r n_{inc}(r) dr = m, \\ p_{inc}(n_{inc} - 1) = 0. \end{cases}$$

The limiting ODE is satisfied on  $(0, R)$  because the ODE for  $n_k$  is satisfied on  $(0, \inf_{l \geq k} R_k)$ . Passing to the limit, we obtain the ODE on  $(0, \lim_{k \rightarrow \infty} \inf_{l \geq k} R_k) = (0, R)$  because  $R = \lim_{k \rightarrow \infty} R_k$ .

We claim that  $n_{inc}$  reaches the value 1. By contradiction, if  $n_{inc} < 1$  on  $[0, R]$ , then  $p_{inc} = 0$  so that  $n_{inc}$  is a  $C^1$  solution to the following ODE on  $[0, R]$  :

$$-\frac{\delta}{r} n'_{inc} - \delta n''_{inc} = R^2 - r^2 - \lambda, \quad n_{inc}(R) = n'_{inc}(R) = n'_{inc}(0) = 0.$$

By Proposition 3.4.2 (A) such a solution does not exist.

By monotonicity and the fact that  $n_{inc}$  reaches value 1 we deduce that there are two zones. In the zone  $\{p_{inc} > 0\}$  we have  $n_{inc} = 1$ , and thus  $p_{inc} = R^2 - r^2 - \lambda$ . Then, when  $n_{inc} < 1$  ( $n_{inc}$  is decreasing), let us say at  $r = R_0$  we have  $p = 0$ . The pressure jump is equal to  $\llbracket p_{inc} \rrbracket = R^2 - R_0^2 - \lambda$ .

Finally, the convergence of the whole sequence follows from uniqueness of the limiting profile as stated in Proposition 3.4.1.

### 3.5 Conclusion and numerical simulations

Motivated by the pressure jump imposed in free boundary problems of tissue growth, [177, 289, 238, 261], we included surface tension in such compressible models. We established that radially symmetric stationary solutions of the Cahn-Hilliard system with a confining potential  $V(r)$  exist and are decreasing. In the incompressible limit, they present a jump of pressure at the boundary of the saturation set  $\{n = 1\}$ . We computed explicitly this pressure jump which is proportional to  $\delta^{1/3}V'(R)^{2/3}$ . There is a vacuum zone  $\{n = 0\}$  that induces a degeneracy which is the main difficulty when establishing the a priori estimates.

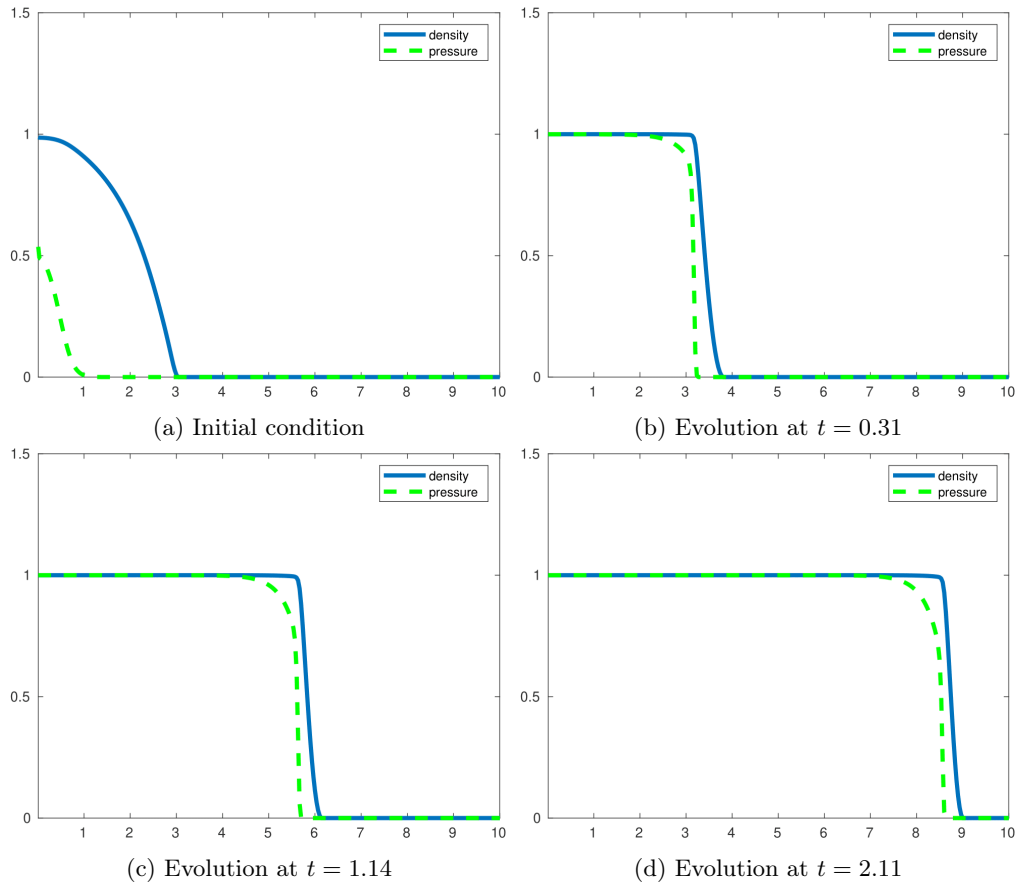
It is an open question to prove a similar result, for a propagating wave, when the system is driven by a source term rather than a confining potential, as in [151] for instance. However, we provide numerical simulations in radial coordinates. More precisely, we focus on the system

$$\begin{aligned} \frac{\partial(rn)}{\partial t} - \frac{\partial}{\partial r} \left( rn \frac{\partial \mu}{\partial r} \right) &= nG(p), \quad \text{in } (0, +\infty) \times I_{R_b}, \\ \mu &= p - \frac{\delta}{r} \frac{\partial}{\partial r} \left( r \frac{\partial n}{\partial r} \right), \quad p = n^\gamma. \end{aligned} \tag{3.5.1}$$

When  $\gamma \rightarrow \infty$ , we expect to find the incompressible limit

$$\begin{cases} -\frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) = G(p), & p(n-1) = 0, \quad \text{in } \{n = 1\}, \\ \llbracket p \rrbracket = -\delta \llbracket \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial n}{\partial r} \right) \rrbracket & \text{on } \partial\{n = 1\}. \end{cases}$$

These equations are obtained formally after setting  $n = 1$  in (3.5.1) and using the relation  $p(n-1) = 0$ . The main open question is to link the value of the pressure jump to the other parameters of the model, *i.e.* the source term  $G$ , the parameter  $\delta$  and boundary's curvature. In radial settings the curvature is  $\frac{1}{R(t)}$  where  $R(t)$  is the radius of the tumor. We present below some numerical simulations for the evolution of the density and the pressure of the tumor. If the pressure jump seems to be decreasing as the tumor grows, it is not numerically clear how to determine the pressure jump.



**Numerical settings.** For the source term, we take  $G(p) = 10(1 - p)$ . We use an explicit scheme, with time step  $dt = 1e-7$ , final time  $t = 2.11$  and the interval is  $I_{R_b} = [0, 10]$  with 300 points. The initial condition is a truncated arctangent. To remove the degeneracy  $r = 0$  in the numerical scheme, we consider  $r + \varepsilon$  instead of  $r$  for some small  $\varepsilon > 0$ .

The pressure  $p$  reaches the value 1, as the density, because we choose the homeostatic pressure  $p_h = 1$  in the source term  $G(p) = 10(p_h - p)$ . The homeostatic pressure is interpreted as the lowest level of pressure that prevents cell multiplication due to contact-inhibition.

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## 3.6 Appendix

### 3.6.1 Limit profile for general force in dimension 2

We generalize the pressure jump formula obtained when  $V(r) = r^2$  and, for a general strictly increasing force  $V$ , we establish that

$$\lambda \approx \frac{\sqrt[3]{12}}{2} \delta^{1/3} (V'(R))^{2/3}. \quad (3.6.1)$$

Hence, we consider the solution in  $(0, R)$  of

$$\delta n''(r) + \delta \frac{n'(r)}{r} = V(R) - V(r) - \lambda, \quad n'(R) = n(R) = 0. \quad (3.6.2)$$

*Step 1 : An expression of the solution.* The solution is given by

$$\begin{cases} n(r) = \frac{R^2}{2\delta} (V(R) - \lambda) \log\left(\frac{r}{R}\right) + \frac{R^2 - r^2}{4\delta} (V(R) - \lambda) + \frac{1}{\delta} \int_r^R \frac{\mathcal{H}(z)}{z} dz, \\ \mathcal{H}(z) := \int_z^R u V(u) du. \end{cases} \quad (3.6.3)$$

Indeed, we immediately verify that  $n(R) = 0$ . Moreover, we have

$$n'(r) = \frac{R^2}{2\delta} (V(R) - \lambda) \frac{1}{r} - \frac{r}{2\delta} (V(R) - \lambda) - \frac{\mathcal{H}(r)}{\delta r}. \quad (3.6.4)$$

As  $\mathcal{H}(R) = 0$ , we have  $n'(R) = 0$ . Finally, we compute  $n''$  using (3.6.4)

$$n''(r) = -\frac{R^2}{2\delta} (V(R) - \lambda) \frac{1}{r^2} - \frac{1}{2\delta} (V(R) - \lambda) + \frac{1}{\delta} V(r) + \frac{\mathcal{H}(r)}{\delta r^2}. \quad (3.6.5)$$

Therefore, combining (3.6.4) and (3.6.5), we obtain (3.6.2).

*Step 2 : Limit profile.* We are looking for the solution  $n$  of (3.6.2) such that  $n(R_0) = 1$ ,  $n'(R_0) = 0$  for some  $R_0 < R$  and some  $\lambda$ , i.e. we have two parameters  $R_0$  and  $\lambda$  to be found.

The condition  $n'(R_0) = 0$  is immediately obtained from (3.6.4). It is given by

$$(R^2 - R_0^2) (V(R) - \lambda) - 2 \mathcal{H}(R_0) = 0. \quad (3.6.6)$$

For the condition  $n(R_0) = 1$ , from (3.6.3) we obtain

$$\frac{R^2}{2\delta} (V(R) - \lambda) \log\left(\frac{R_0}{R}\right) + \frac{R^2 - R_0^2}{4\delta} (V(R) - \lambda) + \frac{1}{\delta} \int_{R_0}^R \frac{\mathcal{H}(z)}{z} dz = 1.$$

Then, we use (3.6.6) to remove the term  $(V(R) - \lambda)$ , and multiply by  $2\delta$  to get an equation for  $R_0$

$$R^2 \frac{\mathcal{H}(R_0)}{R^2 - R_0^2} \log\left(\frac{R_0}{R}\right) + \mathcal{H}(R_0) + 2 \int_{R_0}^R \frac{\mathcal{H}(z)}{z} dz = 2\delta.$$



We introduce the variable  $\tau := \frac{R^2 - R_0^2}{R^2}$  so that the equation reads

$$\mathcal{H}(\sqrt{1-\tau}R) \left( \frac{\log(1-\tau)}{\tau} + 1 \right) + 2 \int_{\sqrt{1-\tau}R}^R \frac{\mathcal{H}(z)}{z} dz = 2\delta, \quad \tau := \frac{R^2 - R_0^2}{R^2}. \quad (3.6.7)$$

*Step 3 : Existence and uniqueness of  $\tau$  and  $R_0$ .* We define the function

$$\mathcal{F}(\tau) := \mathcal{H}(\sqrt{1-\tau}R) \left( \frac{\log(1-\tau)}{\tau} + 1 \right) + 2 \int_{\sqrt{1-\tau}R}^R \frac{\mathcal{H}(z)}{z} dz.$$

As  $\mathcal{H}(R) = 0$  and  $\frac{\log(1-\tau)}{\tau}$  is bounded near  $\tau = 0$ , we have  $\mathcal{F}(\tau) = 0$ . Now, we want to compute  $\mathcal{F}'(\tau)$ . First,

$$\begin{aligned} \frac{d}{d\tau} \mathcal{H}(\sqrt{1-\tau}R) &= \sqrt{1-\tau}R V(\sqrt{1-\tau}R) \frac{R}{2\sqrt{1-\tau}} = \frac{R^2 V(\sqrt{1-\tau}R)}{2}, \\ \frac{d}{d\tau} \left( 2 \int_{\sqrt{1-\tau}R}^R \frac{\mathcal{H}(z)}{z} dz \right) &= 2 \left( -\frac{\mathcal{H}(\sqrt{1-\tau}R)}{\sqrt{1-\tau}R} \right) \frac{R}{2\sqrt{1-\tau}} = \frac{\mathcal{H}(\sqrt{1-\tau}R)}{1-\tau}, \\ \frac{d}{d\tau} \left( \frac{\log(1-\tau)}{\tau} + 1 \right) &= \frac{1}{\tau(\tau-1)} - \frac{\log(1-\tau)}{\tau^2}. \end{aligned} \quad (3.6.8)$$

Therefore, by the product rule, we find

$$\begin{aligned} \mathcal{F}'(\tau) &= \\ &= \frac{R^2 V(\sqrt{1-\tau}R)}{2} \left( \frac{\log(1-\tau)}{\tau} + 1 \right) + \mathcal{H}(\sqrt{1-\tau}R) \left( \frac{1}{1-\tau} + \frac{1}{\tau(\tau-1)} - \frac{\log(1-\tau)}{\tau^2} \right) \\ &= \frac{1}{\tau} \left( \frac{\log(1-\tau)}{\tau} + 1 \right) \left( \frac{R^2 \tau V(\sqrt{1-\tau}R)}{2} - \mathcal{H}(\sqrt{1-\tau}R) \right) \end{aligned} \quad (3.6.9)$$

Since  $\frac{\log(1-\tau)}{\tau} + 1 < 0$  for  $\tau \in (0, 1)$ , to prove  $\mathcal{F}'(\tau) > 0$ , it is sufficient that for  $\tau \in (0, 1)$

$$\frac{R^2 \tau V(\sqrt{1-\tau}R)}{2} - \mathcal{H}(\sqrt{1-\tau}R) < 0. \quad (3.6.10)$$

This function vanishes in  $\tau = 0$ . Moreover, its derivative with respect to  $\tau$  is equal to

$$\frac{R^2 V(\sqrt{1-\tau}R)}{2} + \frac{R^2 \tau V'(\sqrt{1-\tau}R)}{2} \frac{-R}{2\sqrt{1-\tau}} - \frac{R^2 V(\sqrt{1-\tau}R)}{2} = -\frac{R^3 \tau V'(\sqrt{1-\tau}R)}{4\sqrt{1-\tau}} \quad (3.6.11)$$

where we used (3.6.8). As  $V' > 0$ , we conclude (3.6.10) which implies  $\mathcal{F}'(\tau) > 0$ . Hence, in some neighborhood of 0 we can find exactly one  $\tau$  that solves the equation. Moreover, it is unique as  $\mathcal{F}$  is strictly increasing. Then the uniqueness of  $R_0$  and  $\lambda$  follows.

*Step 4 : Taylor expansion of  $\tau$ .* As  $\mathcal{F}$  is strictly increasing, we expect the solution  $\tau$  to be small (if  $\delta$  is small). This justifies to use Taylor expansion around  $\tau = 0$ . We already know that

$\mathcal{F}(0) = \mathcal{F}'(0) = 0$ . Now, we claim  $\mathcal{F}''(0) = 0$ . Indeed,

$$\frac{d}{d\tau} \left( \frac{\log(1-\tau)}{\tau^2} + \frac{1}{\tau} \right) = -\frac{2\log(1-\tau)}{\tau^3} + \frac{\tau-2}{(1-\tau)\tau^2} = -\frac{2\log(1-\tau)(1-\tau) + (2-\tau)\tau}{(1-\tau)\tau^3}$$

so that using (3.6.9) and (3.6.11) we compute

$$\begin{aligned} \mathcal{F}''(\tau) = & -\frac{1}{\tau} \left( \frac{\log(1-\tau)}{\tau} + 1 \right) \frac{R^3 \tau V'(\sqrt{1-\tau}R)}{4\sqrt{1-\tau}} \\ & - \frac{2\log(1-\tau)(1-\tau) + (2-\tau)\tau}{(1-\tau)\tau^3} \tau \left( \frac{R^2 \tau V(\sqrt{1-\tau}R)}{2} - \mathcal{H}(\sqrt{1-\tau}R) \right) \end{aligned} \quad (3.6.12)$$

Since  $2\log(1-\tau)(1-\tau) + (2-\tau)\tau \approx \tau^3$  for small  $\tau$ , the expressions above are bounded in the neighborhood of 0. Evaluating them at  $\tau = 0$ , we obtain  $\mathcal{F}''(0) = 0$ .

Now, we claim that  $\mathcal{F}^{(3)}(0) \neq 0$ . To see this, we write expression (3.6.12) in the form

$$\mathcal{F}''(\tau) = A(\tau) B(\tau) + C(\tau) D(\tau)$$

so that

$$\mathcal{F}^{(3)}(\tau) = A'(\tau) B(\tau) + A(\tau) B'(\tau) + C'(\tau) D(\tau) + C(\tau) D'(\tau).$$

We study the four terms above separately.

- $A'(\tau) B(\tau)|_{\tau=0} = 0$ . Indeed,  $A'(\tau) = -C(\tau)$  and the latter is bounded in the neighbourhood of 0 (see Taylor's expansion above). Moreover,  $B(0) = 0$ .
- $C(\tau) D'(\tau)|_{\tau=0} = 0$ . Indeed,  $C(\tau)$  is bounded around  $\tau = 0$ , while  $D'(0) = B(0) = 0$ .
- $C'(\tau) D(\tau)|_{\tau=0} = 0$ . In fact,  $D(0) = 0$ , so it is sufficient to prove that  $C'(\tau)$  is bounded near  $\tau = 0$ . We have

$$C'(\tau) = \frac{\tau(2\tau^2 - 9\tau + 6) + 6\log(1-\tau)(1-\tau)^2}{\tau^4(1-\tau^2)}.$$

Using expansion  $\log(1-\tau) \approx -\tau - \frac{1}{2}\tau^2 - \frac{1}{3}\tau^3$  we have around  $\tau = 0$  :

$$\begin{aligned} \tau(2\tau^2 - 9\tau + 6) + 6\log(1-\tau)(1-\tau)^2 & \approx \\ & \approx 2\tau^3 - 9\tau^2 + 6\tau - (6\tau + 3\tau^2 + 2\tau^3)(1 + \tau^2 - 2\tau) \\ & = 2\tau^3 - 9\tau^2 + 6\tau - 6\tau - 3\tau^2 - 2\tau^3 - 6\tau^3 - 3\tau^4 - 2\tau^5 + 12\tau^2 + 6\tau^3 + 4\tau^4 \\ & = \tau^4 - 2\tau^5. \end{aligned}$$

- $A(\tau)B'(\tau)|_{\tau=0} = \frac{R^3 V'(R)}{8}$ . Indeed,  $A(0) = \frac{1}{2}$ . Moreover, we have

$$B'(\tau) = \frac{R^3 V'(\sqrt{1-\tau}R)}{4\sqrt{1-\tau}} + \frac{R^3 \tau V''(\sqrt{1-\tau}R)}{4\sqrt{1-\tau}} \frac{(-R)}{2\sqrt{1-\tau}} + \frac{R^3 \tau V''(\sqrt{1-\tau}R)}{8(1-\tau)^{3/2}}$$

which implies  $B'(0) = \frac{R^3 V'(R)}{8}$ .

Therefore, equation (3.6.7) can be approximated with

$$\frac{1}{6} \frac{R^3 V'(R)}{8} \tau^3 = 2\delta \iff \tau^3 = \frac{96\delta}{R^3 V'(R)} \iff \tau = \frac{2\sqrt[3]{12} \delta^{1/3}}{R\sqrt[3]{V'(R)}}.$$

*Step R : Taylor expansion for  $\lambda$ .* Small value of  $\tau$  means that  $R_0$  is close to  $R$ . Therefore, we expand in Taylor series equation (3.6.6) around  $R_0 = R$ . Writing

$$\mathcal{G}(R_0) := (R^2 - R_0^2)(V(R) - \lambda) - 2\mathcal{H}(R_0) = 0,$$

we have

$$\mathcal{G}'(R_0) = -2R_0(V(R) - \lambda) + 2R_0 V(R_0) = 2R_0(V(R_0) - V(R) + \lambda),$$

$$\mathcal{G}''(R_0) = 2(V(R_0) - V(R) + \lambda) + 2R_0 V'(R_0).$$

It follows that  $\mathcal{G}(R) = 0$ ,  $\mathcal{G}'(R) = 2R_0\lambda$  and  $\mathcal{G}''(R) = 2\lambda + 2RV'(R)$ . Therefore, (3.6.6) can be approximated with

$$2R_0\lambda(R_0 - R) + \frac{1}{2}(2\lambda + 2RV'(R))(R_0 - R)^2 = 0,$$

which can be rewritten as

$$\lambda = \frac{R(R - R_0)V'(R)}{R_0 + R}.$$

Using  $\tau = \frac{R^2 - R_0^2}{R^2}$  we have  $R - R_0 = \frac{\tau R^2}{R + R_0}$  so that

$$\lambda = \frac{R^3 \tau V'(R)}{(R_0 + R)^2} \approx \frac{R \tau V'(R)}{4} = \frac{\sqrt[3]{12}}{2} \delta^{1/3} (V'(R))^{2/3}.$$

Notice that when  $V(r) = r^2$ , we have  $V'(R) = 2R$ . Therefore,  $\lambda \approx \sqrt[3]{6} \delta^{1/3} R^{2/3}$  as in (3.4.2).

Deuxième partie

Nonlocal to local convergence of the  
Cahn-Hilliard equation



## Chapitre 4

# Degenerate Cahn-Hilliard equation : From nonlocal to local

### Abstract

There has been recently an important interest in deriving rigorously the Cahn-Hilliard equation from the nonlocal equation, also called aggregation equation. So far, only non-degenerate mobilities were treated. Since we are motivated by models for the biomechanics of living tissues, it is useful to include degenerate mobilities. In this framework, we present a new method to show the convergence of the nonlocal to the local *degenerate* Cahn-Hilliard equation. The method includes the use of nonlocal Poincaré and compactness inequalities.

### 4.1 Introduction

Several recent papers [273, 121, 123, 122] have addressed the problem of deriving rigorously the Cahn-Hilliard equation from the nonlocal equation, also called aggregation equation [85]. In these works, only the case of non-degenerate mobilities is treated, which avoids the delicate question of defining the limit of low-order products that one encounters for nonlocal degenerate mobility that we present now. The degenerate model is written

$$\begin{aligned}\partial_t u &= \operatorname{div}(u \nabla \mu), & \text{in } (0, +\infty) \times \mathbb{T}^d, \\ \mu &= B[u] + F'(u), & \text{in } (0, +\infty) \times \mathbb{T}^d,\end{aligned}$$

equipped with an initial datum  $u^0 \geq 0$ . Here,  $\mathbb{T}^d$  is the  $d$ -dimensional flat torus,  $B$  is the nonlocal operator  $B = B_\varepsilon$  defined with

$$B_\varepsilon[u](x) = \frac{1}{\varepsilon^2}(u(x) - \omega_\varepsilon * u(x)) = \frac{1}{\varepsilon^2} \int_{\mathbb{T}^d} \omega_\varepsilon(y)(u(x) - u(x-y)) \, dy \quad (4.1.1)$$

for  $\varepsilon$  small enough and  $\omega_\varepsilon$  is the usual radial mollification kernel  $\omega_\varepsilon(x) = \frac{1}{\varepsilon^d} \omega(\frac{x}{\varepsilon})$  with  $\omega$  compactly supported in the unit ball of  $\mathbb{R}^d$  satisfying

$$\int_{\mathbb{R}^d} \omega(y) \, dy = 1, \quad \int_{\mathbb{R}^d} y \omega(y) \, dy = 0, \quad \int_{\mathbb{R}^d} y_i y_j \omega \, dy = \delta_{i,j} \frac{2D}{d}, \quad (4.1.2)$$

for some constant  $D > 0$ . Our target is to prove that as  $\varepsilon \rightarrow 0$ , the constructed solutions of

$$\partial_t u = \operatorname{div}(u \nabla \mu), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (4.1.3)$$

$$\mu = B_\varepsilon[u] + F'(u), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (4.1.4)$$

tend to the weak solution of the degenerate Cahn-Hilliard equation

$$\partial_t u = \operatorname{div}(u \nabla \mu), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (4.1.5)$$

$$\mu = -D\Delta u + F'(u), \quad \text{in } (0, +\infty) \times \mathbb{T}^d. \quad (4.1.6)$$

The product  $u \nabla \mu$  in the limiting system is not a priori well defined since we cannot control third-order derivatives. Passing to the limit is thus the challenge we overcome here.

Our motivation for this work is fourfold.

- Firstly, the interest for the nonlocal Cahn-Hilliard equation is an old problem that can be traced back to Giacomini and Lebowitz [189, 190]. These seminal works establish the derivation of the degenerate nonlocal Cahn-Hilliard equation departing from stochastic systems of particles. However, they left open the question of deriving the local degenerate Cahn-Hilliard equation from the nonlocal one. This is the challenge we overcome here.
- Secondly, a revival of interest for this problem appeared in the last years with several papers [273, 121, 123, 122] deriving the local from the nonlocal Cahn-Hilliard equation in the non-degenerate case.
- Third, the nonlocal Cahn-Hilliard equation can be seen as a porous medium equation with a smooth advection term that is well understood, conversely to the local degenerate Cahn-Hilliard equation.
- Finally, the nonlocal Cahn-Hilliard equation (4.1.3)–(4.1.4) is in fact an aggregation-diffusion equation with a nonlocal term corresponding to the aggregation effect [85]. Thus, in this paper, we show that if the nonlocal effect is appropriately scaled, one approaches Cahn-Hilliard equation. This limit was formally stated for instance in [132, 45, 165] and our work provides a rigorous mathematical argument for this approximation.

### 4.1.1 Main result

We make the following assumptions on the potential  $F$ .

*Assumption 4.1.1* (potential  $F$ ). For the interaction potential we assume that there exists  $k \geq 2$  and a decomposition  $F = F_1 + F_2$  such that

- (A)  $F_1, F_2$  are of class  $C^2$ ,
- (B)  $F_1 = 0$  or  $F_1$  is a convex function which has  $k$ -growth in the sense that for some nonnegative constants  $C_1, \dots, C_8$  we have

$$C_1|u|^k - C_2 \leq F_1(u) \leq C_3|u|^k + C_4.$$

$$C_5|u|^{(k-2)} - C_6 \leq F_1''(u) \leq C_7|u|^{(k-2)} + C_8,$$

- (C)  $F_2$  has bounded second derivative *i.e.*  $\|F_2''\|_\infty < \infty$  and  $F_2(u) \geq -C_9 - C_{10}u^2$  where  $C_{10}$  is sufficiently small : more precisely  $4C_{10} < C_p$  with  $C_p$  being the constant in Lemma 4.4.6.

**Example 4.1.2.** *The following potentials satisfy Assumption 4.1.1.*

- (1) *power-type potential*  $F(u) = |u|^\gamma$ ,  $\gamma > 2$  used in the context of tumour growth models [289, 120, 151, 125],

(2) double-well potential  $F(u) = u^2(u-1)^2$  which is an approximation of logarithmic double-well potential often used in Cahn-Hilliard equation, see [277, Chapter 1],

(3) any  $F \in C^2$  such that for some interval  $I \subset \mathbb{R}$  we have  $F''(u) > a > 0$  for  $u \in \mathbb{R} \setminus I$  and

$$\begin{aligned} C|u|^k - C &\leq F(u) \leq C|u|^k + C && \text{for all } u \in \mathbb{R} \setminus I, \\ C|u|^{k-2} - C &\leq F''(u) \leq C|u|^{k-2} + C && \text{for all } u \in \mathbb{R} \setminus I, \end{aligned}$$

see Lemma 4.4.3 for details.

Note that (3) is a more general version of (2).

**Notation 4.1.3** (exponents  $s$  and  $k$ ). In what follows we write

$$k = \begin{cases} 2 & \text{if } F_1 = 0, \\ k & \text{if } F_1 \neq 0. \end{cases}$$

We also define  $s = \frac{2k}{k-1}$  and  $s'$  its conjugate exponent.

Now, we define weak solutions of the nonlocal and local degenerate Cahn-Hilliard equation.

**Definition 4.1.4.** We say that  $u_\varepsilon$  is a weak solution of (4.1.3)-(4.1.4) if

$$\begin{aligned} u_\varepsilon &\in L^\infty(0, T; L^k(\mathbb{T}^d)), & \partial_t u_\varepsilon &\in L^2(0, T; W^{-1, s'}(\mathbb{T}^d)) \\ \nabla u_\varepsilon &\in L^2((0, T) \times \mathbb{T}^d), & \sqrt{F_1''(u_\varepsilon)} \nabla u_\varepsilon &\in L^2((0, T) \times \mathbb{T}^d), \end{aligned}$$

$u(0, x) = u^0(x)$  a.e. in  $\mathbb{T}^d$  and for all  $\varphi \in L^2(0, T; W^{1, \infty}(\mathbb{T}^d))$

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi \rangle_{(W^{-1, s'}(\mathbb{T}^d), W^{1, s}(\mathbb{T}^d))} = - \int_0^T \int_{\mathbb{T}^d} u_\varepsilon \nabla B_\varepsilon[u_\varepsilon] \cdot \nabla \varphi - \int_0^T \int_{\mathbb{T}^d} u_\varepsilon F''(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi. \quad (4.1.7)$$

**Definition 4.1.5.** We say that  $u$  is a weak solution of (4.1.5)-(4.1.6) if

$$\begin{aligned} u &\in L^\infty(0, T; L^k(\mathbb{T}^d)) \cap L^2(0, T; H^2(\mathbb{T}^d)) \cap L^\infty(0, T; H^1(\mathbb{T}^d)), & \partial_t u &\in L^2(0, T; W^{-1, s'}(\mathbb{T}^d)), \\ \sqrt{F_1''(u)} \nabla u &\in L^2((0, T) \times \mathbb{T}^d), \end{aligned}$$

$u(0, x) = u^0(x)$  a.e. in  $\mathbb{T}^d$  and if for all  $\varphi \in L^2(0, T; W^{2, \infty}(\mathbb{T}^d))$  we have

$$\begin{aligned} \int_0^T \langle \partial_t u, \varphi \rangle_{(W^{-1, s'}(\mathbb{T}^d), W^{1, s}(\mathbb{T}^d))} &= -D \int_0^T \int_{\mathbb{T}^d} \Delta u \nabla u \cdot \nabla \varphi - D \int_0^T \int_{\mathbb{T}^d} u \Delta u \Delta \varphi \\ &\quad - \int_0^T \int_{\mathbb{T}^d} u F''(u) \nabla u \cdot \nabla \varphi. \end{aligned}$$

**Remark 4.1.6** (initial condition). In Definitions 4.1.4 and 4.1.5 we can evaluate pointwise value  $u(0, x)$  because by [302, Lemma 7.1], we know that  $u \in C(0, T; W^{-1, s'}(\mathbb{T}^d))$ .

With these assumptions we can construct solutions to (4.1.3)-(4.1.4).

**Theorem 4.1.7** (Existence of solutions for the nonlocal system). Let  $\varepsilon_0$  be given by

$$\varepsilon_0 := \min \left( \varepsilon_0^A, \varepsilon_0^B, \frac{1}{\sqrt{2 \|F_2''\|_\infty}} \right) \quad (4.1.8)$$



where  $\varepsilon_0^A$  and  $\varepsilon_0^B$  are given in Lemma 4.4.6 and 4.4.8 respectively. Let  $\varepsilon < \varepsilon_0$ . Let  $u^0 \geq 0$  be an initial datum with finite energy and entropy  $E_\varepsilon(u^0), \Phi(u^0) < \infty$  defined in (4.1.9)-(4.1.10). There exists a global weak solution  $u_\varepsilon$  of (4.1.3)-(4.1.4) in the sense defined by Definition 4.1.4. It satisfies the dissipation of energy and entropy (4.1.11)-(4.1.12) with  $u = u_\varepsilon, \mu = \mu_\varepsilon$ . Moreover,  $u_\varepsilon \geq 0$ .

Our main result reads as follows.

**Theorem 4.1.8** (Convergence of nonlocal to local Cahn-Hilliard equation on the torus). *Let  $u^0 \geq 0$  be an initial datum with finite energy and entropy  $E(u^0), \Phi(u^0) < \infty$  defined in (4.1.13) and (4.1.10). Let  $\{u_\varepsilon\}$  be a sequence of solutions of the degenerate nonlocal Cahn-Hilliard equation (4.1.3)-(4.1.4) from Theorem 4.1.7. Then, up to a subsequence,*

$$u_\varepsilon \rightarrow u \text{ in } L^2(0, T; H^1(\mathbb{T}^d))$$

where  $u$  is a weak solution of the degenerate Cahn-Hilliard equation (4.1.5)-(4.1.6) as in Definition 4.1.5.

*Remark 4.1.9.* Note that by Lemma 4.4.7 condition  $E(u^0) < \infty$  implies that  $E_\varepsilon(u^0) < \infty$ .

## 4.1.2 Important components of the proof.

There are three main ingredients of the proof.

— Compactness for the system 4.1.3–4.1.4 is obtained from the energy  $E_\varepsilon$  and entropy  $\Phi$

$$E_\varepsilon[u] := \int_{\mathbb{T}^d} F(u) dx + \frac{1}{4\varepsilon^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |u(x) - u(x-y)|^2 dx dy, \quad (4.1.9)$$

$$\Phi[u] := \int_{\mathbb{T}^d} u(\log(u) - 1) + 1 dx \quad (4.1.10)$$

Their dissipation is formally controlled by the identities

$$E_\varepsilon[u](t) + \int_0^t \int_{\mathbb{T}^d} u |\nabla \mu_\varepsilon|^2 \leq E_\varepsilon[u^0], \quad (4.1.11)$$

$$\Phi[u](t) + \frac{1}{2\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |\nabla u(x) - \nabla u(x-y)|^2 + \int_{\mathbb{T}^d} F''(u) |\nabla u|^2 \leq \Phi[u^0]. \quad (4.1.12)$$

According to the result of Bourgain-Brézis-Mironescu [53] which was improved later by Ponce [294], uniform bounds from (4.1.11), (4.1.12) together with Lions-Aubin lemma, yields strong convergence of  $\{u_\varepsilon\}$  and  $\{\nabla u_\varepsilon\}$  to  $u, \nabla u$  in  $L^2((0, T) \times \mathbb{T}^d)$ . We note that in the limit  $\varepsilon \rightarrow 0$ , the energy  $E_\varepsilon[u_\varepsilon]$  satisfy (see [53, Theorem 4] and [294, Theorem 1.2])

$$E[u] = \int_{\mathbb{T}^d} F(u) dx + \frac{C(d)}{2} \int_{\mathbb{T}^d} |\nabla u(x)|^2 dx \leq \liminf_{\varepsilon \rightarrow 0} E_\varepsilon[u_\varepsilon] \quad (4.1.13)$$

for some constant  $C(d)$  depending only on the dimension  $d$ . Similarly, for the nonlocal term in the dissipation of the entropy we have

$$C(d) \sum_{i,j=1}^d \int_0^t \int_{\mathbb{T}^d} |\partial_{x_i} \partial_{x_j} u|^2 \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(x-y)|^2$$

so in the limit  $\varepsilon \rightarrow 0$  we gain one more derivative. We also point out that one can prove rigorously that (4.1.5)–(4.1.6) is a gradient flow of (4.1.13) [71, 270].

- In passing to the limit, we exploit the appropriate definition of weak solutions to (4.1.5)–(4.1.6). Indeed, first we prove convergence to the formulation

$$\begin{aligned} \int_0^T \langle \partial_t u, \varphi \rangle_{(W^{-1,s'}(\mathbb{T}^d), W^{1,s}(\mathbb{T}^d))} &= D \int_0^T \int_{\mathbb{T}^d} (\nabla u \otimes \nabla u) : D^2 \varphi + \\ &+ \frac{D}{2} \int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta \varphi + D \int_0^T \int_{\mathbb{T}^d} u \nabla u \cdot \nabla \Delta \varphi - \int_0^T \int_{\mathbb{T}^d} u F''(u) \nabla u \cdot \nabla \varphi. \end{aligned}$$

Formally, it is obtained by integrating by parts twice using the formula

$$\nabla u \Delta u = \operatorname{div}(\nabla u \otimes \nabla u) - \frac{1}{2} \nabla |\nabla u|^2. \quad (4.1.14)$$

Its main advantage is that it exploits at most first-order derivatives so that we do not need any estimates on the second-order derivatives. This is important as they are not available for nonlocal degenerate Cahn-Hilliard. More precisely, the main difficulty is non-degeneracy of (4.1.3)–(4.1.4), that is we loose estimates on  $\nabla \mu_\varepsilon$  whenever  $u_\varepsilon$  is approaching the zone  $\{u_\varepsilon = 0\}$ . For the non-degenerate equation studied in [273, 121, 123, 122],

$$\partial_t u_\varepsilon = \operatorname{div} \nabla \mu_\varepsilon, \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (4.1.15)$$

$$\mu_\varepsilon = B_\varepsilon[u_\varepsilon] + F'(u_\varepsilon), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (4.1.16)$$

one obtains immediately an estimate on  $\nabla \mu_\varepsilon$  (by multiplying by  $\mu_\varepsilon$ ) and then one can identify its limit. Nevertheless, we point out that in [273, 121, 123, 122] the difficulty is rather the regularity of the potential and the kernel which we do not address in our work, assuming that  $F$  and  $\omega$  are sufficiently smooth.

- For the nonlocal Laplacian operator given by  $B_\varepsilon$  defined in (4.1.1), we find an operator  $S_\varepsilon$  given in (4.3.4) which resembles gradient operator. It satisfies the integration by parts formula (S3) in Lemma 4.3.4 as well as the product rule (S2) in Lemma 4.3.4 with an error that vanishes when  $\varepsilon \rightarrow 0$ . This is necessary to perform usual calculus operations before sending  $\varepsilon \rightarrow 0$ , that is when we do not have Laplace operator in the equation.

### 4.1.3 Literature review and relevancy of the system

**The Cahn Hilliard equation** The equation represents a mathematical model which is widely used to describe phase transitions in fluids and living tissues. In biology, the equation can model the morphological evolution of a growing solid tumor. There are many factors that come into play when approaching a tumor growth model, e.g. cell-cell and cell-matrix adhesion, as well as cell motility and mechanical stress. Nevertheless, the mathematical study and implementation of a model may provide very interesting information on tumor progression. Let us consider a bounded, open tissue domain, in which a tumor is evolving. One possible approach is to describe the tumoral and healthy tissues with volume fractions. By remarking that internal adhesive forces tend to bind the tumor cells together, we should take into account that phase separation may occur between healthy and tumoral tissue domains. As a result a boundary layer of finite thickness  $\sqrt{D}$ , where  $D$  is defined below, may form between them.

Being of fourth-order, the (local) Cahn-Hilliard equation is often rewritten in a system of two

second-order equations, *i.e.*

$$\partial_t u = \operatorname{div}(m(u)\nabla(F'(u) - D\Delta u)) \rightarrow \begin{cases} \partial_t u &= \operatorname{div}(m(u)\nabla\mu), \\ \mu &= -D\Delta u + F'(u), \end{cases} \quad (4.1.17)$$

where  $u$  is the concentration of a phase and  $\mu$  is called the chemical potential in material sciences but is often used as an effective pressure for living tissues [130, 151]. Also, the interaction potential  $F(u)$  contained in this effective pressure term comprises the effects of attraction and repulsion between cells. The physically relevant form of this potential is a double-well logarithmic potential and is often approximated by a smooth polynomial function. However, recent studies show that for the modeling of living tissues and for the particular application where only one of the components of the mixture experiences attractive and repulsive forces, a single-well logarithmic potential is more relevant [67].

The existence and uniqueness of solutions for the Cahn-Hilliard system (4.1.17) strictly depends on the properties of the mobility term  $m(u)$  and the potential  $F(u)$ , as well as the conditions assigned on the boundary. More specifically, the presence of *degeneration* on the mobility, *i.e.* the possibility for it to vanish, can turn the analysis of solutions into a rather complex problem.

**From nonlocal to local Cahn-Hilliard** The nonlocal Cahn-Hilliard equation was first obtained by Giacomini and Lebowitz [189, 190] by starting from a microscopic description. The model is a  $d$ -dimensional lattice gas evolving via Kawasaki exchange dynamics, which is a Poisson nearest neighbor exchange process. In the hydrodynamic limit, they find that the empirical average of the occupation numbers over a small macroscopic volume element tends to a solution of a nonlocal Cahn-Hilliard equation. This latter equation is an approximation of the local Cahn-Hilliard equation, as shown in Theorem 4.1.8. Let us also remark that there are possibly different variants of non-local Cahn-Hilliard equation, see for instance [150] where a version of nonlocal Cahn-Hilliard equation is derived starting from a kinetic description inspired by [317]. The literature concerning the nonlocal Cahn-Hilliard equation is quite well developed and we refer for instance to [181, 183, 14, 241, 298, 103] and [178] for the cases of non-degenerate and degenerate mobilities respectively. On the other hand, for the passage to the limit of the nonlocal Cahn-Hilliard equation to the local Cahn-Hilliard equation, the existing results [273, 121, 123, 122, 274] cover only the case of constant mobility. In [273] the authors prove the convergence on the torus. In [121] a wide class of potentials is considered and the study is made on the torus. In [123] the convergence is obtained in the case of a bounded domain with Neumann boundary conditions and a viscosity term. Finally, in [122], the limit is achieved with a  $W^{1,1}$  kernel and a wide class of singular potentials, with Neumann boundary conditions.

#### 4.1.4 Open problems

We discuss below two problems which we leave open.

Open problem concerning bounded domains. One can ask if the same results hold when  $\mathbb{T}^d$  is replaced with some general bounded domain  $\Omega$ . More precisely, we focus on the system

$$\partial_t u_\varepsilon = \operatorname{div}(u_\varepsilon \nabla \mu_\varepsilon), \quad \text{in } (0, +\infty) \times \Omega, \quad (4.1.18)$$

$$\mu_\varepsilon = B_\varepsilon[u_\varepsilon] + F'(u_\varepsilon), \quad \text{in } (0, +\infty) \times \Omega. \quad (4.1.19)$$

Defining  $\vec{n}$  the outward normal vector to  $\partial\Omega$  we impose the Neumann boundary condition

$$u_\varepsilon \frac{\partial \mu_\varepsilon}{\partial \vec{n}} = 0 \quad \text{on } \partial\Omega. \quad (4.1.20)$$

The operator  $B_\varepsilon$  satisfies

$$B_\varepsilon[u_\varepsilon](x) = \frac{1}{\varepsilon^2} \left( \int_\Omega \omega_\varepsilon(x-y) \, dy \, u_\varepsilon(x) - \omega_\varepsilon * u_\varepsilon(x) \right) = \frac{1}{\varepsilon^2} \int_\Omega \omega_\varepsilon(x-y) (u_\varepsilon(x) - u_\varepsilon(y)) \, dy. \quad (4.1.21)$$

Notice that in the case  $\Omega = \mathbb{T}^d$ , this definition is the same than (4.1.1) up to a change of variable in the integral. However, since  $u_\varepsilon$  is not a priori defined outside  $\Omega$  we need to put the argument  $(x-y)$  on  $\omega_\varepsilon$ .

In the limit, we expect to obtain solutions to

$$\partial_t u = \operatorname{div}(u \nabla \mu), \quad \text{in } (0, +\infty) \times \Omega, \quad (4.1.22)$$

$$\mu = -D\Delta u + F'(u), \quad \text{in } (0, +\infty) \times \Omega \quad (4.1.23)$$

$$\frac{\partial u}{\partial \vec{n}} = u \frac{\partial \mu}{\partial \vec{n}} = 0, \quad \text{on } \partial\Omega. \quad (4.1.24)$$

However, there are two difficult problems related to the equation posed on a bounded domain.

- **Lack of the entropy estimate.** In the case of bounded domain, we cannot use entropy estimate as in (4.1.12). This is because the nonlocal operator is defined as (4.1.21) rather than (4.1.1). As a consequence, we cannot symmetrize the expression with gradients and obtain the term  $\frac{1}{2\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |\nabla u(x) - \nabla u(x-y)|^2$  in the dissipation of the entropy.
- **Recovery of the Neumann boundary conditions.** The question is whether we can prove that in the limit  $\frac{\partial u}{\partial \vec{n}} = 0$  on  $\partial\Omega$ . This is possible for the equation with constant mobility. More precisely, in [122], Authors were discussing the problem of nonlocal to local convergence for the Cahn-Hilliard equation with constant mobility. The constant mobility allows to obtain uniform bound on  $\|B_\varepsilon(u_\varepsilon)\|_2$  which allows to conclude that  $\frac{\partial u}{\partial \vec{n}} = 0$  on  $\partial\Omega$ . This is an extremely interesting phenomenon as this new boundary condition appears only in the limit. In our case, the estimate  $\|B_\varepsilon(u_\varepsilon)\|_2$  seems unavailable.

A possible approach to overcome this problem is to apply Serfaty-Sandier approach on the convergence of gradient flows [304, 308].

Open problem concerning irregular potentials and general mobilities. One could also wonder about the case of more irregular potentials like a logarithmic potential  $F(u) = (1-u) \log(1-u) + u \log(u) - \frac{\theta}{2}(u - \frac{1}{2})^2 + k$ . Our assumptions on  $F$  do not cover this case. The problem is open already at the level of existence of weak solutions as the degenerate mobility  $u$  cannot compensate the blow-up of  $F''(u)$  at  $u = 1$ , as it was for instance used in [157]. Also, we cannot use the same reasoning as in [121] because it is heavily based on the fact that the considered mobility is constant which allows to obtain estimates on the chemical potential  $\mu$ . Nevertheless, it seems that one could prove the same results for the logarithmic potential as above by considering the mobility  $m(u) = u(1-u)$  which allows to compensate the blow-up of  $F''$  at  $u = 1$ . Such generalizations will be the topic of future works.

## 4.2 Existence of weak solutions to the nonlocal problem

The existence of weak solutions for the local Cahn-Hilliard equation with degenerate mobility usually follows the method from [157]. The idea is to apply a Galerkin scheme with a non-degenerate regularized mobility, *i.e.*, calling  $m(n)$  the mobility, then one considers an approximation  $m_\varepsilon(n) \geq \varepsilon$ . Finally, using standard compactness methods one can prove the existence of weak solutions for the initial system. The uniqueness of the weak solutions is still an open question.

In the case of the nonlocal Cahn-Hilliard equation, we have to rely on a fixed point method. We first consider a nondegenerate mobility, and the fixed point argument is put on the nonlocality. Then, we pass to the limit to obtain the nonlocal Cahn-Hilliard equation with degenerate mobility.

For clarity, we recall the definition of Hölder spaces which come naturally when trying to prove existence of parabolic equations.

**Notation 4.2.1.** We write  $C^{k+\alpha, k'+\beta}(\bar{\Omega}_T)$  for the space of  $k$  (resp.  $k'$ ) times continuously differentiable functions in time (resp. in space) with partial derivative of order  $k$  (resp.  $k'$ ) which is Hölder continuous with exponent  $\alpha$  (resp.  $\beta$ ).

### 4.2.1 Approximating solutions

Following the scheme above, we first consider a nondegenerate mobility and prove the existence of solutions to the following system

$$\partial_t u_\delta = \operatorname{div}(T_\delta(u_\delta)\nabla\mu_\delta) \quad (0, +\infty) \times \mathbb{T}^d, \quad (4.2.1)$$

$$\mu_\delta = B_\varepsilon[u_\delta] + F'(u_\delta) \quad (0, +\infty) \times \mathbb{T}^d, \quad (4.2.2)$$

equipped with smooth initial condition  $u_\delta^0(x) = u^0 * \omega_\delta(x)$  which is a smooth approximation of  $u^0$  as  $\delta \rightarrow 0$ . Here,  $\delta > 0$  is a small parameter such that  $2\delta < \frac{1}{\delta} - 1$ ,  $\delta < \frac{1}{4}$  and

$$T_\delta(u) = \begin{cases} \delta & \text{for } u \leq \delta, \\ \text{smooth monotone interpolation} & \text{for } u \in [\delta, 2\delta], \\ u & \text{for } u \in [2\delta, \frac{1}{\delta} - 1], \\ \text{smooth monotone interpolation} & \text{for } u \in [\frac{1}{\delta} - 1, \frac{1}{\delta}], \\ \frac{1}{\delta} & \text{for } u \geq \frac{1}{\delta}. \end{cases} \quad (4.2.3)$$

The estimates for the sequence  $\{u_\delta\}$  will be obtained from the dissipation of energy and entropy. The definition of energy  $E_\varepsilon$  remains the same as in (4.1.11). However, the definition of entropy has to be adapted to take into account the fact that we don't know if the solution remains nonnegative. To this end, we define a function  $\phi_\delta$  by an explicit formula

$$\phi_\delta(x) = \int_1^x \int_1^y \frac{1}{T_\delta(z)} \, dz \, dy. \quad (4.2.4)$$

**Lemma 4.2.2.** Let  $\phi_\delta$  be defined with (4.2.4) and  $\phi(x) = x(\log(x) - 1) + 1$ . Then,

(P1)  $\phi_\delta''(x) = \frac{1}{T_\delta(x)}$  and  $\phi_\delta(1) = \phi_\delta'(1) = 0$ ,

(P2)  $\phi_\delta(x) \rightarrow \phi(x)$  for  $x \geq 0$  as  $\delta \rightarrow 0$ ,

(P3)  $\phi_\delta(x) \geq 0$  for all  $x \in \mathbb{R}$ ,

(P4)  $\phi_\delta(x) \leq \phi(x) + \frac{\delta}{2(1-\delta)}x^2 + 3$  for  $x \geq 0$ ,

(P5)  $\phi_\delta(x) \rightarrow \infty$  when  $\delta \rightarrow 0$  for all  $x < 0$ .

The proof is presented in Appendix 4.4.4.

**Theorem 4.2.3.** *Let  $\delta > 0$ ,  $\varepsilon_0$  be as in (4.1.8) and  $F \in C^4$ . For  $\varepsilon < \varepsilon_0$  there exists classical solution (4.2.1)–(4.2.2). Moreover, they satisfy the mass, energy, and entropy conservation : for all  $t > 0$*

$$\int_{\mathbb{T}^d} u_\delta(t, \cdot) dx = \int_{\mathbb{T}^d} u_\delta^0 dx, \quad (4.2.5)$$

$$E_\varepsilon[u_\delta](t) + \int_0^t \int_{\mathbb{T}^d} T_\delta(u_\delta) |\nabla \mu_\delta|^2 = E_\varepsilon[u_\delta^0], \quad (4.2.6)$$

$$\Phi_\delta[u_\delta](t) + \frac{1}{2\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |\nabla u_\delta(x) - \nabla u_\delta(x-y)|^2 + \int_0^t \int_{\mathbb{T}^d} F''(u_\delta) |\nabla u_\delta|^2 = \Phi_\delta[u_\delta^0]. \quad (4.2.7)$$

**Theorem 4.2.4.** *Let  $\varepsilon_0$  be as in (4.1.8). Let  $F$  satisfy Assumption 4.1.1 with an additional constraint  $2C_{10} < C_p$ . Then, the following sequences are bounded uniformly in  $\delta \in (0, \frac{1}{2})$  and  $\varepsilon \in (0, \varepsilon_0)$*

(A1)  $\{u_\delta\}_\delta$  in  $L^\infty(0, T; L^k(\mathbb{T}^d))$ ,

(A2)  $\{u_\delta\}_\delta$  in  $L^k(0, T; L^{\frac{k}{d-2}}(\mathbb{T}^d))$ ,

(A3)  $\{\sqrt{T_\delta(u)} \nabla \mu_\delta\}_\delta$  in  $L^2((0, T) \times \mathbb{T}^d)$ ,

(A4)  $\{\nabla u_\delta\}_\delta$  in  $L^2((0, T) \times \mathbb{T}^d)$ ,

(A5)  $\{\partial_t u_\delta\}_\delta$  in  $L^2(0, T; W^{-1, s'}(\mathbb{T}^d))$ ,

(A6)  $\{\partial_t \nabla u_\delta\}_\delta$  in  $L^2(0, T; W^{-2, s'}(\mathbb{T}^d))$ ,

(A7)  $\{\sqrt{F_1''(u_\delta)} \nabla u_\delta\}_\delta$  in  $L^2((0, T) \times \mathbb{T}^d)$ ,

(A8)  $\Phi_\delta[u_\delta]$  in  $L^\infty(0, T)$ ,

where  $k$  and  $s$  have been defined in Notation 4.1.3.

To prove Theorem 4.2.3, we need to assume that  $F \in C^4$  which allows us to use known results about classical solutions to uniformly parabolic equations.

*Proof of Theorem 4.2.3.* As  $\delta > 0$  is fixed in this result, we write  $u$  instead of  $u_\delta$ . Given  $w$  we consider an auxiliary equation

$$\partial_t u = \operatorname{div} \left( T_\delta(u) \nabla u \left( \frac{1}{\varepsilon^2} + F''(u) \right) \right) - \operatorname{div} \left( T_\delta(u) \frac{w * \nabla \omega_\varepsilon}{\varepsilon^2} \right). \quad (4.2.8)$$

Let  $\alpha, \sigma, M, \kappa$  be parameters to be specified later. We want to apply Schauder fixed point theorem to the map

$$\begin{aligned} P & : X \rightarrow X \\ P & : w \mapsto u \text{ solution of (4.2.8),} \end{aligned}$$

where  $X$  is defined as the set

$$X = \{w \in C^{\alpha/2, \alpha}([0, T] \times \mathbb{T}^d), \|w\|_{\infty, \sigma} \leq M\}$$

with the norm

$$\|w\|_X := \|w\|_{\infty, \sigma} + \kappa \|w\|_{\alpha/2, \alpha}$$

and the norm  $\|\cdot\|_{\alpha/2, \alpha}$  is the usual Hölder seminorm of functions being Hölder continuous with exponent  $\alpha/2$  with respect to time and  $\alpha$  with respect to space . We also define

$$\|w\|_{\infty, \sigma} := \sup_{[0, T] \times \mathbb{T}^d} |u(t, x)| e^{-\sigma t}. \quad (4.2.9)$$

Note that the new norm is equivalent to the usual supremum norm so all topological properties do not change. We need to prove that  $P$  is continuous,  $P$  maps in fact  $X$  to  $X$ , and that  $P(X)$  is relatively compact in  $X$ . First, we prove that  $P(w) = u$  is the unique classical solution of equation (4.2.8) so that  $P$  is well defined and find Hölder estimates which will be useful to prove the continuity of the operator as well as its relative compactness.

Step 1 :  $P$  is well defined and Hölder estimates. Equation (4.2.8) is equivalent to saying that  $u$  solves parabolic equation

$$\partial_t u = \operatorname{div} A(t, x, u, \nabla u) + B(t, x, u, \nabla u), \quad (u)_{\mathbb{T}^d} = (u_\delta^0)_{\mathbb{T}^d}$$

with

$$A(t, x, z, p) = T_\delta(z) p \left( \frac{1}{\varepsilon^2} + F''(z) \right), \quad B(t, x, z, p) = -T'_\delta(z) p \cdot \frac{w * \nabla \omega_\varepsilon}{\varepsilon^2} - T_\delta(z) \frac{w * \Delta \omega_\varepsilon}{\varepsilon^2},$$

and we recall that  $w \in X$  is Hölder continuous. The function  $A$  satisfies the strong parabolicity condition for sufficiently small  $\varepsilon > 0$ , i.e.

$$A(t, x, z, p) \cdot p \geq \delta p^2 \frac{1}{2\varepsilon_0^2}$$

for all  $\varepsilon < \varepsilon_0$  (this uses Assumptions (B), (C) and (4.1.8)). Since the derivatives  $A_p, A_z, A_t, A_x$  and function  $B$  are Hölder continuous as functions of  $(t, x, z, p)$ , [253, Theorems 12.10, 12.14] asserts that there exists a unique classical solution to (4.2.8). Moreover, this solution satisfies the estimate

$$\|u\|_{C^{1+\alpha/2, 1+\alpha}} \leq C(\delta, \varepsilon_0, \|w\|_{C^{\alpha/2, \alpha}}).$$

With this estimate, (4.2.8) can be considered as a linear equation so that the linear theory for parabolic equations [253, Theorem 5.14] implies

$$\|u\|_{C^{1+\alpha/2, 2+\alpha}} \leq C(\delta, \varepsilon_0, \|w\|_{C^{\alpha/2, \alpha}}). \quad (4.2.10)$$

Therefore  $u$  is a classical solution of (4.2.8) and it admits the Hölder bound (4.2.10).

Step 2 : The operator  $P$  is continuous. We consider a sequence  $\{w_n\}_n$  in  $X$  such that  $\|w_n - w\|_X \rightarrow 0$ . Then  $u_n = P(w_n)$  is compact in  $C^{1,2}$  from estimate (4.2.10) and Arzelà-Ascoli's Theorem. We choose subsequence such that  $u_{n_k} \rightarrow u$  in  $C^{1,2}$ . These functions satisfy

$$\partial_t u_{n_k} = \operatorname{div} \left( T_\delta(u_{n_k}) \nabla u_{n_k} \left( \frac{1}{\varepsilon^2} + F''(u_{n_k}) \right) \right) - \operatorname{div} \left( T_\delta(u_{n_k}) \frac{w_{n_k} * \nabla \omega_\varepsilon}{\varepsilon^2} \right). \quad (4.2.11)$$

Passing to the limit in (4.2.11) and using uniqueness of solutions to (4.2.8) from [253], we obtain that for every subsequence of  $\{u_n\}_n$  we can extract a subsequence which converges to a unique limit  $u = P(w)$ . By a standard subsequence argument, this means that the whole sequence  $\{u_n\}_n$  converges to  $u = P(w)$ . Therefore  $P$  is continuous.

Step 3 :  $P$  maps  $X$  to  $X$ . We write the equation (4.2.8) in the form

$$\begin{aligned} \partial_t u &= T'_\delta(u) |\nabla u|^2 \left( \frac{1}{\varepsilon^2} + F''(u) \right) + T_\delta(u) \Delta u \left( \frac{1}{\varepsilon^2} + F''(u) \right) + T_\delta(u) |\nabla u|^2 F^{(3)}(u) \\ &\quad - T'_\delta(u) \nabla u \cdot \frac{w * \nabla \omega_\varepsilon}{\varepsilon^2} - T_\delta(u) \frac{w * \Delta \omega_\varepsilon}{\varepsilon^2}. \end{aligned}$$

We substitute  $u = v e^{\sigma t}$  and we compute PDE satisfied by  $v$  :

$$\begin{aligned} \partial_t v e^{\sigma t} + v \sigma e^{\sigma t} &= T'_\delta(u) |\nabla v|^2 \left( \frac{1}{\varepsilon^2} + F''(u) \right) e^{2\sigma t} + T_\delta(u) \Delta v \left( \frac{1}{\varepsilon^2} + F''(u) \right) e^{\sigma t} \\ &\quad + T_\delta(u) |\nabla v|^2 F^{(3)}(u) e^{2\sigma t} - T'_\delta(u) \nabla v \cdot \frac{w * \nabla \omega_\varepsilon}{\varepsilon^2} e^{\sigma t} - T_\delta(u) \frac{w * \Delta \omega_\varepsilon}{\varepsilon^2}. \end{aligned} \quad (4.2.12)$$

Now, we multiply by  $v$  and we consider the function  $v^2$ . Let  $(t_*, x_*) \in [0, T] \times \mathbb{T}^d$  be the point where  $v^2$  attains its maximum. First, we consider the case  $t_* > 0$ . Obviously,  $\nabla v^2(t_*, x_*) = 0$  (note that  $\mathbb{T}^d$  does not have a boundary). In particular,  $|\nabla v(t_*, x_*)| = 0$  as  $|\nabla v|^2 v = \nabla v \cdot \nabla v^2 / 2$  and we can assume that  $v(t_*, x_*) \neq 0$  (otherwise  $v$  is constantly 0). Equation (4.2.12) evaluated at  $(t_*, x_*)$  reads

$$\frac{1}{2} \partial_t v^2 e^{\sigma t_*} + v^2 \sigma e^{\sigma t_*} = T_\delta(u) v \Delta v \left( \frac{1}{\varepsilon^2} + F''(u) \right) e^{\sigma t_*} - v T_\delta(u) \frac{w * \Delta \omega_\varepsilon}{\varepsilon^2}. \quad (4.2.13)$$

Moreover, as  $v^2$  attains its maximum in  $(t_*, x_*)$  and  $t_* > 0$ ,  $\partial_t v^2(t_*, x_*) \geq 0$  and  $\Delta v^2(t_*, x_*) \leq 0$ . In particular, at  $(t_*, x_*)$  we have  $v \Delta v = -|\nabla v|^2 + \frac{1}{2} \Delta v^2 \leq 0$  so that from (4.2.13) we obtain

$$v^2 \sigma e^{\sigma t_*} \leq -v T_\delta(u) \frac{w * \Delta \omega_\varepsilon}{\varepsilon^2},$$

so that

$$v^2(t_*, x_*) \sigma e^{\sigma t_*} \leq |v(t_*, x_*)| \frac{\|\Delta \omega_\varepsilon\|_1 \|w(t_*, \cdot)\|_\infty}{\delta \varepsilon^2}.$$

where we used the definition of  $T_\delta$ . As  $v^2$  attains maximum at  $(t^*, x^*)$ ,  $|v(t_*, x_*)|$  also attains maximum at  $(t^*, x^*)$ . Therefore, taking into account the possibility that  $t^* = 0$

$$\|v\|_\infty \leq \max \left( \frac{\|\Delta \omega_\varepsilon\|_1 \|w(t_*, \cdot)\|_\infty}{\delta \varepsilon^2 \sigma} e^{-\sigma t_*}, \|u_\delta^0\|_\infty \right) \leq \max \left( \frac{\|\Delta \omega_\varepsilon\|_1 \|w e^{-\sigma t}\|_\infty}{\delta \varepsilon^2 \sigma}, \|u_\delta^0\|_\infty \right).$$

Choosing  $\sigma = 2\|\Delta \omega_\varepsilon\|_1 / (\delta \varepsilon^2)$ , we obtain estimate

$$\|v\|_\infty \leq \max \left( \frac{1}{2} \|w e^{-\sigma t}\|_\infty, \|u_\delta^0\|_\infty \right).$$

By definition of the norm

$$\|Pw\|_{\infty, \sigma} \leq \max \left( \frac{1}{2} \|w\|_{\infty, \sigma}, \|u_\delta^0\|_\infty \right). \quad (4.2.14)$$

Moreover, the parabolic version of de Giorgi-Nash-Moser theory, see [246, Chap. V, Theorem



1.1], implies that there exists  $\alpha = \alpha(\|w\|_{\infty, \sigma})$  such that the solution of (4.2.8) satisfy

$$\|u\|_{C^{\alpha/2, \alpha}} \leq f(\|w\|_{\infty, \sigma}),$$

where  $f(\|w\|_{\infty, \sigma})$  is a constant which depends on  $\|w\|_{\infty, \sigma}$ . Without loss of generality we may assume that  $f(\|w\|_{\infty, \sigma})$  does not decrease and  $\alpha(\|w\|_{\infty, \sigma})$  does not increase when  $\|w\|_{\infty, \sigma}$  increases.

We proceed to choosing values of parameters  $M$ ,  $\alpha$ ,  $\kappa$  and concluding the proof. We choose

$$M = 3\|u_\delta^0\|_{L^\infty}, \quad \alpha = \alpha(M), \quad \kappa = \frac{\|u_\delta^0\|_{L^\infty}}{2f(M)}.$$

Since  $w$  is in  $X$  and  $f$  is nondecreasing we obtain

$$\|Pw\|_X \leq \frac{1}{2}\|w\|_{\infty, \sigma} + \|u_\delta^0\|_\infty + \kappa f(\|w\|_{\infty, \sigma}) \leq \frac{M}{2} + \|u_\delta^0\|_\infty + \kappa f(M) \leq 3\|u_\delta^0\|_{L^\infty} = M.$$

This means that  $P$  maps  $X$  to  $X$ .

*Step 4 :  $P(X)$  is relatively compact in  $X$ .* The relative compactness of  $P(X)$  follows from (4.2.10). The proof is concluded.  $\square$

*Proof of Theorem 4.2.4.* First, we prove that  $E_\varepsilon[u_\delta^0]$  and  $\Phi_\delta[u_\delta^0]$  are uniformly bounded (with respect to  $\delta$ ) in terms of  $E_\varepsilon[u^0]$  and  $\Phi[u^0]$ . Recall that  $u_\delta^0 = u^0 * \omega_\delta$ . Concerning the energy, this is the consequence of the convexity and the uniform bound on  $F_2$ . Indeed, we have

$$\begin{aligned} & \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |u_\delta^0(x) - u_\delta^0(x-y)|^2 dx dy = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) \left| \int_{\mathbb{T}^d} (u^0(x-z) - u^0(x-y-z)) \omega_\delta(z) dz \right|^2 dx dy \\ & \leq \int_{(\mathbb{T}^d)^3} \omega_\varepsilon(y) |u^0(x-z) - u^0(x-y-z)|^2 \omega_\delta(z) dz dx dy = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |u^0(x) - u^0(x-y)|^2 dx dy \end{aligned}$$

and similarly for the potential terms

$$\int_{\mathbb{T}^d} F_1(u^0 * \omega_\delta) dx \leq \int_{\mathbb{T}^d} F_1(u^0) * \omega_\delta dx = \int_{\mathbb{T}^d} F_1(u^0) dx, \quad \int_{\mathbb{T}^d} |F_2(u^0)| dx \leq \|F_2\|_\infty |\mathbb{T}^d|.$$

Concerning the entropy, as  $\phi_\delta$  is convex, we have for the term  $\Phi_\delta[u_\delta^0]$  :

$$\begin{aligned} \Phi_\delta[u_\delta^0] &= \int_{\mathbb{T}^d} \phi_\delta(u^0 * \omega_\delta) dx \leq \int_{\mathbb{T}^d} \phi_\delta(u^0) * \omega_\delta dx = \int_{\mathbb{T}^d} \phi_\delta(u^0) dx \leq \\ & \leq \int_{\mathbb{T}^d} (\phi(u^0) + \frac{\delta}{2(1-\delta)} |u^0|^2 + 3) dx \leq \Phi[u^0] + \frac{1}{2} \|u^0\|_2^2 + 3|\mathbb{T}^d|, \end{aligned}$$

where we used (P4) in Lemma 4.2.2. The norm  $\|u^0\|_2^2$  can be controlled by the nonlocal term appearing in  $E_\varepsilon[u^0]$  due to Lemma 4.4.6 (note that  $u^0$  belongs at least to  $L^2(\mathbb{T}^d)$  as  $F_1(u^0)$  is integrable thanks to  $E_\varepsilon[u^0] < \infty$ ).

Now, we proceed to the main part of the proof. To prove (A1) and (A3) we want to apply (4.2.6) and Assumption 4.1.1 on the potential. The energy identity reads :

$$\int_{\mathbb{T}^d} F(u_\delta) dx + \frac{1}{4\varepsilon^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |u_\delta(x) - u_\delta(x-y)|^2 dx dy + \int_0^t \int_{\mathbb{T}^d} T_\delta(u_\delta) |\nabla \mu_\delta|^2 = E_\varepsilon[u_\delta^0],$$

Applying Lemma 4.4.6, we deduce

$$\int_{\mathbb{T}^d} F(u_\delta) dx + C_p \int_{\mathbb{T}^d} |u - (u)_{\mathbb{T}^d}|^2 + \int_0^t \int_{\mathbb{T}^d} T_\delta(u_\delta) |\nabla \mu_\delta|^2 \leq E_\varepsilon[u_\delta^0]$$

Splitting  $F = F_1 + F_2$  and applying Assumption 4.1.1 we obtain

$$\int_{\mathbb{T}^d} F_1(u_\delta) dx + C_p \int_{\mathbb{T}^d} |u_\delta - (u_\delta)_{\mathbb{T}^d}|^2 + \int_0^t \int_{\mathbb{T}^d} T_\delta(u_\delta) |\nabla \mu_\delta|^2 \leq E_\varepsilon[u_\delta^0] + C_9 + C_{10} \int_{\mathbb{T}^d} |u_\delta|^2$$

Note that by conservation of mass,  $(u_\delta)_{\mathbb{T}^d} = (u^0)_{\mathbb{T}^d}$ . Therefore, applying the simple inequality  $|a + b|^2 \leq 2|a|^2 + 2|b|^2$  and  $C_p > 2C_{10}$ , we obtain an  $L^\infty(0, T; L^2(\mathbb{T}^d))$  estimate on  $\{u_\delta\}$  which can be improved to  $L^\infty(0, T; L^k(\mathbb{T}^d))$  if  $F_1 \neq 0$  cf. (B) in Assumption 4.1.1. Then, (A1) and so, (A3) is easily implied by the energy as all possibly negative terms are bounded.

Now, to prove (A4) we want to use the entropy equality (4.2.7) :

$$\Phi_\delta[u_\delta](t) + \frac{1}{2\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |\nabla u_\delta(x) - \nabla u_\delta(x - y)|^2 + \int_0^t \int_{\mathbb{T}^d} F''(u_\delta) |\nabla u_\delta|^2 = \Phi_\delta[u_\delta^0].$$

To exploit it, for  $\gamma$  to be chosen later,  $\varepsilon \in (0, \tilde{\varepsilon}_0(\gamma))$  we have by Lemma 4.4.8

$$\begin{aligned} \Phi_\delta[u_\delta](t) + \frac{1}{\gamma} \int_0^t \int_{\mathbb{T}^d} |\nabla u_\delta|^2 + \int_0^t \int_{\mathbb{T}^d} F_1''(u_\delta) |\nabla u_\delta|^2 &\leq \\ &\leq \Phi_\delta[u_\delta^0] + C(\gamma) \int_0^t \int_{\mathbb{T}^d} \|u_\delta\|_{L^2(\mathbb{T}^d)}^2 + \|F_2''\|_\infty \int_0^t \int_{\mathbb{T}^d} |\nabla u_\delta|^2. \end{aligned}$$

We choose  $\gamma = \frac{1}{1 + \|F_2''\|_\infty}$  which yields estimates (A4), (A7) and (A8). Now, to see (A5) we take a smooth test function  $\varphi$  and write thanks to the Hölder inequality

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{T}^d} \partial_t u_\delta \varphi dx dt \right| &= \left| \int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta)^{1/2} T_\delta(u_\delta)^{1/2} \nabla \mu_\delta \cdot \nabla \varphi dx dt \right| \\ &\leq \|T_\delta(u_\delta)^{1/2}\|_{L^\infty(0, T; L^{2k}(\mathbb{T}^d))} \|T_\delta(u_\delta)^{1/2} \nabla \mu_\delta\|_{L^2((0, T) \times \mathbb{T}^d)} \|\nabla \varphi\|_{L^2(0, T; L^s(\mathbb{T}^d))} \\ &\leq C \|\nabla \varphi\|_{L^2(0, T; L^s(\mathbb{T}^d))}. \end{aligned}$$

In the last line we used estimates (A1), (A3) and the definition of  $T_\delta$ . This concludes the proof for estimates (A5) and then (A6) easily follows.

Finally, we prove (A2). We note from (A7) that  $\{\nabla u_\delta^{k/2}\}$  is bounded in  $L^2(0, T; L^2(\mathbb{T}^d))$  and from (A1) that  $\{u_\delta^{k/2}\}$  is bounded in  $L^\infty(0, T; L^2(\mathbb{T}^d))$ . Therefore, by Sobolev embedding, we obtain that  $\{u_\delta^{k/2}\}$  is bounded in  $L^2(0, T; L^{\frac{2d}{d-2}}(\mathbb{T}^d))$  so that  $\{u_\delta\}$  is bounded in  $L^k(0, T; L^{\frac{d}{d-2}}(\mathbb{T}^d))$ .  $\square$

## 4.2.2 Proof of Theorem 4.1.7

*Proof of Theorem 4.1.7. Step 1 : Approximation of the potential.*

For  $F$  as in Assumption 4.1.1, we consider its mollification  $\bar{F}_\delta = F * \eta_\delta$  where  $\{\eta_\delta\}$  is the usual mollifier. We note that  $\bar{F}_\delta$  is  $C^4$  and that  $F, \bar{F}_\delta$  satisfy Assumption 4.1.1 with comparable constants  $C_1, \dots, C_{10}$ , see Lemma 4.4.2. The most important is constant  $C_{10}$  because there is a

constraint on it in terms of  $C_p$ . More precisely,  $F$  satisfies Assumption 4.1.1 with  $C_{10} < C_p/4$  so that from Lemma 4.4.2 we have that  $F_\delta$  satisfies it with  $2C_{10} < C_p/2$ . This allows to apply Theorem 4.2.4 to obtain uniform estimates. Moreover  $F_\delta = F_{\delta,1} + F_{\delta,2}$  with  $F_{\delta,(1,2)}^{(p)} \xrightarrow[\delta \rightarrow 0]{\text{pointwise}} F_{(1,2)}^{(p)}$  where  $p = 0, 1, 2$  is the order of derivative.

*Step 2 : Compactness.* Using Theorem 4.2.3, we can obtain  $u_\delta$  such that for all  $\varphi \in L^\infty(0, T; W^{1,\infty}(\mathbb{T}^d))$

$$\begin{aligned} & \int_0^T \langle \partial_t u_\delta, \varphi \rangle_{(W^{1,s}(\mathbb{T}^d))', W^{1,s}(\mathbb{T}^d)} + \\ & + \int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta) \nabla B_\varepsilon[u_\delta] \cdot \nabla \varphi + \int_0^T \int_{\mathbb{T}^d} u_\delta F_\delta''(u_\delta) \nabla u_\delta \cdot \nabla \varphi = 0. \end{aligned} \quad (4.2.15)$$

The plan is to send  $\delta \rightarrow 0$  in (4.2.15). By Theorem 4.2.4 and standard compactness results we can extract a subsequence (not relabelled) such that

$$(B1) \quad u_\delta \rightarrow u \text{ a.e. and in } L^2((0, T) \times \mathbb{T}^d) \cap L^k((0, T) \times \mathbb{T}^d),$$

$$(B2) \quad \nabla u_\delta \rightharpoonup \nabla u \text{ in } L^2((0, T) \times \mathbb{T}^d),$$

$$(B3) \quad \partial_t u_\delta \rightharpoonup \partial_t u \text{ in } L^2(0, T; W^{-1,s'}(\mathbb{T}^d)),$$

$$(B4) \quad \sqrt{F_{1,\delta}''(u_\delta)} \nabla u_\delta \rightharpoonup \xi \text{ in } L^2((0, T) \times \mathbb{T}^d) \text{ for some } \xi \in L^2((0, T) \times \mathbb{T}^d).$$

Only (B1) needs some justification. From (A1), (A4), (A5) and Aubin-Lions lemma, we obtain the strong convergence  $u_\delta \rightarrow u$  a.e. and in  $L^2((0, T) \times \mathbb{T}^d)$ . To see the second strong convergence, we interpolate between  $L^\infty(0, T; L^k(\mathbb{T}^d))$  and  $L^k(0, T; L^{k \frac{d}{d-2}}(\mathbb{T}^d))$  to prove that  $\{u_\delta\}$  is bounded in  $L^{k+\kappa}(0, T; L^{k+\kappa}(\mathbb{T}^d))$  for some  $\kappa > 0$  because  $k \frac{d}{d-2} > k$ . Now, interpolating between  $L^{k+\kappa}(0, T; L^{k+\kappa}(\mathbb{T}^d))$  and  $L^2((0, T) \times \mathbb{T}^d)$  we obtain strong convergence in  $L^k((0, T) \times \mathbb{T}^d)$ .

*Step 3 : Nonnegativity of  $u$ .* The plan is to obtain a contradiction with the uniform estimate of the entropy. For  $\alpha > 0$ , we define the sets

$$V_{\alpha,\delta} = \{(t, x) \in (0, T) \times \mathbb{T}^d : u_\delta(t, x) \leq -\alpha\},$$

$$V_{\alpha,0} = \{(t, x) \in (0, T) \times \mathbb{T}^d : u(t, x) \leq -\alpha\}.$$

By nonnegativity of  $\phi_\delta$  (see (4.2.4) as well as the properties below) and (A8) in Theorem 4.2.4, there is a constant  $C(T)$  such that

$$\int_{V_{\alpha,\delta}} \phi_\delta(u_\delta) \, dx \, dt \leq \int_{(0,T) \times \mathbb{T}^d} \phi_\delta(u_\delta) \, dx \, dt \leq C(T).$$

For  $u_\delta \leq -\alpha$ , we have  $0 \leq \phi_\delta(-\alpha) \leq \phi_\delta(u_\delta)$  because  $\phi_\delta'(x) \leq 0$  for  $x \leq 0$ , see (4.2.4). Therefore,

$$0 \leq \phi_\delta(-\alpha) \int_{V_{\alpha,\delta}} 1 \, dx \, dt = \int_{V_{\alpha,\delta}} \phi_\delta(x) \, dx \, dt \leq C(T).$$

Sending  $\delta \rightarrow 0$ , exploiting (P5) in Lemma 4.2.2 and using the strong convergence of  $u_\delta \rightarrow u$  we discover

$$\int_{V_{\alpha,0}} 1 \, dx \, dt = \lim_{\delta \rightarrow 0} \int_{V_{\alpha,\delta}} 1 \, dx \, dt = 0$$

(we use here the fact from measure theory asserting that on the measure space  $(X, \mu)$  if  $f_n, f : X \rightarrow \mathbb{R}$  and  $f_n \rightarrow f$  in  $L^1(X, \mu)$  then for  $\alpha \in \mathbb{R}$  we have  $\int_{f_n < \alpha} d\mu \rightarrow \int_{f < \alpha} d\mu$  as  $n \rightarrow \infty$ ). This means that  $V_{\alpha,0}$  is a null set for each  $\alpha > 0$ , concluding the proof of the nonnegativity.

*Step 4 : Identification*  $\xi = \sqrt{F_1''(u)}\nabla u$ . We want to use (B4) so we have to identify  $\xi$ . For that purpose, we use the convergence a.e. of  $u_\delta$  in (B1) and the pointwise convergence  $F_{\delta,1}'' \rightarrow F_1''$  to deduce that  $F_{\delta,1}(u_\delta) \rightarrow F_1(u)$  a.e. Next, using Assumption (B) for  $F_{\delta,1}$  and estimate (A1)

$$\left| \sqrt{F_{\delta,1}''(u_\delta)} \right|^2 \leq C_3 |u_\delta|^{k-2} + C_4.$$

As (RHS) is uniformly integrable by strong convergence (B1), we deduce that  $\left| \sqrt{F_{\delta,1}''(u_\delta)} \right|^2$  is uniformly integrable so that the Vitali convergence theorem implies

$$\sqrt{F_{\delta,1}''(u_\delta)} \rightarrow \sqrt{F_1''(u)} \quad \text{in } L^2((0, T) \times \mathbb{T}^d)$$

Using weak convergence of gradient (B2), we finally obtain  $\xi = \sqrt{F_1''(u)}\nabla u$ .

*Step 5 : Passing to the limit in the first two terms of (4.2.15).* Using (B3) it is easy to pass to the limit in the first term of (4.2.15). Now we focus on the second term. Note that

$$\nabla B_\varepsilon[u_\delta](x) = \frac{1}{\varepsilon^2}(\nabla u_\delta - \omega_\varepsilon * \nabla u_\delta).$$

The two terms of  $\nabla B_\varepsilon$  are treated in the same way. We focus only on the harder term  $\nabla u_\delta$  which does not have regularizing properties of the convolution. For this term it is sufficient to prove that  $T_\delta(u_\delta)\nabla u_\delta \rightharpoonup u\nabla u$  weakly in  $L^1(0, T; L^1(\mathbb{T}^d))$  because  $\nabla\varphi \in L^\infty((0, T) \times \mathbb{T}^d)$ . We first note that by definition of  $T_\delta$ , the strong convergence (B1) and the nonnegativity of  $u$ , we obtain  $T_\delta(u_\delta) \rightarrow u$  strongly in  $L^2((0, T) \times \mathbb{T}^d)$ . Hence, the result follows from weak convergence of the gradient (B2).

*Step 6 : Passing to the limit in the third term of (4.2.15).* For the third term we write  $F_\delta'' = F_{\delta,1}'' + F_{\delta,2}''$  as discussed in Step 1. Then we decompose

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta) F_\delta''(u_\delta) \nabla u_\delta \cdot \nabla \varphi &= \int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta) F_{\delta,1}''(u_\delta) \nabla u_\delta \cdot \nabla \varphi + \int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta) F_{\delta,2}''(u_\delta) \nabla u_\delta \cdot \nabla \varphi \\ &= I_1 + I_2. \end{aligned}$$

For  $I_1$  we write

$$I_1 = \int_0^T \int_{\mathbb{T}^d} T_\delta(u_\delta) \sqrt{F_{\delta,1}''(u_\delta)} \sqrt{F_{\delta,1}''(u_\delta)} \nabla u_\delta \cdot \nabla \varphi.$$

It remains to prove that  $T_\delta(u_\delta) \sqrt{F_{\delta,1}''(u_\delta)}$  converges strongly in  $L^2((0, T) \times \mathbb{T}^d)$  because  $\nabla\varphi \in L^\infty((0, T) \times \mathbb{T}^d)$ . Note that since  $u_\delta \rightarrow u \geq 0$  we have  $T_\delta(u_\delta) \sqrt{F_{\delta,1}''(u_\delta)} \rightarrow u \sqrt{F_1''(u)}$  a.e. Moreover,

$$\left( T_\delta(u_\delta) \sqrt{F_{\delta,1}''(u_\delta)} \right)^2 \leq C_3 |u_\delta|^k + C_4$$

As the (RHS) is uniformly integrable by strong convergence, we deduce that (LHS) is uniformly integrable. Hence, the Vitali convergence theorem implies Assumption (B) and Estimate (A1) show that

$$T_\delta(u_\delta) \sqrt{F_{\delta,1}''(u_\delta)} \rightarrow u \sqrt{F_1''(u)} \quad \text{in } L^2((0, T) \times \mathbb{T}^d)$$

so that  $I_1 \rightarrow \int_0^T \int_{\mathbb{T}^d} u F_1''(u) \nabla u \cdot \nabla \varphi$ . For  $I_2$ , as  $\nabla u_\delta \rightharpoonup \nabla u$  weakly in  $L^2((0, T) \times \mathbb{T}^d)$ , it is sufficient to prove the strong convergence of  $T_\delta(u_\delta) F_{\delta,2}''(u_\delta)$  in  $L^2((0, T) \times \mathbb{T}^d)$  because  $\nabla\varphi \in$

$L^\infty((0, T) \times \mathbb{T}^d)$ . Thanks to Assumption (C) on  $F''_{\delta,2}$ , this term is uniformly bounded so that trivially  $|T_\delta(u_\delta)F''_{\delta,2}(u_\delta)| \leq \|F''_2\|_\infty |T_\delta(u_\delta)|$ . Therefore, Vitali convergence theorem implies that  $T_\delta(u_\delta)F''_{\delta,2}(u_\delta)$  is convergent in  $L^2((0, T) \times \mathbb{T}^d)$  and so

$$I_2 \rightarrow \int_0^T \int_{\mathbb{T}^d} u F''_2(u) \nabla u \cdot \nabla \varphi.$$

*Step 7 : Energy and entropy estimates.* We pass to the limit  $\delta \rightarrow 0$  in (4.2.6)-(4.2.7). With the above convergences and properties of the weak limit, we obtain the result. This ends the proof of Theorem 4.1.7.

*Step 8 : Weak formulation for  $\varphi \in L^2(0, T; W^{1,\infty}(\mathbb{T}^d))$ .* We know that the weak formulation

$$\int_0^T \langle \partial_t u, \varphi \rangle_{(W^{-1,s'}(\mathbb{T}^d), W^{1,s}(\mathbb{T}^d))} = - \int_0^T \int_{\mathbb{T}^d} u \nabla B_\varepsilon[u] \cdot \nabla \varphi - \int_0^T \int_{\mathbb{T}^d} u F''(u) \nabla u \cdot \nabla \varphi \quad (4.2.16)$$

is satisfied for all  $\varphi \in L^\infty(0, T; W^{1,\infty}(\mathbb{T}^d))$ . By a usual density argument we can extend it to a bigger class  $\varphi \in L^2(0, T; W^{1,\infty}(\mathbb{T}^d))$ . Indeed, for the term  $\langle \partial_t u, \varphi \rangle_{(W^{-1,s'}(\mathbb{T}^d), W^{1,s}(\mathbb{T}^d))}$  it is clear as  $\partial_t u$  is a functional on  $L^2(0, T; W^{1,s}(\mathbb{T}^d))$ . For the terms on the (RHS) of (4.2.16) it is sufficient to prove that :

- $u \nabla B_\varepsilon[u] \in L^2(0, T; L^1(\mathbb{T}^d))$  which follows from the fact that  $u \in L^\infty(0, T; L^2(\mathbb{T}^d))$  and  $\nabla u \in L^2((0, T) \times \mathbb{T}^d)$ ,
- $u F''(u) \nabla u \in L^2(0, T; L^1(\mathbb{T}^d))$ . Here, we split  $F = F_1 + F_2$ . For the term  $u F''_2(u) \nabla u$  we use that  $F''_2$  is bounded and  $u \nabla u \in L^2(0, T; L^1(\mathbb{T}^d))$ . For the term  $u F''_1(u) \nabla u$  we write it as  $u \sqrt{F''_1(u)} \sqrt{F''_1(u)} \nabla u$ . By the growth of  $F''_1(u)$  (cf. (B) in Assumption 4.1.1) and  $u \in L^\infty(0, T; L^k(\mathbb{T}^d))$ , we see that  $u \sqrt{F''_1(u)} \in L^\infty(0, T; L^2(\mathbb{T}^d))$ . As  $\sqrt{F''_1(u)} \nabla u \in L^2((0, T) \times \mathbb{T}^d)$ , the conclusion follows. □

Now that weak solutions of the nonlocal Cahn-Hilliard equation have been constructed for a given initial datum, it remains to prove the convergence of the nonlocal system to the local one. This is the purpose of the next section.

### 4.3 Limit $\varepsilon \rightarrow 0$

Weak solutions of the local Cahn-Hilliard equation are understood in the sense of Definition 4.1.5. In order to prove the convergence of the nonlocal system to these solutions, we first collect the necessary estimates uniform in  $\varepsilon$ . Then we pass to the limit  $\varepsilon \rightarrow 0$  to conclude the proof of Theorem 4.1.8.

#### 4.3.1 Uniform estimates in $\varepsilon$

We recall that in the previous section we had obtained the energy and entropy inequalities as well as estimates uniform in  $\varepsilon$ .

**Lemma 4.3.1** (Mass, energy, entropy). *The following identities hold true*

$$\int_{\mathbb{T}^d} u_\varepsilon(t, \cdot) dx = \int_{\mathbb{T}^d} u^0 dx, \quad (4.3.1)$$

$$\frac{d}{dt} E[u_\varepsilon] + \int_{\mathbb{T}^d} u_\varepsilon |\nabla \mu_\varepsilon|^2 \leq 0, \quad (4.3.2)$$

$$\frac{d}{dt} \Phi[u_\varepsilon] + \frac{1}{2\varepsilon^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(x-y)|^2 dx dy + \int_{\mathbb{T}^d} F''(u_\varepsilon) |\nabla u_\varepsilon|^2 dx \leq 0. \quad (4.3.3)$$

**Lemma 4.3.2** (Uniform estimates). *The following sequences are bounded :*

- (A)  $\{u_\varepsilon\}_\varepsilon$  in  $L^\infty(0, T; L^k(\mathbb{T}^d))$ ,
- (B)  $\{u_\varepsilon\}_\varepsilon$  in  $L^k(0, T; L^{k\frac{d}{d-2}}(\mathbb{T}^d))$ ,
- (C)  $\{\nabla u_\varepsilon\}_\varepsilon$  in  $L^2((0, T) \times \mathbb{T}^d)$ ,
- (D)  $\{\sqrt{u_\varepsilon} \nabla \mu_\varepsilon\}_\varepsilon$  in  $L^2((0, T) \times \mathbb{T}^d)$ ,
- (E)  $\{\partial_t u_\varepsilon\}_\varepsilon$  in  $L^2(0, T; W^{-1, s'}(\mathbb{T}^d))$ ,
- (F)  $\{\partial_t \nabla u_\varepsilon\}_\varepsilon$  in  $L^2(0, T; W^{-2, s'}(\mathbb{T}^d))$ ,
- (G)  $\{\sqrt{F_1''(u_\varepsilon)} \nabla u_\varepsilon\}_\varepsilon$  in  $L^2((0, T) \times \mathbb{T}^d)$ .

Our last ingredient for the proof of Theorem 4.1.8 is about the compactness of  $u_\varepsilon$  and its gradient.

**Lemma 4.3.3** (Compactness). *Sequences  $\{u_\varepsilon\}_\varepsilon$  and  $\{\nabla u_\varepsilon\}_\varepsilon$  are strongly compact in  $L^2((0, T) \times \mathbb{T}^d)$ .*

*Démonstration.* The compactness of  $\{u_\varepsilon\}$  follows from the Lions-Aubin lemma applied to estimates (A), (C) and (E). Then, for the compactness of  $\{\nabla u_\varepsilon\}$ , the detailed proof is presented in Appendix 4.4.7. Roughly speaking, from (F) we know that the sequence is compact in time. Compactness in space follows from Theorem 4.4.4 together with the estimate provided by the entropy on the quantity :

$$\frac{1}{4\varepsilon^2} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(x-y)|^2 dx dy dt \leq C.$$

An application of the Fréchet-Kolmogorov theorem yields the result.  $\square$

Now we are ready to prove our main theorem.

### 4.3.2 Proof of Theorem 4.1.8 : convergence $\varepsilon \rightarrow 0$

We want to pass to the limit  $\varepsilon \rightarrow 0$  in Equations (4.1.5)-(4.1.6) and obtain weak solutions of the local Cahn-Hilliard equation. We have at most bounds on the gradient of  $u_\varepsilon$  and the limit equation has four derivatives. That means we need to mimic at the epsilon level integration by parts for nonlocal operators. For that purpose, we define the operator

$$S_\varepsilon[\varphi](x, y) := \frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2\varepsilon}} (\varphi(x-y) - \varphi(x)) \quad (4.3.4)$$

which has the following properties :

**Lemma 4.3.4.** *The operator  $S_\varepsilon$  satisfies :*

(S1)  $S_\varepsilon$  is a linear operator that commutes with derivatives with respect to  $x$ ,

(S2) for all functions  $f, g : \mathbb{T}^d \rightarrow \mathbb{R}$  we have

$$\begin{aligned} S_\varepsilon[fg](x, y) - S_\varepsilon[f](x, y)g(x) - S_\varepsilon[g](x, y)f(x) &= \\ &= \frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2\varepsilon}} [(f(x-y) - f(x))(g(x-y) - g(x))]. \end{aligned}$$

(S3) for all  $u, \varphi \in L^2(\mathbb{T}^d)$

$$\langle B_\varepsilon[u](\cdot), \varphi(\cdot) \rangle_{L^2(\mathbb{T}^d)} = \langle S_\varepsilon[u](\cdot, \cdot), S_\varepsilon[\varphi](\cdot, \cdot) \rangle_{L^2(\mathbb{T}^d \times \mathbb{T}^d)}.$$

(S4) if  $\{u_\varepsilon\}$  is strongly compact in  $L^2(0, T; H^1(\mathbb{T}^d))$  and  $\varphi \in L^\infty((0, T) \times \mathbb{T}^d)$  we have

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} (S_\varepsilon[u_\varepsilon])^2 \varphi(t, x) \rightarrow D \int_0^T \int_{\mathbb{T}^d} |\nabla u(t, x)|^2 \varphi(t, x)$$

$$\text{where } D = \frac{1}{2} \int_{B_1} \omega(y) |y|^2 dy.$$

*Démonstration.* The first one is trivial. For the second one, we just observe

$$\begin{aligned} (f(x-y) - f(x))(g(x-y) - g(x)) &= \\ &= -(f(x-y) - f(x))g(x) - g(x-y)f(x) + f(x-y)g(x-y) = \\ &= -(f(x-y) - f(x))g(x) - (g(x-y) - g(x))f(x) + (f(x-y)g(x-y) - f(x)g(x)). \end{aligned}$$

For the third one, we compute

$$\langle B_\varepsilon[u](\cdot), \varphi(\cdot) \rangle_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (u(x) - u(x-y)) \varphi(x) dy dx.$$

Changing variables  $x' = x - y$ ,  $y' = -y$  and using symmetry of the kernel

$$\langle B_\varepsilon[u](\cdot), \varphi(\cdot) \rangle_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (u(x' - y') - u(x')) \varphi(x' - y') dy' dx'.$$

Therefore,

$$\begin{aligned} 2 \langle B_\varepsilon[u](\cdot), \varphi(\cdot) \rangle_{L^2(\mathbb{T}^d)} &= \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (u(x) - u(x-y)) (\varphi(x) - \varphi(x-y)) dy dx \\ &= \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\sqrt{\omega_\varepsilon(y)}}{\varepsilon} (u(x) - u(x-y)) \frac{\sqrt{\omega_\varepsilon(y)}}{\varepsilon} (\varphi(x) - \varphi(x-y)) dy dx. \end{aligned}$$

Finally, to prove (S4) we use the definition of  $\omega_\varepsilon$  and change variables with respect to  $y$  :

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} (S_\varepsilon[u_\varepsilon])^2 \varphi(t, x) = \int_{B_1} \omega(y) \int_0^T \int_{\mathbb{T}^d} \varphi(t, x) \frac{|u_\varepsilon(t, x) - u_\varepsilon(t, x - \varepsilon y)|^2}{2\varepsilon^2} dx dt dy$$

For fixed  $y$ ,

$$\int_0^T \int_{\mathbb{T}^d} \varphi(t, x) \frac{|u_\varepsilon(t, x) - u_\varepsilon(t, x - \varepsilon y)|^2}{\varepsilon^2} dx dt \rightarrow \int_0^T \int_{\mathbb{T}^d} \varphi(t, x) |\nabla u(x)|^2 |y|^2 dx dt,$$

$$\left| \int_0^T \int_{\mathbb{T}^d} \varphi(t, x) \frac{|u_\varepsilon(t, x) - u_\varepsilon(t, x - \varepsilon y)|^2}{\varepsilon^2} dx dt \right| \leq \|\varphi\|_\infty \sup_\varepsilon \|Du_\varepsilon\|_2^2 |y|^2$$

due to Lemma 4.4.1. As the majorant is integrable, the dominated convergence theorem concludes the proof.  $\square$

Since  $B_\varepsilon$  has a similar behavior as the Laplace operator, one can expect that  $S_\varepsilon$  acts like a gradient (in  $L^2(\mathbb{T}^d)$ ). Nevertheless, note that  $S_\varepsilon[\varphi](x, y)$  is a scalar. From now on, we write  $\nabla S_\varepsilon$  for the gradient of  $S_\varepsilon$  with respect to the variable  $x$  i.e.

$$\nabla S_\varepsilon[\varphi](x, y) := \frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2\varepsilon}} (\nabla\varphi(x - y) - \nabla\varphi(x)).$$

*Proof of Theorem 4.1.8.* We only have to pass to the limit in the term  $\int_0^T \int_{\mathbb{T}^d} \operatorname{div}(u_\varepsilon \nabla \mu_\varepsilon) \varphi dx dt$  where  $\varphi \in C^3([0, T] \times \mathbb{T}^d)$ . Integrating by parts, we obtain

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} \operatorname{div}(u_\varepsilon \nabla \mu_\varepsilon) \varphi dx dt &= - \int_0^T \int_{\mathbb{T}^d} u_\varepsilon \nabla \mu_\varepsilon \cdot \nabla \varphi dx dt = \int_0^T \int_{\mathbb{T}^d} B_\varepsilon[u_\varepsilon] \nabla u_\varepsilon \cdot \nabla \varphi dx dt \\ &+ \int_0^T \int_{\mathbb{T}^d} B_\varepsilon[u_\varepsilon] u_\varepsilon \Delta \varphi dx dt - \int_0^T \int_{\mathbb{T}^d} u_\varepsilon F''(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi dx dt \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{4.3.5}$$

Step 1 : Compactness. Using Lemma 4.3.2 and Lemma 4.3.3 we can choose a subsequence of  $\{u_\varepsilon\}_\varepsilon$  such that

(D1)  $\partial_t u_\varepsilon \rightharpoonup \partial_t u$  weakly in  $L^2(0, T; W^{-1, s'}(\mathbb{T}^d))$ ,

(D2)  $u_\varepsilon \rightarrow u$  strongly in  $L^2((0, T) \times \mathbb{T}^d)$ ,

(D3)  $\nabla u_\varepsilon \rightarrow \nabla u$  strongly in  $L^2((0, T) \times \mathbb{T}^d)$ ,

(D4)  $\sqrt{F_1''(u_\varepsilon)} \nabla u_\varepsilon \rightharpoonup \xi$  weakly in  $L^2((0, T) \times \mathbb{T}^d)$ .

