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SORBONNE UNIVERSITÉ
École Doctorale de Sciences Mathématiques de Paris Centre
Laboratoire de Probabilités, Statistique et Modélisation

THÈSE DE DOCTORAT
Discipline : **Mathématiques**

présentée par
Benjamin BONNEFONT

Champs gaussiens hiérarchiques et mesures de Gibbs

sous la direction d'Olivier ZINDY

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Résumé

Ce travail est consacré à l'étude des processus branchants gaussiens et leur lien avec une classe de modèle en physique statistique appelés champs gaussiens hiérarchiques.

Le premier chapitre est dédié à l'étude de l'overlap à deux températures lorsque les énergies sont données par les positions des particules d'un mouvement brownien branchant. On montre notamment que l'overlap à deux températures surcritiques diffère du cas indépendant (Random Energy Model) -différence que l'on n'observe pas à une seule température- en établissant une inégalité stricte entre leurs valeurs moyennes.

Le deuxième chapitre est issu d'un travail en commun avec Michel Pain et Olivier Zindy. Nous prolongeons l'analyse des effets des processus de décoration sur l'overlap à deux températures dans l'esprit de Derrida et Mottishaw [44] et sur la susceptibilité en température. Nous montrons qu'au voisinage de la température critique, l'overlap moyen a un comportement plus régulier que son homologue indépendant. Nous établissons également des estimées fines concernant le comportement de la susceptibilité au voisinage de la température critique.

Enfin, le dernier chapitre est l'exposé d'un travail avec Vincent Vargas qui concerne la martingale dérivée sous-critique de la marche aléatoire branchante binaire gaussienne et dans lequel nous répondons à la conjecture [72, Conjecture 1] dans le cadre d'un champ gaussien hiérarchique. Nous obtenons des estimées précises sur le comportement de la queue à gauche de leur distribution dans la phase dite L^4 .

Abstract

This work is devoted to the study of Gaussian branching processes and their link with a class of models in statistical physics, called hierarchical Gaussian fields.

The first chapter is dedicated to the study of the overlap at two temperatures when the energies are given by the positions of the particles in a branching Brownian motion. In particular, it is shown that the overlap at two supercritical temperatures differs from the independent case (Random Energy Model) - a difference not observed at a single temperature - by establishing a strict inequality between their mean values.

The second chapter is the result of a joint work with Michel Pain and Olivier Zindy. We extend our analysis of the repercussions of the decoration processes on two-temperature overlaps in the spirit of Derrida and Mottishaw [44] and temperature susceptibility. We show that in the vicinity of the critical temperature, the average overlap behaves more regularly than its independent counterpart. We also establish fine estimates on the behavior of the susceptibility in the vicinity of the critical temperature.

Finally, in the last chapter, we present a joint work with Vincent Vargas on the subcritical derivative martingale of the binary Gaussian branching random walk and in which we answer the conjecture [72, Conjecture 1] in the framework of a hierarchical Gaussian field. There, we obtain precise estimates of the behavior of the left tail of their distribution in the so-called L^4 phase.

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Introduction

0.1 Statistique des valeurs extrêmes

Etant donnée une suite de variables aléatoires réelles $(X_k)_{k \geq 1}$, la théorie des valeurs extrêmes se propose d'étudier le maximum des n premiers termes

$$Y_n := \max_{k \leq n} X_k$$

lorsque n tend vers l'infini. Ce problème été longuement analysé au début du vingtième siècle dans le cas de variables aléatoires indépendantes et identiquement distribuées (noté i.i.d. par la suite). La théorie atteint son apogée avec le théorème de Fisher-Tippett-Gnedenko [48, 49, 54, 101] qui réduit à trois différents types possibles la limite en loi du maximum correctement renormalisé : la loi de Gumbel, notée \mathcal{G} dans ce manuscrit, la loi de Fréchet et celle de Weibull. La loi de Gumbel concerne notamment les variables aléatoires qui admettent une queue de distribution à droite sous-polynomiale (et satisfont certaines autres hypothèses techniques). Sa fonction de répartition est donnée par $x \mapsto \exp(-e^{-x})$. Les deux autres lois concernent respectivement les queues polynomiales et les variables aléatoires bornées. La loi normale centrée réduite $\mathcal{N}(0, 1)$ est un exemple de loi appartenant au domaine d'attraction de la loi de Gumbel. En effet, si les variables aléatoires réelles $(X_k)_{k \geq 1}$ sont supposées i.i.d. de loi $\mathcal{N}(0, 1)$, alors, en posant

$$a_n := \sqrt{2 \log n} \quad \text{et} \quad b_n := \sqrt{2 \log n} - \frac{1}{2\sqrt{2 \log n}} (\log \log n + \log 4\pi), \quad (0.1.1)$$

on a la convergence en loi¹

$$a_n(Y_n - b_n) \Rightarrow \mathcal{G}, \quad n \rightarrow \infty.$$

De nombreux ouvrages sur la théorie des valeurs extrêmes existent, on pourra se référer à [25, 56, 74, 93].

Une question naturelle est de savoir dans quelle mesure ces résultats peuvent s'étendre au cas où les variables aléatoires X_k ne sont plus indépendantes. Considérons par exemple un processus gaussien centré standard et stationnaire (X_k) . Sa loi est caractérisée par la

¹La notation \Rightarrow sera utilisée dans la suite pour désigner la convergence en loi d'une suite de variables aléatoires.

suite $r_k := \text{Cov}(X_0, X_k)$. Le résultat suivant est dû à Berman [10] et la discussion est tirée de [26].

Proposition 0.1.1 (Berman [10]). *Si $r_n = o\left(\frac{1}{\log n}\right)$, alors*

$$a_n(Y_n - b_n) \Rightarrow \mathcal{G}, \quad n \rightarrow \infty.$$

Cependant cette condition n'est pas loin d'être nécessaire au sens suivant, voir par exemple [74].

Proposition 0.1.2 (McCormick et Mittal [79]). *Si $r_n \log n \rightarrow \infty$ de façon croissante à partir d'un certain rang et si $r_n \downarrow 0$, alors il existe (c_n) et (d_n) telles que*

$$c_n(Y_n - d_n) \Rightarrow \mathcal{N}(0, 1), \quad n \rightarrow \infty.$$

Le cas log-corrélé apparaît ainsi comme le cas critique:

Proposition 0.1.3 (Mittal et Ylvisaker [83]). *Si $r_n \log n \rightarrow \gamma > 0$, alors pour (a_n) et (b_n) définies à l'Equation (0.1.1), on a*

$$a_n(Y_n - b_n) \Rightarrow -\gamma + \sqrt{2\gamma} \mathcal{N}(0, 1) + \mathcal{G}, \quad n \rightarrow \infty,$$

où γ est la constante d'Euler et où les deux variables aléatoires sont indépendantes.

Ce résultat sur un modèle à une dimension constitue un premier indicateur de l'intérêt des champs gaussiens log-corrélés.

Dans les modèles à venir, il sera plus naturel de s'intéresser au cas de 2^n variables aléatoires indexées par $\mathcal{N}_n := \{0, 1\}^n$ et de loi $\mathcal{N}(0, n)$ non nécessairement indépendantes, que l'on notera $(X_u)_{u \in \mathcal{N}_n}$. Les exposants "REM" dans la suite font référence au Random Energy Model et correspondent au cas i.i.d. Ce modèle introduit par Derrida dans le cadre de l'étude des verres de spin, est défini en détail dans la Section 0.3 qui suit. Si $\beta_c := \sqrt{2 \log 2}$,

$$m_n^{\text{REM}} := \beta_c n - \frac{1}{2\beta_c} \log n + \frac{1}{2\beta_c} \log(4\pi \log 2), \quad (0.1.2)$$

et

$$M_n^{\text{REM}} := \max_{u \in \mathcal{N}_n} X_u,$$

la convergence en loi devient, voir par exemple Bovier [26],

$$M_n^{\text{REM}} - m_n^{\text{REM}} \Rightarrow \frac{1}{\beta_c} \mathcal{G}, \quad n \rightarrow \infty.$$

Cette normalisation présente plusieurs avantages : en plus d'être le cadre naturel pour les différents modèles à venir, il n'y a plus besoin de scaling multiplicatif, la particule la plus haute se déplace à vitesse constante et l'énergie libre est extensive.

L'étape suivante consiste à étudier le processus extrémal qui encode la position des particules les plus hautes. Ce dernier est défini par

$$\mathcal{E}_n := \sum_{u \in \mathcal{N}_n} \delta_{X_u - m_n^{\text{REM}}},$$

et, dans le cas du REM mentionné précédemment, converge en loi vers un processus de Poisson exponentiel d'intensité $\beta_c e^{-\beta_c x} dx$, que l'on notera par la suite PPP($\beta_c e^{-\beta_c x} dx$).

0.2 Processus branchants

0.2.1 La marche aléatoire branchante

L'exemple probablement le plus simple de champ gaussien log-corrélé (CGLC) après celui de la Proposition 0.1.3 est la marche aléatoire branchante (MAB) binaire gaussienne. Les marches aléatoires branchantes sont une extension spatiale des processus de Galton-Watson au sens suivant : au temps 0 une particule notée \emptyset est placée à l'origine, cette particule donne naissance au temps 1 à un nombre aléatoire de particules-enfants dont les positions spatiales relatives au parent sont données par la réalisation de la loi d'un certain processus ponctuel, noté ici \mathcal{L} disons, puis les particules-enfants donnent chacune naissance à un nombre aléatoire de particules dont les positions spatiales relatives sont données de nouveau par une réalisation de \mathcal{L} et ainsi de suite. Dans le cas particulier de la MAB binaire gaussienne, chaque particule donne naissance à deux enfants dont les positions relatives au parent sont données par deux lois gaussiennes centrées réduites et indépendantes. Pour une présentation complète des MAB, on pourra consulter Shi [97]. On attribue généralement l'origine des processus branchants spatiaux homogènes à Kolmogorov [70]. Pour plus de détails historiques, le lecteur pourra consulter [66].

Nous nous intéressons maintenant au cas plus spécifique de la MAB binaire gaussienne. Le processus est indexé par l'arbre binaire $\mathbb{T} := \bigcup_{n \geq 0} \mathcal{N}_n$ où $\mathcal{N}_0 := \{\emptyset\}$ et $\mathcal{N}_n := \{0, 1\}^n$ pour $n \geq 1$. Si $u, v \in \mathbb{T}$, on écrit $v \leq u$ lorsque v est un ancêtre de u , la longueur de u , appelée génération, est notée $|u|$ et la génération du dernier ancêtre commun à $u, v \in \mathbb{T}$ est notée $u \wedge v$. La MAB binaire gaussienne $(X_u)_{u \in \mathbb{T}}$ est alors définie par³

$$X_u := \sum_{\emptyset < v \leq u} G_v, \quad \forall u \in \mathbb{T},$$

où les $(G_v)_{v \in \mathbb{T}}$ sont des variables aléatoires i.i.d. normales centrées réduites. La Figure 1 représente une réalisation du processus jusqu'à la génération $n = 7$.

On obtient ainsi un processus gaussien centré $(X_u)_{u \in \mathbb{T}}$ dont la covariance est reliée à la structure de l'arbre via

$$\mathbb{E}[X_u X_v] = u \wedge v, \quad \forall u, v \in \mathbb{T}.$$

Il s'agit d'un champ log-corrélé pour la raison suivante : si $u, v \in \mathcal{N}_n$, on peut réécrire la covariance de X_u et X_v comme

$$\mathbb{E}[X_u X_v] = \log_2 \frac{1}{d(u, v)}, \quad \forall u, v \in \mathbb{T},$$

où $d(u, v) := 2^{-u \wedge v}$. La fonction $d : \mathbb{T} \rightarrow [0, 1]$ n'est pas tout à fait une distance car elle ne vérifie pas l'axiome de séparation, cependant elle le devient sur la frontière de l'arbre $\partial \mathbb{T} := \{0, 1\}^{\mathbb{N}}$. Le champ gaussien sur \mathcal{N}_n peut alors être vu comme l'approximation d'un CGLC sur les feuilles de l'arbre.

²La convention que nous choisissons est différente de celle habituellement utilisée où $u \wedge v$ désigne le dernier ancêtre commun à u et v , il s'agit ici d'alléger les notations.

³Ici et par la suite, on adopte les conventions usuelles $\sum_{\emptyset} = 0$ et $\prod_{\emptyset} = 1$.

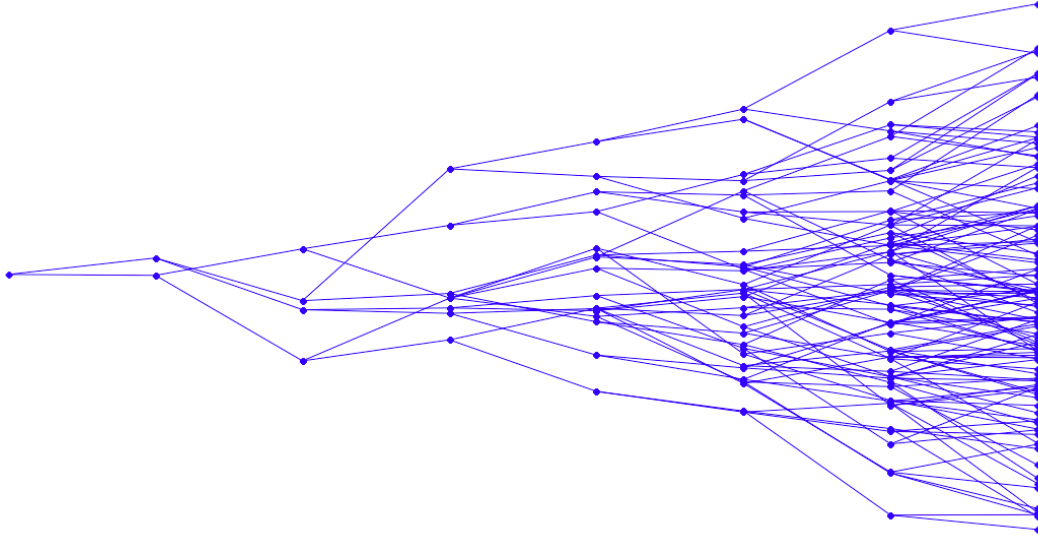


Figure 1: Réalisation des sept premières générations d'une marche aléatoire binaire et gaussienne.

Remark 0.2.1. Le processus obtenu en posant $L_u := \exp(X_u)$ est une forme particulière de cascade multiplicative. Introduites par Mandelbrot [78] puis étudiées notamment par Kahane [62] et Peyrière [92], la motivation initiale de ces objets était de comprendre la théorie de la turbulence de Kolmogorov [69,86]. A un noeud de l'arbre binaire correspond un intervalle dyadique de $[0, 1]$ sur lequel le processus $(L_u)_{u \in \mathcal{N}_n}$ définit une mesure aléatoire qui est le principal objet d'étude.

Le maximum

A la génération $n \in \mathbb{N}$, le maximum est donné par

$$M_n := \max_{u \in \mathcal{N}_n} X_u.$$

La section précédente nous a appris que dans le cas 2^n variables aléatoires i.i.d. de loi $\mathcal{N}(0, n)$, l'ordre du maximum est $\beta_c n - \frac{1}{2\beta_c} \log n$. Ici, on constate une première différence : elle se loge au deuxième ordre comme le montre la proposition suivante prouvée par Bachmann [8] pour des déplacements i.i.d. et log-concaves puis pour une MAB en toute généralité par Aïdékon [1].

Proposition 0.2.2. *Soit $m_n := \beta_c n - \frac{3}{2\beta_c} \log n$, on a la convergence en loi*

$$M_n - m_n \Rightarrow \frac{1}{\beta_c} G + \frac{1}{\beta_c} \log(cZ), \quad n \rightarrow \infty,$$

où $c > 0$ et Z est une variable aléatoire strictement positive indépendante de $G \sim \mathcal{G}$.

En comparant l'expression de m_n ci-dessus avec le terme m_n^{REM} de l'Équation (0.1.2), on observe en effet que la corrélation ralentit le maximum mais seulement au second ordre :

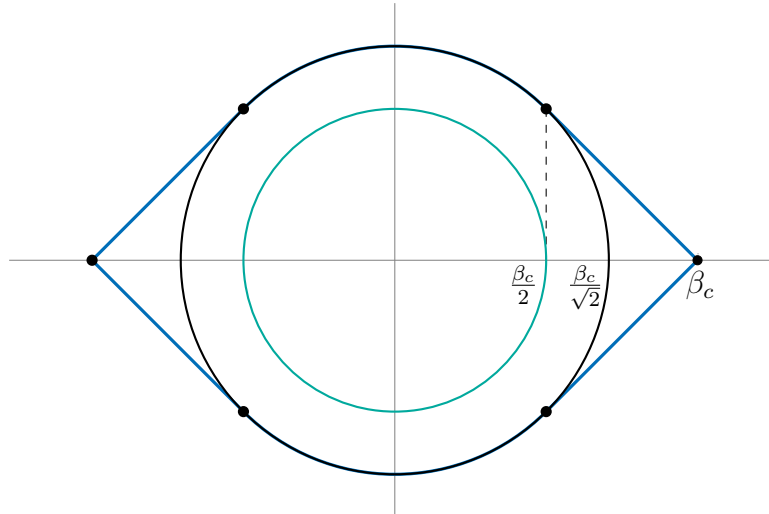


Figure 2: Le domaine de convergence Λ de la martingale additive est délimité par les lignes bleues. Le cercle noir délimite le régime L^2 et le cercle intérieur le régime L^4 .

le $-\frac{1}{2} \log n$ pour le REM est remplacé par $-\frac{3}{2} \log n$ pour la MAB. Ce changement a priori innocent est en fait une des marques de fabrique des champs log-corrélés. Cette correction se manifeste dans de nombreux autres modèles, voir par exemple [3].

La variable aléatoire Z de la proposition précédente joue un rôle fondamental dans l'étude des extrêmes du processus. Il s'agit de la limite d'une certaine martingale du modèle. L'existence de nombreuses martingales dans les MAB est un des outils puissants pour les étudier, nous présentons maintenant ces familles.

Martingales additives

Si $\beta \in \mathbb{C}$, les variables aléatoires

$$W_n(\beta) := \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - n \frac{\beta^2}{2}}, \quad n \in \mathbb{N},$$

forment une martingale par rapport à la filtration naturelle (\mathcal{F}_n) appelée martingale de Biggins. Si $\beta \in \mathbb{R}$, elle est positive et converge donc presque sûrement vers une variable aléatoire W_β . La limite est non triviale si et seulement si $\beta \in [0, \beta_c)$ où $\beta_c = \sqrt{2 \log 2}$, voir Biggins [11]. Dans le cas complexe, Biggins [12] montre qu'il y a en fait convergence uniforme par rapport à β dans tous les compacts inclus dans le domaine Λ de la Figure 2. La fonction $\Lambda \ni \beta \mapsto W_\beta$ est ainsi analytique presque sûrement. Pour la même raison, la dérivée par rapport à β de la martingale W

$$Z_n(\beta) := \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - n \frac{\beta^2}{2}} (X_u - \beta n), \quad n \in \mathbb{N},$$

converge presque sûrement vers une limite Z_β dans ce même domaine.

Lorsque $\beta \in (0, \beta_c)$, la limite Z_β est une variable aléatoire signée et sa loi est particulièrement asymétrique. En effet, la queue à gauche est de type Weibull comme on

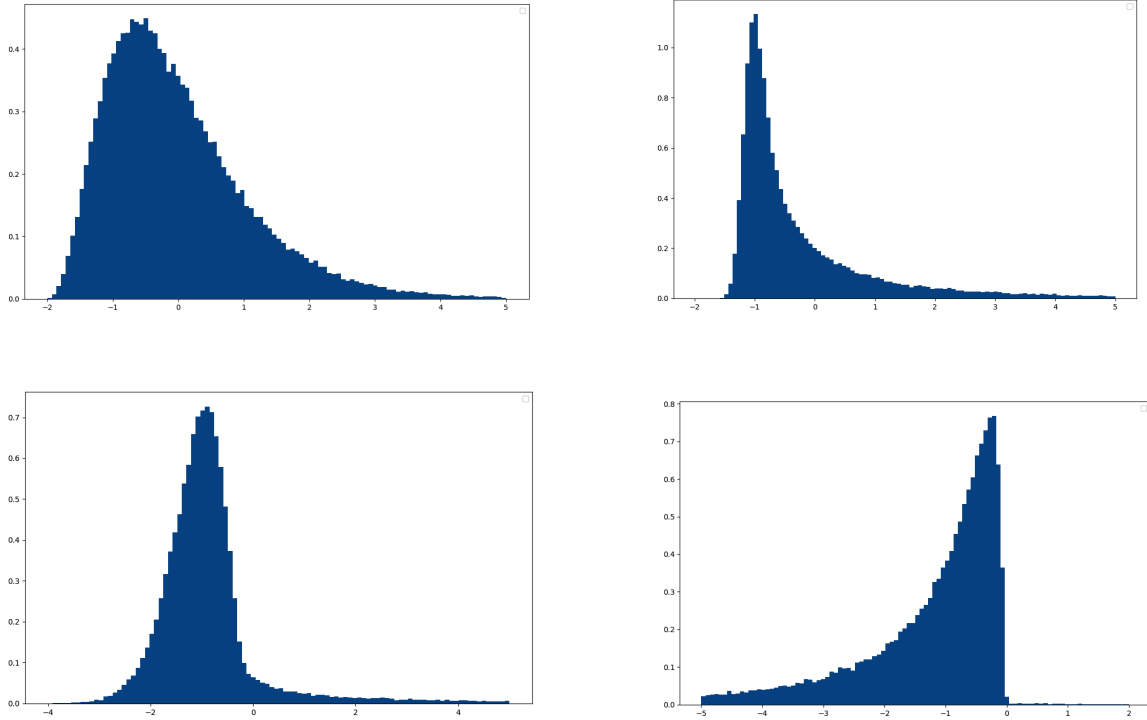


Figure 3: Mesure empirique de $Z_n(\beta)$ pour $n = 20, 10^5$ réalisations, $\beta \in \{0.2, 0.5\}$ et $\beta \in \{0.8, 1.1\}$ respectivement. La valeur critique du modèle est $\beta_c = \sqrt{2 \log 2} \simeq 1.177$.⁴

va le voir alors que le comportement de la queue à droite attendu est celui W_β , à des corrections logarithmiques près. La conjecture de Lacoïn & al. [72] transposée dans le cadre branchant est la suivante. Pour $\beta \in (0, \beta_c)$, il existe des constantes $c, c', C, C' > 0$ telles que

$$C e^{-c x^\gamma} \leq \mathbb{P}(Z_\beta < -x) \leq C' e^{-c' x^\gamma}, \quad \forall x \geq 0,$$

où γ est donnée par

$$\gamma := \left(\frac{\beta_c}{\beta} \right)^2.$$

Les résultats de l'article reproduit dans le Chapitre 3 de ce manuscrit, co-écrit avec Vincent Vargas, concernent la MAB binaire gaussienne. Nous avons obtenu une borne inférieure dans toute la phase $\beta \in (0, \beta_c)$.

Theorem 0.2.3. *Pour $\beta \in (0, \beta_c)$, il existe $c, C > 0$ telles que*

$$\mathbb{P}(Z_\beta < -x) \geq C e^{-c x^\gamma}, \quad \forall x \geq 0.$$

Nous obtenons une borne supérieure dans le régime L^4 qui correspond à $\beta \in (0, \beta_c/2)$.

⁴Pour $\beta = \beta_c$, Z_β est négative et sa queue à gauche est équivalente à Cx^{-1} : la raison pour laquelle la forme semble différente pour $\beta = 1.1$ est probablement la présence d'une grande constante C et d'une petite pour la queue à droite. La plus grande valeur obtenue dans cette réalisation est 336334.8 quand la plus faible est -32.6 .

Theorem 0.2.4. *Pour $\beta \in (0, \frac{\beta_c}{2})$, il existe $c', C' > 0$ telles que*

$$\mathbb{P}(Z_\beta < -x) \leq C' e^{-c' x^\gamma}, \quad \forall x \geq 0.$$

0.2.2 Le mouvement brownien branchant

L'équivalent continu du modèle précédent est le mouvement brownien branchant (MBB). Dans un certain sens, parmi tous les modèles log-corrélés, c'est celui qui possède le plus de symétries. On pourra se référer à [9, 26] pour davantage de détails.

Le MBB décrit un système de particules qui se comporte de la façon suivante : une particule issue de l'origine se déplace suivant un mouvement brownien, puis après un temps exponentiel de paramètre 1, elle se scinde en deux particules filles qui poursuivent leur trajectoire depuis la position de la particule mère et ce de façon indépendante et brownienne.

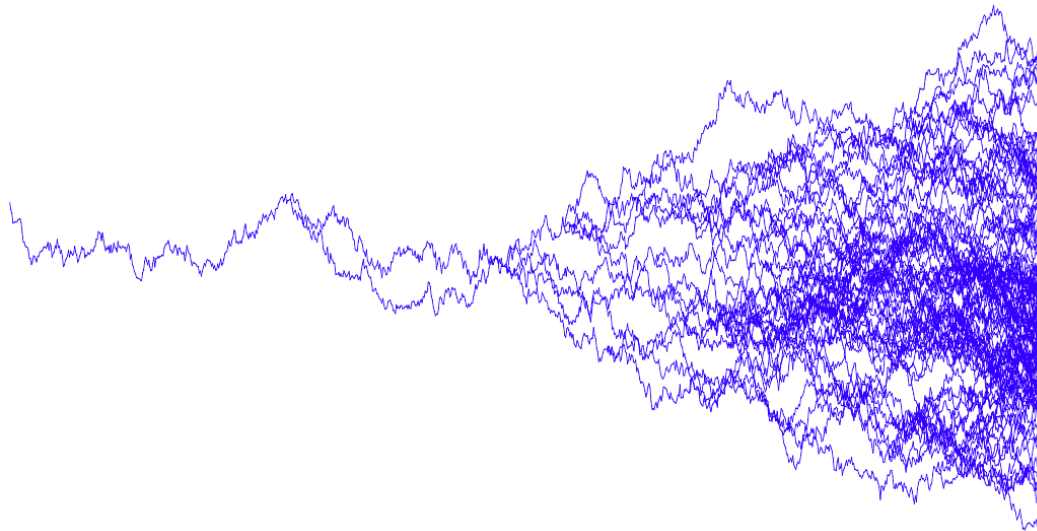


Figure 4: Réalisation d'un mouvement brownien branchant jusqu'au temps $t = 7$.

Pour chaque $u \in \mathbb{T}$, notons respectivement b_u et d_u les instants de naissance et de mort et pour $t \geq 0$, $\mathcal{N}_t := \{u \in \mathbb{T} : b_u \leq t < d_u\}$ l'ensemble des particules vivantes au temps t et $x_u(t)$ leurs positions respectives. Ikeda & al. [57–59] ont construit un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$ sur lequel les trajectoires sont browniennes et l'arbre sous-jacent $T = (T_t)_{t \geq 0}$ est un arbre binaire continu de Galton-Watson avec taux de branchement 1, également appelé processus de Yule.

La position de la particule la plus haute $x(t) := \max_{u \in \mathcal{N}_t} x_u(t)$ a donné lieu à une littérature très fournie depuis McKean [80] qui l'a reliée à l'équation aux dérivées partielles F-KPP. Bramson [32] a obtenu l'ordre du maximum $m_t := \beta_c t - (3/2\beta_c) \log_+ t$, où $\beta_c = \sqrt{2}$, puis Lalley et Sellke [73] ont démontré une représentation intégrale pour la limite en

utilisant la martingale dérivée critique⁵

$$Z := \lim_{t \rightarrow \infty} \sum_{u \in \mathcal{N}_t} (\beta_c t - x_u(t)) e^{-\beta_c(\beta_c t - x_u(t))},$$

au sens où

$$x(t) - m_t \Rightarrow \frac{1}{\beta_c} G + \frac{1}{\beta_c} \log(CZ), \quad t \rightarrow +\infty, \quad (0.2.1)$$

avec $C > 0$ et $G \sim \mathcal{G}$ indépendante de Z .

Processus extrémal

Si l'on s'intéresse à la position des particules autour du maximum, l'objet à étudier est le processus ponctuel

$$\mathcal{E}_t := \sum_{u \in \mathcal{N}_t} \delta_{x_u(t) - m_t},$$

appelé *processus extrémal*.

La preuve de la convergence du processus extrémal (\mathcal{E}_t) est due à Arguin & al. [5] et simultanément Aidékon & al. [2] pour le mouvement brownien branchant, Madaule [75] pour la marche branchante et Biskup & al. [15] pour le champ libre gaussien discret en dimension 2. Pour le MBB, on peut l'énoncer de la façon suivante.

Proposition 0.2.5. *Pour la topologie vague, on a*

$$\mathcal{E}_t \Rightarrow \mathcal{E} := \sum_{i,j} \delta_{\frac{1}{\beta_c} \log(CZ) + p_i + \Delta_j^i}, \quad t \rightarrow +\infty,$$

où C et Z ont été définies dans l'Équation (0.2.1), les p_i sont les points d'un processus ponctuel de Poisson $\text{PPP}(\beta_c e^{-\beta_c x} dx)$ et les Δ^i sont des copies *i.i.d.* d'un processus ponctuel sur \mathbb{R}_- , les trois étant mutuellement indépendants. Les processus Δ^i sont appelées *décorations*.

Il n'est pas difficile de voir que pour le cas indépendant du REM le processus extrémal converge en loi vers $\sum_i \delta_{c+p_i}$ où c est une certaine constante réelle et les p_i sont les atomes d'un processus ponctuel de Poisson $\text{PPP}(\beta_c e^{-\beta_c x} dx)$. D'un PPP exponentiel dans le cas du REM, on passe à un PPP exponentiel, translaté aléatoirement et décoré⁶ pour la MAB. Ce nouvel ingrédient, que constituent les décorations Δ^i , décrit la structure interne des blocs de valeurs extrêmes qui partagent un ancêtre proche : les points de ce processus sont ainsi fortement corrélés les uns avec les autres. L'influence que ces décorations peuvent exercer sur le processus limite fait l'objet de l'article reproduit au Chapitre 2 où l'on s'intéresse, avec Michel Pain et Olivier Zindy, à l'incidence des décorations sur certaines quantités issues de la physique statistique.

⁵Noter que dans cette littérature, la martingale limite est positive et correspond à l'opposée de celle que nous avons introduite dans la section précédente.

⁶Voir [98] pour davantage de détails sur ces processus.

0.3 Physique statistique

On suppose donné un ensemble de configurations de spins $\Sigma_n = \{\pm 1\}^n$ et d'un hamiltonien pour les énergies de ces configurations $(H_\sigma)_{\sigma \in \Sigma_n}$. La fonction de partition du système est définie par

$$Z_n(\beta) := \sum_{\sigma \in \Sigma_n} e^{\beta H_\sigma},$$

où $\beta > 0$ est l'inverse de la température⁷. La mesure de Gibbs associée est donnée par

$$\mathcal{G}_{\beta,n} := \frac{1}{Z_\beta} \sum_{\sigma} e^{\beta H_\sigma} \delta_\sigma,$$

où δ_σ désigne la masse de Dirac en σ . Enfin, l'énergie libre par particule⁸ est donnée par

$$f_n(\beta) := \frac{1}{n \log 2} \log Z_n(\beta).$$

C'est une grandeur qui recèle quantité d'informations sur le système, et prouver l'existence de sa limite thermodynamique (lorsque $n \rightarrow \infty$) constitue souvent un point de départ dans l'étude d'un système de spins.

Un des objectifs de la mécanique statistique est de comprendre la géométrie des mesures de Gibbs. A basse température (β grand), la mesure de Gibbs favorise les états de haute énergie.⁹ C'est pour tenter de comprendre les verres de spin et leurs propriétés déroutantes que les physiciens ont introduit les modèles désordonnés dans lesquels les énergies sont aléatoires, voir par exemple Mezard & al [82]. Deux niveaux d'aléatoire se superposent : le premier se loge dans la mesure de Gibbs et le deuxième provient des niveaux d'énergies qui sont à présent des variables aléatoires. Il en fallait moins pour susciter l'intérêt des probabilistes ! Et lorsque les valeurs extrêmes de l'hamiltonien deviennent prépondérantes, la théorie des valeurs extrêmes exposée dans la Section 0.1 entre en jeu.

Random Energy Model

Le modèle le plus simple de système désordonné est le Random Energy Model (REM) dans lequel les énergies H_σ sont 2^n variables aléatoires i.i.d. gaussiennes et centrées de variance n . Ce modèle introduit par Derrida [42] a pour but de comprendre les mécanismes à l'oeuvre dans les verres de spins. Il apparaît notamment comme la limite $p \rightarrow \infty$ des modèles de Sherrington-Kirkpatrick à p spins. Pour une introduction à ce modèle, voir le chapitre dédié dans [36].

⁷On adopte la convention probabiliste qui ne fait pas apparaître le $-$ devant l'hamiltonien.

⁸ $-\frac{1}{\beta} \log Z_\beta$ est la véritable énergie libre et $\log Z_\beta$ est plutôt appelée entropie libre, voir [81]. Nous nous conformons à l'usage dans la littérature probabiliste ici.

⁹Le paramètre β joue ainsi le rôle de scanner pour reprendre les mots de Talagrand [99].

Le REM présente une transition de phase¹⁰ à $\beta_c = \sqrt{2 \log 2}$. En particulier son énergie libre satisfait à la limite (presque-sûre et L^1)

$$f(\beta) = \begin{cases} 1 + \left(\frac{\beta}{\beta_c}\right)^2, & \text{si } \beta \leq \beta_c, \\ 2 \frac{\beta}{\beta_c}, & \text{si } \beta > \beta_c. \end{cases} \quad (0.3.1)$$

Une façon de le voir est de raisonner à n suffisamment grand. Pour $\beta \leq \beta_c$, il existe un nombre exponentiellement grand de configurations, avec des niveaux d'énergie strictement inférieurs aux extrêmes, qui contribuent à l'énergie libre et à la mesure de Gibbs à peu près uniformément, alors que pour $\beta > \beta_c$, les configurations pertinentes sont celles qui ont les plus grandes énergies et l'énergie libre est dominée par un nombre relativement faible de configurations. Dans le langage de la physique statistique on parle de compétition énergie/entropie : à haute température, l'entropie gagne et à basse température, ce sont les niveaux d'énergie les plus hauts qui l'emportent. On pourra également consulter Bolthausen [21] pour davantage de résultats sur le REM. Précisons simplement ici que le REM est un objet fondamental qui joue un rôle important bien au delà du contexte des verres de spin pour lesquels il a été introduit. Mentionnons que l'étude des verres de spin s'est révélée extrêmement difficile et a culminé avec la solution de Parisi [90, 91] pour l'expression de l'énergie libre du modèle de Sherrington-Kirkpatrick avec l'emploi de techniques heuristiques remarquables. Pour le versant mathématique, on pourra consulter [17, 18, 22, 88, 100].

Les champs log-corrélés comme modèle critique

Dans les modèles désordonnés, les physiciens parlent d'interaction à longue portée ou à faible portée suivant la décroissance spatiale des corrélations. Les potentiels à corrélation logarithmique jouent le rôle de classe intermédiaire ou *critique*, comme le fait apparaître la discussion de Fyodorov et Sommers [53]. Dans des paysages d'énergie uni-dimensionnels, il ressort de l'analyse de Carpentier et Le Doussal [34] que le cas le plus intéressant et qui présente une véritable transition de phase est le cas log-corrélé, qui correspond au cas $\alpha = \frac{1}{2}$ dans l'exemple à venir. Pour reprendre le vocabulaire des physiciens, les corrélations logarithmiques produisent exactement le bon équilibre entre le niveau des puits d'énergie et leur nombre (entropie)¹¹. On pourra également consulter [3]. Les paysages énergétiques dont il est question dans [34], peuvent être visualisés avec la série de Fourier suivante

$$\sum_{n \geq 1} \frac{1}{n^\alpha} (X_n \cos 2\pi n x + Y_n \sin 2\pi n x), \quad (0.3.2)$$

où $\{(X_n, Y_n), n \geq 1\}$ est une famille de variables aléatoires i.i.d. de loi $\mathcal{N}(0, 1)$.

¹⁰On pourra se référer à [16].

¹¹C'est également pour ce type de corrélations que le chaos multiplicatif gaussien n'est pas trivial, voir les travaux précurseurs de Kahane [63].

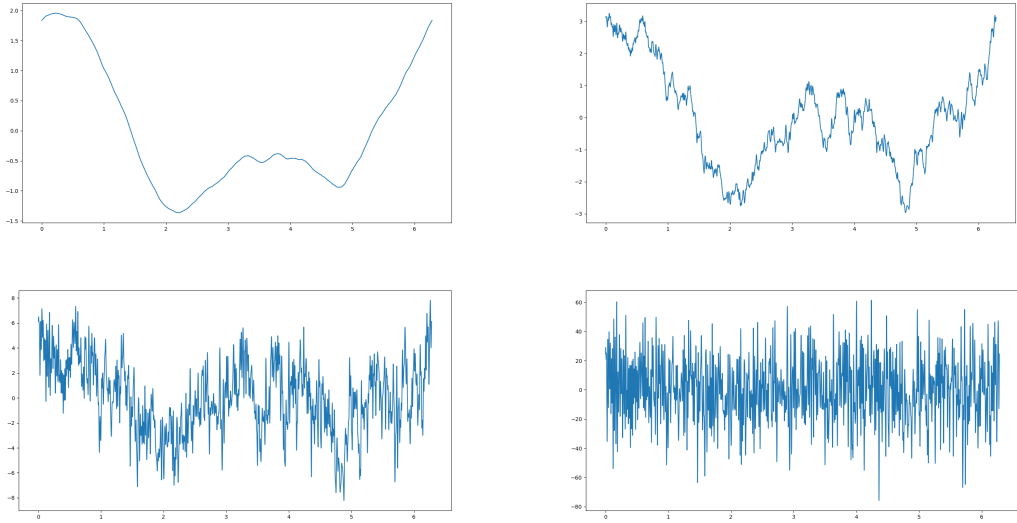


Figure 5: Réalisations de la série de Fourier aléatoire (0.3.2) tronquée à $N = 400$ pour les valeurs de $\alpha \in \{2, 1\}$ et $\alpha \in \{0.5, 0\}$ respectivement. La première est l'intégrale d'un brownien, la deuxième un brownien, la troisième un LCGF sur le cercle (plus précisément un CLG restreint au cercle) et la dernière un bruit blanc. Pour plus de détails, on pourra consulter les travaux de Kahane [64, Chapitre 16].

Dans le contexte branchant, c'est l'article fondateur de Derrida et Spohn [45] qui nous invite à interpréter les positions des particules d'une marche aléatoire branchante comme les énergies d'un système désordonné.

Overlap

Un outil fondamental pour l'étude des verres de spin est la notion d'overlap qui, pour la MAB considérée dans ce manuscrit, est défini par

$$R_{uv} := \frac{1}{n} u \wedge v, \quad \forall u, v \in \mathcal{N}_n.$$

En particulier, une façon d'étudier les extrêmes avec une approche physique statistique, est de considérer la mesure de probabilité aléatoire suivante

$$\mathcal{G}_{\beta, n}^{\otimes 2} (R_{uv} \in \cdot),$$

où l'overlap R_{uv} concerne ici deux particules u et v tirées indépendamment sous la mesure de Gibbs $\mathcal{G}_{\beta, n}$. Pour le REM comme pour la MAB, la convergence de l'overlap est connue [61]:

$$\mathcal{G}_{\beta, n}^{\otimes 2} (R_{uv} \in \cdot) \Rightarrow \begin{cases} \delta_0, & \text{si } \beta \leq \beta_c, \\ (1 - Q_\beta) \delta_0 + Q_\beta \delta_1, & \text{si } \beta > \beta_c. \end{cases} \quad (0.3.3)$$

Q_β est distribuée comme la somme des carrés d'une variable de Poisson-Dirichlet $\text{PD}(\beta_c/\beta)$. On rappelle ici la définition de variable aléatoire de Poisson-Dirichlet : si $\alpha \in (0, 1)$ et

(η_i) sont les atomes d'un PPP $(\mathbf{1}_{s>0} s^{-\alpha-1} ds)$ rangés par ordre décroissant, une variable aléatoire de Poisson-Dirichlet de paramètre α PD(α) est une variable aléatoire sur l'espace des poids décroissants $\vec{s} = (s_1, s_2, \dots)$ avec $1 \geq s_1 \geq s_2 \geq \dots \geq 0$ et $\sum_i s_i \leq 1$ qui a la même loi que $(\eta_i / \sum_j \eta_j)$.

Il est alors intéressant de remarquer que si on pense à la convergence du processus extrémal de ces deux modèles (REM et MAB), on peut écrire

$$Q_\beta^{\text{REM}} = \frac{\sum_i e^{2\beta p_i}}{\left(\sum_i e^{\beta p_i}\right)^2}, \quad Q_\beta = \frac{\sum_i e^{2\beta p_i} \left(\sum_j e^{\beta \Delta_j^i}\right)^2}{\left(\sum_{i,j} e^{\beta(p_i + \Delta_j^i)}\right)^2}.$$

Une propriété des PPP exponentiels permet de constater que les variables aléatoires Q_β^{REM} et Q_β ont la même loi (voir par exemple la discussion précédent l'énoncé du Théorème 1.2 dans [87]) : ainsi les décorations ne sont plus vues à la limite lorsque les deux particules sont tirées à une même température.

De façon surprenante, lorsque l'on passe à deux températures, une différence apparaît entre le REM et la MAB. L'objet d'étude est maintenant la mesure aléatoire

$$\mathcal{G}_{\beta,n} \otimes \mathcal{G}_{\beta',n} (R_{uv} \in \cdot),$$

qui a été étudiée pour le REM par Kurkova [71] et plus récemment dans la littérature physique par Derrida et Mottishaw [44]. Les résultats obtenus dans l'article [23] et présentés dans le Chapitre 1 concernent le MBB. En raison de nos choix de normalisation, $\beta_c = \sqrt{2}$ ici et l'overlap est défini, pour $u, v \in \mathcal{N}_t$, par

$$q_t(u, v) := \frac{1}{t} \mathbb{E} [x_u(t) x_v(t)] = \frac{1}{t} \sup\{s \geq 0 : u, v \text{ partage un ancêtre commun dans } \mathcal{N}_s\}.$$

Le premier résultat concerne la convergence de l'overlap à deux températures.

Theorem 0.3.1. *Soient $\beta, \beta' > 0$.*

(i) *Si $\beta \leq \beta_c$ ou $\beta' \leq \beta_c$ et $a \in (0, 1)$,*

$$\mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t} (q_t(u, v) \geq a) \xrightarrow[t \rightarrow \infty]{} 0, \quad \text{dans } L^1.$$

(ii) *Si $\beta, \beta' > \beta_c$, et $a \in (0, 1)$,*

$$\mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t} (q_t(u, v) \geq a) \xrightarrow[t \rightarrow \infty]{(d)} Q(\beta, \beta'),$$

où

$$Q(\beta, \beta') := \frac{\sum_i \left(\sum_j e^{\beta(p_i + \Delta_j^i)}\right) \left(\sum_j e^{\beta'(p_i + \Delta_j^i)}\right)}{\left(\sum_{i,j} e^{\beta(p_i + \Delta_j^i)}\right) \left(\sum_{i,j} e^{\beta'(p_i + \Delta_j^i)}\right)}. \quad (0.3.4)$$

Le second compare cet overlap limite avec celui du REM.

Theorem 0.3.2. *Si $\beta \neq \beta' > \beta_c$, on a $\mathbb{E}[Q(\beta, \beta')] < \mathbb{E}[Q^{\text{REM}}(\beta, \beta')]$.*

Ce qui prouve d'une part que l'overlap à deux températures distinctes du MBB n'a plus la même loi que celui du REM et qu'en un sens, le MBB est plus chaotique en température que son analogue i.i.d.

Décorations

Comme nous l'avons vu précédemment, la limite du processus extrémal d'un certain nombre de CGLC est un PPP translaté et décoré. Du point de vue de la physique statistique, la translation n'a que peu d'importance : la mesure de Gibbs est invariante par translation des énergies. En revanche, ce n'est pas le cas des décorations comme nous avons pu le mettre en évidence avec Michel Pain et Olivier Zindy dans le travail reproduit dans le Chapitre 2.

Les décorations pour lesquelles on dispose du plus d'informations sont celles du MBB. La description d'Aïdékon & al [2], qui est notamment explicitée dans la Section 2.3.2 du Chapitre 1 de ce manuscrit, la description comme un MBB conditionné à se déplacer anormalement haut obtenue par Arguin & al [5] et enfin, celle utilisée par Cortines & al [39, 40] afin d'obtenir des estimées particulièrement précises.

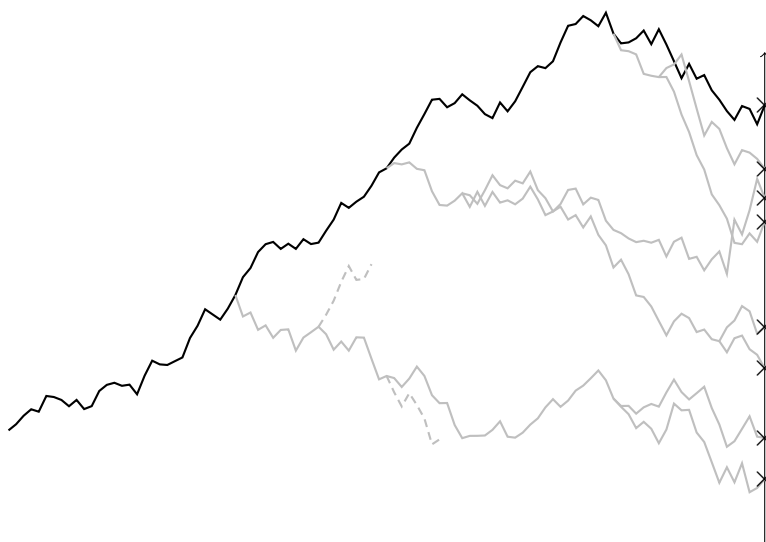


Figure 6: La décoration du MBB obtenue avec la description de Aïdékon & al dans [2].

Nos résultats concernent d'abord l'overlap moyen $F(\beta, \beta') := \mathbb{E}[Q(\beta, \beta')]$. Le comportement pour le REM au voisinage de la température critique est le suivant.

Theorem 0.3.3 (REM). *Pour tout $\beta' > \beta_c = 1$, lorsque $\beta \downarrow 1$,*

$$F(\beta, \beta') \sim (\beta - 1) \log \frac{1}{\beta - 1}.$$

Et de façon surprenante, les décorations ont un effet non trivial : le comportement de F_d , qui est l'équivalent de F dans le cas décoré, est plus lisse au voisinage de la température critique.

