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THÈSE DE DOCTORAT DE

L'UNIVERSITÉ DE RENNES

ÉCOLE DOCTORALE N° 601

*Mathématiques, Télécommunications, Informatique, Signal, Systèmes,
Électronique*

Spécialité : *Mathématiques et leurs interactions*

Par

Shahnaz FARHAT

Analyse multi-échelle des interactions entre les ondes et les particules

Thèse présentée et soutenue à Rennes, le 21/09/2023

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TABLE OF CONTENTS

Remerciements	iii
Aperçu de la thèse	v
Introduction	xix
0.1 General context and main goals	xix
0.2 Part I: the deterministic approach	xxii
0.2.1 General ODEs framework	xxiii
0.2.2 The first application: Hamilton-Jacobi equations	xxvi
0.2.3 The second application: Long-time gyrokinetic equations	xxviii
0.3 Part II: the statistical approach	xxx
0.3.1 Classical limit of the quantum particle-field interactions	xxxi
0.3.2 Abstract nonlinear initial value problems	xxxvi
I <i>The deterministic approach</i>	1
1 Creation of scales and phases in nonlinear evolution equations	2
1.1 Introduction	2
1.1.1 A class of Hamilton-Jacobi equations	3
1.1.2 Onset of smaller scales and new phases	5
1.1.3 From Hamilton-Jacobi equations to nonlinear differential equations	6
1.2 A class of nonlinear differential equations	7
1.2.1 Assumptions and results	10
1.2.2 A few comments on the results	16
1.2.3 Plan and motivations	19
1.3 Blow-up procedure	20
1.3.1 Desingularization method	21
1.3.2 General structure of the lifting	24
1.3.2.1 Admissible functions Ξ_0	25

1.3.2.2	Properties of Ξ	25
1.3.2.3	Passage from the original field z to the lifting \mathfrak{z}	27
1.3.3	Transformation of the equations	28
1.3.4	Proof of Proposition 1.3.2	29
1.4	Three-scale WKB calculus	34
1.4.1	Profile formulation	34
1.4.2	Three-scale analysis	38
1.4.2.1	Formal calculus	39
1.4.2.2	Problem-solving strategy	42
1.4.3	Construction of approximate solutions	44
1.4.3.1	Base case	44
1.4.3.2	Induction step	47
1.5	Stability estimates	50
1.5.1	Justification of the formal computations	51
1.5.2	Description of the redressed field in terms of the frozen phase	54
1.5.3	Back to the original field	58
1.6	Application to Hamilton-Jacobi equations	60
1.6.1	Oscillating framework	61
1.6.1.1	Data adjustment	61
1.6.1.2	Strategy, intermediate results and notations	62
1.6.2	Connection to differential equations	63
1.6.2.1	Method of characteristics	63
1.6.2.2	Comments on assumptions and main result	65
1.6.3	Spatial component of the characteristics \mathbf{x}	68
1.6.3.1	Asymptotic expansion of \mathbf{x}	68
1.6.3.2	Role of transparency conditions	73
1.6.4	Inverse map \mathbf{x}^{-1}	75
1.6.4.1	Formal equations for \mathbf{x}^{-1}	76
1.6.4.2	Nonlinear modulation equation	78
1.6.4.3	Proof of Lemma 1.6.10	80
1.6.5	Proof of Theorem 1.1.5	80
2	Long time Gyrokinetic equations	85
2.1	Introduction	85

2.1.1	Motivations	85
2.1.2	Notions of average and mean flows	88
2.1.3	Assumptions on the external field	91
2.1.3.1	Toroidal description of tokamaks	91
2.1.3.2	Adjusting the magnetic field	92
2.1.3.3	Adjusting the electric field E	92
2.1.4	Results, outcomes and plan	94
2.1.4.1	Coherent structures	94
2.1.4.2	Outcomes	95
2.1.4.3	Plan of the text	97
2.2	Modeling of the dynamics	98
2.2.1	Nondimensionalization	98
2.2.2	Geometric reformulation of the equations	100
2.2.2.1	Change of coordinates for x and v	100
2.2.2.2	Equations in toroidal coordinates for x	102
2.2.2.3	Equations in spherical coordinates for v	102
2.3	Work of preparation	106
2.3.1	Structure of the average flow	106
2.3.1.1	Equations for the average flow	106
2.3.1.2	Two invariant quantities	108
2.3.1.3	Study of the lifespan	109
2.3.1.4	Solving the average flow in the parallel case	110
2.3.2	Periodic properties of the average flow	111
2.3.2.1	Foliation by Hamiltonian subsystems	111
2.3.2.2	Reminders about the pendulum	115
2.3.2.3	Periodic properties away from separatrices	118
2.3.2.4	Linear growths plus periodic oscillations	119
2.4	Applications	121
2.4.1	A manageable framework	121
2.4.1.1	A singular system of odes with multiscale periodic features	121
2.4.1.2	Verification of the prerequisites	123
2.4.2	Coherent structures	125
2.4.2.1	Coherent structures associated with (2.1.8)	125

2.4.2.2	Coherent structures associated with the long time study	
	of (2.1.3)	126
2.4.2.3	Asymptotic behavior of the toroidal angle	130
2.4.2.4	Proof of Theorem 2.1.8	137
2.4.3	The long time gyrokinetic equations	137
2.4.3.1	Computation of the source term $\langle \bar{A}_1 \rangle$	139
2.4.3.2	Confinement properties	142
2.4.3.3	The case of a radial electric field	144
2.4.3.4	Appendix	146

II *The statistical approach* 151

3	Quantum-classical motion of charged particles interacting with scalar fields	152
3.1	Introduction	152
3.1.1	General framework	155
3.1.1.1	The particle-field equation	155
3.1.1.2	The Nelson model	156
3.1.2	Assumptions and main results	159
3.2	The classical system	161
3.2.1	The interaction representation	161
3.2.2	Properties of the particle-field equation	164
3.2.3	Local well-posedness in special subspaces	169
3.3	The quantum system	172
3.3.1	Quantum estimates	172
3.3.2	Self-adjointness of Nelson Hamiltonian	177
3.3.3	The dynamical equation	178
3.3.3.1	Duhamel formula	178
3.3.3.2	The commutator expansion	181
3.4	Existence of Wigner measure	186
3.4.1	Propagation of assumptions	187
3.4.1.1	Position and Momentum operator estimates	187
3.4.1.2	Field operator's estimates	193
3.4.1.3	Propagation of estimates uniformly for all times	197

3.4.2	Existence of unique Wigner measure	199
3.5	Derivation of the characteristic equations	203
3.5.1	Convergence	203
3.5.2	The characteristic equation	207
3.5.3	The Liouville equation	208
3.6	Proof of the main result	211
3.7	Appendix	215
3.7.1	Prokhorov theorem	215
3.7.2	Useful results	215
3.7.3	Technical results about convergence	216
4	Almost sure existence of global solutions	221
4.1	Introduction and main results	221
4.1.1	General framework	225
4.1.2	Main result	228
4.1.3	Application to ODEs	229
4.1.4	Application to PDEs	230
4.2	Global superposition principle	236
4.2.1	Weak topologies	237
4.2.2	Projective argument	240
4.2.3	Analysis on finite dimensions	243
4.2.3.1	The Lipschitz case	243
4.2.3.2	The Borel case	246
4.2.4	Analysis on Banach spaces	251
4.3	Proof of main results	259
4.3.1	The globalization argument	259
4.3.2	Analysis of ODEs and PDEs	262
4.4	Appendices	265
4.4.1	Results in finite dimensions	265
4.4.2	Radon spaces, Tightness	266
4.4.3	Equi-integrability and Dunford-Pettis theorem	267
4.4.4	Compactness argument	268
4.4.5	Disintegration theorem	271
4.4.6	Density argument	271

Conclusion and Perspectives	273
Bibliography	277

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APERÇU DE LA THÈSE

Contexte général et principaux objectifs

L'interaction entre la matière et les champs est un problème fondamental de la physique mathématique. Elle intervient dans de nombreux phénomènes et leur implémentation est à la source d'importantes technologies (parmi lesquelles on peut citer les plasmas de fusion, l'effet photoélectrique, ...). Observée à de très grandes échelles, dans les plasmas [Jac99, Pie17], elle peut y être décrite de manière satisfaisante en ayant recours à une description classique des particules et des champs. Par contre, à de plus petites échelles, l'interaction particule-champ revêt d'autres modalités. Elle joue là encore un rôle majeur. Elle explique notamment le décalage de Lamb dans le spectre de l'atome d'hydrogène [FGA10, Fey61, Wei05]. Elle est impliquée en électrodynamique quantique [Jac99] et en théorie quantique des champs QFT [FGA10] où elle permet de mieux cerner l'interaction entre la matière et les champs.

Cette thèse est consacrée à l'étude de certaines propriétés des interactions entre matière et champs. Nous abordons ce sujet d'un point de vue à la fois classique et quantique. Dans chaque cas, nous considérons des structures mathématiques différentes; nous travaillons avec des outils adaptés; et nous mettons en oeuvre des stratégies ad hoc. Tout au long du texte, nous obtenons des résultats qui, bien qu'abstraites, s'appliquent à des situations physiques concrètes, permettant ainsi d'apporter des renseignements allant au-delà de ce qui est connu. D'une part, nous résolvons des problèmes précédemment ouverts liés à la construction et à la modélisation de solutions fortement oscillantes d'équations aux dérivées partielles non linéaires. D'autre part, nous justifions rigoureusement le principe de correspondance de Bohr pour certains modèles de champs de particules et progressons de façon notable sur le problème connexe de l'existence de solutions globales à faible régularité pour certaines EDP non linéaires.

En mécanique classique, le terme "matière" désigne généralement des particules qui sont décrites par leurs positions et leurs moments, et qui satisfont à l'équation de Newton. On peut également représenter les particules par leur densité f qui vérifie l'équation de Vlasov. Les champs se réfèrent au champ électrique E et au champ magnétique B . Les

quantités f , E et B satisfont le système de Vlasov-Maxwell relativiste, voir la récente thèse de Brigouleix [Bri20] pour une présentation actualisée ou l'article [CI23] pour des résultats récents dans cette direction. Dans la partie I de cette thèse, nous considérons des solutions stationnaires $(E, B)(x)$ des équations de Maxwell. Les problèmes viennent du fait que (E, B) dépend de x , avec de plus B grand (de taille ε^{-1} avec $0 < \varepsilon \ll 1$), comme c'est le cas dans les applications. La force de Lorentz, qui est déterminée par (E, B) et qui donc est fixée, a un impact significatif sur la structure mésoscopique de la densité f . L'objectif est de déterminer son influence sur l'organisation globale de la matière, abordée d'un point de vue eulérien.

En mécanique quantique, la matière est constituée de particules quantiques qui satisfont l'équation de Schrödinger, non relativiste ou semi-relativiste. Dans la partie II de cette thèse, la dynamique des particules est décrite par l'évolution temporelle de leurs opérateurs de position et de quantité de mouvement, tandis que les champs sont des champs quantifiés (bosons). Les interactions et les champs sont généralement exprimés à l'aide d'opérateurs de création et d'annihilation dans les espaces de Fock par la méthode de seconde quantification [Ber66]. Le système complet de particules et de champs est alors décrit par un hamiltonien linéaire (opérateur auto-adjoint non borné) défini sur le produit tensoriel d'un espace de Lebesgue L^2 et d'un espace de Fock.

Cette thèse est organisée en deux parties distinctes, I et II. Elle se concentre sur des systèmes d'équations différentielles ordinaires (ODEs) ayant des structures différentes mais comportant néanmoins un point commun: le rôle crucial joué en toile de fond par des propriétés d'intégrabilité. Les deux parties sont indépendantes. Elles mettent en jeu des systèmes particuliers provenant de mises à l'échelle spécifiques.

- Partie I : nous adoptons une approche déterministe. Nous traitons un système d'équations différentielles non linéaires de dimension finie. Nous examinons en fait une catégorie particulière d'EDO non linéaires caractérisée par la présence de très fortes oscillations. Dans ce contexte, nous parvenons à mettre en ouvre la méthode des caractéristiques pour développer deux applications :
 - La première application concerne l'étude d'une famille d'équations de Hamilton-Jacobi fortement oscillantes;
 - La deuxième application obtient et justifie un modèle réduit (appelé *équations gyrocinétiques en temps longs*) pour le comportement en temps très longs de solutions oscillantes de l'équation de Vlasov.

-
- Partie II : nous adoptons une approche statistique. Nous considérons une classe de systèmes d'équations différentielles non linéaires de dimension infinie. Nous nous concentrons sur deux sujets interconnectés que nous développons en détail :
 - Le premier sujet examine la transition entre l'équation quantique et l'équation classique du champ de particules, grâce à l'utilisation des mesures de Wigner;
 - Le deuxième sujet se concentre sur la construction de solutions globales pour des problèmes de Cauchy non linéaires abstraits, en utilisant des arguments s'appuyant sur la théorie de la mesure. Il convient de noter que ces solutions ne sont pas nécessairement uniques.

Après avoir défini le contexte, nous exposons les principaux objectifs de la thèse. Notre objectif principal dans les deux parties est d'identifier des solutions à faible régularité (ou disons moins régulières qu'à l'usuel) d'équations aux dérivées partielles.

- Objectifs de la première partie : nous recherchons des familles de solutions qui sont “hautement oscillantes”, c'est à dire qui sont caractérisées par de fortes oscillations (ou des variations rapides). Plus précisément, nous cherchons à décrire des pertes de régularité sous la forme de développements asymptotiques oscillants. Le but est en particulier d'identifier et de justifier le contenu des termes principaux (qui capturent le comportement principal en temps longs). Bien que les solutions impliquées soient régulières (lisses), leurs amplitudes ou celles de leurs dérivées (représentées par des puissances négatives d'un petit paramètre ε qui est destiné à tendre vers zéro) sont significativement grandes. Elles dérogent aux régimes habituellement considérés.
- Objectifs de la partie II : nous recherchons des solutions “de faible régularité”, qui sont construites à l'aide de méthodes statistiques et de techniques issues de la théorie de la mesure. La faible régularité de ces solutions provient de la faible régularité des champs de vecteurs utilisés dans leur construction. Les techniques standards comme les arguments de point fixe ne sont pas efficaces dans ce scénario. Au lieu de cela, nous utilisons des arguments statistiques pour construire des solutions globales. Par conséquent, le flux global est de type Borel et peut manquer de continuité.

Les équations différentielles considérées posent des problèmes dans les deux parties, car les arguments habituels ne suffisent pas à fournir des solutions.

- Difficultés dans la partie I : les difficultés proviennent des inhomogénéités des champs et de leur grande taille, ainsi que de la longue durée considérée. En particulier, le

mélange d'oscillations et de fortes non-linéarités implique des régimes supercritiques (grande amplitude) pour la résolution. Bien que les arguments standards puissent fournir des solutions sur un petit intervalle de temps, ils se révèlent incapables d'en produire à plus long terme, et de capturer leur contenu;

- Difficultés de la partie II : Dans les deux grands axes de cette partie, nous sommes confrontés à des défis communs - la faible régularité des champs vectoriels concernés et le fait de travailler dans un espace de dimension infinie, ce qui rend les arguments usuels inefficaces. De plus, le premier sujet porte sur la démonstration de la validité du principe de correspondance de Bohr, ce qui ajoute un cran de difficulté.

Étant donné la portée limitée des arguments habituels pour avancer sur les problèmes évoqués, d'autres approches sont les bienvenues. Dans la thèse, nous développons des méthodes plus sophistiquées qui exploitent certaines spécificités de chaque problème, ceci afin d'apporter des éléments d'information malgré la complexité sous-jacente. Nous espérons ainsi jeter un nouvel éclairage sur des sujets importants et contribuer aux efforts en cours pour approfondir leur compréhension. Ce qui suit est une vue d'ensemble des principaux outils et des diverses stratégies que nous avons utilisés.

Partie I : l'approche déterministe

Nous considérons le système suivant d'équations différentielles ordinaires non linéaires comportant un large terme source oscillant (de forçage):

$$\partial_\tau \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0; z; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right), \quad \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} (0) = \begin{pmatrix} z_0 \\ \mathbf{v}_0 \end{pmatrix}, \quad \varepsilon \ll 1. \quad (0.0.1)$$

Le passage de [\(0.0.1\)](#) à des EDPs (de type Hamilton-Jacobi et Vlasov) repose sur des méthodes ad hoc (passage du point de vue Lagrangien à Eulérien). Côté fluide ou cinétique, il s'agit de comprendre comment un régime laminaire peut donner lieu soudainement à des mouvements pouvant être qualifiés de turbulents. Les solutions sont en fait régulières mais leurs dérivées sont grandes, calibrées par des puissances négatives du paramètre ε (qui est destiné à tendre vers zéro). La difficulté consiste à expliquer un mécanisme de création de grandes variations, c'est-à-dire de rendre compte de l'apparition de nouvelles (petites) échelles. En filigrane, cela demande de pouvoir gérer les interactions non linéaires des oscillations multi-phases et multi-échelles ainsi générées.

Cette thématique apparaît déjà dans des articles antérieurs [Che17, Che15] mais c'était dans un cadre plus restreint ne permettant pas d'incorporer les deux applications obtenues d'une part dans le chapitre 1 pour des équations de Hamilton-Jacobi et d'autre part dans le chapitre 2 pour des équations de Vlasov. Une des pièces manquantes dans [Che17, Che15] était le rôle pouvant être joué par des conditions d'intégrabilité sous-jacentes. Cela a motivé dans le chapitre 1 un travail théorique de mise à plat des hypothèses minimales à imposer sur A et V en vue de pouvoir accéder à :

- **L'existence de solutions** pour (0.0.1) sur un intervalle de temps $[0, \mathcal{T}(\varepsilon, z_0, \mathbf{v}_0)]$ qui est uniforme en le choix du paramètre de raideur ε :

$$\exists \mathcal{T}(z_0, \mathbf{v}_0) \in \mathbb{R}_+^* ; \quad \forall \varepsilon \in]0, 1], \quad 0 < \mathcal{T}(z_0, \mathbf{v}_0) \leq \mathcal{T}(\varepsilon, z_0, \mathbf{v}_0). \quad (0.0.2)$$

C'est plus long que ce que prédit les arguments usuels menant à $\mathcal{T}(\varepsilon, z_0, \mathbf{v}_0) \lesssim \varepsilon$.

- Des **développements asymptotiques** qui décrivent avec une grande précision la forme oscillante des solutions et qui reposent sur des équations effectives (eiconales) pour les phases et (de modulation) pour les profils. L'accent est mis sur le point de vue Eulérien. Etant donné un instant t fixé, on s'intéresse à la vision spatiale globale du flot pour tout ε et x . Cela amène à dégager des *structures cohérentes*.

Pour aboutir à ces informations, il a fallu mettre en place dans le chapitre 1 :

- Une **procédure de relèvement** qui prolonge les méthodes de forme normale. En introduisant de nouvelles inconnues pour décrire l'espace des phases, on arrive à mettre de côté une partie des termes singuliers.
- Un **calcul WKB à trois échelles sur-critique**. Pour pousser au-delà des régimes classiques, l'idée est de regarder la composante \mathbf{v} de la solution comme une phase. Cela signifie que \mathbf{v} est vue comme oscillant par rapport à elle-même au travers d'une relation implicite. Cela conduit à une nouvelle notion d'équation de profil permettant de filtrer l'action de \mathbf{v} . On travaille modulo la détermination de \mathbf{v} . Cela revient à effectuer la construction de solutions approchées et à étudier leur stabilité dans un espace quotient (piloté de façon non linéaire par la solution exacte \mathbf{v}). Le contenu de \mathbf{v} est au final récupéré via un théorème des fonctions implicites, avec une perte d'information (calibrée par des puissances négatives d' ε).

Première application: les équations de Hamilton-Jacobi.

L'étude asymptotique de (0.0.1) débouche sur une première application concernant l'équation scalaire non linéaire:

$$\partial_\tau u_\varepsilon + \frac{1}{\varepsilon} \mathbf{H}\left(\varepsilon, \frac{\tau}{\varepsilon}, x, \varepsilon u_\varepsilon, \nabla_x u_\varepsilon, \frac{u_\varepsilon}{\varepsilon}\right) = 0, \quad u_\varepsilon(0, x) = \mathcal{U}_0(\varepsilon, x), \quad \varepsilon \ll 1. \quad (0.0.3)$$

Des solutions faibles sont certes accessibles par des arguments de compacité. Mais au contraire du chapitre 1, ces méthodes ne donnent accès ni à la forme oscillante des solutions ni à leur stabilité. Cela nécessite de pouvoir établir un lien entre (0.0.3) et (0.0.1) via la méthode des caractéristiques. C'est loin d'être évident car l'opération de projection spatiale des (bi-)caractéristiques est susceptible d'être multivaluée (croisements donnant lieu à l'apparition de caustiques). Pour avancer, deux points clés sont:

- l'exploitation de **conditions de transparence** qui émanent d'une propriété de complète intégrabilité repérable sur un *flot moyennisé* extrait de (0.0.1) et rendu visible après une *analyse asymptotique*.
- la vérification des hypothèses du **théorème d'inversion globale de Hadamard-Lévy** en vue de leur implémentation.

A noter qu'un phénomène nouveaux remarquable est mis en valeur et justifié dans le chapitre 1, à savoir l'apparition d'une nouvelle échelle ε^3 à partir de l'interaction non linéaire des échelles ε^0 , ε^1 et ε^2 déjà présentes dans (0.0.3).

Seconde application: mise en valeur et justification d'équations gyrocinétiques valables en temps long.

Le chapitre 2 s'intéresse aux mouvements de particules chargées (ce sont typiquement des électrons) qui sont soumises à un champ électromagnétique fixe mais inhomogène. Sous l'influence de la force de Lorentz, de telles particules issues à l'instant $t = 0$ de la position $(x_0, v_0) \in \mathbb{R}^3 \times \mathbb{R}^3$ se déplacent (lorsqu'elles sont relativistes) dans l'espace des phases $\mathbb{R}^3 \times \mathbb{R}^3$ suivant:

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = (1 - |v|^2)^{1/2} \left\{ -\frac{1}{\varepsilon} v \wedge B(x) - E(x) + (v \cdot E(x)) v \right\}. \end{cases} \quad (0.0.4)$$

Les courbes intégrales associées à (0.0.4) tracent les caractéristiques d'une équation de Vlasov. En physique des plasmas, le paramètre ε vient de l'inverse de la gyro fréquence des électrons (fréquence de Larmor). En pratique, ε est petit, de l'ordre de 10^{-4} . On le considère comme tendant vers zéro.

Le régime des temps intermédiaires (pour lequel $t \sim 1$) a été très étudié à la fois en mathématique et en physique. C'est la théorie des équations gyrocinétiques. En revanche, les temps plus longs (pour lesquels $t \sim \varepsilon^{-1}$ ou $\tau := \varepsilon t \sim 1$) a été moins abordé du fait des complications induites par les variations du champ (E, B) . L'accent est souvent mis sur le point de vue de Lagrange (théorie KAM), avec un éclairage de type système dynamique. Le chapitre 2 privilégie quant à lui le point de vue d'Euler. Il réussit dans ce cadre à mettre en valeur et à justifier:

- l'apparition de **structures oscillantes cohérentes**.
- des **équations de modulation** qui sont valables en temps longs (pour $\tau \sim 1$).

Le chapitre 2 repose sur:

- un **travail de modélisation** permettant de formuler le problème physique dans un cadre qui est compatible avec le chapitre 1. Cela demande en amont d'introduire des coordonnées géométriques et d'effectuer un certain nombre de changements de variables qui transforment (0.0.4) en un système ayant la forme (0.0.1).
- de vérifier que tous les prérequis du chapitre 1, en particulier les **propriétés d'intégrabilité**, sont effectivement vérifiés.

En présence d'un champ électrique non trivial ($E \neq 0$), l'énergie cinétique n'est pas conservée. Les particules peuvent être accélérées ou décélérées ce qui peut provoquer leur sortie de tout domaine compact. L'analyse menée dans le chapitre 2 étudie rigoureusement ces possibles mécanismes de déconfinement. Sous les hypothèses retenues, elle aboutit à des critères concrets portant sur le champ (E, B) et permettant de garantir que le plasma reste (pour $\tau \sim 1$) dans une zone bornée de l'espace des phases.

Partie II: l'approche statistique

Dans la seconde partie de la thèse, on étudie deux sujets connexes en utilisant dans les deux cas une approche statistique:

- Le premier axe explore la transition du quantique vers le classique pour des systèmes onde-particules où un nombre fini et fixe de particules quantiques non relativistes

ou semi-relativistes interagissent avec un champ scalaire quantifié dans une limite d'échelle semiclassique. Un tel projet vise à dériver rigoureusement des équations effectives à partir des principes fondamentaux de la mécanique quantique (voir par exemple [AF17, BPS15, LP20] et les références qui s'y trouvent). Lors de l'analyse de la transition quantique-classique, on utilise de manière cruciale une méthode statistique similaire à celle de la deuxième partie pour établir l'existence de solutions globales pour une équation onde-particules spécifique. Dans ce cas, l'interaction entre l'onde et les particules est suffisamment singulière pour empêcher d'utiliser un argument de point fixe standard.

- Le deuxième volet concerne la construction, pour des problèmes de Cauchy abstraits en dimension finie ou infinie, de solutions globales en temps. Ces problématiques sont très vastes et très étudiées. Notre objectif ici est de se concentrer sur la construction de telles solutions globales par des outils statistiques et d'arguments abstraits issus de la théorie de la mesure. Une application remarquable de notre approche concerne l'analyse non linéaire des EDPs et les solutions de faible régularité pour les EDPs dispersives (voir par exemple les travaux marquants de Bourgain [Bou94, Bou96, Bou97], ainsi que les contributions ultérieures [BT08b, BTT18, DNY21, NS19]).

Limite classique des interactions particules-champs quantiques

Le chapitre 3 étudie la dynamique de particules chargées en interaction avec un champ de Boson scalaire. Principalement, on explore la transition quantique-classique d'un point de vue mathématique pour la théorie de Yukawa, dans les deux situations physiques de particules chargées non relativistes et semi-relativistes. Rappelons que la théorie de Yukawa modélise la force nucléaire forte comme une interaction entre les nucléons (particules) et les mésons (champs).

- D'un point de vue classique, la dynamique est régie par un système non linéaire d'équations onde-particules de type EDP-EDO. Plus précisément, on considère un nombre fixe de particules classiques (disons n) dans l'espace de configuration \mathbb{R}^d interagissant

avec un champ de boson scalaire comme suit

$$\left\{ \begin{array}{l} \partial_t p_j = -\frac{\partial H}{\partial q_j} = -\nabla_{q_j} V(q) - \int_{\mathbb{R}^d} 2\pi i k \frac{\chi(k)}{\sqrt{\omega(k)}} \left[\alpha(k) e^{2\pi i k \cdot q_j} - \overline{\alpha(k)} e^{-2\pi i k \cdot q_j} \right] dk; \\ \partial_t q_j = \frac{\partial H}{\partial p_j} = \frac{p_j}{\sqrt{p_j^2 + M_j^2}} \quad \text{ou} \quad \frac{p_j}{M_j}; \\ i\partial_t \alpha = \frac{\partial H}{\partial \alpha} = \omega(k) \alpha(k) + \sum_{j=1}^n \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot q_j}. \end{array} \right. \quad (0.0.5)$$

Ici M_j est la masse de la $j^{\text{ème}}$ particule, $\omega(k) := \sqrt{k^2 + \mu^2}$ est la relation de dispersion avec $\mu > 0$, $V : \mathbb{R}^{dn} \rightarrow \mathbb{R}$ est un potentiel extérieur, $\chi : \mathbb{R}^d \rightarrow \mathbb{R}$ est la transformée inverse de Fourier de la distribution de charge qui devrait, avec le potentiel V , satisfaire certaines hypothèses. L'effet de la fonction de troncature χ (appelée "facteur de forme") est de régulariser l'hamiltonien de manière à ce qu'il devienne bien défini. Les deux vecteurs $q_j = (q_j^1, \dots, q_j^d) \in \mathbb{R}^d$ et $p_j = (p_j^1, \dots, p_j^d) \in \mathbb{R}^d$ sont respectivement les positions et les impulsions des particules. Quant au terme $\alpha \in L^2(\mathbb{R}^d, \mathbb{C})$, il joue le rôle du champ de méson scalaire. L'hamiltonien du système onde-particules prend la forme suivante :

$$\begin{aligned} H(p, q, \alpha) &= \sum_{j=1}^n f_j(p_j) + V(q_1, \dots, q_n) + \int_{\mathbb{R}^d} \overline{\alpha(k)} \omega(k) \alpha(k) dk \\ &\quad + \sum_{j=1}^n \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} \left[\alpha(k) e^{2\pi i k \cdot q_j} + \overline{\alpha(k)} e^{-2\pi i k \cdot q_j} \right] dk, \end{aligned}$$

où nous avons introduit $f_j(p_j) := \sqrt{p_j^2 + M_j^2}$ ou $p_j^2/2M_j$. Le premier choix correspond au cas semi-relativiste et le deuxième correspond au cas non-relativiste. On définit de plus l'espace des phases classique :

$$X^\sigma := \mathbb{R}^{dn} \times \mathbb{R}^{dn} \times \mathcal{G}^\sigma$$

où \mathcal{G}^σ est un espace de Lebesgue L^2 à poids muni de la norme suivante

$$\|\alpha\|_{\mathcal{G}^\sigma}^2 = \int_{\mathbb{R}^d} \omega(k)^{2\sigma} |\alpha(k)|^2 dk.$$

Notons que l'espace d'énergie pour lequel la fonction $H(p, q, \alpha)$ se trouve naturellement bien définie est $X^{1/2}$.

De telles équations classiques ont été étudiées auparavant, voir par exemple [\[KKS99\]](#)

[KS98], [KSK97]. Ces articles insistent sur la construction de solutions approchées et sur le comportement en temps long des solutions. Les auteurs considèrent une fonction χ qui est à support compact. Nous arrivons à adoucir cette contrainte jusqu'à:

$$\begin{cases} \bullet & V \in C_b^2(\mathbb{R}^{dn}, \mathbb{R}), \\ \bullet & \omega^{\frac{3}{2}-\sigma} \chi \in L^2(\mathbb{R}^d, \mathbb{R}), \quad \sigma \in [\frac{1}{2}, 1]. \end{cases} \quad (H_1)$$

Sous ces hypothèses, le champ de vecteurs associé à l'équation (0.0.5) manque de régularité pour pouvoir appliquer un argument standard de point fixe en vue d'établir le caractère globalement bien posé. En vue de pouvoir travailler sous la contrainte relaxée (H₁), notre démarche est d'examiner les relations entre les descriptions classiques et quantiques du système onde-particules.

- D'un point de vue quantique, l'évolution est linéaire et générée par le hamiltonien de Nelson qui est donné comme suit

$$\hat{H}_\hbar = \sum_{j=1}^n f_j(\hat{p}_j) + V(\hat{q}) + d\Gamma(\omega) + \sum_{j=1}^n \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} \left[e^{2\pi i k \cdot \hat{q}_j} a_\hbar(k) + e^{-2\pi i k \cdot \hat{q}_j} a_\hbar^*(k) \right] dk, \quad (0.0.6)$$

où $\hat{H} : \mathcal{H} \rightarrow \mathcal{H}$ avec $\mathcal{H} := L^2(\mathbb{R}^{2dn}, \mathbb{C}) \otimes \mathcal{F}(L_k^2(\mathbb{R}^d, \mathbb{C}))$, \hat{p}_j et \hat{q}_j sont les opérateurs moment et position

$$\hat{p}_j = -i\hbar \nabla_{x_j}, \quad \hat{q}_j = x_j, \quad \hat{q} = (\hat{q}_1, \dots, \hat{q}_n),$$

$d\Gamma(\cdot)$ est la seconde quantification et $a_\hbar(k), a_\hbar^*(k)$ sont les opérateurs d'annihilation-création agissant sur l'espace de Fock symétrique $\mathcal{F}(L_k^2(\mathbb{R}^d, \mathbb{C}))$. Sous certaines hypothèses sur χ et le potentiel V , nous savons que l'opérateur \hat{H}_\hbar est auto-adjoint (voir [AF17] et ses références).

L'objectif principal est d'identifier la dynamique classique du système onde-particules (0.0.5) comme une équation effective du système dynamique microscopique (0.0.6) dans le régime semiclassique (petites valeurs de la constante de Planck $\hbar \rightarrow 0$). Une telle relation est connue en physique sous le nom de principe de correspondance de Bohr. Il s'agit ici d'en fournir un cadre mathématique rigoureux et d'explorer la relation entre dynamiques quantiques et classiques. La difficulté est la suivante. D'une part, le champ de vecteur de l'équation onde-particules (0.0.5) est seulement continu mais pas Lipschitz de sorte que l'argument standard de point fixe ne fonctionne pas naturellement. D'autre

part, la dynamique quantique est globalement bien définie. L'approche suivie consiste à transférer certains effets régularisants quantiques à l'équation classique de telle sorte que l'on puisse établir le caractère globalement bien posé de (0.0.5) tout en étudiant en même temps la transition quantique-classique au moyen des mesures de Wigner. Ces dernières années, ces méthodes de mesures de Wigner ont été utilisées pour les systèmes à grand nombre de particules [AN08] et aussi en théorie quantique des champs [AF17] avec une connaissance a priori du caractère bien posé des équations effectives. Alors que dans notre cas, la stratégie fournit à la fois l'existence de solutions globales uniques et la convergence. Formellement, on a le résultat suivant.

Theorem 0.0.1. *Soit $\sigma \in [\frac{1}{2}, 1]$. Supposons que le potentiel V soit suffisamment régulier et que $\omega^{\frac{3}{2}-\sigma}(\cdot)\chi(\cdot) \in L^2(\mathbb{R}^d, \mathbb{C})$. Alors pour toute condition initiale $u_0 \in X^\sigma$ il existe une unique solution globale $u(\cdot) \in \mathcal{C}(\mathbb{R}, X^\sigma) \cap \mathcal{C}^1(\mathbb{R}, X^{\sigma-1})$ de l'équation onde-particules (0.0.5).*

Le second résultat concerne la convergence de la dynamique quantique de (0.0.6) vers celle classique de (0.0.5). Il se reformule en termes de mesures de Wigner selon le diagramme commutatif suivant :

$$\begin{array}{ccc}
 \varrho_{\hbar} & \xrightarrow{e^{-\frac{i}{\hbar}t\hat{H}_{\hbar}}(\cdot)e^{\frac{i}{\hbar}t\hat{H}_{\hbar}}} & \varrho_{\hbar}(t) \\
 \hbar \rightarrow 0 \downarrow & & \downarrow \hbar \rightarrow 0 \\
 \nu_0 & \xrightarrow{(\Phi_t)_{\#}} & \nu_t
 \end{array}$$

où si ϱ_{\hbar} sont des matrices de densité (états quantiques) sur \mathcal{H} convergeant vers la mesure de probabilité de Wigner ν_0 quand $\hbar \rightarrow 0$ alors pour tous les temps $t \in \mathbb{R}$ les matrices de densité $\varrho_{\hbar}(t)$ converge vers $\nu_t = (\Phi_t)_{\#} \nu_0$ où Φ_t est le flot qui résout (0.0.5).

Pour prouver de tels résultats, on montre d'abord la propriété d'unicité des solutions onde-particules en utilisant des techniques classiques. Dans un deuxième temps, on établit d'importantes estimations de propagation uniforme sur la dynamique quantique. Ensuite, on procède selon le schéma suivant :

1. On dérive une formule de Duhamel au sens quantique puis on la développe en puissances de $\hbar \in (0, 1)$.
2. On prend la limite $\hbar \rightarrow 0$ avec à la clé dans la formule de Duhamel une équation caractéristique.

-
3. On prouve l'équivalence entre l'équation caractéristique obtenue et une équation de Liouville statistique avec un champ de vecteurs spécifique.
 4. On fournit une représentation probabiliste des solutions à valeurs mesures de cette équation de Liouville statistique.
 5. On construit un flot global généralisé pour notre problème avec valeur initiale.
 6. Enfin, on prouve la convergence de la dynamique quantique vers l'évolution de l'équation onde-particules.

Problèmes de Cauchy non linéaires.

Dans le chapitre 4, nous considérons le problème général de Cauchy suivant :

$$\begin{cases} \dot{\gamma}(t) = v(t, \gamma(t)), & t \in \mathbb{R}, \\ \gamma(0) = x \in B, \end{cases} \quad (0.0.7)$$

où B est un espace de Banach (de dimension finie ou infinie) et $v : \mathbb{R} \times B \rightarrow B$ est un champ de vecteurs Borélien. Bien entendu, il s'agit d'un problème englobant les EDOs et les EDPs et lié d'une part aux théorèmes de Cauchy-Lipschitz, Peano, Carathéodory pour les EDOs et d'autre part aux arguments de point fixe dans les espaces de Banach pour les EDPs. En particulier, il est bien connu que lorsque v n'est pas suffisamment régulier, le problème (0.0.7) peut ne pas être localement bien posé. D'autre part, on sait que les lois de conservation peuvent être très utiles pour étendre les solutions locales vers des solutions globales. Cependant, dans de nombreux cas, nous ne disposons pas de telles lois ou elles ne sont pas satisfaites pour les données initiales que nous examinons. Néanmoins, pour certains systèmes Hamiltoniens non linéaires, un substitut raisonnable à ces lois de conservation est fourni par des mesures invariantes (par exemple les mesures de Gibbs). En effet, un tel outil peut être très utile pour montrer presque sûrement le caractère bien posé des équations non linéaires dispersives. Une telle approche a été mise en place par J. Bourgain dans les années 90, conduisant plus tard à un formidable développement combinant la théorie des probabilités, les EDPs non linéaires et l'analyse harmonique.

Dans ce cadre, notre but est de prouver, sous des hypothèses minimales, que le problème (0.0.7) admet des solutions globales pour presque sûrement toute donnée initiale $x \in B$, relativement à une mesure de probabilité donnée μ_0 . Notre idée principale est de supposer l'existence d'une famille $(\mu_t)_{t \in \mathbb{R}}$ de mesures de probabilités Boréliennes sur B vérifiant

l'équation de Liouville statistique :

$$\frac{d}{dt} \int_B F(x) d\mu_t(x) = \int_B \langle v(t, x), \nabla F(x) \rangle_{B, B^*} d\mu_t(x), \quad (0.0.8)$$

pour toutes fonctions cylindriques lisses $F : B \rightarrow \mathbb{R}$. Ici on note le crochet de dualité entre B et son dual B^* par $\langle \cdot, \cdot \rangle_{B, B^*}$. En dimension finie, lorsque le champ de vecteurs v est suffisamment lisse, les équations (0.0.7) et (0.0.8) sont connues pour être liées par la méthode des caractéristiques. Dans un tel cas, nous avons

$$\mu_t = (\Phi_t)_\# \mu_0, \quad (0.0.9)$$

où Φ_t est le flot associé au problème de Cauchy (0.0.7) tandis que $(\Phi_t)_\# \mu_0$ désigne la mesure image de μ_0 par l'application Φ_t (voir par exemple [AGS08]). Rappelons qu'habituellement la méthode des caractéristiques suppose le caractère bien-posé du problème de Cauchy (0.0.7) et récupère la solution de l'équation de Liouville (0.0.8). Dans notre travail, nous soulignons plutôt le fait que c'est l'existence de mesures de probabilités vérifiant (0.0.8) qui implique l'existence de solutions globales pour la problème (0.0.7) pour μ_0 -presque toute donnée initiale $x \in B$ (éventuellement sans unicité).

Theorem 0.0.2 (Existence presque certaine de solutions globales). *Soient B un espace séparable dual d'un Banach et $v : \mathbb{R} \times B \rightarrow B$ un champ de vecteurs Borélien. Supposons qu'il existe une courbe étroitement continue $t \in \mathbb{R} \mapsto \mu_t$ de mesures de probabilités sur B satisfaisant l'équation de Liouville (0.0.8). Supposons en outre que le champ vectoriel v vérifie la condition d'intégrabilité,*

$$\int_{\mathbb{R}} \int_B \|v(t, x)\|_B d\mu_t(x) \frac{dt}{\langle t \rangle^\alpha} < +\infty, \quad (0.0.10)$$

pour un certain $\alpha > 1$. Alors pour μ_0 -presque toute donnée initiale $x \in B$ il existe une solution globale du problème de Cauchy (0.0.7).

Notre résultat est prouvé par une approche statistique, des arguments de la théorie de la mesure, des arguments projectifs et un principe de superposition pour les solutions de Liouville (voir par exemple [AGS08, AL18, Man07]). En particulier, nous construisons au moyen d'arguments de faible compacité une mesure de probabilité étendue qui vit sur des solutions globales de (0.0.7) de sorte que sa projection pour chaque instant t soit donnée par μ_t . Ceci nous permet de construire un sous-ensemble $E \subset B$ de mesure totale par

rapport à μ_0 , tel que pour tout $x \in E$ il existe une solution globale satisfaisant (0.0.7).

Comme application de nos résultats, nous étudions des solutions à faible régularité pour diverses EDP hamiltoniennes non linéaires, notamment les équations de Hartree, NLS, Wave et Klein-Gordon; mais aussi les équations d'Euler et les équations quasi-géostrophique. En fait, de telles EDP peuvent être formulé dans le cadre du problème de Cauchy (0.0.7) en utilisant la représentation d'interaction. Ensuite, au moyen du calcul de Malliavin sur des espaces de probabilité gaussienne, nous montrons que chaque fois que nous avons une mesure de Gibbs bien définie pour une équation non linéaire donnée, il existe une solution globale à l'équation de Liouville (0.0.8) avec la condition (0.3.6) vérifiée. Ainsi, le théorème 0.0.2 donne presque sûrement l'existence de solutions globales pour de telles EDPs. En particulier, nous pouvons ainsi récupérer plusieurs résultats d'existence presque sûrement de solutions globales obtenues par la méthode de Bourgain ou la méthode d'Albeverio et Cruzeiro et nous en avons prouvé de nouveaux.

INTRODUCTION

0.1 General context and main goals

The interaction between matter and fields is a fundamental problem in mathematical physics. It is the basis of many interesting phenomena and technologies (e.g. polar lights, photoelectric effect, fusion plasmas, ...). At large scales such interaction is widely observed in plasma physics [Jac99, Pie17]. At small scales, the particle-field interaction is of major importance: it explains in particular the Lamb shift in the spectrum of the hydrogen atom as shown by quantum electrodynamics [FGA10, Fey61, Wei05]. This thesis is devoted to the study of some features of matter-fields interactions. We investigate this subject from both classical and quantum perspectives. In each case, we consider different mathematical structures, we work with adapted tools, and we implement various strategies. Throughout the text, we derive abstract results, and we apply them to physical situations going beyond usual particle-field results. On one hand, we solve problems related to the construction and modeling of highly oscillating solutions of nonlinear partial differential equations. On the other hand, we rigorously justify the Bohr correspondence principle for some particle-field models and make progress on the related problem of existence of low regularity global solutions for nonlinear PDEs.

In classical mechanics, the term “matter” is generally understood to mean particles which are described by their positions and momenta that satisfy the Newton equation. Alternatively, one can describe these particles through their density f which then satisfies the Vlasov equation. The fields refer to the electric field E and the magnetic field B . The quantities f , E and B satisfy the relativistic Vlasov-Maxwell system, see the recent PhD thesis of Brigouleix [Bri20] for some up-to-date presentation or the article [CI23] for recent results in this direction. In Part I of this thesis, we consider stationary solutions $(E, B)(x)$ to Maxwell equations. Problems occur from the fact that (E, B) may depend on x , with moreover B large (of size ε^{-1} with $0 < \varepsilon \ll 1$). The fixed Lorentz force, which is determined by (E, B) , has a significant impact on the mesoscopic structure of the density f . We want to determine its influence on the global organization of matter, approached from Eulerian viewpoint.

In quantum mechanics, the matter are quantum particles that satisfy the non-relativistic or semi-relativistic Schrödinger equation. In Part II of this thesis, the dynamics of these particles are described through the time evolution of their position and momentum operators, whereas the fields are referred to quantized (boson) fields. The interactions and the fields are usually expressed using creation and annihilation operators in Fock spaces through the method of second quantization [Ber66]. The full system of particles and fields is then described by a well-defined linear Hamiltonian (self-adjoint unbounded operator) over the tensor product of a Lebesgue L^2 -space and a Fock space.

This thesis comprises of two distinct parts I and II, each focusing on systems of ordinary differential equations (ODEs) but in different contexts. In each part, we analyze independently particular systems at specific scales.

- Part I: we adopt a *deterministic approach*. We deal with a finite dimensional system of nonlinear differential equations. Initially, we examine a particular category of nonlinear ODEs that exhibit significant oscillations. Subsequently, we employ the method of characteristics to develop two applications:
 - First application involves the study of a family of highly oscillating Hamilton-Jacobi equations;
 - Second application exhibits the long-time gyrokinetic equations that originate from the Vlasov equation.
- Part II: we adopt a *statistical approach*. We consider a class of infinite dimensional systems of nonlinear differential equations. Our focus in this part revolves around two interconnected subjects that we develop in details:
 - The first subject examines the transition from quantum to classical particle-field equation, through the use of Wigner measures;
 - The second subject focuses on the construction of global solutions to abstract nonlinear initial value problems, by employing measure theoretical arguments. It's worth noting that these solutions may not necessarily be unique.

After defining the context, we outline the main goals of this thesis. Our primary objective in both parts is to identify solutions to specific differential equations.

- Purposes of part I: we look for “highly oscillating” family of solutions, which are characterized by their strong oscillations and rapid variations. Specifically, we aim to determine their asymptotic expansions and to identify the leading terms that

capture their primary behavior during the long times. Although these solutions are regular (smooth), their magnitudes or those of their derivatives are significantly large, often represented as negative powers of ε (which is intended to go to zero).

- Purposes of part II: we look for “low regularity” solutions, which are constructed using statistical methods and measure theoretical techniques. The low regularity of these solutions arises from the low regularity of the vector fields used in their construction. Standard techniques such as fixed point arguments are not effective in this scenario. Instead, we use statistical arguments to construct global solutions. Consequently, the global flow is Borel and may lack continuity.

The considered differential equations pose challenges in both parts as the standard arguments are insufficient in providing solutions.

- Difficulties in Part I: the difficulties come from the inhomogeneities of the fields and their large sizes, and from the long time under consideration. Specifically, the mix of oscillations and strong nonlinearities involves supercritical (large amplitude) regimes for resolution. Although standard arguments can provide solutions for a small time interval, they are unable to furnish long-time solutions and to capture their contents;
- Difficulties in Part II: in both of the two subjects in this part, we are faced with common challenges - the low regularity of the vector fields involved and working in infinite dimensional space, which make standard arguments ineffective. Furthermore, the first subject concentrates on proving the validity of the Bohr’s correspondence principle, adding an extra layer of difficulty.

Given the limitations of standard arguments, it is evident that alternative approaches are necessary to tackle these problems. Therefore, we develop more sophisticated methods that take into account the specific characteristics of each problem, in order to construct effective solutions that account for the complexity of the underlying equations. Through this thesis, we hope to shed new light on these important topics and contribute to the ongoing efforts to deepen our understanding of nonlinear differential equations in both finite and infinite dimensional spaces. The following is an overview of the primary tools and diverse strategies that we have employed for each part to address these issues.

0.2 Part I: the deterministic approach

This section aims to provide a rigorous mathematical description of the problems that we are addressing in Part I, which will be thoroughly explored in Chapters 1 and 2. We exploit the transition between ODEs and PDEs. On the one hand, we use a Lagrangian perspective to analyze ODEs, tracing the characteristics of particles over time. On the other hand, we adopt an Eulerian viewpoint to investigate PDEs, where we examine the features of the flow in a space-time domain. In Paragraph [0.2.1](#), we present the underlying general ODEs framework. We describe a theoretical study explaining how the complexity and the oscillatory nature of the system are transferred to its solutions. Then, the outcomes are applied to study a class of oscillating PDEs of type: Hamilton-Jacobi equations in Paragraph [0.2.2](#); and Vlasov equation in Paragraph [0.2.3](#). More precisely, our focus in the second application is on the derivation of the long-time gyrokinetic equations associated to the characteristics of the Vlasov equation. In each paragraph, we provide a brief history, highlight the main difficulties, outline the key outcomes, and describe the strategies to achieve such results.

A significant amount of research has been conducted for constructing oscillating solutions of nonlinear PDEs. This research area is called *nonlinear geometric optics*, see the books [M09](#), [Rau12](#). A basic way to describe oscillations is to call on WKB expansions having the form

$$z(t, x) \sim \sum_{k=p}^{+\infty} \varepsilon^k \mathcal{Z}_k(t, x, \phi_1(t, x)/\varepsilon, \dots, \phi_m(t, x)/\varepsilon), \quad \mathcal{Z}_p \not\equiv 0. \quad (0.2.1)$$

Here ε^{-1} with $0 < \varepsilon \ll 1$ represents a high frequency; the scalar functions ϕ_j stand for phases (telling about the geometry of propagation); and the \mathcal{Z}_k are profiles (carrying more quantitative information). The choice of p (the minimal value of k such that $\mathcal{Z}_k \not\equiv 0$) reveals the impact of nonlinear effects, which are strengthened when p is small. The standard regime is *weakly* nonlinear geometric optics [JMR93a](#), [JMR93b](#) corresponding to the selection of $p = 1$. Then, when computing one order derivatives of z , the factor ε^{-1} issued from the phases is compensated by the small amplitude ε put in factor of \mathcal{Z}_1 , and uniform lipschitz estimates as well as various kinds of stability properties are available. There is nothing like this in the case $p < 1$. On the contrary, strong instabilities may occur [CGM04](#), [Gre00](#), [Leb01](#). In order to remedy this, we can either investigate specific nonlinear equations (as was done for instance in [CGM04](#) where $p = 0$ is reached under decoupling assumptions), or generalize the ansatz for z in order to take into account

hidden aspects (related to the apparent instabilities). These are the two main ideas that inspire what follows:

- the scope is restricted to PDEs whose time evolution may rely on ODEs (through the method of characteristics);
- we add new scales and extra phases. We implement a superposition of oscillations. The phases can themselves be oscillating.

0.2.1 General ODEs framework

In this paragraph, we study the following nonlinear ordinary differential equations with stiff oscillating source terms

$$\partial_\tau \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0; z; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right), \quad \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} (0) = \begin{pmatrix} z_0 \\ \mathbf{v}_0 \end{pmatrix}, \quad \varepsilon \ll 1, \quad (0.2.2)$$

where the unknowns are $(z, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}$ with $n \in \mathbb{N}^*$. Both \mathbf{A} and \mathbf{V} are assumed to be periodic with respect to the two last variables. Nonlinear flows often exhibit a transition from orderly motion to more irregular. The objective is to comprehend through a deterministic approach how this transition can occur and how a previously stable flow regime can suddenly give way to a kind of “turbulent” behavior. The method of characteristics can be used to transfer from ODEs of type (0.2.2) to PDEs of type Hamilton-Jacobi or Vlasov. However, the study of systems of type (0.2.2) poses significant challenges due to highly oscillating source terms. These terms are of very large size $\mathcal{O}(\varepsilon^{-1})$ and depend non-trivially on both z and \mathbf{v}/ε . Moreover, the situation becomes even more challenging when considering the second equation in (0.2.2), where the source term is oscillating with respect to the unknown \mathbf{v} itself. Consequently, standard techniques such as the Picard-Lindelöf theorem only provide information for a compact time interval of very small size ($\tau \sim \varepsilon$). To obtain long-time solutions ($\tau \sim 1$), a new approach is required.

Multiple scales and phases are commonly observed in various systems. To cover the nonlinear features of our system, we have to consider large amplitude solutions of the form:

$$z(\varepsilon; z_0, \mathbf{v}_0; \tau) \sim \sum_{k=0}^{+\infty} \varepsilon^k Z_k(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}_\varepsilon^f(z_0, \mathbf{v}_0; \tau)}{\varepsilon}), \quad (0.2.3a)$$

$$\mathbf{v}(\varepsilon; z_0, \mathbf{v}_0; \tau) \sim \sum_{k=-1}^{+\infty} \varepsilon^k \mathcal{V}_k(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}_\varepsilon^f(z_0, \mathbf{v}_0; \tau)}{\varepsilon}),$$

$$\mathbf{v}_\varepsilon^f(z_0, \mathbf{v}_0; \tau) := \frac{1}{\varepsilon} \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau) + \overline{\mathcal{V}}_0\left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}\right). \quad (0.2.3b)$$

The notations $\overline{\cdot}$ and $\langle \cdot \rangle$ are used to represent the mean value extraction with respect to the two last variables see (1.2.4)-(1.2.6). By examining equations (0.2.3a) and (0.2.3b) and comparing them to (0.2.1), we observe that the phase $\mathbf{v}_\varepsilon^f(\tau)$ is large and oscillates. Specifically, due to Assumption 1.2.6, we have $\langle \overline{\mathcal{V}}_{-1} \rangle(\tau) > 0$ for all $\tau > 0$, indicating that the rapid phase $\langle \overline{\mathcal{V}}_{-1} \rangle(\tau)$ that corresponds to rapid variations (at frequencies ε^{-2}) is effective. Additionally, the expression $\overline{\mathcal{V}}_0(\cdot; \tau, \tau/\varepsilon)$, which corresponds to rapid variations (at frequencies ε^{-1}), exhibits a non-linear interlocking of oscillations. The phase \mathbf{v}_ε^f appears as the truncated version of \mathbf{v} . It is then necessary to fulfill two primary requirements: constructing WKB solutions of the above types (0.2.3a)-(0.2.3b) and verifying the stability of these solutions by demonstrating their closeness to the exact solutions. However, establishing the latter condition is often a complicated task since there are scenarios where WKB solutions can be constructed at any order, yet it is impossible to confirm their proximity to the exact solutions. As a case in point, the authors in [CGM04, Gre00, Leb01] have proven the possibility of constructing WKB approximate solutions which nevertheless are instable.

Earlier articles [Che15, Che17, Che18] previously studied such types of equations. There, the ODE framework came from the motion of charged particles under the sole influence of a strong external magnetic field. In these preliminary studies, the construction of long-time solutions relied on the conservation of kinetic energy. This is no more applicable in our context since $E \neq 0$. To address our more general situation, we replace the conservation of kinetic energy with a complete integrability condition on the mean flow, which is introduced in Definition 1.2.2. This condition is expressed in terms of the source terms \mathbf{A} and \mathbf{V} as indicated in the Cauchy system (1.2.11). Essentially, it states that the mean flow $\Xi_{\text{mf}}(z_0; \mathfrak{z}, s)$ is globally defined and has a periodicity of 2π with respect to s for all $(z_0, \mathfrak{z}) \in \mathbb{R}^n \times \mathbb{R}^n$. The mean flow mentioned above reflects the main behavior of the system (0.2.2) interpreted in the quick time variable $s = \tau/\varepsilon$.

Our objective is to develop a method that can handle nonlinear interactions resulting from oscillations implying multiple phases and scales.

Our main outcomes in this direction are summarized below:

- Theorem [1.2.5](#) provides the **existence** of a family of solutions $\{z_\varepsilon, \mathbf{v}_\varepsilon\}_{\varepsilon \in]0, \varepsilon_0]}$ to the system of nonlinear differential equations [\(0.2.2\)](#) on the time interval $[0, \mathcal{T}(\varepsilon, z_0, \mathbf{v}_0)]$. The lifespan $\mathcal{T}(\varepsilon, z_0, \mathbf{v}_0)$ is **uniformly** bounded below as follows

$$\exists \mathcal{T}(z_0, \mathbf{v}_0) \in \mathbb{R}_+^*; \quad \forall \varepsilon \in]0, \varepsilon_0], \quad \varepsilon_0 \in [0, 1], \quad 0 < \mathcal{T}(z_0, \mathbf{v}_0) \leq \mathcal{T}(\varepsilon, z_0, \mathbf{v}_0).$$

- Theorem [1.2.7](#) gives the **asymptotic expansions** of these solutions with high sup-norm precision, including the emergence of new scales and new phases resulting from the strong nonlinear interactions. Under Assumption [1.2.6](#), these expansions describe their oscillatory structures through the use of effective equations for phases (eikonal equations) and profiles (modulation equations). This allows to detect **oscillating coherent structures**;
- Theorem [1.2.10](#) gives access to the **long-time modulation equations** which inform about the very long-time behavior of solutions. These **reduced equations** are strongly nonlinear showing what nonlinear effects are saturated. They are locally well-posed.

To access the above results, our strategy is to first remove some singularities from [\(0.2.2\)](#). This is accomplished by using a “**blow-up**” method which can be viewed as an (extended) version of usual normal form procedures. Essentially, a blow-up method sets some singularities aside by adding new degrees of freedom represented as non-trivial oscillating variables. This method is efficient provided that we have the above mentioned complete integrability conditions which, in its turn, is responsible to a foliation of \mathbb{R}^n by closed curves. This leads to a new system preserving the same form as [\(0.2.2\)](#), but with partial desingularization. We can get rid of the large term ε^{-1} in front of \mathbf{A} , but not in front of \mathbf{V} . The new system remains strongly unstable under slight modifications of the initial data. The presence of $\varepsilon^{-1} \mathbf{V}$ does not allow to implement a formal calculus starting from [\(0.2.3a\)](#)-[\(0.2.3b\)](#), due to remaining problems of closure in the hierarchy of equations that could be obtained in this way.

To get around this difficulty, instead of dealing with \mathbf{v}_ε^f , we implement the component \mathbf{v} of the solution as a phase. This means in particular that \mathbf{v} is viewed as oscillating with

respect to itself through some oscillating implicit relation. Then, we perform a **WKB** approximation of supercritical type involving three-scale features. To go beyond the classical regimes, the idea is to extract from (0.2.2) an adequate profile equations, and then to exhibit some kind of stability. This yields a special and new notion of profile equations allowing to get rid of \mathbf{v} . In other words, we work modulo the determination of the unknown function \mathbf{v} . This is like investigating the construction of approximate solutions and their stability in a quotient space. The content of \mathbf{v} is ultimately recovered with a loss of information. This analysis depends on the use of averaging techniques.

As previously mentioned, our focus on the general ODEs framework is primarily for its applications to PDEs. In the following, we will illustrate the relevance of our approach by presenting its application to two specific problems: a special class of Hamilton-Jacobi equations and the characteristics of the Vlasov equation. These applications have been detailed in separate published papers [CF23a, CF23b].

0.2.2 The first application: Hamilton-Jacobi equations

We deal here with a family of highly oscillating Hamilton-Jacobi equations of the form

$$\partial_\tau u_\varepsilon + \frac{1}{\varepsilon} \mathbf{H}\left(\varepsilon, \frac{\tau}{\varepsilon}, x, \varepsilon u_\varepsilon, \nabla_x u_\varepsilon, \frac{u_\varepsilon}{\varepsilon}\right) = 0, \quad u_\varepsilon(0, x) = \mathcal{U}_0(\varepsilon, x). \quad (0.2.4)$$

Such classes of oscillating Hamilton-Jacobi equations have been investigated before in various situations, such as homogenization [AF09b, CLL10, Fir17] where they are applied to traffic flows, light propagation and optics [M09, Sch94], in the presence of rough domains [GVLNR18], and so on. Also, global weak solutions are available [Alv99, CC96, CL83]. In these references, the authors provide the existence of viscosity solutions using compactness arguments, but without having a strong form of stability that would enable the comparison of potential approximate solutions to these weak solutions. In contrast, we want here to describe and control the precise highly oscillating structure of (smooth) solutions, which means to recover some sort of stability.

One possible way to approach the problem is by initially substituting formal expansions into equation (0.2.4). But as already mentioned, this approach leads to closure problems as the hierarchy of equations does not permit the successive determination of phases and profiles. Alternative approach is then required, this is achieved by applying the method of characteristics, allowing to fall into the scope of our general ODEs framework in Paragraph

0.2.1. However, one major challenge of this method is avoiding the crossing of the (spatial) characteristic projections. More precisely, the reconstitution of the solution u_ε to **(0.2.4)** from **(0.2.2)** requires to find the inverse of the spatial part of the characteristics, while the differential of the spatial part $x(\tau, x)$ of characteristics is of large size $\mathcal{O}(\varepsilon^{-1})$. To move forward, we need to

- exploit a **transparency condition** which emanates from the integrability condition imposed on the mean flow in this context. This condition implies that the nonlinear terms, which would otherwise contribute without the integrability condition, vanish. Furthermore, it enables control over the size of the differential $D_x x$;
- implement some specific WKB analysis that allows to determine the oscillatory structure associated to the inverse map of the spatial part of the characteristics. This part requires to check the assumptions of the **Hadamard’s global inverse function theorem**.

The main outcome then is Theorem **1.1.5**, establishing the **uniform existence** of a family of solutions $\{u_\varepsilon\}_{\varepsilon \in]0, \varepsilon_0]}$. Moreover, through this theorem, we can see the influence of the time oscillation τ/ε and the nonlinearities $\varepsilon u_\varepsilon$, $\nabla_x u_\varepsilon$, $u_\varepsilon/\varepsilon$ (occurring in the source term **H**) at the level of the **asymptotic expansion** of the solutions. Notably, these solutions are of significant magnitude, scaling as $\mathcal{O}(\varepsilon^{-1})$, and are described by a multiscale and multiphase expansion. The description of their asymptotic expansion (expressed in terms of powers of ε) is associated with an increasing agitation. To be more precise, the analysis begins with scales ε^0 , ε^1 and ε^2 , as well as the phase τ in **(0.2.4)**, but ultimately leads to an asymptotic expansion that incorporates a new scale ε^3 , along with its associated oscillating phase.

Before addressing the second application, let us first explore some of the existing literature on nonlinear gyrokinetic theory. This theory has been extensively researched, with numerous theoretical (**FC82**, **SD84**, **TBRK14**) and numerical (**GSG⁺06**) applications, as well as other works mentioned in the article **BH07**. Other relevant studies related to the second application include research on magnetically confined plasma discussed in **Boo05**. Additionally, **BV23** contains valuable insights on the long-term behavior of charged particles.

0.2.3 The second application: Long-time gyrokinetic equations

This application deals with plasmas. The unknowns are usually the density $f(t, x, v)$, the electric field $E(t, x)$, and the magnetic field $B(t, x)$, evolving in time according to Relativistic Maxwell Vlasov system. To simplify or to specify some phenomena, intermediate models are often investigated, see for instance [DHKR18] where the speed of light is very large or [JM15] where the field (E, B) is a fixed solution to Maxwell equations (without the feedback provided by the electric current). Similarly, we start from stationary solution (E, B) to Maxwell equations. We consider a fixed non-zero electric field $E \neq 0$ and a strong magnetic field $|B| \gg 1$. This leads to deal with the electron gyro frequency ε^{-1} where the value of ε is inversely proportional to the size $|B|$ of the magnetic field. In particular, this implies that $\varepsilon \ll 1$ for many applications, including fusion plasmas (like in Tokamaks [Dum]). This feature induces oscillating behaviors of charged particles.

Under the Lorentz force built on a fixed external inhomogeneous electromagnetic field $(E, B)(x)$, a relativistic charged particle starting at time $t = 0$ from the position $(x_0, v_0) \in \mathbb{R}^3 \times \mathbb{R}^3$ is moving in the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ according to the dimensionless ordinary differential equation

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = (1 - |v|^2)^{1/2} \left\{ -\frac{1}{\varepsilon} v \wedge B(x) - E(x) + (v \cdot E(x)) v \right\}. \end{cases} \quad (0.2.5)$$

The integral curves associated to (0.2.5) give rise to the characteristics of Vlasov equation. The regime of intermediate time corresponds to $t \sim 1$. Our study of the above system (0.2.5) focuses on its behavior over long-time scales ($\tau := \varepsilon t \sim 1$ or $t \sim 1/\varepsilon$). We want to gain a deeper understanding of its long time dynamics by exploiting the outcomes gained by the theoretical study in Paragraph 0.2.1.

The physical model (0.2.5) is not given in the form of the general ODEs framework, outlined in Paragraph 0.2.1. In order to establish compatibility between the two, significant preparatory work is required. This involves completing two tasks:

- The first task is to **model** the physical problem (0.2.5) as a system of equations in the form (0.2.2). To do this, we need to express the position variable x in toroidal geometric coordinates and decompose the velocity variable v using spherical coordinates.

dinates. By performing two diffeomorphic changes of variables, we obtain

$$x = X(r, \theta, \phi), \quad v = V(r, \theta, \phi, v, \zeta, \nu),$$

where the radial distances r , $v := |v|$, and the angles θ, ϕ, ζ, ν are unknowns. The variable ν represents rotations in the plane perpendicular to the magnetic field direction and undergoes fast variations, while the other coordinates undergo slower variations from the perspective of gyrokinetics. This leads to a system of equations in the form (2.1.3), which can be further reduced to fall within the scope of (0.2.2) by performing a preliminary normal form reduction in Section 2.3.

- The second task is to verify the **prerequisites** of the general ODE framework. In particular, in the context of the Vlasov equation, the integrability condition is obtained as a byproduct of properties of the average flow discussed in Paragraph 2.3.2.

The literature on this topic includes a large amount and diverse range of works. The study of the system (0.2.5) during the intermediate times ($t \sim 1$) is usually accomplished by the gyrokinetic approach [BH07, Bos07, Bos10, Bra81, BS94]. During long times ($\tau \sim 1$), the study of the above system is rarely attained due to the complexity inherited from the variation of the electromagnetic field $(E, B)(x)$. And this is usually investigated from Lagrangian point of view by the KAM method [Bra81, BS94]. Unlike the KAM method, our approach adopts an Eulerian viewpoint and most importantly provides an oscillating form for the solutions.

The strong magnetic field plays a crucial role in confining plasmas within the Tokamak's wall. We aim here to investigate whether the electric field can cause deconfinement and serve as a source of instabilities. The electric field $E \neq 0$ is decomposed into three parts: E_r (the radial part), E_\perp (perpendicular to the direction of the magnetic field) and E_\parallel (in the direction of the magnetic field) as indicated in (2.2.6)-(2.2.7). Then observing the first equation of (2.2.12), we can assert that the inclusion of an electric field cause a breakdown in the conservation of the kinetic energy, potentially leading to the development of instabilities. Additionally, we are particularly interested in investigating how the electric field may affect the oscillating coherent structures and whether we can observe its impact in the long-time modulation equations.

Our approach yields the following results:

-
- Theorem [2.1.8](#) states that a family of smooth long-time ($\tau \sim 1$) solutions $\{x_\varepsilon, v_\varepsilon\}_{\varepsilon \in]0, \varepsilon_0]}$ are available uniformly in ε , for a family of initial data in a compact set. These solutions are associated with **coherent structures** that describe their nature;
 - The primary objective of this application is to derive a model that accurately describes the leading behavior of solutions. Specifically, we focus on the **long-time gyrokinetic equations**, which enable us to quantify the precise influence of the field E . Theorem [2.1.9](#) provides explicit expressions for these equations. Upon examining the leading terms, in contrast with the classical gyrokinetic equations, the long time gyrokinetic ansatz imply large amplitude oscillations. We exhibit also two important concrete consequences:
 - a spatial confinement property (related to x) for a whole range of electric fields E in Paragraph [2.4.3.2](#);
 - a long-time stability property concerning the momentum component v when the electric field is radial see Paragraph [2.4.3.3](#).

0.3 Part II: the statistical approach

We develop two connected subjects in Chapters 3 and 4, both are achieved using statistical approach. The first subject explores the quantum-classical transition in particle-field dynamics where a finite and fixed number of non-relativistic or semi-relativistic quantum particles interact with a quantized scalar field in the scaling limit of small value of Planck constant $\hbar \rightarrow 0$. Such topic aims to rigorously derive effective equations from fundamental first principles of quantum mechanics (see e.g. [AF17](#), [BPS15](#), [LP20](#) and references therein). When analyzing the quantum-classical transition, we crucially use measure theoretical arguments to establish the existence of global solutions for a specific wave-particle equation. In our case, the interaction between the wave and the particles are sufficiently singular to prevent us from using a standard fixed point argument. So that, our statistical method, in its turn, furnishes an essential tool for the study of classical and quantum dynamics. This subject is treated in Subsection [0.3.1](#). Following similar statistical method, the second subject in Subsection [0.3.2](#) establishes the construction of almost sure global in times (not necessary unique) solutions to an abstract nonlinear initial value problem in finite or infinite dimensional spaces with a Borel vector field. This is achieved by using statistical techniques and tools issued from measure theory.