Step 1 : Convergence of  $I_1$ . Using (S3) in Lemma 4.3.4 we write

$$\begin{aligned} I_1 &= \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} S_\varepsilon[u_\varepsilon] S_\varepsilon(\nabla u_\varepsilon \cdot \nabla \varphi) dx dy dt \\ &= \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} S_\varepsilon[u_\varepsilon] S_\varepsilon(\nabla u_\varepsilon) \cdot \nabla \varphi dx dy dt + \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} S_\varepsilon[u_\varepsilon] \nabla u_\varepsilon \cdot S_\varepsilon[\nabla \varphi] dx dy dt + R_\varepsilon \\ &= J_1^{(1)} + J_2^{(1)} + R_\varepsilon^{(1)}, \end{aligned}$$

where  $R_\varepsilon^{(1)}$  is defined as

$$R_\varepsilon^{(1)} = \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} S_\varepsilon[u_\varepsilon] (S_\varepsilon(\nabla u_\varepsilon \cdot \nabla \varphi) - S_\varepsilon(\nabla u_\varepsilon) \cdot \nabla \varphi - \nabla u_\varepsilon \cdot S_\varepsilon[\nabla \varphi]) dx dy dt.$$

For  $J_1^{(1)}$  we use identity

$$S_\varepsilon[u_\varepsilon] S_\varepsilon(\nabla u_\varepsilon) = S_\varepsilon[u_\varepsilon] \nabla S_\varepsilon(u_\varepsilon) = \frac{1}{2} \nabla |S_\varepsilon[u_\varepsilon]|^2$$



so after integration by parts we obtain

$$J_1^{(1)} = -\frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} (S_\varepsilon[u_\varepsilon])^2 \Delta \varphi \, dx \, dy \, dt \rightarrow -\frac{D}{2} \int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta \varphi \, dx \, dt \quad (4.3.6)$$

due to (S4) in Lemma 4.3.4. For  $J_2^{(1)}$  we change variables to have

$$\begin{aligned} J_2^{(1)} &= \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) \frac{u_\varepsilon(x-y) - u_\varepsilon(x)}{\varepsilon} \nabla u_\varepsilon(x) \cdot \frac{\nabla \varphi(x-y) - \nabla \varphi(x)}{\varepsilon} \, dx \, dy \, dt = \\ &= \frac{1}{2} \int_{\mathbb{T}^d} \omega(y) \int_0^T \int_{\mathbb{T}^d} \frac{u_\varepsilon(x-\varepsilon y) - u_\varepsilon(x)}{\varepsilon} \nabla u_\varepsilon(x) \cdot \frac{\nabla \varphi(x-\varepsilon y) - \nabla \varphi(x)}{\varepsilon} \, dx \, dt \, dy. \end{aligned}$$

We are first concerned with the inner integral. With Lemma 4.4.1 we have that for fixed  $y \in \mathbb{T}^d$

$$\frac{u_\varepsilon(x-\varepsilon y) - u_\varepsilon(x)}{\varepsilon} \rightarrow -\nabla u(x) \cdot y \text{ in } L^2((0, T) \times \mathbb{T}^d).$$

Moreover, a Taylor expansion implies that

$$\frac{\nabla \varphi(x-\varepsilon y) - \nabla \varphi(x)}{\varepsilon} \rightarrow -D^2 \varphi(x) y \text{ in } L^\infty((0, T) \times \mathbb{T}^d; \mathbb{R}^d).$$

Combining this with a strong convergence  $\nabla u_\varepsilon \rightarrow \nabla u$  in  $L^2((0, T) \times \mathbb{T}^d)$ , we deduce

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} \frac{u_\varepsilon(x-\varepsilon y) - u_\varepsilon(x)}{\varepsilon} \nabla u_\varepsilon(x) \cdot \frac{\nabla \varphi(x-\varepsilon y) - \nabla \varphi(x)}{\varepsilon} \, dx \, dt &\rightarrow \\ &\rightarrow \int_0^T \int_{\mathbb{T}^d} \nabla u(x) \cdot y \nabla u(x) \cdot (D^2 \varphi(x) y) \, dx \, dt. \end{aligned}$$

Finally, we apply the dominated convergence theorem to the integral with respect to  $y$  with the dominating function  $\|D^2 \varphi\|_\infty \sup_\varepsilon \|\nabla u_\varepsilon\|_2^2 |y|^2$ . We obtain

$$\begin{aligned} J_2^{(1)} &\rightarrow \frac{1}{2} \int_{\mathbb{T}^d} \omega(y) |y|^2 \, dy \int_0^T \int_{\mathbb{T}^d} \nabla u(x) \cdot D^2 \varphi(x) \nabla u(x) \, dx \, dt = \\ &= D \int_0^T \int_{\mathbb{T}^d} (\nabla u(x) \otimes \nabla u(x)) : D^2 \varphi(x) \, dx \, dt, \end{aligned} \quad (4.3.7)$$

where we also used the symmetry of  $D^2 \varphi$  and properties of  $\omega$  defined in (4.1.2). It remains to deal with the error term. Using (S2) in Lemma 4.3.4 we can write

$$R_\varepsilon^{(1)} = \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} S_\varepsilon[u_\varepsilon] \frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2\varepsilon}} [(\nabla u_\varepsilon(x-y) - \nabla u_\varepsilon(x)) \cdot (\nabla \varphi(x-y) - \nabla \varphi(x))] \, dx \, dy \, dt.$$

We want to prove that  $R_\varepsilon^{(1)}$  converges to 0. By Cauchy-Schwarz inequality (in time and space) as well as bounds on  $S_\varepsilon[u_\varepsilon]$  it remains to prove that

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} |\nabla u_\varepsilon(x-y) - \nabla u_\varepsilon(x)|^2 |\nabla \varphi(x-y) - \nabla \varphi(x)|^2 \, dy \, dx \, dt \rightarrow 0. \quad (4.3.8)$$

Using Taylor's expansion we can estimate this integral with

$$\varepsilon \|D^2\varphi\|_{L^\infty}^2 \left( \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} |\nabla u_\varepsilon(x-y) - \nabla u_\varepsilon(x)|^2 dy dx dt \right)$$

which converges to zero by the bound from the entropy (4.3.3) so that (4.3.8) follows. We conclude that

$$I_1 \rightarrow -\frac{D}{2} \int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta\varphi dx dt + D \int_0^T \int_{\mathbb{T}^d} (\nabla u \otimes \nabla u) : D^2\varphi dx dt.$$

*Step 2 : Convergence of  $I_2$ .* We observe that the only differences between  $I_1$  and  $I_2$  are  $u_\varepsilon$  and  $\Delta\varphi$  in place of  $\nabla u_\varepsilon$  and  $\nabla\varphi$  respectively. As we have the same (in fact, better) estimates for these quantities, the proof is the same and we conclude

$$I_2 \rightarrow D \int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta\varphi dx dt + D \int_0^T \int_{\mathbb{T}^d} u \nabla u \cdot \nabla \Delta\varphi.$$

*Step 3 : Convergence of  $I_3$ .* For  $I_3$  the proof is similar to the reasoning in Steps 1, 3 and 6 of the proof of Theorem 4.1.7 because we have to use the same estimates. Roughly speaking, one proves that  $u_\varepsilon \rightarrow u$  strongly in  $L^k((0, T) \times \mathbb{T}^d)$  by interpolation so that one can identify  $\xi = \sqrt{F_1''(u)} \nabla u$ . Next, convergence in  $L^k((0, T) \times \mathbb{T}^d)$  allows also to prove strong convergence  $u_\varepsilon \sqrt{F_1''(u_\varepsilon)} \rightarrow u \sqrt{F_1''(u)}$  in  $L^2((0, T) \times \mathbb{T}^d)$  thanks to growth condition (B) while the convergence  $u_\varepsilon \sqrt{F_2''(u_\varepsilon)} \rightarrow u \sqrt{F_2''(u)}$  in  $L^2((0, T) \times \mathbb{T}^d)$  is trivial because  $F_2'' \in L^\infty$ . This shows that

$$I_3 \rightarrow - \int_0^T \int_{\mathbb{T}^d} u F''(u) \nabla u \cdot \nabla \varphi dx dt.$$

*Conclusion of Steps 1-3.* In the limit  $\varepsilon \rightarrow 0$  we obtain

$$\begin{aligned} \int_0^T \langle \partial_t u, \varphi \rangle_{(W^{-1, s'}(\mathbb{T}^d), W^{1, s}(\mathbb{T}^d))} &= D \int_0^T \int_{\mathbb{T}^d} (\nabla u \otimes \nabla u) : D^2\varphi + \\ &+ \frac{D}{2} \int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta\varphi + D \int_0^T \int_{\mathbb{T}^d} u \nabla u \cdot \nabla \Delta\varphi - \int_0^T \int_{\mathbb{T}^d} u F''(u) \nabla u \cdot \nabla \varphi. \end{aligned}$$

*Step 4 : Regularity of  $u$  and better weak formulation* Now we prove the regularity of the limit function  $u$ . This allows us to perform integration by parts on the different terms using the formula (4.1.14) and recover the Definition 4.1.5. In fact, in the limit  $\varepsilon \rightarrow 0$ , from the entropy we obtain (see [53, Theorem 4] and [294, Theorem 1.2])

$$\sum_{i,j=1}^d \int_0^t \int_{\mathbb{T}^d} |\partial_{x_i} \partial_{x_j} u|^2 \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{4\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |\nabla u_\varepsilon(x) - \nabla u_\varepsilon(x-y)|^2$$

so in the limit  $\varepsilon \rightarrow 0$  we gain one more derivative. A similar reasoning can be performed on the energy to see that in the limit  $u \in L^\infty(0, T; H^1(\mathbb{T}^d))$ . Then, since

$$D \int_0^T \int_{\mathbb{T}^d} u \nabla u \cdot \nabla \Delta\varphi = -D \int_0^T \int_{\mathbb{T}^d} \Delta\varphi |\nabla u|^2 - D \int_0^T \int_{\mathbb{T}^d} u \Delta u \Delta\varphi$$

and using formula (4.1.14), we compute

$$\begin{aligned} I_1 + I_2 &= D \int_0^T \int_{\mathbb{T}^d} (\nabla u \otimes \nabla u) : D^2 \varphi - \frac{D}{2} \int_0^T \int_{\mathbb{T}^d} |\nabla u|^2 \Delta \varphi - D \int_0^T \int_{\mathbb{T}^d} u \Delta u \Delta \varphi \\ &= -D \int_0^T \int_{\mathbb{T}^d} \Delta u \nabla u \cdot \nabla \varphi - D \int_0^T \int_{\mathbb{T}^d} u \Delta u \Delta \varphi. \end{aligned}$$

This ends the proof of Theorem 4.1.8.  $\square$

## 4.4 Appendix

### 4.4.1 Difference quotients

**Lemma 4.4.1.** *Let  $\{u_\varepsilon\}$  be a sequence strongly compact in  $L^2(0, T; H^1(\mathbb{T}^d))$ . Then, for fixed  $y \in \mathbb{T}^d$ ,*

$$\frac{u_\varepsilon(t, x - \varepsilon y) - u_\varepsilon(t, x)}{\varepsilon} \rightarrow -\nabla u(t, x) \cdot y \text{ strongly in } L^2((0, T) \times \mathbb{T}^d).$$

*Démonstration.* We write

$$\begin{aligned} \frac{u_\varepsilon(t, x - \varepsilon y) - u_\varepsilon(t, x)}{\varepsilon} &= -y \cdot \int_0^1 \nabla u_\varepsilon(t, x - \varepsilon \theta y) d\theta \\ &= -y \cdot \int_0^1 (\nabla u_\varepsilon(t, x - \varepsilon \theta y) - \nabla u_\varepsilon(t, x)) d\theta - y \cdot \nabla u_\varepsilon(t, x). \end{aligned}$$

By assumption  $y \cdot \nabla u_\varepsilon \rightarrow y \cdot \nabla u$  strongly in  $L^2((0, T) \times \mathbb{T}^d)$  so we only have to prove that the first term on the (RHS) converges to 0. By Fubini's theorem and Cauchy-Schwarz inequality

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d} \left| \int_0^1 (\nabla u_\varepsilon(t, x - \varepsilon \theta y) - \nabla u_\varepsilon(t, x)) d\theta \right|^2 dx dt \\ &\leq C \int_0^1 \int_0^T \int_{\mathbb{T}^d} |\nabla u_\varepsilon(t, x - \varepsilon \theta y) - \nabla u_\varepsilon(t, x)|^2 dx dt d\theta = C \int_0^1 \|\tau_{\varepsilon \theta y} \nabla u_\varepsilon - \nabla u_\varepsilon\|_{L^2((0, T) \times \mathbb{T}^d)}^2 d\theta, \end{aligned}$$

where  $\tau$  is the translation operator. The last term converges to 0 when  $\varepsilon \rightarrow 0$  by the Fréchet Kolmogorov theorem.  $\square$

### 4.4.2 Growth estimates on mollified nonlinearity

**Lemma 4.4.2.** *Let  $F$  satisfies Assumption 4.1.1 with constants  $C_1, \dots, C_{10}$ . Then,  $F_\delta = F * \eta_\delta$  with  $0 \leq \delta \leq 1$  satisfies Assumption 4.1.1 with constants*

$$\begin{aligned} \widetilde{C}_1 &= 2^{1-k} C_1, & \widetilde{C}_2 &= C_1 + C_2, & \widetilde{C}_3 &= 2^{k-1} C_3, & \widetilde{C}_4 &= \widetilde{C}_3 + C_4, \\ \widetilde{C}_5 &= \min(2^{3-k}, 1) C_5, & \widetilde{C}_6 &= C_5 + C_6, & \widetilde{C}_7 &= \max(2^{k-3}, 1) C_7, & \widetilde{C}_8 &= \widetilde{C}_7 + C_8, \\ \widetilde{C}_9 &= C_9 + 2 C_{10}, & \widetilde{C}_{10} &= 2 C_{10}. \end{aligned}$$

*Démonstration.* We decompose  $F_{\delta,1} = F_1 * \eta_\delta$  and  $F_{\delta,2} = F_2 * \eta_\delta$ . Suppose that  $F_1(u) \leq C_3|u|^k + C_4$ . Then,

$$F_{\delta,1}(u) = \int_{\mathbb{R}} F_1(u-s) \eta_\delta(s) ds \leq C_3 \int_{\mathbb{R}} |u-s|^k \eta_\delta(s) ds + C_4 \leq 2^{k-1} C_3 |u|^k + 2^{k-1} C_3 + C_4$$

where we used inequality valid for  $p \geq 0$

$$|u-s|^p \leq \max(1, 2^{p-1}) (|u|^p + |s|^p). \quad (4.4.1)$$

It follows that  $\widetilde{C}_3 = 2^{k-1} C_3$  and  $\widetilde{C}_4 = 2^{k-1} C_3 + C_4$ . In a similar way, we compute constants  $\widetilde{C}_7, \widetilde{C}_8$ . For  $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_5, \widetilde{C}_6$  the reasoning is the same but we have to use a lower bound of the form

$$|u-s|^p \geq \min(1, 2^{1-p}) |u|^p - |s|^p.$$

so that, for example, if  $F_1 \geq C_1|u|^k - C_2$  we have

$$F_{\delta,1}(u) = \int_{\mathbb{R}} F_1(u-s) \eta_\delta(s) ds \geq C_1 \int_{\mathbb{R}} |u-s|^k \eta_\delta(s) ds - C_2 \geq 2^{1-k} C_1 |u|^p - C_1 - C_2.$$

For the constants  $\widetilde{C}_9, \widetilde{C}_{10}$  we argue using (4.4.1) once again

$$F_{\delta,2}(u) \geq -C_9 - C_{10} \int_{\mathbb{R}} |u-s|^2 \eta_\delta(s) ds \geq -C_9 - 2 C_{10} - 2 C_{10} |u|^2.$$

□

### 4.4.3 Potentials satisfying Assumption 4.1.1

**Lemma 4.4.3.** *Let  $F$  be as in (3) in Example 4.1.2. Then,  $F$  satisfies Assumption 4.1.1.*

*Démonstration.* On  $\mathbb{R} \setminus I$  we define  $F_1(u) = F(u)$ . By [329, Theorem 3.2], there exists a  $C^2$  extension of  $F_1$  to  $\mathbb{R}$  denoted by  $F_1$  which preserves convexity, i.e.  $F_1''(u) > b > 0$  for some  $b > 0$ . Moreover,  $F_1$  has  $k$ -growth on  $\mathbb{R}$  (in fact, by continuity, the behaviour of  $F_1$  on  $I$  can be included in constants  $C_2, C_4, C_6, C_8$  in Assumption 4.1.1). We finally define

$$F_2 = \begin{cases} F(u) - F_1(u) & \text{on } I, \\ 0 & \text{on } \mathbb{R} \setminus I. \end{cases}$$

Function  $F_2$  is  $C^2$  because at the endpoints of interval  $I$  we have  $F'' = F_1''$  as  $F_1$  is  $C^2$  extension of  $F$ . Finally,  $F_2$  satisfies condition (C) in Assumption 4.1.1 with  $F_2(u) \geq -\|F_2\|_\infty$ . □

### 4.4.4 Proof of Lemma 4.2.2

*Démonstration.* First, we note the formula which will be useful

$$\phi(x) = \int_1^x \int_1^y \frac{1}{z} dz dy.$$

Now, we proceed to the proof. First, (1) follows from the definition. Next, (2) follows from writing

$$\phi_\delta(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{T_\delta(z)} \operatorname{sgn}(y-1) \operatorname{sgn}(x-1) \mathbf{1}_{y \in [1,x]} \mathbf{1}_{z \in [1,y]} dz dy, \quad (4.4.2)$$

and dominated convergence (for fixed  $x > 0$ ). Then, (3) follows from  $T_\delta \geq 0$  and the observation that  $x \geq 1$ ,  $x < 1$  implies  $y \geq 1$ ,  $y < 1$  respectively.

To see (4), we distinguish three cases.

- When  $x \geq \frac{1}{\delta} - 1$ , we split the integrals and use the estimate  $T_\delta(x) \geq \frac{1}{\delta} - 1$  so that

$$\begin{aligned} \phi_\delta(x) &\leq \int_1^{\frac{1}{\delta}-1} \int_1^y \frac{1}{z} dz dy + \int_{\frac{1}{\delta}-1}^x \int_1^y \frac{1}{\frac{1}{\delta}-1} dz dy \leq \\ &\leq \phi\left(\frac{1}{\delta}-1\right) + \frac{\delta}{2(1-\delta)}x^2 \leq \phi(x) + \frac{\delta}{1-\delta}(x-1)^2, \end{aligned}$$

because  $\phi(x)$  is non-decreasing for  $x \geq 1$ .

- When,  $x \in (2\delta, \frac{1}{\delta} - 1)$  we have  $\phi_\delta = \phi$  because on this set  $T_\delta(z) = z$ .
- When  $x \in [0, 2\delta]$  we have a lower bound  $T_\delta(x) \geq \delta$  so that

$$\phi_\delta(x) \leq \int_x^{2\delta} \int_y^1 \frac{1}{\delta} dz + \int_{2\delta}^1 \int_y^1 \frac{1}{z} dz \leq 2 + \phi(2\delta) \leq 3$$

as  $\phi(2\delta) \leq \phi(0) = 1$  because  $\phi(x)$  is decreasing for  $x \in (0, 1)$ .

Finally, to see (5), let  $x < 0$ . Then,

$$\phi_\delta(x) \geq \int_x^0 \int_y^0 \frac{1}{\delta} dz dy = \frac{1}{\delta} \int_x^0 -y dy = \frac{x^2}{2\delta}.$$

□

#### 4.4.5 Bourgain-Brézis-Mironescu and Ponce compactness result

We upgrade here the result of [294, Proposition 4.2] and [53, Theorem 4] to the time-space setting. We consider sequence of radial functions  $\{\rho_\varepsilon\}$  such that  $\rho_\varepsilon \geq 0$ ,  $\int_{\mathbb{R}^d} \rho_\varepsilon = 1$  and

$$\lim_{\varepsilon \rightarrow 0} \int_{|x| > \delta} \rho_\varepsilon(x) dx = 0 \text{ for all } \delta > 0.$$

For the formulation of the compactness result, we use another sequence  $\{\varphi_\delta\}_{\delta \in (0,1)} \subset C_c^\infty(\mathbb{R}^d)$  of standard mollifiers with mass 1 such that  $\varphi_\delta(x) = \frac{1}{\delta^d} \varphi(\frac{x}{\delta})$  with  $\varphi$  of mass 1 and compactly supported.

**Theorem 4.4.4.** *Let  $d \geq 2$ . Let  $\{f_\varepsilon\}$  be a sequence bounded in  $L^p((0, T) \times \mathbb{T}^d)$ . Suppose that there exists a sequence  $\{\rho_\varepsilon\}$  as above such that*

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f_\varepsilon(t, x) - f_\varepsilon(t, y)|^p}{|x - y|^p} \rho_\varepsilon(|x - y|) dx dy dt \leq C \quad (4.4.3)$$

for some constant  $C$ . Then,  $\{f_\varepsilon\}$  is compact in space in  $L^p((0, T) \times \mathbb{T}^d)$ , i.e.

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{T}^d} |f_\varepsilon * \varphi_\delta(t, x) - f_\varepsilon(t, x)|^p dx dt = 0. \quad (4.4.4)$$

*Remark 4.4.5.* Let  $\omega : \mathbb{R}^d \rightarrow \mathbb{R}$  be a smooth function, supported in the unit ball such that  $\int_{\mathbb{R}^d} \omega(x) dx = 1$ . Consider  $\omega_\varepsilon = \frac{1}{\varepsilon^d} \omega\left(\frac{x}{\varepsilon}\right)$ . Suppose that

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f_\varepsilon(x) - f_\varepsilon(y)|^p}{\varepsilon^p} \omega_\varepsilon(|x - y|) dx dy dt \leq \tilde{C}.$$

Then, (4.4.3) is satisfied. Indeed, we consider

$$\rho_\varepsilon(x) = \frac{\omega_\varepsilon(|x|) |x|^p}{\varepsilon^p \int_{\mathbb{R}^d} \omega(y) |y|^p dy} \quad (4.4.5)$$

so that (4.4.3) holds true with  $\frac{\tilde{C}}{\int_{\mathbb{R}^d} \omega(y) |y|^p dy}$ .

*Proof of Theorem 4.4.4.* The result for sequences that do not depend on time has been obtained in [53, 294]. To demonstrate that it is sufficient to *integrate in time* the reasoning mentioned above, we make an additional assumption that for every  $\varepsilon$ ,  $\rho_\varepsilon$  is a nonincreasing function as in [53, Theorem 4]. For the general case, one has to proceed as in [294, Theorem 1.2].

We define

$$\begin{aligned} F_\varepsilon(s) &:= \int_0^T \int_{|y|=1} \int_{\mathbb{T}^d} |f_\varepsilon(t, x + sy) - f_\varepsilon(t, x)|^p dx dy dt \\ &= \frac{1}{s^{d-1}} \int_0^T \int_{|y|=s} \int_{\mathbb{T}^d} |f_\varepsilon(t, x + y) - f_\varepsilon(t, x)|^p dx dy dt. \end{aligned}$$

By virtue of the computation above, we can express the assumption 4.4.3 using function  $F_\varepsilon$  as follows

$$\int_0^\delta s^{d-1} \frac{F_\varepsilon(s) \rho_\varepsilon(s)}{s^p} ds \leq C. \quad (4.4.6)$$

Using the triangle inequality

$$|f_\varepsilon(t, x + 2sy) - f_\varepsilon(t, x)| \leq |f_\varepsilon(t, x + 2sy) - f_\varepsilon(t, x + sy)| + |f_\varepsilon(t, x + sy) - f_\varepsilon(t, x)|$$

and change of variables we obtain

$$F_\varepsilon(2s) \leq 2^p F_\varepsilon(s), \quad \frac{F_\varepsilon(2s)}{(2s)^p} \leq \frac{F_\varepsilon(s)}{s^p}. \quad (4.4.7)$$

We estimate by Jensen's inequality

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} |f_\varepsilon * \varphi_\delta - f_\varepsilon|^p dx dt &\leq \frac{C}{\delta^d} \int_0^T \int_{\mathbb{T}^d} \int_{|x-y| \leq \delta} |f_\varepsilon(x) - f_\varepsilon(y)|^p dy dx dt \\ &= \frac{C}{\delta^d} \int_0^T \int_{\mathbb{T}^d} \int_{|h| \leq \delta} |f_\varepsilon(x+h) - f_\varepsilon(x)|^p dh dx dt \\ &= \frac{C}{\delta^d} \int_0^T \int_{\mathbb{T}^d} \int_0^\delta s^{d-1} \int_{|h|=s} |f_\varepsilon(x+h) - f_\varepsilon(x)|^p dh ds dx dt = \frac{C}{\delta^d} \int_0^\delta s^{d-1} F_\varepsilon(s) ds. \end{aligned} \quad (4.4.8)$$

Now, we use functional inequality (which requires doubling condition (4.4.7), cf. [53, Eq. (24)])

$$\delta^{-d} \int_0^\delta s^{d-1} \frac{F_\varepsilon(s)}{s^p} ds \leq C(d) \frac{\int_0^\delta s^{d-1} \frac{F_\varepsilon(s) \rho_\varepsilon(s)}{s^p} ds}{\int_{|x|<\delta} \rho_\varepsilon(x) dx} \quad (4.4.9)$$

For each  $\delta > 0$ , there exists  $\varepsilon(\delta)$  such that for all  $\varepsilon < \varepsilon(\delta)$  we have  $\int_{|x|<\delta} \rho_\varepsilon(x) dx = 1$ . In particular, for  $\varepsilon < \varepsilon(\delta)$  we have by (4.4.9) and (4.4.6)

$$\delta^{-d} \int_0^\delta s^{d-1} \frac{F_\varepsilon(s)}{s^p} ds \leq C(d) \delta^p.$$

In view of (4.4.8), the proof is concluded.  $\square$

#### 4.4.6 Nonlocal Poincaré inequalities

Let  $\omega : \mathbb{R}^d \rightarrow \mathbb{R}$  be a smooth function, supported in the unit ball such that  $\int_{\mathbb{R}^d} \omega(x) dx = 1$ . Consider  $\omega_\varepsilon = \frac{1}{\varepsilon^d} \omega\left(\frac{x}{\varepsilon}\right)$ .

**Lemma 4.4.6.** *There exists  $C_p$  and  $\varepsilon_0^A$  such that*

$$\int_{\mathbb{T}^d} |f - (f)_{\mathbb{T}^d}|^2 \leq \frac{1}{4C_p} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f(t,x) - f(t,y)|^2}{\varepsilon^2} \omega_\varepsilon(|x-y|) dx dy$$

for every  $f \in L^2(\mathbb{T}^d)$  and  $\varepsilon \leq \varepsilon_0^A$ .

For the proof, we refer to Ponce [294, Theorem 1.1] with kernel given by (4.4.5). We also have an opposite inequality from [53, Theorem 1] :

**Lemma 4.4.7.** *For all  $f \in H^1(\mathbb{T}^d)$*

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f(x) - f(y)|^2}{\varepsilon^2} \omega_\varepsilon(x-y) dx dy \leq C(\mathbb{T}^d) \|f\|_{H^1(\mathbb{T}^d)}^2.$$

Finally, we formulate a variant of Lemma 4.4.6 which does not require an average on the left-hand side.

**Lemma 4.4.8.** *For each  $\gamma \in (0, 1)$  there exists  $\varepsilon_0^B$  and constant  $C(\gamma)$  such that for all  $\varepsilon \in (0, \varepsilon_0^B)$  and all  $f \in H^1(\mathbb{T}^d)$  we have*

$$\|f\|_{H^1(\mathbb{T}^d)}^2 \leq \gamma \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|\nabla f(x) - \nabla f(y)|^2}{\varepsilon^2} \omega_\varepsilon(|x-y|) dx dy + C(\gamma) \|f\|_{L^2(\mathbb{T}^d)}^2.$$

*Démonstration.* Aiming at a contradiction, suppose that there exists  $\gamma$  with the following property : there exists sequence  $\{\varepsilon_n\}$  with  $0 < \varepsilon_n < \frac{1}{n}$  and sequence  $\{f_n\}$  such that

$$\|f_n\|_{H^1(\mathbb{T}^d)}^2 > \gamma \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|\nabla f_n(x) - \nabla f_n(y)|^2}{\varepsilon_n^2} \omega_{\varepsilon_n}(|x-y|) dx dy + n \|f_n\|_{L^2(\mathbb{T}^d)}^2.$$

As  $\|f_n\|_{H^1(\mathbb{T}^d)} > 0$ , we may define  $g_n := \frac{f_n}{\|f_n\|_{H^1(\mathbb{T}^d)}}$ . Note that  $\|g_n\|_{H^1(\mathbb{T}^d)} = 1$  and

$$1 > \gamma \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|\nabla g_n(x) - \nabla g_n(y)|^2}{\varepsilon_n^2} \omega_{\varepsilon_n}(|x-y|) dx dy + n \|g_n\|_{L^2(\mathbb{T}^d)}^2.$$

The first term gives compactness of the gradients (because  $\{g_n\}$  is bounded in  $H^1(\mathbb{T}^d)$  so that, together with Rellich-Kondrachov, there exists function  $g$  such that  $g_n \rightarrow g$  in  $H^1(\mathbb{T}^d)$  (after passing to a subsequence). But then  $g = 0$  because  $n \|g_n\|_{L^2(\mathbb{T}^d)} < 1$ . This is however contradiction with  $\|g\|_{H^1(\mathbb{T}^d)} = \lim_{n \rightarrow \infty} \|g_n\|_{H^1(\mathbb{T}^d)} = 1$ .  $\square$

#### 4.4.7 Compactness in time/space with the Fréchet-Kolmogorov theorem

**Lemma 4.4.9.** *Suppose that  $\{f_\varepsilon\}$  is a sequence bounded in  $L^2((0, T) \times \mathbb{T}^d)$  such that*

- $\partial_t f_\varepsilon = \nabla^k(J_\varepsilon)$ , where  $\nabla^k$  is any linear differential operator of order  $k \in \mathbb{N}$  and  $\{J_\varepsilon\}$  uniformly bounded in  $L^1((0, T) \times \mathbb{T}^d)$ ,
- $\{f_\varepsilon\}$  is compact in space in  $L^2((0, T) \times \mathbb{T}^d)$ , i.e.

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{T}^d} |f_\varepsilon * \varphi_\delta(t, x) - f_\varepsilon(t, x)|^2 dx dt = 0. \quad (4.4.10)$$

uniformly for all  $\varepsilon$ .

Then,  $\{f_\varepsilon\}$  is compact in time in  $L^2((0, T) \times \mathbb{T}^d)$ , i.e.

$$\lim_{h \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t, x)|^2 dx dt \rightarrow 0 \text{ as } h \rightarrow 0 \quad (4.4.11)$$

and so, it is compact in  $L^2((0, T) \times \mathbb{T}^d)$ .

We recall that

$$\|\nabla^k \varphi_\delta\|_{L^p(\mathbb{T}^d)} \leq \frac{C}{\delta^{k+d-d/p}},$$

and for any function  $g \in L^p(\mathbb{T}^d)$ ,

$$\|g * \varphi_\delta\|_{L^p(\mathbb{T}^d)} \leq \|\varphi_\delta\|_{L^p(\mathbb{T}^d)} \|g\|_{L^1(\mathbb{T}^d)}.$$

*Démonstration.* Using the mollifiers with  $\delta = \delta(h)$  depending on  $h$  to be specified later in the way that  $\delta(h) \rightarrow 0$  as  $h \rightarrow 0$ , we first split

$$\begin{aligned} \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t, x)|^2 dx dt &\leq 4 \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t, x) - f_\varepsilon(t, \cdot) * \varphi_\delta(x)|^2 dx dt \\ &\quad + 4 \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t+h, \cdot) * \varphi_\delta(x)|^2 dx dt \\ &\quad + 4 \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, \cdot) * \varphi_\delta(x) - f_\varepsilon(t, \cdot) * \varphi_\delta(x)|^2 dx dt. \end{aligned}$$

When we apply limit  $\lim_{h \rightarrow 0} \limsup_{\varepsilon \rightarrow 0}$ , the first and second term vanish due to (4.4.10). It



remains to study the third term which reads

$$\begin{aligned} & \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, \cdot) * \varphi_\delta(x) - f_\varepsilon(t, \cdot) * \varphi_\delta(x)|^2 dx dt = \\ &= \int_0^{T-h} \int_{\mathbb{T}^d} \left| \int_t^{t+h} \partial_t f_\varepsilon(s, \cdot) * \varphi_\delta(x) ds \right|^2 dx dt = \int_0^{T-h} \int_{\mathbb{T}^d} \left| \int_t^{t+h} J * \nabla^k \varphi_\delta(s, x) ds \right|^2 dx dt \\ &\leq Ch \int_0^{T-h} \int_{\mathbb{T}^d} \int_t^{t+h} |J * \nabla^k \varphi_\delta(s, x)|^2 ds dx dt, \end{aligned}$$

where we used Jensen's inequality. We perform the change of variables  $s \mapsto v = \frac{s-t}{h}$ , use Fubini's theorem, and obtain

$$h \int_0^{T-h} \int_{\mathbb{T}^d} \int_t^{t+h} |J * \nabla^k \varphi_\delta(s, x)|^2 ds dx dt = h^2 \int_0^1 \int_{\mathbb{T}^d} \int_0^{T-h} |J * \nabla^k \varphi_\delta(vh + t, x)|^2 dt dx dv.$$

Then we use the change of variables  $t \mapsto \tau = vh + t$  and obtain

$$\begin{aligned} h^2 \int_0^1 \int_{\mathbb{T}^d} \int_0^{T-h} |J * \nabla^k \varphi_\delta(vh + t, x)|^2 dt dx dv &= \\ &= h^2 \int_0^1 \int_{\mathbb{T}^d} \int_{vh}^{T+h(v-1)} |J * \nabla^k \varphi_\delta(\tau, x)|^2 d\tau dx dv \leq \frac{h^2}{\delta^{2k+d}} \|J_\varepsilon\|_{L^1_{t,x}}^2. \end{aligned}$$

Using the  $L^1((0, T) \times \mathbb{T}^d)$  bound on  $\{J_\varepsilon\}$  and choosing  $\delta$  such that  $\delta^{2k+d} = h$  we conclude that

$$\lim_{h \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t, x)|^2 dx dt \leq \theta(h).$$

Combined with the compactness in space (4.4.11) and the Fréchet-Kolmogorov theorem we obtain the compactness of  $\{f_\varepsilon\}$  in  $L^2((0, T) \times \mathbb{T}^d)$ .  $\square$

*Remark 4.4.10.* Compared with the usual version of the Fréchet-Kolmogorov theorem, one would expect that the condition for compactness in space should read

$$\lim_{y \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{T}^d} |f_\varepsilon(t, x+y) - f_\varepsilon(t, x)|^2 dx dt = 0. \quad (4.4.12)$$

However, by a careful inspection of the proof, (4.4.10) is sufficient and in fact, in the proof one deduces (4.4.10) from (4.4.12).

# Chapitre 5

## Degenerate Cahn-Hilliard systems : From nonlocal to local

### Abstract

We provide a rigorous mathematical framework to establish the limit of a nonlocal model of cell-cell adhesion system to a local model. When the parameter of the nonlocality goes to 0, the system tends to a Cahn-Hilliard system with degenerate mobility and cross interaction forces. Our analysis relies on a priori estimates and compactness properties.

### 5.1 Introduction

We consider the nonlocal system of Cahn-Hilliard equation with degenerate mobility derived in [165]

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \nabla (\kappa B[\rho] + \alpha B[\eta] - \gamma \rho - \beta \eta)), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (5.1.1)$$

$$\frac{\partial \eta}{\partial t} = \nabla \cdot (\eta \nabla (\alpha B[\rho] + B[\eta] - \beta \rho - \eta)), \quad \text{in } (0, +\infty) \times \mathbb{T}^d. \quad (5.1.2)$$

equipped with an initial datum  $(\rho_0, \eta_0) \in \mathcal{P}(\mathbb{T}^d)^2$  where  $\mathcal{P}(\mathbb{T}^d)$  is the space of probability measures on the flat torus of dimension  $d$ . Here  $\kappa, \alpha, \gamma > 0$ ,  $\beta \in \mathbb{R}$  while  $B$  is the nonlocal operator  $B = B_\varepsilon$  defined with

$$B_\varepsilon[u_\varepsilon](x) = \frac{1}{\varepsilon^2} (u_\varepsilon(x) - \omega_\varepsilon * u_\varepsilon(x)) = \frac{1}{\varepsilon^2} \int_{\mathbb{T}^d} \omega_\varepsilon(y) (u_\varepsilon(x) - u_\varepsilon(x - y)) dy \quad (5.1.3)$$

for  $\varepsilon$  small enough and  $\omega_\varepsilon$  is the usual radial mollification kernel  $\omega_\varepsilon(x) = \frac{1}{\varepsilon^d} \omega(\frac{x}{\varepsilon})$  with  $\omega$  compactly supported in the unit ball satisfying

$$\int_{\mathbb{R}^d} \omega(y) dy = 1, \quad \int_{\mathbb{R}^d} y \omega(y) dy = 0, \quad \int_{\mathbb{R}^d} y_i y_j \omega dy = \delta_{i,j} \frac{2}{d}. \quad (5.1.4)$$

Our target is to prove that as  $\varepsilon \rightarrow 0$ , the constructed solutions of

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \nabla (\kappa B_\varepsilon[\rho] + \alpha B_\varepsilon[\eta] - \gamma \rho - \beta \eta)), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (5.1.5)$$

$$\frac{\partial \eta}{\partial t} = \nabla \cdot (\eta \nabla (\alpha B_\varepsilon[\rho] + B_\varepsilon[\eta] - \beta \rho - \eta)), \quad \text{in } (0, +\infty) \times \mathbb{T}^d \quad (5.1.6)$$

tend to the weak solution of the local system of degenerate Cahn-Hilliard equations

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \nabla (-\kappa \Delta \rho - \alpha \Delta \eta - \gamma \rho - \beta \eta)), \quad (5.1.7)$$

$$\frac{\partial \eta}{\partial t} = \nabla \cdot (\eta \nabla (-\alpha \Delta \rho - \Delta \eta - \beta \rho - \eta)). \quad (5.1.8)$$

Here,  $\kappa > 0$  and  $\gamma > 0$  represent the relative self-adhesion strength of  $\rho$  with respect to  $\eta$ ; while  $\alpha > 0$  and  $\beta \in \mathbb{R}$  give the relative strength of the cross-attraction forces. We also denote by

$$\begin{aligned} \mu_{\rho,\varepsilon} &= \kappa B_\varepsilon[\rho] + \alpha B_\varepsilon[\eta] - \gamma \rho - \beta \eta, \\ \mu_{\eta,\varepsilon} &= \alpha B_\varepsilon[\rho] + B_\varepsilon[\eta] - \beta \rho - \eta, \end{aligned}$$

the chemical potentials (from the Cahn-Hilliard terminology) related to the nonlocal system. The nonlocal system is associated with the following formal energy/entropy structure

$$\begin{aligned} E_\varepsilon[\rho, \eta] &:= \frac{1}{4} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (\kappa |\rho(x) - \rho(x-y)|^2 + |\eta(x) - \eta(x-y)|^2) dx dy \\ &+ \frac{\alpha}{2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (\rho(x) - \rho(x-y))(\eta(x) - \eta(x-y)) dx dy - \int_{\mathbb{T}^d} \frac{\gamma}{2} \rho^2 + \frac{1}{2} \eta^2 + \beta \rho \eta dx, \end{aligned} \quad (5.1.9)$$

$$\Phi[\rho, \eta] := \int_{\mathbb{T}^d} \rho(\log(\rho) - 1) + \eta(\log(\eta) - 1) dx. \quad (5.1.10)$$

Their dissipation is formally controlled by the identities

$$E_\varepsilon[\rho, \eta](t) + \int_0^t \int_{\mathbb{T}^d} \rho |\nabla \mu_{\rho,\varepsilon}|^2 + \int_0^t \int_{\mathbb{T}^d} \eta |\nabla \mu_{\eta,\varepsilon}|^2 \leq E_\varepsilon[\rho_0, \eta_0], \quad (5.1.11)$$

$$\Phi[\rho, \eta](t) + \mathcal{D}\Phi[\rho, \eta](t) \leq \Phi[\rho_0, \eta_0]. \quad (5.1.12)$$

where  $\mathcal{D}\Phi[\rho, \eta](t)$  is the dissipation of the entropy defined as

$$\begin{aligned} \mathcal{D}\Phi[\rho, \eta](t) &= \frac{1}{2} \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (\kappa |\nabla \rho(x) - \nabla \rho(x-y)|^2 + |\nabla \eta(x) - \nabla \eta(x-y)|^2) dx dy ds \\ &+ \alpha \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (\nabla \rho(x) - \nabla \rho(x-y)) \cdot (\nabla \eta(x) - \nabla \eta(x-y)) dx dy ds \\ &- \int_0^t \int_{\mathbb{T}^d} \gamma |\nabla \rho|^2 + |\nabla \eta|^2 + 2\beta \nabla \rho \cdot \nabla \eta dx ds. \end{aligned}$$

It turns out (see Proposition 5.2.1) that for  $\kappa > \alpha^2$ ,  $\kappa > 0$  they provide strong compactness in space of  $\rho_\varepsilon$ ,  $\eta_\varepsilon$ ,  $\nabla \rho_\varepsilon$ ,  $\nabla \eta_\varepsilon$ .

Our first result states that we can construct solutions to (5.1.5)-(5.1.6) satisfying additional uniform estimates which will be relevant in the sequel.

**Theorem 5.1.1** (Existence of solutions for the nonlocal system). *Suppose that  $\kappa > 0$ ,  $\kappa > \alpha^2$ . Let  $\varepsilon_0 = \min(\varepsilon_0^A, \varepsilon_0^B)$  be given by Proposition 5.2.1 and Lemma 5.3.2. Let  $u_0 = (\rho_0, \eta_0) \in \mathcal{P}(\mathbb{T}^d)^2$  be an initial datum with finite energy and entropy  $E_\varepsilon(u^0), \Phi(u^0) \leq C$  defined in (5.1.9)-(5.1.10) where  $C$  is independent of  $\varepsilon$ . Let  $\varepsilon \leq \varepsilon_0$ . Then, there exists a global weak solution  $u_\varepsilon \in \mathcal{P}(\mathbb{T}^d)^2$  of (5.1.5)-(5.1.6) as defined in Definition 5.3.1. Moreover, it satisfies*

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (|\rho_\varepsilon(x) - \rho_\varepsilon(x-y)|^2 + |\eta_\varepsilon(x) - \eta_\varepsilon(x-y)|^2) dx dy \leq C, \quad (5.1.13)$$

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (|\nabla \rho_\varepsilon(x) - \nabla \rho_\varepsilon(x-y)|^2 + |\nabla \eta_\varepsilon(x) - \nabla \eta_\varepsilon(x-y)|^2) dx dy dt \leq C, \quad (5.1.14)$$

for a constant  $C$  that depends on parameters and the initial condition  $u^0$  but not on  $\varepsilon$ .

The proof of this result follows the argument of [135]. We decided to include the proof to demonstrate estimates (5.1.13)–(5.1.14) which are essential for our main result which reads as follows.

**Theorem 5.1.2** (Convergence of nonlocal to local Cahn-Hilliard equation on the torus). *Suppose that  $\kappa > 0$ ,  $\kappa > \alpha^2$ . Let  $u_0 \geq 0$  be an initial datum with finite energy and entropy  $E_\varepsilon(u^0), \Phi(u^0) \leq C$  defined in (5.1.9) and (5.1.10) where  $C$  is independent of  $\varepsilon$ . Let  $\{u_\varepsilon\}$  be a sequence of solutions of the degenerate nonlocal Cahn-Hilliard equation (5.1.5)-(5.1.6) as defined in Definition 5.3.1. Then, up to a subsequence (not relabelled),*

$$u_\varepsilon \rightarrow u \text{ in } L^2(0, T; H^1(\mathbb{T}^d)),$$

where  $u$  is a weak solution of the degenerate Cahn-Hilliard system (5.1.7)-(5.1.8) as in Definition 5.4.1.

We conclude with a short discussion of applied techniques and considered problems.

Existence of weak solutions. The strategy to prove the existence of weak solutions is based on the gradient flow structure of the equation in the Wasserstein space. We use the JKO scheme first introduced in [234] by Jordan, Kinderlehrer and Otto in the context of the Fokker-Planck equation. The main idea is to use an implicit time discretization of the associated variational problems. The sequence created minimizes movements. When the time step goes to 0, the sequence converges to the associated gradient flow. This strategy has been used for instance in [258, 244, 271] in the context of Cahn-Hilliard equation and in [135] in the context of cross-diffusion systems with nonlocal interaction. The proof of Theorem 5.1.1 is in fact closely related to the one in [135] and we follow to some extent their proof. The difference is the control of the positiveness of the energy and the transport of uniform bounds independently of  $\varepsilon$  needed in the last section. Also, our settings are periodic in space. The proof uses two main components :

- The "gradient flow" structure of the scheme which provides classical discrete energy estimates and Hölder continuity in time and allow narrow convergences of the scheme.
- The weak convergences being not enough, we use the flow interchange lemma [270, 258]. The idea is that we usually obtain better estimates with the entropy in the Cahn-Hilliard equation. This entropy generates a heat-flow. Then the flow interchange lemma allows to exchange the dissipation of one functional along the gradient flow of another one and thus improve the regularity of the scheme.

Nevertheless, since we assume that  $\kappa > \alpha^2$  and  $\kappa > 0$ , the system is strongly parabolic with respect to  $(\rho, \eta)$ . Therefore, to prove existence, one could approximate (5.1.5)–(5.1.6) by

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (T_\delta(\rho) \nabla (\kappa B_\varepsilon[\rho] + \alpha B_\varepsilon[\eta] - \gamma \rho - \beta \eta)), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (5.1.15)$$

$$\frac{\partial \eta}{\partial t} = \nabla \cdot (T_\delta(\eta) \nabla (\alpha B_\varepsilon[\rho] + B_\varepsilon[\eta] - \beta \rho - \eta)), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (5.1.16)$$

where  $T_\delta$  is a function such that  $\delta \leq T_\delta \leq \frac{1}{\delta}$  and  $T_\delta(\varrho) \rightarrow \varrho$  as  $\delta \rightarrow 0$ . Then, one sends  $\delta \rightarrow 0$  and obtains solutions to (5.1.5)–(5.1.6). This method is a standard way of proving existence of solutions to Cahn-Hilliard equation with degenerate mobility [113, 157, 151, 287, 156]. Of course, one has to prove existence to (5.1.15)–(5.1.16) by virtue of fixed point method and Schauder's estimates for parabolic equations which is very technical. Therefore, we preferred to apply gradient flow techniques which are natural for our problem.

*Passage to the limit  $\varepsilon \rightarrow 0$  and nonlocal compactness results.* We use the strategy developed by the second and third author in [155] for the single Cahn-Hilliard equation. The main tool is the compactness result due to Bourgain-Brezis-Mironescu [53] and Ponce [294] which reads as follows :

**Proposition 5.1.3.** *Let  $\{f_\varepsilon\}$  be a sequence bounded in  $L^2(\mathbb{T}^d)$ . Suppose that*

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f_\varepsilon(x) - f_\varepsilon(y)|^2}{|x - y|^2} \omega_\varepsilon(|x - y|) dx dy \leq C \quad (5.1.17)$$

for some constant  $C$ . Then,  $\{f_\varepsilon\}$  is strongly compact in  $L^2(\mathbb{T}^d)$  and the limit  $f \in W^{1,2}(\mathbb{T}^d)$ .

Proposition 5.1.3 is crucial because the difference quotient appearing in (5.1.17) can be controlled by the energy and the dissipation of the entropy of Cahn-Hilliard equation which yields compactness. The novelty comes from the treating of cross-interactions terms. It turns out that we can control possibly negative terms appearing in the energy and the dissipation of entropy by a simple interpolation inequality, see Lemma 5.2.2 and Proposition 5.2.1.

*Aggregation-diffusion systems.* System (5.1.5)–(5.1.6) is an example of aggregation-diffusion system [85] which are attracting a lot of mathematical attention nowadays [313, 86, 33, 77, 164]. To motivate, let us first start with an aggregation equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \quad u = -\nabla W * \rho, \quad (5.1.18)$$

where  $W : \mathbb{T}^d \rightarrow \mathbb{R}$  is a symmetric interaction potential. Over the last years, equation (5.1.18) was applied in the context of biological aggregation [51, 319, 278], materials science [216], granular media [88, 43] and it attracted also a lot of mathematical interest, particularly for non-smooth potentials. We only refer to [47] for the  $L^p$  theory, to [76] for the theory in spaces of measures and to [46] for blow-up conditions. Equation (5.1.18) can be derived from the particle system via mean-field limit [225, 224] : one considers  $N$  particles having positions  $\{X_i\}_{i=1, \dots, N}$  satisfying system of ODEs :

$$X'_i = -\frac{1}{N} \sum_{j \neq i} \nabla W(X_i - X_j);$$

then, under appropriate assumptions on  $W$ , empirical measure  $\rho^N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}$  converges (in the weak\* topology of measures) to a solution of (5.1.18), see [91, Theorem 3.1] for the proof and [84] for some extensions.

To obtain the aggregation-diffusion equation from (5.1.18), one can consider interaction potentials of the form  $W_\nu = W + 2\nu\delta_0$  modeling two effects : repulsion of strength  $\nu$  and non-local attraction. Informally, we obtain

$$\frac{\partial \rho}{\partial t} = \nu \Delta \rho^2 + \nabla \cdot (\rho \nabla W * \rho), \quad (5.1.19)$$

but this can be made rigorous by approximating the Dirac mass with a sequence of mollifiers, see [92, 65] with  $m = 2$ . Of course, (5.1.19) can be extended to include more general diffusion term  $\Delta \rho^m$  instead of  $\Delta \rho^2$  as well as an advection term, see [93].

One can also consider system of particles  $\{X_i\}_{i=1,\dots,N}$ ,  $\{Y_i\}_{i=1,\dots,N}$  representing two populations. Then, repeating the derivation explained above, one arrives at a system of equations of the form (5.1.1)–(5.1.2) which can be used to model cell-cell adhesion [26, 87, 82] to reproduce the Steinberg cell-sorting phenomena. Structure preserving numerical schemes have been derived for equations and systems of aggregation-diffusion type [83, 31, 30] as well as Cahn-Hilliard equations and systems [32, 165] recovering the cell sorting mechanism.

*Gradient flows in the periodic setting.* Due to the presence of a non-local operator  $B_\varepsilon$  in (5.1.5)–(5.1.6), we develop our theory on the  $d$ -dimensional torus  $\mathbb{T}^d$  which makes non-local terms easy to be defined. As already explained, solutions to (5.1.5)–(5.1.6) will be constructed via JKO scheme. In what follows, we briefly review the theory of optimal transport on  $\mathbb{T}^d$  comparing to the usual case of  $\mathbb{R}^d$  or bounded domain  $\Omega \subset \mathbb{R}^d$ . We refer the Reader to [305, Section 1.3.2].

First, we need to define a metric and the natural choice is

$$d(x, y) = \inf_{k \in \mathbb{Z}^d} |x - y + k|, \quad x, y \in \mathbb{T}^d,$$

where  $|\cdot|$  is the Euclidean distance. Then, the Wasserstein distance  $W_2$  is defined as

$$W_2^2(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{T}^d \times \mathbb{T}^d} d(x, y)^2 d\pi(x, y), \quad \mu, \nu \in \mathcal{P}(\mathbb{T}^d),$$

where  $\Pi(\mu, \nu)$  is the set of couplings between  $\mu$  and  $\nu$ . This already implies that, say in dimension  $d = 1$ , the optimal transport maps are not necessarily monotone on the torus. Nevertheless, the optimal map always exists if at least one of the measures is absolutely continuous with respect to the Lebesgue measure. Moreover, the optimal map is given by the gradient of some function which is differentiable a.e. We refer to [111] for the first proof of this fact and to [305, Theorem 1.25] for a modern presentation. We also refer to the general result of McCann [272] who proved existence of the optimal map on the general manifold, including the case of torus.

To conclude, let us mention that optimal transport was used to study several PDEs on the torus via JKO scheme, in particular fractional porous medium equation [173], continuity equation with nonlocal velocity in 1D [174], systems of continuity equations with nonlinear diffusion and nonlocal drifts [73] and certain fourth-order equation in one dimension [89].

## 5.2 Energy and entropy for the non-local system

As already mentioned, system (5.1.5)–(5.1.6) has the energy/entropy structure which can be used to obtain compactness estimates. In order to do so, one has to assure nonnegativity of the

quantities of interest. In our case, we focus on the energy  $E_\varepsilon$  and the dissipation of entropy  $\mathcal{D}\Phi$ .

**Proposition 5.2.1.** *Suppose that  $\kappa > 0$ ,  $\kappa > \alpha^2$ . Then, there exists  $\varepsilon_0^A > 0$  depending on  $\kappa$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$  with the following property : for all  $\varepsilon \in (0, \varepsilon_0^A)$  and  $\rho, \eta \in \mathcal{P}(\mathbb{T}^d)$ , up to a constant, the energy defined by (5.1.9) and the dissipation of the entropy defined in (5.1.12) are nonnegative and provide the estimates on the quantities*

$$\begin{aligned} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (|\rho(x) - \rho(x-y)|^2 + |\eta(x) - \eta(x-y)|^2) dx dy &\leq C + E_\varepsilon[\rho, \eta], \\ \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (|\nabla \rho(x) - \nabla \rho(x-y)|^2 + |\nabla \eta(x) - \nabla \eta(x-y)|^2) dx dy ds &\leq C + \mathcal{D}\Phi[\rho, \eta](t), \end{aligned}$$

where  $C$  depends on  $\kappa$ ,  $\alpha$ ,  $\beta$  and  $\gamma$ .

The main tool to establish nonnegativity (up to a constant) is the following non-local Poincaré inequality with parameter which allows to handle negative terms.

**Lemma 5.2.2** (Poincaré inequality with a parameter). *For all  $\delta > 0$  there exists  $C(\delta)$  and  $\varepsilon_0 = \varepsilon_0(\delta)$  such that for all  $\varepsilon < \varepsilon_0$  and all  $f \in H^1(\mathbb{T}^d)$  we have*

$$\|f\|_2^2 \leq \delta \int_{\mathbb{T}^d \times \mathbb{T}^d} \omega_\varepsilon(y) \frac{|f(x) - f(x-y)|^2}{\varepsilon^2} dx dy + C(\delta) \|f\|_1^2, \quad (5.2.1)$$

$$\|\nabla f\|_2^2 \leq \delta \int_{\mathbb{T}^d \times \mathbb{T}^d} \omega_\varepsilon(y) \frac{|\nabla f(x) - \nabla f(x-y)|^2}{\varepsilon^2} dx dy + C(\delta) \|f\|_1^2. \quad (5.2.2)$$

*Démonstration.* We prove (5.2.2) as (5.2.1) is in fact easier. Aiming at a contradiction, suppose that there exists  $\delta > 0$  with the following property : there exists sequence  $\{\varepsilon_n\}$  with  $0 < \varepsilon_n < \frac{1}{n}$  and sequence  $\{f_n\}$  such that

$$\|\nabla f_n\|_{L^2(\mathbb{T}^d)}^2 > \delta \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|\nabla f_n(x) - \nabla f_n(y)|^2}{\varepsilon_n^2} \omega_{\varepsilon_n}(|x-y|) dx dy + n \|f_n\|_{L^1(\mathbb{T}^d)}^2.$$

As  $\|\nabla f_n\|_{L^2(\mathbb{T}^d)} > 0$ , we may define  $g_n := \frac{f_n}{\|\nabla f_n\|_{L^2(\mathbb{T}^d)}}$ . Note that  $\|\nabla g_n\|_{L^2(\mathbb{T}^d)} = 1$  and

$$1 > \gamma \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|\nabla g_n(x) - \nabla g_n(y)|^2}{\varepsilon_n^2} \omega_{\varepsilon_n}(|x-y|) dx dy + n \|g_n\|_{L^1(\mathbb{T}^d)}^2.$$

By Poincaré inequality with average,  $\{g_n\}$  is bounded in  $L^2(\mathbb{T}^d)$  and so in  $H^1(\mathbb{T}^d)$ . Moreover, the first term gives compactness of the gradients (because  $\{g_n\}$  is bounded in  $H^1(\mathbb{T}^d)$  so that, together with Rellich-Kondrachov, there exists function  $g$  such that  $g_n \rightarrow g$  in  $H^1(\mathbb{T}^d)$  (after passing to a subsequence). But then  $g = 0$  because  $n \|g_n\|_{L^1(\mathbb{T}^d)} < 1$ . This is however contradiction with  $\|\nabla g\|_{L^2(\mathbb{T}^d)} = \lim_{n \rightarrow \infty} \|\nabla g_n\|_{L^2(\mathbb{T}^d)} = 1$ .  $\square$

*Proof of Proposition 5.2.1.* We first focus on the energy. We can estimate

$$-\int_{\mathbb{T}^d} \frac{\gamma}{2} \rho^2 + \frac{1}{2} \eta^2 + \beta \rho \eta \geq -\frac{\gamma + |\beta|}{2} \int_{\mathbb{T}^d} \rho^2 - \frac{1 + |\beta|}{2} \int_{\mathbb{T}^d} \eta^2.$$

Then, we use (5.2.1) in Lemma 5.2.2 with  $\delta := \delta / \max\left(\frac{\gamma + |\beta|}{2}, \frac{1 + |\beta|}{2}\right)$  and  $\delta > 0$  to be chosen

later (this also determines  $\varepsilon_0^A = \varepsilon_0^A(\delta)$ ) so that we obtain for  $\varepsilon \in (0, \varepsilon_0^A)$

$$\begin{aligned} & - \int_{\mathbb{T}^d} \frac{\gamma}{2} \rho^2 + \frac{1}{2} \eta^2 + \beta \rho \eta \geq \\ & \geq - \frac{\delta}{4} \int_{\mathbb{T}^d \times \mathbb{T}^d} \omega_\varepsilon(y) \frac{|\rho(x) - \rho(x-y)|^2}{\varepsilon^2} dx dy - \frac{\delta}{4} \int_{\mathbb{T}^d \times \mathbb{T}^d} \omega_\varepsilon(y) \frac{|\eta(x) - \eta(x-y)|^2}{\varepsilon^2} dx dy - C(\delta). \end{aligned}$$

Therefore, using (5.1.9), we can bound energy as follows

$$\begin{aligned} E_\varepsilon[\rho, \eta] & \geq \frac{1}{4\varepsilon^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) ((\kappa - \delta) |\rho(x) - \rho(x-y)|^2 + (1 - \delta) |\eta(x) - \eta(x-y)|^2) dx dy \\ & \quad + \frac{\alpha}{2\varepsilon^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) (\rho(x) - \rho(x-y)) (\eta(x) - \eta(x-y)) dx dy - C(\delta). \end{aligned}$$

Now, by continuity, we choose  $\delta$  so small so that  $\kappa - \delta > 0$  and  $(\kappa - \delta)(1 - \delta) - \alpha^2 > 0$ , i.e. so that the matrix  $\begin{pmatrix} \kappa - \delta & \alpha \\ \alpha & 1 - \delta \end{pmatrix}$  is positively defined. It follows that the associated quadratic form is bounded from below, that is there exists constant  $C$  (in fact, this constant is the smallest eigenvalue of the matrix) such that

$$E_\varepsilon[\rho, \eta] \geq \frac{C}{\varepsilon^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) (|\rho(x) - \rho(x-y)|^2 + |\eta(x) - \eta(x-y)|^2) dx dy - C(\delta).$$

The proof for the entropy is the same : this time we use (5.2.2) in place of (5.2.1).  $\square$

### 5.3 Existence of weak solutions to the nonlocal problem

To prove the existence of weak solutions for our system, we want to apply the JKO scheme, see [234]. As  $\varepsilon$  is fixed in this Section, we write  $(\rho, \eta)$  for the solution, instead of  $(\rho_\varepsilon, \eta_\varepsilon)$ .

*Definition 5.3.1* (Weak solutions). We say that  $u = (\rho(\cdot), \eta(\cdot)) : [0, +\infty) \rightarrow \mathcal{P}(\mathbb{T}^d)^2$  is a weak solution of (5.1.5)-(5.1.6) with initial condition  $(\rho_0, \eta_0)$  satisfying (5.3.4) if  $\rho, \eta \in C([0, T]; \mathcal{P}(\mathbb{T}^d))$ ,  $\rho, \eta \in L^2(0, T; H^1(\mathbb{T}^d)) \cap L^\infty(0, T; L^2(\mathbb{T}^d))$  for all  $T > 0$ , and if for all  $\varphi, \phi \in C_c^\infty([0, +\infty) \times \mathbb{T}^d)$  we have

$$\begin{aligned} & - \int_0^\infty \int_{\mathbb{T}^d} \rho \partial_t \varphi dx dt - \int_{\mathbb{T}^d} \rho_0 \varphi(0) dx = -\kappa \int_0^\infty \int_{\mathbb{T}^d} \rho \nabla B_\varepsilon[\rho] \cdot \nabla \varphi dx dt \\ & \quad - \alpha \int_0^\infty \int_{\mathbb{T}^d} \rho \nabla B_\varepsilon[\eta] \cdot \nabla \varphi dx dt - \int_0^\infty \int_{\mathbb{T}^d} \rho (\gamma \nabla \rho - \beta \nabla \eta) \cdot \nabla \varphi dx dt, \\ & - \int_0^\infty \int_{\mathbb{T}^d} \eta \partial_t \phi dx dt - \int_{\mathbb{T}^d} \eta_0 \phi(0) dx = - \int_0^\infty \int_{\mathbb{T}^d} \eta \nabla B_\varepsilon[\eta] \cdot \nabla \phi dx dt \\ & \quad - \alpha \int_0^\infty \int_{\mathbb{T}^d} \eta \nabla B_\varepsilon[\rho] \cdot \nabla \phi dx dt - \int_0^\infty \int_{\mathbb{T}^d} \eta (\nabla \eta - \beta \nabla \rho) \cdot \nabla \phi dx dt. \end{aligned}$$



We first rewrite system of equations (5.1.5)-(5.1.6) in the following form

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \nabla (\tilde{\kappa} \rho - (\tilde{\kappa} + \gamma) \omega_\varepsilon * \rho + \tilde{\alpha} \eta - (\tilde{\alpha} + \beta) \omega_\varepsilon * \eta)), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (5.3.1)$$

$$\frac{\partial \eta}{\partial t} = \nabla \cdot (\eta \nabla (\tilde{\alpha} \rho - (\tilde{\alpha} + \beta) \omega_\varepsilon * \rho + \tilde{c} \eta - (\tilde{c} + 1) \omega_\varepsilon * \eta)), \quad \text{in } (0, +\infty) \times \mathbb{T}^d. \quad (5.3.2)$$

where  $\tilde{\kappa} = \frac{\kappa}{\varepsilon^2} - \gamma$ ,  $\tilde{\alpha} = \frac{\alpha}{\varepsilon^2} - \beta$ ,  $\tilde{c} = \frac{1}{\varepsilon^2} - 1$ . Using this notation, the energy can be written as

$$E_\varepsilon[\rho, \eta] = \int_{\mathbb{T}^d} \frac{\tilde{\kappa}}{2} \rho^2 + \frac{\tilde{c}}{2} \eta^2 + \tilde{\alpha} \rho \eta \, dx - \frac{\tilde{\kappa} + \gamma}{2} \int_{\mathbb{T}^d} \omega_\varepsilon * \rho \, d\rho - \frac{\tilde{c} + 1}{2} \int_{\mathbb{T}^d} \omega_\varepsilon * \eta \, d\eta - (\tilde{\alpha} + \beta) \int_{\mathbb{T}^d} \omega_\varepsilon * \eta \, d\rho \quad (5.3.3)$$

for  $\rho, \eta \in \mathcal{P}(\mathbb{T}^d)$ . We note that when we use  $\rho$  and  $\eta$  in expressions of the form  $\rho^2, \eta^2, \rho\eta$ , we mean in fact their Radon-Nikodym derivatives with respect to the Lebesgue measures. If they exist and belong to  $L^2(\mathbb{T}^d)$ , the energy above makes sense. Otherwise, we consider that these quantities are  $+\infty$ .

We denote the Wasserstein distance for vectors  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$  as

$$\mathcal{W}_2^2(u, v) = W_2^2(u_1, v_1) + W_2^2(u_2, v_2),$$

for all  $u, v \in \mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d)$  and where  $W_2$  is the usual Wasserstein distance of order 2. For the initial condition, we assume that  $(\rho_0, \eta_0) \in \mathcal{P}(\mathbb{T}^d)^2$  are absolutely continuous with respect to the Lebesgue measure and satisfy

$$E_\varepsilon[\rho_0, \eta_0] \leq C, \quad \Phi[\rho_0, \eta_0] \leq C, \quad (5.3.4)$$

where  $C$  is independent of  $\varepsilon$ .

In what follows it will be necessary to know when the function  $(\rho, \eta) \mapsto \frac{\tilde{\kappa}}{2} \rho^2 + \frac{\tilde{c}}{2} \eta^2 + \tilde{\alpha} \rho \eta$ . Fortunately, when  $\kappa > 0$ ,  $\kappa > \alpha^2$ , this is always the case for sufficiently small  $\varepsilon$ .

**Lemma 5.3.2.** *Suppose that  $\kappa > 0$ ,  $\kappa > \alpha^2$ . Then, there exists  $\varepsilon_0^B > 0$  depending on  $\kappa, \alpha, \gamma, \beta$  such that for all  $\varepsilon \in (0, \varepsilon_0^B)$  the function  $(\rho, \eta) \mapsto \frac{\tilde{\kappa}}{2} \rho^2 + \frac{\tilde{c}}{2} \eta^2 + \tilde{\alpha} \rho \eta$  is convex.*

*Démonstration.* We note that the Hessian matrix reads  $\begin{pmatrix} \tilde{\kappa} & \tilde{\alpha} \\ \tilde{\alpha} & \tilde{c} \end{pmatrix}$  so by the Sylvester's criterion, the desired convexity is equivalent with  $\tilde{\kappa} > 0$ ,  $\tilde{\kappa} \tilde{c} - \tilde{\alpha}^2 > 0$ . Concerning the first condition, as  $\kappa > 0$  and  $\tilde{\kappa} = \frac{\kappa}{\varepsilon^2} - \gamma$ , we can easily find  $\varepsilon$  such that  $\tilde{\kappa} > 0$ . Concerning the second condition, we observe that the term standing next to the highest order term  $\frac{1}{\varepsilon^4}$  equals  $\kappa - \alpha^2$  and it is positive by assumption. Therefore, the conclusion follows.  $\square$

### 5.3.1 Construction of weak solutions

Let  $T \in (0, \infty)$  be a final time of existence and consider the following scheme : given a time step size  $\tau > 0$  and an initial condition  $u_0 = (\rho_0, \eta_0) \in \mathcal{P}(\mathbb{T}^d)^2$  with  $E[u_0] < +\infty$  we define by induction

$$u_\tau^0 := u_0, \quad u_\tau^{n+1} := \operatorname{argmin}_{u \in \mathcal{P}(\mathbb{T}^d)^2} \left\{ \frac{1}{2\tau} \mathcal{W}_2^2(u_\tau^n, u) + E_\varepsilon[u] \right\}, \quad (5.3.5)$$

and we set  $E_\varepsilon[u] = +\infty$  if  $u \notin L^2(\mathbb{T}^d)^2$ .

**Lemma 5.3.3** (Existence of minimizers). *Let  $\tau > 0$  and  $u_0 = (\rho_0, \eta_0) \in \mathcal{P}_2(\mathbb{T}^d)^2$  with  $E_\varepsilon[u_0] < +\infty$ . Then the scheme defined by (5.3.5) is well-defined. Moreover, we have the following energy estimate*

$$E_\varepsilon[u_\tau^N] + \frac{1}{2\tau} \sum_{n=1}^N \mathcal{W}_2^2(u_\tau^n, u_\tau^{n-1}) \leq E_\varepsilon[u_0]. \quad (5.3.6)$$

*Démonstration. Step 1 : The infimum is bounded.* Provided  $u_\tau^n$  is defined, we notice that  $u_\tau^n \in \mathcal{P}(\mathbb{T}^d)^2$  satisfy  $\frac{1}{2\tau} \mathcal{W}_2^2(u_\tau^n, u_\tau^n) + E_\varepsilon[u_\tau^n] = E_\varepsilon[u_\tau^n] < +\infty$ . Therefore the infimum is bounded from above. Since  $E_\varepsilon[u]$  is bounded from below by Proposition 5.2.1 and  $\mathcal{W}_2^2(u_\tau^n, u)$  is nonnegative for all  $u$  we also know that the infimum is bounded from below.

*Step 2 : Candidate for a minimizer.* We suppose that  $u_\tau^n$  is defined and we want to define  $u_\tau^{n+1}$ . Let  $\{u^k\}_k$  be a minimizing sequence in  $\mathcal{P}(\mathbb{T}^d)^2$  for the problem (5.3.5). Without loss of generality, we can assume that for  $k$  large enough we have

$$\frac{1}{2\tau} \mathcal{W}_2^2(u_\tau^n, u^k) + E_\varepsilon[u^k] \leq 2 E_\varepsilon[u_\tau^n], \quad (5.3.7)$$

since the infimum is bounded by  $E_\varepsilon[u_\tau^n]$ . In particular

$$E_\varepsilon[u^k] \leq C(u_\tau^n)$$

with a constant  $C$  independent of  $k$ . By definition of the energy in (5.3.3), this proves that we can extract from  $(u^k)_k$  a subsequence (still denoted by  $k$ ) which converges weakly to some  $u$  in  $L^2(\mathbb{T}^d)^2$ . Now we consider the function  $f(\rho, \eta) = \frac{\kappa}{2}\rho^2 + \frac{\tilde{c}}{2}\eta^2 + \tilde{\alpha}\rho\eta$  which is convex by Lemma 5.3.2. It follows by Tonelli theorem that the functional defined by the first integral of (5.3.3) is lower semi-continuous with respect to the  $L^2$  weak convergence. Since the other terms of (5.3.3) are defined with convolutions, these terms are continuous with respect to the  $L^2$  weak convergence. In the end,  $u$  minimizes (5.3.5). Finally,  $u \in \mathcal{P}(\mathbb{T}^d)^2$  as a direct consequence of the weak convergence.

*Step 3 : The energy estimate (5.3.6).* Estimate (5.3.6) is a consequence of an induction of the inequality

$$E_\varepsilon[u_\tau^{n+1}] + \frac{1}{2\tau} \mathcal{W}_2^2(u_\tau^{n+1}, u_\tau^n) \leq E_\varepsilon[u_\tau^n],$$

by definition of  $u_\tau^n$ . □

We have constructed a discrete in time sequence. We want to prove that a time-interpolation of this sequence converges to a solution of (5.3.1)-(5.3.2). Let  $T > 0$  be fixed and  $n = \lceil \frac{T}{\tau} \rceil$ . We define the interpolation  $u_\tau(t) = (\rho_\tau(t), \eta_\tau(t))$  by

$$\rho_\tau(t) = \rho_\tau^n, \quad \eta_\tau(t) = \eta_\tau^n, \quad t \in ((n-1)\tau, n\tau],$$

where  $(\rho_\tau^n, \eta_\tau^n) = u_\tau^n$  defined in (5.3.5). We prove that this sequence is compact in the following lemma :

**Lemma 5.3.4** (Compactness of the time interpolation sequence). *The sequence of curves  $\{u_\tau\}_\tau$  is uniformly bounded in  $L^\infty(0, T; L^2(\mathbb{T}^d))$ . Moreover, for all  $T > 0$ , there exists an absolutely continuous curve  $u : [0, T] \rightarrow \mathcal{P}(\mathbb{T}^d)^2$  such that up to a subsequence,  $u_\tau(t, \cdot)$  converges to  $u(t, \cdot)$  weakly in  $L^2(\mathbb{T}^d)$  as  $\tau \rightarrow 0$  for all  $t \in [0, T]$  and  $u_\tau$  converges to  $u$  in  $C([0, T], \mathcal{P}(\mathbb{T}^d)^2)$  as  $\tau \rightarrow 0$  for all  $T > 0$ . The curve  $u$  is globally  $1/2$ -Hölder continuous in time*

$$\mathcal{W}_2(u_t, u_s) \leq \sqrt{2E_\varepsilon[u_0]} \sqrt{|s-t|}, \quad (5.3.8)$$

and we have the estimate

$$\|u\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \leq C(1 + E_\varepsilon[u_0]).$$

*Démonstration.* From (5.3.6) and by definition of  $u_\tau$  which takes discrete values in time we have that

$$E_\varepsilon[u_\tau(t)] + \frac{1}{2} \int_0^{t-\tau} \left( \frac{\mathcal{W}_2(u_\tau(s+\tau), u_\tau(s))}{\tau} \right)^2 ds \leq E_\varepsilon[u_0]. \quad (5.3.9)$$

With Proposition 5.2.1, we obtain

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} |u_\tau(x) - u_\tau(x-y)|^2 dx dy \leq C(1 + E_\varepsilon[u_0]) \quad (5.3.10)$$

so that by Lemma 5.2.1 we deduce uniform estimate in  $L^\infty(0, T; L^2(\Omega))$ . To prove weak compactness for all times  $t \in [0, T]$  and Hölder continuity in time for the limiting curve, it is sufficient to prove

$$\limsup_{\tau \rightarrow 0} \mathcal{W}_2(u_\tau(s), u_\tau(t)) \leq C \sqrt{|t-s|}$$

for some constant  $C$  and apply [21, Proposition 3.3.1] (with  $d$  being  $\mathcal{W}_2$  distance,  $\sigma$  being weak topology on  $L^2(\Omega)$ ,  $K$  being the ball in  $L^2(\mathbb{T}^d)$  such that  $u_\tau(s) \in K$  for all  $s \in [0, T]$ ,  $\mathcal{C} = \emptyset$  and  $\omega(t, s) = \sqrt{t-s}$ ). To this end, we write for  $0 \leq s < t$  such that  $s \in ((m-1)\tau, m\tau]$  and  $t \in ((n-1)\tau, n\tau]$

$$\begin{aligned} \mathcal{W}_2(u_\tau(s), u_\tau(t)) &\leq \sum_{i=m}^{n-1} \mathcal{W}_2(u_\tau^i, u_\tau^{i+1}) \leq \left( \sum_{i=m}^{n-1} \mathcal{W}_2^2(u_\tau^i, u_\tau^{i+1}) \right)^{1/2} |n-m|^{1/2} \\ &\leq \sqrt{2E_\varepsilon[u_0]} \sqrt{|t-s| + \tau}, \end{aligned}$$

where in the last line we used (5.3.6) and  $|n-m| < \frac{|t-s|}{\tau} + 1$ . This concludes the proof.  $\square$

### 5.3.2 $H^1$ estimates for the JKO scheme via flow interchange lemma

The weak convergence of the JKO scheme is not enough to pass to the limit in the definition of weak solutions. We need to obtain better estimates on the solutions. In the Cahn-Hilliard equation, better estimates are derived from considering the functional  $\mathcal{U}[\rho, \eta] = \int \rho \log \rho + \eta \log \eta$ . This functional generates the heat flow with respect to the Wasserstein distance  $\mathcal{W}_2$ . To improve the regularity we, therefore, use the flow interchange lemma which states that *the dissipation of one functional along the gradient flow of another functional equals the dissipation of the second functional along the gradient flow of the first one*.

The main result of this section reads :

**Proposition 5.3.5.** *Each solution of the JKO scheme (5.3.5) satisfies*

$$u_\tau^n \in H^1(\mathbb{T}^d) \quad \text{for all } n \in \mathbb{N}, \tau > 0$$

and the time-interpolation  $u_\tau = (\rho_\tau, \eta_\tau)$  satisfies

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (|\nabla \rho_\tau(t, x) - \nabla \rho_\tau(t, x-y)|^2 + |\nabla \eta_\tau(t, x) - \nabla \eta_\tau(t, x-y)|^2) dx dy dt \leq CT,$$

for all  $T > 0$ . Moreover, for every sequence  $\tau_k \downarrow 0$ , we can extract a subsequence (still denoted by  $\tau_k$ ) such that for all  $T > 0$ ,

$$\begin{aligned} u_{\tau_k} &\rightarrow u \text{ strongly in } L^2((0, T) \times \mathbb{T}^d) \\ u_{\tau_k} &\rightharpoonup u \text{ weakly in } L^2(0, T; H^1(\mathbb{T}^d)). \end{aligned}$$

The main tool to prove Proposition 5.3.5 will be the following lemma.

**Lemma 5.3.6.** *Let  $v_0 = (\rho_0, \eta_0) \in L^2(\mathbb{T}^d)^2$  with  $E_\varepsilon[v_0] < +\infty$ . Let  $v = (\rho, \eta) : [0, +\infty) \rightarrow L^2(\mathbb{T}^d)^2$  be a solution of the heat flow*

$$\begin{cases} \partial_t \rho_t = \Delta \rho_t, & \partial_t \eta_t = \Delta \eta_t, & \text{in } (0, +\infty) \times \mathbb{T}^d \\ (\rho(0), \eta(0)) = (\rho_0, \eta_0). \end{cases} \tag{5.3.11}$$

Suppose that

$$\liminf_{s \downarrow 0} \frac{1}{s} (E_\varepsilon[v_s] - E_\varepsilon[v_0]) > -\infty. \tag{5.3.12}$$

Then,  $v_0 \in H^1(\mathbb{T}^d)$  and for some constant  $C$  depending only on the parameters  $\kappa, \alpha, \gamma, \beta$  we have

$$\begin{aligned} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (|\nabla \rho_0(x) - \nabla \rho_0(x - y)|^2 + |\nabla \eta_0(x) - \nabla \eta_0(x - y)|^2) dx dy &\leq \\ &\leq -C \liminf_{s \downarrow 0} \frac{1}{s} (E_\varepsilon[v_s] - E_\varepsilon[v_0]). \end{aligned} \tag{5.3.13}$$

The plan is to initiate the heat flow at the solutions to JKO scheme. If (5.3.12) is verified, then (5.3.13) will provide the desired  $H^1$  estimate by Lemma 5.2.2. The crucial information here is that dissipation of Cahn-Hilliard energy along heat flow is related to the dissipation of Cahn-Hilliard entropy. The technical assumption (5.3.12) will be verified with the flow interchange lemma which requires a definition of a  $\lambda$ -flow in  $\mathcal{P}(\mathbb{T}^d)^2$ .

*Definition 5.3.7.* Let  $\mathcal{F} : \mathcal{P}(\mathbb{T}^d)^2 \rightarrow (-\infty, +\infty]$  be a proper lower semi-continuous functional and  $\lambda \in \mathbb{R}$ . A continuous semigroup  $\mathbf{S}^t : \text{Dom}(\mathcal{F}) \rightarrow \text{Dom}(\mathcal{F})$ ,  $t \geq 0$  is a  $\lambda$ -flow for  $\mathcal{F}$  if it satisfies the Evolution Variational Inequality (EVI)

$$\frac{1}{2} \limsup_{h \downarrow 0} \left[ \frac{\mathcal{W}_2^2(\mathbf{S}^h u, v) - \mathcal{W}_2^2(u, v)}{h} \right] + \frac{\lambda}{2} \mathcal{W}_2^2(u, v) + \mathcal{F}(u) \leq \mathcal{F}(v), \tag{5.3.14}$$

for all measures  $u, v \in \text{Dom}(\mathcal{F})$  with  $\mathcal{W}_2(u, v) < +\infty$ .

**Lemma 5.3.8** (Flow interchange Lemma [270, 258]). *Assume that  $\mathbf{S}_\mathcal{F}$  is a  $\lambda$ -flow for the proper, lower semi-continuous functional  $\mathcal{F}$  in  $\mathcal{P}(\mathbb{T}^d)^2$  and let  $u_\tau^n$  be a  $n$ -th step approximation constructed by the minimizing movement scheme (5.3.5). If  $u_\tau^n \in \text{Dom}(\mathcal{F})$  then*

$$\mathcal{F}[u_\tau^n] - \mathcal{F}[u_\tau^{n-1}] \leq \tau \liminf_{h \downarrow 0} \left( \frac{E_\varepsilon[\mathbf{S}_\mathcal{F}^h(u_\tau^n)] - E_\varepsilon[u_\tau^n]}{h} \right) - \frac{\lambda}{2} \mathcal{W}_2^2(u_\tau^n, u_\tau^{n-1}). \tag{5.3.15}$$

*Remark 5.3.9.* In [270, Lemma 3.2] Lemma 5.3.8 was formulated for the case of a bounded domain. However, it also works in the case of torus as it is just a combination of scheme (5.3.5) and the definition of  $\lambda$ -flow (5.3.14).

We are concerned with the functional

$$\mathcal{U}[\rho, \eta] = \int_{\mathbb{T}^d} \rho(x) \log \rho(x) + \eta(x) \log \eta(x) \, dx,$$

which generates a gradient flow in the product space  $(\mathcal{P}(\mathbb{T}^d)^2, \mathcal{W}_2)$  defined by (5.3.11). We admit the fairly classical result that the entropy  $\mathcal{U}$  possesses a 0-flow given by the heat semigroup.

*Proof of Lemma 5.3.6.* We recall that  $E_\varepsilon$  can be written as in (5.1.9)

$$\begin{aligned} E_\varepsilon[\rho, \eta] &= \frac{1}{4\varepsilon^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) (\kappa |\rho(x) - \rho(x-y)|^2 + |\eta(x) - \eta(x-y)|^2) \, dx \, dy \\ &\quad + \frac{\alpha}{2\varepsilon^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) (\rho(x) - \rho(x-y)) (\eta(x) - \eta(x-y)) \, dx \, dy \\ &\quad - \int_{\mathbb{T}^d} \frac{\gamma}{2} \rho^2 + \frac{1}{2} \eta^2 + \beta \rho \eta \, dx. \end{aligned}$$

From parabolic theory, we know that the solution of (5.3.11) is smooth for  $t > 0$  and thus we have after integration by parts

$$\begin{aligned} \frac{d}{dt} E_\varepsilon[\rho(t, \cdot), v(t, \cdot)] &= -\frac{1}{2\varepsilon^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) (\kappa |\nabla \rho(x) - \nabla \rho(x-y)|^2 + |\nabla \eta(x) - \nabla \eta(x-y)|^2) \, dx \, dy \\ &\quad - \frac{\alpha}{\varepsilon^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) (\nabla \rho(x) - \nabla \rho(x-y)) \cdot (\nabla \eta(x) - \nabla \eta(x-y)) \, dx \, dy \\ &\quad + \int_{\mathbb{T}^d} \gamma |\nabla \rho|^2 + |\nabla \eta|^2 + 2\beta \nabla \rho \cdot \nabla \eta \, dx. \end{aligned}$$

Note that we recognize the dissipation of the entropy (5.1.10). Since the map  $t \rightarrow v(t, \cdot)$  is continuous in  $L^2(\mathbb{T}^d)^2$ , we get that the map  $t \rightarrow E_\varepsilon[v(t, \cdot)]$  is continuous at  $t = 0$ . By (5.3.12) and Proposition 5.2.1 there exists  $C$  such that for all  $t \leq t_0$  for  $t_0$  sufficiently small,

$$\begin{aligned} -C &\leq \frac{E_\varepsilon[v(t, \cdot)] - E_\varepsilon[v_0]}{t} = \frac{1}{t} \int_0^1 \frac{d}{ds} E_\varepsilon[v(st, \cdot)] \, ds \leq \\ &\leq -\frac{C}{2\varepsilon^2} \int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) (|\nabla \rho(st, x) - \nabla \rho(st, x-y)|^2 + |\nabla \eta(st, x) - \nabla \eta(st, x-y)|^2) \, dx \, dy \, ds. \end{aligned} \tag{5.3.16}$$

Therefore, using Lemma 5.2.2, the family  $\{\nabla v(t, \cdot)\}_{t \leq t_0}$  is bounded in  $L^2((0, 1) \times \mathbb{T}^d)$ . Now, as  $v_0 \in L^2(\mathbb{T}^d)$  and  $v$  is the solution of heat equation with initial condition  $v_0$ ,  $v \in C([0, T]; L^2(\mathbb{T}^d))$ . Therefore, there exists a bounded modulus of continuity  $\pi : [0, T] \rightarrow \mathbb{R}^+$  such that  $\lim_{t \rightarrow 0} \pi(t) = 0$  and

$$\|v(t, \cdot) - v_0\|_{L^2(\mathbb{T}^d)} \leq \pi(t).$$

In particular,

$$\|v(st, \cdot) - v_0\|_{L^2(\mathbb{T}^d)} \leq \pi(st)$$

so that dominated convergence theorem implies that  $v(t, \cdot) \rightarrow v_0$  in  $L^2((0, 1) \times \mathbb{T}^d)$  when  $t \rightarrow 0$ . Finally, choosing a subsequence such that  $\nabla v(t, \cdot) \rightarrow \xi$  in  $L^2((0, 1) \times \mathbb{T}^d)$ , using identity

$$\int_0^1 \int_{\mathbb{T}^d} v(ts, x) \operatorname{div} \varphi(s, x) \, ds \, dx = - \int_0^1 \int_{\mathbb{T}^d} \nabla v(ts, x) \varphi(s, x) \, ds \, dx$$

for a smooth test function  $\varphi$  and passing to the limit  $t \rightarrow 0$ , we obtain that  $\nabla v_0 = \xi$  and so,  $v_0 \in H^1(\mathbb{T}^d)$ .

To obtain uniform estimate on  $v_0$  in  $H^1(\mathbb{T}^d)$  we want to pass to the limit  $t \rightarrow 0$  in (5.3.16). For this, we observe that

$$\begin{aligned} \int_0^1 \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |\nabla \rho(st, x) - \nabla \rho(st, x - y)|^2 dx dy ds &= \\ &= 2 \int_0^1 \int_{\mathbb{T}^d} |\nabla \rho(st, x)|^2 dx ds - 2 \int_0^1 \int_{\mathbb{T}^d} \nabla \rho(st, x) \cdot \nabla (\omega_\varepsilon * \rho(st, x)) dx ds. \end{aligned}$$

The first term is lower semi-continuous with respect to the  $L^2$  weak topology while the second converges to

$$\int_0^1 \int_{\mathbb{T}^d} \nabla \rho(st, x) \cdot \nabla (\omega_\varepsilon * \rho(st, x)) dx ds \rightarrow \int_{\mathbb{T}^d} \nabla \rho_0(x) \cdot \nabla (\omega_\varepsilon * \rho_0(x)) dx$$

as a product of a weakly and strongly convergent sequences. □

With this lemma, we are finally able to prove

*Proof of Proposition 5.3.5.* First, we want to apply the flow interchange lemma to the functional  $\mathcal{U}$ . This is possible since  $\mathcal{U}$  is bounded from below on  $\mathcal{P}(\mathbb{T}^d)$ . Applying Lemma 5.3.8 with  $\mathcal{F} = \mathcal{U}$  and Lemma 5.3.6 with  $v_0 = u_\tau^n$

$$C\tau \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (|\nabla \rho_\tau^n(x) - \nabla \rho_\tau^n(x - y)|^2 + |\nabla \eta_\tau^n(x) - \nabla \eta_\tau^n(x - y)|^2) dx dy \leq \mathcal{U}[u_\tau^{n-1}] - \mathcal{U}[u_\tau^n].$$

Summing from  $n = 1$  to  $n = N$  we obtain

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} (|\nabla \rho_\tau(t, x) - \nabla \rho_\tau(t, x - y)|^2 + |\nabla \eta_\tau(t, x) - \nabla \eta_\tau(t, x - y)|^2) dx dy dt \leq CT.$$

From Lemma 5.2.2 we obtain uniform estimate in  $L^2(0, T; H^1(\mathbb{T}^d))$  and so, weak compactness in this space.

It remains to prove the strong compactness in  $L^2(0, T; L^2(\mathbb{T}^d))$ . First, we want to apply Theorem 5.5.3 with Banach space  $X = L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ , set  $U = \{u_\tau\}_{\tau > 0}$ , pseudometric  $g(u_1, u_2) = \mathcal{W}_2^2(u_1, u_2)$  (extended to  $+\infty$  in case  $u_1$  or  $u_2$  are not probability measures) and functional  $\mathcal{F}$  defined as

$$\mathcal{F}(u) = \begin{cases} \|u\|_{H^1(\mathbb{T}^d)}^2 & \text{if } u \in H^1(\mathbb{T}^d) \times H^1(\mathbb{T}^d) \cap \mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d), \\ +\infty & \text{if } u \notin H^1(\mathbb{T}^d) \times H^1(\mathbb{T}^d) \cap \mathcal{P}(\mathbb{T}^d) \times \mathcal{P}(\mathbb{T}^d). \end{cases}$$

We can do this as  $\mathcal{F}$  is lower semicontinuous and its level sets are compact in  $L^2(\mathbb{T}^d)$  by Reillich-Kondrachov theorem. Furthermore,  $g$  is lower semicontinuous [325, Chapter 6]. Finally, (5.5.1) follows from uniform estimates in  $L^2(0, T; H^1(\mathbb{T}^d))$  and estimate (5.3.9).

Therefore, Theorem 5.5.1 gives us a subsequence (not relabelled) such that

$$\|u_n(t, \cdot) - u(t, \cdot)\|_{L^2(\mathbb{T}^d)}^2 \rightarrow 0 \text{ for a.e. } t \in [0, T].$$

As sequence  $\{u_n\}$  is bounded in  $L^\infty(0, T; L^2(\mathbb{T}^d))$ , the conclusion follows by dominated convergence theorem.  $\square$

### 5.3.3 Weak formulation

To derive the weak formulation, we follow [326, Section 8.4.2]. Since (5.3.5) was derived from the Lagrangian point of view, the idea is to investigate its first variation and prove that it is a time-discrete scheme of (5.3.1). For this, we introduce a suitable perturbation of  $u_\tau^{n+1} = (\rho_\tau^{n+1}, \eta_\tau^{n+1})$  as follows : let  $\xi$  be a smooth periodic vector field and  $T_\delta = \mathbf{1} + \delta\xi$ . It is classical to prove that for  $\delta$  small enough,  $T_\delta$  is a  $C^1$  diffeomorphism and  $\det(\nabla T_\delta) > 0$ . Then, we define

$$\tilde{u}_\delta = (T_\delta \# \rho_\tau^{n+1}, \eta_\tau^{n+1}).$$

Note carefully that only the first component was perturbed. This will result in the first equation in Definition 5.3.1. To obtain the second one, it is sufficient to introduce a similar perturbation on the second component of  $u_\tau^{n+1}$ . This results in analogous computations as outlined below and we do not repeat them.