Theorem 0.3.4 (MBB). *Pour tout $\beta' > \beta_c = 1$, lorsque $\beta \downarrow 1$,*

$$F_d(\beta, \beta') = o\left((\beta - 1) \log \frac{1}{\beta - 1}\right).$$

Dans un deuxième temps, nous nous sommes intéressés à la susceptibilité en température qui est définie de la façon suivante. Le coefficient de corrélation entre les énergies libres à deux températures $\beta, \beta' > \beta_c$ est donné par

$$C(\beta, \beta') := \frac{\text{Cov}(\log Z(\beta), \log Z(\beta'))}{\sigma(\log Z(\beta)) \sigma(\log Z(\beta'))},$$

où $\sigma(X)$ est l'écart type de X . A $\beta' > \beta_c$ fixé, on peut écrire le développement

$$C(\beta, \beta + h) = 1 - \kappa(\beta) h^2, \quad h \rightarrow 0,$$

où $\kappa(\beta)$ est la susceptibilité en température. Elle mesure ainsi l'influence d'un léger changement de température du point de vue de l'énergie libre. Dans le cas du REM, elle est explicite comme le montre le résultat suivant.

Theorem 0.3.5 (REM). *Pour tout $\beta > \beta_c$,*

$$\kappa(\beta) = \frac{1}{2} \left(\frac{1}{\beta^2 - 1} + \frac{6}{\pi^2 \beta^3 (\beta + 1)} \left(\frac{\Gamma''\left(\frac{\beta-1}{\beta}\right)}{\Gamma\left(\frac{\beta-1}{\beta}\right)} - \left(\frac{\Gamma'\left(\frac{\beta-1}{\beta}\right)}{\Gamma\left(\frac{\beta-1}{\beta}\right)} \right)^2 \right) - \frac{\beta^2}{(\beta^2 - 1)^2} \right),$$

et

$$\kappa(\beta) \underset{\beta \downarrow 1}{\sim} \left(\frac{3}{2\pi^2} - \frac{1}{8} \right) \frac{1}{(\beta - 1)^2}, \quad \kappa(\beta) \underset{\beta \rightarrow +\infty}{=} O\left(\frac{1}{\beta^5}\right).$$

Et nous obtenons le résultat suivant lorsque la décoration est celle d'un MBB.

Theorem 0.3.6 (MBB). *Pour tout $\beta > \beta_c$, on a $\kappa_d(\beta) > \kappa(\beta)$. Plus précisément,*

$$\frac{3}{2\pi^2} - \frac{1}{8} < \liminf_{\beta \downarrow 1} (\beta - 1)^2 \kappa_d(\beta) \leq \limsup_{\beta \downarrow 1} (\beta - 1)^2 \kappa_d(\beta) \leq \frac{3}{\pi^2} - \frac{1}{8}.$$

The overlap distribution at two temperatures for the branching Brownian motion

Abstract

We study the overlap distribution of two particles chosen under the Gibbs measure at two temperatures for the branching Brownian motion. We first prove the convergence of the overlap distribution using the extended convergence of the extremal process obtained by Bovier and Hartung [27]. We then prove that the mean overlap of two points chosen at different temperatures is strictly smaller than in Derrida's random energy model. The proof of this last result is achieved with the description of the decoration point process obtained by Aïdékon, Berestycki, Brunet and Shi [2]. To our knowledge, it is the first time that this description is being used.

1.1 Introduction

1.1.1 Overview

The binary branching Brownian motion (BBM) describes a system of particles that starts with a single one at 0 which moves as a standard Brownian motion and splits into two new particles after a mean-one exponential time. These two particles then move independently according to Brownian motions and split with the same rule. It appears that the BBM belongs to a more general class of models, called log-correlated Gaussian fields, for which the dependence between the random variables starts to affect the extreme value statistics. In this class, one finds the branching random walk (BRW) and the two-dimensional discrete Gaussian free field (DGFF). Among all these models, it is commonly accepted that the BBM, with its branching structure and its brownian trajectories, is more manageable. We refer to [3] for more details on log-correlated fields.

For $t \geq 0$, let \mathcal{N}_t be the set of particles alive at time t and let $x_u(t)$ denote the position of a particle $u \in \mathcal{N}_t$ at time t . One can interpret the position of the particles of the BBM as an energy, and in statistical physics, it is common to introduce the partition function

and the corresponding Gibbs measure to study the extreme values. For $\beta > 0$ and $t \geq 0$, they are defined by

$$Z_{\beta,t} := \sum_{u \in \mathcal{N}_t} e^{\beta x_u(t)}, \quad \mathcal{G}_{\beta,t} := \frac{1}{Z_{\beta,t}} \sum_{u \in \mathcal{N}_t} e^{\beta x_u(t)} \delta_u.$$

One crucial quantity to investigate the energy landscape at the thermodynamical limit is the overlap $q_t(u, v)$ between the particles u and v and more specifically the following random probability measure

$$\mathcal{G}_{\beta,t}^{\otimes 2}(q_t(u, v) \in \cdot),$$

which is the distribution of the overlap between particles u and v chosen independently under the Gibbs measure $\mathcal{G}_{\beta,t}$ (see next subsection for precise definitions). In their widely known article [45], Derrida and Spohn showed that this random measure converges to a limit whose support is $\{0, 1\}$ in the low-temperature regime, exhibiting a one-step replica symmetry breaking (1-RSB) in the language of spin glasses. Curiously enough, the limiting overlap behaves in the same way as in the i.i.d. case, the so-called random energy model (REM). This model was introduced by Derrida [42] as a toy-model to understand more complex spin glasses. As we will see, this is no longer the case when the temperatures are different.

In a very recent article, Derrida and Mottishaw [44] gave a thorough analysis of the overlap between two copies of the same REM at two temperatures. The distribution of the overlap between two points sampled independently according to Gibbs measures at temperatures β and β' is in our settings

$$\mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(q_t(u, v) \in \cdot).$$

This object originally appeared in the study of the temperature chaos problem, see for example [95] for a survey. For the DGFF, Pain and Zindy [87] showed the convergence of the distribution of the two-temperature overlap and proved that its mean value is strictly smaller than the REM's, when the temperatures are different and below the critical temperature, contrary to the one-temperature case. In this paper, we are interested in the same questions but for the BBM. Even if this model is hierarchical and usually simpler to study, the difficulty here is to deal with the decoration process whose description is less explicit than for the DGFF.

Let us mention that there are two different approaches to tackle the limiting overlap at one temperature. The first one is to prove 1-RSB and then to recover Poisson-Dirichlet statistics thanks to Ghirlanda-Guerra identities, see [6]. This is the method used by Bovier and Kurkova [28] for the BBM, by Arguin and Zindy [7] for the DGFF, and by Jagannath [61] for the BRW with Gaussian increments. A second method is to use the now established convergence of the extremal process of these models towards a *randomly shifted decorated Poisson point process*, see [2] and [5] for the BBM, [15] for the DGFF and [75] for the BRW. It suggests a candidate for the limiting overlap and gives a strong support to prove the convergence. For instance, Mallein [76] proved the same result for the BRW using the convergence of the extremal process obtained by Madaule [75]. However, it is not clear how the first of these approaches could be adapted in the multiple

temperature case.

In this paper, we follow the same approach for the branching Brownian motion that Pain and Zindy [87] used for the DGFF. We show that the two-temperature overlap distribution converges to the one from the limiting process obtained by Bovier & Hartung in [27] and that its expectation is strictly smaller than the REM's when the two temperatures are different.

1.1.2 Model and results

One way to construct a (binary) BBM is to realize it as a process decorating the infinite binary tree $\mathcal{U} = \cup_{n \in \mathbb{Z}_+} \{0, 1\}^n$, with the convention $\{0, 1\}^0 = \{\emptyset\}$. We use 0 and 1 here instead of the classical Ulam-Harris notation because it will be more convenient for the definition of the genealogical embedding function γ to come. For $u \in \mathcal{U}$, let $|u|$ be the length of u and for $k \leq |u|$, let $u_k \in \{0, 1\}$ be the k -th component of u . Write $u \leq v$ if u is an ancestor of v and $u \wedge v$ for the last common ancestor of u and v . For each $u \in \mathcal{U}$, let b_u be the birth-time of u and d_u its death-time and for all $t \geq 0$, let $\mathcal{N}_t := \{u \in \mathcal{U} : b_u \leq t < d_u\}$ be the set of particles alive at time t and $x_u(t)$ the position of particle u at time t . Ikeda et al. [57–59] proved that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the trajectories are Brownian motions and the underlying tree $T = (T_t)_{t \geq 0}$ is a binary continuous time Galton-Watson tree with branching rate 1.

The position of the highest particle $x(t) := \max_{u \in \mathcal{N}_t} x_u(t)$ has been the subject of intense studies since McKean [80] who linked the distribution function of $x(t)$ with the F-KPP partial differential equation. Then, Bramson [32] obtained the right centering term $m_t := \sqrt{2t} - (3/2\sqrt{2}) \log t$ and Lalley and Sellke [73] obtained an integral representation of the limiting law using the limiting *derivative martingale* $Z := \lim_{t \rightarrow \infty} \sum_{u \in \mathcal{N}_t} (\sqrt{2t} - x_u(t)) e^{-\sqrt{2}(\sqrt{2t} - x_u(t))}$. A new step has been taken with the proof of the convergence of the *extremal process*

$$\mathcal{E}_t := \sum_{u \in \mathcal{N}_t} \delta_{\tilde{x}_u(t)}, \quad \text{where } \tilde{x}_u(t) := x_u(t) - m_t,$$

in the space of Radon measures on \mathbb{R} endowed with the vague topology, to a randomly shifted decorated Poisson point process, simultaneously in [2] and [5]. More precisely,

$$\mathcal{E}_t \xrightarrow[t \rightarrow \infty]{(\text{vd})} \sum_{i,j} \delta_{p_i + \Delta_j^i}, \quad (1.1.1)$$

where the $(p_i)_{i \geq 0}$ are the atoms of a Cox process on \mathbb{R} with intensity measure $CZ e^{-\sqrt{2}x} dx$, C a positive constant, Z is the limiting derivative martingale introduced before and $(\Delta^i)_{i \geq 0}$ are i.i.d. point processes on \mathbb{R}_- called *decoration processes*.

Here and after, the set of summation of index like i , j or k is assumed to be \mathbb{Z}_+ unless otherwise specified and one identifies a simple point process with the set of its atoms. We also use, as above, the superscript (vd) for the convergence in distribution of random measures with respect to the vague topology and (wd) with respect to the weak topology, as in the setting of [65, Chapter 4].

Let us recall, from the previous section, the following quantities defined for $\beta > 0$ and $t > 0$

$$Z_{\beta,t} := \sum_{u \in \mathcal{N}_t} e^{\beta x_u(t)}, \quad f_t(\beta) := \frac{1}{t} \mathbb{E}[\log(Z_{\beta,t})], \quad \mathcal{G}_{\beta,t} := \frac{1}{Z_{\beta,t}} \sum_{u \in \mathcal{N}_t} e^{\beta x_u(t)} \delta_u. \quad (1.1.2)$$

There is a phase transition at $\beta_c := \sqrt{2}$ which is related to the asymptotic speed of the extremal particle: we have the following convergence for the (averaged) free energy, see e.g. [45],

$$\lim_{t \rightarrow \infty} f_t(\beta) = f(\beta) := \begin{cases} 1 + (\beta/\beta_c)^2, & \text{if } \beta \leq \beta_c, \\ 2\beta/\beta_c, & \text{if } \beta > \beta_c. \end{cases} \quad (1.1.3)$$

The overlap between particles $u, v \in \mathcal{N}_t$ is defined by

$$q_t(u, v) := \frac{1}{t} d_{u \wedge v} = \frac{1}{t} \mathbb{E}[x_u(t)x_v(t) | T].$$

It is known since [4] that the overlap of extremal particles is either 0 or 1 at the limit $t \rightarrow \infty$. In the following theorem, we establish the convergence of its distribution according to the Gibbs measure at two temperatures. Below the critical temperature, the limiting distribution is the one obtained from the limit of the extremal process in (1.1.1): the overlap is 1 when the same cluster is chosen and 0 otherwise.

Theorem 1.1.1. *Let $\beta, \beta' > 0$.*

(i) *If $\beta \leq \beta_c$ or $\beta' \leq \beta_c$ and $a \in (0, 1)$,*

$$\mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(q_t(u, v) \geq a) \xrightarrow[t \rightarrow \infty]{} 0, \quad \text{in } L^1.$$

(ii) *If $\beta, \beta' > \beta_c$, and $a \in (0, 1)$,*

$$\mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(q_t(u, v) \geq a) \xrightarrow[t \rightarrow \infty]{(d)} Q(\beta, \beta'),$$

where

$$Q(\beta, \beta') := \frac{\sum_i \left(\sum_j e^{\beta(p_i + \Delta_j^i)} \right) \left(\sum_j e^{\beta'(p_i + \Delta_j^i)} \right)}{\left(\sum_{i,j} e^{\beta(p_i + \Delta_j^i)} \right) \left(\sum_{i,j} e^{\beta'(p_i + \Delta_j^i)} \right)}. \quad (1.1.4)$$

The Part (i) of the theorem is a consequence of the convergence of the free energy and a Gaussian integration by parts. The proof of Part (ii) uses the convergence in Equation (1.1.1) and an additional information on the genealogy of the extremal particles, which is what Bovier and Hartung obtained in [27]. The authors define the following function, in a more general setting than ours,

$$\gamma_r(u) := \sum_{\substack{v \leq u: \\ b_v \leq r}} u_{|v|} e^{-b_v}, \quad u \in \mathcal{N}_t, \quad r \leq t.$$

This function encodes the genealogy of the particles on \mathbb{R}_+ in the following way. First, $\gamma_0(\emptyset) = 0$, then each time a particle $u \in \mathcal{N}_t$ with value $\gamma_t(u)$ splits at time t , one of the children keeps the same value $\gamma_t(u)$ and the other one takes the value $\gamma_t(u) + e^{-t}$. If $r \leq t$ and $u \in \mathcal{N}_t$, $\gamma_r(u)$ is simply $\gamma_r(v)$ where $v \in \mathcal{N}_r$ is the ancestor of u alive at time r . This way when two particles originate from a recent split, their images by the function are close. We refer to [27] for more details about the function γ . Bovier and Hartung obtain the following joint convergence of \mathcal{E}_t with $(\gamma_t(u))_{u \in \mathcal{N}_t}$:

Proposition 1.1.2 (Bovier and Hartung [27]). *Let $\tilde{\mathcal{E}}_t := \sum_{u \in \mathcal{N}_t} \delta_{(\gamma_t(u), \tilde{x}_u(t))}$, then $\tilde{\mathcal{E}}_t$ converges in vague distribution to*

$$\tilde{\mathcal{E}} := \sum_{i,j} \delta_{(q_i, p_i) + (0, \Delta_j^i)},$$

where $(q_i, p_i)_i$ are the atoms of a Cox process on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $Z(d\nu) \otimes C e^{-\sqrt{2}x} dx$, $Z(d\nu)$ is a random measure on \mathbb{R}_+ such that $Z(\mathbb{R}_+) = Z$ and C and $(\Delta^i)_i$ were introduced in Equation (1.1.1).

It is then natural to compare the overlap with the one obtained at the limit for the REM, where the positions at time t are given by $|\mathcal{N}_t|$ independent Brownian motions and is studied in [71]. One notices the same effect of the decoration process as in [87] for the DGFF: the expected value of the overlap is strictly smaller than in the REM case. More precisely, let us define, for $\beta, \beta' > \beta_c$,

$$Q^{\text{REM}}(\beta, \beta') := \frac{\sum_i e^{(\beta+\beta')\eta_i}}{\sum_i e^{\beta\eta_i} \sum_i e^{\beta'\eta_i}},$$

where the $(\eta_i)_i$ are the atoms of a $\text{PPP}(e^{-\sqrt{2}x} dx)$.

Theorem 1.1.3. *For any $\beta \neq \beta' > \beta_c$, we have $\mathbb{E}[Q(\beta, \beta')] < \mathbb{E}[Q^{\text{REM}}(\beta, \beta')]$.*

Remark 1.1.4. When $\beta = \beta'$, one has $Q(\beta, \beta) \stackrel{(d)}{=} Q^{\text{REM}}(\beta, \beta)$ and its expected value is $1 - \frac{\beta_c}{\beta}$, see [71].

What we need to prove this result is the precise description of the decoration point process obtained in [2] and some technical estimates. Let us emphasize here that the main difficulty is to handle the description of this point process, which is less explicit than for the DGFF.

1.1.3 Organization of the paper

In Section 1.2, we prove successively Part (i) and Part (ii) of Theorem 1.1.1. Then, in Section 1.3, we give a proof of Theorem 1.1.3. Appendix 1.4.1 contains some technical results that are used in Section 1.2 and Appendix 1.4.2, 1.4.3 contains technical results that are used in Section 1.3.

1.2 Proof of Theorem 1.1.1: convergence of the overlap distribution

1.2.1 Proof of Part (i) of Theorem 1.1.1

The averaged free energy f_t defined in (1.1.2) is a convex function of β and its limit f is differentiable everywhere. By an argument of convexity known as Griffiths' lemma, see for example [100, page 25], the derivative f' is the pointwise limit of f'_t

$$f'(\beta) = \lim_{t \rightarrow \infty} f'_t(\beta) = \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \left[\sum_{u \in \mathcal{N}_t} x_u(t) \frac{e^{\beta x_u(t)}}{\sum_{w \in \mathcal{N}_t} e^{\beta x_w(t)}} \right].$$

One would like to apply a Gaussian integration by parts to the last term to make appear the overlap (see Lemma 1.4.8). In order to deal with a fixed number of Gaussian variables, we use a conditioning on the underlying tree T . Indeed, conditionally on T , $(x_u(t), u \in \mathcal{N}_t)$ is a Gaussian vector with covariances $(d_{u \wedge v})_{u,v \in \mathcal{N}_t}$ and denoting $\mathbb{E}[\cdot | T]$ by \mathbb{E}_T yields

$$\begin{aligned} \mathbb{E}_T \left[\sum_{u \in \mathcal{N}_t} x_u(t) \frac{e^{\beta x_u(t)}}{\sum_{w \in \mathcal{N}_t} e^{\beta x_w(t)}} \right] &= \sum_{u,v \in \mathcal{N}_t} d_{u \wedge v} \mathbb{E}_T \left[\frac{-\beta e^{\beta(x_u(t)+x_v(t))}}{(\sum_{w \in \mathcal{N}_t} e^{\beta x_w(t)})^2} \right] + \sum_{u \in \mathcal{N}_t} t \mathbb{E}_T \left[\frac{\beta e^{\beta x_u(t)}}{\sum_{w \in \mathcal{N}_t} e^{\beta x_w(t)}} \right] \\ &= \beta t \left(1 - \mathbb{E}_T \left[\mathcal{G}_{\beta,t}^{\otimes 2}(q_t(u,v)) \right] \right). \end{aligned}$$

Taking expectation of both sides and using $f'(\beta) = \beta$ for $\beta \leq \beta_c$, see Equation (1.1.3), yields $\mathbb{E}[\mathcal{G}_{\beta,t}^{\otimes 2}(q_t(u,v))] \rightarrow 0$, as $t \rightarrow \infty$. Now assume that $\beta, \beta' > 0$ with $\beta \leq \beta_c$ without loss of generality and take $a \in (0, 1)$, we have

$$\begin{aligned} \mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(q_t(u,v) \geq a) &= \sum_{w \in \mathcal{N}_{at}} \mathcal{G}_{\beta,t}(u \in \mathcal{N}_t : u \geq w) \mathcal{G}_{\beta',t}(v \in \mathcal{N}_t : v \geq w) \\ &\leq \max_{w \in \mathcal{N}_{at}} \mathcal{G}_{\beta,t}(u \in \mathcal{N}_t : u \geq w) \sum_{w \in \mathcal{N}_{at}} \mathcal{G}_{\beta',t}(v \in \mathcal{N}_t : v \geq w) \\ &= \max_{w \in \mathcal{N}_{at}} \mathcal{G}_{\beta,t}(u \in \mathcal{N}_t : u \geq w). \end{aligned}$$

This last term converges to 0 in L^2 since

$$\max_{w \in \mathcal{N}_{at}} \mathcal{G}_{\beta,t}(u \in \mathcal{N}_t : u \geq w)^2 = \max_{w \in \mathcal{N}_{at}} \mathcal{G}_{\beta,t}^{\otimes 2}(u, v \in \mathcal{N}_t : u, v \geq w) \leq \mathcal{G}_{\beta,t}^{\otimes 2}(q_t(u,v) \geq a),$$

concluding the proof.

1.2.2 Convergence of $(\rho_{\beta,t})_t$

In this subsection, we study the convergence of the following random measures on \mathbb{R}_+ defined by

$$\rho_{\beta,t} := \sum_{u \in \mathcal{N}_t} e^{\beta \bar{x}_u(t)} \delta_{\gamma_t(u)}, \quad \text{for } \beta > \beta_c.$$

We prove that $\rho_{\beta,t} \xrightarrow{\text{(vd)}} \rho_\beta$, when $t \rightarrow \infty$, where ρ_β is the corresponding measure in the limiting process, namely

$$\rho_\beta := \sum_{i,j} e^{\beta(p_i + \Delta_j^i)} \delta_{q_i}.$$

Remark 1.2.1. Observing that $Z(d\nu) \otimes C e^{-\sqrt{2}x} dx = Z(d\nu)/Z \otimes e^{-\sqrt{2}(x - \frac{1}{\sqrt{2}} \log(CZ))} dx$, it is easy to see that the point process whose atoms are $\xi_k = p_k - \frac{1}{\sqrt{2}} \log(CZ)$ is a PPP($e^{-\sqrt{2}x} dx$) independent of Z , and that the (q_i) are i.i.d. with distribution $Z(\cdot)/Z(\mathbb{R}_+)$ and independent from (ξ_k) . It yields to the following expression for ρ_β :

$$\rho_\beta = (CZ(\mathbb{R}_+))^{\frac{\beta}{\beta_c}} \sum_k e^{\beta \xi_k} \left(\sum_j e^{\beta \Delta_j^k} \right) \delta_{q_k},$$

This form will be useful in Subsection 1.3.1.

A natural idea to prove the convergence is to use the relation $\rho_{\beta,t}(f) = \tilde{\mathcal{E}}_t(\tilde{f})$, where $f \in C_c^+(\mathbb{R}_+)$ and $\tilde{f}(x, h) := e^{\beta h} f(x)$ together with the convergence of $(\tilde{\mathcal{E}}_t)$, see Proposition 1.1.2. The problem is that \tilde{f} is no more compactly supported and we thus need to control the high and low values of h .

For this purpose, let us denote $\mathcal{N}_t(D) := \{u \in \mathcal{N}_t : \tilde{x}_u(t) \in D\}$, for $D \subset \mathbb{R}$, and define

$$\rho_{\beta,t}^D := \sum_{u \in \mathcal{N}_t(D)} e^{\beta \tilde{x}_u(t)} \delta_{\gamma_t(u)}, \quad \rho_{\beta}^D := \sum_{i,j} \mathbb{1}_D(p_i + \Delta_j^i) e^{\beta(p_i + \Delta_j^i)} \delta_{q_i}. \quad (1.2.1)$$

When $D = [-A, A]$, it is easy to see that $\rho_{\beta,t}^D \xrightarrow{(\text{vd})} \rho_{\beta}^D$ when $t \rightarrow \infty$. Indeed, for $\varepsilon > 0$, choose continuous functions ϕ_{ε} and ψ_{ε} such that $\mathbb{1}_{[-A+\varepsilon, A-\varepsilon]} \leq \phi_{\varepsilon} \leq \mathbb{1}_{[-A, A]} \leq \psi_{\varepsilon} \leq \mathbb{1}_{[-A-\varepsilon, A+\varepsilon]}$. Let $f \in C_c^+(\mathbb{R}_+)$ and define $g_{\varepsilon}(x, h) := f(x) e^{\beta h} \phi_{\varepsilon}(h)$ and $h_{\varepsilon}(x, h) := f(x) e^{\beta h} \psi_{\varepsilon}(h)$, then we have

$$\mathbb{E} \left[\exp(-\tilde{\mathcal{E}}_t(h_{\varepsilon})) \right] \leq \mathbb{E} \left[\exp(-\rho_{\beta,t}^D(f)) \right] \leq \mathbb{E} \left[\exp(-\tilde{\mathcal{E}}_t(g_{\varepsilon})) \right].$$

Since g_{ε} and h_{ε} are compactly supported, Proposition 1.1.2 yields

$$\mathbb{E} \left[\exp(-\tilde{\mathcal{E}}_t(h_{\varepsilon})) \right] \leq \liminf_{t \rightarrow \infty} \mathbb{E} \left[\exp(-\rho_{\beta,t}^D(f)) \right] \leq \limsup_{t \rightarrow \infty} \mathbb{E} \left[\exp(-\rho_{\beta,t}^D(f)) \right] \leq \mathbb{E} \left[\exp(-\tilde{\mathcal{E}}_t(g_{\varepsilon})) \right],$$

and the dominated convergence theorem, when $\varepsilon \rightarrow 0$, gives

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\exp(-\rho_{\beta,t}^D(f)) \right] = \mathbb{E} \left[\exp(-\rho_{\beta}^D(f)) \right],$$

and the convergence of the Laplace functionals concludes.

The end of this subsection shows that this convergence still holds for $D = \mathbb{R}$. It is again sufficient to prove the convergence of the Laplace functionals

$$\mathbb{E} \left[e^{-\rho_{\beta,t}(f)} \right] \xrightarrow{t \rightarrow \infty} \mathbb{E} \left[e^{-\rho_{\beta}(f)} \right], \quad \forall f \in C_c^+(\mathbb{R}_+).$$

Let $D = [-A, A]$ for $A \geq 0$. Since $\rho_{\beta,t}^D(f) \leq \rho_{\beta,t}(f)$, we have $\limsup_{t \rightarrow \infty} \mathbb{E} \left[e^{-\rho_{\beta,t}^D(f)} \right] \leq \mathbb{E} \left[e^{-\rho_{\beta}^D(f)} \right]$ and the dominated convergence theorem, when $A \rightarrow \infty$, yields $\limsup_{t \rightarrow \infty} \mathbb{E} \left[e^{-\rho_{\beta,t}^D(f)} \right] \leq \mathbb{E} \left[e^{-\rho_{\beta}(f)} \right]$.

The proof of $\liminf_{t \rightarrow \infty} \mathbb{E} \left[e^{-\rho_{\beta,t}^D(f)} \right] \geq \mathbb{E} \left[e^{-\rho_{\beta}(f)} \right]$ needs more work. We need to estimate the density of particles outside $D = [-A, A]$. The top values can be controlled by the following inequality, which can be found in the seminal work by Bramson [32, Proposition 3], where $\tilde{x}(t) := \max_{u \in \mathcal{N}_t} \tilde{x}_u(t)$.

Proposition 1.2.2 (Bramson [32]). *There exists $c > 0$ such that*

$$\mathbb{P}(\tilde{x}(t) > A) \leq c(A+1)^2 e^{-\sqrt{2}A}, \quad \text{for } A \geq 0, t \geq A^2 \text{ and } t \geq 2.$$

To address the low values, we use the following proposition whose proof is postponed to the Appendix, see 1.4.1.

Proposition 1.2.3. *Let $\eta > 0$, then $\lim_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}(\rho_{\beta,t}^{[-\infty, -A]}(\mathbb{R}_+) > \eta) = 0$.*

These two propositions enable to prove the lower bound.

Proposition 1.2.4. *Let $f \in C_c^+(\mathbb{R}_+)$, then $\liminf_{t \rightarrow \infty} \mathbb{E}[e^{-\rho_{\beta,t}(f)}] \geq \mathbb{E}[e^{-\rho_{\beta}(f)}]$.*

Proof. Fix $\eta > 0$, we have

$$\mathbb{P}(\rho_{\beta,t}^{[-A, A]^c}(\mathbb{R}_+) > \eta) \leq \mathbb{P}(\rho_{\beta,t}^{[-\infty, -A]}(\mathbb{R}_+) > \eta) + \mathbb{P}(\tilde{x}(t) > A).$$

Applying Proposition 1.2.3 for the first term and Proposition 1.2.2 for the second one yields

$$\lim_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}(\rho_{\beta,t}^{[-A, A]^c}(\mathbb{R}_+) > \eta) = 0. \quad (1.2.2)$$

The fact that f is bounded implies

$$\lim_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}(\rho_{\beta,t}^{[-A, A]^c}(f) > \eta) = 0.$$

Now fix $\varepsilon > 0$ and let A_0, t_0 be large enough such that $\mathbb{P}(\rho_{\beta,t}^{[-A_0, A_0]^c}(f) > \eta) < \varepsilon$ for $t \geq t_0$. Then, with $D = [-A_0, A_0]$, we obtain, for $t \geq t_0$,

$$\mathbb{E}[e^{-\rho_{\beta,t}(f)}] = \mathbb{E}[e^{-\rho_{\beta,t}^D(f)} e^{-\rho_{\beta,t}^{D^c}(f)}] \geq e^{-\eta} \mathbb{E}[e^{-\rho_{\beta,t}^D(f)} \mathbf{1}_{\rho_{\beta,t}^{D^c}(f) \leq \eta}] \geq e^{-\eta} (\mathbb{E}[e^{-\rho_{\beta,t}^D(f)}] - \varepsilon),$$

which implies

$$\liminf_{t \rightarrow \infty} \mathbb{E}[e^{-\rho_{\beta,t}(f)}] \geq e^{-\eta} (\mathbb{E}[e^{-\rho_{\beta}^D(f)}] - \varepsilon) \geq e^{-\eta} (\mathbb{E}[e^{-\rho_{\beta}(f)}] - \varepsilon).$$

The last inequality is true for every $\varepsilon, \eta > 0$ and the conclusion follows. \square

In order to prove Part (ii) of Theorem 1.1.1, we will need a bit more than the convergence for the vague topology.

Proposition 1.2.5. *When $t \rightarrow \infty$, $\rho_{\beta,t}(\mathbb{R}_+) \xrightarrow{(d)} \rho_{\beta}(\mathbb{R}_+)$.*

Proof. We use Laplace transform again and the argument is very similar to the previous one: we just write down the liminf part. Given $\eta, \varepsilon > 0$ and $\lambda > 0$, there exists, by (1.2.2), $A_0, t_0 > 0$ such that $\mathbb{P}(\lambda \rho_{\beta,t}^{[-A, A]^c}(\mathbb{R}_+) > \eta) < \varepsilon$ for every $t \geq t_0$ and $A \geq A_0$. This gives

$$\mathbb{E}[e^{-\lambda \rho_{\beta,t}(\mathbb{R}_+)}] \geq e^{-\eta} (\mathbb{E}[e^{-\lambda \rho_{\beta,t}^{[-A, A]^c}(\mathbb{R}_+)}] - \varepsilon), \quad \text{for } t \geq t_0 \text{ and } A \geq A_0.$$

This time, using the fact that $\rho_{\beta,t}^{[-A, A]}(\mathbb{R}_+) = \mathcal{E}_t(\exp(\beta \cdot) \mathbf{1}_{[-A, A]})$ together with the convergence of the extremal process, see Equation (1.1.1), concludes the proof of Proposition 1.2.5. \square

This last result shows that in fact the convergence in distribution $\rho_{\beta,t} \rightarrow \rho_{\beta}$ holds for the weak topology by [65, Theorem 4.19]. Note that this convergence has been proved by Madaule [75] for the BRW and that the joint convergence for several β is the main step in the proof of the convergence of the extremal process.

Remark 1.2.6. The arguments of Proposition 1.2.5 show that the convergence in distribution of $\mathcal{E}_t(f)$ to $\mathcal{E}(f)$ holds for continuous function on \mathbb{R} that are $\mathcal{O}(\exp(\alpha x))$ as $x \rightarrow -\infty$ for some $\alpha > \sqrt{2}$, and without any restriction on the behaviour at $+\infty$.

Remark 1.2.7. Following Remark 1.2.1 and using the fact that $\sum_k e^{\beta \xi_k} \sum_j e^{\beta \Delta_j^k}$ has the same distribution as $e^{\beta c_\beta} \sum_k e^{\beta \xi_k}$ with $c_\beta := \beta_c^{-1} \log \mathbb{E}[e^{\beta c X_\beta}]$, see Subsection 1.3.1, the previous convergence can be alternatively stated

$$\sum_{u \in \mathcal{N}_t} e^{\beta \tilde{x}_u(t)} \xrightarrow[t \rightarrow \infty]{(d)} Z(\mathbb{R}_+)^{\frac{\beta}{\beta_c}} e^{\beta c_\beta} \sum_k e^{\beta \xi_k},$$

where the serie in the right term has a stable law.

1.2.3 Convergence of $(\rho_{\beta_1, t} \otimes \rho_{\beta_2, t})_t$

Recall that

$$\rho_\beta = \sum_{i,j} e^{\beta(p_i + \Delta_j^i)} \delta_{q_i}, \quad \forall \beta > \beta_c.$$

Proposition 1.2.8. *For $\beta_1, \beta_2 > \beta_c$, we have $\rho_{\beta_1, t} \otimes \rho_{\beta_2, t} \xrightarrow[t \rightarrow \infty]{(wd)} \rho_{\beta_1} \otimes \rho_{\beta_2}$, when $t \rightarrow \infty$.*

Proof. Let us first prove the convergence for the vague topology: it is sufficient to prove that for any $f_1, f_2 \in C_c^+(\mathbb{R}_+)$,

$$\mathbb{E} \left[e^{-\rho_{\beta_1, t}(f_1) - \rho_{\beta_2, t}(f_2)} \right] \longrightarrow \mathbb{E} \left[e^{-\rho_{\beta_1}(f_1) - \rho_{\beta_2}(f_2)} \right], \quad \text{as } t \rightarrow \infty.$$

As in the beginning of Subsection 1.2.2, a direct consequence of the extended convergence of the extremal process is the following convergence, for any $D = [-A, A]$,

$$\rho_{\beta_1, t}^D \otimes \rho_{\beta_2, t}^D \xrightarrow[t \rightarrow \infty]{(vd)} \rho_{\beta_1}^D \otimes \rho_{\beta_2}^D, \quad \text{as } t \rightarrow \infty,$$

where $\rho_{\beta, t}^D$ and ρ_β^D were defined in (1.2.1). From

$$\rho_{\beta_1, t}^D(f_1) + \rho_{\beta_2, t}^D(f_2) \leq \rho_{\beta_1, t}(f_1) + \rho_{\beta_2, t}(f_2),$$

we deduce, as in the proof of Proposition 1.2.4, that

$$\limsup_{t \rightarrow \infty} \mathbb{E} \left[e^{-\rho_{\beta_1, t}(f_1) - \rho_{\beta_2, t}(f_2)} \right] \leq \mathbb{E} \left[e^{-\rho_{\beta_1}(f_1) - \rho_{\beta_2}(f_2)} \right].$$

And it follows from Equation (1.2.2) that

$$\lim_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P} \left(\rho_{\beta_1, t}^{[-A, A]^c}(f_1) + \rho_{\beta_2, t}^{[-A, A]^c}(f_2) > \eta \right) = 0.$$

Fix $\varepsilon > 0$ and let A_0, t_0 be large enough such that $\mathbb{P} \left(\rho_{\beta_1, t}^{[-A_0, A_0]^c}(f_1) + \rho_{\beta_2, t}^{[-A_0, A_0]^c}(f_2) > \eta \right) < \varepsilon$ for $t \geq t_0$. Then, using $D = [-A_0, A_0]$, we obtain, for $t \geq t_0$,

$$\begin{aligned} \mathbb{E} \left[e^{-\rho_{\beta_1, t}(f_1) - \rho_{\beta_2, t}(f_2)} \right] &= \mathbb{E} \left[e^{-\rho_{\beta_1, t}^D(f_1) - \rho_{\beta_2, t}^D(f_2)} e^{-\rho_{\beta_1, t}^{D^c}(f_1) - \rho_{\beta_2, t}^{D^c}(f_2)} \right] \\ &\geq \mathbb{E} \left[e^{-\rho_{\beta_1, t}^D(f_1) - \rho_{\beta_2, t}^D(f_2)} e^{-\eta} \mathbf{1}_{\rho_{\beta_1, t}^{D^c}(f_1) + \rho_{\beta_2, t}^{D^c}(f_2) \leq \eta} \right] \\ &\geq \left(\mathbb{E} \left[e^{-\rho_{\beta_1, t}^D(f_1) - \rho_{\beta_2, t}^D(f_2)} \right] - \varepsilon \right) e^{-\eta}, \end{aligned}$$

and the conclusion follows in the same way as in the proof of Proposition 1.2.4. In order to obtain the convergence for the weak topology, it is sufficient, thanks to [65, Theorem 4.19], to establish

$$\rho_{\beta_1,t}(\mathbb{R}_+)\rho_{\beta_2,t}(\mathbb{R}_+) \xrightarrow[t \rightarrow \infty]{(d)} \rho_{\beta_1}(\mathbb{R}_+)\rho_{\beta_2}(\mathbb{R}_+).$$

For $\lambda, \mu > 0$, the following equality

$$\mathbb{E} \left[e^{-\lambda \rho_{\beta_1,t}(\mathbb{R}_+) - \mu \rho_{\beta_2,t}(\mathbb{R}_+)} \right] = \mathbb{E} \left[e^{-\mathcal{E}_t(\lambda \exp(\beta_1 \cdot) + \mu \exp(\beta_2 \cdot))} \right],$$

with Remark 1.2.6 prove the joint convergence of $(\rho_{\beta_1,t}(\mathbb{R}_+), \rho_{\beta_2,t}(\mathbb{R}_+))$ towards $(\rho_{\beta_1}(\mathbb{R}_+), \rho_{\beta_2}(\mathbb{R}_+))$, concluding the proof. \square

1.2.4 Proof of Part (ii) of Theorem 1.1.1

The first step of the proof is to show that the convergence of $\mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(q_t(u,v) \geq a)$ can be handled with $\mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(|\gamma_t(u) - \gamma_t(v)| \leq \delta)$: this is the content of the following three propositions. Then we will use the convergence of $(\rho_{\beta,t} \otimes \rho_{\beta',t})_t$ from the previous subsection with the expression

$$\mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(|\gamma_t(u) - \gamma_t(v)| \leq \delta) = \frac{\rho_{\beta,t} \otimes \rho_{\beta',t}(\Delta^\delta)}{\rho_{\beta,t} \otimes \rho_{\beta',t}(\mathbb{R}_+^2)}, \quad \text{where } \Delta^\delta := \{(x,y) \in \mathbb{R}_+^2 : |x-y| \leq \delta\}.$$

Recall that $\mathcal{N}_t(D) = \{u \in \mathcal{N}_t : \tilde{x}_u(t) \in D\}$.

Proposition 1.2.9. *Let $\delta > 0$, $a \in (0,1)$ and $D \subset \mathbb{R}$ compact, then*

$$\lim_{t \rightarrow \infty} \mathbb{P}(\exists u, v \in \mathcal{N}_t(D) : q_t(u,v) \geq a \text{ and } |\gamma_t(u) - \gamma_t(v)| > \delta) = 0.$$

Proof. Fix $\varepsilon > 0$. For $r < t$, define, as in [27, Lemma 4.1], the events

$$\mathcal{A}_{r,t}^\gamma(D) = \left\{ \forall u \in \mathcal{N}_t(D) : \gamma_t(u) - \gamma_r(u) \leq e^{-r/2} \right\}.$$

The same lemma gives the existence of $r(D, \varepsilon) \geq 0$ such that $\mathbb{P}(\mathcal{A}_{r,t}^\gamma(D)) \geq 1 - \varepsilon$ whenever $r > r(D, \varepsilon)$ and $t > 3r$. Pick $r > 0$ such that $2e^{-r/2} < \delta$ and $r > r(D, \varepsilon)$. If $t > \max\{3r, \frac{r}{a}\}$, we have, on $\mathcal{A}_{r,t}^\gamma(D)$,

$$\gamma_t(u) - \gamma_r(u) \leq \frac{\delta}{2}, \quad \forall u \in \mathcal{N}_t(D).$$

Then observe that $q_t(u,v) \geq a$ implies that the trajectories of u and v coincide at least up to time $at > r$ so that $|\gamma_t(u) - \gamma_t(v)| \leq \delta$, on $\mathcal{A}_{r,t}^\gamma(D)$. Therefore $\{\exists u, v \in \mathcal{N}_t(D) : q_t(u,v) \geq a \text{ and } |\gamma_t(u) - \gamma_t(v)| > \delta\} \subset \mathcal{A}_{r,t}^\gamma(D)^c$, which has a probability smaller than ε . \square

The following proposition is direct consequence of Theorem 2.1 in [4] and it shows that the choice of $a \in (0,1)$ has no effect, the overlap being concentrated on 0 and 1 at the limit.

Proposition 1.2.10 (Arguin, Bovier and Kistler [4]). *For any compact set $D \subset \mathbb{R}$ and $a \in (0, 1/2)$,*

$$\lim_{t \rightarrow \infty} \mathbb{P}(\exists u, v \in \mathcal{N}_t(D) : q_t(u, v) \in (a, 1 - a)) = 0.$$

A last result is needed.

Proposition 1.2.11. *Let $a \in (0, 1)$ and $D \subset \mathbb{R}$ compact, we have*

$$\lim_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \mathbb{P}(\exists u, v \in \mathcal{N}_t(D) : q_t(u, v) < a \text{ and } |\gamma_t(u) - \gamma_t(v)| \leq \delta) = 0.$$

Proof. Thanks to Proposition 1.2.10 and by monotonicity in a , we can assume that $a = \frac{1}{4}$. Given $\varepsilon > 0$, Lemma 4.3 in [27] provides $\delta > 0$ and $r(\delta, \varepsilon)$ such that for any $r > r(\delta, \varepsilon)$ and $t > 3r$

$$\mathbb{P}(\exists u, v \in \mathcal{N}_t(D) : q_t(u, v) \leq \frac{r}{t} \text{ and } |\gamma_t(u) - \gamma_t(v)| \leq \delta) < \varepsilon.$$

Taking $t = 4r$ concludes the proof. \square

Proof of Theorem 1.1.1. Let us denote $A_{\delta, t} := \{(u, v) \in \mathcal{N}_t^2 : |\gamma_t(u) - \gamma_t(v)| \leq \delta\}$, we have

$$\begin{aligned} & \mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(q_t(u, v) \geq a) - \mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(A_{\delta, t}) \\ &= \mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(q_t(u, v) \geq a; A_{\delta, t}^c) - \mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(q_t(u, v) < a; A_{\delta, t}). \end{aligned}$$

Then, with $D = [-A, A]$, one gets

$$\begin{aligned} & |\mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(q_t(u, v) \geq a) - \mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(A_{\delta, t})| \\ & \leq \mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(q_t(u, v) \geq a; A_{\delta, t}^c \cap \mathcal{N}_t(D)^2) + \mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(q_t(u, v) < a; A_{\delta, t} \cap \mathcal{N}_t(D)^2) \\ & \quad + \mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}((u, v) \notin \mathcal{N}_t(D)^2) \\ & \leq \mathbf{1}_{\{\exists u, v \in \mathcal{N}_t(D) : q_t(u, v) \geq a, |\gamma_t(u) - \gamma_t(v)| > \delta\}} + \mathbf{1}_{\{\exists u, v \in \mathcal{N}_t(D) : q_t(u, v) < a, |\gamma_t(u) - \gamma_t(v)| \leq \delta\}} \\ & \quad + \mathcal{G}_{\beta, t}(\mathcal{N}_t(D)^c) + \mathcal{G}_{\beta', t}(\mathcal{N}_t(D)^c). \end{aligned}$$

We thus have, for every $\eta > 0$,

$$\begin{aligned} & \mathbb{P}(|\mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(q_t(u, v) \geq a) - \mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(A_{\delta, t})| > \eta) \\ & \leq \mathbb{P}(\exists u, v \in \mathcal{N}_t(D) : q_t(u, v) \geq a, |\gamma_t(u) - \gamma_t(v)| > \delta) + \\ & \quad \mathbb{P}(\exists u, v \in \mathcal{N}_t(D) : q_t(u, v) < a, |\gamma_t(u) - \gamma_t(v)| \leq \delta) + \mathbb{P}(\mathcal{G}_{\beta, t}(\mathcal{N}_t(D)^c) \\ & \quad + \mathcal{G}_{\beta', t}(\mathcal{N}_t(D)^c) > \eta/3). \end{aligned}$$

The first term can be handled with Proposition 1.2.9, the second one with Proposition 1.2.11 and the following lemma, whose proof is postponed to Appendix 1.4.1, deals with the third term.

Lemma 1.2.12. *Let $\eta > 0$, then $\lim_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}(\mathcal{G}_{\beta, t}(\mathcal{N}_t([-A, A]^c)) > \eta) = 0$.*

We finally obtain:

$$\lim_{\delta \rightarrow 0} \limsup_{t \rightarrow \infty} \mathbb{P}(|\mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(q_t(u, v) \geq a) - \mathcal{G}_{\beta, t} \otimes \mathcal{G}_{\beta', t}(|\gamma_t(u) - \gamma_t(v)| \leq \delta)| > \eta) = 0. \quad (1.2.3)$$

Now for every $\delta > 0$, choose a continuous function f_δ on \mathbb{R}_+^2 such that $\mathbf{1}_{\Delta^\delta} \leq f_\delta \leq \mathbf{1}_{\Delta^{2\delta}}$. We have

$$\frac{\rho_{\beta,t} \otimes \rho_{\beta',t}(f_{\delta/2})}{\rho_{\beta,t} \otimes \rho_{\beta',t}(\mathbb{R}_+^2)} \leq \mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(|\gamma_t(u) - \gamma_t(v)| \leq \delta) \leq \frac{\rho_{\beta,t} \otimes \rho_{\beta',t}(f_\delta)}{\rho_{\beta,t} \otimes \rho_{\beta',t}(\mathbb{R}_+^2)}.$$

Then, for $\lambda > 0$, the convergence from Proposition (1.2.8) gives

$$\mathbb{E} \left[\exp \left(-\lambda \frac{\rho_\beta \otimes \rho_{\beta'}(f_\delta)}{\rho_\beta \otimes \rho_{\beta'}(\mathbb{R}_+^2)} \right) \right] \leq \liminf_{t \rightarrow \infty} \mathbb{E}[\exp\{-\lambda \mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(|\gamma_t(u) - \gamma_t(v)| \leq \delta)\}],$$

and

$$\limsup_{t \rightarrow \infty} \mathbb{E}[\exp\{-\lambda \mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(|\gamma_t(u) - \gamma_t(v)| \leq \delta)\}] \leq \mathbb{E} \left[\exp \left(-\lambda \frac{\rho_\beta \otimes \rho_{\beta'}(f_{\delta/2})}{\rho_\beta \otimes \rho_{\beta'}(\mathbb{R}_+^2)} \right) \right].$$

The dominated convergence theorem with $\delta \rightarrow 0$ and Equation (1.2.3) yield

$$\lim_{t \rightarrow \infty} \mathbb{E} \left[\exp\{-\lambda \mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(q_t(u, v) \geq a)\} \right] = \mathbb{E} \left[\exp \left(-\lambda \frac{\rho_\beta \otimes \rho_{\beta'}(\Delta)}{\rho_\beta \otimes \rho_{\beta'}(\mathbb{R}_+^2)} \right) \right].$$

Finally, observe that almost surely

$$\frac{\rho_\beta \otimes \rho_{\beta'}(\Delta)}{\rho_\beta \otimes \rho_{\beta'}(\mathbb{R}_+^2)} = Q(\beta, \beta'),$$

since $Z(d\nu)$ is a.s. non-atomic by [27, Proposition 3.2]. This concludes the proof of Part (ii) of Theorem 1.1.1. \square

1.3 Proof of Theorem 1.1.3: the mean overlap is smaller than the REM's

The aim of this section is to prove Theorem 1.1.3. In Subsection 1.3.1, we notice that it is sufficient to prove that a certain functional of the decoration can take arbitrary small values. Then, in Subsection 1.3.2, we use the description of the decoration process obtained in [2] to study the support of this functional.