0.3.1 Classical limit of the quantum particle-field interactions

This subject is concerned with quantum-classical transition in particle-field dynamics. From a classical standpoint, the dynamics are governed by the particle-field equation (Newton-Klein-Gordon equation), a coupled PDE-ODE nonlinear system. We consider a fixed number of classical particles (namely n) in the configuration space \mathbb{R}^d , $d \in \mathbb{N}^*$, interacting with a scalar field as follows

$$\begin{cases} \partial_t p_j = -\nabla_{q_j} V(q) - \int_{\mathbb{R}^d} 2\pi i k \frac{\chi(k)}{\sqrt{\omega(k)}} [\alpha(k) e^{2\pi i k \cdot q_j} - \overline{\alpha(k)} e^{-2\pi i k \cdot q_j}] dk, \\ \partial_t q_j = \nabla f_j(p_j), \\ \partial_t \alpha = -i\omega(k) \alpha(k) - i \sum_{j=1}^n \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot q_j}, \end{cases} \quad (0.3.1)$$

The dynamics of n particles, denoted by their positions q_j and momenta p_j , are described by the time evolution equations in (0.3.1). The dispersion relation ω is given by $\omega(k) = \sqrt{k^2 + m_f^2}$, where $m_f \neq 0$ is the mass of the meson field. The external potential is represented by the map $V : \mathbb{R}^{dn} \rightarrow \mathbb{R}$, while the form factor $\chi : \mathbb{R}^d \rightarrow \mathbb{R}$ describes the coupling between the particles and the field. Both maps must satisfy certain conditions. To further clarify the situation, we define the function $f_j(p_j)$ as either $\sqrt{p_j^2 + M_j^2}$ (for the semi-relativistic case) or $p_j^2/2M_j$ (for the non-relativistic case), where M_j are the masses.

The solution $u = (p, q, \alpha)$ of the above particle-field equation belongs to the following classical phase space

$$X^\sigma := \mathbb{R}^{dn} \times \mathbb{R}^{dn} \times \mathcal{G}^\sigma$$

where \mathcal{G}^σ is the weighted L^2 lebesgue space endowed with the following norm

$$\|\alpha\|_{\mathcal{G}^\sigma}^2 = \int_{\mathbb{R}^d} \omega(k)^{2\sigma} |\alpha(k)|^2 dk.$$

The Hamiltonian of the particle-field equation takes the following form

$$\begin{aligned} H(p, q, \alpha) &= \sum_{j=1}^n f_j(p_j) + V(q_1, \dots, q_n) + \int_{\mathbb{R}^d} \overline{\alpha(k)} \omega(k) \alpha(k) dk \\ &\quad + \sum_{j=1}^n \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} [\alpha(k) e^{2\pi i k \cdot q_j} + \overline{\alpha(k)} e^{-2\pi i k \cdot q_j}] dk. \end{aligned}$$

Such type of classical equations has been studied before, see for instance the articles

[KKS99, KS98, KSK97]. These studies focus on finding approximate solutions and analyzing the long-term behavior of the solutions to a particle-field equation. The authors in these articles consider a form factor that is compactly supported. Whereas, we consider a more general framework by imposing less restrictions on the form factor. We make the following hypothesis:

$$\left\{ \begin{array}{l} \bullet \quad V \in C_b^2(\mathbb{R}^{dn}, \mathbb{R}), \\ \bullet \quad \omega^{\frac{3}{2}-\sigma} \chi \in L^2(\mathbb{R}^d, \mathbb{R}), \quad \sigma \in [\frac{1}{2}, 1]. \end{array} \right. \quad (H_1)$$

Given the hypothesis mentioned above, the vector field associated with equation (0.3.1) lacks the regularity required for standard fixed point arguments to establish global well-posedness. Therefore, it is interesting to investigate whether global solutions to (0.3.1) can still be exhibited under the weaker assumption (H₁). To address this issue, we examine the relationship between classical and quantum dynamics of the particle-field system.

From a quantum standpoint, the linear evolution is generated by the so-called Nelson Hamiltonian. The Hilbert space that we will consider is \mathcal{H} . It is defined as follows

$$\mathcal{H} := L^2(\mathbb{R}^{2dn}, \mathbb{C}) \otimes \Gamma_s(L_k^2(\mathbb{R}^d, \mathbb{C})),$$

where $\Gamma_s(L_k^2(\mathbb{R}^d, \mathbb{C}))$ is the symmetric Fock space which could be identified with

$$\Gamma_s(L_k^2(\mathbb{R}^d, \mathbb{C})) := \bigoplus_{m=0}^{+\infty} L^2(\mathbb{R}^d, \mathbb{C}) \otimes_s^m \simeq \bigoplus_{m=0}^{+\infty} L_s^2(\mathbb{R}^{dm}, \mathbb{C}).$$

Let \hat{p}_j and \hat{q}_j be the quantized momentum and position operators such that for all $j \in \{1, \dots, n\}$

$$\hat{p}_j = -i\hbar \nabla_{x_j}, \quad \hat{q}_j = x_j.$$

The Nelson-Hamiltonian takes then the following form

$$\hat{H}_\hbar = \sum_{j=1}^n f_j(\hat{p}_j) + V(\hat{q}) + d\Gamma(\omega) + \sum_{j=1}^n \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} \left[e^{2\pi i k \cdot \hat{q}_j} \hat{a}_\hbar(k) + e^{-2\pi i k \cdot \hat{q}_j} \hat{a}_\hbar^*(k) \right] dk, \quad (0.3.2)$$

where $\hat{a}_\hbar(k)$ and $\hat{a}_\hbar^*(k)$ are the creation-annihilation operator-valued distributions defined in Paragraph 3.1.1.2. The term $d\Gamma(\omega)$ refers to the free field Hamiltonian and is defined

as follows:

$$d\Gamma(\omega) = \int_{\mathbb{R}^d} \hat{a}_h^*(k) \omega(k) \hat{a}_h(k) dk.$$

The Nelson model was introduced by Edward Nelson in [Nel64a, Nel64b] to describe the interaction between particles (nucleons) and meson field (strong nuclear force). Since then, it has been extensively studied by numerous researchers, as detailed in works such as [AH12, Alb73, Amm00, Ara01, BFS99, CHPS15, CP14, EGS00, Ger02, GHPS11, Hir96, Hø69, Teu02].

The inclusion of a form factor χ in the interaction leads to a well-defined quantum dynamics and a self-adjoint Nelson Hamiltonian. Indeed, it has been proven that under certain mild assumptions on χ and the potential V , the unbounded operator \hat{H}_h is self-adjoint (see [AF17] and references therein).

The quantum dynamics are globally well-defined. Our approach then consists in transferring some quantum regularizing effects to the classical dynamics. The main purpose is to derive the classical dynamics of the particle-field system as an effective equation of a specific microscopic dynamical system in the classical limit of small value of the Planck constant $\hbar \rightarrow 0$. Such convergence is known in physics under the name of Bohr's correspondence principle. We want to provide a mathematical ground to such fundamental principle and to deeply study the relationship between the quantum and classical dynamics. Previous literature has discussed a similar type of convergence known as Hepp's method [Fal12, GNV06], but this method is typically applied to a specific selection of density matrices, namely coherent states. Another study [AFH22] has also explored this convergence for non-relativistic case, considering the Pauli-Fierz Hamiltonian, which converges to the Newton-Maxwell equation. In pursuit of this, our main goals are to:

- prove the validity of the Bohr's correspondence principle;
- establish the global well-posedness of the particle-field equation under (H_1) .

To achieve this scenario, we study such transition by means of Wigner measures in infinite dimensional bosonic quantum field theory. In recent years, these Wigner measure method have been used in many-body theory [AN08] and in quantum field theory [AF17] with an a priori knowledge of GWP for effective equations. Whereas in this work, our strategy furnishes global well-posedness and convergence at the same time.

To define the quantum dynamical system, we let $(\rho_h)_{h \in (0,1)}$ be a family of density matrices

on \mathcal{H} of the particle-field quantum system satisfying

$$\begin{cases} \bullet & \exists C_0 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_\hbar d\Gamma(\omega^{2\sigma})] \leq C_0, \\ \bullet & \exists C_1 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_\hbar (\hat{q}^2 + \hat{p}^2)] \leq C_1. \end{cases} \quad (H_2)$$

The map

$$(z, \alpha) \in X^0 \longmapsto \mathcal{W}(z, \alpha) := e^{i\Im m(\hat{q} + i\hat{p}, z)} \otimes e^{\frac{i}{\sqrt{2}}[\hat{a}_\hbar(\alpha) + \hat{a}_\hbar^*(\alpha)]}$$

defines a strongly continuous irreducible representation of the Weyl's commutation relations over the Hilbert space \mathcal{H} .

Definition 0.3.1 (Wigner measures). A Borel probability measure μ in X^0 is a Wigner measure of a family of density matrices $(\varrho_\hbar)_{\hbar \in (0,1)}$ on the Hilbert space \mathcal{H} if and only if there exists a countable subset $\mathcal{A} \subset (0, 1)$ with $0 \in \overline{\mathcal{A}}$ such that for any $\xi = (p_0, q_0, \alpha_0) \in X^0$:

$$\lim_{\hbar \rightarrow 0, \hbar \in \mathcal{A}} \text{Tr} \left[\mathcal{W}(2\pi q_0, -2\pi p_0, \sqrt{2\pi} \alpha_0) \varrho_\hbar \right] = \int_{X^0} e^{2\pi i \Re e \langle \xi, u \rangle_{X^0}} d\mu(u).$$

Denote by $\mathcal{M}(\varrho_\hbar, \hbar \in \mathcal{A})$ the set of all Wigner measure associated to the family of density matrices $(\varrho_\hbar)_{\hbar \in \mathcal{A}}$. One approach to extend the concept of Wigner measures to systems with an infinite number of degrees of freedom is to utilize cylindrical spaces, which facilitate projection onto finite dimensional spaces. Accordingly, we define Wigner measures in infinite dimensions by employing the same definition as before. Consequently, the evolution of the quantum dynamics, given an initial normal state ϱ_\hbar , corresponds to:

$$\varrho_\hbar(t) := e^{-i\frac{t}{\hbar}\hat{H}} \varrho_\hbar e^{i\frac{t}{\hbar}\hat{H}}.$$

Subsequently, we must determine the Wigner measures of the evolved state $\varrho_\hbar(t)$. Our subsequent action is to define the quantum dynamical system that is fulfilled by the interaction representation of $\varrho_\hbar(t)$. Once this is completed, we proceed to take the classical limit ($\hbar \rightarrow 0$) of the derived quantum dynamics. This yields the following convergence outcome.

Theorem 0.3.2 (Classical limit). *Let $\sigma \in [\frac{1}{2}, 1]$ and assume the hypothesis (H_1) hold true. Let $(\varrho_\hbar)_{\hbar \in (0,1)}$ be a family of density matrices on \mathcal{H} satisfying the hypothesis (H_2) . Assume that $\mathcal{M}(\varrho_\hbar, \hbar \in (0, 1)) = \{\mu_0\}$. Then for all times $t \in \mathbb{R}$, we have*

$$\mathcal{M}(e^{-i\frac{t}{\hbar}\hat{H}} \varrho_\hbar e^{i\frac{t}{\hbar}\hat{H}}, \hbar \in (0, 1)) = \{\mu_t\},$$

where $\mu_t \in \mathcal{P}(X^0)$ satisfies

- (i) μ_t is concentrated on X^σ i.e. $\mu_t(X^\sigma) = 1$;
- (ii) $\mu_t = (\Phi_t)_\# \mu_0$, where $u_0 \mapsto \Phi_t(u_0) = u(t)$ is the generalized Borel measurable global flow of the particle-field equation [\(0.3.1\)](#).

The result above indicates that as \hbar approaches 0, the density matrices ϱ_\hbar on \mathcal{H} converge to the Wigner probability measure μ_0 . Additionally, for all times t , the evolved density matrices $\varrho_\hbar(t)$ also converge to $\mu_t = (\Phi_t)_\# \mu_0$, where Φ_t represents the flow that solves equation [\(0.3.1\)](#).

In general, the convergence mentioned above is non-trivial, and the resulting limits are not necessarily unique. However, under the hypothesis [\(H₂\)](#), we can assert that for any family of states $(\varrho_\hbar)_{\hbar \in (0,1)}$ satisfying [\(H₂\)](#), and for any sequence $\hbar_n \rightarrow 0$, we can extract a subsequence $\hbar_{n_\ell} \rightarrow 0$ such that the set of Wigner measures is non-empty and, most importantly, singleton for all times $t \in \mathbb{R}$. Subsequently, taking the limit of the quantum dynamical system leads to a classical dynamic on the inverse Fourier transform of the Wigner measure, also known as a special characteristic equation. Notably, this characteristic equation is equivalent to a particular Liouville's equation. Consequently, we obtain the following well-posedness result.

Theorem 0.3.3 (Global well-posedness of the particle-field equation). *Let $\sigma \in [\frac{1}{2}, 1]$. Assume the hypothesis [\(H₁\)](#) hold. Then for any initial condition $u_0 \in X^\sigma$ there exists a unique global strong solution $u(\cdot) \in \mathcal{C}(\mathbb{R}, X^\sigma) \cap \mathcal{C}^1(\mathbb{R}, X^{\sigma-1})$ of the particle-field equation. Moreover, the global flow map $u_0 \rightarrow \Phi_t(u_0) = u(t)$ associated to the particle-field equation [\(0.3.1\)](#) is Borel measurable.*

In our situation, the global flow we are considering is not obtained through the fixed point argument. Instead, we construct it using statistical arguments, specifically employing measure theoretic techniques. As a result, the resulting flow is only Borel measurable, and not necessarily continuous.

To prove such results, in view of [\(H₁\)](#), we first show the uniqueness property of the particle-field solutions using some classical techniques. Then, we provide a probabilistic representation of measure-valued solutions for the related Liouville's equation [\[AFH22, AL18, ALR20\]](#) to construct a generalized global flow for the particle-field equation.

As a conclusion, we establish by means of Wigner measures the global well-posedness for the particle-field equation and the Bohr's correspondence principle for the Nelson model.

0.3.2 Abstract nonlinear initial value problems

We consider the following general initial value problem:

$$\begin{cases} \dot{\gamma}(t) = v(t, \gamma(t)), & t \in \mathbb{R}, \\ \gamma(t_0) = x \in B, \end{cases} \quad (0.3.3)$$

where $B = E^*$ is a separable dual of infinite dimensional Banach space E and $v : \mathbb{R} \times B \rightarrow B$ is a Borel vector field. Of course, this is an old problem encompassing ODEs and PDEs and related on one hand to the Cauchy-Lipschitz, Peano, Carathéodory theorems for ODEs and on the other hand to fixed point arguments in Banach spaces for PDEs. In particular, it is well known that when v is not sufficiently smooth then (0.3.3) may not be locally well-posed. On the other hand, conservation laws can be very useful to extend local existence of solutions to a global one. However, in many cases we do lack such laws or they are not satisfied for all initial data that we are considering. Fortunately, for some nonlinear Hamiltonian systems, a reasonable substitute for conservation laws is provided by invariant measures (e.g. Gibbs measures). Moreover, such ingredient can be very useful to construct even LWP almost surely. Such approach was put in a firm mathematical ground by Bourgain in the 90's, leading later to a tremendous development combining probability theory, nonlinear PDEs and harmonic analysis.

In this perspective, our goal is to prove, under minimal assumptions, that the problem (0.3.3) admits global solutions for almost sure any initial data $x \in B$ with respect to a given probability measure μ_{t_0} . Our main idea is to assume the existence of a global in time measure-valued solution to the Liouville type equation:

$$\frac{d}{dt} \int_B F(x) d\mu_t(x) = \int_B \langle v(t, x), \nabla F(x) \rangle_{E, E^*} d\mu_t(x), \quad (0.3.4)$$

satisfied for all $t \in \mathbb{R}$ and for all $F : B \rightarrow \mathbb{R}$ smooth cylindrical functions. Here $\langle \cdot, \cdot \rangle_{E, E^*}$ denotes the duality bracket between E and its dual E^* . The probability measures μ_t which could be stationary or time-dependent satisfy some mild regularity conditions. In finite dimension, when the vector field v is sufficiently smooth, the equations (0.3.3) and (0.3.4) are known to be related through the characteristics method. In such case, we have

$$\mu_t = (\Phi_t)_\# \mu_{t_0}, \quad (0.3.5)$$

where Φ_t is the flow solving the initial value problem (0.3.3) and the r.h.s of (0.3.5) is the push-forward measure [AGS08]. Recall that usually the characteristics method assumes the well-posedness of the initial value problem (0.3.3) and recovers the well-posedness of the Liouville equation (0.3.4). In our work, we underline instead the fact that the existence of global in time measure solutions to (0.3.4) implies the existence of global solutions for the initial value problem (0.3.3) for μ_{t_0} -almost initial data (possibly without uniqueness).

Theorem 0.3.4 (Almost sure existence of global solutions). *Let B be a separable dual Banach space and $v : \mathbb{R} \times B \rightarrow B$ a Borel vector field. Suppose that there exists a narrowly continuous curve $t \in \mathbb{R} \mapsto \mu_t$ of probability measures on B satisfying the Liouville equation (0.3.4). Assume furthermore that the vector field v satisfies the integrability condition,*

$$\int_{\mathbb{R}} \int_B \|v(t, x)\|_B d\mu_t(x) \frac{dt}{\omega(|t|)} < +\infty, \quad (0.3.6)$$

for some non-decreasing positive function $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$. Then for μ_{t_0} -almost any initial data $x \in B$ there exists a global mild solution of the initial value problem (0.3.3).

Our result is proved by measure theoretical techniques, projective arguments and superposition principles for Liouville solutions (see e.g. [AGS08, AL18, Man07]). Similar technique was used recently in [ALR20] to address the problem of uniqueness of solutions to the Gross-Pitaevskii hierarchies. Recall that such Gross-Pitaevskii and Hartree hierarchies are well-studied in the recent ten years and they are related to the mean-field limit of quantum many-body mechanics (see e.g. [CHPS15, CP14]). We construct by means of statistical arguments a path probability measure that leaves on global solutions of the initial value problem (0.3.3) so that its projection for each time t is given by μ_t . This allows us to construct a subset $\mathcal{G} \subset B$ of total measure with respect to μ_{t_0} , such that for all $x \in \mathcal{G}$ there exists a global solution satisfying (0.3.3).

As an application of our result, we study low regularity solutions for various Hamiltonian nonlinear PDEs including Hartree, NLS, Wave and Klein-Gordon, Euler and the modified quasi-geostrophic (mSQG) equations which interpolate between the Euler and the SQG equation. In fact, such PDEs can be recast into the initial value problem (0.3.3) using some interaction representations or specific reductions. Then by means of Malliavin calculus on Gaussian probability spaces, we show that whenever we have a well-defined Gibbs measure of the given nonlinear Hamiltonian equations (Hartree, NLS, Wave), there exists a global solution to the Liouville equation (0.3.4) with (0.3.6) satisfied. Thus, Theorem 0.3.4 yields the existence of global mild solutions almost surely for such dispersive PDEs. In

particular, we recovered several results of almost sure existence of global solutions obtained by Bourgain’s method and prove new ones. On the other hand in Fluid Mechanics, the work of Albeverio and Cruzeiro [AC90] for the Euler equation and the work of Nahmod–Pavlović–Staffilani–Totz [NPST18] on the mSQG equations constructed low-regularity global or arbitrary lifespan solutions. Using our Theorem 0.3.4, we are able to prove existence of global solutions for the mSQG equations.

PART I

The deterministic approach

CREATION OF SCALES AND PHASES IN NONLINEAR EVOLUTION EQUATIONS

Abstract. *The transition from regular to apparently chaotic motions is often observed in nonlinear flows. The purpose of this article is to describe a deterministic mechanism by which several smaller scales (or higher frequencies) and new phases can arise suddenly under the impact of a forcing term. This phenomenon is derived from a multiscale and multiphase analysis of nonlinear differential equations involving stiff oscillating source terms. Under integrability conditions, we show that the blow-up procedure (a type of normal form method) and the Wentzel-Kramers-Brillouin approximation (of supercritical type) introduced in [Che15, Che17] still apply. This allows to obtain the existence of solutions during long times, as well as asymptotic descriptions and reduced models. Then, by exploiting transparency conditions (coming from the integrability conditions), by implementing the Hadamard's global inverse function theorem and by involving some specific WKB analysis, we can justify in the context of Hamilton-Jacobi equations the onset of smaller scales and new phases.*

1.1 Introduction

The aim is to exhibit fundamental mechanisms explaining how the complexity of a nonlinear flow can suddenly increase. The phenomenon is illustrated by Theorem [1.1.5](#) and Figure [1.1](#). We explain how we could have transfer and creation of oscillations. In Subsection [1.1.1](#), we present a class of Hamilton-Jacobi equations and related assumptions. In Subsection [1.1.2](#), we state our main result. Then, in Subsection [1.1.3](#), we explain the link between this class of Hamilton-Jacobi equations and a special class of nonlinear differential equations. The main outcomes concerning the latter class of nonlinear differential equations are detailed in Section [1.2](#).

1.1.1 A class of Hamilton-Jacobi equations

Given some $\varepsilon_0 \in]0, 1]$, the effects will be assessed quantitatively by a small positive real parameter $\varepsilon \in]0, \varepsilon_0]$ which is intended to go to 0. We consider functions u depending on the time variable $\tau \in \mathbb{R}_+$ and on the space variable $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ with $d \in \mathbb{N}^*$. Denote by \mathbb{T} the torus $\mathbb{R}/(2\pi\mathbb{Z})$. Select two smooth scalar functions

$$\mathcal{U}_0 : [0, \varepsilon_0] \times \mathbb{R}^d \rightarrow \mathbb{R}, \quad (1.1.1)$$

$$\mathbf{H} : [0, \varepsilon_0] \times \mathbb{T} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \rightarrow \mathbb{R}. \quad (1.1.2)$$

We can expand \mathcal{U}_0 and \mathbf{H} in powers of ε near $\varepsilon = 0$ up to the order $N \in \mathbb{N}^*$ according to

$$\mathcal{U}_0(\varepsilon, x) = \sum_{j=0}^N \varepsilon^j \mathcal{U}_{0j}(x) + \mathcal{O}(\varepsilon^{N+1}), \quad (1.1.3)$$

$$\mathbf{H}(\varepsilon, \theta_\tau, x, u, p, \theta_u) = \sum_{j=0}^N \varepsilon^j \mathbf{H}_j(\theta_\tau, x, u, p, \theta_u) + \mathcal{O}(\varepsilon^{N+1}). \quad (1.1.4)$$

Then, we consider the family of local solutions $u \equiv u_\varepsilon$ (indexed by ε) to the Cauchy problems

$$\partial_\tau u + \frac{1}{\varepsilon} \mathbf{H}\left(\varepsilon, \frac{\tau}{\varepsilon}, x, \varepsilon u, \nabla_x u, \frac{u}{\varepsilon}\right) = 0, \quad u(0, x) = u_0(x) := \mathcal{U}_0(\varepsilon, x). \quad (1.1.5)$$

The above equation involves fast variations in τ (due to the substitution of θ_τ for τ/ε) and strong nonlinear effects linked to the substitution of εu for $u \in \mathbb{R}$, $\nabla_x u$ for $p \in \mathbb{R}^d$ and of u/ε for $\theta_u \in \mathbb{T}$. The Hamilton-Jacobi equation (1.1.5) is highly oscillating, and therefore the same should apply to its solution u_ε . It follows that the asymptotic description of u_ε should require the use of several *scales* and *phases*:

- (i) A *scale* is a power ε^ℓ with $\ell \in \mathbb{R}_+$. The *scale* ε^ℓ appears in the solution (resp. in the Cauchy problem) when it is needed (even in composite form) for the multiscale description of u_ε (resp. of u_0 or \mathbf{H}). For instance, the three scales ε^0 , ε^1 and ε^2 are needed when dealing with the function $g_\varepsilon : (\tau, x) \mapsto \sin\left[\left(\cos(x + \tau/\varepsilon)\right)/\varepsilon^2\right]$.
- (ii) A *phase* φ is a smooth real valued scalar function occurring in the solution (or in the Cauchy problem) when, after multiplication by some negative power of ε , it is substituted for a periodic variable (like θ_τ or θ_u) in a profile. The phase $\varphi \equiv \varphi_\varepsilon$ may depend on ε and contain oscillations (like in the case of *chirped pulses* [CL09]) but it must be (locally) uniformly bounded with bounded first order derivatives.

For example, the expression $\varepsilon \cos(x + \tau/\varepsilon)$ acts as a phase (associated with the frequency ε^{-3}) in the above function g_ε . The weight ε is here important to recover a time bounded derivative.

To tackle (1.1.5), restrictions on the leading term \mathbf{H}_0 issued from (1.1.4) are needed. To this end, we first consider the following simplification of the nonlinear interaction in \mathbf{H}_0 .

Assumption 1.1.1 (Independence on the periodic variable θ_u). The leading term \mathbf{H}_0 does not depend on the last periodic variable θ_u . In other words

$$\partial_{\theta_u} \mathbf{H}_0 \equiv 0. \quad (1.1.6)$$

We also impose the following positivity condition on \mathbf{H}_0 .

Assumption 1.1.2 (Positivity condition). We have for all $(\theta_\tau, x, u, p) \in \mathbb{T} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$

$$[(p \cdot \nabla_p) \mathbf{H}_0 - \mathbf{H}_0](\theta_\tau, x, u, p) > 0. \quad (1.1.7)$$

Note that Assumption 1.1.2 implies that $\mathbf{H}_0 \not\equiv 0$. In addition, we implement an integrability condition which will serve a lot, for instance to prove the uniform (in ε) local existence of the solutions u_ε .

Assumption 1.1.3 (Complete integrability). For all position $\mathfrak{z} = {}^t(\mathfrak{z}_x, \mathfrak{z}_u, \mathfrak{z}_p) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, the solution ${}^t(\Xi_{0x}, \Xi_{0p})(\mathfrak{z}; \cdot)$ to the Cauchy problem

$$\partial_s \begin{pmatrix} \Xi_{0x} \\ \Xi_{0p} \end{pmatrix} = \begin{pmatrix} +\nabla_p \mathbf{H}_0 \\ -\nabla_x \mathbf{H}_0 \end{pmatrix} (s, \Xi_{0x}, \mathfrak{z}_u, \Xi_{0p}), \quad {}^t(\Xi_{0x}, \Xi_{0p})(\mathfrak{z}; 0) = {}^t(\mathfrak{z}_x, \mathfrak{z}_p), \quad (1.1.8)$$

is globally defined in s , and it is periodic in s of period 2π .

For reasons that will become apparent later in Subsection 1.6.4, we need to impose the following smallness restriction on some second order derivatives of the scalar function \mathbf{H}_0 .

Assumption 1.1.4 (Smallness condition on \mathbf{H}_0). For a well-adjusted small parameter $\delta \in \mathbb{R}_+^*$, we have

$$\|D_{x,u,p} \nabla_p \mathbf{H}_0\| := \sup \left\{ \|D_{x,u,p} \nabla_p \mathbf{H}_0(\theta_\tau, z)\|; \theta_\tau \in \mathbb{T}, z \in K \right\} \leq \delta, \quad (1.1.9)$$

where $\|\cdot\|$ is a matrix norm and K is a compact set which is identified in (1.6.19).

The initial data in the right hand side of (1.1.5) is of amplitude 1. It is smooth with uniformly bounded derivatives. But the source term \mathbf{H} is large, of size $1/\varepsilon$ due to (1.1.7), and it implies very rapid oscillations (involving both τ and u). By this way, strong nonlinear processes are implemented when solving (1.1.5), see Remark 1.6.1. Such aspects are often studied in an isolated or partial manner. They appear for instance in the references [CLL10, CDI01, Fir17, FFI20, GIM15, IM08].

1.1.2 Onset of smaller scales and new phases

The next result proves that the above nonlinear interactions can suddenly generate additional scales and phases. Denote by $B(0, R]$ the closed ball in \mathbb{R}^d of center 0 and radius $R \in \mathbb{R}_+^*$.

Theorem 1.1.5 (WKB description of the flow). *Under Assumptions 1.1.1, 1.1.2, 1.1.3, and 1.1.4, for all $\varepsilon \in]0, \varepsilon_0]$ with ε_0 small enough, there exists a local smooth solution $u \equiv u_\varepsilon$ to (1.1.5) on the product of a time interval and a spatial domain having the form $[0, \mathcal{T}] \times B(0, R]$ with $0 < \mathcal{T}$ and $R > 0$. Moreover, on this uniform region, for all $N \in \mathbb{N}$ with $N \geq 3$, the expression u_ε can be described in the sup norm by the following multiscale and multiphase expansion*

$$\begin{aligned} u_\varepsilon(\tau, x) &= \frac{1}{\varepsilon} \overline{\mathcal{U}}_{-1}\left(x; \tau, \frac{\tau}{\varepsilon}\right) + \overline{\mathcal{U}}_0\left(x; \tau, \frac{\tau}{\varepsilon}\right) \\ &\quad + \sum_{j=1}^N \varepsilon^j \mathcal{U}_j\left(x; \tau, \frac{\tau}{\varepsilon}, \frac{\psi_\varepsilon(x; \tau)}{\varepsilon^3}\right) + \mathcal{O}(\varepsilon^{N+1}), \end{aligned} \tag{1.1.10}$$

where ψ_ε is a phase in the sense of (ii), looking like

$$\psi_\varepsilon(x; \tau) := \varepsilon \overline{\mathcal{U}}_{-1}\left(x; \tau, \frac{\tau}{\varepsilon}\right) + \varepsilon^2 \overline{\mathcal{U}}_0\left(x; \tau, \frac{\tau}{\varepsilon}\right). \tag{1.1.11}$$

The two profiles $\overline{\mathcal{U}}_{-1}(x, \tau, \theta_\tau)$ and $\overline{\mathcal{U}}_0(x, \tau, \theta_\tau)$ which appear both in (1.1.10) and (1.1.11) are smooth functions on the domain $B(0, R] \times [0, \mathcal{T}] \times \mathbb{T}$. For $j \geq 1$, the profiles $\mathcal{U}_j(x; \tau, \theta_\tau, \hat{\theta}_\tau^\circ)$ are smooth with respect to the variables $(x, \tau) \in B(0, R] \times [0, \mathcal{T}]$ and smooth (periodic) with respect to the two last variables $\theta_\tau \in \mathbb{T}$ and $\hat{\theta}_\tau^\circ \in \mathbb{T}$.

When solving (1.1.5), a number of new patterns are generated (creation of oscillations). When looking at smaller details that are at smaller amplitudes, the flow is growing in complexity. This cascade towards more and more scales and phases is put in concrete form

at the level of (1.1.10) both quantitatively in terms of frequencies (larger derivatives) and qualitatively in terms of phases (extra directions of fast variations). Theorem 1.1.5 shows clearly that the description of the solution u_ε with an incrementally precision (expressed in terms of powers of ε) is associated with an increasing agitation. This principle may also be illustrated through the Table 1.1.

Table 1.1 – Growing complexity of the flow as time passes and when looking at smaller details

Aspect of the flow modulo a precision of size ε^j with	Involved profile	Number of scales	Number of phases
$\tau = 0$ and $j \in \mathbb{N}$	\mathcal{U}_{0j}	1	0
$\tau > 0$ and $j = -1, 0$	$\overline{\mathcal{U}}_{-1}, \overline{\mathcal{U}}_0$	2	1
$\tau > 0$ and $j \geq 1$	\mathcal{U}_j	3	2

While it is commonly believed that nonlinear evolution equations can instantaneously develop microstructures (like in turbulent flows), concrete mechanisms for this and rigorous proofs are rarely exhibited (or in very specific contexts due to subsequent instabilities). Theorem 1.1.5 is a step forward in this direction. It is proved in Section 1.6, and appears as a difficult corollary of a WKB analysis which is developed in Sections 1.3, 1.4, and 1.5.

1.1.3 From Hamilton-Jacobi equations to nonlinear differential equations

For smooth solutions, Hamilton-Jacobi equations can be solved by the method of characteristics. In the context of (1.1.5), this yields the following system of nonlinear differential equations

$$\partial_\tau \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0; z; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right), \quad \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} (0) = \begin{pmatrix} z_0 \\ \mathbf{v}_0 \end{pmatrix}, \quad (1.1.12)$$

where, given $n \in \mathbb{N}^*$, the dependent variables $z \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}$ depend on the time $\tau \in \mathbb{R}$. Here, $\varepsilon \in]0, \varepsilon_0]$ is a small positive parameter which is intended to go to zero, whereas $z_0 \in \mathbb{R}^n$ and $\mathbf{v}_0 \in \mathbb{R}$ stand for initial data (which may depend smoothly on $\varepsilon \in [0, \varepsilon_0]$). The precise content of (1.1.12) is described in next Section 1.2.

The strategy to study (1.1.12) is inspired from [Che15, Che17]. But the works [Che15,

[Che17] are devoted to the characteristics of the Vlasov equation, and they rely crucially on the conservation of the kinetic energy (of charged particles). They cannot be applied directly to the system (1.1.12) or in the presence of an electric field (see [CF23a] to this end). With this in mind, the framework of [Che15, Che17] needs to be extended; and the tools of [Che15, Che17] must be revisited. In fact, in comparison to [Che15, Che17], we replace some invariant quantities by (more general) integrability conditions (which will operate at all levels to move forward). As a consequence, we have to adapt (in Sections 1.3, 1.4 and 1.5) our preceding arguments to the new difficulties thus generated.

The proof of Theorem 1.1.5 depends on studying the above system of nonlinear differential equations (1.1.12). It is derived from an existence result (Theorem 1.2.5), asymptotic descriptions (Theorem 1.2.7) and reduced models (Theorem 1.2.10) that apply to the above class of differential equations (1.1.12). The contents of the latter theorems are identified in the next section.

1.2 A class of nonlinear differential equations

In this section, we present the underlying framework. In Subsection 1.2.1, we state the assumptions and the main results. In Subsection 1.2.2, we make some general comments about the content of our theorems. In Subsection 1.2.3, we outline the plan, we highlight some innovative ideas, and we come back to the possible applications.

The expressions \mathbf{A} and \mathbf{V} in the source term of (1.1.12) depend on ε . They are defined up to $\varepsilon = 0$, and they are smooth near $\varepsilon = 0$. Thus, for all $N \in \mathbb{N}$, they can be expressed as Taylor series expansions of order N in terms of ε , near $\varepsilon = 0$:

$$C^\infty([0, \varepsilon_0] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}; \mathbb{R}^n) \ni \mathbf{A}(\varepsilon, z_0; z; \theta_\tau, \theta_r) = \sum_{j=0}^N \varepsilon^j \mathbf{A}_j(z_0; z; \theta_\tau, \theta_r) + \mathcal{O}(\varepsilon^{N+1}),$$

$$C^\infty([0, \varepsilon_0] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}; \mathbb{R}_+^*) \ni \mathbf{V}(\varepsilon, z_0; z; \theta_\tau, \theta_r) = \sum_{j=0}^N \varepsilon^j \mathbf{V}_j(z_0; z; \theta_\tau, \theta_r) + \mathcal{O}(\varepsilon^{N+1}).$$

As indicated, the functions \mathbf{A} and \mathbf{V} may imply $z_0 \in \mathbb{R}^n$. By this way, they can take into account the influence of the component z_0 in the initial condition, as it may be required in the applications. They may imply the unknown z . But they do not involve \mathbf{v} (the scalar variable \mathbf{v} appears in the right hand side only after substitution of θ_r by \mathbf{v}/ε). Most importantly, they are periodic with respect to the two last variables θ_τ and θ_r , of period

respectively 2π and $T_r(z_0)$. More precisely

$$\theta_\tau \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z}), \quad (1.2.1)$$

$$\theta_r \in \mathbb{T}_{r,z_0} := \mathbb{R}/(T_r(z_0)\mathbb{Z}), \quad T_r \in C^\infty(\mathbb{R}^n; \mathbb{R}_+^*). \quad (1.2.2)$$

Remark 1.2.1. With no loss of generality, we work with (1.2.1). We can also start with source terms \mathbf{A} and \mathbf{V} that are periodic in θ_τ of period $T_\tau(z_0)$ with $T_\tau \in C^\infty(\mathbb{R}^n; \mathbb{R}_+^*)$. This means to deal with $\theta_\tau \in \mathbb{T}_{\tau,z_0} := \mathbb{R}/(T_\tau(z_0)\mathbb{Z})$. Suppose this is the case, then we can always replace θ_τ by $2\pi\theta_\tau/T_\tau(z_0)$. This modification has the effect to substitute \mathbf{A} and \mathbf{V} for

$$\frac{T_\tau(z_0)}{2\pi} \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0; z; \frac{T_\tau(z_0)}{2\pi} \theta_\tau, \theta_r \right).$$

This gives rise to (1.2.1) without affecting the general form that has been introduced at the level of (1.1.12). But, when the original functions \mathbf{A} and \mathbf{V} do not imply z_0 , this does produce a (smooth) dependence on z_0 . This is why, care is taken to incorporate z_0 when defining \mathbf{A} and \mathbf{V} .

The source term inside (1.1.12) is:

- stiff since the large weight ε^{-1} is put in factor of the (locally bounded) functions \mathbf{A} and \mathbf{V} which may satisfy $\mathbf{A}_0 \not\equiv 0$ and $\mathbf{V}_0 \not\equiv 0$. In fact, we will assume that \mathbf{V}_0 is positive.
- strongly oscillating due to the large amplitude oscillations which are generated by \mathbf{A} and \mathbf{V} after substitution of θ_τ and θ_r for respectively τ/ε and ν/ε ;
- nonlinear because \mathbf{A} and \mathbf{V} depend on z and θ_r (and therefore ν/ε) in a non-trivial way.

Let us examine more precisely what happens at the level of ν/ε . To this end, fix $\varepsilon \in]0, \varepsilon_0]$ small enough, and look at the scalar component $\nu(\tau)$. Since the function \mathbf{V}_0 will be assumed to be positive (see Assumption 1.2.6), by the mean value theorem, we can find some $c = c(\tau) \in]0, \tau[$ such that

$$\begin{aligned} \frac{\nu(\tau) - \nu_0}{\varepsilon} &= \nu_{\varepsilon,\tau} \frac{\tau}{\varepsilon^2}, \\ 0 < \nu_{\varepsilon,\tau} &:= \varepsilon \partial_\tau \nu(c) = \mathbf{V} \left(\varepsilon, z_0; z(c); \frac{c}{\varepsilon}, \frac{\nu(c)}{\varepsilon} \right) = \mathcal{O}(1). \end{aligned} \quad (1.2.3)$$

This indicates that the description of the solutions ${}^t(z, \nu)(\varepsilon, z_0, \nu_0; \tau)$ to the system

(1.1.12) should involve (at least) three time scales:

- τ for the current time variable and normal variations (in the case of electrons in tokamaks, the value $\tau \sim 1$ represents a few seconds);
- $s := \varepsilon^{-1}\tau$ for the quick time variable and for quick variations (in comparison to changes in τ). The substitution of the periodic variable θ_τ by $\varepsilon^{-1}\tau$ furnishes oscillations at high frequencies of size ε^{-1} ;
- $t := \varepsilon^{-2}\tau$ for the rapid time variable and for rapid variations (in comparison to changes in τ and s). In view of line (1.2.3), the substitution of the periodic variable θ_r by $\varepsilon^{-1}\mathbf{v}(\tau)$ should produce oscillations at very high frequencies of size ε^{-2} . The subscript r in θ_r is introduced to refer to these rapid variations.

To state our results, some basic operations on periodic functions $Z(\theta_\tau, \theta_r) \in L^1(\mathbb{T} \times \mathbb{T}_{r,z_0})$, like above $\mathbf{A}(\varepsilon, z_0; z; \cdot)$ or $\mathbf{V}(\varepsilon, z_0; z; \cdot)$, must be introduced. As a preliminary point, to sort out the different oscillating features, we need to define:

- The *rapid mean value* of Z is the periodic function \bar{Z} given by

$$\bar{Z}(\theta_\tau) := \frac{1}{T_r(z_0)} \int_0^{T_r(z_0)} Z(\theta_\tau, \theta_r) d\theta_r. \quad (1.2.4)$$

- The *rapid oscillating part* of Z is the periodic function Z^* defined by

$$Z^*(\theta_\tau, \theta_r) := Z(\theta_\tau, \theta_r) - \bar{Z}(\theta_\tau). \quad (1.2.5)$$

- The *double mean value* of Z (in both variables θ_τ and θ_r) or the *quick mean value* of \bar{Z} is the constant

$$\langle \bar{Z} \rangle := \frac{1}{2\pi T_r(z_0)} \int_0^{2\pi} \int_0^{T_r(z_0)} Z(\theta_\tau, \theta_r) d\theta_\tau d\theta_r = \frac{1}{2\pi} \int_0^{2\pi} \bar{Z}(\theta_\tau) d\theta_\tau. \quad (1.2.6)$$

- The *quick oscillating part* of \bar{Z} is the periodic function \bar{Z}^* given by

$$\bar{Z}^*(\theta_\tau) := \bar{Z}(\theta_\tau) - \langle \bar{Z} \rangle. \quad (1.2.7)$$

Recall that this induces a decomposition of Z according to

$$Z(\theta_\tau, \theta_r) = \langle \bar{Z} \rangle + \bar{Z}^*(\theta_\tau) + Z^*(\theta_\tau, \theta_r). \quad (1.2.8)$$

Our purpose in Sections [1.3](#), [1.4](#) and [1.5](#) is to find a setting under which the system [\(1.1.12\)](#) can be solved on a time interval that is uniform in $\varepsilon \in]0, \varepsilon_0]$. It is also to exhibit conditions leading to a *three-scale* asymptotic description (when ε goes to zero) of the flow, showing the three frequencies of size 1, ε^{-1} and ε^{-2} that are associated to the underlying presence of the time variables τ , s and t .

1.2.1 Assumptions and results

When solving [\(1.1.12\)](#), the oscillations at frequencies 1, ε^{-1} , and ε^{-2} are closely interlinked. They are difficult to disentangle. To guess what happens, as a first step, it is interesting to interpret [\(1.1.12\)](#) in terms of the quick time variable s . By this way, with

$$\begin{pmatrix} \dot{z} \\ \dot{\mathbf{v}} \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; s) := \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; \varepsilon s),$$

we obtain

$$\partial_s \begin{pmatrix} \dot{z} \\ \dot{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0, \dot{z}; s, \frac{\dot{\mathbf{v}}}{\varepsilon} \right), \quad \begin{pmatrix} \dot{z} \\ \dot{\mathbf{v}} \end{pmatrix} (0) = \begin{pmatrix} z_0 \\ \mathbf{v}_0 \end{pmatrix}. \quad (1.2.9)$$

By Picard-Lindelöf theorem, a local solution to the Cauchy problem [\(1.2.9\)](#) exists on a maximal time interval $[0, S(\varepsilon, z_0, \mathbf{v}_0)[$ with $S(\varepsilon, z_0, \mathbf{v}_0) \in \mathbb{R}_+^* \cup \{+\infty\}$. Fix z_0 and \mathbf{v}_0 . The right hand side of [\(1.2.9\)](#) is no more stiff and it is periodic with respect to θ_r . Thus, it is (locally in \dot{z}) uniformly bounded with respect to $\varepsilon \in]0, \varepsilon_0]$. Taking into account the explosion behavior that a maximal solution must have at the boundary of its domain of definition, it follows that

$$\exists S(z_0, \mathbf{v}_0) \in \mathbb{R}_+^*; \quad \forall \varepsilon \in]0, \varepsilon_0], \quad 0 < S(z_0, \mathbf{v}_0) \leq S(\varepsilon, z_0, \mathbf{v}_0). \quad (1.2.10)$$

Now, the source term of the system [\(1.2.9\)](#) is still oscillating. In view of [\(1.2.3\)](#) with τ replaced by εs , it contains oscillations at high frequencies of size

$$\varepsilon^{-1} \dot{\mathbf{v}}(s) = \varepsilon^{-1} \mathbf{v}(\varepsilon s) = \varepsilon^{-1} (\mathbf{v}_0 + \mathbf{v}_{\varepsilon, \varepsilon s} s) = \mathcal{O}(\varepsilon^{-1}).$$

Since \mathbf{V}_0 (and therefore \mathbf{V} for small values of ε) is assumed to be a positive valued function, the function $\dot{\mathbf{v}}$ is strictly increasing, and the oscillations with respect to θ_r are certainly effective. After integration in time (in s), these oscillations compensate each other to deliver some average evolution. During quick times s , at leading order, the behavior of \dot{z}

inside (1.2.9) may be approximated by the mean flow introduced below.

Definition 1.2.2 (Mean flow associated with the quick time evolution of z). We define

$$\mathbf{A}_{\text{mf}}(z_0; z; \theta_\tau) := \frac{1}{T_{\text{r}}(z_0)} \left(\frac{1}{T_{\text{r}}(z_0)} \int_0^{T_{\text{r}}(z_0)} \frac{1}{V_0(z_0; z; \theta_\tau, \theta_r)} d\theta_r \right)^{-1} \times \int_0^{T_{\text{r}}(z_0)} \frac{\mathbf{A}_0(z_0; z; \theta_\tau, \theta_r)}{V_0(z_0; z; \theta_\tau, \theta_r)} d\theta_r.$$

The *mean flow* associated with (1.2.9) is the mapping $\Xi_{\text{mf}}(z_0; \cdot) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ obtained by solving

$$\partial_s \Xi_{\text{mf}}(z_0; \mathfrak{z}; s) = \mathbf{A}_{\text{mf}}(z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; s); s), \quad \Xi_{\text{mf}}(z_0; \mathfrak{z}; 0) = \mathfrak{z} \in \mathbb{R}^n \quad (1.2.11)$$

on its maximal interval of existence $[0, S_{\text{mf}}(z_0, \mathfrak{z})[$ where $S_{\text{mf}}(z_0, \mathfrak{z}) \in \mathbb{R}_+^* \cup \{+\infty\}$.

Equation (1.2.11) is derived in Subsection 1.3.4 from some homological equation (Lemma 1.3.10). Observe that when V_0 does not depend on θ_r , the function \mathbf{A}_{mf} is extracted from $\mathbf{A}(0, z_0; z; \cdot)$ as it is indicated in (1.2.4), that is

$$\mathbf{A}_{\text{mf}}(z_0; z; \theta_\tau) = \bar{\mathbf{A}}_0(z_0; z; \theta_\tau) = \frac{1}{T_{\text{r}}(z_0)} \int_0^{T_{\text{r}}(z_0)} \mathbf{A}_0(z_0; z; \theta_\tau, \theta_r) d\theta_r. \quad (1.2.12)$$

In general, the lifespan $S_{\text{mf}}(z_0, \mathfrak{z})$ is finite, and the same holds concerning $S(\varepsilon, z_0, \mathbf{v}_0)$.

Example 1.2.3. For instance, just take $n = 1$ and $\mathbf{A} \equiv z^2$ together with $\partial_{\theta_r} V_0 \equiv 0$ so that

$$\mathbf{A} \equiv \mathbf{A}_0 \equiv \mathbf{A}_{\text{mf}} \equiv z^2, \quad \dot{z}(\varepsilon, z_0, \mathbf{v}_0; s) = \Xi_{\text{mf}}(z_0; z_0; s). \quad (1.2.13)$$

In this simple case, we can see that

$$\forall (z_0, \mathbf{v}_0) \in \mathbb{R}_+^* \times \mathbb{R}, \quad S(\varepsilon, z_0, \mathbf{v}_0) = S_{\text{mf}}(z_0, z_0) = z_0^{-1} < +\infty.$$

It follows that the lifespan $\mathcal{T}(\varepsilon, z_0, \mathbf{v}_0) = \varepsilon S(\varepsilon, z_0, \mathbf{v}_0)$ associated with (1.1.12) is finite, and shrinks to 0 like εz_0^{-1} when ε goes to zero. Then, there is no way to guarantee that

$$\exists \mathcal{T}(z_0, \mathbf{v}_0) \in \mathbb{R}_+^*; \quad \forall \varepsilon \in]0, \varepsilon_0], \quad 0 < \mathcal{T}(z_0, \mathbf{v}_0) \leq \mathcal{T}(\varepsilon, z_0, \mathbf{v}_0). \quad (1.2.14)$$

When dealing with (1.2.13), the discussion about what could happen during current times $\tau \sim 1$ is over. We would like to avoid such situations.

To obtain (1.2.14), supplementary conditions on \mathbf{A}_{mf} are clearly needed. As a prerequisite we have to impose $S_{\text{mf}} = +\infty$ on $\mathbb{R}^n \times \mathbb{R}^n$. However, this condition may not be enough.

And it is certainly not sufficient to separate the quick and rapid oscillations, and then to obtain a complete description of them. To this end, we need more restrictive conditions.

Assumption 1.2.4 (Complete integrability of the mean flow). For each position $(z_0, \mathfrak{z}) \in \mathbb{R}^n \times \mathbb{R}^n$, the function $\Xi_{\text{mf}}(z_0, \mathfrak{z}, \cdot)$ issued from (1.2.11) is globally defined and periodic in s of period 2π .

The above assumption implies a strong geometrical restriction concerning (1.2.11). Indeed, this means that the integral curves associated with (1.2.11) must form (by varying the initial data $\mathfrak{z} \in \mathbb{R}^n$) a foliation of \mathbb{R}^n by circles. After a normal form procedure (see Remark 1.3.4), the equation on \hat{z} inside (1.2.9) can be transformed into a perturbed version of (1.2.11). From this perspective, Assumption 1.2.4 says that the dynamical system thus obtained is nearly integrable.

By construction, the function $\mathbf{A}_{\text{mf}}(z_0; z; \cdot)$ is periodic (with respect to θ_τ) with period 2π , and therefore the same holds (with respect to the quick time variable s) for the source term inside (1.2.11). Assumption 1.2.4 requires also that all solutions to (1.2.11) share the same period 2π . This second condition is natural (but it is far from being systematically verified). Next we have a result on the uniform lifespan with respect to $\varepsilon \in]0, \varepsilon_0]$ of the flow generated by (1.1.12).

Theorem 1.2.5. *Under Assumption 1.2.4, the lifespan $\mathcal{T}(\varepsilon, z_0, \mathbf{v}_0)$ that is associated with (1.1.12) is uniformly bounded below by some $\mathcal{T}(z_0, \mathbf{v}_0) \in \mathbb{R}_+^*$, as indicated in (1.2.14).*

From there, the issue about the oscillating structure of ${}^t(z, \mathbf{v})$ during current times $\tau \sim 1$ becomes meaningful. Now, to obtain a precise asymptotic description of the flow ${}^t(z, \mathbf{v})$, we need to impose supplementary restrictions on \mathbf{V}_0 .

Assumption 1.2.6 (Positivity condition on the component \mathbf{V}_0). The function \mathbf{V}_0 is positive and it does not depend on θ_r . Recall that

$$\forall (z_0, z, \theta_\tau, \theta_r) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{r, z_0}, \quad 0 < \mathbf{V}_0(z_0; z; \theta_\tau, \theta_r) \equiv \mathbf{V}_0(z_0; z; \theta_\tau). \quad (1.2.15)$$

This simplifies the self-interactions at the level of the equation on \mathbf{v} . Indeed, at leading order, the source term \mathbf{V} is not impacted by the rapid variations (in \mathbf{v}/ε). Next we have a WKB expansion at all orders in $\varepsilon \in]0, \varepsilon_0]$ of the flow induced by the system (1.1.12).

Theorem 1.2.7. Under Assumptions [1.2.4](#) and [1.2.6](#), we can find profiles

$$\begin{aligned} Z_j(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) &\in C^\infty\left(\mathbb{R}^n \times \mathbb{R} \times [0, \mathcal{T}(z_0, \mathbf{v}_0)] \times \mathbb{T} \times \mathbb{T}_{r, z_0}; \mathbb{R}^n\right), \quad j \in \mathbb{N}, \\ \mathcal{V}_j(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) &\in C^\infty\left(\mathbb{R}^n \times \mathbb{R} \times [0, \mathcal{T}(z_0, \mathbf{v}_0)] \times \mathbb{T} \times \mathbb{T}_{r, z_0}; \mathbb{R}\right), \quad j \in \{-1\} \cup \mathbb{N}, \end{aligned}$$

which are determined through a hierarchy of well-posed modulation equations (starting from $j = -1$ up to any integer value of j), which are such that

$$\mathcal{V}_{-1} \equiv \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau), \quad \mathcal{V}_0 \equiv \overline{\mathcal{V}}_0(z_0, \mathbf{v}_0; \tau, \theta_\tau), \quad Z_0 \equiv \overline{Z}_0(z_0; \tau, \theta_\tau), \quad (1.2.16)$$

and which are adjusted in such a way that, in terms of the sup norm, for all $N \in \mathbb{N}^*$, we have

$$z(\varepsilon, z_0, \mathbf{v}_0; \tau) = \sum_{j=0}^{N-2} \varepsilon^j Z_j\left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\right) + \mathcal{O}(\varepsilon^{N-1}), \quad (1.2.17)$$

$$\mathbf{v}(\varepsilon, z_0, \mathbf{v}_0; \tau) = \sum_{j=-1}^{N-2} \varepsilon^j \mathcal{V}_j\left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\right) + \mathcal{O}(\varepsilon^{N-1}). \quad (1.2.18)$$

The two expansions [\(1.2.17\)](#) and [\(1.2.18\)](#) shed light on the time oscillations but also on the spatial oscillations (encoded in the variations with respect to z_0 and \mathbf{v}_0), thus revealing *collective* aspects of the motion (which are important in many applications). This is achieved through different types of phases, including mainly:

- The *time phase* τ which is associated with quick variations.
- The *exact phase* \mathbf{v} . It has ε^{-1} in factor at the level of the source term of [\(1.1.12\)](#). There, the scalar component \mathbf{v}/ε comes to replace the periodic variable θ_r and therefore it indeed plays the role of a phase.
- The *frozen phase* \mathbf{v}^f which is defined by

$$\mathbf{v}^f(\tau) \equiv \mathbf{v}^f(\varepsilon, z_0, \mathbf{v}_0; \tau) := \frac{1}{\varepsilon} \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau) + \overline{\mathcal{V}}_0\left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}\right). \quad (1.2.19)$$

It is a truncated version of \mathbf{v} which, like \mathbf{v} , operates with ε^{-1} in factor.

- The *rapid phase* $\langle \overline{\mathcal{V}}_{-1} \rangle$ which is associated with rapid variations (at frequencies of size ε^{-2}). As a consequence of Assumption [1.2.6](#), we will find that $\langle \overline{\mathcal{V}}_{-1} \rangle(\tau) > 0$ for

all $\tau > 0$. Thus, the presence of rapid oscillations is sure to happen.

Remark 1.2.8 (About supercritical features). It is worth underlying that in (1.2.18), the same profiles \mathcal{V}_{-1} and \mathcal{V}_0 take part in the description of amplitudes and phases. This is typical of supercritical regimes for quasilinear equations.

Remark 1.2.9 (About the notion of phase). Definition (ii) of a phase φ (in terms of its *bounded* aspects) is open to various interpretations. Indeed, it is based on the dependent or independent variables that are implied. For example:

- The functions $\varphi \equiv \mathbf{v}$ and $\varphi \equiv \mathbf{v}^f$ viewed as depending on the state variable \mathbf{v} or \mathbf{v}^f and from the perspective of the profiles \mathbf{A} and \mathbf{V} inside (1.1.12) can be viewed as phases.
- The functions $\varphi \equiv \varepsilon \mathbf{v}$ and $\varphi \equiv \varepsilon \mathbf{v}^f$ viewed this time as depending on $(\varepsilon, z_0, \mathbf{v}_0, \tau)$ and from the perspective of the profiles \mathbf{A} and \mathbf{V} inside (1.1.12) or Z_j and \mathcal{V}_j inside (1.2.17) or (1.2.18) play the role of phases.

On the other hand, even if \mathbf{v}/ε and \mathbf{v}^f/ε come to replace the periodic variable θ_r , the functions $\varphi \equiv \mathbf{v}$ and $\varphi \equiv \mathbf{v}^f$ viewed as depending on $(\varepsilon, z_0, \mathbf{v}_0, \tau)$ are not (strictly speaking) phases because they are obviously not uniformly bounded. It must be clear that the above names of *exact phase* and *frozen phase* are a matter of convention.

Looking at (1.2.17) and (1.2.18), at the end, we can recognize the simultaneous presence of oscillations implying the frequencies ε^{-1} and ε^{-2} with corresponding phases τ and $\langle \overline{\mathcal{V}}_{-1} \rangle$. There is also a nonlinear imbrication of oscillations carried by the (slightly unusual) expression $\overline{\mathcal{V}}_0(\cdot, \tau, \varepsilon^{-1}\tau)/\varepsilon$. The study of multiscale oscillations has been intensively developed in the past decades in various contexts including the topic of geometric optics [Dum06, Lan13, M09, Rau12, Sch94], the theory of homogenization [AF09b, MV10], two-scaled Wigner measures [FKL03, HKRVuN16] or microlocal Birkhoff normal forms [HKRVuN16]. However, the coverage of situations which can mix oscillations of the above different types is relatively new. This is technically a difficult challenge (with potential extensions in the domain of PDEs) which apparently is not directly within reach of the aforementioned methods.

Asymptotic expansions similar to (1.2.17) and (1.2.18) already appear in the articles [Che15, Che17]. There, they were motivated by questions arising in the study of magnetized plasmas. The actual approach is much broader than in [Che15, Che17]. The purpose is indeed to achieve a comprehensive analysis in more general situations than before. It is to extend the preceding tools and also to explain them more briefly and clearly. We

exhibit integrability and positivity conditions (Assumptions [1.2.4](#) and [1.2.6](#)) allowing to progress. These conditions encompass and extend the framework of [\[Che15, Che17\]](#). They are both intrinsic, easy to test, and suitable for many applications. They are considered in Section [1.6](#) in the case of Hamilton Jacobi equations, and they are designed [\[CF23a\]](#) to incorporate the influence of an external electric field (in addition to the magnetic field) on the long time dynamical behavior of charged particles. The main outcomes of our work concerning [\(1.1.12\)](#) are the following:

- A model for the leading behavior of the flow provided by

$$z(\varepsilon, z_0, \mathbf{v}_0; \tau) = \bar{Z}_0\left(z_0; \tau, \frac{\tau}{\varepsilon}\right) + \varepsilon Z_1\left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(\tau)}{\varepsilon}\right) + \mathcal{O}(\varepsilon^2), \quad (1.2.20)$$

$$\mathbf{v}(\varepsilon, z_0, \mathbf{v}_0; \tau) = \mathbf{v}^f(\varepsilon, z_0, \mathbf{v}_0; \tau) + \mathcal{O}(\varepsilon). \quad (1.2.21)$$

- A careful analysis of the underlying stability and instability properties. This aspect is more subtle and less easy to expose since it appears throughout the text. Let us just outline some difficulties. The precise knowledge of \mathbf{v} (or \mathbf{v}^f) is essential to obtain the L^∞ -precision. As a matter of fact, a perturbation of size ε^2 at the level of $\langle \sqrt{\mathcal{V}}_{-1} \rangle$ may have an impact of size 1 in the calculation of the $\mathcal{O}(\varepsilon^j)$ -terms of [\(1.2.17\)](#) and [\(1.2.18\)](#). This means that a very precise access to \mathbf{v} is crucial to govern the stability properties of the flow.

From the above perspective, our strategy is based on two main arguments:

- First, we implement a blow-up procedure. As explained in a series of remarks (see [1.3.4](#), [1.3.14](#) and [1.5.7](#)), this is a kind of normal form method adapted to our context. In Section [1.3](#), this already leads to the uniform local existence of solutions (z, \mathbf{v}) .
- Then, in Section [1.4](#), we perform a *three-scale WKB calculus* with supercritical attributes. The idea, as is typical in geometric optics [\[CG08, CGW11\]](#), is to replace [\(1.1.12\)](#) by profile equations. But this time, we implement the component \mathbf{v} of the solution previously obtained as a phase (see Remarks [1.4.3](#) and [1.4.4](#) for this unusual trick). This means in particular that \mathbf{v} is viewed as oscillating with respect to itself through the oscillating implicit relation [\(1.4.1\)](#). This yields a special notion of profile equations. This is like investigating the stability issue in a quotient space: we work modulo the determination of the *unknown* function \mathbf{v} .

Because of its importance from an application standpoint, we focus below on the content of [\(1.2.20\)](#). The behavior of z is mainly governed by the profile \bar{Z}_0 which reveals some

kind of (large amplitude oscillating) reduced dynamics during current times $\tau \sim 1$. The determination of \bar{Z}_0 may be achieved as indicated bellow.

Theorem 1.2.10 (Reduced equations). *The function $\bar{Z}_0(\cdot)$ does not depend on ν_0 . It can be expressed in terms of $\Xi_{\text{mf}}(\cdot)$ according to*

$$\bar{Z}_0(z_0; \tau, \theta_\tau) = \Xi_{\text{mf}}\left(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau); \theta_\tau\right). \quad (1.2.22)$$

The function $\langle \bar{\mathfrak{Z}}_0 \rangle$ in the right hand side of (1.2.22) can be determined by solving

$$\partial_\tau \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau) = \langle \bar{A}_1 \rangle\left(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)\right), \quad \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; 0) = z_0, \quad (1.2.23)$$

where, with $\partial_{\theta_r}^{-1}$ as in (1.3.24) (by identifying $\theta \equiv \theta_r$ and $T \equiv T_{r, z_0}$), we have introduced

$$\begin{aligned} A_1(z_0; \mathfrak{z}; \theta_r, \theta_r) &:= D_{\mathfrak{z}} \Xi_{\text{mf}}(z_0; \mathfrak{z}; \theta_r)^{-1} \\ &\times \left\{ \left[A_1 + V_0^{-1} \left(\partial_{\theta_r}^{-1} A_0^* \cdot \nabla_z \right) A_0 - \partial_{\theta_r} \left(V_0^{-1} \partial_{\theta_r}^{-1} A_0^* \right) - V_1 V_0^{-1} A_0^* \right. \right. \\ &\left. \left. + \left(\partial_{\theta_r}^{-1} A_0^* \cdot \nabla_z \right) \left(V_0^{-1} A_0^* \right) \right] \left(z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; \theta_r); \theta_r, \theta_r \right) \right\} \end{aligned} \quad (1.2.24)$$

and where the access to the double mean value $\langle \bar{A}_1 \rangle$ is furnished by (1.2.6).

Recall that the two symbols A_1 and \mathbf{A}_1 are different (we have $A_1 \neq \mathbf{A}_1$). The notations A_1 and \mathbf{A}_1 will be used with different meanings. Given the numerous difficulties of understanding the complex interplay between the various types of oscillations, Theorem 1.2.10 produces a rather easy and explicit way to determine what remains in a first approximation. In fact, it gives access to effective equations which are amenable to numerical computations.

1.2.2 A few comments on the results

The aim of this subsection is to help the reader understand the position, content and significance of the four preceding theorems. This is done below through a list of remarks.

Remark 1.2.11 (About the effective content of $\langle \bar{\mathfrak{Z}}_0 \rangle$). The expression $\langle \bar{A}_1 \rangle$ is issued from the double averaging procedure (1.2.6) which may go hand in hand with a number of cancellations. It follows that all the components of $\langle \bar{\mathfrak{Z}}_0 \rangle$ are not necessarily activated when solving (1.2.23). In general, there remains a reduced number of unknowns. These are the so-called adiabatic (or guiding-center) invariants in the case of charged particles.

Remark 1.2.12 (About the determination of \bar{Z}_0). The expression \bar{Z}_0 consists of two distinct parts: Ξ_{mf} and $\langle \bar{\mathfrak{z}}_0 \rangle$. As explained before, the mean flow Ξ_{mf} can be extracted from (1.2.9). In fact, this amounts to a multiplication of (1.1.12) by ε , and then to the extraction of a mean value involving A_0 and V_0 , as in Definition 1.2.2. Observe that

$$\dot{z}(\varepsilon, z_0, \mathbf{v}_0; s) = \Xi_{\text{mf}}(z_0; z_0; s) + \mathcal{O}(\varepsilon). \quad (1.2.25)$$

Thus, in coherence with what has been said before, the mean flow does furnish the leading behavior of z during quick times s near the current time $\tau = 0$. But, near other current times $\tau \in \mathbb{R}_+^*$, the use of $\langle \bar{\mathfrak{z}}_0 \rangle(z_0; \tau)$ is needed to well describe $\bar{Z}_0(\cdot)$. The access to $\langle \bar{\mathfrak{z}}_0 \rangle$ is much more complicated than to Ξ_{mf} . It involves the determination of A_1 which, in view of (1.2.24), is built with various derivatives and integrations of A_0 and V_0 , as well as terms of size ε inside A , like A_1 . Now, such information should be invisible (or vanishing) when performing (even multiscale) weak limits at the level of (1.1.12) or (1.2.9).

Remark 1.2.13 (About the geometrical interpretation of the content of \bar{Z}_0). Under Assumption 1.2.4, the integral curves associated to (1.2.11) draw a family of circles

$$\mathcal{C}(z_0; \mathfrak{z}) := \left\{ \Xi_{\text{mf}}(z_0; \mathfrak{z}; s); s \in \mathbb{R} \right\} \subset \mathbb{R}^n, \quad (z_0, \mathfrak{z}) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Given $z_0 \in \mathbb{R}^n$, these circles form a foliation of \mathbb{R}^n parameterized by \mathfrak{z} . The role of \mathfrak{z} is twofold. First, since $\mathfrak{z} \in \mathcal{C}(z_0; \mathfrak{z})$, the parameter \mathfrak{z} points to the circle $\mathcal{C}(z_0; \mathfrak{z})$. Secondly, it specifies some origin on this circle. This allows to make sense of the numerical value θ_τ which at the level of the expression $\Xi_{\text{mf}}(z_0; \mathfrak{z}; \theta_\tau)$ is comparable to the number of turns performed around $\mathcal{C}(z_0; \mathfrak{z})$, departing from the position \mathfrak{z} . In this context, the geometrical interpretation of (1.2.22) is the following. The value of $\bar{Z}_0(z_0; \tau, \theta_\tau)$ is obtained by selecting the circle $\mathcal{C}(z_0; \langle \bar{\mathfrak{z}}_0 \rangle(z_0; \tau))$ and by carrying out on it a rotation which starts from the initial point $\langle \bar{\mathfrak{z}}_0 \rangle(z_0; \tau)$ and whose numerical value is θ_τ . Now, in view of (1.2.23), the position $\mathfrak{z} = \langle \bar{\mathfrak{z}}_0 \rangle(z_0; \tau)$ is in general different from z_0 as soon as $\tau \neq 0$. Thus, to obtain a representation formula of \bar{Z}_0 which is in line with (1.2.22), it does not suffice to work with $\mathfrak{z} = z_0$. This remark justifies the introduction at the level of (1.2.11) of the extra variable \mathfrak{z} . In practice, we have indeed to solve (1.2.11) with \mathfrak{z} other than z_0 .

Remark 1.2.14 (About the content of the leading order term). At the end, the main limit behavior

$$z(\varepsilon, z_0, \mathbf{v}_0; \tau) = \Xi_{\text{mf}}\left(z_0; \langle \bar{\mathfrak{z}}_0 \rangle(z_0; \tau); \frac{\tau}{\varepsilon}\right) + \mathcal{O}(\varepsilon) \quad (1.2.26)$$

is built with the help of $\Xi_{\text{mf}}(z_0; \cdot; \cdot)$ through a *current* time variation corresponding to the passage from $\mathfrak{z} = z_0$ to $\mathfrak{z} = \langle \overline{\mathfrak{z}}_0 \rangle(z_0; \tau)$, combined with a *quick* time variation (made of rotations) when θ_τ is replaced by τ/ε (with a number of turns that becomes very large when $\tau \in \mathbb{R}_+^*$ is fixed and ε tends to zero). At the level of (1.2.26), the rapid oscillations which are associated with the phase \mathbf{v}^f do not yet appear.