Using standard properties of push-forward measure we obtain from (5.3.3)

$$\begin{aligned} E_\varepsilon[\tilde{u}_\delta] &= \int_{\mathbb{T}^d} \frac{\tilde{\kappa}}{2} \frac{(\rho_\tau^{n+1}(x))^2}{\det(\mathbf{1} + \delta\nabla\xi)} + \frac{\tilde{c}}{2} (\eta_\tau^{n+1}(x))^2 + \tilde{\alpha} \rho_\tau^{n+1}(x) \eta(x + \delta\xi) dx \\ &\quad - \frac{\tilde{\kappa} + \gamma}{2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(x - y + \delta(\xi(x) - \xi(y))) \rho_\tau^{n+1}(y) \rho_\tau^{n+1}(x) dy dx \\ &\quad - (\tilde{\alpha} + \beta) \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(x + \delta\xi(x) - y) \eta_\tau^{n+1}(y) \rho_\tau^{n+1}(x) dy dx \\ &\quad - \frac{\tilde{c} + 1}{2} \int_{\mathbb{T}^d} \omega_\varepsilon * \eta_\tau^{n+1}(x) \eta_\tau^{n+1}(x) dx. \end{aligned} \tag{5.3.17}$$

Using the minimizing property of  $u_\tau^{n+1}$  in (5.3.5) gives

$$0 \leq \frac{1}{2\tau} [\mathcal{W}_2^2(u_\tau^n, \tilde{u}_\delta) - \mathcal{W}_2^2(u_\tau^n, u_\tau^{n+1})] + E_\varepsilon[\tilde{u}_\delta] - E_\varepsilon[u_\tau^{n+1}]. \tag{5.3.18}$$

The plan is to expand this inequality in terms of  $\delta$ , send  $\delta \rightarrow 0$  and then  $\tau \rightarrow 0$  which will provide the weak formulation in Definition 5.3.1. We consider three types of terms separately.

*Step 1 : the nonlocal terms in  $E_\varepsilon[\tilde{u}_\delta] - E_\varepsilon[u_\tau^{n+1}]$ .* When taking the difference  $E_\varepsilon[\tilde{u}_\delta] - E_\varepsilon[u_\tau^{n+1}]$ , there are two types of nonlocal terms. The first one reads

$$-\frac{\tilde{\kappa} + \gamma}{2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} [\omega_\varepsilon(x - y + \delta(\xi(x) - \xi(y))) - \omega_\varepsilon(x - y)] \rho_\tau^{n+1}(y) \rho_\tau^{n+1}(x) dy dx.$$

Similarly to [135, Proof of Theorem 3.3] we perform Taylor's expansion and using uniform  $L^2$  estimates we obtain that this term is equal to

$$-\delta \frac{\tilde{\kappa} + \gamma}{2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla \omega_\varepsilon(x - y) \cdot (\xi(x) - \xi(y)) \rho_\tau^{n+1}(y) \rho_\tau^{n+1}(x) dy dx + o(\delta).$$

The second term comes from the cross-interaction potentials and we obtain similarly

$$-\delta(\tilde{\alpha} + \beta) \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla \omega_\varepsilon(x - y) \cdot \xi(x) \eta_\tau^{n+1}(y) \rho_\tau^{n+1}(x) \, dy \, dx + o(\delta).$$

Step 2 : the local terms. As in Step 1, there are only two differences. The first one reads

$$\frac{\tilde{\kappa}}{2} \int_{\mathbb{T}^d} \frac{(\rho_\tau^{n+1})^2}{\det(\mathbf{1} + \delta \nabla \xi)} - (\rho_\tau^{n+1})^2 \, dx.$$

Using  $\det(\mathbf{1} + \delta \nabla \xi) = 1 + \delta \operatorname{div} \xi(x) + o(\delta)$ , we obtain that this term is equal to

$$-\delta \frac{\tilde{\kappa}}{2} \int_{\mathbb{T}^d} (\rho_\tau^{n+1})^2 \operatorname{div} \xi(x) \, dx + o(\delta).$$

The second term reads

$$\int_{\mathbb{T}^d} \tilde{\alpha} \rho_\tau^{n+1}(x) (\eta_\tau^{n+1}(x + \delta \xi) - \eta_\tau^{n+1}(x)) \, dx.$$

As  $\nabla \eta_\tau^{n+1} \in L^2(\mathbb{T}^d)$ , the second term is equal to

$$\delta \tilde{\alpha} \int_{\mathbb{T}^d} \rho_\tau^{n+1}(x) \nabla \eta_\tau^{n+1}(x) \cdot \xi \, dx + o(\delta).$$

Step 3 : The Wasserstein terms. Since  $\rho_\tau^n$  and  $\rho_\tau^{n+1}$  are absolutely continuous measures, we know that there exists an optimal map  $\nabla \varphi$  such that  $\nabla \varphi \# \rho_\tau^n = \rho_\tau^{n+1}$  [305, Theorem 1.25] and

$$\mathcal{W}_2^2(\rho_\tau^n, \rho_\tau^{n+1}) = \int_{\mathbb{T}^d} \rho_\tau^n(x) |x - \nabla \varphi(x) + k(x)|^2 \, dx. \quad (5.3.19)$$

where  $k(x) \in \mathbb{Z}^d$ . Moreover, we have  $T_\delta \# \rho_\tau^{n+1} = [(\mathbf{1} + \delta \xi) \circ \nabla \varphi] \# \rho_\tau^n$ . Therefore by definition of the Wasserstein distance, and for  $\delta$  small enough we have

$$\mathcal{W}_2^2(\rho_\tau^n, T_\delta \# \rho_\tau^{n+1}) \leq \int_{\mathbb{T}^d} \rho_\tau^n(x) |x - \nabla \varphi(x) - \delta \xi \circ \nabla \varphi(x) + k(x)|^2 \, dx. \quad (5.3.20)$$

Using (5.3.19)–(5.3.20) and performing Taylor's expansion we obtain

$$\frac{1}{2\tau} [\mathcal{W}_2^2(u_\tau^n, \tilde{u}_\delta) - \mathcal{W}_2^2(u_\tau^n, u_\tau^{n+1})] \leq \frac{\delta}{\tau} \int_{\mathbb{T}^d} \rho_\tau^n(x) (\nabla \varphi(x) - x + k(x)) \cdot (\xi \circ \nabla \varphi(x)) \, dx + o(\delta).$$

Weak formulation. We plug inequalities from Steps 1-3 to (5.3.18). As  $\xi$  can be replaced with  $-\xi$



we obtain equality

$$\begin{aligned}
\frac{1}{\tau} \int_{\mathbb{T}^d} \rho_\tau^n(x) (\nabla\varphi(x) - x + k(x)) \cdot (\xi \circ \nabla\varphi(x)) \, dx &= \\
&= \frac{\tilde{\kappa}}{2} \int_{\mathbb{T}^d} (\rho_\tau^{n+1}(x))^2 \operatorname{div} \xi(x) \, dx - \tilde{\alpha} \int_{\mathbb{T}^d} \rho_\tau^{n+1}(x) \nabla\eta_\tau^{n+1}(x) \cdot \xi \, dx \\
&+ (\tilde{\alpha} + \beta) \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla\omega_\varepsilon(x-y) \cdot \xi(x) \eta_\tau^{n+1}(y) \rho_\tau^{n+1}(x) \, dy \, dx \\
&+ \frac{\tilde{\kappa} + \gamma}{2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla\omega_\varepsilon(x-y) \cdot (\xi(x) - \xi(y)) \rho_\tau^{n+1}(y) \rho_\tau^{n+1}(x) \, dy \, dx.
\end{aligned} \tag{5.3.21}$$

Now we consider test function  $\xi = \nabla\zeta$  for some  $\zeta \in C^\infty(\mathbb{T}^d)$ . By periodicity we have  $\zeta(\nabla\varphi(x)) - \zeta(x) = \zeta(\nabla\varphi(x)) - \zeta(x - k(x))$ . Therefore, we have

$$\zeta(\nabla\varphi(x)) - \zeta(x) = (\nabla\varphi(x) - x + k(x)) \cdot (\nabla\zeta \circ \nabla\varphi(x)) + O(|x - k(x) - \nabla\varphi(x)|^2)$$

and  $\rho_\tau^{n+1} = \nabla\varphi \# \rho_\tau^n$ , we obtain that the (LHS) of (5.3.21) is equal to

$$\frac{1}{\tau} \left( \int_{\mathbb{T}^d} \rho_\tau^{n+1} \zeta - \int_{\mathbb{T}^d} \rho_\tau^n \zeta \right) + O\left( \frac{\mathcal{W}_2^2(\rho_\tau^n, \rho_\tau^{n+1})}{\tau} \right).$$

Now, let  $t_1, t_2$  be arbitrary. As the curve  $\rho_\tau$  is piecewisely constant, we can sum up from  $n = n_1 = \lceil t_1/\tau \rceil$  to  $n_2 = \lceil t_2/\tau \rceil + 1$  and obtain

$$\begin{aligned}
\int_{\mathbb{T}^d} \rho_\tau(t_2) \zeta \, dx - \int_{\mathbb{T}^d} \rho_\tau(t_1) \zeta \, dx + O\left( \sum_{n=n_1}^{n_2} \mathcal{W}_2^2(\rho_\tau^n, \rho_\tau^{n+1}) \right) &= \\
&= \frac{\tilde{\kappa}}{2} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \rho_\tau(t)^2 \Delta\zeta(x) \, dx \, dt - \tilde{\alpha} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \rho_\tau(t, x) \nabla\eta_\tau(t, x) \cdot \nabla\zeta \, dx \, dt \\
&+ (\tilde{\alpha} + \beta) \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla\omega_\varepsilon(x-y) \cdot \nabla\zeta(x) \eta_\tau(t, y) \rho_\tau(t, x) \, dy \, dx \, dt \\
&+ \frac{\tilde{\kappa} + \gamma}{2} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla\omega_\varepsilon(x-y) \cdot (\nabla\zeta(x) - \nabla\zeta(y)) \rho_\tau(t, y) \rho_\tau(t, x) \, dy \, dx \, dt + O(\tau),
\end{aligned}$$

where the term  $O(\tau)$  appears because each term has at least one term ( $\rho_\tau$  or  $\eta_\tau$ ) which is bounded in  $L^\infty(0, T; L^2(\mathbb{T}^d))$ . Using the energy estimate (5.3.6) on the Wasserstein distance, we can incorporate the term  $O(\sum_{n=n_1}^{n_2} \mathcal{W}_2^2(\rho_\tau^n, \rho_\tau^{n+1}))$  into  $O(\tau)$ . Sending  $\tau \rightarrow 0$ , using Proposition 5.3.5 and pointwise (in time) weak convergence (in space) from Lemma 5.3.4 yields

$$\begin{aligned}
\int_{\mathbb{T}^d} \rho(t_2, x) \zeta(x) \, dx - \int_{\mathbb{T}^d} \rho(t_1, x) \zeta(x) \, dx &= \\
&= \frac{\tilde{\kappa}}{2} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \rho(t, x)^2 \Delta\zeta(x) \, dx \, dt - \tilde{\alpha} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \rho(t, x) \nabla\eta(t, x) \cdot \nabla\zeta \, dx \, dt \\
&+ (\tilde{\alpha} + \beta) \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla\omega_\varepsilon(x-y) \cdot \nabla\zeta(x) \eta(t, y) \rho(t, x) \, dy \, dx \, dt \\
&+ \frac{\tilde{\kappa} + \gamma}{2} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla\omega_\varepsilon(x-y) \cdot (\nabla\zeta(x) - \nabla\zeta(y)) \rho(t, y) \rho(t, x) \, dy \, dx \, dt.
\end{aligned} \tag{5.3.22}$$

Performing integration by parts we obtain

$$\frac{\tilde{\kappa}}{2} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \rho(t, x)^2 \Delta \zeta(x) \, dx \, dt = \tilde{\kappa} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \rho(t, x) \nabla \rho(t, x) \cdot \nabla \zeta(x) \, dx \, dt,$$

while changing variables we obtain

$$\begin{aligned} \frac{\tilde{\kappa} + \gamma}{2} \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla \omega_\varepsilon(x - y) \cdot (\nabla \zeta(x) - \nabla \zeta(y)) \rho(t, y) \rho(t, x) \, dy \, dx \, dt = \\ = -(\tilde{\kappa} + \gamma) \int_{t_1}^{t_2} \int_{\mathbb{T}^d} \omega_\varepsilon * \rho(t, x) \cdot \nabla \zeta(x) \, dx \, dt. \end{aligned}$$

Having these two observations in mind, we obtain the weak formulation with test function  $\zeta(x)$  depending only on  $x$ . The general weak formulation with test functions depending on  $t$  and  $x$  as in Definition 5.3.1 follows from multiplying with  $\partial_t \psi(t)$ , integrating in time and using the classical density of functions of the form  $\psi(t) \zeta(x)$  over the set of test functions  $\varphi(t, x)$  (see [142, Theorem D.5]).

Regularity estimates (5.1.13)–(5.1.14). Clearly, it is sufficient to prove these estimates for the first component of  $u = (\rho, \eta)$ . For (5.1.13) we note that from Lemma 5.3.4 we have that  $u_\tau(t, \cdot) \rightharpoonup u(t, \cdot)$  for all fixed  $t \in [0, T]$ . Moreover, we can write the quantity of interest as

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\varepsilon(y) |\rho_\tau(t, x) - \rho_\tau(t, x - y)|^2 \, dx \, dy = 2 \int_{\mathbb{T}^d} |\rho_\tau(t, x)|^2 \, dx - 2 \int_{\mathbb{T}^d} \rho_\tau(t, x) \omega_\varepsilon * \rho_\tau(t, x) \, dx$$

so that the first term is weakly lower semicontinuous while the second is the product of weakly and strongly converging sequences. Applying  $\liminf_{\varepsilon \rightarrow 0}$  to (5.3.10) we deduce (5.1.13). Concerning (5.1.14), the proof is carried out in the similar way : this time we use strong compactness of  $\rho_\tau$  and weak compactness of  $\nabla \rho_\tau$  in  $L^2((0, T) \times \mathbb{T}^d)$  from Lemma 5.3.5 which allow to handle integral with respect to time.

## 5.4 Limit $\varepsilon \rightarrow 0$

We want to send  $\varepsilon \rightarrow 0$  and obtain convergence of weak solutions of the nonlocal Cahn-Hilliard system (5.1.5)–(5.1.6) to weak solutions of the local version of the Cahn-Hilliard system. We define weak solutions of the latter.

*Definition 5.4.1.* We say that  $u = (\rho(\cdot), \eta(\cdot)) : [0, \infty) \rightarrow \mathcal{P}(\mathbb{T}^d)^2$  is a weak solution of (5.1.7)–(5.1.8) with initial condition  $(\rho_0, \eta_0)$  if  $\rho, \eta \in L^2(0, T; H^2(\mathbb{T}^d))$  for all  $T > 0$ , and if for all  $\varphi, \phi \in C_c^\infty([0, +\infty) \times \mathbb{T}^d)$  we have

$$\begin{aligned} - \int_0^\infty \int_{\mathbb{T}^d} \rho \partial_t \varphi \, dx \, dt - \int_{\mathbb{T}^d} \rho_0 \varphi(0, x) \, dx &= -\kappa \int_0^\infty \int_{\mathbb{T}^d} \Delta \rho \nabla \rho \cdot \nabla \varphi \, dx \, dt \\ - \kappa \int_0^\infty \int_{\mathbb{T}^d} \rho \Delta \rho \Delta \varphi \, dx \, dt - \alpha \int_0^\infty \int_{\mathbb{T}^d} D^2 \eta : (\nabla \rho \otimes \nabla \varphi + \rho D^2 \varphi) \, dx \, dt &- \int_0^\infty \int_{\mathbb{T}^d} \rho (\gamma \nabla \rho - \beta \nabla \eta) \cdot \nabla \varphi \, dx \, dt, \\ - \int_0^\infty \int_{\mathbb{T}^d} \eta \partial_t \phi \, dx \, dt - \int_{\mathbb{T}^d} \eta_0 \phi(0, x) \, dx &= - \int_0^\infty \int_{\mathbb{T}^d} \Delta \eta \nabla \eta \cdot \nabla \phi \, dx \, dt \\ - \int_0^\infty \int_{\mathbb{T}^d} \eta \Delta \eta \Delta \phi \, dx \, dt - \alpha \int_0^\infty \int_{\mathbb{T}^d} D^2 \rho : (\nabla \eta \otimes \nabla \phi + \eta D^2 \phi) \, dx \, dt &- \int_0^\infty \int_{\mathbb{T}^d} \eta (\nabla \eta - \beta \nabla \rho) \cdot \nabla \phi \, dx \, dt. \end{aligned}$$

As we will see (Lemma 5.4.3), we have bounds at most on the gradient of  $\nabla\rho_\varepsilon, \nabla\eta_\varepsilon$ , and the limit equation has four derivatives. That means we need to mimic at the epsilon level integration by parts for nonlocal operators. For that purpose, we define the operator

$$S_\varepsilon[\varphi](x, y) := \frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2\varepsilon}}(\varphi(x-y) - \varphi(x)) \quad (5.4.1)$$

which has the following properties, see [155, Lemma 3.4] :

**Lemma 5.4.2.** *The operator  $S_\varepsilon$  satisfies :*

(S1)  $S_\varepsilon$  is a linear operator that commutes with derivatives with respect to  $x$ ,

(S2) for all functions  $f, g : \mathbb{T}^d \rightarrow \mathbb{R}$  we have

$$\begin{aligned} S_\varepsilon[fg](x, y) - S_\varepsilon[f](x, y)g(x) - S_\varepsilon[g](x, y)f(x) &= \\ &= \frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2\varepsilon}}[(f(x-y) - f(x))(g(x-y) - g(x))]. \end{aligned}$$

(S3) for all  $u, \varphi \in L^2(\mathbb{T}^d)$

$$\langle B_\varepsilon[u](\cdot), \varphi(\cdot) \rangle_{L^2(\mathbb{T}^d)} = \langle S_\varepsilon[u](\cdot, \cdot), S_\varepsilon[\varphi](\cdot, \cdot) \rangle_{L^2(\mathbb{T}^d \times \mathbb{T}^d)}.$$

(S4) if  $\{u_\varepsilon\}$  is strongly compact in  $L^2(0, T; H^1(\mathbb{T}^d))$  and  $\varphi \in L^\infty((0, T) \times \mathbb{T}^d)$  we have

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} (S_\varepsilon[u_\varepsilon])^2 \varphi(t, x) \rightarrow \int_0^T \int_{\mathbb{T}^d} |\nabla u(t, x)|^2 \varphi(t, x).$$

### 5.4.1 Uniform estimates in $\varepsilon$ and compactness

We collect here estimates for the solutions of (5.1.5)–(5.1.6) which are uniform in  $\varepsilon > 0$ .

**Lemma 5.4.3.** *Let  $(\rho_\varepsilon, \eta_\varepsilon)$  be the solution of (5.1.5)–(5.1.6) constructed in Theorem 5.1.1. Then, the following sequences are bounded :*

1.  $\{\rho_\varepsilon\}, \{\eta_\varepsilon\}$  in  $L^\infty(0, T; L^2(\Omega))$  and  $L^2(0, T; H^1(\Omega))$ ,
2.  $\{\partial_t \rho_\varepsilon\}, \{\partial_t \eta_\varepsilon\}$  in  $L^2(0, T; H^{-2-\frac{d}{2}}(\Omega))$ .

Moreover, the sequences  $\{\rho_\varepsilon\}, \{\eta_\varepsilon\}$  are strongly compact in  $L^2(0, T; H^1(\Omega))$  and the limits belong to  $L^2(0, T; H^2(\Omega))$ .

*Remark 5.4.4.* We note that we cannot use estimates coming from dissipation of the energy (5.1.11) which can provide better estimates on time derivatives because this information is lost in the JKO scheme. We proceed with a different approach.

*Proof of Lemma 5.4.3.* The first estimate follows from (5.1.13)–(5.1.14) and Lemma 5.2.2. To see the second, we consider equation for  $\rho_\varepsilon$  and test it with a smooth and compactly supported function  $\varphi(t, x)$ . We need to control the following terms

$$\int_0^T \int_{\mathbb{T}^d} \rho_\varepsilon \nabla B_\varepsilon[\rho_\varepsilon] \cdot \nabla \varphi \, dx \, dt, \quad \int_0^T \int_{\mathbb{T}^d} \rho_\varepsilon \nabla B_\varepsilon[\eta_\varepsilon] \cdot \nabla \varphi \, dx \, dt, \quad \int_0^T \int_{\mathbb{T}^d} \rho_\varepsilon (\gamma \nabla \rho_\varepsilon - \beta \nabla \eta_\varepsilon) \cdot \nabla \varphi \, dx \, dt.$$

The first two terms are controlled in the same way so that we focus on the second. Using (S3) in Lemma 5.4.2 and  $\nabla B_\varepsilon[\eta_\varepsilon] = B_\varepsilon[\nabla\eta_\varepsilon]$  we have

$$\int_0^T \int_{\mathbb{T}^d} \rho_\varepsilon \nabla B_\varepsilon[\eta_\varepsilon] \cdot \nabla \varphi \, dx \, dt = \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} S_\varepsilon[\nabla\eta_\varepsilon] \cdot S_\varepsilon[\rho_\varepsilon \nabla\varphi] \, dy \, dx \, dt$$

which is bounded if  $\varphi \in L^2(0, T; H^2(\mathbb{T}^d))$ . Concerning the third term, it is controlled when  $\nabla\varphi \in L^2(0, T; L^\infty(\mathbb{T}^d))$  which is implied by  $\varphi \in L^2(0, T; H^{2+\frac{d}{2}}(\mathbb{T}^d))$  by Sobolev embedding. The conclusion follows.

The strong compactness follows from Lemma 5.5.5 and estimates (5.1.13)–(5.1.14).  $\square$

### 5.4.2 Convergence $\varepsilon \rightarrow 0$

*Proof of Theorem 5.1.2.* As Equations (5.1.5) and (5.1.6) have a similar structure, we focus only on Equation (5.1.5). More precisely, we pass to the limit in the term  $\int_0^\infty \int_{\mathbb{T}^d} \operatorname{div}(\rho_\varepsilon \nabla \mu_{\rho_\varepsilon, \varepsilon}) \varphi \, dx \, dt$  where  $\varphi \in C^3([0, \infty) \times \mathbb{T}^d)$ . Integrating by parts, we obtain

$$\begin{aligned} \int_0^\infty \int_{\mathbb{T}^d} \operatorname{div}(\rho_\varepsilon \nabla \mu_{\rho_\varepsilon, \varepsilon}) \varphi \, dx \, dt &= - \int_0^\infty \int_{\mathbb{T}^d} \rho_\varepsilon \nabla \mu_{\rho_\varepsilon, \varepsilon} \cdot \nabla \varphi \, dx \, dt \\ &= \kappa \int_0^\infty \int_{\mathbb{T}^d} B_\varepsilon[\rho_\varepsilon] \nabla \rho_\varepsilon \cdot \nabla \varphi \, dx \, dt + \kappa \int_0^\infty \int_{\mathbb{T}^d} B_\varepsilon[\rho_\varepsilon] \rho_\varepsilon \Delta \varphi \, dx \, dt \\ &\quad - \alpha \int_0^\infty \int_{\mathbb{T}^d} \rho_\varepsilon B_\varepsilon[\nabla \eta_\varepsilon] \cdot \nabla \varphi \, dx \, dt - \int_0^\infty \int_{\mathbb{T}^d} \rho_\varepsilon (\gamma \nabla \rho_\varepsilon - \beta \nabla \eta_\varepsilon) \cdot \nabla \varphi \, dx \, dt \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{5.4.2}$$

Concerning the term  $I_4$ , its convergence is straightforward because all the sequences  $\{\rho_\varepsilon\}$ ,  $\{\eta_\varepsilon\}$ ,  $\{\nabla \rho_\varepsilon\}$  are compact in  $L^2((0, T) \times \mathbb{T}^d)$ . For the passage to the limit in  $I_1, I_2$  we refer to [155, Steps 1, 2; Proof of Theorem 1.8] (these are exactly the terms that appear for analysis of a single equation). We now prove the convergence of the term  $I_3$ . Due to Lemma 5.4.2 (S2), we have (omitting the constant  $\alpha$ )

$$-I_3 = \int_0^\infty \int_{\mathbb{T}^d} S_\varepsilon[\rho_\varepsilon] S_\varepsilon[\nabla \eta_\varepsilon] \cdot \nabla \varphi \, dx \, dt + \int_0^\infty \int_{\mathbb{T}^d} \rho_\varepsilon S_\varepsilon[\nabla \eta_\varepsilon] \cdot S_\varepsilon[\nabla \varphi] \, dx \, dt + R_\varepsilon =: J_1 + J_2 + R_\varepsilon,$$

where  $R_\varepsilon$  is defined as

$$R_\varepsilon = \int_0^\infty \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} S_\varepsilon[\nabla \eta_\varepsilon] \cdot \frac{\sqrt{\omega_\varepsilon(y)}}{\sqrt{2\varepsilon}} [(\nabla \varphi(x-y) - \nabla \varphi(x)) \cdot (\rho_\varepsilon(x-y) - \rho_\varepsilon(x))] \, dx \, dy \, dt.$$

After a change of variables  $y \rightarrow \frac{y}{\varepsilon}$  and using the definition of  $\omega_\varepsilon$  we obtain

$$J_1 = \frac{1}{2} \int_0^\infty \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega(y) \frac{\rho_\varepsilon(x) - \rho_\varepsilon(x - \varepsilon y)}{\varepsilon} \frac{\nabla \eta_\varepsilon(x) - \nabla \eta_\varepsilon(x - \varepsilon y)}{\varepsilon} \cdot \nabla \varphi(x) \, dx \, dy \, dt. \tag{5.4.3}$$

From Lemma 5.5.1 and 5.5.2 we obtain strong convergence of  $\frac{\rho_\varepsilon(x) - \rho_\varepsilon(x - \varepsilon y)}{\varepsilon} \rightarrow \nabla \rho(x) \cdot y$  and

weak convergence of  $\sqrt{\omega(y)} \frac{\nabla \eta_\varepsilon(x) - \nabla \eta_\varepsilon(x - \varepsilon y)}{\varepsilon} \rightharpoonup \sqrt{\omega(y)} D^2 \eta(x) \cdot y$  (both in  $L^2((0, T) \times \mathbb{T}^d \times \mathbb{T}^d)$ ) so that we easily conclude

$$J_1 \rightarrow \frac{1}{2} \int_{\mathbb{T}^d} \omega(y) \int_0^\infty \int_{\mathbb{T}^d} (\nabla \rho(x) \cdot y) (D^2 \eta(x) y) \cdot \nabla \varphi(x) \, dy \, dx \, dt.$$

where  $D^2 \eta$  denotes the Hessian matrix  $(\partial_{ij} \eta)_{i,j}$ .

For  $J_2$  we similarly write

$$J_2 = \frac{1}{2} \int_0^\infty \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega(y) \rho_\varepsilon(x) \frac{\nabla \eta_\varepsilon(x) - \nabla \eta_\varepsilon(x - \varepsilon y)}{\varepsilon} \cdot \frac{\nabla \varphi(x) - \nabla \varphi(x - \varepsilon y)}{\varepsilon} \, dx \, dy \, dt. \quad (5.4.4)$$

and the same argument as for  $J_1$  shows that

$$J_2 \rightarrow \frac{1}{2} \int_{\mathbb{T}^d} \omega(y) \int_0^\infty \int_{\mathbb{T}^d} \rho(x) (D^2 \eta(x) y) \cdot (D^2 \varphi(x) y) \, dy \, dx \, dt.$$

By properties of  $\omega$ , we obtain

$$J_1 + J_2 = \int_0^\infty \int_{\mathbb{T}^d} D^2 \eta : (\nabla \rho \otimes \nabla \varphi + \rho D^2 \varphi) \, dx \, dt.$$

It remains to show  $R_\varepsilon \rightarrow 0$ . By Cauchy-Schwarz inequality (in time and space) as well as bounds on  $S_\varepsilon[\nabla \eta_\varepsilon]$  from the entropy it remains to prove that

$$\int_0^\infty \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} |\rho_\varepsilon(x - y) - \rho_\varepsilon(x)|^2 |\nabla \varphi(x - y) - \nabla \varphi(x)|^2 \, dy \, dx \, dt \rightarrow 0.$$

Using Taylor's expansion we can estimate this integral with

$$\varepsilon \|D^2 \varphi\|_{L^\infty} \int_0^\infty \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\omega_\varepsilon(y)}{\varepsilon^2} |\rho_\varepsilon(x - y) - \rho_\varepsilon(x)|^2 \, dy \, dx \, dt$$

which converges to zero by the bound from the entropy so that  $R_\varepsilon \rightarrow 0$ .  $\square$

## 5.5 Appendix

### 5.5.1 Results on difference quotients

**Lemma 5.5.1.** *Let  $\{u_\varepsilon\}$  be a sequence strongly compact in  $L^2(0, T; H^1(\mathbb{T}^d))$ . Then, for fixed  $y \in \mathbb{T}^d$ ,*

$$\frac{u_\varepsilon(t, x - \varepsilon y) - u_\varepsilon(t, x)}{\varepsilon} \rightarrow -\nabla u(t, x) \cdot y \text{ strongly in } L^2((0, T) \times \mathbb{T}^d \times \mathbb{T}^d).$$

*Démonstration.* Clearly, the sequence converges in  $L^2((0, T) \times \mathbb{T}^d)$ , that is

$$\int_0^T \int_{\mathbb{T}^d} \left| \frac{u_\varepsilon(t, x - \varepsilon y) - u_\varepsilon(t, x)}{\varepsilon} - \nabla u(t, x) \cdot y \right|^2 \, dx \, dt \rightarrow 0.$$

To see the convergence in  $L^2((0, T) \times \mathbb{T}^d \times \mathbb{T}^d)$ , it is sufficient to apply dominated convergence

theorem as we have the estimate

$$\int_0^T \int_{\mathbb{T}^d} \left| \frac{u_\varepsilon(t, x - \varepsilon y) - u_\varepsilon(t, x)}{\varepsilon} \right|^2 \leq \int_0^T \int_{\mathbb{T}^d} |\nabla u^\varepsilon \cdot y|^2 dx dt \leq |y| \sup_\varepsilon \|\nabla u_\varepsilon\|_{L^2_{t,x}}^2.$$

□

**Lemma 5.5.2.** *Let  $\varphi \in L^\infty(\mathbb{T}^d)$  and  $\{\eta_\varepsilon\}$  be a sequence such that*

- $\varphi(y) \frac{\nabla \eta_\varepsilon(x) - \nabla \eta_\varepsilon(x - \varepsilon y)}{\varepsilon}$  is bounded in  $L^2((0, T) \times \mathbb{T}^d \times \mathbb{T}^d)$ ,
- $\eta_\varepsilon(t, x) \rightharpoonup \eta(t, x)$  in  $L^2(0, T; H^1(\mathbb{T}^d))$  and  $\eta \in L^2(0, T; H^2(\mathbb{T}^d))$ .

Then,

$$\varphi(y) \frac{\nabla \eta_\varepsilon(x) - \nabla \eta_\varepsilon(x - \varepsilon y)}{\varepsilon} \rightharpoonup \varphi(y) D^2 \eta(t, x) \cdot y \text{ weakly in } L^2((0, T) \times \mathbb{T}^d \times \mathbb{T}^d).$$

*Démonstration.* Clearly, after passing to a subsequence, the limit exists and we only need to identify it. For this, we consider a smooth and compactly supported test function  $\psi(t, x, y)$  and compute

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi(y) \frac{\nabla \eta_\varepsilon(t, x) - \nabla \eta_\varepsilon(t, x - \varepsilon y)}{\varepsilon} \psi(t, x, y) dy dx dt &= \\ &= \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi(y) \frac{\psi(t, x, y) - \psi(t, x + \varepsilon y, y)}{\varepsilon} \nabla \eta_\varepsilon(t, x) dy dx dt \end{aligned}$$

which converges to

$$- \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi(y) \nabla \psi(t, x, y) \cdot y \nabla \eta(t, x) dy dx dt = \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi(y) \psi(t, x, y) D^2 \eta(t, x) \cdot y dy dx dt$$

because  $\eta \in L^2(0, T; H^2(\mathbb{T}^d))$ .

□

## 5.5.2 Compactness results

**A version of Lions-Aubin for JKO scheme.** We recall here from [135, Theorem 2.1] a version of Lions-Aubin lemma useful for establishing compactness of a sequence of solutions to JKO scheme. For the proof we refer to [301, Theorem 2].

**Theorem 5.5.3.** *Let  $(X, \|\cdot\|_X)$  be a Banach space. We consider*

- a lower semi-continuous functional  $\mathcal{F} : X \rightarrow [0, +\infty]$  with relatively compact sublevels in  $X$ ,
- a pseudo-distance  $g : X \times X \rightarrow [0, +\infty]$ , that is  $g$  is lower semicontinuous and  $g(\rho, \eta) = 0$  for some  $\rho, \eta \in X$  such that  $\mathcal{F}(\rho), \mathcal{F}(\eta) < \infty$  implies  $\rho = \eta$ .

Let  $U$  be a set of measurable functions  $u : (0, T) \times X$  with  $T > 0$  fixed. Assume further that

$$\sup_{u \in U} \int_0^T \mathcal{F}(u(t)) dt < \infty, \quad \lim_{h \rightarrow 0} \sup_{u \in U} g(u(t+h), u(t)) dt = 0. \quad (5.5.1)$$

Then,  $U$  contains a sequence  $\{u_n\}$  converging in measure to some  $u \in X$ , i.e.

$$\forall \varepsilon > 0 \quad |\{t \in [0, T] : \|u_n - u\|_X > \varepsilon\}| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In particular, there exists a subsequence (not relabelled) such that

$$u_n(t) \rightarrow u(t) \text{ in } X \text{ for a.e. } t \in [0, T].$$

### Nonlocal version of Lions-Aubin lemma

The following result was proved in [155, Theorem B.1] based on [53] and [294]. In fact, the proof of Theorem 5.5.4 is the proof of Proposition 5.1.3 integrated in time.

**Theorem 5.5.4.** *Let  $d \geq 2$ . Let  $\{f_\varepsilon\}$  be a sequence bounded in  $L^p((0, T) \times \mathbb{T}^d)$ . Suppose that there exists a sequence  $\{\rho_\varepsilon\}$  as above such that*

$$\int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f_\varepsilon(t, x) - f_\varepsilon(t, y)|^p}{\varepsilon^p} \omega_\varepsilon(|x - y|) \, dx \, dy \, dt \leq C \quad (5.5.2)$$

for some constant  $C$ . Then,  $\{f_\varepsilon\}$  is compact in space in  $L^p((0, T) \times \mathbb{T}^d)$ , i.e.

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{T}^d} |f_\varepsilon * \varphi_\delta(t, x) - f_\varepsilon(t, x)|^p \, dx \, dt = 0 \quad (5.5.3)$$

for all families of mollifiers  $\{\varphi_\delta\}_{0 < \delta < 1}$ .

We prove here :

**Lemma 5.5.5.** *Suppose that  $\{f_\varepsilon\}$  is a sequence bounded in  $L^2((0, T) \times \mathbb{T}^d)$  such that*

- $\{\partial_t f_\varepsilon\}$  is uniformly bounded in  $L^2(0, T; H^{-k}(\mathbb{T}^d))$  for some  $k \in \mathbb{N}$ ,
- $\{f_\varepsilon\}$  is compact in space in  $L^2((0, T) \times \mathbb{T}^d)$ , i.e.

$$\lim_{\delta \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{T}^d} |f_\varepsilon * \varphi_\delta(t, x) - f_\varepsilon(t, x)|^2 \, dx \, dt = 0 \quad (5.5.4)$$

for some family of mollifiers  $\{\varphi_\delta\}_{0 < \delta < 1}$ .

Then,  $\{f_\varepsilon\}$  is compact in time in  $L^2((0, T) \times \mathbb{T}^d)$ , i.e.

$$\lim_{h \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t, x)|^2 \, dx \, dt \rightarrow 0 \text{ as } h \rightarrow 0 \quad (5.5.5)$$

and so, it is compact in  $L^2((0, T) \times \mathbb{T}^d)$ .

*Démonstration.* Using the mollifiers with  $\delta = \delta(h)$  depending on  $h$  to be specified later in the way that  $\delta(h) \rightarrow 0$  as  $h \rightarrow 0$ , we first split

$$\begin{aligned} \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t, x)|^2 \, dx \, dt &\leq 4 \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t, x) - f_\varepsilon(t, \cdot) * \varphi_\delta(x)|^2 \, dx \, dt \\ &\quad + 4 \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t+h, \cdot) * \varphi_\delta(x)|^2 \, dx \, dt \\ &\quad + 4 \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, \cdot) * \varphi_\delta(x) - f_\varepsilon(t, \cdot) * \varphi_\delta(x)|^2 \, dx \, dt. \end{aligned}$$

When we apply limit  $\lim_{h \rightarrow 0} \limsup_{\varepsilon \rightarrow 0}$ , the first and second term vanish due to (5.5.4). It remains to study the third term. For this, suppose first that  $f_\varepsilon(t, \cdot)$  is smooth in the time variable. Then,

$$\int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, \cdot) * \varphi_\delta(x) - f_\varepsilon(t, \cdot) * \varphi_\delta(x)|^2 dx dt = \int_0^{T-h} \int_{\mathbb{T}^d} \left| \int_t^{t+h} \partial_t f_\varepsilon(s, \cdot) * \varphi_\delta(x) ds \right|^2 dx dt.$$

Now, we can estimate the convolution as follows

$$\partial_t f_\varepsilon(s, \cdot) * \varphi_\delta(x) = \int_{\mathbb{T}^d} \partial_t f_\varepsilon(s, y) \varphi_\delta(x-y) dy \leq \|\partial_t f_\varepsilon(s, \cdot)\|_{H^{-k}(\mathbb{T}^d)} \|\varphi_\delta(x-\cdot)\|_{H^k(\mathbb{T}^d)}.$$

Using this and applying invariance in space of the  $H^k(\mathbb{T}^d)$  norm we obtain

$$\int_0^{T-h} \int_{\mathbb{T}^d} \left| \int_t^{t+h} \partial_t f_\varepsilon(s, \cdot) * \varphi_\delta(x) ds \right|^2 dx dt = h^2 \|\varphi_\delta\|_{H^k(\mathbb{T}^d)}^2 \int_0^{T-h} \left| \frac{1}{h} \int_t^{t+h} \|\partial_t f_\varepsilon(s, \cdot)\|_{H^{-k}(\mathbb{T}^d)} ds \right|^2 dt.$$

Applying Jensen's inequality we obtain

$$h^2 \|\varphi_\delta\|_{H^k(\mathbb{T}^d)}^2 \int_0^{T-h} \left| \frac{1}{h} \int_t^{t+h} \|\partial_t f_\varepsilon(s, \cdot)\|_{H^{-k}(\mathbb{T}^d)} ds \right|^2 dt \leq Th \|\varphi_\delta\|_{H^k(\mathbb{T}^d)}^2 \|\partial_t f_\varepsilon(s, \cdot)\|_{L^2(0, T; H^{-k}(\mathbb{T}^d))}^2.$$

Using that  $\|\varphi_\delta\|_{H^k(\mathbb{T}^d)} \leq \frac{C}{\delta^{k+d/2}}$  we finally obtain

$$\int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, \cdot) * \varphi_\delta(x) - f_\varepsilon(t, \cdot) * \varphi_\delta(x)|^2 dx dt \leq CT \frac{h}{\delta^{k+d/2}} \|\partial_t f_\varepsilon(s, \cdot)\|_{L^2(0, T; H^{-k}(\mathbb{T}^d))}^2.$$

Now, if  $f_\varepsilon(t, x)$  is not smooth in time, we extend it with  $f_\varepsilon(0, x)$  for  $t < 0$ ,  $f_\varepsilon(T, x)$  for  $t > T$  and apply usual regularization to obtain the same estimate. Hence, if we choose  $h(\delta) = \delta^{2k+d}$  we conclude

$$\lim_{h \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \int_0^{T-h} \int_{\mathbb{T}^d} |f_\varepsilon(t+h, x) - f_\varepsilon(t, x)|^2 dx dt \leq \theta(h).$$

Combined with the compactness in space (5.5.5) and the Fréchet-Kolmogorov theorem we obtain the compactness of  $\{f_\varepsilon\}$  in  $L^2((0, T) \times \mathbb{T}^d)$ .  $\square$





Troisième partie

Fluid models with surface tension



## Chapitre 6

# From nonlocal Euler-Korteweg to local Cahn-Hilliard via the high-friction limit

### Abstract

Several recent papers considered the high-friction limit for systems arising in fluid mechanics. Following this approach, we rigorously derive the nonlocal Cahn-Hilliard equation as a limit of the nonlocal Euler-Korteweg equation using the relative entropy method. Applying the recent result by the first and third author, we also derive rigorously the local degenerate Cahn-Hilliard equation. The proof is formulated for dissipative measure-valued solutions of the nonlocal Euler-Korteweg equation which are known to exist on arbitrary intervals of time. Our work provides a new method to derive equations not enjoying classical solutions via relative entropy method by introducing the nonlocal effect in the fluid equation.

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## 6.1 Introduction

We consider the nonlocal Euler-Korteweg system re-scaled in time *i.e.*  $t \rightarrow \frac{t}{\varepsilon}$  and with high friction coefficient  $\frac{1}{\varepsilon}$

$$\partial_t \rho + \frac{1}{\varepsilon} \operatorname{div}(\rho \mathbf{u}) = 0, \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (6.1.1)$$

$$\partial_t(\rho \mathbf{u}) + \frac{1}{\varepsilon} \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = -\frac{1}{\varepsilon^2} \rho \mathbf{u} - \frac{1}{\varepsilon} \rho \nabla(F'(\rho) + B_\eta[\rho]), \quad \text{in } (0, +\infty) \times \mathbb{T}^d. \quad (6.1.2)$$

This equation models the long-time asymptotics of the motion of a compressible fluid with density  $\rho$ , velocity  $\mathbf{u}$  which is in fact a liquid-vapor mixture. The fluid experiences high friction (due to the term  $-\frac{1}{\varepsilon^2} \rho \mathbf{u}$ ) and additional capillary effects in the transition zone between liquid and

vapor (due to the term  $-\frac{1}{\varepsilon}\rho\nabla(F'(\rho) + B_\eta[\rho])$  as proposed by Korteweg [243]).

Concerning the notation,  $\mathbb{T}^d$  is the  $d$ -dimensional flat torus,  $\varepsilon > 0$ ,  $B_\eta$  is the nonlocal operator approximating  $-\Delta$  operator, defined by

$$B_\eta[\rho](x) = \frac{1}{\eta^2}(\rho(x) - \omega_\eta * \rho(x)) = \frac{1}{\eta^2} \int_{\mathbb{T}^d} \omega_\eta(y)(\rho(x) - \rho(x - y)) \, dy \quad (6.1.3)$$

for  $\eta > 0$  small enough and  $\omega_\eta$  is the usual radial mollification kernel  $\omega_\eta(x) = \frac{1}{\eta^d}\omega(\frac{x}{\eta})$  with  $\omega$  compactly supported in the unit ball of  $\mathbb{R}^d$  satisfying

$$\int_{\mathbb{R}^d} \omega(y) \, dy = 1, \quad \int_{\mathbb{R}^d} y \omega(y) \, dy = 0, \quad \int_{\mathbb{R}^d} y_i y_j \omega \, dy = \delta_{i,j} \frac{2D}{d} < \infty. \quad (6.1.4)$$

When  $\varepsilon$  is very small, the friction is so big, that we mostly observe a phase separation phenomenon between the liquid and the vapor. More rigorously, when  $\varepsilon \rightarrow 0$ , we prove that the constructed solution of (6.1.1)-(6.1.2) converge to solutions of the nonlocal Cahn-Hilliard

$$\partial_t \rho = \operatorname{div}(\rho \nabla \mu), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (6.1.5)$$

$$\mu = B_\eta[\rho] + F'(\rho), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (6.1.6)$$

see Theorem 6.1.5. Furthermore, when  $\varepsilon, \eta \rightarrow 0$  in some scaling to be determined, we prove the convergence of (6.1.1)-(6.1.2) to the local Cahn-Hilliard equation

$$\partial_t \rho = \operatorname{div}(\rho \nabla \mu), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (6.1.7)$$

$$\mu = -D\Delta\rho + F'(\rho), \quad \text{in } (0, +\infty) \times \mathbb{T}^d, \quad (6.1.8)$$

which describes the dynamics of phase separation, see Theorem 6.1.4.

Our proof relies on the relative entropy method, which is for instance often used in the context of weak-strong uniqueness. It relies on certain regularity of solutions of the limit system, which is not available in the case of the local Cahn-Hilliard equation. Therefore, we introduce an intermediate step, which is interesting by itself, and consider the nonlocal Cahn-Hilliard equation by introducing the parameter  $\eta$ . Since we know from [155] that the solutions to the nonlocal Cahn-Hilliard equation converge to the weak solutions of the local Cahn-Hilliard equation (see Definition 6.1.3) when  $\eta \rightarrow 0$ , it remains to prove that the nonlocal Euler-Korteweg system tends to the nonlocal Cahn-Hilliard equation when  $\varepsilon \rightarrow 0$ . Then, sending  $\varepsilon$  and  $\eta$  to 0 with the appropriate scaling, we prove the result.

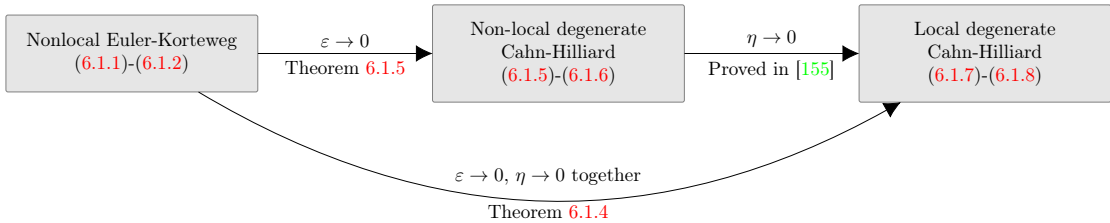


FIGURE 6.1 – Relation between the three equations considered in this article.

The main motivation for our work is the paper of Lattanzio and Tzavaras [249], who prove the

convergence of the local Euler Korteweg system to the local Cahn-Hilliard equation. They assume the existence of admissible weak solutions of the first system and classical solutions of the second one. The first assumption is a drawback as dissipative (that is, satisfying energy inequality) weak solutions existing on arbitrary intervals of time are not known to exist for most models in fluids dynamics. One can try to construct the solutions via the convex integration method but these solutions will have a jump in the energy at the initial time so they will not be dissipative. The second assumption of [249] is also difficult to be satisfied as so far, there is no theory of classical solutions to the local Cahn-Hilliard equation with degenerate mobility on arbitrary intervals of time. Similarly, there is no maximum principle that is necessary in [249] to deduce that the classical solution is strictly positive using positivity of the initial condition.

We propose to overcome the first problem by the concept of dissipative measure-valued solutions, introduced by DiPerna [138] in the context of hyperbolic conservation laws in one dimension and by DiPerna and Majda [139] for the incompressible Euler equations. Roughly speaking, they are defined as the weak limit of classical solutions of appropriate approximating problems. As weak compactness is not sufficient to pass to the limit in nonlinear terms, the definition of the measure-valued solution includes the Young measure  $\nu_{t,x}$  and the concentration measure  $m$  to represent weak limits as in (6.2.1).

While measure-valued solutions are weaker than the usual weak solutions, they are dissipative and they are known to exist. Moreover, their importance comes from the fact that they enjoy the property called weak-strong uniqueness : they coincide with the strong solution whenever the latter exists. The dissipativity is important both for the weak-strong uniqueness and application of the relative entropy method : the weak-strong uniqueness does not hold for weak or measure-valued solutions without any condition on energy as demonstrated by solutions arising by the convex integration method [315, 124].

Since the weak-strong uniqueness property was observed by Brenier, De Lellis and Székelyhidi in [55], measure-valued solutions were studied for several systems including compressible fluid models [207], isentropic Euler system [185], polyconvex elastodynamics [133], Euler-Poisson system [79], general hyperbolic conservation laws [204]. Moreover, for many equations describing compressible fluids, the measure-valued formulation has been significantly simplified [168, 39, 1] : it boils down to the usual distributional identity modulo the so-called Reynolds stress tensor.

Concerning the problem of the existence of classical solutions, we propose to introduce nonlocality in the equation and introduce an intermediate step in the convergence analysis as outlined in Figure 6.1. The advantage is that the nonlocal Cahn-Hilliard equation is in fact a porous medium equation. In particular, it satisfies the maximum principle and so, if the initial condition is positive, the solution remains positive and one can prove the existence and uniqueness of a classical solution, see Section 6.4. Furthermore, it is known that the nonlocal Cahn-Hilliard equation converges to the local one [155] so that at the end, the nonlocality can be removed.

To prove the convergence, we use the relative entropy method. The method is based on introducing a functional called relative entropy (or energy), which measures the dissipation between two solutions of the system. Essentially, the same method is used to prove the aforementioned weak-strong uniqueness when the relative entropy measures the distance between weak (measure-valued) and strong solutions. This strategy has been applied for several singular limits [107, 90, 19, 94, 223, 249, 250] and we also refer to the excellent review on weak-strong uniqueness [327].

Our proof via the relative entropy method is based on an important assumption that the initial datum is well-prepared. In our case, this means that the initial velocity  $\mathbf{u}_0$  vanishes as the parameter  $\varepsilon \rightarrow 0$  cf. (6.1.9) and (6.1.10) so that the initial kinetic energy is very small. Such an assumption is necessary to guarantee that the relative entropy  $\Theta(0)$  at time  $t = 0$  converges to 0 as  $\varepsilon \rightarrow 0$  so that  $\Theta(t) \rightarrow 0$ , cf. (6.5.5), which implies the main result. Let us however remark that one can also study similar problems via compactness methods and this approach is also effective for ill-prepared initial data. Nevertheless, its applicability is restricted to some special cases like one spatial dimension (which allows to use the div-curl lemma in the time-space setting) [268] or the presence of viscosity terms yielding compactness [169].

### 6.1.1 Rigorous formulation of the main result

We make the following assumptions on the potential  $F$ .

*Assumption 6.1.1* (potential  $F$ ). For the interaction potential we assume that there exists  $k \geq 2$  and constant  $C$  such that  $F$  can be written as  $F = F_1 + F_2$  where

1.  $F_1 \in C^4(\mathbb{R})$  is a convex, nonnegative function having  $k$ -growth

$$\frac{1}{C}|u|^k - C \leq F_1(u) \leq C|u|^k + C,$$

$$\frac{1}{C}|u|^{(k-2)} - C \leq F_1''(u) \leq C|u|^{(k-2)} + C$$

and satisfying  $|uF_1'(u)| \leq C(F_1(u) + 1)$ ,  $|uF_1^{(3)}(u)| \leq C(F_1''(u) + 1)$ ,

2.  $F_2 \in C^4(\mathbb{R})$  is such that  $F_2, F_2', F_2'', sF_2^{(3)}(s) \in L^\infty(\mathbb{R})$  are bounded on the whole line. Moreover,  $\|F_2''\|_\infty < \frac{1}{C_P}$  where  $C_P$  is a constant in Lemma 6.7.2.

We also define  $s := \frac{2k}{k-1}$  and  $s'$  its conjugate exponent.

**Example 6.1.2.** *The following potentials satisfy Assumption 6.1.1.*

- (1) *power-type potential*  $F(u) = |u|^\gamma$ ,  $\gamma > 2$  used in the context of tumor growth models [289, 120, 151, 125],
- (2) *double-well potential*  $F(u) = u^2(u-1)^2$  which is an approximation of logarithmic double-well potential often used in Cahn-Hilliard equation, see [277, Chapter 1].

Before stating the main result, we define solutions of the local degenerate Cahn-Hilliard equation.

*Definition 6.1.3.* We say that  $\rho$  is a weak solution of (6.1.7)-(6.1.8) if

$$\begin{aligned} \rho \in L^\infty(0, T; L^k(\mathbb{T}^d)) \cap L^2(0, T; H^2(\mathbb{T}^d)), \quad \partial_t \rho \in L^2(0, T; W^{-1, s'}(\mathbb{T}^d)), \\ \sqrt{F_1''(\rho)} \nabla \rho \in L^2((0, T) \times \mathbb{T}^d), \end{aligned}$$

$\rho(0, x) = \rho_0(x)$  a.e. in  $\mathbb{T}^d$  and if for all  $\varphi \in L^2(0, T; W^{2, \infty}(\mathbb{T}^d))$  we have

$$\begin{aligned} \int_0^T \langle \partial_t \rho, \varphi \rangle_{(W^{-1, s'}(\mathbb{T}^d), W^{1, s}(\mathbb{T}^d))} = -D \int_0^T \int_{\mathbb{T}^d} \Delta \rho \nabla \rho \cdot \nabla \varphi - D \int_0^T \int_{\mathbb{T}^d} \rho \Delta \rho \Delta \varphi \\ - \int_0^T \int_{\mathbb{T}^d} \rho F''(\rho) \nabla \rho \cdot \nabla \varphi. \end{aligned}$$

The definition of dissipative measure-valued solutions to (6.1.1)-(6.1.2) is quite technical and will be presented in Definitions 6.3.1 and 6.3.4. The main theorem reads as follows.

**Theorem 6.1.4.** *Let  $\rho_0$  be an initial density satisfying*

$$\rho_0 \geq \sigma > 0, \quad \rho_0 \in C^3(\mathbb{T}^d)$$

*for some  $\sigma > 0$ . Let  $\mathbf{u}_{0,\varepsilon}$  be an initial velocity satisfying*

$$\|\mathbf{u}_{0,\varepsilon}\|_{L^2(\mathbb{T}^d)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (6.1.9)$$

*Let  $(\rho_{\eta,\varepsilon}, \sqrt{\rho_{\eta,\varepsilon}}\mathbf{u}_{\eta,\varepsilon}, \nu^{\eta,\varepsilon}, m_{\eta,\varepsilon})$  be a dissipative measure-valued solution of (6.1.1)–(6.1.2) with the initial condition  $(\rho_0, \mathbf{u}_{0,\varepsilon})$  and parameters  $\varepsilon, \eta$  satisfying Poincaré inequality (6.3.23). Then, for each sequence  $\eta_k \rightarrow 0$ , there exists a subsequence  $\{\eta_k\}$  (not relabelled) and a sequence  $\{\varepsilon_k\}$  depending on  $\eta_k$  and the final time  $T$  such that  $\varepsilon_k \rightarrow 0$  and  $\rho_{\eta_k, \varepsilon_k} \rightarrow \rho$  in  $L^2(0, T; L^2(\mathbb{T}^d))$ , where  $\rho$  is a weak solution of (6.1.7)–(6.1.8) with initial condition  $\rho_0$  as defined in Definition 6.1.3.*

Let us briefly comment that the measure-valued solution has in fact four components. While the first component  $\rho_{\eta,\varepsilon}$  is the most important since it converges to the Cahn-Hilliard equation, we can also characterize what happens with the other ones, see Theorem 6.6.2. Roughly speaking, the second one converges to 0 in  $L^\infty(0, T; L^2(\mathbb{T}^d))$  which represents that in the high-friction limit, the kinetic energy converges to 0. The parametrized measure  $\nu^{\eta,\varepsilon}$  converges in the second Wasserstein metric  $\mathcal{W}_2$  to the Dirac mass  $\delta_{\rho(t,x)} \otimes \delta_0$  :

$$\int_0^T \int_{\mathbb{T}^d} [\mathcal{W}_2(\nu^{\eta_k, \varepsilon_k}, \delta_{\rho(t,x)} \otimes \delta_0)]^2 dx dt \rightarrow 0 \text{ as } \varepsilon_k, \eta_k \rightarrow 0$$

while the concentration measure  $m_{\eta_k, \varepsilon_k}$  converges to 0 in the total variation norm. The estimate in the Wasserstein metric is in the spirit of [175].

Theorem 6.1.4 is valid only for a subsequence as the convergence from non-local Cahn-Hilliard to the local one is based on the compactness arguments (and there is no uniqueness for the limit equation). On the other hand, the passage from the nonlocal Euler-Korteweg equation to the nonlocal Cahn-Hilliard equation is based on the relative entropy method and so the convergence is satisfied for any sequence. We state this result below.

**Theorem 6.1.5.** *Let  $\eta \in (0, \eta_0)$  where  $\eta_0$  is defined in Lemma 6.7.2. Let  $\rho_0$  be an initial density satisfying*

$$\rho_0 \geq \sigma > 0, \quad \rho_0 \in C^3(\mathbb{T}^d)$$

*for some  $\sigma > 0$ . Let  $\mathbf{u}_{0,\varepsilon}$  be an initial velocity satisfying*

$$\|\mathbf{u}_{0,\varepsilon}\|_{L^2(\mathbb{T}^d)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (6.1.10)$$

*Let  $(\rho_{\eta,\varepsilon}, \sqrt{\rho_{\eta,\varepsilon}}\mathbf{u}_{\eta,\varepsilon}, \nu^{\eta,\varepsilon}, m_{\eta,\varepsilon})$  be a dissipative measure-valued solution of (6.1.1)–(6.1.2) with initial condition  $(\rho_0, \mathbf{u}_{0,\varepsilon})$  and parameters  $\varepsilon, \eta$  satisfying Poincaré inequality (6.3.23). Let  $\rho_\eta$  be the solution of non-local Cahn-Hilliard (6.1.5)–(6.1.6) with the same initial condition  $\rho_0$ . Then,  $\rho_{\eta,\varepsilon} \rightarrow \rho_\eta$  in  $L^\infty(0, T; L^2(\mathbb{T}^d))$  as  $\varepsilon \rightarrow 0$ .*

Similarly as for Theorem 6.1.4, we can prove convergence of the other components of the measure-valued solution  $\sqrt{\rho_{\eta,\varepsilon}}\mathbf{u}_{\eta,\varepsilon}, \nu^{\eta,\varepsilon}, m_{\eta,\varepsilon}$ , see Theorem 6.6.1.

## 6.1.2 Relevancy of the system

**The Euler–Korteweg equation** The compressible Euler–Korteweg equation models the motion of liquid-vapor mixtures with possible phase transitions. It combines the classical Euler



equation with Korteweg tensor introduced in [243]. The equation reads

$$\begin{aligned}\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) &= 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla(p(\rho)) &= -\zeta \rho \mathbf{u} + \rho \nabla(K(\rho) \Delta \rho + \frac{1}{2} K'(\rho) |\nabla \rho|^2).\end{aligned}\tag{6.1.11}$$

Here,  $\rho$  is the density of the fluid,  $\mathbf{u}$  is its velocity,  $K(\rho)$  corresponds to the capillary coefficient,  $\zeta$  is the friction coefficient and  $p$  is the pressure function. In a liquid-vapor system, the tensor  $K$  takes into account that the liquid and vapour are separated by a thin layer of finite thickness and describes the capillary effects in this transition zone. There are numerous mathematical results concerning well-(and ill-)posedness of solutions to (6.1.11), see [141, 29, 59, 24, 44, 58]. For instance, for some particular choice of  $K(\rho)$ , an approach to prove existence of global solutions is to relate the Euler-Korteweg and the Schrödinger equation through the Madelung transform [72, 24]. For more general cases, only local existence [44] and global existence for small irrotational data [29] is known. For the physical background of (6.1.11) (in particular, the form of the Korteweg tensor) we refer to [143, 229, 214] but it is a fairly complicated matter.

The viscous version of (6.1.11), that is the Navier-Stokes-Korteweg system, was also studied in the mathematical literature [25, 188]. In particular, several papers are concerned with the case of the nonlocal equation, where  $-\Delta \rho$  is approximated by the nonlocal operator  $B_\eta$ . In [300], the author proves the short-time well-posedness while in [96], the global well-posedness as well as the convergence of the nonlocal Navier-Stokes-Korteweg to the local one is established. We also refer to [95] for a variant of this system.

**The high-friction limit** The high-friction limit (also referred to in the literature as the relaxation limit) is a part of a long research program of establishing a connection between nonlinear hyperbolic systems and degenerate diffusion equations. One of the first results in this direction [268] states that the solutions to the compressible Euler equations in one dimension

$$\begin{aligned}\partial_t \rho + \partial_x(\rho u) &= 0, \\ \varepsilon^2 \partial_t(\rho u) + \partial_x(\varepsilon^2 \rho u^2 + p(\rho)) &= -u\end{aligned}\tag{6.1.12}$$

converge, as  $\varepsilon \rightarrow 0$ , to the porous media equation

$$\partial_t \rho = \partial_x(\rho \partial_x p(\rho))$$

where  $p(\rho)$  is the pressure function of the form  $p(\rho) = \rho^\gamma$ . To connect (6.1.12) with our system (6.1.1)–(6.1.2), it is sufficient to rescale  $\tilde{u} = \varepsilon u$  so that we have

$$\begin{aligned}\partial_t \rho + \frac{1}{\varepsilon} \partial_x(\rho \tilde{u}) &= 0, \\ \partial_t(\rho \tilde{u}) + \frac{1}{\varepsilon} \partial_x(\rho \tilde{u}^2 + p(\rho)) &= -\frac{\tilde{u}}{\varepsilon^2}.\end{aligned}\tag{6.1.13}$$

Intuitively, it is easy to understand from (6.1.13) that the flow of the fluid with big damping or friction (caused by the term  $-\frac{\tilde{u}}{\varepsilon^2}$ ) and very small kinetic energy (caused by the initial condition) resembles a flow through a porous media. Several other limit passages have been studied between porous medium equation and hyperbolic equations [323, 267, 27]. The revival of interest in this type of problem appeared recently with an observation that one can study these problems by the

relative entropy method [250, 191, 107, 249, 184].

In our case, we consider (6.1.11) with  $K(\rho) = 1$ , large friction coefficient  $\zeta = \frac{1}{\varepsilon}$ , we approximate the Laplace operator  $-\Delta$  by the nonlocal operator  $B_\eta$  with  $\eta$  small enough, and we perform a rescaling in time  $t \rightarrow \frac{t}{\varepsilon}$ . Then, we let both  $\varepsilon, \eta \rightarrow 0$ , and in the limit, we obtain the Cahn-Hilliard equation. Again, it is intuitive that due to the very large damping and small kinetic energy, we observe mostly a phase separation process. The latter is described by the Cahn-Hilliard equation so it is not surprising that it is the limiting PDE.

**The Cahn-Hilliard equation** In their publications [69] and [70] J.W. Cahn and J. E. Hilliard proposed the equation in 1958. It represents now a commonly used mathematical model for describing phase changes in fluids, although the equation was primarily developed in material sciences to explore phase separation processes under isotropy and constant temperature circumstances.

Being of fourth-order, the (local) Cahn-Hilliard equation is often rewritten in a system of two second-order equations, *i.e.*

$$\partial_t \rho = \operatorname{div} (m(\rho) \nabla (F'(\rho) - D\Delta \rho)) \rightarrow \begin{cases} \partial_t \rho &= \operatorname{div} (m(\rho) \nabla \mu), \\ \mu &= -D\Delta \rho + F'(\rho), \end{cases} \quad (6.1.14)$$

where  $\rho$  is the concentration of a phase and  $\mu$  is called the chemical potential in material sciences but is often used as an effective pressure. The interaction potential  $F(\rho)$  contained in this effective pressure term comprises the effects of attraction and repulsion between particles. Finally, the Laplace operator takes into account surface tension effects.

The existence and uniqueness of solutions for the Cahn-Hilliard system (6.1.14) strictly depends on the properties of the mobility term  $m(\rho)$  and the potential  $F(\rho)$ , as well as the conditions assigned on the boundary. More specifically, the presence of *degeneration* on the mobility, *i.e.* the possibility for it to vanish, can turn the analysis of solutions into a rather complex problem. We refer to [157] for the first existence result of weak solutions in the case of degenerate mobility and to [113] for some improvements. The uniqueness and the existence of classical solutions are open questions for this type of mobility. Since we use a relative entropy argument between the Euler-Korteweg equation and the Cahn-Hilliard equation, the existence of a classical solution of the latter is a crucial point. In fact, we need an  $L^\infty$  bound on the second derivative of  $\rho$ , and that the solution remains positive for all times in a finite time interval. Since the Cahn-Hilliard equation does not satisfy the maximum principle, it is impossible to get these estimates. For that purpose, we introduce the nonlocal Cahn-Hilliard equation, which is a second-order equation with a nonlocal smooth advection term. With classical arguments, we are able to prove that the latter admits a unique positive classical solution. Let us finally remark that the nonlocal Cahn-Hilliard equation is nowadays a topic of intense research activity, see for instance [293, 123, 178, 181, 81].

**Mobilities** This work focuses on the mobility case where  $m(\rho) = \rho$ , which is a result of deriving the Cahn-Hilliard equation from fluid models. This mobility is also obtained from Vlasov equation via hydrodynamic limit [150] and is also observed in the nonlocal Cahn-Hilliard equation, which can be derived from systems of interacting particles as an aggregation-diffusion equation (see [92, 65, 93]). Furthermore, it would be of interest to investigate whether this work can be extended

to the mobility case of  $m(\rho) = \rho(1 - \rho)$ , as studied in the original works of Giacomini-Lebowitz [189, 190]. They developed a model based on a  $d$ -dimensional lattice gas that evolves through Kawasaki exchange dynamics, which is a Poisson process that exchanges nearest neighbors. In the hydrodynamic limit, they observed that the average occupation numbers over a small macroscopic volume element tends towards a solution of a non-local Cahn-Hilliard equation with mobility  $m(\rho) = \rho(1 - \rho)$ .

## 6.2 Generalised Young Measures

We introduce the framework of Young measures to define the solutions of the nonlocal Euler-Korteweg equation. This framework is necessary since in the usual approximation schemes we cannot pass to the limit in the terms term of type  $f(z_j)$  where  $f$  is nonlinear and  $\{z_j\}$  is only a weakly star convergent sequence. The idea of Young measures is to embed the problem in a larger space and gain linearity. We write  $f(z_j(y)) = \langle f, \delta_{z_j(y)} \rangle$ , and if  $f \in C_0(\mathbb{R}^n)$ , using the duality  $(L^1(Q; C_0(\mathbb{R}^n)))^* = L_w^\infty(Q; \mathcal{M}(\mathbb{R}^n))$ , Banach-Alaoglu theorem and weak-star continuity of linear operators, we can pass to the limit. The same is true if  $\{f(z_j)\}$  is weakly compact in  $L^1(Q)$  and  $\{z_j\}$  does not grow too fast as the following theorem states :

**Theorem 6.2.1** (Fundamental Theorem of Young Measures). *Let  $Q \subset \mathbb{R}^d$  be a measurable set and let  $z_j : Q \rightarrow \mathbb{R}^n$  be measurable functions such that*

$$\sup_{j \in \mathbb{N}} \int_Q g(|z_j(y)|) \, dy < +\infty$$

for some continuous, nondecreasing function  $g : [0, +\infty) \rightarrow [0, +\infty)$  with  $\lim_{t \rightarrow +\infty} g(t) = +\infty$ . Then, there exists a subsequence (not relabeled) and a weakly star measurable family of probability measures  $\nu = \{\nu_y\}_{y \in Q}$  with the property that whenever the sequence  $\{\psi(y, z_j(y))\}_{j \in \mathbb{N}}$  is weakly compact in  $L^1(Q)$  for a Carathéodory function (measurable in the first and continuous in the second argument)  $\psi : Q \times \mathbb{R}^n \rightarrow \mathbb{R}$ , we have

$$\psi(y, z_j(y)) \rightharpoonup \int_{\mathbb{R}^n} \psi(y, \lambda) \, d\nu_y(\lambda) \quad \text{in } L^1(Q).$$

We say that the sequence  $\{z_j\}_{j \in \mathbb{N}}$  generates the sequence of Young measures  $\{\nu_y\}_{y \in Q}$ .

Note that in the above theorem, we require that the sequence  $\{\psi(y, z_j(y))\}$  is weakly compact in  $L^1(Q)$ . This prevents the concentration effect to appear (think about the family of standard mollifiers). When we do not have weak compactness, we use the following proposition which follows from the Banach-Alaoglu theorem. We formulate it with a distinguishment between time and space variables (that is,  $Q = (0, T) \times \Omega$ ,  $y = (t, x)$  with  $t \in (0, T)$  and  $x \in \Omega$ ) as usually in applications one has better integrability in time which results in better characterization of the resulting measure. The following proposition is a consequence of the Banach-Alaoglu theorem and the Radon-Nikodym theorem, see [55].

**Proposition 6.2.2.** *Let  $f$  be a continuous function and a sequence  $\{f(t, x, z_j(t, x))\}_{j \in \mathbb{N}}$  be bounded in  $L^p(0, T; L^1(\Omega))$  with  $p \geq 1$ . Let  $\{\nu_{t,x}\}_{t,x}$  be the Young measure generated by  $\{z_j\}_j$ . Then there exists a measure  $m^f$  such that (up to a subsequence not relabelled)*

$$f(t, x, z_j(t, x)) - \langle \nu_{t,x}, f \rangle \xrightarrow{*} m^f \quad \text{in } L^p(0, T; \mathcal{M}(\Omega)) \text{ if } p > 1,$$

$$f(t, x, z_j(t, x)) - \langle \nu_{t,x}, f \rangle \xrightarrow{*} m^f \quad \text{in } \mathcal{M}((0, T) \times \Omega) \text{ if } p = 1.$$

Moreover, if  $p > 1$ , the measure  $m^f$  is absolutely continuous with respect to time : for a.e.  $t \in (0, T)$ , there exists measure  $m^f(t, \cdot)$  such that

$$\int_{(0,T) \times \Omega} \psi(t, x) dm^f(t, x) = \int_0^T \int_{\Omega} \psi(t, x) m^f(t, dx) dt.$$

Let us remark that by the fundamental theorem, we have  $m^f = 0$  when the sequence  $\{f(z_j)\}_{j \in \mathbb{N}}$  is weakly compact in  $L^1((0, T) \times \Omega)$ . We use the notation :

$$\bar{f} = \langle f(\lambda), \nu_{t,x} \rangle + m^f \tag{6.2.1}$$

to represent weak limit of  $f(t, x, z_j(t, x))$ . We also need the following result which allows comparing two concentration measures  $m^{f_1}$  and  $m^{f_2}$  for two different nonlinearities  $f_1, f_2$ . For the proof, we refer to [167, Lemma 2.1].

**Proposition 6.2.3.** *Let  $\{\nu_{t,x}\}_{(t,x) \in (0,T) \times \Omega}$  be a Young measure generated by a sequence  $\{z_j\}_{j \in \mathbb{N}}$ . If two continuous functions  $f_1 : (0, T) \times \Omega \rightarrow \mathbb{R}^d$  and  $f_2 : (0, T) \times \Omega \rightarrow \mathbb{R}^+$  satisfy  $|f_1(z)| \leq f_2(z)$  for every  $z$ , and if  $\{f_2(z_j)\}$  is uniformly bounded in  $L^1((0, T) \times \Omega)$ , then we have*

$$|m^{f_1}(A)| \leq m^{f_2}(A),$$

for any borel set  $A \subset (0, T) \times \Omega$ .

Here,  $|\mu|$  is the total variation measure defined as  $|\mu|(A) = \mu^+(A) + \mu^-(A)$  where  $\mu^+, \mu^-$  are positive and negative parts of  $\mu$ .

Let us conclude with a few comments about the measure  $m^f$  which captures concentration effects. One can describe it more precisely. The first attempts to do so by some generalizations of the Young measures were initiated by DiPerna and Majda in the case of the incompressible Euler equations [139]. Then, Alibert and Bouchitté extended the result to a more general class of nonlinearities in [15]. They proved that there exists a subsequence (not relabeled) as well as a parametrized probability measure  $\nu \in L_w^\infty(Q; \mathcal{P}(\mathbb{R}^n))$  (which is identical with the "classical" Young measure), a non-negative measure  $m \in \mathcal{M}^+(Q)$ , and a parametrized probability measure  $\nu^\infty \in L_w^\infty(Q, m; \mathcal{P}(\mathbb{S}^{n-1}))$  such that for any Carathéodory function  $f$  such that  $f(x, z)/(1 + |z|)$  is bounded and uniformly continuous with respect to  $z$ ,

$$f(y, z_j(y)) \xrightarrow{*} \int_{\mathbb{R}^d} f(y, \lambda) d\nu_y(\lambda) + \int_{\mathbb{S}^{n-1}} f^\infty(y, \beta) d\nu_y^\infty(\beta) m(y)$$

weakly\* in the sense of measures. Here,

$$f^\infty(y, \beta) := \lim_{s \rightarrow \infty} \frac{f(y, t\beta)}{t}.$$

Their result was also extended to the case when  $f$  has different growth with respect to different variables, see for instance [207].

## 6.3 Measure-valued solutions to the nonlocal Euler-Korteweg equation

### 6.3.1 Definition of dissipative measure-valued solutions

Let us motivate the definition of a measure-valued solution by their construction. We will consider a sequence of approximating solutions  $\{(\rho_\delta, \mathbf{u}_\delta)\}$ , see Section 6.3.2, satisfying the estimates (uniform in  $\delta$ )

$$\{\rho_\delta\} \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)), \quad \{F(\rho_\delta)\} \text{ in } L^\infty(0, T; L^1(\mathbb{T}^d)), \quad \{\sqrt{\rho_\delta} \mathbf{u}_\delta\} \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)),$$

which will be a consequence of energy inequality (6.3.36). As we do not have estimates on  $\{\mathbf{u}_\delta\}$  itself, we will consider in fact the sequence  $\{(\rho_\delta, \sqrt{\rho_\delta} \mathbf{u}_\delta)\}$ . Up to a subsequence, we have as  $\delta \rightarrow 0$

$$\rho_\delta \overset{*}{\rightharpoonup} \rho \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)) \quad \sqrt{\rho_\delta} \mathbf{u}_\delta \overset{*}{\rightharpoonup} \sqrt{\rho \mathbf{u}} \text{ in } L^\infty(0, T; L^2(\mathbb{T}^d)), \quad (6.3.1)$$

where  $\sqrt{\rho \mathbf{u}}$  is a *definition* of a weak limit of  $\sqrt{\rho_\delta} \mathbf{u}_\delta$ . Let  $\{\nu_{t,x}\}$  be the Young measure generated by this sequence as in Theorem 6.2.1. We will use dummy variables  $(\lambda_1, \lambda') \in \mathbb{R}^+ \times \mathbb{R}^d$  when integrating with respect to  $\nu_{t,x}$ :

$$\langle F(\lambda_1, \lambda'), \nu_{t,x} \rangle := \int_{\mathbb{R}^+ \times \mathbb{R}^d} F(\lambda_1, \lambda') d\nu_{x,t}(\lambda_1, \lambda'), \quad (6.3.2)$$

with  $\lambda_1$  representing  $\rho$  variable and  $\lambda'$  as representing  $\sqrt{\rho \mathbf{u}}$  variable. In terms of Young measures we write weak convergence (6.3.1) as

$$\rho = \langle \lambda_1, \nu \rangle, \quad \sqrt{\rho \mathbf{u}} = \langle \lambda', \nu \rangle, \quad (6.3.3)$$

as there is no concentration measure because of integrability in  $L^2((0, T) \times \mathbb{T}^d)$ . Using notation (6.2.1) we represent weak limits (as  $\delta \rightarrow 0$ ) of all the terms that should appear in the weak formulation and the energy

$$\overline{\rho^2} = \langle \lambda_1^2, \nu \rangle + m^{\rho^2}, \quad (6.3.4)$$

$$\overline{\rho \mathbf{u}} = \langle \sqrt{\lambda_1} \lambda', \nu \rangle, \quad (6.3.5)$$

$$\overline{\rho \mathbf{u} \otimes \mathbf{u}} = \langle \lambda' \otimes \lambda', \nu \rangle + m^{\rho \mathbf{u} \otimes \mathbf{u}}, \quad (6.3.6)$$

$$\overline{\rho |\mathbf{u}|^2} = \langle |\lambda'|^2, \nu \rangle + m^{\rho |\mathbf{u}|^2}, \quad (6.3.7)$$

$$\overline{F(\rho)} = \langle F(\lambda_1), \nu \rangle + m^{F(\rho)} \quad (6.3.8)$$

$$\overline{\rho F'(\rho)} = \langle \lambda_1 F'(\lambda_1), \nu \rangle + m^{\rho F'(\rho)}, \quad (6.3.9)$$

$$\overline{p(\rho)} = \overline{\rho F'(\rho)} - \overline{F(\rho)} + \frac{1}{2\eta^2} \overline{\rho^2}, \quad (6.3.10)$$

where  $p(\rho) := \rho F'(\rho) - F(\rho) + \frac{\rho^2}{2\eta^2}$ .

Moreover, we will identify weak limits of several nonlinearities which will be used in this work. By linearity of weak limits, we have the following identities:

$$\int_{\mathbb{T}^d} \omega_\eta(y) |\rho(x) - \rho(x-y)|^2 dy = \overline{\rho^2} + \overline{\rho^2} * \omega_\eta - 2\rho \omega_\eta * \rho \quad (6.3.11)$$

Similarly, for all bounded  $P : (0, T) \times [0, +\infty) \rightarrow \mathbb{R}^+$  and  $\mathbf{U} : (0, T) \times \mathbb{T}^d \rightarrow \mathbb{R}^d$  we have

$$|\overline{\rho - P}|^2 = \overline{\rho^2} + P^2 - 2\rho P \quad (6.3.12)$$

$$\overline{\rho|\mathbf{u} - \mathbf{U}|^2} = \overline{\rho|\mathbf{u}|^2} + \rho|\mathbf{U}|^2 - 2\overline{\rho\mathbf{u}} \cdot \mathbf{U}, \quad (6.3.13)$$

$$\overline{\rho(\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U})} = \langle (\lambda' - \sqrt{\lambda_1}\mathbf{U}) \otimes (\lambda' - \sqrt{\lambda_1}\mathbf{U}), \nu_{t,x} \rangle + m^{\rho\mathbf{u}\otimes\mathbf{u}}, \quad (6.3.14)$$

$$\begin{aligned} \overline{\int_{\mathbb{T}^d} \omega_\eta(y) |(\rho - P)(x) - (\rho - P)(x - y)|^2 dy} &= \overline{\int_{\mathbb{T}^d} \omega_\eta(y) |\rho(x) - \rho(x - y)|^2 dy} + \\ &+ \overline{\int_{\mathbb{T}^d} \omega_\eta(y) |P(x) - P(x - y)|^2 dy} - 2 \overline{\int_{\mathbb{T}^d} \omega_\eta(y) (P(x) - P(x - y)) (\rho(x) - \rho(x - y)) dy}, \end{aligned} \quad (6.3.15)$$

$$\overline{F(\rho|P)} := \overline{F(\rho)} - F(P) - F'(P)(\rho - P), \quad \overline{p(\rho|P)} := \overline{p(\rho)} - p(P) - p'(P)(\rho - P) \quad (6.3.16)$$

where nonlinearities are defined as

$$F(\rho|P) = F(\rho) - F(P) - F'(P)(\rho - P), \quad p(\rho|P) = p(\rho) - p(P) - p'(P)(\rho - P). \quad (6.3.17)$$

Now, we define measure-valued solutions by *inverting* this discussion.

*Definition 6.3.1* (Measure-valued solution). We say that  $(\rho, \sqrt{\rho\mathbf{u}}, \nu, m)$  where

$$\begin{aligned} \nu &= \{\nu_{t,x}\} \in L_{\text{weak}}^\infty((0, T) \times \mathbb{T}^d; \mathcal{P}([0, +\infty) \times \mathbb{R}^d)) \\ \rho &= \langle \lambda_1, \nu \rangle = \int_{\mathbb{R}^+ \times \mathbb{R}^d} \lambda_1 d\nu_{x,t}(\lambda_1, \lambda') \in L^\infty(0, T; L^2(\mathbb{T}^d)), \\ \sqrt{\rho\mathbf{u}} &= \langle \lambda', \nu \rangle = \int_{\mathbb{R}^+ \times \mathbb{R}^d} \lambda' d\nu_{x,t}(\lambda_1, \lambda') \in L^\infty(0, T; L^2(\mathbb{T}^d)), \\ m &= (m^{\rho^2}, m^{\rho\mathbf{u}\otimes\mathbf{u}}, m^{\rho|\mathbf{u}|^2}, m^{F(\rho)}, m^{\rho F'(\rho)}) \end{aligned}$$

with

$$\begin{aligned} m^{\rho^2}, m^{\rho|\mathbf{u}|^2}, m^{F(\rho)} &\in L^\infty((0, T); \mathcal{M}^+(\mathbb{T}^d)), & m^{\rho F'(\rho)} &\in L^\infty((0, T); \mathcal{M}(\mathbb{T}^d)), \\ m^{\rho\mathbf{u}\otimes\mathbf{u}} &\in L^\infty((0, T); \mathcal{M}(\mathbb{T}^d)^{d \times d}) \end{aligned}$$

and

$$|m^{\rho\mathbf{u}\otimes\mathbf{u}}| \leq m^{\rho|\mathbf{u}|^2} \quad (6.3.18)$$

$$|m^{\rho F'(\rho)}| \leq C_F m^{F(\rho)} + C_F m^{\rho^2}, C_F \text{ defined in (6.8.2)} \quad (6.3.19)$$

is a measure-valued solution of (6.1.1)-(6.1.2) with initial data  $(\rho_0, \mathbf{u}_0)$  if for every  $\psi \in C_c^1([0, T) \times \mathbb{T}^d; \mathbb{R})$ ,  $\phi \in C_c^1([0, T) \times \mathbb{T}^d; \mathbb{R}^d)$  it holds that

$$\int_0^T \int_{\mathbb{T}^d} \partial_t \psi \rho + \frac{1}{\varepsilon} \nabla \psi \cdot \overline{\rho\mathbf{u}} dx dt + \int_{\mathbb{T}^d} \psi(x, 0) \rho_0 dx = 0, \quad (6.3.20)$$

$$\int_0^T \int_{\mathbb{T}^d} \partial_t \phi \cdot \overline{\rho \mathbf{u}} + \frac{1}{\varepsilon} \nabla \phi : \overline{\rho \mathbf{u} \otimes \mathbf{u}} - \frac{1}{\varepsilon^2} \phi \cdot \overline{\rho \mathbf{u}} + \frac{1}{\varepsilon} \operatorname{div} \phi \overline{p(\rho)} + \frac{1}{\varepsilon \eta^2} \phi \cdot \rho \nabla \omega_\eta * \rho \, dx \, dt + \int_{\mathbb{T}^d} \phi(x, 0) \cdot \rho_0 u_0 \, dx = 0, \quad (6.3.21)$$

where  $p(\rho) = \rho F'(\rho) - F(\rho) + \frac{\rho^2}{2\eta^2}$  and all the terms are defined in (6.3.3)–(6.3.10).

*Definition 6.3.2* (nonlinear functions). Given a measure-valued solution  $(\rho, \sqrt{\rho \mathbf{u}}, \nu, m)$  and bounded  $P : (0, T) \times [0, +\infty) \rightarrow \mathbb{R}^+$ ,  $\mathbf{U} : (0, T) \times \mathbb{T}^d \rightarrow \mathbb{R}^d$ , we define nonlinear quantities

$$\overline{\int_{\mathbb{T}^d} \omega_\eta(y) |\rho(x) - \rho(x-y)|^2 \, dy}, \quad \overline{|\rho - P|^2}, \quad \overline{\rho |\mathbf{u} - \mathbf{U}|^2}, \quad \overline{F(\rho|P)}, \quad \overline{p(\rho|P)},$$

$$\overline{\rho(\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U})}, \quad \overline{\int_{\mathbb{T}^d} \omega_\eta(y) |(\rho - P)(x) - (\rho - P)(x-y)|^2 \, dy}$$

by formulas (6.3.11)–(6.3.16).

*Definition 6.3.3* (energy). Given a measure-valued solution  $(\rho, \sqrt{\rho \mathbf{u}}, \nu, m)$  for a.e.  $t \in (0, T)$  we define the energy as

$$E_{mvs}(t) := \int_{\mathbb{T}^d} \frac{1}{2} \overline{\rho |\mathbf{u}|^2} + \overline{F(\rho)} \, dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y) |\rho(x) - \rho(x-y)|^2} \, dy \, dx,$$

where the nonlocal term is defined by (6.3.11). We also define

$$E_0 := \int_{\mathbb{T}^d} \frac{1}{2} \rho_0 |u_0|^2(x) + F(\rho_0) \, dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y) |\rho_0(x) - \rho_0(x-y)|^2 \, dx \, dy.$$

This energy is well-defined because, by Proposition 6.2.2, a concentration measure  $m \in L^\infty(0, T; \mathcal{M}(\mathbb{T}^d))$  admits disintegration  $dm(t, x) = m(t, dx) dt$  where  $m(t, \cdot)$  is a well-defined measure on  $\mathbb{T}^d$  for a.e.  $t \in (0, T)$ .

We now introduce two properties which allows to select the *right* measure-valued solutions.

*Definition 6.3.4* (Dissipative measure-valued solution). We say that a measure-valued solution  $(\rho, \sqrt{\rho \mathbf{u}}, \nu, m)$  is dissipative if

$$E_{mvs}(t) + \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \overline{\rho |\mathbf{u}|^2} \, dx \, dt \leq E_0 \quad (6.3.22)$$

for almost every  $t \in (0, T)$ .

*Definition 6.3.5* (Poincaré inequality). A measure-valued solution  $(\rho, \sqrt{\rho \mathbf{u}}, \nu, m)$  with initial condition  $\rho_0$  satisfies the nonlocal Poincaré inequality if for a.e.  $t \in (0, T)$  and all bounded  $P : \Omega_T \rightarrow [0, +\infty)$  such that  $(P)_{\mathbb{T}^d} = (\rho_0)_{\mathbb{T}^d}$  we have

$$\int_{\mathbb{T}^d} \overline{|\rho - P|^2} \, dx \leq \frac{C_P}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y) |(\rho - P)(x) - (\rho - P)(x-y)|^2} \, dy \, dx. \quad (6.3.23)$$

where the constant  $C_P$  is given by Lemma 6.7.2.

Let us remark that in Lemma 6.5.2, we will prove that any measure-valued solution satisfies

$$\int_{\mathbb{T}^d} |\rho - P|^2 dx \leq \frac{C_P}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y) |(\rho - P)(x) - (\rho - P)(x - y)|^2} dy dx.$$

which is a weaker version of (6.3.23). Nevertheless, (6.3.23) will be necessary to estimate several terms appearing in the application of the relative entropy method in Section 6.5. Let us also point out that similar Poincaré-type inequalities are usually assumed for measure-valued solutions to several different PDEs, see for instance [167, eq. (2.23)].

We conclude with a simple observation concerning the energy.

**Lemma 6.3.6.** *The energy  $E_{mvs}$  defined by (6.3.22) is nonnegative.*

*Démonstration.* The lemma seems to be trivial from the point of view of our discussion about weak limits at the beginning of this section. However, the measure-valued solution is defined by Definition 6.3.1 so that we can argue only using Definitions 6.3.1 and 6.3.2. Clearly,  $\frac{1}{2}\overline{\rho|\mathbf{u}|^2}$  and  $\overline{F(\rho)}$  are nonnegative so that we only have to study the nonlocal term. By (6.3.11),

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y) |\rho(x) - \rho(x - y)|^2} dy = 2 \int_{\mathbb{T}^d} \overline{\rho^2} - 2 \int_{\mathbb{T}^d} \rho \omega_\eta * \rho.$$

By Cauchy-Schwarz and Young convolution inequalities :

$$2 \int_{\mathbb{T}^d} \rho \omega_\eta * \rho dx \leq 2 \int_{\mathbb{T}^d} \rho^2 dx.$$

Using Jensen's inequality (measure  $\nu_{t,x}$  is the probability measure with respect to both coordinates)

$$\int_{\mathbb{T}^d} \rho^2 dx = \int_{\mathbb{T}^d} \langle \lambda_1, \nu_{t,x} \rangle^2 dx \leq \int_{\mathbb{T}^d} \langle \lambda_1^2, \nu_{t,x} \rangle dx \leq \int_{\mathbb{T}^d} \overline{\rho^2} dx \quad (6.3.24)$$

so that the nonlocal term is nonnegative.  $\square$

### 6.3.2 The approximating system

To construct a measure-valued solution we use a method as outlined in [266, Section 5.5], see also [79, 203]. This is a fairly standard procedure based on regularizing density by a positive parameter

$$\rho_{0,\delta} = \rho_0 + \delta, \quad \rho_0 \in C^1(\mathbb{T}^d), \rho_0 > 0, \quad \mathbf{u}_{0,\delta}(x) = \mathbf{u}_0(x) \in W^{3,2}(\mathbb{T}^d)^d, \quad (6.3.25)$$

which makes the density  $\rho_\delta$  globally bounded from below. We will only discuss the main steps and for the full presentation, we refer to [266, Section 5.5].

We work in  $W^{3,2}(\mathbb{T}^d)^d$  (but for dimensions  $d$  higher than 3, we need to work even in  $W^{1+d,2}(\mathbb{T}^d)$ ) because of the embedding  $W^{3,2}(\mathbb{T}^d) \subset C^1(\mathbb{T}^d)$  which will be important for certain estimates. We use notation  $(\cdot, \cdot)$  for the standard scalar product in  $W^{3,2}(\mathbb{T}^d)^d$ . By [266, Appendix, Theorem 4.11], we take  $\{\boldsymbol{\omega}_i\}$  to be an orthonormal basis of  $W^{3,2}(\mathbb{T}^d)^d$  which are  $C^\infty(\mathbb{T}^d)^d$  functions. Finally, we define  $\Pi^N$  to be the projection operator into  $\text{span}\{\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_N\}$  which satisfies  $\|\Pi^N \mathbf{u}\|_{W^{3,2}} \leq \|\mathbf{u}\|_{W^{3,2}}$  and  $\|\Pi^N \mathbf{u}\|_{L^2} \leq \|\mathbf{u}\|_{L^2}$ .



We will find solution  $(\rho_\delta, \mathbf{u}_\delta)$  such that

$$\begin{aligned} \rho_\delta &\in L^\infty((0, T) \times \mathbb{T}^d) \cap L^2(0, T; W^{1,2}(\mathbb{T}^d)), & \frac{\partial \rho_\delta}{\partial t} &\in L^2((0, T) \times \mathbb{T}^d) \\ \mathbf{u}_\delta &\in L^\infty(0, T; W^{3,2}(\mathbb{T}^d)), & \frac{\partial \mathbf{u}_\delta}{\partial t} &\in L^2((0, T) \times \mathbb{T}^d), \end{aligned} \quad (6.3.26)$$

to the following problem : for all  $\psi \in C_c^1([0, T] \times \mathbb{T}^d; \mathbb{R})$ ,  $\phi \in C_c^1([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$  it holds that

$$\int_0^T \int_{\mathbb{T}^d} \partial_t \psi \rho_\delta + \frac{1}{\varepsilon} \nabla \psi \cdot \rho_\delta \mathbf{u}_\delta \, dx \, dt + \int_{\mathbb{T}^d} \psi(x, 0) \rho_{0,\delta} \, dx = 0, \quad (6.3.27)$$

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} \partial_t \phi \cdot \rho_\delta \mathbf{u}_\delta + \frac{1}{\varepsilon} \nabla \phi : \rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta - \frac{1}{\varepsilon^2} \phi \cdot \rho_\delta \mathbf{u}_\delta + \frac{1}{\varepsilon} \operatorname{div} \phi p(\rho_\delta) \, dx \, dt \\ + \int_0^T \int_{\mathbb{T}^d} \frac{1}{\varepsilon \eta^2} \phi \cdot \rho_\delta \nabla \omega_\eta * \rho_\delta \, dx \, dt + \int_{\mathbb{T}^d} \phi(x, 0) \cdot \rho_{0,\delta} \mathbf{u}_{0,\delta} \, dx = \delta \int_0^T ((\mathbf{u}_\delta, \phi)) \, dt. \end{aligned} \quad (6.3.28)$$

To find the solution to (6.3.27)–(6.3.28), we use the method of Galerkin approximations. We look for  $\mathbf{u}^N$  of the form

$$\mathbf{u}^N = \sum_{j=1}^N c_j^N(t) \boldsymbol{\omega}_j$$

solving

$$\frac{\partial \rho^N}{\partial t} + \frac{1}{\varepsilon} \operatorname{div}(\rho^N \mathbf{u}^N) = 0, \quad (6.3.29)$$

$$\begin{aligned} \int_{\mathbb{T}^d} \left( \rho^N \partial_t \mathbf{u}^N + \frac{1}{\varepsilon} \rho^N \mathbf{u}^N \nabla \mathbf{u}^N + \frac{1}{\varepsilon^2} \rho^N \mathbf{u}^N + \frac{1}{\varepsilon} \nabla p(\rho^N) - \frac{1}{\varepsilon \eta^2} \rho^N \nabla \omega_\eta * \rho^N \right) \cdot \boldsymbol{\omega}_i \, dx + \\ + \delta((\mathbf{u}^N, \boldsymbol{\omega}_i)) = 0, \end{aligned} \quad (6.3.30)$$

for  $i = 1, \dots, N$  with initial conditions  $\rho^N(0) = \rho_{0,\delta}$ ,  $\mathbf{u}^N(0) = \Pi^N \mathbf{u}_{0,\delta}$ .