1.3.1 A functional of the decoration

Let $\mathcal{C} := \sum_{j \geq 0} \delta_{\Delta_j}$ be a point process distributed as the decoration process arising in (1.1.1). We assume that its atoms are ranked in non-increasing order, so that $\Delta_0 = 0$ and $\Delta_j \leq 0$ for all $j \geq 1$. Denote, for $\beta > \beta_c$, $X_\beta := \frac{1}{\beta} \log \sum_{j \geq 0} e^{\beta \Delta_j}$, which is well defined a.s. thanks to the following result.

Lemma 1.3.1. *For every $\beta > \beta_c$, we have $\mathbb{E} \left[\sum_{j \geq 0} e^{\beta \Delta_j} \right] < \infty$.*

Proof. Observe that the following integral representation holds:

$$\sum_{j \geq 0} e^{\beta \Delta_j} = \beta \int_0^\infty \mathcal{C}([-s, 0]) e^{-\beta s} ds.$$

And Fubini-Tonelli theorem gives

$$\mathbb{E} \left[\sum_{j \geq 0} e^{\beta \Delta_j} \right] = \beta \int_0^\infty \mathbb{E} [\mathcal{C}([-s, 0])] e^{-\beta s} ds.$$

Proposition 1.5 in [39] gives the asymptotic $\mathbb{E} [\mathcal{C}([-s, 0])] \sim C_\star e^{\beta c s}$ as $s \rightarrow \infty$, for some $C_\star > 0$, concluding the proof. \square

Recall the expression of the overlap from (1.1.4)

$$Q(\beta, \beta') := \frac{\sum_i \left(\sum_j e^{\beta(p_i + \Delta_j^i)} \right) \left(\sum_j e^{\beta'(p_i + \Delta_j^i)} \right)}{\left(\sum_{i,j} e^{\beta(p_i + \Delta_j^i)} \right) \left(\sum_{i,j} e^{\beta'(p_i + \Delta_j^i)} \right)}, \quad \beta, \beta' > \beta_c. \quad (1.3.1)$$

Introducing $X_{\beta,i} := \frac{1}{\beta} \log \sum_{j \geq 0} e^{\beta \Delta_j^i}$ and $\xi_i = p_i - \frac{1}{\sqrt{2}} \log(CZ)$ which form a PPP($e^{-\sqrt{2}x} dx$), see Remark 1.2.1, the overlap can be rewritten in the following manner:

$$Q(\beta, \beta') = \frac{\sum_i e^{\beta(\xi_i + X_{\beta,i})} e^{\beta'(\xi_i + X_{\beta',i})}}{\left(\sum_i e^{\beta(\xi_i + X_{\beta,i})} \right) \left(\sum_i e^{\beta'(\xi_i + X_{\beta',i})} \right)}.$$

Then, Lemma 2.1 in [89] shows that the point process $(\xi_i + X_{\beta,i}, \xi_i + X_{\beta',i})_i$ has the same distribution as $(\xi_i + c_\beta, \xi_i + c_\beta + Y_i)_i$ where $(Y_i)_{i \geq 1}$ are i.i.d. and independent of $(\xi_i)_{i \geq 1}$, and $c_\beta := \beta_c^{-1} \log \mathbb{E}[e^{\beta c X_\beta}]$, so that

$$Q(\beta, \beta') \stackrel{(d)}{=} \frac{\sum_i e^{\beta \xi_i} e^{\beta'(\xi_i + Y_i)}}{\left(\sum_i e^{\beta \xi_i} \right) \left(\sum_i e^{\beta'(\xi_i + Y_i)} \right)}.$$

In order to prove that $\mathbb{E}[Q(\beta, \beta')] < \mathbb{E}[Q^{\text{REM}}(\beta, \beta')]$ when $\beta \neq \beta'$, we stick to the strategy of Pain and Zindy in [87, Section 3] with the following lemma which shows that the Y_i play a negative role in the expected value of the overlap.

Lemma 1.3.2 (Pain and Zindy [87]). *Let $(p_n)_{n \geq 1}$ and $(q_n)_{n \geq 1}$ be nonincreasing deterministic sequences of nonnegative real numbers such that $\sum_{n \geq 1} p_n = 1$. Let $(A_n)_{n \geq 1}$ be a sequence of i.i.d. positive random variables. We set*

$$\tilde{p}_n := \frac{A_n p_n}{\sum_{k \geq 1} A_k p_k}, \quad \forall n \geq 1.$$

Then, we have

$$\mathbb{E} \left[\sum_{n \geq 1} \tilde{p}_n q_n \right] \leq \sum_{n \geq 1} p_n q_n. \quad (1.3.2)$$

Moreover, if A_1 is not almost surely constant, $(q_n)_{n \geq 1}$ is not constant and, for any $n \geq 1$, $p_n > 0$, then the inequality in (1.3.2) is strict.

Assuming the (ξ_k) are ranked in decreasing order and using the previous lemma, conditionally on (ξ_k) , with

$$p_n := \frac{e^{\beta' \xi_n}}{\sum_{k \geq 1} e^{\beta' \xi_k}}, \quad q_n := \frac{e^{\beta \xi_n}}{\sum_{k \geq 1} e^{\beta \xi_k}}, \quad \text{and} \quad A_n := e^{\beta' Y_n},$$

shows that $\mathbb{E}[Q(\beta, \beta')] < \mathbb{E}[Q^{\text{REM}}(\beta, \beta')]$ on the condition that Y_1 is not almost surely constant.

The final step is to prove that Y_1 is not constant almost surely. Assume that it is not the case, then $X_\beta - X_{\beta'}$ is also constant almost surely and therefore there exists $c > 0$ such that, almost surely

$$\left(\sum_{j \geq 0} e^{\beta \Delta_j} \right)^{\frac{1}{\beta}} = c \left(\sum_{j \geq 0} e^{\beta' \Delta_j} \right)^{\frac{1}{\beta'}}. \quad (1.3.3)$$

If we can prove that $\sum_{j \geq 1} e^{\beta \Delta_j}$ can be arbitrary small with positive probability, the following lemma concludes the proof of Theorem 1.1.3 by showing that (1.3.3) is impossible.

Lemma 1.3.3. *Let $\alpha > 1$, $c > 0$ and $(a_{k,j})_{k,j \geq 1}$ be non-negative real numbers such that $\sum_{j \geq 1} a_{k,j} < \infty$. Assume that, for all $k \geq 1$,*

$$1 + \sum_{j \geq 1} a_{k,j} = c \left(1 + \sum_{j \geq 1} a_{k,j}^\alpha \right)^{\frac{1}{\alpha}} \quad \text{and} \quad \sum_{j \geq 1} a_{k,j} \xrightarrow[k \rightarrow \infty]{} 0.$$

Then $c = 1$ and all the $a_{k,j}$ are equal to 0.

Proof. For a non-negative sequence $x = (x_j)_j$, let us denote $|x|_p = (\sum_{j \geq 1} x_j^p)^{1/p}$ for $p \geq 1$ when the sum is finite. The assumption $|a_k|_1 \rightarrow 0$, when $k \rightarrow \infty$, implies that $|a_k|_\alpha \rightarrow 0$, when $k \rightarrow \infty$. Taking the limit k tends to infinity in the equality yields $c = 1$. The second assertion is a consequence of the fact: $|x|_p = |x|_{p'} \Rightarrow x$ is zero except perhaps on one point. \square

Next section is devoted to the last result we need for the proof of Theorem 1.1.3.

Proposition 1.3.4. *Let $\beta > \beta_c$, then 0 is in the support of the law of $\sum_{j \geq 1} e^{\beta \Delta_j}$.*

1.3.2 Laplace transform along the backward path Y

In order to prove Proposition 1.3.4, we will use the description of the decoration process \mathcal{Q} obtained in [2]. It is obtained with conditioned branching Brownian motions issued from a certain path Y , the *backward path*, which we will explicit now. We refer to the article for more details. The authors adopt a different normalization in their article: each of the particles follows a Brownian motion with drift 2 and variance $\sigma^2 := 2$, which is equivalent to consider $X_u(t) := \sqrt{2}(\sqrt{2}t - x_u(t))$ for $u \in \mathcal{N}_t$ in our settings. And instead of looking at the highest particles, they study the extremal process seen from the lowest or leftmost particle $X(t) := \min_{u \in \mathcal{N}_t} X_u(t) = \sqrt{2}(\sqrt{2}t - x(t))$.

For $b > 0$, define the process $\Gamma^{(b)}$ by

$$\Gamma_s^{(b)} := \begin{cases} B_s, & \text{if } s \in [0, T_b], \\ b - R_{s-T_b}, & \text{if } s \geq T_b, \end{cases}$$

where B is a standard Brownian motion, $T_b := \inf\{t \geq 0 : B_t = b\}$ and R is a three-dimensional Bessel process started from 0 and independent from B . If A is a measurable set of $C(\mathbb{R}_+, \mathbb{R})$ and $b > 0$, we consider Y whose law is given by

$$\mathbb{P}(Y \in A, -\inf_{s \geq 0} Y(s) \in db) = \frac{1}{c_1} \mathbb{E} \left[e^{-2 \int_0^\infty G_v(\sigma \Gamma_v^{(b)}) dv} \mathbf{1}_{-\sigma \Gamma^{(b)} \in A} \right], \quad (1.3.4)$$

where G_t is the distribution function of $X(t)$ and $c_1 := \int_0^\infty \mathbb{E}[e^{-2 \int_0^\infty G_v(\sigma \Gamma_v^{(b)}) dv}] db$, which is finite by [2, Equation 6.7].

Remark 1.3.5. If we denote $Y_t(s) := X_t(t-s) - X(t)$ for $s \in [0, t]$, where $s \mapsto X_t(s)$ is the path followed by the leftmost particule at time t , the law of Y is actually the limit of the law of Y_t as $t \rightarrow \infty$, see [2, Theorem 2.3], but we won't need this fact here.

Conditionally on Y distributed as in (1.3.4), let π be a Poisson point process on $[0, \infty)$ with intensity $2(1 - G_t(-Y(t)))dt$. For each point $t \in \pi$, start an independent branching Brownian motion $(\mathcal{N}_t^*(s), s \geq 0)$ at position $Y(t)$ conditioned to $\min \mathcal{N}_t^*(t) > 0$ (here these BBM are considered as point processes). By [2, Theorem 2.3], we have the following representation for the decoration process

$$\mathcal{Q} := \delta_0 + \sum_{t \in \pi} \mathcal{N}_t^*(t).$$

And \mathcal{Q} is related to the previous section's \mathcal{C} by

$$\mathcal{C} \stackrel{(d)}{=} \sum_{x \in \mathcal{Q}} \delta_{-\frac{1}{\sqrt{2}}x}.$$

We need to explicit a bit more the processes $(\mathcal{N}_t^*, t \in \pi)$. Let $(X_u^y(s), s \geq 0, u \in \mathcal{N}_s^y)_{y \geq 0}$ be a family of independent branching Brownian motion starting at 0 with drift 2 and variance σ^2 . Define $X^y(s) := \min_{u \in \mathcal{N}_s^y} X_u^y(s)$ the position of the leftmost particle at time s . Then, conditionally on Y , introduce, for $t > 0$, the processes $\mathcal{N}_t^*(s) := \sum_{u \in \mathcal{N}_s^t} \delta_{Y(t)+X_u^t(s)}$ where the BBM are conditioned to $X^t(t) + Y(t) > 0$. The functional of Proposition 1.3.4 may now be expressed as

$$\begin{aligned} R_\beta &:= \sum_{x \in \mathcal{C} - \{0\}} e^{\beta x} = \sum_{x \in \mathcal{Q} - \{0\}} e^{-\frac{\beta}{\sqrt{2}}x} \\ &= \sum_{t \in \pi} \sum_{y \in \mathcal{N}_t^*(t)} e^{-\frac{\beta}{\sqrt{2}}y} = \sum_{t \in \pi} e^{-\frac{\beta}{\sqrt{2}}Y(t)} \sum_{u \in \mathcal{N}_t^t} e^{-\frac{\beta}{\sqrt{2}}X_u^t(t)} \\ &= \sum_{t \in \pi} e^{-\frac{\beta}{\sqrt{2}}Y(t)} C_t, \end{aligned}$$

where $C_t := \sum_{u \in \mathcal{N}_t^t} e^{-\frac{\beta}{\sqrt{2}}X_u^t(t)}$.

We want to prove that $0 \in \text{supp}(R_\beta)$. The two main ingredients are a rough estimation of the asymptotic of the paths of Y , see Appendix 1.4.2, and a coupling with a simpler case where the BBMs arising in C_t are not conditioned.

Let us begin with this simpler case: we omit the last conditioning $X^t(t) + Y(t) > 0$ for the BBM, more precisely, let us define

$$S_\beta := \sum_{t \in \pi} e^{-\frac{\beta}{\sqrt{2}}Y(t)} D_t,$$

where $D_t := \sum_{u \in \mathcal{N}_t^t} e^{-\frac{\beta}{\sqrt{2}}X_u^t(t)}$ and the BBM $(X_u^y(s), s \geq 0, u \in \mathcal{N}_s^y)_{y \geq 0}$ are independent (and not conditioned as before) and also independent of Y and π . The following lemma gives a sufficient condition for 0 being in the support of $\sum_{t \in \pi} e^{-\beta/\sqrt{2}Y(t)} D_t$. Its proof is postponed to Appendix 1.4.3.

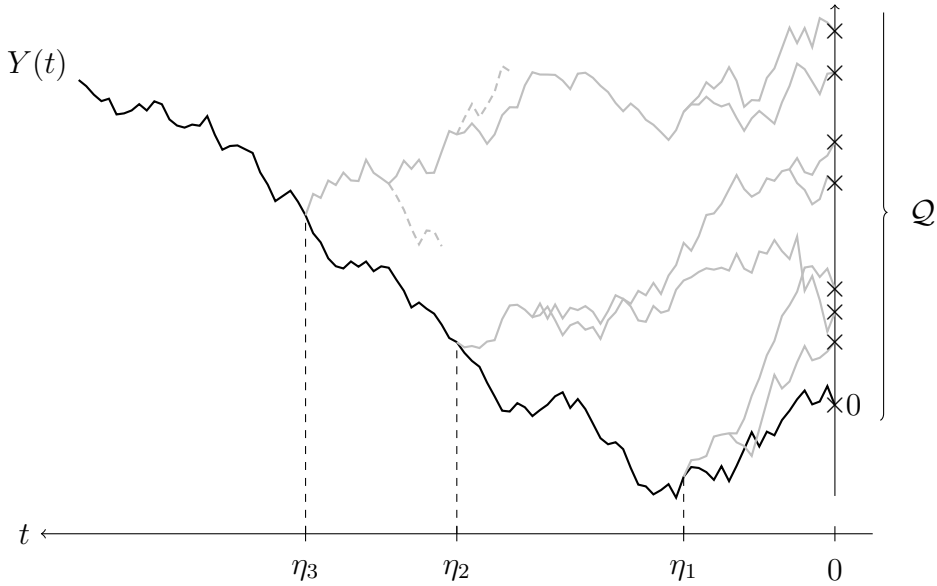


Figure 1.1: The point process \mathcal{Q} obtained with independent BBM issued from the backward path Y at Poissonian times (η_k) and conditioned to stay above 0 at time 0.

Lemma 1.3.6. *Let μ be a Radon measure on \mathbb{R}_+ and \mathcal{P} a PPP(μ). Let $(A_t)_{t \in \mathbb{R}_+}$ be independent positive random variables, independent of \mathcal{P} , and f a positive and measurable function on \mathbb{R}_+ , then*

$$\int_0^\infty \mathbb{E}[(f(t)A_t) \wedge 1] \mu(dt) < \infty \Rightarrow 0 \in \text{supp}\left(\sum_{t \in \mathcal{P}} f(t)A_t\right).$$

The idea is to apply Lemma 1.3.6 conditionally on a given path Y with $A_t = D_t$, $f_Y(t) := e^{-\frac{\beta}{\sqrt{2}}Y(t)}$ and $\mu_Y(dt) := 2(1 - G_t(-Y(t)))dt$: if conditionally on almost every path Y , 0 is in the support of S_β , it is therefore in the support of the unconditional law. Let us prove that the following expectation is integrable w.r.t. dt on \mathbb{R}_+

$$\mathbb{E}[(f_Y(t)D_t) \wedge 1 | Y] = \mathbb{P}\left(D_t > \frac{1}{f_Y(t)} | Y\right) + f_Y(t) \mathbb{E}[D_t \mathbf{1}_{D_t \leq 1/f_Y(t)} | Y]. \quad (1.3.5)$$

It will be more convenient for the computations to come to use the normalization of the previous section with $X_u^y(t) = \sqrt{2}(\sqrt{2}t - x_u^y(t))$. This way, we have $D_t = \sum_{u \in \mathcal{N}_t^t} e^{\beta(x_u^t(t) - \sqrt{2}t)}$ and the superadditivity of the function $x \mapsto x^{\beta/\sqrt{2}}$ gives

$$D_t = \sum_{u \in \mathcal{N}_t^t} e^{\beta(x_u^t(t) - \sqrt{2}t)} \leq \left(\sum_{u \in \mathcal{N}_t^t} e^{\sqrt{2}(x_u^t(t) - \sqrt{2}t)}\right)^{\frac{\beta}{\sqrt{2}}},$$

then

$$\begin{aligned} \mathbb{P}\left(D_t > \frac{1}{f_Y(t)} \mid Y\right) &\leq \mathbb{P}\left(\left(\sum_{u \in \mathcal{N}_t^t} e^{\sqrt{2}(x_u^t(t) - \sqrt{2}t)}\right)^{\frac{\beta}{\sqrt{2}}} > \frac{1}{f_Y(t)} \mid Y\right) \\ &= \mathbb{P}\left(\sum_{u \in \mathcal{N}_t^t} e^{\sqrt{2}(x_u^t(t) - \sqrt{2}t)} > e^{Y(t)} \mid Y\right) \\ &\leq e^{-Y(t)}, \end{aligned}$$

the last inequality being a consequence of Markov's inequality and $\mathbb{E}[\sum e^{\sqrt{2}(x_u^t(t) - \sqrt{2}t)} \mid Y] = 1$. The last term is integrable almost surely by Proposition 1.4.4.

For the second term in (1.3.5), observe that $D_t \leq 1/f_Y(t)$ implies $e^{\beta(x_u^t(t) - \sqrt{2}t)} \leq e^{\frac{\beta}{\sqrt{2}}Y(t)}$ for every $u \in \mathcal{N}_t^t$ and the many-to-one lemma, see [97, Theorem 1.1] for example, yields

$$\begin{aligned} \mathbb{E}[D_t \mathbf{1}_{D_t \leq 1/f_Y(t)} \mid Y] &\leq \mathbb{E}\left[\sum_{u \in \mathcal{N}_t^t} e^{\beta(x_u^t(t) - \sqrt{2}t)} \mathbf{1}_{x_u^t(t) \leq \sqrt{2}t + \frac{1}{\sqrt{2}}Y(t)} \mid Y\right] \\ &= e^t \mathbb{E}\left[e^{\beta(\sqrt{t}G - \sqrt{2}t)} \mathbf{1}_{\sqrt{t}G \leq \sqrt{2}t + \frac{1}{\sqrt{2}}Y(t)} \mid Y\right] \\ &= e^{t - \sqrt{2}\beta t} \mathbb{E}\left[e^{\beta\sqrt{t}G} \mathbf{1}_{G \leq \sqrt{2t} + \frac{1}{\sqrt{2t}}Y(t)} \mid Y\right], \end{aligned}$$

where $G \sim \mathcal{N}(0, 1)$ is independent from Y . The last expectation can be handled with the Gaussian estimate

$$\mathbb{E}\left[e^{\lambda G} \mathbf{1}_{G \leq x}\right] \leq e^{\lambda x - \frac{x^2}{2}}, \quad \forall \lambda > x > 0,$$

whose proof can be found in Appendix 1.4.3. Proposition 1.4.4 ensures that almost surely there exists $t_0 = t_0(Y)$ such that for $t \geq t_0$, $\beta\sqrt{t} > \sqrt{2t} + \frac{1}{\sqrt{2t}}Y(t)$, which implies

$$\mathbb{E}[D_t \mathbf{1}_{D_t \leq 1/f_Y(t)} \mid Y] \leq e^{t - \sqrt{2}\beta t} e^{\beta\sqrt{t}(\sqrt{2t} + \frac{1}{\sqrt{2t}}Y(t)) - \frac{1}{2}(\sqrt{2t} + \frac{1}{\sqrt{2t}}Y(t))^2} \leq e^{(\frac{\beta}{\sqrt{2}} - 1)Y(t)},$$

and thus, for t large enough,

$$f_Y(t) \mathbb{E}[D_t \mathbf{1}_{D_t \leq 1/f_Y(t)} \mid Y] \leq e^{-Y(t)}.$$

We finally conclude that the two terms in (1.3.5) are almost surely integrable w.r.t to dt on \mathbb{R}_+ thanks to Proposition 1.4.4. Therefore, recalling that $\mu_Y(dt) = 2(1 - G_t(-Y(t)))dt$ on \mathbb{R}_+ , we have

$$\int_0^\infty \mathbb{E}[(f_Y(t)D_t) \wedge 1 \mid Y] \mu_Y(dt) \leq \int_0^\infty \mathbb{E}[(f_Y(t)D_t) \wedge 1 \mid Y] 2dt < \infty \quad \text{a.s.} \quad (1.3.6)$$

and Lemma 1.3.6 shows that 0 is in the support of S_β .

Let us go back to the original case with R_β . We now construct a coupling between R_β and S_β . For $t > 0$, denote $D_t^{(1)} := D_t$ and $X_t^{(1)} := X^t(t)$, and let $((D_t^{(i)}, X_t^{(i)}), i \geq 1)$ be an i.i.d. sequence. We then redefine C_t as the first $D_t^{(i)}$ such that $X_t^{(i)} + Y(t) > 0$. The idea of the following is to use the fact that C_t and D_t coincides with large probability for

t large enough in π . Let $\eta_1 \leq \eta_2 \leq \dots$ denote the atoms of π ranked in non-decreasing order and define $T := \sup\{\eta_i : C_{\eta_i} \neq D_{\eta_i}\}$, we have

$$\begin{aligned} & \int_0^\infty \mathbb{E}[(f_Y(t)C_t) \wedge 1 | Y] \mu_Y(dt) \\ & \leq \int_0^\infty \mathbb{E}[(f_Y(t)C_t \mathbf{1}_{t < T}) \wedge 1 | Y] 2dt + \int_0^\infty \mathbb{E}[(f_Y(t)D_t \mathbf{1}_{t \geq T}) \wedge 1 | Y] 2dt \\ & \leq 2 \int_0^\infty \mathbb{P}(T > t | Y) dt + \int_0^\infty \mathbb{E}[(f_Y(t)D_t) \wedge 1 | Y] 2dt, \end{aligned}$$

where the second integral is almost surely finite thanks to Equation (1.3.6). We end this section by proving that the first one is also finite almost surely.

Lemma 1.3.7. $\int_0^\infty \mathbb{P}(T > t | Y) dt$ is finite almost surely.

Proof. $C_t \neq D_t$ occurs when $Y(t) + X^t(t) \leq 0$, we thus need to control the probability of such an event. Using $\tilde{x}^t(t) := \max_{u \in \mathcal{N}_t^t} x_u^t(t) - m_t$, one gets

$$\begin{aligned} \mathbb{P}(Y(t) + X^t(t) \leq 0 | Y) &= \mathbb{P}\left(\tilde{x}^t(t) \geq \frac{1}{\sqrt{2}}Y(t) + \frac{3}{2\sqrt{2}}\log(t) | Y\right) \\ &\leq \mathbb{P}\left(\tilde{x}^t(t) \geq \frac{1}{\sqrt{2}}Y(t) \wedge \sqrt{t} | Y\right), \end{aligned}$$

where \sqrt{t} is here to fulfill the condition $t \geq A^2$ of Proposition 1.2.2. Choose t_0 large enough, depending on Y , such that $Y(t) \geq 0$ for $t \geq t_0$. Then Proposition 1.2.2 implies

$$\mathbb{P}(Y(t) + X^t(t) \leq 0 | Y) \leq c \left(\frac{1}{\sqrt{2}}Y(t) \wedge \sqrt{t} + 1\right)^2 e^{-Y(t) \wedge \sqrt{2t}}, \quad \text{for } t \geq t_0.$$

We thus have, for $c' > 0$ large enough,

$$\mathbb{P}(C_t \neq D_t | Y) \leq c' t e^{-Y(t) \wedge \sqrt{2t}}, \quad \text{for } t \geq t_0.$$

Applying the union bound, with $t > t_0$, gives

$$\begin{aligned} \mathbb{P}(T > t | Y, \pi) &= \mathbb{P}(\exists i \geq 1 : \eta_i > t, C_{\eta_i} \neq D_{\eta_i} | Y, \pi) \leq \sum_{i: \eta_i > t} \mathbb{P}(C_{\eta_i} \neq D_{\eta_i} | Y, \pi) \\ &\leq c' \sum_{i: \eta_i > t} \eta_i e^{-Y(\eta_i) \wedge \sqrt{2\eta_i}}. \end{aligned}$$

Taking expectation with respect to π together with Campbell's theorem, see [67, Section 3.2] for example, yields

$$\mathbb{P}(T > t | Y) \leq c' \int_t^\infty s e^{-Y(s) \wedge \sqrt{2s}} \mu_Y(ds) \leq c' \int_t^\infty s e^{-Y(s) \wedge \sqrt{2s}} 2 ds,$$

and finally

$$\int_{t_0}^\infty \mathbb{P}(T > t | Y) dt \leq c' \int_{t_0}^\infty \left(\int_t^\infty s e^{-Y(s) \wedge \sqrt{2s}} 2 ds \right) dt = 2c' \int_{t_0}^\infty s(s - t_0) e^{-Y(s) \wedge \sqrt{2s}} ds,$$

which is almost surely finite thanks to Proposition 1.4.4. \square

Finally, Lemma 1.3.6 shows that $0 \in \text{supp}(R_\beta)$ and Theorem 1.1.3 is proved.

1.4 Appendix

1.4.1 $\rho_{\beta,t}$ and extremal particles

Let us denote N_t^A the number of particles whose centered positions are above level $-A$ at time t :

$$N_t^A := \mathcal{E}_t([-A, +\infty]) = |\{u \in \mathcal{N}_t : \tilde{x}_u(t) \geq -A\}|, \quad \text{for } A, t \geq 0.$$

We will need the following consequence of [39, Lemma 4.2], where the authors obtained a detailed description of the extreme level sets of the BBM.

Proposition 1.4.1 (Cortines, Hartung and Louidor [39]). *There exists $C > 0$ such that, for all $A \geq 0$, $\mathbb{E}[N_t^A ; \tilde{x}(t) \leq A] \leq C(A+1)^2 e^{\sqrt{2}A}$.*

We can now prove Proposition 1.2.3.

Proposition 1.4.2. *Let $\eta > 0$, then $\lim_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}(\rho_{\beta,t}^{]-\infty, -A]}(\mathbb{R}_+) > \eta) = 0$.*

Proof. Fix $\delta > 0$ such that $\sqrt{2} + \delta < \beta$ and define the event $E_{A,t} := \{\exists n \geq 0 : N_t^{A+n} \geq e^{(\sqrt{2}+\delta)(A+n)}\}$ for $A, t \geq 0$. We can break down the probability in the proposition the following way

$$\begin{aligned} \mathbb{P}(\rho_{\beta,t}^{]-\infty, -A]}(\mathbb{R}_+) > \eta) &\leq \mathbb{P}(\rho_{\beta,t}^{]-\infty, -A]}(\mathbb{R}_+) > \eta, \tilde{x}(t) \leq A) + \mathbb{P}(\tilde{x}(t) > A) \\ &\leq \mathbb{P}(E_{A,t}, \tilde{x}(t) \leq A) + \mathbb{P}(\rho_{\beta,t}^{]-\infty, -A]}(\mathbb{R}_+) > \eta, E_{A,t}^c) \end{aligned} \quad (1.4.1)$$

$$+ c(A+1)^2 e^{-\sqrt{2}A}, \quad (1.4.2)$$

whenever $t \geq A^2$ thanks to Proposition 1.2.2.

Let us begin with the first term in (1.4.1). Proposition 1.4.1 and Markov's inequality yield

$$\mathbb{P}(N_t^A \geq e^{(\sqrt{2}+\delta)A}, \tilde{x}(t) \leq A) \leq C(A+1)^2 e^{-\delta A}, \quad (1.4.3)$$

such that

$$\begin{aligned} \mathbb{P}(E_{A,t}, \tilde{x}(t) \leq A) &\leq \sum_{n \geq 0} \mathbb{P}(N_t^{A+n} \geq e^{(\sqrt{2}+\delta)(A+n)}, \tilde{x}(t) \leq A) \\ &\leq \sum_{n \geq 0} \mathbb{P}(N_t^{A+n} \geq e^{(\sqrt{2}+\delta)(A+n)}, \tilde{x}(t) \leq A+n) \\ &\leq C e^{-\delta A} \sum_{n \geq 0} (A+n+1)^2 e^{-\delta n}, \end{aligned} \quad (1.4.4)$$

where the last term does not depend on t and tends to 0 when $A \rightarrow \infty$.

For the second term in (1.4.1), observe that on the event $E_{A,t}^c$, one has

$$\begin{aligned} \rho_{\beta,t}^{]-\infty, -A]}(\mathbb{R}_+) &\leq \sum_{n \geq 0} N_t^{A+n+1} e^{-\beta(A+n)} \leq \sum_{n \geq 0} e^{(\sqrt{2}+\delta)(A+n+1)} e^{-\beta(A+n)} \\ &\leq \frac{e^{\sqrt{2}+\delta}}{1 - e^{\sqrt{2}+\delta-\beta}} e^{(\sqrt{2}+\delta-\beta)A} \xrightarrow{A \rightarrow \infty} 0. \end{aligned}$$

Therefore, for A large enough, and independent of t , $\mathbb{P}(\rho_{\beta,t}^{]-\infty, -A]}(\mathbb{R}_+) > \eta, E_{A,t}^c)$ vanishes. Combining this fact with (1.4.4) in the previous inequality (1.4.1) concludes the proof. \square

The following lemma shows that the Gibbs measure under $\beta > \beta_c$ is concentrated on the extremal particles.

Lemma 1.4.3. *Let $\eta > 0$, then $\lim_{A \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}(\mathcal{G}_{\beta,t}(\mathcal{N}_t([-A, A]^c)) > \eta) = 0$.*

Proof. We can express $\mathcal{G}_{\beta,t}$ with $\rho_{\beta,t}$ as follows

$$\mathcal{G}_{\beta,t}(\mathcal{N}_t([-A, A]^c)) = \frac{\rho_{\beta,t}^{[-A, A]^c}(\mathbb{R}_+)}{\rho_{\beta,t}(\mathbb{R}_+)}.$$

Recall, from Proposition 1.2.5, that $\rho_{\beta,t}(\mathbb{R}_+) \xrightarrow{(d)} \rho_\beta(\mathbb{R}_+)$, when $t \rightarrow \infty$, where the limit is a positive random variable. If $\varepsilon > 0$, one can therefore find $\delta > 0$ and $t_0 > 0$ such that, for all $t > t_0$, $\mathbb{P}(\rho_{\beta,t}(\mathbb{R}_+) > \delta) > 1 - \varepsilon$. Then $\mathbb{P}(\mathcal{G}_{\beta,t}(\mathcal{N}_t([-A, A]^c)) > \eta) \leq \varepsilon + \mathbb{P}(\rho_{\beta,t}^{[-A, A]^c}(\mathbb{R}_+) > \eta\delta)$. Taking the limsup in t and then the limit in A using (1.2.2) concludes the proof of the lemma. \square

1.4.2 Asymptotics of Y and Bessel Processes

The aim of this section is to prove the following proposition.

Proposition 1.4.4. *Let Y be the backward path introduced in Section 1.3.2 and $\varepsilon > 0$, then*

$$\mathbb{P}(\exists t_0 \in \mathbb{R}_+ : t^{\frac{1}{2}-\varepsilon} < Y(t) < t^{\frac{1}{2}+\varepsilon} \text{ for all } t \geq t_0) = 1.$$

Proof. We begin by proving the aforementioned statement for a three-dimensional Bessel process R started at 0, represented as $R_t := |W_t|$, W being a standard Brownian motion in \mathbb{R}^3 . Let f be an increasing function on $(0, \infty)$, the Dvoretzky-Erdős test, see [84, Theorem 3.22], states that

$$\int_1^\infty f(t)t^{-\frac{3}{2}}dt < \infty \quad \text{if and only if} \quad \liminf_{t \rightarrow \infty} \frac{|W_t|}{f(t)} = \infty \quad \text{a.s.}$$

Considering the function $f(t) := t^{1/2-\varepsilon}$ yields

$$\liminf_{t \rightarrow \infty} \frac{|W_t|}{t^{\frac{1}{2}-\varepsilon}} = \infty \quad \text{a.s.},$$

thus $\mathbb{P}(\exists t_0 > 0 : t^{1/2-\varepsilon} < R_t \text{ for } t \geq t_0) = 1$. The law of the iterated logarithm applied to the components of W , see [46, Theorem 8.5.1] for instance, gives

$$\limsup_{t \rightarrow \infty} \frac{|W_t|}{t^{\frac{1}{2}+\varepsilon}} = 0 \quad \text{a.s.},$$

and therefore

$$\mathbb{P}(\exists t_0 \in \mathbb{R}_+ : t^{\frac{1}{2}-\varepsilon} < R_t < t^{\frac{1}{2}+\varepsilon} \text{ for } t \geq t_0) = 1. \tag{1.4.5}$$

Now recall that $\Gamma^{(b)}$ is defined as follows, for $b > 0$,

$$\Gamma_s^{(b)} = \begin{cases} B_s, & \text{if } s \in [0, T_b], \\ b - R_{s-T_b}, & \text{if } s \geq T_b, \end{cases}$$

where $T_b := \inf\{t \geq 0 : B_t = b\}$ and R is a three-dimensional Bessel process started at 0 and independent from B . Moreover

$$\mathbb{P}(Y \in A, -\inf_{s \geq 0} Y(s) \in db) = \frac{1}{c_1} \mathbb{E} \left[e^{-2 \int_0^\infty G_v(\sigma \Gamma_v^{(b)}) dv} \mathbf{1}_{-\sigma \Gamma^{(b)} \in A} \right],$$

for any measurable set A of $C(\mathbb{R}_+, \mathbb{R})$. Consequently, conditionally on $-\inf_{s \geq 0} Y(s) = b$, the law of Y is absolutely continuous with respect to the law of $-\sigma \Gamma^{(b)}$. And using the fact that $\Gamma^{(b)}(s) = b - R_{s-T_b}$ for $s \geq T_b$, it is not hard to see that if we let $A = \{f \in C(\mathbb{R}_+, \mathbb{R}) : \exists t_0 \in \mathbb{R}_+ \forall t \geq t_0, t^{1/2-\varepsilon} < f(t) < t^{1/2+\varepsilon}\}$, we have $\mathbb{P}(-\sigma \Gamma^{(b)} \in A) = 1$ for every $b > 0$, thanks to (1.4.5). We finally obtain

$$\mathbb{P}(Y \in A) = \int_0^\infty \mathbb{P} \left(Y \in A \mid -\inf_{s \geq 0} Y(s) \in db \right) \mathbb{P} \left(-\inf_{s \geq 0} Y(s) \in db \right) = 1.$$

□

1.4.3 Auxiliary results

Let X be a non-negative random variable and define the log-Laplace transform of X , which is the real number $\phi(\lambda)$ verifying

$$e^{-\phi(\lambda)} = \mathbb{E} \left[e^{-\lambda X} \right], \quad \forall \lambda \geq 0.$$

The following lemma characterizes the fact that $\mathbb{P}(X < \varepsilon) > 0$ for all $\varepsilon > 0$, i.e. that 0 belongs to the support of the law of X denoted $\text{supp}(X)$.

Lemma 1.4.5. *If X is a non-negative random variable then $0 \in \text{supp}(X) \Leftrightarrow \phi(\lambda) = o(\lambda)$ when $\lambda \rightarrow \infty$.*

Proof. Assume that $0 \in \text{supp}(X)$, then for every $\varepsilon > 0$, $\mathbb{E} \left[e^{-\lambda X} \right] \geq \mathbb{P}(X < \varepsilon) e^{-\varepsilon \lambda} > 0$, therefore $\phi(\lambda) \leq \lambda \varepsilon - \log \mathbb{P}(X < \varepsilon)$. And if $0 \notin \text{supp}(X)$, then pick $\varepsilon > 0$ such that $X > \varepsilon$ a.s. so that $\mathbb{E} \left[e^{-\lambda X} \right] \leq e^{-\lambda \varepsilon}$ and $\phi(\lambda) \geq \lambda \varepsilon$. □

Lemma 1.4.6. *Let μ be a Radon measure on \mathbb{R}_+ and \mathcal{P} a PPP(μ). Let $(A_t)_{t \in \mathbb{R}_+}$ be independent positive random variables, independent of \mathcal{P} , and f a positive and measurable function on \mathbb{R}_+ , then*

$$\int_0^\infty \mathbb{E} \left[(f(t) A_t) \wedge 1 \right] \mu(dt) < \infty \Rightarrow 0 \in \text{supp} \left(\sum_{t \in \mathcal{P}} f(t) A_t \right).$$

Proof. Taking expectation with respect to (A_t) first and using Campbell's theorem, see [67, Section 3.2], leads to the Laplace transform

$$\mathbb{E} \left[e^{-\lambda \sum_{t \in \mathcal{P}} f(t) A_t} \right] = e^{-\int_0^\infty \mathbb{E} [1 - e^{-\lambda f(t) A_t}] \mu(dt)}.$$

Let $\phi(\lambda)$ denote the log-Laplace transform of $\sum_{t \in \mathcal{P}} f(t) A_t$ so that

$$\frac{1}{\lambda} \phi(\lambda) = \int_0^\infty \mathbb{E} \left[\frac{1 - e^{-\lambda f(t) A_t}}{\lambda} \right] \mu(dt).$$

The dominated convergence theorem used twice together with the inequality

$$\mathbb{E} \left[\frac{1 - e^{-\lambda f(t)A_t}}{\lambda} \right] \leq \mathbb{E}[(f(t)A_t) \wedge 1], \quad \forall \lambda \geq 1,$$

show that $\phi(\lambda)/\lambda$ tends to 0 when $\lambda \rightarrow \infty$ and Lemma 1.4.5 concludes the proof. \square

Lemma 1.4.7. *Let $G \sim \mathcal{N}(0, 1)$ and $\lambda > x > 0$, then $\mathbb{E} \left[e^{\lambda G} \mathbf{1}_{G \leq x} \right] \leq e^{\lambda x - \frac{x^2}{2}}$.*

Proof. Observe that

$$\mathbb{E} \left[e^{\lambda G} \mathbf{1}_{G \leq x} \right] = \int_{-\infty}^x e^{\lambda u} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}} = e^{\lambda^2/2} \int_{-\infty}^x e^{-\frac{(u-\lambda)^2}{2}} \frac{du}{\sqrt{2\pi}} = e^{\lambda^2/2} \int_{\lambda-x}^{\infty} e^{-\frac{t^2}{2}} \frac{dt}{\sqrt{2\pi}},$$

and the last integral is simply $\mathbb{P}(G > \lambda - x)$ which is bounded by $e^{-\frac{(\lambda-x)^2}{2}}$. \square

The following lemma is called the Gaussian integration by parts, see [100, Equation (A.17)] for a proof.

Lemma 1.4.8. *Let $X = (X_i)_{i \in I}$ be a centered Gaussian vector where I is finite. Then, for any C^1 function $F: \mathbb{R}^I \rightarrow \mathbb{R}$ whose partial derivatives have at most exponential growth, we have for every $i \in I$*

$$\mathbb{E} [X_i F(X)] = \sum_{j \in I} \mathbb{E} [X_i X_j] \mathbb{E} [\partial_j F(X)].$$

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Branching Brownian motion versus Random Energy Model: overlap distribution and temperature susceptibility

Abstract

In comparison with Derrida's REM, we investigate the influence of the so-called decoration processes arising in the limiting extremal processes of numerous log-correlated Gaussian fields. In particular one focuses on two specific quantities from statistical physics in the vicinity of the critical temperature. The first one is the two-temperature overlap, whose behavior is smoothed by the decoration process -unlike the one-temperature overlap which is identical- and the second one is the temperature susceptibility, introduced by Fisher and Huse [47], which is strictly larger in the presence of decorations.

2.1 Introduction and results

In order to shed some light on the mysteries of the Parisi theory for mean field spin glasses, Derrida introduced in the 80's the *random energy models* (REMs) [42], where the Gaussian energy levels are assumed to be independent, and its generalizations, the *generalized random energy models* (GREMs) [43], whose correlations are given by a tree structure of finite depth. One question of central interest in glass theory is to understand the structure of pure states, which translates into the analysis of the extremal process in the language of extreme value theory of stochastic processes.

These two tractable models have been extensively studied and allowed, in particular, to investigate the phenomenon of *replica symmetry breaking*. Indeed the REM seems to be the foremost representative of a universality class: in spin glass terminology one may call this the *1-step replica symmetry breaking* (1-RSB) class, or REM-class. More precisely a spin glass model displays a 1-RSB if there exists some critical $\beta_c > 0$ such

that, asymptotically, the overlap between two points chosen independently according to the Gibbs measure at inverse temperature $\beta > 0$ is concentrated at 0 for $\beta \leq \beta_c$, but is supported by 0 and 1, when $\beta > \beta_c$. This phenomenon is a consequence of the fact that the REM-class undergo what physicists refer to as the *REM-freezing transition*: there is a phase transition for the free energy at some $\beta_c > 0$ meaning that, for $\beta \leq \beta_c$, there is an exponentially large number of configurations, with energy level strictly less than the extremes, contributing to free energy and the Gibbs measure is roughly uniformly distributed among such configurations while, for $\beta > \beta_c$, the relevant configurations are the ones with the largest energies and the free energy becomes dominated by a relatively small set of configurations. Another striking fact characterizing the REM-class is that, for $\beta > \beta_c$, the ordered weights of the pure states under the Gibbs measure at inverse temperature $\beta > 0$ follow asymptotically a *Poisson–Dirichlet distribution* of parameter β_c/β . We refer to Bolthausen [16, 22] and Kistler [68] for surveys on the REMs, GREMs and connections to spin glass theory. Finally, let us mention that, despite the simplicity of the freezing transition, rather sophisticated models are known, or conjectured, to belong to the REM-class, such is the case for the extremes of the Riemann zeta-function along the critical line, see Arguin [3] for a survey.

Natural hierarchical models with an infinite number of levels are the branching Brownian motion (BBM) and the branching random walk (BRW), see e.g. the seminal paper by Derrida and Spohn [45], who introduced directed polymers on trees (a BRW with i.i.d. displacements) as an infinite hierarchical extension of the GREMs for spin glasses. Physicists suggested that Gaussian BRW and BBM should belong to a universality class called *log-correlated Gaussian fields*, or *log-REMs*, which is in some sense a “subclass” of the REM-class. These models are not necessarily hierarchical but admit correlations that decay approximately like the logarithm of the inverse of the distance between index points. We refer to the works by Carpentier and Le Doussal [35], Fyodorov and Bouchaud [50, 51] and Fyodorov & al [52] for connection between log-REMs and spin glass theory. Furthermore, these processes play an essential role in Liouville quantum gravity as well as models of three-dimensional turbulence or finance, see the review on Gaussian Multiplicative Chaos by Rhodes and Vargas [94] for discussions.

Another line of research heavily relies on relations between log-REMs and traveling wave equations of Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) type. Such a relation is exact for a particular instance of log-REM: BBM mentioned above. Therefore BBM was studied over the last 50 years as a subject of interest in its own right, with contributions by McKean [80], Bramson [29, 32], Lalley and Sellke [73], Chauvin and Rouault [37, 38]. From the probabilistic point of view, the full picture was recently obtained by Aïdékon & al [2] and Arguin & al [5] separately, while Cortines & al [39, 40] obtained a third description of the decorations’ distribution. Indeed it is now known that, in the thermodynamic limit, the extremal process tends to a randomly shifted decorated Poisson point process (SDPPP), see [98] for a precise definition. With respect to the Poisson point process which describes the extremes of the (uncorrelated) REMs, the novel ingredient of SDPPP, i.e. mainly the decoration process, describes the internal structure of blocks of extreme values which share a near *ancestor*, and thus are highly correlated. For instance, this confirms the observation made by Bovier and Kurkova [28] that BBM is a particularly interesting example, lying right at the borderline where correlations begin to influence the behavior of the extremes of the process. Finally one can say that BBM is the prototype of hierarchical

log-REMs and therefore will naturally be the model of interest in this paper.

Let us also mention that another important and famous example of log-correlated Gaussian fields is the two-dimensional discrete Gaussian Free Field (GFF), which possesses a complicated (non-hierarchical) structure of extreme values, but it turns out to be possible to compare it with that of the branching random walk. By comparison to analogous results for branching random walks, many deep results have been recently established by Bolthausen, Deuschel and Giacomin [19], Daviaud [41], Bolthausen, Deuschel and Zeitouni [20], Bramson and Zeitouni [31], Bramson, Ding and Zeitouni [30], Biskup and Luidor [14, 15]. We refer to the excellent notes by Biskup [13] for more details.