Remark 1.2.15 (About the structure of the whole asymptotic expansions). Look at the contribution which at the level of (1.2.20) has ε in factor. In general, we have $\partial_{\theta_\tau} Z_1 \neq 0$. This means that the *rapid* variations are activated with a *small amplitude* ε and corresponding frequency ε^{-2} . These rapid oscillations lead to many technical problems. Physically, when dealing with magnetized plasmas, they come from a fast gyromotion. At the end, the structure of z is made of the superposition of two regimes which are built with:

- Large amplitude oscillations. We find that $\partial_{\theta_\tau} Z_0 \neq 0$ when the mean flow is not constant. Then, there are quick variations (involving θ_τ) of amplitude 1;
- Strong oscillations. We find that $\partial_{\theta_\tau} Z_1 \neq 0$ when $A_0^* \neq 0$. Then, if we adopt the terminology of [CGM03] (with ε replaced by $\sqrt{\varepsilon}$), there are *rapid* variations (involving θ_τ) of amplitude ε .

Remark 1.2.16 (About the imbrication between the averaging procedures). The extraction of (1.2.17) and (1.2.18) results from two averaging procedures. The first (in θ_τ) is revealed by (1.2.12); the second (in θ_τ) occurs along the circles generated by Ξ_{mf} . There is no evident order between these highly interconnected operations. In the blow-up Section 1.3, priority is given to θ_τ . But, in the WKB Section 1.4, the integration is first in θ_τ and then in θ_τ , as when passing from (1.2.4) to (1.2.6).

Remark 1.2.17 (About the origin of Theorem 1.1.5). The Hamilton-Jacobi equations will be solved by the method of characteristics. When doing this, the expansion (1.1.10) appears through a *composition* of the oscillations involved by (1.2.17) or (1.2.18), roughly speaking by replacing (z_0, \mathbf{v}_0) inside (1.2.18) by $(z_0, \mathbf{v}_0)(\varepsilon, x^{-1})$ where x^{-1} (Lemmas 1.6.9 and 1.6.10) is the inverse of the spatial characteristic x (Lemma 1.6.5). This is why it is very important to keep track of the dependence on the initial data (z_0, \mathbf{v}_0) at the level of (1.2.18). This also explains how the complexity of the oscillating structures may increase. The chain rule indicates that the number of scales could become larger. In Subsections 1.6.3 and 1.6.4, we will show that this number does indeed increase.

1.2.3 Plan and motivations

The plan of the paper is as follows.

- In Section [1.3](#), we introduce a *lifting* (or *blow-up*) *procedure* which may be regarded as an adapted kind of the (more classical) *normal form* procedure. The purpose is to remove from the right hand side of [\(1.1.12\)](#) as much non-significant singular terms as possible. The idea is to absorb some artificial oscillations by changing the unknowns. This is done by (the inverse of) a nonlinear oscillating transformation. As a corollary, in Subsection [1.3.2.3](#), we can already prove Theorem [1.2.5](#).
 - In Section [1.4](#), we develop a three-scale WKB analysis involving the exact phase \mathfrak{v} . We work at the level of the profile equations [\(1.4.3\)](#) which allow to obtain rid of \mathfrak{v} . The idea is to seek approximate solutions ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ of [\(1.4.3\)](#) in the form of expansions in powers of ε , like in [\(1.4.11\)](#) or [\(1.4.12\)](#). Formal computations lead to a hierarchy (indexed by $j \in \{-1\} \cup \mathbb{N}$) of well-posed equations which are highly interconnected and which allow to determine successively all the profiles \mathfrak{Z}_j and \mathcal{V}_j that constitute \mathfrak{Z}^a and \mathcal{V}^a . In particular, in Paragraph [1.4.3.1](#), we identify the leading profile Z_0 which can be described as in Theorem [1.2.10](#).
 - In Section [1.5](#), we justify the interest of the preceding procedure (the formal calculus) by showing that the approximate solutions ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ lead indeed to exact solutions of the system [\(1.1.12\)](#). This means passing from the \mathfrak{Z}_j and \mathcal{V}_j of [\(1.4.11\)](#) and [\(1.4.12\)](#) to the Z_j and \mathcal{V}_j of [\(1.2.17\)](#) and [\(1.2.18\)](#). This implies the freezing of the phase \mathfrak{v} into \mathfrak{v}^f (through the implicit function theorem, see Subsection [1.5.2](#)) as well as a coming back to the original field (Subsection [1.5.3](#)). At the end, this yields the proof of Theorem [1.2.7](#).
 - In Section [1.6](#), we implement our analysis to construct classical solutions for the Cauchy problem associated with a class of oscillating Hamilton-Jacobi equations. Of course, weak solutions may exist [\[CL83\]](#). But the vanishing viscosity method does not furnish a precise description of their oscillating structures. By contrast, the method of characteristics does apply and it makes such accurate information available. Still, to this end, we need to implement some specific nontrivial arguments. Indeed, the differential of x is apparently highly singular which indicates that supercritical phenomena are achieved. However, this can be overcome through the Hadamard's global inverse function theorem by exploiting transparency conditions (Paragraph [1.6.3.2](#)) emanating from Assumption [1.2.4](#) (or [1.1.3](#)). At the end, we achieve the proof of Theorem [1.1.5](#) in Subsection [1.6.5](#).
- Hamilton-Jacobi equations like [\(1.1.5\)](#), implementing a small parameter $\varepsilon \rightarrow 0$, can ap-

pear in many situations which inspire our interest in this topic, like: homogenization theory [AF09b, CLL10, CDI01, Fir17, FF120, GIM15, MV10] where they are applied to traffic flows, light propagation and optics [M09, Sch94], plasma physics [Che17, CF23a], in the presence of rough domains [GVLNR18], and so on. In these references, the above different multiscale aspects are often discussed separately and partially. We provide here an extensive overview and we investigate new facets. Most importantly, we achieve a better comprehension of the nonlinear specificities induced by the influence of the oscillating term u/ε inside \mathbf{H} . Such aspects have already been raised (for instance in [IM08]) but without going as far as we do.

The present approach is also motivated by the need for an accurate long time ($\tau \sim 1$) description of the dynamics of charged particles in strongly magnetized plasmas. Recall that the gyrokinetic equations [Bos07] deal with systems of the type (1.1.12) during quick times, for $\tau \sim \varepsilon$ or $s \sim 1$. In fact, the mean flow can be related to the guiding center motion, while Theorem 1.2.7 goes far beyond this. It significantly enhances the information content of standard ray tracing methods [Bri13] by justifying asymptotic descriptions which prevail over longer times (namely during *current times* $\tau \sim 1$) and which are valid with any order of precision (expressed in powers of ε). Given the potential implications, there is a very abundant literature (both in physics and mathematics [BS94, Bos07, Bri13]) related to this subject. Usual approaches are however limited because they do not capture the imbrication of oscillations revealed by (1.2.17) and (1.2.18). The introduction of the preceding three-scale framework is necessary to progress. Historically, the structure of (1.1.12) is already implicit in [GSR05], and it becomes more visible in the two following contributions [Che15, Che17] where it is studied in the purely magnetic case. The formulation and assumptions retained here are much more general, and they are designed to take into account the (potentially disruptive) influence of electric fields. But this requires a long preparatory work and a number of specific considerations. This is why this important aspect is developed in the separate contribution [CF23a] with in perspective an analysis of the dynamical confinement properties inside fusion devices during long times.

1.3 Blow-up procedure

The main goal of this section is to obtain rid of the irrelevant oscillations which are put in factor of the large weight ε^{-1} in the source term of the system (1.1.12). As stated in Subsection 1.3.1, at the level of Proposition 1.3.2, we can exchange (1.1.12) with (1.3.3).

In this procedure, the general form of the equations is not modified but (A, V) is replaced by (A, \mathbf{V}) , with A satisfying the simplified condition (1.3.4). When doing this, the crucial tool is a change of variables involving a map Ξ . The general structure of Ξ is specified in Subsection 1.3.2. The transformation of (1.1.12) under the blow-up procedure is detailed in Subsection 1.3.3, where it is explained how A and V can be deduced from A and \mathbf{V} . By this way, it becomes possible to exhibit necessary and sufficient conditions on Ξ leading to (1.3.4). These conditions are the gateway to Assumption 1.2.4. The proof of Proposition 1.3.2 is achieved at the end, in Subsection 1.3.4.

Remark 1.3.1. As commented in Remark 1.2.1, we can always start with some $\theta_\tau \in \mathbb{T}_{\tau, z_0} := \mathbb{R}/(T_\tau(z_0)\mathbb{Z})$. It bears noting that, in such a case, Assumption 1.2.4 implies that the function $\Xi_{\text{mf}}(z_0, \mathfrak{z}, \cdot)$ issued from (1.2.11) should be periodic in s of period $T_\tau(z_0)$ (instead of period 2π).

1.3.1 Desingularization method

The aim of this subsection is to replace the original field z by some auxiliary field \mathfrak{z} . In practice, the new unknown \mathfrak{z} cannot be directly expressed in terms of z . Instead, it is revealed after a *blow-up* procedure on z . The term *blow-up* must be understood here in the sense of *lifting*. More precisely, given $z(\varepsilon, z_0, \mathbf{v}_0; \cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$ and

$$\Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) \in C^\infty([0, \varepsilon_0] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{r, z_0}; \mathbb{R}^n), \quad (1.3.1)$$

a lifting ℓ of z to $\mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{r, z_0}$ is an application $\ell(\varepsilon, z_0, \mathbf{v}_0; \cdot) : \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{r, z_0}$

$$\ell(\varepsilon, z_0, \mathbf{v}_0; \tau) = \left(\mathfrak{z}(\tau), \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right)$$

leading to the commutative diagram

$$\begin{array}{ccc} & \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{r, z_0} & \\ \ell(\varepsilon, z_0, \mathbf{v}_0; \cdot) \nearrow & & \searrow \Xi(\varepsilon, z_0; \cdot) \\ \mathbb{R} & \xrightarrow{z} & \mathbb{R}^n \end{array}$$

or equivalently to the formula

$$z(\tau) = \Xi\left(\varepsilon, z_0; \mathfrak{z}(\tau); \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon}\right). \quad (1.3.2)$$

In this process, the function \mathbf{v} is viewed as an input. The transformation (1.3.2) is driven by \mathbf{v} . The correspondence through (1.3.2) between \mathfrak{z} and z (and conversely) makes sense only on condition that \mathbf{v} is identified and, for the moment, it is supposed to be the local solution of (1.1.12). Knowing this, the key tool is the map $\Xi(\cdot)$ which must be adjusted first. Then, we can pass from \mathfrak{z} to z by following the two arrows at the top of the preceding diagram, or equivalently by using (1.3.2).

The interest of a lifting is to put (a part of) the singularities (of z) aside by raising the number of variables. Here, oscillations are put in factor inside $\Xi(\cdot)$, at the level of the periodic variables θ_τ and θ_r . Note again that the whole procedure is very sensitive to the choice of both \mathbf{v} and Ξ . The purpose is to adjust \mathbf{v} and Ξ in such a way that \mathfrak{z} solves a system of ODEs which is inherited from (1.1.12) but which is less complicated than (1.1.12). In practice, this should manifest as a simplification of the source term A .

Proposition 1.3.2 (Desingularization). *Under Assumption 1.2.4, there exists a map $\Xi(\cdot)$ allowing to convert (1.1.12) through (1.3.2) into the following redressed system*

$$\partial_\tau \begin{pmatrix} \mathfrak{z} \\ \mathbf{v} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} A \\ V \end{pmatrix} \left(\varepsilon, z_0; \mathfrak{z}; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right), \quad \begin{pmatrix} \mathfrak{z} \\ \mathbf{v} \end{pmatrix} (0) = \begin{pmatrix} \mathfrak{z}_0 \\ \mathbf{v}_0 \end{pmatrix} \quad (1.3.3)$$

which takes the same form (1.1.12), with new expressions A and V satisfying (for all $N \in \mathbb{N}$)

$$C^\infty([0, \varepsilon_0] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{r, z_0}; \mathbb{R}^n) \ni A(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) = \sum_{j=0}^N \varepsilon^j A_j(z_0; \mathfrak{z}; \theta_\tau, \theta_r) + \mathcal{O}(\varepsilon^{N+1}),$$

$$C^\infty([0, \varepsilon_0] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{r, z_0}; \mathbb{R}_+^*) \ni V(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) = \sum_{j=0}^N \varepsilon^j V_j(z_0; \mathfrak{z}; \theta_\tau, \theta_r) + \mathcal{O}(\varepsilon^{N+1}),$$

but which now involves a first component A that is subject to the crucial property

$$A_0(z_0; \mathfrak{z}; \cdot) = A(0, z_0; \mathfrak{z}; \cdot) \equiv 0. \quad (1.3.4)$$

Recall that A and V are likely distinct from A and V . These functions are defined up to $\varepsilon = 0$, and they are smooth near $\varepsilon = 0$. As well as for A and V from which they are issued, they do not involve \mathbf{v} , but only \mathfrak{z} . This nonlinearity will be sometimes marked by the notations $A(\mathfrak{z})$ and $V(\mathfrak{z})$, which focus on the dependence on \mathfrak{z} and simply dismiss the role of $(\varepsilon, z_0, \theta_\tau, \theta_r)$.

The proof of Proposition [1.3.2](#) is postponed to Subsection [1.3.4](#).

Corollary 1.3.3 (Uniform time of existence for [\(1.3.3\)](#)). *The lifespan $\mathcal{T}(\varepsilon, \mathfrak{z}_0, \mathbf{v}_0)$ that is associated with [\(1.3.3\)](#) is uniformly (in ε when ε goes to 0) bounded below by some $\mathcal{T}(\mathfrak{z}_0) \in \mathbb{R}_+^*$.*

In other words, we have [\(1.2.14\)](#) with z_0 replaced by \mathfrak{z}_0 .

Proof. Taking into account [\(1.3.4\)](#), the source term $\varepsilon^{-1}A$ in front of $\partial_\tau \mathfrak{z}$ at the level of [\(1.3.3\)](#) is of size $A_1 + O(\varepsilon) = O(1)$ instead of being of size $O(\varepsilon^{-1})$. As long as \mathfrak{z} remains in a compact set, say $\mathfrak{z} \in B(0, r]$ where r is adjusted in such a way that $\mathfrak{z}_0 \in B(0, r/2]$, the two expressions $\varepsilon^{-1}A$ and V are bounded uniformly with respect to ε , \mathfrak{z} and τ by

$$M := \sup_{\varepsilon \in]0, \varepsilon_0]} \sup_{\mathfrak{z} \in B(0, r)} \sup_{\theta_\tau \in \mathbb{T}} \sup_{\theta_r \in \mathbb{T}_{r, z_0}} \max \left(\varepsilon^{-1} |A(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r)|; |V(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r)| \right) < +\infty.$$

Moreover, they are Lipschitz continuous with respect to \mathfrak{z} and θ_r . By Cauchy-Lipschitz theorem, there exists a unique local solution ${}^t(\mathfrak{z}, \mathbf{v})$ to [\(1.3.3\)](#). Since A and V are periodic with respect to θ_r , the explosion, if any, can only occur at the level of the component \mathfrak{z} . Let $\bar{\tau}(\varepsilon) \in \mathbb{R}_+^*$ be such that

$$\bar{\tau}(\varepsilon) := \sup \left\{ \tilde{\tau} \in [0, \mathcal{T}(\varepsilon, \mathfrak{z}_0, \mathbf{v}_0)]; |\mathfrak{z}(\tau)| \leq r, \forall \tau \in [0, \tilde{\tau}] \right\}.$$

By construction, we have $\bar{\tau}(\varepsilon) < \mathcal{T}(\varepsilon, \mathfrak{z}_0, \mathbf{v}_0)$. Thus, if $\bar{\tau}(\varepsilon) = +\infty$, there is nothing to prove. Now, assume that $\bar{\tau}(\varepsilon) < +\infty$. Then we should have

$$|\mathfrak{z}(\bar{\tau}(\varepsilon))| = r. \tag{1.3.5}$$

On the other hand, by the mean value theorem, the solution is such that

$$\begin{aligned} \forall \tau \in [0, \bar{\tau}(\varepsilon)], \quad |\mathfrak{z}(\tau)| &\leq |\mathfrak{z}(\tau) - \mathfrak{z}_0| + |\mathfrak{z}_0| \leq M\tau + r/2, \\ \forall \tau \in [0, \bar{\tau}(\varepsilon)], \quad |\mathbf{v}(\tau)| &\leq |\mathbf{v}(\tau) - \mathbf{v}_0| + |\mathbf{v}_0| \leq \varepsilon^{-1}M\tau + |\mathbf{v}_0|. \end{aligned}$$

It follows that for all $\tau \leq \min(\bar{\tau}(\varepsilon); r/4M)$,

$$|\mathfrak{z}(\tau)| \leq M\tau + r/2 \leq 3r/4 < r.$$

If $\bar{\tau}(\varepsilon) \leq r/4M$, we obtain $|\mathfrak{z}(\tau)| < r$ for all $\tau \in [0, \bar{\tau}(\varepsilon)]$. This is a contradiction with (1.3.5). Thus, we have $\bar{\tau}(\varepsilon) > r/4M$. Hence, the lifespan $\mathcal{T}(\varepsilon, \mathfrak{z}_0, \mathbf{v}_0)$ of the solution to (1.3.3) is uniformly (in ε) bounded below by $\mathcal{T}(\mathfrak{z}_0) := r/4M$. The scenario of Example 1.2.3 is avoided. \square

1.3.2 General structure of the lifting

The map Ξ is built as a small perturbation (of size ε) of some map Ξ_0 . More precisely

$$\Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) = \Xi_0(z_0; \mathfrak{z}; \theta_\tau) + \varepsilon \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r), \quad (1.3.6)$$

with

$$\Xi_0 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}; \mathbb{R}^n), \quad (1.3.7)$$

$$\Xi_1 \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T} \times \mathbb{T}_{r, z_0}; \mathbb{R}^n). \quad (1.3.8)$$

Observe that the form of Ξ inside (1.3.1) and (1.3.6) is the same as the one of \mathbf{A} and \mathbf{V} . For illustration purposes and to assist the reader in the understanding of the text, we will explain through a series of remarks what happens in the case of a standard *normal form* procedure. We start below by recalling what is meant by this.

Remark 1.3.4 (Normal form procedure: definition). This is when

$\Xi_0(z_0; \mathfrak{z}; \theta_\tau) \equiv \Xi_{0nf}(z_0; \mathfrak{z}; \theta_\tau) := \mathfrak{z}$ and when moreover Ξ_1 does not depend on θ_τ , that is when

$$\Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) \equiv \Xi_{nf}(\varepsilon, z_0; \mathfrak{z}; \theta_r) := \mathfrak{z} + \varepsilon \Xi_{1nf}(z_0; \mathfrak{z}; \theta_r). \quad (1.3.9)$$

In fact, the change (1.3.9) is just a small perturbation of the identity map. The formula (1.3.9) may suffice during quick times but certainly not (always) during longer times, see Remark 1.3.15.

When $\varepsilon = 0$ inside (1.3.6), we recover $\Xi_0(z_0; \mathfrak{z}; \theta_\tau)$. In the general case, as will be seen, we need some freedom on Ξ_0 in order to absorb singular terms. Thus, we do not take $\Xi_0 \equiv \Xi_{0nf}$. Extra admissible functions Ξ_0 are presented in Paragraph 1.3.2.1, while basic properties of Ξ are detailed in Paragraph 1.3.2.2. At the end, in Paragraph 1.3.2.3, we explain how to pass from the initial data z_0 to \mathfrak{z}_0 . We also describe how to go from the field z to its corresponding lifting \mathfrak{z} , that is how to obtain the inverse function of (1.3.2).

1.3.2.1 Admissible functions Ξ_0

From now on, we suppose that $\Xi_0(z_0; \cdot; \theta_\tau) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ generates a one-to-one correspondence. Under Assumption [1.2.4](#), as a consequence of Lemma [1.3.10](#), this condition will be verified for the choice $\Xi_0 = \Xi_{\text{mf}}$ that we have in mind.

Condition 1.3.5 (Ξ_0 is a global smooth diffeomorphism). The expression Ξ_0 is subject to [\(1.3.7\)](#). Moreover, for all $(z_0, \theta_\tau) \in \mathbb{R}^n \times \mathbb{T}$, the function $\Xi_0(z_0; \cdot; \theta_\tau) : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

$$\mathfrak{z} \mapsto \Xi_0(z_0; \mathfrak{z}; \theta_\tau) \tag{1.3.10}$$

is a diffeomorphism from \mathbb{R}^n onto \mathbb{R}^n . The corresponding inverse is denoted by $\Xi_0^{-1}(z_0; \cdot; \theta_\tau)$. It is a smooth function of (z_0, z, θ_τ) on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$.

1.3.2.2 Properties of Ξ

We look here more closely at the characteristics of the map Ξ , viewed as a perturbation of Ξ_0 .

Next we have a family of diffeomorphisms indexed by $(\varepsilon, z_0, \tau, \theta_\tau, \theta_r) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{T} \times \mathbb{T}_{r, z_0}$.

Lemma 1.3.6. *Select two functions Ξ_0 and Ξ_1 satisfying respectively [\(1.3.7\)](#) and [\(1.3.8\)](#). Assume Condition [1.3.5](#) and define Ξ as it is indicated in [\(1.3.6\)](#). Fix a compact set $K_0 \subset \mathbb{R}^n$ and a positive real number $R \in \mathbb{R}_+^*$. Then, we can find $\varepsilon_0 \in]0, \varepsilon_0]$ such that, for all $(\varepsilon, z_0, \theta_\tau, \theta_r) \in [0, \varepsilon_0] \times K_0 \times \mathbb{T} \times \mathbb{T}_{r, z_0}$, the map $\Xi(\varepsilon, z_0; \cdot; \theta_\tau, \theta_r) : B(0, R] \rightarrow \mathbb{R}^n$,*

$$\mathfrak{z} \mapsto \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) \tag{1.3.11}$$

is a diffeomorphism from $B(0, R]$ onto its image

$$K \equiv K(\varepsilon, z_0, R, \theta_\tau, \theta_r) := \left\{ \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r); \mathfrak{z} \in B(0, R] \right\}.$$

The corresponding inverse is denoted by $\Xi^{-1}(\varepsilon, z_0; \cdot; \theta_\tau, \theta_r)$. It is a smooth function of $(\varepsilon, z_0, z, \theta_\tau, \theta_r)$ chosen in $[0, \varepsilon_0] \times K_0 \times K \times \mathbb{T} \times \mathbb{T}_{r, z_0}$ which can be expanded in powers of ε according to

$$\Xi^{-1}(z_0; \mathfrak{z}, \theta_\tau, \theta_r) = \Xi_0^{-1}(z_0; \mathfrak{z}, \theta_\tau) + \sum_{j=1}^{+\infty} \varepsilon^j \Xi_j^{-1}(z_0; \mathfrak{z}, \theta_\tau, \theta_r) \tag{1.3.12}$$

where Ξ_0^{-1} is the inverse of the map $\mathfrak{z} \mapsto \Xi_0(z_0; \mathfrak{z}, \theta_\tau)$ and where Ξ_1^{-1} is given by

$$\Xi_1^{-1}(z_0; \mathfrak{z}, \theta_\tau, \theta_r) := -D_{\mathfrak{z}}\Xi_0(z_0; \mathfrak{z}, \theta_\tau)^{-1}\Xi_1(z_0; \Xi_0^{-1}(z_0; \mathfrak{z}, \theta_\tau), \theta_\tau, \theta_r). \quad (1.3.13)$$

Moreover, given K_0 , by adjusting R large enough and ε_0 small enough, we can always ensure that

$$\forall(\varepsilon, z_0, \theta_r) \in [0, \varepsilon_0] \times K_0 \times \mathbb{T}_{r, z_0}, \quad K_0 \subset \Xi(\varepsilon, z_0; B(0, R]; 0, \theta_r). \quad (1.3.14)$$

Proof. Consider the auxiliary map

$$B(0, R] \ni \mathfrak{z} \mapsto \Xi_0^{-1}(z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r); \theta_\tau). \quad (1.3.15)$$

Applying (1.3.6) and (1.3.7), the mean value theorem (in several variables) guarantees that

$$\Xi_0^{-1}(z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r); \theta_\tau) = \mathfrak{z} + O(\varepsilon).$$

More precisely, this is a modification of the identity map $Id_{B(0, R]}$ which is of size $O(\varepsilon)$ in terms of the C^1 -norm on $B(0, R]$. Moreover, by compactness, this holds true uniformly with respect to $z_0 \in K_0$, $\theta_\tau \in \mathbb{T}$ and $\theta_r \in \mathbb{T}_{r, z_0}$. Since the set of C^1 -diffeomorphisms on $B(0, R]$ is open, by restricting ε_0 if necessary, the map inside (1.3.15) is sure to be a diffeomorphism for all values of $(\varepsilon, z_0, \theta_\tau, \theta_r)$ in $[0, \varepsilon_0] \times K_0 \times \mathbb{T} \times \mathbb{T}_{r, z_0}$. Composing (1.3.15) by Ξ_0 on the left, we recover as expected that $\Xi(\varepsilon, z_0; \cdot; \theta_\tau, \theta_r)$ is a diffeomorphism from $B(0, R]$ onto its image. Now, the set

$$\left\{ \Xi_0^{-1}(z_0; z_0; 0); z_0 \in K_0 \right\}$$

is compact as the image of $K_0 \times K_0$ by the continuous function $\Xi_0^{-1}(\cdot; 0)$. Thus, for R sufficiently large, it can be included in a ball of radius R . And thereby, we have

$$\forall z_0 \in K_0, \quad z_0 = \Xi_0(z_0; \Xi_0^{-1}(z_0; z_0; 0); 0) \in \Xi_0(z_0; B(0, R]; 0).$$

This inclusion is none other than (1.3.14) when $\varepsilon = 0$. The general case follows by compactness and perturbative arguments, by restricting $\varepsilon_0 \in]0, \varepsilon_0]$ again if necessary. Moreover, by the definitions of Ξ and then Ξ^{-1} , we must have

$$\Xi(z_0; \Xi^{-1}, \theta_\tau, \theta_r) = \Xi_0(z_0; \Xi^{-1}, \theta_\tau) + \varepsilon \Xi_1(z_0; \Xi^{-1}, \theta_\tau, \theta_r) = \mathfrak{z}. \quad (1.3.16)$$

We can seek Ξ^{-1} in the form of the asymptotic expansion (1.3.12). Then, we can exploit the formal expansion of (1.3.16) in powers of ε to successively determine the Ξ_j^{-1} with $j \geq 0$. By this way, we can extract Ξ_0^{-1} (term with ε^0 in factor) and Ξ_1^{-1} (term with ε in factor) as indicated. The inverse function theorem allows to justify this calculus. \square

1.3.2.3 Passage from the original field z to the lifting \mathfrak{z}

In practice, we fix the compact K_0 , and we consider a collection of initial conditions $z_0 \in K_0$. Then, we adjust R and ε_0 to obtain (1.3.14). The property (1.3.14) is essential to guarantee that all positions in K_0 has a unique preimage inside $B(0, R]$. More precisely, for all $(\varepsilon, \mathbf{v}_0) \in [0, \varepsilon_0] \times \mathbb{R}$, we can now assert that

$$\exists! \mathfrak{z}_0 \equiv \mathfrak{z}_0(\varepsilon, z_0, \mathbf{v}_0) \in B(0, R]; \quad z_0 = \Xi\left(\varepsilon, z_0; \mathfrak{z}_0; 0, \frac{\mathbf{v}_0}{\varepsilon}\right) \quad (1.3.17)$$

or equivalently

$$\begin{aligned} \mathfrak{z}_0 &\equiv \mathfrak{z}_0(\varepsilon, z_0, \mathbf{v}_0) = \Xi^{-1}\left(\varepsilon, z_0; z_0; 0, \frac{\mathbf{v}_0}{\varepsilon}\right) \\ &= \Xi_0^{-1}(z_0; z_0; 0) + \varepsilon \Xi_1^{-1}(z_0; z_0; 0, \frac{\mathbf{v}_0}{\varepsilon}) + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (1.3.18)$$

Note that \mathfrak{z}_0 does depend on ε (even if z_0 does not) and it does oscillate in ε due to the presence of \mathbf{v}_0/ε . The change from z_0 to \mathfrak{z}_0 through Ξ^{-1} introduces high frequencies since in general $\partial_{\theta_r} \Xi_1^{-1} \neq 0$. However, the positions \mathfrak{z}_0 remain uniformly in ε in a compact neighborhood of

$$\left\{ \Xi_0^{-1}(z_0; z_0; 0); z_0 \in K_0 \right\} \subset B(0, R].$$

By the way of (1.3.18), all the initial data z_0 contained in K_0 can be converted into corresponding initial data $\mathfrak{z}_0 \in B(0, R]$ for the forthcoming system of ODEs on \mathfrak{z} . Now, let K be a compact set containing K_0 in its interior ($K_0 \subset \overset{\circ}{K} \subset K \Subset \mathbb{R}^n$). By applying Lemma 1.3.6 with K_0 replaced by K , we can guarantee that

$$\forall (\varepsilon, z, \theta_r) \in [0, \tilde{\varepsilon}_0] \times K \times \mathbb{T}_{r, z_0}, \quad K \subset \Xi(\varepsilon, z; B(0, \tilde{R}); 0, \theta_r), \quad (1.3.19)$$

for some $\tilde{\varepsilon}_0 \leq \varepsilon_0$ and $R \leq \tilde{R}$. By continuity, a solution $z(\varepsilon, z_0, \mathbf{v}_0; \cdot)$ of (1.1.12) issued from $z_0 \in K_0$ will remain in K for sufficiently small values of τ . Thus, at least locally in time, on some open time interval which may be not uniform with respect to $\varepsilon \in]0, \varepsilon_0]$, we can

define

$$\mathfrak{z}(\varepsilon, z_0, \mathbf{v}_0; \tau) := \Xi^{-1}\left(\varepsilon, z_0; z(\varepsilon, z_0, \mathbf{v}_0; \tau); \theta_\tau, \frac{\mathbf{v}(\varepsilon, z_0, \mathbf{v}_0; \tau)}{\varepsilon}\right) \in B(0, \tilde{R}). \quad (1.3.20)$$

At this stage, we have collected enough information to show Theorem [1.2.5](#).

Proof of Theorem [1.2.5](#). We can pass from the local in time solution z of [\(1.1.12\)](#) to some associated field \mathfrak{z} through [\(1.3.20\)](#), with inverse formula [\(1.3.2\)](#). Now, Proposition [1.3.2](#) indicates that \mathfrak{z} can be characterized by the equation [\(1.3.3\)](#), and therefore that it can be determined through [\(1.3.3\)](#) independently from [\(1.1.12\)](#). This means that any local solution to [\(1.1.12\)](#) gives rise to a local solution to [\(1.3.3\)](#), and conversely. Since by Corollary [1.3.3](#), the lifespan associated with [\(1.3.3\)](#) is uniformly bounded below (with some stability in the sup norm), the same applies concerning [\(1.1.12\)](#). \square

Now, the challenge is to derive the equation [\(1.3.3\)](#) on \mathfrak{z} , allowing to identify \mathfrak{z} .

1.3.3 Transformation of the equations

Given a map Ξ as in Subsection [1.3.2](#), the matter here is to show that the system [\(1.1.12\)](#) is transformed under the blow-up procedure into the system [\(1.3.3\)](#). It is also to determine how the new source terms A and V inside [\(1.3.3\)](#) can be deduced from the original A and V of [\(1.1.12\)](#).

Lemma 1.3.7 (Identification of the new source terms A and V from A , V and Ξ).

Let ${}^t(z, \mathbf{v})(\varepsilon, z_0, \mathbf{v}_0; \cdot)$ be a local solution of [\(1.1.12\)](#). Assume Condition [1.3.5](#). Define \mathfrak{z}_0 through [\(1.3.17\)](#) and $\mathfrak{z}(\varepsilon, z_0, \mathbf{v}_0; \cdot)$ locally in time through [\(1.3.20\)](#). Then, the field ${}^t(\mathfrak{z}, \mathbf{v})$ is the unique (local) solution of [\(1.3.3\)](#), with A and V determined as indicated below:

$$A(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) := D_{\mathfrak{z}}\Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r)^{-1} \left\{ A(\varepsilon, z_0; \Xi(\cdot); \theta_\tau, \theta_r) - \partial_{\theta_\tau}\Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) - \varepsilon^{-1}V(\varepsilon, z_0; \Xi(\cdot); \theta_\tau, \theta_r)\partial_{\theta_r}\Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) \right\}, \quad (1.3.21)$$

$$V(\varepsilon; \mathfrak{z}; \theta_\tau, \theta_r) := V(\varepsilon, z_0; \Xi(\cdot); \theta_\tau, \theta_r), \quad (1.3.22)$$

where the point \cdot must be replaced by $(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r)$.

Proof. Recall that the component \mathbf{v} remains unchanged under the blow-up procedure. It is the solution to (1.1.12). In view of (1.3.2), this forces us to define \mathbf{V} as in (1.3.22). Incidentally, this means that the amplitude is maintained when transferring from \mathbf{V} to \mathbf{V} . There remains to prove that \mathfrak{z} is a solution to the first line of (1.3.3) with \mathbf{A} adjusted as in (1.3.21). To this end, combine (1.3.2) with the first equation of (1.1.12) to see that we have to guarantee that

$$\partial_\tau \left\{ \Xi(\varepsilon, z_0; \mathfrak{z}(\tau); \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon}) \right\} = \frac{1}{\varepsilon} \mathbf{A}(\varepsilon, z_0; z; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon}).$$

In other words, exploiting again (1.1.12), we must impose

$$(D_3 \Xi \mathbf{A} + \partial_{\theta_r} \Xi + \varepsilon^{-1} \mathbf{V} \partial_{\theta_r} \Xi)(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) = \mathbf{A}(\varepsilon, z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r); \theta_\tau, \theta_r). \quad (1.3.23)$$

By inverting the matrix $D_3 \Xi$, we recover exactly (1.3.21). \square

1.3.4 Proof of Proposition 1.3.2

In what follows, we need to invert the derivative ∂_{θ_i} for $i \in \{\tau, r\}$. To this end, define $\mathbb{T}_T := \mathbb{R}/(T\mathbb{Z})$ and introduce the sets $L_*^1(\mathbb{T}_T)$ made of periodic functions with zero mean, namely

$$L_*^1(\mathbb{T}_T) := \left\{ \mathcal{Z} \in L^1(\mathbb{T}_T); \int_0^T \mathcal{Z}(\theta) d\theta = 0 \right\};$$

We can define the operators $\partial_\theta^{-1} : L_*^1(\mathbb{T}_T) \rightarrow L_*^1(\mathbb{T}_T)$ according to

$$\partial_\theta^{-1} \mathcal{Z}(\theta) := \int_0^\theta \mathcal{Z}(r) dr - \frac{1}{T} \int_0^T \left(\int_0^s \mathcal{Z}(r) dr \right) ds. \quad (1.3.24)$$

Recall that

$$\forall \mathcal{Z} \in L^1(\mathbb{T}_T), \quad \partial_\theta^{-1} \partial_\theta \mathcal{Z} = \mathcal{Z}^* := \mathcal{Z}(\theta) - \frac{1}{T} \int_0^T \mathcal{Z}(r) dr, \quad (1.3.25a)$$

$$\forall \mathcal{Z} \in L_*^1(\mathbb{T}_T), \quad \partial_\theta \partial_\theta^{-1} \mathcal{Z} = \mathcal{Z}. \quad (1.3.25b)$$

Then, we apply the above arguments to define the inverse of the operators ∂_{θ_i} for $i \in \{\tau, r\}$. We seek conditions on \mathbf{A} and \mathbf{V} allowing to obtain rid of the problematic term of size ε^{-1} which may appear when looking at the first line of (1.3.3). This requires to separate inside (1.3.21) the leading term from the terms with ε in factor. We work at the level of (1.3.23). Assuming as expected that $\mathbf{A}_0 \equiv 0$, with Ξ as in (1.3.6) so that $\partial_{\theta_r} \Xi = \varepsilon \partial_{\theta_r} \Xi_1$,

with \mathbf{A} as decomposed in the beginning, we can expand [\(1.3.23\)](#) according to

$$\begin{aligned} & \left[(D_{\mathfrak{z}}\Xi_0 + \varepsilon D_{\mathfrak{z}}\Xi_1) \left(\varepsilon \mathbf{A}_1 + \mathcal{O}(\varepsilon^2) \right) + \partial_{\theta_\tau} \Xi_0 + \varepsilon \partial_{\theta_\tau} \Xi_1 + \mathbf{V} \partial_{\theta_r} \Xi_1 \right] (\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) \\ & = \mathbf{A} \left(z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r); \theta_\tau, \theta_r \right). \end{aligned}$$

On the one hand, from [\(1.3.22\)](#), we have

$$\begin{aligned} \mathbf{V} & = \mathbf{V}(\varepsilon, z_0; \Xi_0 + \varepsilon \Xi_1; \cdot) \\ & = \mathbf{V}_0(z_0; \Xi_0 + \varepsilon \Xi_1; \cdot) + \varepsilon \mathbf{V}_1(z_0; \Xi_0 + \varepsilon \Xi_1; \cdot) + \mathcal{O}(\varepsilon^2) \\ & = \mathbf{V}_0(z_0; \Xi_0; \cdot) + \varepsilon (\Xi_1 \cdot \nabla_z) \mathbf{V}_0(z_0; \Xi_0; \cdot) + \varepsilon \mathbf{V}_1(z_0; \Xi_0; \cdot) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \mathbf{A} \left(z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r); \cdot \right) \\ & = \left(\mathbf{A}_0 + \varepsilon \mathbf{A}_1 \right) \left(z_0; \Xi(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r); \cdot \right) + \mathcal{O}(\varepsilon^2) \\ & = \mathbf{A}_0(z_0; \Xi_0; \cdot) + \varepsilon (\Xi_1 \cdot \nabla_z) \mathbf{A}_0(z_0; \Xi_0; \cdot) + \varepsilon \mathbf{A}_1(z_0; \Xi_0; \cdot) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Thus, we have to deal with the condition

$$\begin{aligned} & \varepsilon D_{\mathfrak{z}} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) \mathbf{A}_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r) + \partial_{\theta_\tau} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) + \varepsilon \partial_{\theta_\tau} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r) \\ & + \left(\mathbf{V}_0(z_0; \Xi_0; \theta_\tau, \theta_r) + \varepsilon (\Xi_1 \cdot \nabla_z) \mathbf{V}_0(z_0; \Xi_0; \theta_\tau, \theta_r) \right) \partial_{\theta_r} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r) \\ & + \varepsilon \mathbf{V}_1(z_0; \Xi_0; \theta_\tau, \theta_r) \partial_{\theta_r} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r) \\ & = \mathbf{A}_0(z_0; \Xi_0; \theta_\tau, \theta_r) + \varepsilon (\Xi_1 \cdot \nabla_z) \mathbf{A}_0(z_0; \Xi_0; \theta_\tau, \theta_r) \\ & + \varepsilon \mathbf{A}_1(z_0; \Xi_0; \theta_\tau, \theta_r) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{1.3.26}$$

We bring together the terms with the same power of ε in factor to obtain

$$\begin{aligned} & \partial_{\theta_\tau} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) + \mathbf{V}_0(z_0; \Xi_0; \theta_\tau, \theta_r) \partial_{\theta_r} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r) - \mathbf{A}_0(z_0; \Xi_0; \theta_\tau, \theta_r) \\ & + \varepsilon \left[D_{\mathfrak{z}} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) \mathbf{A}_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r) + \partial_{\theta_\tau} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r) \right. \\ & + \left((\Xi_1 \cdot \nabla_z) \mathbf{V}_0(z_0; \Xi_0; \theta_\tau, \theta_r) + \mathbf{V}_1(z_0; \Xi_0; \theta_\tau, \theta_r) \right) \partial_{\theta_r} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r) \\ & \left. - (\Xi_1 \cdot \nabla_z) \mathbf{A}_0(z_0; \Xi_0; \theta_\tau, \theta_r) - \mathbf{A}_1(z_0; \Xi_0; \theta_\tau, \theta_r) \right] + \mathcal{O}(\varepsilon^2) = 0. \end{aligned} \tag{1.3.27}$$

The first line must be zero, which is the same as

$$\mathbf{V}_0(z_0; \Xi_0; \theta_\tau, \theta_r)^{-1} \partial_{\theta_\tau} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) + \partial_{\theta_r} \Xi_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r) = (\mathbf{V}_0^{-1} \mathbf{A}_0)(z_0; \Xi_0; \theta_\tau, \theta_r).$$

First, take the mean value with respect to θ_r in order to obtain rid of Ξ_1 and to identify $\partial_{\theta_\tau} \Xi_0$. Then, subtract the result thus obtained to deduce $\partial_{\theta_r} \Xi_1$. Following these lines, we can exhibit two separate conditions, namely

$$\partial_{\theta_\tau} \Xi_0 = \left((\overline{\mathbf{V}_0^{-1}})^{-1} \overline{\mathbf{V}_0^{-1} \mathbf{A}_0} \right) (z_0; \Xi_0; \theta_\tau), \quad (1.3.28)$$

$$\partial_{\theta_r} \Xi_1 = \left(\mathbf{V}_0^{-1} \mathbf{A}_0 - \mathbf{V}_0^{-1} (\overline{\mathbf{V}_0^{-1}})^{-1} \overline{\mathbf{V}_0^{-1} \mathbf{A}_0} \right) (z_0; \Xi_0; \theta_\tau, \theta_r). \quad (1.3.29)$$

Definition 1.3.8 (Homological equation). The nonlinear ordinary differential equation (1.3.28) in the variable θ_τ is called the homological equation (or sometimes the first modulation equation).

We can complete (1.3.28) with some initial data $\mathfrak{z} \in \mathbb{R}^n$ whose introduction has been motivated by Remark 1.2.13. Hence, the expression Ξ_0 becomes a function of z_0 , \mathfrak{z} and θ_τ , together with

$$\Xi_0(z_0; \mathfrak{z}; 0) = \mathfrak{z}. \quad (1.3.30)$$

Remark 1.3.9 (About the passage from z_0 to \mathfrak{z}_0). Condition (1.3.30) implies that $\Xi_0^{-1}(z_0; \mathfrak{z}; 0) = \mathfrak{z}$. Thus, when $\Xi_1 \equiv 0$, we find that $\Xi^{-1}(\varepsilon, z_0; \mathfrak{z}; 0, \theta_r) = \mathfrak{z}$. Then, in view of (1.3.18), we have

$$\mathfrak{z}_0 = \sum_{j=0}^{+\infty} \varepsilon^j \mathfrak{z}_{0j} = z_0 = \sum_{j=0}^{+\infty} \varepsilon^j z_{0j},$$

so that $\mathfrak{z}_{0j} = z_{0j}$ for all $j \in \mathbb{N}$.

Lemma 1.3.10 (Mean flow is the solution to the homological equation). *We have*

$$\forall s \in \mathbb{R}, \quad \Xi_{\text{mf}}(z_0; \mathfrak{z}; s) = \Xi_0(z_0; \mathfrak{z}; s). \quad (1.3.31)$$

As a consequence, under Assumption 1.2.4, the solution to the Cauchy problem (1.3.28)-(1.3.30) is global and it is periodic with respect to θ_τ of period 2π .

Proof. In view of Definition 1.2.2, the vector fields Ξ_{mf} and Ξ_0 are solutions to the same system of ODEs. By Cauchy-Lipschitz theorem, local solutions do exist and (by uniqueness) they must coincide. The content of Assumption 1.2.4 allows the conclusion. \square

Remark 1.3.11 (Impact of Assumption [1.2.4](#)). Lemma [1.3.10](#) makes the connection between the notion of *mean flow* (Definition [1.2.2](#) appearing in the introduction after heuristic considerations) and the map Ξ_0 (derived from formal computations). The role of Assumption [1.2.4](#) is clearly to furnish global solutions to [\(1.3.28\)](#)-[\(1.3.30\)](#). It is also essential to stay in the periodic framework.

Remark 1.3.12 (About the verification of Condition [1.3.5](#)). Since the map $\Xi_0(z_0; \mathfrak{z}; \cdot)$ can be viewed as a flow, Condition [1.3.5](#) is automatically verified with $\Xi_0^{-1}(z_0; \mathfrak{z}; \theta_\tau) = \Xi_0(z_0; \mathfrak{z}; -\theta_\tau)$.

Remark 1.3.13 (Some hidden constraint on Ξ_{mf} under Assumption [1.2.4](#)).

The right-hand side of [\(1.2.11\)](#) is periodic in s of period 2π . Thus, it can be decomposed like in [\(1.2.7\)](#) into its mean value and its quick oscillating part. It follows that the solution to [\(1.2.11\)](#) is the sum of a linear function plus a periodic function. The resulting expression may indeed be periodic only if

$$\left\langle \mathbf{A}_{\text{mf}}\left(z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; \cdot); \cdot\right) \right\rangle = 0, \quad (1.3.32)$$

which may appear as some *a posteriori* condition which must be satisfied by Ξ_{mf} .

By construction, the right-hand side of [\(1.3.29\)](#) is periodic with respect to the variable θ_r , and it is of mean zero. It can be integrated as indicated in [\(1.3.24\)](#). As a consequence, the part Ξ_1^* is completely determined from [\(1.3.29\)](#). We fix $\bar{\Xi}_1 \equiv 0$, so that

$$\begin{aligned} \Xi_1 &\equiv \Xi_1^*(z_0; \mathfrak{z}; \theta_\tau, \theta_r) \\ &:= \partial_{\theta_r}^{-1} \left(\frac{(\mathbf{A}_0 - (\mathbf{V}_0^{-1})^{-1} \overline{(\mathbf{V}_0^{-1} \mathbf{A}_0)})(z_0; \Xi_0(z_0; \mathfrak{z}; \theta_\tau); \theta_\tau, \theta_r)}{\mathbf{V}_0(z_0; \Xi_0(z_0; \mathfrak{z}; \theta_\tau); \theta_\tau, \theta_r)} \right). \end{aligned} \quad (1.3.33)$$

At this stage, we have exhibited necessary conditions on \mathbf{A}_0 , Ξ_0 , and $\Xi_1 \equiv \Xi_1^*$ to obtain $\mathbf{A}_0 \equiv 0$. We have now to show that these conditions are sufficient. By Lemma [1.3.6](#), the matrix $D_{\mathfrak{z}}\Xi$ is invertible for the data under consideration. Coming back to [\(1.3.23\)](#), we can therefore deduce the value of \mathbf{A} , with $\mathbf{A} = \varepsilon \mathbf{A}_1 + \mathcal{O}(\varepsilon^2)$. Looking at [\(1.3.27\)](#), we find

that

$$\begin{aligned}
 & A_1(z_0; \mathfrak{z}; \theta_\tau, \theta_r) \\
 &= D_{\mathfrak{z}} \Xi_0(z_0; \mathfrak{z}; \theta_\tau)^{-1} \left\{ A_1(z_0; \Xi_0; \theta_\tau, \theta_r) \right. \\
 &\quad + (\Xi_1 \cdot \nabla_z) A_0(z_0; \Xi_0; \theta_\tau, \theta_r) - \partial_{\theta_r} \Xi_1(z_0, \mathfrak{z}; \theta_\tau, \theta_r) \\
 &\quad \left. - \left[V_1(z_0; \Xi_0; \theta_\tau, \theta_r) + (\Xi_1 \cdot \nabla_z) V_0(z_0; \Xi_0; \theta_\tau, \theta_r) \right] \partial_{\theta_r} \Xi_1(z_0, \mathfrak{z}; \theta_\tau, \theta_r) \right\}.
 \end{aligned} \tag{1.3.34}$$

Remark 1.3.14 (Limited framework of the normal form procedure). The restriction on Ξ which is imposed at the level of (1.3.9) strongly reduces the class of systems (1.1.12) which can be managed by the blow up procedure. Indeed, it generates compatibility conditions on A :

- In view of (1.3.28), the selection of $\Xi_0(z_0; \mathfrak{z}; \theta_\tau) = \mathfrak{z}$ is coherent with the homological equation if and only if $A_{\text{mf}} \equiv 0$. This requires that $V_0^{-1} A_0$ has a zero mean (in θ_r), which is very restrictive. In particular, when A_0 do not depend on θ_r , this simply means that we start already with $A_0 \equiv 0$.
- In view of (1.3.29), the function Ξ_1 does in general depend on θ_τ when V_0 and A_0 do depend on θ_τ . Now, this would not be compatible with (1.3.9) which implies that $\partial_{\theta_r} \Xi_1 = 0$. In the normal form procedure, since $D_{\mathfrak{z}} \Xi_0 = Id$ and $\partial_{\theta_r} \Xi_1 = 0$, the expression leading to A_1 must be related to A_1 through

$$\begin{aligned}
 A_1(z_0; \mathfrak{z}; \theta_r) &= A_1(z_0; \mathfrak{z}; \theta_r) + (\Xi_1 \cdot \nabla_z) A_0(z_0; \mathfrak{z}; \theta_r) \\
 &\quad - V_1(z_0; \mathfrak{z}; \theta_r) \partial_{\theta_r} \Xi_1 - (\Xi_1 \cdot \nabla_z) V_0(z_0; \mathfrak{z}) \partial_{\theta_r} \Xi_1,
 \end{aligned}$$

where Ξ_1 does not depend on θ_τ and is as in (1.3.33).

In this section, we have seen that the study of the system (1.1.12) is under Assumption 1.2.4 completely equivalent to the analysis of (1.3.3). The challenge now is to exploit (1.3.4) in order to derive a description of ${}^t(\mathfrak{z}, \nu)$ in terms of asymptotic oscillating series in powers of $\varepsilon \in]0, \varepsilon_0]$. In fact, this requires in addition to Assumption 1.2.4 imposing Assumption 1.2.6.

Remark 1.3.15 (Simplified framework inherited from Assumption 1.2.6). Under Assumption 1.2.6, we find that V_0 does not depend on θ_r . More precisely, we have

$$0 < V_0(z_0; \mathfrak{z}; \theta_\tau, \theta_r) \equiv V_0(z_0; \mathfrak{z}; \theta_\tau) := V_0(z_0; \Xi_0(z_0; \mathfrak{z}; \theta_\tau); \theta_\tau). \tag{1.3.35}$$

Moreover, as seen in (1.2.12), the function A_{mf} is simplified into \bar{A}_0 . On the other hand, the formulas (1.3.28) and (1.3.29) can be replaced by

$$\partial_{\theta_\tau} \Xi_0(z_0; \mathfrak{z}; \theta_\tau) = \bar{A}_0(z_0; \Xi_0(z_0; \mathfrak{z}; \theta_\tau); \theta_\tau), \quad (1.3.36)$$

$$\Xi_1^*(z_0; \mathfrak{z}; \theta_\tau, \theta_r) = V_0(z_0; \Xi_0(z_0; \mathfrak{z}; \theta_\tau); \theta_\tau)^{-1} \partial_{\theta_r}^{-1} A_0^*(z_0; \Xi_0(z_0; \mathfrak{z}; \theta_\tau); \theta_\tau, \theta_r). \quad (1.3.37)$$

1.4 Three-scale WKB calculus

We work here under Assumptions 1.2.4 and 1.2.6. The purpose is to construct approximate solutions through formal computations. The preceding work of preparation (in Section 1.3) allows to formulate the problem in terms of ${}^t(\mathfrak{z}, \mathbf{v})$. Thus, we can consider (1.3.3) and we can benefit from (1.3.4). We can also exploit the content of Remark 1.3.15. We proceed in several stages. In Subsection 1.4.1, we replace ${}^t(\mathfrak{z}, \mathbf{v})$ by some corresponding profile ${}^t(\mathfrak{z}, \mathcal{V})$; we define a notion of profile equations with associated approximate solutions ${}^t(\mathfrak{z}^a, \mathcal{V}^a)$; we also state the main result (Proposition 1.4.2) of this section. The construction of ${}^t(\mathfrak{z}^a, \mathcal{V}^a)$ is clarified in Subsection 1.4.2. The proof of Proposition 1.4.2 is achieved in Subsection 1.4.3.

1.4.1 Profile formulation

The first step is to seek solutions to (1.3.3) in the form

$$\begin{pmatrix} \mathfrak{z} \\ \mathbf{v} \end{pmatrix}(\tau) = \begin{pmatrix} \mathfrak{z} \\ \mathcal{V} \end{pmatrix} \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right), \quad (1.4.1)$$

where $\mathbf{v}(\tau)$ stands for the *exact solution* to (1.3.3). This multi-scale approach allows to separate the rapid variations (which have not yet been identified due to the presence of \mathbf{v}) from the slower (current and quick) variations (which must be determined first and foremost). This is like knowing \mathfrak{z} and \mathbf{v} modulo the action of a one-parameter group of rotations (associated with $\theta_r \in \mathbb{R}$) which are aimed to be ultimately specified through the replacement of θ_r by \mathbf{v}/ε .

At the level of (1.4.1), the *profile* ${}^t(\mathfrak{z}, \mathcal{V})(\tau, \theta_\tau, \theta_r)$ may depend on the parameters ε, z_0

and \mathbf{v}_0 (which will not be always indicated). Assuming (1.4.1), observe that

$$\partial_\tau \begin{pmatrix} \mathfrak{z} \\ \mathbf{v} \end{pmatrix} (\tau) = \left[Op(\mathfrak{z}; \partial) \begin{pmatrix} \mathfrak{z} \\ \mathbf{v} \end{pmatrix} \right] \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right),$$

where we have introduced the partial differential operator

$$Op(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) \equiv Op(\mathfrak{z}; \partial) := \partial_\tau + \varepsilon^{-1} \partial_{\theta_\tau} + \varepsilon^{-2} V(\varepsilon, z_0; \mathfrak{z}; \theta_\tau, \theta_r) \partial_{\theta_r} \quad (1.4.2)$$

which involves the parameters $(\varepsilon, z_0) \in [0, \varepsilon_0] \times \mathbb{R}^n$, is non-linear with respect to \mathfrak{z} , and implies the coefficient V which undergoes variations in $(\theta_\tau, \theta_r) \in \mathbb{T} \times \mathbb{T}_{r, z_0}$. Consider the *profile equations* which are associated to (1.3.3). These are the relaxed version of (1.3.3) made of the transport equations

$$\left[Op(\mathfrak{z}; \partial) \begin{pmatrix} \mathfrak{z} \\ \mathbf{v} \end{pmatrix} - \frac{1}{\varepsilon} \begin{pmatrix} A \\ V \end{pmatrix} (\mathfrak{z}) \right] (\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) = 0 \quad (1.4.3)$$

together with the initial data (at time $\tau = 0$)

$$\begin{pmatrix} \mathfrak{z} \\ \mathbf{v} \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; 0, \theta_\tau, \theta_r) = \begin{pmatrix} \mathfrak{z}_0 \\ \mathbf{v}_0 \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; \theta_\tau, \theta_r). \quad (1.4.4)$$

To recover the initial data of (1.3.3) with \mathfrak{z} and \mathbf{v} as in (1.4.1), in view of (1.3.18), we have to impose

$$\begin{pmatrix} \mathfrak{z} \\ \mathbf{v} \end{pmatrix} \left(\varepsilon, z_0, \mathbf{v}_0; 0, 0, \frac{\mathbf{v}_0}{\varepsilon} \right) = \begin{pmatrix} \Xi^{-1}(\varepsilon, z_0; z_0, 0, \frac{\mathbf{v}_0}{\varepsilon}) \\ \mathbf{v}_0 \end{pmatrix}. \quad (1.4.5)$$

To this end, it suffices to work with the (relaxed) condition

$$\begin{pmatrix} \mathfrak{z}_0 \\ \mathbf{v}_0 \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; \theta_\tau, \theta_r) = \begin{pmatrix} \Xi^{-1}(\varepsilon, z_0; z_0, 0, \theta_r) \\ \mathbf{v}_0 \end{pmatrix}. \quad (1.4.6)$$

The component \mathfrak{z}_0 may be expanded in powers of ε . The same applies to Ξ^{-1} . Assuming

that z_0 does not depend on ε , this yields

$$\begin{aligned}\mathfrak{Z}_0(\varepsilon, z_0, \mathbf{v}_0; \theta_\tau, \theta_r) &= \sum_{j=0}^N \varepsilon^j \mathfrak{Z}_{0j}(z_0; \theta_r) + \mathcal{O}(\varepsilon^{N+1}) \\ &= \sum_{j=0}^N \varepsilon^j \Xi_j^{-1}(z_0; z_0, 0, \theta_r) + \mathcal{O}(\varepsilon^{N+1}).\end{aligned}\tag{1.4.7}$$

In particular, taking into account (1.3.12) and (1.3.30), we find that

$$\mathfrak{Z}_{00}(z_0, \theta_r) = \Xi_1^{-1}(z_0; z_0, 0) = z_0 \equiv \mathfrak{Z}_{00}(z_0), \quad \mathfrak{Z}_{01}(z_0, \theta_r) = \Xi_1^{-1}(z_0; z_0, 0, \theta_r).$$

When z_0 depends (smoothly) on ε , we have $z_0 = z_{00} + \varepsilon z_{01} + \dots$, and (1.4.7) can be further expanded in powers of ε to obtain

$$\mathfrak{Z}_0(\varepsilon, z_0, \mathbf{v}_0; \theta_\tau, \theta_r) = \sum_{j=0}^N \varepsilon^j \tilde{\Xi}_j^{-1}(z_{00}, \dots, z_{0j}, \theta_r) + \mathcal{O}(\varepsilon^{N+1}).\tag{1.4.8}$$

By the way, we can observe that the oscillations of \mathfrak{z}_0 are easily absorbed at time $\tau = 0$ by the profile formulation, just because $\mathbf{v}(0) = \mathbf{v}_0$ and because \mathfrak{Z}_0 may depend on θ_r . Note also that the condition (1.4.4) has the effect of introducing at the level of \mathcal{V} (and therefore \mathfrak{Z}) a dependence on \mathbf{v}_0 . Neither \mathfrak{Z}_0 nor \mathcal{V}_0 depend on θ_r . But the variable θ_r appears at the level of \mathfrak{Z} and \mathcal{V} through the variations with respect to θ_r of the coefficient V inside (1.4.2).

By looking at the Cauchy problem (1.4.3)-(1.4.4), we can see that:

- The presence of \mathbf{v} has completely disappeared;
- The equation on \mathfrak{Z} is now decoupled from the one on \mathcal{V} .

These two properties are major assets in what follows. In fact, knowing what the component $\mathbf{v}(\tau)$ is, the solution to (1.3.3) can be directly deduced from (1.4.3) and (1.4.4) together with the substitution formula (1.4.1). Now, we would like to find good candidates for solving (1.4.3)-(1.4.4).

Definition 1.4.1 (Formal solutions to the profile equations).

Fix $(z_0, \mathbf{v}_0) \in \mathbb{R}^n \times \mathbb{R}$. Given some time $\mathcal{T} \in \mathbb{R}_+^*$ and some integer $N \in \mathbb{N}^*$, we say that ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)(\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r)$ is an approximate solution on $[0, \mathcal{T}]$ of order N to the Cauchy problem (1.4.3)-(1.4.4) if it satisfies (1.4.3)-(1.4.4) modulo some remainder which is of size ε^N in the supremum norm.

More precisely, Definition [1.4.1](#) means that ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)(\cdot, z_0, \mathbf{v}_0; \cdot)$ is a smooth (C^∞) function of

$$(\varepsilon, \tau, \theta_\tau, \theta_r) \in]0, \varepsilon_0] \times [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r, z_0}$$

which is such that

$$\left[\text{Op}(\mathfrak{Z}^a; \partial) \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} - \frac{1}{\varepsilon} \begin{pmatrix} A \\ V \end{pmatrix} (\mathfrak{Z}^a) - \varepsilon^N \begin{pmatrix} \mathcal{R}_N^{\mathfrak{Z}} \\ \mathcal{R}_N^{\mathcal{V}} \end{pmatrix} \right] (\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) = 0 \quad (1.4.9)$$

together with

$$\sup_{\varepsilon \in]0, \varepsilon_0]} \sup_{\tau \in [0, \mathcal{T}]} \sup_{\theta_\tau \in \mathbb{T}} \sup_{\theta_r \in \mathbb{T}_{r, z_0}} (|\mathcal{R}_N^{\mathfrak{Z}}| + |\mathcal{R}_N^{\mathcal{V}}|) (\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) < +\infty. \quad (1.4.10)$$

In what follows, we seek ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ through a finite series like

$$\mathfrak{Z}^a(\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) = \sum_{j=0}^N \varepsilon^j \mathfrak{Z}_j(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r), \quad (1.4.11)$$

$$\mathcal{V}^a(\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) = \sum_{j=-1}^N \varepsilon^j \mathcal{V}_j(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r), \quad (1.4.12)$$

with

$${}^t(\mathfrak{Z}_j, \mathcal{V}_j) \in C^\infty(\mathbb{R}^n \times \mathbb{R} \times [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r, z_0}; \mathbb{R}^n \times \mathbb{R}). \quad (1.4.13)$$

Moreover, in the same vein as [\(1.2.16\)](#), we impose

$$\mathcal{V}_{-1} \equiv \langle \bar{\mathcal{V}}_{-1} \rangle, \quad \mathcal{V}_0 \equiv \bar{\mathcal{V}}_0, \quad \mathfrak{Z}_0 \equiv \bar{\mathfrak{Z}}_0. \quad (1.4.14)$$

Proposition 1.4.2 (Existence of formal solutions to the profile equations). *Under Assumptions [1.2.4](#) and [1.2.6](#), we can find some time $\mathcal{T} \in \mathbb{R}_+^*$ and, for all $N \in \mathbb{N}^*$, an approximate solution ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ on $[0, \mathcal{T}]$ of order N to the Cauchy problem [\(1.4.3\)](#)-[\(1.4.4\)](#), which takes the form of [\(1.4.11\)](#)-[\(1.4.12\)](#) together with [\(1.4.13\)](#)-[\(1.4.14\)](#). The parts $\langle \bar{\mathfrak{Z}}_j \rangle$ and $\langle \bar{\mathcal{V}}_j \rangle$ are uniquely determined by a sequence of well-posed evolution equations associated with initial data issued from [\(1.4.4\)](#), while the expressions $\bar{\mathfrak{Z}}_j^*$, \mathfrak{Z}_j^* , $\bar{\mathcal{V}}_j^*$ and \mathcal{V}_j^* are derived from elliptic equations. We also find that $\mathfrak{Z}_0 \equiv \langle \bar{\mathfrak{Z}}_0 \rangle$ and $\mathfrak{Z}_1 \equiv \bar{\mathfrak{Z}}_1$.*

Remark 1.4.3 (Meaning of the WKB hierarchy on the profiles). Readers may wonder why the formal calculus is not performed directly on the system [\(1.3.3\)](#), with an expansion of \mathbf{v} in powers of ε . This is because a small error on the determination of \mathbf{v} , even of size ε , can

completely shuffle (through the substitution of θ_r for \mathbf{v}/ε) sequences which are expressed in powers of ε . In other words, a small change of \mathbf{v} at the level of (1.4.1) can strongly modify the asymptotic representations (1.4.11) and (1.4.12) of the profiles \mathfrak{Z} and \mathcal{V} . It can mix the terms \mathfrak{Z}_j and \mathcal{V}_j and then cause intractable closure problems. This is why it is so important to work with the exact phase \mathbf{v} . We return to this point in Subsection 1.5.1.

Next we have a halfway to formal solutions of the redressed system of Proposition 1.3.2

Remark 1.4.4. Because of the influence of \mathbf{v} , in the continuation of Remark 1.4.3, it bears noting that

$$\begin{pmatrix} \mathfrak{z}^a \\ \mathbf{v}^a \end{pmatrix}(\tau) := \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix}\left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon}\right) \quad (1.4.15)$$

does not provide, strictly speaking, with an approximate solution to (1.3.3). Indeed

$$\begin{aligned} & \partial_\tau \mathfrak{z}^a - \frac{1}{\varepsilon} A(\mathfrak{z}^a) \\ &= \varepsilon^N \mathcal{R}_N^{\mathfrak{z}}\left(\varepsilon, z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon}\right) \\ & \quad + \varepsilon^{-2} \left\{ \left[V(\varepsilon, z_0; \mathfrak{z}; \cdot) - V(\varepsilon, z_0; \mathfrak{z}^a; \cdot) \right] \partial_{\theta_r} \mathfrak{Z}^a(\varepsilon, z_0, \mathbf{v}_0; \cdot) \right\} \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right). \end{aligned}$$

We will see later that $\partial_{\theta_r} \mathfrak{Z}^a = \mathcal{O}(\varepsilon^2)$. To estimate the right hand side, we need to control the difference between the exact solution \mathfrak{z} and its model \mathfrak{z}^a . However, there is no assurance for the moment that $\mathfrak{z} - \mathfrak{z}^a$ is small, of size ε^N (or less). This issue is considered in Subsection 1.5.1.

1.4.2 Three-scale analysis

The expression ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ obtained through (1.4.11) and (1.4.12) is plugged into (1.4.3). The various contributions are ordered in increasing powers of ε . This yields in Subsection 1.4.2.1 a cascade of successive equations on ${}^t(\mathfrak{Z}_j, \mathcal{V}_j)$. The methodology for solving these equations is explained in Subsection 1.4.2.2.

1.4.2.1 Formal calculus

The matter is to list a cascade of successive equations on ${}^t(\mathfrak{Z}_j, \mathcal{V}_j)$. To this end, we perform a formal analysis at the level of (1.4.9) which can be expanded according to

$$\begin{aligned} & Op(\mathfrak{Z}^a; \partial) \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) - \frac{1}{\varepsilon} \begin{pmatrix} A \\ V \end{pmatrix} (\mathfrak{Z}^a) (\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) \\ &= \sum_{j=-2}^{N-1} \varepsilon^j \left(\begin{array}{c} \mathcal{L}_j(\bar{\mathfrak{Z}}_0, \mathfrak{Z}_1, \dots) \\ \mathcal{M}_j(\bar{\mathfrak{Z}}_0, \mathfrak{Z}_1, \dots; \mathcal{V}_{-1}, \mathcal{V}_0, \dots) \end{array} \right) + O(\varepsilon^N). \end{aligned} \quad (1.4.16)$$

We can resume (1.4.11), (1.4.12) and (1.4.14) in the form

$$\begin{aligned} & \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} (\tau, \theta_\tau, \theta_r) \\ &= \varepsilon^{-1} \left(\begin{array}{c} 0 \\ \langle \bar{\mathcal{V}}_{-1} \rangle \end{array} \right) (\tau) + \sum_{j=0}^{N-1} \varepsilon^j \begin{pmatrix} \bar{\mathfrak{Z}}_j \\ \bar{\mathcal{V}}_j \end{pmatrix} (\tau, \theta_\tau) + \sum_{j=1}^N \varepsilon^j \begin{pmatrix} \mathfrak{Z}_j^* \\ \mathcal{V}_j^* \end{pmatrix} (\tau, \theta_\tau, \theta_r) \end{aligned} \quad (1.4.17)$$

where the operations $\bar{\cdot}$, $\langle \cdot \rangle$ and \cdot^* are furnished by (1.2.4)-(1.2.5)-(1.2.6). Taking into account (1.4.17) when dealing with the left part of (1.4.16), we find that

$$\begin{aligned} & Op(\mathfrak{Z}^a; \partial) \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) - \frac{1}{\varepsilon} \begin{pmatrix} A \\ V \end{pmatrix} (\mathfrak{Z}^a) (\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) \\ &= \varepsilon^{-1} \partial_\tau \left(\begin{array}{c} 0 \\ \langle \bar{\mathcal{V}}_{-1} \rangle \end{array} \right) (\tau) + \sum_{j=0}^{N-1} \varepsilon^j \partial_\tau \begin{pmatrix} \bar{\mathfrak{Z}}_j \\ \bar{\mathcal{V}}_j \end{pmatrix} (\tau, \theta_\tau) \\ &+ \sum_{j=-1}^{N-2} \varepsilon^j \partial_{\theta_\tau} \begin{pmatrix} \bar{\mathfrak{Z}}_{j+1} \\ \bar{\mathcal{V}}_{j+1} \end{pmatrix} (\tau, \theta_\tau) + \sum_{j=1}^N \varepsilon^j \partial_\tau \begin{pmatrix} \mathfrak{Z}_j^* \\ \mathcal{V}_j^* \end{pmatrix} (\tau, \theta_\tau, \theta_r) \\ &+ \sum_{j=0}^{N-1} \varepsilon^j \partial_{\theta_\tau} \begin{pmatrix} \mathfrak{Z}_{j+1}^* \\ \mathcal{V}_{j+1}^* \end{pmatrix} (\tau, \theta_\tau, \theta_r) \\ &+ \sum_{k=-1}^{+\infty} \sum_{i=-1}^k \varepsilon^k V_{k-i}(z_0; \mathfrak{Z}^a; \theta_\tau, \theta_r) \partial_{\theta_r} \begin{pmatrix} \mathfrak{Z}_{i+2}^* \\ \mathcal{V}_{i+2}^* \end{pmatrix} (\tau, \theta_\tau, \theta_r) \\ &- \sum_{k=-1}^{+\infty} \varepsilon^k \begin{pmatrix} A_{k+1} \\ V_{k+1} \end{pmatrix} (z_0; \mathfrak{Z}^a; \theta_\tau, \theta_r). \end{aligned} \quad (1.4.18)$$

By definition, the expressions \mathcal{L}_j and \mathcal{M}_j are independent of $\varepsilon \in]0, \varepsilon_0]$. They are obtained by collecting the terms which, for $j < N$, appear in factor of ε^j . From this perspective,

the right hand side of (1.4.18) is still not in convenient form. The difficulties come from the two last lines in (1.4.18) which involve in factor of ε^k expressions that still depend on ε . We seek the profile ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ to be the approximate solution to the profile equation (1.4.3) in the sense of preceding Definition 1.4.1. To this end, \mathcal{L}_j and \mathcal{M}_j should be zero.