The proof of existence to (6.3.29)–(6.3.30) follows 3 steps : using a fixed point argument to prove the existence on a small interval, deriving a priori estimates on this interval, extending the procedure on the whole interval. The crucial point is the lower bound on  $\rho^N$  in terms of  $\delta$ . This is obtained by the method of characteristics. Indeed,

$$\rho^N(t, x) \geq \operatorname{ess\,inf}_{x \in \mathbb{T}^d} \rho_{0,\delta} \exp \left( -\frac{1}{\varepsilon} \int_0^t \|\operatorname{div} u^N\|_\infty \, dt \right) \geq \delta \exp \left( -\frac{1}{\varepsilon} \int_0^t \|u^N\|_{W^{3,2}} \, dt \right) \quad (6.3.31)$$

by the well-known formula for the continuity equation. On the other hand, thanks to the regularizing term,  $\|u^N\|_{L^2(0,T;W^{3,2}(\mathbb{T}^d))} \leq \frac{C}{\delta}$ . This gives uniform lower (and also upper) bound on  $\rho^N$  and allows us to look at (6.3.30) as a system of ODEs. We refer to [266, Section 5.5] and omit the details. We obtain the following lemma :

**Lemma 6.3.7.** *For fixed  $N$ , there exists a solution to (6.3.29)–(6.3.30) such that  $\rho^N \in C^1([0, T] \times$*

$\mathbb{T}^d$ ),  $\mathbf{u}^N \in C^1([0, T]; W^{3,2}(\mathbb{T}^d)^d)$ . Moreover, we have the energy estimate : for all times  $\tau \in [0, T]$

$$\begin{aligned} & \int_{\mathbb{T}^d} \frac{1}{2} \rho^N |\mathbf{u}^N|^2 + F(\rho^N) \, dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y) |\rho^N(x) - \rho^N(x-y)|^2 \, dx \, dy \\ & + \delta \int_0^\tau \|u^N\|_{W^{3,2}}^2 \, dt + \frac{1}{\varepsilon^2} \int_0^\tau \int_{\mathbb{T}^d} \rho^N |\mathbf{u}^N|^2 \, dx \leq \\ & \leq \int_{\mathbb{T}^d} \frac{1}{2} \rho_{0,\delta} |\mathbf{u}_0|^2 + F(\rho_{0,\delta}) \, dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y) |\rho_{0,\delta}(x) - \rho_{0,\delta}(x-y)|^2 \, dx \, dy, \end{aligned} \quad (6.3.32)$$

as well as the following estimates

$$\rho^N(t, x) \geq C \left( \frac{1}{\delta} \right) \quad (6.3.33)$$

$$\|\rho^N\|_{L^\infty((0,T) \times \Omega)} + \int_0^\tau \|\partial_t \rho^N\|_{L^2(\mathbb{T}^d)}^2 + \int_0^\tau \|\nabla \rho^N\|_{L^2(\mathbb{T}^d)}^2 \leq C \left( \frac{1}{\delta} \right), \quad (6.3.34)$$

$$\int_0^T \|\partial_t \mathbf{u}^N\|_{L^2(\mathbb{T}^d)}^2 + \delta \|\mathbf{u}^N\|_{L^\infty((0,T); W^{3,2}(\mathbb{T}^d)^d)} \leq C \left( \frac{1}{\delta} \right), \quad (6.3.35)$$

where  $C \left( \frac{1}{\delta} \right)$  is a constant depending on  $\frac{1}{\delta}$  and other fixed parameters (like  $\varepsilon$ ).

*Démonstration.* The energy estimate follows by testing (6.3.30) by  $\mathbf{u}^N$  (in the Galerkin sense : we multiply (6.3.30) by  $c_i^N$  and sum for  $i = 1, \dots, N$ ). Estimate (6.3.33) follows from the characteristics as explained in (6.3.31). Similarly, we obtain the upper bound. Concerning the estimates on derivatives of  $\rho^N$ , they follow by differentiating the formula from the method of characteristics and using the bound  $\|\mathbf{u}^N\|_{L^2(0,T; W^{3,2}(\mathbb{T}^d))} \leq \frac{C}{\delta}$ . Finally, (6.3.35) is a consequence of testing (6.3.30) by  $\partial_t \mathbf{u}^N$ .  $\square$

Using the estimates in Lemma 6.3.7, up to a subsequence, we can pass to the limit  $N \rightarrow \infty$

$$\begin{aligned} \rho^N & \rightarrow \rho_\delta \quad \text{strongly in } L^2((0, T) \times \mathbb{T}^d), \\ \mathbf{u}^N & \rightarrow \mathbf{u}_\delta \quad \text{strongly in } L^2((0, T) \times \mathbb{T}^d)^d \end{aligned}$$

(the convergence holds even in better spaces). We also have an energy inequality :

$$\begin{aligned} & \int_{\mathbb{T}^d} \frac{1}{2} \rho_\delta |\mathbf{u}_\delta|^2 + F(\rho_\delta) \, dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y) |\rho_\delta(x) - \rho_\delta(x-y)|^2 \, dx \, dy \\ & + \delta \int_0^\tau \|u_\delta\|_{W^{3,2}}^2 \, dt + \frac{1}{\varepsilon^2} \int_0^\tau \int_{\mathbb{T}^d} \rho_\delta |\mathbf{u}_\delta|^2 \, dx \leq \\ & \leq \int_{\mathbb{T}^d} \frac{1}{2} \rho_{0,\delta} |\mathbf{u}_0|^2 + F(\rho_{0,\delta}) \, dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y) |\rho_{0,\delta}(x) - \rho_{0,\delta}(x-y)|^2 \, dx \, dy, \end{aligned} \quad (6.3.36)$$

This concludes the proof of existence of  $(\rho_\delta, \mathbf{u}_\delta)$  satisfying (6.3.27)–(6.3.28).

### 6.3.3 Existence of dissipative measure-valued solutions

It remains to pass to the limit  $\delta \rightarrow 0$  in (6.3.27)–(6.3.28). First we gather some uniform bounds in  $\delta$ , being a simple consequence of (6.3.33) and (6.3.36), in the following lemma :

**Lemma 6.3.8.** *Let  $(\rho^\delta, \mathbf{u}^\delta)$  be weak solutions of (6.3.27)–(6.3.28) as constructed above. Then, there exists a constant  $C > 0$  independent of  $\delta$  such that*

$$\begin{aligned} \rho_\delta &\geq 0 \quad \text{a.e. in } (0, T) \times \mathbb{T}^d, \\ \|\sqrt{\rho_\delta} \mathbf{u}_\delta\|_{L^\infty(0, T; L^2(\mathbb{T}^d))} &\leq C, \quad \|F(\rho_\delta)\|_{L^\infty(0, T; L^1(\mathbb{T}^d))} \leq C, \quad \|\rho_\delta\|_{L^\infty(0, T; L^2(\mathbb{T}^d))} \leq C, \\ \delta \|\mathbf{u}_\delta\|_{L^2(0, T; W^{3,2}(\mathbb{T}^d))}^2 &\leq C, \\ \|\partial_t \rho_\delta\|_{L^2(0, T; (W^{1,4}(\mathbb{T}^d))')} &\leq C. \end{aligned}$$

In fact, the proof of the existence of dissipative measure-valued solution follows now the method described at the beginning of Section 6.3.1. By Lemma 6.3.8, we have sufficient estimates to have convergence (6.3.1) which allows us to define the Young measure  $\{\nu_{t,x}\}$  as in (6.3.2)–(6.3.3). Then, the representations formulas for weak limits of nonlinearities (6.3.4)–(6.3.7) are a consequence of Lemma 6.2.2 and the estimate on  $\|\sqrt{\rho_\delta} \mathbf{u}_\delta\|_{L^\infty(0, T; L^2(\mathbb{T}^d))}$  which guarantees that all of the considered quantities are at least in  $L^\infty(0, T; L^1(\mathbb{T}^d))$ . Note that  $m^{\rho^\mathbf{u}} = 0$  because we have a uniform bound  $\|\rho_\delta \mathbf{u}_\delta\|_{L^\infty(0, T; L^{\frac{4}{3}}(\mathbb{T}^d))} \leq C$ . Next, (6.3.8) follows from the estimate on  $\|F(\rho_\delta)\|_{L^\infty(0, T; L^1(\mathbb{T}^d))}$ . Here, the measure  $m^{F(\rho)}$  is nonnegative because  $F = F_1 + F_2$  where  $F_1 \geq 0$  while  $F_2$  is bounded so that the only concentration effect can arise from  $F_1$ . Similarly, by Assumption 6.1.1,  $\|\rho_\delta F'(\rho_\delta)\|_{L^\infty(0, T; L^1(\mathbb{T}^d))} \leq C$  so that (6.3.9) follows. Finally, (6.3.10) is a consequence of the linearity and uniqueness of weak limits. This allows to pass to the limit  $\delta \rightarrow 0$  in almost all of the terms in formulation (6.3.27)–(6.3.28).

Concerning the regularizing term on the (RHS) of (6.3.28), we observe that

$$\left| \delta \int_0^T ((\mathbf{u}_\delta, \phi)) dt \right| \leq \delta \|\mathbf{u}_\delta\|_{L^2((0, T); W^{3,2}(\mathbb{T}^d))} \|\phi\|_{L^2((0, T); W^{3,2}(\mathbb{T}^d))} \leq C\sqrt{\delta} \|\phi\|_{L^2((0, T); W^{3,2}(\mathbb{T}^d))} \rightarrow 0.$$

When it comes to the nonlocal terms, we observe that we can identify their weak limits because the convolution upgrades a weak convergence to the strong one. More precisely, if  $\rho_\delta \xrightarrow{*} \rho$  in  $L^\infty(0, T; L^2(\mathbb{T}^d))$ , then  $\rho_\delta * \omega_\eta \rightarrow \rho * \omega_\eta$  in  $L^p(0, T; L^p(\mathbb{T}^d))$  strongly, for all  $1 \leq p < \infty$ . This follows by the Lions-Aubin lemma and a standard subsequence argument as the sequence  $\{\rho_\delta * \omega_\eta\}_\delta$  has uniformly bounded derivatives in the spatial derivatives while its time derivative is bounded in some negative Sobolev space by Lemma 6.3.8.

Concerning (6.3.18), we notice that it is a consequence of the inequality

$$|\lambda' \otimes \lambda'| = \left( \sum_{i,j=1}^d (\lambda'_i \lambda'_j)^2 \right)^{1/2} = \sum_{i=1}^d |\lambda'_i|^2 = |\lambda'|^2$$

and Lemma 6.2.3. Similarly, (6.3.19) follows from by virtue of Proposition 6.2.3 and inequality (6.8.2).

Next, the constructed measure-valued solution is dissipative in the sense of Definition 6.3.4 because we can pass to the limit in (6.3.36) using identified weak limits (rigorously, one multiplies (6.3.36) with a nonnegative test function of time, passes to the limit and then performs a standard localization argument).

Finally, the constructed solution satisfies Poincaré inequality as in Definition 6.3.5. Indeed, by

Lemma 6.7.2 we have for all bounded and nonnegative  $\varphi : [0, T] \rightarrow [0, \infty)$

$$\int_0^T \int_{\mathbb{T}^d} \varphi(t) |(\rho_\delta - P) - \delta|^2 \leq \frac{C_P}{4\eta^2} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi(t) |(\rho_\delta - P)(x) - (\rho_\delta - P)(x - y)|^2 \omega_\eta(y) \, dx \, dy \, dt$$

because  $(\rho_\delta - P)_{\mathbb{T}^d} = \delta$ . The (LHS) can be written as

$$\int_0^T \int_{\mathbb{T}^d} \varphi(t) |(\rho_\delta - P) - \delta|^2 = \int_0^T \int_{\mathbb{T}^d} \varphi(t) ((\rho_\delta - P)^2 + \delta^2 - 2\delta(\rho_\delta - P)).$$

As  $\rho_\delta - P$  is bounded in  $L^\infty(0, T; L^2(\mathbb{T}^d))$ , the last two terms vanish in the limit  $\delta \rightarrow 0$ . Finally, the term  $(\rho_\delta - P)^2$  has weak limit  $\overline{\rho^2} + P^2 - 2P\rho$  which is exactly  $\overline{(\rho - P)^2}$ , cf. (6.3.12). Similarly, we consider the term on the (RHS) so that we obtain

$$\int_0^T \int_{\mathbb{T}^d} \varphi(t) \overline{|\rho - P|^2} \leq \frac{C_P}{4\eta^2} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi(t) \overline{\omega_\eta(y) |(\rho - P)(x) - (\rho - P)(x - y)|^2} \, dy \, dx \, dt.$$

As this inequality holds for all  $\varphi$ , we conclude the proof.

## 6.4 Classical solutions to the nonlocal Cahn-Hilliard equation

To prove the convergence of the measure-valued solution of the nonlocal Euler-Korteweg to a solution of the Cahn-Hilliard equation, we use arguments similar to weak-strong uniqueness. Therefore, we study below the classical solutions of the nonlocal Cahn-Hilliard equation. More precisely, we consider the equation (6.1.5)–(6.1.6). The initial condition is a smooth positive function, more precisely we consider for some  $\alpha, \sigma > 0$

$$\rho(0, x) = \rho_0(x), \quad \rho_0 \in C^{2+\alpha}(\mathbb{T}^d), \quad \rho_0(x) \geq \sigma \quad \forall x \in \mathbb{T}^d. \quad (6.4.1)$$

We also suppose that  $F \in C^4$  which is required by the parabolic regularity theory exploited in Lemma 6.4.2. Equations (6.1.5)–(6.1.6) can be rewritten as

$$\partial_t \rho - \Delta(\phi(\rho)) + \operatorname{div}(\rho b(\rho)) = 0, \quad \phi(\rho) = \frac{\rho^2}{2\eta^2} + \int_0^\rho s F''(s) \, ds, \quad b(\rho) = \frac{\nabla \omega_\eta * \rho}{\eta^2}. \quad (6.4.2)$$

**Theorem 6.4.1.** *Equation (6.4.2) with initial condition  $u_0$  satisfying (6.4.1) admits a classical unique solution.*

To prove this theorem we first consider an approximate problem and we define  $T_\delta$  a smooth function such that

$$T_\delta(0) = \frac{\delta}{2}, \quad T_\delta(\rho) = \rho \text{ if } u \geq \delta, \quad T_\delta \text{ is increasing.}$$

The plan is to approximate (6.1.5) with

$$\partial_t \rho = \operatorname{div}(T_\delta(\rho) \nabla \mu). \quad (6.4.3)$$

We also define

$$\phi_\delta(\rho) := \int_0^\rho \frac{T_\delta(s)}{\eta^2} ds + \int_0^\rho T_\delta(s) F''(s) ds = \int_0^\rho T_\delta(s) \left( \frac{1}{\eta^2} + F''(s) \right) ds \quad (6.4.4)$$

so that equation (6.4.3) can be rewritten as a porous media equation

$$\partial_t \rho - \Delta(\phi_\delta(\rho)) + \operatorname{div}(\rho b(\rho)) = 0 \quad \rho(0, x) = \rho_0(x). \quad (6.4.5)$$

From the properties of  $F$  we note that  $\phi_\delta \geq 0$  and  $\phi'_\delta \geq 0$ .

**Lemma 6.4.2** (existence). *There exists a classical solution to (6.4.5). Moreover, the solution obeys the maximum principle*

$$\underline{\rho}(t) := \sigma \exp\left(-\int_0^t \|\operatorname{div} b(\rho)\|_{L^\infty}(s) ds\right) \leq \rho(t, x) \leq \sigma \exp\left(\int_0^t \|\operatorname{div} b(\rho)\|_{L^\infty}(s) ds\right).$$

*Démonstration.* The existence follows from [155]. To prove the maximum principle, we denote  $w = \rho - \underline{\rho}$  so that

$$\partial_t w - \Delta(\phi_\delta(\rho)) + \operatorname{div}(w b(\rho)) + \underline{\rho}(\operatorname{div}(b(\rho)) - \|\operatorname{div}(b(\rho))\|_{L^\infty}) = 0, \quad w(0, x) = \rho_0(x) - \underline{\rho} \geq 0.$$

We multiply this equation by  $\operatorname{sgn}^-(w) := \begin{cases} -1 & \text{if } w < 0 \\ 0 & \text{if } w \geq 0. \end{cases}$ . We obtain, with  $w^- = \min\{w, 0\}$ ,  $|w^-| = -\min\{w, 0\}$ .

$$\partial_t |w^-| + \Delta(\phi_\delta(\rho)) \operatorname{sgn}^-(w) + \operatorname{div}(|w^-| b(\rho)) \leq 0.$$

Therefore integrating in space and using the inequality

$$\int_{\mathbb{T}^d} \Delta \phi_\delta(\rho) \operatorname{sgn}^-(w) \geq 0,$$

we obtain

$$\partial_t \int_{\mathbb{T}^d} |w^-| \leq 0.$$

Using the initial condition we conclude  $|w^-| = 0$ .

Since the solutions to (6.4.3) satisfy uniform lower bound, we obtain  $T_\delta(\rho) = \rho$  for sufficiently small  $\delta$  and thus classical solutions of Theorem 6.4.1.  $\square$

**Lemma 6.4.3** (uniqueness). *Classical nonnegative solutions to (6.4.2) are unique.*

*Démonstration.* We want to adapt usual  $L^1$  contraction principle [324, Proposition 3.5] to the case with additional continuity equation term. Let  $\rho_1, \rho_2$  be solutions to (6.4.5) and let  $w = \rho_1 - \rho_2$ . Equation for  $w$  reads

$$\partial_t w - \Delta(\phi(\rho_1) - \phi(\rho_2)) + \operatorname{div}(\rho_1 b(\rho_1) - \rho_2 b(\rho_2)) = 0.$$

We multiply this equation by  $p_\varepsilon(\phi(\rho_1) - \phi(\rho_2))$  where  $p_\varepsilon$  approximates  $p(u) = \mathbb{1}_{u>0}$  and  $p'_\varepsilon \geq 0$ .

Then,

$$\int_{\mathbb{T}^d} \Delta(\phi(\rho_1) - \phi(\rho_2)) p_\varepsilon(\phi(\rho_1) - \phi(\rho_2)) \, dx = - \int_{\mathbb{T}^d} p'_\varepsilon |\nabla(\phi(\rho_1) - \phi(\rho_2))|^2 \, dx \leq 0.$$

Concerning the other terms we notice that after sending  $\varepsilon \rightarrow 0$  we obtain  $p(\phi(\rho_1) - \phi(\rho_2)) = p(\rho_1 - \rho_2)$  by monotonicity of  $\phi$ . Therefore,

$$\int_{\mathbb{T}^d} \partial_t w p(\rho_1 - \rho_2) \, dx = \partial_t \int_{\mathbb{T}^d} |w|_+ \, dx.$$

Now, we split the divergence term into two parts :

$$\operatorname{div}(\rho_1 b(\rho_1) - \rho_2 b(\rho_2)) = [\rho_1 \operatorname{div} b(\rho_1) - \rho_2 \operatorname{div} b(\rho_2)] + [\nabla \rho_1 b(\rho_1) - \nabla \rho_2 b(\rho_2)] = A + B.$$

The term  $A$  can be estimated in  $L^1(\mathbb{T}^d)$  with

$$\begin{aligned} \|A\|_1 &\leq \|\rho_1 \operatorname{div} b(\rho_1) - \rho_2 \operatorname{div} b(\rho_1)\|_1 + \|\rho_2 \operatorname{div} b(\rho_1) - \rho_2 \operatorname{div} b(\rho_2)\|_1 \\ &\leq \|\rho_1 - \rho_2\|_1 \|\operatorname{div} b(\rho_1)\|_\infty + \frac{1}{\eta^2} \|\rho_2\|_1 \|D^2 \omega_\eta\|_\infty \|\rho_1 - \rho_2\|_1 \\ &\leq \frac{\|D^2 \omega_\eta\|_\infty}{\eta^2} (\|\rho_1\|_1 + \|\rho_2\|_1) \|\rho_1 - \rho_2\|_1 \end{aligned}$$

where we used Young's convolutional inequality. Therefore,

$$\int_{\mathbb{T}^d} p A \, dx \leq \|p A\|_1 \leq \frac{\|D^2 \omega_\eta\|_\infty}{\eta^2} (\|\rho_1\|_1 + \|\rho_2\|_1) \|\rho_1 - \rho_2\|_1.$$

where we denoted for simplicity  $p = p(\rho_1 - \rho_2)$ . Concerning term  $B$  we write similarly

$$B = (\nabla \rho_1 b(\rho_1) - \nabla \rho_2 b(\rho_1)) + (\nabla \rho_2 b(\rho_1) - \nabla \rho_2 b(\rho_2)) =: B_1 + B_2.$$

As above, we easily obtain

$$\|B_2\|_1 \leq \frac{\|\nabla \omega_\eta\|_\infty}{\eta^2} \|\nabla \rho_2\|_1 \|\rho_1 - \rho_2\|_1, \quad \int_{\mathbb{T}^d} p B_2 \, dx \leq \frac{\|\nabla \omega_\eta\|_\infty}{\eta^2} \|\nabla \rho_2\|_1 \|\rho_1 - \rho_2\|_1.$$

The term  $B_1$  is more tricky. Keeping in mind that everything is multiplied by  $p(\rho_1 - \rho_2)$  we have

$$\begin{aligned} \int_{\mathbb{T}^d} (\nabla \rho_1 - \nabla \rho_2) p(\rho_1 - \rho_2) b(\rho_1) \, dx &= \int_{\mathbb{T}^d} \nabla |\rho_1 - \rho_2|_+ b(\rho_1) \, dx = \\ &= - \int_{\mathbb{T}^d} |\rho_1 - \rho_2|_+ \operatorname{div} b(\rho_1) \, dx \leq \|\rho_1 - \rho_2\|_1 \|\rho_1\|_1 \frac{\|D^2 \omega_\eta\|_\infty}{\eta^2}. \end{aligned}$$

We conclude that for some constant  $C$  depending on  $L^1$  norms of  $\rho_1$ ,  $\rho_2$  and  $\nabla \rho_2$  we have

$$\partial_t \int_{\mathbb{T}^d} |\rho_1 - \rho_2|_+ \, dx \leq \int_{\mathbb{T}^d} |\rho_1 - \rho_2| \, dx.$$

Replacing  $\rho_1$  and  $\rho_2$  we obtain

$$\partial_t \int_{\mathbb{T}^d} |\rho_1 - \rho_2| dx \leq \int_{\mathbb{T}^d} |\rho_1 - \rho_2| dx.$$

so that we conclude  $\rho_1 = \rho_2$ .  $\square$

## 6.5 Convergence of nonlocal Euler-Korteweg to nonlocal Cahn-Hilliard

To prove convergence of nonlocal Euler-Korteweg equation to the nonlocal Cahn-Hilliard equation, we first rewrite the latter as a nonlocal Euler-Korteweg equation with an error term :

$$\partial_t P + \frac{1}{\varepsilon} \operatorname{div}(P\mathbf{U}) = 0, \quad (6.5.1)$$

$$\partial_t(P\mathbf{U}) + \frac{1}{\varepsilon} \operatorname{div}(P\mathbf{U} \otimes \mathbf{U}) = -\frac{1}{\varepsilon^2} P\mathbf{U} - \frac{1}{\varepsilon} P\nabla(F'(P) + B_\eta[P]) + e(P, \mathbf{U}). \quad (6.5.2)$$

Here, velocity  $\mathbf{U}$  is given by

$$\mathbf{U} = -\varepsilon \nabla(F'(P) - B_\eta[P]) \quad (6.5.3)$$

and the error term is given by

$$\begin{aligned} e(P, \mathbf{U}) &= \partial_t(P\mathbf{U}) + \frac{1}{\varepsilon} \operatorname{div}(P\mathbf{U} \otimes \mathbf{U}) \\ &= \varepsilon \operatorname{div}(P\nabla(F'(P) + B_\eta[P])) \otimes \nabla(F'(P) + B_\eta[P]) - \varepsilon \partial_t(P\nabla(F'(P) + B_\eta[P])). \end{aligned}$$

Finally, given strong solution  $(P, \mathbf{U})$  and measure-valued solution represented by  $(\rho, \sqrt{\rho}\mathbf{u}, \nu, m)$  we define the relative entropy as

$$\Theta(t) = \int_{\mathbb{T}^d} \frac{1}{2} \overline{\rho|\mathbf{u} - \mathbf{U}|^2} + \overline{F(\rho|P)} dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y)|(\rho - P)(x) - (\rho - P)(x - y)|^2} dy dx. \quad (6.5.4)$$

where nonlinearity  $F(\rho|P)$  is defined in (6.3.17) and measure-valued terms are defined by (6.3.13), (6.3.15) and (6.3.16). The main result reads :

**Theorem 6.5.1.** *Let  $(\rho, \sqrt{\rho}\mathbf{u}, \nu, m)$  be a dissipative measure valued solution of (6.1.1)–(6.1.2) satisfying Poincaré inequality (6.3.23) and let  $(P, \mathbf{U})$  be classical solutions of (6.5.1)–(6.5.2). Then, for a constant independent of  $\varepsilon$  and  $\eta$  we have*

$$\Theta(t) \leq \left( \Theta(0) + \varepsilon^4 C(\|P\|_{C^{2,1}}) \left\| \frac{1}{P} \right\|_\infty^2 \right) e^{TC(\|P\|_{C^{2,1}})/\eta^{d+3}}. \quad (6.5.5)$$

**Lemma 6.5.2.** *Let  $\eta \in (0, \eta_0)$ . Then, the relative entropy defined by (6.5.4) is nonnegative : there exists a  $\kappa \in (0, 1)$  such that*

$$\int_{\mathbb{T}^d} \overline{\rho|\mathbf{u} - \mathbf{U}|^2} dx + \frac{\kappa}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y)|(\rho - P)(x) - (\rho - P)(x - y)|^2} dy dx \leq \Theta(t) \quad (6.5.6)$$

where both terms on the (LHS) are nonnegative. Moreover, for the constant  $C_P$  (defined in

Lemma 6.7.2) we have an estimate

$$\|\rho - \mathbf{P}\|_{L^2(\mathbb{T}^d)}^2 \leq \frac{C_P}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y)|(\rho - \mathbf{P})(x) - (\rho - \mathbf{P})(x - y)|^2} dy dx. \quad (6.5.7)$$

*Proof of Lemma 6.5.2.* We study the three terms appearing in (6.5.4) separately. First, for (6.3.13) we write by Fubini theorem

$$\overline{\rho|\mathbf{u} - \mathbf{U}|^2} = \langle |\lambda'|^2 + \lambda_1 |\mathbf{U}|^2 - 2\sqrt{\lambda_1} \lambda' \mathbf{U}, \nu_{t,x} \rangle + m^{\rho|\mathbf{u}|^2} = \langle |\lambda' - \sqrt{\lambda_1} \mathbf{U}|^2, \nu_{t,x} \rangle + m^{\rho|\mathbf{u}|^2},$$

so that, after integration in space, it is positive (for a.e.  $t \in (0, T)$ ). Now, we study the nonlocal term. We claim that (after integration)

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y)|\rho(x) - \rho(x - y)|^2} dy dx \geq \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y)|\rho(x) - \rho(x - y)|^2 dy dx. \quad (6.5.8)$$

Indeed, by definition (6.3.11), the (LHS) equals  $2 \int_{\mathbb{T}^d} \overline{\rho^2} - \int_{\mathbb{T}^d} 2\rho\omega_\eta * \rho$ . By (6.3.24), we know that  $\int_{\mathbb{T}^d} \overline{\rho^2} \geq \int_{\mathbb{T}^d} \rho^2$ . To conclude the proof of (6.5.8), it is sufficient to observe

$$2 \int_{\mathbb{T}^d} \rho^2 - \int_{\mathbb{T}^d} 2\rho\omega_\eta * \rho = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta |\rho(x) - \rho(x - y)|^2 dy dx.$$

Now, combining (6.3.15) and (6.5.8), we obtain

$$\begin{aligned} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y)|(\rho - \mathbf{P})(x) - (\rho - \mathbf{P})(x - y)|^2} dy dx &\geq \\ &\geq \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y)|(\rho - \mathbf{P})(x) - (\rho - \mathbf{P})(x - y)|^2 dy dx. \end{aligned}$$

Using Lemma 6.7.2, we conclude the proof of (6.5.7) and nonnegativity of the nonlocal term.

It remains to study the term  $\overline{F(\rho|\mathbf{P})}$ . The concentration measure  $m^{F(\rho)}$  is nonnegative and will be neglected in the estimate below. We split  $F = F_1 + F_2$  (where  $F_1, F_2$  are defined in Assumption 6.1.1) in (6.3.16) so that from (6.3.8) and (6.3.16)

$$\begin{aligned} \overline{F(\rho|\mathbf{P})} &= \langle F_1(\lambda_1) - F_1(\mathbf{P}) - F_1'(\mathbf{P})(\lambda_1 - \mathbf{P}), \nu_{t,x} \rangle \\ &\quad + \langle F_2(\lambda_1) - F_2(\mathbf{P}) - F_2'(\mathbf{P})(\lambda_1 - \mathbf{P}), \nu_{t,x} \rangle + m^{F(\rho)} \end{aligned} \quad (6.5.9)$$

The first term is nonnegative by convexity of  $F_1$ . The second can be estimated from below (by Taylor's expansion) with  $-\|F_2''\|_\infty \langle (\lambda_1 - \mathbf{P})^2, \nu_{t,x} \rangle$ . Now, recall that  $\|F_2''\|_\infty < \frac{1}{C_P}$  (cf. Assumption 6.1.1). In particular, there exists  $\kappa \in (0, 1)$  such that  $\|F_2''\|_\infty < \frac{1-\kappa}{C_P}$ . Using Poincaré inequality (6.3.23) and the fact that the concentration measure  $m^{\rho^2}$  is nonnegative we have

$$\begin{aligned} -\|F_2''\|_\infty \int_{\mathbb{T}^d} \langle (\lambda_1 - \mathbf{P})^2, \nu_{t,x} \rangle dx &\geq -\|F_2''\|_\infty \int_{\mathbb{T}^d} \overline{(\rho - \mathbf{P})^2} dx \geq -\frac{(1-\kappa)}{C_P} \int_{\mathbb{T}^d} \overline{(\rho - \mathbf{P})^2} dx \\ &\geq -\frac{1-\kappa}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y)|(\rho - \mathbf{P})(x) - (\rho - \mathbf{P})(x - y)|^2} dy dx. \end{aligned} \quad (6.5.10)$$



Therefore, we can compensate a possibly negative term with the positive nonlocal term appearing in (6.5.4).  $\square$

*Proof of Theorem 6.5.1.* We split the reasoning into several steps.

*Step 1 : Energy identities.* First, we recall that the dissipative measure valued solutions satisfy

$$\begin{aligned} & \int_{\mathbb{T}^d} \frac{1}{2} \overline{\rho |\mathbf{u}|^2} + \overline{F(\rho)} \, dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y) |\rho(x) - \rho(x-y)|^2} \, dy \, dx + \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \overline{\rho |\mathbf{u}|^2} \, dx \, dt \\ & \leq \int_{\mathbb{T}^d} \frac{1}{2} \rho_0 |u_0|^2(x) + F(\rho_0) \, dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y) |\rho_0(x) - \rho_0(x-y)|^2 \, dy \, dx. \end{aligned} \quad (6.5.11)$$

where the quantities on the (LHS) of (6.5.11) are evaluated at time  $t$ . Similarly, the classical solutions  $(\mathbf{P}, \mathbf{U})$  satisfy

$$\begin{aligned} & \int_{\mathbb{T}^d} \frac{1}{2} \mathbf{P} |\mathbf{U}|^2 + F(\mathbf{P}) \, dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y) |\mathbf{P}(x) - \mathbf{P}(x-y)|^2 \, dx \, dy = \\ & = \int_{\mathbb{T}^d} \frac{1}{2} \mathbf{P}_0 |u_0|^2(x) + F(\mathbf{P}_0) \, dx + \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \omega_\eta(y) |\mathbf{P}_0(x) - \mathbf{P}_0(x-y)|^2 \, dx \, dy \\ & \quad - \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \mathbf{P} |\mathbf{U}|^2 \, dx \, dt + \int_0^t \int_{\mathbb{T}^d} \mathbf{U} \cdot e(\mathbf{P}, \mathbf{U}) \, dt \, dx. \end{aligned} \quad (6.5.12)$$

Identity (6.5.12) can be obtained from testing (6.5.1)–(6.5.2) by  $\mathbf{U}$  and performing several integration by parts.

*Step 2 : Estimate for the mixed terms  $F'(\mathbf{P})(\rho - \mathbf{P})$ ,  $B_\eta[\mathbf{P}]$  and  $\rho |\mathbf{U}|^2$ .* We consider weak solutions of the mass equation satisfied by the differences between the measure valued solutions and classical solutions :

$$\int_0^T \int_{\mathbb{T}^d} \partial_t \psi (\rho - \mathbf{P}) + \frac{1}{\varepsilon} \nabla \psi \cdot (\overline{\rho \mathbf{u}} - \mathbf{P} \mathbf{U}) \, dx \, dt + \int_{\mathbb{T}^d} \psi(x, 0) (\rho_0 - \mathbf{P}_0) \, dx = 0. \quad (6.5.13)$$

We set

$$\theta_\delta(t) := \begin{cases} 1 & \text{for } 0 \leq \tau < t, \\ \frac{t-\tau}{\delta} + 1 & \text{for } t \leq \tau < t + \delta, \\ 0 & \text{for } \tau \geq t + \delta. \end{cases}$$

Note that  $\theta'(t)$  is an approximation of the dirac mass  $-\delta_t$ . We consider test function in (6.5.13) defined as  $\psi = \theta_\delta(t) (F'(\mathbf{P}) + B_\eta[\mathbf{P}] - \frac{1}{2} |\mathbf{U}|^2)$  so that after letting  $\delta \rightarrow 0$  we obtain

$$\begin{aligned} & \int_{\mathbb{T}^d} \left( F'(\mathbf{P}) + B_\eta[\mathbf{P}] - \frac{1}{2} |\mathbf{U}|^2 \right) (\rho - \mathbf{P}) \Big|_{\tau=0}^t \, dx = \\ & = + \int_0^t \int_{\mathbb{T}^d} \partial_\tau \left( F'(\mathbf{P}) + B_\eta[\mathbf{P}] - \frac{1}{2} |\mathbf{U}|^2 \right) (\rho - \mathbf{P}) \, dx \, d\tau \\ & \quad + \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^d} \nabla \left( F'(\mathbf{P}) + B_\eta[\mathbf{P}] - \frac{1}{2} |\mathbf{U}|^2 \right) \cdot (\overline{\rho \mathbf{u}} - \mathbf{P} \mathbf{U}) \, dx \, d\tau. \end{aligned} \quad (6.5.14)$$

*Step 3 : Estimate for the mixed term  $\overline{\rho \mathbf{u}} \mathbf{U}$ .* We consider weak solutions of the momentum equa-

tion satisfied by the differences between the measure valued solutions and classical solutions :

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{T}^d} \partial_t \phi \cdot (\overline{\rho \mathbf{u}} - \mathbf{P}\mathbf{U}) + \frac{1}{\varepsilon} \nabla \phi : (\overline{\rho \mathbf{u} \otimes \mathbf{u}} - \mathbf{P}\mathbf{U} \otimes \mathbf{U}) - \frac{1}{\varepsilon^2} \phi \cdot (\overline{\rho \mathbf{u}} - \mathbf{P}\mathbf{U}) + \frac{1}{\varepsilon} \operatorname{div} \phi (\overline{p(\rho)} - p(\mathbf{P})) \\ & + \frac{1}{\varepsilon \eta^2} \phi \cdot (\rho \nabla \omega_\eta * \rho - \mathbf{P} \cdot \nabla \omega_\eta * \mathbf{P}) + \int_{\mathbb{T}^d} \phi(x, 0) \cdot (\rho_0 \mathbf{u}_0 - \mathbf{P}_0 \mathbf{U}_0) dx - \int_0^\infty \int_{\mathbb{T}^d} \phi \cdot e(\mathbf{P}, \mathbf{U}) = 0. \end{aligned}$$

We consider test function  $\phi = \theta_\delta(t) \mathbf{U}$  so that after letting  $\delta \rightarrow 0$  we obtain

$$\begin{aligned} & \int_{\mathbb{T}^d} \mathbf{U} \cdot (\overline{\rho \mathbf{u}} - \mathbf{P}\mathbf{U}) \Big|_{\tau=0}^t dx = \int_0^t \int_{\mathbb{T}^d} \partial_\tau \mathbf{U} \cdot (\overline{\rho \mathbf{u}} - \mathbf{P}\mathbf{U}) dx d\tau \\ & + \int_0^t \int_{\mathbb{T}^d} \frac{1}{\varepsilon} \nabla \mathbf{U} : (\overline{\rho \mathbf{u} \otimes \mathbf{u}} - \mathbf{P}\mathbf{U} \otimes \mathbf{U}) + \frac{1}{\varepsilon} \operatorname{div}(\mathbf{U})(\overline{p(\rho)} - p(\mathbf{P})) dx d\tau \\ & - \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \mathbf{U} \cdot (\overline{\rho \mathbf{u}} - \mathbf{P}\mathbf{U}) dx d\tau + \frac{1}{\varepsilon \eta^2} \int_0^t \int_{\mathbb{T}^d} \mathbf{U} \cdot (\rho \nabla \omega_\eta * \rho - \mathbf{P} \cdot \nabla \omega_\eta * \mathbf{P}) dx d\tau \\ & - \int_0^t \int_{\mathbb{T}^d} \mathbf{U} \cdot e(\mathbf{P}, \mathbf{U}) dx d\tau. \end{aligned} \tag{6.5.15}$$

*Step 4 : First estimate on the relative entropy.* Let us observe that when we subtract (6.5.12), (6.5.14) and (6.5.15) from (6.5.11) we obtain an estimate for  $\Theta(t) - \Theta(0)$ . To see this, let us write explicitly the (LHS) after the subtraction (we omit integral with respect to  $x$  for simplicity and we consider only terms at time  $\tau = t$ ; of course, for  $\tau = 0$ , they will be analogous) :

$$\begin{aligned} & \frac{1}{2} \overline{\rho |\mathbf{u}|^2} + \overline{F(\rho)} + \frac{1}{4\eta^2} \overline{\int_{\mathbb{T}^d} \omega_\eta(y) |\rho(x) - \rho(x-y)|^2 dy} - \frac{1}{2} \mathbf{P} |\mathbf{U}|^2 - F(\mathbf{P}) \\ & - \frac{1}{4\eta^2} \int_{\mathbb{T}^d} \omega_\eta(y) |\mathbf{P}(x) - \mathbf{P}(x-y)|^2 dy - \left( F'(\mathbf{P}) + B_\eta[\mathbf{P}] - \frac{1}{2} |\mathbf{U}|^2 \right) (\rho - \mathbf{P}) - \mathbf{U} \cdot (\overline{\rho \mathbf{u}} - \mathbf{P}\mathbf{U}). \end{aligned}$$

We claim that this expression equals  $\Theta(t)$ . Indeed, the terms containing both density and velocity sum up to the term  $\overline{\rho |\mathbf{u} - \mathbf{U}|^2}$  as in (6.3.13). Similarly, terms with the potential  $F$  and its derivative  $F'$  can be combined to (6.3.16). Finally, for the nonlocal term, the claim is the consequence of two identities :

$$B_\eta[\mathbf{P}] \rho = \frac{1}{2\eta^2} \int_{\mathbb{T}^d} \omega_\eta(y) (\mathbf{P}(x) - \mathbf{P}(x-y)) (\rho(x) - \rho(x-y)) dy$$

and the similar one for  $B_\eta[\mathbf{P}] \mathbf{P}$  we can easily see that this expression equals  $\Theta(t)$ . Subtracting

all the terms on the (RHS) of (6.5.12), (6.5.14), (6.5.15) from (RHS) of (6.5.11) we obtain

$$\begin{aligned}
\Theta(t) - \Theta(0) &\leq -\frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \overline{\rho|\mathbf{u}|^2} - \mathbf{P}|\mathbf{U}|^2 - \mathbf{U} \cdot (\overline{\rho\mathbf{u}} - \mathbf{P}\mathbf{U}) \, dx \, d\tau \\
&\quad - \int_0^t \int_{\mathbb{T}^d} \partial_\tau \left( F'(\mathbf{P}) + B_\eta[\mathbf{P}] - \frac{1}{2}|\mathbf{U}|^2 \right) (\rho - \mathbf{P}) + \partial_\tau \mathbf{U} \cdot (\overline{\rho\mathbf{u}} - \mathbf{P}\mathbf{U}) \, dx \, d\tau \\
&\quad - \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^d} \nabla \left( F'(\mathbf{P}) + B_\eta[\mathbf{P}] - \frac{1}{2}|\mathbf{U}|^2 \right) \cdot (\overline{\rho\mathbf{u}} - \mathbf{P}\mathbf{U}) \, dx \, d\tau \\
&\quad - \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^d} \nabla \mathbf{U} : (\overline{\rho\mathbf{u}} \otimes \overline{\mathbf{u}} - \mathbf{P}\mathbf{U} \otimes \mathbf{U}) + \operatorname{div}(\mathbf{U})(\overline{p(\rho)} - p(\mathbf{P})) \, dx \, d\tau \\
&\quad - \frac{1}{\varepsilon} \frac{1}{\eta^2} \int_0^t \int_{\mathbb{T}^d} \mathbf{U} \cdot (\rho \nabla \omega_\eta * \rho - \mathbf{P} \cdot \nabla \omega_\eta * \mathbf{P}) \, dx \, d\tau.
\end{aligned} \tag{6.5.16}$$

*Step 5 : Terms with  $\partial_\tau \mathbf{U}$  in (6.5.16).* To estimate the right-hand side of (6.5.16) we first try to eliminate time derivative from (6.5.16). To this end, we compute  $\partial_t \mathbf{U}$  from the equations (6.5.1)-(6.5.2) to obtain that  $\mathbf{U}$  satisfies

$$\partial_t \mathbf{U} + \frac{1}{\varepsilon} (\mathbf{U} \cdot \nabla) \mathbf{U} = -\frac{1}{\varepsilon^2} \mathbf{U} - \frac{1}{\varepsilon} \nabla (F'(\mathbf{P}) + B_\eta[\mathbf{P}]) + \frac{e(\mathbf{P}, \mathbf{U})}{\mathbf{P}}. \tag{6.5.17}$$

We take the scalar product of this equation with  $\overline{\rho\mathbf{u}} - \rho \mathbf{U}$  which yields

$$\begin{aligned}
&\partial_t \mathbf{U} \cdot (\overline{\rho\mathbf{u}} - \mathbf{P}\mathbf{U}) + \frac{1}{2} \partial_t |\mathbf{U}|^2 (\mathbf{P} - \rho) + \frac{1}{\varepsilon} \nabla \mathbf{U} : (\overline{\rho\mathbf{u}} \otimes \mathbf{U} - \rho \mathbf{U} \otimes \mathbf{U}) \\
&= \frac{1}{\varepsilon^2} (\rho |\mathbf{U}|^2 - \mathbf{U} \cdot \overline{\rho\mathbf{u}}) - \frac{1}{\varepsilon} \nabla (F'(\mathbf{P}) + B_\eta[\mathbf{P}]) \cdot (\overline{\rho\mathbf{u}} - \rho \mathbf{U}) + \frac{e(\mathbf{P}, \mathbf{U})}{\mathbf{P}} \cdot (\overline{\rho\mathbf{u}} - \rho \mathbf{U}),
\end{aligned}$$

where we used identities

$$\frac{1}{\varepsilon} (\mathbf{U} \cdot \nabla) \mathbf{U} \cdot (\overline{\rho\mathbf{u}} - \rho \mathbf{U}) = \frac{1}{\varepsilon} \nabla \mathbf{U} : (\overline{\rho\mathbf{u}} \otimes \mathbf{U} - \rho \mathbf{U} \otimes \mathbf{U}),$$

$$\partial_t \mathbf{U} \cdot (\overline{\rho\mathbf{u}} - \rho \mathbf{U}) = \partial_t \mathbf{U} \cdot (\overline{\rho\mathbf{u}} - \mathbf{P}\mathbf{U}) + \frac{1}{2} \partial_t |\mathbf{U}|^2 (\mathbf{P} - \rho).$$

Finally, using matrix identity  $x A y = A : x \otimes y$  where  $x, y \in \mathbb{R}^d$  and  $A \in \mathbb{R}^{d \times d}$  we easily deduce the formula

$$\begin{aligned}
\nabla \mathbf{U} : (\overline{\rho\mathbf{u}} \otimes \mathbf{U} - \rho \mathbf{U} \otimes \mathbf{U}) &= \nabla \mathbf{U} : (\overline{\rho\mathbf{u}} \otimes \overline{\mathbf{u}} - \mathbf{P}\mathbf{U} \otimes \mathbf{U}) \\
&\quad - \nabla \mathbf{U} : \overline{\rho(\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U})} - \nabla \left( \frac{1}{2} |\mathbf{U}|^2 \right) (\overline{\rho\mathbf{u}} - \mathbf{P}\mathbf{U}),
\end{aligned}$$

where

$$\overline{\rho(\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U})} := \overline{\rho\mathbf{u}} \otimes \overline{\mathbf{u}} - \overline{\rho\mathbf{u}} \otimes \mathbf{U} - \mathbf{U} \otimes \overline{\rho\mathbf{u}} + \rho \mathbf{U} \otimes \mathbf{U}.$$

We obtain

$$\begin{aligned}
& \frac{1}{2} \partial_t |\mathbf{U}|^2 (\mathbf{P} - \rho) + \partial_t (\mathbf{U}) \cdot (\overline{\rho \mathbf{u}} - \mathbf{P} \mathbf{U}) - \frac{1}{\varepsilon} \nabla \left( \frac{1}{2} |\mathbf{U}|^2 \right) \cdot (\overline{\rho \mathbf{u}} - \mathbf{P} \mathbf{U}) + \\
& + \frac{1}{\varepsilon} \nabla \mathbf{U} : (\overline{\rho \mathbf{u}} \otimes \mathbf{u} - \mathbf{P} \mathbf{U} \otimes \mathbf{U}) = \frac{1}{\varepsilon} \nabla \mathbf{U} : \overline{\rho (\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U})} + \\
& = \frac{1}{\varepsilon^2} (\rho |\mathbf{U}|^2 - \mathbf{U} \cdot \overline{\rho \mathbf{u}}) - \frac{1}{\varepsilon} \nabla (F'(\mathbf{P}) + B_\eta[\mathbf{P}]) \cdot (\overline{\rho \mathbf{u}} - \rho \mathbf{U}) + \frac{e(\mathbf{P}, \mathbf{U})}{\mathbf{P}} \cdot (\overline{\rho \mathbf{u}} - \rho \mathbf{U}).
\end{aligned} \tag{6.5.18}$$

Note that this gives us an estimate on four terms appearing on the (RHS) of (6.5.16).

*Step 6 : Terms with  $F'$  and  $B_\eta$  in (6.5.16).* We now consider the expression

$$\begin{aligned}
& - \int_0^t \int_{\mathbb{T}^d} \partial_\tau (F'(\mathbf{P}) + B_\eta[\mathbf{P}]) (\rho - \mathbf{P}) + \frac{1}{\varepsilon} \nabla (F'(\mathbf{P}) + B_\eta[\mathbf{P}]) \cdot (\overline{\rho \mathbf{u}} - \mathbf{P} \mathbf{U}) \, dx \, d\tau \\
& \quad + \int_0^t \int_{\mathbb{T}^d} \frac{1}{\varepsilon} \nabla (F'(\mathbf{P}) + B_\eta[\mathbf{P}]) \cdot (\overline{\rho \mathbf{u}} - \rho \mathbf{U}) \, dx \, d\tau.
\end{aligned}$$

The first integral comes from (6.5.16) while the second from (6.5.18) plugged into (6.5.16). We can simplify this to get

$$- \int_0^t \int_{\mathbb{T}^d} \partial_\tau (F'(\mathbf{P}) + B_\eta[\mathbf{P}]) (\rho - \mathbf{P}) + \frac{1}{\varepsilon} \nabla (F'(\mathbf{P}) + B_\eta[\mathbf{P}]) \cdot \mathbf{U} (\rho - \mathbf{P}) \, dx \, d\tau. \tag{6.5.19}$$

We split the term with  $B_\eta[\mathbf{P}] = \frac{\mathbf{P}}{\eta^2} - \frac{\mathbf{P} * \omega_\eta}{\eta^2}$  for the local and non-local parts. Now, concerning the terms with potential  $F$ , we use (6.5.1) to deduce

$$\partial_t F'(\mathbf{P}) = F''(\mathbf{P}) \partial_t \mathbf{P} = -\frac{1}{\varepsilon} F''(\mathbf{P}) \nabla \mathbf{P} \cdot \mathbf{U} - \frac{1}{\varepsilon} F''(\mathbf{P}) \mathbf{P} \operatorname{div} \mathbf{U} = -\frac{1}{\varepsilon} \nabla F'(\mathbf{P}) \cdot \mathbf{U} - \frac{1}{\varepsilon} F''(\mathbf{P}) \mathbf{P} \operatorname{div} \mathbf{U}.$$

Similarly,

$$\frac{1}{\eta^2} \partial_t \mathbf{P} = -\frac{1}{\varepsilon \eta^2} \nabla \mathbf{P} \cdot \mathbf{U} - \frac{1}{\varepsilon \eta^2} \mathbf{P} \operatorname{div} \mathbf{U}.$$

Therefore, the local parts of (6.5.19) sum up to

$$-\frac{1}{\varepsilon} \left( F''(\mathbf{P}) \mathbf{P} + \frac{1}{\eta^2} \mathbf{P} \right) \operatorname{div} \mathbf{U} (\rho - \mathbf{P}) = -\frac{1}{\varepsilon} p'(\mathbf{P}) \operatorname{div} \mathbf{U} (\rho - \mathbf{P})$$

which together with  $-\frac{1}{\varepsilon} \operatorname{div}(\mathbf{U})(\overline{p(\rho)} - p(\mathbf{P})) \operatorname{div} \mathbf{U}$  from (6.5.16) gives  $\overline{p(\rho|\mathbf{P})} \operatorname{div} \mathbf{U}$ , where

$$\overline{p(\rho|\mathbf{P})} := \overline{p(\rho)} - p(\mathbf{P}) - p'(\mathbf{P}) (\rho - \mathbf{P}).$$

Now, we consider the nonlocal parts in (6.5.19) and the last nonlocal term coming from (6.5.16).

which equals

$$\begin{aligned} \frac{1}{\eta^2} \int_0^t \int_{\mathbb{T}^d} \partial_\tau (\mathbf{P} * \omega_\eta) (\rho - \mathbf{P}) + \frac{1}{\varepsilon \eta^2} \nabla (\mathbf{P} * \omega_\eta) \cdot \mathbf{U} (\rho - \mathbf{P}) \, dx \, d\tau \\ - \frac{1}{\varepsilon \eta^2} \int_0^t \int_{\mathbb{T}^d} \mathbf{U} \cdot (\rho \nabla \omega_\eta * \rho - \mathbf{P} \cdot \nabla \omega_\eta * \mathbf{P}) \, dx \, d\tau. \end{aligned} \quad (6.5.20)$$

Using (6.5.1) and properties of the convolution we can rewrite the first term in (6.5.20) :

$$\begin{aligned} \frac{1}{\eta^2} \int_0^t \int_{\mathbb{T}^d} \partial_\tau (\mathbf{P} * \omega_\eta) (\rho - \mathbf{P}) \, dx \, d\tau &= -\frac{1}{\varepsilon} \frac{1}{\eta^2} \int_0^t \int_{\mathbb{T}^d} \operatorname{div}(\mathbf{P}\mathbf{U}) (\omega_\eta * (\rho - \mathbf{P})) \, dx \, d\tau \\ &= \frac{1}{\varepsilon} \frac{1}{\eta^2} \int_0^t \int_{\mathbb{T}^d} \mathbf{P}\mathbf{U} \cdot (\nabla \omega_\eta * (\rho - \mathbf{P})) \, dx \, d\tau. \end{aligned}$$

so that (6.5.20) boils down to

$$-\frac{1}{\varepsilon} \frac{1}{\eta^2} \int_0^t \int_{\mathbb{T}^d} (\rho - \mathbf{P}) \mathbf{U} \cdot \nabla \omega_\eta * (\rho - \mathbf{P}) \, dx \, d\tau.$$

*Step 7 : Final estimate on the relative entropy.* Using the steps above and (6.5.16) we obtain

$$\begin{aligned} \Theta(t) - \Theta(0) &\leq -\frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \overline{\rho |\mathbf{u} - \mathbf{U}|^2} \, dx \, d\tau - \int_0^t \int_{\mathbb{T}^d} \frac{e(\mathbf{P}, \mathbf{U})}{\mathbf{P}} (\overline{\rho \mathbf{u}} - \rho \mathbf{U}) \, dx \, d\tau \\ &\quad - \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^d} \nabla \mathbf{U} : \overline{\rho (\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U})} \, dx \, d\tau - \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^d} \operatorname{div}(\mathbf{U}) \overline{\rho |\rho|} \, dx \, d\tau \\ &\quad - \frac{1}{\varepsilon} \frac{1}{\eta^2} \int_0^t \int_{\mathbb{T}^d} (\rho - \mathbf{P}) \mathbf{U} \cdot \nabla \omega_\eta * (\rho - \mathbf{P}) \, dx \, d\tau =: A + B + C + D + E. \end{aligned} \quad (6.5.21)$$

By definition of  $\mathbf{U}$  we notice that

$$\|\mathbf{U}\|_\infty, \|\nabla \mathbf{U}\|_\infty, |e(\mathbf{P}, \mathbf{U})| \leq \varepsilon C(\|\mathbf{P}\|_{C^{2,1}}),$$

where  $C(\|\mathbf{P}\|_{C^{2,1}})$  is a numerical constant which depends on  $\|\mathbf{P}\|_{C^{2,1}}$  and blows up when  $\eta \rightarrow 0$  since we do not have estimates in  $C^2$  of the solutions of the local Cahn-Hilliard equation. Now, we estimate the terms appearing on the (RHS) of (6.5.21).

*Term E.* For the nonlocal term  $E$  we use boundedness of  $\mathbf{U}$  to have

$$\left| \frac{1}{\varepsilon} \frac{1}{\eta^2} \int_0^t \int_{\mathbb{T}^d} (\rho - \mathbf{P}) \mathbf{U} \cdot \nabla \omega_\eta * (\rho - \mathbf{P}) \, dx \, d\tau \right| \leq \frac{C \|\mathbf{U}\|_\infty}{\eta^2} \|\rho - \mathbf{P}\|_2 \|\nabla \omega_\eta * (\rho - \mathbf{P})\|_2 \leq \frac{C \|\mathbf{U}\|_\infty}{\eta^{d+3}} \|\rho - \mathbf{P}\|_2^2.$$

Using (6.5.7) for  $\eta \in (0, \eta_0)$  we obtain

$$E \leq \frac{C(\|\mathbf{P}\|_{C^{2,1}})}{4\eta^{d+3}} \int_0^t \Theta(\tau) \, d\tau.$$

Term B. Using (6.3.3) and (6.3.5) we can write

$$B = - \int_0^t \int_{\mathbb{T}^d} \left\langle \frac{e(\mathbf{P}, \mathbf{U})}{\mathbf{P}} \sqrt{\lambda_1} (\lambda' - \sqrt{\lambda_1} \mathbf{U}), \nu_{t,x} \right\rangle dx d\tau$$

Using Cauchy-Schwartz with a parameter

$$B \leq \int_0^t \int_{\mathbb{T}^d} \left\langle \frac{\varepsilon^2}{2} \left| \frac{e(\mathbf{P}, \mathbf{U})}{\mathbf{P}} \right|^2 \lambda_1 + \frac{1}{2\varepsilon^2} |\lambda' - \sqrt{\lambda_1} \mathbf{U}|^2, \nu_{t,x} \right\rangle dx d\tau$$

Now,  $\left| \frac{e(\mathbf{P}, \mathbf{U})}{\mathbf{P}} \right| \leq \varepsilon C(\|\mathbf{P}\|_{C^{2,1}}) \left\| \frac{1}{\mathbf{P}} \right\|_\infty$ . Moreover, expanding the square in  $|\lambda' - \sqrt{\lambda_1} \mathbf{U}|^2$  and using (6.3.5), (6.3.7) we recognize that

$$\int_0^t \int_{\mathbb{T}^d} \langle |\lambda' - \sqrt{\lambda_1} \mathbf{U}|^2, \nu_{t,x} \rangle \leq \int_0^t \int_{\mathbb{T}^d} \overline{\rho |\mathbf{u} - \mathbf{U}|^2}.$$

Therefore, we have the estimate

$$B \leq \varepsilon^4 C(\|\mathbf{P}\|_{C^{2,1}}) \left\| \frac{1}{\mathbf{P}} \right\|_\infty^2 + \frac{1}{2\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} \overline{\rho |\mathbf{u} - \mathbf{U}|^2}.$$

Term C. We have

$$\left| \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^d} \nabla \mathbf{U} : \overline{\rho (\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U})} dx d\tau \right| \leq \frac{C \|\nabla U\|_\infty}{\varepsilon} \int_0^t \int_{\mathbb{T}^d} \left| \overline{\rho (\mathbf{u} - \mathbf{U}) \otimes (\mathbf{u} - \mathbf{U})} \right| dx d\tau$$

Estimating directly under the integral in (6.3.14)

$$\left| \langle (\lambda' - \sqrt{\lambda_1} \mathbf{U}) \otimes (\lambda' - \sqrt{\lambda_1} \mathbf{U}), \nu_{t,x} \rangle \right| \leq \langle |\lambda' - \sqrt{\lambda_1} \mathbf{U}|^2, \nu_{t,x} \rangle$$

and using (6.3.18) we arrive at

$$C \leq C(\|\mathbf{P}\|_{C^{2,1}}) \int_0^t \int_{\mathbb{T}^d} \overline{\rho |\mathbf{u} - \mathbf{U}|^2} dx d\tau$$

Term D. Using (6.3.16) and (6.3.10), we can write

$$|\overline{p(\rho|\mathbf{P})}| \leq \langle p(\lambda_1) - p(\mathbf{P}) - p'(\mathbf{P})(\lambda_1 - \mathbf{P}), \nu_{t,x} \rangle + |m^{\rho F'(\rho)}| + m^{F(\rho)} + \frac{1}{\eta^2} m^{\rho^2}.$$

The first part can be estimated using (6.8.1) :

$$\langle p(\lambda_1|\mathbf{P}), \nu_{t,x} \rangle \leq C_{F,R} \langle F(\lambda_1|\mathbf{P}), \nu_{t,x} \rangle + \left( C_{F,R} + \frac{1}{\eta^2} \right) \langle (\lambda_1 - \mathbf{P})^2, \nu_{t,x} \rangle \quad (6.5.22)$$

The concentration measures part can be estimated using (6.3.19) :

$$|m^{\rho F'(\rho)}| + m^{F(\rho)} + \frac{1}{\eta^2} m^{\rho^2} \leq (C_F + 1) m^{F(\rho)} + \left( C_F + \frac{1}{\eta^2} \right) m^{\rho^2}. \quad (6.5.23)$$

Summing up (6.5.22) and (6.5.23) we obtain

$$|\overline{p(\rho|\mathbf{P})}| \leq C \overline{F(\rho|\mathbf{P})} + C \left(1 + \frac{1}{\eta^2}\right) \overline{|\rho - \mathbf{P}|^2}.$$

The last term can be estimated by the nonlocal term appearing in the definition of  $\Theta$  due to the Poincaré inequality (6.3.23). As  $\overline{F(\rho|\mathbf{P})}$  also appears in the definition of  $\Theta$  we obtain

$$D \leq \left| \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^d} \operatorname{div}(\mathbf{U}) \overline{p(\rho|\mathbf{P})} \, dx \, d\tau \right| \leq C(\|\mathbf{P}\|_{C^{2,1}}) \left(1 + \frac{1}{\eta^2}\right) \int_0^t \Theta(\tau) \, d\tau.$$

We conclude that for  $\eta < 1$  :

$$\Theta(t) \leq \Theta(0) + \frac{C(\|\mathbf{P}\|_{C^{2,1}})}{4\eta^{d+3}} \int_0^t \Theta(\tau) \, d\tau + \varepsilon^4 C(\|\mathbf{P}\|_{C^{2,1}}) \left\| \frac{1}{\mathbf{P}} \right\|_{\infty}^2$$

Using Gronwall's lemma, we obtain (6.5.5).  $\square$

*Proof of Theorem 6.1.5.* The proof is a direct consequence of (6.5.5). Indeed, we consider the relative entropy  $\Theta$  as in (6.5.4) with  $\rho = \rho_{\eta,\varepsilon}$ ,  $\mathbf{u} = \mathbf{u}_{\eta,\varepsilon}$ ,  $\mathbf{P} = \rho_{\eta}$  and  $\mathbf{U} = -\varepsilon \nabla(F'(\rho_{\eta}) - B_{\eta}(\rho_{\eta}))$ . As  $\eta \in (0, \eta_0)$  is fixed,  $\rho_{\eta}$  (which depends on  $\eta$ !) is a  $C^{2,1}$  function bounded away from 0 (Theorem 6.4.1, Lemma 6.4.2). Furthermore,

$$\Theta(0) \leq C(\varepsilon^2 + \|\mathbf{u}_{0,\varepsilon}\|_{L^2(\mathbb{T}^d)}^2) \rightarrow 0 \quad (6.5.24)$$

(here, we use that the initial density  $\rho_0$  belongs to  $C^3$  so that  $\|\mathbf{U}(0, x)\|_{L^{\infty}(\mathbb{T}^d)} \leq C\varepsilon$ , cf. (6.5.3)). Therefore, we get that  $\Theta(t) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . By (6.5.6) and (6.5.7), we obtain convergence in  $L^2(\mathbb{T}^d)$ , even uniformly in time.  $\square$

*Proof of Theorem 6.1.4.* We write  $\rho_{\eta}$  (note that it does not depend on  $\varepsilon$ , cf. (6.5.1) and (6.5.3)) for solutions to (6.5.1)–(6.5.2) and we note that they depend on  $\eta$ . From [155] we know that there exists a subsequence  $\eta_k \rightarrow 0$  such that

$$\|\rho_{\eta_k} - \rho\|_{L^2((0,T) \times \mathbb{T}^d)} \rightarrow 0,$$

where  $\rho$  is a weak solution to the local Cahn-Hilliard equation. Now, let  $\rho_{\eta_k, \varepsilon_k}$  be a measure-valued solution of non-local Euler-Korteweg equation. Using (6.5.24), (6.5.6) and (6.5.7), we have

$$\|\rho_{\eta_k} - \rho_{\eta_k, \varepsilon_k}\|_{L^2((0,T) \times \mathbb{T}^d)} \leq C \left( \varepsilon_k^2 + \|\mathbf{u}_{0, \varepsilon_k}\|_{L^2(\mathbb{T}^d)}^2 + \varepsilon_k^4 \|\rho_{\eta_k}\|_{C^{2,1}}^2 \left\| \frac{1}{\rho_{\eta_k}} \right\|_{\infty}^2 \right) e^{CT \|\rho_{\eta_k}\|_{C^{2,1}} / \eta_k^{d+3}}.$$

Of course, the quantity  $\|\rho_{\eta_k}\|_{C^{2,1}}^2 \left\| \frac{1}{\rho_{\eta_k}} \right\|_{\infty}^2 e^{CT \|\rho_{\eta_k}\|_{C^{2,1}} / \eta_k^{d+3}}$  is blowing up as  $\eta_k \rightarrow 0$  (because we lose parabolicity), nevertheless we can choose  $\varepsilon_k$  so small to obtain convergence to 0. The conclusion follows by triangle inequality.  $\square$

## 6.6 Convergence result for the parametrized measure $\nu^{\eta,\varepsilon}$ and the concentration measures $m_{\eta,\varepsilon}$

Theorems 6.1.4 and 6.1.5 answer the question of what happens with the function  $\rho_{\eta,\varepsilon}$  when  $\eta, \varepsilon \rightarrow 0$ . However, the measure-valued solution  $(\rho_{\eta,\varepsilon}, \overline{\sqrt{\rho_{\eta,\varepsilon}} \mathbf{u}_{\eta,\varepsilon}}, \nu^{\eta,\varepsilon}, m_{\eta,\varepsilon})$  is in fact a collection of four components. Below, we address the question of convergence of the other components :  $\overline{\sqrt{\rho_{\eta,\varepsilon}} \mathbf{u}_{\eta,\varepsilon}}, \nu^{\eta,\varepsilon}, m_{\eta,\varepsilon}$ . We provide a detailed proof only for the situation in Theorem 6.1.5. Adaptation to the case analyzed in Theorem 6.1.4 is straightforward.

We first recall some basic notions from measure theory. We consider the set  $\mathbb{R}^+ \times \mathbb{R}^d$  and we write  $(\lambda_1, \lambda')$  for a given element of this set where  $\lambda_1 \in \mathbb{R}^+$  and  $\lambda' \in \mathbb{R}^d$  as in Section 6.3.1. For two probability measures  $\mu, \nu$  on  $\mathbb{R}^+ \times \mathbb{R}^d$  with a finite second moment, that is,

$$\int_{\mathbb{R}^+ \times \mathbb{R}^d} (|\lambda_1|^2 + |\lambda'|^2) d\mu(\lambda_1, \lambda') < \infty, \quad \int_{\mathbb{R}^+ \times \mathbb{R}^d} (|\lambda_1|^2 + |\lambda'|^2) d\nu(\lambda_1, \lambda') < \infty,$$

the Wasserstein distance  $\mathcal{W}_2(\mu, \nu)$  is defined as

$$\mathcal{W}_2(\mu, \nu)^2 = \inf_{\pi \in \Pi(\mu, \nu)} \int_{(\mathbb{R}^+ \times \mathbb{R}^d)^2} \left[ |\lambda_1 - \tilde{\lambda}_1|^2 + |\lambda' - \tilde{\lambda}'|^2 \right] d\pi(\lambda_1, \lambda', \tilde{\lambda}_1, \tilde{\lambda}'), \quad (6.6.1)$$

where the set  $\Pi(\mu, \nu)$  is the set of couplings between  $\mu, \nu$ ; that is, the set of measures  $\pi$  on the product  $(\mathbb{R}^+ \times \mathbb{R}^d)^2$  such that

$$\pi(A \times (\mathbb{R}^+ \times \mathbb{R}^d)) = \mu(A), \quad \pi((\mathbb{R}^+ \times \mathbb{R}^d) \times B) = \nu(B).$$

Furthermore, for a measure  $\mu$  on some space  $X$ , the total variation of  $\mu$  is defined as

$$\|\mu\|_{TV} = |\mu|(X),$$

where  $|\mu|(A) = \mu^+(A) - \mu^-(A)$  and  $\mu^+, \mu^-$  are positive and negative parts of  $\mu$ , respectively. Note that if  $\mu$  is a nonnegative measure,  $\|\mu\|_{TV} = \mu(X)$ . For more on spaces of measures and related norms, we refer to [142, Chapter 1].

**Theorem 6.6.1.** *Under the notation of Theorem 6.1.5, the function  $\overline{\sqrt{\rho_{\eta,\varepsilon}} \mathbf{u}_{\eta,\varepsilon}}$  converges to 0 in  $L^\infty(0, T; L^2(\mathbb{T}^d))$  :*

$$\operatorname{esssup}_{t \in (0, T)} \int_{\mathbb{T}^d} |\overline{\sqrt{\rho_{\eta,\varepsilon}} \mathbf{u}_{\eta,\varepsilon}}|^2 dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (6.6.2)$$

*Moreover, the parametrized measure  $\nu^{\eta,\varepsilon} \in L^\infty_{weak}((0, T) \times \mathbb{T}^d; \mathcal{P}([0, +\infty) \times \mathbb{R}^d))$  converges to  $\delta_{\rho_\eta(t,x)} \otimes \delta_{\mathbf{0}}$  in the following sense*

$$\operatorname{esssup}_{t \in (0, T)} \int_{\mathbb{T}^d} [\mathcal{W}_2(\nu^{\eta,\varepsilon}, \delta_{\rho_\eta(t,x)} \otimes \delta_{\mathbf{0}})]^2 dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (6.6.3)$$

*Furthermore, the concentration measures vector  $m_{\eta,\varepsilon}$  converges to 0 in the total variation norm, uniformly in time :*

$$\operatorname{esssup}_{t \in (0, T)} \|m_{\eta,\varepsilon}(t, \cdot)\|_{TV} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (6.6.4)$$

*Démonstration.* From the proof of Theorem 6.1.5, we know that  $\sup_{t \in (0, T)} \Theta(t) \rightarrow 0$  where  $\Theta(t)$



is defined as in (6.5.4) with  $\rho := \rho_{\eta,\varepsilon}$ ,  $P := \rho_\eta$ ,  $\mathbf{u} := \mathbf{u}_{\eta,\varepsilon}$  and

$$\mathbf{U} := -\varepsilon \nabla(F'(\rho_\eta) - B_\eta(\rho_\eta)). \quad (6.6.5)$$

Due to Lemma 6.5.2 this yields

$$\sup_{t \in (0, T)} \int_{\mathbb{T}^d} \frac{1}{2} \overline{\rho_{\eta,\varepsilon} |\mathbf{u}_{\eta,\varepsilon} - \mathbf{U}|^2} dx + \frac{\kappa}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y) |(\rho_{\eta,\varepsilon} - \rho_\eta)(x) - (\rho_{\eta,\varepsilon} - \rho_\eta)(x - y)|^2} dy dx \rightarrow 0 \quad (6.6.6)$$

and these two quantities are nonnegative. First, by Poincaré inequality (6.3.23) and (6.3.12), we have

$$\operatorname{esssup}_{t \in (0, T)} \int_{\mathbb{T}^d} \int_{\mathbb{R}^+ \times \mathbb{R}^d} |\lambda_1 - \rho_\eta(t, x)|^2 d\nu_{t,x}^{\eta,\varepsilon}(\lambda_1, \lambda') dx + m_{\eta,\varepsilon}^{\rho^2}(t, \mathbb{T}^d) \rightarrow 0. \quad (6.6.7)$$

In particular,

$$\operatorname{esssup}_{t \in (0, T)} \int_{\mathbb{T}^d} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+ \times \mathbb{R}^d} |\lambda_1 - \widetilde{\lambda}_1(t, x)|^2 d\nu_{t,x}^{\eta,\varepsilon}(\lambda_1, \lambda') d\delta_{\rho_\eta(t,x)}(\widetilde{\lambda}_1) dx \rightarrow 0. \quad (6.6.8)$$

Second, due to (6.3.13), we can expand the term  $\int_{\mathbb{T}^d} \frac{1}{2} \overline{\rho_{\eta,\varepsilon} |\mathbf{u}_{\eta,\varepsilon} - \mathbf{U}|^2}$  into three integrals :

$$\frac{1}{2} \int_{\mathbb{T}^d} \overline{\rho_{\eta,\varepsilon} |\mathbf{u}_{\eta,\varepsilon}|^2} dx - \int_{\mathbb{T}^d} \overline{\rho_{\eta,\varepsilon} \mathbf{u}_{\eta,\varepsilon}} \cdot \mathbf{U} dx + \frac{1}{2} \int_{\mathbb{T}^d} \rho_{\eta,\varepsilon} |\mathbf{U}|^2 dx. \quad (6.6.9)$$

We claim that the second and third term converge to 0. For the third term, this follows easily from (6.6.5), nonnegativity of  $\rho_{\eta,\varepsilon}$  and the conservation of mass  $\int_{\mathbb{T}^d} \rho_{\eta,\varepsilon} dx = \int_{\mathbb{T}^d} \rho_0 dx$ . Concerning the second term, by the dissipativity (Definition 6.3.4) and nonnegativity of the energy (Lemma 6.3.6), we have the uniform estimate

$$\begin{aligned} \left| \int_{\mathbb{T}^d} \overline{\rho_{\eta,\varepsilon} \mathbf{u}_{\eta,\varepsilon}} dx \right| &= \left| \int_{\mathbb{T}^d} \langle \sqrt{\lambda_1} \lambda', \nu^{\eta,\varepsilon} \rangle dx \right| \leq \frac{1}{2} \int_{\mathbb{T}^d} \langle \lambda_1, \nu^{\eta,\varepsilon} \rangle dx + \frac{1}{2} \int_{\mathbb{T}^d} \langle |\lambda'|^2, \nu^{\eta,\varepsilon} \rangle dx \leq \\ &\leq \frac{1}{2} \int_{\mathbb{T}^d} \rho_{\eta,\varepsilon} dx + \frac{1}{2} \int_{\mathbb{T}^d} \overline{|\sqrt{\rho_{\eta,\varepsilon}} \mathbf{u}_{\eta,\varepsilon}|^2} dx = \frac{1}{2} \int_{\mathbb{T}^d} \rho_0 dx + \frac{1}{2} \int_{\mathbb{T}^d} \overline{|\sqrt{\rho_{\eta,\varepsilon}} \mathbf{u}_{\eta,\varepsilon}|^2} dx \leq C \end{aligned}$$

As  $|\mathbf{U}| \leq C\varepsilon$ , we conclude that  $\operatorname{esssup}_{t \in (0, T)} \left| \int_{\mathbb{T}^d} \overline{\rho_{\eta,\varepsilon} \mathbf{u}_{\eta,\varepsilon}} \cdot \mathbf{U} dx \right| \rightarrow 0$  as  $\varepsilon \rightarrow 0$  so that (6.6.9) implies

$$\operatorname{esssup}_{t \in (0, T)} \frac{1}{2} \int_{\mathbb{T}^d} \overline{\rho_{\eta,\varepsilon} |\mathbf{u}_{\eta,\varepsilon}|^2} dx \rightarrow 0.$$

Again, we can write it as

$$\operatorname{esssup}_{t \in (0, T)} \int_{\mathbb{T}^d} \int_{\mathbb{R}^+ \times \mathbb{R}^d} |\lambda'|^2 d\nu_{t,x}^{\eta,\varepsilon}(\lambda_1, \lambda') dx + m_{\eta,\varepsilon}^{\rho} |\mathbf{u}|^2(t, \mathbb{T}^d) \rightarrow 0 \quad (6.6.10)$$

which implies

$$\operatorname{esssup}_{t \in (0, T)} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^+ \times \mathbb{R}^d} |\lambda' - \widetilde{\lambda}'|^2 d\nu_{t,x}^{\eta,\varepsilon}(\lambda_1, \lambda') d\delta_{\mathbf{0}}(\widetilde{\lambda}') dx \rightarrow 0. \quad (6.6.11)$$

Now, as the product measure  $\nu_{t,x}^{\eta,\varepsilon}(\lambda_1, \lambda') \otimes \delta_{\rho_\eta}(\widetilde{\lambda}_1) \otimes \delta_{\mathbf{0}}(\widetilde{\lambda}')$  is an admissible coupling between

$\nu_{t,x}^{\eta,\varepsilon}$  and  $\delta_{\rho_\eta} \otimes \delta_{\mathbf{0}}$  we can estimate the infimum in (6.6.1) by

$$\begin{aligned} [\mathcal{W}_2(\nu_{t,x}^{\eta,\varepsilon}, \delta_{\rho_\eta} \otimes \delta_{\mathbf{0}})]^2 &\leq \int_{\mathbb{R}^+} \int_{\mathbb{R}^+ \times \mathbb{R}^d} |\lambda_1 - \widetilde{\lambda}_1(t, x)|^2 d\nu_{t,x}^{\eta,\varepsilon}(\lambda_1, \lambda') d\delta_{\rho_\eta(t,x)}(\widetilde{\lambda}_1) + \\ &\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^+ \times \mathbb{R}^d} |\lambda' - \widetilde{\lambda}'|^2 d\nu_{t,x}^{\eta,\varepsilon}(\lambda_1, \lambda') d\delta_{\mathbf{0}}(\widetilde{\lambda}') \end{aligned}$$

so that integrating over  $\mathbb{T}^d$  and taking  $\text{esssup}_{t \in (0, T)}$  we conclude the proof of (6.6.3) due to (6.6.8) and (6.6.11). Furthermore, by Jensen's inequality

$$\int_{\mathbb{T}^d} |\sqrt{\rho_{\eta,\varepsilon} \mathbf{u}_{\eta,\varepsilon}}|^2 = \int_{\mathbb{T}^d} |\langle \lambda', \nu^{\eta,\varepsilon} \rangle|^2 dx \leq \int_{\mathbb{T}^d} \langle |\lambda'|^2, \nu^{\eta,\varepsilon} \rangle.$$

Taking  $\text{esssup}_{t \in (0, T)}$  and using (6.6.10), we arrive at (6.6.2).

Finally, we study the concentration measures. From (6.6.7) and (6.6.10) we know that

$$\text{esssup}_{t \in (0, T)} m_{\eta,\varepsilon}^{\rho^2}(t, \mathbb{T}^d), \quad \text{esssup}_{t \in (0, T)} m_{\eta,\varepsilon}^{\rho |\mathbf{u}|^2}(t, \mathbb{T}^d) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Using (6.3.18) we obtain the same for  $|m_{\eta,\varepsilon}^{\rho \mathbf{u} \otimes \mathbf{u}}|$ . It remains to study  $m_{\eta,\varepsilon}^{F(\rho)}$  and  $m_{\eta,\varepsilon}^{\rho F'(\rho)}$ . In fact, if we prove that  $\text{esssup}_{t \in (0, T)} m_{\eta,\varepsilon}^{F(\rho)}(t, \mathbb{T}^d)$  converges to 0 as  $\varepsilon \rightarrow 0$ , the same will be true for  $|m_{\eta,\varepsilon}^{\rho F'(\rho)}|$  due to (6.3.19).

By  $\sup_{t \in (0, T)} \Theta(t) \rightarrow 0$  and (6.6.6), we have that

$$\sup_{t \in (0, T)} \int_{\mathbb{T}^d} \overline{F(\rho_{\eta,\varepsilon} | \rho_\eta)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

We can write  $\overline{F(\rho_{\eta,\varepsilon} | \rho_\eta)}$  as (cf. (6.5.9))

$$\begin{aligned} \overline{F(\rho_{\eta,\varepsilon} | \rho_\eta)} &= \langle F_1(\lambda_1) - F_1(\rho_\eta) - F'_1(\rho_\eta)(\lambda_1 - \rho_\eta), \nu^{\eta,\varepsilon} \rangle + \\ &\quad + \langle F_2(\lambda_1) - F_2(\rho_\eta) - F'_2(\rho_\eta)(\lambda_1 - \rho_\eta), \nu^{\eta,\varepsilon} \rangle + m_{\eta,\varepsilon}^{F(\rho)}. \end{aligned} \quad (6.6.12)$$

The first term is nonnegative while the second converges to 0. Indeed, it can be bounded by  $\|F_2''\|_\infty \langle (\lambda_1 - \rho_\eta)^2, \nu_{t,x} \rangle$  which can be estimated due to inequality (cf. (6.5.10)) :

$$\|F_2''\|_\infty \int_{\mathbb{T}^d} \langle (\lambda_1 - \rho_\eta)^2, \nu_{t,x} \rangle dx \leq \frac{1 - \kappa}{4\eta^2} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \overline{\omega_\eta(y) |(\rho - \rho_\eta)(x) - (\rho - \rho_\eta)(x - y)|^2} dy dx$$

for some  $\kappa \in (0, 1)$ . Thanks to (6.6.6),

$$\text{esssup}_{t \in (0, T)} \int_{\mathbb{T}^d} |\langle F_2(\lambda_1) - F_2(\rho_\eta) - F'_2(\rho_\eta)(\lambda_1 - \rho_\eta), \nu^{\eta,\varepsilon} \rangle| dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Due to (6.6.12), the proof of (6.6.4) is concluded.  $\square$

We can also formulate a similar result to Theorem 6.6.1 in the context of Theorem 6.1.4. The proof is the same as the one of Theorem 6.6.1.

**Theorem 6.6.2.** *Under the notation of Theorem 6.1.4, the function  $\overline{\sqrt{\rho_{\eta_k, \varepsilon_k}} \mathbf{u}_{\eta_k, \varepsilon_k}}$  converges to 0 in  $L^\infty(0, T; L^2(\mathbb{T}^d))$  :*

$$\operatorname{esssup}_{t \in (0, T)} \int_{\mathbb{T}^d} |\overline{\sqrt{\rho_{\eta_k, \varepsilon_k}} \mathbf{u}_{\eta_k, \varepsilon_k}}|^2 dx \rightarrow 0 \text{ as } \varepsilon_k, \eta_k \rightarrow 0.$$

Moreover, the parametrized measure  $\nu^{\eta_k, \varepsilon_k} \in L^\infty_{\text{weak}}((0, T) \times \mathbb{T}^d; \mathcal{P}([0, +\infty) \times \mathbb{R}^d))$  converges to  $\delta_{\rho(t, x)} \otimes \delta_0$  in the following sense :

$$\int_0^T \int_{\mathbb{T}^d} [\mathcal{W}_2(\nu^{\eta_k, \varepsilon_k}, \delta_{\rho(t, x)} \otimes \delta_0)]^2 dx dt \rightarrow 0 \text{ as } \varepsilon_k, \eta_k \rightarrow 0.$$

Furthermore, the concentration measures vector  $m_{\eta_k, \varepsilon_k}$  converges to 0 in the total variation norm, uniformly in time :

$$\operatorname{esssup}_{t \in (0, T)} \|m_{\eta_k, \varepsilon_k}(t, \cdot)\|_{TV} \rightarrow 0 \text{ as } \varepsilon_k, \eta_k \rightarrow 0.$$

## 6.7 Some inequalities

**Lemma 6.7.1.** *Let  $\sigma > 0$  and a final time  $T > 0$ . Let  $\underline{u}$  be defined by  $\underline{u}(t) = \sigma \exp\left(-\int_0^t \|\operatorname{div} b\|_{L^\infty}(s) ds\right)$  and  $\phi_\delta$  defined in (6.4.4). Then*

$$\int_{\mathbb{T}^d} \Delta \phi_\delta(u) \operatorname{sgn}^-(u - \underline{u}) \geq 0.$$

*Démonstration.* We note  $f_\tau$  a concave approximation as  $\tau \rightarrow 0$  of the function  $f : x \mapsto \min\{x, 0\}$ . Then  $f'_\tau$  approximates  $f' : x \mapsto \operatorname{sgn}^-(x)$ . We have

$$\int_{\mathbb{T}^d} \Delta \phi_\delta(u) f'_\tau(u - \underline{u}) = - \int_{\mathbb{T}^d} \phi'_\delta(u) f''_\tau(u - \underline{u}) |\nabla u|^2.$$

Since  $\phi'_\delta \geq 0$ ,  $f''_\tau \leq 0$  by concavity and we conclude by sending  $\tau \rightarrow 0$ .  $\square$

**Lemma 6.7.2.** *There exists  $\eta_0 > 0$  and constant  $C_P$  such that for all  $\eta \in (0, \eta_0)$  and all  $f \in L^2(\mathbb{T}^d)$  we have*

$$\|f - (f)_{\mathbb{T}^d}\|_{L^2(\mathbb{T}^d)}^2 \leq C_P \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|f(x) - f(y)|^2}{4\eta^2} \omega_\eta(|x - y|) dx dy.$$

## 6.8 Bound on the relative pressure

**Lemma 6.8.1.** *Let  $F$  satisfy Assumption (6.1.1),  $p(\rho) = \rho F'(\rho) - F(\rho) + \frac{\rho^2}{2\eta^2}$  and  $F(\rho|P)$ ,  $p(\rho|P)$  be defined by (6.3.17). Then there exists a constant  $C_{F,R}$  such that  $p(\rho|P)$  is bounded in terms of  $F(\rho|P)$  and  $|\rho - P|^2$  i.e.*

$$p(\rho|P) \leq C_{F,R} F(\rho|P) + \left(C_{F,R} + \frac{1}{\eta^2}\right) |\rho - P|^2. \quad (6.8.1)$$

Similarly, there exists constant  $C_F$  such that

$$|\rho F'(\rho)| \leq C_F F(\rho) + C_F \rho^2 + C_F. \quad (6.8.2)$$

*Démonstration.* We write

$$p(\rho|P) = (\rho - P)^2 \int_0^1 \int_0^\tau p''(s\rho + (1-s)P) ds d\tau,$$

$$F(\rho|P) = (\rho - P)^2 \int_0^1 \int_0^\tau F''(s\rho + (1-s)P) ds d\tau.$$

We note  $h(s) = s\rho + (1-s)P$  to simplify the notations. By definition  $p'(\rho) = \rho(F''(\rho) + \frac{1}{\eta^2})$ . Therefore we obtain

$$\begin{aligned} p(\rho|P) &= (\rho - P)^2 \int_0^1 \int_0^\tau F_1''(h(s)) + F_2''(h(s)) + h(s)F_1^{(3)}(h(s)) + h(s)F_2^{(3)}(h(s)) ds d\tau + \frac{1}{\eta^2} |\rho - P|^2 \\ &= F(\rho|P) + (\rho - P)^2 \int_0^1 \int_0^\tau h(s)F_1^{(3)}(h(s)) + h(s)F_2^{(3)}(h(s)) ds d\tau + \frac{1}{\eta^2} |\rho - P|^2. \end{aligned}$$

We note  $I_1 = \int_0^1 \int_0^\tau h(s)F_1^{(3)}(h(s)) ds d\tau$  and  $I_2 = \int_0^1 \int_0^\tau h(s)F_2^{(3)}(h(s)) ds d\tau$ . By assumptions on  $|uF_1^{(3)}|$  we obtain

$$I_1 \leq C + C \int_0^1 \int_0^\tau F_1''(h(s)) ds d\tau \leq C + C \int_0^1 \int_0^\tau F_1''(h(s)) + F_2''(h(s)) ds d\tau,$$

where the value of  $C$  changed in the last inequality, using the boundedness assumption on  $F_2''$ . For  $I_2$  we simply use boundedness of  $|uF_2^{(3)}(u)|$  so that

$$I_2 \leq C.$$

This concludes the proof of (6.8.1). Concerning (6.8.2), we have

$$\rho F'(\rho) = \rho F_1'(\rho) + \rho F_2'(\rho) \leq C(1 + F_1(\rho)) + C\rho \leq C(1 + F(\rho)) + C\left(\frac{1}{2} + \frac{\rho^2}{2}\right) + C,$$

where we used estimate on  $\rho F_1'(\rho)$ , boundedness of  $F_2'$ ,  $F_2$  and inequality  $2\rho \leq 1 + \rho^2$ . The proof is concluded.  $\square$



## Chapitre 7

# Analysis and numerical simulation of a generalized compressible Cahn-Hilliard-Navier-Stokes model with friction effects

### Abstract

Motivated by the mathematical modeling of tumor invasion in healthy tissues, we propose a generalized compressible diphasic Navier-Stokes Cahn-Hilliard model that we name G-NSCH. We assume that the two phases of the fluid represent two different populations of cells : cancer cells and healthy tissue. We include in our model possible friction and proliferation effects. The model aims to be as general as possible to study the possible mechanical effects playing a role in invasive growth of a tumor. In the present work, we focus on the analysis and numerical simulation of the G-NSCH model. Our G-NSCH system is derived rigorously and satisfies basic mechanics of fluids and thermodynamics of particles. Under simplifying assumptions, we prove the existence of global weak solutions. We also propose a structure preserving numerical scheme based on the scalar auxiliary variable method to simulate our system and present some numerical simulations validating the properties of the numerical scheme and illustrating the solutions of the G-NSCH model.

## 7.1 Introduction

We derive, analyze and simulate numerically the generalized compressible Navier-Stokes-Cahn-Hilliard variant (*G-NSCH* in short)

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (7.1.1)$$

$$\frac{\partial(\rho c)}{\partial t} + \operatorname{div}(\rho c \mathbf{v}) = \operatorname{div}(b(c) \nabla \mu) + F_c, \quad (7.1.2)$$

$$\rho \mu = -\gamma \Delta c + \rho \frac{\partial \psi_0}{\partial c}, \quad (7.1.3)$$

$$\begin{aligned} \frac{\partial(\rho \mathbf{v})}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = & - \left[ \nabla p + \gamma \operatorname{div} \left( \nabla c \otimes \nabla c - \frac{1}{2} |\nabla c|^2 \mathbf{1} \right) \right] + \operatorname{div}(\nu(c) (\nabla \mathbf{v} + \nabla \mathbf{v}^T)) \\ & - \frac{2}{3} \operatorname{div}(\nu(c) \operatorname{div}(\mathbf{v}) \mathbf{1}) + \operatorname{div}(\eta(c) \operatorname{div}(\mathbf{v}) \mathbf{1}) - \kappa(\rho, c) \mathbf{v}, \end{aligned} \quad (7.1.4)$$

stated in  $(0, T) \times \Omega$ , where  $T > 0$  is finite time horizon, and  $\Omega \subset \mathbb{R}^d$  ( $d = 1, 2, 3$ ) is an open bounded domain with a smooth boundary  $\partial\Omega$ .

Interested by the modeling of invasive growth of tumors in healthy tissues, we motivate the different terms of the model with this biological application in mind. However, we emphasize that the model is a general compressible diphasic fluid model that could be used for other applications.

System (7.1.1)–(7.1.4) models the motion of a diphasic fluid composed of two immiscible components, *i.e.* two different cell types (*e.g.* tumor and healthy cells), and comprises viscosity effects, surface tension, and friction on rigid fibers representing the extracellular matrix (*ECM* in short). In System (7.1.1)–(7.1.4),  $\rho$  is the total density of the mixture (*i.e.* the sum of the two partial densities),  $c$  is the relative mass fraction of one component (*e.g.* the cancer cells),  $\mathbf{v}$  is the mass averaged total velocity,  $\mu$  is called the chemical potential,  $p$  is the pressure. The coefficient  $\gamma$  is related to the surface tension and is equal to the square of the width of the diffuse interface existing between the two populations. The friction coefficient  $\kappa(\rho, c)$  is a non-negative function of the density and the mass fraction, and takes into account the possible difference of friction strength between the two populations. We use this friction term to model possible adhesive effects of the cells on the ECM. The coefficients  $\nu(c)$  and  $\eta(c)$  represents the viscosity coefficients (shear and dilatational, respectively) of the mixture. Possible differences in viscosities could be considered for the two populations. The function  $\psi_0$  represents the separation of the two components of the mixture and phenomenologically models the behavior of cells (*i.e.* cells tend to form aggregates of the same cell type). The function  $F_c(\cdot)$  accounts for the possible proliferation and death of cells and these two effects are assumed to be modelled as an exchange of mass between the populations. The non-negative function  $b(\cdot)$  models the mobility of cells. This function models the probability for a cell of any of the two populations to find an available neighboring spot to which it can move. More details about the general assumptions and precise forms of the different functions will be given in the next sections.

The motivation of our model stands from the modeling of tumor progression and invasion in healthy tissues. Indeed, as explained in Appendix 7.7.8, under suitable choices of functionals, our model can be viewed as a representation of a proliferating population of cells, *i.e.* the tumor cells, in a domain filled with a non-proliferating population, *i.e.* the healthy cells and the rest of tissue (ECM, extracellular fluid, etc.). The proliferation of cells happens by consuming mass from the other phase (we are not injecting mass in the system). Both cell populations move in an

ECM constituted of rigid fibers on which they can adhere. As we only focus on the mechanical effects generated by the properties of the cells that could play a role during invasion, we do not consider in the model other effects that are known to be important in tumor progression : *e.g.* angiogenesis, digestion of the ECM by proteolic enzymes, role of helping cells located in the stroma.

We emphasize that this article only concerns the analysis and the numerical simulation of the G-NSCH model (7.1.1)–(7.1.4). This latter comprises effects that are negligible in biological situations, *e.g.* inertia effects. We propose here an analysis of the model and a structure preserving numerical scheme for the G-NSCH model.

**Literature review** The motion of a binary mixture of two immiscible and compressible fluids can be described by the Navier-Stokes equation coupled to the Cahn-Hilliard model : the Navier-Stokes-Cahn-Hilliard model (*NSCH model* in short). The well-known incompressible variant of the compressible NSCH model has been denominated model H (see *e.g.* [215, 202]). Model H has been proposed to represent viscous fluid flow in an incompressible binary mixture undergoing phase separation. This model assumes matching densities, *i.e.*  $\rho_1 = \rho_2$  and, hence, constant total density  $\rho$ . To consider non-matching densities, Lowengrub and Truskinovsky [263] proposed the compressible NSCH model. Expanding the divergence term in the mass balance equation, the authors found a relation denoting the quasi-compressible nature of the fluid. Concomitantly, Anderson, FcFadden, and Wheeler [22] proposed a similar system. In the present work, we use a similar system. We also remark that a very recent work [148] proposed a unified framework for the incompressible NSCH system and shows that the different NSCH models found in the literature only differ from their general modelling framework by specific constitutive hypotheses. Under some simplifying assumptions compared to the system proposed in [263] but being closer to the system in [22], the analysis of the compressible NSCH model with no-flux boundary conditions has been realized by Abels and Feireisl [5]. Their analysis requires to simplify the model proposed in [263] to avoid zones with zero density which would make this analysis a lot more difficult since the control from certain estimates would be lost. In another article, for the same system, Abels proved the existence of strong solutions for short times [3]. Considering the same assumptions and dynamic boundary conditions, Cherfils *et al.* [104] proved the well-posedness of the compressible NSCH model with these special boundary conditions. These latter allow to model the interaction of the fluid components and the walls of the domain.