While from a probabilistic point of view, the difference between BBM and REM is perfectly known, it is not the same when quantities from statistical physics are considered. Indeed it is not clear when the decorations of the log-correlated models are felt at the level of the Gibbs measure. For example, Bonenfant [23] recently proved for the BBM that the distribution of the overlap between two points sampled independently according to Gibbs measures at different temperatures is different than the REM's one, while it is the same for both models if the two points are sampled at the same temperature. This raises questions about the influence of these decorations. Motivated by the recent work of Derrida and Mottishaw [44], one seeks to compare both models by studying carefully the overlap distribution in the neighborhood of the critical temperature. Another quantity of interest is the notion of temperature susceptibility introduced by Fisher and Huse [47] and studied for (an approximation of) the REM by Sales and Bouchaud [96].

2.1.1 Motivations

On one side, the (binary) branching Brownian motion (BBM) is a branching Markov process defined on some general probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as follows. Initially, there is one single particle at the origin. Each particle moves according to a standard Brownian motion, during an exponentially distributed time of parameter $1/2^1$ and then splits into two new particles. These new particles start the same process from their place of birth, behaving independently of the others and the system goes on indefinitely.

Let L_t denote the set of alive particles at time $t \geq 0$ and $h_t(x)$ the position of the particle x at time t . The position of the highest particle has been studied by Bramson [29, 32] and Lalley and Sellke [73], who proved the following convergence in law, for some positive constant $c_d > 0$,

$$\mathbb{P} \left(\max_{x \in L_t} h_t(x) \leq t - \frac{3}{2} \log t + x \right) \xrightarrow{t \rightarrow \infty} \mathbb{E} \left[e^{-c_d Z_\infty e^{-x}} \right], \quad \forall x \in \mathbb{R}, \quad (2.1.1)$$

where

$$D_\infty := \lim_{t \rightarrow +\infty} \sum_{x \in L_t} (t - h_t(x)) e^{h_t(x) - t} > 0, \quad \text{a.s.},$$

is the limiting *derivative martingale*. Notice that the limiting law appearing in Equation (2.1.1) corresponds to a randomly shifted Gumbel distribution. A new step has recently been taken with the proof of the convergence of the extremal process in the space of

¹This choice is made such that the critical inverse temperature introduced later in this article and denoted by β_c equals 1.

Radon measures on \mathbb{R} endowed with the vague topology, to a randomly shifted decorated Poisson point process. More precisely, Aïdékon et al. [2] and Arguin et al. [5] proved simultaneously that

$$\sum_{x \in L_t} \delta_{h_t(x) - t + \frac{3}{2} \log t - \log(c_d D_\infty)} \xrightarrow{(vd)} \sum_{i,j} \delta_{\xi_i + d_{ij}}, \quad t \rightarrow +\infty, \quad (2.1.2)$$

where $\xi = \sum_{i \geq 0} \delta_{\xi_i}$ is a Poisson point process on \mathbb{R} with intensity measure $e^{-x} dx$ independent of $(\sum_{j \geq 0} \delta_{d_{ij}})_{i \geq 0}$, which are i.i.d. copies of a point process \mathcal{D} on $(-\infty, 0]$ which has a.s. an atom at 0. The point process \mathcal{D} is called *decoration process*.

Here and after, the set of summation of index like i, j is assumed to be \mathbb{Z}_+ unless otherwise specified and one identifies a simple point process with the set of its atoms. We also use, as above, the superscript (vd) for the convergence in distribution of random measures with respect to the vague topology.

On the other side, the REM of interest is defined as follows in order to be comparable to the BBM introduced above: it consists of $n_t := \lfloor e^{t/2} \rfloor$ i.i.d. centered Gaussian random variables of variance t , denoted by $(g_t(k); 1 \leq k \leq n_t)$. It is well known that the maximum of the REM satisfies the following convergence in distribution:

$$\mathbb{P} \left(\max_{1 \leq k \leq n_t} g_t(k) \leq t - \frac{1}{2} \log t + x \right) \xrightarrow[t \rightarrow \infty]{} e^{-c_0 e^{-x}}, \quad \forall x \in \mathbb{R}, \quad (2.1.3)$$

for some numerical positive constant $c_0 > 0$. In comparison with the BBM, one observes that the limiting law here is simply a standard Gumbel distribution. Furthermore, the extremal process for the REM satisfies the following convergence in the space of Radon measures on \mathbb{R} endowed with the vague topology:

$$\sum_{1 \leq k \leq n_t} \delta_{g_t(k) - t + \frac{1}{2} \log t - \log c_0} \xrightarrow{(vd)} \sum_i \delta_{\xi_i}, \quad t \rightarrow +\infty, \quad (2.1.4)$$

where again the $(\xi_i)_{i \geq 0}$ are the atoms of a Poisson point process on \mathbb{R} with intensity measure $e^{-x} dx$. We refer to Kistler [68] for a recent survey on the REM.

Looking first at the convergences in Equations (2.1.1) and (2.1.3) allows to compare the BBM and the REM from a probabilistic point of view: both model's maxima share the same first order t but the correlations for the BBM start to affect the second order, namely $-\frac{3}{2} \log t$ for the BBM is smaller than $-\frac{1}{2} \log t$ for the REM. And finally Equations (2.1.2) and (2.1.4) complete the picture by telling that the limiting extremal process for the REM is a standard Poisson point process while the BBM's one is a randomly shifted decorated Poisson point process. The additional ingredient for the BBM is mainly the decoration process, which describes the internal structure of blocks of particles of extreme values sharing a near ancestor and thus highly correlated.

In statistical physics, it is common to consider first the *partition function* $Z_{t,d}(\beta)$ of the model, here the BBM (β stands for the inverse-temperature):

$$Z_{t,d}(\beta) := \sum_{x \in L_t} e^{\beta h_t(x)}, \quad \forall \beta > 0,$$

and the *free energy*

$$f_{t,d}(\beta) := \frac{1}{t} \log Z_{t,d}(\beta), \quad \forall \beta > 0.$$

It is well known, see [45], that the BBM exhibits a phase transition at the level of the free energy and that this latter is the same as for the REM, namely

$$f_d(\beta) := \lim_{t \rightarrow \infty} f_{t,d}(\beta) = \begin{cases} 1 + \beta^2, & \text{if } \beta < \beta_c := 1, \\ 2\beta, & \text{if } \beta \geq \beta_c, \end{cases} \quad \text{a.s. and in } L^1.$$

In particular, the model undergoes *freezing* above β_c in the sense that the quantity $f(\beta)/\beta$ is constant. More importantly, one considers the *Gibbs measure*; this is the random probability measure $\mathcal{G}_{\beta,t,d}$ on L_t defined, at inverse temperature $\beta > 0$, by

$$\mathcal{G}_{\beta,t,d}(x) := \frac{e^{\beta h_t(x)}}{Z_{t,d}(\beta)}, \quad \forall x \in L_t.$$

By design, the Gibbs measure concentrates at low temperature, i.e. when $\beta > \beta_c$, on the high points of the Gaussian field. With spin glasses in mind, one also considers the normalized covariance or *overlap* defined for $x, y \in L_t$ by

$$q_{t,d}(x, y) := \frac{1}{t} \mathbb{E} [h_t(x)h_t(y)] = \frac{1}{t} \sup\{s \geq 0 : x, y \text{ share a common ancestor in } L_s\}.$$

A fundamental object, that records the correlations of high points, is the *distribution function of the overlap* sampled from the Gibbs measure, i.e. $\mathcal{G}_{\beta,t,d} \otimes \mathcal{G}_{\beta',t,d}(q_{t,d}(u, v) \geq a)$ for any $a \in (0, 1)$, where u (respectively v) is sampled according to $\mathcal{G}_{\beta,t,d}$ (respectively $\mathcal{G}_{\beta',t,d}$). Bonnefont [23] recently proved that, if $\beta \leq \beta_c$ or $\beta' \leq \beta_c$, then $\mathcal{G}_{\beta,t,d} \otimes \mathcal{G}_{\beta',t,d}(q_{t,d}(u, v) \geq a)$ tends to 0 in L^1 for all $a \in (0, 1)$, while if $\beta > \beta_c$ and $\beta' > \beta_c$, then, for all $a \in (0, 1)$,

$$\mathcal{G}_{\beta,t,d} \otimes \mathcal{G}_{\beta',t,d}(q_{t,d}(u, v) \geq a) \xrightarrow[t \rightarrow \infty]{(\text{law})} \frac{\sum_i (e^{\beta \xi_i} \sum_j e^{\beta d_{ij}}) (e^{\beta' \xi_i} \sum_j e^{\beta' d_{ij}})}{(\sum_{i,j} e^{\beta \xi_i} e^{\beta d_{ij}}) (\sum_{i,j} e^{\beta' \xi_i} e^{\beta' d_{ij}})} =: Q_d(\beta, \beta'),$$

where the $(\xi_i)_i$ and the $(d_{ij})_{i,j}$ were introduced for the description of the limiting extremal process of the BBM, see Equation (2.1.2). In other words, this result proves the convergence of the pushforward of the measure $\mathcal{G}_{\beta,t,d} \otimes \mathcal{G}_{\beta',t,d}$ on L_t^2 by the function $q_{t,d}$, which is a random measure on $[0, 1]$. The limit is either δ_0 if $\beta \wedge \beta' \leq \beta_c$, or $(1 - Q_d(\beta, \beta'))\delta_0 + Q_d(\beta, \beta')\delta_1$ otherwise. Note that, when $\beta > \beta_c$ and $\beta' > \beta_c$, the random variables $(e^{\beta \xi_i} \sum_j e^{\beta d_{ij}} / \sum_i e^{\beta \xi_i} \sum_j e^{\beta d_{ij}})_{i \geq 1}$ are the asymptotic weights of the clusters under $\mathcal{G}_{\beta,t,d}$, such that $Q_d(\beta, \beta')$ is simply the probability of choosing two points in the same cluster (when they are chosen proportionally to their Gibbs weights with inverse temperature β and β' respectively).

For the REM, the overlap is simply given by

$$q_t(k, \ell) := \frac{1}{t} \mathbb{E} [g_t(k)g_t(\ell)] = \mathbb{1}_{\{k=\ell\}}, \quad \forall 1 \leq k, \ell \leq n_t,$$

and the Gibbs measure at inverse temperature $\beta > 0$ is defined by

$$\mathcal{G}_{\beta,t}(k) := \frac{e^{\beta g_t(k)}}{Z_t(\beta)}, \quad \forall 1 \leq k \leq n_t,$$

where $Z_t(\beta) := \sum_{1 \leq k \leq n_t} e^{\beta g_t(k)}$. Kurkova [71] proved that, if $\beta \leq \beta_c$ or $\beta' \leq \beta_c$, then $\mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(q_t(u, v) \geq a)$ tends to 0 in L^1 for all $a \in (0, 1)$, while if $\beta > \beta_c$ and $\beta' > \beta_c$, then, for all $a \in (0, 1)$,

$$\mathcal{G}_{\beta,t} \otimes \mathcal{G}_{\beta',t}(q_t(u, v) \geq a) \xrightarrow[t \rightarrow \infty]{(d)} \frac{\sum_i e^{\beta \xi_i} e^{\beta' \xi_i}}{(\sum_i e^{\beta \xi_i})(\sum_i e^{\beta' \xi_i})} =: Q(\beta, \beta').$$

It is well known that in the case $\beta' = \beta$, the random variables $Q_d(\beta, \beta)$ and $Q(\beta, \beta)$ have the same distribution (see [87, Equation (1.2)] for more details). Therefore a natural question one may ask is whether $Q_d(\beta, \beta')$ and $Q(\beta, \beta')$ still have the same distribution when $\beta \neq \beta' > \beta_c$. Following the work by Pain and Zindy [87] for the two-dimensional discrete Gaussian free field, Bonnefont [23] proved, for the BBM, that

$$F_d(\beta, \beta') := \mathbb{E}[Q_d(\beta, \beta')] < \mathbb{E}[Q(\beta, \beta')] =: F(\beta, \beta'),$$

meaning that the answer is negative. In this paper, our aim is to study and compare the functions $\beta \mapsto F_d(\beta, \beta')$ and $\beta \mapsto F(\beta, \beta')$, when $\beta' > \beta_c$ is fixed. We will specially focus on their behavior when β tends to β_c^+ .

Before introducing the second quantity of interest let us define, for any $\beta > \beta_c$, the partition functions associated with both limiting extremal processes

$$Z_d(\beta) := \sum_{i,j} e^{\beta(\xi_i + d_{ij})}, \quad Z(\beta) := \sum_i e^{\beta \xi_i}.$$

Then, let us consider the correlation coefficient between the free energies at two temperatures given for each model by

$$C_d(\beta, \beta') := \frac{\text{Cov}(\log Z_d(\beta), \log Z_d(\beta'))}{\sigma(\log Z_d(\beta)) \sigma(\log Z_d(\beta'))}, \quad C(\beta, \beta') := \frac{\text{Cov}(\log Z(\beta), \log Z(\beta'))}{\sigma(\log Z(\beta)) \sigma(\log Z(\beta'))},$$

where $\sigma(X) := \sqrt{\text{Var}(X)}$. The *susceptibility* to temperature changes we are interested in is defined as the coefficient $\kappa_d(\beta)$ for the BBM and $\kappa(\beta)$ for the REM such that

$$\begin{aligned} C_d(\beta, \beta + h) &= 1 - \kappa_d(\beta) h^2, & h \rightarrow 0, \\ C(\beta, \beta + h) &= 1 - \kappa(\beta) h^2, & h \rightarrow 0. \end{aligned}$$

2.1.2 Main results

Our first results concern the behavior of the mean overlap $F(\beta, \beta')$ at two supercritical inverse temperatures β and β' , in the regime where β' is fixed but β approaches the critical inverse temperature. It is not hard to see that the function $\beta \mapsto F(\beta, \beta')$ has an infinite right-derivative at 1. We establish a sharper estimate in the following result.

Theorem 2.1.1 (REM case). *For any $\beta' > \beta_c = 1$, as $\beta \downarrow 1$,*

$$F(\beta, \beta') \sim (\beta - 1) \log \frac{1}{\beta - 1}.$$

We compare this behavior to the one arising for the BBM. Surprisingly, the decorations have the effect of smoothing the function $\beta \mapsto F_d(\beta, \beta')$ close to 1, as shown in the following result. Moreover, this phenomenon is a consequence of some fine properties of the decoration of the BBM and it seems that this decoration is critical for this question in some sense made more precise in Section 2.3.3.

Theorem 2.1.2 (BBM case). *For any $\beta' > \beta_c = 1$, as $\beta \downarrow 1$,*

$$F_d(\beta, \beta') = o\left((\beta - 1) \log \frac{1}{\beta - 1}\right).$$

We now turn to results concerning the temperature susceptibility. We first compute explicitly the temperature susceptibility for the REM in the supercritical phase and study its asymptotic behavior close to the critical temperature, as well as in the low temperature regime.

Theorem 2.1.3 (REM case). *For any $\beta > \beta_c$,*

$$\kappa(\beta) = \frac{1}{2} \left(\frac{1}{\beta^2 - 1} + \frac{6}{\pi^2 \beta^3 (\beta + 1)} \left(\frac{\Gamma''\left(\frac{\beta-1}{\beta}\right)}{\Gamma\left(\frac{\beta-1}{\beta}\right)} - \left(\frac{\Gamma'\left(\frac{\beta-1}{\beta}\right)}{\Gamma\left(\frac{\beta-1}{\beta}\right)} \right)^2 \right) - \frac{\beta^2}{(\beta^2 - 1)^2} \right),$$

and

$$\kappa(\beta) \underset{\beta \downarrow 1}{\sim} \left(\frac{3}{2\pi^2} - \frac{1}{8} \right) \frac{1}{(\beta - 1)^2}, \quad \kappa(\beta) \underset{\beta \rightarrow +\infty}{=} O\left(\frac{1}{\beta^5}\right).$$

Then, we study the temperature susceptibility of the BBM. We first prove that it is strictly larger than the one of the REM: this is actually true in much more generality, see Proposition 2.4.2. We also study the behavior of the temperature susceptibility close to the critical temperature, showing that it diverges at the same speed as for the REM but with a different multiplicative constant. This fact is again specific to the decoration of the BBM and does not hold in general in the decorated case.

Theorem 2.1.4 (BBM case). *For any $\beta > \beta_c$, one has $\kappa_d(\beta) > \kappa(\beta)$. Moreover, we have*

$$\frac{3}{2\pi^2} - \frac{1}{8} < \liminf_{\beta \downarrow 1} (\beta - 1)^2 \kappa_d(\beta) \leq \limsup_{\beta \downarrow 1} (\beta - 1)^2 \kappa_d(\beta) \leq \frac{3}{\pi^2} - \frac{1}{8}.$$

2.1.3 Organization of the paper

Section 2.2 is dedicated to the proofs of preliminary results on the REM and the BBM's decoration process which will be used afterwards. Section 2.3 is dedicated to the proofs of Theorem 2.1.1 and Theorem 2.1.2, while in Section 2.4, one investigates the effect on the correlation between free energies when one temperature becomes critical. Most estimates are relegated to the appendix.

2.2 Preliminaries

2.2.1 Properties of Z

In this section, we will make use of the distribution of the (ξ_k) . Let us denote again the partition function $Z(\beta) := \sum e^{\beta \xi_k}$ where $(\xi_k)_{k \geq 1}$ are the atoms ranked in non-increasing order of a PPP($e^{-x} dx$). It will also be convenient to use $\eta_k := e^{-\xi_k}$ because these are the atoms of a homogeneous Poisson point process on \mathbb{R}_+ and allows us to use the law of large numbers. Hence $Z(\beta)$ can be rewritten

$$Z(\beta) = \sum \frac{1}{\eta_k^\beta}, \quad \forall \beta > 1.$$

The following basic properties are proved in the Appendix 2.2.1 and Appendix 2.2.2.

Proposition 2.2.1. *The family of partitions functions $(Z(\beta))_{\beta > 1}$ satisfy the following properties:*

1. For all $\beta > 1$, $\mathbb{E} [Z(\beta)] = +\infty$.
2. Almost surely, $\beta \mapsto Z(\beta)$ is infinitely differentiable on $(1, \infty)$ and

$$Z(\beta) \sim \frac{1}{\beta - 1}.$$

3. The Laplace transform of $Z(\beta)$ is given by

$$\mathbb{E} [e^{-tZ(\beta)}] = \exp \left\{ -\Gamma \left(1 - \frac{1}{\beta} \right) t^{\frac{1}{\beta}} \right\},$$

in other words, $Z(\beta)$ is a stable random variable, whose law is $S_{\frac{1}{\beta}} \left(\left(\Gamma \left(1 - \frac{1}{\beta} \right) \cos \frac{\pi}{2\beta} \right)^\beta, 1, 0 \right)$.

Proposition 2.2.2. *The family of free energies $(\log Z(\beta))_{\beta > 1}$ satisfy the following properties:*

1. Almost surely and in L^1 , $\frac{1}{\beta} \log Z(\beta) \rightarrow \xi_1$, when $\beta \rightarrow \infty$.
2. For any $\beta > 1$,

$$\mathbb{E} \log Z(\beta) = \beta \log \Gamma \left(1 - \frac{1}{\beta} \right) + \gamma (\beta - 1),$$

$$\text{Var} \log Z(\beta) = \frac{\pi^2}{6} (\beta^2 - 1).$$

2.2.2 The decoration of the BBM

In this section, we recall some results on the law of the decoration point process

$$\mathcal{D} = \sum_{k \geq 0} \delta_{d_k},$$

appearing in the limit of the extremal process of the BBM. As mentioned in the introduction (see Equation (2.1.2)), convergence of this extremal process has been established in [2] and [5], which give two different descriptions of the law of \mathcal{D} . However, we use here a third description proved in [39, Lemma 5.1], as well as several other results shown in this paper and its sequel [40]. Note that the authors work with a BBM with branching rate 1 (instead of 1/2 for us here), but that both processes have the same distribution up to a time-space scaling. In particular, one can check that the decoration process \mathcal{C} appearing in this case (branching rate 1) can be related to \mathcal{D} as follows:

$$\mathcal{C} = \sum_{k \geq 0} \delta_{d_k/\sqrt{2}}.$$

For the sake of clarity, we give below the description of the law of \mathcal{C} (instead of \mathcal{D}) so that we can work exactly in the same setting as [39] when we adapt some of their proofs.

We consider a BBM with a spine defined under a new probability measure $\tilde{\mathbb{P}}$ as follows. It starts with one particle at 0 at time 0 which is part of the spine. Particles along the spine branch at rate 2 and move according to a standard Brownian motion. When they branch into two particles, one of them, chosen uniformly at random, is part of the spine and the other one starts a standard BBM with branching rate 1. We denote by L_t the set of particles alive at time t , by X_t the particle at time t which is part of the spine, and by $h_t(x)$ for $x \in L_t$ the position of particle x at time t .

Moreover, we set

$$m_t := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t,$$

which is the position of the maximum of the BBM (with branching rate 1) at time t up to $O(1)$ fluctuations, and, for $0 < r \leq t$ and $x \in L_t$,

$$\mathcal{C}_{t,r}(x) := \sum_{y \in L_t : d(x,y) < r} \delta_{h_t(y) - h_t(x)},$$

where $d(x, y) := \inf\{s \geq 0 : x \text{ and } y \text{ share a common ancestor in } L_{t-s}\}$. For brevity, we introduce the new probability measure

$$\tilde{\mathbb{P}}_t(\cdot) := \tilde{\mathbb{P}}\left(\cdot \mid h_t(X_t) = \max_{x \in L_t} h_t(x) = m_t\right), \quad (2.2.1)$$

as well as

$$\mathcal{C}_{t,r_t}^* := \mathcal{C}_{t,r_t}(X_t).$$

Then, for any positive function $t \mapsto r_t$ such that both r_t and $t - r_t$ tend to ∞ as $t \rightarrow \infty$, Lemma 5.1 in [39] (with $u = 0$) establishes that, for the vague convergence on the set of Radon measures on \mathbb{R} ,

$$\mathcal{C}_{t,r_t}^* \text{ under } \tilde{\mathbb{P}}_t \xrightarrow[t \rightarrow \infty]{(d)} \mathcal{C} \text{ under } \mathbb{P}. \quad (2.2.2)$$

In their papers [39, 40], Cortines, Hartung and Louidor develop tools to study \mathcal{C}_{t,r_t}^* under $\tilde{\mathbb{P}}_t$ and therefore obtain results on the distribution of \mathcal{C} .

Some of the main results in [39] concern the level sets of the decoration point process itself. Recall that \mathcal{C} is supported on $(-\infty, 0]$ and has a.s. an atom at 0. In [39, Proposition 1.5], they prove the existence of constants $C_*, C > 0$ such that

$$\mathbb{E} [\mathcal{D}([-\sqrt{2}x, 0])] = \mathbb{E} [\mathcal{C}([-x, 0])] \sim C_* e^{\sqrt{2}x}, \quad \text{as } x \rightarrow \infty, \quad (2.2.3)$$

and, for any $x \geq 0$,

$$\mathbb{E} [\mathcal{D}([-\sqrt{2}x, 0])^2] = \mathbb{E} [\mathcal{C}([-x, 0])^2] \leq C(x+1) e^{2\sqrt{2}x}. \quad (2.2.4)$$

Note also that, as a consequence of Equation (2.2.3), there exists $C > 0$ such that, for any $x \geq 0$,

$$\mathbb{E} [\mathcal{D}([-\sqrt{2}x, 0])] = \mathbb{E} [\mathcal{C}([-x, 0])] \leq C e^{\sqrt{2}x}. \quad (2.2.5)$$

From this, we can deduce a first moment estimate and a second moment bound for $S_\beta = \sum_{k \geq 0} e^{\beta d_k}$ stated in the following proposition.

Proposition 2.2.3. *Let $C_* > 0$ be the constant appearing in Equation (2.2.3). As $\beta \downarrow 1$, we have*

$$\mathbb{E} S_\beta \sim \frac{C_*}{\beta - 1}, \quad (2.2.6)$$

and there exists a constant $C > 0$ such that, for any $\beta \in (1, 2]$,

$$\mathbb{E} [S_\beta^2] \leq \frac{C}{(\beta - 1)^3}. \quad (2.2.7)$$

Proof. We write $S_\beta = \int_0^\infty \mathcal{D}([-x, 0]) \beta e^{-\beta x} dx$. Therefore, by Fubini's theorem and Equation (2.2.3),

$$\mathbb{E} S_\beta = \int_0^\infty (C_* e^x + o(e^x)) \beta e^{-\beta x} dx \underset{\beta \downarrow 1}{\sim} \frac{C_*}{\beta - 1}.$$

Similarly, applying Fubini's theorem and then Cauchy-Schwarz inequality together with Equation (2.2.4) yields

$$\begin{aligned} \mathbb{E} [S_\beta^2] &= \int_0^\infty \int_0^\infty \mathbb{E} [\mathcal{D}([-x, 0]) \mathcal{D}([-y, 0])] \beta^2 e^{-\beta(x+y)} dx dy \\ &\leq C \int_0^\infty \int_0^\infty \sqrt{(x+1)(y+1)} e^{x+y} \beta^2 e^{-\beta(x+y)} dx dy \\ &= C \left(\int_0^\infty \sqrt{x+1} e^x \beta e^{-\beta x} dx \right)^2 \leq \frac{C}{(\beta - 1)^3}, \end{aligned}$$

which concludes the proof. \square

Corollary 2.2.4. *Let $C_* > 0$ be the constant appearing in Equation (2.2.3). As $\beta \downarrow 1$, we have*

$$\mathbb{E} [S_\beta^{1/\beta}] \sim \mathbb{E} [S_\beta^{2-1/\beta}] \sim \frac{C_*}{\beta - 1}.$$

Proof. This is a consequence of the following inequalities: for any real random variables $X \geq 1$ and $Y \geq 0$ and any $\varepsilon \in (0, 1)$,

$$\mathbb{E}[Y]^{1+\varepsilon} \mathbb{E}[X]^{-\varepsilon} \leq \mathbb{E}[X^{-\varepsilon}Y] \leq \mathbb{E}[Y] \quad \text{and} \quad \mathbb{E}[Y] \leq \mathbb{E}[X^\varepsilon Y] \leq \mathbb{E}[Y]^{1-\varepsilon} \mathbb{E}[XY]^\varepsilon, \quad (2.2.8)$$

which follows on one side simply from the fact that $X \geq 1$ and on the other side from Hölder's inequality. These inequalities with $X = Y = S_\beta$ and $\varepsilon = 1 - 1/\beta$ together with Proposition 2.2.3 yield the desired result. \square

Moreover, we also need bounds on the first and second moment of the level sets at finite t contained in the following lemma.

Lemma 2.2.5. *There exists $C > 0$ such that for any $v \geq 0$ and $t \geq 1$,*

$$\begin{aligned} \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*([-v, 0]) \right] &\leq C e^{\sqrt{2}v}, \\ \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*([-v, 0])^2 \right] &\leq C(v+1) e^{2\sqrt{2}v}. \end{aligned}$$

Proof. We explain here how these bounds can be deduced from the proof of Lemma 5.2 and Lemma 5.3 in [39], which respectively establish an asymptotic equivalent of the first moment of $\mathcal{C}_{t,r_t}^*([-v, 0])$ as $t \rightarrow \infty$ and then $v \rightarrow \infty$ and an asymptotic upper bound for the second moment.

Both proofs begin by writing, for $k \in \{1, 2\}$,

$$\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*([-v, 0])^k \right] = \frac{\tilde{\mathbb{E}} \left[\mathcal{C}_{t,r_t}^*([-v, 0])^k \mathbb{1}_{\{\max_{x \in L_t} h_t(x) \leq m_t\}} \mid h_t(X_t) = m_t \right]}{\tilde{\mathbb{P}} \left(\max_{x \in L_t} h_t(x) \leq m_t \mid h_t(X_t) = m_t \right)}, \quad (2.2.9)$$

where the denominator satisfies, for some constant $C_1 > 0$, as $t \rightarrow \infty$,

$$\tilde{\mathbb{P}} \left(\max_{x \in L_t} h_t(x) \leq m_t \mid h_t(X_t) = m_t \right) \sim \frac{C_1}{t}, \quad (2.2.10)$$

by Lemmas 3.1 and Lemma 3.4 in [39] (the constant C_1 equals $2f^{(0)}(0)g(0)$ with notation of Lemma 3.4 in [39]). In particular, there exists a constant $c > 0$ such that the probability in Equation (2.2.10) is at most c/t for any $t \geq 1$. Therefore, it remains to prove that there exist $C > 0$ and $t_0 \geq 0$ such that, for any $t \geq t_0$ and $v \geq 0$,

$$\tilde{\mathbb{E}} \left[\mathcal{C}_{t,r_t}^*([-v, 0])^k \mathbb{1}_{\max_{x \in L_t} h_t(x) \leq m_t} \mid h_t(X_t) = m_t \right] \leq C(v+1)^{k-1} e^{k\sqrt{2}v}. \quad (2.2.11)$$

Indeed, for $t \leq t_0$, one can bound the indicator function by 1 and $\mathcal{C}_{t,r_t}^*([-v, 0]) \leq \#L_{t_0}$, which has finite first and second moment. We now focus on proving Equation (2.2.11).

For $k = 2$, the statement of Lemma 5.3 in [39] says that (2.2.11) holds for any $v \geq 0$ and for t large enough depending on v . However, by checking its proof, one can see that the requirement “ t large enough” is used only before Equation (5.44) in order to get $r_t \leq t/2$ (to apply Lemma 5.5), but this is a condition independent of x , such that Equation (2.2.11) is true.

For the case $k = 1$, by applying successively Lemma 5.4 and Lemma 5.5 in [39] (note that $j_{t,v}(s) = j_{t,v}^{\geq 0}(s)$), we get for any t such that $r_t \leq t/2$ and any $x \geq 0$,

$$\begin{aligned} \tilde{\mathbb{E}} \left[\mathcal{C}_{t,r_t}^*([-v, 0]) \mathbb{1}_{\max_{x \in L_t} h_t(x) \leq m_t} \mid h_t(X_t) = m_t \right] &= 2 \int_0^{r_t} j_{t,v}(s) \, ds \\ &\leq \frac{C}{t} (v+1) e^{\sqrt{2}v} \int_0^\infty \frac{e^{-v^2/(16s)} + e^{-v/2}}{(s+1)\sqrt{s}} \, ds. \end{aligned}$$

This last integral is smaller than $C((v+1)^{-1} + e^{-v/2})$ for any $v \geq 0$ (see [39, Equations (5.44)-(5.45)] for details) and so we get Equation (2.2.11). \square

We conclude the section with a bound for crossed moments of level sets.- Its proof is postponed to Section 2.5.3.

Proposition 2.2.6. *There exists $C > 0$ such that, for any $v \geq v' \geq 0$,*

$$\mathbb{E} [\mathcal{C}([-v, 0])\mathcal{C}([-v', 0])] \leq C(v'+1)e^{\sqrt{2}(v+v')}.$$

2.3 Overlap distribution at two temperatures: proofs

2.3.1 A preliminary result for the REM

Proposition 2.3.1 (REM case). *For any $\beta, \beta' > \beta_c$, $F(\beta, \beta')$ can be written*

$$F(\beta, \beta') = \int_0^\infty dx \, \mathbb{E} \frac{1}{(1+x^\beta Z(\beta))(1+x^{\beta'} Z(\beta'))}.$$

Proof. This is an application of Palm formula (see Proposition 2.5.2):

$$\begin{aligned} F(\beta, \beta') &= \mathbb{E} \frac{Z(\beta + \beta')}{Z(\beta)Z(\beta')} \\ &= \sum_k \mathbb{E} \frac{1}{\eta_k^{\beta+\beta'}} \frac{1}{\left(\frac{1}{\eta_k^\beta} + \sum_{j \neq k} \frac{1}{\eta_j^\beta}\right) \left(\frac{1}{\eta_k^{\beta'}} + \sum_{j \neq k} \frac{1}{\eta_j^{\beta'}}\right)} \\ &= \int_0^\infty dx \, \mathbb{E} \frac{1}{(x^{-\beta} + Z(\beta))(x^{-\beta'} + Z(\beta'))} \frac{1}{x^{\beta+\beta'}}, \end{aligned}$$

and the result follows. \square

Remark 2.3.2. An alternative form for F can be derived with equality $\frac{1}{Z(\beta)} = \int_0^\infty e^{-tZ(\beta)} \, dt$, as it is done by Derrida and Mottishaw [44]. Indeed one can write

$$\frac{Z(\beta + \beta')}{Z(\beta)Z(\beta')} = \int_{[0, \infty]^2} dt \, dt' \sum_{k \geq 1} \frac{1}{\eta_k^{\beta+\beta'}} e^{-tZ(\beta) - t'Z(\beta')}.$$

Palm formula with the function

$$F(x, \Pi) := \frac{1}{x^{\beta+\beta'}} \exp \left\{ -\frac{t}{x^\beta} - \frac{t'}{x^{\beta'}} \right\} \exp \left\{ -\sum_{y \in \Pi} \frac{t}{y^\beta} + \frac{t'}{y^{\beta'}} \right\},$$

gives

$$\begin{aligned} \mathbb{E} \sum_{k \geq 1} \frac{1}{\eta_k^{\beta+\beta'}} e^{-tZ(\beta) - t'Z(\beta')} &= \mathbb{E} \sum_{k \geq 1} \frac{1}{\eta_k^{\beta+\beta'}} \exp \left\{ -\frac{t}{\eta_k^\beta} - \frac{t'}{\eta_k^{\beta'}} \right\} \exp \left\{ -\sum_{j \neq k} \frac{t}{\eta_j^\beta} + \frac{t'}{\eta_j^{\beta'}} \right\} \\ &= \int_0^\infty dx \frac{1}{x^{\beta+\beta'}} \exp \left\{ -\frac{t}{x^\beta} - \frac{t'}{x^{\beta'}} \right\} \mathbb{E} e^{-tZ(\beta) - t'Z(\beta')}, \end{aligned}$$

and

$$\begin{aligned} F(\beta, \beta') &= \int_{[0, \infty]^2} dt dt' \int_0^\infty \frac{1}{x^{\beta+\beta'}} \exp \left\{ -\frac{t}{x^\beta} - \frac{t'}{x^{\beta'}} \right\} dx e^{-\int_0^\infty (1 - e^{-(tu^{-\beta} + t'u^{-\beta'})}) du} \\ &= \beta \int_0^\infty dx \frac{\int_0^\infty e^{-y - xy^{\beta'/\beta}} y^{\frac{\beta'-1}{\beta}} dy}{\int_0^\infty (1 - e^{-u - xu^{\beta'/\beta}}) u^{-1 - \frac{1}{\beta}} du}, \end{aligned}$$

where the last line computation is detailed in Proposition 2.5.5.

2.3.2 The REM case

Proof. Using the first expression for $F(\beta, \beta')$ as in Proposition 2.3.1, one can estimate its behavior near the critical temperature. Without loss of generality, one assumes that $1 < \beta < \beta'$, so that $Z(\beta)^{1/\beta} > Z(\beta')^{1/\beta'}$. Then

$$\begin{aligned} F(\beta, \beta') &= \mathbb{E} \int_0^\infty \frac{dx}{(1 + x^\beta Z(\beta))(1 + x^{\beta'} Z(\beta'))} \\ &= \underbrace{\mathbb{E} \int_0^{Z(\beta)^{-\frac{1}{\beta}}} \frac{dx}{(1 + x^\beta Z(\beta))(1 + x^{\beta'} Z(\beta'))}}_{= O(\beta-1)} + \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^\infty \frac{dx}{(1 + x^\beta Z(\beta))(1 + x^{\beta'} Z(\beta'))} \\ &= \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{Z(\beta')^{-\frac{1}{\beta'}}} \frac{dx}{(1 + x^\beta Z(\beta))(1 + x^{\beta'} Z(\beta'))} + \underbrace{\mathbb{E} \int_{Z(\beta')^{-\frac{1}{\beta'}}}^\infty \frac{dx}{(1 + x^\beta Z(\beta))(1 + x^{\beta'} Z(\beta'))}}_{= O(\beta-1)} \\ &\quad + O(\beta - 1). \end{aligned}$$

The first term in the second line is indeed smaller than $\mathbb{E}[Z(\beta)^{-\frac{1}{\beta}}]$ which is a $O(\beta - 1)$ thanks to Equation (2.5.4) and the second term in the third line is smaller than

$\mathbb{E} \frac{1}{Z(\beta)Z(\beta')} \int_{Z(\beta')^{-1/\beta'}}^{\infty} x^{-\beta-\beta'} dx$ which is again smaller than $\mathbb{E}[Z(\beta)^{-\frac{1}{\beta}}]$. Then

$$\begin{aligned} & \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{Z(\beta')^{-\frac{1}{\beta'}}} \frac{dx}{(1+x^\beta Z(\beta))(1+x^{\beta'} Z(\beta'))} \\ &= \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{Z(\beta')^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta Z(\beta)(1+x^{\beta'} Z(\beta'))} - \underbrace{\mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{Z(\beta')^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta Z(\beta)(1+x^\beta Z(\beta))(1+x^{\beta'} Z(\beta'))}}_{=O(\beta-1)}, \end{aligned}$$

since the last term is smaller than $\mathbb{E} \frac{1}{Z(\beta)^2} \int_{Z(\beta)^{-1/\beta}}^{\infty} x^{-2\beta} dx \leq \mathbb{E}[Z(\beta)^{-\frac{1}{\beta}}]$. Moreover,

$$\begin{aligned} & \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{Z(\beta')^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta Z(\beta)(1+x^{\beta'} Z(\beta'))} \\ &= \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{Z(\beta')^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta Z(\beta)} - \underbrace{\mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{Z(\beta')^{-\frac{1}{\beta'}}} \frac{x^{\beta'} Z(\beta') dx}{x^\beta Z(\beta)(1+x^{\beta'} Z(\beta'))}}_{=O(\beta-1)}. \end{aligned}$$

because the last term is smaller than $\mathbb{E} \frac{Z(\beta')}{Z(\beta)} \int_0^{Z(\beta')^{-1/\beta'}} x^{\beta'-\beta} dx$ which is smaller than $\mathbb{E}[Z(\beta)^{-\frac{1}{\beta}}]$. We finally have

$$\begin{aligned} F(\beta, \beta') &= \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{Z(\beta')^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta Z(\beta)} + O(\beta-1) \\ &= \frac{1}{\beta-1} \mathbb{E} \left[\frac{1}{Z(\beta)} \left(Z(\beta)^{\frac{\beta-1}{\beta}} - Z(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] + O(\beta-1) \\ &= \frac{1}{\beta-1} \mathbb{E} \left[\frac{1}{Z(\beta)} \left(Z(\beta)^{\frac{\beta-1}{\beta}} - 1 \right) \right] + \frac{1}{\beta-1} \mathbb{E} \left[\frac{1}{Z(\beta)} \left(1 - Z(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] + O(\beta-1). \end{aligned}$$

The second term is $O(\beta-1)$ since Cauchy-Schwarz inequality and Equation (2.5.3) yield

$$\mathbb{E} \left[\frac{1}{Z(\beta)} \left(1 - Z(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] \leq \underbrace{\mathbb{E} \left[\frac{1}{Z(\beta)^2} \right]}_{\sim_{\beta \downarrow 1} (\beta-1)}^{\frac{1}{2}} \mathbb{E} \left[\left(1 - Z(\beta')^{\frac{\beta-1}{\beta'}} \right)^2 \right]^{\frac{1}{2}},$$

and Equation (2.5.5) implies

$$\begin{aligned} \mathbb{E} \left[\left(1 - Z(\beta')^{\frac{\beta-1}{\beta'}} \right)^2 \right] &= 1 - 2 \mathbb{E} \left[Z(\beta')^{\frac{\beta-1}{\beta'}} \right] + \mathbb{E} \left[Z(\beta')^{2\frac{\beta-1}{\beta'}} \right] \\ &\underset{\beta \downarrow 1}{=} O\left((\beta-1)^2\right). \end{aligned}$$

The first term is explicit thanks to Lemma 2.5.3:

$$\begin{aligned} \mathbb{E} \left[\frac{1}{Z(\beta)^{\frac{1}{\beta}}} - \frac{1}{Z(\beta)} \right] &= \frac{1}{\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(1-\frac{1}{\beta}\right)} - \frac{\Gamma(\beta+1)}{\Gamma\left(1-\frac{1}{\beta}\right)^\beta} \\ &= (\beta-1)^2 \log \frac{1}{\beta-1} + O\left((\beta-1)^2\right). \end{aligned}$$

□

Remark 2.3.3. Here are some comments on this calculation. Recall we start from

$$\begin{aligned} F(\beta, \beta') &= \mathbb{E} \sum_k \frac{1}{\left(\frac{1}{\eta_k^\beta} + \sum_{j \neq k} \frac{1}{\eta_j^\beta}\right) \left(\frac{1}{\eta_k^{\beta'}} + \sum_{j \neq k} \frac{1}{\eta_j^{\beta'}}\right)} \frac{1}{\eta_k^{\beta+\beta'}} \\ &= \int_0^\infty dx \mathbb{E} \frac{1}{(x^{-\beta} + Z(\beta))(x^{-\beta'} + Z(\beta'))} \frac{1}{x^{\beta+\beta'}}. \end{aligned}$$

Restricting ourselves to the overlap created by particles $\eta_k \in I$ means restricting the integral to I . It turns out that in the calculation above we show that the main contribution in the integral is the part

$$\int_{Z(\beta)^{-\frac{1}{\beta}}}^{Z(\beta')^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta Z(\beta)} = \frac{1}{Z(\beta)} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{Z(\beta')^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta}.$$

Moreover, it does not change the first order behavior of this integral to replace the upper endpoint by 1, or even, up to a $1 - \varepsilon$ factor, by $Z(\beta)^{-\frac{\varepsilon}{\beta}}$. This means that the event contributing mainly to $F(\beta, \beta') = \mathbb{E}[Q(\beta, \beta')]$ is the one where

$$\xi_1 \in \left[\frac{\varepsilon}{\beta} \log Z(\beta), \frac{1}{\beta} \log Z(\beta) \right] \simeq \left[\varepsilon \log \frac{1}{\beta-1}, \log \frac{1}{\beta-1} \right].$$

We now discuss how to find heuristically this result. In order to make $Q(\beta, \beta')$ larger, it is interesting to make ξ_1 large, so that it becomes a more likely state for both Gibbs measures. As soon as ξ_1 is large (larger than some large fixed K), it has a weight close to 1 under the Gibbs measure at β' and therefore

$$Q(\beta, \beta') \simeq \frac{e^{\beta \xi_1}}{Z(\beta)}.$$

Note that ξ_1 starts having an order 1 weight under the Gibbs measure at β when $\xi_1 \asymp \log Z(\beta)$. It is not interesting to take ξ_1 much larger than $K \log \frac{1}{\beta-1}$ because then $Q(\beta, \beta')$ stays stuck at 1. Therefore, we restrict ourselves to $\xi_1 \in [K, K \log \frac{1}{\beta-1}]$ and get

$$\mathbb{E} Q(\beta, \beta') \simeq \int_K^{K \log \frac{1}{\beta-1}} (\beta-1) e^{\beta x} \mathbb{P}(\xi_1 \in dx) \simeq \int_K^{K \log \frac{1}{\beta-1}} (\beta-1) e^{\beta-1x} dx \simeq (\beta-1) \log \frac{1}{\beta-1}.$$

Note that, in this integral, we could restrict ourselves to the interval $[\varepsilon \log \frac{1}{\beta-1}, \log \frac{1}{\beta-1}]$ without changing the first order (up to a $1 - \varepsilon$ factor).

Remark 2.3.4. In fact, using the almost sure $Z(\beta) \sim \frac{1}{\beta-1}$, we get

$$Q(\beta, \beta') \sim (\beta-1) \frac{Z(1+\beta')}{Z(\beta')}, \quad \beta \rightarrow 1^+,$$

proving that the function $Q(\cdot, \beta')$ is almost surely right differentiable at 1. There is a simple way to show that however $F(\beta', \cdot)$ has an infinite derivative at $\beta = 1$ by using Fatou's lemma, indeed

$$\liminf_{\beta \downarrow 1} \frac{F(\beta, \beta')}{\beta-1} = \liminf_{\beta \downarrow 1} \mathbb{E} \frac{Q(\beta, \beta')}{\beta-1} \geq \mathbb{E} \liminf_{\beta \downarrow 1} \frac{Q(\beta, \beta')}{\beta-1} = \mathbb{E} \frac{Z(1+\beta')}{Z(\beta')},$$

and it is not hard to see that this last expectation is infinite.

2.3.3 The general decorated case

We study here the influence of the decorations on the behavior of $F_d(\beta, \beta') = \mathbb{E} Q_d(\beta, \beta')$, when $\beta' > \beta_c = 1$ is fixed and $\beta \rightarrow 1^+$. We will see that they can change the leading order. We first work in a more general setting than for the BBM and introduce some quantities needed to state the main result of this section: Proposition 2.3.6. In the remainder of this section, we use the terms h and $\beta - 1$ interchangeably. For any $\gamma \geq -1$, let us define

$$\delta(\gamma) := \limsup_{\beta \rightarrow 1^+} \frac{\log \mathbb{E} [S_\beta^{1+\gamma}]}{\log \mathbb{E} [S_\beta]} - 1. \quad (2.3.1)$$

Observe that the function δ is convex as a limsup of convex functions. Define also

$$\alpha_+ := \limsup_{\beta \rightarrow 1^+} \frac{\mathbb{E} [S_\beta \log S_\beta]}{\mathbb{E} [S_\beta] \log \mathbb{E} [S_\beta]}, \quad \alpha_- := \liminf_{\beta \rightarrow 1^+} \frac{\mathbb{E} [S_\beta \log S_\beta]}{\mathbb{E} [S_\beta] \log \mathbb{E} [S_\beta]},$$

which are related to δ with the following result.

Proposition 2.3.5. *Assume that $\mathbb{E} [S_\beta] > 1$ for β in a neighborhood of 1, then*

$$\delta'(0^-) \leq \alpha_- \leq \alpha_+ \leq \delta'(0^+).$$

Proof. Let us prove the last inequality, the first one being similar. Pick $\gamma > 0$ and remark that

$$\frac{\mathbb{E} [S_\beta \log S_\beta]}{\mathbb{E} [S_\beta] \log \mathbb{E} [S_\beta]} = \frac{1}{\gamma} \frac{\mathbb{E} [S_\beta \log (S_\beta^\gamma)]}{\mathbb{E} [S_\beta] \log \mathbb{E} [S_\beta]},$$

and use Jensen's inequality with $\mathbb{E}_\beta [\cdot] := \frac{\mathbb{E} [S_\beta \cdot]}{\mathbb{E} [S_\beta]}$ to obtain

$$\frac{\mathbb{E} [S_\beta \log (S_\beta^\gamma)]}{\mathbb{E} [S_\beta]} = \mathbb{E}_\beta [\log (S_\beta^\gamma)] \leq \log \mathbb{E}_\beta [S_\beta^\gamma] = \log \frac{\mathbb{E} [S_\beta^{1+\gamma}]}{\mathbb{E} [S_\beta]}.$$

Therefore,

$$\frac{\mathbb{E} [S_\beta \log S_\beta]}{\mathbb{E} [S_\beta] \log \mathbb{E} [S_\beta]} \leq \frac{1}{\gamma} \left(\frac{\log \mathbb{E} [S_\beta^{1+\gamma}]}{\log \mathbb{E} [S_\beta]} - 1 \right).$$

Taking limsup of both sides, we deduce that $\alpha_+ \leq \frac{\delta(\gamma)}{\gamma}$, and the fact that δ is convex together with $\delta(0) = 0$ ends the proof. \square

The general statement for any decoration is the given in the next proposition.