By means of (1.3.4) and (1.3.35), it is easy to see that, for $j = -1$, we find

$$\mathcal{L}_{-1}(\bar{\mathfrak{Z}}_0, \mathfrak{Z}_1) = \partial_{\theta_\tau} \bar{\mathfrak{Z}}_0 + V_0(z_0; \bar{\mathfrak{Z}}_0; \theta_\tau) \partial_{\theta_\tau} \mathfrak{Z}_1^*. \quad (1.4.19)$$

Integrate this with respect to θ_τ to deduce that $\partial_{\theta_\tau} \bar{\mathfrak{Z}}_0 = 0$, and therefore (since V_0 is positive) that $\partial_{\theta_\tau} \mathfrak{Z}_1^* = 0$. This implies that $\mathfrak{Z}_1^* = 0$. In fact, we have (1.4.33). These relations are used below in order to exhibit the expressions of \mathcal{L}_j for $j \geq 0$.

- For $j = 0$:

$$\mathcal{L}_0(\bar{\mathfrak{Z}}_0, \mathfrak{Z}_1, \mathfrak{Z}_2) = \partial_\tau \bar{\mathfrak{Z}}_0 + \partial_{\theta_\tau} \bar{\mathfrak{Z}}_1 + V_0(z_0; \bar{\mathfrak{Z}}_0; \theta_\tau) \partial_{\theta_\tau} \mathfrak{Z}_2^* - A_1(z_0; \bar{\mathfrak{Z}}_0; \theta_\tau, \theta_\tau). \quad (1.4.20)$$

- For $j = 1$:

$$\begin{aligned} \mathcal{L}_1(\bar{\mathfrak{Z}}_0, \mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3) &= \partial_\tau \bar{\mathfrak{Z}}_1 + \partial_{\theta_\tau} \bar{\mathfrak{Z}}_2 + \partial_{\theta_\tau} \mathfrak{Z}_2^* \\ &+ [V_1 + (\mathfrak{Z}_1 \cdot \nabla_{\mathfrak{z}})(V_0)] \partial_{\theta_\tau} \mathfrak{Z}_2^* + V_0 \partial_{\theta_\tau} \mathfrak{Z}_3^* - A_2 - (\mathfrak{Z}_1 \cdot \nabla_{\mathfrak{z}})(A_1). \end{aligned} \quad (1.4.21)$$

- For $j \geq 2$:

$$\begin{aligned} \mathcal{L}_j(\bar{\mathfrak{Z}}_0, \dots, \mathfrak{Z}_{j+2}) &= \partial_\tau \bar{\mathfrak{Z}}_j + \partial_{\theta_\tau} \bar{\mathfrak{Z}}_{j+1} + \partial_\tau \mathfrak{Z}_j^* + \partial_{\theta_\tau} \mathfrak{Z}_{j+1}^* \\ &+ \sum_{(i,m,k,l_1,\dots,l_k,b_1,\dots,b_k) \in S_j} \frac{1}{k!} \frac{\partial^k V_m}{\partial_{\mathfrak{z}^{b_1}} \dots \partial_{\mathfrak{z}^{b_k}}}(z_0; \bar{\mathfrak{Z}}_0; \theta_\tau, \theta_\tau) \prod_{t=1}^k \mathfrak{Z}_{l_t}^{b_t} \partial_{\theta_\tau} \mathfrak{Z}_i^* \\ &- \sum_{(m,k,l_1,\dots,l_k,b_1,\dots,b_k) \in S'_j} \frac{1}{k!} \frac{\partial^k A_m}{\partial_{\mathfrak{z}^{b_1}} \dots \partial_{\mathfrak{z}^{b_k}}}(z_0; \bar{\mathfrak{Z}}_0; \theta_\tau, \theta_\tau) \prod_{t=1}^k \mathfrak{Z}_{l_t}^{b_t} \end{aligned} \quad (1.4.22)$$

where

$$\begin{aligned} S_j &= \{(i, m, k, l_1, \dots, l_k, b_1, \dots, b_k); 2 \leq i \leq N, 0 \leq m < +\infty, k \leq j, 1 \leq l_t \leq N, \\ &\quad l_1 = \dots = l_k, 0 \leq b_t \leq n, t \in \{1, \dots, k\}, m + i + l_1 + \dots + l_k = j + 2\}, \\ S'_j &= \{(m, k, l_1, \dots, l_k, b_1, \dots, b_k); 1 \leq m < +\infty, k \leq j, 1 \leq l_t \leq N, \\ &\quad l_1 = \dots = l_k, 0 \leq b_t \leq n, t \in \{1, \dots, k\}, m + l_1 + \dots + l_k = j + 1\}. \end{aligned}$$

The expression \mathcal{L}_j can also be put in the form

$$\begin{aligned} \mathcal{L}_j(\bar{\mathfrak{Z}}_0, \dots, \bar{\mathfrak{Z}}_{j+2}) &= \partial_\tau \bar{\mathfrak{Z}}_j + \partial_{\theta_\tau} \bar{\mathfrak{Z}}_{j+1} + V_0 \partial_{\theta_r} \bar{\mathfrak{Z}}_{j+2}^* - (\bar{\mathfrak{Z}}_j \cdot \nabla_{\mathfrak{z}})(A_1) \\ &\quad + (\bar{\mathfrak{Z}}_j \cdot \nabla_{\mathfrak{z}})(V_0) \partial_{\theta_r} \bar{\mathfrak{Z}}_2^* + G_j(\bar{\mathfrak{Z}}_0, \bar{\mathfrak{Z}}_1, \dots, \bar{\mathfrak{Z}}_{j-1}, \bar{\mathfrak{Z}}_j^*, \bar{\mathfrak{Z}}_{j+1}^*), \end{aligned} \quad (1.4.23)$$

where the expression of G_j can be deduced from (1.4.22) as follows

$$\begin{aligned} G_j(\bar{\mathfrak{Z}}_0, \bar{\mathfrak{Z}}_1, \dots, \bar{\mathfrak{Z}}_{j-1}, \bar{\mathfrak{Z}}_j^*, \bar{\mathfrak{Z}}_{j+1}^*) &= +\partial_\tau \bar{\mathfrak{Z}}_j^* + \partial_{\theta_\tau} \bar{\mathfrak{Z}}_{j+1}^* \\ &\quad + \sum_{(i, m, k, l_1, \dots, l_k, b_1, \dots, b_k) \in \mathfrak{S}_j} \frac{1}{k!} \frac{\partial^k V_m}{\partial_{\mathfrak{z}^{b_1}} \dots \partial_{\mathfrak{z}^{b_k}}} (z_0; \bar{\mathfrak{Z}}_0; \theta_\tau, \theta_r) \prod_{t=1}^k \mathfrak{z}_{l_t}^{b_t} \partial_{\theta_r} \bar{\mathfrak{Z}}_i^* \\ &\quad - \sum_{(m, k, l_1, \dots, l_k, b_1, \dots, b_k) \in \mathfrak{S}'_j} \frac{1}{k!} \frac{\partial^k A_m}{\partial_{\mathfrak{z}^{b_1}} \dots \partial_{\mathfrak{z}^{b_k}}} (z_0; \bar{\mathfrak{Z}}_0; \theta_\tau, \theta_r) \prod_{t=1}^k \mathfrak{z}_{l_t}^{b_t} \end{aligned} \quad (1.4.24)$$

with

$$\begin{aligned} \mathfrak{S}_j &= S_j \setminus \{(i, m, k) = (j + 2, 0, 0) \text{ and } (i, m, k, l_1) = (2, 0, 1, j)\}, \\ \mathfrak{S}'_j &= S'_j \setminus \{(m, k, l_1) = (1, 1, j)\}. \end{aligned}$$

In a similar fashion, we can define the expressions \mathcal{M}_j as indicated below.

- For $j = -1$:

$$\begin{aligned} \mathcal{M}_{-1}(\bar{\mathfrak{Z}}_0, \langle \bar{\mathcal{V}}_{-1} \rangle, \bar{\mathcal{V}}_0, \mathcal{V}_1) &= \partial_\tau \langle \bar{\mathcal{V}}_{-1} \rangle + \partial_{\theta_\tau} \bar{\mathcal{V}}_0 + V_0(z_0; \bar{\mathfrak{Z}}_0; \theta_\tau) \partial_{\theta_r} \mathcal{V}_1^* - V_0(z_0; \bar{\mathfrak{Z}}_0; \theta_\tau). \end{aligned} \quad (1.4.25)$$

- For $j \geq 0$:

$$\begin{aligned}
 & \mathcal{M}_j(\bar{\mathfrak{Z}}_0, \dots, \mathfrak{Z}_{j+1}, \bar{\mathcal{V}}_0, \dots, \mathcal{V}_{j+2}) \\
 &= \partial_\tau \bar{\mathcal{V}}_j + \partial_{\theta_\tau} \bar{\mathcal{V}}_{j+1} + \partial_\tau \mathcal{V}_j^* + \partial_{\theta_\tau} \mathcal{V}_{j+1}^* \\
 &+ \sum_{(i,m,k,l_1,\dots,l_k,b_1,\dots,b_k) \in S_j} \frac{1}{k!} \frac{\partial^k V_m}{\partial_{\mathfrak{Z}^{b_1}} \dots \partial_{\mathfrak{Z}^{b_k}}}(z_0; \bar{\mathfrak{Z}}_0; \theta_\tau, \theta_r) \prod_{t=1}^k \mathfrak{Z}_{l_t}^{b_t} \partial_{\theta_r} \mathcal{V}_i^* \\
 &- \sum_{(m,k,l_1,\dots,l_k,b_1,\dots,b_k) \in S_j''} \frac{1}{k!} \frac{\partial^k V_m}{\partial_{\mathfrak{Z}^{b_1}} \dots \partial_{\mathfrak{Z}^{b_k}}}(z_0; \bar{\mathfrak{Z}}_0; \theta_\tau, \theta_r) \prod_{t=1}^k \mathfrak{Z}_{l_t}^{b_t}
 \end{aligned} \tag{1.4.26}$$

with

$$\begin{aligned}
 S_j'' = \{ & (m, k, l_1, \dots, l_k, b_1, \dots, b_k); 0 \leq m < +\infty, k \leq j+1, 1 \leq l_t \leq N, \\
 & l_1 = \dots = l_k, 0 \leq b_t \leq n, t \in \{1, \dots, k\}, m + l_1 + \dots + l_k = j+1 \}.
 \end{aligned}$$

The above expression can be put in the form

$$\begin{aligned}
 & \mathcal{M}_j(\bar{\mathfrak{Z}}_0, \dots, \mathfrak{Z}_{j+1}, \bar{\mathcal{V}}_0, \dots, \mathcal{V}_{j+2}) \\
 &= \partial_\tau \bar{\mathcal{V}}_j + \partial_{\theta_\tau} \bar{\mathcal{V}}_{j+1} + V_0 \partial_{\theta_r} \mathcal{V}_{j+2}^* + K_j(\bar{\mathfrak{Z}}_0, \dots, \mathfrak{Z}_{j+1}, \mathcal{V}_1^*, \dots, \mathcal{V}_{j+1}^*),
 \end{aligned}$$

where again the expression K_j may be deduced from (1.4.26). Looking at (1.4.16), to obtain (1.4.9), we have to solve for $j \in \{-1, \dots, N-1\}$ the following cascade of equations

$$\begin{pmatrix} \mathcal{L}_j(\bar{\mathfrak{Z}}_0, \dots, \mathfrak{Z}_{j+2}) \\ \mathcal{M}_j(\bar{\mathfrak{Z}}_0, \dots, \mathfrak{Z}_{j+1}, \bar{\mathcal{V}}_0, \dots, \mathcal{V}_{j+2}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.4.27}$$

1.4.2.2 Problem-solving strategy

The purpose here is to explain how we can solve equations $\mathcal{L}_j \equiv 0$ and $\mathcal{M}_j \equiv 0$ to determine the profiles \mathfrak{Z}_j and \mathcal{V}_j . This is a survey of the method that will be used (in next Subsection 1.4.3) to find approximate solutions to (1.4.3). In fact, this means manipulating the equations and expressions according to certain rules that we would like to emphasize and make explicit now.

The well-posedness of the hierarchy of equations $\mathcal{L}_j \equiv 0$ and $\mathcal{M}_j \equiv 0$ is the gateway to the existence of the profiles ${}^t(\mathfrak{Z}_j, \mathcal{V}_j)$. Because of the decoupling property which is highlighted in the previous paragraph, we can primarily determine \mathfrak{Z}^a . Then we can explain how to

recover the remaining component \mathcal{V}^a .

Let us begin with the base case (initialization case) concerning \mathcal{L} . This means to examine the following system (S_0) ,

$$\begin{aligned}\mathcal{L}_{-1}(\mathfrak{Z}_0, \mathfrak{Z}_1) &= 0, \\ \mathcal{L}_0(\mathfrak{Z}_0, \mathfrak{Z}_1, \mathfrak{Z}_2) &= 0.\end{aligned}\tag{1.4.28}$$

We will see that solving this system furnishes $\bar{\mathfrak{Z}}_0 \equiv \langle \bar{\mathfrak{Z}}_0 \rangle$, $\bar{\mathfrak{Z}}_1^*$ and \mathfrak{Z}_2^* . Now, we address the strategy of solving every single equation $\mathcal{L}_j = 0$ for $j \geq 1$ through the following three points:

- (1) First we take the average of the equation $\mathcal{L}_j = 0$ in both variables $\theta_\tau \in \mathbb{T}$ and $\theta_r \in \mathbb{T}_{r,z_0}$ to obtain a well-posed (linear for $j \geq 1$) Cauchy differential equation on $\langle \bar{\mathfrak{Z}}_j \rangle$, where the profiles \mathfrak{Z}_0 up to \mathfrak{Z}_{j-1} , $\bar{\mathfrak{Z}}_j^*$, \mathfrak{Z}_j^* and \mathfrak{Z}_{j+1}^* are viewed as input.
- (2) Second we substitute the expression of $\partial_\tau \langle \bar{\mathfrak{Z}}_j \rangle$ obtained from 1 and take the mean value of the equation $\mathcal{L}_j = 0$ with respect to $\theta_r \in \mathbb{T}_{r,z_0}$ to obtain the expression of $\bar{\mathfrak{Z}}_{j+1}^*$.
- (3) Third we subtract the expressions of $\partial_\tau \langle \bar{\mathfrak{Z}}_j \rangle$ and $\partial_{\theta_r} \bar{\mathfrak{Z}}_{j+1}^*$ in $\mathcal{L}_j = 0$ to obtain \mathfrak{Z}_{j+2}^* .

Using such argument for $j \geq 1$ gives rise to \mathfrak{Z}_j by combining the three equations (S_j) ,

$$\begin{aligned}\mathcal{L}_{j-2}(\mathfrak{Z}_0, \dots, \mathfrak{Z}_j) &= 0 \\ \mathcal{L}_{j-1}(\mathfrak{Z}_0, \dots, \mathfrak{Z}_{j+1}) &= 0 \\ \mathcal{L}_j(\mathfrak{Z}_0, \dots, \mathfrak{Z}_{j+2}) &= 0\end{aligned}\tag{1.4.29}$$

through the following steps

Step 1: Apply items (1)–(3) to the equation $\mathcal{L}_{j-2} = 0$; this leads to \mathfrak{Z}_j^* .

Step 2: Apply items (1) and (2) to the equation $\mathcal{L}_{j-1} = 0$; this yields $\bar{\mathfrak{Z}}_j^*$.

Step 3: Apply item (1) to the equation $\mathcal{L}_j = 0$; this allows to identify $\langle \bar{\mathfrak{Z}}_j \rangle$.

Steps (1)–(3) allow us to determine \mathfrak{Z}_j^* , $\bar{\mathfrak{Z}}_j^*$, and $\langle \bar{\mathfrak{Z}}_j \rangle$, and therefore \mathfrak{Z}_j . This way, it becomes possible to access all the profiles by solving successively the systems (S_j) for $j \in \{0, \dots, N-1\}$. Looking at (S_j) gives also access to $\bar{\mathfrak{Z}}_{j+1}^*$ (through \mathcal{L}_j), \mathfrak{Z}_{j+1}^* (through \mathcal{L}_{j-1}) as well as \mathfrak{Z}_{j+2}^* (through again \mathcal{L}_j). The arising claim (for $j \geq 2$) is thus as follows: (H_j) ,

$$\begin{aligned}\mathfrak{Z}_k &\text{ are known on the domain } [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r,z_0} \text{ for } 0 \leq k \leq j-2, \\ \bar{\mathfrak{Z}}_{j-1}^*, \mathfrak{Z}_{j-1}^*, \text{ and } \mathfrak{Z}_j^* &\text{ are known on the domain } [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r,z_0}.\end{aligned}\tag{1.4.30}$$

This means that we argue by induction. We have first to find the constraints required to validate the starting point of the induction, which is (H_2) . Then, we proceed successively. We prove that (H_{j+1}) holds true given that (H_k) is verified for $k \in \{2, \dots, j\}$. By this way, we can recover \mathfrak{Z}_{j-1} , $\overline{\mathfrak{Z}}_j^*$, \mathfrak{Z}_j^* and \mathfrak{Z}_{j+1}^* , and so on. This program is achieved in the next section.

A similar strategy is repeated concerning the profiles \mathcal{V}_j . We start by analyzing the basic case. Then, we propagate to the higher one on the basis of the following hypothesis (\tilde{H}_j) ,

$$\begin{aligned} \mathcal{V}_k \text{ are known on the domain } [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r,z_0} \text{ for } -1 \leq k \leq j-2, \\ \overline{\mathcal{V}}_{j-1}^*, \mathcal{V}_{j-1}^* \text{ and } \mathcal{V}_j^* \text{ are known on the domain } [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r,z_0}. \end{aligned} \quad (1.4.31)$$

From the profiles \mathfrak{Z}_j and \mathcal{V}_j thus obtained, we can build the approximate solution ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ as indicated in (1.4.11) and (1.4.12).

1.4.3 Construction of approximate solutions

The strategy of constructing an approximate solution was clarified in the previous section. We are concerned here with the verification of the preceding arguments. Following this line, we achieve the proof of Proposition 1.4.2. In Subsection 1.4.3.1 we explain the base case where it becomes possible to access the proof of Theorem 1.2.10, whereas in Subsection 1.4.3.2 we check the validity of hypotheses (H_j) and (\tilde{H}_j) using the induction argument.

1.4.3.1 Base case

We want to prove here that the initialization case is well-posed. To this end, we can limit ourselves to looking at the equations $\mathcal{L}_{-1} \equiv 0$, $\mathcal{L}_0 \equiv 0$ and $\mathcal{M}_{-1} \equiv 0$. As already seen, exploiting Assumption 1.2.6, the restriction $\mathcal{L}_{-1} \equiv 0$ reduces to

$$\mathcal{L}_{-1}(\overline{\mathfrak{Z}}_0, \mathfrak{Z}_1) = \partial_{\theta_\tau} \overline{\mathfrak{Z}}_0 + V_0(z_0; \overline{\mathfrak{Z}}_0; \theta_\tau) \partial_{\theta_r} \mathfrak{Z}_1^* = 0. \quad (1.4.32)$$

This amounts to

$$\overline{\mathfrak{Z}}_0(\tau, \theta_\tau) = \langle \overline{\mathfrak{Z}}_0 \rangle(\tau), \quad \mathfrak{Z}_1(\tau, \theta_\tau, \theta_r) = \overline{\mathfrak{Z}}_1(\tau, \theta_\tau). \quad (1.4.33)$$

The expression \mathcal{L}_0 from (1.4.20) becomes (in view of Assumption 1.2.6)

$$\partial_\tau \langle \overline{\mathfrak{Z}}_0 \rangle + \partial_{\theta_r} \overline{\mathfrak{Z}}_1^* + V_0(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle; \theta_\tau) \partial_{\theta_r} \mathfrak{Z}_2^* - A_1(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle; \theta_\tau, \theta_r) = 0. \quad (1.4.34)$$

At this stage, we are able to complete the proof of Theorem [1.2.10](#).

Proof of Theorem [1.2.10](#). Take the average of [\(1.4.34\)](#) in both variables θ_τ and θ_r to obtain

$$\partial_\tau \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau) - \langle \bar{A}_1 \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)) = 0, \quad \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; 0) = \Xi_0^{-1}(z_0; z_0; 0) = z_0. \quad (1.4.35)$$

This is exactly [\(1.2.23\)](#). By restricting \mathcal{T} if necessary, the Cauchy-Lipschitz theorem provides with the existence on the interval $[0, \mathcal{T}]$ of a local solution $\langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \cdot)$ to the above non-linear differential equation. The formula [\(1.2.24\)](#) comes from [\(1.3.34\)](#) where $\Xi_0 \equiv \Xi_{\text{mf}}$ and $\Xi_1 \equiv \Xi_1^*$ is replaced by [\(1.3.37\)](#). On the other hand, by construction, we have [\(1.3.18\)](#) and [\(1.3.6\)](#) which lead to [\(1.2.22\)](#) after the use of the first line inside [\(1.4.17\)](#). \square

Formula [\(1.4.35\)](#) can be viewed as a second modulation equation. It plays an important role concerning the long time gyrokinetic equation. There are other outcomes which are issued from [\(1.4.34\)](#) and which are given by the following lemma.

Lemma 1.4.5 (Determination of $\bar{\mathfrak{Z}}_1^*$ and \mathfrak{Z}_2^*). *Under Assumption [1.2.6](#), the profiles $\bar{\mathfrak{Z}}_1^*$ and \mathfrak{Z}_2^* under the constraint $\mathcal{L}_0 \equiv 0$ are given by*

$$\bar{\mathfrak{Z}}_1^*(\tau, \theta_\tau) = (\partial_{\theta_\tau}^{-1} \bar{A}_1^*)(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau), \quad (1.4.36)$$

$$\mathfrak{Z}_2^*(\tau, \theta_\tau, \theta_r) = V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau)^{-1} \partial_{\theta_r}^{-1} A_1^*(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau, \theta_r). \quad (1.4.37)$$

Proof. We take into account Assumption [1.2.6](#) in the following analysis. First, we average the equation [\(1.4.34\)](#) with respect to θ_r to obtain

$$\partial_\tau \langle \bar{\mathfrak{Z}}_0 \rangle(\tau) + \partial_{\theta_\tau} \bar{\mathfrak{Z}}_1^*(\tau, \theta_\tau) - \bar{A}_1(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau) = 0. \quad (1.4.38)$$

We also plug the ansatz [\(1.4.35\)](#) into [\(1.4.38\)](#) to obtain

$$\begin{aligned} \partial_{\theta_\tau} \bar{\mathfrak{Z}}_1^*(\tau, \theta_\tau) &= \bar{A}_1(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau) - \langle \bar{A}_1 \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau)) \\ &= \bar{A}_1^*(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau). \end{aligned} \quad (1.4.39)$$

Since $\bar{A}_1^*(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \cdot) \in L_*^1(\mathbb{T})$, we can invert the operator ∂_{θ_τ} as indicated in [\(1.3.24\)](#). Thus, we obtain [\(1.4.36\)](#). Similarly to obtain \mathfrak{Z}_2^* , substitute [\(1.4.38\)](#) in [\(1.4.34\)](#). This gives

$$V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau) \partial_{\theta_r} \mathfrak{Z}_2^*(\tau, \theta_\tau, \theta_r) = A_1^*(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau, \theta_r). \quad (1.4.40)$$

Again, as $A_1^*(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau, \cdot) \in L_*^1(\mathbb{T}_{r,z_0})$, we can invert the operator ∂_{θ_τ} and the positive source term $V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau)$ to recover (1.4.37). \square

It is obvious that to solve $\mathcal{L}_0 \equiv 0$, it is enough to adjust $\langle \bar{\mathfrak{Z}}_0 \rangle$, $\bar{\mathfrak{Z}}_1^*$ and $\bar{\mathfrak{Z}}_2^*$ as indicated in (1.4.35), (1.4.36), and (1.4.37). By the way, this furnishes (H2).

Similarly, we can determine basic parts of the profile \mathcal{V}^a through the following lemma.

Lemma 1.4.6 (Determination of $\langle \bar{\mathcal{V}}_{-1} \rangle$ and $\bar{\mathcal{V}}_0^*$). *Under Assumption 1.2.6, the expressions of the profiles $\langle \bar{\mathcal{V}}_{-1} \rangle$ and $\bar{\mathcal{V}}_0^*$ under the constraint $\mathcal{M}_{-1} \equiv 0$ are given by*

$$\langle \bar{\mathcal{V}}_{-1} \rangle(z_0; \tau) = \int_0^\tau \langle V_0 \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(r)) dr, \quad (1.4.41)$$

$$\bar{\mathcal{V}}_0^*(z_0; \tau, \theta_\tau) = (\partial_{\theta_\tau}^{-1} \bar{V}_0^*)(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau). \quad (1.4.42)$$

Moreover, we have $\mathcal{V}_1^* \equiv 0$. This means that $\mathcal{V}_1 \equiv \bar{\mathcal{V}}_1$.

Proof. With the aid of expression (1.4.25), the restriction $\mathcal{M}_{-1} \equiv 0$ reduces to

$$\partial_\tau \langle \bar{\mathcal{V}}_{-1} \rangle - V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau) + \partial_{\theta_\tau} \bar{\mathcal{V}}_0^* + V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau) \partial_{\theta_\tau} \mathcal{V}_1^* = 0. \quad (1.4.43)$$

Take the average in θ_r to obtain rid of the last term, and then in θ_τ to suppress the penultimate term. Complete with the initial data coming from (1.4.4). We find that

$$\partial_\tau \langle \bar{\mathcal{V}}_{-1} \rangle(\tau) - \langle V_0 \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau)) = 0, \quad \langle \bar{\mathcal{V}}_{-1} \rangle(0) = 0 \quad (1.4.44)$$

which implies (1.4.41). Now take the mean value with respect to θ_r to obtain rid of the term $V_0 \partial_{\theta_r} \mathcal{V}_1^*$ in (1.4.43). Then exploit (1.4.44) and subtract the result thus obtained to deduce

$$\begin{aligned} \partial_{\theta_\tau} \bar{\mathcal{V}}_0^*(\tau, \theta_\tau) &= V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau) - \langle V_0 \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau)) \\ &= \bar{V}_0^*(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau). \end{aligned} \quad (1.4.45)$$

Since $\bar{V}_0^*(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \cdot) \in L_*^1(\mathbb{T})$, we can invert the operator ∂_{θ_τ} in (1.4.45) to obtain (1.4.42). Finally, replace in (1.4.43) the expressions $\partial_\tau \langle \bar{\mathcal{V}}_{-1} \rangle$ and $\partial_{\theta_\tau} \bar{\mathcal{V}}_0^*$ as indicated in (1.4.44) and (1.4.45), this gives

$$V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau) \partial_{\theta_\tau} \mathcal{V}_1^*(\tau, \theta_\tau, \theta_r) = 0.$$

Since V_0 is positive function, we must have $\mathcal{V}_1^* \equiv 0$. \square

Recall that $\mathcal{V}_0^* \equiv 0$. Thus, from Lemma [1.4.6](#), we have (\tilde{H}_1) . Let us clarify here what happens at the initial time $\tau = 0$. From [\(1.4.7\)](#), we have to impose

$$\mathfrak{Z}_j(0, 0, \theta_r) = \langle \bar{\mathfrak{Z}}_j \rangle(0) + \bar{\mathfrak{Z}}_j^*(0, 0) + \mathfrak{Z}_j^*(0, 0, \theta_r) = \Xi_j^{-1}(z_0; z_0, 0, \theta_r).$$

This means the cauchy problem [\(1.4.53\)](#) is accompanied with the following initial data

$$\langle \bar{\mathfrak{Z}}_j \rangle(0) = \Xi_j^{-1}(z_0; z_0, 0, \theta_r) - \bar{\mathfrak{Z}}_j^*(0, 0) - \mathfrak{Z}_j^*(0, 0, \theta_r). \quad (1.4.46)$$

Similarly for \mathcal{V} , we have

$$\mathcal{V}(0, 0, \theta_r) = \mathfrak{v}_0 = \langle \bar{\mathcal{V}} \rangle(0) + \bar{\mathcal{V}}^*(0, 0) + \mathcal{V}^*(0, 0, \theta_r). \quad (1.4.47)$$

1.4.3.2 Induction step

We want here to verify the validity of the hypotheses presented in Paragraph [1.4.2.2](#) concerning the induction argument. We have already checked from Lemma [1.4.5](#) and the equation [\(1.4.35\)](#) the validity of (H_2) . Assume that (H_k) is valid for all $k \in \{2, \dots, j\}$. We have to prove that (H_{j+1}) :

$$\begin{aligned} \mathfrak{Z}_k \text{ are known on the domain } [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r, z_0} \text{ for } 0 \leq k \leq j-1, \\ \bar{\mathfrak{Z}}_j^*, \mathfrak{Z}_j^* \text{ and } \mathfrak{Z}_{j+1}^* \text{ are known on the domain } [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r, z_0}, \end{aligned} \quad (1.4.48)$$

holds. From the validity of (H_j) , the profiles \mathfrak{Z}_k for $0 \leq k \leq j-2$, $\bar{\mathfrak{Z}}_{j-1}^*$, \mathfrak{Z}_{j-1}^* and \mathfrak{Z}_j^* have been identified. We still need to determine $\bar{\mathfrak{Z}}_{j-1}$, $\bar{\mathfrak{Z}}_j^*$ and \mathfrak{Z}_{j+1}^* . To this end, consider the expression \mathcal{L}_{j-1} as defined in [\(1.4.23\)](#), that is

$$\begin{aligned} & \partial_\tau \bar{\mathfrak{Z}}_{j-1} + \partial_{\theta_\tau} \bar{\mathfrak{Z}}_j + V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau) \partial_{\theta_\tau} \mathfrak{Z}_{j+1}^* \\ & - (\mathfrak{Z}_{j-1} \cdot \nabla_{\mathfrak{z}}) A_1(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau, \theta_r) \\ & + (\mathfrak{Z}_{j-1} \cdot \nabla_{\mathfrak{z}}) V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau) \partial_{\theta_\tau} \mathfrak{Z}_2^* \\ & + G_{j-1}(\bar{\mathfrak{Z}}_0, \mathfrak{Z}_1, \dots, \mathfrak{Z}_{j-2}, \mathfrak{Z}_{j-1}^*, \mathfrak{Z}_j^*) = 0. \end{aligned} \quad (1.4.49)$$

Substituting $\bar{\mathfrak{Z}}_{j-1} = \langle \bar{\mathfrak{Z}}_{j-1} \rangle + \bar{\mathfrak{Z}}_{j-1}^*$ at the level of (1.4.49), we obtain

$$\begin{aligned} & \partial_\tau \langle \bar{\mathfrak{Z}}_{j-1} \rangle + \partial_\tau \bar{\mathfrak{Z}}_{j-1}^* + \partial_{\theta_r} \bar{\mathfrak{Z}}_{j-1}^* - \left((\langle \bar{\mathfrak{Z}}_{j-1} \rangle + \bar{\mathfrak{Z}}_{j-1}^* + \mathfrak{Z}_{j-1}^*) \cdot \nabla_{\mathfrak{z}} \right) A_1(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau, \theta_r) \\ & + V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau) \partial_{\theta_r} \mathfrak{Z}_{j+1}^* + \left((\langle \bar{\mathfrak{Z}}_{j-1} \rangle + \bar{\mathfrak{Z}}_{j-1}^* + \mathfrak{Z}_{j-1}^*) \cdot \nabla_{\mathfrak{z}} \right) V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau) \partial_{\theta_r} \mathfrak{Z}_2^* \\ & + G_{j-1}(\bar{\mathfrak{Z}}_0, \mathfrak{Z}_1, \dots, \mathfrak{Z}_{j-2}, \mathfrak{Z}_{j-1}^*, \mathfrak{Z}_j^*) = 0. \end{aligned} \quad (1.4.50)$$

We obtain a linearized version of (1.4.34), which is

$$\begin{aligned} & \partial_\tau \langle \bar{\mathfrak{Z}}_{j-1} \rangle + \partial_{\theta_r} \bar{\mathfrak{Z}}_{j-1}^* + (\langle \bar{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau) \partial_{\theta_r} \mathfrak{Z}_2^* \\ & + V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau) \partial_{\theta_r} \mathfrak{Z}_{j+1}^* - (\langle \bar{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) A_1(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau, \theta_r) = \mathcal{G}_{j-1} \end{aligned} \quad (1.4.51)$$

where

$$\begin{aligned} & \mathcal{G}_{j-1}(\bar{\mathfrak{Z}}_0, \mathfrak{Z}_1, \dots, \mathfrak{Z}_{j-2}, \bar{\mathfrak{Z}}_{j-1}^*, \mathfrak{Z}_{j-1}^*, \mathfrak{Z}_j^*) \\ & = -\partial_\tau \bar{\mathfrak{Z}}_{j-1}^* - G_{j-1}(\bar{\mathfrak{Z}}_0, \mathfrak{Z}_1, \dots, \mathfrak{Z}_{j-2}, \bar{\mathfrak{Z}}_{j-1}^*, \mathfrak{Z}_j^*) \\ & + \left((\bar{\mathfrak{Z}}_{j-1}^* + \mathfrak{Z}_{j-1}^*) \cdot \nabla_{\mathfrak{z}} \right) A_1(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau, \theta_r) \\ & - \left((\bar{\mathfrak{Z}}_{j-1}^* + \mathfrak{Z}_{j-1}^*) \cdot \nabla_{\mathfrak{z}} \right) (V_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle; \theta_\tau)) \partial_{\theta_r} \mathfrak{Z}_2^*. \end{aligned} \quad (1.4.52)$$

It is obvious that the expression $\mathcal{G}_{j-1} \equiv \mathcal{G}_{j-1}(\bar{\mathfrak{Z}}_0, \mathfrak{Z}_1, \dots, \mathfrak{Z}_{j-2}, \bar{\mathfrak{Z}}_{j-1}^*, \mathfrak{Z}_{j-1}^*, \mathfrak{Z}_j^*)$ is known function, since by induction, the profiles \mathfrak{Z}_{j-1}^* , \mathfrak{Z}_j^* , $\bar{\mathfrak{Z}}_{j-1}^*$ and \mathfrak{Z}_k where $0 \leq k \leq j-2$ are known functions on the domain $[0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r,z_0}$. With the aid of Assumption 1.2.6, average (1.4.51) in both variables θ_τ and θ_r to exhibit

$$\partial_\tau \langle \bar{\mathfrak{Z}}_{j-1} \rangle - (\langle \bar{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) \langle \bar{A}_1(\langle \bar{\mathfrak{Z}}_0 \rangle) \rangle = \langle \bar{\mathcal{G}}_{j-1} \rangle. \quad (1.4.53)$$

This may be completed by the initial data, namely $\langle \bar{\mathfrak{Z}}_{0(j-1)} \rangle$ issued from (1.4.46) where, because of (H_j) , the expressions $\bar{\mathfrak{Z}}_{j-1}^*(0, 0)$ and $\mathfrak{Z}_{j-1}^*(0, 0, \theta_r)$ are known. The equation (1.4.53) is *linear*. It has therefore a solution $\langle \bar{\mathfrak{Z}}_{j-1} \rangle$ on the whole interval $[0, \mathcal{T}]$. By validity of (H_j) , $\bar{\mathfrak{Z}}_{j-1}^*$ and \mathfrak{Z}_{j-1}^* are known. And thereby, in view of (1.2.8), this implies that the whole of \mathfrak{Z}_{j-1} has been identified. Substitute $\partial_\tau \langle \bar{\mathfrak{Z}}_{j-1} \rangle$ as indicated in (1.4.53) inside (1.4.51). Then average with respect to θ_r to obtain

$$\partial_{\theta_r} \bar{\mathfrak{Z}}_j^* = \bar{\mathcal{G}}_{j-1}^* + (\langle \bar{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) \bar{A}_1^*(\langle \bar{\mathfrak{Z}}_0 \rangle), \quad (1.4.54)$$

or equivalently

$$\bar{\mathfrak{Z}}_j^* = \partial_{\theta_r}^{-1} \bar{\mathcal{G}}_{j-1}^* + (\langle \bar{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_{\mathfrak{z}}) \bar{A}_1^*(\langle \bar{\mathfrak{Z}}_0 \rangle). \quad (1.4.55)$$

Again substitute in (1.4.51) the expressions $\partial_\tau \langle \bar{\mathfrak{Z}}_{j-1} \rangle$ and $\partial_{\theta_\tau} \bar{\mathfrak{Z}}_j^*$ as indicated in (1.4.53) and (1.4.54). This gives

$$\begin{aligned} \mathfrak{Z}_{j+1}^* &= (V_0)^{-1} \partial_{\theta_\tau}^{-1} \left[\mathcal{G}_{j-1}^* + (\langle \bar{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_3) A_1^*(\langle \bar{\mathfrak{Z}}_0 \rangle) \right. \\ &\quad \left. - (\langle \bar{\mathfrak{Z}}_{j-1} \rangle \cdot \nabla_3) V_0(\langle \bar{\mathfrak{Z}}_0 \rangle) \partial_{\theta_\tau} \mathfrak{Z}_2^* \right]. \end{aligned} \quad (1.4.56)$$

This discussion determines (H_{j+1}) .

The same strategy applies concerning the construction of the profiles \mathcal{V}_j , for $j \in \{-1, \dots, N+1\}$ under Assumption 1.2.6. Lemma 1.4.6 gives rise to (\tilde{H}_1) . Assume now that the hypotheses (\tilde{H}_k) are valid for all $k \in \{1, \dots, j-1\}$. We have to prove that (\tilde{H}_j) holds true. To this end, pick the expression \mathcal{M}_{j-2} from (1.4.26) and solve the equation $\mathcal{M}_{j-2} \equiv 0$. We find

$$\partial_\tau \langle \bar{\mathcal{V}}_{j-2} \rangle + \partial_{\theta_\tau} \bar{\mathcal{V}}_{j-1}^* + V_0 \partial_{\theta_\tau} \mathcal{V}_j^* = \mathcal{K}_{j-2} \quad (1.4.57)$$

where \mathcal{K}_{j-2} is known function by induction. Average (1.4.57) with respect to θ_τ and θ_r , we obtain

$$\partial_\tau \langle \bar{\mathcal{V}}_{j-2} \rangle = \langle \bar{\mathcal{K}}_{j-2} \rangle. \quad (1.4.58)$$

The equation (1.4.58) together with the initial data coming from the second equation of (1.4.4) allows to determine $\langle \bar{\mathcal{V}}_{j-2} \rangle$ by time integration on the interval $[0, \mathcal{T}]$. Substitute (1.4.58) in the equation (1.4.57). Then, average with respect to θ_r to obtain

$$\bar{\mathcal{V}}_{j-1}^* = \partial_{\theta_\tau}^{-1} \bar{\mathcal{K}}_{j-2}^*. \quad (1.4.59)$$

Again, use (1.4.58) and (1.4.59) in (1.4.57), we obtain

$$\mathcal{V}_j^* = (V_0)^{-1} \partial_{\theta_\tau}^{-1} \mathcal{K}_{j-2}^*.$$

Briefly, we have just verified (\tilde{H}_j) .

Proof of Proposition 1.4.2. We select some $N \in \mathbb{N}^*$, and we build the profiles ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ as it is indicated in (1.4.11) and (1.4.12), with the aid of the profiles ${}^t(\mathfrak{Z}_j, \mathcal{V}_j)$ which have been determined in the previous Paragraphs 1.4.3.1 and 1.4.3.2. It is easy then to see that ${}^t(\mathfrak{Z}^a, \mathcal{V}^a)$ is an approximate solution to the profile equation (1.4.3). Indeed, by construction, we have reset to zero all the terms composing the sum in (1.4.16). \square

1.5 Stability estimates.

Our purpose here is to show that the formal solutions to (1.4.3)-(1.4.4) can be used to approximate through (1.4.15) the exact solutions of the redressed system (1.3.3). Note that we do not compare ${}^t(\mathfrak{z}^a, \mathcal{V}^a)$ and the solution ${}^t(\mathfrak{z}, \mathcal{V})$ to the profile equation (1.4.3), which would be relatively easy. Instead, we want to associate the expression ${}^t(\mathfrak{z}^a, \mathbf{v}^a)$ of (1.4.15) and the solution ${}^t(\mathfrak{z}, \mathbf{v})$ of (1.3.3). To this end, we can always consider the expression ${}^t(r^\mathfrak{z}, r^\mathbf{v})$ defined by the weighted difference

$$\begin{pmatrix} r^\mathfrak{z} \\ r^\mathbf{v} \end{pmatrix}(\tau) := \varepsilon^{-N} \begin{pmatrix} \mathfrak{z} - \mathfrak{z}^a \\ \mathbf{v} - \mathbf{v}^a \end{pmatrix}(\tau) \quad (1.5.1)$$

or equivalently by the relation

$$\begin{pmatrix} \mathfrak{z} \\ \mathbf{v} \end{pmatrix}(\tau) = \varepsilon^{-1} \begin{pmatrix} 0 \\ \langle \bar{\mathcal{V}}_{-1} \rangle \end{pmatrix}(\tau) + \sum_{j=0}^N \varepsilon^j \begin{pmatrix} \mathfrak{z}_j \\ \mathcal{V}_j \end{pmatrix} \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right) + \varepsilon^N \begin{pmatrix} r^\mathfrak{z} \\ r^\mathbf{v} \end{pmatrix}(\tau). \quad (1.5.2)$$

In Subsection 1.5.1, we prove that ${}^t(\varepsilon^N r^\mathfrak{z}, \varepsilon^N r^\mathbf{v})$ can indeed be viewed as a remainder, as suggested by (1.5.2). In Subsection 1.5.2, we highlight the role of the *frozen phase* \mathbf{v}^f which is given by

$$\mathbf{v}^f(\tau) \equiv \mathbf{v}^f(\varepsilon, z_0, \mathbf{v}_0; \tau) := \frac{1}{\varepsilon} \langle \bar{\mathcal{V}}_{-1} \rangle(z_0, \mathbf{v}_0; \tau) + \bar{\mathcal{V}}_0(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}). \quad (1.5.3)$$

The frozen phase is a known quantity because it is built from $\langle \bar{\mathcal{V}}_{-1} \rangle(\tau)$ and $\bar{\mathcal{V}}_0(\tau, \theta_\tau)$ which have been already determined. It is constructed by collecting the two first terms of the expansion (1.5.2) of \mathbf{v} . Note that the definition (1.5.3) coincides with (1.2.19) since, at the end, we will find that $\mathcal{V}_{-1} \equiv \mathcal{V}_{-1}$ and $\mathcal{V}_0 \equiv \mathcal{V}_0$. By construction, we have

$$\mathbf{v}(\tau) = \mathbf{v}^f(\tau) + \sum_{j=1}^N \varepsilon^j \mathcal{V}_j \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right) + \varepsilon^N r^\mathbf{v}(\tau). \quad (1.5.4)$$

For $j \geq 2$, in general, we have $\mathcal{V}_j^* \neq 0$ while θ_τ must be replaced by \mathbf{v}/ε . This means that the access to \mathbf{v} is necessary to construct the right hand side of (1.5.4), or that the knowledge of \mathbf{v} is required to obtain a precision of size ε or more. Now, we would like to derive a self-contained representation of \mathfrak{z} and \mathbf{v} (which does not call for \mathbf{v}). In Subsection 1.5.2, we explain how \mathbf{v}^f/ε can become a substitute for \mathbf{v}/ε in the right part of (1.5.2). By this way, we end up with a justified WKB expansion of the redressed field ${}^t(\mathfrak{z}, \mathbf{v})$

whose all components can be determined by formal computations. Finally, there remains to interpret this result in terms of the original field z . This is done in Subsection [1.5.3](#) where the proof of Theorem [1.2.7](#) is complete. \square

1.5.1 Justification of the formal computations

The aim here is to compare ${}^t(\mathfrak{z}, \mathbf{v})$ and ${}^t(\mathfrak{z}^a, \mathbf{v}^a)$. This can be done by estimating the size of the weighted difference ${}^t(r^\mathfrak{z}, r^\mathbf{v})$. To this end, the strategy is to first exhibit a non-linear differential equation satisfied by ${}^t(r^\mathfrak{z}, r^\mathbf{v})$.

Lemma 1.5.1. *For all $N \in \mathbb{N}$, the expression ${}^t(r^\mathfrak{z}, r^\mathbf{v})$ issued from [\(1.5.1\)](#) is subject to*

$$\partial_\tau \begin{pmatrix} r^\mathfrak{z} \\ r^\mathbf{v} \end{pmatrix} (\tau) = \begin{pmatrix} \mathcal{R}^\mathfrak{z} \\ \mathcal{R}^\mathbf{v} \end{pmatrix} \left(\varepsilon, z_0, \mathbf{v}_0, r^\mathfrak{z}; \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right), \quad \begin{pmatrix} r^\mathfrak{z} \\ r^\mathbf{v} \end{pmatrix} (0) = O(1) \quad (1.5.5)$$

where

$$\begin{aligned} & \begin{pmatrix} \mathcal{R}^\mathfrak{z} \\ \mathcal{R}^\mathbf{v} \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0, r; \tau, \theta_\tau, \theta_r) \\ & := \varepsilon^{-N-2} \left[V(\varepsilon, z_0; \mathfrak{Z}^a; \theta_\tau, \theta_r) - V(\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r; \theta_\tau, \theta_r) \right] \partial_{\theta_r} \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \\ & + \varepsilon^{-N-1} \left[\begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} (\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r; \theta_\tau, \theta_r) - \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} (\varepsilon, z_0; \mathfrak{Z}^a; \theta_\tau, \theta_r) \right] \\ & - \begin{pmatrix} \mathcal{R}_N^\mathfrak{z} \\ \mathcal{R}_N^\mathbf{v} \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r). \end{aligned} \quad (1.5.6)$$

Proof. Taking into account [\(1.5.2\)](#), equation [\(1.3.3\)](#) can be reformulated as

$$\partial_\tau \left[\begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right) + \varepsilon^N \begin{pmatrix} r^\mathfrak{z} \\ r^\mathbf{v} \end{pmatrix} (\tau) \right] = \frac{1}{\varepsilon} \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0; \mathfrak{z}; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right).$$

This is the same as

$$\begin{aligned}
 & \partial_\tau \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right) + \varepsilon^{-1} \partial_{\theta_\tau} \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right) \\
 & + \varepsilon^{-2} \mathbf{V} \left(\varepsilon, z_0; \mathfrak{Z}; \frac{\mathbf{v}(\tau)}{\varepsilon} \right) \partial_{\theta_\tau} \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right) + \varepsilon^N \partial_\tau \begin{pmatrix} r^{\mathfrak{J}} \\ r^\mathbf{v} \end{pmatrix} (\tau) \\
 & = \frac{1}{\varepsilon} \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0; \mathfrak{Z}; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right)
 \end{aligned} \tag{1.5.7}$$

or equivalently

$$\begin{aligned}
 \varepsilon^N \partial_\tau \begin{pmatrix} r_\varepsilon^{\mathfrak{J}} \\ r_\varepsilon^\mathbf{v} \end{pmatrix} (\tau) & = -Op(\mathfrak{Z}^a; \partial) \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right) + \frac{1}{\varepsilon} \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0; \mathfrak{Z}^a; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right) \\
 & + \varepsilon^{-1} \left[\begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r_\varepsilon^{\mathfrak{J}}; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right) - \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0; \mathfrak{Z}^a; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right) \right] \\
 & + \varepsilon^{-2} \left[\mathbf{V} \left(\varepsilon, z_0; \mathfrak{Z}^a; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right) - \mathbf{V} \left(\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r_\varepsilon^{\mathfrak{J}}; \frac{\mathbf{v}}{\varepsilon} \right) \right] \partial_{\theta_\tau} \begin{pmatrix} \mathfrak{Z}^a \\ \mathcal{V}^a \end{pmatrix} \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right).
 \end{aligned}$$

Using (1.4.9) and then dividing by ε^N we see (1.5.6) appear. On the other hand, at the time $t = 0$, all has been done in the WKB construction to ensure that the initial data for ${}^t(\mathfrak{z}, \mathbf{v})$ in (1.3.3) matches up to the order ε^N with ${}^t(\mathfrak{z}^a, \mathbf{v}^a)$. This is why we have the right part of (1.5.5). \square

Proposition 1.5.2 (Weighted estimates on the difference ${}^t(r^{\mathfrak{J}}, r^\mathbf{v})$). *In the sense of the sup norm, uniformly with respect to $\varepsilon \in]0, \varepsilon_0]$ and $\tau \in [0, \mathcal{T}]$, we can assert that ${}^t(r^{\mathfrak{J}}, \varepsilon r^\mathbf{v}) = \mathcal{O}(1)$.*

Coming back to (1.5.2), this means that the contribution ${}^t(\varepsilon^N r^{\mathfrak{J}}, \varepsilon^N r^\mathbf{v})$ can indeed be viewed as a remainder, namely of size ε^{N-1} .

Proof. It suffices to show that we can find two constants $C \in \mathbb{R}_+^*$ and $\tilde{C} \in \mathbb{R}_+^*$ such that

$$\sup_{\varepsilon \in]0, \varepsilon_0]} \sup_{\tau \in [0, \mathcal{T}]} |r^{\mathfrak{J}}(\tau)| \leq C e^{C\mathcal{T}}, \tag{1.5.8}$$

$$\sup_{\varepsilon \in]0, \varepsilon_0]} \sup_{\tau \in [0, \mathcal{T}]} |\varepsilon r^\mathbf{v}(\tau)| \leq \tilde{C} (e^{C\mathcal{T}} + \mathcal{T}). \tag{1.5.9}$$

We start with information which is helpful to estimate the size of (1.5.6). We have:

- $\partial_{\theta_r} \mathfrak{Z}^a = \mathcal{O}(\varepsilon^2)$ since $\mathfrak{Z}_0^* = 0$ and $\mathfrak{Z}_1^* = 0$ according respectively to (1.4.14) and (1.4.33);

- The function $V(\cdot)$ is locally lipschitz with respect to \mathfrak{z} . Thus, by the mean-value theorem and because \mathfrak{Z}^a , θ_τ and θ_r stay in compact sets, we can find some $L \in \mathbb{R}_+^*$ such that

$$\sup_{\theta_\tau \in \mathbb{T}} \sup_{\theta_r \in \mathbb{T}_{r,z_0}} |V(\varepsilon, z_0; \mathfrak{Z}^a; \theta_\tau, \theta_r) - V(\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r; \theta_\tau, \theta_r)| \leq L \varepsilon^N |r|;$$

- For the same reasons and due to (1.3.4), we can find some $L' \in \mathbb{R}_+^*$ such that

$$\begin{aligned} & \sup_{\theta_\tau \in \mathbb{T}} \sup_{\theta_r \in \mathbb{T}_{r,z_0}} |A(\varepsilon, z_0; \mathfrak{Z}^a; \theta_\tau, \theta_r) - A(\varepsilon, z_0; \mathfrak{Z}^a + \varepsilon^N r; \theta_\tau, \theta_r)| \\ & \leq L' \varepsilon^{N+1} |r|; \end{aligned} \tag{1.5.10}$$

- Recall that from (1.4.10) we have

$$\sup_{\varepsilon \in]0, \varepsilon_0]} \sup_{\tau \in [0, \mathcal{T}]} \sup_{\theta_\tau \in \mathbb{T}} \sup_{\theta_r \in \mathbb{T}_{r,z_0}} |\mathcal{R}_N^{\mathfrak{z}}(\varepsilon; \tau, \theta_\tau, \theta_r)| < +\infty.$$

When computing the component $\mathcal{R}^{\mathfrak{z}}$, observe in the second line of (1.5.6) the compensation between the loss ε^{-2} and the gain ε^2 brought by $\partial_{\theta_r} \mathfrak{Z}^a$. The same applies in the third line of (1.5.6) between the loss ε^{-N-1} and the gain ε^{N+1} given by (1.5.10). It follows that

$$\begin{aligned} \sup_{\varepsilon \in]0, \varepsilon_0]} \sup_{\tau \in [0, \mathcal{T}]} |\partial_\tau r^{\mathfrak{z}}(\tau)| &= \sup_{\varepsilon \in]0, \varepsilon_0]} \sup_{\tau \in [0, \mathcal{T}]} \left| \mathcal{R}^{\mathfrak{z}} \left(\varepsilon, z_0, \mathbf{v}_0, r^{\mathfrak{z}}; \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right) \right| \\ &\leq C(|r^{\mathfrak{z}}| + 1). \end{aligned}$$

On the other hand, from (1.5.5), we know (say with the same constant C) that

$$\sup_{\varepsilon \in]0, \varepsilon_0]} |r^{\mathfrak{z}}(0)| \leq C.$$

Then, by Gronwall's lemma, we can recover (1.5.8). Now, the situation is quite different concerning the other component $\mathcal{R}^{\mathbf{v}}$. This is due to the contribution of

$$\varepsilon^{-N-1} \left[V_0 \left(z_0; \mathfrak{Z}^a + \varepsilon^N r; \theta_\tau, \theta_r \right) - V_0 \left(z_0; \mathfrak{Z}^a; \theta_\tau, \theta_r \right) \right]$$

which can actually be of large size ε^{-1} . Taking this into account, we can only assert that

$$\begin{aligned} \sup_{\varepsilon \in]0, \varepsilon_0]} \sup_{\tau \in [0, \mathcal{T}]} |\varepsilon \partial_\tau r^\nu(\tau)| &\leq \sup_{\varepsilon \in]0, \varepsilon_0]} \sup_{\tau \in [0, \mathcal{T}]} |\varepsilon \mathcal{R}^\nu \left(\varepsilon, z_0, \mathbf{v}_0, r^j; \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}(\tau)}{\varepsilon} \right)| \\ &\leq \tilde{C}(|r^j| + 1). \end{aligned}$$

The right-hand side does not depend on \mathbf{v} . It can be bounded as indicated in (1.5.8). Then, after integration in time, this yields (1.5.9). \square

The drawback with (1.5.2) is the presence in the right hand side of the unknown function $\mathbf{v}(\tau)$. This is remedied in the next subsection where \mathbf{v} is replaced by \mathbf{v}^f .

1.5.2 Description of the redressed field in terms of the frozen phase

The second line of (1.5.2) can be interpreted as an implicit relation on \mathbf{v} . This is not very informative because the explicit oscillating content of $\mathbf{v}(\cdot)$ remains to be clarified. Now, this may be achieved by cutting \mathbf{v} into the (well determined) frozen phase \mathbf{v}^f introduced at the level of (1.5.3). Next we have a description of the exact phase \mathbf{v} through a WKB expansion involving only the frozen phase \mathbf{v}^f .

Lemma 1.5.3. *Fix $N \geq 2$. Then there exist profiles*

$$\mathcal{V}_j(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) \in C^\infty(\mathbb{R}^n \times \mathbb{R} \times [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r, z_0}; \mathbb{R}), \quad j \in \{1, \dots, N-2\}, \quad (1.5.11)$$

which can be computed from the \mathcal{V}_k with $k \leq j$ with in particular

$$\mathcal{V}_1(\tau, \theta_\tau, \theta_r) = \bar{\mathcal{V}}_1(\tau, \theta_\tau), \quad (1.5.12)$$

$$\mathcal{V}_2(\tau, \theta_\tau, \theta_r) = \mathcal{V}_2\left(\tau, \theta_\tau, \theta_r + \bar{\mathcal{V}}_1(\tau, \theta_\tau)\right), \quad (1.5.13)$$

and which are adjusted in such a way that, in terms of the sup norm, we have

$$\mathbf{v}(\tau) = \mathbf{v}^f(\tau) + \sum_{j=1}^{N-2} \varepsilon^j \mathcal{V}_j \left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(z_0, \mathbf{v}_0; \tau)}{\varepsilon} \right) + \mathcal{O}(\varepsilon^{N-1}). \quad (1.5.14)$$

Note a loss of precision from ε^N to ε^{N-1} when passing from the description (1.5.2) to (1.5.14). This is coherent with the loss recorded in (1.5.9). By convention, we set $\mathcal{V}_{-1} :=$

\mathcal{V}_{-1} and $\mathcal{V}_0 := \mathcal{V}_0$. By this way, the formulation (1.5.14) becomes compatible with (1.2.18). Moreover, from (1.2.15) and (1.4.44), we can infer that the value of $\mathcal{V}_{-1}(\tau)$ is positive as soon as $\tau > 0$. Thus, frequencies of size ε^{-2} are created at time $\tau = 0$, and then they persist.

Remark 1.5.4 (Exact phase vs. frozen phase). It is important to point out that neither \mathbf{v} nor \mathbf{v}^f are phases in the usual sense of the term, since they both still depend on ε . The difference is that \mathbf{v} is (a component of) the unknown solution whereas \mathbf{v}^f can be derived explicitly from the WKB calculus. At the end, there remains

$$\mathbf{v}(\tau) = \mathbf{v}^f(\tau) + \sum_{j=1}^{N-2} \varepsilon^j \mathcal{V}_j \left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \bar{\mathcal{V}}_{-1} \rangle(\tau)}{\varepsilon^2} + \frac{\bar{\mathcal{V}}_0(\tau, \frac{\tau}{\varepsilon})}{\varepsilon} \right) + \mathcal{O}(\varepsilon^{N-1}). \quad (1.5.15)$$

The sole use of the time phase τ and of the rapid phase $\langle \bar{\mathcal{V}}_{-1} \rangle(\tau)$ would not be consistent with an expansion of \mathfrak{z} and \mathbf{v} in terms of profiles (not depending on ε) due to the presence of the extra (large) shift $\bar{\mathcal{V}}_0(\tau, \frac{\tau}{\varepsilon})/\varepsilon$. We had a choice of whether to make some phases or some profiles depend on ε . We have selected the first option.

Proof of Lemma 1.5.3. The idea is to seek an expression $\mathcal{V}^e(\varepsilon, \tau, \theta_\tau, \theta_r, r)$ that is adjusted in such a way that

$$\mathbf{v}(\tau) = \mathbf{v}^f(\tau) + \varepsilon \mathcal{V}^e \left(\varepsilon, \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(\tau)}{\varepsilon}, \varepsilon^{N-1} r^{\mathbf{v}(\tau)} \right). \quad (1.5.16)$$

In view of (1.5.2), this amounts to finding \mathcal{V}^e so that

$$\begin{aligned} & \mathcal{V}^e \left(\varepsilon, \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(\tau)}{\varepsilon}, \varepsilon^{N-1} r^{\mathbf{v}(\tau)} \right) - \bar{\mathcal{V}}_1(\tau, \theta_\tau) \\ & - \sum_{j=2}^N \varepsilon^{j-1} \mathcal{V}_j \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(\tau)}{\varepsilon} + \mathcal{V}^e \left(\varepsilon, \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(\tau)}{\varepsilon}, \varepsilon^{N-1} r^{\mathbf{v}(\tau)} \right) \right) - \varepsilon^{N-1} r^{\mathbf{v}(\tau)} \\ & = 0. \end{aligned} \quad (1.5.17)$$

To this end, it suffices to achieve the relaxed condition

$$\mathcal{H}(\varepsilon, \tau, \theta_\tau, \theta_r, r; \mathcal{V}^e(\varepsilon, \tau, \theta_\tau, \theta_r, r)) = 0 \quad (1.5.18)$$

where we have introduced

$$\mathcal{H}(\varepsilon, \tau, \theta_\tau, \theta_r, r; y) := y - \bar{\mathcal{V}}_1(\tau, \theta_\tau) - \sum_{j=2}^N \varepsilon^{j-1} \mathcal{V}_j(\tau, \theta_\tau, \theta_r + y) - r. \quad (1.5.19)$$

The expression $\mathcal{H}(x; y)$ may be viewed as a nonlinear functional in $y \in \mathbb{R}$ depending on the multidimensional parameter $x = (\varepsilon, \tau, \theta_\tau, \theta_r, r)$. From this perspective, the implicit relation (1.5.18) may define \mathcal{V}^e as a function of x by applying the implicit function theorem. Here, it is possible (and more efficient to obtain global results) to work directly. Compute

$$\partial_y \mathcal{H}(\varepsilon, \tau, \theta_\tau, \theta_r, r; y) = 1 - \sum_{j=2}^N \varepsilon^{j-1} \partial_y \mathcal{V}_j(\tau, \theta_\tau, \theta_r + y).$$

By construction, we can assert that

$$\sup_{\tau \in [0, \mathcal{T}]} \sup_{\theta_\tau \in \mathbb{T}} \sup_{\theta_r \in \mathbb{T}_{r, z_0}} \sup_{y \in \mathbb{R}} |\partial_y \mathcal{V}_j(\tau, \theta_\tau, \theta_r + y)| < +\infty. \quad (1.5.20)$$

Note that the compactness of $[0, \mathcal{T}]$, \mathbb{T} and \mathbb{T}_{r, z_0} , as well as the periodic behavior of \mathcal{V}_j with respect to θ_r , are crucial to obtain (1.5.20) notwithstanding the lack of compactness concerning $y \in \mathbb{R}$. From the bound (1.5.20), we can deduce that for all

$$(\varepsilon, \tau, \theta_\tau, \theta_r, r, y) \in [0, \varepsilon_0] \times [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r, z_0} \times \mathbb{R} \times \mathbb{R}$$

with $\varepsilon_0 \in \mathbb{R}_+^*$ small enough, we have

$$\partial_y \mathcal{H}(\varepsilon, \tau, \theta_\tau, \theta_r, r; y) > 0.$$

On the other hand, it is evident that

$$\lim_{y \rightarrow \pm\infty} \mathcal{H}(\varepsilon, \tau, \theta_\tau, \theta_r, r; y) = \pm\infty.$$

This means that we can find a unique position $\mathcal{V}^e(\varepsilon, \tau, \theta_\tau, \theta_r, r) \in \mathbb{R}$ leading to (1.5.18), with \mathcal{V}^e depending smoothly on its arguments. In particular, expanding \mathcal{V}^e near $r = 0$ and using (1.5.9), we obtain from (1.5.16) that

$$\mathbf{v}(\tau) = \mathbf{v}^f(\tau) + \varepsilon \mathcal{V}^e\left(\varepsilon, \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(\tau)}{\varepsilon}, 0\right) + \mathcal{O}(\varepsilon^{N-1}). \quad (1.5.21)$$

Since $\mathcal{V}^e(\varepsilon, \tau, \theta_\tau, \theta_r, 0)$ is also smooth in ε near $\varepsilon = 0$, by expanding \mathcal{V}^e in powers of ε , we can recover (1.5.14) in the form

$$\begin{aligned} \frac{1}{\varepsilon} [\mathbf{v}(\tau) - \mathbf{v}^f(\tau)] &= \mathcal{V}^e\left(\varepsilon, \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(\tau)}{\varepsilon}, 0\right) + \mathcal{O}(\varepsilon^{N-2}) \\ &= \sum_{j=0}^{N-2} \frac{1}{j!} \varepsilon^j \partial_\varepsilon^j \mathcal{V}^e\left(0, \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(\tau)}{\varepsilon}, 0\right) + \mathcal{O}(\varepsilon^{N-2}). \end{aligned}$$

After comparison with (1.5.14), this furnishes

$$\mathcal{V}_j(\tau, \theta_\tau, \theta_r) := \frac{1}{(j-1)!} \partial_\varepsilon^{j-1} \mathcal{V}^e(0, \tau, \theta_\tau, \theta_r, 0), \quad \forall j \in \{1, \dots, N-2\}. \quad (1.5.22)$$

From (1.5.18) and (1.5.19) written with $r = 0$, it is easy to infer that

$$\begin{aligned} \mathcal{V}^e(\varepsilon, \tau, \theta_\tau, \theta_r, 0) - \bar{\mathcal{V}}_1(\tau, \theta_\tau) \\ - \sum_{j=2}^N \varepsilon^{j-1} \mathcal{V}_j(\tau, \theta_\tau, \theta_r + \mathcal{V}^e(\varepsilon, \tau, \theta_\tau, \theta_r, 0)) = 0. \end{aligned} \quad (1.5.23)$$

The term with ε^0 in factor yields $\mathcal{V}^e(0, \tau, \theta_\tau, \theta_r, 0) = \bar{\mathcal{V}}_1(\tau, \theta_\tau)$. Applying (1.5.22) with $j = 1$, we find (1.5.12). The next derivatives of (1.5.23) with respect to ε , taken at $\varepsilon = 0$, allow to deduce successively how the profiles \mathcal{V}_j can be expressed in terms of the \mathcal{V}_k with $k \leq j$, just by applying (1.5.22). For instance, we find (1.5.13) and so on. \square

The preceding description (1.5.2) of \mathfrak{z} is not fully satisfactory. Indeed, it still involves the unknown \mathbf{v} . However, using Lemma 1.5.3, this difficulty can now easily be overcome.

Lemma 1.5.5 (Description of \mathfrak{z} through the frozen phase \mathbf{v}^f). *Fix $N \geq 2$. There exist profiles*

$$\mathcal{Z}_j(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) \in C^\infty(\mathbb{R}^n \times \mathbb{R} \times [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r, z_0}; \mathbb{R}^n), \quad (1.5.24)$$

for $j \in \{0, \dots, N-2\}$, which can be computed from the \mathfrak{Z}_k and \mathcal{V}_k (or \mathcal{V}_k) with $k \leq j$ with in particular

$$\mathcal{Z}_0(\tau, \theta_\tau, \theta_r) = \langle \bar{\mathfrak{Z}}_0 \rangle(\tau), \quad (1.5.25)$$

$$\mathcal{Z}_1(\tau, \theta_\tau, \theta_r) = \bar{\mathfrak{Z}}_1(\tau, \theta_\tau), \quad (1.5.26)$$

and which are adjusted in such a way that, in terms of the supremum norm, we have

$$\mathfrak{z}(\tau) = \sum_{j=0}^{N-2} \varepsilon^j \mathcal{Z}_j \left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(z_0, \mathbf{v}_0; \tau)}{\varepsilon} \right) + \mathcal{O}(\varepsilon^{N-1}). \quad (1.5.27)$$

Proof. We substitute the phase \mathbf{v} as described by (1.5.14) in the first component of (1.5.2). Since $\mathfrak{z}_0 \equiv \langle \bar{\mathfrak{z}}_0 \rangle$ and $\partial_{\theta_r} \mathfrak{z}_1 = 0$, we find that

$$\begin{aligned} \mathfrak{z}(\tau) &= \langle \bar{\mathfrak{z}}_0 \rangle(\tau) + \varepsilon \bar{\mathfrak{z}}_1(\tau, \theta_\tau) \\ &+ \sum_{j=2}^{N-2} \varepsilon^j \mathfrak{z}_j \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(\tau)}{\varepsilon} + \sum_{i=1}^{N-2} \varepsilon^{i-1} \mathcal{V}_i \left(\tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(\tau)}{\varepsilon} \right) \right) + \mathcal{O}(\varepsilon^{N-1}). \end{aligned} \quad (1.5.28)$$

Apply Taylor expansion to the function \mathfrak{z}_j with respect to the last variable to obtain

$$\begin{aligned} &\mathfrak{z}_j \left(\tau, \theta_\tau, \theta_r + \bar{\mathcal{V}}_1 + \sum_{i=2}^{N-2} \varepsilon^{i-1} \mathcal{V}_i \right) \\ &= \sum_{k=0}^{+\infty} \frac{1}{k!} \partial_{\theta_r}^k \mathfrak{z}_j(\tau, \theta_\tau, \theta_r + \bar{\mathcal{V}}_1) \left(\sum_{i=2}^{N-2} \varepsilon^{i-1} \mathcal{V}_i \right)^k. \end{aligned} \quad (1.5.29)$$

Then we plug (1.5.29) into (1.5.28), and collect the terms with the same power of ε in factor. Then, compare the result with the expansion (1.5.27). This allows us to determine inductively the profiles \mathcal{Z}_j from the \mathfrak{z}_k and \mathcal{V}_k with $k \leq j$. In particular, the terms which have ε^0 and ε^1 in factor furnish respectively (1.5.25) and (1.5.26). \square

1.5.3 Back to the original field

The final stage is to provide a WKB expansion concerning the original field z .