Results on the analysis of the incompressible variant of the NSCH model, *i.e.* the model H, are numerous and we here mention only a few of them since a complete review would be out of the scope of the present article. With a non-degenerate mobility coefficient and a physically relevant choice of potential, the well-posedness and regularity analysis of model H has been performed by Abels [2] using tools both from the analysis of Navier-Stokes model and the Cahn-Hilliard model. It is worth mentioning that the non-degeneracy of the mobility coefficient leads to non-physical effects, *i.e.* Ostwald ripening effects (see [6]). For this reason, Abels, Depner and Garcke studied model H with a degenerate mobility [4]. Their analysis relies on a regularization of the mobility and singular potential into, respectively, a non-degenerate and non-singular potential. Then, suitable *a-priori* estimates uniform in the regularization parameter allow to pass to the limit in the regularization and show the existence of weak solutions to the degenerate model H. We now review partially the extensive literature about the Cahn-Hilliard equation and its use for the modelling of tumors. The Cahn-Hilliard equation has been initially used to represent the phase separation in binary mixtures and has been applied to the spinodal decomposition of binary alloys under a sudden cooling [70, 69]. The model represents the two phases of the fluids as continua separated by a diffuse interface. This equation has been used later in many different applications and we do not intend here to give an overview of all these. However,



we refer the reader interested in the topic to the presentation of the Cahn-Hilliard equation and its applications to the review book [277]. We are interested here in the application of the Cahn-Hilliard framework to tumor modelling (see *e.g.* [265]). Latter, different variants of the Cahn-Hilliard model appeared : *e.g.* (without giving a complete overview) its coupling to Darcy's law [187], Brinkman's law [145], chemotaxis [299]. Recently, a variant of the CH equation has been used to better represent the growth and organization of tumors. The main change is the use of a single-well logarithmic degenerate potential instead of a double-well potential [97, 11, 287]. This type of potential has been proposed in [20] to represent the action of the cells depending only on the local density, *i.e.* attraction at low cell density and repulsion for large cell density representing the tendency of cells to avoid overcrowding. The Cahn-Hilliard framework has also been utilized in systems representing invasive growth of tumors. The interested reader can find a lot of information about phase-field type systems modelling tumor growth and invasion in the very recent survey paper [180] and references therein.

We now review some of the literature about the numerical simulation of NSCH models. The numerical simulation of Model H for binary fluids with non-matching densities has been the subject of numerous works (see *e.g.* [219] and references therein). However, in part due to its complexity, the numerical simulation of the compressible NSCH system has been less explored. A  $C^0$  finite element numerical scheme for a variant of the quasi-compressible NSCH model proposed in [263] has been proposed in [199]. Around the same time, Giesselmann and Pryer [13, 192] designed a discontinuous Galerkin finite element scheme to simulate the quasi-incompressible NSCH system which preserves the total mass and the dissipation of energy. A numerical method has also been proposed in [210] in the case of constant mobility  $b(c)$  and smooth polynomial potential  $\psi(c)$ . However, the system simulated in [210] is a simplification of the compressible NSCH system since the pressure does not appear in the definition of the chemical potential  $\mu$  in their system.

The previous works we presented for the simulation of the compressible or quasi-compressible NSCH systems deal with constant mobility combined with a smooth polynomial potential. We aim to simulate the compressible NSCH model with choices of mobility and potential relevant for biology (but also relevant for material sciences and fluid mechanics), *i.e.* degenerate mobility combined with a logarithmic potential. We now review briefly some relevant discretization methods for the Cahn-Hilliard equation with degenerate mobility and singular potentials. Considering a degenerate mobility and a double-well logarithmic potential, we mention the work of Barrett, Blowey and Garcke [37]. In this article the authors proposed a finite element scheme with a variational inequality to preserve the bounds of the solution. Based on these ideas, Agosti *et al.* [11] proposed a similar finite element scheme for the single-well logarithmic potential case. The difficulty in this latter case lies in the fact that the degeneracy and the singularity sets do not coincide and, considering an order parameter that must remain within the bounds  $[0, 1)$ , negative solutions can appear if a standard discretization method is used. The numerical scheme proposed in [11] solves this issue but does not preserve the mass. In a more recent work, Agosti [10] proposed a discontinuous Galerkin finite element scheme that preserves the bounds  $[0, 1)$  and preserves the exact mass. However, the main drawback of the previously mentioned methods is that they are computationally expensive : they solve a strongly coupled nonlinear system and resort to the use of iterative algorithms.

Since the Cahn-Hilliard equation is a gradient flow (see *e.g.* [258]), a structure-preserving linear scheme can be constructed using the Scalar Auxiliary Variable (*SAV* in short) method [311]. The *SAV* method is a very powerful tool to design unconditionally energy-stable numerical schemes for models possessing a gradient-flow (see *e.g.* [312, 333] and references therein) or Hamiltonian structure (see *e.g.* [23, 295] and references therein). The *SAV* method has evolved during the past 6 years starting from the original *SAV* method [311, 310] to improved variants such as the

generalized version GSAV (see *e.g.* [333, 220]) and the relaxed RSAV method [231]. In our work, we use the GSAV method that has been already used in [222] for the Cahn-Hilliard equation. In this latter work, the scheme is structure-preserving from the use of a scalar variable that represents the discrete energy, and an additional equation is solved to ensure dissipation at the discrete level. The bounds of the order parameter are ensured using a transformation that maps  $\mathbb{R}$  to the physical relevant interval  $((0, 1)$  in the case of a double-well potential). Hence, compared to other techniques, the SAV method has the advantage to allow for the design of a linear, efficient, structure-preserving scheme and can easily be used for our G-NSCH system. We also emphasize that the SAV method has been used for the simulation of the incompressible NSCH model with positive mobility and polynomial potential in [252]. In the present work, we use the GSAV method to design a numerical scheme for the G-NSCH model. Our numerical scheme allows to use degenerate mobility and singular potential functionals which is more physically relevant. To the best of our knowledge, our G-NSCH model is new because it comprises the friction force term and the exchange between the two phases of the fluid. Moreover, the use of the GSAV method for a compressible NSCH system is new, especially with our choice of functionals (*i.e.* degenerate mobility and singular potential).

**Objectives of our work** The first objective of our work is to study the well-posedness of the G-NSCH model under some simplifying assumptions (*i.e.* smooth potential and positive mobility). The second objective is the design of an efficient and structure-preserving numerical scheme for the G-NSCH model with singular double-well potential and degenerate mobility. The third focus of the present work concerns the rigorous derivation of the G-NSCH model that is presented in the Appendix.

**Outline of the paper** Section 7.2 presents the notations, functional spaces and assumptions we use in our work for the analytical part but also for the numerical part. Section 7.3 concerns the proof of the existence of weak solutions for the G-NSCH system (7.1.1)–(7.1.4) under simplifying assumptions. A structure preserving numerical scheme based on the GSAV method is then proposed in Section 7.4 and some numerical results are presented in Section 7.5. Our model's equations come from a thermodynamically consistent derivation of the compressible Navier-Stokes-Cahn-Hilliard model including friction effects and source terms. The derivation is described in Appendix 7.7.1. From the general model, we propose in Appendix 7.7.8 two reductions : The G-NSCH studied and simulated in the present work and one biologically relevant reduction that will be the focus of a forthcoming work.

## 7.2 General assumptions, notations and functional setting

The equations are set in a domain  $\Omega_T = \Omega \times (0, T)$  with  $\Omega$  an open and bounded subset of  $\mathbb{R}^d$  ( $d = 1, 2, 3$ ). We assume that the boundary  $\partial\Omega$  is sufficiently smooth. We indicate the usual Lebesgue and Sobolev spaces by respectively  $L^p(\Omega)$ ,  $W^{m,p}(\Omega)$  with  $H^m(\Omega) := W^{m,2}(\Omega)$ , where  $1 \leq p \leq +\infty$  and  $m \in \mathbb{N}$ . For  $q \in [1, +\infty]$ , we indicate the Bochner spaces by  $L^q(0, T; X)$  (where  $X$  is a Banach space). Finally,  $C$  denotes a generic constant that appears in inequalities and whose value can change from one line to another. This constant can depend on various parameters unless specified otherwise.

### 7.2.1 Assumptions on functionals

We divide the assumptions on the different terms appearing in system (7.1.1)–(7.1.4) into two parts : analytical and numerical assumptions. Indeed we are not able to prove the existence of weak solutions in the general setting used for the numerical simulations. For instance, the case of the usual logarithmic double-well potential in the Cahn-Hilliard equation is not treated but can be implemented in our numerical scheme. However, we can analyze our system with a polynomial approximation of the double well. We also consider non-degenerate mobilities to obtain estimates on the chemical potential  $\mu$  directly. The case of degenerate mobility, see for instance [158], seems unavailable as we do not have anymore the classical “entropy” estimates of the Cahn-Hilliard equation that provide bound on second-order derivatives of the mass fraction  $c$ .

**Framework for numerical simulations** We assume that the viscosity  $\nu(c), \eta(c)$  and permeability  $\kappa(\rho, c)$  coefficients are smooth non-negative functions. The mobility is a non-negative function of the order parameter (mass fraction)  $c$ . Hence, we assume that

$$b \in C^1([0, 1]; \mathbb{R}^+), \quad \text{and} \quad b(c) \geq 0 \quad \text{for} \quad 0 \leq c \leq 1. \quad (7.2.1)$$

In agreement with the literature (see e.g [104]), the homogeneous free energy  $\psi_0(\rho, c)$  is assumed to be of the form

$$\psi_0(\rho, c) = \psi_e(\rho) + \psi_{\text{mix}}(\rho, c), \quad (7.2.2)$$

with  $\psi_{\text{mix}}(\rho, c) = H(c) \log \rho + Q(c)$  and  $Q(c)$  is a double-well (or single-well) potential. Then, using the constitutive relation for the pressure, we have

$$p(\rho, c) = \rho^2 \frac{\partial \psi_0}{\partial \rho} = p_e(\rho) + \rho H(c), \quad (7.2.3)$$

where  $p_e = \rho^2 \psi'_e(\rho)$  and is assumed to satisfy

$$p_1 \rho^{a-1} - p_2 \leq p'_e(\rho) \leq p_3(1 + \rho^{a-1}), \quad \text{for} \quad a > 3/2, \quad p_1, p_2, p_3 > 0. \quad (7.2.4)$$

We assume that the exchange term  $F_c$  (that can depend on the mass fraction and the density) is bounded,

$$|F_c(\rho, c)| + \left| \frac{F_c(\rho, c)}{\rho} \right| \leq C, \quad \forall (\rho, c) \in \mathbb{R}^2. \quad (7.2.5)$$

**Remark 7.2.1** (Double-well logarithmic potential). In the present work, we aim to use a double-well logarithmic potential in the definition of the mixing potential. A relevant example of potential is

$$\psi_{\text{mix}} = \frac{1}{2} (\alpha_1(1-c) \log(\rho(1-c)) + \alpha_2 c \log(\rho c)) - \frac{\theta}{2} (c - \frac{1}{2})^2. \quad (7.2.6)$$

This potential gives

$$H(c) = \frac{1}{2} (\alpha_1(1-c) + \alpha_2 c), \quad Q(c) = \frac{1}{2} (\alpha_1(1-c) \log(1-c) + \alpha_2 c \log(c)) - \frac{\theta}{2} (c - \frac{1}{2})^2,$$

where  $\theta > 1$ .

**Additional assumptions for the existence of weak solutions and analysis of the numerical scheme.** Concerning the existence of weak solutions and analysis of the numerical scheme, we need to strengthen our assumptions. The viscosity coefficients  $\nu(c), \eta(c)$  are assumed

to be bounded from below by a positive constant and the friction coefficient  $\kappa(\rho, c)$  is assumed to be nonnegative. Moreover,  $\nu(c), \eta(c)$  and  $\kappa(\rho, c)$  are functions bounded in  $L^2(0, T; L^2(\Omega))$  whenever  $c$  is bounded in  $L^\infty(0, T; H^1(\Omega))$  and  $\rho$  is smooth (for instance  $C(0, T; C^2(\bar{\Omega}))$ ). We consider  $a > 2$  the exponent of the pressure law. In the numerical simulations, we take degenerate mobilities of the form  $b(c) = c(1-c)^\alpha$ . However, in the analysis, we consider a non-degenerate mobility by truncating the previous mobility. For instance, using a small parameter  $0 < \varepsilon_b \ll 1$ , we approximate the mobility  $b(\cdot)$  by

$$b_{\varepsilon_b}(c) = \begin{cases} b(1 - \varepsilon_b), & \text{if } c \geq 1 - \varepsilon_b, \\ b(\varepsilon_b), & \text{if } c \leq \varepsilon_b, \\ b(c), & \text{otherwise,} \end{cases}$$

and consider the case of a fixed  $\varepsilon_b$ . Dropping the  $\varepsilon_b$  subscript, we obtain that

$$b \in C^1(\mathbb{R}; \mathbb{R}^+), \quad \text{and} \quad b(c) \geq C > 0 \quad \forall c \in \mathbb{R}. \quad (7.2.7)$$

Concerning the functionals appearing in the definition of the free energy  $\psi_0$  we assume that  $H$  and  $H'$  are bounded and that  $Q$  is a polynomial approximation of the double well potential. More precisely we take

$$\begin{aligned} H_1 &\leq H'(c), \quad H(c) \leq H_2, \quad c \in \mathbb{R}, \quad H_1, H_2 > 0, \\ Q(c) &= \frac{1}{4}c^2(1-c)^2. \end{aligned} \quad (7.2.8)$$

The case of the double-well logarithmic potential has not been tackled yet even though this is the main motivation for the decomposition of  $\psi_{mix}$  as in the works [5] and [104].

Also, to make the computations simpler, we assume that

- $a > 6$  where  $a$  is the pressure exponent,
- $\psi_\varepsilon(\rho) = \frac{\rho^{a-1}}{a-1}$  and therefore  $p_\varepsilon(\rho) = \rho^a$ .

These two assumptions are not necessary and could be removed but simplify the analysis. We refer for instance to [5, 166] for the more general setting. For instance, the condition  $a > 6$  is used to not introduce another parameter in the approximating scheme which would make the article longer. Note that the assumptions on  $\psi_0$  imply in particular the following lemma which is essential to obtain estimates on the energy dissipation :

**Lemma 7.2.2.** *There exists a constant  $C$  such that*

$$\left| \rho \frac{\partial \psi_0}{\partial c} \right| \leq C \rho \psi_0 + C.$$

Its proof uses the assumption on  $H$  and the fact that for  $c$  large,  $Q'(c) \approx c^3 \leq c^4 + 1 \approx Q(c) + 1$ .

### 7.3 Existence of weak solutions

We now turn to the proof of the existence of weak solutions for the G-NSCH model (7.1.1)–(7.1.4) subjected to boundary conditions

$$\mathbf{v} = 0, \quad \frac{\partial c}{\partial \mathbf{n}} = b(c) \frac{\partial \mu}{\partial \mathbf{n}} = 0, \quad \text{on } \partial \Omega, \quad (7.3.1)$$

and initial conditions

$$\rho(0, x) = \rho_0 \geq 0 \in L^a(\Omega), \quad c(0, x) = c_0 \in H^1(\Omega) \quad \rho_0 \mathbf{v}(0, x) = \mathbf{m}_0, \text{ with } \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega). \quad (7.3.2)$$

Also, we suppose  $\rho_0 \neq 0$ . In this section we take  $d = 3$ . The proof of the result is quite long and technical. Therefore, when possible and for the sake of clarity, we omit some proofs and give instead appropriate references.

**Outline of the analysis** For readability reasons, we here present the outline of the analysis of the G-NSCH model. We first start with the analysis of a "truncated" version of G-NSCH model in the sense that the double-well is truncated for large values of  $c$  with a parameter  $\varepsilon_Q$ . Then, for this fixed truncation, we prove the existence of weak solutions using the ideas of [256, 166, 5, 104]. Then, we pass to the limit  $\varepsilon_Q \rightarrow 0$ . Namely, recalling that  $Q(c) = \frac{1}{4}c^2(1-c)^2$  we first consider  $Q_{\varepsilon_Q}(c)$  a smooth truncated approximation of  $Q$  that satisfies

$$|Q_{\varepsilon_Q}|, |Q'_{\varepsilon_Q}|, |Q''_{\varepsilon_Q}| \leq C \left( \frac{1}{\varepsilon_Q} \right). \quad (7.3.3)$$

In the first subsections, we drop the  $\varepsilon_Q$  notation and work with the regularized problem. We will use the  $\varepsilon_Q$  notation when we pass to the limit. For the moment, we benefit from the properties of the regularization.

### 7.3.1 Energy estimates

The G-NSCH system comes with an energy structure which is useful to obtain first a priori estimates.

**Proposition 7.3.1.** *Smooth solutions of the system (7.1.1)–(7.1.4) satisfy the following energy relation*

$$\frac{d}{dt} E + D = \int_{\Omega} \mu F_c \, dx, \quad (7.3.4)$$

where  $E$  is the energy, and  $D$  is the dissipation defined as

$$E = \int_{\Omega} \rho \frac{|\mathbf{v}|^2}{2} + \rho \psi_0 + \frac{\gamma}{2} |\nabla c|^2 \, dx, \quad (7.3.5)$$

$$D = \int_{\Omega} \frac{\nu(c)}{2} \left| \nabla \mathbf{v} + \nabla \mathbf{v}^T - \frac{2}{3} \operatorname{div}(\mathbf{v}) \mathbb{I} \right|^2 + \eta(c) |\operatorname{div}(\mathbf{v}) \mathbf{1}|^2 + b(c) |\nabla \mu|^2 + \kappa(\rho, c) |\mathbf{v}|^2 \, dx. \quad (7.3.6)$$

This yields a priori estimates on the solution i.e. there exists a positive constant  $C$  such that

$$E(t) + \int_0^t D(s) \, ds \leq C + CE(0).$$

Note that the energy is bounded from below since  $\rho \log \rho H(c)$  is bounded from below with (7.2.8). Also, the purpose of the assumptions  $\nu(c), \eta(c)$  and  $b(c)$  bounded from below by a positive constant becomes clear, they are crucial to obtain estimates on the  $H^1(\Omega)$  norm of  $\mu$  and  $\mathbf{v}$ .

*Démonstration.* We recall the formula

$$\nabla c \Delta c = \operatorname{div}(\nabla c \otimes \nabla c) - \frac{1}{2} \nabla |\nabla c|^2. \quad (7.3.7)$$

We denote by  $\mathbb{T}$  the tensor

$$\mathbb{T} = \nu(c)(\nabla \mathbf{v} + \nabla \mathbf{v}^T - \frac{2}{3}\operatorname{div}(\mathbf{v})\mathbb{I}) + \eta(c)\operatorname{div}(\mathbf{v})\mathbf{1}. \quad (7.3.8)$$

Then we multiply Equation (7.1.1) by  $\frac{|\mathbf{v}|^2}{2}$  and sum it with the scalar product of Equation (7.1.4) with  $\mathbf{v}$ . We obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left( \rho \frac{|\mathbf{v}|^2}{2} \right) + \operatorname{div} \left( \frac{1}{2} \rho |\mathbf{v}|^2 \mathbf{v} + p(\rho, c) \mathbf{v} - \mathbb{T} \cdot \mathbf{v} \right) + \mathbb{T} : \nabla \mathbf{v} + \kappa(\rho, c) \mathbf{v}^2 = p(\rho, c) \operatorname{div}(\mathbf{v}) \\ + \gamma \operatorname{div} \left( \frac{1}{2} |\nabla c|^2 \mathbb{I} - (\nabla c \otimes \nabla c) \right) \cdot \mathbf{v}, \end{aligned}$$

which is equivalent to

$$\frac{\partial}{\partial t} \left( \rho \frac{|\mathbf{v}|^2}{2} \right) + \operatorname{div} \left( \frac{1}{2} \rho |\mathbf{v}|^2 \mathbf{v} + p(\rho, c) \mathbf{v} - \mathbb{T} \cdot \mathbf{v} \right) + \mathbb{T} : \nabla \mathbf{v} + \kappa(\rho, c) \mathbf{v}^2 = p(\rho, c) \operatorname{div} \mathbf{v} - \gamma \Delta c \nabla c \cdot \mathbf{v}. \quad (7.3.9)$$

Then, we multiply Equation (7.1.2) by  $\mu$  and obtain using also (7.1.1)

$$\rho \mu (\partial_t c + \mathbf{v} \cdot \nabla c) = \operatorname{div}(b(c) \nabla \mu) \mu + \mu F_c.$$

And, using (7.1.3) we obtain

$$\rho \frac{\partial \psi_0}{\partial c} (\partial_t c + \mathbf{v} \cdot \nabla c) = \operatorname{div}(b(c) \nabla \mu) \mu + \gamma \Delta c (\partial_t c + \mathbf{v} \cdot \nabla c) + \mu F_c.$$

The previous equation can be rewritten using the chain rule as

$$\begin{aligned} \partial_t(\rho \psi_0) + \operatorname{div}(\rho \psi_0 \mathbf{v}) - \psi_0(\partial_t \rho + \operatorname{div}(\rho \mathbf{v})) - \rho \frac{\partial \psi_0}{\partial \rho} (\partial_t \rho + \mathbf{v} \cdot \nabla \rho) \\ = \operatorname{div}(b(c) \nabla \mu) \mu + \gamma \Delta c (\partial_t c + \mathbf{v} \cdot \nabla c) + \mu F_c. \end{aligned}$$

We have  $\rho \frac{\partial \psi_0}{\partial \rho} (\partial_t \rho + \mathbf{v} \cdot \nabla \rho) = \rho \frac{\partial \psi_0}{\partial \rho} (-\rho \operatorname{div}(\mathbf{v})) = -p \operatorname{div}(\mathbf{v})$  (see Equation (7.2.3) for the definition of the pressure). Moreover, we know that  $\Delta c \partial_t c = \operatorname{div}(\partial_t c \nabla c) - \partial_t \left( \frac{|\nabla c|^2}{2} \right)$  and, hence,

$$\begin{aligned} \partial_t(\rho \psi_0) + \operatorname{div}(\rho \psi_0 \mathbf{v}) + p \operatorname{div}(\mathbf{v}) = \operatorname{div}(b(c) \nabla \mu) \mu + \gamma \left[ \operatorname{div}(\partial_t c \nabla c) - \partial_t \left( \frac{|\nabla c|^2}{2} \right) + \Delta c \mathbf{v} \cdot \nabla c \right] \\ + \mu F_c. \quad (7.3.10) \end{aligned}$$

Summing (7.3.9) and (7.3.10) we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left( \rho \frac{|\mathbf{v}|^2}{2} + \rho \psi_0 + \frac{\gamma}{2} |\nabla c|^2 \right) + \operatorname{div} \left( \rho \psi_0 \mathbf{v} + \frac{1}{2} \rho |\mathbf{v}|^2 \mathbf{v} + p(\rho, c) \mathbf{v} - \mathbb{T} : \mathbf{v} - \gamma \partial_t c \nabla c \right) - \operatorname{div}(b(c) \nabla \mu) \mu \\ + \mathbb{T} : \nabla \mathbf{v} + \kappa(\rho, c) |\mathbf{v}|^2 = \mu F_c. \end{aligned}$$

Now we use the fact that

$$\mathbb{T} : \nabla \mathbf{v} = \frac{\nu(c)}{2} \left| \nabla \mathbf{v} + \nabla \mathbf{v}^T - \frac{2}{3} \operatorname{div}(\mathbf{v}) \mathbb{I} \right|^2 + \eta(c) |\operatorname{div}(\mathbf{v}) \mathbf{1}|^2. \quad (7.3.11)$$

Integrating in space and using the boundary conditions (7.3.1) ends the proof of the first part of the proposition. To prove the second part, we integrate the equation in time and control the right-hand side. Indeed, due to the assumption on the source term (7.2.5), we have

$$\left| \int_0^t \int_{\Omega} \mu F_c \, dx \, dt \right| \leq C \int_0^t \int_{\Omega} |\mu|.$$

We want to use Lemma 7.3.7 to control the  $L^1$  norm of  $\mu$ . Integrating the equations on  $\rho$  to obtain  $\int_{\Omega} \rho \, dx = \int_{\Omega} \rho_0 \, dx > M_0$  we satisfy the first assumption of the lemma. For the second, we notice that we can consider a variant of this lemma such that instead of asking  $\rho$  to be in  $L^{6/5}$  we have the inequality

$$\left\| \mathbf{u} - \frac{1}{|\Omega|} \int_{\Omega} \rho \mathbf{u} \right\|_{L^2} \leq C \|\nabla \mathbf{u}\|_{L^2} + \|\rho\|_{L^{6/5}}.$$

Using Young's inequality, the fact that in the energy  $\rho\psi_0$  contains a term of the form  $\rho^{a+1}$  we obtain for  $\tilde{C}$  small enough

$$\int_0^t \int_{\Omega} |\mu| \, dx \leq C + \tilde{C} \int_0^t \int_{\Omega} |\mu|^2 \, dx \leq C + CE(t) + \frac{\inf_c b(c)}{2} \int_{\Omega} |\nabla \mu|^2 \, dx + C \left| \int_{\Omega} \rho \mu \, dx \right|.$$

Since the energy dissipation controls the third term of the right-hand side, it remains to control the last term of the right-hand side. We recall that  $\rho\mu = \rho \frac{\partial \psi_0}{\partial c} - \gamma \Delta c$ . Using the Neumann boundary conditions on  $c$ , it remains to control  $\left| \int_{\Omega} \rho \frac{\partial \psi_0}{\partial c} \right|$ . Using Lemma 7.2.2, we obtain

$$\left| \int_{\Omega} \rho \frac{\partial \psi_0}{\partial c} \, dx \right| \leq C + CE(t).$$

We conclude using Gronwall's lemma. □

### 7.3.2 Existence of weak solutions for fixed $\varepsilon_Q$

The weak solutions of system (7.1.1)–(7.1.4) are defined as follows

**Definition 7.3.2.** We say that  $(\rho, \mathbf{v}, c, \mu)$  is a weak of system (7.1.1)–(7.1.4) provided :

- $\rho \geq 0$  and we have the regularity

$$\begin{aligned} \rho &\in L^\infty(0, T; L^a(\Omega)), \\ \mathbf{v} &\in L^2(0, T; H_0^1(\mathbb{R}^3)), \quad \sqrt{\rho} \mathbf{v} \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)), \quad \mathbb{T} : \nabla \mathbf{v} \in L^1(0, T; L^1(\Omega)), \\ c &\in L^\infty(0, T; H^1(\Omega)), \\ \mu &\in L^2(0, T; H^1(\Omega)). \end{aligned}$$

- Equations (7.1.1)–(7.1.4) are satisfied in the distributional sense.
- The initial conditions (7.3.2) are satisfied a.e. in  $\Omega$ .
- The boundary conditions (7.3.1) are satisfied.

We state our main theorem about the existence of weak solutions

**Theorem 7.3.3** (Existence of weak solutions). *There exist  $(\rho, \mathbf{v}, c, \mu)$  weak solutions of (7.1.1)–(7.1.4) in the sense of Definition 7.3.2.*

In order to prove the existence of weak solutions, we use an approximating scheme with a small parameter  $\varepsilon > 0$  borrowing the idea from [256, 166]. More precisely, let  $X_n = \text{span}\{\eta_i\}_{i=1,\dots,n}$  be the set of the first  $n$  vectors of a basis of  $H_0^1(\Omega; \mathbb{R}^3)$  such that  $X_n \subset C^2(\bar{\Omega}; \mathbb{R}^3)$ . We consider the following problem for  $(\rho, \mathbf{v}_n, c)$  with  $\mathbf{v}_n \in X_n$  (with coordinates depending on time) :

$$\partial_t \rho + \text{div}(\rho \mathbf{v}_n) = \varepsilon \Delta \rho, \tag{7.3.12}$$

and for every  $\eta \in X_n$ ,

$$\begin{aligned} & \int_{\Omega} \rho \mathbf{v}_n(t) \cdot \eta \, dx - \int_{\Omega} \mathbf{m}_0 \cdot \eta \, dx - \int_0^t \int_{\Omega} \rho \mathbf{v}_n \otimes \mathbf{v}_n : \nabla \eta \, dx \, ds - \int_0^t \int_{\Omega} p(\rho, c) \text{div}(\eta) \, dx \, ds \\ & + \varepsilon \int_0^t \int_{\Omega} (\nabla \mathbf{v}_n \nabla \rho) \cdot \eta \, dx \, ds + \int_0^t \int_{\Omega} \mathbb{T} : \nabla \eta \, dx \, ds + \gamma \int_0^t \int_{\Omega} \left( \frac{1}{2} |\nabla c|^2 \mathbb{I} - (\nabla c \otimes \nabla c) \right) : \nabla \eta \, dx \, ds \\ & \quad + \int_{\Omega} \int_0^t \kappa(\rho, c) \mathbf{v}_n \cdot \eta \, dx \, ds = 0. \end{aligned} \tag{7.3.13}$$

And for the equation on the mass fraction

$$\partial_t c + \mathbf{v}_n \cdot \nabla c = \frac{1}{\rho} \text{div}(b(c) \nabla \mu) + \frac{F_c}{\rho}, \quad \mu = \frac{\partial \psi_0}{\partial c} - \gamma \frac{\Delta c}{\rho}. \tag{7.3.14}$$

We consider Neumann boundary conditions

$$\nabla \rho \cdot \mathbf{n} = b(c) \nabla \mu \cdot \mathbf{n} = \nabla c \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega, \tag{7.3.15}$$

and the Dirichlet boundary condition for  $\mathbf{v}_n$  is included in the definition of  $X_n$ . Finally, we consider the initial conditions

$$\rho(0, \cdot) = \rho_{0,\varepsilon} > 0, \quad c(0, \cdot) = c_{0,\varepsilon}, \quad \rho \mathbf{v}_n(0, \cdot) = \mathbf{m}_0, \tag{7.3.16}$$

where  $\rho_{0,\varepsilon}, c_{0,\varepsilon}$  satisfy the Neumann boundary conditions and they are smooth approximations of  $\rho_0, c_0$  (when  $\varepsilon \rightarrow 0$ ).

We now comment on the scheme used above and detail the strategy of the proof. We add the artificial diffusion in (7.3.12) with the parameter  $\varepsilon > 0$ . Here,  $\mathbf{v}_n$  is fixed and we can conclude the global in time existence of classical solutions to (7.3.12) which are positive since the initial condition is positive (and using maximum principle). Using this positivity, we conclude the existence of a strong solution to Equation (7.3.14) which is in fact a fourth-order parabolic equation. Having obtained  $c$ , we focus on Equation (7.3.13) and we prove existence for a small time with Schauder’s fixed point theorem. Note the presence of the additional term  $\varepsilon \int (\nabla \mathbf{v}_n \nabla \rho) \cdot \eta$  which is useful to cancel energy terms introduced by  $\varepsilon \Delta \rho$  in (7.3.12). Having obtained existence on a short time interval we compute the energy of the system and obtain global existence. Then, we pass to the limit  $n \rightarrow \infty$ . It remains to send  $\varepsilon$  and  $\varepsilon_Q$  to 0 and obtain solutions of system (7.1.1)–(7.1.4).

We first turn our attention to Equation (7.3.12). From [166], we obtain the following proposition, and lemma

**Proposition 7.3.4.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain of class  $C^{2+\beta}$  for some  $\beta > 0$ . For a fixed  $\mathbf{v}_n \in X_n$ , there exists a unique solution to Equation (7.3.12) with Neumann boundary conditions (7.3.15) and initial data conditions (7.3.16). Furthermore, the mapping  $\mathbf{v}_n \mapsto \rho[\mathbf{v}_n]$ , that assigns to any  $\mathbf{v}_n \in X_n$  the unique solution of (7.3.12), takes bounded sets in the space*



$C(0, T; C_0^2(\bar{\Omega}, \mathbb{R}^3))$  into bounded sets in the space

$$V := \{\partial_t \rho \in C(0, T; C^\beta(\bar{\Omega})), \rho \in C(0, T; C^{2+\beta}(\bar{\Omega}))\}.$$

**Lemma 7.3.5.** *The solutions of (7.3.12) satisfy*

$$\begin{aligned} \left( \inf_{x \in \Omega} \rho(0, x) \right) \exp \left( - \int_0^t \|\operatorname{div} \mathbf{v}_n(s)\|_{L^\infty(\Omega)} \, ds \right) &\leq \rho(t, x) \\ &\leq \left( \sup_{x \in \Omega} \rho(0, x) \right) \exp \left( \int_0^t \|\operatorname{div} \mathbf{v}_n(s)\|_{L^\infty(\Omega)} \, ds \right), \end{aligned}$$

for all  $t \in [0, T]$  and  $x \in \Omega$ .

Using the latter lemma, if the velocity field is in  $W^{1,\infty}$ , the density is bounded from below by a positive constant (provided the initial condition is positive). We now focus on Equation (7.3.14).

**Proposition 7.3.6.** *Let  $\rho$  be given such that  $\rho \in C(0, T; C^2(\bar{\Omega}))$  and  $\rho \geq \underline{\rho} > 0$ . Then Equation (7.3.14) with Neumann boundary conditions (7.3.15) admits a strong solution. Moreover, the mapping  $\mathbf{v}_n \mapsto c[\mathbf{v}_n]$  takes bounded sets in the space  $C(0, T; C_0^2(\bar{\Omega}, \mathbb{R}^3))$  into bounded sets in the space*

$$W := \{c \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega))\}. \quad (7.3.17)$$

The existence of a strong solution is based on the remark that the highest order term of this equation is  $-\gamma \frac{b(c)}{\rho} \Delta^2 c$ . Using  $b(c), \rho \geq C > 0$  we obtain a fourth-order parabolic equation with smooth coefficients and with zero Neumann boundary conditions. Therefore, we can admit the global in time strong solution than can be achieved through a Galerkin scheme and we focus on the estimates (7.3.17). In the proof, we need the following two lemmas

**Lemma 7.3.7** (Lemma 3.1 in [169]). *Let  $\Omega \in \mathbb{R}^3$  be a bounded Lipschitz domain and let  $M_0 > 0$ ,  $K > 0$ . Assume that  $\rho$  is a nonnegative function such that*

$$0 < M_0 \leq \int_{\Omega} \rho \, dx, \quad \int_{\Omega} \rho^a \, dx \leq K, \quad \text{with } a > \frac{6}{5}.$$

*Then, there exists a positive constant  $C = C(M_0, K, a)$  such that the inequality*

$$\left\| \mathbf{u} - \frac{1}{|\Omega|} \int_{\Omega} \rho \mathbf{u} \right\|_{L^2(\Omega; \mathbb{R}^3)} \leq C \|\nabla \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^{3 \times 3})},$$

*holds for any  $\mathbf{u} \in W^{1,2}(\Omega; \mathbb{R}^3)$ .*

**Lemma 7.3.8** (Theorem 10.17 in [171]). *Let  $\Omega \subset \mathbb{R}^3$  be a bounded Lipschitz domain, and let  $1 < p < +\infty$ ,  $M_0 > 0$ ,  $K > 0$ ,  $a > 1$ . Then there exists a positive constant  $C = C(p, M_0, K, a)$  such that the inequality*

$$\|\mathbf{u}\|_{W^{1,p}(\Omega; \mathbb{R}^3)} \leq C \left( \|\nabla \mathbf{u} + \nabla^T \mathbf{u} - \frac{2}{3} \operatorname{div} \mathbf{u} \mathbb{I}\|_{L^p(\Omega; \mathbb{R}^{3 \times 3})} + \int_{\Omega} \rho |\mathbf{u}| \, dx \right),$$

*holds for any  $\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^3)$  and any non-negative function  $\rho$  such that*

$$0 < M_0 \leq \int_{\Omega} \rho \, dx, \quad \int_{\Omega} \rho^a \, dx \leq K.$$

*Proof of Proposition 7.3.6.* We admit the existence of solutions and focus on a priori estimates. We multiply Equation (7.3.14) by  $-\Delta c$ . Using the boundary conditions and integrating in space yields

$$\begin{aligned} & \partial_t \int_{\Omega} \frac{|\nabla c|^2}{2} dx + \gamma \int_{\Omega} b(c) \left| \nabla \left( \frac{\Delta c}{\rho} \right) \right|^2 dx \\ &= \int_{\Omega} \frac{1}{2} \operatorname{div}(\mathbf{v}_n) |\nabla c|^2 - \nabla \mathbf{v}_n : \nabla c \otimes \nabla c dx + \int_{\Omega} b(c) \nabla \left( \frac{\partial \psi_0}{\partial c} \right) \cdot \nabla \left( \frac{\Delta c}{\rho} \right) dx - \int_{\Omega} \frac{F_c}{\rho} \Delta c. \end{aligned}$$

Here, we have also used the formula (7.3.7). We use the  $L^\infty$  bounds on  $\mathbf{v}_n$ ,  $\operatorname{div}(\mathbf{v}_n)$ ,  $b(c)$ ,  $\rho$  the fact that  $\frac{F_c}{\rho}$  is also bounded in  $L^\infty$ , properties on  $\partial_c \psi_0$  (7.3.3), and obtain

$$\partial_t \int_{\Omega} \frac{|\nabla c|^2}{2} dx + \gamma \int_{\Omega} b(c) \left| \nabla \left( \frac{\Delta c}{\rho} \right) \right|^2 dx \leq C \int_{\Omega} |\nabla c|^2 dx + C \int_{\Omega} \left| \nabla \frac{\Delta c}{\rho} \right| dx + C \int_{\Omega} |\Delta c|.$$

We want to control the last term on the right-hand side. We use Lemma 7.3.7 with  $\mathbf{u} = \frac{\Delta c}{\rho}(1, 0, 0)^T$  and obtain, together with Neumann boundary conditions on  $c$ ,

$$\left\| \frac{\Delta c}{\rho} \right\|_{L^2(\Omega)} \leq C \left\| \nabla \left( \frac{\Delta c}{\rho} \right) \right\|_{L^2(\Omega; \mathbb{R}^3)}. \quad (7.3.18)$$

Then, writing  $\Delta c = \rho \frac{\Delta c}{\rho}$  and using the  $L^\infty$  bound on  $\rho$ ,

$$\int_{\Omega} |\Delta c| \leq C \left\| \nabla \left( \frac{\Delta c}{\rho} \right) \right\|_{L^2(\Omega; \mathbb{R}^3)}.$$

Finally, using Young's inequality and Gronwall's lemma, we obtain

$$\sup_{t \in (0, T)} \int_{\Omega} |\nabla c|^2 dx + \gamma \int_0^T \int_{\Omega} b(c) \left| \nabla \left( \frac{\Delta c}{\rho} \right) \right|^2 dx \leq C. \quad (7.3.19)$$

With Lemma 7.3.7 (and integrating the equation on  $\rho c$  using also the boundary conditions) we obtain the bound

$$c \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega)). \quad (7.3.20)$$

□

Having defined  $\rho$  and  $c$ , we now solve Equation (7.3.13) with a fixed point argument. We define the operator

$$\mathcal{M}[\rho] : X_n \rightarrow X_n^*, \quad \langle \mathcal{M}[\rho] \mathbf{v}, \mathbf{w} \rangle := \int_{\Omega} \rho \mathbf{v} \cdot \mathbf{w} dx, \quad \mathbf{v}, \mathbf{w} \in X_n.$$

This operator ([166])  $\mathcal{M}[\rho]$  is invertible, and

$$\|\mathcal{M}^{-1}[\rho]\|_{\mathcal{L}(X_n^*; X_n)} \leq \frac{1}{\inf_{\Omega} \rho}, \quad \|\mathcal{M}^{-1}[\rho_1] - \mathcal{M}^{-1}[\rho_2]\|_{\mathcal{L}(X_n^*; X_n)} \leq C(n, \underline{\rho}) \|\rho_1 - \rho_2\|_{L^1(\Omega)}, \quad (7.3.21)$$

for any  $\rho_1, \rho_2 \geq \rho$ . Finally, Equation (7.3.13) can be reformulated as

$$\mathbf{v}_n(t) = \mathcal{M}^{-1}[\rho(t)] \left( \mathbf{m}_0^* + \int_0^t \mathcal{N}[\mathbf{v}_n(s), \rho(s), c(s)] ds \right), \quad (7.3.22)$$

with

$$\langle \mathbf{m}_0^*, \eta \rangle = \int_{\Omega} \mathbf{m}_0 \cdot \eta dx,$$

and

$$\begin{aligned} \langle \mathcal{N}[\mathbf{v}_n, \rho, c], \eta \rangle = \int_{\Omega} \left( \rho \mathbf{v}_n \otimes \mathbf{v}_n - \mathbb{T} - \frac{\gamma}{2} |\nabla c|^2 \mathbb{I} + \gamma \nabla c \otimes \nabla c \right) : \nabla \eta + p(\rho, c) \operatorname{div}(\eta) \\ - (\varepsilon \nabla \mathbf{v}_n \nabla \rho + \kappa(\rho, c) \mathbf{v}_n) \cdot \eta dx. \end{aligned}$$

To prove that Equation (7.3.22) has a solution, we apply Schauder's fixed-point theorem in a short time interval  $[0, T(n)]$ . Then, we need uniform estimates to iterate the procedure.

**Lemma 7.3.9** (Schauder Fixed Point Theorem). *Let  $X$  be a Hausdorff topological vector space and  $S$  be a closed, bounded, convex, and non-empty subset of  $X$ . Then, any compact operator  $A : S \rightarrow S$  has at least one fixed point.*

With notation of the lemma 7.3.9, we call  $A$  the operator from Equation (7.3.22) and  $S = B(\mathbf{u}_{0,n})$  the unit ball with center  $\mathbf{u}_{0,n}$  in  $C([0, T]; X_n)$ ,  $\mathbf{u}_{0,n}$  is defined by

$$\int_{\Omega} \rho_0 \mathbf{u}_{0,n} \cdot \eta dx = \int_{\Omega} \mathbf{m}_0 \cdot \eta dx, \quad \forall \eta \in X_n.$$

More precisely, we consider

$$\begin{aligned} A : S &\rightarrow C([0, T]; X_n), \\ \mathbf{u} &\mapsto \mathcal{M}^{-1}[\rho(t)] \left( \mathbf{m}_0^* + \int_0^t \mathcal{N}[\mathbf{u}(s), \rho(s), c(s)] ds \right). \end{aligned}$$

**Lemma 7.3.10.** *There exists a time  $T = T(n)$  small enough such that the operator  $A$  maps  $S$  into itself. Moreover, the mapping is continuous.*

*Démonstration.* By definition of  $A$  and  $\mathbf{m}_0^*$ , we need to prove that  $\|\mathcal{M}^{-1}[\rho(t)] \int_0^t \mathcal{N}(s) ds\|_{C(0, T; X_n)} \leq 1$ . With properties (7.3.21), it is sufficient to prove that there exists a final time  $T$  small enough such that

$$\left\| \int_0^t \mathcal{N}(s) ds \right\|_{C(0, T; X_n^*)} \leq \inf_{\Omega_T} \rho.$$

Note that the infimum of  $\rho$  needs to be taken over the set  $\Omega_T = (0, T) \times \Omega$  as  $\rho$  depends on time. But, since we only consider small times, using Lemma 7.3.5 we see that this infimum is bounded by below. More precisely, for every  $T_0$ , there exists  $C(T_0) > 0$  such that for every  $T \leq T_0$ ,  $\inf_{\Omega_T} \rho \geq C(T_0)$ . We recall that  $X_n \subset C^2(\Omega; \mathbb{R}^3)$  is finite-dimensional. With the definition of the

tensor  $\mathbb{T}$  and the pressure  $p(\rho, c)$  given by (7.3.11)-(7.2.3) we estimate by Hölder's inequality :

$$\begin{aligned} & \int_0^t \int_{\Omega} (\rho \mathbf{u} \otimes \mathbf{u} - \mathbb{T} - \frac{\gamma}{2} |\nabla c|^2 \mathbb{I} + \gamma \nabla c \otimes \nabla c) : \nabla \eta + p(\rho, c) \operatorname{div}(\eta) - (\varepsilon \nabla \mathbf{u} \nabla \rho + \kappa(\rho, c) \mathbf{u}) \cdot \eta \, dx \, ds \\ & \leq C(\sqrt{T} + T)(\|\eta\|_{X_n} + \|\nabla \eta\|_{X_n})(\|\rho\|_{L^\infty} \|\mathbf{u}\|_{L^\infty}^2 + C\|\nu(c)\|_{L^2} \|\nabla \mathbf{u}\|_{L^\infty} + C\|\nabla c\|_{L^4}^2 + \|\rho\|_{L^\infty}^a \\ & + \|\rho\|_{L^\infty} \|H(c)\|_{L^\infty} + \varepsilon \|\mathbf{u}\|_{X_n} \|\nabla \rho\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty} \|\kappa(\rho, c)\|_{L^2}). \end{aligned}$$

Using assumptions of the subsection 7.2.1 and Propositions 7.3.4-7.3.6, we prove that all the quantities on the right-hand side are bounded, with a bound that may depend on  $n$ , except  $\|\nabla c\|_{L^4}$  which needs an argument. Note that from (7.3.17), we deduce  $\nabla c$  is bounded in  $L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$  (by a constant which depends on  $\rho$ , and also on  $\|\mathbf{u}\|_{L^\infty}, \|\nabla \mathbf{u}\|_{L^\infty}$ ). By Sobolev embedding with  $d = 3$ ,  $\nabla c$  is bounded in  $L^2(0, T; L^\infty(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ . By Hölder inequality (or interpolation), we obtain an  $L^4(0, T; L^4(\Omega))$  bound :  $\|\nabla c\|_{L^4 L^4}^4 \leq \|\nabla c\|_{L^\infty L^2}^2 \|\nabla c\|_{L^2 L^\infty}^2$ . With the previous estimates, and for  $T$  small enough, we obtain the result.  $\square$

**Lemma 7.3.11.** *The image of  $S$  under  $A$  is in fact a compact subset of  $S$ . Therefore,  $A$  admits a fixed point.*

*Démonstration.* We want to apply the Arzelà-Ascoli theorem to deduce the relative compactness of  $A(S)$ . From the previous computation, and using the fact that  $X_n$  is finite-dimensional, we can prove that  $A(S)$  is pointwise relatively compact. It remains to prove its equicontinuity. We want to estimate for  $t' \leq t$  the  $X_n$  norm of  $\mathcal{M}^{-1}[\rho(t)] \left( \mathbf{m}_0^* + \int_0^t \mathcal{N}[\mathbf{u}(s), \rho(s), c(s)] \, ds \right) - \mathcal{M}^{-1}[\rho(t')] \left( \mathbf{m}_0^* + \int_0^{t'} \mathcal{N}[\mathbf{u}(s), \rho(s), c(s)] \, ds \right)$ . For simplicity, we write  $\mathcal{N}(s) := \mathcal{N}[\mathbf{u}(s), \rho(s), c(s)]$ , and rewrite the previous difference as

$$\mathcal{M}^{-1}[\rho(t) - \rho(t')] \left( \mathbf{m}_0^* + \int_0^t \mathcal{N}(s) \, ds \right) + \mathcal{M}^{-1}[\rho(t')] \left( \mathbf{m}_0^* + \int_{t'}^t \mathcal{N}(s) \, ds \right).$$

For the first term, we use (7.3.21) and the Hölder continuity of  $\rho$  given by Proposition 7.3.4. For the second term, we repeat the computations in the proof of Lemma 7.3.10. This ends the result.  $\square$

We have the existence of a small interval  $[0, T(n)]$ . To iterate the procedure in order to prove that  $T(n) = T$ , it remains to find a bound on  $\mathbf{v}_n$  independent of  $T(n)$ .

**Lemma 7.3.12.**  *$\mathbf{v}_n$  is bounded in  $X_n$  independently of  $T(n)$ .*

*Démonstration.* Note that we do not ask for a bound independent of  $n$  but only of  $T(n)$  since we use in the proof the fact that  $X_n$  is finite-dimensional. The proof uses the energy structure of the equation. We differentiate Equation (7.3.13) in time and take  $\eta = \mathbf{v}_n$  as a test function. This yields

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \rho \frac{|\mathbf{v}_n|^2}{2} \, dx + \frac{1}{2} \int_{\Omega} (\partial_t \rho + \operatorname{div}(\rho \mathbf{v}_n)) |\mathbf{v}_n|^2 \, dx - \int_{\Omega} p(\rho, c) \operatorname{div}(\mathbf{v}_n) \, dx - \frac{\varepsilon}{2} \int_{\Omega} \Delta \rho |\mathbf{v}_n|^2 \, dx \\ & + \int_{\Omega} \mathbb{T} : \nabla \mathbf{v}_n \, dx + \gamma \int_{\Omega} \left( \frac{1}{2} |\nabla c|^2 \mathbb{I} - (\nabla c \otimes \nabla c) \right) : \nabla \mathbf{v}_n \, dx + \int_{\Omega} \kappa(\rho, c) |\mathbf{v}_n|^2 \, dx = 0. \quad (7.3.23) \end{aligned}$$

Here we used

$$\begin{aligned}\int_{\Omega} \partial_t(\rho \mathbf{v}_n) \cdot \mathbf{v}_n &= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\mathbf{v}_n|^2 dx + \frac{1}{2} \int_{\Omega} \partial_t \rho |\mathbf{v}_n|^2 dx, \\ \int_{\Omega} \operatorname{div}(\rho \mathbf{v}_n \otimes \mathbf{v}_n) \cdot \mathbf{v}_n dx &= \frac{1}{2} \int_{\Omega} \operatorname{div}(\rho \mathbf{v}_n) |\mathbf{v}_n|^2 dx, \\ \varepsilon \int_{\Omega} (\nabla \mathbf{v}_n \nabla \rho) \cdot \mathbf{v}_n dx &= -\frac{\varepsilon}{2} \int_{\Omega} \Delta \rho |\mathbf{v}_n|^2 dx.\end{aligned}$$

With (7.3.12), we see that (7.3.23) reads

$$\begin{aligned}\frac{d}{dt} \int_{\Omega} \rho \frac{|\mathbf{v}_n|^2}{2} dx - \int_{\Omega} p(\rho, c) \operatorname{div}(\mathbf{v}_n) dx + \int_{\Omega} \mathbb{T} : \nabla \mathbf{v}_n dx \\ + \gamma \int_{\Omega} \left( \frac{1}{2} |\nabla c|^2 \mathbb{I} - (\nabla c \otimes \nabla c) \right) : \nabla \mathbf{v}_n dx + \int_{\Omega} \kappa(\rho, c) |\mathbf{v}_n|^2 dx = 0.\end{aligned}\quad (7.3.24)$$

Now as in (7.3.10), we obtain with the artificial viscosity

$$\begin{aligned}\partial_t(\rho \psi_0) + \operatorname{div}(\rho \psi_0 \mathbf{v}_n) + p \operatorname{div}(\mathbf{v}_n) - \psi_0 \varepsilon \Delta \rho - \varepsilon \rho \frac{\partial \psi_0}{\partial \rho} \Delta \rho = \operatorname{div}(b(c) \nabla \mu) \mu \\ + \operatorname{div}(\partial_t c \nabla c) - \partial_t \left( \frac{|\nabla c|^2}{2} \right) + \gamma \Delta c \mathbf{v}_n \cdot \nabla c + \mu F_c.\end{aligned}$$

Integrating this equation in space, and summing with (7.3.24), we obtain

$$\begin{aligned}\frac{d}{dt} \int_{\Omega} \rho \left( \frac{|\mathbf{v}_n|^2}{2} + \psi_0 \right) + \gamma \int_{\Omega} \frac{|\nabla c|^2}{2} dx + \varepsilon \int_{\Omega} \nabla \left( \psi_0 + \rho \frac{\partial \psi_0}{\partial \rho} \right) \cdot \nabla \rho dx \\ + \int_{\Omega} \mathbb{T} : \nabla \mathbf{v}_n dx + \int_{\Omega} b(c) |\nabla \mu|^2 dx + \int_{\Omega} \kappa(\rho, c) |\mathbf{v}_n|^2 dx = \int_{\Omega} \mu F_c dx.\end{aligned}\quad (7.3.25)$$

By definition of  $\psi_0$ , we obtain

$$\begin{aligned}\varepsilon \int_{\Omega} \nabla \left( \psi_0 + \rho \frac{\partial \psi_0}{\partial \rho} \right) \cdot \nabla \rho dx = \varepsilon \int_{\Omega} \left( ((a-1) + (a-1)^2) \rho^{a-2} + \frac{H(c)}{\rho} \right) |\nabla \rho|^2 dx \\ + \varepsilon \int_{\Omega} (H'(c)(\log(\rho) + 1) + Q'(c)) \nabla c \cdot \nabla \rho dx.\end{aligned}$$

Therefore, the energy reads

$$\begin{aligned}\frac{d}{dt} \int_{\Omega} \rho \left( \frac{|\mathbf{v}_n|^2}{2} + \psi_0 \right) + \gamma \int_{\Omega} \frac{|\nabla c|^2}{2} dx + \varepsilon \int_{\Omega} \left( ((a-1) + (a-1)^2) \rho^{a-2} + \frac{H(c)}{\rho} \right) |\nabla \rho|^2 dx \\ + \int_{\Omega} \mathbb{T} : \nabla \mathbf{v}_n dx + \int_{\Omega} b(c) |\nabla \mu|^2 dx + \int_{\Omega} \kappa(\rho, c) |\mathbf{v}_n|^2 dx = \int_{\Omega} \mu F_c dx \\ - \varepsilon \int_{\Omega} (H'(c)(\log(\rho) + 1) + Q'(c)) \nabla c \cdot \nabla \rho dx.\end{aligned}\quad (7.3.26)$$

We need to prove that the right-hand side can be controlled in term of the left-hand side to obtain estimates. For the first term on the right-hand side, we treat it as in the proof of Proposition 7.3.1.

For the second term, we know by assumption on  $H$  and  $Q$ , and the fact that  $(\log(\rho) + 1)^2$  is bounded by a constant times  $\frac{1}{\rho} + (a-1)\rho^{a-2}$  that it can be bounded in terms of the left-hand side. Note that we used the hypothesis  $|Q'(c)| \leq C$ . This is based on the fact that  $Q$  is in fact  $Q_{\varepsilon_Q}$  so that we have  $|Q'(c)| \leq C(\frac{1}{\varepsilon_Q})$  with a constant that blows up when  $\varepsilon_Q$  is sent to 0. As we intend to send  $\varepsilon_Q \rightarrow 0$  in the next step, it is important to notice that we can still manage to have this energy inequality since in fact the term  $\varepsilon \int_{\Omega} Q'(c) \nabla c \cdot \nabla \rho \, dx$  can be estimated by  $\frac{\varepsilon}{4} \int_{\Omega} \left( ((a-1) + (a-1)^2)\rho^{a-2} + \frac{H(c)}{\rho} \right) |\nabla \rho|^2 \, dx$  and  $\int_{\Omega} \varepsilon C(\frac{1}{\varepsilon_Q}) |\nabla c|^2 \, dx$ . Since  $\varepsilon$  will be sent to 0 before  $\varepsilon_Q$ , the energy inequality will still hold independently of  $\varepsilon_Q$  in the limit  $\varepsilon \rightarrow 0$ . With Gronwall's lemma, and properties of the tensor  $\mathbb{T}$ , we deduce that  $\mathbf{v}_n$  is bounded in  $L^2(0, T(n); H^1(\Omega; \mathbb{R}^3))$  independently of  $T(n)$ . Also, the previous bounds do not depend on  $n$ . Since all the norms are equivalent, it is also bounded in  $L^1(0, T(n); W^{1,\infty}(\Omega, \mathbb{R}^3))$ . Therefore, we can apply the maximum principle stated in Lemma 7.3.5, and obtain that the density  $\rho$  is bounded from below by a constant independent of  $T(n)$ . Then, using once again the energy inequality, we obtain that  $\mathbf{v}_n$  is bounded uniformly in time in  $L^2(\Omega; \mathbb{R}^3)$ . This procedure can be repeated for every final time  $T$ .  $\square$

Finally, we are left with the following proposition

**Proposition 7.3.13.** *For any fixed  $n$  and  $T$ , there exists a solution  $(\rho, c, \mathbf{v}_n)$  defined on  $(0, T)$  (with appropriate regularity) to (7.3.12)-(7.3.14)-(7.3.13) subject to boundary conditions (7.3.15) and initial conditions (7.3.12). Moreover, this solution satisfies the energy dissipation inequality*

$$E(t) + \varepsilon \int_{\Omega_t} \left( (a + a^2)\rho^{a-1} + \frac{H(c)}{\rho} \right) |\nabla \rho|^2 \, dx \, dt + \int_{\Omega_t} \mathbb{T} : \nabla \mathbf{v}_n \, dx \, dt + \int_{\Omega_t} b(c) |\nabla \mu|^2 \, dx \, dt + \int_{\Omega_t} \kappa(\rho, c) |\mathbf{v}_n|^2 \, dx \, dt \leq C + CE(0), \quad (7.3.27)$$

where

$$E(t) = \int_{\Omega} \rho \left( \frac{|\mathbf{v}_n|^2}{2} + \psi_0 \right) + \gamma \frac{|\nabla c|^2}{2} \, dx,$$

and with a constant  $C = C\left(1, \frac{\varepsilon}{\varepsilon_Q}\right)$  that does not depend on  $n$ .

Now, we need to find estimates, independent of  $n$ , to pass to the limit  $n \rightarrow \infty$ . Since  $\rho$  and  $c$  depend on  $n$ , we write  $\rho_n$  and  $c_n$  from now on.

**Proposition 7.3.14.** *We have the following estimates uniformly in  $n$  and  $\varepsilon$  :*

- (A1)  $\{\rho_n \psi_0\}$  in  $L^\infty(0, T; L^1(\Omega))$ ,
- (A2)  $\{\rho_n\}$  in  $L^\infty(0, T; L^a(\Omega))$ ,
- (A3)  $\{\mathbb{T} : \nabla \mathbf{v}_n\}$  in  $L^1(0, T; L^1(\Omega))$ ,
- (A4)  $\{\sqrt{\rho_n} \mathbf{v}_n\}$  in  $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$ ,
- (A5)  $\{\sqrt{b(c_n)} \nabla \mu_n\}$  in  $L^2(0, T; L^2(\Omega; \mathbb{R}^3))$ ,
- (A6)  $\{\mathbf{v}_n\}$  in  $L^2(0, T; H_0^1(\Omega; \mathbb{R}^3))$ ,
- (A7)  $\{\sqrt{\varepsilon} \nabla \rho_n\}$  in  $L^2(0, T; L^2(\Omega))$ ,
- (A8)  $\{c_n\}$  in  $L^\infty(0, T; H^1(\Omega))$ ,
- (A9)  $\{\rho_n \partial_c \psi_0\}$  in  $L^\infty(0, T; L^r(\Omega))$  for  $r < \frac{6a}{6+a}$ ,
- (A10)  $\{\mu_n\}$  in  $L^2(0, T; H^1(\Omega))$ ,

- (A11)  $\{\rho_n \mu_n\}$  in  $L^2(0, T; L^{6a/(6+a)})$ ,
- (A12)  $\{c_n\}$  in  $L^2(0, T; W^{2,r}(\Omega)) \cap L^{2+\nu}(0, T; W^{1,2+\nu})$  for some  $\nu > 0$ ,
- (A13)  $\{\rho_n c_n\}$  in  $L^\infty(0, T; L^{\frac{6a}{6+a}}(\Omega))$ ,
- (A14)  $\{\rho_n c_n \mathbf{v}_n\}$  in  $L^2(0, T; L^{\frac{6a}{3+4a}}(\Omega))$ ,
- (A15)  $\{p(\rho_n, c_n)\}$  in  $L^{1+\tilde{\nu}}((0, T) \times \Omega)$  for some  $\tilde{\nu} > 0$ .

*Démonstration.* Estimates (A1)-(A2)-(A3)-(A4)-(A5) follow immediately from the energy equality (7.3.27). Estimate (A6) is the result of Lemma 7.3.8 and estimates (A2)-(A3)-(A4). To obtain estimate (A7), we multiply Equation (7.3.12) by  $\rho_n$ , and using integration by parts, we obtain

$$2\varepsilon \int_0^T \int_\Omega |\nabla \rho_n|^2 \, dx \, dt \leq \|\rho_0\|_{L^2(\Omega)}^2 + \|\rho_n\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\rho_n\|_{L^2(0,T;L^4(\Omega))}^2 \|\nabla \mathbf{v}_n\|_{L^2(0,T;L^2(\Omega)^d)}.$$

Using (A2) and (A6), we deduce (A7). To prove Estimate (A8), we first notice that equality (7.3.27) provides the uniform bound on  $\{\nabla c_n\}$  in  $L^2(0, T; L^2(\Omega))$ . To conclude with Lemma 7.3.7, we need to bound  $\int_\Omega \rho_n c_n$ . Combining Equations (7.3.12)-(7.3.14), we obtain

$$\partial_t(\rho_n c_n) + \operatorname{div}(\rho_n c_n \mathbf{v}_n) = -\varepsilon c \Delta \rho + \operatorname{div}(b(c) \nabla \mu) + F_c.$$

Integrating in space, using the boundary conditions, and Estimate (A7), the  $L^2$  bound on  $\{\nabla c_n\}$ , assumption 7.2.5 yields  $\{\int_\Omega \rho_n c_n\}$  is in  $L^\infty(0, T)$ . We deduce Estimate (A8). Estimate (A9) follows from the definition of  $\psi_0$  and Estimate (A1). Estimate (A10) follows from Estimates (A5)-(A9) and Lemma 7.3.7. Estimate (A11) follows from Estimates (A2)-(A10). Estimate (A12) is a consequence of Equation (7.1.3), the previous estimates and interpolation. The two next estimates are a consequence of the other estimates and Sobolev embeddings. Finally, the last estimate on the pressure can be adapted from [104, Subsection 2.5]. This estimate is useful when we obtain the convergence a.e. of  $\rho_n$  and  $c_n$  so we can obtain strong convergence of  $p(\rho_n, c_n)$  in  $L^1$  by Vitali's convergence theorem.  $\square$

From [166], we also obtain the following Proposition

**Proposition 7.3.15.** *There exists  $r > 1$  and  $p > 2$  such that*

$$\begin{aligned} \partial_t \rho_n, \Delta \rho_n & \text{ are bounded in } L^r((0, T) \times \Omega), \\ \nabla \rho_n & \text{ is bounded in } L^p(0, T; L^2(\Omega, \mathbb{R}^3)), \end{aligned}$$

*independently of  $n$  (but not independently of  $\varepsilon$ ).*

With all the previous bound, we can pass to the limit when  $n \rightarrow \infty$  and obtain the different equation and energy estimates in a weak formulation. Since the passage to the limit  $n \rightarrow \infty$  is simpler than the next passage  $\varepsilon \rightarrow 0$ , we only detail the latter. Indeed, as  $n \rightarrow \infty$  we can obtain easily strong convergence of  $\rho$  which helps a lot in the different limits. So we assume that we can pass to the limit and that the bounds obtained in Proposition 7.3.14 still hold independently of  $\varepsilon$ . It remains now to send  $\varepsilon$  to 0.

We recall the equations that we want to pass to the limit into :

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = \varepsilon \Delta \rho_\varepsilon, \tag{7.3.28}$$

$$\partial_t(\rho_\varepsilon c_\varepsilon) + \operatorname{div}(\rho_\varepsilon c_\varepsilon \mathbf{v}_\varepsilon) = -\varepsilon c_\varepsilon \Delta \rho_\varepsilon + \operatorname{div}(b(c_\varepsilon) \nabla \mu_\varepsilon) + F_{c_\varepsilon}, \tag{7.3.29}$$

and for every  $\eta$  (sufficiently regular)

$$\begin{aligned} & \int_{\Omega} \rho_{\varepsilon} \mathbf{v}_{\varepsilon}(t) \cdot \eta \, dx - \int_{\Omega} \mathbf{m}_0 \cdot \eta \, dx - \int_0^t \int_{\Omega} \rho_{\varepsilon} \mathbf{v}_{\varepsilon} \otimes \mathbf{v}_{\varepsilon} : \nabla_x \eta \, dx \, ds - \int_0^t \int_{\Omega} p(\rho_{\varepsilon}, c_{\varepsilon}) \operatorname{div}(\eta) \, dx \, ds \\ & + \varepsilon \int_0^t \int_{\Omega} (\nabla \mathbf{v}_{\varepsilon} \nabla \rho_{\varepsilon}) \cdot \eta \, dx \, ds + \int_0^t \int_{\Omega} \mathbb{T}_{\varepsilon} : \nabla \eta \, dx \, ds + \gamma \int_0^t \int_{\Omega} \left( \frac{1}{2} |\nabla c_{\varepsilon}|^2 \mathbb{I} - (\nabla c_{\varepsilon} \otimes \nabla c_{\varepsilon}) \right) : \nabla \eta \, dx \, ds \\ & \quad + \int_{\Omega} \int_0^t \kappa(c_{\varepsilon}) \mathbf{v}_{\varepsilon} \cdot \eta \, dx \, ds = 0. \quad (7.3.30) \end{aligned}$$

Using Proposition 7.3.14, which yields uniform estimates in  $\varepsilon$ , we pass to the limit in the previous equations. The difficult terms are the one involving nonlinear combinations. Indeed, it is not clear that we can obtain strong convergence of  $\rho_{\varepsilon}$  as we have no estimates on higher order derivatives. We use the following lemma, see [256].

**Lemma 7.3.16.** *Let  $g_n, h_n$  converge weakly to  $g, h$  respectively in  $L^{p_1}(0, T; L^{p_2}(\Omega))$ ,  $L^{q_1}(0, T; L^{q_2}(\Omega))$  where  $1 \leq p_1, p_2 \leq +\infty$  and*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

*We assume in addition that*

$$\frac{\partial g_n}{\partial t} \quad \text{is bounded in } L^1(0, T; W^{-m, 1}(\Omega)) \text{ for some } m \geq 0 \text{ independent of } n, \quad (7.3.31)$$

*and*

$$\|h_n - h_n(t, \cdot + \xi)\|_{L^{q_1}(0, T; L^{q_2}(\Omega))} \rightarrow 0 \quad \text{as } |\xi| \rightarrow 0, \text{ uniformly in } n. \quad (7.3.32)$$

*Then,  $g_n h_n$  converges to  $gh$  in the sense of distributions.*

**Remark 7.3.17.** This lemma admits many variants, and it is possible to identify the weak limit of the products with lower regularity, we refer for instance to [280].

We want to apply the previous lemma to the terms  $\rho_{\varepsilon} \mathbf{v}_{\varepsilon}$ ,  $\rho_{\varepsilon} c_{\varepsilon}$ ,  $\rho_{\varepsilon} \mu_{\varepsilon}$ ,  $\rho_{\varepsilon} c_{\varepsilon}^2$ ,  $\rho_{\varepsilon} \mathbf{v}_{\varepsilon}$ ,  $\rho_{\varepsilon} \mathbf{v}_{\varepsilon} \otimes \mathbf{v}_{\varepsilon}$ ,  $\rho_{\varepsilon} \mathbf{v}_{\varepsilon} c_{\varepsilon}$ . We admit that  $\frac{\partial \rho_{\varepsilon}}{\partial t}$ ,  $\frac{\partial \rho_{\varepsilon} \mathbf{v}_{\varepsilon}}{\partial t}$  and  $\frac{\partial \rho_{\varepsilon} c_{\varepsilon}}{\partial t}$  satisfy (7.3.31) by using Proposition 7.3.14 and Equations (7.3.28)-(7.3.29)-(7.3.30). The compactness in space required in (7.3.32) also uses Proposition 7.3.14. We refer also to [104, Subsection 3.1] for similar results. The terms  $\varepsilon c_{\varepsilon} \Delta \rho_{\varepsilon}$  and  $\varepsilon \int_0^t \int_{\Omega} (\nabla \mathbf{v}_{\varepsilon} \nabla \rho) \cdot \eta \, dx \, ds$  converge to 0 (the first one in the distributional sense) thanks to estimates (A7)-(A8).

It remains to pass to the limit in (i.e identifying the weak limits)

$$\begin{aligned} & p(\rho_{\varepsilon}, c_{\varepsilon}), \quad \frac{1}{2} |\nabla c_{\varepsilon}|^2, \quad \nabla c_{\varepsilon} \otimes \nabla c_{\varepsilon}, \\ & b(c_{\varepsilon}) \nabla \mu_{\varepsilon}, \quad F_{c_{\varepsilon}}(\rho_{\varepsilon}, c_{\varepsilon}), \quad \rho_{\varepsilon} \partial_c \psi_0. \end{aligned}$$

The convergence of the last term is used to identify  $\rho \mu$ . To prove the previous convergences, we need to prove strong compactness in  $L^2$  of  $c_{\varepsilon}$ ,  $\nabla c_{\varepsilon}$  and convergence a.e. of  $\rho_{\varepsilon}$  to use Vitali's convergence theorem. But they follow from the arguments in [5] and [104, Section 3.3 and 3.4]. We obtain



**Lemma 7.3.18.** *Up to a subsequence (not relabeled),*

$$\rho_\varepsilon \rightarrow \rho \text{ a.e.} \quad (7.3.33)$$

$$c_\varepsilon \rightarrow c \text{ a.e. and strongly in } L^2(0, T; L^2(\Omega)) \quad (7.3.34)$$

$$\nabla c_\varepsilon \rightarrow \nabla c \text{ a.e. and strongly in } L^2(0, T; L^2(\Omega)) \quad (7.3.35)$$

Altogether, we can pass to the limit in every term of the equations :

- $p(\rho_\varepsilon, c_\varepsilon)$  : we use (7.3.33), (7.3.34) and (A15)
- $\frac{1}{2}|\nabla c_\varepsilon|^2$  and  $\nabla c_\varepsilon \otimes \nabla c_\varepsilon$  : (7.3.35)
- $b(c_\varepsilon)\nabla \mu_\varepsilon$  : (7.3.34) and (A10)
- $F_{c_\varepsilon}(\rho_\varepsilon, c_\varepsilon)$  : (7.3.33), (7.3.34) and (7.2.5)
- $\rho_\varepsilon \partial_c \psi_0$  : (7.3.33), (7.3.34) and (A9).

This concludes the argument.

### 7.3.3 Sending $\varepsilon_Q \rightarrow 0$

The last step in our proof is to let  $\varepsilon_Q$  vanishes and recover the existence of weak solutions for the double well potential  $Q(c) = \frac{1}{4}c^2(1-c)^2$ . Since we have the energy estimates from before, that still hold by properties of the weak convergence, the work is essentially the same but we have to be careful about two points. The first one is to indeed have an energy estimate independent of  $\varepsilon_Q$ . We discussed this point after Equation (7.3.26) and, hence, we do not repeat it here. The second point are the estimates obtained in Proposition 7.3.14. However, the estimates are essentially the same, except for estimate (A9) (that is the only one containing  $Q$ ) which becomes

$$\{\rho \partial_c \psi_0\} \text{ in } L^\infty(0, T; L^{\frac{2a}{a+2}}(\Omega)). \quad (7.3.36)$$

This can be proved knowing that, when  $\varepsilon_Q \approx 0$ , we have that for  $c$  large  $\rho Q'_{\varepsilon_Q}(c) \approx \rho c^3$ , and we use estimates (A2)-(A8). Altogether, the reasoning to pass to the limit is the same and we conclude.

## 7.4 Numerical scheme for the G-NSCH model

We propose a numerical scheme for the G-NSCH model (7.1.1)–(7.1.4) subjected to periodic boundary conditions.

We combine ideas from the numerical scheme for the variant of the compressible NSCH system in [210] and fast structure-preserving scheme for degenerate parabolic equations [221, 222]. Namely, we adapt the relaxation [232] of the Navier-stokes part as used in [210]. The part of the scheme for the Cahn-Hilliard part of the system is designed using the GSAV method. More precisely, a variant used for degenerate parabolic models that preserves the physical bounds of the solution [221, 222].

Indeed, we expect that the volume fraction  $c$  remains within the physically (or biologically) relevant bounds  $c \in (0, 1)$ . Thus, following [221, 222], we construct the invertible mapping  $T : \mathbb{R} \rightarrow (0, 1)$ , with  $c = T(v)$ , transforming Equations (7.1.2)–(7.1.3) into

$$\begin{aligned} \rho (\partial_t v + (\mathbf{v} \cdot \nabla) v) &= \frac{1}{T'(v)} (\operatorname{div}(b(c)\nabla \mu) + F_c), \\ \rho \mu &= -\gamma T'(v)\Delta v - \gamma T''(v)|\nabla v|^2 + \rho \frac{\partial \psi_0}{\partial c}. \end{aligned} \quad (7.4.1)$$

Following [221] and [222], we can choose

$$T(v) = \frac{1}{2} \tanh(v) + \frac{1}{2}, \text{ or } T(v) = \frac{1}{1 + \exp(-v)},$$

thus, preserving the bounds  $c \in (0, 1)$ .

The SAV method allows to solve efficiently (and also linearly) the nonlinear Cahn-Hilliard part while preserving the dissipation of a modified energy. In the following, we assume that it exists a positive constant  $\underline{C}$  such that the energy associated with the Cahn-Hilliard part, *i.e.*

$$E[t](\rho, c) = \int_{\Omega} \frac{\gamma}{2} |\nabla c|^2 + \rho \psi_0(\rho, c) = E_0[t] + E_1[t],$$

with  $E_1$  the nonlinear part of the energy, and  $E_0$  the linear part, is bounded from below, *i.e.*  $E_1 + \underline{C} \geq 1$

We define

$$r(t) = E(t) + C_0, \quad \text{with } C_0 = 2\underline{C} + \|E(\rho^0, c^0)\|_{L^\infty(\Omega)},$$

and apply the SAV method. System (7.4.1) becomes

$$\begin{aligned} \rho (\partial_t v + (\mathbf{v} \cdot \nabla) v) &= \frac{1}{T'(v)} (\operatorname{div}(b(c) \nabla \mu) + F_c), \\ \rho \mu &= -\gamma T'(v) \Delta v - \gamma T''(v) |\nabla v|^2 + \rho \frac{\partial \psi_0}{\partial c}, \\ \frac{dr}{dt} &= -\frac{r(t)}{E[t] + C_0} \int_{\Omega} b(c) |\nabla \mu|^2 - \mu F_c \, dx, \end{aligned} \quad (7.4.2)$$

One can easily see that the previous modifications do not change our system at the continuous level.

### 7.4.1 One-dimensional scheme

We consider our problem in a one-dimensional domain  $\Omega = (0, L)$ . Even though  $\mathbf{v}$  is now a scalar, we still denote it in bold font to not make the confusion with  $v$  from the transformation  $c = T(v)$ . As mentioned previously, we relax the Navier-Stokes part of our system. Namely, we introduce a relaxation parameter  $\iota \geq 0$  and write  $U = (\rho, \rho \mathbf{v})$ . We rewrite Equation (7.1.4) as

$$\begin{cases} \partial_t U + \partial_x V = G(U), \\ \partial_t V + A \partial_x U = -\frac{1}{\iota} (V - F(U)), \end{cases} \quad (7.4.3)$$

in which  $G(U) = (0, -\kappa \mathbf{v})$ ,  $F(U) = (\rho \mathbf{v}, \rho \mathbf{v}^2 + p - (\frac{4}{3} \nu(c) + \eta(c)) \partial_x \mathbf{v} + \frac{\gamma}{2} |\partial_x c|^2)$  and  $A = \operatorname{diag}(a_1, a_2)$  satisfying Liu's subcharacteristic condition

$$A \geq F'(U), \quad \forall U.$$

In what follows, and following [210], we use

$$a_1 = a_2 = \max \left\{ \sup \left( \mathbf{v} + \sqrt{\partial_{\rho p}} \right)^2, \sup \left( \mathbf{v} - \sqrt{\partial_{\rho p}} \right)^2 \right\}.$$

We discretize the domain using a set of  $N_x$  nodes located at the center of control volumes of size

$\Delta x$  such that  $\Omega = \bigcup_{j=0, \dots, N_x-1} [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ .

Our scheme follows the discrete set of equations

$$U_j^* = U_j^n, \quad (7.4.4)$$

$$V_j^* = V_j^n - \frac{\Delta t}{\iota} (V_j^* - F(U_j^*)), \quad (7.4.5)$$

$$U_j^{n+1} = U_j^* - \frac{\Delta t}{\Delta x} (V_{j+\frac{1}{2}}^* - V_{j-\frac{1}{2}}^*) + \Delta t G(U_j^{n+1}), \quad (7.4.6)$$

$$V_j^{n+1} = V_j^* - \frac{\Delta t}{\Delta x} A (U_{j+\frac{1}{2}}^* - U_{j-\frac{1}{2}}^*), \quad (7.4.7)$$

$$\rho_j^{n+1} T'(v_j^n) \left( \frac{\bar{v}_j^{n+1} - v_j^n}{\Delta t} + \mathbf{v}_j^{n+1} \cdot (\nabla \bar{v}^{n+1})_j \right) = g(c^n, \mu^{n+1}, \rho^{n+1})_j, \quad (7.4.8)$$

$$g(c^n, \mu^{n+1}, \rho^{n+1})_j = \left( \frac{1}{\Delta x} \left( (b(c^n) \nabla \mu^{n+1})_{j+\frac{1}{2}} - (b(c^n) \nabla \mu^{n+1})_{j-\frac{1}{2}} \right) \right) + F_c(\rho_j^n, c_j^n), \quad (7.4.9)$$

$$\rho_j^{n+1} \mu_j^{n+1} = (-\gamma T'(v_j^n) (\Delta \bar{v}^{n+1})_j - \gamma T''(v_j^n) (\nabla v^n)_j \cdot (\nabla \bar{v}^{n+1})_j) + \rho_j^{n+1} \left( \frac{\partial \psi_0}{\partial c} \right)_j^n, \quad (7.4.10)$$

$$\sum_j \Delta x T(\lambda \bar{v}_j^{n+1}) = \sum_j \Delta x c^0 + \sum_{r=1}^n \Delta t \sum_j \Delta x F_c(\rho_j^r, c_j^r), \quad (7.4.11)$$

$$\bar{c}^{n+1} = T(\lambda \bar{v}^{n+1}), \quad (7.4.12)$$

$$\begin{aligned} \frac{1}{\Delta t} (r^{n+1} - r^n) &= -\frac{r^{n+1}}{E(\bar{c}^{n+1}) + C_0} \Delta x \sum_j b(\bar{c}_j^{n+1}) |(\nabla \mu^{n+1})_j|^2 \\ &\quad + \frac{r^{n+1}}{E(\bar{c}^{n+1}) + C_0} \Delta x \sum_j \mu_j^{n+1} F_c(\rho_j^{n+1}, \bar{c}_j^{n+1}), \end{aligned} \quad (7.4.13)$$

$$\xi^{n+1} = \frac{r^{n+1}}{E(\bar{c}^{n+1}) + C_0}, \quad (7.4.14)$$

$$c_j^{n+1} = \nu^{n+1} \bar{c}_j^{n+1}, \quad \text{with } \nu^{n+1} = 1 - (1 - \xi^{n+1})^2, \quad (7.4.15)$$

$$v_j^{n+1} = \lambda \nu^{n+1} \bar{v}_j^{n+1}. \quad (7.4.16)$$

**Remark 7.4.1** (Computation of interface values). To obtain the interface values  $U_{j+\frac{1}{2}}^*, U_{j-\frac{1}{2}}^*$  and  $V_{j+\frac{1}{2}}^*, V_{j-\frac{1}{2}}^*$ , we use the upwind method, *i.e.*

$$U_{j+\frac{1}{2}} = \frac{1}{2} (U_j + U_{j+1}) - \frac{\sqrt{a_1}}{2} (V_{j+1} - V_j), \quad V_{j+\frac{1}{2}} = \frac{1}{2} (V_j + V_{j+1}) - \frac{1}{2\sqrt{a_2}} (U_{j+1} - U_j).$$

We also mention that similarly to [210], one can implement a MUSCL scheme (see *e.g.* [251]) to obtain a higher order reconstruction. The upwind method permits to rewrite Equations (7.4.6)–(7.4.7) as

$$U_j^{n+1} = U_j^* - \frac{\Delta t}{2\Delta x} (V_{j+1}^* - V_{j-1}^*) + \frac{\Delta t}{2\Delta x} \sqrt{a} (\delta_x^2 U_j^*) + \Delta t G(U_j^{n+1}), \quad (7.4.17)$$

$$V_j^{n+1} = V_j^* - \frac{a\Delta t}{2\Delta x} (U_{j+1}^* - U_{j-1}^*) + \frac{\Delta t}{2\Delta x} \sqrt{a} (\delta_x^2 V_j^*), \quad (7.4.18)$$

where we used the notation  $\delta_x^2 U = U_{j+1} - 2U_j + U_{j-1}$ . In Equations (7.4.17)–(7.4.18), we emphasize that  $U^* = U^n$  and  $V^* = V^n - \frac{\Delta t}{\iota} (V^* - F(U^n))$ .

**Remark 7.4.2** (Algorithm to compute the solution of the discrete equations' system). Equations (7.4.4) to (7.4.7) are solved from Equations (7.4.17)–(7.4.18), hence, a solution  $(U^{n+1}, V^{n+1})$  is computed just from vector computations. The coupling between Equation (7.4.8) and Equation (7.4.10) is also linear (nonlinear terms are taken at the previous time step to linearize the equations). We solve this coupled system using the GMRES algorithm but we emphasize that other iterative solver could work as long as they allow the matrix of the linear system to be non-symmetric. The coefficient  $\lambda$  is computed using an iterative method. Then, the discrete solution  $(\bar{\rho}^{n+1}, \mu^{n+1})$ , together with the coefficient  $\lambda$ , is used in Equation (7.4.13) to find  $r^{n+1}$  and, in Equation (7.4.14),  $\xi^{n+1}$ . At this point, we solve Equation (7.4.15) and (7.4.16) from the previous steps.

In the following, we use the notations,

$$\langle U, V \rangle = \Delta x \sum_j U_j V_j, \quad \text{and} \quad \|U\|^2 = \langle U, U \rangle.$$

We also use  $\Delta_{0,x} U := \frac{1}{2}(U_{j+1} - U_{j-1})$ .

Our numerical scheme possesses the following important properties :

**Proposition 7.4.3** (Energy stability, bounds and mass preserving). *Assuming the CFL-like condition  $\frac{\Delta t}{\Delta x} \sqrt{a_1} \leq 1$  and the condition*

$$\Delta t \leq C \frac{C_0}{E[\bar{c}^n]}, \quad (7.4.19)$$

*our numerical scheme satisfies the energy dissipation-like inequality*

$$\|\sqrt{a}U^{n+1}\|^2 + \|V^{n+1}\|^2 + r^{n+1} \leq \|\sqrt{a}U^n\|^2 + \|V^*\|^2 + C^{n+1}r^n, \quad (7.4.20)$$

where  $r^{n+1} \geq 0$  and

$$C^{n+1} = \frac{1}{1 + \frac{\Delta t}{E(\bar{c}^{n+1}) + C_0} \sum_{j=1}^{N_x} b(\bar{c}_j^{n+1}) |\nabla \mu_j^{n+1}|^2 - \mu_j^{n+1} F_c(\rho_j^{n+1}, \bar{c}_j^{n+1})}.$$

*The previous constant can be estimated only in terms of  $E[\bar{c}^n]$  and therefore do not depend on the step  $n + 1$ . Furthermore, the numerical scheme preserves the physically relevant bounds of the mass fraction, i.e.*

$$0 < c^{n+1} < 1.$$

**Remark 7.4.4.** Note that the constant  $C^{n+1}$  is smaller than 1 whenever the nonnegative part of the dissipation of the energy is greater than the increase of energy induced by the source term  $F_c$ . This of course satisfied when we have  $F_c = 0$  for instance.

*Démonstration.* We start with Equation (7.4.17), and using the definition of the function  $G(U_j^{n+1})$  as well as assuming  $\kappa(c) \geq 0$  (for  $c \in \mathbb{R}$ ), after taking the square on both sides, multiplying by

$\Delta x$  and summing over the nodes  $j = 0, \dots, N_x$ , we have

$$\begin{aligned} \|U^{n+1}\|^2 \leq & \|U^n\|^2 + \left(\frac{\Delta t}{2\Delta x}\right)^2 \|\Delta_{0,x} V^*\|^2 + \left(\frac{\Delta t\sqrt{a}}{2\Delta x}\right)^2 \|\delta_x^2 U^n\|^2 - \frac{\Delta t}{\Delta x} \langle \Delta_{0,x} V^*, U^n \rangle \\ & + \frac{\Delta t\sqrt{a}}{\Delta x} \langle U^n, \delta_x^2 U^n \rangle - \frac{\sqrt{a}\Delta t^2}{2\Delta x^2} \langle \Delta_{0,x} V^*, \delta_x^2 U^n \rangle. \end{aligned}$$

Repeating the same computations for Equation (7.4.18), we have

$$\begin{aligned} \|V^{n+1}\|^2 \leq & \|V^n\|^2 + \left(\frac{a\Delta t}{2\Delta x}\right)^2 \|\Delta_{0,x} U^n\|^2 + \left(\frac{\Delta t\sqrt{a}}{2\Delta x}\right)^2 \|\delta_x^2 V^*\|^2 - \frac{a\Delta t}{\Delta x} \langle \Delta_{0,x} U^n, V^* \rangle \\ & + \frac{\Delta t\sqrt{a}}{\Delta x} \langle U^*, \delta_x^2 V^* \rangle - \frac{a^{\frac{3}{2}}\Delta t^2}{2\Delta x^2} \langle \Delta_{0,x} U^n, \delta_x^2 V^* \rangle. \end{aligned}$$

At this point, the proof is similar to the proof of [210, Theorem 4.1] (these steps use the periodic boundary conditions and the summation by parts formula to cancel some terms when summing both of the previous equations together), to obtain for a constant  $C > 0$ ,

$$\|\sqrt{a}U^{n+1}\|^2 + \|V^{n+1}\|^2 \leq C (\|\sqrt{a}U^n\|^2 + \|V^*\|^2).$$

Then, for the Cahn-Hilliard part, we easily obtain from Equation (7.4.13)

$$r^{n+1} \left( 1 + \Delta t \frac{\Delta x \sum_j b(\bar{c}_j^{n+1}) |(\nabla \mu^{n+1})_j|^2 - \mu_j^{n+1} F_c(\rho_j^{n+1}, \bar{c}_j^{n+1})}{E[\bar{c}^{n+1}] + C_0} \right) = r^n.$$

Therefore, as long as

$$E(\bar{c}^{n+1}) + C_0 + \Delta t \left( \Delta x \sum_j b(\bar{c}_j^{n+1}) |(\nabla \mu^{n+1})_j|^2 - \mu_j^{n+1} F_c(\rho_j^{n+1}, \bar{c}_j^{n+1}) \right) \geq 0,$$

so does  $r^{n+1}$ . Assuming  $\|F_c\|_{L^\infty} < C$ , it remains to control the discrete  $L^1$  norm of  $\mu^{n+1}$ . Performing the same computations as in the proof of Proposition 7.3.1 in continuous case, it follows that

$$\left| \Delta x \sum_j \mu_j^{n+1} F_c(\rho_j^{n+1}, \bar{c}_j^{n+1}) \right| \leq C + CE[\bar{c}^n] + \frac{1}{2} \Delta x \sum_j b(\bar{c}_j^{n+1}) |(\nabla \mu^{n+1})_j|^2.$$

Of course, one first needs to prove that a discrete version of Lemma 7.3.7 holds. At the continuous level, this theorem is proved by contradiction using Rellich's theorem. Hence, a similar proof can be obtained at the discrete level, in the spirit of the Poincaré-Wirtinger inequality, see [163, Lemma 3.8, Remark 3.16]. Based on these evidences we use a discrete version of Lemma 7.3.7 to conclude that there exists  $C$  a universal constant such that provided  $r^n \geq 0$  and

$$\Delta t \leq C \frac{C_0}{E[\bar{c}^n]},$$

so does  $r^{n+1} \geq 0$ , and (7.4.20) follows.

Finally, from the definition of  $\xi^{n+1}$  and  $C_0$ , we have

$$0 < \xi^{n+1} < \frac{r^0}{E(\bar{c}^{n+1}) + C_0} \leq 2.$$

The bounds for the mass fraction  $c$  are ensured by the transformation  $T(v)$ . This finishes the proof.  $\square$

**Remark 7.4.5.** We observe during numerical simulations that the condition (7.4.19) is obtained for reasonably small  $\Delta t$ . We also note that if we do not consider any source term, *i.e.*  $F_c = 0$ , the scheme satisfies the dissipation relation

$$\|\sqrt{a}U^{n+1}\|^2 + \|V^{n+1}\|^2 + r^{n+1} \leq C (\|\sqrt{a}U^n\|^2 + \|V^*\|^2) + r^n,$$

with the stability condition

$$\frac{\Delta t}{\Delta x} \sqrt{a_1} \leq 1.$$

## 7.5 Numerical experiments

In this section, we use the assumptions on the functionals stated in the "Framework for numerical simulations" paragraph in subsection 7.2.1. Throughout this section we use the double-well logarithmic potential

$$\psi_{\text{mix}} = \frac{1}{2} (\alpha_1(1-c) \log(\rho(1-c)) + \alpha_2 c \log(\rho c)) - \frac{\theta}{2} (c - \frac{1}{2})^2 + k,$$

with  $k = 100$ , and  $\theta = 4$  ( $\alpha_1$  and  $\alpha_2$  are specified later). We also use a degenerate mobility, *i.e.*

$$b(c) = c(1-c).$$

We start by using the one-dimensional scheme (7.4.4)–(7.4.16) with no exchange term and friction, *i.e.*  $\kappa(\rho, c) = 0$  and  $F_c(\rho, c) = 0$ , and we verify that the scheme preserves all the properties stated in Proposition 7.4.3. We then use a non-zero exchange term and we compare the solution with same friction forces for the two phases or contrast of friction forces. Then, we perform two-dimensional simulations with friction forces contrast.

Finally, we verify the spatial and temporal convergence orders of the scheme.

**Remark 7.5.1** (Implementation details). All numerical schemes are implemented using Python 3 and the Numpy and Scipy modules. The linear system for the Cahn-Hilliard part of the model is solved using the Generalized Minimal RESidual iteration (GMRES) iterative solver (function available in the `scipy.sparse.linalg` module). The tolerance on the convergence of the residual is indicated in each of the following subsections. To find the  $\lambda$  that allows to compute the correct mass, we use the function `fsolve` of the `scipy.optimize` module which uses a modification of the Powell's conjugate direction method.

### 7.5.1 One dimensional numerical test cases

**Comparison between matching and non-matching densities.** We start with a one-dimensional test cases to show the spatiotemporal evolution of the density, mass fraction, and

velocity. We also verify numerically the properties stated in Proposition 7.4.3. We compare numerical results for matching and non-matching densities for the phases of the fluid. For this comparison, we set  $\kappa(\rho, c) = 0$  and  $F_c(\rho, c) = 0$ .

We use the computational domain  $\Omega = (0, 1)$  discretized in  $N_x = 128$  cells. We take  $T = 0.5$  (this has been chosen because the system reaches a meta-stable state by that time) and use the initial time step  $\Delta t = 1 \times 10^{-5}$  (this time step size is adapted from the CFL-like condition stated in Proposition 7.4.3).

We choose the width of the diffuse interface to be  $\gamma = 1/600$ , the viscosity to be constant  $\nu(c) = 1 \times 10^{-2}$ , and  $\eta = 2 \times 10^{-2}$ , the relaxation parameter to be  $\iota = 1 \times 10^{-5}$ , and the exponent for the barotropic pressure equals to  $a = 3$ .

To model matching densities for the two phases of the fluid, we choose  $\alpha_1 = \alpha_2 = 1$ . To represent non-matching densities for the two phases, we can choose  $\alpha_1 \neq \alpha_2$ . This allows us to model a fluid for which the phase denoted by the index 1 is denser compared to the phase indicated by the index 2. Indeed, this can be seen on the effect of the values  $\alpha_1$  and  $\alpha_2$  on the potential. Taking  $\alpha_1 < \alpha_2$  shifts the well corresponding to phase 1 very close to 0 compared to the other phase. This models the fact that the fluid 1 is in fact more compressible and thus aggregates of pure phase 1 appear denser.

We choose constant initial conditions for the density and the pressure, *i.e.*

$$\rho_j^0 = 0.8, \quad \mathbf{v}_j^0 = 0.5, \quad j = 0, \dots, N_x - 1.$$

The initial mass fraction is assumed to be a constant with a small random noise, *i.e.*

$$c_j^0 = \underline{c} + 0.05r_j, \quad j = 0, \dots, N_x - 1,$$

with  $\underline{c} = 0.5$  and  $r$  is a vector of random values between 0 and 1 given by the uniform distribution. We choose a tolerance for the convergence of the residual for the GMRES algorithm of  $\text{rtol} = 10^{-10}$ .

Figure 7.1 compares the results obtained for matching and non-matching densities for the two phases of the fluid. For the two cases, we report the evolution of the density  $\rho$ , the mass fraction  $c$ , the velocity  $\mathbf{v}$  and the pressure  $p$  at different times. We observe that, for both cases, after an initial regularization of the initial condition, the separation of the two phases of the fluid occurs and small aggregates appear (see first and second columns of Figure 7.1). Then, the coarsening of the small aggregates into larger ones occurs. We arrive at the end of the simulation to the solution depicted in the two figures on the last column of Figure 7.1. Hence, we can conclude that our numerical scheme catches well the spinodal decomposition of the binary fluid while it is transported to the right (since the velocity  $\mathbf{v}$  is positive during these simulations).

A difference between the two simulations is observed on the densities and pressures. Indeed, for matching densities, we observe that  $\rho$  organizes such that it is equal in aggregates of each phases and drops at the interfaces between the aggregates. We also observe a drop of pressure  $p$  at the interface, probably explained by capillary effects. For non-matching densities, as expected, there is a density difference between aggregates of phase 1 and 2. Indeed, selecting  $\alpha_1 < \alpha_2$  makes the aggregates of phase 1 denser compared to aggregates of phase 2. Our explanation is that, due to the fact that attractive effects are stronger in phase 1, more mass is allowed to move inside aggregates of phase 1. However, as the pressure function accounts for the difference  $\alpha_1 \neq \alpha_2$ , the pressure equilibrates to a field similar to the matching density case (*i.e.*  $p$  varies from an equilibrium value and depicts a drop at the interface between the aggregates of the different phases).