Proposition 2.3.6. *If the following assumptions hold,*

1. $\mathbb{E} S_\beta \sim \frac{C_*}{\beta-1}$, when $\beta \rightarrow 1^+$,

2. For any $\beta' > 1$, $\mathbb{E}[S_\beta \log S_{\beta'}] = O\left(\frac{1}{\beta-1}\right)$, when $\beta \rightarrow 1^+$,
3. $\delta'(0^+) < 2$ or $\delta'(0^+) = 2$ and there exists $\zeta > 0$ such that $\mathbb{E}S_\beta^{1+\zeta} \leq \frac{C}{(\beta-1)^{1+2\zeta}}$ for some $C > 0$,
4. $\alpha_+ = \alpha_- =: \alpha$,

then,

$$F_d(\beta, \beta') = (2 - \alpha)(\beta - 1) \log \frac{1}{\beta - 1} + o\left((\beta - 1) \log \frac{1}{\beta - 1}\right).$$

Remark 2.3.7. The case $\alpha = 2$ corresponds to an extremal case where the behavior of $F_d(\cdot, \beta')$ at 1^+ smoothes out. This is what happens for the BBM as will be seen in Lemma 2.3.8.

Proof. Assume that $\beta < \beta'$. Applying Palm formula gives

$$\begin{aligned} F_d(\beta, \beta') &= \mathbb{E} \frac{\sum e^{\beta \xi_k} S_{\beta,i} e^{\beta' \xi_k} S_{\beta',i}}{\sum e^{\beta \xi_k} S_{\beta,i} \sum e^{\beta' \xi_k} S_{\beta',i}} \\ &= \mathbb{E} \frac{\sum e^{\beta(\xi_k + c_\beta)} e^{\beta'(\xi_k + c_\beta + Y_k)}}{\sum e^{\beta(\xi_k + c_\beta)} \sum e^{\beta'(\xi_k + c_\beta + Y_k)}} \\ &= \mathbb{E} \frac{\sum \frac{R_k}{\eta_k^{\beta+\beta'}}}{\sum \frac{1}{\eta_k} \sum \frac{R_k}{\eta_k^{\beta'}}}, \quad \text{with } R_k := e^{\beta' Y_k}, \\ &= \mathbb{E} \sum \frac{R_k}{\eta_k^{\beta+\beta'}} \frac{1}{\left(\frac{1}{\eta_k} + \sum_{j \neq k} \frac{1}{\eta_j}\right) \left(\frac{R_k}{\eta_k^{\beta'}} + \sum_{j \neq k} \frac{R_j}{\eta_j^{\beta'}}\right)} \\ &= \int_0^1 \mathbb{P}_{R_1}(dz) \mathbb{E} \int_0^\infty \frac{1}{1 + x^\beta Z(\beta)} \frac{1}{1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')} dx, \end{aligned}$$

where $\tilde{Z}(\beta') := \sum \frac{R_k}{\eta_k^{\beta'}}$. In the third line, we used [87, Lemma 3.1] which provides the equality in distribution between the random variables under expectation in the first and second line thanks to a family of i.i.d. (Y_k) whose law is given by

$$\mathbb{E} f(Y_1) = \frac{1}{\mathbb{E}\left[S_\beta^{\frac{1}{\beta}}\right]} \mathbb{E} \left[S_\beta^{\frac{1}{\beta}} f \left(\frac{1}{\beta'} \log S_{\beta'} - \frac{1}{\beta} \log S_\beta \right) \right].$$

(R_k) is therefore a sequence of i.i.d. random variables with values in $(0, 1)$, since $Y_1 \leq 0$ almost surely. If $\eta < 1 < \nu$, Hölder's inequality yields

$$\mathbb{E}[X^\eta] \geq \mathbb{E}[X] \left(\frac{\mathbb{E}[X]}{\mathbb{E}[X^\nu]} \right)^{\frac{1-\eta}{\nu-1}},$$

thus, with $\eta = \frac{1}{\beta}$ and ν such that $\delta(\nu - 1) < \infty$, which exists thanks to Assumption 3, and h small enough,

$$\mathbb{E}\left[S_\beta^{\frac{1}{\beta}}\right] \geq \mathbb{E} S_\beta \left(\frac{\frac{C_*}{2} \frac{1}{\beta-1}}{\frac{1}{(\beta-1)^{2+\delta(\nu-1)}}} \right)^{\frac{\beta-1}{\beta(\nu-1)}}.$$

The parenthesis tends to 1 when $\beta \rightarrow 1^+$ and therefore

$$\mathbb{E} \left[S_\beta^{\frac{1}{\beta}} \right] \sim \mathbb{E} [S_\beta] . \quad (2.3.2)$$

Let us introduce the following function for $z \in [0, 1]$,

$$F_z(\beta, \beta') := \mathbb{E} \int_0^\infty \frac{dx}{(1 + x^\beta Z(\beta)) \left(1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')\right)} ,$$

so that

$$\begin{aligned} F_z(\beta, \beta') &= \mathbb{E} \underbrace{\int_0^{Z(\beta)^{-\frac{1}{\beta}}} \frac{dx}{(1 + x^\beta Z(\beta)) \left(1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')\right)}}_{\leq \mathbb{E} Z(\beta)^{-1/\beta} = O(\beta-1)} + \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^\infty \frac{dx}{(1 + x^\beta Z(\beta)) \left(1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')\right)} \\ &= \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{(\tilde{Z}(\beta')/z)^{-\frac{1}{\beta'}}} \frac{dx}{(1 + x^\beta Z(\beta)) \left(1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')\right)} \\ &\quad + \underbrace{\mathbb{E} \int_{(\tilde{Z}(\beta')/z)^{-\frac{1}{\beta'}}}^\infty \frac{dx}{(1 + x^\beta Z(\beta)) \left(1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')\right)}}_{\leq z^{\frac{1-\beta}{\beta'}} O(\beta-1)} + O(\beta-1) , \end{aligned}$$

since the last integral is smaller than $\mathbb{E} \left[\frac{1}{Z(\beta)^2} \right]^{1/2} \mathbb{E} \left[\tilde{Z}(\beta')^{2(\beta-1)/\beta'} \right]^{1/2} z^{(1-\beta)/\beta'}$, which is as announced thanks to Equations (2.5.3) and (2.5.8). And the first integral is

$$\begin{aligned} &\mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{(\tilde{Z}(\beta')/z)^{-\frac{1}{\beta'}}} \frac{dx}{(1 + x^\beta Z(\beta)) \left(1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')\right)} \\ &= \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{(\tilde{Z}(\beta')/z)^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta Z(\beta) \left(1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')\right)} - \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{(\tilde{Z}(\beta')/z)^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta Z(\beta) (1 + x^\beta Z(\beta)) \left(1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')\right)} , \end{aligned}$$

this last term being smaller than

$$\mathbb{E} \frac{1}{Z(\beta)^2} \int_{Z(\beta)^{-1/\beta}}^\infty x^{-2\beta} dx + \mathbb{E} \frac{1}{Z(\beta) \tilde{Z}(\beta')} \int_{(\tilde{Z}(\beta')/z)^{-\frac{1}{\beta'}}}^\infty \frac{dx}{x^{\beta+\beta'}} ,$$

which is $z^{\frac{1-\beta}{\beta'}} O(\beta-1)$. Pay attention to the fact that the order of the integration bounds matters here, contrary to the undecorated case. Now,

$$\begin{aligned} \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{(\tilde{Z}(\beta')/z)^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta Z(\beta) \left(1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')\right)} &= \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{(\tilde{Z}(\beta')/z)^{-\frac{1}{\beta'}}} \frac{dx}{x^\beta Z(\beta)} \\ &\quad - \mathbb{E} \int_{Z(\beta)^{-\frac{1}{\beta}}}^{(\tilde{Z}(\beta')/z)^{-\frac{1}{\beta'}}} \frac{\frac{x^{\beta'}}{z} \tilde{Z}(\beta') dx}{x^\beta Z(\beta) \left(1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta')\right)} . \end{aligned}$$

The last term in the right hand side is smaller than

$$\frac{1}{z} \mathbb{E} \frac{\tilde{Z}(\beta')}{Z(\beta)} \int_0^{(\tilde{Z}(\beta')/z)^{-1/\beta'}} x^{\beta'-\beta} dx + \mathbb{E} \int_0^{Z(\beta)^{-1/\beta}} \frac{\frac{x^{\beta'}}{z} \tilde{Z}(\beta') dx}{x^\beta Z(\beta) (1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta'))}$$

where the first term is treated in a similar fashion as before. A change of variables in the integral of the second term yields

$$\mathbb{E} \int_0^{Z(\beta)^{-1/\beta}} \frac{\frac{x^{\beta'}}{z} \tilde{Z}(\beta') dx}{x^\beta Z(\beta) (1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta'))} = \mathbb{E} \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \frac{1}{Z(\beta)} z^{\frac{1-\beta}{\beta'}} \int_0^{Z(\beta)^{-1/\beta} \tilde{Z}(\beta')^{\frac{1}{\beta'}} z^{-1/\beta'}} \frac{u^{\beta'} du}{u^\beta (1 + u^{\beta'})},$$

and using the fact that $\int_0^x \frac{u^{\beta'} du}{u^\beta (1 + u^{\beta'})} \leq 2 \log(1 + x) \leq C_\varepsilon x^\varepsilon$, for some $\varepsilon \in (0, 1/2)$ to be determined later, one obtains, for some constants $C_1, C_2 > 0$,

$$\begin{aligned} & \mathbb{E} \int_0^{Z(\beta)^{-1/\beta}} \frac{\frac{x^{\beta'}}{z} \tilde{Z}(\beta') dx}{x^\beta Z(\beta) (1 + \frac{x^{\beta'}}{z} \tilde{Z}(\beta'))} \\ & \leq C_\varepsilon \mathbb{E} \left[\frac{1}{Z(\beta)^{1+\frac{\varepsilon}{\beta}}} \tilde{Z}(\beta')^{\frac{\beta-1+\varepsilon}{\beta'}} \right] z^{\frac{1-\beta-\varepsilon}{\beta'}} \\ & \leq C_\varepsilon \mathbb{E} \left[\frac{1}{Z(\beta)^{3+\frac{3\varepsilon}{\beta}}} \right]^{\frac{1}{3}} \mathbb{E} \left[\tilde{Z}(\beta')^{\frac{3}{2} \frac{\beta-1+\varepsilon}{\beta'}} \right]^{\frac{2}{3}} z^{\frac{1-\beta-\varepsilon}{\beta'}} \\ & \leq C_\varepsilon \underbrace{\mathbb{E} \left[\frac{1}{Z(\beta)^{3+\frac{3\varepsilon}{\beta}}} \right]^{\frac{1}{3}}}_{\sim h^{1+\varepsilon}} \underbrace{\left(\left(\frac{\mathbb{E} [S_{\beta'}^{1/\beta'}]}{\mathbb{E} [S_\beta^{1/\beta}]} \right)^{\frac{3}{2}(\beta-1+\varepsilon)} \mathbb{E} \left[Z_{\beta'}^{\frac{3}{2} \frac{\beta-1+\varepsilon}{\beta'}} \right] \right)^{\frac{2}{3}}}_{\sim C_1 h^\varepsilon} z^{\frac{1-\beta-\varepsilon}{\beta'}} \\ & \leq C_2 h^{1+2\varepsilon} z^{\frac{1-\beta-\varepsilon}{\beta'}}, \end{aligned}$$

where we used Hölder's inequality in the second line. Then,

$$\begin{aligned} F_z(\beta, \beta') &= \mathbb{E} \int_{Z(\beta)^{-1/\beta}}^{(\tilde{Z}(\beta')/z)^{-1/\beta'}} \frac{dx}{x^\beta Z(\beta)} + z^{\frac{1-\beta}{\beta'}} O(\beta - 1) + z^{\frac{1-\beta-\varepsilon}{\beta'}} O((\beta - 1)^{1+2\varepsilon}) \\ &= \frac{1}{\beta - 1} \mathbb{E} \left[\frac{1}{Z(\beta)} \left(Z(\beta)^{\frac{\beta-1}{\beta}} - \left(\frac{\tilde{Z}(\beta')}{z} \right)^{\frac{\beta-1}{\beta'}} \right) \right] \\ & \quad + z^{\frac{1-\beta}{\beta'}} O(\beta - 1) + z^{\frac{1-\beta-\varepsilon}{\beta'}} O((\beta - 1)^{1+2\varepsilon}) \\ &= \frac{1}{\beta - 1} \mathbb{E} \left[\frac{1}{Z(\beta)} \left(Z(\beta)^{\frac{\beta-1}{\beta}} - 1 \right) \right] + \frac{1}{\beta - 1} \mathbb{E} \left[\frac{1}{Z(\beta)} \left(1 - z^{\frac{1-\beta}{\beta'}} \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] \\ & \quad + z^{\frac{1-\beta}{\beta'}} O(\beta - 1) + z^{\frac{1-\beta-\varepsilon}{\beta'}} O((\beta - 1)^{1+2\varepsilon}). \end{aligned}$$

We thus have

$$\begin{aligned} F(\beta, \beta') &= \int_0^1 F_z(\beta, \beta') \mathbb{P}_{R_1}(dz) \\ &= (\beta - 1) \log \frac{1}{\beta - 1} + \frac{1}{\beta - 1} \int_0^1 \mathbb{P}_{R_1}(dz) \mathbb{E} \left[\frac{1}{Z(\beta)} \left(1 - z^{\frac{1-\beta}{\beta'}} \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] \\ &\quad + O(\beta - 1) \int_0^1 \mathbb{P}_{R_1}(dz) z^{\frac{1-\beta}{\beta'}} + O((\beta - 1)^{1+2\varepsilon}) \int_0^1 \mathbb{P}_{R_1}(dz) z^{\frac{1-\beta-\varepsilon}{\beta'}}. \end{aligned}$$

Let us start by the two last terms. The first one verifies

$$1 \leq \int_0^1 z^{\frac{1-\beta}{\beta'}} \mathbb{P}_{R_1}(dz) = \mathbb{E} e^{(1-\beta)Y_1} = \frac{\mathbb{E} \left[S_\beta S_{\beta'}^{\frac{1-\beta}{\beta'}} \right]}{\mathbb{E} \left[S_\beta^{\frac{1}{\beta}} \right]} \leq \frac{\mathbb{E} S_\beta}{\mathbb{E} \left[S_\beta^{\frac{1}{\beta}} \right]} \underset{\beta \downarrow 1}{\sim} 1.$$

The second one is controlled by

$$\int_0^1 z^{\frac{1-\beta-\varepsilon}{\beta'}} \mathbb{P}_{R_1}(dz) = \frac{1}{\mathbb{E} \left[S_\beta^{\frac{1}{\beta}} \right]} \mathbb{E} \left[S_\beta^{1+\frac{\varepsilon}{\beta}} S_{\beta'}^{\frac{1-\beta-\varepsilon}{\beta'}} \right] \leq C_3 h \mathbb{E} \left[S_\beta^{1+\varepsilon} \right]$$

for some constants $C_3 > 0$. Assumption 3 guarantees that $\mathbb{E} \left[S_\beta^{1+\varepsilon} \right] = O\left(\frac{1}{h^{1+2\varepsilon}}\right)$ as soon as ε is small enough, so that

$$\begin{aligned} F(\beta, \beta') &= (\beta - 1) \log \frac{1}{\beta - 1} + \frac{1}{\beta - 1} \int_0^1 \mathbb{P}_{R_1}(dz) \mathbb{E} \left[\frac{1}{Z(\beta)} \left(1 - z^{\frac{1-\beta}{\beta'}} \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] \\ &\quad + O(\beta - 1). \end{aligned} \quad (2.3.3)$$

Let us split the integrand in the previous expression

$$\mathbb{E} \left[\frac{1}{Z(\beta)} \left(1 - z^{\frac{1-\beta}{\beta'}} \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] = \left(1 - z^{\frac{1-\beta}{\beta'}} \right) \mathbb{E} \frac{1}{Z(\beta)} + z^{\frac{1-\beta}{\beta'}} \mathbb{E} \left[\frac{1}{Z(\beta)} \left(1 - \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right) \right].$$

Begin with the second term

$$\mathbb{E} \left[\frac{1}{Z(\beta)} \left(1 - \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] = \mathbb{E} \left[\left(\frac{1}{Z(\beta)} - \mathbb{E} \frac{1}{Z(\beta)} \right) \left(1 - \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] + \mathbb{E} \frac{1}{Z(\beta)} \mathbb{E} \left[1 - \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right],$$

and

$$\left| \mathbb{E} \left[\left(\frac{1}{Z(\beta)} - \mathbb{E} \frac{1}{Z(\beta)} \right) \left(1 - \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] \right| \leq \underbrace{\text{Var} \left(\frac{1}{Z(\beta)} \right)^{\frac{1}{2}}}_{\underset{\beta \downarrow 1}{\sim} (\beta-1)^{3/2}} \underbrace{\mathbb{E} \left[\left(1 - \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right)^2 \right]^{\frac{1}{2}}}_{= o((\beta-1)^{1/2})},$$

where we applied Lemma 2.5.8 and Lemma 2.5.9 to conclude. Thus, using $h = \beta - 1$, one obtains

$$\begin{aligned}
 \mathbb{E} \left[\frac{1}{Z(\beta)} \left(1 - \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] &= \mathbb{E} \frac{1}{Z(\beta)} \mathbb{E} \left[1 - \tilde{Z}(\beta')^{\frac{h}{\beta'}} \right] + o(h^2) \\
 &= \mathbb{E} \frac{1}{Z(\beta)} \left(1 - \left(\frac{\mathbb{E} [S_{\beta'}^{1/\beta'}]}{\mathbb{E} [S_{\beta}^{1/\beta}]} \right)^h \mathbb{E} \left[Z_{\beta'}^{\frac{h}{\beta'}} \right] \right) + o(h^2) \\
 &= (h + o(h)) \left(h \log \frac{1}{h} + O(h) \right) + o(h^2) \\
 &= h^2 \log \frac{1}{h} + o \left(h^2 \log \frac{1}{h} \right).
 \end{aligned}$$

Therefore,

$$\frac{1}{\beta - 1} \int_0^1 \mathbb{P}_{R_1}(dz) z^{\frac{1-\beta}{\beta'}} \mathbb{E} \left[\frac{1}{Z(\beta)} \left(1 - \tilde{Z}(\beta')^{\frac{\beta-1}{\beta'}} \right) \right] \underset{\beta \downarrow 1}{\sim} (\beta - 1) \log \frac{1}{\beta - 1}. \quad (2.3.4)$$

For the first term, we need an extra order for the asymptotic of

$$\int_0^1 z^{\frac{1-\beta}{\beta'}} \mathbb{P}_{R_1}(dz) = \frac{\mathbb{E} \left[S_{\beta} S_{\beta'}^{\frac{1-\beta}{\beta'}} \right]}{\mathbb{E} \left[S_{\beta}^{\frac{1}{\beta}} \right]}. \quad (2.3.5)$$

For the numerator, a similar decomposition as before holds, more precisely

$$\mathbb{E} \left[S_{\beta} S_{\beta'}^{\frac{1-\beta}{\beta'}} \right] = \mathbb{E} \left[(S_{\beta} - \mathbb{E} S_{\beta}) \left(S_{\beta'}^{\frac{1-\beta}{\beta'}} - 1 \right) \right] + \mathbb{E} [S_{\beta}] \mathbb{E} \left[S_{\beta'}^{\frac{1-\beta}{\beta'}} \right],$$

but works only in the case $\mathbb{E}[S_{\beta}^2] = O\left(\frac{1}{(\beta-1)^2}\right)$. We have nevertheless

$$\begin{aligned}
 \mathbb{E} \left[S_{\beta} S_{\beta'}^{-\frac{h}{\beta'}} \right] &= \mathbb{E} \left[S_{\beta} e^{-\frac{h}{\beta'} \log S_{\beta'}} \right] \\
 &\geq \mathbb{E} S_{\beta} - \frac{h}{\beta'} \mathbb{E} [S_{\beta} \log S_{\beta'}] \\
 &= \mathbb{E} S_{\beta} - O(1),
 \end{aligned}$$

thanks to Assumption 2.

Therefore,

$$\mathbb{E} \left[S_{\beta} S_{\beta'}^{-\frac{h}{\beta'}} \right] = \mathbb{E} [S_{\beta}] + O(1).$$

Similarly for the denominator in Equation (2.3.5), write

$$\mathbb{E} \left[S_{\beta}^{\frac{1}{\beta}} \right] = \mathbb{E} \left[S_{\beta} S_{\beta}^{-\frac{h}{\beta}} \right] = \mathbb{E} \left[S_{\beta} \left(1 - \frac{h}{\beta} \log S_{\beta} \right) \right] + \mathbb{E} \left[S_{\beta} \left(S_{\beta}^{-\frac{h}{\beta}} - \left(1 - \frac{h}{\beta} \log S_{\beta} \right) \right) \right].$$

Remark that

$$0 \leq \mathbb{E} \left[S_\beta \left(S_\beta^{-\frac{h}{\beta}} - \left(1 - \frac{h}{\beta} \log S_\beta \right) \right) \right] \leq \mathbb{E} \left[S_\beta \left(\frac{h}{\beta} \log S_\beta \right)^2 \right] \leq h^2 O \left(\mathbb{E} [S_\beta^{1+\epsilon}] \right) = o(1),$$

which implies, thanks to Assumption 4,

$$\begin{aligned} \mathbb{E} \left[S_\beta^{\frac{1}{\beta}} \right] &= \mathbb{E} S_\beta - \frac{h}{\beta} \mathbb{E} [S_\beta \log S_\beta] + o(1) = \mathbb{E} S_\beta \left(1 - \frac{\mathbb{E} [S_\beta \log S_\beta]}{\mathbb{E} S_\beta \log \mathbb{E} [S_\beta]} h \log \mathbb{E} S_\beta \right) + o(1) \\ &= \mathbb{E} S_\beta \left(1 - \alpha h \log \frac{1}{h} + o \left(h \log \frac{1}{h} \right) \right). \end{aligned}$$

We can now conclude that

$$\begin{aligned} \int_0^1 z^{\frac{1-\beta}{\beta'}} \mathbb{P}_{R_1}(dz) &= \frac{\mathbb{E} \left[S_\beta S_{\beta'}^{\frac{1-\beta}{\beta'}} \right]}{\mathbb{E} \left[S_\beta^{\frac{1}{\beta}} \right]} = \frac{\mathbb{E} S_\beta + O(1)}{\mathbb{E} S_\beta \left(1 - \alpha h \log \frac{1}{h} + o \left(h \log \frac{1}{h} \right) \right)} \\ &= 1 + \alpha h \log \frac{1}{h} + o \left(h \log \frac{1}{h} \right), \end{aligned}$$

and

$$\frac{1}{\beta-1} \int_0^1 \left(1 - z^{\frac{1-\beta}{\beta'}} \right) \mathbb{P}_{R_1}(dz) \mathbb{E} \frac{1}{Z(\beta)} \sim -\alpha(\beta-1) \log \frac{1}{\beta-1}. \quad (2.3.6)$$

Finally, combining Equation (2.3.3) with Equations (2.3.4) and (2.3.6) concludes the proof. \square

2.3.4 The BBM case

Proof of Theorem 2.1.2. We are going to apply Proposition 2.3.6. Assumption 1 and 3 follow from Proposition 2.2.3, where one can take $\zeta = 1$ for Assumption 3. Assumption 4 is the content of the next lemma.

Lemma 2.3.8. *For the decoration arising in the BBM, $\alpha_- = \alpha_+ = \delta'(0^+) = 2$.*

Proof of Lemma 2.3.8. As a consequence of Proposition 2.2.3, we have $\delta(1) \leq 2$. By convexity of the function δ together with $\delta(0) = 0$, we have $\delta'(0^+) \leq \delta(1)/1 \leq 2$. Hence, by Proposition 2.3.5, it remains to prove that $\alpha_- \geq 2$. For this, we fix some $\varepsilon > 0$ and show that $\alpha_- \geq 2(1 - \varepsilon)$.

Consider the continuous step function $\chi: x \in \mathbb{R} \mapsto (x-1)\mathbb{1}_{x \in [1,2]} + \mathbb{1}_{x > 2}$. For some $K > 0$ which will be chosen later in terms of ε , we define, for any $x \in \mathbb{R}$,

$$f_K(x) := \chi(-2x(\beta-1)K) [1 - \chi(-x(\beta-1)/K)] e^{\beta\sqrt{2}x},$$

which is chosen such that $f_K(x) = e^{\beta\sqrt{2}x}$ if $-x(\beta-1) \in [K^{-1}, K]$ and $f_K(x) = 0$ if $-x(\beta-1) \notin [(2K)^{-1}, 2K]$. Then, we have

$$S_\beta = \sum_{k \geq 0} e^{\beta d_k} \geq \sum_{k \geq 0} f_K(d_k/\sqrt{2}) = \mathcal{C}(f_K),$$

as well as $S_\beta \geq 1$ because $d_0 = 0$. Therefore, using that $x \log x \geq y \log y$ for any $y \geq 0$ and $x \geq y \vee 1$, we get

$$\mathbb{E}[S_\beta \log S_\beta] \geq \mathbb{E}[\mathcal{C}(f_K) \log \mathcal{C}(f_K)]. \quad (2.3.7)$$

Since f_K is continuous with compact support, we know by Equation (2.2.2) that $\mathcal{C}_{t,r_t}^*(f_K)$ under $\tilde{\mathbb{P}}_t$ converges in distribution to $\mathcal{C}(f_K)$ under \mathbb{P} . Moreover, $1 \leq \mathcal{C}_{t,r_t}^*(f_K) \leq \mathcal{C}_{t,r_t}^*([-2K, 0])$ which is bounded in L^2 by Lemma 2.2.5, so $\mathcal{C}_{t,r_t}^*(f_K) \log \mathcal{C}_{t,r_t}^*(f_K)$ is uniformly integrable and we get

$$\mathbb{E}[\mathcal{C}(f_K) \log \mathcal{C}(f_K)] = \lim_{t \rightarrow \infty} \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \log \mathcal{C}_{t,r_t}^*(f_K) \right], \quad (2.3.8)$$

so we now aim at lower bounding the expectation on right-hand side.

For some other parameters $L, M > 0$ to be chosen later in terms of ϵ , we define the event

$$A_{\beta,t} := \left\{ \max_{s(1-\beta)^2 \in [L^{-1}, L]} (h_{t-s}(X_{t-s}) - m_t + m_s) > -M \right\}.$$

It follows from [40, Lemma 2.6] (together with [39, Lemma 3.2] to rewrite the probability as written in [40, Lemma 2.6]) that

$$\sup_{\beta > 1} \limsup_{t \rightarrow \infty} \frac{1}{\beta - 1} \tilde{\mathbb{P}}_t(A_{\beta,t}) < \infty. \quad (2.3.9)$$

On the other hand, we get, using Jensen's inequality with the convex function $x \mapsto x \log x$ in the 2nd inequality,

$$\begin{aligned} \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \log \mathcal{C}_{t,r_t}^*(f_K) \right] &\geq \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \log \mathcal{C}_{t,r_t}^*(f_K) \mid A_{\beta,t} \right] \cdot \tilde{\mathbb{P}}_t(A_{\beta,t}) \\ &\geq \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \mid A_{\beta,t} \right] \cdot \log \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \mid A_{\beta,t} \right] \cdot \tilde{\mathbb{P}}_t(A_{\beta,t}) \\ &= \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \mathbb{1}_{A_{\beta,t}} \right] \cdot \left(\log \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \mathbb{1}_{A_{\beta,t}} \right] - \log \tilde{\mathbb{P}}_t(A_{\beta,t}) \right). \end{aligned} \quad (2.3.10)$$

Now it remains to prove that $K, L, M > 0$ can be chosen such that, for any $\beta > 1$ close enough to 1,

$$\liminf_{t \rightarrow \infty} \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \mathbb{1}_{A_{\beta,t}} \right] \geq (1 - \epsilon) \frac{C_\star}{\beta - 1}, \quad (2.3.11)$$

where C_\star is the constant appearing in (2.2.3). Indeed, assume that (2.3.11) holds. Then, using Equation (2.2.6) for the denominator and Equations (2.3.7), (2.3.8) and (2.3.10) for the numerator, we get

$$\begin{aligned} \alpha_- &= \liminf_{\beta \rightarrow 1^+} \frac{\mathbb{E}[S_\beta \log S_\beta]}{\mathbb{E}[S_\beta] \log \mathbb{E}[S_\beta]} \\ &\geq \liminf_{\beta \rightarrow 1^+} \frac{\beta - 1}{C_\star \log \frac{1}{\beta - 1}} \cdot \liminf_{t \rightarrow \infty} \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \mathbb{1}_{A_{\beta,t}} \right] \left(\log \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \mathbb{1}_{A_{\beta,t}} \right] - \log \tilde{\mathbb{P}}_t(A_{\beta,t}) \right). \end{aligned}$$

Applying Equations (2.3.11) and (2.3.9) yields $\alpha_- \geq 2(1 - \epsilon)$, which gives the desired result by letting $\epsilon \rightarrow 0$.

We now prove Equation (2.3.11). On the one hand, note that, using (2.2.2), we have

$$\begin{aligned}
 \lim_{t \rightarrow \infty} \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \right] &= \mathbb{E} \left[\mathcal{C}(f_K) \right] \\
 &\geq \mathbb{E} \left[\sum_{k \geq 0} e^{\beta d_k} \mathbb{1}_{-(\beta-1)d_k/\sqrt{2} \in [K^{-1}, K]} \right] \\
 &= \int_{K^{-1}(\beta-1)^{-1}}^{\infty} \mathbb{E} \left[\mathcal{C} \left(\left[- \left(x \wedge \frac{K}{\beta-1} \right), -\frac{1}{K(\beta-1)} \right] \right) \right] \beta \sqrt{2} e^{-\beta \sqrt{2} x} dx,
 \end{aligned} \tag{2.3.12}$$

using Fubini's theorem. Keeping only the part $x \leq K(\beta-1)^{-1}$, we get that the right-hand side of Equation (2.3.12) is at least

$$\begin{aligned}
 &\int_{K^{-1}(\beta-1)^{-1}}^{K(\beta-1)^{-1}} \mathbb{E} \left[\mathcal{C}([-x, 0]) \right] \beta \sqrt{2} e^{-\beta \sqrt{2} x} dx - \mathbb{E} \left[\mathcal{C} \left(\left[-\frac{1}{K(\beta-1)}, 0 \right] \right) \right] \int_{K^{-1}(\beta-1)^{-1}}^{K(\beta-1)^{-1}} \beta \sqrt{2} e^{-\beta \sqrt{2} x} dx \\
 &\sim \frac{C_\star}{\beta-1} \left(e^{-\sqrt{2}/K} - e^{-\sqrt{2}K} \right), \quad \text{as } \beta \rightarrow 1^+,
 \end{aligned}$$

recalling that $\mathbb{E} \left[\mathcal{C}([-x, 0]) \right] \sim C_\star e^x$ as $x \rightarrow \infty$ (see Equation (2.2.3)). Therefore, we can now choose $K > 0$ large enough such that, for any $\beta > 1$ close enough to 1,

$$\lim_{t \rightarrow \infty} \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \right] \geq \left(1 - \frac{\varepsilon}{2} \right) \frac{C_\star}{\beta-1}. \tag{2.3.13}$$

On the other hand, using now that $f_K(x) \leq e^{\beta \sqrt{2} x} \mathbb{1}_{-(\beta-1)x \in [(2K)^{-1}, 2K]}$ and then Fubini's theorem as before (but without taking the limit $t \rightarrow \infty$), we get

$$\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \mathbb{1}_{A_{\beta,t}^c} \right] \leq \int_{(2K)^{-1}(\beta-1)^{-1}}^{\infty} \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^* \left(\left[- \left(x \wedge \frac{2K}{\beta-1} \right), 0 \right] \right) \mathbb{1}_{A_{\beta,t}^c} \right] \beta \sqrt{2} e^{-\beta \sqrt{2} x} dx.$$

Then, let $\varepsilon' = \varepsilon C_\star/4$. By [40, Lemma 3.1], there exists $x_0, \eta, M > 0$ such that, for any $x \geq x_0$,

$$\limsup_{t \rightarrow \infty} \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*([-x, 0]); \max_{s \in [\eta x^2, \eta^{-1} x^2]} (h_{t-s}(X_{t-s}) - m_t + m_s) \leq -M \right] \leq \varepsilon' e^{\sqrt{2} x}. \tag{2.3.14}$$

We fix these $x_0, \eta, M > 0$ and set $L = (2K)^2/\eta$. Then, for any $x \in [(2K)^{-1}(\beta-1)^{-1}, 2K(\beta-1)^{-1}]$, we have $[\eta x^2, \eta^{-1} x^2] \subset [L^{-1}(\beta-1)^{-2}, L(\beta-1)^{-2}]$, hence

$$A_{\beta,t}^c \subset \left\{ \max_{s \in [\eta x^2, \eta^{-1} x^2]} (h_{t-s}(X_{t-s}) - m_t + m_s) \leq -M \right\}$$

and Equation (2.3.14) still holds if we replace the expectation by $\tilde{\mathbb{E}}_t[\mathcal{C}_{t,r_t}^*([-x, 0]) \mathbb{1}_{A_{\beta,t}^c}]$. Therefore, using reversed Fatou's lemma with $\tilde{\mathbb{E}}_t[\mathcal{C}_{t,r_t}^*([-x, 0]) \mathbb{1}_{A_{\beta,t}^c}] \leq C e^{\sqrt{2} x}$ as a domination (by Lemma 2.2.5), we get, for any $\beta > 1$ close enough to 1 such that $(2K)^{-1}(\beta-1)^{-1} \geq x_0$,

$$\begin{aligned}
 \limsup_{t \rightarrow \infty} \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_K) \mathbb{1}_{A_{\beta,t}^c} \right] &\leq \int_{(2K)^{-1}(\beta-1)^{-1}}^{\infty} \varepsilon' \exp \left(\sqrt{2} \left(x \wedge \frac{2K}{\beta-1} \right) \right) \beta \sqrt{2} e^{-\beta \sqrt{2} x} dx \\
 &\leq \frac{\varepsilon}{2} \cdot \frac{C_\star}{\beta-1}.
 \end{aligned} \tag{2.3.15}$$

Combining Equations (2.3.13) and (2.3.15) proves (2.3.11) and concludes the proof. \square

It remains to prove that Assumption 2 is fulfilled by the decoration process of the BBM. With the notations of the proof of Proposition 2.3.5, Jensen's inequality yields

$$\mathbb{E}[S_\beta \log S_{\beta'}] = \mathbb{E} S_\beta \mathbb{E}_\beta \log S_{\beta'} \leq \mathbb{E} S_\beta \log \frac{\mathbb{E}[S_\beta S_{\beta'}]}{\mathbb{E} S_\beta},$$

and by noticing that

$$\begin{aligned} \mathbb{E}[S_\beta S_{\beta'}] &= \beta \beta' \int_0^\infty \int_0^\infty \mathbb{E}[\mathcal{D}[-x, 0] \mathcal{D}[-y, 0]] e^{-\beta x} e^{-\beta' y} dx dy \\ &\leq \beta \beta' \int_{x=0}^\infty \int_{y=x}^\infty C(x+1) e^{x+y} e^{-\beta x} e^{-\beta' y} dx dy \\ &\quad + \beta \beta' \int_{x=0}^\infty \int_{y=0}^x C(y+1) e^{x+y} e^{-\beta x} e^{-\beta' y} dx dy, \end{aligned}$$

where we used Proposition 2.2.6, the conclusion follows by a simple computation that shows that the first term remains bounded while the second is $O\left(\frac{1}{\beta-1}\right)$. \square

2.4 Temperature susceptibility: proofs

2.4.1 The REM case

Proof of Theorem 2.1.3. Expanding $C(\beta, \beta + h)$ in h gives, according to Lemma 2.5.11,

$$\begin{aligned} C(\beta, \beta + h) &= \frac{\text{Cov}(\log Z(\beta), \log Z(\beta + h))}{\sigma(\log Z(\beta)) \sigma(\log Z(\beta + h))} \\ &= \frac{\text{Cov}\left(\log Z, \log Z + \frac{Z'}{Z} h + \left(\frac{Z''}{Z} - \left(\frac{Z'}{Z}\right)^2\right) h^2\right)}{\sigma(\log Z(\beta)) \sigma\left(\log Z + \frac{Z'}{Z} h + \left(\frac{Z''}{Z} - \left(\frac{Z'}{Z}\right)^2\right) h^2\right)} + o(h^2) \\ &= 1 - \frac{1}{2} \left(\frac{\text{Var}\left(\frac{Z'}{Z}\right)}{\text{Var}(\log Z)} - \left(\frac{\text{Cov}\left(\log Z, \frac{Z'}{Z}\right)}{\text{Var}(\log Z)} \right)^2 \right) h^2 + o(h^2), \end{aligned}$$

and all these quantities are explicit (the computations are detailed in Proposition 2.5.6). One finds:

$$\kappa(\beta) = \frac{1}{2} \left(\frac{1}{\beta^2 - 1} + \frac{6}{\pi^2 \beta^3 (\beta + 1)} \left(\frac{\Gamma''\left(\frac{\beta-1}{\beta}\right)}{\Gamma\left(\frac{\beta-1}{\beta}\right)} - \left(\frac{\Gamma'\left(\frac{\beta-1}{\beta}\right)}{\Gamma\left(\frac{\beta-1}{\beta}\right)} \right)^2 \right) - \frac{\beta^2}{(\beta^2 - 1)^2} \right),$$

and

$$\kappa(\beta) \underset{\beta \downarrow 1}{\sim} \frac{1}{2} \left(\frac{3}{\pi^2} - \frac{1}{4} \right) \frac{1}{(\beta - 1)^2}, \quad \kappa(\beta) \underset{\beta \rightarrow \infty}{=} O\left(\frac{1}{\beta^5}\right).$$

\square

2.4.2 The decorated case

In this section, our goal is to prove Theorem 2.1.4 which concerns the BBM. However, we first work in the general decorated case. Let $\beta > \beta_c = 1$. Recall that we are interested in

$$Z_d(\beta) := \sum_{i,k \geq 0} e^{\beta(\xi_i + d_{ik})},$$

where the $(\xi_i)_{i \geq 0}$ are the atoms of a Poisson point process on \mathbb{R} with intensity measure $e^{-x} dx$ independent of $(\sum_{k \geq 0} \delta_{d_{ik}})_{i \geq 0}$, which are i.i.d. copies of a point process $\mathcal{D} = \sum_{k \geq 0} \delta_{d_k}$ on $(-\infty, 0]$ which has a.s. an atom at 0. Recall also that we set $S_\beta = \sum_{k \geq 0} e^{\beta d_k}$. We first assume only that

$$\forall \beta > 1, \quad \mathbb{E}[S_\beta] < \infty. \quad (2.4.1)$$

which holds true for the BBM (see [23, Lemma 3.1] or the proof of Proposition 2.2.3).

Under this assumption, we introduce a new probability measure \mathbb{P}_β such that

- the distribution of \mathcal{D} under \mathbb{P}_β is characterized by

$$\mathbb{E}_\beta [F(\mathcal{D})] = \frac{\mathbb{E} [S_\beta^{1/\beta} F(\mathcal{D})]}{\mathbb{E} [S_\beta^{1/\beta}]},$$

for any measurable bounded function F from the space of Radon measure on \mathbb{R} to \mathbb{R} ;

- under \mathbb{P}_β , the $(\xi_i)_{i \geq 1}$ are still the atoms of a Poisson point process on \mathbb{R} with intensity measure $e^{-x} dx$ independent of $(\sum_{k \geq 0} \delta_{d_{ik}})_{i \geq 1}$, which are i.i.d. copies of \mathcal{D} .

With this definition in hand, the following fact holds (see e.g. [89, Lemma 2.1]).

Lemma 2.4.1. *Let $\beta > 1$. Assume (2.4.1), then*

$$\left(\sum_{i \geq 0} \delta_{\xi_i + \frac{1}{\beta} \log S_{\beta,i}}, \left(\sum_{k \geq 0} \delta_{d_{ik}} \right)_{i \geq 0} \right) \text{ under } \mathbb{P} \stackrel{(d)}{=} \left(\sum_{i \geq 0} \delta_{\xi_i + c_\beta}, \left(\sum_{k \geq 0} \delta_{d_{ik}} \right)_{i \geq 0} \right) \text{ under } \mathbb{P}_\beta. \quad (2.4.2)$$

This lemma is used to prove the following comparison result between the decorated case and the REM case.

Proposition 2.4.2. *Assume that (2.4.1) holds for any $\beta > 1$. Then, for any $\beta > 1$,*

$$\kappa_d(\beta) = \kappa(\beta) + \frac{3}{\pi^2 \beta(\beta + 1)} \text{Var}_\beta \left(\frac{1}{\beta} \log S_\beta - \frac{S'_\beta}{S_\beta} \right),$$

where Var_β denotes the variance under \mathbb{P}_β and $S'_\beta := \sum_{k \geq 0} d_k e^{\beta d_k}$.

Proof. The formula

$$\kappa_d(\beta) = \frac{1}{2} \left(\frac{\text{Var} \left(\frac{Z'_d(\beta)}{Z_d(\beta)} \right)}{\text{Var}(\log Z_d(\beta))} - \left(\frac{\text{Cov} \left(\log Z_d(\beta), \frac{Z'_d(\beta)}{Z_d(\beta)} \right)}{\text{Var}(\log Z_d(\beta))} \right)^2 \right)$$

is still true thanks to Lemma 2.5.11. Writing $Z_d(\beta) = \sum_{i \geq 0} e^{\beta(\xi_i + \frac{1}{\beta} \log S_{\beta,i})}$, with $S_{\beta,i} := \sum_{k \geq 0} e^{\beta d_{ik}}$ for $i \geq 0$, it follows from Lemma 2.4.1 that $Z_d(\beta)$ has the same distribution as $e^{\beta c_\beta} Z(\beta)$. Hence, we have

$$\text{Var}(\log Z_d(\beta)) = \text{Var}(\log Z(\beta)) = \frac{\pi^2}{6}(\beta^2 - 1)$$

and, by differentiating with respect to β ,

$$\begin{aligned} \text{Cov} \left(\log Z_d(\beta), \frac{Z'_d(\beta)}{Z_d(\beta)} \right) &= \frac{1}{2} \frac{d}{d\beta} \text{Var}(\log Z_d(\beta)) \\ &= \frac{1}{2} \frac{d}{d\beta} \text{Var}(\log Z(\beta)) \\ &= \text{Cov} \left(\log Z(\beta), \frac{Z'(\beta)}{Z(\beta)} \right). \end{aligned}$$

Therefore, we get

$$\kappa_d(\beta) = \kappa(\beta) + \frac{3}{\pi^2(\beta^2 - 1)} \left(\text{Var} \frac{Z'_d(\beta)}{Z_d(\beta)} - \text{Var} \frac{Z'(\beta)}{Z(\beta)} \right). \quad (2.4.3)$$

We shall now focus on $\text{Var} \left(\frac{Z'_d(\beta)}{Z_d(\beta)} \right)$. Notice first that

$$Z'_d(\beta) = \sum_{i,k \geq 0} (\xi_i + d_{ik}) e^{\beta(\xi_i + d_{ik})}.$$

Therefore, it follows from Lemma 2.4.1 that

$$\frac{Z'_d(\beta)}{Z_d(\beta)} \text{ under } \mathbb{P} \stackrel{(d)}{=} \frac{\sum_{i,k} (\xi_i + c_\beta - \frac{1}{\beta} \log S_{\beta,i} + d_{ik}) e^{\beta(\xi_i - \frac{1}{\beta} \log S_{\beta,i} + d_{ik})}}{\sum_i e^{\beta \xi_i}} \text{ under } \mathbb{P}_\beta.$$

Moreover, the quantity on the right-hand side of the last equation equals

$$c_\beta + \frac{\sum_i \xi_i e^{\beta \xi_i}}{\sum_i e^{\beta \xi_i}} + \frac{\sum_i e^{\beta \xi_i} (S'_{\beta,i}/S_{\beta,i} - \frac{1}{\beta} \log S_{\beta,i})}{\sum_i e^{\beta \xi_i}}.$$

Hence, using the law of total variance, we get

$$\begin{aligned} \text{Var} \left(\frac{Z'_d(\beta)}{Z_d(\beta)} \right) &= \text{Var}_\beta \left(\mathbb{E}_\beta \left[\frac{\sum_i \xi_i e^{\beta \xi_i}}{\sum_i e^{\beta \xi_i}} + \frac{\sum_i e^{\beta \xi_i} (S'_{\beta,i}/S_{\beta,i} - \frac{1}{\beta} \log S_{\beta,i})}{\sum_i e^{\beta \xi_i}} \middle| \xi \right] \right) \\ &\quad + \mathbb{E}_\beta \left[\text{Var}_\beta \left(\frac{\sum_i \xi_i e^{\beta \xi_i}}{\sum_i e^{\beta \xi_i}} + \frac{\sum_i e^{\beta \xi_i} (S'_{\beta,i}/S_{\beta,i} - \frac{1}{\beta} \log S_{\beta,i})}{\sum_i e^{\beta \xi_i}} \middle| \xi \right) \right]. \end{aligned}$$

Then, using the fact that the $(S_{\beta,i}, S'_{\beta,i})_{i \geq 0}$ are i.i.d. and independent of ξ , we get

$$\begin{aligned} \text{Var} \left(\frac{Z'_d(\beta)}{Z_d(\beta)} \right) &= \text{Var}_\beta \left(\frac{\sum_i \xi_i e^{\beta \xi_i}}{\sum_i e^{\beta \xi_i}} \right) + \mathbb{E}_\beta \left[\frac{\sum_i e^{2\beta \xi_i}}{(\sum_i e^{\beta \xi_i})^2} \right] \text{Var}_\beta \left(\frac{S'_\beta}{S_\beta} - \frac{1}{\beta} \log S_\beta \right) \\ &= \text{Var} \left(\frac{Z'(\beta)}{Z(\beta)} \right) + \left(1 - \frac{1}{\beta} \right) \text{Var}_\beta \left(\frac{S'_\beta}{S_\beta} - \frac{1}{\beta} \log S_\beta \right). \end{aligned}$$

Coming back to (2.4.3), this concludes the proof. \square

Lemma 2.4.3. *For the decoration arising in the BBM and the constant C_\star appearing in Equation (2.2.3), we have, as $\beta \downarrow 1$,*

$$\mathbb{E} [S'_\beta] \sim -\frac{C_\star}{(\beta-1)^2} \quad \text{and} \quad \mathbb{E} [S''_\beta] \sim \frac{2C_\star}{(\beta-1)^3}.$$

Proof. This follows from Equation (2.2.3) in a similar way as the proof of Proposition 2.2.3. \square

Proof of Theorem 2.1.4. We first prove that $\kappa_d(\beta) > \kappa(\beta)$ for some fixed $\beta > 1$. By Proposition 2.4.2, it is sufficient to show that

$$\text{Var}_\beta \left(\frac{1}{\beta} \log S_\beta - \frac{S'_\beta}{S_\beta} \right) > 0. \quad (2.4.4)$$

We proceed by contradiction: assume that this variance equals zero. Then, there exists $c \in \mathbb{R}$ such that, $\tilde{\mathbb{P}}$ -a.s.,

$$\frac{1}{\beta} \log S_\beta = \frac{S'_\beta}{S_\beta} + c, \quad (2.4.5)$$

But \mathbb{P} is absolutely continuous w.r.t. $\tilde{\mathbb{P}}$ such that Equation (2.4.5) holds also \mathbb{P} -a.s. On the one hand, using $S_\beta = \sum_{k \geq 0} e^{\beta d_k}$, we have

$$S_\beta \log S_\beta > \sum_{k \geq 0} e^{\beta d_k} \beta d_k = \beta S'_\beta.$$

Thus, we necessarily have $c > 0$. On the other hand, fix some $\beta' \in (\beta_c, \beta)$. Then, there exists $C > 0$ such that

$$(1 \vee |x|) e^{\beta x} \leq C e^{\beta' x}, \quad \forall x \leq 0. \quad (2.4.6)$$

Now, for some $\varepsilon > 0$, consider the event

$$\left\{ \sum_{k \geq 1} e^{\beta' d_k} \leq \varepsilon \right\}.$$

This event has positive \mathbb{P} -probability by [23, Proposition 3.4]. So, on this event intersected with the one where Equation (2.4.5) holds (this intersection being non-empty), we have

$$c = \frac{1}{\beta} \log S_\beta - \frac{S'_\beta}{S_\beta} \leq \frac{1}{\beta} \log \left(1 + \sum_{k \geq 1} e^{\beta d_k} \right) + \sum_{k \geq 1} d_k e^{\beta d_k} \leq \frac{1}{\beta} \log (1 + C\varepsilon) + C\varepsilon,$$

using Equation (2.4.6). Letting $\varepsilon \rightarrow 0$ shows $c \leq 0$, which contradicts $c > 0$ and concludes our proof of Equation (2.4.4).