Lemma 1.5.6 (Asymptotic description of the original field). *Fix $N \geq 2$. Then there exist profiles*

$$\mathcal{Z}_j(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) \in C^\infty(\mathbb{R}^n \times \mathbb{R} \times [0, \mathcal{T}] \times \mathbb{T} \times \mathbb{T}_{r, z_0}; \mathbb{R}^n), \quad (1.5.30)$$

with $j \in \{0, \dots, N-2\}$ which can be computed from the \mathcal{Z}_k and \mathcal{V}_k (or \mathfrak{z}_k and \mathcal{V}_k) with

$k \leq j$ with in particular

$$Z_0(\tau, \theta_\tau) \equiv \bar{Z}_0(\tau, \theta_\tau) = \Xi_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau), \quad (1.5.31)$$

$$\begin{aligned} Z_1(\tau, \theta_\tau, \theta_r) &= (\bar{\mathfrak{Z}}_1(\tau, \theta_\tau) \cdot \nabla_{\mathfrak{z}}) \Xi_0(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau) \\ &\quad + \Xi_1(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau); \theta_\tau, \theta_r + \bar{\mathcal{V}}_1(\tau, \theta_\tau)). \end{aligned} \quad (1.5.32)$$

and which are adjusted in such a way that, in terms of the sup norm, we have

$$z(\varepsilon, z_0, \mathbf{v}_0; \tau) = \sum_{j=0}^{N-2} \varepsilon^j Z_j\left(z_0, \mathbf{v}_0; \tau, \frac{\tau}{\varepsilon}, \frac{\mathbf{v}^f(z_0, \mathbf{v}_0; \tau)}{\varepsilon}\right) + \mathcal{O}(\varepsilon^{N-1}). \quad (1.5.33)$$

Remark 1.5.7 (Normal form procedure: implication). When Ξ is as in (1.3.9), we have $\partial_{\theta_\tau} \Xi_0 \equiv 0$, and therefore the dependence of \bar{Z}_0 on θ_τ is not activated. Then, there remains $Z_0 \equiv \langle \bar{Z}_0 \rangle$.

Proof. Using (1.3.2), the expression $z(\tau)$ can be recovered from Ξ , \mathbf{v} and \mathfrak{z} which can be extracted respectively through (1.3.6), (1.5.14) and (1.5.27). By combining this information, we find the constraint

$$\begin{aligned} &\sum_{j=0}^{N-2} \varepsilon^j Z_j(\tau, \theta_\tau, \theta_r) \\ &= \Xi_0\left(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau) + \sum_{j=1}^{N-2} \varepsilon^j \mathcal{Z}_j(\tau, \theta_\tau, \theta_r); \theta_\tau\right) + \mathcal{O}(\varepsilon^{N-1}) + \varepsilon \Xi_1\left(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(\tau) \right. \\ &\quad \left. + \sum_{j=1}^{N-2} \varepsilon^j \mathcal{Z}_j(\tau, \theta_\tau, \theta_r); \theta_\tau, \theta_r + \bar{\mathcal{V}}_1 + \sum_{j=2}^{N-2} \varepsilon^{j-1} \mathcal{V}_j(\tau, \theta_\tau, \theta_r)\right). \end{aligned} \quad (1.5.34)$$

Recall that Ξ_0 is the mean flow (Lemma 1.3.10) and that Ξ_1 can be deduced from (1.3.37). Taylor's Theorem in both variables \mathfrak{z} and θ_r can be applied to develop the right hand side of (1.5.34) in powers of ε . Then, by identifying the terms with the same power of ε in factor, we can obtain explicit formulas yielding the Z_j in terms of the \mathcal{Z}_k and \mathcal{V}_k .

For ε^0 , we obviously obtain (1.5.31). The expression having ε in factor at the level of (1.5.34) is composed of two contributions. The first coming from Ξ_0 yields the first line of (1.5.32); the second issued from Ξ_1 leads to the second line of (1.5.32). \square

Proof of Theorem 1.2.7. To conclude, it suffices to compile what has been done before. The expansion (1.2.17) is the same as (1.5.33) in Lemma 1.5.6. On the other hand, the description of the exact phase \mathbf{v} is achieved in (1.5.14), at the level of Lemma 1.5.3. The

$\mathcal{O}(\varepsilon^\infty)$ in (1.2.17) and (1.2.18) can be obtained by just varying the choice of N , with an arbitrary remainder of size ε^{N-1} which is controlled at the level of Proposition 1.5.2. \square

1.6 Application to Hamilton-Jacobi equations

Let $d \in \mathbb{N}^*$. We work with the scalar function $H : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$,

$$(\tau, x, u, p) \mapsto H(\tau, x, u, p).$$

Given some initial data $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$, the Hamilton-Jacobi equation built with H and u_0 takes the following form

$$\partial_\tau u + H(\tau, x, u, \nabla_x u) = 0, \quad u(0, x) = u_0(x). \quad (1.6.1)$$

The study of evolution equations like (1.6.1) is fundamental in classical mechanics. It is a long-standing concern [Ben77] which has motivated many contributions. The discussion depends heavily on the functional framework retained on H and u . Roughly speaking:

- (GW) Global weak solutions u can be constructed by compactness arguments (CA), see [CL83] and subsequent works. But uniqueness and stability require additional constraints on both H (typically convexity conditions with respect to p) and u (entropic conditions);
- (LS) Local smooth solutions u are available by the method of characteristics (MC). But this approach can work only under minimal smoothness conditions (say H and u_0 in \mathcal{C}^2) and as long as the spatial projections of the (phase space) characteristics do not cross.

In Subsection 1.6.1, we introduce a small parameter $\varepsilon \in]0, \varepsilon_0]$ (going to zero) at the level of the system (1.6.1) in order to obtain a family of Cauchy problems of the type (1.1.5). In addition, we comment the effects of introducing such parameter in this system. In Subsection 1.6.2, we connect the resolution of the PDE (1.1.5) to the one of the ODE (1.1.12). Then, we show Theorem 1.1.5 in three steps: in Subsection 1.6.3, we detail the behavior of the spatial characteristic $x(\varepsilon, \cdot)$; in Subsection 1.6.4, we construct its inverse map $x^{-1}(\varepsilon, \cdot)$; and in the last Subsection 1.6.5, we complete the proof.

1.6.1 Oscillating framework

To go beyond the standard results, a natural strategy is to implement a parameter (here $\varepsilon \in]0, \varepsilon_0]$) whose effect is to break (asymptotically when ε goes to zero) the usual assumptions. By this way, it can be possible to better target some underlying difficulties, and then to solve them. This is done in Paragraph [1.6.1.1](#) by introducing oscillations. In Paragraph [1.6.1.2](#), we describe the strategy which allows to achieve the proof of Theorem [1.1.5](#).

1.6.1.1 Data adjustment

Recall that we have introduced the variables u and p in the introduction which serve to replace respectively the terms εu and $\nabla_x u$. In addition, for $\star \in \{\tau, u\}$, we have denoted by θ_\star the periodic variable which is aimed to be replaced with the quotient \star/ε . We also had that the expression \mathbf{H} introduced at the level of [\(1.1.2\)](#) is indeed a function of the variables $(\varepsilon, \theta_\tau, x, u, p, \theta_u)$, which is assumed to be smooth on its domain of definition. As indicated in [\(1.1.4\)](#), the function $\mathbf{H}(\varepsilon, \cdot)$ can be expanded in powers of ε near $\varepsilon = 0$. Now, replace H and u_0 inside [\(1.6.1\)](#) by H_ε and $u_{0\varepsilon}$ as indicated below:

- The hamiltonian $H_\varepsilon(\cdot) \equiv H(\varepsilon, \cdot)$ may depend on $\varepsilon \in]0, \varepsilon_0]$ according to

$$H_\varepsilon(\tau, x, u, p) \equiv H(\varepsilon, \tau, x, u, p) = \frac{1}{\varepsilon} \mathbf{H}\left(\varepsilon, \frac{\tau}{\varepsilon}, x, \varepsilon u, p, \frac{u}{\varepsilon}\right). \quad (1.6.2)$$

When $\mathbf{H}_0 \neq 0$, the source term H_ε is of large amplitude ε^{-1} , and it does imply oscillations (in both τ and u) at frequencies of size ε^{-1} . The role of ε is precisely to impact the \mathcal{C}^1 -estimates (on H) which become non uniform in $\varepsilon \in]0, \varepsilon_0]$;

- The initial data u_0 inside [\(1.6.1\)](#) may depend on ε . More precisely, with \mathcal{U}_0 smooth and as in [\(1.1.1\)](#)-[\(1.1.3\)](#), we impose $u_{0\varepsilon}(x) = \mathcal{U}_0(\varepsilon, x)$.

And thus, the initial value problems [\(1.6.1\)](#) with $H \equiv H_\varepsilon$ as in [\(1.6.2\)](#) and $u_\varepsilon(0, \cdot) = u_{0\varepsilon}$ as above is exactly the Hamilton-Jacobi equation [\(1.1.5\)](#). From now on, we consider the smooth solutions $u_\varepsilon \equiv u_\varepsilon(\tau, x) \equiv u(\varepsilon, \tau, x)$, sometimes simply denoted by u , to the the initial value problems [\(1.1.5\)](#).

Remark 1.6.1 (Return to a more standard regime). Applying the method of characteristics, a smooth solution u_ε does exist (locally in space) on a maximal time interval $[0, \mathcal{T}_\varepsilon[$, with $\mathcal{T}_\varepsilon \in \mathbb{R}_+^*$ that may shrink to zero when ε goes to 0. Assuming that $\mathcal{U}_{00} \equiv 0$, changing

τ into $\varepsilon^2 \tilde{\tau}$ and u into $\varepsilon \tilde{u}$, we obtain

$$\partial_{\tilde{\tau}} \tilde{u}_\varepsilon + \mathbf{H}(\varepsilon, \varepsilon \tilde{\tau}, x, \varepsilon^2 \tilde{u}, \varepsilon \nabla_x \tilde{u}, \tilde{u}) = 0, \quad \tilde{u}_\varepsilon(0, \cdot) = \varepsilon^{-1} u_{0\varepsilon} = O(1). \quad (1.6.3)$$

It is clear that the lifespan associated with (1.6.3) is uniform in $\varepsilon \in]0, \varepsilon_0]$, and therefore that $\varepsilon^2 \lesssim \mathcal{T}_\varepsilon$. When $\mathcal{U}_{00} \neq 0$, such a lower bound is no more evident. This means that, in comparison with the first well understood situation (1.6.3), the study of (1.1.5) for $\tau \sim 1$ corresponds to a very long time investigation $\tilde{\tau} \sim \varepsilon^{-2}$ for large initial data of size ε^{-1} . The latter difficulty is of course partly offset by the (nonlinear) periodic behavior of \mathbf{H} with respect to θ_u .

1.6.1.2 Strategy, intermediate results and notations

To construct solutions u_ε (uniformly in ε) and to justify asymptotic results (when $\varepsilon \rightarrow 0$), the above-mentioned approaches (GW) and (LS) face two significant barriers:

- (i) Compactness arguments (CA) are not accompanied by a (strong form of) stability allowing to compare exact and approximate solutions;
- (ii) The method of characteristics (MC) may be subjected (after spatial projection) to crossing problems at times \mathcal{T}_ε vanishing when ε goes to zero.

The first challenge (i) seems difficult to overcome. Indeed, the potential lack of control makes it impossible to compare the weak and approximate solutions with each other. In particular, in the continuation of the indent (i), a method relying on the absorption of small error terms cannot be implemented (with quantitative estimates). On the other hand, for reasons that have been already discussed in Remark 1.2.12, other more direct tools like homogenization [MV10] or multiscale young measures [AF09b] are not amenable to capture the leading behavior of u_ε . And they cannot be implemented to compare the exact solution to the approximate one with a high degree of precision. To obtain accurate descriptions of the solution u_ε , we must stick to the approach (ii).

The link between (1.1.5) and (1.1.12) is achieved through Lemma 1.6.2 together with the representation formula (1.6.8). Then, we have to detail inside (1.6.8) the content of $u(\varepsilon, \cdot)$ and $x^{-1}(\varepsilon, \cdot)$. To this end, our strategy is to:

- Exhibit (Lemma 1.6.5) the asymptotic expansions of the spatial projections $x(\varepsilon, \tau, x)$ of the (phase space) characteristics. This requires to check the validity of Assumptions 1.2.4 and 1.2.6 (of Subsection 1.2.1) in the contexts inherited from (1.1.12);

- Prove (this is done in Lemmas [1.6.9](#) and [1.6.10](#)) that the map $x \mapsto \tilde{x} := \mathbf{x}(\varepsilon, \tau, x)$ is (at least for τ small enough) uniformly in $\varepsilon \in]0, \varepsilon_0]$, a local diffeomorphism and find the oscillating description of the corresponding inverse \mathbf{x}^{-1} ;
- Exploit (Theorem [1.1.5](#)) the formula [\(1.6.8\)](#) to reveal the final oscillating structure of u_ε .

Now, we make a compilation of some notations that have been or will be involved. It is also to provide additional clarification (to avoid confusion and misunderstandings). The symbol θ is always used for a periodic (scalar) variable. But, it may be connected to the PDE setting (like θ_τ and θ_u in Section [1.1](#)) or to the ODE context (like θ_τ and θ_r in Section [1.2](#)). The connection in the next Subsection [1.6.2](#) will imply some identifications between these variables. Keep in mind that

$$\theta_u \equiv \theta_r \quad (\text{for } u/\varepsilon \text{ or } \mathbf{v}/\varepsilon).$$

On the other hand, the distinction between θ_r° (in Lemma [1.6.5](#)) and $\hat{\theta}_r^\circ$ (Theorem [1.1.5](#) and Lemma [1.6.10](#)) comes from the fact that different phases come to replace θ_r° and $\hat{\theta}_r^\circ$.

1.6.2 Connection to differential equations

In Paragraph [1.6.2.1](#), we will apply the method of characteristics in the context of [\(1.1.5\)](#) in order to deduce a system of ordinary differential equations which allows to solve [\(1.1.5\)](#) and which may fit with [\(1.1.12\)](#). In Paragraph [1.6.2.2](#), we provide the readers with some comments on assumptions and main result. Indeed, we comment the assumptions related to the PDE [\(1.1.5\)](#) in the context inherited from the ODE [\(1.1.12\)](#) and we give some interpretations of Theorem [1.1.5](#).

1.6.2.1 Method of characteristics

Assume that $u(\varepsilon, \cdot)$ is a local smooth solution to [\(1.1.5\)](#). Then, we consider a local solution $\mathbf{x} \equiv \mathbf{x}(\varepsilon, \tau, x)$ to the self-contained system

$$\begin{aligned} \dot{\mathbf{x}} &:= \frac{d\mathbf{x}}{d\tau} = \frac{1}{\varepsilon} \nabla_p \mathbf{H} \left(\varepsilon, \frac{\tau}{\varepsilon}, \mathbf{x}, \varepsilon u(\varepsilon, \tau, \mathbf{x}), \nabla_x u(\varepsilon, \tau, \mathbf{x}), \frac{u(\varepsilon, \tau, \mathbf{x})}{\varepsilon} \right), \\ \mathbf{x}(\varepsilon, 0, x) &= x. \end{aligned} \tag{1.6.4}$$

Define

$$\mathbf{p} \equiv \mathbf{p}(\varepsilon, \tau, x) := \nabla_x u\left(\varepsilon, \tau, \mathbf{x}(\varepsilon, \tau, x)\right), \quad \mathbf{u} \equiv u(\varepsilon, \tau, x) := u\left(\varepsilon, \tau, \mathbf{x}(\varepsilon, \tau, x)\right). \quad (1.6.5)$$

Lemma 1.6.2 (Tracking the characteristics). *The time evolution of $(\mathbf{x}, \mathbf{p}, u) \equiv (\mathbf{x}, \mathbf{p}, u)(\varepsilon, \tau, x)$ is governed by the following system of coupled equations*

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{1}{\varepsilon} \nabla_{\mathbf{p}} \mathbf{H}\left(\varepsilon, \frac{\tau}{\varepsilon}, \mathbf{x}, \varepsilon \mathbf{u}, \mathbf{p}, \frac{\mathbf{u}}{\varepsilon}\right), \\ \dot{\mathbf{p}} &= -\left(\frac{1}{\varepsilon} \nabla_x \mathbf{H} + \partial_u \mathbf{H} \mathbf{p} + \frac{1}{\varepsilon^2} \partial_{\theta_u} \mathbf{H} \mathbf{p}\right)\left(\varepsilon, \frac{\tau}{\varepsilon}, \mathbf{x}, \varepsilon \mathbf{u}, \mathbf{p}, \frac{\mathbf{u}}{\varepsilon}\right), \\ \dot{u} &= \frac{1}{\varepsilon} (\nabla_{\mathbf{p}} \mathbf{H} \cdot \mathbf{p} - \mathbf{H})\left(\varepsilon, \frac{\tau}{\varepsilon}, \mathbf{x}, \varepsilon \mathbf{u}, \mathbf{p}, \frac{\mathbf{u}}{\varepsilon}\right), \end{aligned} \quad (1.6.6)$$

together with

$$(\mathbf{x}, \mathbf{p}, u)(\varepsilon, 0, x) = \left(x, \nabla_x \mathcal{U}_0(\varepsilon, x), \mathcal{U}_0(\varepsilon, x)\right). \quad (1.6.7)$$

In view of (1.6.7), x is the spatial foot of the characteristic emanating from $(\mathbf{x}, \mathbf{p}, u)(\varepsilon, 0, x)$. The spatial projection of this characteristic is located at the time τ at the position $\mathbf{x}(\varepsilon, \tau, x)$. When the map $x \mapsto \tilde{x} = \mathbf{x}(\varepsilon, \tau, x)$ is a local diffeomorphism, the inverse $x^{-1}(\varepsilon, \tau, \tilde{x})$ does exist and it furnishes a unique feedback allowing to solve (1.1.5). Then, we can simply recover $u(\varepsilon, \cdot)$ through

$$u(\varepsilon, \tau, \tilde{x}) = u\left(\varepsilon, \tau, x^{-1}(\varepsilon, \tau, \tilde{x})\right). \quad (1.6.8)$$

The inversion formula (1.6.8) makes the transition from the Lagrangian point of view in (1.6.6), where \mathbf{u} and \mathbf{x} are functions of (τ, x) , to the Eulerian perspective where u is a function measuring a quantity at the location \tilde{x} through which the motion flows as time passes. In view of (1.6.8), to determine $u(\varepsilon, \cdot)$ at the position (τ, \tilde{x}) , we need to compose the (oscillating) quantity $u(\varepsilon, \tau, x)$ with the (oscillating) position $x \equiv x^{-1}(\varepsilon, \tau, \tilde{x})$. In doing so, we must face a *composition* of oscillations, where it is crucial to know precisely how the functions $u(\cdot)$ and $x^{-1}(\cdot)$ depend respectively on (ε, τ, x) and $(\varepsilon, \tau, \tilde{x})$.

Proof of Lemma 1.6.2. In view of (1.6.5), the first equation of (1.6.6) is just a reformulation of (1.6.4). Now, the definition (1.6.5) leads to

$$\dot{\mathbf{p}} = (\nabla_x \partial_\tau u)(\tau, x) + \left((\dot{\mathbf{x}} \cdot \nabla_x) \nabla_x u\right)(\tau, x), \quad \dot{u} = \partial_\tau u(\tau, x) + (\dot{\mathbf{x}} \cdot \nabla_x) u(\tau, x). \quad (1.6.9)$$

Compute the spatial gradeant of (1.1.5) to obtain

$$\nabla_x \partial_\tau u + \frac{1}{\varepsilon} (\nabla_p \mathbf{H} \cdot \nabla_x) \nabla_x u + \frac{1}{\varepsilon} \nabla_x \mathbf{H} + \partial_u \mathbf{H} \nabla_x u + \frac{1}{\varepsilon^2} \partial_{\theta_u} \mathbf{H} \nabla_x u = 0. \quad (1.6.10)$$

Taking into account (1.6.4), the first two terms of (1.6.10) coincide with \dot{p} . By this way, we can recognize the second equation inside (1.6.6). Finally, combining the second part of (1.6.9) with (1.1.5), (1.6.4) and (1.6.5), we find the third equation of (1.6.6). \square

1.6.2.2 Comments on assumptions and main result

Starting from (1.6.6), there are different ways of falling within the context of (1.1.12). Indeed, the connection between (1.1.5) and (1.1.12) can be achieved through (1.6.6) by specifying the values of z and \mathbf{v} in terms of \mathbf{x} , \mathbf{p} and u . When doing this, care must be taken to recover the special structure of (1.1.12). The selection of $\mathbf{v} := u$ is a natural choice. Now, one is tempted to simply take $z = {}^t(\mathbf{x}, \mathbf{p})$. But, to ensure that the profiles \mathbf{A} and \mathbf{V} do not depend on $\mathbf{v} \equiv u$ as required in (1.1.12), we must incorporate εu as a component of z . For this reason, we work with

$$\begin{aligned} z &= {}^t(z_x, z_p, z_u) := {}^t(\mathbf{x}, \mathbf{p}, \varepsilon u) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \equiv \mathbb{R}^n, \\ \mathbf{v} &:= u, \quad n = 2d + 1. \end{aligned} \quad (1.6.11)$$

We want to be consistent with the notation used in Sections 1.3, 1.4, and 1.5. To this end, we compare the system on (z, \mathbf{v}) that is issued from (1.6.6)-(1.6.11) with (1.1.12). With $\mathbf{A} = {}^t(\mathbf{A}_x, \mathbf{A}_p, \mathbf{A}_u)$ as well as $\theta_u \equiv \theta_r$, these two systems can be identified on condition that

$$\begin{aligned} \mathbf{A}_x(\varepsilon; z; \theta_\tau, \theta_r) &:= \nabla_p \mathbf{H}(\varepsilon, \theta_\tau, z_x, z_u, z_p, \theta_r), \\ \mathbf{A}_p(\varepsilon; z; \theta_\tau, \theta_r) &:= -(\nabla_x \mathbf{H} + \varepsilon \partial_u \mathbf{H} z_p + \varepsilon^{-1} \partial_{\theta_u} \mathbf{H} z_p)(\varepsilon, \theta_\tau, z_x, z_u, z_p, \theta_r), \\ \mathbf{A}_u(\varepsilon; z; \theta_\tau, \theta_r) &:= \varepsilon [(z_p \cdot \nabla_p) \mathbf{H} - \mathbf{H}](\varepsilon, \theta_\tau, z_x, z_u, z_p, \theta_r), \\ \mathbf{V}(\varepsilon; z; \theta_\tau, \theta_r) &:= [(z_p \cdot \nabla_p) \mathbf{H} - \mathbf{H}](\varepsilon, \theta_\tau, z_x, z_u, z_p, \theta_r). \end{aligned} \quad (1.6.12)$$

The function \mathbf{A}_p must be smooth near $\varepsilon = 0$. This could be inconsistent with the weight ε^{-1} remaining in the second line of (1.6.12). On the other hand, in coherence with Assumption 1.2.6, the expression \mathbf{V}_0 must be positive. These considerations lead to Assumptions 1.1.1 and 1.1.2. Then, with the convention $\mathbf{H}_{-1} \equiv 0$ and $\mathbf{A}_j = {}^t(\mathbf{A}_{jx}, \mathbf{A}_{jp}, \mathbf{A}_{ju})$, for all $j \in \mathbb{N}$, we

find that

$$\begin{aligned}
 A_{jx} &:= \nabla_p H_j, \\
 A_{jp} &:= -\nabla_x H_j - \partial_u H_{j-1} z_p - \partial_{\theta_u} H_{j+1} z_p, \\
 A_{ju} &:= (z_p \cdot \nabla_p) H_{j-1} - H_{j-1}, \\
 V_j &:= (z_p \cdot \nabla_p) H_j - H_j.
 \end{aligned} \tag{1.6.13}$$

In particular $A_0 = {}^t(\nabla_p H_0, -\nabla_x H_0 - \partial_{\theta_u} H_1 z_p, 0)$. Since V_0 is defined in terms of H_0 , in view of Assumption [1.1.1](#), the function V_0 does not depend on θ_r . Thus, we can apply [\(1.2.12\)](#) to see that

$$A_{\text{mf}} = \bar{A}_0 = {}^t(\nabla_p H_0, -\nabla_x H_0 - \overline{\partial_{\theta_u} H_1 z_p}, 0) = {}^t(\nabla_p H_0, -\nabla_x H_0, 0).$$

This implies that the mean flow (Definition [1.2.2](#)) which is denoted by $\Xi_{\text{mf}} \equiv \Xi_0(\mathfrak{z}; s) = {}^t(\Xi_{0x}, \Xi_{0p}, \Xi_{0u})$ is such that

$$\Xi_0(\mathfrak{z}; s) = {}^t(\Xi_{0x}, \Xi_{0p}, \mathfrak{z}_u), \quad \forall \mathfrak{z} = {}^t(\mathfrak{z}_x, \mathfrak{z}_p, \mathfrak{z}_u) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}, \tag{1.6.14}$$

where ${}^t(\Xi_{0x}, \Xi_{0p})$ satisfies (with $\theta_r \equiv s$ as in the introduction) the Hamiltonian system [\(1.1.8\)](#). Then, Assumption [1.2.4](#) amounts to the same thing as Assumption [1.1.3](#)

Remark 1.6.3 (Common situations leading to Assumption [1.1.3](#)). In general, it is not easy to test the periodic condition presented in Assumption [1.1.3](#). We furnish below a list of situations where $\Xi_0(\mathfrak{z}; \cdot)$ is indeed periodic.

Suppose that the function H_0 does not depend on x . Then, the mean flow is given by

$$\Xi_0(\mathfrak{z}; s) = {}^t(\Xi_{0x}(\mathfrak{z}; s), \mathfrak{z}_p, \mathfrak{z}_u), \quad \Xi_{0x}(\mathfrak{z}; s) = \mathfrak{z}_x + \int_0^s \nabla_p H_0(\tilde{s}, \mathfrak{z}_u, \mathfrak{z}_p) d\tilde{s}. \tag{1.6.15}$$

When $\langle \nabla_p H_0 \rangle \equiv 0$, it is obvious that the mean flow Ξ_0 is periodic in s of period 2π . When moreover $\nabla_p H_0^* \neq 0$, the function Ξ_0 is non constant (the dynamic is nontrivial).

For similar reasons, if we assume that H_0 does not depend on p and is such that $\langle \nabla_x H_0 \rangle \equiv 0$, the function $\Xi_0(\mathfrak{z}; \cdot)$ is periodic in s of period 2π .

When $\partial_s H_0 \equiv 0$, the system [\(1.1.8\)](#) is autonomous. Then, Assumption [1.1.3](#) is satisfied on condition that the level curves of $H_0(\cdot, \mathfrak{z}_u, \cdot)$ are (locally) diffeomorphic to a family of circles (existence of Liouville torus and thereby of action-angle variables).

In line with [\(1.3.37\)](#), we find that the lifting Ξ is given by

$$\Xi(\mathfrak{z}; \theta_\tau, \theta_r) = \Xi_0(\mathfrak{z}; \theta_\tau) + \varepsilon \Xi_1^*(\mathfrak{z}; \theta_\tau, \theta_r), \quad \Xi_1^* = {}^t(\Xi_{1x}^*, \Xi_{1p}^*, \Xi_{1u}^*)$$

together with

$$\begin{aligned} \Xi_{1x}^* &\equiv 0, & \Xi_{1u}^* &\equiv 0, \\ \Xi_{1p}^* &= -V_0(\Xi_0(\mathfrak{z}; \theta_\tau); \theta_\tau)^{-1} H_1^*(\theta_\tau, \Xi_{0x}, \mathfrak{z}_u, \Xi_{0p}, \theta_\tau) \Xi_{0p}(\mathfrak{z}; \theta_\tau). \end{aligned} \quad (1.6.16)$$

From definition (1.6.11) of z and \mathbf{v} together with (1.6.5) and the initial data of (1.6.4), at time $\tau = 0$, we must start with

$$(z_0, \mathbf{v}_0)(\varepsilon, x) = \left(x, \nabla_x \mathcal{U}_0(\varepsilon, x), \varepsilon \mathcal{U}_0(\varepsilon, x), \mathcal{U}_0(\varepsilon, x) \right) = \sum_{j=0}^{+\infty} \varepsilon^j (z_{0j}, \mathbf{v}_{0j})(x). \quad (1.6.17)$$

Observe in particular that

$$\begin{aligned} z_{00}(x) &= (x, \nabla_x \mathcal{U}_{00}(x), 0), \\ z_{01}(x) &= \left(0, \nabla_x \mathcal{U}_{01}(x), \mathcal{U}_{00}(x) \right), \\ \mathbf{v}_{00}(x) &= \mathcal{U}_{00}(x). \end{aligned} \quad (1.6.18)$$

From now on, we select x in a ball $B(0, R]$ for some $R \in \mathbb{R}_+^*$. Knowing what \mathbf{A} , \mathbf{V} and $\Xi_0 \equiv \Xi_{\text{mf}}$ (Lemma 1.3.10) are, we can deduce the value of \mathbf{A}_1 through (1.3.34), and then we have access to $\langle \bar{\mathfrak{z}}_0 \rangle$ through (1.4.34). Now, consider

$$K := \left\{ \Xi_0(\langle \bar{\mathfrak{z}}_0 \rangle(z_{00}(x); \tau); r); x \in B(0, R], \tau \in [0, 1], r \in \mathbb{R} \right\} \subset \mathbb{R}^n. \quad (1.6.19)$$

Since $\Xi_0(\mathfrak{z}; \cdot)$ is periodic, the set K (which is presented in Assumption 1.1.4) is compact. The main purpose of Section 1.6 is to prove Theorem 1.1.5 which is an important consequence of Theorem 1.2.7. Looking at the asymptotic description (1.1.10) of the solution u_ε , it bears noting that:

- The function ψ_ε is a phase in the sense (ii) of Section 1.1: it is smooth scalar function and its first derivatives are uniformly bounded of size at most $\mathcal{O}(1)$;
- The rapid variable $\hat{\theta}_r^0$ is activated at the level of the profiles \mathcal{U}_j as soon as $j \geq 1$;
- Recall that $\mathcal{V}_{-1} \equiv \mathcal{V}_{-1}$ and look at (1.4.44). Since V_0 is a positive function, in view of (1.6.23), (1.6.45) and (1.6.61), we can assert that $\bar{\mathcal{V}}_{-1}$ is not zero for $\tau > 0$. Thus (time) oscillations at frequency ε^{-3} do occur inside (1.1.10);
- Recall that $\mathcal{V}_0 \equiv \mathcal{V}_0 = \langle \bar{\mathcal{V}}_0 \rangle(\tau) + \bar{\mathcal{V}}_0^*(\tau, \theta_\tau)$. At time $\tau = 0$, taking into account (1.4.47), this is just $\mathbf{v}_{00} = \mathcal{U}_{00}$ which may be chosen non-zero. The same remains

true for $\tau \in \mathbb{R}_+^*$ (small enough). In view of (1.6.24), (1.6.46) and (1.6.62), we find in general that $\overline{\mathcal{W}}_0 \neq 0$. This means that the $O(\varepsilon^2)$ terms inside (1.1.11) is also essential.

The construction of the phase ψ_ε appearing in (1.1.11) is explained in what follows. In the next Section 1.6.3, we start the proof of Theorem 1.1.5 by looking at the component $x(\varepsilon, \cdot)$ of (1.6.6), which is the spatial projection of z .

1.6.3 Spatial component of the characteristics x

The first thing to check is the uniform local existence of x . Below, we prove that the map $x(\varepsilon; \cdot)$ exists locally uniformly in ε .

Lemma 1.6.4. *Select any $R \in \mathbb{R}_+^*$. Under Assumptions 1.1.1, 1.1.2 and 1.1.3, we can find some $\mathcal{T} \in \mathbb{R}_+^*$ such that the solution $(z, \mathbf{v})(\varepsilon, \tau, x)$ to (1.1.12) with \mathbf{A} and \mathbf{V} as in (1.6.12) and initial data z_0 and \mathbf{v}_0 as in (1.6.17) is, for all $\varepsilon \in]0, \varepsilon_0]$, defined on $[0, \mathcal{T}] \times B(0, R)$. In particular, the two components $x(\varepsilon, \tau, x)$ and $u(\varepsilon, \tau, x)$ exist on a uniform domain.*

Proof. Recall that we have selected positions x inside $B(0, R]$. From (1.6.17), we find that (z_0, \mathbf{v}_0) stays in the compact set $B(0, R] \times B(0, M_1] \times B(0, M_0]^2$ with

$$M_j := \|\mathcal{U}_0\|_{W^{j,\infty}([0,\varepsilon_0] \times B(0,R]; \mathbb{R})} < +\infty, \quad j \in \{0, 1\}. \quad (1.6.20)$$

This means that the initial data coming from (1.6.17) remain uniformly in a compact set. On the other hand, we have seen that Assumptions 1.1.1, 1.1.2 and 1.1.3 imply Assumption 1.2.4 when dealing with the system on (z, \mathbf{v}) which is issued from (1.6.6)-(1.6.11). Thus, it suffices to apply Theorem 1.2.5. \square

From there, the aspects about the oscillating structure of x make sense. To this end, we clarify the asymptotic expansion of x in Paragraph 1.6.3.1. Then, in Paragraph 1.6.3.2, we explore the effect of a transparency condition emanating from Assumption 1.1.3 on the differential of x : $D_x x(\cdot)$. Indeed, the latter condition furnishes a control on the size of the Jacobian matrix $D_x x(\cdot)$.

1.6.3.1 Asymptotic expansion of x

Note that Assumptions 1.1.1 and 1.1.2 also imply Assumption 1.2.6. We can apply Theorems 1.2.7 to obtain a description of (z, \mathbf{v}) which is coherent with (1.2.17) and (1.2.18).

We find that $\mathbf{x} \equiv \mathbf{x}(\varepsilon, \tau, x)$ is given by

$$\begin{aligned} \mathbf{x} = & \sum_{j=0}^{N-2} \varepsilon^j Z_{jx} \left((z_0, \mathbf{v}_0)(\varepsilon, x); \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0(\varepsilon, x); \tau)}{\varepsilon^2} \right) \\ & + \frac{\overline{\mathcal{V}}_0 \left((z_0, \mathbf{v}_0)(\varepsilon, x); \tau, \frac{\tau}{\varepsilon} \right)}{\varepsilon} + \mathcal{O}(\varepsilon^{N-1}) \end{aligned} \quad (1.6.21)$$

where the Z_{jx} , $\langle \overline{\mathcal{V}}_{-1} \rangle$ and $\overline{\mathcal{V}}_0$ are issued from the procedure of Section 1.5. Before proceeding, the expansion of $\mathbf{x}(\varepsilon, \tau, x)$ must be further simplified. Next we a result on the asymptotic oscillating description of the spatial component of the characteristics.

Lemma 1.6.5. *Under Assumptions 1.1.1, 1.1.2, and 1.1.3, the map $(\tau, x) \mapsto \mathbf{x}(\varepsilon, \tau, x)$ can be expressed according to the following asymptotic expansion (which is valid for all $N \in \mathbb{N}$ with $N \geq 2$)*

$$\begin{aligned} \mathbf{x}(\varepsilon, \tau, x) &= Z_{0x}^\circ(x; \tau, \frac{\tau}{\varepsilon}) + \varepsilon^1 Z_{1x}^\circ(x; \tau, \frac{\tau}{\varepsilon}) \\ &+ \sum_{j=2}^N \varepsilon^j Z_{jx}^\circ \left(x; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1}^\circ \rangle(x; \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0^\circ(x; \tau, \frac{\tau}{\varepsilon})}{\varepsilon} \right) + \mathcal{O}(\varepsilon^{N+1}), \end{aligned} \quad (1.6.22)$$

where, with z_{00} , z_{01} and \mathbf{v}_{00} as in (1.6.18), we have introduced

$$\langle \overline{\mathcal{V}}_{-1}^\circ \rangle(x; \tau) := \langle \overline{\mathcal{V}}_{-1} \rangle(z_{00}(x); \tau), \quad (1.6.23)$$

$$\overline{\mathcal{V}}_0^\circ(x; \tau, \theta_\tau) := (z_{01}(x) \cdot \nabla_{z_0}) \langle \overline{\mathcal{V}}_{-1} \rangle(z_{00}(x); \tau) + \overline{\mathcal{V}}_0(z_{00}(x), \mathbf{v}_{00}(x); \tau, \theta_\tau), \quad (1.6.24)$$

and where, for $j \geq 0$, the profiles $Z_{jx}^\circ(x; \tau, \theta_\tau, \theta_r^\circ)$ are periodic with respect to the two last variables $\theta_\tau \in \mathbb{T}$ and $\theta_r^\circ \in \mathbb{T}$.

Recall that the three couples of phases

$$\langle \overline{\mathcal{V}}_{-1} \rangle, \overline{\mathcal{V}}_0 \text{ in (1.6.21)}, \quad \langle \overline{\mathcal{V}}_{-1}^\circ \rangle, \overline{\mathcal{V}}_0^\circ \text{ in (1.6.22)}, \quad \widehat{\overline{\mathcal{V}}_{-1}^\circ}, \widehat{\overline{\mathcal{V}}_0^\circ} \text{ in (1.6.44)}$$

are (in general) distinct from one another. This is why the symbols θ_r , θ_r° , and $\widehat{\theta}_r^\circ$ are not the same. This is aimed at highlighting the difference between the various phases that are involved.

Proof of Lemma 1.6.5. Comparing (1.6.21) with (1.6.22), there are two improvements:

- (A) The first two terms of the expansion - that is the first line of (1.6.22) have been clarified;
- (B) The structure of the phase - that is what comes to replace θ_r° in the second line of (1.6.22) has been reduced.

We consider one item at the time.

- (A) Recall (1.5.34) and (1.6.16). Since $\Xi_{1x}^* \equiv 0$, there remains

$$\begin{aligned} & \sum_{j=0}^{N-2} \varepsilon^j Z_{jx} \left((z_0, \mathbf{v}_0)(\varepsilon, x); \tau, \theta_\tau, \theta_r \right) \\ &= \Xi_{0x} \left(\sum_{j=0}^{N-2} \varepsilon^j \mathcal{Z}_j \left((z_0, \mathbf{v}_0)(\varepsilon, x); \tau, \theta_\tau, \theta_r \right); \theta_\tau \right) + \mathcal{O}(\varepsilon^{N-1}). \end{aligned} \quad (1.6.25)$$

Keep in mind that θ_τ and θ_r must be replaced as indicated in (1.6.21). After this substitution, we can assert that

$$\begin{aligned} & \sum_{j=0}^{N-2} \varepsilon^j \mathcal{Z}_j \left((z_0, \mathbf{v}_0)(\varepsilon, x); \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1} \rangle (z_0(\varepsilon, x); \tau)}{\varepsilon^2} \right. \\ & \quad \left. + \frac{\overline{\mathcal{V}}_0 \left((z_0, \mathbf{v}_0)(\varepsilon, x); \tau, \frac{\tau}{\varepsilon} \right)}{\varepsilon} \right) \\ &= \sum_{j=0}^{N-2} \varepsilon^j \tilde{\mathcal{Z}}_j \left(x; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1}^\circ \rangle (x; \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0^\circ \left(x; \tau, \frac{\tau}{\varepsilon} \right)}{\varepsilon} \right) + \mathcal{O}(\varepsilon^{N-1}). \end{aligned} \quad (1.6.26)$$

To elucidate the origin of (1.6.26), we expand $\langle \overline{\mathcal{V}}_{-1} \rangle(\cdot; \tau)$ and $\overline{\mathcal{V}}_0(\cdot; \tau, \theta_\tau)$ composed with $(z_0, \mathbf{v}_0)(\varepsilon, x)$ in powers of ε . Knowing (1.2.16), this yields (1.6.23) and (1.6.24) as well as

$$\begin{aligned} & \langle \overline{\mathcal{V}}_{-1} \rangle (z_0(\varepsilon, x); \tau) + \varepsilon \overline{\mathcal{V}}_0 \left((z_0, \mathbf{v}_0)(\varepsilon, x); \tau, \frac{\tau}{\varepsilon} \right) \\ &= \langle \overline{\mathcal{V}}_{-1}^\circ \rangle (x; \tau) + \varepsilon \overline{\mathcal{V}}_0^\circ \left(x; \tau, \frac{\tau}{\varepsilon} \right) + \mathcal{O}(\varepsilon^2). \end{aligned} \quad (1.6.27)$$

Then, we stick to the following strategy in order to recover the profiles $\tilde{\mathcal{Z}}_j$ introduced in (1.6.26):

- (A.1) Use (1.6.27) to localize the oscillations at frequencies ε^{-2} at the position $\langle \overline{\mathcal{V}}_{-1}^\circ \rangle + \varepsilon \overline{\mathcal{V}}_0^\circ$ and incorporate the $O(1)$ -remainder inside a $O(1)$ -shift in θ_r of the $\mathcal{Z}_j \left((z_0, \mathbf{v}_0)(\varepsilon, x); \tau, \theta_\tau, \cdot \right)$. Note that this shift depends smoothly on x , τ and τ/ε , and therefore it can be incorporated inside the \mathcal{Z}_j by modifying their description;

(A.2) Expand the (new) preceding profiles $\mathcal{Z}_j(\cdot; \tau, \theta_\tau, \theta_r)$ composed with $(z_0, \mathbf{v}_0)(\varepsilon, x)$ in powers of ε by using (1.6.17);

(A.3) Gather the ε^j -terms coming from the left-hand side of (1.6.26) after applying the above two steps to recover the final expressions $\tilde{\mathcal{Z}}_j$.

In particular, we can consider (1.5.25), (1.5.26), (1.5.28), and (1.5.29) to see that

$$\begin{aligned}\mathcal{Z}_0(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) &= \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau), \\ \mathcal{Z}_1(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) &= \bar{\mathfrak{Z}}_1(z_0; \tau, \theta_\tau), \\ \mathcal{Z}_2(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r) &= \mathfrak{Z}_2(z_0, \mathbf{v}_0; \tau, \theta_\tau, \theta_r + \bar{\mathcal{V}}_1(z_0, \mathbf{v}_0; \tau, \theta_\tau)).\end{aligned}\tag{1.6.28}$$

Following the methodology of the above explanations, (1.6.28) reveals that the first three terms of the right hand side of (1.6.26) are defined as follows:

$$\tilde{\mathcal{Z}}_0(x; \tau) := \langle \bar{\mathfrak{Z}}_0 \rangle(z_{00}(x); \tau) \tag{1.6.29}$$

$$\tilde{\mathcal{Z}}_1(x; \tau, \theta_\tau) := (z_{01}(x) \cdot \nabla_{z_0}) \langle \bar{\mathfrak{Z}}_0 \rangle(z_{00}(x); \tau) + \bar{\mathfrak{Z}}_1(z_{00}(x); \tau, \theta_\tau) \tag{1.6.30}$$

$$\begin{aligned}\tilde{\mathcal{Z}}_2(x; \tau, \theta_\tau, \theta_r^\circ) &:= \mathfrak{Z}_2(z_{00}(x), \mathbf{v}_{00}(x); \tau, \theta_\tau, \theta_r^\circ + b(x; \tau, \theta_\tau)) \\ &\quad + (z_{01}(x) \cdot \nabla_{z_0}) \bar{\mathfrak{Z}}_1(z_{00}(x); \tau, \theta_\tau) + (z_{02}(x) \cdot \nabla_{z_0}) \langle \bar{\mathfrak{Z}}_0 \rangle(z_{00}(x); \tau) \\ &\quad + \frac{1}{2} D_{z_0}^2 \langle \bar{\mathfrak{Z}}_0 \rangle(z_{00}(x); \tau) (z_{01}(x), z_{01}(x))\end{aligned}\tag{1.6.31}$$

where

$$\begin{aligned}b(x; \tau, \theta_\tau) &:= \bar{\mathcal{V}}_1(z_{00}(x), \mathbf{v}_{00}(x); \tau, \theta_\tau) + \left[(z_{01}(x) \cdot \nabla_{z_0}) + \mathbf{v}_{01}(x) \partial_{v_0} \right] \bar{\mathcal{V}}_0(z_{00}(x), \mathbf{v}_{00}(x); \tau, \theta_\tau) \\ &\quad + (z_{02}(x) \cdot \nabla_{z_0}) \langle \bar{\mathcal{V}}_{-1} \rangle(z_{00}(x); \tau) + \frac{1}{2} D_{z_0}^2 \langle \bar{\mathcal{V}}_{-1} \rangle(z_{00}(x); \tau) (z_{01}(x), z_{01}(x)).\end{aligned}$$

By using (1.5.31) together with (1.6.29), the leading term issued (after expansion in powers of ε) from the right hand side of (1.6.25) is

$$Z_{0x}^\circ(x; \tau, \theta_\tau) := \Xi_{0x}(\tilde{\mathcal{Z}}_0(x; \tau); \theta_\tau) = \Xi_{0x}(\langle \bar{\mathfrak{Z}}_0 \rangle(z_{00}(x); \tau); \theta_\tau). \tag{1.6.32}$$

Remark 1.6.6 (About the content of Z_{0x}°). The expression $\langle \bar{\mathfrak{Z}}_0 \rangle$ is given by (1.4.35) with A_1 as in (1.3.34). Since $\Xi_{0u}(\mathfrak{z}; \theta_\tau) = \mathfrak{z}_u$ and because $\Xi_{1u}^* \equiv 0$, we have

$$\langle \bar{A}_{1u} \rangle = \langle \bar{A}_{1u} \rangle = \left[(\mathfrak{z}_p \cdot \nabla_p) \langle H_0 \rangle - \langle H_0 \rangle \right] (\mathfrak{z}).$$

Because of (1.1.7) we know that $\langle \bar{A}_{1u} \rangle > 0$. The component $\langle \bar{\mathfrak{z}}_{0u} \rangle$ is therefore strictly increasing (in τ), and these variations can affect (by a coupling effect) the two components $\langle \bar{\mathfrak{z}}_{0x} \rangle$ and $\langle \bar{\mathfrak{z}}_{0p} \rangle$. On the other hand, in view of Remark 1.6.3, the map Ξ_{0x} can depend on $s \equiv \theta_\tau$. In general, the function Z_{0x}° is subject to variations in both τ and θ_τ .

By applying (1.5.32) and (1.6.30) to the component Z_{1x} , since $\Xi_{1x} \equiv \Xi_{1x}^* \equiv 0$, we can collect the ε -terms from (1.6.25) after expansion to see that

$$Z_{1x}^\circ(x; \tau, \theta_\tau) := \left[\tilde{\mathfrak{Z}}_1(x; \tau, \theta_\tau) \cdot \nabla_{\mathfrak{z}} \right] \Xi_{0x} \left(\langle \bar{\mathfrak{z}}_0 \rangle(z_{00}(x); \tau); \theta_\tau \right). \quad (1.6.33)$$

In the above process, the variable θ_τ is not requested, as it could be. This is the first important simplification.

(B) Let us now consider the second improvement. To elucidate what happens for $j \geq 2$, in view of (1.6.25), it suffices to compose $\Xi_{0x}(\cdot; \theta_\tau)$ with (1.6.26). The main things have already been said at the level of A.1). There remains the form exhibited in (1.6.22) with in particular:

$$\begin{aligned} Z_{2x}^\circ(x; \tau, \theta_\tau, \theta_\tau^0) &:= \left(\tilde{\mathfrak{Z}}_2(x; \tau, \theta_\tau, \theta_\tau^0) \cdot \nabla_{\mathfrak{z}} \right) \Xi_{0x} \left(\langle \bar{\mathfrak{z}}_0 \rangle(z_{00}(x); \tau); \theta_\tau \right) \\ &+ \frac{1}{2} D_{\mathfrak{z}}^2 \Xi_{0x} \left(\langle \bar{\mathfrak{z}}_0 \rangle(z_{00}(x); \tau); \theta_\tau \right) \left(\tilde{\mathfrak{Z}}_1(x; \tau, \theta_\tau), \tilde{\mathfrak{Z}}_1(x; \tau, \theta_\tau) \right). \end{aligned} \quad (1.6.34)$$

□

Remark 1.6.7 (About the matching of initial data). It is instructive to compare the value $x(\varepsilon, 0, x)$ given by (1.6.17), that is $z_{0x}(\varepsilon, x) \equiv x$ with the formula (1.6.22) at time $\tau = 0$. Since $\mathcal{V}_{-1} \equiv \mathcal{V}_{-1}$, taking into account (1.4.44), (1.6.18), and (1.6.23), we find $\langle \bar{\mathcal{V}}_{-1}^\circ \rangle(x; 0) = \langle \bar{\mathcal{V}}_{-1} \rangle(x, \nabla_x \mathcal{U}_{00}(x), 0; 0) = 0$. On the other hand, from (1.6.24) and (1.4.47), we have

$$\bar{\mathcal{V}}_0^\circ(x; 0, 0) = \bar{\mathcal{V}}_0(x, \nabla_x \mathcal{U}_{00}(x), 0, \mathcal{U}_{00}(x); 0, 0) = \mathfrak{v}_{00}(x) \equiv \mathcal{U}_{00}(x).$$

Then, from (1.6.22), we can infer that

$$x(\varepsilon, 0, x) = \sum_{j=0}^N \varepsilon^j Z_{jx}^\circ \left(x; 0, 0, \frac{\mathfrak{v}_{00}(x)}{\varepsilon} \right) + \mathcal{O}(\varepsilon^{N+1}).$$

But, in line with Remark 1.3.9, since $\Xi_{1x}^* \equiv 0$, the initial data \mathfrak{z}_{0x} must reduce to

$$\mathfrak{z}_{0x}(\varepsilon, x) = \Xi_{0x}^{-1} \left(z_0(\varepsilon, x), 0 \right) = z_{0x}(\varepsilon, x) = x.$$

This means that $Z_{0x}^\circ(x, 0, 0) \equiv x$ and $Z_{jx}^\circ(x, 0, 0, \theta_r^\circ) \equiv 0$ for all $j \geq 1$. These properties could be deduced from the preceding construction of the Z_{jx}° . The absence of the variable θ_r° is specific to \mathfrak{z}_x and to $\tau = 0$. Since $\Xi_{1p}^* \neq 0$, the component $\mathfrak{z}_{0p}(\varepsilon, x)$ can indeed oscillate with respect to $\nu_{00}(x)/\varepsilon$. Moreover, due to coupling effects, the expressions Z_{jx}° can depend on θ_r° for $\tau > 0$.

1.6.3.2 Role of transparency conditions

The aim of this paragraph is to compute $D_x x$. It is also to show how some kind of transparency conditions (emanating from Assumption [1.1.3](#)) leads to have a control on the size of this differential. In view of the expansion [\(1.6.22\)](#), we can compute the Jacobian matrix $D_x x(\varepsilon, \tau, x)$ according to

$$\begin{aligned} D_x x(\varepsilon, \tau, x) &= D_x Z_{0x}^\circ(x; \tau, \frac{\tau}{\varepsilon}) \\ &+ \partial_{\theta_r^\circ} Z_{2x}^\circ\left(x; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(x; \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(x; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\right) \otimes \nabla_x \langle \overline{\mathcal{V}}_{-1} \rangle(x; \tau) + \mathcal{O}(\varepsilon). \end{aligned} \quad (1.6.35)$$

Recalling [\(1.1.8\)](#) and [\(1.6.32\)](#) we see that

$$\begin{aligned} Z_{0x}^\circ(x; \tau, s) &= \Xi_{0x}(\langle \overline{\mathfrak{Z}}_0 \rangle(z_{00}(x); \tau); s) \\ &= \langle \overline{\mathfrak{Z}}_{0x} \rangle(z_{00}(x); \tau) + \int_0^s \nabla_p \mathbf{H}_0\left(r, \Xi_{0x}(\langle \overline{\mathfrak{Z}}_0 \rangle(z_{00}(x); \tau); r), \right. \\ &\quad \left. \langle \overline{\mathfrak{Z}}_{0u} \rangle(z_{00}(x); \tau), \Xi_{0p}(\langle \overline{\mathfrak{Z}}_0 \rangle(z_{00}(x); \tau); r)\right) dr. \end{aligned} \quad (1.6.36)$$

In view of [\(1.6.35\)](#), the expression $D_x Z_{0x}^\circ(x; \tau, s)$ must be computed at the position $s = \tau/\varepsilon$. This means that s must be replaced by τ/ε in [\(1.6.36\)](#). At first sight, the integral (in r) from 0 up to τ/ε should furnish a contribution of size ε^{-1} , which would indicate that $D_x x(\varepsilon, \tau, x)$ is very large (and therefore out of control). However, as mentioned in Remark [1.3.13](#), Assumption [1.1.3](#) implies that

$$\langle \nabla_p \mathbf{H}_0(\cdot, \Xi_{0x}(\mathfrak{z}; \cdot), \mathfrak{z}_u, \Xi_{0p}(\mathfrak{z}; \cdot)) \rangle \equiv 0. \quad (1.6.37)$$

This identity can be viewed as a *transparency condition*: nonlinear terms that should contribute (without Assumption [1.1.3](#)) disappear in practice. As a matter of fact, denoting

by $\lfloor s \rfloor$ the integer part of the real number s , we can assert that

$$\begin{aligned} Z_{0x}^\circ(x; \tau, s) &= \langle \bar{\mathfrak{J}}_{0x} \rangle(z_{00}(x); \tau) + \int_{2\pi\lfloor s/2\pi \rfloor}^s \nabla_p \mathbf{H}_0 \left(r, \Xi_{0x}(\langle \bar{\mathfrak{J}}_0 \rangle(z_{00}(x); \tau); r), \right. \\ &\quad \left. \langle \bar{\mathfrak{J}}_{0u} \rangle(z_{00}(x); \tau), \Xi_{0p}(\langle \bar{\mathfrak{J}}_0 \rangle(z_{00}(x); \tau); r) \right) dr. \end{aligned} \quad (1.6.38)$$

We see here why the cancelation property (1.6.37) is crucial. It allows to reduce the long time integration (when s is replaced by τ/ε with $\tau > 0$ and $\varepsilon \ll 1$) in the second line of (1.6.36) to an integration over some interval of uniformly bounded size (in s), as indicated in (1.6.38). We have the following interesting expression of the differential $D_x \mathbf{x}$.

Lemma 1.6.8 (Differential of the characteristic \mathbf{x}). *Select $R \in \mathbb{R}_+^*$. Under Assumption 1.1.3, for all $\varepsilon \in [0, \varepsilon_0]$, for all $(\tau, x) \in [0, \mathcal{T}] \times B(0, R]$, we have*

$$D_x \mathbf{x}(\varepsilon, \tau, x) = \text{Id} + \mathcal{O}(\tau) + \mathcal{O}(\|D_{x,p,u} \nabla_p \mathbf{H}_0\|) + \mathcal{O}(\varepsilon). \quad (1.6.39)$$

Formula (1.6.39) with Assumption 1.1.4 are the gate to prove that the map $x \mapsto \mathbf{x}(\varepsilon; \cdot)$ is uniformly invertible. We will prove this fact in the next Section 1.6.4

Proof of Lemma 1.6.4. From (1.4.35) with (1.6.18) we have

$$\langle \bar{\mathfrak{J}}_{0x} \rangle(z_{00}(x); \tau) = x + \int_0^\tau \langle \bar{\mathfrak{A}}_{1x} \rangle(\langle \bar{\mathfrak{J}}_0 \rangle(x, \nabla_x \mathcal{U}_{00}(x), 0; r)) dr. \quad (1.6.40)$$

From (1.6.38) and (1.6.40), since $\Xi_0(\mathfrak{z}; \cdot)$ is globally bounded (since it is a periodic function), with $\|\cdot\|$ as in (1.1.9), we can already infer that

$$D_x Z_{0x}^\circ(x; \tau, s) = \text{Id} + \mathcal{O}(\tau) + \mathcal{O}(\|D_{x,p,u} \nabla_p \mathbf{H}_0\|). \quad (1.6.41)$$

On the other hand, we can exploit (1.4.41) together with $\mathcal{V}_{-1} \equiv \mathcal{V}_{-1}$, as well as (1.6.18), (1.6.13) and (1.6.23) to deduce that

$$\begin{aligned} \langle \bar{\mathcal{V}}_{-1}^\circ \rangle(x; \tau) &= \int_0^\tau \left[(\langle \bar{\mathfrak{J}}_{0p} \rangle(x, \nabla_x \mathcal{U}_{00}(x), 0; r) \cdot \nabla_p) \langle \mathbf{H}_0 \rangle \right. \\ &\quad \left. - \langle \mathbf{H}_0 \rangle \right] (\langle \bar{\mathfrak{J}}_0 \rangle(x, \nabla_x \mathcal{U}_{00}(x), 0; r)) dr. \end{aligned} \quad (1.6.42)$$

In view of (1.6.35) and taking into account (1.6.38), with $\langle \bar{\mathfrak{J}}_0 \rangle \equiv \langle \bar{\mathfrak{J}}_0 \rangle(z_{00}(x); \tau)$, we obtain

that

$$\begin{aligned}
 & D_x \mathbf{x}(\varepsilon, \tau, x) \\
 &= \text{Id} + \int_0^\tau D_x \left[\langle \bar{\mathbf{A}}_{1x} \rangle \left(\langle \bar{\mathbf{3}}_0 \rangle(x, \nabla_x \mathcal{U}_{00}(x), 0; r) \right) \right] dr \\
 &+ \int_{2\pi \lfloor \tau/2\pi\varepsilon \rfloor}^{\tau/\varepsilon} D_x \left[\nabla_p \mathbf{H}_0 \left(r, \Xi_{0x}(\langle \bar{\mathbf{3}}_0 \rangle; r), \langle \bar{\mathbf{3}}_{0u} \rangle(z_{00}(x); \tau), \Xi_{0p}(\langle \bar{\mathbf{3}}_0 \rangle; r) \right) \right] dr \\
 &+ \partial_{\theta_\tau^\circ} Z_{2x}^\circ \left(x; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \bar{\mathcal{V}}_{-1}^\circ \rangle(x; \tau)}{\varepsilon^2} + \frac{\bar{\mathcal{V}}_0^\circ(x; \tau, \frac{\tau}{\varepsilon})}{\varepsilon} \right) \otimes \nabla_x \langle \bar{\mathcal{V}}_{-1}^\circ \rangle(x; \tau) + \mathcal{O}(\varepsilon).
 \end{aligned} \tag{1.6.43}$$

Using (1.6.42), we can control the fourth line of (1.6.43) by some $\mathcal{O}(\tau)$. On the other hand, we have $0 \leq \tau/\varepsilon - 2\pi \lfloor \tau/2\pi\varepsilon \rfloor \leq 2\pi$ in such a way that $D_x \mathbf{x}(\varepsilon, \tau, x)$ is as in (1.6.39). \square

1.6.4 Inverse map \mathbf{x}^{-1}

The inverse of the spatial characteristic $\mathbf{x}(\varepsilon, \tau, x)$ is denoted by $\mathbf{x}^{-1}(\varepsilon, \tau, \tilde{x})$. The existence part of Theorem 1.1.5 relies on (1.6.8). To this end, it is important to show that $\mathbf{x}^{-1}(\varepsilon, \tau, \tilde{x})$ is uniformly defined. Below, we present the statement of the uniform local existence of \mathbf{x}^{-1} .

Lemma 1.6.9 (Uniform local existence of \mathbf{x}^{-1}). *Select $R \in \mathbb{R}_+^*$. Under Assumptions 1.1.3 and 1.1.4, for $|\tau|$ small enough, the map $x \in B(0, R] \mapsto \tilde{x} = \mathbf{x}(\varepsilon, \tau, x)$ is for all $\varepsilon \in]0, \varepsilon_0]$ (by restricting ε_0 if necessary) locally uniformly (in ε) invertible.*

Proof. The proof is based on the expansion (1.6.39). We take $\tau \leq \mathcal{T}$ and $\varepsilon \leq \varepsilon_0$ with \mathcal{T} and ε_0 small enough. We work under Assumption 1.1.4 with δ small enough. In view of (1.6.41) and (1.6.39), both $D_x Z_{0x}^\circ(x; \tau, s)$ and $D_x \mathbf{x}(\varepsilon, \tau, x)$ are of the form $\text{Id} + \mathbf{B}$ with $\|\mathbf{B}\| < 1$. Thus, for all $\tau \in [0, \mathcal{T}]$, the maps $x \mapsto Z_{0x}^\circ(x; \tau, s)$ and $x \mapsto \mathbf{x}(\varepsilon, \tau, x)$ are locally invertible (uniformly in $s \equiv \theta_\tau$ for the first map and in $\varepsilon \in]0, \varepsilon_0]$ for the second). \square

The next step is to find the asymptotic expansion of the inverse \mathbf{x}^{-1} . We have the following asymptotic description of the inverse map of the spatial component of the characteristics.

Lemma 1.6.10. *Under Assumptions 1.1.1, 1.1.2, 1.1.3, and 1.1.4, for $|\tau|$ small enough, for all $\varepsilon \in]0, \varepsilon_0]$ and for all $N \geq 2$, the inverse map $\mathbf{x}^{-1}(\varepsilon, \tau, \tilde{x})$ can be expanded according*

to

$$\begin{aligned}
 & x^{-1}(\varepsilon, \tau, \tilde{x}) \\
 &= \hat{Z}_{0x}^{\circ} \left(\tilde{x}; \tau, \frac{\tau}{\varepsilon} \right) + \varepsilon \hat{Z}_{1x}^{\circ} \left(\tilde{x}; \tau, \frac{\tau}{\varepsilon} \right) \\
 &+ \sum_{j=2}^N \varepsilon^j \hat{Z}_{jx}^{\circ} \left(\tilde{x}; \tau, \frac{\tau}{\varepsilon}, \frac{\overline{\mathcal{V}}_{-1}^{\circ}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0^{\circ}(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon} \right) + \mathcal{O}(\varepsilon^{N+1}),
 \end{aligned} \tag{1.6.44}$$

where $\hat{Z}_{0x}^{\circ} \equiv (Z_{0x}^{\circ})^{-1}$ is the local inverse of the map $x \mapsto Z_{0x}^{\circ}(x; \tau, \theta_{\tau})$, where we have introduced

$$\overline{\mathcal{V}}_{-1}^{\circ}(\tilde{x}; \tau, \theta_{\tau}) := \langle \overline{\mathcal{V}}_{-1}^{\circ} \rangle \left(\hat{Z}_{0x}^{\circ}(\tilde{x}, \tau, \theta_{\tau}); \tau \right), \tag{1.6.45}$$

$$\begin{aligned}
 & \overline{\mathcal{V}}_0^{\circ}(\tilde{x}; \tau, \theta_{\tau}) \\
 &:= \left(\hat{Z}_{1x}^{\circ}(\tilde{x}; \tau, \theta_{\tau}) \cdot \nabla_x \right) \langle \overline{\mathcal{V}}_{-1}^{\circ} \rangle \left(\hat{Z}_{0x}^{\circ}(\tilde{x}, \tau, \theta_{\tau}); \tau \right) + \overline{\mathcal{V}}_0^{\circ} \left(\hat{Z}_{0x}^{\circ}(\tilde{x}, \tau, \theta_{\tau}); \tau, \theta_{\tau} \right),
 \end{aligned} \tag{1.6.46}$$

and where, for $j \geq 0$, the profiles $\hat{Z}_{jx}^{\circ}(\tilde{x}; \tau, \theta_{\tau}, \hat{\theta}_{\tau}^{\circ})$ are periodic with respect to the two last variables $\theta_{\tau} \in \mathbb{T}$ and $\hat{\theta}_{\tau}^{\circ} \in \mathbb{T}$.

The rest of this section is devoted to the proof of Lemma [1.6.10](#). The proof is achieved in three steps: in Paragraph [1.6.4.1](#), we give the formal expansion of the inverse x^{-1} ; at ε^2 -order, we face a strong nonlinearity which is overcome by implementing the Hadamard's global inverse function theorem in Paragraph [1.6.4.2](#); finally, in Paragraph [1.6.4.3](#), we complete the proof of Lemma [1.6.10](#) by justifying the formal WKB expansion.

1.6.4.1 Formal equations for x^{-1}

We turn now to the proof of [\(1.6.44\)](#). By the definition of x^{-1} , we have

$$x \left(\varepsilon, \tau, x^{-1}(\varepsilon, \tau, \tilde{x}) \right) = \tilde{x}. \tag{1.6.47}$$

We can seek x^{-1} in the form of an asymptotic expansion similar to [\(1.6.22\)](#), like in [\(1.6.44\)](#), that is with a hat “ $\hat{\cdot}$ ” on each expression to make the distinction. In other words, we can postulate [\(1.6.44\)](#) and use [\(1.6.47\)](#) to check that [\(1.6.44\)](#) is indeed convenient. This means to deal with

$$\sum_{j=0}^N \varepsilon^j \hat{Z}_{jx}^{\circ} \left(x^{-1}; \tau, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1}^{\circ} \rangle(x^{-1}; \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0^{\circ}(x^{-1}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon} \right) + \mathcal{O}(\varepsilon^{N+1}) = \tilde{x}. \tag{1.6.48}$$

The leading order term inside (1.6.48) gives rise to $Z_{0x}^\circ(\hat{Z}_{0x}^\circ; \tau, \theta_\tau) = \tilde{x}$. This relation can be achieved with $\hat{Z}_{0x}^\circ \equiv (Z_{0x}^\circ)^{-1}$. The next term, the one which has ε in factor inside (1.6.48), leads to

$$\hat{Z}_{1x}^\circ \equiv \hat{Z}_{1x}^\circ(\tilde{x}; \tau, \theta_\tau) := -D_x Z_{0x}^\circ(\hat{Z}_{0x}^\circ(\tilde{x}; \tau, \theta_\tau); \tau, \theta_\tau)^{-1} Z_{1x}^\circ(\hat{Z}_{0x}^\circ(\tilde{x}; \tau, \theta_\tau); \tau, \theta_\tau).$$

As foreseen, the rapid variable $\hat{\theta}_r^\circ$ can be activated at the level of the profiles \hat{Z}_{jx}° only for $j \geq 2$. Now, in coherence with (1.6.45) and (1.6.46), we can assert that

$$\begin{aligned} & \frac{1}{\varepsilon^2} \langle \overline{\mathcal{V}}_{-1}^\circ \rangle(x^{-1}(\varepsilon, \tau, \tilde{x}); \tau) + \frac{1}{\varepsilon} \overline{\mathcal{V}}_0^\circ(x^{-1}(\varepsilon, \tau, \tilde{x}); \tau, \theta_\tau) \\ &= \frac{1}{\varepsilon^2} \overline{\mathcal{V}}_{-1}^\circ(\tilde{x}; \tau, \theta_\tau) + \frac{1}{\varepsilon} \overline{\mathcal{V}}_0^\circ(\tilde{x}; \tau, \theta_\tau) \\ & \quad + (\hat{Z}_{2x}^\circ \cdot \nabla_x) \langle \overline{\mathcal{V}}_{-1}^\circ \rangle(\hat{Z}_{0x}^\circ(\tilde{x}, \tau, \theta_\tau); \tau) + \frac{1}{2} D_x^2 \langle \overline{\mathcal{V}}_{-1}^\circ \rangle(\hat{Z}_{1x}^\circ, \hat{Z}_{1x}^\circ) \\ & \quad + (\hat{Z}_{1x}^\circ \cdot \nabla_x) \overline{\mathcal{V}}_0^\circ(\hat{Z}_{0x}^\circ(\tilde{x}, \tau, \theta_\tau); \tau, \theta_\tau) + \mathcal{O}(\varepsilon). \end{aligned}$$

We may see (1.6.48) as a consequence of a relaxed condition involving τ , \tilde{x} , θ_τ and $\hat{\theta}_r^\circ$. Then, we can work with \tilde{x} , θ_τ and $\hat{\theta}_r^\circ$ fixed in compact sets. In what follows, these variables are mentioned only when it is necessary to avoid confusion. In this perspective, the contribution which has ε^2 in factor inside (1.6.48) can be written

$$\hat{Z}_{2x}^\circ + \mathcal{F}(\tau, \hat{Z}_{2x}^\circ) = 0 \tag{1.6.49}$$

with by construction

$$\begin{aligned} & \mathcal{F}(\tau, \hat{Z}_{2x}^\circ) \\ &:= D_x Z_{0x}^\circ(\hat{Z}_{0x}^\circ; \tau, \theta_\tau)^{-1} Z_{2x}^\circ(\hat{Z}_{0x}^\circ; \tau, \theta_\tau, \hat{\theta}_r^\circ + (\hat{Z}_{2x}^\circ \cdot \nabla_x) \langle \overline{\mathcal{V}}_{-1}^\circ \rangle(\hat{Z}_{0x}^\circ; \tau) + \mathcal{D}^0) + \mathcal{D}^1, \end{aligned}$$

where \mathcal{D}^0 and \mathcal{D}^1 are entirely determined (since they depend on the already known functions \hat{Z}_{0x}° and \hat{Z}_{1x}°). Remark that (1.6.49) is a *nonlinear* equation. This means that the actual asymptotic calculus is *critical*. We come back to this point in Paragraph 1.6.4.2.