Figure 7.2 shows that, for both cases (*i.e.* matching and non-matching densities), our numerical scheme preserved the properties presented in Proposition 7.4.3. We defined the discrete

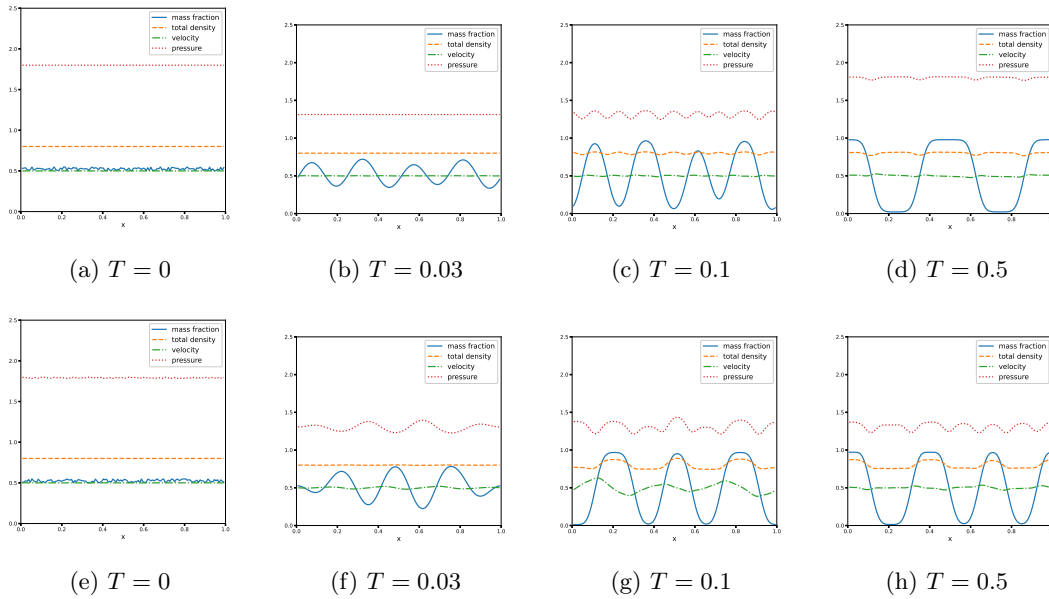


FIGURE 7.1 – Simulation of compressible Navier-Stokes-Cahn-Hilliard model (with  $\kappa = F_c = 0$ ). Matching densities (Top row) and non-matching densities (bottom row) for the two phases of the fluid.

dissipation of energy

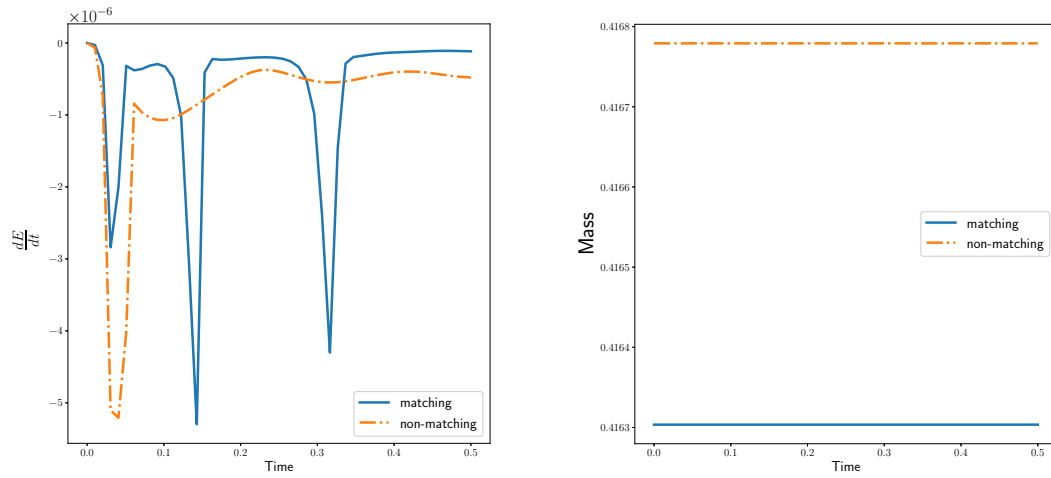
$$\frac{dE}{dt} = \|\sqrt{a}U^{n+1}\|^2 + \|V^{n+1}\|^2 + r^{n+1} - [\|\sqrt{a}U^n\|^2 + \|V^*\|^2 + C^{n+1}r^n]. \quad (7.5.1)$$

We emphasize that as no exchange term was present in the previous simulation  $C^{n+1}$  is bounded from above by 1, hence we used  $C^{n+1} = 1$  for the simulations in this paragraph. Figure 7.2 presents the temporal evolution of the dissipation  $\frac{dE}{dt}$ , the mass  $\int_{\Omega} \rho c dx$ , the minimum and maximum values of  $c$ , and the value of  $\xi$ . We observe for both cases that the dissipation (7.5.1) is strictly negative, as expected by proposition 7.4.3. The mass of fluid 1 is preserved up to a small numerical error (we emphasize that the error on the initial mass at the end of the simulation is less than  $10^{-9}$  for both simulations). This latter result is expected as we set  $F_c(\rho, c) = 0$  for these simulations. We observe that the physical bounds of the mass fraction are ensured, *i.e.* maximum and minimum values for  $c$  lie the interval  $(0, 1)$ . The scalar variable is very close to 1 (up to an error of order  $10^{-5}$ ) as observed in Figure 7.2c. This verifies that the modified energy  $r^{n+1}$  and the real energy of the Cahn-Hilliard part of the model

$$E^{n+1} = \Delta x \sum_j \frac{\gamma}{2} |(\nabla c)_j|^2 + \rho^{n+1} \left( \frac{\partial \psi_0}{\partial c^{n+1}} \right)_j,$$

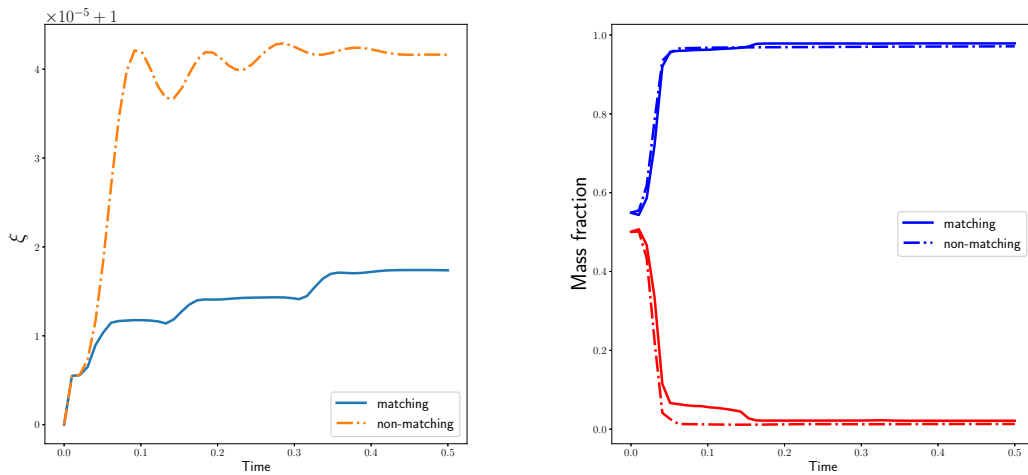
are close.





(a) Dissipation  $\frac{dE}{dt}$ . The value of the dissipation remains negative ensuring that Inequality (7.4.20) is satisfied at the discrete level.

(b) Mass  $\int_{\Omega} \rho c dx$  of fluid 1. Difference of initial mass is due to the random initial conditions for the two simulations.



(c) Scalar variable  $\xi$ . The value remains close to 1 (error is of order  $10^{-5}$ ), ensuring that the modified energy  $r^{n+1}$  is close to the real energy  $E^{n+1}$ .

(d) minimum and maximum of  $c$ . The mass fraction lies in the physically relevant interval  $(0, 1)$  throughout the simulation.

FIGURE 7.2 – Temporal evolution of the dissipation of the energy  $\frac{dE}{dt}$ , mass of the fluid 1 given by  $\int_{\Omega} \rho c dx$ , scalar variable  $\xi$ , and of the minimum and maximal values of the mass fraction  $c$  for matching densities (solid lines) and non-matching densities (dash-dotted lines).

**Mass exchange and contrast of friction forces.** In this test case, we consider mass exchange between the two phases and friction effects. We choose

$$F_c(\rho, c) = r_{\text{trans}}\rho c(1 - c/c_{\text{max}}), \quad \kappa(\rho, c) = \rho c\kappa_1 + \rho(1 - c)\kappa_2,$$

where  $0 < c_{\text{max}} < 1$  denotes the mass fraction at which we have an equilibrium for exchange of mass and  $r_{\text{trans}}$  is the rate of mass exchange. In this test case, we use  $c_{\text{max}} = 0.9$ , and  $r_{\text{trans}} = 1$ . We compare the solution obtained with no contrast of friction effect, *i.e.*  $\kappa_1 = \kappa_2 = 10$ , and the solution obtained with  $\kappa_1 = 0$ ,  $\kappa_2 = 10$ . To study the long time behavior of the numerical simulations, we set  $T = 5$ . The rest of the parameters and the initial conditions are chosen as for the previous non-matching densities test case.

Figure 7.3 compares the solutions for the two cases. In both cases, we observe that the separation of the two phases occurs and that the velocity decreases in time due to friction effects. Furthermore, as time passes, phase 1 of the fluid increases due to the exchange term  $F_c(\rho, c) \neq 0$  and, hence, zones of mass fraction close to the value 1 enlarge. At the end of both simulations, there is one large aggregate of fluid 1. The difference between the two simulations appears clearly at time  $t = 0.1$ . When the friction forces are stronger in phase two compared to phase 1, *i.e.*  $\kappa_2 > \kappa_1$ , zones of larger density appear destabilized, *i.e.* the shape of the aggregates is not symmetric (compare the solution for Figure 7.3b and Figure 7.3f). For each aggregate of fluid 1, the density at the right of the aggregate is larger compared to the left. We conclude that the contrast in friction forces is captured well by the model as simulations depict a contrast of velocity for the two phases of the fluid and leading to less regular patterns for the densities. Furthermore, we emphasize that even with non-zero mass exchange and friction forces, the numerical scheme ensures the properties stated in Proposition 7.4.3 as observed in Figure 7.4. We emphasize that compared to the simulation without source in which  $\xi$  seems to converge to a constant value, the variable  $\xi$  increases slightly with time (compare Figures 7.2c and 7.4c). A possible remedy to this issue is discussed in the conclusion of this article.

## 7.5.2 Two-dimensional numerical test cases

We now simulate the G-NSCH system (7.1.1)–(7.1.4) in two dimensions. Details about the two-dimensional numerical scheme can be found in the Appendix.

### Phase separation with non-matching densities and contrast of friction strengths.

We use  $N_x = N_y = 64$  cells in each direction. We fix the final time at  $T = 1$ . We set up two simulations, both with no exchange terms  $F_c(\rho, c) = 0$ . For the first, we consider no contrast of friction effects, *i.e.*  $\kappa_1 = \kappa_2 = 10$ ,  $\nu_1 = \nu_2 = 0.01$ , and  $\eta_1 = \eta_2 = 0.02$ . For the second simulation, we take  $\kappa_1 = 0$ ,  $\kappa_2 = 10$ ,  $\nu_1 = \nu_2 = 0.01$ , and  $\eta_1 = \eta_2 = 0.02$ . Hence, fluid 2 has stronger friction effects.

The other parameters are  $\gamma = \frac{1}{800}$ ,  $\theta = 4$ ,  $\alpha_1 = 0.8$ ,  $\alpha_2 = 1.2$ ,  $\iota = 10^{-4}$ ,  $C_0 = 100$ ,  $a = 1.5$ . The tolerance of GMRES solver is set to  $\text{tol} = 10^{-10}$ .

The initial velocities in both directions are constants in space  $\mathbf{v}_x^0 = 0.5$  and  $\mathbf{v}_y^0 = 0.5$ . The initial density is also constant in space  $\rho^0 = 0.8$ . The initial mass fraction is set to a perturbed constant  $c^0 = 0.3 - 0.05r$ , with  $r$  a random uniform number for each cell center.

Figure 7.5 compares the temporal evolution of the density of fluid 1 given by  $\rho c$  for both cases. We observe that both solutions depict phase separation and progressive coarsening of small aggregates into larger ones. This phenomenon occurs as the fluid is transported to the top right corner (we recall that we implemented periodic boundary conditions). Careful inspection of the relative density  $\rho c$  distribution inside each aggregates reveals the effect of the contrast of friction

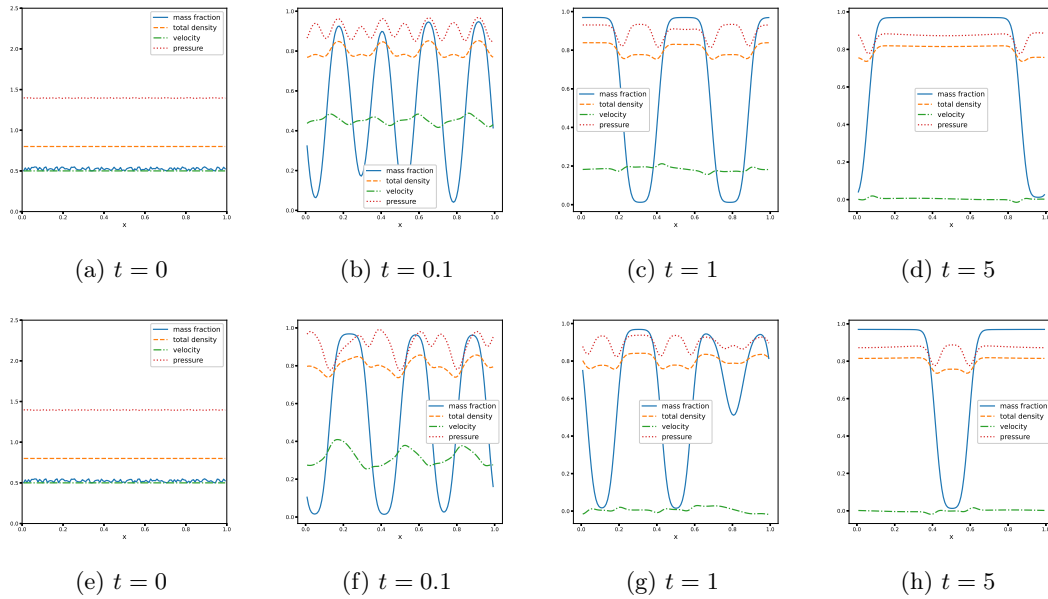


FIGURE 7.3 – Simulation of compressible Navier-Stokes-Cahn-Hilliard model nonmatching densities, exchange term  $(F_c(\rho, c) \neq 0)$ , same friction effects for both fluids (top row) and contrast of friction forces (bottom row).

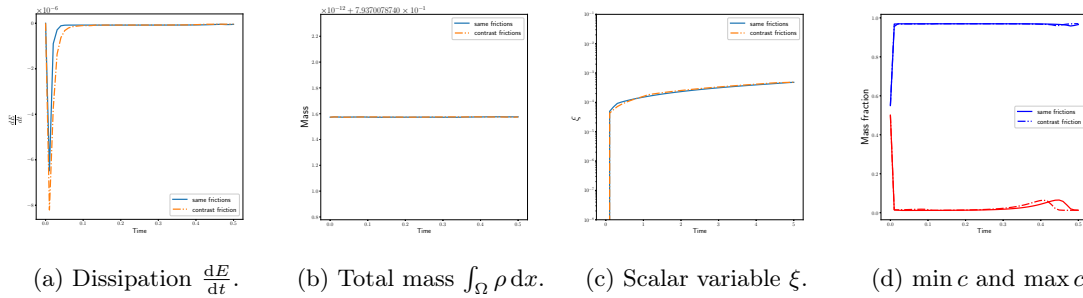


FIGURE 7.4 – Temporal evolution of the dissipation of the energy  $\frac{dE}{dt}$ , mass of the fluid 1 given by  $\int_{\Omega} \rho c \, dx$ , scalar variable  $\xi$ , and of the minimum and maximal values of the mass fraction  $c$  for matching densities (solid lines) and non-matching densities (dash-dotted lines).

between the two solutions. Indeed, as the fluid 1 encounters a resistance when transported by the flow (because it pushes a fluid that experiences more friction), the mass of fluid 1 seems to concentrate in the top right corner of each aggregate. This can be observed inspecting the level lines depicted on Figure 7.6. Indeed on this figure, we see that the top of each aggregate is not localized in the center of the aggregates but is shifted to the top-right. This indicates that the 2D scheme captures correctly the effect seen with the 1D numerical scheme when contrast of friction between the two phases is considered.

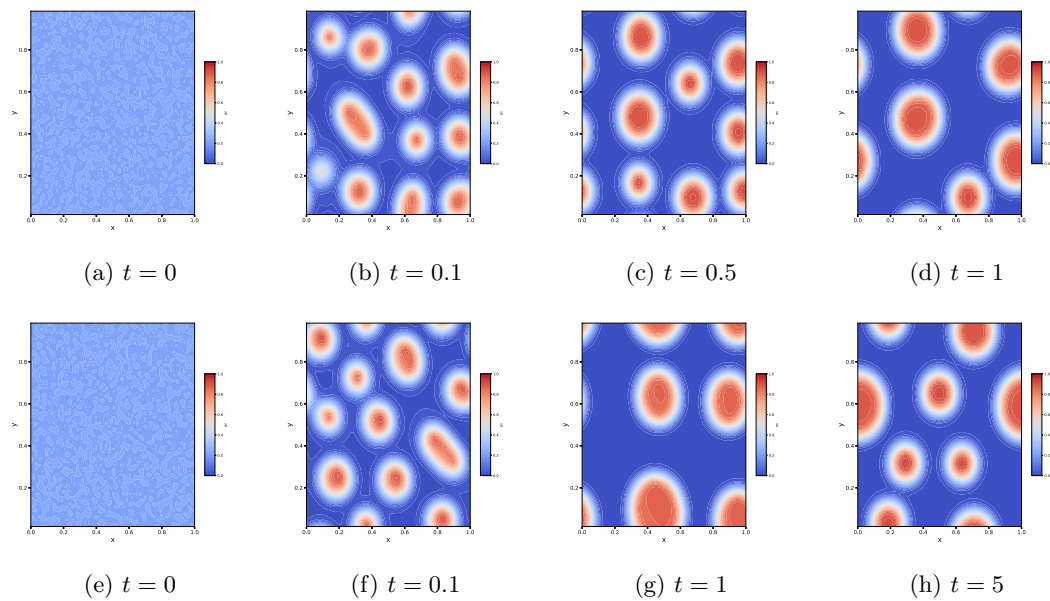
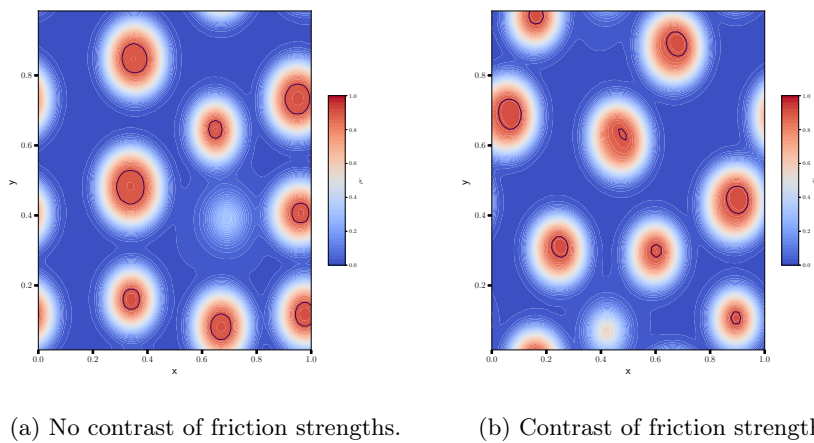


FIGURE 7.5 – Two dimensional simulations of compressible Navier-Stokes-Cahn-Hilliard model with nonmatching densities, same friction effects for both fluids (top row) and contrast of friction (bottom row).



(a) No contrast of friction strengths.

(b) Contrast of friction strengths.

FIGURE 7.6 – Relative density of fluid 1 at time  $t = 0.25$  considering a contrast of friction strengths between the two fluids. The black circles represent the level  $\rho c = \max(\rho c) - 0.03$ . This corresponds to the tops of each aggregate.

### 7.5.3 Convergence tests

We study the numerical convergence of the one dimensional scheme (7.4.4)–(7.4.16) with  $\kappa(\rho, c) = 0$  and  $F_c(\rho, c) = 0$ . The computational domain is  $\Omega = (0, 1)$ . The final time is  $T = 0.05$ . The other parameters  $\gamma, \beta, \eta, \nu, a, \alpha_1, \alpha_2$  are chosen as for the previous 1D test case with non-matching densities (see Subsection 7.5.1). The initial condition for the mass fraction is given by

$$c^0 = 0.4 + 0.01 \cos(6\pi x).$$

The initial conditions for the velocity and the total density are chosen as in the previous 1D test cases.

We set the tolerance  $\text{rtol}$  of the GMRES algorithm to  $\text{rtol} = 10^{-6}$  for the spatial convergence test and  $\text{rtol} = 10^{-10}$  for the temporal convergence test.

#### Convergence in space

We fix the time step to  $\Delta t = 1 \times 10^{-5}$  and we vary the grid size. We choose an increasing number of cells  $N_x = \{64, 128, 256, 512, 1024, 2048\}$ . For each quantity  $c, \mathbf{v}, \rho$ , we compute the discrete errors

$$\begin{aligned} \text{error}(\rho_{\Delta x}, \rho_{\Delta x/2}) &= \|\rho_{\Delta x} - \rho_{\Delta x/2}\|_{L^\infty(0, T; L^a(\Omega))}, \\ \text{error}(c_{\Delta x}, c_{\Delta x/2}) &= \|c_{\Delta x} - c_{\Delta x/2}\|_{L^2(0, T; L^2(\Omega))}, \\ \text{error}(\mathbf{v}_{\Delta x}, \mathbf{v}_{\Delta x/2}) &= \|\mathbf{v}_{\Delta x} - \mathbf{v}_{\Delta x/2}\|_{L^2(0, T; L^2(\Omega))}, \end{aligned} \quad (7.5.2)$$

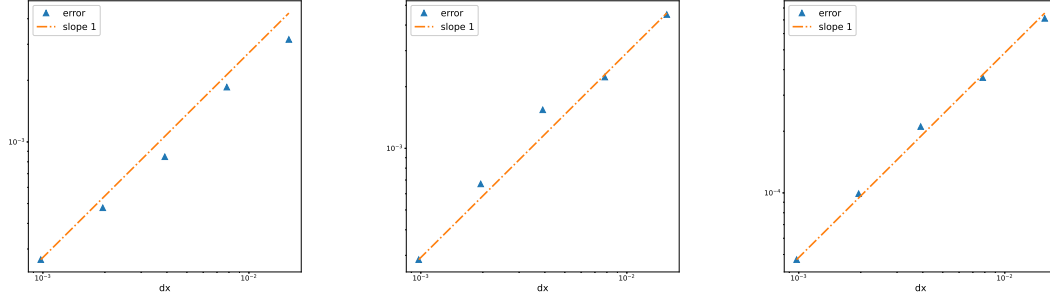
where  $(\rho_{\Delta x/2}, c_{\Delta x/2}, \mathbf{v}_{\Delta x/2})$  denotes the solution computed using twice the number of cells of the simulation that computes the solution  $(\rho_{\Delta x}, c_{\Delta x}, \mathbf{v}_{\Delta x})$ .

To compute the discrete norms, we save the solution every  $\Delta t_{\text{save}} = 0.001$ . The norms in (7.5.2) are computed following

$$\begin{aligned} \|\rho_{\Delta x} - \rho_{\Delta x/2}\|_{L^\infty(0, T; L^a(\Omega))} &= \max_{t_{\text{save}}} \left( \frac{\Delta x}{2} \sum_{j=1}^{N_x} (\rho_{\Delta x}(x_j) - \rho_{\Delta x/2}(x_j))^a \right)^{1/a}, \\ \|c_{\Delta x} - c_{\Delta x/2}\|_{L^2(0, T; L^2(\Omega))} &= \left( \sum_{t_{\text{save}}} \Delta t_{\text{save}} \left( \left( \frac{\Delta x}{2} \sum_{j=1}^{N_x} (c_{\Delta x}(x_j) - c_{\Delta x/2}(x_j))^2 \right)^{1/2} \right)^2 \right)^{1/2}, \\ \|\mathbf{v}_{\Delta x} - \mathbf{v}_{\Delta x/2}\|_{L^2(0, T; L^2(\Omega))} &= \left( \sum_{t_{\text{save}}} \Delta t_{\text{save}} \left( \left( \frac{\Delta x}{2} \sum_{j=1}^{N_x} (\mathbf{v}_{\Delta x}(x_j) - \mathbf{v}_{\Delta x/2}(x_j))^2 \right)^{1/2} \right)^2 \right)^{1/2}, \end{aligned}$$

with  $N_x$  the number of points on the  $\Delta x/2$  grid, and  $t_{\text{save}}$  the array of times at which snapshots of the solutions have been taken. Hence, the solution on the coarse grid  $\Delta x$  is extended on the fine grid  $\Delta x/2$  using the nearest solution from the coarse grid.

We arrive at the results given in Figure 7.7. As expected by the upwind scheme, the spatial order of convergence is a little less than 1 for the total density  $\rho$  (see Figure 7.7a) and the velocity  $\mathbf{v}$  (see Figure 7.7c). We recover first order for the mass fraction.



(a) Spatial convergence of density  $\rho$ . (b) Spatial convergence of mass fraction  $c$ . (c) Spatial convergence of velocity  $\mathbf{v}$ .

FIGURE 7.7 – Convergence in space for the total density  $\rho$ , the mass fraction  $c$  and the velocity  $\mathbf{v}$ . The orange dashed line represents the slope 1.

### Convergence in time

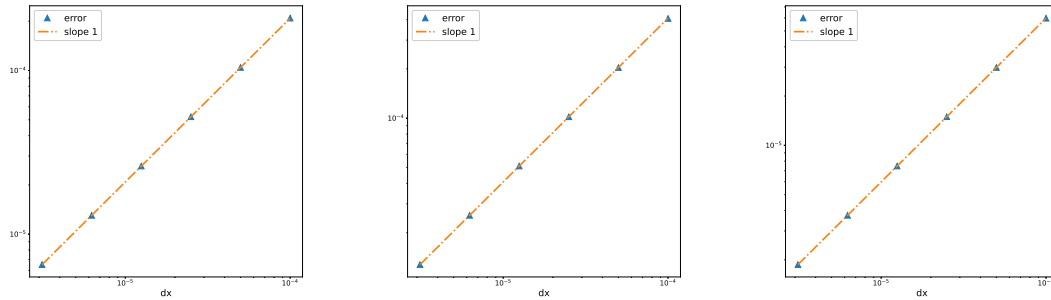
We here fix the grid size and select  $N_x = 128$  points. We choose  $\Delta t = 1 \times 10^{-4}$ , and decrease the time steps according to  $\Delta t_{\text{array}} = \{\Delta t, \frac{\Delta t}{2}, \frac{\Delta t}{4}, \frac{\Delta t}{8}, \frac{\Delta t}{16}, \frac{\Delta t}{32}, \frac{\Delta t}{64}\}$ . We deactivate the time step adaptive strategy from the CFL condition. The other parameters and initial conditions are chosen as in the spatial convergence test (see Subsection 7.5.3). To check the convergence in time of our scheme, we compute the errors between two solutions computed with two time steps  $\Delta t$  that differ only from a factor  $\frac{1}{2}$ . We denote these two different solutions by  $(\rho_{\Delta t}, c_{\Delta t}, \mathbf{v}_{\Delta t})$  and  $(\rho_{\Delta t/2}, c_{\Delta t/2}, \mathbf{v}_{\Delta t/2})$ . We use the same method as for the spatial convergence computations, we save the solutions every  $\Delta t_{\text{save}} = 0.001$ . We compute the norms

$$\begin{aligned} \text{error}(\rho_{\Delta t}, \rho_{\Delta t/2}) &= \|\rho_{\Delta t} - \rho_{\Delta t/2}\|_{L^\infty(0,T;L^a(\Omega))}, \\ \text{error}(c_{\Delta t}, c_{\Delta t/2}) &= \|c_{\Delta t} - c_{\Delta t/2}\|_{L^2(0,T;L^2(\Omega))}, \\ \text{error}(\mathbf{v}_{\Delta t}, \mathbf{v}_{\Delta t/2}) &= \|\mathbf{v}_{\Delta t} - \mathbf{v}_{\Delta t/2}\|_{L^2(0,T;L^2(\Omega))}. \end{aligned} \quad (7.5.3)$$

with

$$\begin{aligned} \|\rho_{\Delta t} - \rho_{\Delta t/2}\|_{L^\infty(0,T;L^a(\Omega))} &= \max_{t_{\text{save}}} \left( \Delta x \sum_{j=1}^{N_x} (\rho_{\Delta t}(x_j) - \rho_{\Delta t/2}(x_j))^a \right)^{1/a}, \\ \|c_{\Delta t} - c_{\Delta t/2}\|_{L^2(0,T;L^2(\Omega))} &= \left( \sum_{t_{\text{save}}} \Delta t_{\text{save}} \left( \left( \Delta x \sum_{j=1}^{N_x} (c_{\Delta t}(x_j) - c_{\Delta t/2}(x_j))^2 \right)^{1/2} \right)^2 \right)^{1/2}, \\ \|\mathbf{v}_{\Delta t} - \mathbf{v}_{\Delta t/2}\|_{L^2(0,T;L^2(\Omega))} &= \left( \sum_{t_{\text{save}}} \Delta t_{\text{save}} \left( \left( \Delta x \sum_{j=1}^{N_x} (\mathbf{v}_{\Delta t}(x_j) - \mathbf{v}_{\Delta t/2}(x_j))^2 \right)^{1/2} \right)^2 \right)^{1/2}, \end{aligned}$$

We obtain the results depicted in Figure 7.8. We observe that the order of convergence in time for our scheme is exactly 1 for the three quantities.



(a) Temporal convergence of density  $\rho$ . (b) Temporal convergence of mass fraction  $c$ . (c) Temporal convergence of velocity  $\mathbf{v}$ .

FIGURE 7.8 – Convergence in time for the total density  $\rho$ , the mass fraction  $c$  and the velocity  $\mathbf{v}$ . The orange dashed line represents the slope 1.

## 7.6 Conclusion and perspectives

We presented a generalized model of diphasic compressible fluid termed G-NSCH, that comprises possible mass transfer between the two phases and friction effects. Under simplifying assumptions, summarized in Section 7.2.1, we proved the existence of weak solutions of the G-NSCH system. We also proposed a numerical scheme and prove, under the same simplifying assumptions, that it is stable and structure preserving (*i.e.* it ensures the physically relevant bounds for the mass fraction  $c$ , and it satisfies an energy dissipation inequality). For the numerical simulations, we chose relevant functionals, thus, relaxing the simplifying assumptions that were necessary for the analysis. We presented numerical simulations showing that our numerical scheme possesses the robustness found analytically. The numerical simulations allowed us to show the ability of our model to represent diphasic fluids with matching or non-matching densities for the two phases. Furthermore, we computed numerically the spatial and temporal convergences our numerical scheme.

Our model and numerical scheme allow to use physically relevant choice of functional and to consider contrast of properties between the two phases of the fluid. Our aim is to perform efficient simulation of general compressible diphasic fluids while being able to capture instabilities that could emerge considering contrasts of properties such as Saffman-Taylor or Rayleigh-Taylor instabilities. However, we emphasize that to achieve this latter goal, we have to be able to capture accurately the possible fine structures appearing during the numerical simulations. We plan to improve our numerical scheme in several ways. First, we plan to adapt the Relaxed version of the Generalized SAV method. Indeed, we observed during our numerical simulation that the variable  $\xi$  is between  $10^{-5}$  and  $10^{-3}$ . However, with the transfer term  $F_c(\rho, c) \neq 0$ , as the simulation progresses, the gap between  $r$  and the real energy increases, *i.e.*  $\xi$  becomes larger. As shown in [333], this problem is solved with the relaxed G-SAV method and the use of this method is one of our further developments. In the same work [333], it is mentioned that this relaxed method works well even when an *external force* is comprised in the model. In our case, this *external force* will take the form of a mass transfer or mass source term. The second improvement concerns the accuracy of our scheme. As shown in our work, the temporal and spatial orders of our scheme do not exceed 1. Thus, we will aim to design a high-order finite element scheme for the generalized compressible NSCH system that will remain structure preserving taking advantage of

the flexibility of the relaxed GSAV method. On another aspect, we plan to use the reduced version of the G-NSCH model presented in the Appendix 7.7.8 to represent tumor growth while removing non-necessary effects such as inertia. Our goal is to present a model and numerical simulations capturing Saffman-Taylor-like instabilities depicted by the protrusions of the tumor in the healthy tissue and commonly observed in the context of, *e.g.*, skin cancer [97]. Furthermore, analytical aspects of this work can also be improved. This direction is challenging because as pointed out in the present work, necessary tools to perform the solutions' existence proof do not work with physically or biologically relevant potentials or mobility functions. In fact, singular potentials, degenerate mobilities and degenerate viscosity functions are not allowed. One possible solution is to derive a Bresch-Desjardins entropy estimate [56, 57] for the compressible NSCH as it has been done recently by Vasseur and Yu [321], and Bresch, Vasseur and Yu [62], for the compressible Navier-Stokes model with degenerate viscosities.

To conclude, we emphasize that the G-NSCH model is the basis of a reduced system that takes into account only biologically relevant physical effects that play a role in tumor evolution (presented in the present article in Appendix 7.7.8 as *Problem 2*). Therefore, this work has to be seen as the first part. In a subsequent work, relying heavily on the present one, we will focus on numerical simulations, and sensitivity analysis of the reduced model.

## 7.7 Appendix

### 7.7.1 Derivation of the model

In this Appendix, we present the rigorous derivation of our G-NSCH model.

We formulate our problem in Eulerian coordinates and in a smooth bounded domain  $\Omega \subset \mathbb{R}^d$  (where  $d = \{1, 2, 3\}$  is the dimension). The balance laws derived in the following sections are in local form.

We have two fluids in the model where  $\rho_1, \rho_2$  are the relative densities of respectively fluid 1 and 2. Thus,  $\rho_i$  represents the mass  $M_i$  of the fluid per volume occupied by the  $i$ -th phase  $V_i$ , *i.e.*

$$\rho_i = \frac{M_i}{V_i}.$$

Then, we define the volume fractions  $\varphi_1, \varphi_2$  which are defined by the volume occupied by the  $i$ -th phase over the total volume of the mixture

$$\varphi_i = \frac{V_i}{V}.$$

Therefore, the mass density of population  $i$  which is the mass of population  $i$  in volume  $V$  is given by

$$\phi_i = \rho_i \varphi_i.$$

We further assume that the fluid is saturated, *i.e.*

$$\varphi_1 + \varphi_2 = 1.$$

The total density of the mixture is then given by

$$\rho = \phi_1 + \phi_2.$$



We also introduce the mass fractions  $c_i = M_i/M$  and we have the relations

$$\rho c_i = \phi_i, \quad \text{and} \quad c_1 = (1 - c_2). \quad (7.7.1)$$

We denote by  $p$  the pressure inside the mixture and  $\mathbf{v}_1, \mathbf{v}_2$  are the velocities of the different phases. We use a mass-average mixture velocity

$$\mathbf{v} = \frac{1}{\rho} (\phi_1 \mathbf{v}_1 + \phi_2 \mathbf{v}_2). \quad (7.7.2)$$

We define the material derivative for a generic function  $g$  (scalar or vector-valued) by

$$\frac{Dg}{Dt} = \frac{\partial g}{\partial t} + \mathbf{v} \cdot \nabla g, \quad (7.7.3)$$

and indicate the definition of the differential operator

$$\mathbf{v} \cdot \nabla g = \sum_{j=1}^d \mathbf{v}_j \frac{\partial g}{\partial x_j}.$$

In the following, we denote vectors by bold roman letters and we use bold Greek letters to denote second-order tensors.

## 7.7.2 Mass balance equations

We have the mass balance equations

$$\begin{cases} \frac{\partial \phi_1}{\partial t} + \operatorname{div}(\phi_1 \mathbf{v}_1) = F_1(\rho, c_1, c_2), \\ \frac{\partial \phi_2}{\partial t} + \operatorname{div}(\phi_2 \mathbf{v}_2) = F_2(\rho, c_1, c_2). \end{cases} \quad (7.7.4)$$

The functions  $F_i(\rho, c_1, c_2)$  ( $i = 1, 2$ ) act as source or exchange terms of mass.

Summing the two equations, we obtain the continuity equation for the total density of the mixture, using  $c = c_1$ , and the relations (7.7.1), we obtain the balance equation for the total density of the mixture

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = F_1 + F_2 =: F_\rho. \quad (7.7.5)$$

To obtain a system analogous to (7.7.4), we rewrite the first equation of (7.7.4) using the definition of the mass fraction (7.7.1) to obtain

$$\frac{\partial \rho c}{\partial t} + \operatorname{div}(\rho c \mathbf{v}_1) = F_1(\rho, c, 1 - c) =: F_c. \quad (7.7.6)$$

The mass of the component 1 is transported by the average velocity  $\mathbf{v}$  and the remaining diffusive flux  $\mathbf{J}_1 = \rho c (\mathbf{v} - \mathbf{v}_1)$ . Therefore, we can replace the previous equation by

$$\frac{\partial \rho c}{\partial t} + \operatorname{div}(\rho c \mathbf{v}) = \operatorname{div}(\mathbf{J}_1) + F_c.$$

Then, using the definition of the material derivative (7.7.3) and the mass balance equation for

the total mixture (7.7.5), the left-hand side of the previous equation reads

$$\frac{\partial \rho c}{\partial t} + \operatorname{div}(\rho c \mathbf{v}) = \rho \frac{Dc}{Dt} + c \left[ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) \right] = \rho \frac{Dc}{Dt} + c F_\rho.$$

Altogether, we obtain the balance equation for the mass fraction of the component 1

$$\rho \frac{Dc}{Dt} = \operatorname{div}(\mathbf{J}_1) + F_c - c F_\rho. \quad (7.7.7)$$

Since  $c_2 = 1 - c$ , solving the equations (7.7.5) and (7.7.7) is equivalent to solving the system (7.7.4). In the following, we refer to  $c$  as the order parameter (terminology often used in the framework of the Cahn-Hilliard model [70, 69]).

### 7.7.3 Balance of linear momentum

We write the balance of linear momentum [146], which describes the evolution of the velocity  $\mathbf{v}$  due to internal stresses and external forces. Following continuum mechanics, the Cauchy stress tensor gives the stresses acting inside the mixture due to viscous and non-viscous effects. An additional stress must be taken into account to represent the effect of concentration gradients [159]. Altogether, we assume that the stress tensor is a function of the total density  $\rho$ , the order parameter  $c$  (i.e. the mass fraction of fluid 1), its gradient  $\nabla c$ , and the total velocity of the mixture  $\mathbf{v}$ , *i.e.*

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\rho, c, \nabla c, \mathbf{v}).$$

The friction around the pores of the medium is modeled by a drag term in the balance equation [282] with a permeability coefficient  $\kappa(\rho, c) = \kappa_1(\rho, c) + \kappa_2(\rho, c)$  (the sum of the two friction coefficients for each component of the mixture). The permeability coefficient relates the properties of the fluid and the porous medium.

For each dimension (for example if  $d = 3$ , then  $j = \{x, y, z\}$ ), the balance of linear momentum reads [146]

$$\frac{\partial \rho \mathbf{v}_j}{\partial t} + \operatorname{div}(\rho \mathbf{v}_j \mathbf{v}) = \operatorname{div}(\boldsymbol{\sigma})_j - \kappa(\rho, c) \mathbf{v}_j + F_{\mathbf{v}_j},$$

where  $F_{\mathbf{v}_j}(\mathbf{v}_j, \rho)$  represents the gain or loss of velocity in the  $j$ -th direction from different effects such as external forces. Then, using the continuity equation (7.7.5), we can rearrange the left-hand side to obtain

$$\frac{\partial \rho \mathbf{v}_j}{\partial t} + \operatorname{div}(\rho \mathbf{v}_j \mathbf{v}) = \rho \frac{D\mathbf{v}_j}{Dt} + \mathbf{v}_j \left[ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) \right] = \rho \frac{D\mathbf{v}_j}{Dt} + \mathbf{v}_j F_\rho + F_{\mathbf{v}_j}.$$

Therefore, we have

$$\rho \frac{D\mathbf{v}_j}{Dt} = \operatorname{div}(\boldsymbol{\sigma})_j - (\kappa(\rho, c) + F_\rho) \mathbf{v}_j + F_{\mathbf{v}_j}.$$

We can rewrite the balance of linear momentum in a more compact form

$$\rho \frac{D\mathbf{v}}{Dt} = \operatorname{div}(\boldsymbol{\sigma}) - (\kappa(\rho, c) + F_\rho) \mathbf{v} + F_{\mathbf{v}}, \quad (7.7.8)$$

where  $F_{\mathbf{v}}(\mathbf{v}, \rho)$  is the vector of coordinates  $F_{\mathbf{v}_j}$ .

### 7.7.4 Energy balance

The total energy of the mixture is the sum of the kinetic energy  $\rho \frac{1}{2} |\mathbf{v}|^2$  and of the internal energy  $\rho u$ , where  $u = u(\rho, c, \nabla c)$  is a specific internal energy. Compared to the classical conservation law for the total energy, we have an additional energy flux  $\boldsymbol{\tau} \frac{Dc}{Dt}$ . Indeed, due to the interface region, surface effects must be taken into account. Following this direction, Gurtin [200] proposed to include in the second law of thermodynamics, the effect of an additional force called the *microscopic-stress* which is related to forces acting at the microscopic scale. We denote this supplementary stress by  $\boldsymbol{\tau}$ .

Since we assume that the system is maintained in an isothermal state, the balance equation for the energy is given by [146]

$$\begin{aligned} \frac{\partial}{\partial t} \left( \rho \frac{1}{2} |\mathbf{v}|^2 + \rho u \right) + \operatorname{div} \left( \rho \left( \frac{1}{2} |\mathbf{v}|^2 + u \right) \mathbf{v} \right) \\ = \operatorname{div} (\boldsymbol{\sigma}^T \mathbf{v}) + \operatorname{div} \left( \boldsymbol{\tau} \frac{Dc}{Dt} \right) - \operatorname{div} (\mathbf{q}) + \rho g + c_\rho F_\rho + c_c F_c + c_v F_v, \end{aligned} \quad (7.7.9)$$

where  $\mathbf{q}$  is the heat flux and  $\rho g$  is the density of heat sources to maintain the temperature constant. The last three terms in Equation (7.7.9) account for the energy supply coming from the mass and velocity sources (see *e.g.* [201, 248]). The prefactors  $c_\rho, c_c, c_v$  will be determined later to satisfy the free energy imbalance. Then, repeating the same calculations on the left-hand side to use the balance of mass (7.7.5), we have

$$\frac{\partial}{\partial t} \left( \rho \frac{1}{2} |\mathbf{v}|^2 + \rho u \right) + \operatorname{div} \left( \rho \left( \frac{1}{2} |\mathbf{v}|^2 + u \right) \mathbf{v} \right) = \rho \left[ \frac{D}{Dt} \left( \frac{1}{2} |\mathbf{v}|^2 + u \right) \right] + \left( \frac{1}{2} |\mathbf{v}|^2 + u \right) F_\rho.$$

Applying the chain rule to the kinetic part, we obtain

$$\rho \frac{D}{Dt} \left( \frac{1}{2} |\mathbf{v}|^2 \right) = \rho \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt},$$

and using the balance of linear momentum (7.7.8), we arrive to

$$\rho \mathbf{v} \cdot \frac{D\mathbf{v}}{Dt} = \mathbf{v} \cdot \operatorname{div} (\boldsymbol{\sigma}) - (\kappa(\rho, c) + F_\rho) |\mathbf{v}|^2 + F_v \cdot \mathbf{v}.$$

Using these previous equations inside (7.7.9), we obtain the balance equation for the internal energy

$$\begin{aligned} \rho \frac{Du}{Dt} = \operatorname{div} (\boldsymbol{\sigma}^T \mathbf{v}) - \mathbf{v} \cdot \operatorname{div} (\boldsymbol{\sigma}) + \operatorname{div} \left( \boldsymbol{\tau} \frac{Dc}{Dt} \right) + (\kappa(\rho, c) + F_\rho) |\mathbf{v}|^2 - F_v \cdot \mathbf{v} \\ - \operatorname{div} (\mathbf{q}) + \rho g - \left( \frac{1}{2} |\mathbf{v}|^2 + u \right) F_\rho + c_\rho F_\rho + c_c F_c + c_v F_v. \end{aligned}$$

However, since

$$\mathbf{v} \cdot (\operatorname{div} (\boldsymbol{\sigma})) - \operatorname{div} (\boldsymbol{\sigma}^T \mathbf{v}) = -\boldsymbol{\sigma} : \nabla \mathbf{v},$$

where  $\nabla \mathbf{v} = (\partial_{x_j} \mathbf{v}_i)_{i,j=1,\dots,d}$  is the Jacobi matrix and, we have  $A : B = \sum_{i,j} A_{ij} B_{ij}$ , for two

matrices  $A, B$ . Altogether, we have the balance equation for the internal energy

$$\begin{aligned} \rho \frac{Du}{Dt} &= \boldsymbol{\sigma} : \nabla \mathbf{v} + \operatorname{div} \left( \boldsymbol{\tau} \frac{Dc}{Dt} \right) + (\kappa(\rho, c) + F_\rho) |\mathbf{v}|^2 - F_v \mathbf{v} \\ &\quad - \operatorname{div}(\mathbf{q}) + \rho g - \left( \frac{1}{2} |\mathbf{v}|^2 + u \right) F_\rho + c_\rho F_\rho + c_c F_c + c_v F_v. \end{aligned} \quad (7.7.10)$$

### 7.7.5 Entropy balance and Clausius-Duhem inequality

We aim to apply the second law of thermodynamics. To do so, we define the entropy  $s = s(\rho, c, \nabla c)$  and the Helmholtz free energy  $\mathcal{F} = \mathcal{F}(\rho, c, \nabla c)$ , both related through the equation

$$\mathcal{F} = u - Ts, \quad (7.7.11)$$

where  $T$  denotes the temperature.

From the mass balance equation (7.7.5), we have the entropy balance equation

$$\frac{\partial \rho s}{\partial t} + \operatorname{div}(s \rho \mathbf{v}) = \rho \frac{Ds}{Dt} + s \left[ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) \right] = \rho \frac{Ds}{Dt} + s F_\rho. \quad (7.7.12)$$

Then, using the definition of the Helmholtz free energy (7.7.11) and the balance of energy (7.7.10), we obtain

$$\begin{aligned} \rho \frac{Ds}{Dt} &= -\frac{\rho}{T} \frac{D\mathcal{F}}{Dt} + \frac{\rho}{T} \frac{Du}{Dt} \\ &= -\frac{\rho}{T} \frac{D\mathcal{F}}{Dt} + \frac{1}{T} \left[ \boldsymbol{\sigma} : \nabla \mathbf{v} + \operatorname{div} \left( \boldsymbol{\tau} \frac{Dc}{Dt} \right) + (\kappa(\rho, c) + F_\rho) |\mathbf{v}|^2 - F_v \mathbf{v} \right. \\ &\quad \left. - \operatorname{div}(\mathbf{q}) + \rho g - \left( \frac{1}{2} |\mathbf{v}|^2 + u \right) F_\rho + c_\rho F_\rho + c_c F_c + c_v F_v \right], \end{aligned} \quad (7.7.13)$$

where we have replaced the material derivative of the internal energy using its balance equation (7.7.10).

The constitutive relations for the functions constituting the Navier-Stokes-Cahn-Hilliard model are often derived to satisfy the Clausius-Duhem inequality (Coleman-Noll Procedure) [146]. Indeed, this inequality provides a set of restrictions for the dissipative mechanisms occurring in the system. However, in our case, due to the presence of source terms, we can not ensure that this inequality holds without some assumptions on the proliferation and friction of the fluid around the pores. Therefore, we use here a different method : the Lagrange multipliers method. Indeed, the Liu [259] and Müller [281] method is based on using Lagrange multipliers to derive a set of restrictions on the constitutive relations that can be applied even in the presence of source terms.

Following classical Thermodynamics [281], we state the second law as an entropy inequality, *i.e.*, the Clausius-Duhem inequality in the local form [146]

$$\rho \frac{Ds}{Dt} \geq -\operatorname{div} \left( \frac{\mathbf{q}}{T} \right) + \frac{\rho g}{T} + \operatorname{div}(\mathcal{J}), \quad (7.7.14)$$

where  $\mathcal{J}$  is the entropy flux. The inequality (7.7.14) results from the fact that the entropy of the

mixture can only increase. Using the equation (7.7.13), we obtain

$$\begin{aligned} \frac{\rho}{T} \frac{D\mathcal{F}}{Dt} - \frac{1}{T} [\boldsymbol{\sigma} : \nabla \mathbf{v} + \operatorname{div} \left( \boldsymbol{\tau} \frac{Dc}{Dt} \right) + (\kappa(\rho, c) + F_\rho) |\mathbf{v}|^2 \\ - F_v \mathbf{v} - \left( \frac{1}{2} |\mathbf{v}|^2 + u \right) F_\rho + c_\rho F_\rho + c_c F_c + c_v F_v] + \operatorname{div} (\mathcal{J}) \leq 0. \end{aligned} \quad (7.7.15)$$

Then, using the chain rule

$$\frac{D\mathcal{F}}{Dt} = \frac{D\rho}{Dt} \frac{\partial \mathcal{F}}{\partial \rho} + \frac{Dc}{Dt} \frac{\partial \mathcal{F}}{\partial c} + \frac{D\nabla c}{Dt} \cdot \frac{\partial \mathcal{F}}{\partial \nabla c},$$

and

$$\frac{D\nabla c}{Dt} = \nabla \left[ \frac{Dc}{Dt} \right] - (\nabla \mathbf{v})^T \nabla c, \quad \frac{D\rho}{Dt} = -\rho \operatorname{div}(\mathbf{v}) + F_\rho,$$

in the entropy inequality (7.7.15), we obtain

$$\begin{aligned} \rho \left[ (-\rho \operatorname{div}(\mathbf{v}) + F_\rho) \frac{\partial \mathcal{F}}{\partial \rho} + \frac{Dc}{Dt} \frac{\partial \mathcal{F}}{\partial c} + \left( \nabla \left[ \frac{Dc}{Dt} \right] - (\nabla \mathbf{v})^T \nabla c \right) \cdot \frac{\partial \mathcal{F}}{\partial \nabla c} \right] - \operatorname{div} \left( \boldsymbol{\tau} \frac{Dc}{Dt} \right) - \boldsymbol{\sigma} : \nabla \mathbf{v} \\ - \left[ (\kappa(\rho, c) + F_\rho) |\mathbf{v}|^2 - F_v \mathbf{v} - \left( \frac{1}{2} |\mathbf{v}|^2 + u \right) F_\rho + c_\rho F_\rho + c_c F_c + c_v F_v \right] + T \operatorname{div} (\mathcal{J}) \leq 0. \end{aligned} \quad (7.7.16)$$

By the chain rule, we have

$$\operatorname{div} \left( \boldsymbol{\tau} \frac{Dc}{Dt} \right) = \boldsymbol{\tau} \nabla \left[ \frac{Dc}{Dt} \right] + \frac{Dc}{Dt} \operatorname{div}(\boldsymbol{\tau}).$$

Furthermore, we know that

$$-\rho^2 \operatorname{div}(\mathbf{v}) \frac{\partial \mathcal{F}}{\partial \rho} = -\rho^2 \frac{\partial \mathcal{F}}{\partial \rho} \mathbf{1} : \nabla \mathbf{v},$$

and

$$-\rho \left( (\nabla \mathbf{v})^T \nabla c \right) \cdot \frac{\partial \mathcal{F}}{\partial \nabla c} = -\rho \left( \nabla c \otimes \frac{\partial \mathcal{F}}{\partial \nabla c} \right) : \nabla \mathbf{v}.$$

Gathering the previous three relations and reorganizing the terms of (7.7.16), we obtain

$$\begin{aligned} \left( -\rho^2 \frac{\partial \mathcal{F}}{\partial \rho} \mathbf{1} - \rho \nabla c \otimes \frac{\partial \mathcal{F}}{\partial \nabla c} - \boldsymbol{\sigma} \right) : \nabla \mathbf{v} + \left( \rho \frac{\partial \mathcal{F}}{\partial c} - \operatorname{div}(\boldsymbol{\tau}) \right) \frac{Dc}{Dt} \\ + \left( \rho \frac{\partial \mathcal{F}}{\partial \nabla c} - \boldsymbol{\tau} \right) \nabla \left[ \frac{Dc}{Dt} \right] + T \operatorname{div} (\mathcal{J}) \\ - \left[ (\kappa(\rho, c) + F_\rho) |\mathbf{v}|^2 - F_v \mathbf{v} - \left( \frac{1}{2} |\mathbf{v}|^2 + u - \rho \frac{\partial \mathcal{F}}{\partial \rho} \right) F_\rho + c_\rho F_\rho + c_c F_c + c_v F_v \right] \leq 0. \end{aligned} \quad (7.7.17)$$

Then, we use Liu's Lagrange multipliers method [259]. We denote by  $L_c$  the Lagrange multiplier associated with the mass fraction equation (7.7.7). The method of Lagrange multipliers consists in setting the following local dissipation inequality that has to hold for arbitrary values

of  $(\rho, c, \nabla\rho, \nabla c, \mathbf{v}, p)$

$$\begin{aligned}
-D_{\text{iss}} &:= \left( -\rho^2 \frac{\partial \mathcal{F}}{\partial \rho} \mathbf{1} - \rho \nabla c \otimes \frac{\partial \mathcal{F}}{\partial \nabla c} - \boldsymbol{\sigma} \right) : \nabla \mathbf{v} \\
&+ \left( \rho \frac{\partial \mathcal{F}}{\partial c} - \text{div}(\boldsymbol{\tau}) \right) \frac{Dc}{Dt} + \left( \rho \frac{\partial \mathcal{F}}{\partial \nabla c} - \boldsymbol{\tau} \right) \nabla \left[ \frac{Dc}{Dt} \right] + T \text{div}(\mathcal{J}) \\
&- \left[ (\kappa(\rho, c) + F_\rho) |\mathbf{v}|^2 - F_v \mathbf{v} - \left( \frac{1}{2} |\mathbf{v}|^2 + u - \rho \frac{\partial \mathcal{F}}{\partial \rho} \right) F_\rho + c_\rho F_\rho + c_c F_c + c_v F_v \right] \\
&- L_c \left( \rho \frac{Dc}{Dt} - \text{div}(\mathbf{J}_1) - F_c - c F_\rho \right) \leq 0.
\end{aligned} \tag{7.7.18}$$

Since,

$$\text{div}(L_c \mathbf{J}_1) = L_c \text{div}(\mathbf{J}_1) + \nabla L_c \cdot \mathbf{J}_1,$$

we reorganize the terms of (7.7.18) to obtain

$$\begin{aligned}
-D_{\text{iss}} &:= \left( -\rho^2 \frac{\partial \mathcal{F}}{\partial \rho} \mathbf{1} - \rho \nabla c \otimes \frac{\partial \mathcal{F}}{\partial \nabla c} - \boldsymbol{\sigma} \right) : \nabla \mathbf{v} \\
&+ \left( \rho \frac{\partial \mathcal{F}}{\partial c} - \text{div}(\boldsymbol{\tau}) - \rho L_c \right) \frac{Dc}{Dt} + \left( \rho \frac{\partial \mathcal{F}}{\partial \nabla c} - \boldsymbol{\tau} \right) \nabla \left[ \frac{Dc}{Dt} \right] + \text{div}(T\mathcal{J} + L_c \mathbf{J}_1) \\
&- \nabla L_c \cdot \mathbf{J}_1 \\
&- \left[ (\kappa(\rho, c) + F_\rho) |\mathbf{v}|^2 - F_v \mathbf{v} - \left( \frac{1}{2} |\mathbf{v}|^2 + u - \rho \frac{\partial \mathcal{F}}{\partial \rho} \right) F_\rho \right. \\
&\quad \left. + c_\rho F_\rho + c_c F_c + c_v F_v - L_c(F_c + c F_\rho) \right] \leq 0.
\end{aligned} \tag{7.7.19}$$

### 7.7.6 Constitutive assumptions and model equations

First of all, we assume that the free energy density  $\mathcal{F}$  is of Ginzburg-Landau type and has the following form [70, 69]

$$\mathcal{F}(\rho, c, \nabla c) := \psi_0(\rho, c) + \frac{\gamma}{2} |\nabla c|^2, \tag{7.7.20}$$

where  $\psi_0$  is the homogeneous free energy accounting for the processes of phase separation and the gradient term  $\frac{\gamma}{2} |\nabla c|^2$  represents the surface tension between the two phases. This free energy is the basis of the Cahn-Hilliard model which describes the phase separation occurring in binary mixtures. Furthermore, as obtained in Wise *et al.* [176], the adhesion energy between different cell species is indeed well represented by such a choice of the free energy functional.

To satisfy the inequality (7.7.19), we first choose

$$\boldsymbol{\tau} := \rho \frac{\partial \mathcal{F}}{\partial \nabla c} = \gamma \rho \nabla c.$$

Then, we define the chemical potential  $\mu(\rho, c, \nabla c)$  by

$$\mu := \frac{\partial \mathcal{F}}{\partial c} - \frac{1}{\rho} \text{div}(\boldsymbol{\tau}) = \frac{\partial \mathcal{F}}{\partial c} - \frac{1}{\rho} \text{div}\left(\rho \frac{\partial \mathcal{F}}{\partial \nabla c}\right) = \frac{\partial \psi_0}{\partial c} - \frac{\gamma}{\rho} \text{div}(\rho \nabla c),$$

which in turn gives a condition for the Lagrange multiplier

$$L_c = \mu. \tag{7.7.21}$$

Using these previous constitutive relations, we have already canceled some terms in the entropy inequality, *i.e.*

$$\left( \rho \frac{\partial \mathcal{F}}{\partial c} - \operatorname{div}(\boldsymbol{\tau}) - \rho L_c \right) \frac{Dc}{Dt} + \left( \rho \frac{\partial \mathcal{F}}{\partial \nabla c} - \boldsymbol{\tau} \right) \nabla \left[ \frac{Dc}{Dt} \right] = 0.$$

Then, using classical results on isothermal diffusion [263, 146], we have

$$\mathcal{J} := -\frac{\mu \mathbf{J}_1}{T}, \quad (7.7.22)$$

and, using a generalized Fick's law, we have

$$\mathbf{J}_1 := b(c) \nabla \mu, \quad (7.7.23)$$

where  $b(c)$  is a nonnegative mobility function that we will specify in the following. The two constitutive relations for the diffusive fluxes (7.7.22) and (7.7.23) together with (7.7.21), we obtain

$$\operatorname{div}(T\mathcal{J} + L_c \mathbf{J}_1) - \nabla L_c \cdot \mathbf{J}_1 = -b(c) |\nabla \mu|^2 \leq 0.$$

Following [263, 5], we define the pressure inside the mixture

$$p := \rho^2 \frac{\partial \psi_0}{\partial \rho}. \quad (7.7.24)$$

From standard rheology, we assume that the fluid satisfies Newton's rheological laws. The stress tensor is composed of two parts for the viscous  $\tilde{\mathbf{P}}$  and non-viscous  $\mathbf{P}$  contributions of stress

$$\boldsymbol{\sigma} := \mathbf{P} + \tilde{\mathbf{P}}, \quad (7.7.25)$$

and we have by standard continuum mechanics (see *e.g.* [22, 146, 5])

$$\begin{cases} \mathbf{P} = -\left(p - \frac{\gamma}{2} |\nabla c|^2\right) \mathbf{1} - \gamma \rho \nabla c \otimes \nabla c, \\ \tilde{\mathbf{P}} = \nu(c) (\nabla \mathbf{v} + \nabla \mathbf{v}^T) + \lambda(c) \operatorname{div}(\mathbf{v}) \mathbf{1}. \end{cases} \quad (7.7.26)$$

In (7.7.26),  $\nu(c)$  denotes the shear viscosity and  $\lambda(c) = \eta(c) - \frac{2}{3}\nu(c)$  where  $\eta(c)$  is the dilatational viscosity that encodes the response of the fluid to volume changes. The second term in the non-viscous part of the stress (namely  $-\gamma(\rho \nabla c \otimes \nabla c)$ ) represents capillary stresses that act at the interface of the two populations.

Using (7.7.26), we can cancel terms in (7.7.19)

$$\left( -\rho^2 \frac{\partial \mathcal{F}}{\partial \rho} \mathbf{1} - \rho \nabla c \otimes \frac{\partial \mathcal{F}}{\partial \nabla c} - \boldsymbol{\sigma} \right) : \nabla \mathbf{v} = 0.$$

The remaining terms of the entropy inequality are the ones associated with proliferation and friction. The last step to satisfy the entropy inequality is to choose arbitrarily a value for  $c_\rho$ , such that

$$\begin{aligned} & -\left[ (\kappa(\rho, c) + F_\rho) |\mathbf{v}|^2 - F_v \mathbf{v} - \left( \frac{1}{2} |\mathbf{v}|^2 + u - \rho \frac{\partial \mathcal{F}}{\partial \rho} \right) F_\rho \right. \\ & \quad \left. + c_\rho F_\rho + c_c F_c + c_v F_v - L_c (F_c + c F_\rho) \right] \leq 0. \end{aligned}$$

Reorganizing the terms we have

$$-\kappa(\rho, c)|\mathbf{v}|^2 - F_\rho \left[ c_\rho + |\mathbf{v}|^2 - \left( \frac{1}{2}|\mathbf{v}|^2 + u - \rho \frac{\partial \mathcal{F}}{\partial \rho} \right) - \mu c \right] - F_{\mathbf{v}} [c_{\mathbf{v}} - \mathbf{v}] - F_c [c_c - \mu] \leq 0.$$

The obvious choices are

$$\begin{cases} c_\rho = -|\mathbf{v}|^2 + \left( \frac{1}{2}|\mathbf{v}|^2 + u - \rho \frac{\partial \mathcal{F}}{\partial \rho} \right) + \mu c, \\ c_{\mathbf{v}} = \mathbf{v}, \\ c_c = \mu. \end{cases}$$

From the previous constitutive relations, we satisfy the dissipation inequality (7.7.19).

### 7.7.7 Summary of the model's equations

Using the previous constitutive relations our general model is the following compressible Navier-Stokes-Cahn-Hilliard system

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\operatorname{div}(\rho \mathbf{v}) + F_\rho, \\ \rho \frac{Dc}{Dt} &= \operatorname{div}(b(c)\nabla \mu) + F_c - cF_\rho, \\ \rho \mu &= -\gamma \operatorname{div}(\rho \nabla c) + \rho \frac{\partial \psi_0}{\partial c}, \\ \rho \frac{D\mathbf{v}}{Dt} &= -[\nabla p + \gamma \operatorname{div}(\rho \nabla c \otimes \nabla c)] + \operatorname{div}(\nu(c)(\nabla \mathbf{v} + \nabla \mathbf{v}^T)) \\ &\quad - \frac{2}{3} \operatorname{div}(\nu(c)\operatorname{div}(\mathbf{v})\mathbf{1}) + \operatorname{div}(\eta(c)\operatorname{div}(\mathbf{v})\mathbf{1}) - (\kappa(\rho, c) + F_\rho)\mathbf{v} + F_{\mathbf{v}}, \end{aligned} \tag{7.7.27}$$

with  $p$  defined in (7.7.24).

### 7.7.8 Model reductions, general assumptions and biologically relevant choices of the model's functions

#### 7.7.9 Specific choices of functionals and model reductions

**Problem 1 : General compressible NSCH with friction term and mass transfer.**

Assuming no creation of mass nor transfer of mass from the exterior of the system we have

$$F_c = -F_{1-c}, \tag{7.7.28}$$

leading to mass conservation

$$F_\rho = 0. \tag{7.7.29}$$

Furthermore, we assume no external source of velocity and energy, leading to

$$F_{\mathbf{v}} = 0, \text{ and } F_u = 0. \tag{7.7.30}$$

Furthermore, using the same simplifying assumption as in Abels and Feireisl [5] to avoid vacuum



zones, our final reduced system of equations is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (7.7.31)$$

$$\frac{\partial \rho c}{\partial t} + \operatorname{div}(\rho c \mathbf{v}) = \operatorname{div}(b(c) \nabla \mu) + F_c, \quad (7.7.32)$$

$$\rho \mu = -\gamma \Delta c + \rho \frac{\partial \psi_0}{\partial c}, \quad (7.7.33)$$

$$\begin{aligned} \frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) = & - \left[ \nabla p + \gamma \operatorname{div} \left( \nabla c \otimes \nabla c - \frac{1}{2} |\nabla c|^2 \mathbf{1} \right) \right] + \operatorname{div}(\nu(c) (\nabla \mathbf{v} + \nabla \mathbf{v}^T)) \\ & - \frac{2}{3} \operatorname{div}(\nu(c) \operatorname{div}(\mathbf{v}) \mathbf{1}) + \operatorname{div}(\eta(c) \operatorname{div}(\mathbf{v}) \mathbf{1}) - \kappa(\rho, c) \mathbf{v}, \end{aligned} \quad (7.7.34)$$

**Problem 2 : Biologically relevant variant of the system.** We here refer to the two phases of the mixture as cell populations and not fluids. For this variant of the system, we assume the production of mass and neglect certain effects. Namely, we neglect inertia effects, and the viscosity of the fluid, and assume no external source of velocity. This leads to the momentum equation

$$\nabla p + \kappa(\rho, c) \mathbf{v} = -\gamma \operatorname{div} \left( \nabla c \otimes \nabla c - \frac{1}{2} |\nabla c|^2 \mathbf{1} \right) - F_\rho \mathbf{v}.$$

Assuming that one cell population proliferates while the other does not leads to

$$F_c = F_\rho = \rho c P_c(p), \quad \text{and} \quad F_{1-c} = 0,$$

with a pressure-dependent proliferation rate  $P_c(p) \geq 0$ . The growth function  $P_c(p)$  is used to represent the capacity of cells to divide accordingly to the pressure exerted on them. It is well known that cells are able to divide as long as the pressure is not too large. Once a certain pressure  $p_{\max}$  is reached cells enter a quiescent state. Therefore, we assume that

$$P'_c(p) \leq 0, \quad \text{and} \quad P_c(p) = 0 \quad \text{for} \quad p > p_{\max}. \quad (7.7.35)$$

Combining these changes, the model becomes

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = \rho c P_c(p), \\ \frac{\partial \rho c}{\partial t} + \operatorname{div}(\rho c \mathbf{v}) = \operatorname{div}(b(c) \nabla \mu) + \rho c P_c(p), \\ \rho \mu = -\gamma \Delta c + \rho \frac{\partial \psi_0}{\partial c}, \\ \nabla p + \kappa(\rho, c) \mathbf{v} = -\gamma \operatorname{div} \left( \nabla c \otimes \nabla c - \frac{1}{2} |\nabla c|^2 \mathbf{1} \right) - \rho c P_c(p) \mathbf{v}. \end{cases} \quad (7.7.36)$$

### 7.7.10 Biologically consistent choices of functions

As said in the derivation of the model, the free energy density  $\mathcal{F}$  is the sum of two terms :  $\frac{\gamma}{2} |\nabla c|^2$  taking into account the surface tension effects existing between the phases of the mixture and the potential  $\psi_0(\rho, c)$  representing the cell-cell interactions and pressure. Thus, we choose

$$\psi_0(\rho, c) = \psi_e(\rho) + \psi_{\text{mix}}(\rho, c), \quad (7.7.37)$$

with  $\psi_{\text{mix}}(\rho, c) = H(c) \log \rho + Q(c)$ . Then, using the constitutive relation for the pressure we have

$$p(\rho, c) = \rho^2 \frac{\partial \psi_0}{\partial \rho} = p_e(\rho) + \rho H(c). \quad (7.7.38)$$

The function  $b(c)$  is the active mobility of the cells.

Let us explain how the choices of functions for the free energy density and mobility are motivated by biological observations.

To satisfy the conditions (7.2.7), we propose to choose

$$b(c) = C_b c(1-c)^\alpha, \quad \alpha \geq 1, \quad (7.7.39)$$

where  $C_b$  is a positive constant.

We use for the pressure a power law such that

$$p_e(\rho) = \frac{1}{a-1} \rho^{a-1}. \quad (7.7.40)$$

For  $H(c)$  and  $Q(c)$ , two choices can be considered depending on the behavior of the cells we want to represent. If the two cell populations exert attractive forces when they recognize cells of the same type and repulsion with the other type, the potential has to take a form of a double-well for which the two stable phases are located at the bottom of the two wells (see e.g. Figure 7.9a). This is a situation close to the phase separation in binary fluids. Thermodynamically consistent potentials are of Ginzburg-Landau type with the presence of logarithmic terms. An example of double-well potential is given by

$$\psi_{\text{mix}} = \frac{1}{2} (\alpha_1(1-c) \log(\rho(1-c)) + \alpha_2 c \log(\rho c)) - \frac{\theta}{2} (c - \frac{1}{2})^2 + k, \quad (7.7.41)$$

thus giving

$$H(c) = \frac{1}{2} (\alpha_1(1-c) + \alpha_2 c), \quad Q(c) = \frac{1}{2} (\alpha_1(1-c) \log(1-c) + \alpha_2 c \log(c)) - \frac{\theta}{2} (c - \frac{1}{2})^2 + k,$$

where  $\theta > 1$ , and  $k, \alpha_1, \alpha_2 > 0$  are an arbitrary constants.

To meet the phenomenological observations of the interaction between cells when the mixture is composed of only one cell population, a single-well potential seems more appropriate [67, 98].

Indeed, when the distance between cells falls below a certain value (i.e. if the cell density is large enough), cells are attracted to each other. Then, it exists a threshold value called the mechanical equilibrium for which  $\partial_c \psi_0 = 0$  i.e. there is an equilibrium between attractive and repulsive forces. For larger cell densities, cells are packed too close to each other, they thus experience a repulsive force. When cells are so packed that they fill the whole control volume, then the repulsive force becomes infinite due to the pressure. The representation of such functional is depicted in Figure 7.9b. A typical example of single-well potential which has been used for the modeling of living tissue and cancer [98, 11] is

$$\psi_{\text{mix}}(\rho, c) = -(1-c_e) \log(\rho(1-c)) - \frac{c^3}{3} - (1-c_e) \frac{c^2}{2} - (1-c_e)c + k, \quad (7.7.42)$$

thus giving

$$H(c) = -(1-c_e), \quad Q(c) = -(1-c_e) \log(1-c) - \frac{c^3}{3} - (1-c_e) \frac{c^2}{2} - (1-c_e)c + k, \quad (7.7.43)$$

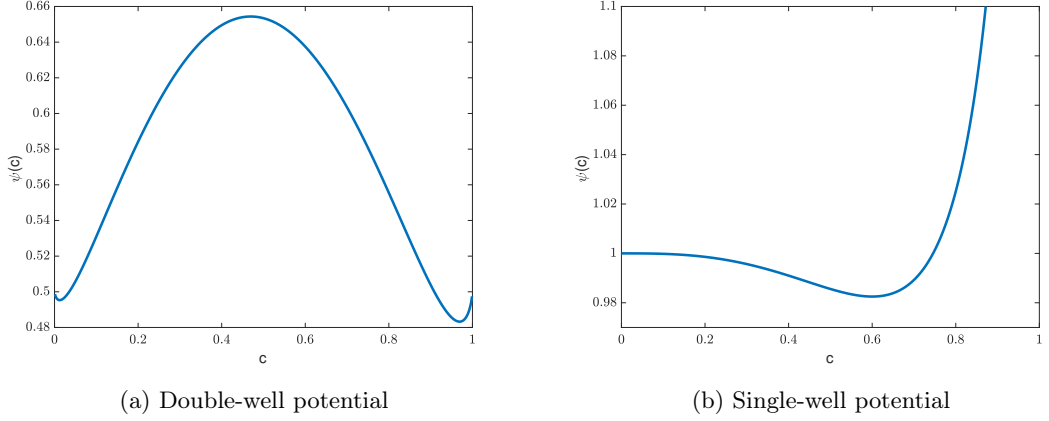


FIGURE 7.9 – For both figures  $\rho = 1$ . Double-well logarithmic potential (left) with  $\alpha_1 = 1.2$  and  $\alpha_2 = 0.8$  and single-well logarithmic potential (right)

where  $k$  is an arbitrary constant.

### 7.7.11 Description of the two-dimensional numerical scheme

We describe the two-dimensional scheme. This scheme possesses the same properties as the one-dimensional scheme.

We write the velocity field  $\mathbf{v} = (u_x, u_y)$ . System (7.1.1)–(7.1.4) with the transformation proposed at the beginning of this section, reads

$$\partial_t \rho + \partial_x(\rho u_x) + \partial_y(\rho u_y) = 0, \quad (7.7.44)$$

$$\begin{aligned} \partial_t \left( \rho \begin{bmatrix} u_x \\ u_y \end{bmatrix} \right) + \begin{bmatrix} \partial_x(\rho u_x^2 + p) \\ \partial_y(\rho u_y^2 + p) \end{bmatrix} + \begin{bmatrix} \partial_y(\rho u_x u_y) \\ \partial_x(\rho u_x u_y) \end{bmatrix} &= 2 \begin{bmatrix} \partial_x(\nu(c) \partial_x u_x) \\ \partial_y(\nu(c) \partial_y u_y) \end{bmatrix} + \begin{bmatrix} \partial_y(\nu(c) (\partial_y u_x + \partial_x u_y)) \\ \partial_x(\nu(c) (\partial_y u_x + \partial_x u_y)) \end{bmatrix} \\ + \begin{bmatrix} \partial_x \left( (\eta(c) - \frac{2}{3} \nu(c)) (\partial_x u_x + \partial_y u_y) \right) \\ \partial_y \left( (\eta(c) - \frac{2}{3} \nu(c)) (\partial_x u_x + \partial_y u_y) \right) \end{bmatrix} - \frac{\gamma}{2} \begin{bmatrix} \partial_x((\partial_x c)^2 - (\partial_y c)^2) \\ \partial_y((\partial_y c)^2 - (\partial_x c)^2) \end{bmatrix} - \gamma \begin{bmatrix} \partial_y(\partial_x c \partial_y c) \\ \partial_x(\partial_x c \partial_y c) \end{bmatrix} \\ - \kappa(\rho, c) \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \end{aligned} \quad (7.7.45)$$

$$\rho(\partial_t v + u_x \partial_x v + u_y \partial_y v) = \frac{1}{T'(v)} (\partial_x(b(c) \partial_x \mu) + \partial_y(b(c) \partial_y \mu)) + \frac{1}{T'(v)} F_c, \quad (7.7.46)$$

$$\rho \mu = -\gamma T'(v) (\partial_{xx} c + \partial_{yy} c) - \gamma T''(v) ((\partial_x v)^2 + (\partial_y v)^2) + \rho \frac{\partial \psi_0}{\partial c}, \quad (7.7.47)$$

$$\frac{dr}{dt} = -\frac{r(t)}{E[t] + C_0} \int_{\Omega} b(c) |\nabla \mu|^2 - \mu F_c \, dx. \quad (7.7.48)$$

We introduce the notations  $U = (\rho, \rho u_x, \rho u_y)$ ,  $G(U) = (0, -\kappa u_x, -\kappa u_y)$  and

$$\begin{aligned} F(U) &= (\rho u_x, \rho u_x^2 + p - 2\nu(c) \partial_x u_x + \left( \frac{2}{3} \nu(c) - \eta(c) \right) (\partial_x u_x + \partial_y u_y) + \frac{1}{2} \gamma ((\partial_x c)^2 - (\partial_y c)^2), \\ &\quad \rho u_x u_y - \nu(c) (\partial_y u_x + \partial_x u_y) + \gamma \partial_x c \partial_y c, \end{aligned}$$

$$K(U) = (\rho u_y, \rho u_x u_y - \nu(c) (\partial_y u_x + \partial_x u_y) + \gamma \partial_x c \partial_y c, \\ \rho u_y^2 + p - 2\nu(c) \partial_y u_y + \left( \frac{2}{3} \nu(c) - \eta(c) \right) (\partial_x u_x + \partial_y u_y) + \frac{1}{2} \gamma ((\partial_y c)^2 - (\partial_x c)^2)).$$

The stabilization (see [233, 210]) of the Navier-Stokes part of our system reads, with  $\iota > 0$  the relaxation parameter,

$$\begin{cases} \partial_t U + \partial_x V + \partial_y W = G(U), \\ \partial_t V + A \partial_x U = -\frac{1}{\iota} (V - F(U)), \\ \partial_t W + B \partial_y U = -\frac{1}{\iota} (W - K(U)), \end{cases} \quad (7.7.49)$$

in which  $A = \text{diag}(a_1, a_2, a_3)$  and  $B = \text{diag}(b_1, b_2, b_3)$ . In the following, we choose

$$a_1 = a_2 = a_3 = \max\{\sup (u_x + \sqrt{\partial_\rho p})^2, \sup u_x^2, \sup (u_x - \sqrt{\partial_\rho p})^2\}, \\ b_1 = b_2 = b_3 = \max\{\sup (u_y + \sqrt{\partial_\rho p})^2, \sup u_y^2, \sup (u_y - \sqrt{\partial_\rho p})^2\}.$$

We assume that our two-dimensional domain is a square  $[0, L] \times [0, L]$ . We discretize the domain using square control volumes of size  $\Delta x \times \Delta y$ . The cell centers are located at positions  $(x_j, y_j)$ , and we approximate the value of a variable at the cell center by its mean, *e.g.*

$$\rho_{j,i} = \frac{1}{\Delta x \Delta y} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \rho(\mathbf{x}, t) \, d\mathbf{x}.$$

Simply employing a first-order time discretization, the numerical scheme becomes

$$U_{j,i}^* = U_{j,i}^n, \quad (7.7.50)$$

$$V_{j,i}^* = V_{j,i}^n - \frac{\Delta t}{\eta} (V_{j,i}^* - F(U_{j,i}^*)), \quad (7.7.51)$$

$$W_{j,i}^* = W_{j,i}^n - \frac{\Delta t}{\eta} (W_{j,i}^* - K(U_{j,i}^*)), \quad (7.7.52)$$

$$U_{j,i}^{n+1} = U_{j,i}^* - \frac{\Delta t}{\Delta x} (V_{j+\frac{1}{2},i}^* - V_{j-\frac{1}{2},i}^*) - \frac{\Delta t}{\Delta y} (W_{j,i+\frac{1}{2}}^* - W_{j,i-\frac{1}{2}}^*) + \Delta t G(U_{j,i}^{n+1}), \quad (7.7.53)$$

$$V_{j,i}^{n+1} = V_{j,i}^* - \frac{\Delta t}{\Delta x} A (U_{j+\frac{1}{2},i}^* - U_{j-\frac{1}{2},i}^*), \quad (7.7.54)$$

$$W_{j,i}^{n+1} = W_{j,i}^* - \frac{\Delta t}{\Delta y} B (U_{j,i+\frac{1}{2}}^* - U_{j,i-\frac{1}{2}}^*), \quad (7.7.55)$$

$$\frac{\bar{v}_{j,i}^{n+1} - v_{j,i}^n}{\Delta t} + \mathbf{v}_{j,i}^{n+1} \cdot (\nabla \bar{v}^{n+1})_{j,i} = g(c^n, \mu^{n+1}, \rho^{n+1})_{j,i}, \quad (7.7.56)$$

$$\begin{aligned} g(c^n, \mu^{n+1}, \rho^{n+1})_{j,i} &= \frac{1}{T'(v_{j,i}^n) \rho_{j,i}^{n+1} \Delta x} \left( (b(c^n) \nabla \mu^{n+1})_{j+\frac{1}{2},i} - (b(c^n) \nabla \mu^{n+1})_{j-\frac{1}{2},i} \right) \\ &\quad + \frac{1}{T'(v_{j,i}^n) \rho_{j,i}^{n+1} \Delta y} \left( (b(c^n) \nabla \mu^{n+1})_{j,i+\frac{1}{2}} - (b(c^n) \nabla \mu^{n+1})_{j,i-\frac{1}{2}} \right) \\ &\quad + \frac{F_c(\rho_{j,i}^n, c_{j,i}^n)}{T'(v_{j,i}^n) \rho_{j,i}^{n+1}}, \end{aligned} \quad (7.7.57)$$

$$\mu_{j,i}^{n+1} = \frac{1}{\rho_{j,i}^n} \left( -\gamma T'(v_{j,i}^n) (\Delta \bar{v}^{n+1})_{j,i} - \gamma T''(v_{j,i}^n) |(\nabla v^n)_{j,i}|^2 \right) + \left( \frac{\partial \psi_0}{\partial c} \right)_{j,i}^n, \quad (7.7.58)$$

$$\int_{\Omega} T(\lambda \bar{v}^{n+1}) \, dx = \int_{\Omega} c^n + \Delta t F_c \, dx, \quad (7.7.59)$$

$$\bar{c}_{j,i}^{n+1} = T(\lambda_j \bar{v}_{j,i}^{n+1}), \quad (7.7.60)$$

$$\begin{aligned} \frac{1}{\Delta t} (r^{n+1} - r^n) &= -\frac{r^{n+1}}{E(\bar{c}^{n+1}) + C_0} \int_{\Omega} b(\bar{c}^{n+1}) |\nabla \mu^{n+1}|^2 \, dx + \\ &\quad + \frac{r^{n+1}}{E(\bar{c}^{n+1}) + C_0} \int_{\Omega} \mu^{n+1} F_c(\rho^{n+1}, \bar{c}^{n+1}) \, dx, \end{aligned} \quad (7.7.61)$$

$$\xi^{n+1} = \frac{r^{n+1}}{E(\bar{c}^{n+1}) + C_0}, \quad (7.7.62)$$

$$c_{j,i}^{n+1} = \nu^{n+1} \bar{c}_{j,i}^{n+1}, \quad \text{with } \nu^{n+1} = 1 - (1 - \xi^{n+1})^2, \quad (7.7.63)$$

$$v_{j,i}^{n+1} = \nu^{n+1} \bar{v}_{j,i}^{n+1}. \quad (7.7.64)$$

Quatrième partie

Kinetic theory and the  
Cahn-Hilliard equation



## Chapitre 8

# From Vlasov equation to degenerate nonlocal Cahn-Hilliard equation

### Abstract

We provide a rigorous mathematical framework to establish the hydrodynamic limit of the Vlasov model introduced in [317] by Noguchi and Takata in order to describe phase transition of fluids by kinetic equations. We prove that, when the scale parameter tends to 0, this model converges to a nonlocal Cahn-Hilliard equation with degenerate mobility. For our analysis, we introduce appropriate forms of the short and long range potentials which allow us to derive Helmholtz free energy estimates. Several compactness properties follow from the energy, the energy dissipation and kinetic averaging lemmas. In particular we prove a new weak compactness bound on the flux.

### 8.1 Introduction

We consider the following Vlasov-Cahn-Hilliard equation (VCH in short)

$$\begin{cases} \varepsilon^2 \partial_t f_\varepsilon + \varepsilon \xi \cdot \nabla_x f_\varepsilon + \varepsilon F_\varepsilon \cdot \nabla_\xi f_\varepsilon = \varrho_\varepsilon(t, x) M(\xi) - f_\varepsilon, & t \geq 0, x \in \mathbb{R}^d, \xi \in \mathbb{R}^d, \\ \varrho_\varepsilon(t, x) = \int_{\mathbb{R}^d} f_\varepsilon(t, x, \xi) d\xi, \end{cases} \quad (8.1.1)$$

with an initial data  $f_\varepsilon(0, x, \xi) = f^0(x, \xi) \geq 0$ . The unknown is the function

$$f_\varepsilon \equiv f_\varepsilon(t, x, \xi), \quad t \in (0, T), x \in \mathbb{R}^d, \xi \in \mathbb{R}^d,$$

such that, for every infinitesimal volume  $dx d\xi$  around the point  $(x, \xi)$  in the phase space, the quantity  $f_\varepsilon(t, x, \xi) dx d\xi$  is the number of particles which have position  $x$  and velocity  $\xi$  at fixed time  $t$ . The small parameter  $\varepsilon > 0$  arises from physical dimensions of the system and we are interested in the limit when it tends to 0. Following [317], the force field  $F_\varepsilon(t, x)$  is decomposed as long-range attractive and short-range repulsive

$$F_\varepsilon = F_\varepsilon^L + F_\varepsilon^S, \quad F_\varepsilon^{L,S}(t, x) = -\nabla \Phi_\varepsilon^{L,S}(t, x). \quad (8.1.2)$$



We define the convolution in the space variable as  $f \star g = \int_{\mathbb{R}^d} f(y)g(x-y) dy$  and set

$$\Phi_{\alpha,\varepsilon}^S(t, x) = \frac{1}{\alpha^2} \omega^S \star \omega^S \star \varrho_\varepsilon,$$

where  $\omega^S \geq 0$  is a function that may be thought of as a centered Gaussian. We use a double convolution in order to enforce positivity of the corresponding operator as it appears in energy considerations. We assume that  $\omega^S$  satisfies

$$\int_{\mathbb{R}^d} \omega^S(y) dy = 1, \quad \int_{\mathbb{R}^d} y \omega^S(y) dy = 0, \quad \int_{\mathbb{R}^d} |y|^2 \omega^S(y) dy < \infty. \quad (8.1.3)$$

The long-range potential is of the form

$$\Phi_{\alpha,\varepsilon}^L(t, x) = -\frac{1}{\alpha^2} \omega_\alpha^L \star \omega^S \star \omega^S \star \varrho_\varepsilon, \quad (8.1.4)$$

where  $\omega_\alpha^L(x) = \frac{1}{\alpha^d} \omega^L\left(\frac{x}{\alpha}\right)$  may be thought of as a high temperature Gaussian and  $\omega^L$  is a smooth, nonnegative, symmetric, compactly supported function such that, for some  $\delta > 0$ ,

$$\int_{\mathbb{R}^d} \omega^L(y) dy = 1, \quad \int_{\mathbb{R}^d} y \omega^L(y) dy = 0, \quad \int_{\mathbb{R}^d} y_i y_j \omega^L dy = \delta_{i,j} \delta, \quad \int_{\mathbb{R}^d} \omega^L(y) |y|^3 dy < \infty. \quad (8.1.5)$$

The equilibrium distribution  $M(\xi) \geq 0$  is a Maxwellian that we normalize as

$$M(\xi) := \left(\frac{1}{2\pi D}\right)^{d/2} \exp\left(-\frac{|\xi|^2}{2D}\right), \quad (8.1.6)$$

and we have, for  $i = 1, \dots, d$ ,

$$\int_{\mathbb{R}^d} M(\xi) d\xi = 1, \quad \int_{\mathbb{R}^d} \xi_i M(\xi) d\xi = 0, \quad \int_{\mathbb{R}^d} \xi_i^2 M(\xi) d\xi = D < \infty, \quad (8.1.7)$$

so that  $D$  can be interpreted as the diffusion coefficient.

### 8.1.1 The macroscopic limit

The right-hand side of Equation (8.1.1) is a relaxation term that conserves mass but neither momentum nor energy since we aim at using a diffusive scaling. Formally one can guess that

$$f_\varepsilon(t, x, \xi) \rightarrow \varrho(t, x) M(\xi), \quad \text{as } \varepsilon \rightarrow 0. \quad (8.1.8)$$

The mass conservation equation on  $\varrho_\varepsilon$  is obtained by integrating Equation (8.1.1) with respect to  $\xi$  against 1,

$$\partial_t \varrho_\varepsilon(t, x) + \operatorname{div} J_\varepsilon(t, x) = 0, \quad J_\varepsilon(t, x) = \int_{\mathbb{R}^d} \frac{\xi}{\varepsilon} f_\varepsilon(t, x, \xi) d\xi. \quad (8.1.9)$$

Then, integrating against  $\xi$ , we obtain the flux equation

$$\varepsilon^2 \partial_t J_\varepsilon(t, x) + \nabla_x \cdot \int_{\mathbb{R}^d} \xi \otimes \xi f_\varepsilon(t, x, \xi) d\xi - F_\varepsilon \varrho_\varepsilon = -J_\varepsilon(t, x). \quad (8.1.10)$$

Combined with (8.1.8), this flux equation allows us to identify the limit of  $J_\varepsilon$  and to prove that as  $\varepsilon, \alpha \rightarrow 0$ , the macroscopic densities tend to a solution of a degenerate nonlocal Cahn-Hilliard equation type. More precisely, we have the

**Theorem 8.1.1** (Limit  $\varepsilon \rightarrow 0$ ). *With the assumptions and notations (8.1.2)–(8.1.6), let  $\alpha = \varepsilon$ . Let  $f^0$  be a non-negative distribution that satisfies (8.2.1)–(8.2.2) and let  $f_\varepsilon$  be a solution of (8.1.1) with initial condition  $f^0$ . Then, we can extract a subsequence (not relabelled) such that  $\varrho_\varepsilon \rightarrow \varrho$  in  $L_t^p L_x^1$  strongly for  $1 \leq p < \infty$  where  $\varrho$  solves in the distributional sense the equation*

$$\partial_t \varrho - D\Delta \varrho - \operatorname{div}(\varrho \nabla \Phi) = 0, \quad \Phi = -\delta \Delta [\omega^S \star \omega^S \star \varrho], \quad (8.1.11)$$

with initial data  $\varrho^0 = \int_{\mathbb{R}^d} f^0(x, \xi) d\xi$ .

In fact, [317] obtains formally a more complete description which we cannot prove at the moment (see Section 8.4).

**Remark 8.1.2.** • Writing formally  $\Delta \varrho = \operatorname{div}(\varrho \nabla \log(\varrho))$ , this term can be added to the potential so as to obtain a kind of Cahn-Hilliard equation.

- Different scaling between  $\alpha$  and  $\varepsilon$  can be considered,  $\alpha$  constant is also possible
- Uniqueness can be proved in the class of uniformly bounded densities, see Appendix 8.6.3.
- The proof of this result uses compactness arguments, therefore we do not have any explicit control on the error between the limit solution and the system in terms of  $\varepsilon$ .

### 8.1.2 Contents

In Section 8.2, we collect various uniform estimates  $\varepsilon$ . Section 8.3 is devoted to passing to the limit  $\varepsilon \rightarrow 0$ . Some open problems are drawn in Section 8.4. The Appendix contains different mathematical tools and lemmas used throughout the proofs.

### 8.1.3 Literature review and relevancy of the system

**Phase transitions in fluids** In [317], Noguchi and Takata consider a kinetic model to capture the dynamics of phase transition for the Van der Waals fluid. The model reads as follows

$$\begin{aligned} \partial_t f + \xi \cdot \nabla_x f + F \cdot \nabla_\xi f &= A(\varrho)(\varrho M - f), \quad A(\varrho) > 0, \\ F &= F^1 + F^2, \quad F^{1,2} = -\nabla \Phi^{1,2}, \\ \Phi^1 &= \begin{cases} \varrho - \omega^L \star \varrho \\ \text{or } -\kappa \Delta \varrho \end{cases}, \quad \Phi^2 = -C_1 \log(1 - \varrho) + \frac{\varrho}{1 - \varrho} - C_2 \varrho, \quad C_1, C_2 \in \mathbb{R}, \end{aligned}$$

for some kernel  $\omega^L$  and  $\kappa > 0$ .  $\Phi^1$  is a combination of short range repulsion and long range attraction.  $\Phi^2$  is a short range interaction potential.

The authors state that full details of intermolecular collisions are not considered and that the collision term on the right-hand side plays just a thermal bath role which is less desirable for a physical/mechanical justification of the system. This problem has been addressed formally in [193, 316]. From the mathematical point of view, it raises new and interesting different difficulties and it can be definitely seen as a next step in our work. Nevertheless, the authors show that with this thermal bath term, the system exhibits the essential features of phase transition dynamics, both theoretically and numerically. By placing themselves in the framework of the strong interaction, they find a rescaling of the first equation of the system and obtain Equation (8.1.1) of VCH.

Then, setting  $A \equiv 1$  and letting  $\varepsilon \rightarrow 0$  they obtain formally that in the limit (that we refer to as the hydrodynamic limit), the macroscopic density  $\varrho$  satisfy

$$\partial_t \varrho - \Delta \varrho - \operatorname{div}(\varrho \nabla(\Phi^1 + \Phi^2)) = 0.$$

Noting that  $\Delta \varrho = \operatorname{div}(\varrho \nabla \log(\varrho))$  they obtain the Cahn-Hilliard equation with degenerate mobility

$$\partial_t \varrho - \operatorname{div}(\varrho \nabla(\Phi^1 + \tilde{\Phi}^2)) = 0, \quad \tilde{\Phi}^2 = \Phi^2 + \log(\varrho).$$

This model presents several mathematical difficulties. First of all, we are not aware of any existence result concerning the Vlasov equation when the potential  $\Phi$  is a function of the density  $\varrho$ . In the Vlasov-Poisson system, one has  $\Phi = \Delta^{-1} \varrho$  and there is a gain of two derivatives. For the existence of classical solutions for Vlasov-Poisson we refer to [255, 292, 306, 218]. A second difficulty comes from the rigorous passage to the limit. Indeed, the bound provided by the energy do not provide enough compactness. For instance, one cannot apply the averaging lemma 8.3.4 on this system because the functions are not bounded in  $L^1$  uniformly in  $\varepsilon$ . For these reasons, we add the convolutions  $\omega^S$  in (8.1.1)–(8.1.4) and provide a rigorous mathematical framework to establish the hydrodynamic limit of this model when  $\Phi^2 = 0$ . It would be possible to prove a similar result when  $\Phi^2 = \omega^S \star f'(\varrho)$  where  $|f(\varrho)| \leq C|\varrho \log \varrho|$  for  $C$  small enough.