We study now the regime $\beta \downarrow 1$. Since the asymptotics of $\kappa(\beta)$ are given by Theorem 2.1.3, it remains to study the behavior of

$$\text{Var}_\beta \left(\frac{1}{\beta} \log S_\beta - \frac{S'_\beta}{S_\beta} \right) = \frac{1}{\beta^2} \text{Var}_\beta \log S_\beta - \frac{2}{\beta} \text{Cov}_\beta \left(\log S_\beta, \frac{S'_\beta}{S_\beta} \right) + \text{Var}_\beta \left(\frac{S'_\beta}{S_\beta} \right). \quad (2.4.7)$$

As we will see, the main term is the third one, which is of order $(\beta - 1)^{-2}$, while the other terms are negligible.

For the first term on the right-hand side of Equation (2.4.7), observe that

$$\text{Var}_\beta (\log S_\beta) \leq \mathbb{E}_\beta [\log^2 S_\beta] \leq \mathbb{E}_\beta [\log^2(e + S_\beta)] \leq \log^2(e + \mathbb{E}_\beta [S_\beta]),$$

by Jensen's inequality. Then, by definition of \mathbb{E}_β , we get

$$\mathbb{E}_\beta S_\beta = \frac{\mathbb{E} [S_\beta^{1+1/\beta}]}{\mathbb{E} [S_\beta^{1/\beta}]} \leq \frac{\mathbb{E} [S_\beta^2]}{\mathbb{E} [S_\beta^{1/\beta}]} = O \left(\frac{1}{(1 - \beta)^2} \right),$$

as $\beta \downarrow 1$ by Equation (2.2.7) and Corollary 2.2.4. This proves

$$\text{Var}_\beta \log S_\beta = O \left(\log^2 \frac{1}{\beta - 1} \right). \quad (2.4.8)$$

We now consider the third term on the right-hand side of Equation (2.4.7). Using the definition of \mathbb{E}_β and then that $S_\beta \geq 1$, we get

$$\mathbb{E}_\beta \left[\frac{S'_\beta}{S_\beta} \right] = \frac{\mathbb{E} [S_\beta^{1/\beta-1} S'_\beta]}{\mathbb{E} [S_\beta^{1/\beta}]} \sim -\frac{1}{\beta - 1}, \quad (2.4.9)$$

as $\beta \downarrow 1$ by Corollary 2.2.4 and the fact that $\mathbb{E} [S_\beta^{1/\beta-1} (-S'_\beta)] \sim C_*(\beta - 1)^{-2}$ as a consequence of the first part of (2.2.8) (with $X = S_\beta$ and $Y = -S'_\beta$) together with Lemma 2.4.3. On the other hand, it follows from Cauchy–Schwarz inequality that

$$\frac{S_\beta'^2}{S_\beta} = S_\beta \cdot \left(\sum_{k \geq 0} d_k \frac{e^{\beta d_k}}{S_\beta} \right)^2 \leq S_\beta \cdot \sum_{k \geq 0} d_k^2 \frac{e^{\beta d_k}}{S_\beta} = S_\beta'',$$

and therefore

$$\mathbb{E}_\beta \left[\left(\frac{S'_\beta}{S_\beta} \right)^2 \right] = \frac{\mathbb{E} [S_\beta^{1/\beta-2} S_\beta'^2]}{\mathbb{E} [S_\beta^{1/\beta}]} \leq \frac{\mathbb{E} [S_\beta'^2 / S_\beta]}{\mathbb{E} [S_\beta^{1/\beta}]} \leq \frac{\mathbb{E} [S_\beta'']}{\mathbb{E} [S_\beta^{1/\beta}]} \sim \frac{2}{(\beta - 1)^2}, \quad (2.4.10)$$

using Lemma 2.4.3 and Corollary 2.2.4. For the lower bound, using the first part of (2.2.8) with $X = S_\beta$ and $Y = (S'_\beta)^2 / S_\beta$, we get

$$\mathbb{E}_\beta \left[\left(\frac{S'_\beta}{S_\beta} \right)^2 \right] = \frac{\mathbb{E} [S_\beta^{1/\beta-2} S_\beta'^2]}{\mathbb{E} [S_\beta^{1/\beta}]} \geq \frac{\mathbb{E} [S_\beta'^2 / S_\beta]^{2-1/\beta}}{\mathbb{E} [S_\beta^{1/\beta}] \mathbb{E} [S_\beta]^{1-1/\beta}}.$$

Hence, applying Lemma 2.4.4, Proposition 2.2.3 and Corollary 2.2.4, we get

$$\liminf_{\beta \downarrow 1} (\beta - 1)^2 \mathbb{E}_\beta \left[\left(\frac{S'_\beta}{S_\beta} \right)^2 \right] > 1. \quad (2.4.11)$$

Combining Equations (2.4.9), (2.4.10) and (2.4.11) yields

$$0 < \liminf_{\beta \downarrow 1} (\beta - 1)^2 \text{Var}_\beta \left(\frac{S'_\beta}{S_\beta} \right) \leq \limsup_{\beta \downarrow 1} (\beta - 1)^2 \text{Var}_\beta \left(\frac{S'_\beta}{S_\beta} \right) \leq 1. \quad (2.4.12)$$

Finally, for the second term on the right-hand side of Equation (2.4.7), by Cauchy–Schwarz inequality and then Equations (2.4.8) and (2.4.12), we get

$$\left| \text{Cov}_\beta \left(\log S_\beta, \frac{S'_\beta}{S_\beta} \right) \right| \leq \left(\text{Var}_\beta (\log S_\beta) \text{Var}_\beta \left(\frac{S'_\beta}{S_\beta} \right) \right)^{1/2} = O \left(\frac{\log \frac{1}{\beta-1}}{\beta-1} \right),$$

which proves that this term is negligible in (2.4.7) and concludes the proof. \square

Lemma 2.4.4. *We have*

$$\liminf_{\beta \downarrow 1} (\beta - 1)^3 \mathbb{E} \left[\frac{(S'_\beta)^2}{S_\beta} \right] > C_\star.$$

Remark 2.4.5. A lower bound with a weak inequality could be easily obtained via Cauchy–Schwarz inequality:

$$\mathbb{E} \left[\frac{(S'_\beta)^2}{S_\beta} \right] \geq \frac{\mathbb{E}[S'_\beta]^2}{\mathbb{E}[S_\beta]} \sim \frac{C_\star}{(\beta - 1)^3},$$

using Lemma 2.4.3 and Proposition 2.2.3. Equality at the first order in this inequality would suggest that S'_β and S_β are colinear at first order (on the events that are significant for the first moment). Therefore, the idea of the proof below is to find an event such that the first moments of S'_β and S_β given this event have a different ratio than the one for the non-conditional first moments.

Proof. Step 1: Working at finite t . Fix some $\beta > 1$. Setting $f_\beta(x) := e^{\beta\sqrt{2}x}$ and $\partial_\beta f_\beta(x) := \sqrt{2}x e^{\beta\sqrt{2}x}$, note that $S_\beta = \mathcal{C}(f_\beta)$ and $S'_\beta = \mathcal{C}(\partial_\beta f_\beta)$. Our first aim in this step is to show

$$\mathbb{E} \left[\frac{(S'_\beta)^2}{S_\beta} \right] = \lim_{t \rightarrow \infty} \tilde{\mathbb{E}}_t \left[\frac{(\mathcal{C}_{t,r_t}^*(\partial_\beta f_\beta))^2}{\mathcal{C}_{t,r_t}^*(f_\beta)} \right]. \quad (2.4.13)$$

For $K > 0$, let χ_K denote a continuous function such that $\mathbb{1}_{[-K,K]} \leq \chi_K \leq \mathbb{1}_{[-(K+1),K+1]}$. It follows from the vague convergence stated in Equation (2.2.2) that

$$\left(\mathcal{C}_{t,r_t}^*(\chi_K f_\beta), \mathcal{C}_{t,r_t}^*(\chi_K \partial_\beta f_\beta) \right) \text{ under } \tilde{\mathbb{P}}_t \xrightarrow[t \rightarrow \infty]{(d)} \left(\mathcal{C}(\chi_K f_\beta), \mathcal{C}(\chi_K \partial_\beta f_\beta) \right) \text{ under } \mathbb{P}.$$

Moreover, recalling \mathcal{C}_{t,r_t}^* is supported on $(-\infty, 0]$ under $\tilde{\mathbb{P}}_t$ we get

$$\left| \frac{\mathcal{C}_{t,r_t}^*(\chi_K \partial_\beta f_\beta)^2}{\mathcal{C}_{t,r_t}^*(\chi_K f_\beta)} \right| \leq 2(K+1)^2 \mathcal{C}_{t,r_t}^*(\chi_K f_\beta) \leq 2(K+1)^2 \mathcal{C}_{t,r_t}^*([-(K+1), 0]),$$

which is bounded in L^2 by Lemma 2.2.5. Hence, $\mathcal{C}_{t,r_t}^*(\chi_K \partial_\beta f_\beta)^2 / \mathcal{C}_{t,r_t}^*(\chi_K f_\beta)$ is uniformly integrable and we get

$$\mathbb{E} \left[\frac{\mathcal{C}(\chi_K \partial_\beta f_\beta)^2}{\mathcal{C}(\chi_K f_\beta)} \right] = \lim_{t \rightarrow \infty} \tilde{\mathbb{E}}_t \left[\frac{\mathcal{C}_{t,r_t}^*(\chi_K \partial_\beta f_\beta)^2}{\mathcal{C}_{t,r_t}^*(\chi_K f_\beta)} \right]. \quad (2.4.14)$$

We now control the differences between the expectations in Equation (2.4.13) and Equation (2.4.14) and show they are small when K is large. We have

$$\begin{aligned} \left| \frac{\mathcal{C}(\partial_\beta f_\beta)^2}{\mathcal{C}(f_\beta)} - \frac{\mathcal{C}(\chi_K \partial_\beta f_\beta)^2}{\mathcal{C}(\chi_K f_\beta)} \right| &\leq \frac{|\mathcal{C}(\partial_\beta f_\beta)^2 - \mathcal{C}(\chi_K \partial_\beta f_\beta)^2|}{\mathcal{C}(f_\beta)} + \mathcal{C}(\chi_K \partial_\beta f_\beta)^2 \left| \frac{1}{\mathcal{C}(f_\beta)} - \frac{1}{\mathcal{C}(\chi_K f_\beta)} \right| \\ &\leq 2\mathcal{C}(\partial_\beta f_\beta) \mathcal{C}(\mathbb{1}_{(-\infty, -K]} \partial_\beta f_\beta) + (K+1)^2 \mathcal{C}(\mathbb{1}_{(-\infty, -K]} \partial_\beta f_\beta), \end{aligned} \quad (2.4.15)$$

where, for the 1st term, we used $\mathcal{C}(f_\beta) \geq 1$ and the fact that $\partial_\beta f_\beta$ is of constant sign on $(-\infty, 0]$ which is the support of \mathcal{C} and, for the 2nd term, we used $|\mathcal{C}(\chi_K \partial_\beta f_\beta)| / \mathcal{C}(f_\beta) \leq |\mathcal{C}(\chi_K \partial_\beta f_\beta)| / \mathcal{C}(\chi_K f_\beta) \leq K+1$. Then, writing $\partial_\beta f_\beta(x) = \int_{-\infty}^x \sqrt{2}(\beta\sqrt{2}y+1)e^{\beta\sqrt{2}y} dy$ and using Fubini's theorem, we have

$$\begin{aligned} &\mathbb{E} \left[\mathcal{C}(\partial_\beta f_\beta) \mathcal{C}(\mathbb{1}_{(-\infty, -K]} \partial_\beta f_\beta) \right] \\ &= \int_{-\infty}^0 \int_{-\infty}^{-K} \mathbb{E} [\mathcal{C}([-y, 0]) \mathcal{C}([-z, -K])] 2(\beta\sqrt{2}z+1) e^{\beta\sqrt{2}z} (\beta\sqrt{2}y+1) e^{\beta\sqrt{2}y} dz dy \\ &\leq C(\beta)(K+1)^{3/2} e^{-(\beta-1)\sqrt{2}K}, \end{aligned}$$

where we bounded the last expectation by $(|y|+1)^{1/2}(|z|+1)^{1/2} e^{-\sqrt{2}(y+z)}$ using Cauchy–Schwarz inequality and Equation (2.2.4). Similarly, we have $\mathbb{E}[\mathcal{C}(\mathbb{1}_{(-\infty, -K]} \partial_\beta f_\beta)] \leq C(\beta)(K+1)e^{-(\beta-1)\sqrt{2}K}$ by using (2.2.5). Coming back to Equation (2.4.15), we get

$$\left| \mathbb{E} \left[\frac{\mathcal{C}(\partial_\beta f_\beta)^2}{\mathcal{C}(f_\beta)} \right] - \mathbb{E} \left[\frac{\mathcal{C}(\chi_K \partial_\beta f_\beta)^2}{\mathcal{C}(\chi_K f_\beta)} \right] \right| \leq C(\beta)(K+1)^3 e^{-(\beta-1)\sqrt{2}K},$$

and the same holds true for \mathcal{C}_{t,r_t}^* instead of \mathcal{C} , uniformly in $t \geq 1$, by replacing (2.2.4) and (2.2.5) by Lemma 2.2.5 in the proof. Therefore, letting $K \rightarrow \infty$ in (2.4.14) yields (2.4.13).

Step 2: Using Cauchy–Schwarz inequality conditionally on a well-chosen event. We fix some parameters $a > 0$ and $0 < b < B$. For any $\beta > 1$, letting $r = r(\beta) := a(\beta-1)^{-2}$, and for any $t > r$, we introduce the event

$$B_{r,t} = \left\{ h_{t-r}(X_{t-r}) - m_t + m_r \in [-B\sqrt{r}, -b\sqrt{r}] \right\}. \quad (2.4.16)$$

Then, we use Cauchy–Schwarz inequality given $B_{r,t}$ or given $B_{r,t}^c$ to get

$$\begin{aligned} \tilde{\mathbb{E}}_t \left[\frac{(\mathcal{C}_{t,r_t}^*(\partial_\beta f_\beta))^2}{\mathcal{C}_{t,r_t}^*(f_\beta)} \right] &= \tilde{\mathbb{E}}_t \left[\frac{(\mathcal{C}_{t,r_t}^*(\partial_\beta f_\beta))^2}{\mathcal{C}_{t,r_t}^*(f_\beta)} \Big| B_{r,t} \right] \tilde{\mathbb{P}}_t(B_{r,t}) + \tilde{\mathbb{E}}_t \left[\frac{(\mathcal{C}_{t,r_t}^*(\partial_\beta f_\beta))^2}{\mathcal{C}_{t,r_t}^*(f_\beta)} \Big| B_{r,t}^c \right] \tilde{\mathbb{P}}_t(B_{r,t}^c) \\ &\geq \frac{\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(\partial_\beta f_\beta) \Big| B_{r,t} \right]^2}{\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_\beta) \Big| B_{r,t} \right]} \tilde{\mathbb{P}}_t(B_{r,t}) + \frac{\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(\partial_\beta f_\beta) \Big| B_{r,t}^c \right]^2}{\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_\beta) \Big| B_{r,t}^c \right]} \tilde{\mathbb{P}}_t(B_{r,t}^c) \\ &= \frac{\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(\partial_\beta f_\beta) \mathbb{1}_{B_{r,t}} \right]^2}{\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_\beta) \mathbb{1}_{B_{r,t}} \right]} + \frac{\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(\partial_\beta f_\beta) \mathbb{1}_{B_{r,t}^c} \right]^2}{\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_\beta) \mathbb{1}_{B_{r,t}^c} \right]}, \end{aligned} \quad (2.4.17)$$

which yields a lower bound for the right-hand side of (2.4.13), that we now have to estimate.

Step 3: Estimating the expectations. Let $\varepsilon > 0$. Our aim in this step consists in proving that, for $\beta > 1$ close enough to 1, there exists $t_0 > r$ such that, for any $t \geq t_0$

$$\begin{aligned} |(\beta - 1)\tilde{\mathbb{E}}_t [\mathcal{C}_{t,r_t}^*(f_\beta)\mathbb{1}_{B_{r,t}}] - C_\star\kappa| &\leq \varepsilon, \\ |(\beta - 1)^2\tilde{\mathbb{E}}_t [\mathcal{C}_{t,r_t}^*(\partial_\beta f_\beta)\mathbb{1}_{B_{r,t}}] + C_\star\kappa'| &\leq \varepsilon, \end{aligned} \quad (2.4.18)$$

where κ, κ' are constants depending only on the parameters a, b, B defined as follows

$$\begin{aligned} \kappa &:= \frac{1}{2\sqrt{\pi}} \int_0^\infty \varphi_{b,B} \left(\frac{a \vee w}{|a - w|} \right) \left(\int_0^\infty u e^{-u - u^2/(4w)} du \right) \frac{dw}{w^{3/2}}, \\ \kappa' &:= \frac{1}{2\sqrt{\pi}} \int_0^\infty \varphi_{b,B} \left(\frac{a \vee w}{|a - w|} \right) \left(\int_0^\infty u^2 e^{-u - u^2/(4w)} du \right) \frac{dw}{w^{3/2}}, \end{aligned} \quad (2.4.19)$$

with $\varphi_{b,B}$ defined by

$$\varphi_{b,B}(v) := \int_{b\sqrt{v}}^{B\sqrt{v}} \sqrt{\frac{2}{\pi}} y^2 e^{-y^2/2} dy, \quad v \geq 0. \quad (2.4.20)$$

Note that $\kappa, \kappa' \in (0, 1)$ and they tend to 1 as $b \rightarrow 0$ and $B \rightarrow \infty$ (which means $\mathbb{1}_{B_{r,t}} \rightarrow 1$). To prove the first inequality in (2.4.18), we write

$$\tilde{\mathbb{E}}_t [\mathcal{C}_{t,r_t}^*(f_\beta)\mathbb{1}_{B_{r,t}}] = \int_0^\infty \tilde{\mathbb{E}}_t [\mathcal{C}_{t,r_t}^*([-x, 0])\mathbb{1}_{B_{r,t}}] \beta\sqrt{2} e^{-\beta\sqrt{2}x} dx.$$

Then, we apply Lemma 2.4.6 for some $\theta \in (0, 1)$ and with $\varepsilon > 0$ introduced earlier to estimate the part $x \in [\theta\sqrt{r}, \theta^{-1}\sqrt{r}]$ of the integral and use Lemma 2.2.5 to bound the remaining part. Note also that $\varphi_{b,B} \leq 1$ and $\int_0^\infty \frac{x e^{-x^2/(2s)}}{\sqrt{2\pi s^{3/2}}} ds = 1$. Therefore, for $\beta > 1$ close enough to 1 (equivalently r large enough), there exists $t_0 > r$ such that, for any $t \geq t_0$,

$$\begin{aligned} &\left| \tilde{\mathbb{E}}_t [\mathcal{C}_{t,r_t}^*(f_\beta)\mathbb{1}_{B_{r,t}}] - C_\star \int_0^\infty \left(\int_0^\infty \varphi_{b,B} \left(\frac{r \vee s}{|r - s|} \right) \frac{x e^{-x^2/(2s)}}{\sqrt{2\pi s^{3/2}}} ds \right) \beta\sqrt{2} e^{-(\beta-1)\sqrt{2}x} dx \right| \\ &\leq \varepsilon \int_0^\infty \beta\sqrt{2} e^{-(\beta-1)\sqrt{2}x} dx + C \int_{[\theta\sqrt{r}, \theta^{-1}\sqrt{r}]^c} \beta\sqrt{2} e^{-(\beta-1)\sqrt{2}x} dx \\ &= \frac{\beta}{\beta - 1} \left(\varepsilon + C \left(1 - e^{-\theta\sqrt{a}} + e^{-\theta^{-1}\sqrt{a}} \right) \right), \end{aligned} \quad (2.4.21)$$

recalling that $r = a(\beta - 1)^{-2}$. Choosing θ small enough and considering $\beta < 2$, the right-hand side of Equation (2.4.21) is smaller than $3\varepsilon/(\beta - 1)$. Then, we rewrite the double integral on the left-hand side of Equation (2.4.21) by using Fubini's theorem and changing variables with $u = (\beta - 1)\sqrt{2}x$ and $w = s(\beta - 1)^2$, which shows that this double integral equals $\beta\kappa/(\beta - 1)$. This proves the first inequality in Equation (2.4.18) (with 4ε instead of ε). The second inequality is proved by writing

$$\tilde{\mathbb{E}}_t [\mathcal{C}_{t,r_t}^*(f_\beta)\mathbb{1}_{B_{r,t}}] = - \int_0^\infty \tilde{\mathbb{E}}_t [\mathcal{C}_{t,r_t}^*([-x, 0])\mathbb{1}_{B_{r,t}}] \sqrt{2}(\beta\sqrt{2}x - 1) e^{-\beta\sqrt{2}x} dx$$

and then proceeding similarly (note that $\beta\sqrt{2}x - 1$ can be replaced by $\beta\sqrt{2}x$ up to a negligible error as $\beta \downarrow 1$).

Step 4: Conclusion. We first claim that for $\theta \in (0, 1)$ and $\epsilon > 0$, there exists $r_0 > 0$ such that, for any $r \geq r_0$, there exists $t_0 > 0$ such that, for any $t \geq t_0$ and any $v \in [\theta\sqrt{r}, \theta^{-1}\sqrt{r}]$,

$$\left| \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*([-v, 0]) \right] - C_\star e^{\sqrt{2}v} \right| \leq \epsilon e^{\sqrt{2}v}.$$

This is a slightly stronger version of [39, Lemma 5.2] where no uniformity for $v \in [\theta\sqrt{r}, \theta^{-1}\sqrt{r}]$ is stated: however, the aforementioned claim follows from their proof (in a similar way as the proof of Lemma 2.4.7 below). Then, we deduce from this claim that, for $\beta > 1$ close enough to 1, up to a modification of t_0 , we also have, for any $t \geq t_0$,

$$\begin{aligned} |(\beta - 1)\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(f_\beta) \right] - C_\star| &\leq \epsilon, \\ |(\beta - 1)^2\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*(\partial_\beta f_\beta) \right] + C_\star| &\leq \epsilon, \end{aligned}$$

where these inequalities are obtained in the same way as (2.4.18) are obtained from Lemma 2.4.7. Combining this with (2.4.13), (2.4.17) and (2.4.18), we get

$$\mathbb{E} \left[\frac{(S'_\beta)^2}{S_\beta} \right] \geq \frac{1}{(\beta - 1)^3} \left(\frac{(C_\star\kappa' - \epsilon)^2}{C_\star\kappa + \epsilon} + \frac{(C_\star(1 - \kappa') - \epsilon)^2}{C_\star(1 - \kappa) + \epsilon} \right).$$

Letting $\epsilon \rightarrow 0$, this proves

$$\liminf_{\beta \downarrow 1} (\beta - 1)^3 \mathbb{E} \left[\frac{(S'_\beta)^2}{S_\beta} \right] \geq C_\star \left(\frac{(\kappa')^2}{\kappa} + \frac{(1 - \kappa')^2}{1 - \kappa} \right). \quad (2.4.22)$$

By Lemma 2.4.8, we can choose a, b, B such that $\kappa \neq \kappa'$ and, together with the fact that $\kappa, \kappa' \in (0, 1)$, this implies that the right-hand side of (2.4.22) is larger than C_\star . \square

Lemma 2.4.6. *Let $\theta \in (0, 1)$, $\epsilon, a > 0$ and $0 < b < B$. There exists $r_0 > 0$ such that, for any $r \geq r_0$, there exists $t_0 > 0$ such that, for any $t \geq t_0$ and any $v \in [\theta\sqrt{r}, \theta^{-1}\sqrt{r}]$,*

$$\left| \tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*([-v, 0]) \mathbb{1}_{B_{r,t}} \right] - C_\star e^{\sqrt{2}v} \int_0^\infty \varphi_{b,B} \left(\frac{r \vee s}{|r - s|} \right) \frac{ve^{-v^2/(2s)}}{\sqrt{2\pi s^3/2}} ds \right| \leq \epsilon e^{\sqrt{2}v}.$$

Proof. This is mainly a refinement of the proof of Lemma 5.2 in [39], therefore we explain how to adapt their argument and use their notation without introducing it. Recalling the definition of $\tilde{\mathbb{P}}_t$ in (2.2.1), we first have

$$\tilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*([-v, 0]) \mathbb{1}_{B_{r,t}} \right] = \frac{\tilde{\mathbb{E}} \left[\mathcal{C}_{t,r_t}^*([-v, 0]) \mathbb{1}_{B_{r,t}} \mathbb{1}_{\{\max_{x \in L_t} h_t(x) \leq m_t\}} \mid h_t(X_t) = m_t \right]}{\tilde{\mathbb{P}}(\max_{x \in L_t} h_t(x) \leq m_t \mid h_t(X_t) = m_t)}. \quad (2.4.23)$$

Recall from (2.2.10) that the denominator in (2.4.23) is asymptotically equivalent to C_1/t , so we now focus on the numerator.

We introduce the event

$$\mathcal{B}_{r,t} := \left\{ \widehat{W}_{t,r} \in [-B\sqrt{r}, -b\sqrt{r}] \right\},$$

which is analog to $B_{r,t}$ but for the process $(\widehat{W}_{t,s})_{s \in [0,t]}$ defined in [39, Eq. (3.1)] as

$$\widehat{W}_{t,s} := W_s - \gamma_{t,s}, \quad \text{with } \gamma_{t,s} := \frac{3}{2\sqrt{2}} \left(\log_+ s - \frac{s}{t} \log_+ t \right), \quad (2.4.24)$$

for any $0 \leq s \leq t$, with W a standard Brownian motion under \mathbb{P} . Defining, for $v \geq 0$ and $0 \leq s, r \leq t$,

$$j_{t,v,r}(s) := \mathbb{E} \left[J_{t,v}(s) \mathbb{1}_{\mathcal{B}_{r,t}} \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0 \right],$$

where $J_{t,v}(s)$ is introduced in [39, Eq. (5.9)], it follows from the proof of Lemma 5.4 in [39], that

$$\tilde{\mathbb{E}} \left[\mathcal{C}_{t,r,t}^*([-v, 0]) \mathbb{1}_{\mathcal{B}_{r,t}} \mathbb{1}_{\max_{x \in L_t} h_t(x) \leq m_t} \mid h_t(X_t) = m_t \right] = 2 \int_0^{r_t} j_{t,v,r}(s) ds. \quad (2.4.25)$$

As in [39, Eq. (5.17)], for any $M \geq 0$, we split $j_{t,v,r}(s)$ into

$$\begin{aligned} j_{t,v,r}^{<M}(s) &:= \mathbb{E} \left[J_{t,v}(s) \mathbb{1}_{\mathcal{B}_{r,t}} \mathbb{1}_{|\widehat{W}_{t,s}| < M} \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0 \right], \\ j_{t,v,r}^{\geq M}(s) &:= \mathbb{E} \left[J_{t,v}(s) \mathbb{1}_{\mathcal{B}_{r,t}} \mathbb{1}_{|\widehat{W}_{t,s}| \geq M} \mid \widehat{W}_{t,0} = \widehat{W}_{t,t} = 0 \right]. \end{aligned}$$

We postpone the estimate of $j_{t,v,r}^{<M}(s)$ to Lemma 2.4.7 (which replace Lemma 5.6 in [39]). With this lemma in hand, we can conclude the proof. Let $\delta \in (0, \theta \wedge \frac{1}{2})$ and $M > 0$. Considering t large enough such that $\delta^{-1}r \leq r_t$, we decompose the right-hand side of (2.4.25) as

$$2 \int_{[\delta r, \delta^{-1}r]} j_{t,v,r}^{<M}(s) ds + 2 \int_{[\delta r, \delta^{-1}r]} j_{t,v,r}^{\geq M}(s) ds + 2 \int_{[\delta r, \delta^{-1}r]^c} j_{t,v,r}(s) ds,$$

where the two last terms are negligible. Indeed, it is proved in [39, Eq. (5.41)] (and the paragraph around) that

$$\limsup_{\eta \rightarrow 0} \limsup_{M \rightarrow \infty} \limsup_{v \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{t}{e^{\sqrt{2}v}} \cdot \left(2 \int_{[\eta v^2, \eta^{-1}v^2]} j_{t,v}^{\geq M}(s) ds + 2 \int_{[\eta v^2, \eta^{-1}v^2]^c} j_{t,v}(s) ds \right) = 0,$$

and therefore

$$\limsup_{\delta \rightarrow 0} \limsup_{M \rightarrow \infty} \limsup_{r \rightarrow \infty} \limsup_{t \rightarrow \infty} \sup_{v \in [\theta\sqrt{r}, \theta^{-1}\sqrt{r}]} \left(2 \int_{[\delta r, \delta^{-1}r]} j_{t,v,r}^{\geq M}(s) ds + 2 \int_{[\delta r, \delta^{-1}r]^c} j_{t,v,r}(s) ds \right) = 0. \quad (2.4.26)$$

Now, setting $S_\delta := [\delta r, (1 - \delta)r] \cup [(1 + \delta)r, \delta^{-1}r]$, we have

$$2 \int_{[\delta r, \delta^{-1}r]} j_{t,v,r}^{<M}(s) ds = 2 \int_{S_\delta} j_{t,v,r}^{<M}(s) ds + 2 \int_{(1-\delta)r}^{(1+\delta)r} j_{t,v,r}^{<M}(s) ds.$$

This last integral is also negligible for δ small enough (in the same sense as in Equation (2.4.26)), because

$$j_{t,v,r}^{<M}(s) \leq j_{t,v}(s) = j_{t,v}^{\geq 0}(s) \leq C \frac{(v+1)e^{\sqrt{2}v}}{s^{3/2}t} \leq C(\theta) \frac{e^{\sqrt{2}v}}{rt},$$

for any $M > 0$, $0 \leq r \leq t/4$, $s \in [(1 - \delta)r, (1 + \delta)r]$ and $v \in [\theta\sqrt{r}, \theta^{-1}\sqrt{r}]$ by [39, Lemma 5.5]. Finally, using Lemma 2.4.7, we have, as $t \rightarrow \infty$, then $r \rightarrow \infty$ and then $M \rightarrow \infty$, uniformly in $v \in [\theta\sqrt{r}, \theta^{-1}\sqrt{r}]$,

$$2 \int_{S_\delta} j_{t,v,r}^{<M}(s) ds \sim \frac{2C_2}{t} ve^{\sqrt{2}v} \int_{S_\delta} \frac{e^{-v^2/(2s)}}{s^{3/2}} \varphi_{b,B} \left(\frac{r \vee s}{|r - s|} \right) ds.$$

Coming back to (2.4.23) and letting $\delta \rightarrow 0$, this proves that there exists $r_0 > 0$ such that, for any $r \geq r_0$, there exists $t_0 > 0$ such that, for any $t \geq t_0$ and any $v \in [\theta\sqrt{r}, \theta^{-1}\sqrt{r}]$,

$$\left| \widetilde{\mathbb{E}}_t \left[\mathcal{C}_{t,r_t}^*([-v, 0]) \mathbb{1}_{B_{r,t}} \right] - \frac{2C_2}{C_1} ve^{\sqrt{2}v} \int_0^\infty \varphi_{b,B} \left(\frac{r \vee s}{|r - s|} \right) \frac{e^{-v^2/(2s)}}{s^{3/2}} ds \right| \leq \varepsilon e^{\sqrt{2}v}. \quad (2.4.27)$$

Using the relation² $C_\star = 2\sqrt{2\pi} C_2/C_1$ gives the desired result. \square

Lemma 2.4.7. *Let $a > 0$, $0 < b < B$ and $\delta \in (0, 1/2)$. As $t \rightarrow \infty$, then $r \rightarrow \infty$ and then $M \rightarrow \infty$, we have*

$$j_{t,v,r}^{<M}(s) \sim \frac{C_2}{ts^{3/2}} ve^{\sqrt{2}v - v^2/(2s)} \varphi_{b,B} \left(\frac{r \vee s}{|r - s|} \right),$$

uniformly $s \in [\delta r, (1 - \delta)r] \cup [(1 + \delta)r, \delta^{-1}r]$ and $v \in [\delta\sqrt{r}, \delta^{-1}\sqrt{r}]$, with

$$C_2 := \frac{2\sqrt{2}}{\pi} f^{(0)}(0)g(0) \int_{\mathbb{R}} f(z)g(z)^2 e^{\sqrt{2}z} dz \in (0, \infty),$$

where $f^{(0)}$, f , g are positive functions introduced in [39, Lemma 3.4].

Proof. We follow ideas from the proof of Lemma 5.6 in [39], but instead of only distinguishing according to the value of $\widehat{W}_{t,s}$, we also distinguish according to the value of $\widehat{W}_{t,r}$. For comparison, the constant C appearing in the statement of Lemma 5.6 in [39] equals C_2 introduced here.

We start with the case $r \leq s$, that is $s \in [(1 + \delta)r, \delta^{-1}r]$. Then, we have, with the notations from [39],

$$j_{t,v,r}^{<M}(s) = \int_{-B\sqrt{r}}^{-b\sqrt{r}} \int_{-M}^M p_t((r, y); (s, z)) q_t((0, 0); (r, y)) q_t((r, y); (s, z)) q_t((s, z); (t, 0)) e_{s,v}(z) dz dy.$$

The function $(y, z) \mapsto p_t((r, y); (s, z))$ is the density of $(\widehat{W}_{t,r}, \widehat{W}_{t,s})$ given $\widehat{W}_{t,0} = \widehat{W}_{t,t} = 0$. It is explicitly given by (recall the definition of $\gamma_{t,s}$ in Equation (2.4.24))

$$\begin{aligned} p_t((r, y); (s, z)) &= \frac{1}{2\pi} \sqrt{\frac{t}{r(s-r)(t-s)}} \exp \left(-\frac{(s(y + \gamma_{t,r}) - r(z + \gamma_{t,s}))^2}{2rs(s-r)} - \frac{t(z + \gamma_{t,s})^2}{2s(t-s)} \right) \\ &\sim \frac{1}{2\pi\sqrt{r(s-r)}} \exp \left(-\frac{sy^2}{2r(s-r)} \right), \end{aligned}$$

²The authors of [39] do not keep precisely track of constants but this relation can be deduced from a careful reading of their paper. Alternatively letting $b \rightarrow 0$ and $B \rightarrow \infty$, we have $\mathbb{1}_{B_{r,t}} \rightarrow 1$ and $\varphi_{b,B} \rightarrow 1$ and (2.4.27) gives an alternative proof of their Lemma 5.2, showing that the constant C appearing there equals $2\sqrt{2\pi}C_2/C_1$. Then, a quick look at the proof of Proposition 1.5 in [39] ensures that this constant C is actually C_\star .

as $t \rightarrow \infty$ and then $r \rightarrow \infty$, uniformly in $s \in [(1 + \delta)r, \delta^{-1}r]$, $y \in [-B\sqrt{r}, -b\sqrt{r}]$ and $z \in [-M, M]$. Furthermore, it follows from [39, Lemma 3.4] that

$$\begin{aligned} q_t((0, 0); (r, y)) &= q_r((0, 0); (r, y)) \sim \frac{2(-y)f^{(0)}(0)}{r}, \\ q_t((r, y); (s, z)) &= q_s((r, y); (s, z)) \sim \frac{2(-y)g(z)}{s - r}, \\ q_t((s, z); (t, 0)) &\sim \frac{2f^{(s)}(z)g(0)}{t - s} \sim \frac{2f(z)g(0)}{t}, \end{aligned}$$

as $t \rightarrow \infty$ and then $r \rightarrow \infty$, uniformly in s, y and z as before. Finally, it follows from [39, Lemma 4.2] (see also Eq. (5.37) in [39]) that

$$e_{s,v}(z) \sim ve^{\sqrt{2}v - v^2/(2s)} \frac{g(z)}{\sqrt{\pi}} e^{\sqrt{2}z},$$

as $r \rightarrow \infty$, uniformly in $v \in [\delta\sqrt{r}, \delta^{-1}\sqrt{r}]$ and s, z as before. Altogether, this proves

$$j_{t,v,r}^{<M}(s) \sim \frac{4}{\pi^{3/2}} \frac{f^{(0)}(0)g(0)}{tr^{3/2}(s-r)^{3/2}} ve^{\sqrt{2}v - v^2/(2s)} \int_{-B\sqrt{r}}^{-b\sqrt{r}} y^2 e^{-sy^2/(2r(s-r))} dy \int_{-M}^M f(z)g(z)^2 e^{\sqrt{2}z} dz,$$

as $t \rightarrow \infty$ and then $r \rightarrow \infty$, uniformly in s and v as before. A change of variable in the first integral and letting $M \rightarrow \infty$ in the second integral (the fact that this integral converges to a finite limit is justified at the end of the proof of Lemma 5.6 in [39]) yields the result.

The case $s \leq r$, that is $s \in [\delta r, (1 - \delta)r]$ is similar: we write

$$j_{t,v,r}^{<M}(s) = \int_{-B\sqrt{r}}^{-b\sqrt{r}} \int_{-M}^M p_t((s, z); (r, y)) q_t((0, 0); (s, z)) q_t((s, z); (r, y)) q_t((r, y); (t, 0)) e_{s,v}(z) dz dy$$

and use the same asymptotics as before, the main difference being

$$\begin{aligned} p_t((s, z); (r, y)) &= \frac{1}{2\pi} \sqrt{\frac{t}{s(r-s)(t-r)}} \exp\left(-\frac{(r(z + \gamma_{t,s}) - s(y + \gamma_{t,r}))^2}{2rs(r-s)} - \frac{t(y + \gamma_{t,r})^2}{2r(t-r)}\right) \\ &\sim \frac{1}{2\pi\sqrt{s(r-s)}} \exp\left(-\frac{y^2}{2(s-r)}\right). \end{aligned}$$

This yields the result in that case. □

Lemma 2.4.8. *Recall the definition of κ and κ' in (2.4.19). There exist $a > 0$ and $0 < b < B$ such that $\kappa \neq \kappa'$.*

Proof. We proceed by contradiction: assume that, for any $a > 0$ and $0 < b < B$, $\kappa = \kappa'$. Then, differentiating the relation $\kappa = \kappa'$ w.r.t. B , we get, for any $a, B > 0$, the following quantity is the same for $k = 1$ and $k = 2$:

$$\int_0^\infty \frac{a \vee w}{|a - w|} \exp\left(-\frac{B^2}{2} \frac{a \vee w}{|a - w|}\right) \left(\int_0^\infty u^k e^{-u - u^2/(4w)} du\right) \frac{dw}{w^{3/2}}.$$

On the other hand, a direct calculation shows that, for $k \in \{1, 2\}$,

$$\int_0^\infty \left(\int_0^\infty u^k e^{-u-u^2/(4w)} du \right) \frac{dw}{w^{3/2}} = 2\sqrt{\pi}$$

and therefore is the same for $k = 1$ and $k = 2$. Therefore, we deduce that the following quantity is the same for $k = 1$ and $k = 2$:

$$\int_0^\infty \left(\exp \left(-\frac{B^2}{2} \frac{a \vee w}{|a-w|} \right) - e^{-B^2/2} \right) \left(\int_0^\infty u^k e^{-u-u^2/(4w)} du \right) \frac{dw}{w^{3/2}}. \quad (2.4.28)$$

Our goal is now to study the behavior as $a \rightarrow 0$ of this quantity, to find a contradiction. We decompose the main integral in Equation (2.4.28) into a part $w \in [0, a]$ in which we change w to $x = a/(a-w)$ and u to $v = u(x/(a(x-1)))^{1/2}$, and a part $w \in (a, \infty)$ in which we change w to $x = w/(w-a)$ and u to $v = u((x-1)/(ax))^{1/2}$. This yields that the following quantity is the same for $k = 1$ and $k = 2$:

$$\begin{aligned} a^{k/2} \int_1^\infty \left(x e^{-xB^2/2} - e^{-B^2/2} \right) & \left(\frac{(x-1)^{k/2-1}}{x^{k/2+1}} \int_0^\infty v^k e^{-v^2/4-v(\frac{a(x-1)}{x})^{1/2}} dv \right. \\ & \left. + \frac{x^{k/2-1}}{(x-1)^{k/2+1}} \int_0^\infty v^k e^{-v^2/4-v(\frac{ax}{x-1})^{1/2}} dv \right) dx. \end{aligned} \quad (2.4.29)$$

For $k = 1$, both integrals w.r.t. v in Equation (2.4.29) converge as $a \rightarrow 0$ towards $\int_0^\infty v e^{-v^2/4} dv = 2$ by dominated convergence. Hence, by dominated convergence again but in the integral w.r.t. x , (2.4.29) equals

$$2a^{1/2} \int_1^\infty \left(x e^{-xB^2/2} - e^{-B^2/2} \right) \left(\frac{(x-1)^{-1/2}}{x^{3/2}} + \frac{x^{-1/2}}{(x-1)^{3/2}} \right) dx + o(a^{1/2}),$$

as $a \rightarrow 0$. For $k = 2$, we cannot use the same argument: the second domination cannot be justified. Instead, we bound the first integral w.r.t. v by a constant and, for the second one, for some $\varepsilon > 0$, we write

$$\int_0^\infty v^2 e^{-v^2/4-v(\frac{ax}{x-1})^{1/2}} dv = \left(\frac{x-1}{ax} \right)^{3\varepsilon} \int_0^\infty u^2 e^{-u^2(\frac{x-1}{ax})^{2\varepsilon}/4-u(\frac{ax}{x-1})^{1/2-\varepsilon}} du \leq C \left(\frac{x-1}{ax} \right)^{3\varepsilon},$$

by bounding the last integral by $\int_0^\infty u^2 e^{-u^2/4} du$ if $(x-1)/ax \geq 1$ and by $\int_0^\infty u^2 e^{-u} du$ otherwise. With $\varepsilon < 1/6$, this proves that (2.4.29) is a $o(a^{1/2})$ for $k = 2$. Since (2.4.29) is the same for $k = 1$ and $k = 2$, this implies

$$\int_1^\infty \left(x e^{-xB^2/2} - e^{-B^2/2} \right) \left(\frac{(x-1)^{-1/2}}{x^{3/2}} + \frac{x^{-1/2}}{(x-1)^{3/2}} \right) dx = 0, \quad (2.4.30)$$

for any $B > 0$. But the left-hand side of Equation (2.4.30) tends to infinity as $B \rightarrow \infty$, so this is a contradiction and concludes the proof. \square

2.5 Appendix

2.5.1 Generic properties for the REM

Proof of Proposition 2.2.1

Proof. (i) $\mathbb{E} Z(\beta) > \mathbb{E} \frac{1}{\eta_1^\beta} = +\infty$ since $\eta_1 \sim \mathcal{E}(1)$.

(ii) For $\varepsilon \in (0, 1)$, the law of large numbers for the (η_k) provides a $N = N(\varepsilon)$ such that $(1 - \varepsilon)k \leq \eta_k \leq (1 + \varepsilon)k$ for $n > N$, thus

$$\sum_{k=1}^N \frac{1}{\eta_k^\beta} + \frac{1}{(1 + \varepsilon)^\beta} \sum_{k>N} \frac{1}{k^\beta} \leq Z(\beta) \leq \sum_{k=1}^N \frac{1}{\eta_k^\beta} + \frac{1}{(1 - \varepsilon)^\beta} \sum_{k>N} \frac{1}{k^\beta},$$

therefore,

$$\frac{1}{1 + \varepsilon} \leq \liminf_{\beta \downarrow 1} (\beta - 1)Z(\beta) \leq \limsup_{\beta \downarrow 1} (\beta - 1)Z(\beta) \leq \frac{1}{1 - \varepsilon},$$

and letting $\varepsilon \rightarrow 0$ concludes the proof.