For $j \geq 3$, the situation is easier since we have to deal with a linearized version of (1.6.49), which looks like

$$\begin{aligned} & \hat{Z}_{jx}^\circ + (\hat{Z}_{jx}^\circ \cdot \nabla_x) \langle \overline{\mathcal{V}}_{-1}^\circ \rangle(\hat{Z}_{0x}^\circ; \tau) D_x Z_{0x}^\circ(\hat{Z}_{0x}^\circ; \tau, \theta_\tau)^{-1} \partial_{\theta_r^\circ} Z_{2x}^\circ(\hat{Z}_{0x}^\circ; \tau, \theta_\tau, \hat{\theta}_r^\circ \\ & \quad + \tilde{\mathcal{D}}^0) + \mathcal{D}^j = 0, \end{aligned} \tag{1.6.50}$$

where $\tilde{\mathcal{D}}^0$ and \mathcal{D}^j are known functions since they depend on the \hat{Z}_{kx}° with $k < j$. Due again to (1.6.42), this may be formulated as $(\text{Id} + \mathcal{O}(\tau))\hat{Z}_{jx}^\circ + \mathcal{D}^j = 0$ which has obviously a unique solution for $|\tau|$ small enough.

1.6.4.2 Nonlinear modulation equation

We now come back to solve the nonlinearity inherited from (1.6.49). Since Z_{2x}° is periodic in θ_r° , the function \mathcal{F} is (locally in time) uniformly bounded (say by some $R \in \mathbb{R}_+^*$) with respect to the variable $\hat{Z}_{2x}^\circ \in \mathbb{R}^d$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$ with $\chi \equiv 1$ for $|r| \leq 1$ and $\chi \equiv 0$ for $2 \leq r$. We define $\chi_R(r) := \chi(r/R)$. Let $\eta \in \mathbb{R}_+^*$, introduce the following auxiliary expression (in the new unknown \tilde{Z}_{2x}°)

$$f_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ) := \tilde{Z}_{2x}^\circ + \mathcal{F}(\chi_\eta(\tau)\tau, \chi_{R^2}(|\tilde{Z}_{2x}^\circ|^2)\tilde{Z}_{2x}^\circ). \quad (1.6.51)$$

Then, consider the smooth map $F_{\eta,R} : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \times \mathbb{R}^d$,

$$(\tau, \tilde{Z}_{2x}^\circ) \mapsto \left(\tau, f_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ) - \mathcal{F}(0, 0) \right).$$

We want to apply the Hadamard's global inverse function theorem to the \mathcal{C}^2 mapping $F_{\eta,R}$. To this end, we have to check the needed assumptions:

- find $F_{\eta,R}(0, 0) = (0, f_{\eta,R}(0, 0) - \mathcal{F}(0, 0)) = (0, \mathcal{F}(0, 0) - \mathcal{F}(0, 0)) = (0, 0)$;
- The limit of $|F_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ)|$ is $+\infty$ when $|\tau|$ goes to $+\infty$ is infinite because the first component of $F_{\eta,R}$ is just τ . The limit of $|F_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ)|$ is also $+\infty$ when $|\tilde{Z}_{2x}^\circ|$ goes to $+\infty$ because, because of the cutoff χ_{R^2} , it is bounded below by the limit of $|\tilde{Z}_{2x}^\circ + \mathcal{F}(\chi_\eta(\tau)\tau, 0)|$ when $|\tilde{Z}_{2x}^\circ|$ goes to $+\infty$. The map F is proper;
- Let us study the structure of the Jacobian matrix of $F_{\eta,R}$. To this end, we have to control $\partial_\tau F_{\eta,R} = (1, \partial_\tau f_{\eta,R})$ and $D_{\tilde{Z}_{2x}^\circ} F_{\eta,R} = (0, D_{\tilde{Z}_{2x}^\circ} f_{\eta,R})$. We start by looking at the region of $\mathbb{R} \times \mathbb{R}^d$ where $\sqrt{2}R \leq |\tilde{Z}_{2x}^\circ|$. Then, we consider the ball $|\tilde{Z}_{2x}^\circ| \leq \sqrt{2}R$ with first $2\eta \leq |\tau|$ and finally $|\tau| \leq 2\eta$.

For $\sqrt{2}R \leq |\tilde{Z}_{2x}^\circ|$, we have just to deal with

$$f_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ) := \tilde{Z}_{2x}^\circ + \mathcal{F}(\chi_\eta(\tau)\tau, 0)$$

so that $\partial_\tau f_{\eta,R} = \mathcal{O}(1)$ and $D_{\tilde{Z}_{2x}^\circ} f_{\eta,R} = \text{Id}$.

For $|\tilde{Z}_{2x}^\circ| \leq \sqrt{2}R$, we find that

$$\begin{aligned} & |\partial_\tau f_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ)| \\ & \leq (\|\tau\chi'(\tau)\|_\infty + \|\chi\|_\infty) \max_{|\tau| \leq 2\eta, \|\tilde{Z}_{2x}^\circ\| \leq \sqrt{2}R} \|\partial_\tau \mathcal{F}(\tau, \tilde{Z}_{2x}^\circ)\|_\infty < +\infty. \end{aligned}$$

Now, we look at $D_{\tilde{Z}_{2x}^\circ} f_{\eta,R}$. For $2\eta \leq |\tau|$, since $\langle \overline{\mathcal{V}}_{-1}^\circ \rangle|_{\tau=0} \equiv 0$, there remains

$$\begin{aligned} & f_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ) \\ & = \tilde{Z}_{2x}^\circ + D_x Z_{0x}^\circ (\hat{Z}_{0x}^\circ|_{\tau=0}; 0, \theta_\tau)^{-1} Z_{2x}^\circ (\hat{Z}_{0x}^\circ|_{\tau=0}; 0, \theta_\tau, \hat{\theta}_r^\circ + \mathcal{D}^0|_{\tau=0}) + \mathcal{D}^1|_{\tau=0}, \end{aligned}$$

so that $D_{\tilde{Z}_{2x}^\circ} f_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ) = \text{Id}$. In the same vein, for $|\tau| \leq 2\eta$, exploiting the structure of \mathcal{F} and (1.6.42) again, we find that

$$\begin{aligned} D_{\tilde{Z}_{2x}^\circ} f_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ) & = \text{Id} + (D_{\hat{Z}_{2x}^\circ} \mathcal{F})(\chi_\eta(\tau)\tau, \chi_{R^2}(|\tilde{Z}_{2x}^\circ|^2)\tilde{Z}_{2x}^\circ) D_{\tilde{Z}_{2x}^\circ} (\chi_{R^2}(|\tilde{Z}_{2x}^\circ|^2)\tilde{Z}_{2x}^\circ) \\ & = \text{Id} + \mathcal{O}(1) |\nabla_x \langle \overline{\mathcal{V}}_{-1}^\circ \rangle(\hat{Z}_{0x}^\circ; \tau)| \\ & = \text{Id} + \mathcal{O}(\eta). \end{aligned}$$

The final outcome is

$$D_{\tau, \tilde{Z}_{2x}^\circ} F_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ) = \begin{pmatrix} 1 & 0 \\ \mathcal{O}(1) & \text{Id} + \mathcal{O}(\eta) \end{pmatrix}.$$

For $|\eta|$ chosen small enough, the Jacobian matrix of $F_{\eta,R}$ is bounded and the corresponding Jacobian determinant is nonzero at each point.

Thus, we can assert that $F_{\eta,R}$ is one-to-one and onto. In particular, the position $(\tau, -\mathcal{F}(0,0))$ has a unique preimage. This furnishes some \tilde{Z}_{2x}° such that $f_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ) = 0$. In view of (1.6.51), this can be achieved only by some \tilde{Z}_{2x}° satisfying $|\tilde{Z}_{2x}^\circ| \leq R$. Now, for $|\tau| \leq \eta$, knowing that $|\tilde{Z}_{2x}^\circ| \leq R$, the relation (1.6.49) is satisfied, since it is exactly the same as $F_{\eta,R}(\tau, \tilde{Z}_{2x}^\circ) = (\tau, -\mathcal{F}(0,0))$. At the end, we can say that the solution \hat{Z}_{2x}° to (1.6.49) exists and it is just the restriction of \tilde{Z}_{2x}° for $|\tau| \leq \eta$.

1.6.4.3 Proof of Lemma 1.6.10

Select some $N \in \mathbb{N}^*$ with $N \geq 2$. Define the formal approximate solution x_a^{-1} as follows

$$\begin{aligned} x_a^{-1}(\varepsilon; \tau, \tilde{x}, \theta_\tau, \hat{\theta}_r^\circ) \\ := \hat{Z}_{0x}^\circ(\tilde{x}; \tau, \theta_\tau) + \varepsilon \hat{Z}_{1x}^\circ(\tilde{x}; \tau, \theta_\tau) + \sum_{j=2}^N \varepsilon^j \hat{Z}_{jx}^\circ(\tilde{x}; \tau, \theta_\tau, \hat{\theta}_r^\circ), \end{aligned} \quad (1.6.52)$$

where the profiles \hat{Z}_{jx}° are the ones constructed in the previous Paragraphs 1.6.4.1 and 1.6.4.2. To summarize, we have to show that this formal solution can be exploited to approximate the exact solution x^{-1} of (1.6.47). Indeed, we consider the error term R_{inv} (corresponds to the inverse) defined through the relation

$$x^{-1}(\varepsilon; \tau, \tilde{x}) = x_a^{-1}(\varepsilon; \tau, \tilde{x}, \frac{\tau}{\varepsilon}, \frac{\overline{\mathcal{V}}_{-1}^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}) + \varepsilon^{N+1} R_{\text{inv}}.$$

It is sufficient then to prove that R_{inv} can be viewed as remainder. Now, since θ_τ and $\hat{\theta}_r^\circ$ belong to compact sets (torus), as well as \tilde{x} (we work locally in space, with \tilde{x} in a compact set), the preceding arguments (in Paragraphs 1.6.4.1 and 1.6.4.2) can be applied uniformly with respect to these variables, yielding

- the determination of the profiles \hat{Z}_{0x}° , \hat{Z}_{1x}° and \hat{Z}_{jx}° for all $j \geq 3$ and the specification of the nonlinear equation (1.6.49) at ε^2 -order in Paragraph 1.6.4.1;
- the determination of some \hat{Z}_{2x}° as a solution to the above mentioned nonlinear modulation equation (1.6.49) in Paragraph 1.6.4.2.

Moreover, the linearized version of (1.6.47) along this approximate solution furnishes for the error term R_{inv} an equation similar to (1.6.50). By this way, we can construct an approximate solution to (1.6.47) which takes indeed the form of (1.6.44) and which inherits a precision at any order (in terms of powers of ε). Hence, we obtain the stability and thereby (1.6.44) is proved. \square

1.6.5 Proof of Theorem 1.1.5

With $u \equiv v$ as in Lemma 1.6.4 and x^{-1} as in Lemma 1.6.9, the formula (1.6.8) can be applied to recover the existence part of Theorem 1.1.5. We can now turn to the proof of (1.1.10). Knowing that $u \equiv v$ is given by an expansion similar to (1.2.18) and that x^{-1} is as in (1.6.44), the formula (1.6.8) reveals that u_ε can be obtained through a composition

of three-scale oscillations. More precisely, at the level of (1.2.18), the values of z_0 and \mathbf{v}_0 must be computed as indicated in (1.6.17) as functions of x , and then x must be replaced by the expression \mathbf{x}^{-1} of (1.6.44). In other words, with $\mathbf{x}^{-1} \equiv \mathbf{x}^{-1}(\varepsilon, \tau, \tilde{x})$ as in (1.6.44), we have

$$\begin{aligned} u(\varepsilon, \tau, \tilde{x}) &= \frac{1}{\varepsilon} \langle \overline{\mathcal{V}}_{-1} \rangle (z_0(\varepsilon, \mathbf{x}^{-1}); \tau) + \overline{\mathcal{V}}_0 \left((z_0, \mathbf{v}_0)(\varepsilon, \mathbf{x}^{-1}); \tau, \frac{\tau}{\varepsilon} \right) \\ &\quad + \mathcal{O}(\varepsilon^{N+1}) + \sum_{j=1}^N \varepsilon^j \mathcal{V}_j \left((z_0, \mathbf{v}_0)(\varepsilon, \mathbf{x}^{-1}); \tau, \frac{\tau}{\varepsilon}, \right. \\ &\quad \left. \frac{\langle \overline{\mathcal{V}}_{-1} \rangle (z_0(\varepsilon, \mathbf{x}^{-1}); \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0 \left((z_0, \mathbf{v}_0)(\varepsilon, \mathbf{x}^{-1}); \tau, \frac{\tau}{\varepsilon} \right)}{\varepsilon} \right). \end{aligned} \quad (1.6.53)$$

This asymptotic description of u_ε is not yet in suitable form. We can further improve it in order to recover the oscillatory structure (1.1.10). Before proceeding, we have to consider the following preliminary steps:

- (i) Expand the initial data $(z_0, \mathbf{v}_0)(\varepsilon; \cdot)$ composed with \mathbf{x}^{-1} in powers of ε ;
 - (ii) Clarify the structure of the phase that comes to replace $\hat{\theta}_r^\circ$ in (1.1.10). More precisely, explain how $\hat{\theta}_r^\circ$ can become a substitute for θ_r in the right hand side of (1.6.53).
- (i) The initial data $(z_0, \mathbf{v}_0)(\varepsilon; \cdot)$ composed with $\mathbf{x}^{-1} \equiv \mathbf{x}^{-1}(\varepsilon, \tau, \tilde{x})$ can be expanded in powers of ε according to

$$\begin{aligned} &(z_0, \mathbf{v}_0)(\varepsilon; \mathbf{x}^{-1}) \\ &= \sum_{j=0}^N \varepsilon^j (\hat{z}_{0j}, \hat{\mathbf{v}}_{0j}) \left(\tilde{x}; \tau, \frac{\tau}{\varepsilon}, \frac{\overline{\hat{\mathcal{V}}}_{-1}^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\hat{\mathcal{V}}}_0^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon} \right) + \mathcal{O}(\varepsilon^{N+1}). \end{aligned} \quad (1.6.54)$$

In particular, taking into account (1.6.17), (1.6.18), and (1.6.44), we find that

$$\hat{z}_{00} \equiv \bar{\hat{z}}_{00}(\tilde{x}; \tau, \theta_\tau) := z_{00} \circ \hat{Z}_{0x}^\circ(\tilde{x}; \tau, \theta_\tau); \quad (1.6.55)$$

$$\hat{z}_{01} \equiv \bar{\hat{z}}_{01}(\tilde{x}; \tau, \theta_\tau) := \left((\hat{Z}_{1x}^\circ(\tilde{x}; \tau, \theta_\tau) \cdot \nabla_x) z_{00} + z_{01} \right) \circ \hat{Z}_{0x}^\circ(\tilde{x}; \tau, \theta_\tau); \quad (1.6.56)$$

$$\hat{\mathbf{v}}_{00} \equiv \bar{\hat{\mathbf{v}}}_{00}(\tilde{x}; \tau, \theta_\tau) := \mathbf{v}_{00} \circ \hat{Z}_{0x}^\circ(\tilde{x}; \tau, \theta_\tau). \quad (1.6.57)$$

- (ii) We need now to clarify the expansion of the part which comes to replace θ_r in (1.6.53). Exploit (1.6.55), (1.6.56) and (1.6.57) in order to expand the parts involving $\langle \overline{\mathcal{V}}_{-1} \rangle$ and

$\overline{\mathcal{V}}_0$ according to

$$\begin{aligned}
 & \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0(\varepsilon, \mathbf{x}^{-1}); \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0((z_0, \mathbf{v}_0)(\varepsilon, \mathbf{x}^{-1}); \tau, \frac{\tau}{\varepsilon})}{\varepsilon} \\
 &= \frac{1}{\varepsilon^2} \langle \overline{\mathcal{V}}_{-1} \rangle(\tilde{z}_{00}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}); \tau) + \frac{1}{\varepsilon} \left\{ \overline{\mathcal{V}}_0(\tilde{z}_{00}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}), \tilde{\mathbf{v}}_{00}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}); \tau, \frac{\tau}{\varepsilon}) \right. \\
 & \quad \left. + [\tilde{z}_{01}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}) \cdot \nabla_{z_0}] \langle \overline{\mathcal{V}}_{-1} \rangle(\tilde{z}_{00}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}); \tau) \right\} \\
 & \quad + \sum_{j=0}^N \varepsilon^j \mathcal{U}_j^\circ\left(\tilde{x}; \tau, \frac{\tau}{\varepsilon}, \frac{\overline{\mathcal{V}}_{-1}^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\right) + \mathcal{O}(\varepsilon^{N+1}).
 \end{aligned} \tag{1.6.58}$$

In view of (1.6.23), we have

$$D_x \langle \overline{\mathcal{V}}_{-1}^\circ \rangle(x; \tau) = D_x z_{00}(x) \nabla_{z_0} \langle \overline{\mathcal{V}}_{-1} \rangle(z_{00}(x); \tau). \tag{1.6.59}$$

Consider then (1.6.23), (1.6.24), (1.6.45), and plug (1.6.59) in (1.6.46), the expression (1.6.58) becomes

$$\begin{aligned}
 & \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0(\varepsilon, \mathbf{x}^{-1}); \tau)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0((z_0, \mathbf{v}_0)(\varepsilon, \mathbf{x}^{-1}); \tau, \frac{\tau}{\varepsilon})}{\varepsilon} \\
 &= \frac{\overline{\mathcal{V}}_{-1}^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon} \\
 & \quad + \sum_{j=0}^N \varepsilon^j \mathcal{U}_j^\circ\left(\tilde{x}; \tau, \frac{\tau}{\varepsilon}, \frac{\overline{\mathcal{V}}_{-1}^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon})}{\varepsilon}\right) + \mathcal{O}(\varepsilon^{N+1}).
 \end{aligned} \tag{1.6.60}$$

In view of (1.6.53) and (1.6.60), we can assert that the first two terms of the expansion (1.1.10) are identified as follows

$$\overline{\mathcal{U}}_{-1}(\tilde{x}; \tau, \frac{\tau}{\varepsilon}) := \overline{\mathcal{V}}_{-1}^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon}); \tag{1.6.61}$$

$$\overline{\mathcal{U}}_0(\tilde{x}; \tau, \frac{\tau}{\varepsilon}) := \overline{\mathcal{V}}_0^\circ(\tilde{x}; \tau, \frac{\tau}{\varepsilon}). \tag{1.6.62}$$

This explains how the phase $\psi_\varepsilon = \varepsilon \overline{\mathcal{U}}_{-1} + \varepsilon^2 \overline{\mathcal{U}}_0$ does appear inside (1.1.10). The strategy of recovering the remaining oscillatory structure (1.1.10) is as follows:

- (1) Use (1.6.60) to localize the oscillations at frequencies ε^{-3} at the position $\varepsilon \overline{\mathcal{U}}_{-1} + \varepsilon^2 \overline{\mathcal{U}}_0$ and incorporate the $\mathcal{O}(1)$ -remainder inside $\mathcal{O}(1)$ -shift in θ_r of the profiles \mathcal{V}_j in (1.6.53);

- (2) Expand the new profiles $\mathcal{V}_j(\cdot; \tau, \theta_\tau, \theta_r)$ thus obtained after step (1) composed with $(z_0, \mathbf{v}_0)(\varepsilon, \mathbf{x}^{-1})$ in powers of ε using the expansion (1.6.54);
- (3) Gather the ε^j -terms coming from (1.6.53) after applying the above two steps to recover the expressions \mathcal{U}_j for $j \geq 1$.

Of course, the rapid oscillations (implying $\overline{\mathcal{V}}_{-1}^\circ$ and $\overline{\mathcal{V}}_0^\circ$) involved by $(z_0, \mathbf{v}_0)(\varepsilon; \mathbf{x}^{-1})$ at the level of (1.6.54) and those appearing in the sum inside (1.6.58) are still present. But they can be incorporated inside the profiles \mathcal{U}_j with $j \geq 1$. \square

Remark 1.6.11 (About the matching of initial data). We have $\mathbf{x}(\varepsilon, 0, x) = x$ and $\mathbf{x}^{-1}(\varepsilon, 0, x) = x$. As prescribed by (1.1.3) and the initial data inside (1.1.5), we find that

$$\begin{aligned} u(\varepsilon, 0, x) &= u(\varepsilon, 0, \mathbf{x}^{-1}(\varepsilon, 0, x)) = u(\varepsilon, 0, x) = \mathbf{v}_0(\varepsilon, x) \\ &= (\mathcal{U}_{00} + \varepsilon \mathcal{U}_{01} + \cdots + \varepsilon^N \mathcal{U}_{0N})(x) + \mathcal{O}(\varepsilon^{N+1}). \end{aligned}$$

Compare this with (1.1.10) at time $\tau = 0$. The above line implies that

$$\overline{\mathcal{U}}_{-1}(x; 0, 0) = 0, \quad \mathcal{U}_j(x; 0, 0, \hat{\theta}_r^\circ) = \mathbf{v}_{0j}(x) = \mathcal{U}_{0j}(x), \quad \forall j \in \mathbb{N}.$$

In view of (1.6.61), knowing that $\mathcal{V}_{-1}|_{\tau=0} \equiv \mathcal{V}_{-1}|_{\tau=0} \equiv 0$, this is consistent. The same applies for $\overline{\mathcal{U}}_0$ and so on.

LONG TIME GYROKINETIC EQUATIONS

Abstract. *The aim of this text is to elucidate the oscillating patterns [Che18] which are generated in a toroidal plasma by a strong external magnetic field and a nonzero electric field. It is also to justify and then study new modulation equations which are valid for longer times than before. Oscillating coherent structures are induced by the collective motions of charged particles which satisfy a system of ODEs implying a large parameter, the gyrofrequency $\varepsilon^{-1} \gg 1$. By exploiting the properties of underlying integrable systems, we can complement the KAM picture [BS94, Bra81] and go beyond the classical results about gyrokinetics [BS94, Bos10, Bra81, BH07]. The purely magnetic situation was addressed in [Che15, Che17]. We are concerned here with the numerous additional difficulties due to the influence of a nonzero electric field.*

2.1 Introduction

In Subsection 2.1.1, we present the general ideas. In Subsection 2.1.2, we define the two key concepts of *average flow* and *mean flow*. In Subsection 2.1.3, we clarify the assumptions made on the external electromagnetic field. In Subsection 2.1.4, we specify the coherent structures; we highlight the applications to toroidal plasmas; and we detail our plan.

2.1.1 Motivations

We study the motions of charged particles in the presence of a fixed external electromagnetic field (E, B) with $E \neq 0$ and $|B| \gg 1$. The case where $E \equiv 0$ was recently published in [Che15, Che17]. As is well-known, the electric and magnetic forces affect the trajectories of charged particles through the Lorentz force. They can be exploited in order to confine plasmas through for instance toroidal devices (like tokamaks). The purpose is to ensure that the plasma stays away from the walls.

Understanding the transport processes in axisymmetric configurations is an old chal-

lenging goal. The subject has a long history with a huge number of contributions. There are many approaches, including the KAM method [BS94, Bra81] and gyrokinetics [Bos10, BH07]. The difference of our viewpoint with KAM theory is - away from a set of zero measure in the phase space (corresponding to the presence of separatrices) - a global quantitative space-time description [Che18] of plasma oscillations, in terms of explicit phases and profiles. Its main novelty compared to gyrokinetics is the deterministic rigorous investigation of longer times. And, in connection with [Che15, Che18, Che17], the originality of the present paper is the addition of a non trivial electric field $E \neq 0$.

The main tool to confine plasmas is a large fixed external magnetic field B . But it is also essential to explore what happens under the simultaneous presence of a smaller electric field $E \neq 0$. Indeed, the influence of such $E \neq 0$ cannot be ignored. First, due to the coupling between the Vlasov and Maxwell equations, a nonzero electric field is spontaneously generated. Secondly, in experiments, such non trivial E is a known cause of disruptions. When $E \neq 0$, the kinetic energy is no more conserved. The particles can be accelerated or decelerated, leading to the development of instabilities. Such instabilities often proceed from the production of whistler waves [Che15, Che18], which have their origins in the collective organization of charged particles [CC22]. It is therefore important to detect the oscillating coherent structures which are induced by the *inhomogeneous* features of the electromagnetic field (E, B) .

In the case $E \equiv 0$, this program has been achieved in [Che15, Che17] with difficulties coming from the variations of B . When $E \neq 0$, the discussion is even more complicated (with new geometrical structures appearing at the level of Paragraph 2.3.2.1). Still, it falls within the scope of the WKB analysis introduced in [CF23b]. This is not obvious, far from it. Thus, our first task is to make sure that all prerequisites of [CF23b] are satisfied when dealing with concrete data emanating from the modeling of tokamaks. Once done, the results of [CF23b] furnish long time modulation equations (Theorem 2.1.9) where it is possible to measure the impact of E . Then, our second motivation is to investigate the stability issues at the level of these reduced equations. In particular, we want to examine the potential effects of the condition $E \neq 0$.

A relativistic charged particle starting at time $t = 0$ from the position $(x_0, v_0) \in \mathbb{R}^3 \times \mathbb{R}^3$ is moving in the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ according to the dimensionless ordinary differential

equation

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = (1 - |v|^2)^{1/2} \left\{ -\frac{1}{\varepsilon} v \wedge B(x) - E(x) + (v \cdot E(x)) v \right\}. \end{cases} \quad (2.1.1)$$

The integral curves associated to (2.1.1) give rise to the characteristics of the Vlasov equation. The origin of (2.1.1) is recalled in Subsection 2.2.1. In physics, the parameter ε comes from the inverse of the electron gyrofrequency. In practice, it is small, such that $\varepsilon \simeq 10^{-4} \ll 1$. This is why it is often viewed as going to zero. The regime of intermediate times (related to gyrokinetics) is when $t \sim 1$. We are concerned here with the study of (2.1.1) for longer times, when $\tau := \varepsilon t \sim 1$ (or $t \sim \varepsilon^{-1}$). Thus, with ${}^t(\tilde{x}, \tilde{v})(x_0, v_0; \tau) := {}^t(x, v)(x_0, v_0; \tau/\varepsilon)$ we have to consider:

$$\begin{cases} \frac{d\tilde{x}}{d\tau} = \frac{1}{\varepsilon} \tilde{v}, \\ \frac{d\tilde{v}}{d\tau} = \frac{1}{\varepsilon} (1 - |\tilde{v}|^2)^{1/2} \left\{ -\frac{1}{\varepsilon} \tilde{v} \wedge B(\tilde{x}) - E(\tilde{x}) + (\tilde{v} \cdot E(\tilde{x})) \tilde{v} \right\}. \end{cases} \quad (2.1.2)$$

Knowing that $\varepsilon \ll 1$, the right hand side of (2.1.2) is very large. We can recognize a leading term of size ε^{-2} yielding the (well-known for $t \sim 1$) repercussions of the strong external magnetic field (the gyromotions). But there are also terms of size ε^{-1} which take into account the impact of the electric field. This leads to a number of extra effects that we aimed to better understand. Here, for some $\varepsilon_0 \in]0, 1[$, we assume that $\varepsilon \in]0, \varepsilon_0]$. To clearly exhibit the hierarchy of effects in terms of powers of ε , the parameter ε is intended to go to zero.

In contrast to preceding works [Che15, Che17], in the present situation, the kinetic energy is in general not conserved. Still, we can describe the leading behavior of the solution (\tilde{x}, \tilde{v}) to (2.1.2). On this basis, in Section 2.4, we can exhibit a spatial confinement property (related to \tilde{x}) for a whole range of electric fields E adjusted as in Assumption 2.1.7. In addition, when the electric field is radial, that is when the scalar electric potential Φ depends only on r , we can also prove a long time ($\tau \sim 1$) stability property concerning the momentum component \tilde{v} .

2.1.2 Notions of average and mean flows

For realistic choices of (E, B) , we would like to deduce from (2.1.2) simplified models which are valid for $\tau \sim 1$. To fit with the toroidal geometry, in Subsection 2.2.2, we change the phase space $\mathbb{R}^3 \times \mathbb{R}^3$ into $(\mathbb{R} \times \mathbb{T} \times \mathbb{T})^2$, where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is the torus. We also extract from (2.1.1) a special class of a nonlinear differential equations having the following form

$$\partial_t \begin{pmatrix} \varkappa \\ \nu \end{pmatrix} = \begin{pmatrix} \mathcal{A} \\ \mathcal{V} \end{pmatrix} \left(\varepsilon; \varkappa; \frac{\nu}{\varepsilon} \right), \quad \begin{pmatrix} \varkappa \\ \nu \end{pmatrix} (0) = \begin{pmatrix} z_0 \\ \nu_0 \end{pmatrix}, \quad (2.1.3)$$

where the new dependent variables are $\varkappa = {}^t(r, \theta, \nu, \zeta) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T}$ and $\nu \in \mathbb{T}$, and where $z_0 \in \mathbb{R}^4$ and $\nu_0 \in \mathbb{T}$ stand for initial data (which may depend smoothly on $\varepsilon \in [0, \varepsilon_0]$). The vector valued function \mathcal{A} does not involve ε , whereas the scalar component \mathcal{V} does. More precisely, the source term ${}^t(\mathcal{A}, \mathcal{V})$ looks like

$$C^\infty(\mathbb{R}_+^* \times \mathbb{T} \times \mathbb{R} \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^4) \ni \mathcal{A}(\varepsilon; \varkappa; \nu) \equiv \mathcal{A}(\varkappa; \nu) = {}^t(\mathcal{A}_r, \mathcal{A}_\theta, \mathcal{A}_\nu, \mathcal{A}_\zeta)(\varkappa; \nu), \quad (2.1.4)$$

$$C^\infty([0, \varepsilon_0] \times \mathbb{R}_+^* \times \mathbb{T} \times \mathbb{R} \times \mathbb{T} \times \mathbb{T}; \mathbb{R}) \ni \mathcal{V}(\varepsilon; \varkappa; \nu) = \mathcal{V}_0(\varkappa) + \varepsilon \mathcal{V}_1(\varkappa; \nu). \quad (2.1.5)$$

Most importantly, the source terms \mathcal{A} and \mathcal{V} are periodic in the last variable ν of period 2π . This means that, when passing from (2.1.1) to (2.1.3), the singular weight ε^{-1} has been converted into fast oscillations carried by ν . The profile $\mathcal{A}(\varkappa; \nu)$ can be decomposed according to

$$\mathcal{A}(\varkappa; \nu) = \bar{\mathcal{A}}(\varkappa) + \mathcal{A}^*(\varkappa; \nu), \quad \bar{\mathcal{A}}(\varkappa) := \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}(\varkappa; \nu) d\nu, \quad (2.1.6)$$

where $\bar{\mathcal{A}} = {}^t(\bar{\mathcal{A}}_r, \bar{\mathcal{A}}_\theta, \bar{\mathcal{A}}_\nu, \bar{\mathcal{A}}_\zeta)$ is the mean part and $\mathcal{A}^* = {}^t(\mathcal{A}_r^*, \mathcal{A}_\theta^*, \mathcal{A}_\nu^*, \mathcal{A}_\zeta^*)$ is the oscillating part. The same applies to $\mathcal{V} = \bar{\mathcal{V}} + \mathcal{V}^*$. The function $\varkappa(\cdot)$ is the sum of the *average flow* $Z(z_0; t)$ defined below plus deviations.

Definition 2.1.1 (The *average flow* associated with the intermediate time evolution in t of \varkappa). The *average flow* $Z(z_0; t)$ on \mathbb{R}^4 that is associated with (2.1.3) is the (locally or globally defined) mapping $Z : \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}^4$ obtained by solving

$$\partial_t Z = \bar{\mathcal{A}}(Z), \quad Z(z_0; 0) = z_0, \quad Z = {}^t(Z_r, Z_\theta, Z_\nu, Z_\zeta). \quad (2.1.7)$$

The average flow issued from (2.1.1) is studied in Section 2.3. It reflects the main behavior of the dynamics during intermediate times. It is described in Subsection 2.3.1. It plays a key role in what happens next, when $\tau \sim 1$. Indeed, to have access to [CF23b], one of the difficulties is to prove that, for all fixed z_0 , the average flow $Z(z_0; \cdot)$ is globally defined and periodic in t (with a period which may depend on z_0). Subsection 2.3.2 is designated to this purpose.

Once this is done, we can normalize the period of the average flow. To this end, in Paragraph 2.4.1.1, we change adequately (in a way depending on z_0) the time variable t into $s \sim t$ (with $s \neq t$ but $s \sim t$ still representing intermediate times). Then, we can interpret the long time behavior in terms of $\tau := \varepsilon s$ (where $\tau \neq \tau$ but $\tau \sim \tau$). Instead of ${}^t(z, \nu)$, we work with ${}^t(z, \mathbf{v})$ as in (2.4.4). By this way, in Section 2.4, we can enter into the framework of [CF23b]. As a matter of fact, the system (2.1.2) is recast as follows

$$\partial_\tau \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} \left(\varepsilon, z_0; z; \frac{\tau}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right), \quad \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} (0) = \begin{pmatrix} z_0 \\ \mathbf{v}_0 \end{pmatrix}, \quad (2.1.8)$$

where the functions \mathbf{A} and \mathbf{V} satisfy

$$C^\infty(\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^4) \ni \mathbf{A}(z_0; z; s, \nu) = {}^t(\mathbf{A}_r, \mathbf{A}_\theta, \mathbf{A}_v, \mathbf{A}_\zeta)(z_0; z; s, \nu), \quad (2.1.9)$$

$$C^\infty([0, \varepsilon_0] \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{T} \times \mathbb{T}; \mathbb{R}) \ni \mathbf{V}(\varepsilon, z_0; z; s, \nu) = (\mathbf{V}_0 + \varepsilon \mathbf{V}_1)(z_0; z; s, \nu). \quad (2.1.10)$$

Compare (2.1.2) with (2.1.8). The singular weight ε^{-2} has disappeared but there remains a large factor ε^{-1} multiplied by a double periodic function. Indeed, the source term ${}^t(\mathbf{A}, \mathbf{V})$ is periodic of period 2π in both variables s and ν .

Definition 2.1.2 (The *mean flow* associated with the intermediate time evolution in s of z). Let $\bar{\mathbf{A}}(z_0; z; s)$ be defined from \mathbf{A} as in (2.1.6). Given a position $z_0 \in \mathbb{R}^4$, the mean flow $\Xi_{\text{mf}}(z_0; \mathfrak{z}; s)$ on \mathbb{R}^4 associated with (2.1.8) is the (locally or globally defined) mapping $\Xi_{\text{mf}}(z_0; \cdot) : \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}^4$ which is obtained by solving

$$\partial_s \Xi_{\text{mf}}(z_0; \mathfrak{z}; s) = \bar{\mathbf{A}}(z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; s); s), \quad \Xi_{\text{mf}}(z_0; \mathfrak{z}; 0) = \mathfrak{z}. \quad (2.1.11)$$

Remark 2.1.3. As already explained, the average flow is the gate to the construction of the mean flow. It should not be confused with the mean flow. The two equations (2.1.7) and (2.1.11) are distinct. We have $\bar{\mathcal{A}} \neq \bar{\mathbf{A}}$, not least because $\bar{\mathcal{A}}$ and $\bar{\mathbf{A}}$ depend on distinct variables. The above notion of mean flow is consistent with the one of [CF23b], Definition

1]. ◦

In accordance with the above remark, we give below an overview about the distinct time variables which are used in this introduction. We have mentioned the following time variables: t , $\tau = \varepsilon t$, s and $\tau = \varepsilon s$. Each time variable owns its special class of notations.

◁ For the time variable t , we implement:

- (x, v) which solves the characteristics equations (2.1.1) of the Vlasov system;
- (z, v) which solves the system (2.1.3). This system is extracted from the system (2.1.1) on (x, v) in Subsection 2.2.2. The latter is achieved by reformulating the equations for x after writing it in toroidal coordinates and reformulating the equations for v after decomposing it in the spherical coordinates. It is obvious then that $v \neq v = \varepsilon \nu$;
- the average flow Z which solves (2.1.7). Its equation is extracted from the equation on z in (2.1.3) after applying the averaging operator (2.1.6) with respect to ν .

◁ For the time variable τ , we use (\tilde{x}, \tilde{v}) which solves the characteristic equations (2.1.2). The latter system is just the system (2.1.1) interpreted in the long time variable $\tau = \varepsilon t$.

◁ For the time variable s and $\tau = \varepsilon s$, we use:

- (z, ν) which solves the system (2.1.8). The latter system is the system (2.1.3) on (z, ν) interpreted in the time variable τ . This is clarified in Subsection 2.4.1;
- the mean flow Ξ_{mf} which solves (2.1.11). We will show in Subsection 2.4.1 that the mean flow Ξ_{mf} is related to the average flow Z .

The oscillating coherent structures can be detected on the WKB approximation in (τ, z_0, ν_0) of the solutions ${}^t(z_\varepsilon, \nu_\varepsilon)(\tau, z_0, \nu_0)$ to (2.1.8). As shown in [CF23b], such expansions become available provided that the two following criterions are verified.

Condition 2.1.4 (Complete integrability of the *mean* flow). For all $(z_0, \mathfrak{z}) \in \mathbb{R}^4 \times \mathbb{R}^4$, the mean flow $\Xi_{\text{mf}}(z_0; \mathfrak{z}; \cdot)$ is globally defined and periodic of period 2π .

Condition 2.1.5 (Constraints on V_0). The function V_0 is positive and does not depend on $\nu \in \mathbb{T}$.

The validity of Conditions [2.1.4](#) and [2.1.5](#) in the case of functions A and V issued from [\(2.1.1\)](#) is verified in Subsection [2.4.1](#). This means that the results of [\[CF23b\]](#) can be applied concerning [\(2.1.1\)](#). Related applications are developed in Section [2.4](#).

2.1.3 Assumptions on the external field

In Paragraph [2.1.3.1](#), we implement the toroidal coordinates. In Paragraph [2.1.3.2](#), we describe the structure of the external magnetic field B . In Paragraph [2.1.3.3](#), we detail the content of the electric field E . By doing so, we take care to adjust B and E in a manner which is consistent with what could be expected in tokamaks.

2.1.3.1 Toroidal description of tokamaks

To simplify the discussion, we introduce geometrical coordinates. It is easy to see that every point $x \in \mathbb{R}^3$ in the cartesian coordinates can also be identified by a radial coordinate $r \in \mathbb{R}_+$ (the distance from the magnetic axis) and by the two rotational invariances $\theta \in \mathbb{T}$ (the poloidal angle) and $\phi \in \mathbb{T}$ (the toroidal angle). These new coordinates are associated with the following decomposition

$$x = X(r, \theta, \phi) = \left(R(r, \theta) \cos \phi, R(r, \theta) \sin \phi, r \sin \theta \right), \quad R(r, \theta) = R_0 + r \cos \theta. \tag{2.1.12}$$

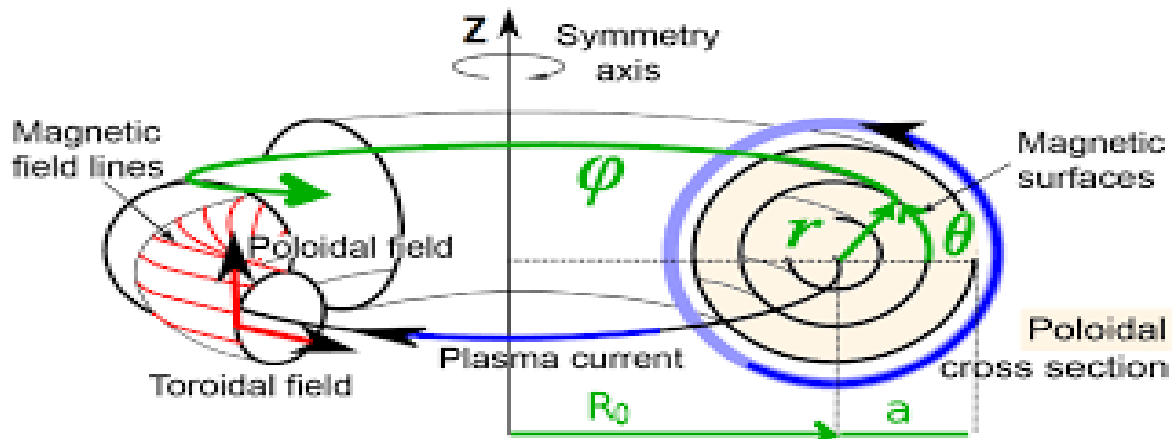


Figure 2.1 – Schematic diagram of the tokamak-plasma geometry.

Retain that $0 \leq r \leq a < R_0$ where a and R_0 are respectively the minor radius and major radius of the tokamak, see Figure [2.1](#). We denote by e_r , e_θ and e_ϕ the orthonormal basis

corresponding to the selection of (r, θ, ϕ) , which is given by

$$\begin{aligned} e_r(\theta, \phi) &= (\cos \theta \cos \phi, \cos \theta \sin \phi, \sin \theta), \\ e_\theta(\theta, \phi) &= (-\sin \theta \cos \phi, -\sin \theta \sin \phi, \cos \theta), \\ e_\phi(\phi) &= (-\sin \phi, \cos \phi, 0). \end{aligned} \tag{2.1.13}$$

Remark that $e_r \wedge e_\theta = -e_\phi$ and recall that $\nabla = e_r \partial_r + e_\theta \frac{1}{r} \partial_\theta + e_\phi \frac{1}{R_0} \partial_\phi$.

2.1.3.2 Adjusting the magnetic field

The function $B(\cdot)$ inside (2.1.1) has ε^{-1} in factor (with $\varepsilon \ll 1$) for reasons clarified in Subsection 2.2.1. It represents the dimensionless version of the external magnetic field, and therefore it is of amplitude one. We impose common restrictions on B , compatible with all the contributions [CSL05, DIK⁺94, DGS⁺18, Hu, Jac99, WC11, ZFG81]. In particular, the reader can refer to the Notes on Tokamak equilibrium [Hu].

Assumption 2.1.6 (Toroidal external magnetic field). The magnetic field B is issued from a scalar magnetic potential $A(\cdot)$ through the relation $B = \nabla \wedge A$ where, given some auxiliary function $I \in C^1([0, a]; \mathbb{R}_+^*)$ which must be associated with a poloidal current, we have

$$A(r, \theta) := -r \nabla \phi - g(r, \theta) \nabla \theta, \quad g(r, \theta) := \int_0^r \frac{\tilde{r} I(\tilde{r})}{R_0 + \tilde{r} \cos \theta} d\tilde{r}.$$

The above magnetic field $B(\cdot)$ can always be decomposed (see Figure 2.2) according to

$$B = \underbrace{B_t}_{\text{Toroidal magnetic field}} + \underbrace{B_p}_{\text{Poloidal magnetic field}}, \quad B_t \parallel e_\phi, \quad B_p \parallel e_\theta, \quad B_t \perp B_p.$$

In practice, the poloidal magnetic field is produced by the plasma current flowing in the toroidal direction. It helps to confine the plasma and to get some equilibrium. As seen in Figure 2.2, the magnetic field lines follow helical paths around the torus.

2.1.3.3 Adjusting the electric field E

In the journal [DIK⁺94], the authors take measurements that fit (after nondimensionalization) with the condition $|E| \simeq 1$. Thus, from now on, the electric field E inside (2.1.1) is supposed to be of amplitude one. Due to the cylindrical symmetry, it is reasonable to assume some invariance with respect to ϕ .

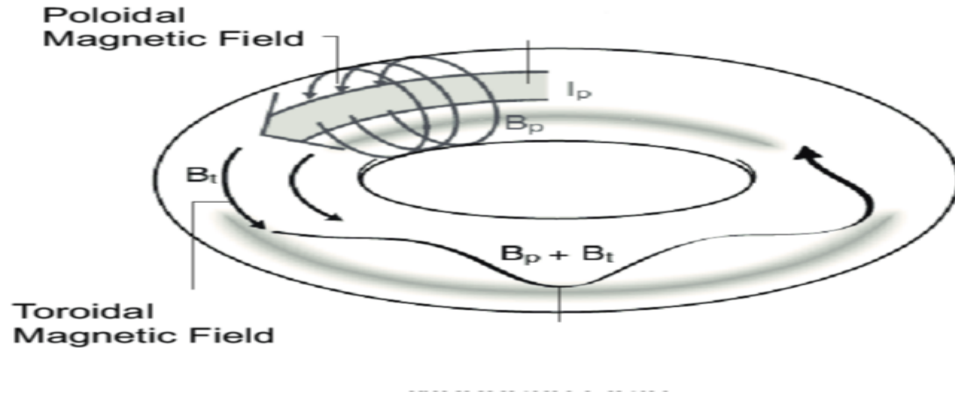


Figure 2.2 – The decomposition of the magnetic field.

Assumption 2.1.7 (Axisymmetric external electric field). The electric field E is issued from a scalar electric potential Φ which does not depend on ϕ . More precisely, we can find a smooth (non constant) function $\Phi : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$E(r, \theta, \phi) = \nabla\Phi = \partial_r\Phi(r, \theta) e_r(\theta, \phi) + r^{-1} \partial_\theta\Phi(r, \theta) e_\theta(\theta, \phi). \quad (2.1.14)$$

Even though $\partial_\phi\Phi \equiv 0$, the above field $E(\cdot)$ does depend on ϕ through $e_r(\cdot)$ and $e_\theta(\cdot)$. When moreover $\Phi(r, \theta) \equiv \Phi(r)$, the electric field $E \equiv E_r(r, \theta, \phi) \equiv \partial_r\Phi e_r$ is said to be *radial*. There is no definitive consensus concerning the choice of E . But the formula (2.1.14) is sufficiently general to take into account most of the phenomena.

The model (2.1.14) is selected in accordance with the authoritative publications [WC11, Section 2.18] and [ZFG81] where, as in the next Paragraph 2.2.2.1, there are three nontrivial components E_r , E_\parallel and E_\perp emanating from $\nabla\Phi$. In [ZFG81], the authors consider the relativistic guiding center motion of charged particles in magnetic fields consistent with toroidal Tokamak equilibrium. They take into account that runaway electrons increase their kinetic energy due to the accelerating force of the toroidal electric field. They try to know what is the orbit of a runaway electron which is produced on a given magnetic surface after it has performed a very large number of toroidal revolutions. They search for whether the observed phenomena can be explained by the change of the runaway electron orbits due to a flattening of the current and pressure profiles. In this process, the acceleration due to the toroidal electric field is essential. We share here the same concerns but from a mathematical viewpoint.

From knowledge of E , we can define the *separatrix region* \mathcal{S} according to a procedure that is described in Subsection 2.3.2. The (closed) set \mathcal{S} is used in Theorem 2.1.8 below. It is

properly defined at the level of (2.3.29).

2.1.4 Results, outcomes and plan

Paragraph 2.1.4.1 is devoted to a first application about coherent structures. Paragraph 2.1.4.2 focusses on the properties of the long time gyrokinetic equations. Paragraph 2.1.4.3 details the plan of the article.

2.1.4.1 Coherent structures

As long as $t \sim 1$, the trajectory of charged particles is a helix that winds rapidly around the magnetic field lines. The gyrokinetic theory describes perfectly the (relatively) slow motion of the guiding center. But what happens next? For $t \gg 1$, that is for $t \sim \varepsilon^{-1}$ or $\tau \sim 1$, the motions become much more complicated. As explained below, their collective behaviors are organized around multiscale structures which can be completely described through WKB expansions.

Theorem 2.1.8 (Long time global picture of the flow: emergence of coherent structures). *Suppose that Assumptions 2.1.6 and 2.1.7 hold. Select in the phase space any bounded connected open set Ω whose closure $\bar{\Omega}$ does not intersect the separatrix region \mathcal{S} . Then, for all $\varepsilon \in]0, \varepsilon_0]$ with ε_0 small enough, there exists some $\tilde{\mathcal{T}} \in \mathbb{R}_+^*$ such that for all initial data $(x_0, v_0) \in \Omega$, the system (2.1.2) has a solution on $[0, \tilde{\mathcal{T}}]$. In addition, we can find smooth profiles*

$$\begin{aligned} \tilde{X}_j(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}) &\in C^\infty(\Omega \times [0, \tilde{\mathcal{T}}] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^3), & j \in \mathbb{N}, \\ \tilde{V}_j(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}) &\in C^\infty(\Omega \times [0, \tilde{\mathcal{T}}] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^3), & j \in \mathbb{N}, \end{aligned}$$

adjusted in such a way that the long time behavior of the solution $(\tilde{x}_\varepsilon, \tilde{v}_\varepsilon)$ of (2.1.2) can be described in the sup-norm as follows

$$\begin{pmatrix} \tilde{x}_\varepsilon \\ \tilde{v}_\varepsilon \end{pmatrix} (x_0, v_0; \tau) \underset{\varepsilon \rightarrow 0}{\sim} \sum_{j=0}^{+\infty} \varepsilon^j \begin{pmatrix} \tilde{X}_j \\ \tilde{V}_j \end{pmatrix} \left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2} \right). \quad (2.1.15)$$

The profiles \tilde{X}_j and \tilde{V}_j are identified by modulation equations (which can be made explicit).

Given $\mathbb{P}(\cdot)$ as in Lemma [2.3.7](#), the expression Ψ^1 is given by

$$\Psi^1(x_0, v_0; \tau) := \frac{2\pi\tau}{\mathbb{P}(z_0(x_0, v_0))}. \quad (2.1.16)$$

The function Ψ_ε^2 plays the role of a phase; it is oscillating; and it is defined by

$$\begin{aligned} \Psi_\varepsilon^2(x_0, v_0; \tau) := & \langle \overline{\mathcal{V}}_{-1} \rangle (z_0(x_0, v_0), \Psi^1(x_0, v_0; \tau)) \\ & + \varepsilon \left\{ [\partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle] (z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)) s^* \left(z_0(x_0, v_0); \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon} \right) \right. \\ & \left. + \overline{\mathcal{V}}_0 \left(z_0(x_0, v_0), 0; \Psi^1(x_0, v_0; \tau), \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon} + s^* \left(z_0(x_0, v_0); \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon} \right) \right) \right\}, \end{aligned} \quad (2.1.17)$$

where s^* is introduced in Lemma [2.3.8](#), whereas $\langle \overline{\mathcal{V}}_{-1} \rangle$ and $\overline{\mathcal{V}}_0$ are computed in Proposition [2.4.2](#).

Gyrokinetics deals with $t \sim 1$ (that is with x and v) and fast gyromotions around the field lines, which occur at the frequency ε^{-1} in terms of the time scale t . On the other hand, Theorem [2.1.8](#) addresses longer times $\tau \sim 1$ (which imply \tilde{x} and \tilde{v}) and extra fast oscillating behaviors, which arise at the frequency ε^{-1} in terms of the time scale τ . Depending on the chosen set Ω , we are faced with librations (in the case of trapped particles in banana orbits) or rotations (in the case of passing particles), see Figure [2.3](#). The expansion [\(2.1.15\)](#) makes visible the complex macroscopic juxtaposition of all these oscillations.

2.1.4.2 Outcomes

By analogy with the usual gyrokinetic equations, which are valid for $t \sim 1$, the modulation equations leading to the main profiles \tilde{X}_0 and \tilde{V}_0 are referred to as the *long time gyrokinetic equations*. They are of practical prime interest because they reveal the leading behavior of the solutions (for $\tau \sim 1$). This is why they are studied below in detail.

Theorem 2.1.9 (Long time gyrokinetic equations). *The profile $\tilde{X}_0(\cdot)$ does not depend on \tilde{v} , while the profile $\tilde{V}_0(\cdot)$ does. These functions can be expressed in terms of the average*

flow Z , which is the solution to (2.1.7), as follows

$$\tilde{X}_0(x_0, v_0; \tau, \tilde{s}) := X(\langle \bar{\mathfrak{Z}}_{0r} \rangle; Z_\theta(\langle \bar{\mathfrak{Z}}_0 \rangle; \mathbb{P} \tilde{s}/2\pi), \bar{\Phi}_0(\tau, \tilde{s})), \quad (2.1.18)$$

$$\begin{aligned} \tilde{V}_0(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}) := & V(\langle \bar{\mathfrak{Z}}_{0r} \rangle; Z_\theta(\langle \bar{\mathfrak{Z}}_0 \rangle; \mathbb{P} \tilde{s}/2\pi), \bar{\Phi}_0(\tau, \tilde{s}), \\ & Z_\nu(\langle \bar{\mathfrak{Z}}_0 \rangle; \mathbb{P} \tilde{s}/2\pi), Z_\zeta(\langle \bar{\mathfrak{Z}}_0 \rangle; \mathbb{P} \tilde{s}/2\pi), \tilde{\nu} + \bar{\mathcal{V}}_1(\tau, \tilde{s})), \end{aligned} \quad (2.1.19)$$

where X and V are explicit functions given by (2.2.4) and (2.2.8), \mathbb{P} is as in Lemma 2.3.7, $\bar{\Phi}_0$ is issued from (2.4.51) and (2.4.53), and the scalar function $\bar{\mathcal{V}}_1$ is as in (2.4.23). In the right hand side of (2.1.18) and (2.1.19), $\langle \bar{\mathfrak{Z}}_0 \rangle$ is for the function $\langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau) = {}^t(\langle \bar{\mathfrak{Z}}_{0r} \rangle, \langle \bar{\mathfrak{Z}}_{0\theta} \rangle, \langle \bar{\mathfrak{Z}}_{0\nu} \rangle, \langle \bar{\mathfrak{Z}}_{0\zeta} \rangle)(z_0; \tau)$ which is determined by solving the nonlinear Cauchy problem

$$\partial_\tau \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau) = \langle \bar{\mathfrak{A}}_1 \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)), \quad \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; 0) = z_0, \quad (2.1.20)$$

where $z_0 = {}^t(r_0, \theta_0, v_0, \zeta_0)$ may be computed from (x_0, v_0) through (2.2.4) and (2.2.8), and where the expression of $\langle \bar{\mathfrak{A}}_1 \rangle = {}^t(\langle \bar{\mathfrak{A}}_{1r} \rangle, \langle \bar{\mathfrak{A}}_{1\theta} \rangle, \langle \bar{\mathfrak{A}}_{1\nu} \rangle, \langle \bar{\mathfrak{A}}_{1\zeta} \rangle)$ is given by (2.4.68). Moreover, as claimed in Lemma 2.4.11, we find that $\langle \bar{\mathfrak{A}}_{1r} \rangle(z_0; \mathfrak{z}) = 0$ for all $(z_0, \mathfrak{z}) \in \mathbb{R}^d \times \mathbb{R}^d$, so that $\langle \bar{\mathfrak{Z}}_{0r} \rangle(z_0; \tau) = r_0$.

The above theorem has multiple implications:

- A model for the leading behavior of the flow (\tilde{x}, \tilde{v}) is provided by:

$$\begin{aligned} \tilde{x}(x_0, v_0; \tau) &= \tilde{X}_0\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}\right) + \mathcal{O}(\varepsilon), \\ \tilde{v}(x_0, v_0; \tau) &= \tilde{V}_0\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2}\right) + \mathcal{O}(\varepsilon). \end{aligned} \quad (2.1.21)$$

The behavior of \tilde{x} and \tilde{v} are mainly governed respectively by the profiles \tilde{X}_0 and \tilde{V}_0 ;

- Looking at (2.1.18), we find $\partial_{\tilde{s}} \tilde{X}_0 \not\equiv 0$ and $\partial_{\tilde{\nu}} \tilde{X}_0 \equiv 0$. This implies that the propagation of the oscillating singularities at the leading order of the spatial part \tilde{x} is completely achieved by $\Psi^1(x_0, v_0; \tau)$. On the other hand, we have $\partial_{\tilde{\nu}} \tilde{X}_1 \not\equiv 0$. This implies that the spatial part \tilde{x} involves small amplitude oscillations related to $\Psi_\varepsilon^2(x_0, v_0; \tau)$ associated with higher frequency ε^{-2} . The rapid variation with respect to $\tilde{\nu}$ is activated at the level of \tilde{X}_j as soon as $j \geq 1$;
- Looking at (2.1.19), we find $\partial_{\tilde{s}} \tilde{V}_0 \not\equiv 0$ and $\partial_{\tilde{\nu}} \tilde{V}_0 \not\equiv 0$. The large amplitude oscillations related to $\Psi^1(x_0, v_0; \tau)$ and $\Psi_\varepsilon^2(x_0, v_0; \tau)$ emerge in the velocity part \tilde{v} at the level of

\tilde{V}_j as soon as $j \geq 0$;

- The expressions \tilde{X}_0 and \tilde{V}_0 consist of distinct parts. There are first the two diffeomorphic maps X as in (2.2.4) and V as in (2.2.8) composed with the average flow Z , the scalar function $\tilde{\mathcal{V}}_1$, and the toroidal angle $\tilde{\Phi}_0$. Secondly, there is the contribution $\langle \bar{\mathfrak{3}}_0 \rangle$. Now, the access to $\langle \bar{\mathfrak{3}}_0 \rangle$ is complicated. It requires the determination of $\langle \bar{\mathfrak{A}}_1 \rangle$. Computing or even finding the simplified expression of $\langle \bar{\mathfrak{A}}_1 \rangle$ requires long computations which are achieved in Paragraph 2.4.3.1;
- There is a cancellation effect induced by the action of the operator $\langle \cdot \rangle$ on $\bar{\mathfrak{A}}_1(\cdot)$. Indeed, in view of Theorem 2.1.9, we have $\langle \bar{\mathfrak{A}}_{1r} \rangle \equiv 0$. This implies that the spatial component $\langle \bar{\mathfrak{3}}_{0r} \rangle$ is not activated when solving (2.1.20). And thus, it suffices to look at the remaining three components $\langle \bar{\mathfrak{3}}_{0\theta} \rangle$, $\langle \bar{\mathfrak{3}}_{0v} \rangle$ and $\langle \bar{\mathfrak{3}}_{0\zeta} \rangle$. We find also that $\langle \bar{\mathfrak{A}}_{1v} \rangle \equiv 0$ when the electric field is radial, that is when $\Phi = \Phi(r)$, leading in this simplified case to a stability property concerning the component $\langle \bar{\mathfrak{3}}_{0v} \rangle$.

2.1.4.3 Plan of the text

The present text is organized as follows.

Section 2.2 furnishes useful tools to deal with the axisymmetric configuration.

- In Subsection 2.2.1, we clarify the origin of (2.1.1).
- In Subsection 2.2.2, the space coordinate x is changed into the toroidal coordinates, while the velocity v is expressed in spherical coordinates coming from a basis $(e_r, e_\perp, e_\parallel)$ where the vector e_\parallel represents the (unitary) direction of the magnetic field. The system (2.1.1) is then interpreted in terms of these new coordinates.

Section 2.3 describes the properties of the average flow.

- In Subsection 2.3.1, we specify the time evolution equations for the average flow.
- In Subsection 2.3.2, we study the periodic properties of the average flow.

Section 2.4 is devoted to applications.

- In Subsection 2.4.1, we show that our analysis is consistent with the one presented in [CF23b].
- In Subsection 2.4.2, we prove Theorem 2.1.8.
- In Subsection 2.4.3, we exhibit and study the long time gyrokinetic equations.

2.2 Modeling of the dynamics

In the present chapter, we are concerned with the equations satisfied by a charged particle in the presence of a prescribed electromagnetic field (which is not affected by the repartition of the charged particles). We consider the motion of an electron with mass m and electric charge $-e$ (of negative sign) in a given electric field \mathbf{E} and magnetic field \mathbf{B} . In Subsection 2.2.1, we explain the origin of (2.1.1). In Subsection 2.2.2, we adapt the phase space and the equations in accordance with the geometries induced by the shape of the toroidal device and by the variations of \mathbf{B} .

2.2.1 Nondimensionalization

Under the influence of a Lorentz force \mathbf{F} , a relativistic charged particle is moving according to

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{v}, \\ \frac{d\mathbf{p}}{dt} = \mathbf{F} := -e [\mathbf{E}(\mathbf{x}) + \mathbf{v} \wedge \mathbf{B}(\mathbf{x})], \end{cases} \quad (2.2.1)$$

with the relations

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m}, \quad \gamma = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} = \left(1 + \frac{\mathbf{p}^2}{m^2 c^2}\right)^{1/2}, \quad |\mathbf{v}| < c,$$

where $c \simeq 3 \times 10^8 \text{ ms}^{-1}$ is the speed of light, $e \simeq 1.6 \times 10^{-19} \text{ C}$ is the elementary charge, and $m \simeq 9.1 \times 10^{-31} \text{ kg}$ is the electron mass.

Lemma 2.2.1 (Equivalent system). *The system (2.2.1) amounts to the same thing as*

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{v}, \\ \frac{d\mathbf{v}}{dt} = -\frac{e}{m} \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{1/2} [\mathbf{E}(\mathbf{x}) + \mathbf{v} \wedge \mathbf{B}(\mathbf{x}) - c^{-2} (\mathbf{v} \cdot \mathbf{E}(\mathbf{x})) \mathbf{v}]. \end{cases} \quad (2.2.2)$$

Proof. The chain rule yields

$$\frac{\gamma^3}{c^2} m \left(\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \right) \mathbf{v} + \gamma m \frac{d\mathbf{v}}{dt} = -e [\mathbf{E} + \mathbf{v} \wedge \mathbf{B}].$$

We can decompose the acceleration $a := d\mathbf{v}/dt$ into the sum of two parts a_{\parallel} and a_{\perp} with a_{\parallel} parallel to the velocity \mathbf{v} and a_{\perp} perpendicular to it. Thus, we have $\mathbf{v} \cdot a_{\perp} = 0$ and

$\mathbf{v} \cdot a_{\parallel} = \mathbf{v} \cdot a$. This furnishes $(\mathbf{v} \cdot a_{\parallel})\mathbf{v} = \mathbf{v}^2 a_{\parallel}$ and therefore

$$\begin{aligned} \mathbf{F} &= \frac{\gamma^3 m \mathbf{v}^2}{c^2} a_{\parallel} + \gamma m (a_{\parallel} + a_{\perp}) = \gamma^3 m \left(\frac{\mathbf{v}^2}{c^2} + \frac{1}{\gamma^2} \right) a_{\parallel} + \gamma m a_{\perp} \\ &= \gamma^3 m \left(\frac{\mathbf{v}^2}{c^2} + 1 - \frac{\mathbf{v}^2}{c^2} \right) a_{\parallel} + \gamma m a_{\perp} = \gamma^3 m a_{\parallel} + \gamma m a_{\perp}. \end{aligned}$$

This can be used to calculate acceleration from force, yielding

$$a = \frac{1}{m \gamma} \left(\mathbf{F} - \frac{(\mathbf{v} \cdot \mathbf{F})\mathbf{v}}{c^2} \right). \quad (2.2.3)$$

Since $\mathbf{v} \cdot \mathbf{F} = -e \mathbf{v} \cdot \mathbf{E}$, after substitution inside (2.2.3), we find (2.2.2). \square

Let us assume that $\mathbf{x} \in \Omega \subset \mathbb{R}^3$ where Ω is a bounded open domain of size \mathbf{L} . We then consider the nondimensional variable $x := \mathbf{x}/\mathbf{L}$ with $|x| < 1$. We also introduce the normalized velocity $v := \mathbf{v}/c$ with $|v| < 1$. Taking into account these selections, it is more convenient to work with the time variable $t := c\mathbf{t}/\mathbf{L}$. It remains to interpret the electric and magnetic fields in terms of these variables. So, we select $E(x) := \underline{\mathbf{E}}^{-1} \mathbf{E}(Lx)$ and $B(x) := \underline{\mathbf{B}}^{-1} \mathbf{B}(Lx)$ where $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ are typical amplitudes of respectively \mathbf{E} and \mathbf{B} . The system (2.2.2) becomes

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = (1 - v^2)^{1/2} \left\{ -\frac{e \mathbf{L} \underline{\mathbf{B}}}{m c} v \wedge B(x) - \frac{e \mathbf{L} \underline{\mathbf{E}}}{m c^2} [E(x) - (v \cdot E(x)) v] \right\}. \end{cases}$$

The inverse of the electron gyrofrequency is the dimensionless parameter $\varepsilon := (m c)/(e \mathbf{L} \underline{\mathbf{B}})$. Common tokamaks are built with $L \simeq 5 m$ and $\underline{\mathbf{B}} \simeq 5 T$ so that $\varepsilon \simeq 10^{-4}$. On the other hand, in line with [FW02, Neg18, ZFG81], we can consider that

$$\frac{e \mathbf{L} \underline{\mathbf{E}}}{m c^2} = 1.$$

This is consistent with the realistic selections of $\underline{\mathbf{E}} \simeq 5 \times 10^5 V m^{-1}$. This furnishes (2.1.1) as a starting point for our analysis.

2.2.2 Geometric reformulation of the equations

The cartesian coordinates are not suitable to differentiate the various types of motion. In Paragraph [2.2.2.1](#), we present a toroidal description of x , and we decompose v in spherical coordinates emanating from an orthonormal basis following the variations of B . In Paragraph [2.2.2.2](#), we reformulate the equations for x accordingly. Then, in Paragraph [2.2.2.3](#), we do the same for v .