Our work also provides a generic model to obtain different nonlocal and degenerate equations of Cahn-Hilliard/thin-film type as the hydrodynamic limit of kinetic models. For other kinetic models modeling phase transitions, we refer to [193, 197, 242].

**Kinetic theory** The main purpose of kinetic theory is to provide a description of the evolution of a gas or plasma, and more generally a many-particle system made up of  $N$  similar individual elements, in the limit when  $N$  tends to infinity which corresponds to the so-called thermodynamical limit.

In the kinetic theory, the density of particles is described with the probability measure

$$f \equiv f(t, x, \xi), \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^d,$$

such that, for every infinitesimal volume  $dx d\xi$  around the point  $(x, \xi)$  in the phase space, the quantity  $f(t, x, \xi) dx d\xi$  is the number of particles which have position  $x$  and velocity  $\xi$  at fixed time  $t$ . For this reason,  $f$  is a nonnegative function and integrable in both space and velocity variables, but it is not directly observable. Nevertheless, at each point of the domain it provides all measurable macroscopic quantities which can be expressed in terms of microscopic averages :

$$\begin{aligned} \varrho(t, x) &= \int_{\mathbb{R}^d} f(t, x, \xi) d\xi && \text{(macroscopic density),} \\ J(t, x) &= \int_{\mathbb{R}^d} \xi f(t, x, \xi) d\xi && \text{(flux).} \end{aligned}$$

It is clear that such a statistical description makes sense only with a very large number of particles, and as a consequence, all kinetic equations are expected to approximate the true dynamics of gases just in the thermodynamical limit. Rescaling the time and space with a parameter  $\varepsilon$ , *i.e.*  $t \rightarrow \varepsilon^2 t$ ,  $x \rightarrow \varepsilon x$  and sending  $\varepsilon \rightarrow 0$  is called the hydrodynamic limit. It allows us to find a rigorous derivation of macroscopic models from a microscopic description of matter. For hydrodynamics on the Vlasov-Poisson-Fokker-Planck system, we refer to [149, 195].

Our aim is to obtain an equation on the macroscopic density and to relate it to a known model that has applications in fluid dynamics or biology, *i.e.* the Cahn-Hilliard equation.

**The Cahn-Hilliard equation** Equation (8.1.11) is an example of a Cahn-Hilliard type equation that is widely used nowadays to represent phase transitions in fluids and living tissues [328, 176, 187, 186, 179, 144, 145, 287, 151, 264, 154]. Originally introduced in the context of materials sciences [70, 69], it is currently applied in numerous fields, including complex fluids, polymer science, and mathematical biology. For the overview of mathematical theory, we refer to [277].

Cahn-Hilliard equation takes the form of

$$\partial_t \varrho = \operatorname{div} (b(\varrho) \nabla (\Phi(\varrho) - \delta \Delta \varrho)) \rightarrow \begin{cases} \partial_t \varrho &= \operatorname{div} (b(\varrho) \nabla \mu), \\ \Phi &= -\delta \Delta \varrho + \Phi^I(\varrho), \end{cases} \tag{8.1.12}$$

where  $\varrho$  represents the relative density of one component  $\varrho = \varrho_1 / (\varrho_1 + \varrho_2)$ ,  $b(\varrho)$  is the mobility,  $\Phi^I$  is the interaction potential while  $\Phi$  is the quantity of chemical potential.

We obtain a nonlocal version of the Cahn-Hilliard equation. The nonlocality comes from the convolution of the Laplace operator with a smooth kernel  $\omega^S$  concentrated around the origin. There is a different possibility to approximate this operator nonlocally, we refer for instance to [123, 273], where the authors prove the convergence of a nonlocal Cahn-Hilliard equation with constant mobility to a local Cahn-Hilliard equation and to [155] for a similar result with degenerate mobility. In our case, because of the degenerate mobility, it is not clear that we can pass from the nonlocal Equation (8.1.11) to a local one by sending  $\omega^S$  to a Dirac mass.

## 8.2 Entropy, energy, and uniform estimates

The analysis relies on various uniform bounds in  $\varepsilon$  which use an initial data that satisfies

$$\int_{\mathbb{R}^{2d}} (1 + |x| + |\xi|^2 + |\log f^0|) f^0(x, \xi) \, dx \, d\xi < +\infty, \tag{8.2.1}$$

$$\sup_{\alpha \leq 1} \frac{1}{\alpha^2} \int_{\mathbb{R}^{2d}} \omega_\alpha^L(y) [\varrho^0 \star \omega^S(x) - \varrho^0 \star \omega^S(x - y)]^2 \, dx \, dy < +\infty. \tag{8.2.2}$$

Then, we begin with proving the bounds

**Theorem 8.2.1** (Uniform estimates). *With the assumptions (8.2.1) and (8.2.2), the following uniform estimates hold for  $\varepsilon \in (0, 1)$  :*

- (A)  $\{f_\varepsilon\}$  in  $L_t^\infty L_{x,\xi}^1$  and  $\{\varrho_\varepsilon\}$  in  $L_t^\infty L_x^1$ ,
- (B)  $\{f_\varepsilon |\log(f_\varepsilon)|\}$  and  $\{f_\varepsilon |\xi|^2\}$  in  $L_t^\infty L_{x,\xi}^1$ ,
- (C)  $\{\varrho_\varepsilon |\log(\varrho_\varepsilon)|\}$  in  $L_t^\infty L_x^1$ ,
- (D)  $\left\{ \frac{(\varrho_\varepsilon M - f_\varepsilon)(\log(\varrho_\varepsilon M) - \log(f_\varepsilon))}{\varepsilon^2} \right\}$  in  $L_{t,x,\xi}^1$ ,
- (E)  $\left\{ \frac{\varrho_\varepsilon M - f_\varepsilon}{\varepsilon} \right\}$  in  $L_{t,x,\xi}^1$ ,
- (F)  $\{J_\varepsilon\}$  and  $\{J_\varepsilon \log^{1/2} \log^{1/2} \max(J_\varepsilon, e)\}$  in  $L_{t,x}^1$ ,

(G)  $\{f_\varepsilon|x|\}$ ,  $\{\varrho_\varepsilon|x|\}$  in  $L_t^\infty L_{x,\xi}^1$  and  $L_t^\infty L_x^1$  respectively.

Moreover,  $\{\varrho_\varepsilon\}$  and  $\{J_\varepsilon\}$  are weakly compact in  $L_{t,x}^1$ .

The proof of these estimates uses a fundamental property of energy dissipation. To show that, we define the energy (kinetic+potential) and the Helmholtz free energy respectively as

$$\mathcal{E}(t) := \int_{\mathbb{R}^{2d}} |\xi|^2 f_\varepsilon \, dx \, d\xi + \frac{1}{2\alpha^2} \int_{\mathbb{R}^{2d}} \omega_\alpha^L(y) [\varrho_\varepsilon \star \omega^S(t, x) - \varrho_\varepsilon \star \omega^S(t, x - y)]^2 \, dx \, dy, \quad (8.2.3)$$

$$\mathcal{F}(t) := \int_{\mathbb{R}^{2d}} [2D \log(f_\varepsilon) + |\xi|^2] f_\varepsilon \, dx \, d\xi + \frac{1}{2\alpha^2} \int_{\mathbb{R}^{2d}} \omega_\alpha^L(y) [\varrho_\varepsilon \star \omega^S(t, x) - \varrho_\varepsilon \star \omega^S(t, x - y)]^2 \, dx \, dy. \quad (8.2.4)$$

The Helmholtz free energy satisfies the

**Theorem 8.2.2** (Free energy dissipation). *The free energy  $\mathcal{F}(t)$  is dissipated as*

$$\frac{d}{dt} \mathcal{F}(t) = -\frac{2D}{\varepsilon^2} \int_{\mathbb{R}^{2d}} [f_\varepsilon - \varrho_\varepsilon M(\xi)] [\log f_\varepsilon - \log(\varrho_\varepsilon M(\xi))] \, dx \, d\xi = -2D \int_{\mathbb{R}^{2d}} \mathcal{D}_\varepsilon \, dx \, d\xi, \quad (8.2.5)$$

where the dissipation term is defined as

$$\mathcal{D}_\varepsilon(t, x, \xi) := \frac{1}{\varepsilon^2} [f_\varepsilon - \varrho_\varepsilon M(\xi)] [\log f_\varepsilon - \log(\varrho_\varepsilon M(\xi))] \geq 0. \quad (8.2.6)$$

This theorem can be seen as a combination of relations for both the total energy and the entropy of the system.

**Proposition 8.2.3** (Total energy dissipation). *The total energy  $\mathcal{E}(t)$  is dissipated as*

$$\frac{d}{dt} \mathcal{E}(t) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^{2d}} |\xi|^2 [\varrho_\varepsilon M(\xi) - f_\varepsilon] \, dx \, d\xi. \quad (8.2.7)$$

*Démonstration.* By multiplying (8.1.1) by  $|\xi|^2$  and taking the integrals with respect to  $x$  and  $\xi$  we obtain

$$\begin{aligned} \varepsilon^2 \int_{\mathbb{R}^{2d}} |\xi|^2 \partial_t f_\varepsilon \, dx \, d\xi + \varepsilon \int_{\mathbb{R}^{2d}} |\xi|^2 \xi \cdot \nabla_x f_\varepsilon \, dx \, d\xi + \varepsilon \int_{\mathbb{R}^{2d}} |\xi|^2 F_\varepsilon \nabla_\xi f_\varepsilon \, dx \, d\xi \\ = \int_{\mathbb{R}^{2d}} |\xi|^2 [\varrho_\varepsilon M(\xi) - f_\varepsilon] \, dx \, d\xi. \end{aligned} \quad (8.2.8)$$

For integrable solutions, the second term on the left-hand side vanishes. Furthermore, with integration by parts, the above equation reduces to

$$\varepsilon^2 \frac{d}{dt} \int_{\mathbb{R}^{2d}} |\xi|^2 f_\varepsilon \, dx \, d\xi - 2\varepsilon \int_{\mathbb{R}^{2d}} \xi F_\varepsilon f_\varepsilon \, dx \, d\xi = \int_{\mathbb{R}^{2d}} |\xi|^2 [\varrho_\varepsilon M(\xi) - f_\varepsilon] \, dx \, d\xi. \quad (8.2.9)$$

By recalling (8.1.9), the second term can be rewritten as

$$-2\varepsilon \int_{\mathbb{R}^{2d}} \xi F_\varepsilon f_\varepsilon \, dx \, d\xi = -2\varepsilon^2 \int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon} \operatorname{div} J_\varepsilon \, dx = 2\varepsilon^2 \int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon} \partial_t \varrho_\varepsilon \, dx. \quad (8.2.10)$$

We now want to prove that

$$2 \int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon} \partial_t \varrho_\varepsilon \, dx = \frac{1}{2\alpha^2} \frac{d}{dt} \int_{\mathbb{R}^{2d}} \omega_\alpha^L(y) [\varrho_\varepsilon \star \omega^S(t, x) - \varrho_\varepsilon \star \omega^S(t, x - y)]^2 \, dx \, dy. \quad (8.2.11)$$

First, by recalling (8.1.2),

$$2 \int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}(t, x) \partial_t \varrho_\varepsilon(t, x) \, dx = 2 \int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}^L(t, x) \partial_t \varrho_\varepsilon(t, x) \, dx + 2 \int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}^S(t, x) \partial_t \varrho_\varepsilon(t, x) \, dx. \quad (8.2.12)$$

As regards the first term on the right-hand side

$$\begin{aligned} 2 \int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}^L(t, x) \partial_t \varrho_\varepsilon(t, x) \, dx &= -\frac{2}{\alpha^2} \int_{\mathbb{R}^d} [\omega_\alpha^L \star \omega^S \star \omega^S \star \varrho_\varepsilon](t, x) \partial_t \varrho_\varepsilon(t, x) \, dx \\ &= -\frac{2}{\alpha^2} \int_{\mathbb{R}^d} [\omega_\alpha^L \star \varrho_\varepsilon \star \omega^S](t, x) \partial_t [\varrho_\varepsilon \star \omega^S](t, x) \, dx \\ &= -\frac{1}{\alpha^2} \int_{\mathbb{R}^{2d}} \omega_\alpha^L(y) [\varrho_\varepsilon \star \omega^S](t, x - y) \partial_t [\varrho_\varepsilon \star \omega^S](t, x) \, dx \, dy \\ &\quad - \frac{1}{\alpha^2} \int_{\mathbb{R}^{2d}} \omega_\alpha^L(y) [\varrho_\varepsilon \star \omega^S](t, x) \partial_t [\varrho_\varepsilon \star \omega^S](t, x - y) \, dx \, dy \\ &= -\frac{1}{\alpha^2} \frac{d}{dt} \int_{\mathbb{R}^{2d}} \omega_\alpha^L(y) [[\varrho_\varepsilon \star \omega^S](t, x) \cdot [\varrho_\varepsilon \star \omega^S](t, x - y)] \, dx \, dy \end{aligned}$$

The second term on the right-hand side can be handled similarly and gives

$$\begin{aligned} 2 \int_{\mathbb{R}^d} \Phi_{\alpha,\varepsilon}^S(t, x) \partial_t \varrho_\varepsilon(t, x) \, dx &= \frac{2}{\alpha^2} \int_{\mathbb{R}^d} [\omega^S \star \varrho_\varepsilon](t, x) \partial_t [\varrho_\varepsilon \star \omega^S](t, x) \, dx \\ &= \frac{1}{\alpha^2} \frac{d}{dt} \int_{\mathbb{R}^d} [\varrho_\varepsilon \star \omega^S]^2(t, x) \, dx \\ &= \frac{1}{2\alpha^2} \frac{d}{dt} \int_{\mathbb{R}^{2d}} \omega_\alpha^L(y) [[\varrho_\varepsilon \star \omega^S]^2(t, x) + [\varrho_\varepsilon \star \omega^S]^2(t, x - y)] \, dx \, dy. \end{aligned}$$

By summing up the two previous identities we get (8.2.11), which, inserted in (8.2.9), concludes that

$$\begin{aligned} \varepsilon^2 \frac{d}{dt} \int_{\mathbb{R}^{2d}} |\xi|^2 f_\varepsilon \, dx \, d\xi + \frac{\varepsilon^2}{2\alpha^2} \frac{d}{dt} \int_{\mathbb{R}^{2d}} \omega_\alpha^L(y) [\varrho_\varepsilon \star \omega^S(t, x) - \varrho_\varepsilon \star \omega^S(t, x - y)]^2 \, dx \, dy \\ = \int_{\mathbb{R}^{2d}} |\xi|^2 [\varrho_\varepsilon M(\xi) - f_\varepsilon] \, dx \, d\xi. \end{aligned}$$

□

**Proposition 8.2.4** (Entropy relation). *The following estimate holds :*

$$\frac{d}{dt} \int_{\mathbb{R}^{2d}} f_\varepsilon \log f_\varepsilon \, dx \, d\xi = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^{2d}} [\varrho_\varepsilon M(\xi) - f_\varepsilon] \log f_\varepsilon \, dx \, d\xi. \quad (8.2.13)$$

*Démonstration.* By multiplying (8.1.1) by  $(1 + \log f_\varepsilon)$  we obtain

$$\varepsilon^2 \frac{d}{dt} (f_\varepsilon \log f_\varepsilon) + \varepsilon \xi \cdot \nabla_x f_\varepsilon (1 + \log f_\varepsilon) + \varepsilon F_\varepsilon \nabla_\xi f_\varepsilon (1 + \log f_\varepsilon) = [\varrho_\varepsilon M(\xi) - f_\varepsilon] (1 + \log f_\varepsilon)$$

By taking the integrals with respect to  $x$  and  $\xi$ , the second and third terms in the above equation vanish and we obtain

$$\varepsilon^2 \frac{d}{dt} \int_{\mathbb{R}^{2d}} f_\varepsilon \log f_\varepsilon \, dx \, d\xi = \int_{\mathbb{R}^{2d}} [\varrho_\varepsilon M(\xi) - f_\varepsilon] \log f_\varepsilon \, dx \, d\xi$$

as announced.  $\square$

With these two estimates, we can finally prove Theorem 8.2.2.

*Proof of Theorem 8.2.2.* From Propositions 8.2.3 and 8.2.4, we get the following result :

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(t) &= \frac{1}{\varepsilon^2} \left[ \int_{\mathbb{R}^{2d}} |\xi|^2 [\varrho_\varepsilon M(\xi) - f_\varepsilon] \, dx \, d\xi + 2D \int_{\mathbb{R}^{2d}} [\varrho_\varepsilon M(\xi) - f_\varepsilon] \log f_\varepsilon \, dx \, d\xi \right] \\ &= \frac{1}{\varepsilon^2} \left[ \int_{\mathbb{R}^{2d}} \varrho_\varepsilon M(\xi) |\xi|^2 \, dx \, d\xi - \int_{\mathbb{R}^{2d}} |\xi|^2 f_\varepsilon \, dx \, d\xi + 2D \int_{\mathbb{R}^{2d}} [\varrho_\varepsilon M(\xi) - f_\varepsilon] \log f_\varepsilon \, dx \, d\xi \right]. \end{aligned} \quad (8.2.14)$$

Using (8.1.6), we know that  $\log(\varrho_\varepsilon M(\xi)) = \log \varrho_\varepsilon + C - \frac{|\xi|^2}{2D}$  for some constant  $C$ . Inserting this expression of  $|\xi|^2$  in the first two terms on the righthand side of (8.2.14), we obtain

$$\begin{aligned} \frac{2D}{\varepsilon^2} \int_{\mathbb{R}^{2d}} [\varrho_\varepsilon M(\xi) - f_\varepsilon] [\log \varrho_\varepsilon + C - \log(\varrho_\varepsilon M(\xi))] \, dx \, d\xi \\ = \frac{2D}{\varepsilon^2} \int_{\mathbb{R}^{2d}} [\varrho_\varepsilon M(\xi) - f_\varepsilon] [-\log(\varrho_\varepsilon M(\xi))] \, dx \, d\xi. \end{aligned}$$

Added to the third term on the righthand side of (8.2.14), we obtain the announced result.  $\square$

In order to prove Theorem 8.2.1, a major difficulty is to estimate the flux  $J_\varepsilon$  defined by (8.1.9). We start by establishing a useful inequality, recalling the notation (8.2.6).

**Lemma 8.2.5** (Pointwise estimates on  $J_\varepsilon$ ). *For every  $0 < r \leq 1$  and  $(s, x) \in (0, T) \times \mathbb{R}^d$ , we have*

$$|J_\varepsilon(s, x)| \leq r\varepsilon \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1} + C \frac{1}{r^d} \exp\left(\frac{2C_M}{r^2}\right) \varrho_\varepsilon(s, x)^{1/2} \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1}^{1/2}.$$

*Démonstration.* For  $r > 0$ , we decompose  $J_\varepsilon(s, x) = J_\varepsilon^{(1)}(s, x) + J_\varepsilon^{(2)}(s, x)$ , with

$$J_\varepsilon^{(1)} = \frac{1}{\varepsilon} \int_{\{|\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})| \geq \frac{|\xi|}{r}\}} \xi (f_\varepsilon - \varrho_\varepsilon M(\xi)) \, d\xi, \quad J_\varepsilon^{(2)} = \frac{1}{\varepsilon} \int_{\{|\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})| \leq \frac{|\xi|}{r}\}} \xi (f_\varepsilon - \varrho_\varepsilon M(\xi)) \, d\xi.$$

For  $J_\varepsilon^{(1)}$ , we write

$$|J_\varepsilon^{(1)}(s, x)| \leq \frac{r}{\varepsilon} \int_{\{|\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})| \geq \frac{|\xi|}{r}\}} \left| \log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right) \right| \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right| \, d\xi \leq r\varepsilon \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1}.$$

For  $J_\varepsilon^{(2)}$ , we use the Cauchy-Schwarz inequality and, with  $B(\xi) := \frac{|\xi|}{r(\exp(\frac{|\xi|}{r}) - 1)}$ ,

$$|J_\varepsilon^{(2)}(s, x)| \leq \left( \int_{\mathbb{R}^d} |\xi|^2 \frac{\varrho_\varepsilon M}{B(\xi)} \, d\xi \right)^{1/2} \left( \frac{1}{\varepsilon^2} \int_{\{|\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})| \leq \frac{|\xi|}{r}\}} \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right|^2 B(\xi) \, d\xi \right)^{1/2}.$$

Because  $M(\xi)$  is a Gaussian and  $\varrho_\varepsilon$  depends only on  $(t, x)$ , we obtain

$$|J_\varepsilon^{(2)}(s, x)| \leq \varrho_\varepsilon(s, x)^{1/2} \left( \int_{\mathbb{R}^d} |\xi|^2 \frac{M(\xi)}{B(\xi)} d\xi \right)^{1/2} (I_1 + I_2)^{1/2}.$$

Here we have split the second integral according to the sign of  $\log(\frac{f_\varepsilon}{\varrho_\varepsilon M})$ . When it is negative, we may write, since  $B(\xi) \leq 1$ ,

$$I_1 := \frac{1}{\varepsilon^2} \int_{\{\frac{f_\varepsilon}{\varrho_\varepsilon M} \leq 1\}} \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right|^2 B(\xi) d\xi \leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right| \left| \log \frac{f_\varepsilon}{\varrho_\varepsilon M} \right| d\xi = \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1}.$$

The second term is defined as

$$I_2 := \frac{1}{\varepsilon^2} \int_{\{0 \leq \log(\frac{f_\varepsilon}{\varrho_\varepsilon M}) \leq \frac{|\xi|}{r}\}} \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right|^2 B(\xi) d\xi.$$

Since  $\log$  is a concave function, for  $A > 1$  and  $y \in [1, A]$ , we have  $y - 1 \leq \log(y) \frac{A-1}{\log(A)}$ . We choose  $A = A(\xi) := \exp(\frac{|\xi|}{r})$  and  $y = \frac{f_\varepsilon}{\varrho_\varepsilon M}$  so that  $y \in [1, A]$  means exactly  $0 \leq \log(\frac{f_\varepsilon}{\varrho_\varepsilon M}) \leq \frac{|\xi|}{r}$ . Then,  $I_2$  can be estimated as follows

$$I_2 \leq \frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right| \log \left( \frac{f_\varepsilon}{\varrho_\varepsilon M} \right) \frac{r(\exp(\frac{|\xi|}{r}) - 1)}{|\xi|} B(\xi) d\xi = \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1}.$$

Therefore, for some constant  $C_M$ , defined through  $M(\xi)$ , we have

$$|J_\varepsilon^{(2)}(s, x)| \leq C \varrho_\varepsilon(s, x)^{1/2} \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1}^{1/2} \left( \int_{\mathbb{R}^d} r |\xi| \exp\left(\frac{-|\xi|^2}{C_M}\right) \left( \exp\left(\frac{|\xi|}{r}\right) - 1 \right) d\xi \right)^{1/2}.$$

It remains to treat the integral factor that we denote by  $I_3$  and for  $r$  smaller than 1,

$$I_3 = \int_{\mathbb{R}^d} r |\xi| \exp\left(\frac{-|\xi|^2}{C_M}\right) \left( \exp\left(\frac{|\xi|}{r}\right) - 1 \right) d\xi \leq \frac{C}{r^d} \exp\left(\frac{2C_M}{r^2}\right)$$

where  $C$  does not depend on  $r$ . This can be seen by splitting the integral in the zones  $\{|\xi| \leq \frac{2C_M}{r}\}$  and  $\{|\xi| \geq \frac{2C_M}{r}\}$ . Finally, we obtain

$$|J_\varepsilon| \leq r\varepsilon \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1} + C \frac{1}{r^d} \exp\left(\frac{2C_M}{r^2}\right) \varrho_\varepsilon(s, x)^{1/2} \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1}^{1/2}.$$

□

From this lemma, we deduce the following  $L^1$  bounds on  $J_\varepsilon$

**Proposition 8.2.6** (Estimate on  $J_\varepsilon$  in  $L_x^1$ ). *With the decomposition of Lemma 8.2.5,  $J_\varepsilon(s, x) = J_\varepsilon^{(1)}(s, x) + J_\varepsilon^{(2)}(s, x)$ , we have*

- $|J_\varepsilon^{(1)}(s, x)| \leq \varepsilon \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1},$
- $|J_\varepsilon^{(2)}(s, x)| \leq C \varrho_\varepsilon(s, x)^{1/2} \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1}^{1/2},$
- $\|J_\varepsilon^{(2)}(s, \cdot) \log_+^{1/2} |J_\varepsilon^{(2)}(s, \cdot)|\|_{L_x^1} \leq C \left[ \|\varrho_\varepsilon(s, \cdot) \log_+ \varrho_\varepsilon(s, \cdot)\|_{L_x^1} + \|\mathcal{D}_\varepsilon(s, \cdot, \cdot)\|_{L_{x,\xi}^1} \right],$



$$- \|J_\varepsilon \log^{1/2} \log^{1/2} \max(J_\varepsilon, e)\|_{L^1_{t,x}} \leq C(\|\mathcal{D}_\varepsilon(s, \cdot, \cdot)\|_{L^1_{x,\xi}}, \|\varrho_\varepsilon(s, \cdot) \log_+ \varrho_\varepsilon(s, \cdot)\|_{L^1_{t,x}}).$$

The first two estimates are similar to [149, Proposition 7.1] for the Vlasov-Poisson-Fokker-Planck system. Here, we have additionally included the last two controls and we give a different proof.

*Démonstration.* The first two estimates are a direct consequence of Lemma 8.2.5. The third estimate follows from the inequality, for  $u \geq 1, v \geq 0$  and  $uv \geq 1$ ,

$$(uv)^{1/2} \log^{1/2}(uv) \leq u \log u + \sqrt{2}v.$$

The last result is given for the sake of completeness and its technical proof is postponed to Appendix 8.6.4. This concludes the proof of Proposition 8.2.6.  $\square$

With these estimates, we can now prove the main result of this section.

*Proof of Theorem 8.2.1.* Estimate (A) follows by mass conservation. The next bounds are deduced from the energy equality (8.2.4)-(8.2.5) which we write as

$$\int_{\mathbb{R}^{2d}} [2D \log(f_\varepsilon(t)) + |\xi|^2] f_\varepsilon(t) \, dx \, d\xi + 2D \int_0^t \|\mathcal{D}_\varepsilon(s, \cdot, \cdot)\|_{L^1_{x,\xi}} \, ds \leq \mathcal{F}(0), \tag{8.2.15}$$

where we ignore the nonnegative interaction term as it does not help in this computation. It is standard, see Appendix 8.6.1, to conclude from this inequality that

$$\int_{\mathbb{R}^{2d}} [2D |\log(f_\varepsilon(t))| + \frac{1}{2} |\xi|^2] f_\varepsilon(t) \, dx \, d\xi + D \|\mathcal{D}_\varepsilon\|_{L^1_{t,x,\xi}} \leq \mathcal{F}(0) + C \left( \|\varrho_\varepsilon\|_{L^\infty_t L^1_x}, \|x f^0\|_{L^1_{x,\xi}} \right). \tag{8.2.16}$$

The estimates (B) and (D) follow immediately. Then, estimate (E) follows from estimate (D) and the Csiszár-Kullback Inequality, see Lemma 8.6.1.

Estimate (C) is also very standard and we reproduce the proof from [195, Lemma 2.1]. We consider the convex function  $\psi(\varrho) = \varrho \log(\varrho)$  and apply the Jensen inequality. We obtain

$$\begin{aligned} \varrho_\varepsilon \log(\varrho_\varepsilon) &= \psi(\varrho_\varepsilon) = \psi \left( \int_{\mathbb{R}^d} \frac{f_\varepsilon}{M} M \, d\xi \right) \leq \int_{\mathbb{R}^d} \psi \left( \frac{f_\varepsilon}{M} \right) M \, d\xi = \\ &= \int_{\mathbb{R}^d} \frac{f_\varepsilon}{M} \left[ \log f_\varepsilon - \log M(\xi) \right] M \, d\xi = \int_{\mathbb{R}^d} f_\varepsilon \left[ \log f_\varepsilon + \frac{|\xi|^2}{2D} \right] \, d\xi + C \int_{\mathbb{R}^d} f_\varepsilon \, d\xi. \end{aligned}$$

The conclusion follows by taking the absolute values of both sides and integrating with respect to  $x$ .

Finally, estimate (F) is a direct consequence of Proposition 8.2.6, whereas (G) follows from (8.6.3). Concerning the weak compactness of  $\{\varrho_\varepsilon\}$ , it follows from estimates (C) and (G). Then, the weak local compactness of  $\{J_\varepsilon\}$  is a direct consequence of Proposition 8.2.6 and the Dunford-Pettis theorem. Indeed, with the notations of Lemma 8.2.5,  $J_\varepsilon^1$  converges strongly to 0 in  $L^1_{t,x}$ . For  $J_\varepsilon^2$  we first have the weak local compactness in  $L^1_{t,x}$  thanks to the third estimate of Proposition 8.2.6, bound (C) and the Dunford Pettis theorem. To prove the global weak compactness we only need to prove it for  $J_\varepsilon^{(2)}$ . We recall that, from Lemma 8.2.5, we have

$$|J_\varepsilon^{(2)}(s, x)| \leq C \varrho_\varepsilon(s, x)^{1/2} \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L^1_\xi}^{1/2}.$$

Therefore we can estimate with the Cauchy-Schwarz inequality

$$\|J_\varepsilon^{(2)} |x|^{1/2}\|_{L^1_{t,x}} \leq C \|\varrho_\varepsilon |x|\|_{L^1_{t,x}}^{1/2} \|\mathcal{D}_\varepsilon\|_{L^1_{t,x,\xi}}^{1/2}$$

which yields global weak compactness in  $L^1_{t,x}$  with the Dunford-Pettis theorem. This ends the proof.  $\square$

### 8.3 The limit $\varepsilon \rightarrow 0$

We now perform the analysis allowing us to prove Theorem 8.1.1. We take  $\alpha = \varepsilon$  where the parameter  $\alpha$  defines the long range potential (8.1.4). Note, however, that different scaling between  $\alpha$  and  $\varepsilon$  could possibly be considered.

Recalling the mass balance equation (8.1.9) and the  $\xi$ -moment equation (8.1.10), our aim is to take the limit  $\varepsilon \rightarrow 0$  in these equations, and establish the relations

$$\partial_t \varrho(t, x) + \operatorname{div} J(t, x) = 0, \tag{8.3.1}$$

$$J(t, x) = -D\nabla \varrho(t, x) - \varrho \nabla \Phi(t, x), \quad \Phi = -\delta \Delta[\omega^S \star \omega^S \star \varrho], \tag{8.3.2}$$

which are equivalent to (8.1.11).

A significant contribution comes from Theorem 8.2.1. The entropy bound for  $\varrho_\varepsilon$ , see (C), and the  $L^1$  bound on  $J_\varepsilon$ , see Proposition 8.2.6, we immediately conclude that

- after extractions,  $\varrho_\varepsilon$  and  $J_\varepsilon(t, x)$  admit weak limits in  $L^1_{t,x}$ ,  $\varrho$  and  $J$ , see also Theorem 8.2.1,
- the equation (8.3.1) holds in the distributional sense.

The latter estimate on  $J_\varepsilon$  also tells us that  $\varepsilon^2 \partial_t J_\varepsilon(t, x)$  converges to 0 in the distributional sense. Therefore, establishing the equation (8.3.2) from equation (8.1.10), is reduced to proving the two local weak limits in  $L^1_{t,x}$

$$\int_{\mathbb{R}^d} \xi \otimes \xi f_\varepsilon(t, x, \xi) \, d\xi \rightarrow D\varrho(t, x) \mathbf{I}, \quad \varrho_\varepsilon \nabla \Phi_\varepsilon \rightarrow \varrho \nabla \Phi(t, x).$$

These follow directly from the following three lemmas

**Lemma 8.3.1.** *We have*

$$\int_{(0,T) \times \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \xi \otimes \xi (f_\varepsilon - \varrho_\varepsilon M) \, d\xi \right| dx dt \xrightarrow{\varepsilon \rightarrow 0} 0.$$

**Lemma 8.3.2.** *The sequence  $\{\varrho_\varepsilon\}$  is precompact in  $L^p_t L^1_x$  for every  $1 \leq p < \infty$ .*

**Lemma 8.3.3.** *The potential  $\Phi_\varepsilon(t, x)$  satisfies, uniformly in  $\varepsilon \in (0, 1)$ ,*

$$\|\Phi_\varepsilon\|_\infty \leq C, \quad \|\nabla \Phi_\varepsilon\|_\infty \leq C. \tag{8.3.3}$$

Moreover, we have for every  $1 \leq p < \infty$  the strong convergence in  $L^p_t L^\infty_x$ ,

$$\Phi_\varepsilon(t, x) \rightarrow \Phi(t, x), \quad \nabla \Phi_\varepsilon(t, x) \rightarrow \nabla \Phi(t, x), \quad \Phi(t, x) := -\delta \Delta[\omega^S \star \omega^S \star \varrho(t, x)]. \tag{8.3.4}$$

The end of the proof of Theorem 8.1.1 is thus to establish these results.

*Proof of Lemma 8.3.3.* Recalling the expressions of both long-range and short-range potentials and that  $\alpha = \varepsilon$ , we see that

$$\Phi_\varepsilon(t, x) = -\frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \omega_\varepsilon^L(z) [\omega^S \star \omega^S \star \varrho_\varepsilon(t, x - z) - \omega^S \star \omega^S \star \varrho_\varepsilon(t, x)] dz.$$

Let now set  $y = \frac{z}{\varepsilon}$ , so that from (8.1.5) we deduce that

$$\Phi_\varepsilon(t, x) = -\frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \omega^L(y) [\omega^S \star \omega^S \star \varrho_\varepsilon(t, x - \varepsilon y) - \omega^S \star \omega^S \star \varrho_\varepsilon(t, x)] dy.$$

Because the convolution terms are smooth (say  $W^{3,\infty}$ ), we may use the Taylor expansion and obtain

$$\Phi_\varepsilon(t, x) = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \nabla_x [\omega^S \star \omega^S \star \varrho_\varepsilon(t, x)] \cdot y \omega^L(y) dy - \int_{\mathbb{R}^d} D_x^2 [\omega^S \star \omega^S \star \varrho_\varepsilon(t, x)] y \cdot y \omega^L(y) dy + O(\varepsilon)$$

where the term  $O(\varepsilon)$  converges to 0 in  $L^\infty$  since it is controlled by

$$C\varepsilon \int_{\mathbb{R}^d} |y|^3 \omega^L(y) \|D_x^3 \omega^S \star \omega^S \star \varrho_\varepsilon(t, \cdot)\|_\infty dy,$$

and we recall the uniform bound (A). Moreover, recalling (8.1.5), we see that the first term in the right-hand side vanishes and the Hessian matrix reduces to the Laplacian, so that

$$\Phi_\varepsilon(t, x) = -\delta \Delta_x [\omega^S \star \omega^S \star \varrho_\varepsilon(t, x)] + O(\varepsilon) \quad (8.3.5)$$

from which we directly conclude from (A)

$$\|\Phi_\varepsilon\|_\infty \leq C \quad \text{uniformly in } \varepsilon \in (0, 1).$$

As far as  $\nabla \Phi_\varepsilon$  is concerned, the properties of convolution with respect to derivatives gives

$$\nabla \Phi_\varepsilon(t, x) = -\frac{1}{\varepsilon^2} \int_{\mathbb{R}^d} \omega_\varepsilon^L(z) [\nabla \omega^S \star \omega^S \star \varrho_\varepsilon(t, x - z) - \nabla \omega^S \star \omega^S \star \varrho_\varepsilon(t, x)] dz,$$

so that the  $L_{t,x}^\infty$  bounded on  $\nabla \Phi_\varepsilon$  follows from the previous argument assuming now that  $\omega^S \in W^{4,\infty}$ .

It remains to show that  $\Phi_\varepsilon \rightarrow \Phi$  strongly in  $L_t^p L_x^\infty$ , the convergence of  $\nabla \Phi_\varepsilon$  uses the same arguments. The convergence follows from (8.3.5) since we have

$$\Phi_\varepsilon(t, x) - \Phi(t, x) = -\delta [\Delta \omega^S \star \omega^S \star (\varrho_\varepsilon - \varrho)(t, x)] + O(\varepsilon),$$

so that, thanks to the above control of the term  $O(\varepsilon)$  and properties of the convolution,

$$\|\Phi_\varepsilon - \Phi\|_{L_t^p L_x^\infty} \leq C \|\varrho_\varepsilon - \varrho\|_{L_t^p L_x^1} + C\varepsilon. \quad (8.3.6)$$

Using Lemma 8.3.2, we obtain the result.  $\square$

*Proof of Lemma 8.3.2.* This result is a consequence of the compactness averaging lemma in kinetic theory [194, 286]. Here, we use the following variant from [269, Lemma 4.2].

**Lemma 8.3.4.** *Assume that  $\{h^\varepsilon\}$  is bounded in  $L^2_{t,x,\xi}$ ,  $\{h_0^\varepsilon\}$  and  $\{h_1^\varepsilon\}$  are bounded in  $L^1_{t,x,\xi}$ . Moreover, suppose that*

$$\varepsilon \partial_t h^\varepsilon + \xi \cdot \nabla_x h^\varepsilon = h_0^\varepsilon + \nabla_\xi \cdot h_1^\varepsilon.$$

Then, for all  $\psi \in C_0^\infty(\mathbb{R}^d)$ ,

$$\left\| \int_{\mathbb{R}^d} (h^\varepsilon(t, x + y, \xi) - h^\varepsilon(t, x, \xi)) \psi(\xi) \, d\xi \right\|_{L^1_{t,x}} \rightarrow 0,$$

when  $y \rightarrow 0$  uniformly in  $\varepsilon$ .

To prove Lemma 8.3.2, we cannot apply this averaging lemma directly on  $\{f_\varepsilon\}$  because  $\{f_\varepsilon\}$  is not bounded in  $L^2_{t,x,\xi}$  and we follow the argument in [149] which follows idea of renormalized solutions [136]. We fix  $\nu > 0$  and we consider the functions  $\beta_\nu(f) = \frac{f}{1+\nu f}$  with derivative  $\beta'_\nu(f) = \frac{1}{(1+\nu f)^2}$ . Now we multiply (8.1.1) by  $\beta'_\nu(f)$  and obtain

$$\varepsilon \partial_t \beta_\nu(f_\varepsilon) + \xi \cdot \nabla_x \beta_\nu(f_\varepsilon) = \frac{(\varrho_\varepsilon M - f) \beta'_\nu(f)}{\varepsilon} - \nabla_\xi \cdot (F_\varepsilon \beta_\nu(f_\varepsilon)).$$

We verify assumptions of Lemma 8.3.4. From (A) we see that  $h^\varepsilon = \beta_\nu(f_\varepsilon)$  is bounded in  $L^1_{t,x,\xi} \cap L^\infty_{t,x,\xi}$  and hence in  $L^2_{t,x,\xi}$  by interpolation. The  $L^1_{t,x,\xi}$  bound on  $h_0^\varepsilon = \frac{(\varrho_\varepsilon M - f) \beta'_\nu(f)}{\varepsilon}$  is deduced from (E) and the  $L^\infty_{t,x,\xi}$  bound on  $\beta'_\nu(f_\varepsilon)$ . Finally, since  $F_\varepsilon$  is bounded in  $L^\infty_{t,x}$  and  $\beta_\nu(f_\varepsilon)$  is bounded in  $L^1_{t,x,\xi}$  we see that  $h_1^\varepsilon = -F_\varepsilon \beta_\nu(f_\varepsilon)$  is bounded in  $L^1_{t,x,\xi}$ .

The assumptions of Lemma 8.3.4 are satisfied and we obtain

$$\left\| \int_{\mathbb{R}^d} (\beta_\nu(f_\varepsilon)(t, x + y, \xi) - \beta_\nu(f_\varepsilon)(t, x, \xi)) \psi(\xi) \, d\xi \right\|_{L^1_{t,x}} \rightarrow 0,$$

when  $y \rightarrow 0$ , uniformly in  $\varepsilon$ . As this is true for all  $\nu > 0$ , Lemma 8.6.3 implies

$$\left\| \int_{\mathbb{R}^d} (f_\varepsilon(t, x + y, \xi) - f_\varepsilon(t, x, \xi)) \psi(\xi) \, d\xi \right\|_{L^1_{t,x}} \rightarrow 0, \quad (8.3.7)$$

when  $y \rightarrow 0$ , uniformly in  $\varepsilon$ .

The final step is to remove the weight  $\psi$  in the convergence (8.3.7) using uniform bound on  $\{f_\varepsilon |\xi|^2\}$ . To this end, consider a sequence of functions  $\{\psi_n(\xi)\}_n$  in  $\mathcal{D}(\mathbb{R}^d)$  such that  $\psi_n(\xi) = 1$  for  $|\xi| \leq n$  and  $\psi_n(\xi) = 0$  for  $|\xi| \geq n + 1$ . Then,

$$\left\| \int_{\mathbb{R}^d} (f_\varepsilon(t, x, \xi) (1 - \psi_n(\xi))) \, d\xi \right\|_{L^1_{t,x}} \leq \left\| \int_{|\xi| \geq n} f_\varepsilon(t, x, \xi) \frac{|\xi|^2}{n^2} \, d\xi \right\|_{L^1_{t,x}} \leq \frac{\|f_\varepsilon |\xi|^2\|_{L^1_{t,x,\xi}}}{n^2}$$

and similarly for the term with  $f_\varepsilon(t, x + y, \xi)$ . Hence, we may choose first  $n$  large enough and then for such  $n$  apply (8.3.7) to deduce

$$\|\varrho_\varepsilon(x + y) - \varrho_\varepsilon(x)\|_{L^1_{t,x}} = \left\| \int_{\mathbb{R}^d} (f_\varepsilon(t, x + y, \xi) - f_\varepsilon(t, x, \xi)) \, d\xi \right\|_{L^1_{t,x}} \rightarrow 0, \quad (8.3.8)$$

when  $|y| \rightarrow 0$ , uniformly in  $\varepsilon > 0$ . This yields compactness in space.

From Lemma 8.6.4 we know that  $\{\varrho_\varepsilon\}$  is also compact in time, and as a result

$$\begin{aligned} & \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, x+k) - \varrho_\varepsilon(t, x)| \, dx \, dt \\ & \leq \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, x+k) - \varrho_\varepsilon(t+h, x)| \, dt \, dx + \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, x) - \varrho_\varepsilon(t, x)| \, dt \, dx \leq \theta(h, k), \end{aligned}$$

where  $\theta(h, k) \rightarrow 0$  whenever  $|h|, |k| \rightarrow 0$  uniformly in  $\varepsilon$ . This provides the equicontinuity of  $\{\varrho_\varepsilon\}$  in  $L^1_{t,x}$  which provides us with local compactness in  $x$ .

From (G) in Theorem 8.2.1 we know that

$$\sup_{0 < \varepsilon < 1} \int_{(0, T) \times \mathbb{R}^d} |x \varrho_\varepsilon(t, x)| \, dt \, dx < \infty,$$

and we obtain the strong convergence of the density in  $L^1_{t,x}$  by Fréchet-Kolmogorov theorem, see also [314]. Using Estimate (A) we obtain by interpolation and [314, Theorem 1] the strong convergence in  $L^p_t L^1_x$  for every  $1 \leq p < \infty$  and this concludes the proof of Lemma 8.3.2.  $\square$

*Proof of Lemma 8.3.1.* We adapt the proof of Lemma 8.2.5. We write

$$\begin{aligned} R_\varepsilon & := \left| \int_{\mathbb{R}^d} \xi \otimes \xi (f_\varepsilon - \varrho_\varepsilon M) \, d\xi \right| \leq \int_{\mathbb{R}^d} |\xi|^2 |f_\varepsilon - \varrho_\varepsilon M| \, d\xi \\ & \leq \int_{\left\{ \left| \log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right) \right| \geq \frac{|\xi|^2}{r} \right\}} |\xi|^2 \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right| \, d\xi + \int_{\left\{ \left| \log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right) \right| \leq \frac{|\xi|^2}{r} \right\}} |\xi|^2 \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right| \, d\xi = I_1 + I_2, \end{aligned}$$

where  $r$  is chosen later. For the first term, we just write

$$I_1 \leq r \int_{\mathbb{R}^d} \log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right) \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right| \, d\xi \leq r \varepsilon^2 \|D_\varepsilon\|_{L^1_\xi}.$$

The term  $I_2$  is decomposed in two parts : where  $f_\varepsilon \geq \varrho_\varepsilon M$  and  $f_\varepsilon < \varrho_\varepsilon M$ . The resulting integrals are called  $I_2^A$  and  $I_2^B$ . We only discuss  $I_2^A$  as  $I_2^B$  can be treated similarly as it was discussed in Lemma 8.2.5. We use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} I_2^A & \leq \left( \int_{\left\{ 0 \leq \log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right) \leq \frac{|\xi|^2}{r} \right\}} |\xi|^4 \frac{\varrho_\varepsilon M}{B(\xi)} \, d\xi \right)^{1/2} \\ & \quad \cdot \left( \int_{\left\{ 0 \leq \log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right) \leq \frac{|\xi|^2}{r} \right\}} \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right|^2 B(\xi) \, d\xi \right)^{1/2} =: I_2^{A,1} \cdot I_2^{A,2}, \end{aligned}$$

where, as before,  $B(\xi) = \frac{\log(A)}{A-1} = \frac{|\xi|^2}{r(\exp(\frac{|\xi|^2}{r})-1)}$ , with  $A = A(\xi) := \exp(\frac{|\xi|^2}{r})$ . As in the proof of Lemma 8.2.5, we have the inequality  $\log(y) \geq (y-1) \frac{\log(A)}{A-1}$  which yields with  $y = \frac{f_\varepsilon}{\varrho_\varepsilon M}$

$$I_2^{A,2} \leq \left( \int_{\left\{ 0 \leq \log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right) \leq \frac{|\xi|^2}{r} \right\}} \varrho_\varepsilon M \left| \frac{f_\varepsilon}{\varrho_\varepsilon M} - 1 \right| \log\left(\frac{f_\varepsilon}{\varrho_\varepsilon M}\right) \, d\xi \right)^{1/2} \leq \varepsilon \|D_\varepsilon\|_{L^1_\xi}^{1/2}.$$

Now we choose  $r$  such that  $M(\xi) \exp(\frac{|\xi|^2}{r}) = C \exp(-a|\xi|^2)$  for some  $a > 0$ . Then, we have

$$\int_{\mathbb{R}^d} |\xi|^4 \frac{M(\xi)}{B(\xi)} d\xi \leq r \int_{\mathbb{R}^d} |\xi|^2 M(\xi) \exp\left(\frac{|\xi|^2}{r}\right) d\xi \leq Cr \int_{\mathbb{R}^d} |\xi|^2 \exp(-a|\xi|^2) d\xi =: C^2.$$

It follows that  $I_2^{A,1} \leq C\varrho_\varepsilon^{1/2}$ .

Finally we get

$$R_\varepsilon \leq r\varepsilon^2 \|\mathcal{D}_\varepsilon\|_{L_\xi^1} + C\varepsilon\varrho_\varepsilon^{1/2} \|D_\varepsilon\|_{L_\xi^1}^{1/2}$$

and, using the Cauchy-Schwarz inequality, the proof of Lemma 8.3.1 is concluded. □

This also concludes the proof of Theorem 8.1.1.

### 8.4 Conclusion

We proved that macroscopic densities  $\{\varrho_\varepsilon\}$  formed from solutions of the Vlasov-Cahn-Hilliard equation (8.1.1) converge to the solutions of non-local degenerate Cahn-Hilliard (8.1.11). It is an open question whether one can obtain a local version of this equation by sending short-range interaction kernel  $\omega^S$  to the Dirac mass  $\delta_0$ . One expects in the limit the local degenerate Cahn-Hilliard equation :

$$\partial_t \varrho - D\Delta\varrho - \operatorname{div}(\varrho\nabla\Phi) = 0 \tag{8.4.1}$$

where  $\Phi = -\delta\Delta\varrho$ . One can try to perform this limit either on equation (8.1.11) or directly on (8.1.1), by sending  $\omega_\alpha^L \xrightarrow{*} \delta_0, \omega^S \xrightarrow{*} \delta_0$  together, see Figure 8.1. Passing from (8.1.1) to (8.4.1), the main difficulty is the lack of entropy which gives integrability of second-order derivatives in the nondegenerate Cahn-Hilliard. On the other hand, when one tries to pass to the limit from (8.1.11) to (8.4.1), the entropy is available but it yields estimates only on

$$\Delta(\varrho \star \omega^S) \text{ in } L_t^2 L_x^2, \quad \nabla\sqrt{\varrho} \text{ in } L_t^2 L_x^2.$$

The minimal required information allowing to pass to the limit seems to be strong compactness of  $\{\nabla\varrho\}$  in  $L_t^2 L_x^2$ .

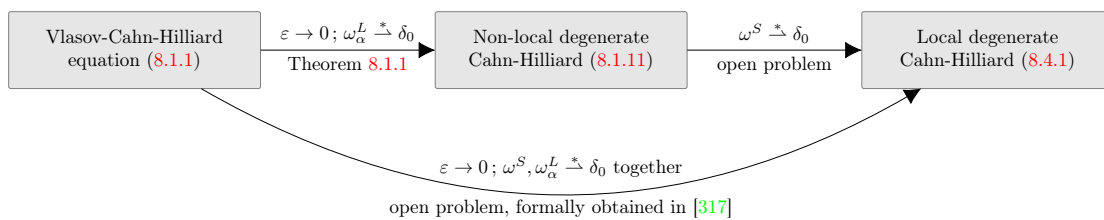


FIGURE 8.1 – Relation between three types of the degenerate Cahn-Hilliard equations.

Moreover, it is also open to prove whether we can add the "usual" double-well Cahn-Hilliard interaction potential in the system. In fact, as far as this modification is concerned, it is not even clear if there exists a solution to the Vlasov-Cahn-Hilliard equation when the potential  $\Phi$  is a function of the density  $\varrho$ .

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## 8.6 Appendix

### 8.6.1 Useful inequality and lower bound on the energy

We recall two lemmas which have been used in the proof of Theorem 8.2.1. The first one is a variant of the Csiszar-Kullback inequality.

**Lemma 8.6.1.** *Let  $f, g \geq 0$  with  $\|f\|_1 = \|g\|_1$ . Then,*

$$\|f - g\|_1^2 \leq \|f\|_1 \int_{\mathbb{R}^d} (f - g) (\log f - \log g)$$

The second lemma is used to control  $f \log_-(f)$  from  $f \log f$ , which immediately establishes the Inequality (8.6.1).

**Lemma 8.6.2.** *Let  $\log_-(f) := \max\{-\log(f), 0\}$ . Then*

$$\int_{\mathbb{R}^{2d}} 2D \log_-(f_\varepsilon(t)) f_\varepsilon(t) dx d\xi \leq C \left( \|\varrho_\varepsilon\|_{L^1_{t,x}}, \|xf^0\|_{L^1_{x,\xi}} \right) + \int_{\mathbb{R}^{2d}} \frac{|\xi|^2}{4} f_\varepsilon(t) d\xi dx + D \int_0^t \|\mathcal{D}_\varepsilon(s, \cdot, \cdot)\|_{L^1_{x,\xi}} ds. \quad (8.6.1)$$

*Proof of Lemma 8.6.1.* Let  $\|f\|_1 = \|g\|_1 = 1$ . Usual the Csiszar-Kullback inequality gives us

$$\|f - g\|_1^2 \leq 2 \int_{\mathbb{R}^d} f \log \left( \frac{f}{g} \right).$$

By symmetry of the (LHS) we have

$$2\|f - g\|_1^2 \leq 2 \int_{\mathbb{R}^d} f \log \left( \frac{f}{g} \right) + 2 \int_{\mathbb{R}^d} g \log \left( \frac{g}{f} \right) = 2 \int_{\mathbb{R}^d} (f - g) (\log f - \log(g)).$$

The general case follows by rescaling. □

*Proof of Lemma 8.6.2.* We proceed as in [149, Proposition 5.1].

We divide the domain in two parts :

$$\Omega_1 := \left\{ f_\varepsilon > \exp \left( -\frac{|x|}{4} - \frac{|\xi|^2}{8D} \right) \right\}, \quad \Omega_2 := \left\{ f_\varepsilon \leq \exp \left( -\frac{|x|}{4} - \frac{|\xi|^2}{8D} \right) \right\},$$

On  $\Omega_1$ ,  $\log_-(f_\varepsilon)$  is bounded so that we have

$$f_\varepsilon \log_-(f_\varepsilon) \leq \left( \frac{|x|}{4} + \frac{|\xi|^2}{8D} \right) f_\varepsilon,$$

while on  $\Omega_2$ ,  $f_\varepsilon \leq 1$  so that  $\sqrt{f_\varepsilon} \log_-(f_\varepsilon)$  is bounded by some constant  $C$ . Hence,

$$f_\varepsilon \log_-(f_\varepsilon) \leq C\sqrt{f_\varepsilon} \leq C \exp\left(-\frac{|x|}{8} - \frac{|\xi|^2}{16D}\right).$$

It follows that

$$\int_{\mathbb{R}^{2d}} \log_-(f_\varepsilon(t)) f_\varepsilon(t) dx d\xi \leq \int_{\mathbb{R}^{2d}} C \exp\left(-\frac{|x|}{8} - \frac{|\xi|^2}{16D}\right) + \left(\frac{|x|}{4} + \frac{|\xi|^2}{8D}\right) f_\varepsilon(t) d\xi dx. \quad (8.6.2)$$

Now, we only need to bound the term  $\int_{\mathbb{R}^{2d}} \frac{|x|}{4} f_\varepsilon(t) d\xi dx$ . For this, we first observe that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |x| f_\varepsilon(t) d\xi = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} f_\varepsilon(t) \frac{x}{|x|} \xi d\xi = \frac{x}{|x|} J_\varepsilon \leq 2 \|\mathcal{D}_\varepsilon(s, \cdot, \cdot)\|_{L^1_\xi} + C \varrho_\varepsilon,$$

where we have used Proposition 8.2.6 and Young's inequality (with  $\varepsilon \leq 1$ ). Therefore, for all  $t \geq 0$

$$\int_{\mathbb{R}^{2d}} |x| f_\varepsilon(t) d\xi dx \leq \int_{\mathbb{R}^{2d}} |x| f^0 d\xi dx + C \|\varrho_\varepsilon\|_{L^1_{t,x}} + 2 \int_0^t \|\mathcal{D}_\varepsilon(s, \cdot, \cdot)\|_{L^1_{x,\xi}} ds. \quad (8.6.3)$$

Finally, equation (8.6.2) simplifies to give the desired result (8.6.1). □

## 8.6.2 Criteria for compactness

**Lemma 8.6.3** (Compactness of  $\beta_\nu(f_n)$  implies compactness of  $f_n$ ). *Let  $\{f_n(t, x, \xi)\}$  be a sequence such that  $\{f_n\}$  and  $\{f_n \log f_n\}$  are bounded in  $L^1_{t,x,\xi}$ . Let  $\psi(\xi) \in C_c^\infty(\mathbb{R}^d)$ . Suppose that for all  $\nu > 0$  and all  $\varepsilon > 0$ , there exists  $\delta(\nu, \varepsilon)$  such that, whenever  $|y| \leq \delta(\nu, \varepsilon)$ ,*

$$\left\| \int_{\mathbb{R}^d} (\beta_\nu(f_n(t, x + y, \xi)) - \beta_\nu(f_n(t, x, \xi))) \psi(\xi) d\xi \right\|_{L^1_{t,x}} \leq \varepsilon.$$

Then, for all  $\varepsilon > 0$  there exists  $\delta(\varepsilon) > 0$  such that

$$\left\| \int_{\mathbb{R}^d} (f_n(t, x + y, \xi) - f_n(t, x, \xi)) \psi(\xi) d\xi \right\|_{L^1_{t,x}} \leq \varepsilon.$$

*Démonstration.* First, we observe that

$$|\beta_\nu(s) - s| \leq \left| \frac{s}{1 + s\nu} - s \right| = \frac{\nu s^2}{1 + \nu s} \leq \min(\nu s^2, s).$$



Therefore, for  $M$  and  $\nu$  to be chosen later

$$\begin{aligned} & \left\| \int_{\mathbb{R}^d} (f_n(t, x + y, \xi) - \beta_\nu(f_n(t, x + y, \xi))) \psi(\xi) d\xi \right\|_{L^1_{t,x}} \leq \\ & \leq \|\psi\|_\infty \nu \int_{f_n(t, x + y, \xi) \leq M} f_n^2(t, x + y, \xi) d\xi dx dt + \|\psi\|_\infty \int_{f_n(t, x + y, \xi) \geq M} f_n(t, x + y, \xi) d\xi dx dt \\ & \leq \|\psi\|_\infty \nu M \|f_n\|_1 + \|\psi\|_\infty \frac{\|f_n \log f_n\|_1}{\log M}. \end{aligned}$$

Similarly,

$$\left\| \int_{\mathbb{R}^d} (f_n(t, x, \xi) - \beta_\nu(f_n(t, x, \xi))) \psi(\xi) d\xi \right\|_{L^1_{t,x}} \leq \|\psi\|_\infty \nu M \|f_n\|_1 + \|\psi\|_\infty \frac{\|f_n \log f_n\|_1}{\log M}.$$

Let  $\varepsilon > 0$ . First, we choose  $\nu$  and  $M$  such that

$$\|\psi\|_\infty \nu M \|f_n\|_1 + \|\psi\|_\infty \frac{\|f_n \log f_n\|_1}{\log M} \leq \frac{\varepsilon}{3}.$$

Then, we take  $\delta(\nu, \varepsilon/3)$  such that

$$\left\| \int_{\mathbb{R}^d} (\beta_\nu(f_n(t, x + y, \xi)) - \beta_\nu(f_n(t, x, \xi))) \psi(\xi) d\xi \right\|_{L^1_{t,x}} \leq \varepsilon/3$$

when  $|y| \leq \delta(\nu, \varepsilon/3)$ . The conclusion follows by the triangle inequality.  $\square$

**Lemma 8.6.4.** *The sequence  $\{\varrho_\varepsilon\}$  from Lemma 8.3.2 is compact in time, i.e.*

$$\lim_{|h| \rightarrow 0} \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, x) - \varrho_\varepsilon(t, x)| dx dt = 0 \quad \text{uniformly in } \varepsilon.$$

The proof of this lemma uses a sequence  $(\varphi_\delta)_{\delta>0} \in C_c^\infty(\mathbb{R}^d)$  of standard mollifiers with mass 1 such that  $\varphi_\delta(x) = \frac{1}{\delta^d} \varphi(\frac{x}{\delta})$  with  $\varphi$  of mass 1 and compactly supported. Moreover

$$\|\nabla^k \varphi_\delta\|_{L^1(\mathbb{R}^d)} \leq \frac{C}{\delta^k},$$

and for any function  $g \in L^p(\mathbb{R}^d)$ ,

$$\|g \star \varphi_\delta\|_{L^p(\mathbb{R}^d)} \leq \|\varphi_\delta\|_{L^1(\mathbb{R}^d)} \|g\|_{L^p(\mathbb{R}^d)}.$$

*Démonstration.* We know that

$$\partial_t \varrho_\varepsilon + \nabla \cdot J_\varepsilon = 0$$

where  $J_\varepsilon$  is bounded uniformly in  $L^1_{t,x}$ , see Proposition 8.2.6.

Using the mollifiers with  $\delta$  depending on  $h$  to be specified later on, we first notice that

$$\begin{aligned} \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, x) - \varrho_\varepsilon(t, x)| dx dt &\leq \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t, x) - \varrho_\varepsilon(t, \cdot) \star \varphi_\delta(x)| dx dt \\ &+ \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, x) - \varrho_\varepsilon(t+h, \cdot) \star \varphi_\delta(x)| dx dt \\ &+ \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, \cdot) \star \varphi_\delta(x) - \varrho_\varepsilon(t, \cdot) \star \varphi_\delta(x)| dx dt. \end{aligned}$$

For the first and second terms, the computations are the same, hence, we only present it for the first term. Using the properties of the mollifiers and the compactness of  $\varrho_\varepsilon$  in space, we want to prove that

$$\int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t, x) - \varrho_\varepsilon(t, \cdot) \star \varphi_\delta(x)| dx dt \leq \theta(\delta).$$

where  $\theta(\delta) \rightarrow 0$  when  $\delta \rightarrow 0$  uniformly in  $\varepsilon$ . We write

$$\int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t, x) - \varrho_\varepsilon(t, \cdot) \star \varphi_\delta(x)| dx dt = \int_0^{T-h} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \varphi(y) (\varrho_\varepsilon(t, x) - \varrho_\varepsilon(t, x - \delta y)) dy \right| dx dt.$$

Then we use Fubini's theorem and the fact that  $\varphi$  is compactly supported in some compact set  $K$  we obtain

$$\int_0^{T-h} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \varphi(y) (\varrho_\varepsilon(t, x) - \varrho_\varepsilon(t, x - \delta y)) dy \right| dx dt \leq \int_K \|\tau_{\delta y} \varrho_\varepsilon - \varrho_\varepsilon\|_{L^1((0, T) \times \mathbb{R}^d)} dy.$$

where  $\tau_x$  is the translation operator in  $x$  variable. Now we use the compactness in space obtained in (8.3.8), so that

$$\int_K \|\tau_{\delta y} \varrho_\varepsilon - \varrho_\varepsilon\|_{L^1((0, T) \times \mathbb{R}^d)} dy \leq |K| \sup_{y \in K} \|\tau_{\delta y} \varrho_\varepsilon - \varrho_\varepsilon\|_{L^1((0, T) \times \mathbb{R}^d)} \leq \theta(\delta).$$

Therefore the first and the second term are bounded by  $\theta(\delta)$  where  $\theta(\delta) \rightarrow 0$  when  $\delta \rightarrow 0$  uniformly in  $\varepsilon$ . It remains to study the third term. The third term reads

$$\begin{aligned} \int_0^{T-h} \int_{\mathbb{R}^d} |\varrho_\varepsilon(t+h, \cdot) \star \varphi_\delta(x) - \varrho_\varepsilon(t, \cdot) \star \varphi_\delta(x)| dx dt &= \int_0^{T-h} \int_{\mathbb{R}^d} \left| \int_t^{t+h} \partial_t \varrho_\varepsilon(s, \cdot) \star \varphi_\delta(x) ds \right| dx dt \\ &= \int_0^{T-h} \int_{\mathbb{R}^d} \left| - \sum_{i=1}^d \int_t^{t+h} J_i \star \partial_i \varphi_\delta(s, x) ds \right| dx dt \leq \sum_{i=1}^d \int_0^{T-h} \int_{\mathbb{R}^d} \int_t^{t+h} |J_i \star \partial_i \varphi_\delta(s, x)| ds dx dt, \end{aligned}$$

where we used  $J_\varepsilon = (J_i)_{i=1, \dots, d}$ . We perform the change of variables  $v = \frac{s-t}{h}$ , use Fubini's theorem and obtain

$$\int_0^{T-h} \int_{\mathbb{R}^d} \int_t^{t+h} |J_i \star \partial_i \varphi_\delta(s, x)| ds dx dt = h \int_0^1 \int_{\mathbb{R}^d} \int_0^{T-h} |J_i \star \partial_i \varphi_\delta(vh + t, x)| dt dx dv.$$

Then we use the change of variables  $\tau = vh + t$  and obtain

$$h \int_0^1 \int_{\mathbb{R}^d} \int_0^{T-h} |J_i \star \partial_i \varphi_\delta(vh + t, x)| dt dx dv = h \int_0^1 \int_{\mathbb{R}^d} \int_{vh}^{T+h(v-1)} |J_i \star \partial_i \varphi_\delta(\tau, x)| d\tau dx dv \leq \frac{h}{\delta} \|J_\varepsilon\|_{L^1_{t,x}}.$$

Using the  $L^1_{t,x}$  bound on  $J_\varepsilon$  and taking  $\delta = h^{1/2}$  we conclude.  $\square$

### 8.6.3 Uniqueness in $L^\infty$

Let  $d \geq 3$ . We are interested in the uniqueness of these solutions in the class of functions such that

$$\varrho \in L^\infty_{t,x} \cap L^\infty_t L^1_x \cap C^w_t L^1_x \quad (8.6.4)$$

where  $C^w_t L^1_x$  denotes the space of weakly continuous in time functions with values in  $L^1_x$ . In this class, the definition of distributional solutions of Theorem 8.1.1 can be formulated as follows : for every test function  $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$  we have, with

$$- \int_{\mathbb{R}^d} \varrho^0 \varphi(0, x) dx - \int_0^T \int_{\mathbb{R}^d} \varrho \partial_t \varphi dx dt = D \int_0^T \int_{\mathbb{R}^d} \varrho \Delta \varphi dx dt - \int_0^T \int_{\mathbb{R}^d} \varrho \nabla \Phi(\varrho) \cdot \nabla \varphi dx dt,$$

where  $\Phi(\varrho) = -\delta \Delta(\omega^S \star \omega^S \star \varrho)$  and  $\varrho \in L^\infty_t L^1_x$ .

By interpolation  $\varrho$  belongs to every  $L^p_{t,x}$ ,  $1 \leq p \leq \infty$  and so is  $\nabla \Phi(\varrho)$ . Therefore this formulation implies

$$\int_0^T \langle \partial_t \varrho, \varphi \rangle = D \int_0^T \int_{\mathbb{R}^d} \varrho \Delta \varphi dx dt - \int_0^T \int_{\mathbb{R}^d} \varrho \nabla \Phi(\varrho) \cdot \nabla \varphi dx dt, \quad (8.6.5)$$

for every  $\varphi \in L^1_t W_x^{1,1} \cap L^1_t \dot{H}_x^2$  where  $\langle \cdot, \cdot \rangle$  denotes the dual pairing between  $\dot{H}^{-2}$  and  $\dot{H}^2$ .

Let  $\varrho_1, \varrho_2$  be two solutions as above with same initial data which satisfy  $\varrho_1, \varrho_2 \in L^\infty_{t,x}$ . The goal is to prove that  $\varrho_1 = \varrho_2$ . We subtract Equation (8.6.5) for  $\varrho_2$  and  $\varrho_1$ . Writing  $\varrho = \varrho_2 - \varrho_1$ , we obtain

$$\int_0^T \langle \partial_t \varrho, \varphi \rangle = D \int_0^T \int_{\mathbb{R}^d} \varrho \Delta \varphi dx dt - \int_0^T \int_{\mathbb{R}^d} \varrho \nabla \Phi(\varrho_2) \cdot \nabla \varphi dx dt - \int_0^T \int_{\mathbb{R}^d} \varrho_1 \nabla \Phi(\varrho) \cdot \nabla \varphi dx dt. \quad (8.6.6)$$

We want to test (8.6.6) with  $\varphi(t) = -\mathcal{N} \star \varrho$  where  $\mathcal{N}$  is the Newtonian potential so that  $-\Delta \varphi = \varrho$ . This is an admissible test function. Indeed,  $\partial_{x_i, x_j} \varphi \in L^\infty_t L^2_x$  by the Calderon-Zygmund theory cf. [102, Theorem 3.5, Chapter 3]. Moreover, as  $\nabla \mathcal{N} \in L^{\frac{d}{d-1}, \infty}$  (i.e. weak  $L^p$  spaces) we can use Young's convolutional inequality to deduce

$$\|\nabla \varphi\|_{L^\infty_t L^2_x} \leq C \|\nabla \mathcal{N}\|_{L^{\frac{d}{d-1}, \infty}} \|\varrho\|_{L^\infty_t L^{\frac{2d}{d+2}}_x}.$$

Finally,  $\varphi \in L^\infty_{t,x}$  cf. [34, Lemma 1]. Therefore, testing (8.6.6) with  $\varphi$  we obtain

$$\frac{1}{2} \int_{\mathbb{R}^d} |\nabla \varphi(T)|^2 dx + D \int_0^T \int_{\mathbb{R}^d} \varrho^2 = - \int_0^T \int_{\mathbb{R}^d} \varrho \nabla \Phi(\varrho_2) \cdot \nabla \varphi - \int_0^T \int_{\mathbb{R}^d} \varrho_1 \nabla \Phi(\varrho) \cdot \nabla \varphi.$$

We denote by  $I_1$  and  $I_2$  the two terms of the right-hand side. Using  $-\Delta\varphi = \varrho$  and the formula  $\Delta\varphi\nabla\varphi = \nabla \cdot (\nabla\varphi \otimes \nabla\varphi) - \frac{1}{2}\nabla|\nabla\varphi|^2$  we obtain

$$I_1 = \int_0^T \int_{\mathbb{R}^d} \Delta\varphi\nabla\varphi \cdot \nabla\Phi(\varrho_2) \leq C \int_0^T \int_{\mathbb{R}^d} |D^2\Phi(\varrho_2)| |\nabla\varphi|^2 \leq C \int_0^T \int_{\mathbb{R}^d} |\nabla\varphi|^2.$$

as  $|D^2\Phi(\varrho_2)|$  can be bounded as in Lemma 8.3.3 only in terms of  $\|\varrho_2\|_{L_{t,x}^\infty}$ . For  $I_2$  we recall that  $\varrho_1$  is bounded in  $L_{t,x}^\infty$ . Using the Cauchy-Schwarz inequality it remains to see that  $\|\nabla\Phi(\varrho)\|_{L^2} \leq C\|\nabla\varphi\|_{L^2}$  which can be achieved by definition of  $\Phi(\varrho)$  and  $\varphi$  and the fact that convolutions commute with derivatives. Therefore

$$I_2 \leq C\|\nabla\varphi\|_{L_{t,x}^2}^2.$$

Combining the previous results we obtain

$$\|\nabla\varphi(T, \cdot)\|_{L^2}^2 \leq C \int_0^T \|\nabla\varphi\|_{L^2}^2,$$

so that  $\|\nabla\varphi\|_{L^2}^2 = 0$  and the proof is concluded.

#### 8.6.4 Estimate on $J_\varepsilon \log^{1/2} \log^{1/2} \max(J_\varepsilon, e)$

From Lemma 8.2.5 we recall that for  $0 < r \leq 1$

$$|J_\varepsilon(s, x)| \leq r\varepsilon \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1} + C \frac{1}{r^d} \exp\left(\frac{2C_M}{r^2}\right) \varrho_\varepsilon(s, x)^{1/2} \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1}^{1/2}.$$

We can make further simplifications : applying a simple rescaling of  $r$ , ignoring  $\varepsilon$ , estimating  $\frac{1}{r^d} \leq \exp(\frac{1}{r^d})$  and changing  $r = \frac{1}{\alpha}$  we can assume

$$|J_\varepsilon(s, x)| \leq \frac{C}{\alpha} \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1} + C \exp(\alpha^2) \varrho_\varepsilon(s, x)^{1/2} \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1}^{1/2}. \quad (8.6.7)$$

To choose the best  $\alpha$  in the inequality above, we let  $u = \varrho_\varepsilon$ ,  $v = \|\mathcal{D}_\varepsilon(s, x, \cdot)\|_{L_\xi^1}$  so that we can estimate

$$|J_\varepsilon(s, x)| \leq C v \min_{1 < \alpha < \infty} \left[ \frac{1}{\alpha} + \exp(\alpha^2) \sqrt{\frac{u}{v}} \right]. \quad (8.6.8)$$

**Lemma 8.6.5.** *Let  $v \geq e$ ,  $u \geq 0$ ,  $v > e^2 u$ . The minimum in (8.6.8) is attained for  $\alpha > 1$  which is the unique solution of*

$$2\alpha^3 \exp(\alpha^2) = \sqrt{\frac{v}{u}}.$$

For such  $\alpha > 1$  we have

$$v \left[ \frac{1}{\alpha} + \exp(\alpha^2) \sqrt{\frac{u}{v}} \right] = v \left[ \frac{1}{\alpha} + \frac{1}{2\alpha^3} \right] \leq \frac{2v}{\alpha}.$$

Then,

$$\frac{2v}{\alpha} \leq \begin{cases} \frac{2\sqrt{2}v}{\log_+^{1/2} \log_+^{1/2} v} & \text{if } v \geq u \log_+^{1/2} v, \\ 2u \log_+^{1/2} v & \text{if } v < u \log_+^{1/2} v. \end{cases} \quad (8.6.9)$$

*Démonstration.* The first statement is a consequence of simple calculus and we only have to prove that the minimum is attained for  $\alpha > 1$ . This follows from

$$\sqrt{\frac{v}{u}} = 2\alpha^3 \exp(\alpha^2) \leq \exp(2\alpha^2) \implies \frac{1}{2} \log\left(\frac{v}{u}\right) \leq \alpha^2. \quad (8.6.10)$$

As  $v > e^2 u$ , we deduce  $\alpha > 1$ .

We proceed to the estimates on  $\frac{v}{2\alpha}$ . Suppose that  $v \geq u \log_+^{1/2} v$ . Then, we have

$$\log v \geq \log u + \log \log_+^{1/2} v \implies \log_+^{1/2}\left(\frac{v}{u}\right) \geq \log_+^{1/2} \log_+^{1/2} v$$

(we use here  $\frac{v}{u} > e^2$  and  $v > e$  to write  $\log_+$  instead of  $\log$ ). In view of (8.6.10), this gives lower bound on  $\alpha$  which implies

$$\frac{2v}{\alpha} \leq \frac{2\sqrt{2}v}{\log_+^{1/2} \log_+^{1/2} v}.$$

We are left with the case  $v < u \log_+^{1/2} v$ . In this case we estimate directly using  $\alpha > 1$  :

$$\frac{2v}{\alpha} \leq 2v \leq 2u \log_+^{1/2} v.$$

□

We proceed to estimating  $J_\varepsilon \log^{1/2} \log^{1/2} \max(J_\varepsilon, e)$  in  $L_{t,x}^1$ . Let us observe that we can always restrict the set of integration to the points  $(t, x)$  where  $\|\mathcal{D}_\varepsilon\|_{L_\xi^1}$  is arbitrarily large. Indeed, given  $M \geq e$ , we estimate

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} J_\varepsilon \log^{1/2} \log^{1/2} \max(J_\varepsilon, e) &\leq \\ &\leq \int_0^T \int_{\mathbb{R}^d} J_\varepsilon \log^{1/2} \log^{1/2} \max(J_\varepsilon, e) \mathbf{1}_{J_\varepsilon \leq M} + \int_0^T \int_{\mathbb{R}^d} J_\varepsilon \log^{1/2} \log^{1/2} \max(J_\varepsilon, e) \mathbf{1}_{\|\mathcal{D}_\varepsilon\|_{L_\xi^1} \leq e^2 \varrho_\varepsilon} \\ &\quad + \int_0^T \int_{\mathbb{R}^d} J_\varepsilon \log^{1/2} \log^{1/2} \max(J_\varepsilon, e) \mathbf{1}_{\|\mathcal{D}_\varepsilon\|_{L_\xi^1} > e^2 \varrho_\varepsilon} \mathbf{1}_{J_\varepsilon > M}. \end{aligned}$$

The first integral is bounded by  $\|J_\varepsilon\|_{L_{t,x}^1} \log^{1/2} \log^{1/2} M$ . For the second integral, we note that (8.6.7) implies that  $J_\varepsilon \leq C \varrho_\varepsilon$  so this integral is finite because we can use Young's inequality and  $\log x \leq x$  to get

$$\varrho_\varepsilon \log^{1/2} \log^{1/2} \max(\varrho_\varepsilon, e) \leq \varrho_\varepsilon + \frac{1}{2} \varrho_\varepsilon \log \max(\varrho_\varepsilon, e).$$

In the third integral, by estimate (8.6.7) with  $\alpha = 2$ , we have  $\|\mathcal{D}_\varepsilon\|_{L_\xi^1} \geq \frac{M}{C}$  for some constant  $C$ . It follows that  $\|\mathcal{D}_\varepsilon\|_{L_\xi^1}$  can be assumed to be arbitrarily large by taking sufficiently large  $M$ . This allows us to apply Lemma 8.6.5.

Splitting the domain of integration for two subsets as in Lemma 8.6.5, it is sufficient to prove

that the following functions

$$P_\varepsilon^1 := \frac{\|\mathcal{D}_\varepsilon\|_{L_\xi^1}}{\log_+^{1/2} \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1}} \log_+^{1/2} \log_+^{1/2} \left( \frac{\|\mathcal{D}_\varepsilon\|_{L_\xi^1}}{\log_+^{1/2} \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1}} \right),$$

$$P_\varepsilon^2 := \varrho_\varepsilon \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1} \log_+^{1/2} \log_+^{1/2} \left( \varrho_\varepsilon \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1} \right).$$

are bounded in  $L_{t,x}^1$  (here, we use that  $\log_+^{1/2} \log_+^{1/2} v = \log_+^{1/2} \log_+^{1/2} \max(v, e)$ ).

For  $P_\varepsilon^1$  (this is the limiting case!), we restrict to the values of  $\|\mathcal{D}_\varepsilon\|_{L_\xi^1}$  so large that  $\log_+^{1/2} \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1} > 1$ . Then,

$$\log_+^{1/2} \log_+^{1/2} \left( \frac{\|\mathcal{D}_\varepsilon\|_{L_\xi^1}}{\log_+^{1/2} \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1}} \right) \leq \log_+^{1/2} \log_+^{1/2} \left( \|\mathcal{D}_\varepsilon\|_{L_\xi^1} \right)$$

so that  $P_\varepsilon^1 \leq \|\mathcal{D}_\varepsilon\|_{L_\xi^1}$ .

For  $P_\varepsilon^2$ , we apply  $\log x \leq x$ ,  $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$  and  $2xy \leq x^2 + y^2$  to get

$$\begin{aligned} P_\varepsilon^2 &\leq \varrho_\varepsilon \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1} \log_+^{1/2} \left( \varrho_\varepsilon \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1} \right) \leq \\ &\leq \varrho_\varepsilon \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1} \log_+^{1/2} \varrho_\varepsilon + \varrho_\varepsilon \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1} \log_+^{1/2} \log_+^{1/2} \|\mathcal{D}_\varepsilon\|_{L_\xi^1} \\ &\leq \varrho_\varepsilon \log_+ \varrho_\varepsilon + \varrho_\varepsilon \log_+ \|\mathcal{D}_\varepsilon\|_{L_\xi^1} + \varrho_\varepsilon \log_+ \|\mathcal{D}_\varepsilon\|_{L_\xi^1} \end{aligned}$$

so it is sufficient to prove that  $\varrho_\varepsilon \log_+ \|\mathcal{D}_\varepsilon\|_{L_\xi^1}$  is bounded in  $L_{t,x}^1$ . This follows from Fenchel-Young's inequality

$$\varrho_\varepsilon \log_+ \|\mathcal{D}_\varepsilon\|_{L_\xi^1} \leq \varrho_\varepsilon \log \varrho_\varepsilon + \varrho_\varepsilon + \|\mathcal{D}_\varepsilon\|_{L_\xi^1}.$$



## Chapitre 9

# On the limit problem arising in the kinetic derivation of the Cahn-Hilliard equation

### Abstract

The non-local degenerate Cahn-Hilliard equation is derived from the Vlasov equation with long range attraction. We study the local limit as the delocalization parameter converges to 0. The difficulty arises from the degeneracy which requires compactness estimates, but all necessary a priori estimates can be obtained only on the nonlocal quantities yielding almost no information on the limiting solution itself. We introduce a novel condition on the nonlocal kernel which allows us to exploit the available nonlocal a priori estimates. The condition, satisfied by most of the kernels appearing in the applications, can be of independent interest. Our approach is flexible and systems can be treated as well.

## 9.1 Introduction

We consider the nonlocal PDE

$$\partial_t \rho_\varepsilon - \Delta \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \nabla \Delta(\rho_\varepsilon * \omega_\varepsilon * \omega_\varepsilon)) = 0 \quad (9.1.1)$$

arising in the kinetic derivation of the degenerate Cahn-Hilliard equation via hydrodynamic limit as in our recent paper [150] which makes the formal approach of Takata and Noguchi [317] fully rigorous. Furthermore, PDEs of the form (9.1.1) appear in the numerical analysis of the particle method as explained below. Concerning the notation,  $\varepsilon > 0$  is a small parameter,  $\rho_\varepsilon : \mathbb{R}^d \times [0, T] \rightarrow [0, \infty)$  is the solution while

$$\omega_\varepsilon(x) = \frac{1}{\varepsilon^d} \omega\left(\frac{x}{\varepsilon}\right)$$

is a usual mollification kernel (one can think of  $\omega$  being smooth, nonnegative and compactly supported function but more general assumption will be presented below). The remaining question



is whether one can consider the limit  $\varepsilon \rightarrow 0$  in (9.1.1) and arrive at

$$\partial_t \rho - \Delta \rho + \operatorname{div}(\rho \nabla \Delta \rho) = 0. \tag{9.1.2}$$

A natural strategy for such problems are so-called commutator estimates, originally studied in the context of renormalized solutions to transport equation [137] and conservation of energy for weak solutions of Euler equation [110] or more general hyperbolic systems [205, 35]. The commutator estimates control the error resulting from the fact that the mollification operator does not commute with nonlinear functions. In our case, they cannot be applied directly, as we do not have sufficient estimates on  $\{\rho_\varepsilon\}$ . Indeed, both energy and entropy identities for (9.1.1) yield estimates on non-local quantities  $\rho_\varepsilon * \omega_\varepsilon$ ,  $\nabla \rho_\varepsilon * \omega_\varepsilon$  and  $\Delta \rho_\varepsilon * \omega_\varepsilon$  (see Section 9.3). In particular, we do not even know that  $\{\rho_\varepsilon\}$  is uniformly bounded in  $L^2_{\text{loc}}((0, T) \times \mathbb{R}^d)$  (except in dimension  $d = 2$ ). In Appendix 9.6.1 we show that for a fairly arbitrary kernel (up to some weak growth assumptions at infinity), the direct commutator estimates work in dimension  $d = 2$ . The same is true if one knows a priori that  $\{\rho_\varepsilon\}$  is uniformly integrable in  $L^2((0, T) \times \mathbb{R}^d)$  (or even  $L^2_{\text{loc}}((0, T) \times \mathbb{R}^d)$  if  $\omega$  is compactly supported).

Being not satisfied as we cannot cover the physical dimension  $d = 3$ , our approach is to look for an appropriate assumption on  $\omega$  so that we can exploit more the aforementioned nonlocal quantities. We were inspired by the recent work on Landau equation [80, 75] where the authors consider the kernel  $\omega(x) = e^{-(1+|x|^2)^{1/2}}$  which satisfies the crucial estimate

$$|\nabla \omega(x)| \leq C \omega(x). \tag{9.1.3}$$

Such inequalities allow to transform difficult-to-estimate terms by better understood ones, for instance  $\varepsilon \rho_\varepsilon * |\nabla \omega_\varepsilon|$  can be estimated pointwisely by  $\rho_\varepsilon * \omega_\varepsilon$  due to nonnegativity of  $\rho_\varepsilon$  and  $\omega_\varepsilon$ . Nevertheless, (9.1.3) is fairly restrictive - for instance, it excludes compactly supported kernels. In our setting, the following assumption turns out to be successful :

*Assumption 9.1.1.* We assume that  $\omega : \mathbb{R}^d \rightarrow [0, \infty)$  is a smooth function such that  $\int_{\mathbb{R}^d} \omega(x) dx = 1$  and  $\omega(x) = \omega(-x)$ . Moreover, there exists an integrable kernel  $f : \mathbb{R}^d \rightarrow [0, \infty)$  such that for all  $x \in \mathbb{R}^d$

$$(|x| + |x|^2) |\nabla \omega(x)| \leq C \omega * f(x). \tag{9.1.4}$$

Furthermore, we assume that  $\omega$  has sufficient decay at  $+\infty$  :

$$\lim_{R \rightarrow \infty} \sup_{|x|=R} |x|^d \omega(x) = 0. \tag{9.1.5}$$

This covers the case of Gaussian  $\omega(x) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2}$  and any nonnegative, compactly supported kernel by choosing  $f = \omega$ , see Lemmas 9.2.1 and 9.2.2. Moreover, we can cover the kernel  $\omega(x) = e^{-(1+|x|^2)^{1/2}}$  by choosing  $f$  more carefully, see Lemma 9.2.3.

For the initial condition we suppose that  $\rho(t = 0) = \rho^0$  where  $\rho^0$  satisfies

$$\rho^0 \geq 0, \quad \rho^0 \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d), \quad \rho^0 |\log \rho^0| \in L^1(\mathbb{R}^d), \quad |x|^2 \rho^0 \in L^1(\mathbb{R}^d). \tag{9.1.6}$$

Our main result reads :

**Theorem 9.1.2.** *Let  $\{\rho_\varepsilon\}_\varepsilon$  be a sequence of solutions to (9.1.1) with initial condition  $\rho^0$  satisfying (9.1.6). Then, up to a subsequence not relabeled,  $\rho_\varepsilon \rightarrow \rho$  in  $L^p(0, T; L^1(\mathbb{R}^d))$  for all  $p \in [1, \infty)$  where  $\rho$  is a weak solution of the degenerate Cahn-Hilliard equation (9.1.2).*

Our methods are quite flexible and they allow to study the same question for systems of the type

$$\partial_t \rho_\varepsilon^i - \Delta \rho_\varepsilon^i - \operatorname{div} \left( \rho_\varepsilon^i \nabla \Delta \sum_{j=1}^N K_\varepsilon^{i,j} * \rho_\varepsilon^j \right) = 0, \quad i = 1, \dots, N, \quad (9.1.7)$$

under some additional structural assumptions. This is discussed in Section 9.5 (see Theorem 9.5.1).

To conclude the introduction, let us mention that similar problems have been studied in the literature for the porous media equation. Up to our knowledge, the first result of this type was obtained by Lions and Mas-Gallic [257] for the PDE

$$\partial_t \rho_\varepsilon = \operatorname{div}(\rho_\varepsilon \nabla \rho_\varepsilon * \omega_\varepsilon * \omega_\varepsilon)$$

Then, the cases of cross-diffusion systems and general nonlinear diffusion equations has been considered in [93] and [65, 212], respectively. These problems are motivated by the numerical algorithms called particle methods. More precisely, consider  $N$  particles moving according to the system of ODEs

$$X_i'(t) = -\frac{1}{N} \sum_{j \neq i} \nabla W(X_i(t) - X_j(t)).$$

Then, the empirical measure  $\mu^N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}$  solves in the sense of distributions

$$\partial_t \mu^N - \operatorname{div}(\mu^N \nabla \mu^N * W) = 0$$

so in the limit  $N \rightarrow \infty$

$$\partial_t \mu - \operatorname{div}(\mu \nabla \mu * W) = 0.$$

If  $W \xrightarrow{*} \delta_0$ , we recover the porous media equation. For numerical experiments based on this method we refer to [257].

Let us also comment that (9.1.1) could be called the nonlocal Cahn-Hilliard equation but it should not be confused with the nonlocal effect in the following PDE

$$\partial_t \rho_\varepsilon = \operatorname{div}(\rho_\varepsilon \nabla (B_\varepsilon[\rho_\varepsilon] + F'(\rho_\varepsilon))), \quad (9.1.8)$$

where  $F$  is the potential and  $B_\varepsilon[u]$  is a nonlocal operator approximating  $-\Delta u$ , i.e.

$$B_\varepsilon[u](x) = \frac{1}{\varepsilon^2} (u(x) - \omega_\varepsilon * u(x)) = \frac{1}{\varepsilon^2} \int_{\mathbb{T}^d} \omega_\varepsilon(y) (u(x) - u(x-y)) \, dy.$$

The equation was obtained by Giacomin and Lebowitz [189] as a derivation of the degenerate Cahn-Hilliard equation

$$\partial_t \rho = \operatorname{div}(\rho \nabla (-\Delta \rho + F'(\rho))), \quad (9.1.9)$$

proposed in [70] to model the dynamics of phase separation in binary mixtures. The question of passing to the limit from (9.1.8) to (9.1.9) was addressed only recently in [155] for a single equation and in [81] for a system. This problem is fairly different from (9.1.1) as energy and entropy yields strong compactness of  $\{\rho_\varepsilon\}$  and  $\{\nabla \rho_\varepsilon\}$  rather than their mollifications  $\{\rho_\varepsilon * \omega_\varepsilon\}$  and  $\{\nabla \rho_\varepsilon * \omega_\varepsilon\}$  as in the case of (9.1.1). We also remark that the same problem was studied in

the context of the nondegenerate Cahn-Hilliard equation [273, 121, 123, 122]

$$\partial_t \rho_\varepsilon = \operatorname{div} \nabla \mu_\varepsilon, \quad \mu_\varepsilon = B_\varepsilon[\rho_\varepsilon] + F'(\rho_\varepsilon).$$

Here, one obtains immediately an estimate on  $\{\nabla \mu_\varepsilon\}$  (by multiplying by  $\mu_\varepsilon$ ) which greatly simplifies identification of the limits. Nevertheless, we point out that in [273, 121, 123, 122] the difficulty is rather the low regularity of the potential and the kernel.

The structure is as follows. In Section 9.2 we show that Assumption 9.1.1 is satisfied by a wide class of kernels. In Section 9.3 we gather the a priori estimates necessary for the proof of the main result, Theorem 9.1.2, which is proved in Section 9.4. In the last Section 9.5, we show how the result can be extended to systems. Finally, Appendix 9.6.1 is dedicated to the proof of Theorem 9.1.2 in dimension 2 for a broader class of kernels.

## 9.2 Examples of kernels satisfying Assumption 9.1.1

Three particular classes of kernels are usually found in the literature and we show they satisfy Assumption 9.1.1. In fact, in all of those examples, we only need to verify condition (9.1.4).

**Lemma 9.2.1.** *Let  $\omega : \mathbb{R}^d \rightarrow [0, \infty)$  be a smooth function such that  $\int_{\mathbb{R}^d} \omega(x) dx = 1$ . Suppose that  $\omega$  is supported on the unit ball  $\{x \in \mathbb{R}^d : |x| \leq 1\}$  and  $\omega > 0$  on the interior  $\{x \in \mathbb{R}^d : |x| < 1\}$ . Then,  $\omega$  satisfies (9.1.4) with  $f = \omega$ .*

*Démonstration.* To prove (9.1.4), we only need to consider  $|x| \leq 1$ . By smoothness and compact support of  $\omega$ , there exists a constant such that  $(|x| + |x|^2) |\nabla \omega(x)| \leq C$  and it remains to prove that  $\inf_{|x| \leq 1} \omega * \omega(x) > 0$ . For any  $|x| \leq 1$ , we see from the formula  $\omega * \omega(x) = \int_{|y| \leq 1} \omega(x - y) \omega(y) dy$  that  $\omega * \omega(x) > 0$ . As any continuous function attains its infimum on a compact set, the conclusion follows.  $\square$

**Lemma 9.2.2.** *Let  $\omega(x) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2}$ . Then,  $\omega$  satisfies (9.1.4) with  $f = \omega$ .*

*Démonstration.* For Gaussians, we know that

$$\omega * \omega(x) = \frac{1}{(2\pi)^d} e^{-|x|^2/4} \int_{\mathbb{R}^d} e^{-\left|\frac{x}{\sqrt{2}} - \sqrt{2}y\right|^2/2} dy = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/4}.$$

Therefore, since the function  $(|x|^2 + |x|^3) e^{-|x|^2/4}$  is globally bounded, we find

$$(|x| + |x|^2) |\nabla \omega(x)| \leq C (|x|^2 + |x|^3) e^{-|x|^2/2} \leq C e^{-|x|^2/4} = C \omega * \omega(x).$$

$\square$

**Lemma 9.2.3.** *Let  $\omega(x) = e^{-(1+|x|^2)^{1/2}}$ . Then,  $\omega$  satisfies (9.1.4) with  $f(x) = e^{-(1+|x|^2/3)^{1/2}}$ .*

*Démonstration.* We need to estimate the convolution  $\omega * f$  from below so we need to estimate the expression  $\sqrt{1 + |x - y|^2} + \sqrt{1 + |y|^2/3}$  from above. Using  $\sqrt{x} + \sqrt{y} \leq \sqrt{2(x + y)}$  and  $\sqrt{x + y} \leq$

$\sqrt{x} + \sqrt{y}$ , we have

$$\begin{aligned} \sqrt{1 + |x - y|^2} + \sqrt{1 + |y|^2/3} &\leq \sqrt{2} \sqrt{2 + |x|^2 - 2x \cdot y + \frac{4}{3}|y|^2} \\ &\leq \sqrt{2} \sqrt{\frac{1 + |x|^2}{4} + \left| \frac{\sqrt{3}}{2}x - \frac{2}{\sqrt{3}}y \right|^2 + \frac{7}{4}} \\ &\leq \frac{\sqrt{2}}{2} \sqrt{1 + |x|^2} + \sqrt{2} \sqrt{\left| \frac{\sqrt{3}}{2}x - \frac{2}{\sqrt{3}}y \right|^2 + \frac{7}{4}}. \end{aligned}$$

Note that the integral  $\int_{\mathbb{R}^d} e^{-\sqrt{2} \sqrt{\left| \frac{\sqrt{3}}{2}x - \frac{2}{\sqrt{3}}y \right|^2 + \frac{7}{4}}} dy$  is a constant independent of  $x$  (by a change of variables). Therefore,

$$e^{-\frac{\sqrt{2}}{2} \sqrt{1 + |x|^2}} \leq C \omega * f(x).$$

We conclude by observing that the function  $(|x| + |x|^2) e^{-(1 - \frac{\sqrt{2}}{2}) \sqrt{1 + |x|^2}}$  is globally bounded

$$\begin{aligned} (|x| + |x|^2) |\nabla \omega(x)| &\leq C (|x| + |x|^2) e^{-\sqrt{1 + |x|^2}} = \\ &= C (|x| + |x|^2) e^{-(1 - \frac{\sqrt{2}}{2}) \sqrt{1 + |x|^2}} e^{-\frac{\sqrt{2}}{2} \sqrt{1 + |x|^2}} \leq C e^{-\frac{\sqrt{2}}{2} \sqrt{1 + |x|^2}} \leq C \omega * f(x). \end{aligned}$$

□

### 9.3 Uniform estimates and compactness

The first immediate estimate is the conservation of mass. Integrating the equation in space we obtain an  $L^\infty(0, T; L^1(\mathbb{R}^d))$  control on the solution. Moreover, the nonlocal equation (9.1.1) comes with an energy/entropy structure. Defining

$$E_\varepsilon[\rho] = \int_{\mathbb{R}^d} \frac{|\nabla \rho * \omega_\varepsilon|^2}{2} dx, \quad \Phi[\rho] = \int_{\mathbb{R}^d} \rho \log(\rho) dx, \tag{9.3.1}$$

we obtain the dissipation equalities :

$$\frac{dE_\varepsilon[\rho]}{dt} + \int_{\mathbb{R}^d} |\Delta \rho * \omega_\varepsilon|^2 dx + \int_{\Omega} \rho |\nabla \Delta \rho * \omega_\varepsilon * \omega_\varepsilon|^2 dx = 0, \tag{9.3.2}$$

$$\frac{d\Phi[\rho]}{dt} + \int_{\mathbb{R}^d} \frac{|\nabla \rho|^2}{\rho} dx + \int_{\mathbb{R}^d} |\Delta \rho * \omega_\varepsilon|^2 dx = 0. \tag{9.3.3}$$

Of course one has to be careful with the entropy equality, as  $\rho \log(\rho)$  can be negative when  $\rho$  is small and one needs to show that its negative part is integrable.