(iii) The Laplace functional of a Poisson point process gives the following expression, for $t \geq 0$,

$$\mathbb{E} e^{-tZ(\beta)} = \exp \left\{ - \int_0^\infty (1 - e^{-tx^{-\beta}}) dx \right\} = \exp \left\{ -\Gamma \left(1 - \frac{1}{\beta} \right) t^{\frac{1}{\beta}} \right\},$$

which is the Laplace transform of a stable law whose parameters are given in the aforementioned proposition. \square

Proof of Proposition 2.2.2

Proof. (i) Observe that

$$\frac{1}{\beta} \log Z(\beta) = \frac{1}{\beta} \log \sum_{k \geq 0} e^{\beta \xi_k} \geq \frac{1}{\beta} \log e^{\beta \xi_1} = \xi_1,$$

and

$$\frac{1}{\beta} \log Z(\beta) = \xi_1 + \frac{1}{\beta} \log \left(1 + \sum_{k \geq 2} e^{\beta(\xi_k - \xi_1)} \right) \leq \xi_1 + \frac{1}{\beta} \sum_{k \geq 2} e^{\beta(\xi_k - \xi_1)}.$$

Since the sum in the last term is finite a.s. and decreasing in β , the conclusion follows. The L^1 case can be proved with the previous inequalities or by noticing that $\left(\frac{1}{\beta} \log Z(\beta) \right)_{\beta > 2}$ is bounded in L^2 .

(ii) The moments of $\log Z(\beta)$ stem from the following lemma.

Lemma 2.5.1. *For $\beta > 1$ and $\alpha < \frac{1}{2\beta e}$, one has*

$$\mathbb{E} [Z(\beta)^\alpha] = \sum_n \frac{\mathbb{E} \log^n Z(\beta)}{n!} \alpha^n.$$

In particular,

$$\mathbb{E} \log Z(\beta) = \gamma(\beta - 1) + \beta \log \Gamma \left(1 - \frac{1}{\beta} \right), \quad (2.5.1)$$

$$\mathbb{E} \log^2 Z(\beta) = \frac{1}{6} \pi^2 (\beta^2 - 1) + \gamma^2 (\beta - 1)^2 + \beta \log \Gamma \left(1 - \frac{1}{\beta} \right) \left(2\gamma(\beta - 1) + \beta \log \Gamma \left(1 - \frac{1}{\beta} \right) \right). \quad (2.5.2)$$

Proof. From the classical inequality $|\log u| \leq u \vee \frac{1}{u}$, one gets for any $\epsilon > 0$, $|\log u| \leq \frac{1}{\epsilon} u^\epsilon \vee \frac{1}{u^\epsilon}$ and therefore,

$$|\log u|^n \leq \frac{1}{\epsilon^n} \left(u^{\epsilon n} + \frac{1}{u^{\epsilon n}} \right),$$

then taking $\epsilon = \frac{1}{2\beta n}$ yields

$$\mathbb{E} \left[|\log Z(\beta)|^n \right] \leq (2\beta n)^n \left(\mathbb{E} Z(\beta)^{\frac{1}{2\beta}} + \mathbb{E} \frac{1}{Z(\beta)^{\frac{1}{2\beta}}} \right),$$

proving that the serie is well defined and that its radius of convergence is greater than $\frac{1}{2\beta e}$ (in fact strictly greater but we won't need it). The two formulae follow by derivating the explicit form in Lemma 2.5.3. \square

\square

REM formulae

Proposition 2.5.2 (Palm formula). *Let Π be a PPP(μ) where μ is a non-zero σ -finite positive measure and F be a positive measurable function, then*

$$\mathbb{E} \left[\sum_{X \in \Pi} F(X, \Pi \setminus \{X\}) \right] = \int \mu(dx) \mathbb{E}[F(x, \Pi)].$$

Lemma 2.5.3. *For any $\beta > 1$ and any $\alpha > -\frac{1}{\beta}$,*

$$\mathbb{E} \frac{1}{Z(\beta)^\alpha} = \frac{\Gamma(\alpha\beta + 1)}{\Gamma(\alpha + 1) \Gamma\left(1 - \frac{1}{\beta}\right)^{\alpha\beta}}.$$

Proof. By taking the expectation of $\frac{1}{Z(\beta)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-tZ(\beta)} dt$, when $\alpha > 0$ and the expectation of $\frac{1}{Z(\beta)^\alpha} = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty t^\alpha Z(\beta) e^{-tZ(\beta)} dt$, when $-1/\beta < \alpha < 0$. \square

This formula yields to the following estimates

$$\mathbb{E} \frac{1}{Z(\beta)^\alpha} \underset{\beta \downarrow 1}{\sim} (\beta - 1)^\alpha, \quad (2.5.3)$$

$$\mathbb{E} \frac{1}{Z(\beta)^{1/\beta}}, \underset{\beta \downarrow 1}{\sim} (\beta - 1), \quad (2.5.4)$$

$$\mathbb{E}[Z(\beta)^\alpha] \underset{\alpha \rightarrow 0}{=} 1 + \left(\gamma(\beta - 1) + \beta \log \Gamma \left(1 - \frac{1}{\beta} \right) \right) \alpha \quad (2.5.5)$$

$$+ \left\{ \frac{1}{6} \pi^2 (\beta^2 - 1) + \gamma^2 (\beta - 1)^2 + \right. \quad (2.5.6)$$

$$\left. \beta \log \Gamma \left(1 - \frac{1}{\beta} \right) \left(2\gamma(\beta - 1) + \beta \log \Gamma \left(1 - \frac{1}{\beta} \right) \right) \right\} \frac{\alpha^2}{2} + O(\alpha^3). \quad (2.5.7)$$

Lemma 2.5.4. For $r, \beta > 1$,

$$\mathbb{E} \frac{Z(r\beta)}{Z(\beta)^r} = \frac{\Gamma \left(r - \frac{1}{\beta} \right)}{\Gamma(r) \Gamma \left(1 - \frac{1}{\beta} \right)}.$$

Proof. Using Palm formula (see Proposition 2.5.2) and Fubini's theorem, one gets

$$\begin{aligned} \mathbb{E} \frac{Z(r\beta)}{Z(\beta)^r} &= \mathbb{E} \int_0^\infty \frac{dx}{(1 + x^\beta Z(\beta))^r} = \mathbb{E} [Z(\beta)^{-1/\beta}] \int_0^\infty \frac{dx}{(1 + x^\beta)^r} \\ &= \frac{1}{\Gamma \left(1 + \frac{1}{\beta} \right) \Gamma \left(1 - \frac{1}{\beta} \right)} \frac{\Gamma \left(1 + \frac{1}{\beta} \right) \Gamma \left(r - \frac{1}{\beta} \right)}{\Gamma(r)}, \end{aligned}$$

where the last integral in the first line is obtained by the change of variables $1/u = 1 + x^\beta$. \square

Proposition 2.5.5. For $\beta, \beta' > 1$, we have

$$F(\beta, \beta') = \beta \int_0^\infty dx \frac{\int_0^\infty e^{-y - xy^{\beta'/\beta}} y^{\frac{\beta'-1}{\beta}} dy}{\int_0^\infty \left(1 - e^{-u - xu^{\beta'/\beta}} \right) u^{-1 - \frac{1}{\beta}} du}.$$

Proof. Let us first observe that

$$\begin{aligned}
 F(\beta, \beta') &= \int_0^\infty dx \mathbb{E} \frac{1}{(1 + x^\beta Z(\beta))(1 + x^{\beta'} Z(\beta'))} \\
 &= \int_0^\infty dx \int_0^\infty \int_0^\infty dt dt' \mathbb{E} e^{-t(1+x^\beta Z(\beta)) - t'(1+x^{\beta'} Z(\beta'))} \\
 &= \int_0^\infty dx \int_0^\infty \int_0^\infty dt dt' e^{-t} e^{-t'} \exp \left\{ -x \int_0^\infty \left(1 - e^{-tu^{-\beta} - t'u^{-\beta'}} \right) du \right\} \\
 &= \int_0^\infty \int_0^\infty dt dt' e^{-t} e^{-t'} \frac{1}{\int_0^\infty \left(1 - e^{-tu^{-\beta} - t'u^{-\beta'}} \right) du} \\
 &= \int_0^\infty \int_0^\infty dt dt' e^{-t} e^{-t'} \frac{\beta t^{-\frac{1}{\beta}}}{\int_0^\infty \left(1 - e^{-u-t't^{-\frac{\beta'}{\beta}} u^{\frac{\beta'}{\beta}}} \right) u^{-1-\frac{1}{\beta}} du}.
 \end{aligned}$$

With the change of variables $x = t' t^{-\frac{\beta'}{\beta}}$ and $y = t$, one gets

$$\begin{aligned}
 F(\beta, \beta') &= \int_0^\infty \int_0^\infty dx dy e^{-y} e^{-xy^{\frac{\beta'}{\beta}}} \frac{\beta y^{\frac{\beta'-1}{\beta}}}{\int_0^\infty \left(1 - e^{-u-xu^{\frac{\beta'}{\beta}}} \right) u^{-1-\frac{1}{\beta}} du} \\
 &= \beta \int_0^\infty dx \frac{\int_0^\infty e^{-y-xy^{\beta'/\beta}} y^{\frac{\beta'-1}{\beta}} dy}{\int_0^\infty \left(1 - e^{-u-xu^{\beta'/\beta}} \right) u^{-1-\frac{1}{\beta}} du}.
 \end{aligned}$$

□

Proposition 2.5.6. *For $\beta > 1$, we have*

$$\text{Var} \frac{Z'(\beta)}{Z(\beta)} = \frac{\pi^2}{6} + \frac{\beta-1}{\beta^3} \left(\frac{\Gamma''\left(\frac{\beta-1}{\beta}\right)}{\Gamma\left(\frac{\beta-1}{\beta}\right)} - \left(\frac{\Gamma'\left(\frac{\beta-1}{\beta}\right)}{\Gamma\left(\frac{\beta-1}{\beta}\right)} \right)^2 \right).$$

Proof. To lighten notations, let us denote $\Gamma = \Gamma\left(\frac{\beta-1}{\beta}\right)$ and similarly $\Gamma' = \Gamma'\left(\frac{\beta-1}{\beta}\right)$, $\Gamma'' = \Gamma''\left(\frac{\beta-1}{\beta}\right)$. Applying again Palm formula allows the following computation

$$\begin{aligned}
 \mathbb{E} \frac{Z''(\beta)}{Z(\beta)} &= \int_0^\infty \log^2 x \mathbb{E} \frac{1}{1 + x^\beta Z(\beta)} dx \\
 &= \int_0^\infty dx \log^2 x \int_0^\infty e^{-t} \mathbb{E} e^{-tx^\beta Z(\beta)} dt \\
 &= \int_0^\infty dt e^{-t} \int_0^\infty \log^2 x e^{-\Gamma t^{\frac{1}{\beta}} x} dx \\
 &= \frac{1}{\Gamma} \int_0^\infty dt t^{-\frac{1}{\beta}} e^{-t} \int_0^\infty \left(\log u - \log t^{\frac{1}{\beta}} \Gamma \right)^2 e^{-u} du \\
 &= \frac{1}{\Gamma} \int_0^\infty dt t^{-\frac{1}{\beta}} e^{-t} \left(\Gamma''(1) - 2 t^{\frac{1}{\beta}} \Gamma'(1) \log \Gamma + t^{\frac{1}{\beta}} \log^2 \Gamma \right) \\
 &= \gamma^2 + \frac{\pi^2}{6} + 2\gamma \log \Gamma + \log^2 \Gamma + \frac{2\gamma}{\beta} \frac{\Gamma'}{\Gamma} + \frac{2}{\beta} \frac{\Gamma'}{\Gamma} \log \Gamma + \frac{1}{\beta^2} \frac{\Gamma''}{\Gamma}.
 \end{aligned}$$

Moreover, differentiating twice on both sides

$$\mathbb{E} \log Z(\beta) = \beta \log \Gamma \left(\frac{\beta - 1}{\beta} \right) + \gamma(\beta - 1),$$

implies

$$\mathbb{E} \left[\frac{Z''(\beta)}{Z(\beta)} - \left(\frac{Z'(\beta)}{Z(\beta)} \right)^2 \right] = \frac{1}{\beta^3} \left(\frac{\Gamma''}{\Gamma} - \left(\frac{\Gamma'}{\Gamma} \right)^2 \right).$$

Therefore

$$\begin{aligned} \mathbb{E} \left[\left(\frac{Z'(\beta)}{Z(\beta)} \right)^2 \right] &= \mathbb{E} \frac{Z''(\beta)}{Z(\beta)} - \mathbb{E} \left[\frac{Z''(\beta)}{Z(\beta)} - \left(\frac{Z'(\beta)}{Z(\beta)} \right)^2 \right] \\ &= \gamma^2 + \frac{\pi^2}{6} + 2\gamma \log \Gamma + \log^2 \Gamma + \frac{2\gamma}{\beta} \frac{\Gamma'}{\Gamma} + \frac{2}{\beta} \frac{\Gamma'}{\Gamma} \log \Gamma + \frac{\beta - 1}{\beta^3} \frac{\Gamma''}{\Gamma} + \frac{1}{\beta^3} \left(\frac{\Gamma'}{\Gamma} \right)^2. \end{aligned}$$

And finally, using

$$\mathbb{E} \frac{Z'(\beta)}{Z(\beta)} = \log \Gamma + \frac{1}{\beta} \frac{\Gamma'}{\Gamma} + \gamma,$$

we obtain

$$\text{Var} \frac{Z'(\beta)}{Z(\beta)} = \mathbb{E} \left[\left(\frac{Z'(\beta)}{Z(\beta)} \right)^2 \right] - \mathbb{E} \left[\frac{Z'(\beta)}{Z(\beta)} \right]^2 = \frac{\pi^2}{6} + \frac{\beta - 1}{\beta^3} \left(\frac{\Gamma''}{\Gamma} - \left(\frac{\Gamma'}{\Gamma} \right)^2 \right).$$

□

Lemma 2.5.7. For any $\beta, \beta' > 1$,

$$\mathbb{E} \left[Z(\beta') e^{-tZ(\beta)} \right] = \frac{1}{\beta} t^{\frac{1-\beta'}{\beta}} \Gamma \left(\frac{\beta' - 1}{\beta} \right) \mathbb{E} e^{-tZ(\beta)}.$$

Proof. It is a consequence of Palm formula again, indeed

$$\begin{aligned} \mathbb{E} \left[Z(\beta') e^{-tZ(\beta)} \right] &= \mathbb{E} \left[\sum_k \frac{1}{\eta_k^{\beta'}} e^{-t/\eta_k^\beta} \exp \left\{ -t \sum_{j \neq k} \frac{1}{\eta_j^\beta} \right\} \right] \\ &= \int_0^\infty \frac{1}{x^{\beta'}} e^{-t/x^\beta} \mathbb{E} e^{-tZ(\beta)} dx \\ &= \mathbb{E} e^{-tZ(\beta)} \int_0^\infty \left(\frac{u}{t} \right)^{\beta'/\beta} e^{-u} t^{1/\beta} \frac{du}{\beta u^{1/\beta+1}}. \end{aligned}$$

□

Lemma 2.5.8.

$$\text{Var} \frac{1}{Z(\beta)} \sim (\beta - 1)^3, \quad \beta \rightarrow 1^+.$$

Proof. Using Lemma 2.5.3, one gets

$$\text{Var} \frac{1}{Z(\beta)} = \Gamma \left(\frac{\beta - 1}{\beta} \right)^{-2\beta} \left(\frac{1}{2} \Gamma(1 + 2\beta) - \Gamma(1 + \beta)^2 \right).$$

and a Taylor expansion concludes the proof. \square

2.5.2 The decorations

General formulae

Using $Z_d(\beta') \stackrel{(d)}{=} \mathbb{E} [e^{Y_1}]^{\beta'} Z_{\beta'}$ and $\mathbb{E} [e^{Y_1}] = \mathbb{E} [S_{\beta'}^{1/\beta'}] / \mathbb{E} [S_{\beta}^{1/\beta}]$ we get

$$\mathbb{E} [Z_d(\beta')^\alpha] = \left(\frac{\mathbb{E} [S_{\beta'}^{1/\beta'}]}{\mathbb{E} [S_{\beta}^{1/\beta}]} \right)^{\alpha\beta'} \mathbb{E} [Z_{\beta'}^\alpha]. \quad (2.5.8)$$

Lemma 2.5.9. *Assume that $\mathbb{E}[S_\beta] \underset{\beta \downarrow 1}{\sim} \frac{C_\star}{\beta-1}$ and $\mathbb{E}[S_\beta^{1/\beta}] \underset{\beta \downarrow 1}{\sim} \mathbb{E}[S_\beta]$. Then*

$$\mathbb{E} \left[\left(1 - Z_d(\beta')^{\frac{\beta-1}{\beta'}} \right)^2 \right] = o(\beta - 1), \quad \beta \rightarrow 1^+.$$

Proof. Let us denote $h = \beta - 1$. Expanding

$$\begin{aligned} \mathbb{E} \left[\left(1 - Z_d(\beta')^{\frac{h}{\beta'}} \right)^2 \right] &= 1 - 2\mathbb{E} \left[Z_d(\beta')^{\frac{h}{\beta'}} \right] + \mathbb{E} \left[Z_d(\beta')^{\frac{2h}{\beta'}} \right] \\ &= 1 - 2 \left(\frac{\mathbb{E} [S_{\beta'}^{1/\beta'}]}{\mathbb{E} [S_{\beta}^{1/\beta}]} \right)^h \mathbb{E} \left[Z_{\beta'}^{\frac{h}{\beta'}} \right] + \left(\frac{\mathbb{E} [S_{\beta'}^{1/\beta'}]}{\mathbb{E} [S_{\beta}^{1/\beta}]} \right)^{2h} \mathbb{E} \left[Z_{\beta'}^{\frac{2h}{\beta'}} \right] \end{aligned}$$

and combining with Equation (2.5.5) concludes the proof. \square

Lemma 2.5.10. *Assume that $\mathbb{E}[S_\beta] \underset{\beta \downarrow 1}{\sim} \mathbb{E}[S_\beta^{1/\beta}] \underset{\beta \downarrow 1}{\sim} \frac{C_\star}{\beta-1}$. Then, for every $\varepsilon > 0$,*

$$\mathbb{E} \left[\frac{Z_d(\beta')^{\frac{1}{\beta'}}}{Z(\beta)^{\frac{1}{\beta}}} \right] = O \left((\beta - 1)^{2-\varepsilon} \right), \quad \beta \rightarrow 1^+.$$

Proof. Let $\nu \in (0, 1)$ and $p, q > 1$ be such that $1/p + 1/q = 1$ and which value will be chosen later. If $\beta < \beta'$, one has

$$\begin{aligned}
 \mathbb{E} \left[\frac{Z_d(\beta')^{\frac{1}{\beta'}}}{Z(\beta)^{\frac{1}{\beta}}} \right] &= \mathbb{E} \left[\left(\frac{Z_d(\beta')^{\frac{1}{\beta'}}}{Z(\beta)^{\frac{1}{\beta}}} \right)^{\nu\beta} Z_d(\beta')^{\frac{1-\nu\beta}{\beta'}} \frac{1}{Z(\beta)^{\frac{1}{\beta}-\nu}} \right] \\
 &\leq \mathbb{E} \left[Z_d(\beta')^{\frac{1-\nu\beta}{\beta'}} \frac{1}{Z(\beta)^{\frac{1}{\beta}-\nu}} \right] \\
 &\leq \mathbb{E} \left[Z_d(\beta')^{p\frac{1-\nu\beta}{\beta'}} \right]^{\frac{1}{p}} \mathbb{E} \left[\frac{1}{Z(\beta)^{\frac{q}{\beta}-q\nu}} \right]^{\frac{1}{q}} \\
 &= \underbrace{\left(\frac{\mathbb{E} [S_{\beta'}^{1/\beta'}]}{\mathbb{E} [S_{\beta}^{1/\beta}]} \right)^{1-\nu\beta}}_{\underset{\beta \downarrow 1}{\sim} (\beta-1)^{1-\nu}} \mathbb{E} \left[Z_{\beta'}^{p\frac{1-\nu\beta}{\beta'}} \right]^{\frac{1}{p}} \underbrace{\mathbb{E} \left[\frac{1}{Z(\beta)^{\frac{q}{\beta}-q\nu}} \right]^{\frac{1}{q}}}_{\underset{\beta \downarrow 1}{\sim} (\beta-1)^{1-\nu}},
 \end{aligned}$$

and the middle term is finite (and uniformly bounded over β) if $p(1-\nu) < 1$. \square

Existence of κ_d

Lemma 2.5.11. *Suppose that $\mathbb{E}S_{\beta} < \infty$ for every $\beta > 1$, then the following function*

$$(1, \infty) \ni \beta \mapsto f(\beta) := \mathbb{E} [\log Z_d(\beta') \log Z_d(\beta)],$$

is C^2 for every $\beta' > 1$ and its second derivative is given by the second derivative under \mathbb{E} .

Proof. Let us write

$$Z_d(\beta) = e^{\beta\xi_1} \sum e^{\beta E_k},$$

where $(E_k)_{k \geq 1}$ is the decreasing reordering of $\{\xi_i + d_{i,j} - \xi_1; i, j\}$. Note that $E_1 = 0$ and rewrite f as

$$\begin{aligned}
 f(\beta) &= \mathbb{E} \left[\log Z_d(\beta') \log \left(e^{\beta\xi_1} \sum e^{\beta E_k} \right) \right] \\
 &= \beta \mathbb{E} [\xi_1 \log Z_d(\beta')] + \mathbb{E} \left[\log Z_d(\beta') \log \sum e^{\beta E_k} \right].
 \end{aligned}$$

Thus f is C^1 with

$$f'(\beta) = \mathbb{E} [\xi_1 \log Z_d(\beta')] + \mathbb{E} \left[\log Z_d(\beta') \frac{\sum E_k e^{\beta E_k}}{\sum e^{\beta E_k}} \right],$$

since for every $\beta > \beta_1$ given a $\beta_1 > 1$, one has

$$\begin{aligned}
 \left| \log Z_d(\beta') \frac{\sum E_k e^{\beta E_k}}{\sum e^{\beta E_k}} \right| &\leq |\log Z_d(\beta')| \sum |E_k| e^{\beta E_k} \\
 &\leq |\log Z_d(\beta')| \sum |E_k| e^{\beta_1 E_k}.
 \end{aligned}$$

And this last term has a finite expectation. For the second derivative, one has

$$f''(\beta) = \mathbb{E} \left[\log Z_d(\beta') \left(\frac{\sum E_k^2 e^{\beta E_k}}{\sum e^{\beta E_k}} - \left(\frac{\sum E_k e^{\beta E_k}}{\sum e^{\beta E_k}} \right)^2 \right) \right],$$

since for every $\beta > \beta_1$, for a given $\beta_1 > 1$, one has

$$\begin{aligned} \left| \log Z_d(\beta') \left(\frac{\sum E_k^2 e^{\beta E_k}}{\sum e^{\beta E_k}} - \left(\frac{\sum E_k e^{\beta E_k}}{\sum e^{\beta E_k}} \right)^2 \right) \right| &\leq |\log Z_d(\beta')| \frac{\sum E_k^2 e^{\beta E_k}}{\sum e^{\beta E_k}} \\ &\leq |\log Z_d(\beta')| \sum E_k^2 e^{\beta_1 E_k}, \end{aligned}$$

which has again finite expectation. \square

2.5.3 Crossed moments of level sets of the decoration of the BBM

Our aim in this section is to prove Proposition 2.2.6. For this, we follow the proof of Proposition 1.5 in [39], which bounds $\mathbb{E}[\mathcal{C}([-v, 0])^2]$. This proof is based on a series of five lemmas, that we re-state here in a new version tuned for our purpose of dealing with two level sets of different levels v and v' .

Lemma 2.5.12. *There exists $C > 0$ such that, for any $0 \leq r \leq t$ and $v \leq v' \leq u$,*

$$\begin{aligned} &\tilde{\mathbb{P}}^{(2)} \left(\hat{h}_t(X_t(1)) \geq v, \hat{h}_t(X_t(2)) \geq v', \hat{h}_t^* \leq u \mid d(X_t(1), X_t(2)) = r \right) \\ &\leq \frac{C e^{-t-r}}{1 + (r \wedge (t-r))^{3/2}} (u_+ + 1) e^{\sqrt{2}u} (u - v + 1) (u - v' + 1) e^{-\sqrt{2}(v+v')} \left(e^{-(u-v)^2/(4t)} + e^{-(u-v)/2} \right). \end{aligned}$$

Proof. This is a new version of Lemma 4.3 in [39] and we explain how to adapt its proof. Similarly as Eq. (4.15) in [39], the probability in the statement equals

$$\begin{aligned} &\int_{\mathbb{R}} \tilde{\mathbb{P}} \left(\hat{h}_r(X_r) \geq v - z, \hat{h}_r^* \leq u - z \right) \tilde{\mathbb{P}} \left(\hat{h}_r(X_r) \geq v - z, \hat{h}_r^* \leq u - z \right) \\ &\quad \times \tilde{\mathbb{P}} \left(\hat{h}_{t-r}(X_{t-r}) - m_{t,r} \in dz, \hat{h}_t^*(B(X_t)^c) \leq u \right). \end{aligned} \tag{2.5.9}$$

Following [39], we split the integral according to $z \leq u$ and $z > u$.

For $z > u$, using [39, Eq. (4.2)] for the first one and [39, Eq. (4.3)] for the second one, we bound the product of the two first probabilities in (2.5.9) by

$$C e^{-2r} (u - v + 1) (u - v' + 1) e^{-\sqrt{2}(v+v')} e^{2\sqrt{2}z - \frac{3}{2}(z-u)} \left(e^{-(v-z)^2/(4t)} + e^{(v-z)/2} \right).$$

This is exactly the same as Eq. (4.20) in [39] up to the factor $(u - v' + 1) e^{-\sqrt{2}v'}$ where primes have been added. Therefore, this part of the integral is dealt with exactly the same way as in [39].

For $z \leq u$, we use [39, Eq. (4.2)] for both first probabilities in (2.5.9) and note that, because $v \leq v'$,

$$\begin{aligned} \left(e^{-(v-z)^2/(4t)} + e^{(v-z)/2} \right) \left(e^{-(v'-z)^2/(4t)} + e^{(v'-z)/2} \right) &\leq \left(e^{-(v-z)^2/(4t)} + e^{(v-z)/2} \right) \left(1 + e^{(v-z)/2} \right) \\ &\leq \left(e^{-(v-z)^2/(4t)} + e^{v-z} + 2e^{(v-z)/2} \right). \end{aligned}$$

This shows the product of the two first probabilities in (2.5.9) is at most

$$Ce^{-2r}(u-v+1)(u-v'+1)e^{-\sqrt{2}(v+v')}e^{2\sqrt{2}z}(u-z+1)^2 \left(e^{-(v-z)^2/(4t)} + e^{(v-z)/2} + e^{(v-z)/2} \right).$$

There are two differences with Eq. (4.19) in [39]: the factor $(u-v'+1)e^{-\sqrt{2}v'}$ where primes have been added (but this adds no new difficulty), and the additional term $e^{(v-z)/2}$ in the last parentheses. This latter gives rise to the following new term, which should be added to the integral in [39, Eq. (4.21)],

$$\int_{-\infty}^u e^{\frac{v}{2}+(\sqrt{2}-\frac{1}{2})z}(u-z+1)^3 dz \leq Ce^{\sqrt{2}u}e^{-(u-v)/2},$$

which can be included in the upper bound of the statement of the lemma after taking care of the other factors (see Eq. (4.22) in [39]: this is exactly how another part of the integral in Eq. (4.21) is bounded). \square

Lemma 2.5.13. *There exists $C > 0$ such that, for any $t \geq 0$ and $v \leq v' \leq u$,*

$$\begin{aligned} \mathbb{E} \left[\mathcal{E}_t([v, \infty)) \mathcal{E}_t([v', \infty)); \widehat{h}_t^* \leq u \right] \\ \leq C(u_+ + 1)e^{\sqrt{2}u}(u-v+1)(u-v'+1)e^{-\sqrt{2}(v+v')} \left(e^{-(u-v)^2/(4t)} + e^{-(u-v)/2} \right). \end{aligned}$$

Proof. This follows from Lemma 2.5.12 in the same way as Lemma 4.4 follows from Lemma 4.3 in [39]. \square

Lemma 2.5.14. *For any $v, v' \geq 0$,*

$$\begin{aligned} \widetilde{\mathbb{E}} \left[\mathcal{C}_{t,r_t}^*([-v, 0]) \mathcal{C}_{t,r_t}^*([-v', 0]); \widehat{h}_t^* \leq 0 \mid \widehat{h}_t(X_t) = 0 \right] \\ = 4 \int_0^{r_t} \int_0^{r_t} j_{t,v,v'}(s, s') ds' ds + 2 \int_0^{r_t} j_{t,v,v'}(s, s) ds, \end{aligned}$$

where $j_{t,v,v'}(s, s') := \widehat{\mathbb{E}}_t[J_{t,v}(s)J_{t,v'}(s')]$.

Proof. This follows directly from the proof of Lemma 5.4 in [39]. \square

Lemma 2.5.15. *There exists $C > 0$ such that, for any $t \geq 0$, $s, s' \in [0, t/2]$ and $v \geq v' \geq 0$,*

$$j_{t,v,v'}(s, s') \leq \frac{C(v+1)(v'+1)e^{\sqrt{2}(v+v')}}{t(s \wedge s' + 1)\sqrt{s \wedge s'}(|s' - s| + 1)\sqrt{|s' - s|} + \mathbb{1}_{s=s'}} \left(e^{-v^2/(16s)} + e^{-v/4} \right).$$

Proof. This is a new version of Lemma 5.5 in [39].

For the case $s \neq s'$, we first assume that $s < s'$ but do not assume $v \geq v'$. Then it follows directly from the proof in [39] that

$$j_{t,v,v'}(s, s') \leq \frac{C(v+1)(v'+1)e^{\sqrt{2}(v+v')}}{t(s+1)\sqrt{s}(s'-s+1)\sqrt{s'-s} + \mathbb{1}_{s=s'}} \left(e^{-v^2/(16s)} + e^{-v/4} \right) \left(e^{-(v')^2/(16s)} + e^{-v'/4} \right).$$

Then, we use $j_{t,v,v'}(s, s') = j_{t,v',v}(s', s)$ to cover the case $s > s'$ (this is fine because we removed the assumption $v \geq v'$). Finally, we bound by 2 the largest of the two last factors in the previous displayed equation. This yields the desired result.

The case $s = s'$ is also identical to the proof in [39], applying Lemma 2.5.13 instead of Lemma 4.4 there. \square

Lemma 2.5.16. *There exists $C > 0$ such that, for any $t \geq 1$ and $v \geq v' \geq 0$,*

$$\tilde{\mathbb{E}} \left[\mathcal{C}_{t,r_t}^*([-v, 0]) \mathcal{C}_{t,r_t}^*([-v', 0]) \mid \hat{h}_t^* = \hat{h}_t(X_t) = 0 \right] \leq C(v'+1)e^{-\sqrt{2}(v+v')}.$$

Proof. This follows from Lemmas 2.5.14 and 2.5.15, in a similar way as the proof of Lemma 5.3 in [39]. See also the proof of Lemma 2.2.5 here to see how to get a uniform bound (not as in [39, Lemma 5.3]). \square

Proof of Proposition 2.2.6. This follows from Lemma 2.5.16 in the same way as Proposition 1.5 follows from Lemma 5.3 in [39]. \square

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The left tail of the subcritical derivative martingale in a branching random walk

Abstract

Motivated by the study of the quantum Mabuchi theory [72], we obtain in this work a sharp estimate on the left tail of the distribution of the so-called derivative martingale in the L^4 phase.

3.1 Introduction

Branching random walks (BRW) have a long history in probability theory and its applications. Given a real parameter β (the inverse temperature in the language of statistical physics), one can associate to a BRW a natural random measure called a multiplicative cascade. Multiplicative cascades were introduced by Mandelbrot in [78] as a toy model for Gaussian multiplicative chaos (GMC), a random measure which models energy dissipation in a turbulent flow. The precise mathematical construction of GMC was established by Kahane in the landmark paper [63] following the works on turbulence by Kolmogorov-Obukhov [69, 86] and Mandelbrot [77]. In both aforementioned models, the main object of interest (a random measure) is obtained as the limit of a positive martingale and it is also very natural to inquire on the existence and the properties of the derivatives of these martingales which form yet another sequence of martingales. In the case of GMC theory, these derivatives appear as a crucial ingredient in the construction of the quantum Mabuchi theory [72] or as so-called logarithmic fields in conformal field theory [102]. In this paper, we will work within the simplified framework of the Gaussian BRW as motivated by [72], we will give the first sharp estimates on the left tail of the derivative.

We consider the case of a BRW with binary splitting and independent standard Gaussian increments. The process is indexed by the binary tree $\mathbb{T} = \cup_{n \geq 0} \{0, 1\}^n$ where by convention $\{0, 1\}^0 = \{\emptyset\}$. For $u \in \mathbb{T}$, let us denote $|u|$ the length of u , $v \leq u$ if v is an ancestor of u and $u \wedge v$ the *length* of the last common ancestor of u and v . The value of

of the BRW at u is given by

$$X_u := \sum_{\emptyset < v \leq u} G_v,$$

where $(G_v)_{v \in \mathbb{T}}$ are i.i.d. standard Gaussian random variables¹. The natural filtration of the process is given by $\mathcal{F}_n = \sigma(X_u, |u| \leq n)$. We can then define the additive martingale with real parameter β

$$W_n(\beta) := \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2} n},$$

which is a positive martingale with respect to the filtration (\mathcal{F}_n) and thus converges almost surely to a limit W_β . It is well known that this limit is non trivial if and only if $\beta \in (-\beta_c, \beta_c)$ where $\beta_c = \sqrt{2 \log 2}$. **In the sequel, we will only consider the case $\beta \in [0, \beta_c)$.** The (almost surely positive) random variable W_β is very well understood. For instance the right tail is known with high precision and follows a power law decay, see for instance [55]. In particular, W_β is in L^p if and only if $p < \left(\frac{\beta_c}{\beta}\right)^2$.

Since for all β , $W_n(\beta)$ is a martingale, all higher order derivatives with respect to β are also martingales. In this paper, we will focus on the first derivative and hence study the following martingale

$$Z_n(\beta) := \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - n \frac{\beta^2}{2}} (X_u - \beta n).$$

Thanks to the almost sure uniform convergence on (complex) compact sets included in the domain Λ of figure 3.1 obtained in [12], the function $\beta \mapsto W_\beta$ is almost surely analytical on Λ .

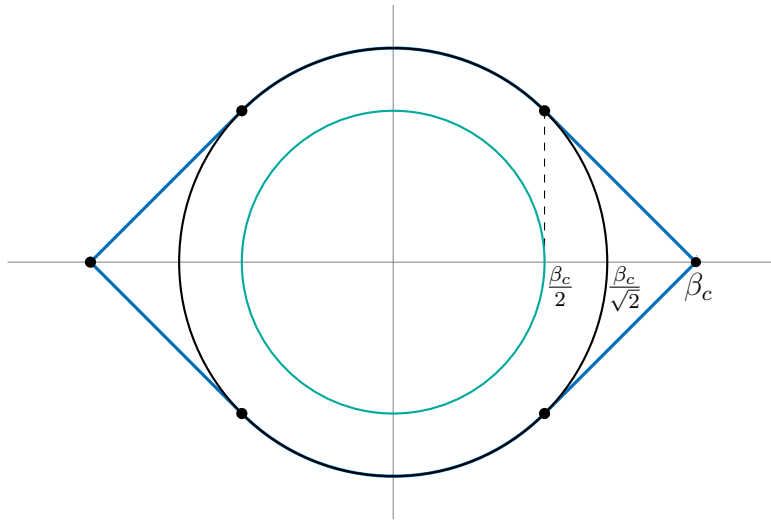


Figure 3.1: The domain of convergence of the additive martingale Λ is delimited by the blue curves. The black circle delimits the L^2 phase and the inner circle the L^4 phase.

Uniform convergence implies that $Z_n(\beta)$ converges also almost surely to a non trivial limit $Z_\beta = \partial_\beta W_\beta$ when $\beta \in (0, \beta_c)^2$. Contrary to the critical case, the limit Z_β is signed

¹Here and hereafter, we adopt the classical conventions $\sum_\emptyset = 0$ and $\prod_\emptyset = 1$.

²In fact, the convergence holds in the whole domain of convergence.

and has a highly non symmetrical distribution. Indeed, the right tail Z_β is expected to behave similarly to the right tail of W_β up to logarithmic corrections whereas the left tail is expected to be very thin. The behavior of the left tail was conjectured for general GMC measures in [72, Conjecture 1]. The purpose of this paper is precisely to prove this conjecture within the framework of the Gaussian BRW. In this context, the conjecture can be rephrased as follows: for $\beta \in (0, \beta_c)$, there exist constants $c, c', C, C' > 0$ such that for all $x \geq 0$

$$Ce^{-cx^\gamma} \leq \mathbb{P}(Z_\beta < -x) \leq C'e^{-c'x^\gamma},$$

where

$$\gamma := \left(\frac{\beta_c}{\beta}\right)^2.$$

The aim of the present paper is to prove the lower bound conjecture for $\beta \in (0, \beta_c)$ and the upper bound for $\beta \in (0, \beta_c/2)$, in the so-called L^4 phase.

3.1.1 Results

We will divide the presentation of our results in two parts. First we give a lower bound on the left tail in the full regime $\beta \in (0, \beta_c)$.

Theorem 3.1.1. *For $\beta \in (0, \beta_c)$, there exist c and $C > 0$ depending on β such that for $x \geq 0$*

$$\mathbb{P}(Z_\beta < -x) \geq Ce^{-cx^\gamma}.$$

Second we give an upper bound on the left tail in the regime $\beta \in (0, \beta_c/2)$.

Theorem 3.1.2. *For $\beta \in (0, \frac{\beta_c}{2})$, there exist c' and $C' > 0$ depending on β such that for $x \geq 0$*

$$\mathbb{P}(Z_\beta < -x) \leq C'e^{-c'x^\gamma}.$$

We will make two remarks on the limiting cases $\beta = 0$ and $\beta = \beta_c$ where much more is known and the behaviour of the left tail is quite different.

Remark 3.1.3. In the case $\beta = 0$, one can check that $Z_0 = \lim_{\frac{1}{2^n} \sum_{|u|=n} X_u$ is a standard Gaussian random variable.

Remark 3.1.4. The case $\beta = \beta_c$ has been thoroughly studied in the literature. In this case, the variable W_{β_c} is trivial; however Z_{β_c} is non trivial, negative almost surely and the left tail follows a power law

$$\mathbb{P}(Z_{\beta_c} < -x) \sim \frac{C}{x}, \quad \text{as } x \rightarrow +\infty,$$

for some $C > 0$, see [33].

3.2 Lower bound for the left tail: proof of Theorem 3.1.1

This section is devoted to proving Theorem 3.1.1.

3.2.1 Heuristics

A quick inspection of $Z_n(\beta)$ as a function of $(X_u)_{|u|=n}$, namely

$$Z_n(\beta) = \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2}n} (X_u - \beta n),$$

shows that

$$\text{ess inf } Z_n(\beta) = -\frac{1}{\beta} e^{\frac{\beta^2}{2}n} =: -m_n.$$

This value is achieved when all the particles are located at $\beta n - \frac{1}{\beta}$. This suggests the following scenario to get a lower bound. Put all the particles around $\beta n - \frac{1}{\beta}$ to minimize $Z_n(\beta)$ and then show that, on this event, the remainder $Z_\beta - Z_n(\beta)$ is negligible using the decomposition of the derivative martingale

$$Z_\beta = Z_n(\beta) + R_n(\beta),$$

where

$$R_n(\beta) = \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2}n} Z_\beta^u + \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2}n} (X_u - \beta n) (W_\beta^u - 1) \quad (3.2.1)$$

and (Z_β^u, W_β^u) are independent copies of (Z_β, W_β) independent of \mathcal{F}_n .

To see why this works, suppose for instance that we are in the L^2 phase ($\beta < \beta_c/\sqrt{2}$). In this case, Z^u and $W^u - 1$ are centered with finite variance, thus the central limit theorem ensures that $\sum_{|u|=n} Z_\beta^u$ fluctuates as $2^{n/2}$. If the X_u are all located near $\beta n - \frac{1}{\beta}$, we have

$$\frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2}n} Z_\beta^u \simeq \left(\frac{e^{\frac{\beta^2}{2}}}{\sqrt{2}} \right)^n \frac{1}{2^{\frac{n}{2}}} \sum_{|u|=n} Z_\beta^u$$

which is exponentially small thanks to the condition $\beta < \beta_c$.

The cost for placing all the particles around $\beta n - \frac{1}{\beta}$ is of order $\exp(-2^n)$ as we will see and noting that $2^n = \left(e^{\frac{\beta^2}{2}} \right)^\gamma$ shows that we obtain the expected order.

Remark 3.2.1. The probability that all the particles are near $\beta n - \frac{1}{\beta}$ at time n in the i.i.d. case is

$$\mathbb{P}\left(\mathcal{N}(0, n) = \beta n - \frac{1}{\beta} + O(1)\right)^{2^n} = e^{-\frac{\beta^2}{2}n2^n + o(n2^n)}.$$

We will see in the proof that the extra n term in the exponential comes from the i.i.d. assumption and it will disappear for the correlated process $(X_u, |u| = n)$.

Following the above heuristic, we start by giving a lower bound for the left tail of $Z_n(\beta)$ and then a lower bound for the left tail of Z_β by showing that $Z_n(\beta)$ and Z_β are close enough when the aforementioned scenario occurs.

3.2.2 Lower bound for the left tail of $Z_n(\beta)$

The process $(X_u, |u| = n)$ is a Gaussian process with covariance given by $\text{Cov}(X_u, X_v) = u \wedge v$, where $u \wedge v$ is the length of the last common ancestor of u and v . By ordering the n -th generation with the lexicographical order, this covariance gives a covariance matrix $\Sigma_n \in \mathcal{M}_{2^n}(\mathbb{R})$. The first ones are

$$\Sigma_1 = I_2, \quad \Sigma_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \quad \Sigma_{n+1} = \left(\begin{array}{c|c} J_n + \Sigma_n & 0 \\ \hline 0 & J_n + \Sigma_n \end{array} \right),$$

where $J_n \in \mathcal{M}_{2^n}(\mathbb{R})$ have all entries equal to 1. The recurrence relation above enables to compute the determinant of Σ_n which is $(2^n - 1) \prod_{k=0}^{n-1} (2^{n-k} - 1)^{2^k}$. And the terms in the product are exactly the eigenvalues of Σ_n as can be seen by using the same recursion for the characteristic polynomials.

Now, recall that $-m_n = -\frac{1}{e\beta} \exp(\frac{\beta^2}{2}n)$ is the essential infimum of $Z_n(\beta)$ and for $\varepsilon \in (0, 1)$, let

$$\mathbf{V}_n^\varepsilon := \left\{ x = (x_u)_{|u|=n} \in \mathbb{R}^{2^n} : \frac{1}{2^n} \sum_{|u|=n} e^{\beta x_u - n \frac{\beta^2}{2}} (x_u - \beta n) < -(1 - \varepsilon)m_n \right\},$$

so that the following equality holds:

$$\begin{aligned} \mathbb{P}(Z_n(\beta) < -(1 - \varepsilon)m_n) &= \mathbb{P}((X_u)_{|u|=n} \in \mathbf{V}_n^\varepsilon) \\ &= \frac{1}{\sqrt{2\pi}^{2^n} \sqrt{\det \Sigma_n}} \int_{\mathbf{V}_n^\varepsilon} e^{-\frac{1}{2}x^t \Sigma_n^{-1} x} dx. \end{aligned}$$

If $0 < \alpha_-^\varepsilon < 1 < \alpha_+^\varepsilon$ are the two solutions of the equation $\alpha e^{-\alpha} = \frac{1}{e}(1 - \varepsilon)$, it is not hard to see that $\Pi_n^\varepsilon := \left[\beta n - \frac{\alpha_-^\varepsilon}{\beta}, \beta n - \frac{\alpha_+^\varepsilon}{\beta} \right]^{2^n} \subset \mathbf{V}_n^\varepsilon$. Then

$$\begin{aligned} \mathbb{P}(Z_n(\beta) < -(1 - \varepsilon)m_n) &\geq \frac{1}{\sqrt{2\pi}^{2^n} \sqrt{\det \Sigma_n}} \int_{\Pi_n^\varepsilon} e^{-\frac{1}{2}x^t \Sigma_n^{-1} x} dx \\ &\geq \frac{1}{\sqrt{2\pi}^{2^n} \sqrt{\det \Sigma_n}} \lambda(\Pi_n^\varepsilon) \exp\left(-\frac{1}{2} \sup_{x \in \Pi_n^\varepsilon} x^t \Sigma_n^{-1} x\right). \end{aligned}$$

It is an exercise to show that $\log \det \Sigma_n = \theta 2^n - 2 \log 2 + o(1)$ for some positive $\theta \simeq 0.9458$, the computations are done in Lemma 3.4.2. Notice that the biggest eigenvalue of Σ_n^{-1} is 1. Therefore if $x \in \Pi_n^\varepsilon$, by writing $x = \left(\beta n - \frac{1}{\beta}\right) \mathbf{1} + h$ (where $\mathbf{1}$ is the vector with all coordinates equal to one), one has

$$\begin{aligned} x^t \Sigma_n^{-1} x &= \left(\beta n - \frac{1}{\beta}\right)^2 \mathbf{1}^t \Sigma_n^{-1} \mathbf{1} + 2 \left(\beta n - \frac{1}{\beta}\right) \mathbf{1}^t \Sigma_n^{-1} h + h^t \Sigma_n^{-1} h \\ &= \left(\beta n - \frac{1}{\beta}\right)^2 \frac{2^n}{2^n - 1} + 2 \left(\beta n - \frac{1}{\beta}\right) \frac{1}{2^n - 1} \sum_i h_i + h^t \Sigma_n^{-1} h \\ &\leq \left(\frac{\alpha_+^\varepsilon - 1}{\beta}\right)^2 2^n + O(n^2), \end{aligned}$$

where we used the fact that $\alpha_+^\varepsilon - 1 > 1 - \alpha^\varepsilon$. And one obtains

$$\mathbb{P}\left(Z_n(\beta) < -(1 - \varepsilon)m_n\right) \geq e^{-\kappa_\varepsilon 2^n + O(n^2)},$$

where $\kappa_\varepsilon := \frac{1}{2}(\log 2\pi + \theta) - \log \frac{\alpha_+^\varepsilon - \alpha^\varepsilon}{\beta} + \frac{1}{2}\left(\frac{\alpha_+^\varepsilon - 1}{\beta}\right)^2$.