2.2.2.1 Change of coordinates for x and v

Recall ([2.1.13](#)). Remark that

$$x = X(r, \theta, \phi) = R_0 (\cos \phi, \sin \phi, 0) + r e_r, \quad \dot{x} = \dot{r} e_r + r \dot{\theta} e_\theta + R(r, \theta) \dot{\phi} e_\phi \quad (2.2.4)$$

as well as

$$\nabla_x = e_r \partial_r + e_\theta \frac{1}{r} \partial_\theta + e_\phi \frac{1}{R} \partial_\phi$$

Picture [2.1](#) represents r, θ, ϕ (and the shape of the confined plasma). From now on, we will work with $(r, \theta, \phi) \in [0, a] \times \mathbb{T}^2$ instead of x . By assumption [2.1.6](#), the function $B(\cdot)$ takes the form

$$B = I(r) \nabla \phi + \nabla \phi \wedge \nabla r, \quad I(r) \in \mathbb{R}_+^*,$$

which leads to

$$B = I(r) R^{-1} e_\phi + R^{-1} e_\phi \wedge e_r = \frac{I(r)}{R(r, \theta)} e_\phi - \frac{1}{R(r, \theta)} e_\theta.$$

Retain that

$$B(r, \theta) = B_\theta(r, \theta) e_\theta + B_\phi(r, \theta) e_\phi, \quad B_\theta(r, \theta) := -\frac{1}{R(r, \theta)}, \quad B_\phi(r, \theta) := \frac{I(r)}{R(r, \theta)}.$$

We aim to straighten out the field lines in an orthogonal way. For that, we define

$$e_{\parallel}(r, \theta, \phi) := -\frac{1}{\sqrt{1 + I(r)^2}} e_\theta(\theta, \phi) + \frac{I(r)}{\sqrt{1 + I(r)^2}} e_\phi(\phi).$$

The magnetic field then can be written

$$B(X(r, \theta, \phi)) = b(r, \theta) e_{\parallel}(r, \theta, \phi), \quad b(r, \theta) := \frac{1}{R(r, \theta)} \sqrt{1 + I(r)^2},$$

where $b > 0$ is the norm of the magnetic field B . Introduce the angle $w(r) \in]-\pi/2, 0[$ between the two directions e_ϕ and e_\parallel . We have

$$e_\parallel(r, \theta, \phi) = \sin \omega(r) e_\theta(\theta, \phi) + \cos \omega(r) e_\phi(\phi).$$

In the basis (e_ϕ, e_θ) , this gives rise to

$$(\cos \omega(r), \sin \omega(r)) = \left(\frac{I(r)}{\sqrt{1 + I(r)^2}}, -\frac{1}{\sqrt{1 + I(r)^2}} \right).$$

The unitary direction e_\parallel can be completed in order to obtain a positively oriented orthonormal basis $(e_r, e_\perp, e_\parallel)$ with $e_r \wedge e_\perp = -e_\parallel$. We find that

$$e_\perp(r, \theta, \phi) := \cos \omega(r) e_\theta(\theta, \phi) - \sin \omega(r) e_\phi(\phi).$$

We can also pass from (e_\perp, e_\parallel) to (e_θ, e_ϕ) through

$$e_\theta = +\cos \omega e_\perp + \sin \omega e_\parallel, \quad e_\phi = -\sin \omega e_\perp + \cos \omega e_\parallel. \quad (2.2.5)$$

From (2.1.14) and (2.2.5), we have

$$E(x) = E_r(r, \theta) e_r + E_\perp(r, \theta) e_\perp + E_\parallel(r, \theta) e_\parallel,$$

with

$$E_r(r, \theta) = \partial_r \Phi(r, \theta), \quad E_\perp(r, \theta) = r^{-1} \cos \omega(r) \partial_\theta \Phi(r, \theta), \quad (2.2.6)$$

and

$$E_\parallel(r, \theta) = r^{-1} \sin \omega(r) \partial_\theta \Phi(r, \theta). \quad (2.2.7)$$

On the other hand, $v(t)$ can be decomposed according to the spherical coordinates associated with the basis $(e_r, e_\perp, e_\parallel)$ as follows

$$\begin{aligned} v &= V(r, \theta, \phi, v, \zeta, \nu) \\ &= v \left[\cos \zeta \cos \nu e_r(\theta, \phi) + \cos \zeta \sin \nu e_\perp(r, \theta, \phi) + \sin \zeta e_\parallel(r, \theta, \phi) \right]. \end{aligned} \quad (2.2.8)$$

with radial distance $v := |v| \in \mathbb{R}_+$, azimuth angle $\zeta \in [-\pi/2, \pi/2]$, and polar angle $\nu \in \mathbb{T}$. The two maps $V(r, \theta, \phi, 0, \cdot, \cdot)$ and $V(r, \theta, \phi, v, \pm\pi/2, \cdot)$ are not one-to-one. Unless otherwise stated, we will work away from $v = 0$ (with $0 < v < 1$) and away from parallel

and antiparallel situations (with $\zeta \notin \{-\pi/2, +\pi/2\}$). The number ν allows to describe rotations in the plane perpendicular to e_{\parallel} . From the perspective of gyrokinetics (as long as $t \simeq 1$), the coordinates r, θ, ϕ, v and ζ undergo slow variations, whereas ν is subjected to fast fluctuations.

2.2.2.2 Equations in toroidal coordinates for x

The purpose of this paragraph is to interpret (2.1.1) in terms of $r, \theta, \phi, v, \zeta$ and ν . To this end, we follow [Che17] with some adaptations due to the presence of the electric field. The spatial part (r, θ, ϕ) can be handled as follows.

Lemma 2.2.2. [Spatial toroidal equations] The first equation inside (2.1.1) furnishes

$$\begin{pmatrix} \frac{dr}{dt} \\ \frac{d\theta}{dt} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_r \\ \mathcal{A}_\theta \end{pmatrix} := v \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r} \cos \omega(r) & \frac{1}{r} \sin \omega(r) \end{pmatrix} \begin{pmatrix} \cos \zeta \cos \nu \\ \cos \zeta \sin \nu \\ \sin \zeta \end{pmatrix}, \quad (2.2.9)$$

together with

$$\frac{d\phi}{dt} = \mathcal{A}_\phi := \frac{v}{R(r, \theta)} \left(-\sin \omega(r) \cos \zeta \sin \nu + \cos \omega(r) \sin \zeta \right). \quad (2.2.10)$$

At time $t = 0$, we start with $(r, \theta, \phi)(0) = (r_0, \theta_0, \phi_0)$.

Proof. We can express the spherical coordinates (2.2.8) of v in the basis (e_r, e_θ, e_ϕ) through

$$\begin{aligned} v &= v \cos \zeta \cos \nu e_r(\theta, \phi) + v \left[\cos \zeta \sin \nu \cos \omega(r) + \sin \zeta \sin \omega(r) \right] e_\theta(\theta, \phi) \\ &+ v \left[\sin \zeta \cos \omega(r) - \cos \zeta \sin \nu \sin \omega(r) \right] e_\phi(\phi). \end{aligned} \quad (2.2.11)$$

Just compare the second part of (2.2.4) to the equation (2.2.11) to find (2.2.9) and (2.2.10). \square

2.2.2.3 Equations in spherical coordinates for v

Below, we interpret the second equation of (2.1.1) in terms of $r, \theta, \phi, v, \zeta$ and ν .

Lemma 2.2.3 (Velocity equations in spherical coordinates). *We have*

$$\begin{cases} \dot{v} = \mathcal{A}_v := -(1-v^2)^{3/2} [\cos \zeta \cos \nu E_r + \cos \zeta \sin \nu E_\perp + \sin \zeta E_\parallel], \\ \dot{\zeta} = \mathcal{A}_\zeta := v \mathcal{F}_\zeta^g + \frac{(1-v^2)^{1/2}}{v} [\sin \zeta \cos \nu E_r + \sin \zeta \sin \nu E_\perp - \cos \zeta E_\parallel], \\ \dot{\nu} = \frac{v \mathcal{F}_\nu^g}{\cos \zeta} + \frac{(1-v^2)^{1/2}}{v \cos \zeta} [\sin \nu E_r - \cos \nu E_\perp - \frac{1}{\varepsilon} v b \cos \zeta], \end{cases} \quad (2.2.12)$$

where \mathcal{F}_ζ^g and \mathcal{F}_ν^g are geometrical coefficients given by

$$\begin{aligned} \mathcal{F}_\zeta^g(r, \zeta, \theta, \nu) &:= \omega'(r) \cos \zeta \cos \nu \sin \nu \\ &\quad - \frac{1}{r} \sin \omega \cos \nu (\cos \omega \cos \zeta \sin \nu + \sin \omega \sin \zeta) \\ &\quad - \frac{1}{R} (\cos \theta \cos \omega \cos \nu - \sin \theta \sin \nu) (-\sin \omega \cos \zeta \sin \nu + \cos \omega \sin \zeta). \end{aligned}$$

$$\begin{aligned} \mathcal{F}_\nu^g(r, \theta, \zeta, \nu) &:= -\omega'(r) \cos \zeta (\cos \nu)^2 \sin \zeta \\ &\quad - \frac{1}{r} (\cos \omega \sin \nu \cos \zeta + \sin \omega \sin \zeta) (\sin \omega \sin \nu \sin \zeta + \cos \omega \cos \zeta) \\ &\quad - \frac{1}{R} (-\sin \omega \cos \zeta \sin \nu + \cos \omega \sin \zeta) (\cos \omega \cos \theta \sin \nu + \sin \theta \cos \nu) \sin \zeta \\ &\quad + \frac{1}{R} (-\sin \omega \cos \zeta \sin \nu + \cos \omega \sin \zeta) \cos \zeta \sin \omega \cos \theta. \end{aligned}$$

Note that the right hand sides of (2.2.12) are not defined when $v = 0$ or when $\zeta \in \{-\pi/2, +\pi/2\}$. For this reason, at time $t = 0$, we start with ${}^t(v, \zeta, \nu)(0) = {}^t(v_0, \zeta_0, \nu_0)$ satisfying

$$0 < v_0 < 1, \quad \zeta_0 \notin \{-\pi/2, +\pi/2\}. \quad (2.2.13)$$

The system (2.2.12) involves all components E_r , E_\perp and E_\parallel of the electric field E . The effects on the motion of these components are interpreted in [WC11, Sections 2.5, 2.6, 3.9]. In particular:

- The perpendicular component E_\perp takes part in the so called $E \times B$ drift which may induce small displacements in perpendicular directions to both fields E and B ;
- The parallel component E_\parallel lies in the magnetic surface. It can be responsible for the acceleration or deceleration of particles along the magnetic field.

Proof. We introduce first the auxilliary orthonormal basis (e_v^1, e_v^2, e_v^3) such that $e_v^1 \wedge e_v^2 = e_v^3$ and built with

$$\begin{aligned}
 e_v^1 &:= +\cos\zeta \cos\nu e_r + \cos\zeta \sin\nu e_\perp + \sin\zeta e_\parallel, \\
 e_v^2 &:= -\sin\zeta \cos\nu e_r - \sin\zeta \sin\nu e_\perp + \cos\zeta e_\parallel, \\
 e_v^3 &:= -\sin\nu e_r + \cos\nu e_\perp.
 \end{aligned}$$

We can also express $(e_r, e_\perp, e_\parallel)$ in terms of this new basis:

$$\begin{aligned}
 e_r &:= \cos\zeta \cos\nu e_v^1 - \sin\zeta \cos\nu e_v^2 - \sin\nu e_v^3, \\
 e_\perp &:= \cos\zeta \sin\nu e_v^1 - \sin\zeta \sin\nu e_v^2 + \cos\nu e_v^3, \\
 e_\parallel &:= \sin\zeta e_v^1 + \cos\zeta e_v^2.
 \end{aligned}$$

By construction, we have $v = v e_v^1$ which can be used to compute the second equation of (2.1.1) in terms of $r, \theta, \phi, v, \zeta$ and ν . To this end, we can exploit the three following intermediate formulas:

1. The first one is

$$\begin{aligned}
 \dot{v} &= \dot{v} e_v^1 + v \dot{\zeta} e_v^2 + v \cos\zeta \dot{\nu} e_v^3 + v \cos\zeta (\dot{\theta} \cos\omega - \dot{\phi} \sin\omega \cos\theta) e_v^3 \\
 &\quad + v (\dot{\theta} \sin\omega + \dot{\phi} \cos\omega \cos\theta) (-\sin\zeta e_r + \cos\zeta \cos\nu e_\parallel) \\
 &\quad + v (\dot{r} \omega'(r) + \dot{\phi} \sin\theta) (\sin\zeta e_\perp - \cos\zeta \sin\nu e_\parallel).
 \end{aligned}$$

2. The second is

$$\begin{pmatrix} \cos\nu & -\sin\nu \\ \sin\nu & +\cos\nu \end{pmatrix} \begin{pmatrix} -\sin\zeta e_r + \cos\zeta \cos\nu e_\parallel \\ +\sin\zeta e_\perp - \cos\zeta \sin\nu e_\parallel \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin\zeta \end{pmatrix} \begin{pmatrix} e_v^2 \\ e_v^3 \end{pmatrix}.$$

3. The third is

$$B(X) \wedge V = -v b(r, \theta) \cos\zeta e_v^3.$$

By combining (1) and (2), we get

$$\begin{aligned}
 \dot{v} &= \dot{v} e_v^1 + v \dot{\zeta} e_v^2 + v \cos\zeta \dot{\nu} e_v^3 + v \cos\zeta (\dot{\theta} \cos\omega - \dot{\phi} \sin\omega \cos\theta) e_v^3 \\
 &\quad + v (\dot{\theta} \sin\omega + \dot{\phi} \cos\omega \cos\theta) \sin\nu \sin\zeta e_v^3 + v (\dot{r} \omega'(r) + \dot{\phi} \sin\theta) \cos\nu \sin\zeta e_v^3 \\
 &\quad + v (\dot{\theta} \sin\omega + \dot{\phi} \cos\omega \cos\theta) \cos\nu e_v^2 - v (\dot{r} \omega'(r) + \dot{\phi} \sin\theta) \sin\nu e_v^2.
 \end{aligned}$$

This allow us to write the second equation of (2.1.1) in the basis (e_v^1, e_v^2, e_v^3) according to

$$\begin{aligned}
 0 &= (1 - v^2)^{-1/2} \dot{v} + E - (v \cdot E) v + \frac{1}{\varepsilon} v \wedge B \\
 &= \left[(1 - v^2)^{-1/2} \dot{v} + (1 - v^2) (\cos\zeta \cos\nu E_r + \cos\zeta \sin\nu E_\perp + \sin\zeta E_\parallel) \right] e_v^1 \\
 &\quad + \left[(1 - v^2)^{-1/2} v \dot{\zeta} + (1 - v^2)^{-1/2} v (\dot{\theta} \sin\omega + \dot{\phi} \cos\omega \cos\theta) \cos\nu \right. \\
 &\quad \left. + (1 - v^2)^{-1/2} v \cos\zeta \dot{\nu} + (1 - v^2)^{-1/2} v \cos\zeta (\dot{\theta} \cos\omega - \dot{\phi} \sin\omega \cos\theta) \right] e_v^2 \\
 &\quad + \left[(1 - v^2)^{-1/2} v \cos\zeta (\dot{\theta} \sin\omega + \dot{\phi} \cos\omega \cos\theta) \sin\nu \sin\zeta \right. \\
 &\quad \left. + (1 - v^2)^{-1/2} v \cos\zeta (\dot{r} \omega'(r) + \dot{\phi} \sin\theta) \cos\nu \sin\zeta \right] e_v^3.
 \end{aligned}$$

$$\begin{aligned}
 & - (1 - v^2)^{-1/2} v \left(\dot{r} \omega'(r) + \dot{\phi} \sin \theta \right) \sin \nu - \sin \zeta \cos \nu E_r - \sin \zeta \sin \nu E_\perp + \\
 & \left. \cos \zeta E_\parallel \right] e_v^2 \\
 & + \left[(1 - v^2)^{-1/2} v \cos \zeta \dot{\nu} + (1 - v^2)^{-1/2} v \left(\dot{\theta} \sin \omega + \dot{\phi} \cos \omega \cos \theta \right) \sin \nu \sin \zeta \right. \\
 & + (1 - v^2)^{-1/2} v \left(\dot{r} \omega'(r) + \dot{\phi} \sin \theta \right) \cos \nu \sin \zeta \\
 & \left. + (1 - v^2)^{-1/2} v \cos \zeta \left(\dot{\theta} \cos \omega - \dot{\phi} \sin \omega \cos \theta \right) - \sin \nu E_r + \cos \nu E_\perp + \right. \\
 & \left. \frac{1}{\varepsilon} v b \cos \zeta \right] e_v^3.
 \end{aligned}$$
 As (e_v^1, e_v^2, e_v^3) is a basis, all components must be zero. The one in factor of e_v^1 leads to the first equation of (2.2.12). Those in factor of e_v^2 and e_v^3 , after replacing the time derivatives of $\dot{r}, \dot{\theta}, \dot{\phi}$ as in (2.2.9) and (2.2.10), give rise respectively to the second and third equation of (2.2.12). \square

To absorb the large factor ε^{-1} in the last equation of (2.2.12), we perform now a sort of filtering method. Recall that we have decomposed the velocity in spherical coordinates where v as well as ζ are intended to be slow scales whereas $\nu \in \mathbb{T}$ is a fast one. This implies that it is natural to seek ν in the form $\nu = \varepsilon^{-1} v$ with, at time $t = 0$, the condition $v(0) = \varepsilon \nu_0$. Then, comparing this to the third equation in (2.2.12), we get

$$\dot{v} = \mathcal{V}_0(r, \theta, v) + \varepsilon \mathcal{V}_1(r, \theta, v, \zeta, \nu), \quad v(0) = \varepsilon \nu_0, \quad (2.2.14)$$

where the two source terms \mathcal{V}_0 and \mathcal{V}_1 are given by

$$\mathcal{V}_0(r, \theta, v) := - (1 - v^2)^{1/2} b(r, \theta) < 0, \quad (2.2.15)$$

$$\mathcal{V}_1(r, \theta, v, \zeta, \nu) := v (\cos \zeta)^{-1} \mathcal{F}_\nu^g + v^{-1} (\cos \zeta)^{-1} (1 - v^2)^{1/2} \left[\sin \nu E_r - \cos \nu E_\perp \right]. \quad (2.2.16)$$

The right hand sides of (2.2.9) and (2.2.12) do not depend on ϕ . Thus, we can put ϕ aside. The strategy to solve (2.1.1) is to focus on (2.2.9)-(2.2.12), and then (ultimately) to consider (2.2.10) to recover ϕ . By construction, the system (2.2.9)-(2.2.12) amounts to the same thing as (2.1.3) with

$$\varkappa := {}^t(r, \theta, v, \zeta), \quad \mathcal{A} := {}^t(\mathcal{A}_r, \mathcal{A}_\theta, \mathcal{A}_v, \mathcal{A}_\zeta), \quad v := \varepsilon \nu, \quad \mathcal{V} := \mathcal{V}_0 + \varepsilon \mathcal{V}_1, \quad \nu_0 = \varepsilon \nu_0. \quad (2.2.17)$$

At time $t = 0$, we start with $\varkappa(0) = z_0 := {}^t(r_0, \theta_0, v_0, \zeta_0)$ where

$$0 < r_0 < a < R_0, \quad 0 \leq \theta_0 < 2\pi, \quad 0 < v_0 < 1, \quad -\frac{\pi}{2} < \zeta_0 < \frac{\pi}{2}. \quad (2.2.18)$$

The nonlinear source term ${}^t(\mathcal{A}, \mathcal{V})(\varepsilon; z; \nu)$ satisfies (2.1.4) and (2.1.5). In particular, it is periodic of period 2π with respect to the last variable ν . The Cauchy-Lipschitz theorem guarantees the existence of unique local solution to (2.1.3). Since both \mathcal{A} and \mathcal{V} are bounded for z in a compact set and $\nu \in \mathbb{T}$, the lifespan T_ε is such that $T_\varepsilon \geq T$ for some $T > 0$.

2.3 Work of preparation

We consider here the average flow (in the sense of Definition 2.1.1) which is associated with the system (2.1.3) on r, θ, v and ζ derived from (2.1.1). Thus, with $Z = {}^t(Z_r, Z_\theta, Z_v, Z_\zeta)$ and $\bar{\mathcal{A}} = {}^t(\bar{\mathcal{A}}_r, \bar{\mathcal{A}}_\theta, \bar{\mathcal{A}}_v, \bar{\mathcal{A}}_\zeta)$, we look at the flow generated by (2.1.7). In Subsection 2.3.1, its structure is made explicit. In Subsection 2.3.2, its periodic properties (which are essential to eliminate problems of secular growth) are investigated.

2.3.1 Structure of the average flow

In Subsection 2.3.1.1, we detail the content of (2.1.7) for the realistic data issued from (2.2.9) and (2.2.12). In Subsection 2.3.1.2, we exhibit two invariant quantities. In Subsection 2.3.1.3, this information is exploited to show that the lifespan associated with (2.1.7) is infinite. Finally, in Subsection 2.3.1.4, we come back to the parallel case.

2.3.1.1 Equations for the average flow

The first step is to clarify the structure of (2.1.7).

Lemma 2.3.1 (The system of odes in $]0, a[\times \mathbb{T} \times]0, 1[\times]-\pi/2, \pi/2[$ satisfied by the average flow). *We find that*

$$\frac{dZ_r}{dt} = 0, \quad Z_r(0) = r_0. \quad (2.3.1)$$

For the remaining components, we obtain the following system of coupled equations

$$\left\{ \begin{array}{l} \frac{dZ_\theta}{dt} = +r_0^{-1} \sin \omega(r_0) Z_v \sin Z_\zeta, \quad Z_\theta(z_0; 0) = \theta_0, \\ \frac{dZ_v}{dt} = -(1 - Z_v^2)^{3/2} \sin Z_\zeta E_{\parallel}(r_0, Z_\theta), \quad Z_v(z_0; 0) = v_0, \\ \frac{dZ_\zeta}{dt} = -Z_v \left(2R(r_0, Z_\theta)\right)^{-1} \sin \omega(r_0) \sin Z_\theta \cos Z_\zeta \\ \quad - Z_v^{-1} (1 - Z_v^2)^{1/2} \cos Z_\zeta E_{\parallel}(r_0, Z_\theta), \quad Z_\zeta(z_0; 0) = \zeta_0. \end{array} \right. \quad (2.3.2)$$

From (2.3.1), we know that $Z_r(t) = r_0$ for all $t \in \mathbb{R}$. In comparison with (2.2.12), observe that both components E_r and E_\perp have disappeared from (2.3.2).

Proof. We have $\bar{\mathcal{A}}_r = 0$ so that (2.3.1) is obvious. By formula (2.1.6), we can see that

◁ The mean value of \mathcal{A}_θ is

$$\begin{aligned} \bar{\mathcal{A}}_\theta(z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{v}{r_0} \left[\cos \zeta \sin \nu \cos \omega(r_0) + \sin \zeta \sin \omega(r_0) \right] d\nu \\ &= r_0^{-1} v \sin \zeta \sin \omega(r_0). \end{aligned}$$

◁ The mean value of \mathcal{A}_v is

$$\begin{aligned} \bar{\mathcal{A}}_v(z) &= \frac{1}{2\pi} \int_0^{2\pi} -(1 - v^2)^{3/2} \left[\cos \zeta \cos \nu E_r + \cos \zeta \sin \nu E_\perp + \sin \zeta E_{\parallel} \right] d\nu \\ &= -(1 - v^2)^{3/2} \sin \zeta E_{\parallel}(r_0, \theta). \end{aligned}$$

◁ The mean value of \mathcal{A}_ζ is

$$\begin{aligned} \bar{\mathcal{A}}_\zeta(z) &= \frac{1}{2\pi} \int_0^{2\pi} \left[v \mathcal{F}_\zeta^g + v^{-1} (1 - v^2)^{1/2} (\sin \zeta \cos \nu E_r + \sin \zeta \sin \nu E_\perp - \cos \zeta E_{\parallel}) \right] d\nu \\ &= -v^{-1} (1 - v^2)^{1/2} \cos \zeta E_{\parallel} - \frac{1}{2\pi} \frac{v}{R} \sin \theta \cos \zeta \sin \omega \int_0^{2\pi} \frac{1 + \cos 2\nu}{2} d\nu \\ &= -v^{-1} (1 - v^2)^{1/2} \cos \zeta E_{\parallel}(r_0, \theta) - \frac{v}{2R(r_0, \theta)} \sin \theta \cos \zeta \sin \omega(r_0). \end{aligned}$$

It is now clear that (2.1.7) implies (2.3.2). □

2.3.1.2 Two invariant quantities

With Φ as in (2.2.7), define

$$H_1(r_0; \theta, v, \zeta) := v (1 - v^2)^{-1/2} (R_0 + r_0 \cos \theta)^{1/2} \cos \zeta, \quad (2.3.3)$$

$$H_2(r_0; \theta, v, \zeta) \equiv \mathcal{H}_2(r_0; \theta, v) := \Phi(r_0, \theta) + (1 - v^2)^{-1/2}. \quad (2.3.4)$$

Lemma 2.3.2 (Constants of motion). *For all $r_0 \in \mathbb{R}_+^*$, the two functions $H_1(r_0; \cdot)$ and $H_2(r_0; \cdot)$ given by (2.3.3) and (2.3.4) are constant along the trajectories induced by (2.3.2), in the sense that*

$$\frac{d}{dt} [H_i(r_0; Z_\theta, Z_v, Z_\zeta)] = 0, \quad \forall i \in \{1, 2\}. \quad (2.3.5)$$

Proof. Compute

$$\begin{aligned} \partial_\theta H_1(r_0; \theta, v, \zeta) &= -\frac{1}{2} r_0 v (1 - v^2)^{-1/2} R(r_0, \theta)^{-1/2} \sin \theta \cos \zeta, \\ \partial_v H_1(r_0; \theta, v, \zeta) &= \sqrt{R(r_0, \theta)} \cos \zeta [(1 - v^2)^{-1/2} + v^2(1 - v^2)^{-3/2}], \\ \partial_\zeta H_1(r_0; \theta, v, \zeta) &= -v (1 - v^2)^{-1/2} \sqrt{R(r_0, \theta)} \sin \zeta, \\ \partial_\theta H_2(r_0; \theta, v) &= r_0 (\sin \omega(r_0))^{-1} E_{||}(r_0, \theta) = \partial_\theta \Phi(r_0, \theta), \\ \partial_v H_2(r_0; \theta, v) &= v (1 - v^2)^{-3/2}. \end{aligned} \quad (2.3.6)$$

We derive now the two function H_1 and H_2 with respect to time to get

$$\begin{aligned} \frac{d}{dt} H_1(r_0; Z_\theta, Z_v, Z_\zeta) &= \partial_\theta H_1(r_0; Z_\theta, Z_v, Z_\zeta) \frac{dZ_\theta}{dt} + \partial_v H_1(r_0; Z_\theta, Z_v, Z_\zeta) \frac{dZ_v}{dt} \\ &\quad + \partial_\zeta H_1(r_0; Z_\theta, Z_v, Z_\zeta) \frac{dZ_\zeta}{dt}, \\ \frac{d}{dt} H_2(r_0; Z_\theta, Z_v) &= \partial_\theta H_2(r_0; Z_\theta, Z_v) \frac{dZ_\theta}{dt} + \partial_v H_2(r_0; Z_\theta, Z_v) \frac{dZ_v}{dt}. \end{aligned} \quad (2.3.7)$$

Substitute in (2.3.7) the terms dZ_θ/dt , dZ_v/dt and dZ_ζ/dt as indicated in (2.3.2) as well as the partial derivatives of H_i with $i \in \{1, 2\}$ as indicated in (2.3.6) to get that (2.3.5) holds true. \square

In general (at least when $\zeta \neq 0$), the level surfaces of H_1 and H_2 intersect transversally in the three dimensional space $\mathbb{T} \times]0, 1[\times \mathbb{T}$ to define curves which correspond to integral curves of (2.3.2).

2.3.1.3 Study of the lifespan

The conservation of H_1 and H_2 allows to extract the following interesting information.

Lemma 2.3.3 (Infinite lifespan). *Fix any $r_0 \in]0, a[$. Then, for all (θ_0, v_0, ζ_0) satisfying (2.2.18), the solution to the nonlinear system (2.3.2) exists for all times $t \in \mathbb{R}$.*

Proof. We denote by $S(z_0) \in \mathbb{R}_+^* \cup \{+\infty\}$ the lifespan associated with (2.3.2). From (2.3.5) with $i = 2$, we can deduce that

$$\left(1 - Z_v(t)^2\right)^{-1/2} = -\frac{r_0}{\sin \omega(r_0)} \int_{\theta_0}^{Z_\theta(t)} E_{\parallel}(r_0, \tilde{\theta}) d\tilde{\theta} + (1 - v_0^2)^{-1/2}. \quad (2.3.8)$$

The function $E_{\parallel}(r_0, \cdot)$ is a periodic function with mean value equal to zero. Define

$$-\infty < m(r_0) := \min_{Z_\theta \in \mathbb{R}} \int_0^{Z_\theta} E_{\parallel}(r_0, \tilde{\theta}) d\tilde{\theta} \leq 0, \quad 0 \leq M(r_0) := \max_{Z_\theta \in \mathbb{R}} \int_0^{Z_\theta} E_{\parallel}(r_0, \tilde{\theta}) d\tilde{\theta} < +\infty.$$

Recall that $\omega(r_0) \in]-\pi/2, 0[$ so that $\sin \omega(r_0) < 0$. We see that

$$\forall t \in [0, S(z_0)[, \quad \left(1 - Z_v(t)^2\right)^{-1/2} \leq -\frac{r_0}{\sin \omega(r_0)} M(r_0) + (1 - v_0^2)^{-1/2}.$$

This implies that

$$\exists \eta_1 \in]0, 1[; \quad \forall t \in [0, S(z_0)[, \quad Z_v(t) \leq 1 - \eta_1. \quad (2.3.9)$$

From (2.3.5) with $i = 1$, we can infer that

$$\begin{aligned} Z_v &= v_0 \frac{(1 - Z_v^2)^{1/2}}{(1 - v_0^2)^{1/2}} \frac{(R_0 + r_0 \cos \theta_0)^{1/2}}{(R_0 + r_0 \cos Z_\theta)^{1/2}} \frac{\cos \zeta_0}{\cos Z_\zeta} \\ &\geq v_0 \frac{\sqrt{\eta_1}}{(1 - v_0^2)^{1/2}} \frac{(R_0 + r_0 \cos \theta_0)^{1/2}}{(R_0 + r_0)^{1/2}} \cos \zeta_0 > 0. \end{aligned}$$

By this way, we get

$$\exists \eta_2 \in]0, 1[; \quad \forall t \in [0, S(z_0)[, \quad \eta_2 \leq Z_v(t). \quad (2.3.10)$$

But we can also assert that

$$\begin{aligned} \cos Z_\zeta &= v_0 \frac{(1 - Z_v^2)^{1/2}}{(1 - v_0^2)^{1/2}} \frac{(R_0 + r_0 \cos \theta_0)^{1/2}}{(R_0 + r_0 \cos Z_\theta)^{1/2}} \frac{\cos \zeta_0}{Z_v} \\ &\geq v_0 \frac{\sqrt{\eta_1}}{(1 - v_0^2)^{1/2}} \frac{(R_0 + r_0 \cos \theta_0)^{1/2}}{(R_0 + r_0)^{1/2}} \cos \zeta_0 > 0, \end{aligned}$$

which means that

$$\exists \eta_3 \in]0, 1[; \quad \forall t \in [0, S(z_0)[, \quad -\frac{\pi}{2} + \eta_3 \leq Z_\zeta(t) \leq \frac{\pi}{2} - \eta_3. \quad (2.3.11)$$

Thus, the flow remains in a compact set of $\mathbb{T} \times]0, 1[\times]-\pi/2, \pi/2[$. By extension theorem, this implies that $S(z_0) = +\infty$. \square

In view of (2.2.8), we have

$$v \cdot e_{\parallel} = v \sin \zeta = v \cos(\pi/2 - \zeta).$$

Thus, the real number $\pi/2 - \zeta \in]0, \pi[$ represents the angle between the direction v and the magnetic field B . The endpoints $\zeta = -\pi/2$ and $\zeta = \pi/2$ correspond respectively to antiparallel and parallel situations.

2.3.1.4 Solving the average flow in the parallel case

As already observed, the system (2.2.12) is not meaningful when $\cos \zeta = 0$. By contrast, the right hand sides of (2.3.2) are defined and smooth with respect to Z_ζ without any restriction on Z_ζ . By continuity, any (local or global if it does exist) solution to (2.3.2) issued from $\zeta_0 = -\pi/2$ or $\zeta_0 = \pi/2$ tells us what happens (approximately) when $\cos \zeta_0 > 0$ is small. We work with $\zeta_0 = \pi/2$, the other case $\zeta_0 = -\pi/2$ being very similar. In the limiting case $\zeta_0 = \pi/2$, we find that $Z_\zeta(t) = \pi/2$ for all time t , and we have just to solve

$$\begin{cases} \frac{dZ_\theta}{dt} = +r_0^{-1} Z_v \sin \omega(r_0), & Z_\theta(z_0; 0) = \theta_0, \\ \frac{dZ_v}{dt} = -(1 - Z_v^2)^{3/2} E_{\parallel}(r_0, Z_\theta), & Z_v(z_0; 0) = v_0. \end{cases} \quad (2.3.12)$$

The functional H_1 (which is zero) does not help. On the other hand, the conservation of

H_2 yields (2.3.9) as before. To simplify, let us assume that

$$-\frac{r_0 m(r_0)}{\sin \omega(r_0)} + (1 - v_0^2)^{-1/2} > 1.$$

Using (2.3.8), this ensures that $Z_v > 0$ remains away from zero. As a consequence, we have

$$\exists \eta_4 > 0; \quad \forall t \in \mathbb{R}_+, \quad \frac{dZ_\theta}{dt} \leq -\eta_4 < 0.$$

Define $T := \inf \{ t \in \mathbb{R}_+^*; Z_\theta(t) = \theta_0 - 2\pi \}$. From (2.3.8), we see that $Z_v(T) = v_0$. Thus, at the time T , we return to the same initial data as in (2.3.12). The solution of (2.3.12) viewed in $\mathbb{T} \times]0, 1[$ is therefore periodic of period $T \in \mathbb{R}_+^*$. This brief discussion suggests that (2.3.2) could generate similar periodic features. The next subsection checks this point.

2.3.2 Periodic properties of the average flow

We start with initial data as in (2.2.18). The next step is to show that (2.3.2) is made of a family of Hamiltonian subsystems sharing the same simplified form, namely (2.3.23). This is achieved in Paragraph 2.3.2.1 after a series of change of variables. In the resulting Hamiltonian subsystems, as observed in Paragraph 2.3.2.2, we can decompose the phase space into the union of distinct subsets leading to different types of motion: libration, rotation or convergence to a fixed point (separatrices). In the two first cases, we can construct action-angle variables. In Paragraph 2.3.2.3, we select a domain Ω that avoids the separatrix region. We observe that the flow issued from positions inside Ω is periodic. In Paragraph 2.3.2.3, we exhibit extra almost periodic features.

2.3.2.1 Foliation by Hamiltonian subsystems

Define

$$\begin{aligned} \mathcal{U}_v :]0, 1[&\longrightarrow]1, +\infty[& \mathcal{U}_\zeta :]-\pi/2, \pi/2[&\longrightarrow \mathbb{R} \\ Z_v &\longmapsto (1 - Z_v^2)^{-1/2}, & Z_\zeta &\longmapsto \sqrt{2} \operatorname{sgn} Z_\zeta (-\ln \cos Z_\zeta)^{1/2}. \end{aligned}$$

It is clear that $\mathcal{U}_v :]0, 1[\rightarrow]1, +\infty[$ is one-to-one. The case of \mathcal{U}_ζ is examined below.

Lemma 2.3.4 (About \mathcal{U}_ζ). *The function $\mathcal{U}_\zeta :]-\pi/2, \pi/2[\rightarrow \mathbb{R}$ is a smooth diffeomorphic map.*

Proof. Due to the factor $\text{sgn } Z_\zeta$, remark that $\mathcal{U}_\zeta(Z_\zeta) \sim Z_\zeta$ near $Z_\zeta = 0$. This indicates that \mathcal{U}_ζ should be a local diffeomorphism near $Z_\zeta = 0$. For a global argument, consider the function

$$\begin{aligned} G :] - \pi/2, \pi/2[&\longrightarrow \mathbb{R}_+ \\ Z_\zeta &\longmapsto - \ln \cos Z_\zeta . \end{aligned}$$

Observe that $G(0) = G'(0) = 0$. On the other hand $G''(Z_\zeta) = (\cos Z_\zeta)^{-2}$. The integral version of Taylor's theorem yields

$$2 G(Z_\zeta) = -2 \ln \cos Z_\zeta = Z_\zeta^2 g(Z_\zeta), \quad g(Z_\zeta) := 2 \int_0^1 (1-s) \left(\cos(Z_\zeta s) \right)^{-2} ds .$$

Remark that $g(\cdot)$ is a smooth positive even function on the interval $] - \pi/2, \pi/2[$, which is strictly increasing on $[0, \pi/2[$ and such that $g(0) = 1$. This implies that $Z_\zeta \longmapsto Z_\zeta \sqrt{g(Z_\zeta)} \equiv \mathcal{U}_\zeta(Z_\zeta)$ is well-defined, smooth and odd. It has a positive derivative, and it tends to $\pm\infty$ when Z_ζ goes to $\pm\pi/2$. Therefore, it gives rise to a diffeomorphism from $] - \pi/2, \pi/2[$ onto \mathbb{R} . \square

To elucidate the internal structure of (2.3.2), the first step is to replace the state variable Z by $\mathcal{Z} = {}^t(\mathcal{Z}_r, \mathcal{Z}_\theta, \mathcal{Z}_v, \mathcal{Z}_\zeta) = \mathcal{U}(Z)$ where \mathcal{U} is the diffeomorphic map

$$\begin{aligned} \mathcal{U} :]0, a] \times \mathbb{T} \times]0, 1[\times] - \pi/2, \pi/2[&\longrightarrow]0, a] \times \mathbb{T} \times]1, +\infty[\times \mathbb{R} \\ (\mathcal{Z}_r, \mathcal{Z}_\theta, \mathcal{Z}_v, \mathcal{Z}_\zeta) &\longmapsto (\mathcal{Z}_r, \mathcal{Z}_\theta, \mathcal{U}_v(\mathcal{Z}_v), \mathcal{U}_\zeta(\mathcal{Z}_\zeta)) . \end{aligned} \quad (2.3.13)$$

By convention, we can set $\mathcal{U}_r(\mathcal{Z}_r) := \mathcal{Z}_r$ and $\mathcal{U}_\theta(\mathcal{Z}_\theta) := \mathcal{Z}_\theta$. The second step is to change the time variable t . To this end, introduce

$$\begin{aligned} f :] - \pi/2, \pi/2[&\longrightarrow \mathbb{R}_+^* \\ Z_\zeta &\longmapsto \mathcal{U}'_\zeta(Z_\zeta) \cos Z_\zeta . \end{aligned}$$

By differentiating the identity $\mathcal{U}_\zeta(Z_\zeta)^2 = -2 \ln \cos Z_\zeta$, we can deduce the extra relation

$$\forall Z_\zeta \in] - \pi/2, \pi/2[, \quad \sin Z_\zeta = f(Z_\zeta) \mathcal{U}_\zeta(Z_\zeta) . \quad (2.3.14)$$

We now follow the evolution of $Z(z_0; \cdot)$ in terms of the new state variable \mathcal{Z} and also according to some adapted internal clock associated with the new time variable

$$\mathbf{t}(z_0; t) := \frac{\sin \omega(r_0)}{r_0} \int_0^t Z_v(z_0; \tilde{t}) f \circ Z_\zeta(z_0; \tilde{t}) d\tilde{t} . \quad (2.3.15)$$

Recall that $\sin \omega(r_0) < 0$. Due to (2.3.9), (2.3.10) and (2.3.11), we have

$$\exists (\eta_5, \eta_6) \in \mathbb{R}^2; \quad 0 < \eta_5 \leq -\frac{dt}{dt} \leq \eta_6. \quad (2.3.16)$$

Thus, we can change the time variable $t \in \mathbb{R}$ into $\mathfrak{t} \in \mathbb{R}$ (with a one-to-one global correspondence). Let $t(z_0; \cdot)$ be the inverse function of $\mathfrak{t}(z_0; \cdot)$. We can define

$$\mathcal{Z}(z_0; \mathfrak{t}) := \mathcal{U} \circ Z(z_0; t(z_0; \mathfrak{t})). \quad (2.3.17)$$

Lemma 2.3.5 (The system of odes in $]0, a[\times \mathbb{T} \times]1, +\infty[\times \mathbb{R}$ satisfied by $\mathcal{Z}(z_0; \cdot)$). *Starting from $(r_0, \theta_0, v_0, \zeta_0)$ as in (2.2.18), define*

$$\mathcal{Z}_0 \equiv \mathcal{Z}_0(z_0) = (\mathcal{Z}_{0r}, \mathcal{Z}_{0\theta}, \mathcal{Z}_{0v}, \mathcal{Z}_{0\zeta})(z_0) := (r_0, \theta_0, \mathcal{U}_v(v_0), \mathcal{U}_\zeta(\zeta_0)).$$

We find that

$$\frac{d\mathcal{Z}_r}{d\mathfrak{t}} = 0, \quad \mathcal{Z}_r(0) = \mathcal{Z}_{0r}. \quad (2.3.18)$$

On the other hand, the system (2.3.2) is equivalent to

$$\left\{ \begin{array}{l} \frac{d\mathcal{Z}_\theta}{d\mathfrak{t}} = +\mathcal{Z}_\zeta, \quad \mathcal{Z}_\theta(0) = \mathcal{Z}_{0\theta}, \\ \frac{d\mathcal{Z}_v}{d\mathfrak{t}} = -\partial_\theta \Phi(r_0, \mathcal{Z}_\theta) \mathcal{Z}_\zeta, \quad \mathcal{Z}_v(0) = \mathcal{Z}_{0v}, \\ \frac{d\mathcal{Z}_\zeta}{d\mathfrak{t}} = -\frac{r_0 \sin \mathcal{Z}_\theta}{2R(r_0, \mathcal{Z}_\theta)} - \frac{\mathcal{Z}_v}{\mathcal{Z}_v^2 - 1} \partial_\theta \Phi(r_0, \mathcal{Z}_\theta), \quad \mathcal{Z}_\zeta(0) = \mathcal{Z}_{0\zeta}. \end{array} \right. \quad (2.3.19)$$

From (2.3.18), we know that $\mathcal{Z}_r(\mathfrak{t}) = r_0$ for all $\mathfrak{t} \in \mathbb{R}$.

Proof. The line (2.3.18) is straightforward. In view of (2.3.2) and (2.3.15), the first equation of (2.3.19) is satisfied if and only if

$$\frac{d\mathcal{Z}_\theta}{d\mathfrak{t}} = \frac{dt}{d\mathfrak{t}} \frac{d\mathcal{Z}_\theta}{dt} = \frac{\sin \mathcal{Z}_\zeta}{f \circ \mathcal{Z}_\zeta} = \mathcal{Z}_\zeta = \mathcal{U}_\zeta(\mathcal{Z}_\zeta)$$

which is guaranteed by (2.3.14). The second equation of (2.3.19) comes from

$$\frac{d\mathcal{Z}_v}{d\mathfrak{t}} = \frac{dt}{d\mathfrak{t}} \mathcal{U}'_v(\mathcal{Z}_v) \frac{d\mathcal{Z}_v}{dt} = -\frac{r_0}{\sin \omega(r_0)} \frac{\sin \mathcal{Z}_\zeta}{f \circ \mathcal{Z}_\zeta} E_{\parallel}(r_0, \mathcal{Z}_\theta)$$

together with (2.2.7) and (2.3.14). The last equation of (2.3.19) can be obtained by following the same lines of computations. \square

From (2.3.9), (2.3.10) and (2.3.11), we know that \mathcal{Z} remains in a compact set of $\mathbb{R} \times \mathbb{T} \times]1, +\infty[\times \mathbb{R}$. On the other hand, the invariant functionals $H_1(r_0; \cdot)$ and $H_2(r_0; \cdot)$ can be expressed in terms of \mathcal{Z} . They simply become

$$\mathcal{H}_1(\mathcal{Z}) := (\mathcal{Z}_v^2 - 1)^{1/2} (R_0 + \mathcal{Z}_r \cos \mathcal{Z}_\theta)^{1/2} e^{-\mathcal{Z}_\zeta^2/2}, \quad \mathcal{H}_2(\mathcal{Z}) := \Phi(\mathcal{Z}_r, \mathcal{Z}_\theta) + \mathcal{Z}_v.$$

Of course, the relation (2.3.5) becomes

$$\frac{d}{dt} [\mathcal{H}_i(\mathcal{Z})] = 0, \quad \forall i \in \{1, 2\}. \quad (2.3.20)$$

The information (2.3.20) for $i = 2$ is particularly interesting. Starting from the initial data \mathcal{Z}_0 having (second) energy $\mathcal{H}_{20} := \Phi(\mathcal{Z}_{0r}, \mathcal{Z}_{0\theta}) + \mathcal{Z}_{0v}$, we can compute the component $\mathcal{Z}_v(\mathbf{t})$ as a function of $\mathcal{Z}_\theta(\mathbf{t})$ according to

$$\mathcal{Z}_v(\mathbf{t}) = \mathcal{F}_{r_0, \mathcal{H}_{20}}(\mathcal{Z}_\theta(\mathbf{t})), \quad \mathcal{F}_{r_0, \mathcal{H}_{20}}(\mathcal{Z}_\theta) := \mathcal{H}_{20} - \Phi(r_0, \mathcal{Z}_\theta). \quad (2.3.21)$$

Consider the two-dimensional manifold

$$\mathcal{E}_{r_0, \mathcal{H}_{20}} := \left\{ \mathcal{Z} \in]0, a] \times \mathbb{T} \times]1, +\infty[\times \mathbb{R}; \mathcal{Z}_r = r_0, \Phi(r_0, \mathcal{Z}_\theta) + \mathcal{Z}_v = \mathcal{H}_{20} \right\}.$$

This surface may be parametrized by $q \equiv \mathcal{Z}_\theta$ and $p \equiv \mathcal{Z}_\zeta$ in the following way

$$\mathcal{E}_{r_0, \mathcal{H}_{20}} := \left\{ \mathcal{P}\mathcal{F}_{r_0, \mathcal{H}_{20}}(q, p); q \in \mathbb{T}, p \in \mathbb{R} \right\}, \quad \mathcal{P}\mathcal{F}_{r_0, \mathcal{H}_{20}}(q, p) := (r_0, q, \mathcal{F}_{r_0, \mathcal{H}_{20}}(q), p).$$

Now, starting from any position $\mathcal{P}\mathcal{F}_{r_0, \mathcal{H}_{20}}(q_0, p_0)$ inside $\mathcal{E}_{r_0, \mathcal{H}_{20}}$, which is associated with the choice of some (q_0, p_0) , we can use (2.3.21) instead of the equation on \mathcal{Z}_v to compute $\mathcal{Z}_v(\mathbf{t})$. As a consequence, limited to $\mathcal{E}_{r_0, \mathcal{H}_{20}}$, the third equation of (2.3.19) implies a right hand side that can be viewed as minus the derivative with respect to \mathcal{Z}_θ of the function

$$\mathcal{K}_{r_0, \mathcal{H}_{20}}(\mathcal{Z}_\theta) := -\frac{1}{2} \ln(R_0 + r_0 \cos \mathcal{Z}_\theta) - \frac{1}{2} \ln(\mathcal{F}_{r_0, \mathcal{H}_{20}}(\mathcal{Z}_\theta)^2 - 1). \quad (2.3.22)$$

By this way, we have extracted from (2.3.19) a two-dimensional systems in $\mathbb{T} \times \mathbb{R}$ looking

like

$$\begin{cases} \frac{dq}{dt} = +p, & q(0) = q_0, \\ \frac{dp}{dt} = -\partial_q \mathcal{K}_{r_0, \mathcal{H}_{20}}(q), & p(0) = p_0. \end{cases} \quad (2.3.23)$$

In so doing, we have highlighted a foliation of the dynamics generated by (2.3.18)-(2.3.19) by self-contained subsystems (which are indexed by r_0 and \mathcal{H}_{20}) on $(q, p) := (\mathcal{Z}_\theta, \mathcal{Z}_\zeta)$, where p is the conjugate variable of q . For the sake of simplicity, we will sometimes skip the two indices r_0 and \mathcal{H}_{20} , and simply use $\mathcal{K} \equiv \mathcal{K}_{r_0, \mathcal{H}_{20}}$.

The reduced equations (2.3.23) describe the motion of a body of mass 1 subject to a conservative potential energy \mathcal{K} . The Hamiltonian is given by $\mathcal{H}(q, p) := (p^2/2) + \mathcal{K}(q)$. Since $\mathcal{K}(\cdot)$ is a periodic function of period 2π , the system (2.3.23) is of pendulum type.

2.3.2.2 Reminders about the pendulum

The system (2.3.23) on $\mathbb{T} \times \mathbb{R}$ is well known in classical dynamics. When $\mathcal{K}(q) = \cos q$, this is just the simple pendulum which is addressed in detail in [Bri13]. The more general case, when \mathcal{K} is for instance as in (2.3.22), is discussed in the lecture notes of M.V.N. Murthy or in [PR82]. For general potentials \mathcal{K} , we cannot solve (2.3.23) explicitly. But still we can explore its properties. In this paragraph, we highlight particulars that are needed for the subsequent discussion.

Definition 2.3.6 (Critical value). A critical value (sometimes called critical energy) is a number E_{crit} such that $E_{crit} = \mathcal{K}(q)$ for some q satisfying $\partial_q \mathcal{K}(q) = 0$.

The set of critical values is denoted by

$$\mathcal{C} := \left\{ E_{crit} \in \mathbb{R}; \exists q \in \mathbb{T} \text{ such that } \partial_q \mathcal{K}(q) = 0 \text{ and } \mathcal{K}(q) = E_{crit} \right\}.$$

The set $\mathcal{C} \subset \mathbb{T}$ is compact with $\#\mathcal{C} \geq 1$. By Sard's lemma, it is of measure zero. For the sake of simplicity, let us first recall what happens when \mathcal{C} is finite of cardinal m , that is when

$$\mathcal{C} = \left\{ E_{crit}^1, \dots, E_{crit}^m \right\}, \quad E_{crit}^1 < \dots < E_{crit}^m.$$

Then, there is a partition of the phase space into $\mathbb{T} \times \mathbb{R} = \mathcal{L} \cup \mathcal{R} \cup \mathcal{S}$ corresponding to the distinct types of motion indicated below:

- **The region of libration motion:** .

$$\mathcal{L} := \cup_{j=0}^{m-1} \mathcal{L}^j, \quad \mathcal{L}^j := \left\{ (q, p) \in \mathbb{T} \times \mathbb{R}; E_{crit}^j < \mathcal{H}(q, p) < E_{crit}^{j+1} \right\}.$$

The particles issued from some $(q_0, p_0) \in \mathcal{L}$ are *trapped*. Define $\mathcal{H}_0 := \mathcal{H}(p_0, q_0)$. Identify the two turning points q_1 and q_2 associated to the energy level \mathcal{H}_0 , which are the largest q_1 and the smallest q_2 satisfying $q_1 \leq q \leq q_2$ together with $\mathcal{K}(q_1) = \mathcal{K}(q_2) = \mathcal{H}_0$. Then, the solution $(q, p)(\cdot)$ is periodic of period

$$0 < \mathcal{P} \equiv \mathcal{P}(\mathcal{H}_0) \equiv \mathcal{P}(z_0) := \int_{q_1}^{q_2} \frac{\sqrt{2}}{\sqrt{\mathcal{H}_0 - \mathcal{K}(q)}} dq. \quad (2.3.24)$$

Physically, the particles are bouncing back and forth between two poloidal angles $\theta_1 \equiv q_1$ and $\theta_2 \equiv q_2$. They explore only a part of the torus (in the θ -variable).

- **The region of rotation motion:**

$$\mathcal{R} \equiv \mathcal{L}^m := \left\{ (q, p) \in \mathbb{T} \times \mathbb{R}; \mathcal{H}(q, p) > E_{crit}^m \right\}.$$

The particles issued from some $(q_0, p_0) \in \mathcal{R}$ are *passing*. They are in transit in q with momentum p not changing sign. This corresponds to high levels of energy \mathcal{H} , with $\partial_t q$ of fixed sign. The position $q(\mathbf{t})$ is going to $\pm\infty$ when $\mathbf{t} \rightarrow \pm\infty$ according to the sign \pm of p_0 . Select some $(q_0, p_0) \in \mathcal{R}$. Then, viewed on $\mathbb{T} \times \mathbb{R}$, the motion is periodic of period

$$0 < \mathcal{P} \equiv \mathcal{P}(\mathcal{H}_0) \equiv \mathcal{P}(z_0) := \frac{1}{\sqrt{2}} \int_0^{2\pi} \frac{1}{\sqrt{\mathcal{H}_0 - \mathcal{K}(q)}} dq. \quad (2.3.25)$$

- **The separatrix region:**

$$\mathcal{S} := \cup_{j=1}^m \mathcal{S}^j, \quad \mathcal{S}^j := \left\{ (q, p) \in \mathbb{T} \times \mathbb{R}; \mathcal{H}(q, p) = E_{crit}^j \right\}.$$

The particles issued from some $(q_0, p_0) \in \mathcal{S}$ are converging to a fixed point. The level sets \mathcal{S}^j of \mathcal{H} may be viewed as boundaries which separate the preceding different types of trajectories (having distinct homotopy classes).

When the cardinal of \mathcal{C} is infinite, by extension, we can still define the (closed) separatrix region $\mathcal{S} := \mathcal{H}^{-1}(\mathcal{C}) \subset \mathbb{T} \times \mathbb{R}$ and the (open) rotation region $\mathcal{R} := \left\{ (q, p) \in \mathbb{T} \times \mathbb{R}; \mathcal{H}(q, p) > \right.$

$\sup \mathcal{C}$. The complement set of $\mathcal{S} \cup \mathcal{R}$ is the (open) libration region \mathcal{L} .

The phase portrait of the simple pendulum is represented below. We find that $\mathcal{C} = \{-1, 1\}$ so that $m = 2$. The region of libration motion contains the closed trajectories (homeomorphic to circles with nonzero radius) which are drawn inside the two red curves. The region of rotation motion contains the trajectories which are outside the red curves. The separatrix region is represented by the two red curves plus a (non visible) equilibrium position placed at $(\pi, 0)$.

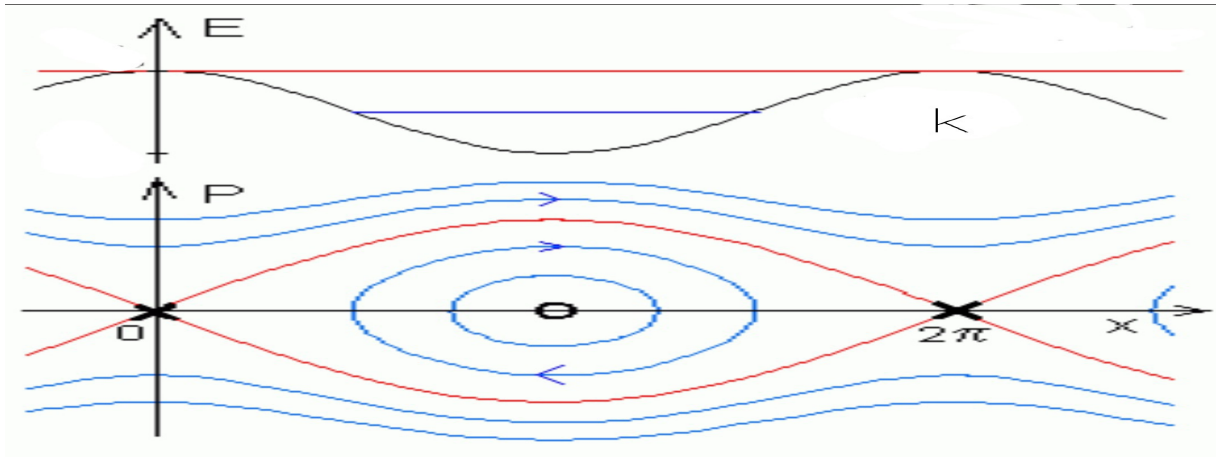


Figure 2.3 – The phase portrait when $\mathcal{K}(q) = \cos q$

Inside \mathcal{L} or \mathcal{R} , we can use a canonical transformation to action-angle variables (I, Θ) . The new Hamiltonian is denoted by W . It is independent of Θ . Thus, it suffices to solve

$$\begin{cases} \partial_t I = 0, & I(0) = I_0, \\ \partial_t \Theta = \partial_I W(I), & \Theta(0) = \Theta_0. \end{cases} \quad (2.3.26)$$

The solution to this Cauchy problem is globally defined. It is simply

$$(I, \Theta)(I_0, \Theta_0; t) = (I_0, \Theta_0 + \partial_I W(I_0) t), \quad 0 < \partial_I W(I_0) = 2\pi/\mathcal{P}, \quad t \in \mathbb{R}. \quad (2.3.27)$$

To normalize the period of the solution, we can change the time variable t (or \mathfrak{t}) into s through

$$s \equiv s(z_0; t) := \partial_I W(I_0) \mathfrak{t}(z_0; t). \quad (2.3.28)$$

Indeed, expressed in terms of s , the flow associated to (2.3.26) just becomes

$$(I, \Theta)(I_0, \Theta_0; s) = (I_0, \Theta_0 + s), \quad s \in \mathbb{R},$$

which is a uniform translation.

2.3.2.3 Periodic properties away from separatrices

In the present situation, the potential \mathcal{K} does depend on (r_0, \mathcal{H}_{20}) , or more generally on z_0 . The same applies to the corresponding separatrix region $\mathcal{S}_{r_0, \mathcal{H}_{20}}$, the period $\mathcal{P} \equiv \mathcal{P}(z_0)$, the construction of $W \equiv W(z_0)$, and so on \dots , including the passage from \mathfrak{t} to s . Look at

$$\begin{aligned} \mathcal{S} := & \left\{ \mathcal{PF}_{r_0, \mathcal{H}_{20}}(\mathcal{S}_{r_0, \mathcal{H}_{20}}); r_0 \in]0, a], \mathcal{H}_{20} \geq \inf \Phi(r_0, \cdot) + 1 \right\} \\ & \subset]0, a] \times \mathbb{T} \times]1, +\infty[\times \mathbb{R}. \end{aligned} \quad (2.3.29)$$

We want to avoid the positions which emerge from \mathcal{S} because they involve qualitatively different behaviors. What happens in the proximity of \mathcal{S} is complicated. On the other hand, we can select in the phase space any bounded connected open domain \mathcal{D} (made of libration or rotation) such that $\bar{\mathcal{D}} \cap \mathcal{S} = \emptyset$. The open set Ω of Theorem 2.1.8 is just the pullback of such domain \mathcal{D} in the original phase space variables (x, v) . In other words

$$\begin{aligned} \Omega := & \left\{ (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3; \exists (r, \theta, \phi, v, \zeta, \nu) \in]0, a[\times \mathbb{T}^2 \times]0, 1[\times] -\frac{\pi}{2}, \frac{\pi}{2}[\times \mathbb{T}; \right. \\ & \left. \mathcal{U}(r, \theta, v, \zeta) \in \mathcal{D}, (x, v) = (X(r, \theta, \phi), V(r, \theta, \phi, v, \zeta, \nu)) \right\}. \end{aligned} \quad (2.3.30)$$

Lemma 2.3.7 (Periodic behaviors of the average flow). *Select $z_0 \in \mathcal{U}^{-1}(\mathcal{D})$. The average flow issued from such z_0 , when it is interpreted in the time variable s , is periodic in s with period 2π . On the other hand*

$$\exists ! \mathbb{P}(z_0) \in \mathbb{R}_+^*; \quad \mathfrak{t}(z_0; -\mathbb{P}(z_0)) = -\mathcal{P}(z_0), \quad (2.3.31)$$

and the flow $Z(z_0, \cdot)$ is periodic in t of period $\mathbb{P}(z_0)$.

Proof. Taking into account (2.3.15) and (2.3.27), looking at $Z(z_0, \cdot)$ in terms of the time variable s means to consider the function

$$s \longmapsto Z\left(z_0; \mathfrak{t}\left(z_0; \mathcal{P}(z_0) s/2\pi\right)\right) = \mathcal{U}^{-1} \circ \mathcal{Z}\left(z_0; \mathcal{P}(z_0) s/2\pi\right). \quad (2.3.32)$$

The periodicity for the component $Z_r(z_0, \cdot)$ is obvious because this function is simply constant. On the other hand, our study of the reduced system (2.3.23) has revealed that $(\mathcal{Z}_\theta, \mathcal{Z}_\zeta)(z_0, \cdot)$ is periodic in \mathfrak{t} with a period $\mathcal{P}(z_0)$ given by (2.3.24) or (2.3.25). Exploiting

(2.3.19), this implies for the remaining component \mathcal{Z}_v that

$$\begin{aligned} \mathcal{Z}_v(\mathbf{t} + \mathcal{P}) &= \mathcal{Z}_v(\mathbf{t}) - \int_{\mathbf{t}}^{\mathbf{t} + \mathcal{P}} \partial_{\theta} \Phi(r_0, \mathcal{Z}_{\theta}(s)) \mathcal{Z}_{\zeta}(s) ds = \mathcal{Z}_v(\mathbf{t}) - \int_0^{\mathcal{P}} \partial_{\theta} \Phi(r_0, \mathcal{Z}_{\theta}(s)) \frac{d\mathcal{Z}_{\theta}}{ds}(s) ds \\ &= \mathcal{Z}_v(\mathbf{t}) - \int_{\theta_0}^{\mathcal{Z}_{\theta}(\mathcal{P})} \partial_{\theta} \Phi(r_0, \theta) d\theta = \mathcal{Z}_v(\mathbf{t}) + \Phi(r_0, \theta_0) - \Phi(r_0, \mathcal{Z}_{\theta}(\mathcal{P})) = \mathcal{Z}_v(\mathbf{t}). \end{aligned}$$

This clearly means that $\mathcal{Z}_v(z_0, \cdot)$, and therefore the field $\mathcal{Z}(z_0, \cdot)$ as a whole, is periodic in \mathbf{t} of period $\mathcal{P} \equiv \mathcal{P}(z_0)$. This period for $\mathcal{Z}(z_0, \cdot)$ becomes 2π at the level of the map (2.3.32).

It is now a matter of transferring this periodic information on \mathcal{Z} to Z . By construction, we have $\mathbf{t}(z_0; 0) = 0$, and the condition (2.3.16) ensures that (2.3.31) is satisfied by a unique $\mathbb{P}(z_0) \in \mathbb{R}_+^*$. Then, using (2.3.17), we can assert that

$$Z(z_0; -\mathbb{P}(z_0)) = \mathcal{U}^{-1} \circ \mathcal{Z}(z_0; -\mathcal{P}(z_0)) = \mathcal{U}^{-1} \circ \mathcal{Z}(z_0; 0) = Z(z_0; 0).$$

This means that $Z(z_0; \cdot)$ is coming back to the initial condition $Z(z_0; 0)$ at the time $t = -\mathbb{P}(z_0)$. By Cauchy-Lipschitz theorem, the flow $Z(z_0; \cdot)$ is periodic of period $\mathbb{P}(z_0)$. \square

2.3.2.4 Linear growths plus periodic oscillations

Given a periodic function F of period \mathbb{P} , we can decompose F into a mean part $\langle F \rangle_{\mathbb{P}}$ plus a contribution F^* with zero mean according to

$$F(t) = \langle F \rangle_{\mathbb{P}} + F^*(t), \quad \langle F \rangle_{\mathbb{P}} := \frac{1}{\mathbb{P}} \int_0^{\mathbb{P}} F(t) dt, \quad \int_0^{\mathbb{P}} F^*(t) dt = 0. \quad (2.3.33)$$

With this convention, we have $\langle F \rangle_{2\pi} \equiv \bar{F}$, where the operator \bar{F} is as in (2.1.6). Moreover

$$\int_0^t F(s) ds = \langle F \rangle_{\mathbb{P}} t + \mathbb{I}F^*(t), \quad \mathbb{I}F^*(t) := \int_0^t F^*(s) ds = \mathbb{I}F^*(t + \mathbb{P}), \quad (2.3.34)$$

where the exponent $*$ serves to highlight the periodicity. The above integration of F reveals a linear part (which is non-trivial as soon as $\langle F \rangle_{\mathbb{P}} \neq 0$) plus a periodic contribution. This principle applies to certain quantities that have been exhibited before. For instance, it allows to further investigate the change of time variable $s(z_0; \cdot)$ introduced at the level of (2.3.28).

Lemma 2.3.8. [Link between t and s] For all time $t \in \mathbb{R}$, with $s(z_0; \cdot)$ as in (2.3.28), we

have

$$s \equiv s(z_0; t) = \frac{2\pi t}{\mathbb{P}(z_0)} + s^*\left(z_0; \frac{2\pi t}{\mathbb{P}(z_0)}\right), \quad s^*(z_0; t + 2\pi) = s^*(z_0; t). \quad (2.3.35)$$

Proof. We can apply (2.3.34) to the product $(Z_v f \circ Z_\zeta)(z_0; t)$ which, in view of Lemma 2.3.7, is periodic of period $\mathbb{P}(z_0)$. This furnishes

$$\mathbf{t}(z_0; t) := \frac{\sin \omega(r_0)}{r_0} \left[\left\langle (Z_v f \circ Z_\zeta)(z_0; \cdot) \right\rangle_{\mathbb{P}(z_0)} t + \mathbf{I}(Z_v f \circ Z_\zeta)(z_0; \cdot)^*(t) \right],$$

with from Lemma 2.3.7

$$\begin{aligned} \left\langle (Z_v f \circ Z_\zeta)(z_0; \cdot) \right\rangle_{\mathbb{P}(z_0)} &= \frac{1}{\mathbb{P}(z_0)} \int_0^{\mathbb{P}(z_0)} (Z_v f \circ Z_\zeta)(z_0; t) dt \\ &= \frac{1}{\mathbb{P}(z_0)} \int_{-\mathbb{P}(z_0)}^0 (Z_v f \circ Z_\zeta)(z_0; t) dt \\ &= -\frac{1}{\mathbb{P}(z_0)} \frac{r_0}{\sin \omega(r_0)} \mathbf{t}(z_0; -\mathbb{P}(z_0)) = \frac{\mathcal{P}(z_0) r_0}{\mathbb{P}(z_0) \sin \omega(r_0)}. \end{aligned}$$

Coming back to (2.3.28), we find (2.3.35) with

$$s^*(z_0; t) = \frac{2\pi}{\mathcal{P}(z_0)} \frac{\sin \omega(r_0)}{r_0} \mathbf{I}(Z_v f \circ Z_\zeta)(z_0; \cdot)^*\left(\frac{\mathbb{P}(z_0) t}{2\pi}\right),$$

which is indeed periodic of period 2π . \square

The two physical quantities ϕ (toroidal angle) and ν (slow gyromotion of the polar angle) associated with the velocity v have been put aside respectively at the level of Lemma 2.2.2 and inside the system (2.1.3). For the sake of completeness, we indicate below their approximate behavior. Looking at (2.2.10) and (2.2.14), neglecting the impact of $\sin \nu$ (due to the cancellations induced by the rapid oscillations) and the $O(\varepsilon)$ -term, it is expected that $\phi \sim \bar{\phi}$ and $\nu \sim \bar{\nu}$ with $\bar{\phi}$ and $\bar{\nu}$ subject to

$$\begin{aligned} \frac{d\bar{\phi}}{dt} &= \bar{\mathcal{A}}_\phi(Z) = R(Z_r, Z_\theta)^{-1} Z_v \cos \omega(Z_r) \sin Z_\zeta, \quad \bar{\phi}(0) = \phi_0, \\ \frac{d\bar{\nu}}{dt} &= \bar{\mathcal{V}}_0(Z) = -\sqrt{1 - Z_v^2} b(Z_r, Z_\theta), \quad \bar{\nu}(0) = 0. \end{aligned}$$

It is worth noting that neither $\bar{\phi}$ nor $\bar{\nu}$ are periodic. Instead, applying again (2.3.34), we

find

$$\begin{aligned}\bar{\phi}(t) &= \left\langle \bar{\mathcal{A}}_\phi \circ Z(z_0; \cdot) \right\rangle_{\mathbb{P}(z_0)} t + \mathbb{I}(\bar{\mathcal{A}}_\phi \circ Z(z_0; \cdot))^*(t), \\ \bar{v}(t) &= \left\langle \mathcal{V}_0 \circ Z(z_0; \cdot) \right\rangle_{\mathbb{P}(z_0)} t + \mathbb{I}(\mathcal{V}_0 \circ Z(z_0; \cdot))^*(t),\end{aligned}$$

where the coefficients in factor of t are (in general) non zero. These expressions are composed with \cos or \sin at the level of (2.2.4) and (2.2.8). As a consequence, even at a rough level of description, the complete flow associated with (2.1.1) inherits almost periodic (and not only purely periodic) features.

2.4 Applications

Subsection 2.4.1 shows that the description of (2.1.2) falls within the scope of [CF23b] in the case of the realistic data that we have introduced. Subsection 2.4.2 describes the coherent structures underlying the flow issued from (2.1.2) with, at the end (Paragraph 2.4.2.4) the proof of Theorem 2.1.8. Subsection 2.4.3 exhibits and studies the long time gyrokinetic equations in order to draw out some implications.