**Proposition 9.3.1.** *Suppose the initial condition  $\rho^0$  satisfies (9.1.6). Then, there exists a unique nonnegative weak solution to (9.1.1) satisfying the following bounds, uniformly with respect to  $\varepsilon$  :*

- (A)  $\{\rho_\varepsilon\}_\varepsilon \in L^\infty(0, T; L^1(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d))$ ,
- (B)  $\{\partial_t \rho_\varepsilon\}_\varepsilon \in L^2(0, T; H^{-k}(\mathbb{R}^d))$  for some  $k$ ,
- (C)  $\{\sqrt{\rho_\varepsilon} \nabla \Delta \rho_\varepsilon * \omega_\varepsilon * \omega_\varepsilon\}_\varepsilon \in L^2((0, T) \times \mathbb{R}^d)$ ,

- (D)  $\{\rho_\varepsilon * \omega_\varepsilon\}_\varepsilon \in L^\infty(0, T; H^1(\mathbb{R}^d)) \cap L^2(0, T; H^2(\mathbb{R}^d))$ ,  
(E)  $\{|x|^2 \rho_\varepsilon\}_\varepsilon \in L^\infty(0, T; L^1(\mathbb{R}^d))$ ,  
(F)  $\{\nabla \sqrt{\rho_\varepsilon}\}_\varepsilon \in L^2(0, T; L^2(\mathbb{R}^d))$ ,  
(G)  $\{\nabla \rho_\varepsilon\}_\varepsilon \in L^2(0, T; L^1(\mathbb{R}^d))$ .

Moreover, we can extract a subsequence such that

$$\rho_\varepsilon \rightarrow \rho \text{ strongly in } L^p(0, T; L^1(\mathbb{R}^d)), \quad p < \infty, \quad (9.3.4)$$

$$\rho_\varepsilon * \omega_\varepsilon \rightharpoonup \rho \text{ weakly in } L^\infty(0, T; H^1(\mathbb{R}^d)) \cap L^2(0, T; H^2(\mathbb{R}^d)), \quad (9.3.5)$$

$$\rho_\varepsilon * \omega_\varepsilon \rightarrow \rho \text{ a.e. and strongly in } L^2(0, T; H_{loc}^1(\mathbb{R}^d)). \quad (9.3.6)$$

*Proof of Proposition 9.3.1.* The existence and uniqueness of solutions is a classical matter as (9.1.1) is an advection-diffusion equation with smooth advection (as  $\omega_\varepsilon$  is smooth). The  $L^1$  bound in (A) is a consequence of mass conservation. (C) follows directly from (9.3.2). Estimate (B) is a consequence of (A), (C), Equation (9.1.1) and splitting

$$\rho_\varepsilon \nabla \Delta \rho_\varepsilon * \omega_\varepsilon * \omega_\varepsilon = \sqrt{\rho_\varepsilon} \sqrt{\rho_\varepsilon} \nabla \Delta \rho_\varepsilon * \omega_\varepsilon * \omega_\varepsilon.$$

To prove (D), we first deduce from (9.3.2) bounds on  $\nabla \rho_\varepsilon * \omega_\varepsilon$  and  $\Delta \rho_\varepsilon * \omega_\varepsilon$ . Then, the  $L^\infty(0, T; L^2(\mathbb{R}^d))$  bound on  $\{\rho_\varepsilon * \omega_\varepsilon\}_\varepsilon$  follows from bounds on  $\{\rho_\varepsilon * \omega_\varepsilon\}_\varepsilon$  in  $L^\infty(0, T; L^1(\mathbb{R}^d))$ ,  $\{\nabla \rho_\varepsilon * \omega_\varepsilon\}_\varepsilon$  in  $L^\infty(0, T; L^2(\mathbb{R}^d))$  and from the Gagliardo-Nirenberg inequality). The estimate in  $L^2(0, T; H^2(\mathbb{R}^d))$  is a consequence of the second-order regularizing property of the operator  $\Delta$  on the whole space.

To see (E), we multiply Equation (9.1.1) by  $|x|^2$  and obtain after integration by parts

$$\frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 \rho_\varepsilon \, dx = 2 \int_{\mathbb{R}^d} \rho_\varepsilon \, dx + 2 \int_{\mathbb{R}^d} \sqrt{\rho_\varepsilon} x \cdot \sqrt{\rho_\varepsilon} \nabla \Delta \rho_\varepsilon * \omega_\varepsilon * \omega_\varepsilon \, dx.$$

Using (A), (C), the Cauchy-Schwarz inequality and the Gronwall lemma, we obtain (E).

It remains to prove the second part of estimate (A) namely the  $L \log(L)$  bound on  $\rho_\varepsilon$ . A small difficulty is that the negative part of  $\rho_\varepsilon \log(\rho_\varepsilon)$  might not be integrable on the whole space. Nevertheless, as in [212], one can prove that  $\rho_\varepsilon |\log \rho_\varepsilon|_-$  is uniformly bounded in  $L^\infty(0, T; L^1(\mathbb{R}^d))$  by splitting  $\mathbb{R}^d$  for  $\{x : |\rho_\varepsilon| \leq e^{-|x|^2}\}$ ,  $\{x : |\rho_\varepsilon| > e^{-|x|^2}\}$  and applying tail estimate (E). Hence, we can use (9.3.3) and deduce Estimate (F). Estimate (G) follows from (A) and (F) by writing  $\nabla \rho_\varepsilon = \frac{\nabla \rho_\varepsilon}{\sqrt{\rho_\varepsilon}} \sqrt{\rho_\varepsilon}$ .

The convergences (9.3.4)–(9.3.6) are a consequence of the Lions-Aubin lemma and the Banach-Alaoglu theorem where convergence (9.3.4) has been upgraded from a local to a global one by the tail estimate (E).  $\square$

## 9.4 The proof of the main result

We only need to study the term  $\int_0^T \int_{\mathbb{R}^d} \nabla \varphi \rho_\varepsilon \nabla \Delta(\rho_\varepsilon * \omega_\varepsilon * \omega_\varepsilon) \, dx \, dt$ , for test functions  $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ . Using the properties of the mollifiers

$$\int_0^T \int_{\mathbb{R}^d} \nabla \varphi \rho_\varepsilon \nabla \Delta(\rho_\varepsilon * \omega_\varepsilon * \omega_\varepsilon) \, dx \, dt = - \int_0^T \int_{\mathbb{R}^d} (\nabla \varphi \rho_\varepsilon) * \nabla \omega_\varepsilon \Delta(\rho_\varepsilon * \omega_\varepsilon) \, dx \, dt.$$

Thanks to the weak convergence (9.3.6), we only need to prove the following strong convergence result

$$(\nabla\varphi\rho_\varepsilon) * \nabla\omega_\varepsilon \rightarrow \nabla\varphi \cdot \nabla\rho + \Delta\varphi\rho = \operatorname{div}(\nabla\varphi\rho) \text{ strongly in } L^2((0, T) \times \mathbb{R}^d). \quad (9.4.1)$$

We write  $\nabla\varphi(y) = \nabla\varphi(y) - \nabla\varphi(x) + \nabla\varphi(x)$  which results in two terms :

$$\begin{aligned} (\nabla\varphi\rho_\varepsilon) * \nabla\omega_\varepsilon(x) &= \int_{\mathbb{R}^d} \nabla\varphi(y)\rho_\varepsilon(y) \cdot \nabla\omega_\varepsilon(x-y) \, dy = \nabla\varphi(x) \cdot \nabla\rho_\varepsilon * \omega_\varepsilon + \\ &+ \int_{\mathbb{R}^d} (\nabla\varphi(y) - \nabla\varphi(x))\rho_\varepsilon(y) \cdot \nabla\omega_\varepsilon(x-y) \, dy =: I + J. \end{aligned}$$

According to (9.3.6), the term  $I$  converges strongly in  $L^2((0, T) \times \mathbb{R}^d)$  (note that  $\varphi$  is compactly supported). The rest of the proof is devoted to the analysis of the term  $J$ .

By Taylor's expansion

$$\nabla\varphi(y) - \nabla\varphi(x) = D^2\varphi(x)(y-x) + R(x, y)$$

where the term  $R$  satisfies  $|R| \leq C|y-x|^2$ . We split  $J = J_1 + J_2$ , where

$$J_1 = \int_{\mathbb{R}^d} D^2\varphi(x)(y-x)\rho_\varepsilon(y)\nabla\omega_\varepsilon(x-y) \, dy, \quad J_2 = \int_{\mathbb{R}^d} R(x, y)\rho_\varepsilon(y)\nabla\omega_\varepsilon(x-y) \, dy.$$

Term  $J_1$ . We prove that  $\lim_{\varepsilon \rightarrow 0} J_1 = \Delta\varphi\rho$  in  $L^2((0, T) \times \mathbb{R}^d)$ . Since  $\varphi$  is compactly supported, it is sufficient to prove that

$$\int_{\mathbb{R}^d} \rho_\varepsilon(y)(x_i - y_i)\partial_j\omega_\varepsilon(x-y) \, dy \rightarrow -\rho(x)\delta_{i,j} \quad \text{in } L^2_{\text{loc}}((0, T) \times \mathbb{R}^d). \quad (9.4.2)$$

Assertion (9.4.2) will be obtained by proving convergence in  $L^1((0, T) \times \mathbb{R}^d)$  and uniform boundedness in  $L^p((0, T) \times \mathbb{R}^d)$  for some  $p > 2$ . Concerning the convergence in  $L^1((0, T) \times \mathbb{R}^d)$ , we first change variables

$$\int_{\mathbb{R}^d} \rho_\varepsilon(y)(x_i - y_i)\partial_j\omega_\varepsilon(x-y) \, dy = \int_{\mathbb{R}^d} \rho_\varepsilon(x - \varepsilon z)z_i\partial_j\omega(z) \, dz$$

We can estimate in  $L^1((0, T) \times \mathbb{R}^d)$  the difference

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (\rho_\varepsilon(x - \varepsilon z) - \rho_\varepsilon(x))z_i\partial_j\omega(z) \, dz \right| dx dt \leq \\ &\leq \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 \varepsilon|\nabla\rho_\varepsilon(x - \varepsilon sz)| |z|^2 |\nabla\omega(z)| \, ds dz dx dt \leq \varepsilon\|\nabla\rho_\varepsilon\|_{L^1_{t,x}} \| |z|^2 \nabla\omega \|_{L^1_x}, \end{aligned}$$

where integrability of  $|z|^2 \nabla\omega(z)$  is a consequence of assumption (9.1.4) :

$$\int_{\mathbb{R}^d} |z|^2 |\nabla\omega(z)| \, dz \leq C \int_{\mathbb{R}^d} \omega * f(z) \, dz = C \|\omega\|_{L^1} \|f\|_{L^1} \leq C.$$

Therefore, it is sufficient to study the term  $\rho_\varepsilon(x) \int_{\mathbb{R}^d} z_i \partial_j \omega(z) dz$  which equals  $-\rho_\varepsilon(x)$  because

$$\int_{\mathbb{R}^d} z_i \partial_j \omega(z) dz = -\delta_{i,j} \int_{\mathbb{R}^d} \omega(z) dz = -\delta_{i,j},$$

where the boundary term vanishes thanks to (9.1.5). The conclusion follows because  $\rho_\varepsilon$  is strongly convergent in  $L^1((0, T) \times \mathbb{R}^d)$ , cf. (9.3.4).

Concerning the uniform boundedness in  $L^p((0, T) \times \mathbb{R}^d)$  with  $p > 2$ , by nonnegativity of  $\rho_\varepsilon$ , definition of  $\omega_\varepsilon(x) = \frac{1}{\varepsilon^d} \omega\left(\frac{x}{\varepsilon}\right)$  and assumption (9.1.4),

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \rho_\varepsilon(y) (x_i - y_i) \partial_j \omega_\varepsilon(x - y) dy \right| &= \left| \int_{\mathbb{R}^d} \rho_\varepsilon(y) \frac{(x_i - y_i)}{\varepsilon^{d+1}} \partial_j \omega\left(\frac{x - y}{\varepsilon}\right) dy \right| \leq \\ &\leq C \int_{\mathbb{R}^d} \rho_\varepsilon(y) \frac{1}{\varepsilon^d} \omega * f\left(\frac{x - y}{\varepsilon}\right) dy. \end{aligned} \quad (9.4.3)$$

A change of variables shows that

$$\begin{aligned} \frac{1}{\varepsilon^d} \omega * f\left(\frac{x - y}{\varepsilon}\right) &= \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \omega(z) f\left(\frac{x - y - z}{\varepsilon}\right) dz = \\ &= \frac{1}{\varepsilon^{2d}} \int_{\mathbb{R}^d} \omega\left(\frac{z}{\varepsilon}\right) f\left(\frac{x - y - z}{\varepsilon}\right) dz = \omega_\varepsilon * f_\varepsilon(x - y), \end{aligned} \quad (9.4.4)$$

where  $f_\varepsilon(x) := \frac{1}{\varepsilon^d} f\left(\frac{x}{\varepsilon}\right)$ . Due to (9.4.3),

$$\left| \int_{\mathbb{R}^d} \rho_\varepsilon(y) (x_i - y_i) \partial_j \omega_\varepsilon(x - y) dy \right| \leq C \rho_\varepsilon * \omega_\varepsilon * f_\varepsilon(x).$$

Note that by the Gagliardo-Nirenberg inequality and uniform bound in  $L^\infty(0, T; H^1(\mathbb{R}^d))$ ,  $\{\rho_\varepsilon * \omega_\varepsilon\}_\varepsilon$  is bounded in  $L^\infty(0, T; L^{\frac{2d}{d-2}}(\mathbb{R}^d))$  where  $\frac{2d}{d-2} > 2$ . The same is true for  $\{\rho_\varepsilon * \omega_\varepsilon * f_\varepsilon\}_\varepsilon$  by the Young convolutional inequality. The conclusion follows.

Term  $J_2$ . We prove that  $\lim_{\varepsilon \rightarrow 0} J_2 = 0$ . By  $|R| \leq C|y - x|^2$ , it is sufficient to prove

$$\int_{\mathbb{R}^d} \rho_\varepsilon(y) |x - y|^2 |\nabla \omega_\varepsilon(x - y)| dy \rightarrow 0 \quad \text{in } L^2((0, T) \times \mathbb{R}^d).$$

Again, we want to use assumption (9.1.4). By definition of  $\omega_\varepsilon$  :

$$\begin{aligned} \int_{\mathbb{R}^d} \rho_\varepsilon(y) |x - y|^2 |\nabla \omega_\varepsilon(x - y)| dy &= \varepsilon \int_{\mathbb{R}^d} \rho_\varepsilon(y) \frac{1}{\varepsilon^d} \left| \frac{x - y}{\varepsilon} \right|^2 \left| \nabla \omega\left(\frac{x - y}{\varepsilon}\right) \right| dy \leq \\ &\leq \varepsilon \int_{\mathbb{R}^d} \rho_\varepsilon(y) \frac{1}{\varepsilon^d} \omega * f\left(\frac{x - y}{\varepsilon}\right) = \varepsilon \rho_\varepsilon * \omega_\varepsilon * f_\varepsilon(x), \end{aligned}$$

where in the last line we applied (9.4.4). By the Young convolutional inequality,  $\rho_\varepsilon * \omega_\varepsilon * f_\varepsilon$  on the (RHS) is bounded in  $L^2((0, T) \times \mathbb{R}^d)$  so the conclusion follows.

## 9.5 Extension to systems

Motivated by [212], we consider the system of  $N$  equations

$$\partial_t \rho_\varepsilon^i - \Delta \rho_\varepsilon^i + \operatorname{div} \left( \rho_\varepsilon^i \nabla \Delta \sum_{j=1}^N K_\varepsilon^{i,j} * \rho_\varepsilon^j \right) = 0, \quad (9.5.1)$$

where  $1 \leq i \leq N$  and the kernels  $K_\varepsilon^{i,j}$  are of the form

$$K_\varepsilon^{i,j} = \sum_{k=1}^N \alpha^{i,k} \alpha^{j,k} \omega_\varepsilon^i * \omega_\varepsilon^j,$$

where the  $\omega^i$  are kernels satisfying Assumption 9.1.1. The coefficients  $\{\alpha^{i,k}\}$  form a matrix  $A$  and we assume it is invertible. Under these assumptions, for any set of functions  $\eta_1, \dots, \eta_N$ :

$$\sum_{i,j=1}^N \int_{\mathbb{R}^d} \eta^i K_\varepsilon^{i,j} * \eta^j = \int_{\mathbb{R}^d} \sum_{k=1}^N \left( \sum_{i=1}^N \alpha^{i,k} \eta^i * \omega_\varepsilon^i \right)^2$$

so that

$$\tilde{C} \|A^{-1}\|^2 \int_{\mathbb{R}^d} \sum_{i=1}^N (\eta^i * \omega_\varepsilon^i)^2 \geq \sum_{i,j=1}^N \int_{\mathbb{R}^d} \eta^i K_\varepsilon^{i,j} * \eta^j \geq C \|A^{-1}\|^2 \int_{\mathbb{R}^d} \sum_{i=1}^N (\eta^i * \omega_\varepsilon^i)^2. \quad (9.5.2)$$

**Theorem 9.5.1.** *Let  $\{\rho_\varepsilon^i\}_\varepsilon$  be a sequence of solutions to (9.5.1) with initial condition  $\rho^{0,i}$  satisfying (9.1.6). Then, for  $i = 1, \dots, N$ , and for a subsequence not relabeled,  $\rho_\varepsilon^i \rightarrow \rho^i$  in  $L^p(0, T; L^1(\mathbb{R}^d))$  for all  $p \in [1, \infty)$  where  $\rho^i$  is a weak solution of*

$$\partial_t \rho^i - \Delta \rho^i + \operatorname{div} \left( \rho^i \nabla \Delta \sum_{j=1}^N K^{i,j} \rho^j \right) = 0, \quad K^{i,j} = \sum_{k=1}^N \alpha^{i,k} \alpha^{j,k}.$$

We first extend the uniform bounds in Proposition 9.3.1 to the case of system (9.5.1).

**Proposition 9.5.2.** *Suppose that for all  $i = 1, \dots, N$ , the initial conditions  $\rho^{0,i}$  satisfy (9.1.6). Then, the nonnegative solution to (9.5.1) satisfies the following bounds, uniformly with respect to  $\varepsilon$ :*

- (A)  $\{\rho_\varepsilon^i\}_\varepsilon \in L^\infty(0, T; L^1(\mathbb{R}^d) \cap L \log L(\mathbb{R}^d))$ ,
- (B)  $\{\partial_t \rho_\varepsilon^i\}_\varepsilon \in L^2(0, T; H^{-k}(\mathbb{R}^d))$  for some  $k$ ,
- (C)  $\{\sqrt{\rho_\varepsilon^i} \nabla \Delta \sum_{j=1}^N K_\varepsilon^{i,j} * \rho_\varepsilon^j\}_\varepsilon \in L^2((0, T) \times \mathbb{R}^d)$ ,
- (D)  $\{\rho_\varepsilon^i * \omega_\varepsilon^i\}_\varepsilon \in L^\infty(0, T; H^1(\mathbb{R}^d)) \cap L^2(0, T; H^2(\mathbb{R}^d))$ ,
- (E)  $\{|x|^2 \rho_\varepsilon^i\}_\varepsilon \in L^\infty(0, T; L^1(\mathbb{R}^d))$ ,
- (F)  $\{\nabla \sqrt{\rho_\varepsilon^i}\}_\varepsilon \in L^2(0, T; L^2(\mathbb{R}^d))$ ,
- (G)  $\{\nabla \rho_\varepsilon^i\}_\varepsilon \in L^2(0, T; L^1(\mathbb{R}^d))$ ,



Moreover, we can extract a subsequence such that for all  $i = 1, \dots, N$

$$\rho_\varepsilon^i \rightarrow \rho^i \text{ strongly in } L^p(0, T; L^1(\mathbb{R}^d)), p < \infty \quad (9.5.3)$$

$$\rho_\varepsilon^i * \omega_\varepsilon^i \rightharpoonup \rho^i \text{ weakly in } L^\infty(0, T; H^1(\mathbb{R}^d)) \cap L^2(0, T; H^2(\mathbb{R}^d)), \quad (9.5.4)$$

$$\rho_\varepsilon^i * \omega_\varepsilon^i \rightarrow \rho^i \text{ a.e. and strongly in } L^2(0, T; H_{loc}^1(\mathbb{R}^d)). \quad (9.5.5)$$

*Démonstration.* The proof is almost the same as the proof of Proposition 9.3.1. The only difficulty is to obtain the energy and entropy identities corresponding to (9.3.2) and (9.3.3), respectively.

Concerning the energy, we multiply (9.5.1) with  $\Delta \sum_{j=1}^N K^{i,j} * \rho_\varepsilon^j$ , integrating in space and summing up for  $i = 1, \dots, N$  yields

$$\begin{aligned} & \frac{C \|A^{-1}\|^2}{2} \sum_{i=1}^N \int_{\mathbb{R}^d} |\nabla \rho_\varepsilon^i * \omega_\varepsilon^i(t, x)|^2 dx + C \|A^{-1}\|^2 \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d} |\Delta \rho_\varepsilon^i * \omega_\varepsilon^i|^2 dx ds + \\ & + \sum_{i=1}^N \int_0^t \int_{\mathbb{R}^d} \rho_\varepsilon^i \left| \nabla \Delta \sum_{j=1}^N K_\varepsilon^{i,j} * \rho_\varepsilon^j \right|^2 dx ds \leq \frac{\tilde{C} \|A\|^2}{2} \sum_{i=1}^N \int_{\mathbb{R}^d} |\nabla \rho^{i,0}|^2 dx. \end{aligned}$$

This identity implies (C) and (D). Estimate (B) follows from the PDE (9.5.1) and (C). Thanks to (C), we also deduce (E).

Concerning the entropy, we multiply (9.5.1) with  $\log \rho_\varepsilon^i$ , integrate in space and sum up to obtain

$$\sum_{i=1}^N \partial_t \int_{\mathbb{R}^d} \rho_\varepsilon^i (\log \rho_\varepsilon^i - 1) dx + \sum_{i=1}^N \int_{\mathbb{R}^d} \frac{|\nabla \rho_\varepsilon^i|^2}{\rho_\varepsilon^i} dx + \sum_{i,j=1}^N \int_{\mathbb{R}^d} K^{i,j} * \Delta \rho_\varepsilon^j \Delta \rho_\varepsilon^i dx = 0.$$

Applying (9.5.2) with  $\eta_i = \Delta \rho_\varepsilon^i$ , we deduce

$$\sum_{i=1}^N \partial_t \int_{\mathbb{R}^d} \rho_\varepsilon^i (\log \rho_\varepsilon^i - 1) dx + \sum_{i=1}^N \int_{\mathbb{R}^d} \frac{|\nabla \rho_\varepsilon^i|^2}{\rho_\varepsilon^i} dx + C \|A^{-1}\| \sum_{i=1}^N \int_{\mathbb{R}^d} |\Delta \rho_\varepsilon^i * \omega_\varepsilon^i|^2 dx \leq 0.$$

As in the proof of Proposition 9.3.1, one may check that  $\rho_\varepsilon^i |\log \rho_\varepsilon^i|_-$  is uniformly bounded in  $L^\infty(0, T; L^1(\mathbb{R}^d))$  which implies (A), (F) and (G). The convergences (9.5.3)–(9.5.5) easily follow from the estimates.  $\square$

*Proof of Theorem 9.5.1.* By linearity, we only need to explain how to pass to the limit in the term

$$\int_0^T \int_{\mathbb{R}^d} \nabla \varphi \rho_\varepsilon^i \nabla \Delta (\rho_\varepsilon^j * \omega_\varepsilon^j * \omega_\varepsilon^i) dx dt = - \int_0^T \int_{\mathbb{R}^d} (\nabla \varphi \rho_\varepsilon^i) * \nabla \omega_\varepsilon^i \Delta (\rho_\varepsilon^j * \omega_\varepsilon^j) dx dt.$$

However, in the proof of Theorem 9.1.2, we proved that

$$(\nabla \varphi \rho_\varepsilon^i) * \nabla \omega_\varepsilon^i \rightarrow \nabla \varphi \cdot \nabla \rho^i + \Delta \varphi \rho^i \text{ strongly in } L^2((0, T) \times \mathbb{R}^d),$$

see (9.4.1). Thanks to the weak convergence  $\Delta (\rho_\varepsilon^j * \omega_\varepsilon^j) \rightharpoonup \Delta \rho_\varepsilon^j$  in (9.5.5), we conclude the proof.  $\square$

## 9.6 Appendix

### 9.6.1 Proof of the convergence for general kernels and $d = 2$

In dimension  $d = 2$  another proof of the main result uses weaker assumptions, namely

$$d = 2, \quad y \omega(y) \in L^1(\mathbb{R}^d), \quad y \nabla \omega(y) \in L^2(\mathbb{R}^d).$$

As in the main proof, we only need to study term  $\int_0^T \int_{\mathbb{R}^d} \nabla \varphi \rho_\varepsilon \nabla \Delta(\rho_\varepsilon * \omega_\varepsilon * \omega_\varepsilon) dx dt$ , where  $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ . Integrating by parts

$$\int_0^T \int_{\mathbb{R}^d} \nabla \varphi \rho_\varepsilon \nabla \Delta(\rho_\varepsilon * \omega_\varepsilon * \omega_\varepsilon) dx dt = - \int_0^T \int_{\mathbb{R}^d} \operatorname{div}(\nabla \varphi \rho_\varepsilon) * \omega_\varepsilon \Delta(\rho_\varepsilon * \omega_\varepsilon) dx dt$$

According to the a priori estimate (D), we need to prove that  $\operatorname{div}(\nabla \varphi \rho_\varepsilon) * \omega_\varepsilon$  converges strongly in  $L^2((0, T) \times \mathbb{R}^d)$ . We introduce the truncation operator

$$T_M(\rho) = \begin{cases} \rho & \text{if } \rho \leq M, \\ M & \text{if } \rho > M, \end{cases}$$

so that splitting  $\rho_\varepsilon = \rho_\varepsilon - T_M(\rho_\varepsilon) + T_M(\rho_\varepsilon)$  we have

$$\begin{aligned} \operatorname{div}(\nabla \varphi \rho_\varepsilon) * \omega_\varepsilon &= \operatorname{div}(\nabla \varphi T_M(\rho_\varepsilon)) * \omega_\varepsilon + \operatorname{div}(\nabla \varphi(\rho_\varepsilon - T_M(\rho_\varepsilon))) * \omega_\varepsilon \\ &= (\Delta \varphi T_M(\rho_\varepsilon)) * \omega_\varepsilon + (\nabla \varphi \cdot \nabla \rho_\varepsilon \mathbf{1}_{\rho_\varepsilon \leq M}) * \omega_\varepsilon + (\nabla \varphi(\rho_\varepsilon - T_M(\rho_\varepsilon))) * \nabla \omega_\varepsilon \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

The parameter  $M$  will be chosen later in terms of  $\varepsilon$  so that  $M \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ . Term  $I_1$ . We write

$$\begin{aligned} I_1(t, x) &= \int_{\mathbb{R}^d} (\Delta \varphi(x - y) - \Delta \varphi(x)) T_M(\rho_\varepsilon)(x - y) \omega_\varepsilon(y) dy + \Delta \varphi(T_M(\rho_\varepsilon)) * \omega_\varepsilon \\ &= I_1^A + I_1^B. \end{aligned}$$

As  $|\Delta \varphi(x - y) - \Delta \varphi(x)| \leq C |y|$ , we can estimate

$$\|I_1^A\|_{L^2_{t,x}} \leq \sqrt{M} \|\sqrt{\rho_\varepsilon}\|_{L^2_{t,x}} \| |y| \omega_\varepsilon(y) \|_{L^1} \leq \varepsilon \sqrt{M} \|\sqrt{\rho_\varepsilon}\|_{L^2_{t,x}} \| |y| \omega(y) \|_{L^1}$$

so that  $\|I_1^A\|_{L^2_{t,x}} \leq C \varepsilon \sqrt{M}$ . Furthermore, note that the term  $I_1^B$  is compact in  $L^2((0, T) \times \mathbb{R}^d)$  whenever  $M \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$ . To see this, first note that it is sufficient to establish local compactness as  $\varphi$  is compactly supported. The latter can be proved by the Vitali theorem : we have convergence in measure (even in  $L^1_{\text{loc}}((0, T) \times \mathbb{R}^d)$ ) of  $T_M(\rho_\varepsilon) * \omega_\varepsilon$  and uniform integrability thanks to the pointwise estimate

$$0 \leq T_M(\rho_\varepsilon) * \omega_\varepsilon \leq \rho_\varepsilon * \omega_\varepsilon$$

since  $\rho_\varepsilon * \omega_\varepsilon$  is compact in  $L^2_{\text{loc}}((0, T) \times \mathbb{R}^d)$ . We conclude that

$$I_1^B \rightarrow \Delta \varphi \rho \text{ in } L^2((0, T) \times \mathbb{R}^d) \quad \text{when } \varepsilon \rightarrow 0, M \rightarrow \infty. \quad (9.6.1)$$

Term  $I_2$ . We have

$$I_2(t, x) = \int_{\mathbb{R}^d} (\nabla\varphi(x-y) - \nabla\varphi(x)) \nabla\rho_\varepsilon(x-y) \mathbb{1}_{\rho_\varepsilon(x-y) \leq M} \omega_\varepsilon(y) \, dy \\ + \nabla\varphi \cdot (\nabla\rho_\varepsilon \mathbb{1}_{\rho_\varepsilon \leq M}) * \omega_\varepsilon =: I_2^A + I_2^B.$$

As  $(\nabla\varphi(x-y) - \nabla\varphi(x)) \leq C|y|$  and  $|\nabla\rho_\varepsilon(x-y)| \mathbb{1}_{\rho_\varepsilon(x-y) \leq M} \leq \sqrt{M} \frac{|\nabla\rho_\varepsilon(x-y)|}{\sqrt{\rho_\varepsilon(x-y)}}$ , we can estimate the term  $I_2^A$  as follows

$$\|I_2^A\|_{L^2_{t,x}} \leq \sqrt{M} \left\| \frac{\nabla\rho_\varepsilon}{\sqrt{\rho_\varepsilon}} \right\|_{L^2_{t,x}} \| |y| \omega_\varepsilon(y) \|_{L^1} \leq \varepsilon \sqrt{M} \left\| \frac{\nabla\rho_\varepsilon}{\sqrt{\rho_\varepsilon}} \right\|_{L^2_{t,x}} \| |y| \omega(y) \|_{L^1}$$

so that  $|I_2^A| \leq C \varepsilon \sqrt{M}$  according to estimate (F).

Term  $I_3$ . We write

$$I_3(t, x) = \int_{\mathbb{R}^d} (\nabla\varphi(x-y) - \nabla\varphi(x)) (\rho_\varepsilon(x-y) - T_M(\rho_\varepsilon(x-y))) \nabla\omega_\varepsilon(y) \, dy \\ + \nabla\varphi (\rho_\varepsilon - T_M(\rho_\varepsilon)) * \nabla\omega_\varepsilon = I_3^A + I_3^B.$$

We observe that  $|\nabla\varphi(x-y) - \nabla\varphi(x)| \leq C|y|$  and  $|\rho_\varepsilon - T_M(\rho_\varepsilon)| \leq 2\rho_\varepsilon \mathbb{1}_{\rho_\varepsilon \geq M}$  so the term  $I_3^A$  can be estimated as

$$\|I_3^A\|_{L^2_{t,x}} \leq C \|(\rho_\varepsilon \mathbb{1}_{\rho_\varepsilon \geq M}) * (|y| |\nabla\omega_\varepsilon(y)|)\|_{L^2_{t,x}}.$$

By the Gagliardo-Nirenberg-Sobolev inequality, we get that  $\{\rho_\varepsilon\}$  is uniformly bounded in  $L^2((0, T) \times \mathbb{R}^d)$ . Therefore,

$$\|I_3^A\|_{L^2_{t,x}} \leq C \|\rho_\varepsilon \mathbb{1}_{\rho_\varepsilon \geq M}\|_{L^2_t L^1_x} \| |y| |\nabla\omega_\varepsilon(y)| \|_{L^2} \leq C \|\rho_\varepsilon\|_{L^2_{t,x}} \|\mathbb{1}_{\rho_\varepsilon \geq M}\|_{L^\infty_t L^2_x} \| |y| |\nabla\omega_\varepsilon(y)| \|_{L^2}.$$

It remains to estimate  $\|\mathbb{1}_{\rho_\varepsilon \geq M}\|_{L^\infty_t L^2_x}$  and  $\| |y| |\nabla\omega_\varepsilon(y)| \|_{L^2}$ . We have

$$\|\mathbb{1}_{\rho_\varepsilon \geq M}\|_{L^\infty_t L^2_x} \leq \sup_{t \in (0, T)} \left( \int_{\mathbb{R}^d} \frac{\rho_\varepsilon \log \rho_\varepsilon}{M \log(M)} \, dx \right)^{1/2} \leq \frac{C}{M^{1/2} \log^{1/2} M},$$

$$\| |y| |\nabla\omega_\varepsilon(y)| \|_{L^2}^2 \leq \int_{\mathbb{R}^d} \frac{1}{\varepsilon^{2d+2}} |y|^2 \left| \nabla\omega\left(\frac{y}{\varepsilon}\right) \right|^2 \, dy = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} |y|^2 |\nabla\omega(y)|^2 \, dy \leq \frac{C}{\varepsilon^2},$$

since  $d = 2$  and using mass conservation (A). We conclude that

$$\|I_3^A\|_{L^2_{t,x}} \leq \frac{C}{\varepsilon M^{1/2} \log^{1/2} M}.$$

The conclusion. Note that the terms  $I_2^B$  and  $I_3^B$  combine to

$$I_2^B + I_3^B = \nabla\varphi \nabla(\rho_\varepsilon * \omega_\varepsilon)$$

which is compact in  $L^2((0, T) \times \mathbb{R}^d)$  and converges to  $\nabla\varphi \nabla\rho$ . Therefore,

$$\operatorname{div}(\nabla\varphi \rho_\varepsilon) * \omega_\varepsilon = \Delta\varphi(T_M(\rho_\varepsilon)) * \omega_\varepsilon + \nabla\varphi \nabla(\rho_\varepsilon * \omega_\varepsilon) + R, \quad (9.6.2)$$

where the first two terms are compact in  $L^2((0, T) \times \mathbb{R}^d)$  (see also (9.6.1)) while

$$\|R\|_{L^2_{t,x}} \leq C \varepsilon \sqrt{M} + \frac{C}{\varepsilon \sqrt{M} \log^{1/2}(M)}.$$

The conclusion follows by choosing  $M$  such that  $\varepsilon^2 M \log^{1/2} M = 1$ .

*Remark 9.6.1.* In arbitrary dimension  $d$ , if we knew that  $\{\rho_\varepsilon\}_\varepsilon$  is uniformly integrable in  $L^2((0, T) \times \mathbb{R}^d)$ , i.e.

$$\lim_{\varepsilon \rightarrow 0} \|\rho_\varepsilon \mathbf{1}_{\rho_\varepsilon > \frac{1}{\varepsilon}}\|_{L^2_{t,x}} = 0, \quad (9.6.3)$$

we could conclude in an easier way. Indeed, assuming that  $y \nabla \omega(y) \in L^1(\mathbb{R}^d)$ , one can estimate

$$\|I_3^A\|_{L^2_{t,x}} \leq \|\psi \rho_\varepsilon \mathbf{1}_{\rho_\varepsilon \geq M}\|_{L^2_{t,x}} \| |y| \nabla \omega_\varepsilon(y) \|_{L^1} \leq C \|\psi \rho_\varepsilon \mathbf{1}_{\rho_\varepsilon \geq M}\|_{L^2_{t,x}}.$$

Choosing  $M = \frac{1}{\varepsilon}$ , we conclude. The condition (9.6.3) can be relaxed to be satisfied locally when  $\omega$  is compactly supported. We stress that we do not have any a priori estimate implying (9.6.3).



## Annexe A

# On the inviscid limit connecting Brinkman's and Darcy's models of tissue growth with nonlinear pressure

### Abstract

Several recent papers have addressed modelling of the tissue growth by the multi-phase models where the velocity is related to the pressure by one of the physical laws (Stoke's, Brinkman's or Darcy's). While each of these models has been extensively studied, not so much is known about the connection between them. In the recent paper (arXiv :2303.10620), assuming the linear form of the pressure, the Authors connected two multi-phase models by an inviscid limit : the viscoelastic one (of Brinkman's type) and the inviscid one (of Darcy's type). Here, we prove that the same is true for a nonlinear, power-law pressure. The new ingredient is that we use relation between the pressure  $p$  and the Brinkman potential  $W$  to deduce compactness in space of  $p$  from the compactness in space of  $W$ .

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## A.1 Introduction

Last years brought deep understanding of mechanical models of tissue growth. These models are based on the continuity equation for the density  $\rho$

$$\partial_t \rho + \operatorname{div}(\rho v) = 0,$$

where the velocity  $v$  is linked to the pressure  $p$  which is assumed to be a power-law function of density  $\rho$  i.e.  $p(\rho) = \rho^\gamma$  for some  $\gamma \geq 1$ . The most widely studied one, Darcy's law, asserts that the velocity  $v = -\nabla p$ . Such approach has been thoroughly studied [[118](#), [289](#), [288](#), [198](#),

227], also in the context of two populations [64, 206, 235, 260, 262, 296], presence of a nutrient [290, 119], more general Patlak-Keller-Segel equation [211] or additional advection effects [120, 237]. Another approach is the Brinkman's law [126, 125, 291, 318, 238]. Here, velocity equals  $v = -\nabla W$  where  $W$  solves an elliptic equation

$$-\sigma \Delta W + W = p,$$

for some small  $\sigma > 0$ . In this approach, the velocity enjoys higher regularity due to elliptic regularity theory. Last but not least, one can consider the Stoke's law where the velocity is given by the corresponding Navier-Stokes(-Korteweg) equation [322, 254, 245] or with additional surface-tension effects [154, 130, 151, 152] by including Cahn-Hilliard-type terms.

Most of the studies discussed above have been carried out to in the context of so-called incompressible limit. This procedure links mechanistic models and free-boundary problems extensively used in the context of tumor growth [66]. Mathematically, the limit corresponds to sending  $\gamma \rightarrow \infty$  in the pressure relation  $p(\rho) = \rho^\gamma$ . In the limit,  $\rho \leq 1$  and the zone  $\Omega_t := \{x : \rho(t, x) = 1\}$  is interpreted as a tumor resulting in the free boundary problem which has been extensively studied, see for instance [228, 240, 275, 236]. In this context, it is worth mentioning another form of the pressure  $p(\rho) = \varepsilon \frac{\rho}{1-\rho}$  which enforces the density to stay below 1 so it is useful for modeling populations with congestion constraints, see [213, 131, 60, 61, 285]. For such pressure laws, one can also study incompressible limit by sending  $\varepsilon \rightarrow 0$ .

In the present work, we are interested in linking the two populations model of Brinkman's type with the one of Darcy's type. Hence, we consider the system of PDEs posed on  $[0, T] \times \mathbb{R}^d$

$$\partial_t u_\sigma - \operatorname{div}(u_\sigma \nabla W_\sigma) = u_\sigma F(p_\sigma), \quad \partial_t v_\sigma - \operatorname{div}(v_\sigma \nabla W_\sigma) = v_\sigma G(p_\sigma), \quad (\text{A.1.1})$$

where  $u_\sigma, v_\sigma$  are densities of two populations of interest,  $p = (u_\sigma + v_\sigma)^\gamma$  is the pressure,  $\gamma > 1$  and  $W_\sigma$  is the solution of the elliptic equation

$$-\sigma \Delta W_\sigma + W_\sigma = p_\sigma \quad (\text{A.1.2})$$

corresponding to the so-called Brinkman's law. Our target is to rigorously justify the limit  $\sigma \rightarrow 0$  where we expect the Darcy's law  $W = p = (u + v)^\gamma$  and the densities  $u, v$  satisfy

$$\partial_t u - \operatorname{div}(u \nabla p) = u F(p), \quad \partial_t v - \operatorname{div}(v \nabla p) = v G(p). \quad (\text{A.1.3})$$

In [117], the case of  $\gamma = 1$  was established. Here, we study the nonlinear case  $\gamma > 1$ . As discussed above, from the point of view of free boundary models, large  $\gamma$  is more physically relevant and this motivates our studies.

We first list the assumptions which are standard in the theory of (A.1.1).

*Assumption A.1.1.* We assume that :

- (A) The nonlinearities  $F, G$  belong to  $C^1(\mathbb{R})$  and they are strictly decreasing :  $F', G' \leq -\alpha < 0$  for some  $\alpha > 0$ . Moreover, there exists  $p_H > 0$  (the so-called homeostatic pressure) such that  $F(p_H) = G(p_H) = 0$ .
- (B) The initial condition  $(u^0, v^0)$  is nonnegative and satisfies the following : the upper bound  $(u^0 + v^0)^\gamma \leq p_H$ , the mass bound  $\int_{\mathbb{R}^d} (u^0 + v^0) dx \leq C$  and the tail estimate  $\int_{\mathbb{R}^d} (u^0 + v^0) |x|^2 dx \leq C$ .

The weak solutions to the systems (A.1.1)-(A.1.2) and (A.1.3) are defined as follows :

*Definition A.1.2* (Weak solutions to the Brinkman system). We say that  $(u_\sigma, v_\sigma)$  is a weak solution of (A.1.1)-(A.1.2) with initial condition  $(u^0, v^0)$  if  $u_\sigma, v_\sigma \in L^\infty(0, T; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$  and for all  $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^d)$  and  $\phi \in C_c^\infty([0, T) \times \mathbb{R}^d)$  :

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} u_\sigma \partial_t \varphi \, dx \, dt + \int_{\mathbb{R}^d} \varphi(0, x) u^0(x) \, dx &= \\ &= \int_0^T \int_{\mathbb{R}^d} u_\sigma \nabla W_\sigma \cdot \nabla \varphi \, dx \, dt - \int_0^T \int_{\mathbb{R}^d} u_\sigma F(p_\sigma) \varphi \, dx \, dt, \end{aligned}$$

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} v_\sigma \partial_t \phi \, dx \, dt + \int_{\mathbb{R}^d} \phi(0, x) v^0(x) \, dx &= \\ &= \int_0^T \int_{\mathbb{R}^d} v_\sigma \nabla W_\sigma \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\mathbb{R}^d} v_\sigma G(p_\sigma) \phi \, dx \, dt, \end{aligned}$$

with  $p_\sigma = (u_\sigma + v_\sigma)^\gamma$  and  $-\Delta W_\sigma + W_\sigma = p_\sigma$  a.e. in  $(0, T) \times \mathbb{R}^d$ .

We note that the terms  $u_\sigma \nabla W_\sigma, v_\sigma \nabla W_\sigma$  make sense. Indeed, one can write  $W_\sigma = K_\sigma * p_\sigma$  where  $K_\sigma$  is a fundamental solution of  $-\sigma \Delta K_\sigma + K_\sigma = \delta_0$ . It is well-known (see, for instance, [291, eq. (2.6)]) that  $K_\sigma \geq 0, \int_{\mathbb{R}^d} K_\sigma \, dx = 1$  and  $\nabla K_\sigma \in L^1(\mathbb{R}^d)$  where the last estimate blows up when  $\sigma \rightarrow 0$ . Therefore, for  $\sigma$  fixed,  $\nabla W_\sigma \in L^\infty(0, T; L^q(\mathbb{R}^d))$  for all  $q \in [1, \infty]$ .

*Definition A.1.3* (Weak solutions to the Darcy system). We say that  $(u, v)$  is a weak solution of (A.1.3) with initial condition  $(u^0, v^0)$  and  $p = (u + v)^\gamma$  if  $u, v \in L^\infty(0, T; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)), \nabla p \in L^2((0, T) \times \mathbb{R}^d)$  and for all  $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^d)$  and  $\phi \in C_c^\infty([0, T) \times \mathbb{R}^d)$  :

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} u \partial_t \varphi \, dx \, dt + \int_{\mathbb{R}^d} \varphi(0, x) u^0(x) \, dx &= \\ &= \int_0^T \int_{\mathbb{R}^d} u \nabla p \cdot \nabla \varphi \, dx \, dt - \int_0^T \int_{\mathbb{R}^d} u F(p) \varphi \, dx \, dt, \end{aligned}$$

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} v \partial_t \phi \, dx \, dt + \int_{\mathbb{R}^d} \phi(0, x) v^0(x) \, dx &= \\ &= \int_0^T \int_{\mathbb{R}^d} v \nabla p \cdot \nabla \phi \, dx \, dt - \int_0^T \int_{\mathbb{R}^d} v G(p) \phi \, dx \, dt. \end{aligned}$$

The existence of weak solutions to the Brinkman system is given by the following result.

**Theorem A.1.4.** *Under Assumption A.1.1, there exists a weak solution  $(u_\sigma, v_\sigma)$  to the system (A.1.1)-(A.1.2) in the sense of Definition A.1.2. Moreover, the solution is uniformly bounded*

$$0 \leq (u_\sigma + v_\sigma)^\gamma \leq p_H. \tag{A.1.4}$$

The existence result is fairly standard and it is based on suitable regularizations. Nevertheless, it contains few interesting technical difficulties, therefore we present the proof in the Appendix A.5.1. Let us point that the uniform bound (A.1.4) is the direct consequence of the maximum principle in [318, Lemma 2.1].

Our main result is the rigorous justification the limit  $\sigma \rightarrow 0$ .



**Theorem A.1.5** (Brinkman to Darcy). *Let  $(u_\sigma, v_\sigma)$  be a weak solution of system (A.1.1)-(A.1.2) as in Theorem A.1.4. Then, as  $\sigma \rightarrow 0$ , we can extract a subsequence (not relabeled) such that*

$$\begin{aligned} u_\sigma &\rightarrow u && \text{weakly}^* \text{ in } L^\infty((0, T) \times \mathbb{R}^d) \text{ and weakly in } L^q((0, T) \times \mathbb{R}^d) \text{ for } 1 \leq q < \infty, \\ v_\sigma &\rightarrow v && \text{weakly}^* \text{ in } L^\infty((0, T) \times \mathbb{R}^d) \text{ and weakly in } L^q((0, T) \times \mathbb{R}^d) \text{ for } 1 \leq q < \infty, \\ p_\sigma &\rightarrow p && \text{strongly in } L^q((0, T) \times \mathbb{R}^d) \text{ for all } 1 \leq q < +\infty, \\ W_\sigma &\rightarrow p && \text{strongly in } L^2(0, T; H^1(\mathbb{R}^d)) \cap L^q((0, T) \times \mathbb{R}^d) \text{ for all } 1 < q < +\infty, \end{aligned}$$

with  $p = (u + v)^\gamma$ . Moreover  $(u, v)$  is a weak solution of (A.1.3) as in Definition A.1.3.

Let us briefly outline the strategy. The main difficulty is to pass to the limit in the terms  $u_\sigma \nabla W_\sigma$  and  $v_\sigma \nabla W_\sigma$ . It seems that there is no hope for the strong compactness of  $u_\sigma$  and  $v_\sigma$  because this requires at least uniform bounds on  $\{D^2 W_\sigma\}$  (cf. [42]) which does not seem to be available (see (A.2.2) for the energy identity). Therefore, we plan to prove strong compactness of  $\{\nabla W_\sigma\}$  in  $L^2((0, T) \times \mathbb{R}^d)$  by proving weak compactness and convergences of norms :

$$\nabla W_\sigma \rightharpoonup \nabla p \text{ in } L^2((0, T) \times \mathbb{R}^d), \quad \lim_{\sigma \rightarrow 0} \|\nabla W_\sigma\|_{L^2((0, T) \times \mathbb{R}^d)}^2 = \|\nabla p\|_{L^2((0, T) \times \mathbb{R}^d)}^2,$$

which was recently applied in several problems of similar nature [226, 116, 260, 296]. This can be achieved if one proves strong compactness of  $\{p_\sigma\}$ . Indeed, weak compactness of  $\{\nabla W_\sigma\}$  is then a consequence of the energy estimate (A.2.2) and the elliptic equation (A.1.2). The convergence of norms follows from the energy : we compare energy (A.2.2) for  $\sigma \rightarrow 0$  with the energy for

$$\partial_t(u + v) - \operatorname{div}((u + v) \nabla p) = u F(p) + v G(p),$$

which can be written because  $\{p_\sigma\}$  is strongly compact.

It remains to explain how we obtain strong compactness of the pressures  $\{p_\sigma\}$ . From a priori estimates (Proposition A.2.1) we know that  $\{p_\sigma\}$  is compact in time while  $\{W_\sigma\}$  is compact in space. Moreover, the term  $\sigma \Delta W_\sigma$  converges strongly to 0. Therefore, we can use the elliptic equation (A.1.2) to translate information about compactness of  $\{W_\sigma\}$  into compactness of  $\{p_\sigma\}$ . Details are given in Lemma A.3.2 and Lemma A.3.4.

We also remark that our method covers the linear case  $\gamma = 1$  studied in [117]. The only difference is that the energy identity used to obtain all the estimates and deduce strong compactness of  $\{\nabla W_\sigma\}$  is deduced by multiplying equation for the sum  $u_\sigma + v_\sigma$  with  $\log(u_\sigma + v_\sigma)$ . Some care is necessary as this function may not be admissible in the vacuum where  $u_\sigma + v_\sigma = 0$  and the details are discussed in [117].

Finally, let us remark that the problem of passing to the limit from (A.1.1) to (A.1.3) can be seen in a much broader context of passing to the limit from the nonlocal equation to the local one. More precisely, (A.1.2) can be written as  $W_\sigma = K_\sigma * p_\sigma$  where  $K_\sigma$  is a kernel approaching Dirac mass  $\delta_0$  so that  $W = p$  in the limit  $\sigma \rightarrow 0$ . Such problems are intensively studied for several PDEs, including porous media equation [93, 65, 257, 212], Cahn-Hilliard equation (both nondegenerate [123, 121] and degenerate [155, 81]) and hyperbolic conservation laws [109].

## A.2 A priori estimates

Here, we prove the following :

**Proposition A.2.1.** *Let  $\sigma > 0$ . Let  $(u_\sigma, v_\sigma)$  be a weak solution of (A.1.1)–(A.1.2). Then, the following sequences are uniformly bounded with respect to  $\sigma \in (0, 1)$  :*

- (A)  $\{p_\sigma\}$  and  $\{W_\sigma\}$  in  $L^\infty(0, T; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ ,
- (B)  $\{\nabla W_\sigma\}$  in  $L^2((0, T) \times \mathbb{R}^d)$ ,
- (C)  $\{\sqrt{\sigma} \Delta W_\sigma\}$  in  $L^2((0, T) \times \mathbb{R}^d)$ ,
- (D)  $\{\partial_t p_\sigma\}$  in  $L^1(0, T; H_{loc}^{-s}(\mathbb{R}^d))$ , for  $s$  large enough,
- (E)  $\{p_\sigma |x|^2\}$  in  $L^\infty(0, T; L^1(\mathbb{R}^d))$ ,

*Démonstration.* First, the  $L^\infty$  estimate for  $p_\sigma$  is a direct consequence of (A.1.4) and the same is true for  $W_\sigma = K_\sigma * p_\sigma$  because  $\int_{\mathbb{R}^d} K_\sigma dx = 1$ . The estimates (A), (B) and (C) are a consequence of energy considerations. The equation for the sum  $u_\sigma + v_\sigma$  reads :

$$\partial_t(u_\sigma + v_\sigma) - \operatorname{div}((u_\sigma + v_\sigma)\nabla W_\sigma) = u_\sigma F(p_\sigma) + v_\sigma G(p_\sigma). \quad (\text{A.2.1})$$

We multiply with  $\gamma(u_\sigma + v_\sigma)^{\gamma-1}$  and integrate in space so that

$$\begin{aligned} \partial_t \int_{\mathbb{R}^d} p_\sigma dx + \gamma(\gamma-1) \int_{\mathbb{R}^d} (u_\sigma + v_\sigma)^{\gamma-1} \nabla(u_\sigma + v_\sigma) \cdot \nabla W_\sigma dx = \\ = \gamma \int_{\mathbb{R}^d} (u_\sigma F(p_\sigma) + v_\sigma G(p_\sigma))(u_\sigma + v_\sigma)^{\gamma-1} dx, \end{aligned}$$

which, with (A.1.2), can be rewritten as

$$\begin{aligned} \partial_t \int_{\mathbb{R}^d} p_\sigma dx + (\gamma-1) \int_{\mathbb{R}^d} |\nabla W_\sigma|^2 + \sigma |\Delta W_\sigma|^2 dx \\ = \gamma \int_{\mathbb{R}^d} (u_\sigma F(p_\sigma) + v_\sigma G(p_\sigma))(u_\sigma + v_\sigma)^{\gamma-1} dx \leq C \int_{\mathbb{R}^d} p_\sigma dx, \end{aligned} \quad (\text{A.2.2})$$

because  $p_\sigma$  is uniformly bounded and  $F, G$  are continuous. Finally, we have  $\|W_\sigma\|_{L^1(\mathbb{R}^d)} = \|K_\sigma * p_\sigma\|_{L^1(\mathbb{R}^d)} = \|p_\sigma\|_{L^1(\mathbb{R}^d)}$  which concludes the proof of (A), (B) and (C).

Next, we establish the bound (D) on  $\partial_t p_\sigma$ . We first write the equation on  $p$  which can be obtained after multiplying (A.2.1) by  $\gamma(u_\sigma + v_\sigma)^{\gamma-1}$  :

$$\partial_t p_\sigma = \nabla p_\sigma \nabla W_\sigma + \gamma p_\sigma \Delta W_\sigma + \gamma (u_\sigma F(p_\sigma) + v_\sigma G(p_\sigma)) (u_\sigma + v_\sigma)^{\gamma-1}.$$

First note that the last term on the right-hand side is bounded in  $L^\infty((0, T) \times \mathbb{R}^d)$  by assumptions on  $F$  and  $G$ . In order to obtain a bound on  $\partial_t p_\sigma$  in a negative Sobolev spaces, we see that, up to integration by parts, it remains to study the term  $p_\sigma \Delta W_\sigma$ . Let  $\varphi$  be a smooth, compactly supported test function. Then, by definition of  $W_\sigma$

$$\int_{\mathbb{R}^d} p_\sigma \Delta W_\sigma \varphi dx = -\sigma \int_{\mathbb{R}^d} |\Delta W_\sigma|^2 \varphi dx - \int_{\mathbb{R}^d} |\nabla W_\sigma|^2 \varphi dx - \int_{\mathbb{R}^d} W_\sigma \nabla W_\sigma \cdot \nabla \varphi dx.$$

The proof of (D) is concluded using (A), (B) and (C). Now, we prove (E). Since  $p_\sigma$  is bounded in  $L^\infty$  we only need to prove that  $u_\sigma$  and  $v_\sigma$  have uniformly bounded second moments. We compute it for  $u_\sigma$  and the proof is similar for  $v_\sigma$

$$\partial_t \int_{\mathbb{R}^d} |x|^2 u_\sigma dx + 2 \int_{\mathbb{R}^d} u_\sigma \nabla W_\sigma \cdot x dx = \int_{\mathbb{R}^d} |x|^2 u_\sigma F(p_\sigma) dx.$$

Integrating in time, using Cauchy-Schwartz inequality, estimate (B), assumptions on  $F$  and Gronwall's inequality, we obtain the result.  $\square$

### A.3 Strong compactness of the pressure

**Proposition A.3.1.** *There exist functions  $u$ ,  $v$  and  $p$  such that  $p = (u + v)^\gamma$  a.e. and such that up to a subsequence (not relabelled) for all  $1 \leq q < \infty$  :*

$$u_\sigma \rightarrow u \quad \text{weakly}^* \text{ in } L^\infty((0, T) \times \mathbb{R}^d) \text{ and weakly in } L^q((0, T) \times \mathbb{R}^d), \quad (\text{A.3.1})$$

$$v_\sigma \rightarrow v \quad \text{weakly}^* \text{ in } L^\infty((0, T) \times \mathbb{R}^d) \text{ and weakly in } L^q((0, T) \times \mathbb{R}^d), \quad (\text{A.3.2})$$

$$\sigma \Delta W_\sigma \rightarrow 0 \quad \text{strongly in } L^2((0, T) \times \mathbb{R}^d), \quad (\text{A.3.3})$$

$$p_\sigma \rightarrow p \quad \text{strongly in } L^q((0, T) \times \mathbb{R}^d), \quad (\text{A.3.4})$$

$$W_\sigma \rightarrow p \quad \text{weakly in } L^2(0, T; H^1(\mathbb{R}^d)), \text{ strongly in } L^q((0, T) \times \mathbb{R}^d) \text{ for } q > 1, \quad (\text{A.3.5})$$

$$u_\sigma + v_\sigma \rightarrow u + v \quad \text{strongly in } L^q((0, T) \times \mathbb{R}^d), \quad (\text{A.3.6})$$

$$(u_\sigma + v_\sigma)(T) \rightarrow (u + v)(T) \quad \text{weakly in } L^\gamma(\mathbb{R}^d). \quad (\text{A.3.7})$$

The crucial step in the proof of Proposition A.3.1 is the strong compactness of the pressure  $p_\sigma$  which will be achieved by the following lemma which in the spirit is a version of Lions-Aubin-Simon's argument [314].

**Lemma A.3.2.** *Suppose that for each compact set  $K \subset \mathbb{R}^d$*

$$\lim_{y \rightarrow 0} \int_0^T \int_K |p_\sigma(t, x + y) - p_\sigma(t, x)| dx dt = 0 \quad \text{uniformly in } \sigma \in (0, 1). \quad (\text{A.3.8})$$

*Moreover, assume that  $\{\partial_t p_\sigma\}$  is bounded in  $L^1(0, T; H_{loc}^{-s}(\mathbb{R}^d))$  for some  $s > 0$  and  $\{p_\sigma |x|^2\}$  is uniformly bounded in  $L^1((0, T) \times \mathbb{R}^d)$ . Then, the sequence  $\{p_\sigma\}$  is strongly compact in  $L^1((0, T) \times \mathbb{R}^d)$ .*

*Remark A.3.3.* Several variants of this result are possible. For instance, one can have more general assumption on the time derivative or one can also formulate it in for the space  $L^p((0, T) \times \mathbb{R}^d)$  space with  $p > 1$ . Another trivial generalization is that the tail estimate could be replaced with more general tightness assumption.

*Proof of Lemma A.3.2.* In view of the Riesz-Kolmogorov-Frechet theorem, to establish local compactness it is sufficient to prove

$$\lim_{h \rightarrow 0} \int_0^{T-h} \int_K |p_\sigma(t + h, x) - p_\sigma(t, x)| dx dt = 0 \quad \text{uniformly in } \sigma \in (0, 1)$$

for each compact set  $K \subset \mathbb{R}^d$ . Using a family of smooth, compactly supported mollifiers  $\{\varphi_\delta\}$

with  $\delta$  depending on  $h$ , we have

$$\begin{aligned} \int_0^{T-h} \int_K |p_\sigma(t+h, x) - p_\sigma(t, x)| \, dx \, dt &\leq \\ &\leq \int_0^{T-h} \int_K |p_\sigma(t+h, x) - p_\sigma(t+h, x) * \varphi_\delta| \, dx \, dt \\ &\quad + \int_0^{T-h} \int_K |p_\sigma(t, x) - p_\sigma(t, x) * \varphi_\delta| \, dx \, dt \\ &\quad + \int_0^{T-h} \int_K |p_\sigma * \varphi_\delta(t+h, x) - p_\sigma * \varphi_\delta(t, x)| \, dx \, dt. \end{aligned}$$

The first two terms converge to 0 when  $\delta \rightarrow 0$ , independently of  $h$ , as a consequence of (A.3.8). Hence, we only need to study the third term. We write

$$p_\sigma * \varphi_\delta(t+h, x) - p_\sigma * \varphi_\delta(t, x) = h \int_0^1 \partial_t p_\sigma * \varphi_\delta(t+sh, x) \, ds.$$

Therefore, the term of interest can be estimated by

$$\begin{aligned} Ch \int_0^{T-h} \int_K \left| \int_0^1 \partial_t p_\sigma * \varphi_\delta(t+sh, x) \, ds \right| \, dx \, dt &\leq \\ &\leq Ch \int_0^1 \int_0^{T-h} \int_K |\partial_t p_\sigma * \varphi_\delta(t+sh, x)| \, dx \, dt \, ds \leq Ch \|\partial_t p_\sigma * \varphi_\delta\|_{L^1((0,T) \times K)}, \end{aligned}$$

where we applied Fubini's theorem. It remains to estimate the convolution. We have

$$\partial_t p_\sigma * \varphi_\delta(t, x) = \int_{\mathbb{R}^d} \partial_t p_\sigma(t, y) \varphi_\delta(x-y) \, dy \leq \|\partial_t p_\sigma(t, \cdot)\|_{H_{loc}^{-s}} \|\varphi_\delta\|_{H^s},$$

Applying the  $L^1((0, T) \times K)$  norm we obtain

$$\|\partial_t p_\sigma * \varphi_\delta\|_{L^1((0,T) \times K)} \leq |K| \|\partial_t p_\sigma\|_{L_t^1 H_{loc,x}^{-s}} \|\varphi_\delta\|_{H^s} \leq \frac{C(K)}{\delta^{s+d/2}}.$$

Choosing  $h = \delta^{1+s+d/2}$  we obtain compactness of  $\{p_\sigma\}$  on  $(0, T) \times K$  for each compact set  $K$ .

To obtain global compactness, we perform a usual argument which uses the decay estimate. Let  $B_n$  be the sequence of balls  $B(0, n)$ . By the diagonal method, we construct a subsequence such that  $p_\sigma \rightarrow p$  in  $L^1((0, T) \times B_n)$  for each  $n \in \mathbb{N}$ . Then,

$$\begin{aligned} \|p_\sigma - p\|_{L^1((0,T) \times \mathbb{R}^d)} &\leq \|p_\sigma - p\|_{L^1((0,T) \times B_n)} + \|p_\sigma - p\|_{L^1((0,T) \times (\mathbb{R}^d \setminus B_n))} \\ &\leq \|p_\sigma - p\|_{L^1((0,T) \times B_n)} + \frac{C}{n^2}, \end{aligned}$$

where  $C = \sup_{\sigma \in (0,1)} \|p_\sigma |x|^2\|_{L^1((0,T) \times \mathbb{R}^d)}$  and we used, that by Fatou lemma,

$$\|p|x|^2\|_{L^1((0,T) \times \mathbb{R}^d)} \leq \liminf_{\sigma \rightarrow 0} \|p_\sigma |x|^2\|_{L^1((0,T) \times \mathbb{R}^d)}.$$

Hence,  $\limsup_{\sigma \rightarrow 0} \|p_\sigma - p\|_{L^1((0,T) \times \mathbb{R}^d)} \leq \frac{C}{n^2}$  for all  $n$  which concludes the proof.  $\square$

By interpolation, we deduce :

**Lemma A.3.4.** *The sequence of pressures  $\{p_\sigma\}$  is strongly compact in  $L^q((0, T) \times \mathbb{R}^d)$  for all  $1 \leq q < +\infty$ .*

It turns out that the sequence of pressures  $\{p_\sigma\}$  satisfies (A.3.8).

*Proof of (A.3.8).* We recall the definition of the Brinkman law

$$p_\sigma = -\sigma \Delta W_\sigma + W_\sigma.$$

First note that  $-\sigma \Delta W_\sigma$  converges strongly to 0 in  $L^2((0, T) \times \mathbb{R}^d)$  as a consequence of (C). In particular it is compact and by the converse of the Riesz-Kolmogorov-Frechet theorem we deduce

$$\lim_{y \rightarrow 0} \int_0^T \int_K |\sigma \Delta W_\sigma(t, x + y) - \sigma \Delta W_\sigma(t, x)| dx dt = 0 \text{ uniformly in } \sigma \in (0, 1).$$

To conclude the proof of (A.3.8) it remains to prove that

$$\lim_{y \rightarrow 0} \int_0^T \int_K |W_\sigma(t, x + y) - W_\sigma(t, x)| dx dt = 0 \text{ uniformly in } \sigma \in (0, 1).$$

This is a simple consequence of the formula

$$W_\sigma(t, x + y) - W_\sigma(t, x) = \int_0^1 \nabla W_\sigma(t, x + sy) \cdot y ds$$

and the uniform bound (B) (note that we work on the compact set  $K$  so that  $L^2(K)$  embeds into  $L^1(K)$ ).

□

*Proof of Proposition A.3.1.* Convergences (A.3.1), (A.3.2), (A.3.3) follow from Proposition A.2.1 and nonnegativity of  $u_\sigma, v_\sigma$ . The strong convergence of the pressure (A.3.4) is a consequence of Lemma A.3.4. Combining this convergence and (A.3.3) we deduce the strong convergence in  $L^2$  of  $W_\sigma$  and then in every  $L^q$  (except  $q = 1, \infty$ ) by interpolation using (A). Using also the estimate (B), we deduce the weak convergence of  $\nabla W_\sigma$  and conclude the proof of (A.3.5). The convergence (A.3.6) is a consequence of (A.3.4) : indeed we can extract a subsequence of pressures that converge a.e. so that  $u_\sigma + v_\sigma$  converges a.e. Due to the uniform  $L^\infty$  bound (A) and tail estimate (E), Vitali convergence theorem implies (A.3.6).

Finally, we prove (A.3.7). We adapt the argument from [54, Lemma II.5.9]. First, we prove that  $(u_\sigma + v_\sigma)(T)$  makes sense as an element of  $L^\gamma(\mathbb{R}^d)$ . Let  $\eta_\delta = \frac{1}{\delta} \mathbf{1}_{[-\delta, 0]}$  and  $f_\delta(t, x) = (u_\sigma + v_\sigma) * \eta_\delta$  where the convolution is a convolution in time. As  $u_\sigma + v_\sigma \in L^\infty(0, T; L^\gamma(\mathbb{R}^d))$ ,  $\|f_\delta(T, \cdot)\|_{L^\gamma(\mathbb{R}^d)} \leq C$  independently of  $\delta > 0$ . Hence, up to a subsequence,  $f_\delta(T) \rightharpoonup f$  in  $L^\gamma(\mathbb{R}^d)$  and it remains to prove  $f = (u_\sigma + v_\sigma)(T)$ . Let  $\psi \in C_c^\infty(\mathbb{R}^d)$ . Since

$$\int_{\mathbb{R}^d} f_\delta(T, x) \psi(x) dx = \left[ \int_{\mathbb{R}^d} (u_\sigma + v_\sigma)(\cdot, x) \psi(x) dx \right] * \eta_\delta(T) \quad (\text{A.3.9})$$

and the function  $t \mapsto \int_{\mathbb{R}^d} (u_\sigma + v_\sigma) \psi(x) dx$  is continuous (it can be easily seen that the sequence  $\{\partial_t(u_\sigma + v_\sigma)\}$  is uniformly bounded in  $L^2(0, T; H_{loc}^{-1}(\mathbb{R}^d))$  and such regularity implies also continuity in  $C(0, T; H_{loc}^{-1}(\mathbb{R}^d))$ , see [302, Lemma 7.1]), the (RHS) of (A.3.9) converges to

$\int_{\mathbb{R}^d} (u_\sigma(T) + v_\sigma(T)) \psi(x) dx$  so that  $f = (u_\sigma + v_\sigma)(T)$  a.e. on  $\mathbb{R}^d$ . Exactly the same argument shows that  $(u + v)(T)$  makes sense because, thanks to convergences (A.3.1), (A.3.2), (A.3.5), (A.3.6), we can pass to the limit  $\sigma \rightarrow 0$  and deduce the same weak formulation

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} (u + v) \partial_t \varphi dx dt + \int_{\mathbb{R}^d} (u^0 + v^0) \varphi(0) dx - \int_0^T \int_{\mathbb{R}^d} (u + v) \nabla p \cdot \nabla \varphi dx dt = \\ = \int_0^T \int_{\mathbb{R}^d} u F(p) dx dt + \int_0^T \int_{\mathbb{R}^d} v G(p) dx dt, \end{aligned} \tag{A.3.10}$$

which implies continuity of  $t \mapsto \int_{\mathbb{R}^d} (u + v) \psi(x) dx$ .

The argument above shows that the sequence  $\{(u_\sigma + v_\sigma)(T)\}$  is bounded in  $L^\gamma(\mathbb{R}^d)$  so it has a subsequence converging to some  $s \in L^\gamma(\mathbb{R}^d)$  when  $\sigma \rightarrow 0$ . We claim that  $s = (u + v)(T)$ . To this end, we consider the weak formulation from Definition A.1.2 with test function of the form  $\psi(x) \eta_\delta(t)$  where  $\eta_\delta = 1$  on  $[0, T - 2\delta]$ ,  $\eta_\delta = 0$  on  $[T - \delta, T]$  and it is linear interpolation on  $[T - 2\delta, T - \delta]$  (such function is admissible by density as it has Sobolev derivative). By weak continuity, as  $\delta \rightarrow 0$ ,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} (u_\sigma + v_\sigma)(t, x) \psi(x) \partial_t \eta_\delta(t) dx dt = \frac{1}{\delta} \int_{T-2\delta}^{T-\delta} \int_{\mathbb{R}^d} (u_\sigma + v_\sigma)(t, x) \psi(x) dx dt \\ \xrightarrow{\delta \rightarrow 0} \int_{\mathbb{R}^d} (u_\sigma + v_\sigma)(T, x) \psi(x) dx \xrightarrow{\sigma \rightarrow 0} \int_{\mathbb{R}^d} s(x) \psi(x) dx. \end{aligned}$$

We can apply the same limiting procedure  $\delta \rightarrow 0$  in the weak formulation (A.3.10). By comparing the results, we deduce that  $s = (u + v)(T)$  a.e. on  $\mathbb{R}^d$  and this concludes the proof.  $\square$

## A.4 Strong convergence of $\nabla W_\sigma$ and conclusion

*Proof of Theorem A.1.5.* Due to the weak convergence of  $u_\sigma$  and  $v_\sigma$ , cf. (A.3.1)–(A.3.2), to pass to the limit in (A.1.1), it is sufficient to prove strong convergence of  $\nabla W_\sigma$ . As  $\nabla W_\sigma \rightharpoonup \nabla p$ , it is sufficient to prove convergence of  $L^2$  norms, i.e.

$$\|\nabla W_\sigma\|_{L^2((0,T) \times \mathbb{R}^d)}^2 \rightarrow \|\nabla p\|_{L^2((0,T) \times \mathbb{R}^d)}^2.$$

By the properties of the weak convergence

$$\|\nabla p\|_{L^2((0,T) \times \mathbb{R}^d)}^2 \leq \liminf_{\sigma \rightarrow 0} \|\nabla W_\sigma\|_{L^2((0,T) \times \mathbb{R}^d)}^2. \tag{A.4.1}$$

so we only need to estimate  $\limsup_{\sigma \rightarrow 0}$ . The idea is to pass to the limit in Equation (A.2.1). First, due to the strong compactness of  $u + v$  and  $p$  in Proposition A.3.1, we can pass to the limit in Equation (A.2.1) and obtain (in the weak sense)

$$\partial_t(u + v) - \operatorname{div}((u + v) \nabla p) = u F(p) + v G(p). \tag{A.4.2}$$

We can test this equation with  $\gamma(u + v)^{\gamma-1}$  (see Remark A.4.1 below for the precise argument)

and we obtain after integrating in time

$$\begin{aligned} \int_{\mathbb{R}^d} p(T, x) \, dx + (\gamma - 1) \int_0^T \int_{\mathbb{R}^d} |\nabla p|^2 \, dx \, dt &= \\ &= \int_{\mathbb{R}^d} p^0(x) \, dx + \gamma \int_0^T \int_{\mathbb{R}^d} (u F(p) + v G(p))(u + v)^{\gamma-1} \, dx \, dt. \end{aligned} \quad (\text{A.4.3})$$

Note that (A.3.7) implies

$$\int_{\mathbb{R}^d} p(T, x) \, dx \leq \liminf_{\sigma \rightarrow 0} \int_{\mathbb{R}^d} p_\sigma(T, x) \, dx \quad (\text{A.4.4})$$

by the weak lower semicontinuity of the norm. Integrating (A.2.2) in time from  $[0, T]$  and applying  $\limsup_{\sigma \rightarrow 0}$  we see that

$$\begin{aligned} (\gamma - 1) \limsup_{\sigma \rightarrow 0} \int_0^T \int_{\mathbb{R}^d} |\nabla W_\sigma|^2 \, dx \, dt &\leq \int_{\mathbb{R}^d} p^0(x) \, dx + \\ &+ \gamma \int_0^T \int_{\mathbb{R}^d} (u F(p) + v G(p))(u + v)^{\gamma-1} \, dx \, dt - \liminf_{\sigma \rightarrow 0} \int_{\mathbb{R}^d} p_\sigma(T, x) \, dx. \end{aligned} \quad (\text{A.4.5})$$

Combining (A.4.3), (A.4.4) and (A.4.5) we see that

$$\limsup_{\sigma \rightarrow 0} \|\nabla W_\sigma\|_{L^2((0, T) \times \mathbb{R}^d)}^2 \leq \|\nabla p\|_{L^2((0, T) \times \mathbb{R}^d)}^2. \quad (\text{A.4.6})$$

which together with (A.4.1) concludes the proof.  $\square$

*Remark A.4.1.* To make integration by parts rigorous, we test equation with

$$\gamma(u + v)^{\gamma-1} \psi_R(x)$$

where  $\psi_R$  is a smooth function such that  $\psi_R(x) = 1$  for  $|x| \leq R$ ,  $\psi_R(x) = 0$  for  $|x| \geq R + 1$  and  $|\psi'_R| \leq 1$ . Then, the integration by parts is justified and we obtain

$$(\gamma - 1) \int_0^T \int_{\mathbb{R}^d} |\nabla p|^2 \psi_R(x) \, dx \, dt + \gamma \int_0^T \int_{R \leq |x| \leq R+1} p \nabla p \psi'_R \, dx \, dt.$$

The first term converges to  $(\gamma - 1) \int_{\mathbb{R}^d} |\nabla p|^2 \psi_R(x) \, dx$  by the dominated convergence theorem. For the second term, we note that  $p \nabla p \in L^1((0, T) \times \mathbb{R}^d)$  (in fact, we have even better) so this term converges to 0 again by the dominated convergence theorem.

*Remark A.4.2.* Similarly, to make testing (A.4.2) with  $\gamma(u + v)^{\gamma-1}$  rigorous, one mollifies (A.4.2) both in time and space with  $\eta_\delta(t) \psi_\varepsilon(x)$ . Then, one tests (A.4.2) with  $\gamma((u + v) * \eta_\delta * \psi_\varepsilon)^{\gamma-1}$  so that the usual chain rule in Sobolev spaces can be applied resulting in the term of the form  $\int_{\mathbb{R}^d} ((u + v) * \eta_\delta * \psi_\varepsilon)^\gamma(T, x) \, dx$ . Then, one sends  $\delta \rightarrow 0$  using weak continuity of the sum  $(u + v)$  as in the proof of Proposition A.3.1 and then  $\varepsilon \rightarrow 0$  using the properties of the mollifiers.

## A.5 Appendix

### A.5.1 Proof of Theorem A.1.4 (existence result)

Here, we prove existence of solutions to the system (A.1.1) by introducing an artificial diffusion. We also recall the uniform (in terms of  $\gamma$  and  $\sigma$ )  $L^\infty$  bounds from [318].

We rewrite (A.1.1) as follows

$$\partial_t u - \nabla u \cdot \nabla W - \frac{1}{\sigma} u (W - p) = u F(p), \quad \partial_t v - \nabla v \cdot \nabla W - \frac{1}{\sigma} v (W - p) = v G(p),$$

where we skipped the lower index  $\sigma$  as  $\sigma$  is fixed. It is useful to write  $W = K_\sigma * p$  where  $K_\sigma$  is a fundamental solution  $-\sigma \Delta K_\sigma + K_\sigma = \delta_0$  and  $p = (u + v)^\gamma$ . We regularize the problem in three ways. First, we introduce diffusion. Second, we mollify  $K_\sigma$  with a usual mollifier  $\omega_\delta$  (i.e.  $\omega_\delta(x) = \frac{1}{\delta^d} \omega(x/\delta)$  where  $\omega$  is smooth, supported in the unit ball and of mass 1). Third, we truncate all nonlinearities by the truncation operator  $Q(p) = p \mathbf{1}_{p \leq 2p_H} + 2p_H \mathbf{1}_{p > 2p_H}$ . The resulting system reads

$$\begin{aligned} \partial_t u - \varepsilon \Delta u - \nabla u \cdot \nabla W_\delta &= \frac{1}{\sigma} u (W_\delta - Q(p)) + u F(Q(p)), \\ \partial_t v - \varepsilon \Delta v - \nabla v \cdot \nabla W_\delta &= \frac{1}{\sigma} v (W_\delta - Q(p)) + v G(Q(p)), \\ W_\delta &= K_\sigma * \omega_\delta * Q(p), \end{aligned} \tag{A.5.1}$$

where  $p = (u + v)^\gamma$ . By properties of convolutions,  $\nabla W_\delta = K_\sigma * \nabla \omega_\delta * Q(p)$  so that (A.5.1) can be considered as a semilinear parabolic system with Lipschitz nonlinearities which are well-understood [246, 253]. Hence, it has a nonnegative solution  $(u, v)$ .

Now, following [318, Lemma 2.1], we claim that  $p \leq p_H$ . To this end, we sum up equations for  $u, v$  and multiply by  $\gamma (u + v)^{\gamma-1}$  to obtain

$$\begin{aligned} \partial_t p - \varepsilon \gamma \Delta (u + v) (u + v)^{\gamma-1} - \nabla p \cdot \nabla W_\delta &= \\ &= \frac{\gamma}{\sigma} p (W_\delta - Q(p)) + \gamma (u + v)^{\gamma-1} (u F(Q(p)) + v G(Q(p))). \end{aligned} \tag{A.5.2}$$

Since at  $t = 0$ ,  $p \leq p_H$ , if the estimate is not satisfied, by continuity there is time  $t > 0$  where  $p(t, \cdot)$  reaches its maximum with value in  $(p_H, 2p_H)$ . Let  $x^*$  be a point where this happens. At this point,  $\nabla p = 0$ ,  $\Delta (u + v) \leq 0$  (because the function  $u + v = p^{1/\gamma}$  reaches its maximum). Furthermore, the source term is strictly negative due to (A) in Assumption A.1.1. Finally, we note that by Young's convolutional inequality

$$W_\delta(t, x^*) - p(t, x^*) \leq \|p(t, \cdot)\|_\infty \|K_\sigma\|_1 \|\omega_\delta\|_1 - \|p(t, \cdot)\|_\infty = 0$$

so that the term  $W_\delta - p$  at  $x^*$  is nonpositive. We conclude that

$$\partial_t p(t, x^*) < 0$$

so that  $p$  cannot become greater than  $p_H$ . We conclude that  $Q(p) = p$  in (A.5.1).

Now, we send  $\delta \rightarrow 0$ . We write  $u_\delta$  and  $v_\delta$  for solutions to (A.5.1),  $p_\delta = (u_\delta + v_\delta)^\gamma$  for the pressure



and  $W_\delta = K_\sigma * \omega_\delta * p_\delta$ . First, thanks to the presence of diffusion and  $\nabla K_\sigma \in L^1(\mathbb{R}^d)$ , the sequences  $\{u_\delta\}$  and  $\{v_\delta\}$  are locally compact in  $L^q((0, T) \times \mathbb{R}^d)$  for all  $q < \infty$  by usual Lions-Aubin lemma and interpolation in Lebesgue spaces. As a consequence, it is easy to pass to the limit in the source terms. The nontrivial part is to pass to the limit in the advection term. To identify the limit we write

$$\int_0^T \int_{\mathbb{R}^d} \nabla u_\delta \cdot \nabla W_\delta \varphi = - \int_0^T \int_{\mathbb{R}^d} u_\delta \Delta W_\delta \varphi + u_\delta \nabla W_\delta \cdot \nabla \varphi.$$

Hence, it is sufficient to prove that  $\Delta W_\delta$  and  $\nabla W_\delta$  converge at least weakly to the appropriate limits. However, these sequences are bounded in  $L^\infty((0, T) \times \mathbb{R}^d)$  because  $\Delta W_\delta = \frac{1}{\sigma}(p_\delta - p_\delta * K_\sigma) * \omega_\delta$  and  $\nabla W_\delta = \nabla K_\sigma * p_\delta * \omega_\delta$ . Therefore, up to a subsequence, they have weak\* limits which equals  $\Delta W$  and  $\nabla W$  with  $W = K_\sigma * p$  due to the strong compactness of  $u_\delta$  and  $v_\delta$ .

In the limit  $\delta \rightarrow 0$ , we obtain the system

$$\partial_t u - \varepsilon \Delta u - \operatorname{div}(u \nabla W) = u F(p), \quad \partial_t v - \varepsilon \Delta v - \operatorname{div}(v \nabla W) = v G(p), \quad (\text{A.5.3})$$

where  $W = K_\sigma * p$  and it remains to remove the diffusion, i.e. send  $\varepsilon \rightarrow 0$ . Again, we write  $u_\varepsilon$  and  $v_\varepsilon$  for solutions to (A.5.3),  $p_\varepsilon = (u_\varepsilon + v_\varepsilon)^\gamma$  for the pressure and  $W_\varepsilon = K_\sigma * p_\varepsilon$ . Clearly,  $u_\varepsilon \xrightarrow{*} u$  and  $v_\varepsilon \xrightarrow{*} v$  in  $L^\infty((0, T) \times \mathbb{R}^d)$ . Moreover, standard computations show that  $\{\partial_t u_\varepsilon\}$ ,  $\{\partial_t v_\varepsilon\}$  are uniformly bounded in  $L^2(0, T; H^{-1}(\mathbb{R}^d))$ . We prove that both sequences  $\{u_\varepsilon\}$  and  $\{v_\varepsilon\}$  are strongly compact in space so that by Lemma A.3.2, we deduce strong compactness. The same will follow for the pressure  $p_\varepsilon$  so that  $\nabla W_\varepsilon = \nabla K_\sigma * p_\varepsilon$  converges in  $L^1((0, T) \times \mathbb{R}^d)$  to  $\nabla K_\sigma * p$  and so, by interpolation, in  $L^q((0, T) \times \mathbb{R}^d)$  for all  $q \in [1, 2)$ . This is sufficient to pass to the limit in (A.5.3).

The proof of compactness in space follows the method of Jabin and Belgacem [42] (the only difference is that we deal with an additional source term). Let us recall that [42] deals with compactness for the conservative equations

$$\partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon - \operatorname{div}(u_\varepsilon a_\varepsilon) = 0,$$

where  $a_\varepsilon$  is the vector field satisfying the following :

1.  $\sup_{\varepsilon \in (0, 1)} \|\operatorname{div} a_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R}^d)} < \infty$ ,
2.  $\sup_{\varepsilon \in (0, 1)} \|a_\varepsilon\|_{L^\infty(0, T; W^{1, p}(\mathbb{R}^d))} < \infty$  for some  $p > 1$ ,
3.  $\operatorname{div} a_\varepsilon = d_\varepsilon + r_\varepsilon$  where  $d_\varepsilon$  is compact in space while  $r_\varepsilon$  is such that  $|r_\varepsilon(x) - r_\varepsilon(y)| \leq C |u_\varepsilon(x) - u_\varepsilon(y)|$ .

In our case,  $a_\varepsilon = \nabla W_\varepsilon$  satisfies (1) and (2). Indeed,  $\operatorname{div} \nabla W_\varepsilon = \Delta W_\varepsilon = \frac{1}{\sigma}(p_\varepsilon - K_\sigma * p_\varepsilon)$  is uniformly bounded. Furthermore, it is easy to see, for instance from (A.5.2), that  $\{p_\varepsilon\}$  is uniformly bounded in  $L^\infty(0, T; L^1(\mathbb{R}^d))$  so that it is bounded in  $L^\infty(0, T; L^2(\mathbb{R}^d))$ . Hence,  $\{\Delta W_\varepsilon\}$  is bounded in  $L^\infty(0, T; L^2(\mathbb{R}^d))$  which easily implies that  $\{W_\varepsilon\}$  is bounded in  $L^\infty(0, T; W^{2, 2}(\mathbb{R}^d))$  so that (2) holds true with  $p = 2$ .

Concerning (3), it is satisfied in a weaker sense. We have  $d_\varepsilon = W_\varepsilon$  (it is compact in space by the estimate on  $\{\nabla W_\varepsilon\}$ ) and  $r_\varepsilon = p_\varepsilon$  which satisfies (by the uniform boundedness of  $\{u_\varepsilon\}$  and  $\{v_\varepsilon\}$ )

$$|r_\varepsilon(t, x) - r_\varepsilon(t, y)| \leq C |u_\varepsilon(t, x) - u_\varepsilon(t, y)| + C |v_\varepsilon(t, x) - v_\varepsilon(t, y)|, \quad (\text{A.5.4})$$

so that the estimate depends on both species. Below, we briefly explain a simple modification of

argument in [42] to cover the case of (A.5.4) as well as how to include the source terms.

The compactness in [42] is established by analysis of the quantity

$$\mathcal{Q}_{u_\varepsilon}(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{K}_h(x-y) |u_\varepsilon(t,x) - u_\varepsilon(t,y)| dx dy,$$

where  $\mathcal{K}_h$  is a smooth, nonnegative kernel, supported in the ball  $B_2(0)$  such that  $\mathcal{K}_h(x) = \frac{1}{(|x|^2+h^2)^{d/2}}$  on  $B_1(0)$ . Similarly, we define  $\mathcal{Q}_{v_\varepsilon}(t)$ . It can be proved, cf. [42, Lemma 3.1], that the sequence  $\{u_\varepsilon\}$  is locally compact in space in  $L^1((0,T) \times \mathbb{R}^d)$  if

$$\lim_{h \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \frac{1}{|\log h|} \int_0^T \mathcal{Q}_{u_\varepsilon}(t) dt = 0. \quad (\text{A.5.5})$$

Then, one computes  $\frac{d}{dt} \mathcal{Q}_{u_\varepsilon}$  using the PDE on  $u_\varepsilon$ . Applying [42, proof of Theorem 1.2] with new assumption (A.5.4) and additional source term we deduce

$$\begin{aligned} \mathcal{Q}_{u_\varepsilon}(t) &\leq \mathcal{Q}_{u_\varepsilon}(0) + C \int_0^t (\mathcal{Q}_{u_\varepsilon}(s) + \mathcal{Q}_{v_\varepsilon}(s)) ds + C \frac{\varepsilon}{h^2} \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{K}_h(x-y) |d_\varepsilon(t,x) - d_\varepsilon(t,y)| dx dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{K}_h(x-y) |u_\varepsilon(x)F(p_\varepsilon(x)) - u_\varepsilon(y)F(p_\varepsilon(y))| dx dy ds. \end{aligned} \quad (\text{A.5.6})$$

The last integral can be bounded by  $C \int_0^t (\mathcal{Q}_{u_\varepsilon}(s) + \mathcal{Q}_{v_\varepsilon}(s)) ds$ . Now, to deduce compactness, we write the same expression as (A.5.6) for  $\mathcal{Q}_{v_\varepsilon}(t)$  and we sum up to deduce

$$\begin{aligned} \mathcal{Q}_{u_\varepsilon}(t) + \mathcal{Q}_{v_\varepsilon}(t) &\leq \mathcal{Q}_{u_\varepsilon}(0) + \mathcal{Q}_{v_\varepsilon}(0) + C \int_0^t (\mathcal{Q}_{u_\varepsilon}(s) + \mathcal{Q}_{v_\varepsilon}(s)) ds + C \frac{\varepsilon}{h^2} \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{K}_h(x-y) |d_\varepsilon(t,x) - d_\varepsilon(t,y)| dx dy ds. \end{aligned}$$

Now, it is easy to see that since  $\{\nabla d_\varepsilon\}$  is uniformly bounded in  $L^1((0,T) \times \mathbb{R}^d)$ , the last term is bounded by a constant. Hence, Gronwall's inequality and (A.5.5) imply compactness in space of  $\{u_\varepsilon\}$  and  $\{v_\varepsilon\}$ .



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ETUDE MATHÉMATIQUE D'ÉQUATIONS  
DE TYPE CAHN-HILLIARD DÉGÉNÉRÉES

Résumé

Nous étudions des équations de type Cahn-Hilliard, équation qui fut introduite pour décrire la séparation de phases dans les systèmes multi-composants. Les résultats obtenus dans ce travail ont été motivés par des applications biologiques, notamment la formation de tissus et la croissance de tumeurs, ainsi que par des applications physiques, tels que les écoulements de fluides impliquant des phénomènes de tension de surface.

La première partie de cette thèse est une analyse de la relation entre l'équation de Cahn-Hilliard et les modèles Hele-Shaw, qui sont fréquemment utilisés pour modéliser l'écoulement des fluides ou l'évolution de tumeurs cancéreuses dans des espaces confinés. Nous examinons en particulier comment obtenir les modèles Hele-Shaw dans la limite dite *incompressible* de l'équation de Cahn-Hilliard. La deuxième partie de la thèse se concentre sur l'étude de l'équation de Cahn-Hilliard non-locale (et ses variations) et sa convergence vers l'équation locale. Cette équation, qui peut être dérivée rigoureusement à partir d'un système de particules en interaction, est obtenue en remplaçant le laplacien, qui est un terme local, par une approximation non-locale prenant en compte les interactions à longue distance entre les composants. Nous montrons que la solution de l'équation non-locale converge vers la solution de l'équation locale dans la limite d'interaction à courte distance. La troisième partie de la thèse se penche sur l'étude des modèles de fluides plus classiques, tels que les équations d'Euler et de Navier-Stokes, qui intègrent des phénomènes de tension de surface. Ces modèles sont utilisés pour décrire les écoulements de fluides ou les mouvements de cellules dans lesquels les forces interfaciales jouent un rôle important. La quatrième partie juxtapose la théorie cinétique, traditionnellement employée pour la représentation de phénomènes physiques à une échelle mésoscopique, avec l'équation de Cahn-Hilliard. Notre étude se concentre spécifiquement sur l'équation de Vlasov-Cahn-Hilliard, qui décrit les processus de transition de phase.

**Mots clés :** Cahn-Hilliard, limite incompressible, tension de surface, analyse asymptotique, flow gradient

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Abstract

We study Cahn-Hilliard type equations, an equation that was introduced to describe phase separation in multi-component systems. The results obtained in this work have been motivated by biological applications, such as tissue formation and tumor growth, as well as physical applications, such as fluid flows involving surface tension phenomena.

The first part of this thesis is an analysis of the relationship between the Cahn-Hilliard equation and the Hele-Shaw models, which are frequently used to model fluid flow or the evolution of cancerous tumors in confined spaces. In particular, we examine how to obtain Hele-Shaw models in the so-called *incompressible* limit of the Cahn-Hilliard equation. The second part of the thesis focuses on the study of the non-local Cahn-Hilliard equation (and its variations) and its convergence to the local equation. This equation, which can be derived rigorously from a system of interacting particles, is obtained by replacing the Laplacian, which is a local term, by a non-local approximation taking into account the long range interactions between the components. We show that the solution of the non-local equation converges to the solution of the local equation in the short range interaction limit. The third part of the thesis focuses on the study of more classical fluid models, such as the Euler and Navier-Stokes equations, which incorporate surface tension phenomena. These models are used to describe fluid flows or cell motions in which interfacial forces play an important role. The fourth part juxtaposes kinetic theory, traditionally used to represent physical phenomena on a mesoscopic scale, with the Cahn-Hilliard equation. Our study focuses specifically on the Vlasov-Cahn-Hilliard equation, which describes phase transition processes.

**Keywords:** Cahn-Hilliard, incompressible limit, surface tension, asymptotic analysis, gradient flows

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