Taking for instance $\varepsilon = 1/2$ gives the existence of a constant $\lambda > 0$ such that

$$\liminf_{n \rightarrow \infty} \frac{\log \mathbb{P}\left(Z_n(\beta) < -0.5m_n\right)}{2^n} \geq -\lambda.$$

3.2.3 Lower bound for the left tail of Z_β

We now show that $Z_n(\beta)$ is a good approximation to Z_β when estimating the left tail below a fraction of m_n . Fix $\delta > 0$, and let us write

$$\begin{aligned} & \mathbb{P}\left(|Z_\beta - Z_n(\beta)| \geq \delta |Z_n(\beta)| \mid \mathcal{F}_n\right) \\ &= \mathbb{P}\left(\left|\frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2}n} Z_\beta^u + \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2}n} (X_u - \beta n)(W_\beta^u - 1)\right| \geq \delta |Z_n(\beta)| \mid \mathcal{F}_n\right) \\ &\leq \mathbb{P}\left(\left|\frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2}n} Z_\beta^u\right| \geq \frac{\delta |Z_n(\beta)|}{2} \mid \mathcal{F}_n\right) \\ &+ \mathbb{P}\left(\left|\frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2}n} (X_u - \beta n)(W_\beta^u - 1)\right| \geq \frac{\delta |Z_n(\beta)|}{2} \mid \mathcal{F}_n\right) \\ &\leq \left(\frac{2}{\delta |Z_n(\beta)|}\right)^p B_p \left(\mathbb{E}|Z_\beta|^p \frac{1}{2^{np}} \sum_{|u|=n} e^{p\beta X_u - \frac{p\beta^2}{2}n}\right. \\ &\quad \left.+ \mathbb{E}|W_\beta - 1|^p \frac{1}{2^{np}} \sum_{|u|=n} e^{p\beta X_u - \frac{p\beta^2}{2}n} |X_u - \beta n|^p\right), \end{aligned}$$

using Lemma 3.4.1 for the last inequality with some $p \in [1, \gamma \wedge 2)$. On the event $\{(X_u)_{|u|=n} \in \Pi_n^\varepsilon\}$, we have

$$\begin{aligned} |Z_n(\beta)| &\geq \frac{1 - \varepsilon}{e\beta} e^{\frac{\beta^2}{2}n}, \\ \frac{1}{2^{np}} \sum_{|u|=n} e^{p\beta X_u - \frac{p\beta^2}{2}n} &\leq e^{-p\alpha^\varepsilon} \frac{1}{2^{(p-1)n}} e^{p\frac{\beta^2}{2}n}, \\ \frac{1}{2^{np}} \sum_{|u|=n} e^{p\beta X_u - \frac{p\beta^2}{2}n} |X_u - \beta n|^p &\leq \left(\frac{e^{-\alpha^\varepsilon} \alpha_+^\varepsilon}{\beta}\right)^p \frac{1}{2^{(p-1)n}} e^{p\frac{\beta^2}{2}n}. \end{aligned}$$

Therefore,

$$\mathbb{P}\left(|Z_\beta - Z_n(\beta)| \geq \delta |Z_n(\beta)| \mid \Pi_n^\varepsilon\right) \leq \left(\frac{2e\beta e^{-\alpha^\varepsilon}}{\delta(1 - \varepsilon)}\right)^p B_p \left(\mathbb{E}[|Z_\beta|^p] \right. \tag{3.2.2}$$

$$\left. + \mathbb{E}[|W_\beta - 1|^p] \left(\frac{\alpha_+^\varepsilon}{\beta}\right)^p\right) \frac{1}{2^{(p-1)n}}. \tag{3.2.3}$$

And the right term decays exponentially to 0 as soon as $p > 1$. Then, on $\Pi_n^\varepsilon \cap \{|Z_\beta - Z_n(\beta)| \leq \delta |Z_n(\beta)|\}$, one has

$$Z_\beta \leq -(1 - (\delta + \varepsilon))m_n,$$

thus

$$\begin{aligned} \mathbb{P}(Z_\beta \leq -(1 - (\delta + \varepsilon))m_n) &\geq \mathbb{P}((X_u) \in \Pi_n^\varepsilon) \mathbb{P}(|Z_\beta - Z_n(\beta)| \leq \delta |Z_n(\beta)| \mid (X_u) \in \Pi_n^\varepsilon) \\ &\geq e^{-\kappa_\varepsilon 2^n + O(n^2)} \left(1 - \frac{C(\varepsilon, \delta)}{2^{(p-1)n}}\right), \end{aligned}$$

where $C(\varepsilon, \delta)$ is the term in the r.h.s of Equation (3.2.2). This last inequality and the continuity of κ_ε for $\varepsilon \in (0, 1)$ lead to

$$\liminf_{n \rightarrow \infty} \frac{\log \mathbb{P}(Z_\beta \leq -(1 - \varepsilon)m_n)}{2^n} \geq -\kappa_\varepsilon.$$

And therefore,

$$\liminf_{x \rightarrow \infty} \frac{\log \mathbb{P}(Z_\beta \leq -x)}{x^\gamma} > -\infty$$

which concludes the proof of Theorem 3.1.1.

3.3 Upper bound for the left tail

3.3.1 Large deviations with the branching property

From now on, we omit the subscript β to alleviate notations. The branching property gives the decomposition

$$Z = \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u). \quad (3.3.1)$$

In order to obtain large deviations bounds, we need to control the Laplace transform of the random variables $Z + aW$. Intuitively, for $a > 0$, the fat right tail of W should help. For the same reason, it seems hopeless at first sight to obtain a nice bound in the case $a < 0$. But there must be some compensation with Z because a typical scenario for a large value for W leads to a positive large value for Z too (the box scenario of Subsection 3.2.2 being very atypical). Let us rewrite things a bit:

$$\begin{aligned} Z_n + aW_n &= \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2}n} (X_u - \beta n + a) \\ &= e^{-\beta a} \frac{1}{2^n} \sum_{|u|=n} e^{\beta(X_u + a) - \frac{\beta^2}{2}n} ((X_u + a) - \beta n). \end{aligned}$$

Therefore $Z + aW = e^{-\beta a} Z^{[a]}$ where $Z^{[a]}$ is the limit of the derivative martingale when the initial ancestor starts at a . Since the large negative values for Z_n are obtained when all the particles lie around βn at time n , it seems reasonable to expect that starting from $a < 0$ makes it harder to achieve. In fact, by exponential tilting, we can shift the mean of the X_u 's and obtain a bound on the Laplace transform of $Z^{[a]}$ with the Laplace transform of Z .

Lemma 3.3.1. *For $a \in \mathbb{R}$ and $\lambda > 0$, ones has*

$$\mathbb{E} \left[e^{-\lambda(Z+aW)} \right] \leq e^{\frac{a^2}{2}} \mathbb{E} \left[e^{-2\lambda e^{-\beta a} Z} \right]^{\frac{1}{2}}.$$

Proof. Recall that $(X_u)_{|u|=n}$ is a centered Gaussian vector with covariance matrix Σ_n and note that $\frac{1}{2^n} \sum_{|u|=n} X_u \sim \mathcal{N}(0, 1 - 2^{-n})$. Under the probability \mathbb{Q}_a defined by

$$\frac{d\mathbb{Q}_a}{d\mathbb{P}} = Y_n^{[a]} := e^{\frac{a}{(1-2^{-n})2^n} \sum_{|u|=n} X_u - \frac{a^2}{2(1-2^{-n})}},$$

the variables $(X_u)_{|u|=n}$ are shifted: we have $\mathbb{E}_{\mathbb{Q}_a}[X_u] = a$ and the covariance matrix remains unchanged. Therefore,

$$\begin{aligned} \mathbb{E} \left[e^{-\lambda Z_n^{[a]}} \right] &= \mathbb{E} \left[e^{-\lambda \frac{1}{2^n} \sum_{|u|=n} e^{\beta(X_u+a) - \frac{\beta^2}{2} n} (X_u + a - \beta n)} \right] \\ &= \mathbb{E} \left[Y_n^{[a]} e^{-\lambda \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2} n} (X_u - \beta n)} \right] \\ &= e^{-\frac{a^2}{2(1-2^{-n})}} \mathbb{E} \left[e^{\frac{a}{(1-2^{-n})2^n} \sum_{|u|=n} X_u} e^{-\lambda \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2} n} (X_u - \beta n)} \right] \\ &\leq e^{-\frac{a^2}{2(1-2^{-n})}} \mathbb{E} \left[e^{\frac{2a}{(1-2^{-n})2^n} \sum_{|u|=n} X_u} \right]^{\frac{1}{2}} \mathbb{E} \left[e^{-2\lambda \frac{1}{2^n} \sum_{|u|=n} e^{\beta X_u - \frac{\beta^2}{2} n} (X_u - \beta n)} \right]^{\frac{1}{2}} \\ &= e^{\frac{a^2}{2(1-2^{-n})}} \mathbb{E} \left[e^{-2\lambda Z_n} \right]^{\frac{1}{2}}. \end{aligned}$$

Then, applying Fatou's lemma and conditional Jensen's inequality yields

$$\mathbb{E} \left[e^{-\lambda Z^{[a]}} \right] \leq \liminf_{n \rightarrow \infty} e^{\frac{a^2}{2}} \mathbb{E} \left[e^{-2\lambda \mathbb{E}[Z | \mathcal{F}_n]} \right]^{\frac{1}{2}} \leq e^{\frac{a^2}{2}} \mathbb{E} \left[e^{-2\lambda Z} \right]^{\frac{1}{2}}.$$

Finally, using $Z + aW = e^{-\beta a} Z^{[a]}$ gives

$$\mathbb{E} \left[e^{-\lambda(Z+aW)} \right] = \mathbb{E} \left[e^{-\lambda e^{-\beta a} Z^{[a]}} \right] \leq e^{\frac{a^2}{2}} \mathbb{E} \left[e^{-2\lambda e^{-\beta a} Z} \right]^{\frac{1}{2}}.$$

□

For high particles and under the sub-Gaussian assumption³, we have the more refined lemma:

Lemma 3.3.2. *For $a > 0$ and $\lambda \geq 0$, we have*

$$\mathbb{E} \left[e^{-\lambda(Z+aW)} \right] \leq e^{C(\lambda e^{-\beta a} a + (\lambda e^{-\beta a})^2)}.$$

³Here we think of sub-Gaussian on the left, meaning that a random variable V is sub-Gaussian if there exist c and $C > 0$ such that for $\lambda \geq 0$, we have $\mathbb{E} e^{-\lambda V} \leq C e^{c\lambda^2}$. This will be proved in Proposition 3.3.3.

Proof. With the sub-Gaussian result of Proposition 3.3.3, we can sharpen the previous lemma using Hölder's inequality instead of Cauchy-Schwarz in the proof of Lemma 3.3.1. If $a > 0$, one has

$$\begin{aligned}\mathbb{E} \left[e^{-\lambda Z^{[a]}} \right] &\leq e^{(p-1)\frac{a^2}{2}} \mathbb{E} \left[e^{-\frac{p}{p-1}\lambda Z} \right]^{\frac{p-1}{p}} \\ &\leq e^{(p-1)\frac{a^2}{2} + C\frac{p}{p-1}\lambda^2}.\end{aligned}$$

By choosing $p = 1 + \frac{\lambda}{a}$, one gets

$$\mathbb{E} \left[e^{-\lambda Z^{[a]}} \right] \leq e^{C(\lambda a + \lambda^2)},$$

for some other constant $C > 0$. Therefore

$$\mathbb{E} \left[e^{-\lambda(Z+aW)} \right] \leq e^{C(\lambda e^{-\beta a} a + (\lambda e^{-\beta a})^2)}.$$

□

We now turn to the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2: We start with representation (3.3.1) and apply a Chernoff bound to the three following terms

$$\begin{aligned}Z &= \frac{1}{2^n} \sum_{X_u < \beta n} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u) \\ &\quad + \frac{1}{2^n} \sum_{X_u \in [\beta n, \beta n + 1]} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u) \\ &\quad + \frac{1}{2^n} \sum_{X_u > \beta n + 1} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u).\end{aligned}$$

From now on, fix some $\alpha > 0$ whose value will be determined later.

For the particles below βn , we have, for $\lambda > 0$,

$$\begin{aligned}\mathbb{P} \left(\frac{1}{2^n} \sum_{X_u < \beta n} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u) < -\alpha e^{\frac{\beta^2}{2}n} \middle| \mathcal{F}_n \right) \\ &= \mathbb{E} \left[\exp \left(-\lambda \frac{1}{2^n} \sum_{X_u < \beta n} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u) \right) > \exp \left(\lambda \alpha e^{\frac{\beta^2}{2}n} \right) \middle| \mathcal{F}_n \right] \\ &\leq \exp \left(-\lambda \alpha e^{\frac{\beta^2}{2}n} \right) \prod_{X_u < \beta n} \mathbb{E} \left[\exp \left(-\lambda \frac{1}{2^n} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u) \right) \middle| \mathcal{F}_n \right].\end{aligned}$$

Then, using Proposition 3.3.4,

$$\begin{aligned}\mathbb{E} \left[\exp \left(-\lambda \frac{1}{2^n} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u) \right) \middle| \mathcal{F}_n \right] \\ &\leq \exp \left(-\lambda \frac{1}{2^n} e^{\beta X_u - \frac{\beta^2}{2}n} (X_u - \beta n) + C \left(\left(\lambda \frac{1}{2^n} e^{\beta X_u - \frac{\beta^2}{2}n} e^{-\beta(X_u - \beta n)} \right)^2 + 1 \right) \right) \\ &\leq \exp \left(\frac{1}{e\beta} \lambda \frac{e^{\frac{\beta^2}{2}n}}{2^n} + C \left(\left(\lambda \frac{e^{\frac{\beta^2}{2}n}}{2^n} \right)^2 + 1 \right) \right).\end{aligned}$$

Therefore, by choosing $\lambda = 2^n e^{-\frac{\beta^2}{2}n}$, one gets

$$\begin{aligned} \mathbb{P}\left(\sum_{X_u < \beta n} \frac{1}{2^n} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u) < -\alpha e^{\frac{\beta^2}{2}n} \middle| \mathcal{F}_n\right) \\ \leq \exp(-\alpha 2^n) \prod_{X_u < \beta n} \exp\left(\frac{1}{e\beta} + 2C\right) \\ \leq \exp\left(-\left(\alpha - \frac{1}{e\beta} - C\right)2^n\right). \end{aligned}$$

For particles between βn and $\beta n + 1$, Lemma 3.3.1 and Proposition 3.3.3 provide $C > 0$ such that, for $\lambda, a \geq 0$,

$$\mathbb{E}\left[e^{-\lambda(Z+aW)}\right] \leq e^{\frac{a^2}{2} + C(\lambda e^{-\beta a})^2}.$$

Then,

$$\begin{aligned} \mathbb{P}\left(\frac{1}{2^n} \sum_{\beta n \leq X_u \leq \beta n + 1} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u) < -\alpha e^{\frac{\beta^2}{2}n} \middle| \mathcal{F}_n\right) \\ \leq \exp\left(-\lambda \alpha e^{\frac{\beta^2}{2}n}\right) \prod_{\beta n \leq X_u \leq \beta n + 1} \mathbb{E}\left[\exp\left(-\lambda \frac{1}{2^n} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u)\right) \middle| \mathcal{F}_n\right] \\ \leq \exp\left(-\lambda \alpha e^{\frac{\beta^2}{2}n}\right) \prod_{\beta n \leq X_u \leq \beta n + 1} \exp\left(\frac{1}{2}(X_u - \beta n)^2 + \left(\lambda \frac{1}{2^n} e^{\beta X_u - \frac{\beta^2}{2}n} e^{-\beta(X_u - \beta n)}\right)^2\right) \\ \leq \exp\left(-\lambda \alpha e^{\frac{\beta^2}{2}n}\right) \prod_{\beta n \leq X_u \leq \beta n + 1} \exp\left(\frac{1}{2} + \left(\lambda \frac{e^{\frac{\beta^2}{2}n}}{2^n}\right)^2\right). \end{aligned}$$

Taking $\lambda = 2^n e^{-\frac{\beta^2}{2}n}$ gives

$$\mathbb{P}\left(\frac{1}{2^n} \sum_{\beta n \leq X_u \leq \beta n + 1} e^{\beta X_u - \frac{\beta^2}{2}n} (Z^u + (X_u - \beta n)W^u) < -\alpha e^{\frac{\beta^2}{2}n} \middle| \mathcal{F}_n\right) \leq \exp\left(-\left(\alpha - \frac{3}{2}\right)2^n\right).$$

It remains to deal with the particles above $\beta n + 1$. An estimate on the Laplace transform of W is needed for this purpose. Recall that

$$W = \frac{1}{2} e^{\beta X_0 - \frac{\beta^2}{2}} W^0 + \frac{1}{2} e^{\beta X_1 - \frac{\beta^2}{2}} W^1,$$

where W^0 and W^1 are independent and have the same law as W . A straightforward computation shows that

$$\lim_{x \rightarrow 0} \frac{\log \log 1/\mathbb{P}\left(\frac{1}{2} e^{\beta X_0 - \frac{\beta^2}{2}} < x\right)}{\log \log 1/x} = 2.$$

And [85, Theorem 1.2] provides $c > 0$ such that for every $\lambda \geq 1$, one has⁴

$$\mathbb{E}\left[e^{-\lambda W}\right] \leq e^{-c \log^{3/2}(\lambda)}. \quad (3.3.2)$$

⁴This is true for any exponent strictly below 2, here we choose 3/2.

When $x > \beta n + 1$, we can use a portion of W to improve the Chernoff bound:

$$\begin{aligned} & \mathbb{E} \left[\exp \left(-\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} (Z + (x - \beta n)W) \right) \right] \\ &= \mathbb{E} \left[\exp \left(-\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} (Z + (x - \beta n - 1)W) \right) \exp \left(-\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} W \right) \right] \\ &\leq \mathbb{E} \left[\exp \left(-2\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} (Z + (x - \beta n - 1)W) \right) \right]^{\frac{1}{2}} \mathbb{E} \left[\exp \left(-2\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} W \right) \right]^{\frac{1}{2}}. \end{aligned}$$

Then using Lemma 3.3.2 for the first term and Equation (3.3.2) for the second term yields

$$\begin{aligned} & \mathbb{E} \left[\exp \left(-\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} (Z + (x - \beta n)W) \right) \right] \\ &\leq \exp \left(C\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} e^{-\beta(x - \beta n - 1)} (x - \beta n - 1) + 2C \left(\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} e^{-\beta(x - \beta n - 1)} \right)^2 \right. \\ &\quad \left. - \frac{c}{2} \log^{3/2} \left(2\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} \right) \right) \\ &= \exp \left(C\lambda \frac{e^{\frac{\beta^2}{2} n}}{2^n} e^{\beta} (x - \beta n - 1) + 2C \left(\lambda \frac{e^{\frac{\beta^2}{2} n}}{2^n} e^{\beta} \right)^2 - \frac{c}{2} \log^{3/2} \left(2\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} \right) \right), \end{aligned}$$

as soon as $2\lambda \frac{1}{2^n} e^{\beta(\beta n + 1) - n \frac{\beta^2}{2}} \geq 1$. For $\lambda = e^{-\beta} 2^n e^{-\frac{\beta^2}{2} n}$, this condition is fulfilled and we obtain

$$\begin{aligned} \mathbb{E} \left[\exp \left(-\lambda \frac{1}{2^n} e^{\beta x - \frac{\beta^2}{2} n} (Z + (x - \beta n)W) \right) \right] &\leq \exp \left(C(x - \beta n - 1) + 2C - \frac{c\beta^{3/2}}{2} (x - \beta n - 1)^{3/2} \right) \\ &\leq \exp(D), \end{aligned}$$

for some $D > 0$ ⁵. Choosing $\lambda = e^{-\beta} 2^n e^{-\frac{\beta^2}{2} n}$ gives

$$\begin{aligned} & \mathbb{P} \left(\frac{1}{2^n} \sum_{X_u > \beta n + 1} e^{\beta X_u - \frac{\beta^2}{2} n} (Z^u + (X_u - \beta n)W^u) < -\alpha e^{\frac{\beta^2}{2} n} \mid \mathcal{F}_n \right) \\ &\leq \exp \left(-\lambda \alpha e^{\frac{\beta^2}{2} n} \right) \prod_{X_u > \beta n + 1} \mathbb{E} \left[\exp \left(-\lambda \frac{1}{2^n} e^{\beta X_u - \frac{\beta^2}{2} n} (Z^u + (X_u - \beta n)W^u) \right) \mid \mathcal{F}_n \right] \\ &\leq \exp \left(-(e^{-\beta} \alpha - D) 2^n \right). \end{aligned}$$

With $\alpha = 3 \max \left(1 + C + \frac{1}{e^\beta}, \frac{5}{2}, (1 + D) e^\beta \right)$, one gets

$$\begin{aligned} \mathbb{P} \left(Z < -\alpha e^{\frac{\beta^2}{2} n} \right) &\leq \mathbb{P} \left(\frac{1}{2^n} \sum_{X_u < \beta n} e^{\beta X_u - \frac{\beta^2}{2} n} (Z^u + (X_u - \beta n)W^u) < -\frac{\alpha}{3} e^{\frac{\beta^2}{2} n} \right) \\ &\quad + \mathbb{P} \left(\frac{1}{2^n} \sum_{\beta n \leq X_u \leq \beta n + 1} e^{\beta X_u - \frac{\beta^2}{2} n} (Z^u + (X_u - \beta n)W^u) < -\frac{\alpha}{3} e^{\frac{\beta^2}{2} n} \right) \\ &\quad + \mathbb{P} \left(\frac{1}{2^n} \sum_{X_u > \beta n + 1} e^{\beta X_u - \frac{\beta^2}{2} n} (Z^u + (X_u - \beta n)W^u) < -\frac{\alpha}{3} e^{\frac{\beta^2}{2} n} \right) \\ &\leq 3 \exp(-2^n). \end{aligned}$$

⁵Here we use the fact that $\beta > 0$.

Now, if $x > 1$, picking $n \in \mathbb{N}$ such that $\alpha e^{\frac{\beta^2}{2}n} \leq x < \alpha e^{\frac{\beta^2}{2}(n+1)}$ provides a $c > 0$ such that

$$\mathbb{P}(Z < -x) \leq 3e^{-cx^\gamma}$$

and concludes the proof.

3.3.2 A continuous analogue

The aim of this section is twofold. First we prove that Z displays a sub-Gaussian left tail in the L^4 phase ($\beta < \beta_c/2$) using a related model. Then, we use the same techniques to recover a uniform bound on the Laplace transform of $Z + aW$ when $a < 0$.

We consider the binary branching Wiener process which is defined the following way: start with one particule at 0 that splits into two particles which diffuse as standard Brownian motions. At time 1, those two particles split into two new particles which diffuse independently from the position of their ancestor and so on. More precisely, if $(B^v)_{v \in \mathbb{T}}$ is a family of i.i.d. Brownian motions and $\mathcal{N}_t := \{0, 1\}^{\lceil t \rceil}$, where $\lceil t \rceil$ is the smallest integer above t , define for $t \geq 0$ and $u \in \mathcal{N}_t$

$$X_t(u) := \sum_{\emptyset < v \leq u} B_1^v + B_{t-\lceil t \rceil}^u,$$

where $\lceil t \rceil$ is the integer part of t . At time t , we thus have $2^{\lceil t \rceil}$ particles and at integer times $n \in \mathbb{N}$ the process is distributed as the original branching random walk. This detour by the continuous case will allow us to perform stochastic calculus.

The analogue of the derivative martingale is given by

$$Z_t := \frac{1}{2^{\lceil t \rceil}} \sum_{u \in \mathcal{N}_t} (X_t(u) - \beta t) e^{\beta X_t(u) - \frac{\beta^2}{2} t}.$$

Note that despite the integer part in the definition of Z_t , it is a continuous martingale with respect to $\mathcal{F}_t := \sigma(X_s(u), u \in \mathcal{N}_s, s \leq t)$. Its quadratic variation is given by

$$\langle Z \rangle_t = \frac{1}{4^{\lceil t \rceil}} \sum_{u, v \in \mathcal{N}_t} \int_0^t (1 + \beta(X_s(u) - \beta s))(1 + \beta(X_s(v) - \beta s)) e^{\beta X_s(u) - \frac{\beta^2}{2} s} e^{\beta X_s(v) - \frac{\beta^2}{2} s} \mathbf{1}_{s \leq u \wedge v} ds,$$

where $u \wedge v$ is the generation of the last common ancestor of u and v .

Let $\partial \mathbb{T} := \{0, 1\}^{\mathbb{N}}$ and for $u \in \partial \mathbb{T}$, let $X_t(u)$ denotes the position of the ancestor of u at time t . $\partial \mathbb{T}$ is endowed with the ultrametric distance d defined by $d(u, v) = 2^{-u \wedge v}$ and the uniform probability measure μ which is characterized by $\mu(\{v : v \geq u\}) = 2^{-|u|}$. The quadratic variation can be reformulated as

$$\begin{aligned} \langle Z \rangle_t &= \int_{\partial \mathbb{T}^2 \times [0, t]} (1 + \beta(X_s(u) - \beta s))(1 + \beta(X_s(v) - \beta s)) e^{\beta(X_s(u) + X_s(v)) - \beta^2 s} \mathbf{1}_{s \leq u \wedge v} \mu(du) \mu(dv) ds \\ &= \int_{\partial \mathbb{T}^2 \times [0, t]} (1 + \beta(X_s(u) - \beta s))^2 e^{2\beta X_s(u) - \beta^2 s} \mathbf{1}_{s \leq u \wedge v} \mu(du) \mu(dv) ds. \end{aligned}$$

The approach laid in [72] is to define an auxiliary martingale \tilde{Z} where the excursions above some well chosen threshold are removed and to show Gaussian concentration for

this new martingale and the difference with the original one by proving that their brackets remain bounded. We are reproducing the main ideas of the proof here since we are going to use them for the proof of Proposition 3.3.4. The following proposition states that the left tail of Z is sub-Gaussian in the L^4 phase.

Proposition 3.3.3. *Let $\beta < \frac{\beta_c}{2}$, then there exists $C > 0$ such that, for $\lambda \geq 0$,*

$$\mathbb{E} \left[e^{-\lambda Z} \right] \leq e^{C\lambda^2}.$$

Proof. Let us define the following stopping times:

$$\begin{aligned} T_k^u &:= \inf \{ t \geq R_{k-1}^u : X_t(u) = (\beta + \eta)t + A \}, \\ R_k^u &:= \inf \{ t \geq T_k^u : X_t(u) = \beta t \}, \end{aligned}$$

where $R_0^u := 0$. Let $\mathcal{R}^u := \cup_k [R_{k-1}^u, T_k^u]$ and define

$$\tilde{Z}_t := \int_{\partial\mathbb{T}} \int_0^t (1 + \beta(X_s(u) - \beta s)) e^{\beta X_s(u) - \frac{\beta^2}{2}s} \mathbf{1}_{s \in \mathcal{R}^u} dX_s(u) \mu(du).$$

It is a martingale and the difference with the original martingale Z is controlled by

$$\begin{aligned} Q &= \int_{\partial\mathbb{T}} \sum_{k \geq 1} \mathbf{1}_{\{T_k^u < +\infty\}} (A + \eta T_k^u) e^{\left(\frac{\beta^2}{2} + \beta\eta\right)T_k^u + \beta A} \mu(du) \\ &=: \int_{\partial\mathbb{T}} Q^u \mu(du), \end{aligned}$$

and the fact that $Z_\infty \geq \tilde{Z}_\infty - Q$, see [72, Section 5.2] for more details.

Now we prove that both right terms exhibit Gaussian concentration. The bracket of the first one is given by

$$\begin{aligned} \langle \tilde{Z} \rangle_t &= \int_{\partial\mathbb{T}^2 \times [0, t]} (1 + \beta(X_s(u) - \beta s)) (1 + \beta(X_s(v) - \beta s)) e^{\beta(X_s(u) + X_s(v)) - \beta^2 s} \\ &\quad \mathbf{1}_{\{s \in \mathcal{R}^u \cap \mathcal{R}^v\}} \mathbf{1}_{\{s \leq u \wedge v\}} \mu(du) \mu(dv) ds \\ &= \int_{\partial\mathbb{T}^2 \times [0, t]} (1 + \beta(X_s(u) - \beta s))^2 e^{2\beta X_s(u) - \beta^2 s} \mathbf{1}_{\{s \in \mathcal{R}^u \cap \mathcal{R}^v\}} \mathbf{1}_{\{s \leq u \wedge v\}} \mu(du) \mu(dv) ds. \end{aligned}$$

Using the fact that $X_s(u) \leq (\beta + \eta)s + A$ when $s \in \mathcal{R}^u$, one gets

$$\langle \tilde{Z} \rangle_t \leq \int_{\partial\mathbb{T}^2 \times [0, t]} (1 + \beta(\eta s + A))^2 e^{(\beta^2 + 2\beta\eta)s + 2\beta A} \mathbf{1}_{\{s \leq u \wedge v\}} \mu(du) \mu(dv) ds.$$

Now, note that the following integral

$$\int_{\partial\mathbb{T}^2} e^{\alpha u \wedge v} \mu(du) \mu(dv) = \sum_{n \geq 0} \frac{e^{\alpha n}}{2^{n+1}}$$

is finite if and only if $\alpha < \log 2$. This guaranties that $\langle \tilde{Z} \rangle_\infty$ is bounded if $\beta^2 + 2\beta\eta < \log 2$ and provides a sub-Gaussian tail for \tilde{Z}_∞ .

To prove Gaussian concentration for Q , the authors in [72] start by showing that $\mathbb{E}Q < \infty$. Then they use the continuous martingale $Q_t^u := \mathbb{E}[Q^u | \mathcal{F}_t]$ and the decomposition

$$dQ_t^u := A_t^u dX_t(u).$$

This way, the bracket is given by

$$\langle Q \rangle_\infty = \int_{\partial\mathbb{T}^2 \times [0, \infty)} A_t^u A_t^v \mathbf{1}_{\{t \leq u \wedge v\}} \mu(du) \mu(dv) dt.$$

The method used in [72] to obtain the expression of A_t^u is «purely Brownian» and does not involve the covariance structure, it is therefore valid in our context. Let us reproduce the results here.

When $t \in (T_k^u, R_k^u)$, the Markov property for $(X_t(u))_{t \geq 0}$ gives

$$Q^u = \sum_{i=1}^k (A + \eta T_i^u) e^{\left(\frac{\beta^2}{2} + \beta\eta\right) T_i^u} + \mathbb{E}_{X_t(u)} \left[\sum_{i \geq 1} \mathbf{1}_{\widehat{T}_i^t < +\infty} (A + \eta(\widehat{T}_i^t + t)) e^{\left(\frac{\beta^2}{2} + \beta\eta\right) (\widehat{T}_i^t + t) + \beta A} \right],$$

where \mathbb{E}_z denotes the expectation with respect to the law of a standard Brownian motion starting at z and $\widehat{T}_0^t := 0$,

$$\begin{aligned} \widehat{R}_k^t &:= \inf\{s \geq \widehat{T}_{k-1}^t : B_s \leq \beta(t+s)\}, \\ \widehat{T}_k^t &:= \inf\{s \geq \widehat{R}_k^t : B_s = A + (\beta + \eta)(t+s)\}. \end{aligned}$$

In this case, this yields to the following

$$A_t^u = \partial_z \left(\mathbb{E}_z \left[\sum_{i \geq 1} \mathbf{1}_{\widehat{T}_i^t < +\infty} (A + \eta(\widehat{T}_i^t + t)) e^{\left(\frac{\beta^2}{2} + \beta\eta\right) (\widehat{T}_i^t + t) + \beta A} \right] \right) \Big|_{z=X_t(u)}.$$

When $t \in (R_k^u, T_{k+1}^u)$, in the same manner,

$$Q^u = \sum_{i=1}^k (A + \eta T_i^u) e^{\left(\frac{\beta^2}{2} + \beta\eta\right) T_i^u} + \mathbb{E}_{X_t(u)} \left[\sum_{i \geq 1} \mathbf{1}_{T_i^t < +\infty} (A + \eta(T_i^t + t)) e^{\left(\frac{\beta^2}{2} + \beta\eta\right) (T_i^t + t) + \beta A} \right],$$

where $R_0^t := 0$ and

$$\begin{aligned} T_k^t &:= \inf\{s \geq R_{k-1}^t : B_s \leq A + (\beta + \eta)(t+s)\}, \\ R_k^t &:= \inf\{s \geq T_k^t : B_s = \beta(t+s)\}. \end{aligned}$$

Then

$$A_t^u = \partial_z \left(\mathbb{E}_z \left[\sum_{i \geq 1} \mathbf{1}_{T_i^t < +\infty} (A + \eta(T_i^t + t)) e^{\left(\frac{\beta^2}{2} + \beta\eta\right) (T_i^t + t) + \beta A} \right] \right) \Big|_{z=X_t(u)}.$$

In a more compact form, this gives

$$A_t^u = \begin{cases} f_1(t, X_t(u)), & \text{if } t \in (T_k^u, R_k^u), \\ f_2(t, X_t(u)), & \text{if } t \in (R_k^u, T_{k+1}^u). \end{cases}$$

If one chooses η such that $2\eta > \beta$ and $\beta^2 + 2\beta\eta < \log 2$ (which is possible in the L^4 phase), [72, Lemma 5.3, Lemma 5.4]⁶ provides $C = C(A, \eta, \beta) > 0$ such that

$$\begin{aligned} |f_1(t, z)| &\leq C(t+1)e^{\left(\frac{\beta^2}{2} + \beta\eta\right)t}, & \text{for } z \geq \beta t, \\ |f_2(t, z)| &\leq C(t+1)e^{\left(\frac{\beta^2}{2} + \beta\eta\right)t}, & \text{for } z \leq (\beta + \eta)t + A. \end{aligned}$$

Thus,

$$\langle Q \rangle_\infty \leq \int_{\partial\mathbb{T}^2 \times [0, \infty)} C^2(t+1)^2 e^{(\beta^2 + 2\beta\eta)t} \mathbf{1}_{t \leq u \wedge v} \mu(du) \mu(dv) dt < \infty,$$

which proves that Q displays Gaussian concentration. \square

The above approach also provides a useful bound on the Laplace transform of low particles.

Proposition 3.3.4. *There exists $C > 0$ such that, for $a \leq 0$ and $\lambda \geq 0$,*

$$\mathbb{E} \left[e^{-\lambda(Z+aW)} \right] \leq e^{-\lambda a + C \left((\lambda e^{-\beta a})^2 + 1 \right)}.$$

Proof. As in the discrete case, define the derivative martingale when the initial ancestor starts at $a \leq 0$:

$$Z_t^{[a]} := \frac{1}{2^{\lfloor t \rfloor}} \sum_{u \in \mathcal{N}_t} (a + X_t(u) - \beta t) e^{\beta(a + X_t(u)) - \frac{\beta^2}{2}t}.$$

Define as previously

$$\begin{aligned} T_k^{u, [a]} &:= \inf \{ t \geq R_{k-1}^u : a + X_t(u) = (\beta + \eta)t + A \}, \\ R_k^{u, [a]} &:= \inf \{ t \geq T_k^u : a + X_t(u) = \beta t \}, \end{aligned}$$

where $R_0^{u, [a]} := 0$. Let $\mathcal{R}^{u, [a]} := \bigcup_k [R_{k-1}^{u, [a]}, T_k^{u, [a]}]$ and define

$$\tilde{Z}_t^{[a]} = ae^{\beta a} + \int_{\partial\mathbb{T}} \int_0^t (1 + \beta(a + X_s(u) - \beta s)) e^{\beta(a + X_s(u)) - \frac{\beta^2}{2}s} \mathbf{1}_{s \in \mathcal{R}^{u, [a]}} dX_s(u) \mu(du).$$

It is a martingale bounded in L^2 by Equation (3.3.3) and we have

$$Z_\infty^{[a]} \geq \tilde{Z}_\infty^{[a]} - Q^{[a]},$$

where

$$\begin{aligned} Q^{[a]} &= \int_{\partial\mathbb{T}} Q_u^{[a]} \mu(du) \\ &= \int_{\partial\mathbb{T}} \sum_{k \geq 1} \mathbf{1}_{T_k^{u, [a]} < +\infty} (A + \eta T_k^{u, [a]}) e^{\left(\frac{\beta^2}{2} + \beta\eta\right)T_k^{u, [a]} + \beta A} \mu(du). \end{aligned}$$

We are going to prove

$$\sup_{a \leq 0} \text{ess sup} \langle \tilde{Z}^{[a]} \rangle_\infty < \infty, \quad \mathbb{E} Q^{[a]} \leq \mathbb{E} Q, \quad \sup_{a \leq 0} \text{ess sup} \langle Q^{[a]} \rangle_\infty < \infty. \quad (3.3.3)$$

⁶You should read $z \leq (\gamma + \eta)t + A$ in the statement of Lemma 5.4.

Let us admit for a moment those bounds. Then, using $\mathbb{E}\tilde{Z}_\infty^{[a]} = ae^{\beta a}$,

$$\begin{aligned} \mathbb{E}\left[e^{-\lambda Z_\infty^{[a]}}\right] &\leq \mathbb{E}\left[e^{-2\lambda\tilde{Z}_\infty^{[a]}}\right]^{\frac{1}{2}} \mathbb{E}\left[e^{2\lambda Q^{[a]}}\right]^{\frac{1}{2}} \\ &\leq e^{-\lambda\mathbb{E}\tilde{Z}_\infty^{[a]} + \text{ess sup}\langle\tilde{Z}^{[a]}\rangle_\infty\lambda^2} e^{\lambda\mathbb{E}Q^{[a]} + \text{ess sup}\langle Q^{[a]}\rangle_\infty\lambda^2} \\ &\leq e^{-\lambda ae^{\beta a} + C(\lambda^2 + 1)}, \end{aligned}$$

for some $C > 0$. And using $Z + aW = e^{-\beta a}Z^{[a]}$ concludes the proof.

Now let us prove the 3 statements in (3.3.3). First, using the fact that $a + X_s(u) \leq (\beta + \eta)s + A$ when $s \in \mathcal{R}^{u,[a]}$, one obtains the same finite bound as in the case $a = 0$, namely

$$\langle\tilde{Z}^{[a]}\rangle_\infty \leq \int_{\partial\mathbb{T}^2 \times [0, \infty)} (1 + \beta(\eta s + A))^2 e^{(\beta^2 + 2\beta\eta)s + 2\beta A} \mathbf{1}_{s \leq u \wedge v} \mu(du)\mu(dv) ds < \infty.$$

Remark 3.3.5. In the case $a > 0$, we lose the control on the bracket for $t \in (0, T_1)$.

The strong Markov property for $(X_t(u), t \geq 0)$ applied at $T_1^{u,[a]}$ shows that the conditional law of $(T_k^{u,[a]})_{k \geq 2}$ given $T_1^{u,[a]}$ is equal to the conditional law of $(T_k^u)_{k \geq 2}$ given T_1^u . We can thus focus on the first hitting time. Now, recall that the hitting time $T_{\alpha,b}$ of a line $s \mapsto \alpha + bs$ with $b > 0$ by a standard Brownian motion has a law given by

$$\mathbf{1}_{t > 0} \frac{\alpha}{\sqrt{2\pi t^{\frac{3}{2}}}} e^{-\alpha b} e^{-\frac{b^2}{2}t - \frac{\alpha^2}{2t}} + (1 - e^{-2\alpha b}) \delta_\infty,$$

see for instance [24, Formula 2.0.2], and observe that the density part is decreasing in α as soon as $\alpha > \frac{1}{b}$. Thus, if we choose $A > \frac{1}{\beta + \eta}$, we have, for $t \in (0, \infty)$,

$$\mathbb{P}(T_1^{u,[a]} = t) \leq \mathbb{P}(T_1^u = t)^7.$$

Then, with $f^{u,[a]}(t) := \mathbb{P}(T_1^{u,[a]} = t)$, one gets

$$\begin{aligned} \mathbb{P}(Q_u^{[a]} > x) &= \int_0^\infty \mathbb{P}(Q_u^{[a]} > x \mid T_1^{u,[a]} = t) \mathbb{P}(T_1^{u,[a]} = t) dt \\ &= \int_0^\infty \mathbb{P}\left(\sum_{k \geq 1} \mathbf{1}_{T_k^{u,[a]} < +\infty} (A + \eta T_k^{u,[a]}) e^{(\frac{\beta^2}{2} + \beta\eta)T_k^{u,[a]} + \beta A} > x \mid T_1^{u,[a]} = t\right) f^{u,[a]}(t) dt \\ &= \int_0^\infty \mathbb{P}\left(\sum_{k \geq 1} \mathbf{1}_{T_k^u < +\infty} (A + \eta T_k^u) e^{(\frac{\beta^2}{2} + \beta\eta)T_k^u + \beta A} > x \mid T_1^u = t\right) f^{u,[a]}(t) dt \\ &\leq \int_0^\infty \mathbb{P}\left(\sum_{k \geq 1} \mathbf{1}_{T_k^u < +\infty} (A + \eta T_k^u) e^{(\frac{\beta^2}{2} + \beta\eta)T_k^u + \beta A} > x \mid T_1^u = t\right) \mathbb{P}(T_1^u = t) dt \\ &= \mathbb{P}(Q_u > x). \end{aligned}$$

We thus have the following stochastic dominance

$$Q_u^{[a]} \leq_s Q_u^8. \tag{3.3.4}$$

⁷Note that we also have $T_1^{u,[a]} \geq T_1^u$ almost surely!

⁸We didn't find a way to extend it to $Q^{[a]} \leq_s Q$.

And by linearity

$$\mathbb{E}Q^{[a]} = \int_{\partial\mathbb{T}} \mathbb{E} [Q_u^{[a]}] \mu(du) \leq \int_{\partial\mathbb{T}} \mathbb{E} [Q_u] \mu(du) = \mathbb{E} [Q].$$

What remains to be proved is a uniform bound for $\text{ess sup} \langle Q^{[a]} \rangle_\infty$ over a . The proof goes the same way as in [72]: with

$$Q_u^{[a]}(t) := \mathbb{E} [Q_u^{[a]} | \mathcal{F}_t],$$

and

$$dQ_u^{[a]}(t) = A_u^{[a]}(t) dX_t(u),$$

the Markov property applied to $(a + X_t(u))_{t \geq 0}$ yields to the same expression for the infinitesimal increment

$$A_u^{[a]}(t) = \begin{cases} f_1(t, a + X_t(u)), & \text{if } t \in (T_k^{u,[a]}, R_k^{u,[a]}), \\ f_2(t, a + X_t(u)), & \text{if } t \in (R_k^{u,[a]}, T_{k+1}^{u,[a]}). \end{cases}$$

And since in the first case $a + X_t(u) \geq \beta t$ and $a + X_t(u) \leq (\beta + \eta)t + A$ in the second case, one has by [72, Lemma 5.3, Lemma 5.4]

$$|A_u^{[a]}(t)| \leq C(t+1)e^{(\frac{\beta^2}{2} + \beta\eta)t}.$$

This yields to the desired uniform bound over a

$$\langle Q^{[a]} \rangle_\infty \leq \int_{\partial\mathbb{T}^2 \times [0, \infty)} C^2(t+1)^2 e^{(\beta^2 + 2\beta\eta)t} \mathbf{1}_{t \leq u \wedge v} \mu(du)\mu(dv)dt < \infty.$$

□

3.4 Appendix

The following lemma helps to control the terms in (3.2.1).

Lemma 3.4.1. *If $p \in [1, \gamma \wedge 2)$, then $Z_\beta, W_\beta \in L^p$ and there exists $B_p > 0$ such that*

$$\mathbb{E} \left[\left| \sum_{|u|=n} e^{\beta X_u - n\kappa_\beta} Z_\beta^u \right|^p \middle| \mathcal{F}_n \right] \leq B_p \mathbb{E} |Z_\beta|^p \sum_{|u|=n} e^{p(\beta X_u - n\kappa_\beta)},$$

$$\mathbb{E} \left[\left| \sum_{|u|=n} e^{\beta X_u - n\kappa_\beta} (X_u - \beta n)(W_\beta^u - 1) \right|^p \middle| \mathcal{F}_n \right] \leq B_p \mathbb{E} |W_\beta - 1|^p \sum_{|u|=n} e^{p(\beta X_u - n\kappa_\beta)} |X_u - \beta n|^p.$$

Proof. If Y_1, \dots, Y_N are centered and independent random variables in L^p , the Marcinkiewicz-Zygmund inequality provides a positive B_p (which does not depend on Y) such that

$$\mathbb{E} \left[\left| \sum_{k=1}^N Y_k \right|^p \right] \leq B_p \mathbb{E} \left[\left(\sum_{k=1}^N Y_k^2 \right)^{p/2} \right].$$

Now the function $x \mapsto x^{p/2}$ is subadditive since $p < 2$, thus

$$\mathbb{E} \left[\left| \sum_{k=1}^N Y_k \right|^p \right] \leq B_p \mathbb{E} \left[\sum_{k=1}^N |Y_k|^p \right],$$

and the proof of the proposition is a consequence of this last inequality applied to the random variables $e^{\beta X_u - n\kappa_\beta} Z_\beta^u$ and $e^{\beta X_u - n\kappa_\beta} (X_u - \beta n)(W_\beta^u - 1)$ conditionally on \mathcal{F}_n . \square

Lemma 3.4.2. *There exists $\theta > 0$, such that, as n tends to ∞ ,*

$$\log \det \Sigma_n = \theta 2^n - 2 \log 2 + o(1).$$

Proof.

$$\begin{aligned} \log \det \Sigma_n &= \log(2^n - 1) + \sum_{k=0}^{n-1} 2^k \log(2^{n-k} - 1) \\ &= \log(2^n - 1) + 2^n \sum_{k=1}^n \frac{1}{2^k} \left(k \log 2 + \log \left(1 - \frac{1}{2^k} \right) \right) \\ &= 2^n \left(\log 2 \sum_{k=1}^n \frac{k}{2^k} + \sum_{k=1}^n \frac{1}{2^k} \log \left(1 - \frac{1}{2^k} \right) + \frac{\log(2^n - 1)}{2^n} \right) \\ &= 2^n \left(\underbrace{\theta - \log 2 \sum_{k=n+1}^{\infty} \frac{k}{2^k} - \sum_{k=n+1}^{\infty} \frac{1}{2^k} \log \left(1 - \frac{1}{2^k} \right) + \frac{\log(2^n - 1)}{2^n}}_{\frac{-2 \log 2}{2^n} + O(4^{-n})} \right), \end{aligned}$$

where we set $\theta = 2 \log 2 + \sum_{k=1}^{\infty} \frac{1}{2^k} \log \left(1 - \frac{1}{2^k} \right)$ and used $\sum_{k=n+1}^{\infty} \frac{k}{2^k} = \frac{n+2}{2^n}$ and $\sum_{k=n+1}^{\infty} \frac{1}{2^k} \log \left(1 - \frac{1}{2^k} \right) = O(4^{-n})$. \square

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