2.4.1 A manageable framework

We want to apply the results of the article [CF23b]. To this end, in Paragraph 2.4.1.1, we explain how to get a system of nonlinear differential equations which is consistent with the one presented in [CF23b]. Then, in Paragraph 2.4.1.2, we check that Conditions 2.1.4 and 2.1.5 (required in [CF23b] to apply Theorems 2, 3 and 4 there) are satisfied.

2.4.1.1 A singular system of odes with multiscale periodic features

Like $\mathfrak{t}(z_0; \cdot)$, the change of time variable $s(z_0; \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ introduced at the level of (2.3.28) is a global diffeomorphism. The associated inverse is denoted by $t(z_0; s)$.

Lemma 2.4.1. *The function $\partial_t s(z_0; t(z_0; \cdot))$ is negative, and it is periodic of period 2π .*

Proof. From (2.3.27) and (2.3.28), the derivative of s is found to be

$$\partial_t s(z_0; t) = \underbrace{\partial_t W(I_0)}_{>0} \frac{dt(z_0; t)}{dt}. \quad (2.4.1)$$

The negative sign of $\partial_t s$ follows then from (2.3.16) where $dt/dt < 0$. Due to (2.3.35), we have

$$s + 2\pi = s(z_0; t(z_0; s)) + 2\pi = \frac{2\pi}{\mathbb{P}(z_0)} [t(z_0; s) + \mathbb{P}(z_0)] + s^*\left(z_0; \frac{2\pi}{\mathbb{P}(z_0)} t(z_0; s)\right).$$

By definition, the left hand side is the same as $s(z_0; t(z_0; s + 2\pi))$ where $t(z_0; s)$ is the inverse of $s(z_0; \cdot)$. On the other hand, due again to (2.3.35) and since $s^*(z_0; \cdot)$ is periodic of period 2π , the right hand side coincides with $s(z_0; t(z_0; s) + \mathbb{P}(z_0))$. This is possible if and only if

$$t(z_0; s + 2\pi) = t(z_0; s) + \mathbb{P}(z_0). \quad (2.4.2)$$

Use again (2.3.15) and (2.3.27), the expression (2.4.1) becomes

$$\partial_t s(z_0; t) = \frac{2\pi}{\mathcal{P}(z_0)} \frac{\sin \omega(r_0)}{r_0} (Z_v f \circ Z_\zeta)(z_0; t) < 0. \quad (2.4.3)$$

Now, knowing from Lemma 2.3.7 that $\partial_t s(z_0; t)$ is periodic in t of period $\mathbb{P}(z_0)$, we can easily deduce the expected result. \square

The system (2.1.3) can be expressed in the time variable s and then in terms of $\tau = \varepsilon s$. To this end, it suffices to consider

$$\begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; \tau) := \begin{pmatrix} \tilde{z} \\ \tilde{\mathbf{v}} \end{pmatrix} (\varepsilon, z_0, \mathbf{v}_0; t(z_0; \tau/\varepsilon)), \quad (2.4.4)$$

and then, with $\mathbf{v}_0 = \varepsilon \nu_0$, to replace (2.1.3) by (2.1.8) where

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{V} \end{pmatrix} (\varepsilon, z_0; z; s, \nu) := \partial_t s(z_0; t(z_0; s))^{-1} \begin{pmatrix} \mathcal{A} \\ \mathcal{V} \end{pmatrix} (\varepsilon, z; \nu). \quad (2.4.5)$$

The above expression \mathbf{A} (like \mathcal{A}) does not depend on ε , whereas \mathbf{V} (like \mathcal{V}) does. The source term ${}^t(\mathbf{A}, \mathbf{V})$ is a smooth function. From Lemma 2.4.1, it is periodic in s with period 2π . In view of the preceding discussion, we can assert that ${}^t(\mathbf{A}, \mathbf{V})$ satisfies (2.1.9) and (2.1.10).

One may wonder why it is necessary to pass from t to s . The reason is that a *uniform* period (not depending on z_0) is crucial with a view to applying [CF23b]. Otherwise, the WKB construction (the determination of the phases and the profiles) does not work due to instabilities with respect to variations in the initial data.

We now deal with the system of ODE (2.1.8). It is of the same type as the one addressed in [CF23b] where in the latter reference we have that

- the solution $(z, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}$. Then since in our case, we have defined $z = {}^t(r, \theta, \mathbf{v}, \zeta)$, this means we work with $n = 4$;
- the source terms \mathbf{A} and \mathbf{V} are periodic in the last two variables of periods respectively 2π and $T_r(z_0)$. This means $T_r(z_0) = 2\pi$ in our case.

In Subsection 2.4.2, we want to apply Theorems 2 and 3 from [CF23b]. In Subsection 2.4.3, we plan to implement Theorem 4 from [CF23b]. But to this end, first and foremost, we need to check that all requirements of [CF23b], which have been reformulated into Conditions 2.1.4 and 2.1.5 in our introduction, are satisfied. This what is done below.

2.4.1.2 Verification of the prerequisites

Given a function $Z(s, \nu)$ which is periodic of period 2π in both variables s and ν , define

$$\bar{Z}(s) := \frac{1}{2\pi} \int_0^{2\pi} Z(s, \nu) d\nu, \quad Z^*(s, \nu) := Z(s, \nu) - \bar{Z}(s), \quad (2.4.6)$$

$$\langle \bar{Z} \rangle := \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} Z(s, \nu) ds d\nu, \quad \bar{Z}^*(s) := \bar{Z}(s) - \langle \bar{Z} \rangle. \quad (2.4.7)$$

Retain that this induces a decomposition of Z in three components according to

$$Z(s, \nu) = \langle \bar{Z} \rangle + \bar{Z}^*(s) + Z^*(s, \nu). \quad (2.4.8)$$

- *Validity of Condition 2.1.4.* This condition is about the mean flow Ξ_{mf} which, as indicated in (2.1.11), is the solution to

$$\partial_s \Xi_{\text{mf}}(z_0; \mathfrak{z}; s) = \bar{\mathbf{A}}(z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; s); s), \quad \Xi_{\text{mf}}(z_0; \mathfrak{z}; 0) = \mathfrak{z},$$

where \mathbf{A} is defined in (2.4.5) and $\bar{\mathbf{A}}$ is as indicated in (2.4.6). Thus, the first thing to do is to make the link between Ξ_{mf} and the average flow Z of Lemma 2.3.1. To this end, we claim that

$$\Xi_{\text{mf}}(z_0; \mathfrak{z}; s) = Z(\mathfrak{z}; t(z_0, s)), \quad (2.4.9)$$

where $t(z_0; \cdot)$ is the inverse of the map $s(z_0; \cdot)$. Then, to prove the claim (2.4.9), we compute

$$\partial_s [Z(\mathfrak{z}; t(z_0; s))] = \partial_s t(z_0; s) \partial_t Z(\mathfrak{z}; t(z_0; s)). \quad (2.4.10)$$

To proceed, we consider the following useful identities.

- ◁ From (2.1.7), we have $\partial_t Z(\mathfrak{z}; t(z_0; s)) = \bar{\mathcal{A}}(Z(\mathfrak{z}; t(z_0; s)))$ where $\mathcal{A} = {}^t(\mathcal{A}_r, \mathcal{A}_\theta, \mathcal{A}_v, \mathcal{A}_\zeta)$ is defined explicitly as indicated in (2.2.9) and (2.2.12);
- ◁ Since $t(z_0; \cdot)$ is the inverse of $s(z_0; \cdot)$, we have $\partial_s t(z_0; s) = \partial_t s(z_0; t(z_0; s))^{-1}$;
- ◁ We apply (2.4.6) to the first equation of (2.4.5) to see that

$$\bar{\mathcal{A}}(z_0; Z(\mathfrak{z}; t(z_0; s)); s) = \partial_t s(z_0; t(z_0; s))^{-1} \bar{\mathcal{A}}(Z(\mathfrak{z}; t(z_0; s)));$$

- ◁ From the proof of Lemma 2.4.9, we have $t(z_0; 0) = 0$. This implies that

$$Z(\mathfrak{z}; t(z_0; 0)) = Z(\mathfrak{z}; 0) = \mathfrak{z}.$$

Exploiting the above identities, the equation (2.4.10) becomes

$$\partial_s [Z(\mathfrak{z}; t(z_0; s))] = \bar{\mathcal{A}}(z_0; Z(\mathfrak{z}; t(z_0; s)); s), \quad Z(\mathfrak{z}; t(z_0; 0)) = \mathfrak{z}. \quad (2.4.11)$$

It bears noting that (2.4.11) coincides with the above equation of the mean flow Ξ_{mf} . Thus, by Cauchy-Lipschitz theorem, we must have (2.4.9).

Since $t(z_0, \cdot)$ and $Z(\mathfrak{z}; \cdot)$ are globally defined (Lemma 2.3.3), the same applies to $\Xi_{\text{mf}}(z_0; \mathfrak{z}; \cdot)$. Moreover, with Lemma 2.3.7 and (2.4.2), we can see on (2.4.9) that, for all (z_0, \mathfrak{z}) , the map $\Xi_{\text{mf}}(z_0; \mathfrak{z}; \cdot)$ is as required periodic of period 2π .

By this way, we have proved that the first requirement to apply Theorems 2 and 3 in [CF23b] is satisfied. Below, we consider the second one which is condition 2.1.5.

- *Validity of Condition 2.1.5.* The second equation in the system (2.1.8) reads

$$\partial_\tau \mathbf{v} = \frac{1}{\varepsilon} \mathbf{V}_0 \left(z_0; z; \frac{\boldsymbol{\tau}}{\varepsilon} \right) + \mathbf{V}_1 \left(z_0; z; \frac{\boldsymbol{\tau}}{\varepsilon}, \frac{\mathbf{v}}{\varepsilon} \right)$$

where, in view of (2.2.15) and (2.4.5), we have

$$\mathbf{V}_0(z_0; z; s) = -\partial_t s(z_0; t(z_0; s))^{-1} \sqrt{1 - v^2} b(r, \theta) > 0,$$

which is positive and does not depend on ν (whereas \mathbf{V}_1 does). This is Condition 2.1.5.

In conclusion, all the prerequisites to apply Theorems 2, 3 and 4 in [CF23b] are in our hands.

2.4.2 Coherent structures

In Paragraph [2.4.2.1](#), we apply [\[CF23b\]](#) to get the oscillating description of the solutions (z, \mathbf{v}) to [\(2.1.8\)](#). Then, to show Theorem [2.1.8](#), we have to rebuild $(\tilde{x}, \tilde{v})(\varepsilon, x_0, v_0; \tau)$ from $(z, \mathbf{v})(\varepsilon, z_0, \mathbf{v}_0; \tau)$. To this end, in Paragraph [2.4.2.2](#), we replace (z, \mathbf{v}) , (z_0, \mathbf{v}_0) and τ in terms of respectively (z, v) , (x_0, v_0) and τ . This is not sufficient since, in view of [\(2.2.4\)](#), [\(2.2.8\)](#) and [\(2.2.17\)](#), the access to (x, v) or (\tilde{x}, \tilde{v}) from these data needs also the knowledge of ϕ . Therefore, in Paragraph [2.4.2.3](#), we exhibit separately the WKB expansion of the toroidal angle ϕ . Finally, in Paragraph [2.4.2.4](#), we can conclude with the proof of Theorem [2.1.8](#).

2.4.2.1 Coherent structures associated with [\(2.1.8\)](#)

Recall that the map \mathcal{U} has been defined at the level of Paragraph [2.3.2.1](#), see [\(2.3.13\)](#). On the other hand, as explained in Paragraph [2.3.2.3](#), a bounded connected set $\mathcal{D} \subset B(0, R]$ with $R \in \mathbb{R}_+^*$ has been selected away from the separatrix region \mathcal{S} . Now, given $\mathcal{Z}_0 \in \mathcal{D}$ as in Lemma [2.3.5](#), we have $z_0 = \mathcal{U}^{-1}(\mathcal{Z}_0) \in \mathcal{U}^{-1}(\mathcal{D}) \subset \mathcal{U}^{-1}(B(0, R])$ where the latter is a compact set of $]0, a] \times \mathbb{T} \times]0, 1[\times]-\pi/2, \pi/2[$. As a direct consequence of [\[CF23b\]](#), Theorems 2 and 3], we have the following result.

Proposition 2.4.2. *[The WKB expansion of the flow induced by the system [\(2.1.8\)](#) from [\[CF23b\]](#)] Suppose that Assumptions [2.1.6](#) and [2.1.7](#) hold. Let $z_0 \in \mathcal{U}^{-1}(\mathcal{D})$ and $\mathbf{v}_0 \in \mathbb{T}$. The lifespan $\mathcal{T}(\varepsilon, z_0, \mathbf{v}_0)$ that is associated with the system [\(2.1.8\)](#) is uniformly bounded below by some $\mathcal{T}(z_0, \mathbf{v}_0) \in \mathbb{R}_+^*$. Moreover, there exist smooth profiles $Z_j = {}^t(Z_{jr}, Z_{j\theta}, Z_{jv}, Z_{j\zeta})$ and \mathcal{V}_j as follows*

$$\begin{aligned} Z_j(z_0, \mathbf{v}_0; \tau, s, \nu) &\in C^\infty\left(\mathcal{U}^{-1}(\mathcal{D}) \times \mathbb{T} \times [0, \mathcal{T}(z_0, \mathbf{v}_0)] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^4\right), \\ \mathcal{V}_j(z_0, \mathbf{v}_0; \tau, s, \nu) &\in C^\infty\left(\mathcal{U}^{-1}(\mathcal{D}) \times \mathbb{T} \times [0, \mathcal{T}(z_0, \mathbf{v}_0)] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}\right), \end{aligned}$$

which, taking into account [\(2.4.6\)](#) and [\(2.4.7\)](#), are such that

$$\mathcal{V}_{-1} \equiv \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \tau), \quad \mathcal{V}_0 \equiv \overline{\mathcal{V}}_0(z_0, \mathbf{v}_0; \tau, s), \quad Z_0 \equiv \overline{Z}_0(z_0; \tau, s), \quad (2.4.12)$$

and which are adjusted in such a way that, in terms of the sup norm, we have

$$\begin{aligned} z(\varepsilon, z_0, \mathbf{v}_0; \boldsymbol{\tau}) &= \sum_{j=0}^{+\infty} \varepsilon^j Z_j \left(z_0, \mathbf{v}_0; \boldsymbol{\tau}, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \boldsymbol{\tau})}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(z_0, \mathbf{v}_0; \boldsymbol{\tau}, \frac{\tau}{\varepsilon})}{\varepsilon} \right), \\ \mathbf{v}(\varepsilon, z_0, \mathbf{v}_0; \boldsymbol{\tau}) &= \sum_{j=-1}^{+\infty} \varepsilon^j \mathcal{V}_j \left(z_0, \mathbf{v}_0; \boldsymbol{\tau}, \frac{\tau}{\varepsilon}, \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \boldsymbol{\tau})}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(z_0, \mathbf{v}_0; \boldsymbol{\tau}, \frac{\tau}{\varepsilon})}{\varepsilon} \right). \end{aligned} \quad (2.4.13)$$

From [CF23b, Theorem 4], with $\langle \overline{\mathfrak{Z}}_0 \rangle$ determined as indicated in this Theorem 4, and using (2.4.9), we have also

$$\overline{Z}_0(z_0; \boldsymbol{\tau}, s) = \Xi_{\text{mf}}(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \boldsymbol{\tau}); s) = \mathcal{Z}(\langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \boldsymbol{\tau}); t(z_0, s)). \quad (2.4.14)$$

As explained in the next paragraphs, Theorem 2.1.8 is a corollary of Proposition 2.4.2. To see why, we have now to reverse the procedure going from (x, v) to (z, \mathbf{v}) . The first step is to replace inside (2.4.13) the time variable $\boldsymbol{\tau}$ by τ and the state variable (z, \mathbf{v}) by (z, v) - in fact by (\tilde{z}, \tilde{v}) with (\tilde{z}, \tilde{v}) as in (2.4.17).

2.4.2.2 Coherent structures associated with the long time study of (2.1.3)

By construction, we have $\tau = \varepsilon t$ and $\boldsymbol{\tau} = \varepsilon s$ as well as $s = s(z_0; t)$. Taking into account Lemma 2.3.8, we can directly pass from τ to $\boldsymbol{\tau}$ through

$$\boldsymbol{\tau} = \varepsilon s \left(z_0; \frac{\tau}{\varepsilon} \right) = \frac{2\pi\tau}{\mathbf{P}(z_0)} + \varepsilon s^* \left(z_0; \frac{2\pi\tau}{\varepsilon \mathbf{P}(z_0)} \right). \quad (2.4.15)$$

On the other hand, the change (2.4.4) can be reformulated as

$$\begin{pmatrix} \tilde{z} \\ v \end{pmatrix} \left(\varepsilon, z_0, \mathbf{v}_0; \frac{\tau}{\varepsilon} \right) = \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} \left(\varepsilon, z_0, \mathbf{v}_0; \varepsilon s \left(z_0; \frac{\tau}{\varepsilon} \right) \right). \quad (2.4.16)$$

Amongst other things, the formula (2.1.15) allows to capture the dependence of the flow $(\tilde{x}, \tilde{v})(\cdot)$ on the initial data (x_0, v_0) . To take this into account, we have to look inside (2.4.4) at (z_0, \mathbf{v}_0) as a function of (x_0, v_0) . According to (2.2.4) and (2.2.8), we have

$$(x_0, v_0) = \left(X(r_0, \theta_0, \phi_0), V(r_0, \theta_0, \phi_0, \mathbf{v}_0, \zeta_0, \nu_0) \right), \quad z_0 = {}^t(r_0, \theta_0, \mathbf{v}_0, \zeta_0).$$

The map $(X, V)(\cdot)$ is smooth and one-to-one. This implies that both z_0 and ν_0 are smoothly determined by (x_0, v_0) . In other words, they can be viewed as functions $z_0(x_0, v_0)$

and $\nu_0(x_0, v_0)$ of (x_0, v_0) where (x_0, v_0) is selected in the bounded connected open set $\Omega \subset \mathbb{R}^3 \times \mathbb{R}^3$ which is introduced in Theorem 2.1.8, see also (2.3.30). Following this perspective, we define Ψ^1 as in (2.1.16) and, since $\nu_0 = \varepsilon\nu_0$, we introduce

$$\begin{pmatrix} \tilde{z} \\ \tilde{v} \end{pmatrix}(\varepsilon, x_0, v_0; \tau) := \begin{pmatrix} z \\ v \end{pmatrix}\left(\varepsilon, z_0(x_0, v_0), \varepsilon\nu_0(x_0, v_0); \frac{\tau}{\varepsilon}\right). \quad (2.4.17)$$

We denote by \tilde{r} , $\tilde{\theta}$, \tilde{v} and $\tilde{\zeta}$ the components of \tilde{z} . Recall (2.3.16), (2.3.27) and (2.3.28) which furnish

$$\tau = \varepsilon \frac{2\pi}{\mathcal{P}(z_0(x_0, v_0))} \mathbf{t}\left(z_0(x_0, v_0); \frac{\tau}{\varepsilon}\right), \quad 0 < \tilde{\eta}_5 \leq \left|\frac{d\tau}{d\tau}\right| \leq \tilde{\eta}_6.$$

By construction, the expression (\tilde{z}, \tilde{v}) is defined on the interval $[0, \tilde{\mathcal{T}}(\varepsilon, x_0, v_0)[$ where $\tilde{\mathcal{T}}$ can be deduced from $\mathcal{T}(\varepsilon, z_0, \nu_0)$ by inverting (uniformly in ε) the above relation from τ to τ . In so doing, we find that

$$\exists \tilde{\mathcal{T}}(x_0, v_0) \in \mathbb{R}_+^*; \quad \forall \varepsilon \in]0, \varepsilon_0], \quad \tilde{\mathcal{T}}(x_0, v_0) \leq \tilde{\mathcal{T}}(\varepsilon, x_0, v_0). \quad (2.4.18)$$

Lemma 2.4.3 (The WKB expansion interpreted at the level of (\tilde{z}, \tilde{v})). *Suppose that Assumptions 2.1.6 and 2.1.7 hold true. Take Ω and $\tilde{\mathcal{T}}$ as indicated above. Then, for all $\varepsilon \in]0, \varepsilon_0]$ with ε_0 small enough, the field ${}^t(\tilde{z}, \tilde{v})(\varepsilon, x_0, v_0; \cdot)$ is defined on $[0, \tilde{\mathcal{T}}(x_0, v_0)]$ and there exist smooth profiles $\tilde{Z}_j = {}^t(\tilde{Z}_{jr}, \tilde{Z}_{j\theta}, \tilde{Z}_{jv}, \tilde{Z}_{j\zeta})$ and $\tilde{\mathcal{V}}_j$ as follows*

$$\begin{aligned} \tilde{Z}_j(x_0, v_0; \tau, \tilde{s}, \tilde{v}) &\in C^\infty\left(\Omega \times [0, \tilde{\mathcal{T}}(x_0, v_0)] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}^4\right), \\ \tilde{\mathcal{V}}_j(x_0, v_0; \tau, \tilde{s}, \tilde{v}) &\in C^\infty\left(\Omega \times [0, \tilde{\mathcal{T}}(x_0, v_0)] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}\right), \end{aligned}$$

and such that

$$\begin{aligned} \tilde{z}(\varepsilon, x_0, v_0; \tau) &= \sum_{j=0}^{+\infty} \varepsilon^j \tilde{Z}_j\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2}\right), \\ \tilde{v}(\varepsilon, x_0, v_0; \tau) &= \sum_{j=-1}^{+\infty} \varepsilon^j \tilde{\mathcal{V}}_j\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2}\right) \\ &= \frac{1}{\varepsilon} \Psi_\varepsilon^2(x_0, v_0; \tau) + \mathcal{O}(\varepsilon), \end{aligned} \quad (2.4.19)$$

where $\Psi^1(x_0, v_0; \tau)$ and $\Psi_\varepsilon^2(x_0, v_0; \tau)$ are respectively as in (2.1.16) and (2.1.17).

Before proceeding to the proof of the above Lemma [2.4.3](#), note that we have used many notations to explain the complicated oscillating structure of the solutions. Let us make distinction between different notations that were and will be used in this article.

- ◁ The function Ψ^1 plays the role of a phase which is linear in τ ;
- ◁ The function Ψ_ε^2 is also a phase which oscillates *weakly* with respect to ε . Indeed, at the level of [\(2.1.17\)](#), the weight ε is in factor of profiles involving the frequency ε^{-1} ;
- ◁ Up to now, we have introduced three distinct couple of variables which solve three distinct systems:
 - (z, v) is related to the time t . It describes the characteristic equations of the Vlasov equation [\(2.1.1\)](#);
 - (z, \mathbf{v}) is related to the time τ . Its expansions are described by the profiles (Z_j, \mathcal{V}_j) in Proposition [2.4.2](#);
 - (\tilde{z}, \tilde{v}) is related to the time τ . It describes the characteristic equations of the Vlasov system [\(2.1.2\)](#). Its expansions are described by the profiles $(\tilde{Z}_j, \tilde{\mathcal{V}}_j)$ in Lemma [2.4.3](#);
- ◁ We have introduced the profiles $(\tilde{X}_j, \tilde{V}_j)$ in Theorem [2.1.8](#). It describes the asymptotic expansion related to (\tilde{x}, \tilde{v}) . This is extracted from the profiles $(\tilde{Z}_j, \tilde{\mathcal{V}}_j)$ as well as the two diffeomorphic maps X as in [\(2.2.4\)](#) and V as in [\(2.2.8\)](#). And thus, we can assert that every notation with the symbol tilde $\tilde{\cdot}$ is designated to describe the system [\(2.1.2\)](#) at time τ ;
- ◁ From the above discussions, it is obvious that $X \neq \tilde{X}_j$ and $V \neq \tilde{\mathcal{V}}_j \neq \tilde{V}_j$ as well as $\tilde{v} \neq \tilde{v}$;
- ◁ The following terms $\mathcal{V}_0, \mathcal{V}_1$ in [\(2.1.5\)](#), V_0, V_1 in [\(2.1.10\)](#), V_0 in [\(2.4.38\)](#), \tilde{V}_j in [\(2.1.15\)](#) and V in [\(2.2.8\)](#) are also distinct. Indeed, we have that
 - $\mathcal{V}_0, \mathcal{V}_1$ and V_0, V_1 are the source terms associated respectively to v and \mathbf{v} ;
 - V_0 is defined in terms of the source term V_0 as indicated in [\(2.4.38\)](#);
 - \tilde{V}_j represents the profiles which are used to describe the oscillatory structure of \tilde{v} ;
 - V is a diffeomorphic map;
- ◁ The distinction between s, ν (Proposition [2.4.2](#)) and $\tilde{s}, \tilde{\nu}$ (Theorem [2.1.8](#) and Lemma [2.4.3](#)) is to highlight that different phases come to replace different variables;

◁ The following terms Z in (2.1.7), Z_j in (2.4.13) and \tilde{Z}_j in (2.4.19) are also distinct:

- Z is the average flow;
- Z_j and \tilde{Z}_j represent the profiles associated to z and \tilde{z} respectively.

Proof. Combining (2.4.15), (2.4.16), (2.4.17) and (2.1.16), we find that

$$\begin{pmatrix} \tilde{z} \\ \tilde{v} \end{pmatrix} (\varepsilon, x_0, v_0; \tau) = \begin{pmatrix} z \\ \mathbf{v} \end{pmatrix} \left(\varepsilon, z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0); \Psi^1(x_0, v_0; \tau) + \varepsilon s^* \left(z_0(x_0, v_0); \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon} \right) \right).$$

To go further, we exploit (2.4.13) where τ is replaced as indicated above by $\Psi^1 + \varepsilon s^*$. We argue with \tilde{z} , the case of \tilde{v} being completely similar. By this way, we find that

$$\tilde{z}(\varepsilon, x_0, v_0; \tau) = \sum_{j=0}^{+\infty} \varepsilon^j Z_j \left(z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0); \Psi^1 + \varepsilon s^*, \frac{\Psi^1}{\varepsilon} + s^*, \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0(x_0, v_0); \Psi^1 + \varepsilon s^*)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0); \Psi^1 + \varepsilon s^*, \frac{\Psi^1}{\varepsilon} + s^*)}{\varepsilon} \right).$$

The first line involves only oscillations with respect to Ψ^1 (including those induced by s^*) which can be incorporated inside profiles \tilde{Z}_j as indicated in (2.4.19). By expanding the second line in powers of ε , we find

$$\begin{aligned} & \frac{\langle \overline{\mathcal{V}}_{-1} \rangle(z_0(x_0, v_0); \Psi^1 + \varepsilon s^*)}{\varepsilon^2} + \frac{\overline{\mathcal{V}}_0(z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0); \Psi^1 + \varepsilon s^*, \frac{\Psi^1}{\varepsilon} + s^*)}{\varepsilon} \\ &= \frac{1}{\varepsilon^2} \Psi_\varepsilon^2(x_0, v_0; \tau) + \mathcal{O}(1), \end{aligned}$$

with Ψ_ε^2 as in (2.1.17) and where again the $\mathcal{O}(1)$ may imply oscillations at the frequency ε^{-1} with respect to Ψ^1 which can contribute to the \tilde{Z}_j . This explains the origin of (2.4.19). \square

Retain that the preceding construction yields

$$\tilde{Z}_0 \equiv \overline{\tilde{Z}}_0(x_0, v_0; \tau, \tilde{s}) = \overline{Z}_0 \left(z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau), \tilde{s} + s^*(z_0(x_0, v_0); \tilde{s}) \right), \quad (2.4.20)$$

$$\tilde{\mathcal{V}}_{-1} \equiv \langle \overline{\tilde{\mathcal{V}}}_{-1} \rangle(x_0, v_0; \tau) = \langle \overline{\mathcal{V}}_{-1} \rangle(z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)), \quad (2.4.21)$$

$$\begin{aligned} \tilde{\mathcal{V}}_0 \equiv \overline{\mathcal{V}}_0(x_0, v_0; \tau, \tilde{s}) &= \partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle (z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)) s^*(z_0(x_0, v_0); \tilde{s}) \\ &\quad + \overline{\mathcal{V}}_0(z_0(x_0, v_0), 0; \Psi^1(x_0, v_0; \tau), \tilde{s} + s^*(z_0(x_0, v_0), \tilde{s})), \end{aligned} \quad (2.4.22)$$

$$\begin{aligned} \tilde{\mathcal{V}}_1 \equiv \overline{\mathcal{V}}_1(x_0, v_0; \tau, \tilde{s}) &= \frac{1}{2} \partial_\tau^2 \langle \overline{\mathcal{V}}_{-1} \rangle (z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)) s^*(z_0(x_0, v_0); \tilde{s})^2 \\ &\quad + \partial_\tau \overline{\mathcal{V}}_0(z_0(x_0, v_0), 0; \Psi^1(x_0, v_0; \tau), \tilde{s} + s^*(z_0(x_0, v_0), \tilde{s})) s^* \\ &\quad + \partial_{v_0} \overline{\mathcal{V}}_0(z_0(x_0, v_0), 0; \Psi^1(x_0, v_0; \tau), \tilde{s} + s^*(z_0(x_0, v_0), \tilde{s})) \nu_0 \\ &\quad + \overline{\mathcal{V}}_1(z_0(x_0, v_0), 0; \Psi^1(x_0, v_0; \tau), \tilde{s} + s^*(z_0(x_0, v_0), \tilde{s})). \end{aligned} \quad (2.4.23)$$

At the level of (2.4.19), we can already recognize the oscillating structure of (2.1.15). But, there are still complications since, as explained in Subsection 2.2.2 - see especially (2.2.4), (2.2.8) and (2.2.17), to deduce (\tilde{x}, \tilde{v}) from (\tilde{z}, \tilde{v}) , we also need access to ϕ .

2.4.2.3 Asymptotic behavior of the toroidal angle

In line with (2.4.17), introduce

$$\tilde{\phi}(\varepsilon, x_0, v_0; \tau) := \phi\left(\varepsilon, z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0), \phi_0(x_0, v_0); \frac{\tau}{\varepsilon}\right). \quad (2.4.24)$$

From (2.2.10), with \mathcal{A}_ϕ as in (2.2.10), we know that

$$\frac{d\tilde{\phi}}{d\tau} = \frac{1}{\varepsilon} \mathcal{A}_\phi\left(\tilde{z}, \frac{\tilde{v}}{\varepsilon}\right) = \frac{1}{\varepsilon} \frac{\tilde{v}}{R(\tilde{r}, \tilde{\theta})} \left(-\sin \omega(\tilde{r}) \cos \tilde{\zeta} \sin\left(\frac{\tilde{v}}{\varepsilon}\right) + \cos \omega(\tilde{r}) \sin \tilde{\zeta} \right). \quad (2.4.25)$$

On the other hand, from the last line of (2.4.19), we can infer that

$$\frac{\tilde{v}}{\varepsilon} = \frac{1}{\varepsilon^2} \Psi_\varepsilon^2(x_0, v_0; \tau) + \sum_{j=1}^{+\infty} \varepsilon^{j-1} \tilde{\mathcal{V}}_j\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2}\right). \quad (2.4.26)$$

Applying Lemma 2.4.3 and (2.4.26) to replace \tilde{v}/ε as well as rules of composition, it follows that the source term of (2.4.25) can be expanded (for adequate profiles \mathcal{A}_{ϕ_j}) according to

$$\frac{1}{\varepsilon} \mathcal{A}_\phi\left(\tilde{z}, \frac{\tilde{v}}{\varepsilon}\right) = \sum_{j=-1}^{+\infty} \varepsilon^j \mathcal{A}_{\phi_j}\left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2}\right). \quad (2.4.27)$$

In view of (2.4.20) and (2.4.23), knowing that $\tilde{Z}_0 \equiv \bar{\tilde{Z}}_0(\tau, \tilde{s})$ and $\tilde{\mathcal{V}}_1 \equiv \bar{\tilde{\mathcal{V}}}_1(\tau, \tilde{s})$, the term with ε^{-1} in factor inside (2.4.27) yields

$$\begin{aligned} \mathcal{A}_{\phi-1}(x_0, v_0; \tau, \tilde{s}, \tilde{v}) &= \mathcal{A}_{\phi}(\bar{\tilde{Z}}_0(x_0, v_0; \tau, \tilde{s}), \tilde{v} + \bar{\tilde{\mathcal{V}}}_1(x_0, v_0; \tau, \tilde{s})) \\ &= \left[\frac{\bar{\tilde{Z}}_{0v}}{R(\bar{\tilde{Z}}_{0r}, \bar{\tilde{Z}}_{0\theta})} \left(-\sin \omega(\bar{\tilde{Z}}_{0r}) \cos \bar{\tilde{Z}}_{0\zeta} \sin(\tilde{v} + \bar{\tilde{\mathcal{V}}}_1) + \cos \omega(\bar{\tilde{Z}}_{0r}) \sin \bar{\tilde{Z}}_{0\zeta} \right) \right] (x_0, v_0; \tau, \tilde{s}). \end{aligned}$$

Lemma 2.4.4 (The double mean of $\mathcal{A}_{\phi-1}$). *With $\mathcal{A}_{\phi-1}$ as above, we have*

$$\langle \bar{\mathcal{A}}_{\phi-1} \rangle(x_0, v_0; \tau) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \mathcal{A}_{\phi-1}(x_0, v_0; \tau, \tilde{s}, \tilde{v}) d\tilde{v} d\tilde{s} = 0. \quad (2.4.28)$$

Proof. Consider (2.4.20) together with (2.4.14) to extract

$$\bar{\tilde{Z}}_0(x_0, v_0; \tau, \tilde{s}) = \mathcal{Z} \left(\langle \bar{\mathfrak{Z}}_0 \rangle(z_0(x_0, v_0); \Psi^1(\tau)); t(z_0(x_0, v_0); \tilde{s} + s^*(z_0(x_0, v_0); \tilde{s})) \right), \quad (2.4.29)$$

where \mathcal{Z} is the solution to (2.3.2) and $t(z_0; \cdot)$ is the inverse of $s(z_0; \cdot)$. In particular, due to (2.3.1), we find that

$$\bar{\tilde{Z}}_{0r} \equiv \langle \bar{\tilde{Z}}_{0r} \rangle(x_0, v_0; \tau) = \langle \bar{\mathfrak{Z}}_{0r} \rangle(z_0(x_0, v_0); \Psi^1(\tau)). \quad (2.4.30)$$

With the aid of (2.4.29) and (2.4.30), the mean of $\mathcal{A}_{\phi-1}$ with respect to $\tilde{v} \in \mathbb{T}$ is

$$\begin{aligned} \bar{\mathcal{A}}_{\phi-1}(\tau, \tilde{s}) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{Z_v}{R(\langle \bar{\mathfrak{Z}}_{0r} \rangle, Z_{\theta})} \left[-\sin \omega(\langle \bar{\mathfrak{Z}}_{0r} \rangle) \cos Z_{\zeta} \sin(\tilde{v} + \bar{\tilde{\mathcal{V}}}_1) + \cos \omega(\langle \bar{\mathfrak{Z}}_{0r} \rangle) \sin Z_{\zeta} \right] d\tilde{v} \\ &= \left[\frac{Z_v}{R(\langle \bar{\mathfrak{Z}}_{0r} \rangle, Z_{\theta})} \cos \omega(\langle \bar{\mathfrak{Z}}_{0r} \rangle) \sin Z_{\zeta} \right] \left(\langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1(\tau)); t(z_0; \tilde{s} + s^*(z_0; \tilde{s})) \right). \end{aligned}$$

Now, we can compute then the mean of $\bar{\mathcal{A}}_{\phi-1}$ with respect to $\tilde{s} \in \mathbb{T}$ to find that

$$\begin{aligned} \langle \bar{\mathcal{A}}_{\phi-1} \rangle(x_0, v_0; \tau) &= \frac{1}{2\pi} \cos \omega(\langle \bar{\mathfrak{Z}}_{0r} \rangle) \int_0^{2\pi} \left[\frac{Z_v}{R(\langle \bar{\mathfrak{Z}}_{0r} \rangle, Z_{\theta})} \sin Z_{\zeta} \right] \left(\langle \bar{\mathfrak{Z}}_0 \rangle; t(z_0; \tilde{s} + s^*(z_0; \tilde{s})) \right) d\tilde{s} \\ &= \frac{1}{\mathbb{P}(z_0)} \cos \omega(\langle \bar{\mathfrak{Z}}_{0r} \rangle) \int_0^{\mathbb{P}(z_0)} \left[\frac{Z_v}{R(\langle \bar{\mathfrak{Z}}_{0r} \rangle, Z_{\theta})} \sin Z_{\zeta} \right] \left(\langle \bar{\mathfrak{Z}}_0 \rangle; t(z_0; \underbrace{\frac{2\pi\tilde{t}}{\mathbb{P}(z_0)} + s^*(z_0; \frac{2\pi\tilde{t}}{\mathbb{P}(z_0)})}_{=s(z_0; \tilde{t})}) \right) d\tilde{t} \\ &= \frac{1}{\mathbb{P}(z_0)} \cos \omega(\langle \bar{\mathfrak{Z}}_{0r} \rangle) \int_0^{\mathbb{P}(z_0)} \left[\frac{Z_v}{R(\langle \bar{\mathfrak{Z}}_{0r} \rangle, Z_{\theta})} \sin Z_{\zeta} \right] \left(\langle \bar{\mathfrak{Z}}_0 \rangle; \tilde{t} \right) d\tilde{t}, \end{aligned}$$

where, to obtain the third line, we took the change of variable $\tilde{s} = 2\pi\tilde{t}/\mathbb{P}(z_0)$ and we exploited (2.3.35). Now, looking at the first equation of (2.3.2), we have

$$\left[Z_v \sin Z_\zeta \right] (\langle \bar{\mathfrak{Z}}_0 \rangle, \tilde{t}) = \frac{dZ_\theta}{d\tilde{t}} \frac{\langle \bar{\mathfrak{Z}}_{0r} \rangle}{\sin \omega(\langle \bar{\mathfrak{Z}}_{0r} \rangle)}. \quad (2.4.31)$$

This furnishes

$$\langle \bar{\mathcal{A}}_{\phi-1} \rangle(x_0, v_0; \tau) = \frac{1}{\mathbb{P}(z_0)} \frac{\langle \bar{\mathfrak{Z}}_{0r} \rangle \cos \omega(\langle \bar{\mathfrak{Z}}_{0r} \rangle)}{\sin \omega(\langle \bar{\mathfrak{Z}}_{0r} \rangle)} \int_0^{\mathbb{P}(z_0)} \frac{1}{R_0 + \langle \bar{\mathfrak{Z}}_{0r} \rangle \cos Z_\theta(\langle \bar{\mathfrak{Z}}_0 \rangle; \tilde{t})} \frac{dZ_\theta(\langle \bar{\mathfrak{Z}}_0 \rangle, \tilde{t})}{d\tilde{t}} d\tilde{t}.$$

This must be zero since the average flow $Z(\langle \bar{\mathfrak{Z}}_0 \rangle; \cdot)$ is periodic of period $\mathbb{P}(z_0)$. \square

We seek a WKB solution $\tilde{\phi}_a^\varepsilon$ to (2.4.25) in a similar form as (2.4.27), that is

$$\tilde{\phi}_a^\varepsilon(x_0, v_0; \tau) = \sum_{j=-1}^{+\infty} \varepsilon^j \tilde{\Phi}_j \left(x_0, v_0; \tau, \frac{\Psi^1(x_0, v_0; \tau)}{\varepsilon}, \frac{\Psi_\varepsilon^2(x_0, v_0; \tau)}{\varepsilon^2} \right). \quad (2.4.32)$$

The idea is to perform a formal analysis at the level of (2.4.25) using (2.4.32) to find constraints on the profiles $\tilde{\Phi}_j$. To this end, we need first to compute the derivative of the oscillating phase Ψ_ε^2 with respect to time variable τ . From (2.1.17), it is found to be

$$\begin{aligned} \partial_\tau \Psi_\varepsilon^2(\tau) &= \frac{2\pi}{\mathbb{P}(z_0)} \left(1 + \partial_t s^*(z_0; \frac{\Psi^1}{\varepsilon}) \right) \left(\partial_s \bar{\mathcal{V}}_0 + \partial_\tau \langle \bar{\mathcal{V}}_{-1} \rangle \right) \left(z_0, 0; \Psi^1, \frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon}) \right) \\ &\quad + \varepsilon \frac{2\pi}{\mathbb{P}(z_0)} \left(\partial_\tau \bar{\mathcal{V}}_0 + \partial_\tau^2 \langle \bar{\mathcal{V}}_{-1} \rangle s^*(z_0; \frac{\Psi^1}{\varepsilon}) \right) \left(z_0, 0; \Psi^1, \frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon}) \right). \end{aligned} \quad (2.4.33)$$

To proceed, we need to simplify (2.4.33). To this end, we exploit two helpful information. The first is issued from (2.3.35) which gives rise to

$$\partial_t s(z_0; \frac{\tau}{\varepsilon}) = \frac{2\pi}{\mathbb{P}(z_0)} \left(1 + \partial_t s^*(z_0; \frac{\Psi^1}{\varepsilon}) \right). \quad (2.4.34)$$

The second tool is given in the following lemma.

Lemma 2.4.5 (Simplification). *With \mathcal{V}_0 as in (2.2.15), we have*

$$\begin{aligned} &\left(\partial_s \bar{\mathcal{V}}_0 + \partial_\tau \langle \bar{\mathcal{V}}_{-1} \rangle \right) \left(z_0, 0; \Psi^1, \frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon}) \right) \\ &= \partial_t s(z_0; \frac{\tau}{\varepsilon})^{-1} \mathcal{V}_0 \left(\Xi_{\text{mf}}(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})) \right). \end{aligned} \quad (2.4.35)$$

Proof. First from (2.3.35), we have

$$s(z_0; \frac{\tau}{\varepsilon}) = \frac{\Psi^1(\tau)}{\varepsilon} + s^*(z_0; \frac{\Psi^1(\tau)}{\varepsilon}). \quad (2.4.36)$$

Exploit (2.4.36) together with [CF23b, Section 3, equations (3.41) and (3.42)] to get

$$\begin{aligned} \partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \Psi^1) &= \langle \overline{V}_0 \rangle(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1)), \\ \partial_s \overline{\mathcal{V}}_0(z_0, 0; \Psi^1, \underbrace{\frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon})}_{=s(z_0; \frac{\tau}{\varepsilon})}) &= \overline{V}_0^*(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})). \end{aligned}$$

Then exploiting the above two expressions leads to

$$\begin{aligned} \partial_s \overline{\mathcal{V}}_0(z_0, 0; \Psi^1, \frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon})) + \partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \Psi^1) \\ = \overline{V}_0(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})) \equiv V_0(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})), \end{aligned} \quad (2.4.37)$$

where we have introduced the term V_0 as indicated in [CF23b, Section 2, equation (2.22)].

We have

$$V_0(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})) := V_0(z_0; \Xi_{\text{mf}}(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})); s(z_0; \frac{\tau}{\varepsilon})). \quad (2.4.38)$$

Then, using (2.4.38), (2.4.37) becomes

$$\begin{aligned} \partial_s \overline{\mathcal{V}}_0(z_0, 0; \Psi^1, \frac{\Psi^1}{\varepsilon} + s^*(z_0; \frac{\Psi^1}{\varepsilon})) + \partial_\tau \langle \overline{\mathcal{V}}_{-1} \rangle(z_0; \Psi^1) \\ = V_0(z_0; \Xi_{\text{mf}}(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon})); s(z_0; \frac{\tau}{\varepsilon})) \\ = \partial_t s(z_0; \underbrace{t(z_0; s(z_0; \frac{\tau}{\varepsilon}))}_{= \frac{\tau}{\varepsilon} \text{ since } t \text{ is inverse of } s})^{-1} \mathcal{V}_0(\Xi_{\text{mf}}(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon}))) \\ = \partial_t s(z_0; \frac{\tau}{\varepsilon})^{-1} \mathcal{V}_0(\Xi_{\text{mf}}(z_0; \langle \overline{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1); s(z_0; \frac{\tau}{\varepsilon}))), \end{aligned} \quad (2.4.39)$$

where in the line before the last, we have used (2.4.5). □

With the aid of the two identities (2.4.34) and (2.4.35), the line (2.4.33) reduces to

$$\partial_\tau \Psi_\varepsilon^2(\tau) = \mathcal{V}_0 + \varepsilon \frac{2\pi}{\mathbb{P}(z_0)} (\partial_\tau \overline{\mathcal{V}}_0 + \partial_\tau^2 \langle \overline{\mathcal{V}}_{-1} \rangle s^*). \quad (2.4.40)$$

In view of (2.4.40), the equation (2.4.25) is formally equivalent to

$$\begin{aligned} \sum_{j=-1}^{+\infty} \varepsilon^j \partial_\tau \tilde{\Phi}_j + \sum_{j=-1}^{+\infty} \varepsilon^{j-1} \frac{2\pi}{\mathbb{P}(z_0)} \left\{ \partial_{\tilde{s}} \tilde{\Phi}_j + \left(\partial_\tau \overline{\mathcal{V}}_0 + \partial_\tau^2 \langle \overline{\mathcal{V}}_{-1} \rangle s^* \right) \partial_{\tilde{\nu}} \tilde{\Phi}_j \right\} \\ + \sum_{j=-1}^{+\infty} \varepsilon^{j-2} \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_j = \sum_{j=-1}^{+\infty} \varepsilon^j \mathcal{A}_{\phi_j}(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}). \end{aligned} \quad (2.4.41)$$

Lemma 2.4.6 (The WKB expansion of the toroidal coordinate $\tilde{\phi}$). *Suppose that Assumptions 2.1.6 and 2.1.7 hold. Let Ω as in Theorem 2.1.8 and $\tilde{\mathcal{T}}$ as in (2.4.18). For all $\varepsilon \in]0, \varepsilon_0]$ with ε_0 small enough, the function $\tilde{\phi}(\varepsilon, x_0, v_0; \cdot)$ is defined on $[0, \tilde{\mathcal{T}}(x_0, v_0)]$. Moreover, there exist smooth profiles*

$$\tilde{\Phi}_j(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}) \in C^\infty\left(\Omega \times [0, \tilde{\mathcal{T}}] \times \mathbb{T} \times \mathbb{T}; \mathbb{R}\right), \quad j \in \mathbb{N},$$

such that the expression $\tilde{\phi}_a^\varepsilon$ which is built as indicated in (2.4.32) with $\tilde{\Phi}_{-1} \equiv 0$ is a solution to the cascade of equations that is issued from (2.4.41). This furnishes through (2.4.32) an approximated solution $\tilde{\phi}_a^\varepsilon$ which is close in the sup norm to the solution $\tilde{\phi}$ of (2.4.25) with, for all $N \in \mathbb{N}$, a precision of order $\mathcal{O}(\varepsilon^N)$.

In what follows, we need to invert the derivative $\partial_{\tilde{s}}$ and $\partial_{\tilde{\nu}}$. To this end, introduce the set $L_*^1(\mathbb{T})$ made of periodic functions with zero mean, namely

$$L_*^1(\mathbb{T}) := \left\{ \mathcal{Z} \in L^1(\mathbb{T}); \int_0^{2\pi} \mathcal{Z}(\theta) d\theta = 0 \right\}.$$

We can define an operator $\partial_{\tilde{s}}^{-1}$ or $\partial_{\tilde{\nu}}^{-1} : L_*^1(\mathbb{T}) \longrightarrow L_*^1(\mathbb{T})$ according to

$$\begin{aligned} \partial_{\tilde{s}}^{-1} \mathcal{Z}(\tilde{s}) &:= \int_0^{\tilde{s}} \mathcal{Z}(r) dr - \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^s \mathcal{Z}(r) dr \right) ds, \\ \partial_{\tilde{\nu}}^{-1} \mathcal{Z}(\tilde{\nu}) &:= \int_0^{\tilde{\nu}} \mathcal{Z}(r) dr - \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^s \mathcal{Z}(r) dr \right) ds. \end{aligned} \quad (2.4.42)$$

Proof. The matter here is to determine the profiles $\tilde{\Phi}_j$ using the constraint (2.4.41). Define

$$\Gamma_{-3}(\tilde{\Phi}_{-1}) := \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_{-1}, \quad (2.4.43)$$

$$\Gamma_{-2}(\tilde{\Phi}_{-1}, \tilde{\Phi}_0) := \frac{2\pi}{\mathbb{P}(z_0)} \left\{ \partial_{\tilde{s}} \tilde{\Phi}_{-1} + \partial_{\tilde{\nu}} \tilde{\Phi}_{-1} \left(\partial_\tau \overline{\mathcal{V}}_0 + \partial_\tau^2 \langle \overline{\mathcal{V}}_{-1} \rangle s^* \right) \right\} + \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_0. \quad (2.4.44)$$

The above two expressions (2.4.43) and (2.4.44) are respectively the ε^{-3} and ε^{-2} terms

collected from (2.4.41). In a similar fashion, we pick the ε^j terms from (2.4.41) and we gather them in the expression Γ_j which, for $j \geq -1$, is defined by

$$\begin{aligned} \Gamma_j(\tilde{\Phi}_j, \tilde{\Phi}_{j+1}, \tilde{\Phi}_{j+2}) := & \partial_\tau \tilde{\Phi}_j + \frac{2\pi}{\mathbb{P}(z_0)} \left\{ \partial_{\tilde{s}} \tilde{\Phi}_{j+1} + \partial_{\tilde{\nu}} \tilde{\Phi}_{j+1} \left(\partial_\tau \overline{\mathcal{V}}_0 + \partial_\tau^2 \langle \overline{\mathcal{V}} - 1 \rangle s^* \right) \right\} \\ & + \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_{j+2} - \mathcal{A}_{\phi_j}(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}). \end{aligned} \quad (2.4.45)$$

Introduce the following hypothesis

$$\mathcal{H}_{\phi_j} : \left\{ \text{The profiles } \tilde{\Phi}_{-1}, \dots, \tilde{\Phi}_j, \overline{\tilde{\Phi}}_{j+1}^*, \tilde{\Phi}_{j+1}^*, \tilde{\Phi}_{j+2}^* \text{ are known on the domain } [0, \tilde{\mathcal{T}}] \times \mathbb{T}^2 \right\}$$

which reflects the outcome of solving successively $\Gamma_k = 0$ from $k = -3$ up to $k = j$. To prove Lemma 2.4.6, it suffices to show that \mathcal{H}_{ϕ_j} is verified for all $j \geq -1$. This is achieved in two steps:

- A) First, we prove that the preliminary condition $\mathcal{H}_{\phi_{-1}}$ is verified. This amounts to the determination of the profiles $\tilde{\Phi}_{-1}$, $\overline{\tilde{\Phi}}_0^*$, $\tilde{\Phi}_0^*$ and $\tilde{\Phi}_1^*$;
- B) Secondly, we prove the validity of the hypothesis \mathcal{H}_{ϕ_j} by using an induction argument.

In what follows, we first check A). Then, we consider B).

A) To obtain the starting point $\mathcal{H}_{\phi_{-1}}$, we can limit ourselves at looking at the constraints $\Gamma_k = 0$ for $k = -3, -2, -1$. The constraint $\Gamma_{-3} = 0$ with Γ_{-3} emanating from (2.4.43) implies that

$$\underbrace{\mathcal{V}_0}_{\neq 0} \partial_{\tilde{\nu}} \tilde{\Phi}_{-1} = 0 \quad \Rightarrow \quad \tilde{\Phi}_{-1} \equiv \overline{\tilde{\Phi}}_{-1}(x_0, v_0; \tau, \tilde{s}). \quad (2.4.46)$$

With the simplification (2.4.46), the constraint $\Gamma_{-2} = 0$ becomes

$$\frac{2\pi}{\mathbb{P}(z_0)} \partial_{\tilde{s}} \overline{\tilde{\Phi}}_{-1} + \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_0^* = 0. \quad (2.4.47)$$

Since \mathcal{V}_0 does not depend on $\tilde{\nu}$, the equation (2.4.47) amounts to the same thing as

$$\overline{\tilde{\Phi}}_{-1} \equiv \langle \overline{\tilde{\Phi}}_{-1} \rangle(x_0, v_0; \tau), \quad \tilde{\Phi}_0 \equiv \overline{\tilde{\Phi}}_0(x_0, v_0; \tau, \tilde{s}). \quad (2.4.48)$$

With the aid of (2.4.48), the equation $\Gamma_{-1} = 0$ reduces to

$$\partial_\tau \langle \bar{\Phi}_{-1} \rangle + \frac{2\pi}{\mathbb{P}(z_0)} \partial_{\tilde{s}} \bar{\Phi}_0^* + \mathcal{V}_0 \partial_{\tilde{\nu}} \tilde{\Phi}_1^* = \mathcal{A}_{\phi_{-1}}(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}). \quad (2.4.49)$$

Take in (2.4.49) the average in both variables $(\tilde{s}, \tilde{\nu}) \in \mathbb{T}^2$ to get

$$\partial_\tau \langle \bar{\Phi}_{-1} \rangle = \langle \bar{\mathcal{A}}_{\phi_{-1}} \rangle(x_0, v_0; \tau). \quad (2.4.50)$$

Due to (2.4.28) and since $\langle \bar{\Phi}_{-1} \rangle(x_0, v_0; 0) = 0$, we find that $\bar{\Phi}_{-1}(x_0, v_0; \tau, \tilde{s}) \equiv \langle \bar{\Phi}_{-1} \rangle(x_0, v_0; \tau) \equiv 0$. The double mean value $\langle \bar{\Phi}_{-1} \rangle$ is not activated so that $\partial_\tau \langle \bar{\Phi}_{-1} \rangle(x_0, v_0; \tau) = 0$. Then, the equation (2.4.49) averaged with respect to $\tilde{\nu} \in \mathbb{T}$ furnishes

$$\frac{2\pi}{\mathbb{P}(z_0)} \partial_{\tilde{s}} \bar{\Phi}_0^* = \bar{\mathcal{A}}_{\phi_{-1}}(x_0, v_0; \tau, \tilde{s}) \equiv \bar{\mathcal{A}}_{\phi_{-1}}^*(x_0, v_0; \tau, \tilde{s}). \quad (2.4.51)$$

Since the right hand side of (2.4.51) belongs to $L_*^1(\mathbb{T})$ with respect to \tilde{s} , we can invert $\partial_{\tilde{s}}$ as indicated in (2.4.42) to recover $\bar{\Phi}_0^*$. Then, substitute $\partial_{\tilde{s}} \bar{\Phi}_0^*$ as in (2.4.51) into (2.4.49) to obtain

$$\partial_{\tilde{\nu}} \tilde{\Phi}_1^* = \frac{1}{\mathcal{V}_0} \mathcal{A}_{\phi_{-1}}^*(x_0, v_0; \tau, \tilde{s}, \tilde{\nu}). \quad (2.4.52)$$

Once more, since the right hand side of (2.4.52) belongs to $L_*^1(\mathbb{T})$ with respect to $\tilde{\nu}$, we can invert $\partial_{\tilde{\nu}}$ as indicated in (2.4.42) to get $\tilde{\Phi}_1^*$. We have determined $\bar{\Phi}_{-1} \equiv 0$, $\bar{\Phi}_0^*$, $\tilde{\Phi}_0^* \equiv 0$ and $\tilde{\Phi}_1^*$ which means that the starting point $\mathcal{H}_{\phi_{-1}}$ is verified.

B) We assume that \mathcal{H}_{ϕ_k} is verified for $k = -1$ up to $k = j - 1$. This means that $\bar{\Phi}_{-1}, \dots, \tilde{\Phi}_{j-1}, \bar{\Phi}_j^*, \tilde{\Phi}_j^*, \tilde{\Phi}_{j+1}^*$ have been identified on the domain $[0, \tilde{\mathcal{T}}] \times \mathbb{T}^2$. The aim is to prove that \mathcal{H}_{ϕ_j} holds true. Recall that $\tilde{\Phi}_j$ can be decomposed according to (2.4.8). To obtain \mathcal{H}_{ϕ_j} , we need to determine $\langle \bar{\Phi}_j \rangle, \bar{\Phi}_{j+1}^*, \tilde{\Phi}_{j+2}^*$. Look at the equation $\langle \bar{\Gamma}_j \rangle = 0$ with Γ_j as in (2.4.45), that is

$$\partial_\tau \langle \bar{\Phi}_j \rangle = \langle \bar{\mathcal{A}}_{\phi_j} \rangle(x_0, v_0; \tau), \quad (2.4.53)$$

which must be completed with the initial data coming from $\tilde{\phi}(x_0, v_0; 0) = \phi_0$. The linear Cauchy differential equation (2.4.53) has a solution on the whole interval $[0, \tilde{\mathcal{T}}]$, leading to $\langle \bar{\Phi}_j \rangle$. Next, we consider the equation $\bar{\Gamma}_j = 0$ where $\bar{\Phi}_j$ is written in the form $\bar{\Phi}_j = \bar{\Phi}_j^* + \langle \bar{\Phi}_j \rangle$ and where $\partial_\tau \langle \bar{\Phi}_j \rangle$ is replaced as indicated in (2.4.53). This furnishes

$$\bar{\Phi}_{j+1}^* = \frac{\mathbb{P}(z_0)}{2\pi} \partial_{\tilde{s}}^{-1} [\bar{\mathcal{A}}_{\phi_j}^*(x_0, v_0; \tau, \tilde{s}) - \partial_\tau \bar{\Phi}_j^*]. \quad (2.4.54)$$

There remains to extract $\tilde{\Phi}_{j+2}^*$ from the equation $\Gamma_j = 0$ through

$$\tilde{\Phi}_{j+2}^* = \frac{1}{\mathcal{V}_0} \partial_{\tilde{v}}^{-1} \left[\mathcal{A}_{\phi_j}^* - \partial_{\tau} \tilde{\Phi}_j^* - \frac{2\pi}{\mathbb{P}(z_0)} \left\{ \partial_{\tilde{s}} \tilde{\Phi}_{j+1}^* + \partial_{\tilde{v}} \tilde{\Phi}_{j+1}^* \left(\partial_{\tau} \overline{\mathcal{V}}_0 + \partial_{\tau}^2 \langle \overline{\mathcal{V}}_{-1} \rangle s^* \right) \right\} \right].$$

In summary, we have \mathcal{H}_{ϕ_j} . Thus, for all $N \in \mathbb{N}$, we know that $\partial_{\tau}(\tilde{\phi} - \tilde{\phi}_a^{\varepsilon}) = \mathcal{O}(\varepsilon^N)$ and therefore (just by integration) that $\tilde{\phi} - \tilde{\phi}_a^{\varepsilon} = \mathcal{O}(\varepsilon^N)$. \square

2.4.2.4 Proof of Theorem 2.1.8

According to (2.2.4), (2.2.8) and (2.2.17), the solution ${}^t(x, v)$ to the system (2.1.1) looks like

$$\begin{pmatrix} x \\ v \end{pmatrix} (\varepsilon, x_0, v_0; t) = \begin{pmatrix} X(r, \theta, \phi) \\ V(r, \theta, \phi, v, \zeta, v/\varepsilon) \end{pmatrix} (z_0(x_0, v_0), \varepsilon \nu_0(x_0, v_0), \phi_0(x_0, v_0); t). \quad (2.4.55)$$

This means, using (2.4.17) as well as $(\tilde{x}, \tilde{v})(x_0, v_0; \tau) = (x, v)(x_0, v_0; \frac{\tau}{\varepsilon})$, that

$$\begin{pmatrix} \tilde{x} \\ \tilde{v} \end{pmatrix} (\varepsilon, x_0, v_0; \tau) = \begin{pmatrix} X(\tilde{r}, \tilde{\theta}, \tilde{\phi}) \\ V(\tilde{r}, \tilde{\theta}, \tilde{\phi}, \tilde{v}, \tilde{\zeta}, \tilde{v}/\varepsilon) \end{pmatrix} (\varepsilon, x_0, v_0; \tau). \quad (2.4.56)$$

The existence part of Theorem 2.1.8 is guaranteed by Lemmas 2.4.3 and 2.4.6. On the other hand, the expansion of \tilde{x} and \tilde{v} can be recovered by applying Taylor expansion in (2.4.56) to the two functions $X(\cdot)$ and $V(\cdot)$ after substituting $\tilde{z} = (\tilde{r}, \tilde{\theta}, \tilde{v}, \tilde{\zeta})$, $\tilde{\phi}$ and \tilde{v}/ε as indicated respectively in Lemma 2.4.3, Lemma 2.4.6 and (2.4.26). By this way, we can recognize (2.1.15).

2.4.3 The long time gyrokinetic equations

The goal of this section is to understand the long time leading behavior of the solution to (2.1.2) under Assumptions 2.1.6 and 2.1.7. In view of (2.4.19), (2.4.20), (2.4.21), (2.4.22), (2.4.23), (2.4.26) and (2.4.48), the identity (2.4.56) gives rise to

$$\tilde{x}(x_0, v_0; \tau) = X\left(\overline{\tilde{Z}}_{0r}, \overline{\tilde{Z}}_{0\theta}, \overline{\tilde{\Phi}}_0\right)(x_0, v_0; \tau, \frac{\Psi^1}{\varepsilon}) + \mathcal{O}(\varepsilon), \quad (2.4.57)$$

$$\tilde{v}(x_0, v_0; \tau) = V\left(\overline{\tilde{Z}}_{0r}, \overline{\tilde{Z}}_{0\theta}, \overline{\tilde{\Phi}}_0, \overline{\tilde{Z}}_{0v}, \overline{\tilde{Z}}_{0\zeta}, \frac{\Psi^2}{\varepsilon^2} + \overline{\mathcal{V}}_1\right)(x_0, v_0; \tau, \frac{\Psi^1}{\varepsilon}) + \mathcal{O}(\varepsilon). \quad (2.4.58)$$

Substitute $s(z_0; \frac{\tau}{\varepsilon})$ as indicated in (2.4.36) inside (2.4.29) to see that

$$\bar{Z}_0(x_0, v_0; \tau, \frac{\Psi^1(\tau)}{\varepsilon}) = Z\left(\langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1(\tau)); t(z_0; s(z_0; \frac{\tau}{\varepsilon}))\right) = Z\left(\langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \Psi^1(\tau)); \frac{\tau}{\varepsilon}\right). \quad (2.4.59)$$

With the aid of (2.4.59) and since we have the simplification (2.4.30), in line with (2.1.15), the equations (2.4.57) and (2.4.58) become (2.1.21), where, with $\langle \bar{\mathfrak{Z}}_0 \rangle \equiv \langle \bar{\mathfrak{Z}}_0 \rangle(z_0(x_0, v_0); \Psi^1(\tau))$ and $\mathbb{P} \equiv \mathbb{P}(z_0(x_0, v_0))$, we have introduced \tilde{X}_0 and \tilde{V}_0 as indicated in (2.1.18) and (2.1.19).

In contrast with the classical gyrokinetic equations, the *long time* gyrokinetic ansatz \tilde{X}_0 implies large amplitude oscillations (carried by \tilde{s}). In the formulas (2.1.18) and (2.1.19) for \tilde{X}_0 and \tilde{V}_0 , the following components can be directly determined from the data: X and V are given by (2.2.4) and (2.2.8); the average flow Z can be deduced from (2.3.1) and (2.3.2); the scalar function $\bar{\mathcal{V}}_1$ comes from (2.4.23) together with the WKB construction of Proposition 2.4.2; and $\bar{\Phi}_0$ is issued from (2.4.51) and (2.4.53) which are based on $\bar{\mathcal{A}}_{\phi-1}^*$ and $\bar{\mathcal{A}}_{\phi 0}$. Once done, the access to the main profiles \tilde{X}_0 and \tilde{V}_0 is entirely driven by $\langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)$.

Now, we recall how to determine $\langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)$ precisely from [CF23b]. The result below is proved in [CF23b, Theorem 4] where the source terms \mathbf{A} and \mathbf{V} in the latter reference are supposed to be expanded in powers of ε up to order $N \in \mathbb{N}$ as follows

$$\mathbf{A} = \sum_{j=0}^N \varepsilon^j \mathbf{A}_j + \mathcal{O}(\varepsilon^{N+1}), \quad \mathbf{V} = \sum_{j=0}^N \varepsilon^j \mathbf{V}_j + \mathcal{O}(\varepsilon^{N+1}). \quad (2.4.60)$$

Proposition 2.4.7 (Theorem 4 in [CF23b]: Reduced equations). *The function $\langle \bar{\mathfrak{Z}}_0 \rangle$ in the right hand side of (2.4.14) can be determined by solving*

$$\partial_\tau \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau) = \langle \bar{\mathbf{A}}_1 \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)), \quad \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; 0) = z_0, \quad (2.4.61)$$

where, with ∂_ν^{-1} as in (2.4.42), we have introduced

$$\begin{aligned} \mathbf{A}_1(z_0; \mathfrak{z}; s, \nu) &:= D_{\mathfrak{z}} \Xi_{\text{mf}}(z_0; \mathfrak{z}; s)^{-1} \\ &\left\{ \left[\mathbf{A}_1 + \mathbf{V}_0^{-1} \left(\partial_\nu^{-1} \mathbf{A}_0^* \cdot \nabla_z \right) \mathbf{A}_0 - \partial_s (\mathbf{V}_0^{-1} \partial_\nu^{-1} \mathbf{A}_0^*) - \mathbf{V}_1 \mathbf{V}_0^{-1} \mathbf{A}_0^* \right. \right. \\ &\quad \left. \left. + \left(\partial_\nu^{-1} \mathbf{A}_0^* \cdot \nabla_z \right) (\mathbf{V}_0^{-1}) \mathbf{A}_0 \right] (z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; s); s, \nu) \right\} \end{aligned} \quad (2.4.62)$$

and where the access to the double mean value $\langle \bar{A}_1 \rangle$ is furnished by (2.4.7).

Then to apply the above proposition, the first thing to do is to adapt the arguments to our context. In our case, due to (2.1.9) and (2.1.10), the source terms \mathbf{A} and \mathbf{V} in (2.4.60) are as follows

$$\mathbf{A} = \mathbf{A}_0, \quad \mathbf{A}_j = 0, \quad \forall j \geq 1, \quad (2.4.63)$$

$$\mathbf{V} = \mathbf{V}_0 + \varepsilon \mathbf{V}_1, \quad \mathbf{V}_j = 0, \quad \forall j \geq 2. \quad (2.4.64)$$

Then, Proposition 2.4.7 with the choice of \mathbf{A} and \mathbf{V} in (2.4.63) and (2.4.64) implies that the function $\langle \bar{\mathfrak{J}}_0 \rangle$ can be determined by solving (2.4.61) where the expression $A_1 = {}^t(A_{1r}, A_{1\theta}, A_{1\nu}, A_{1\zeta})$ in (2.4.62) is reduced to

$$\begin{aligned} A_1(z_0; \mathfrak{z}; s, \nu) &:= D_{\mathfrak{z}} \Xi_{\text{mf}}(z_0; \mathfrak{z}; s)^{-1} \\ &\left\{ \left[\mathbf{V}_0^{-1} \left(\partial_{\nu}^{-1} \mathbf{A}^* \cdot \nabla_z \right) \mathbf{A} - \partial_s (\mathbf{V}_0^{-1} \partial_{\nu}^{-1} \mathbf{A}^*) - \mathbf{V}_1 \mathbf{V}_0^{-1} \mathbf{A}^* \right. \right. \\ &\quad \left. \left. + \left(\partial_{\nu}^{-1} \mathbf{A}^* \cdot \nabla_z \right) (\mathbf{V}_0^{-1}) \mathbf{A}^* \right] (z_0; \Xi_{\text{mf}}(z_0; \mathfrak{z}; s); s, \nu) \right\}. \end{aligned} \quad (2.4.65)$$

In Paragraph 2.4.3.1, we explore the effect on A_1 of the mean operator along $(s, \nu) \in \mathbb{T}^2$. Then, in Paragraph 2.4.3.2, we exhibit a spatial confinement property. In Paragraph 2.4.3.3, we also exhibit a conservation of the kinetic energy in the the special case when the electric field is radial. Finally, in Paragraph 2.4.3.4, we present some computations explicitly as an appendix.

2.4.3.1 Computation of the source term $\langle \bar{A}_1 \rangle$

Let us examine the averaging operation on A_1 along $\nu \in \mathbb{T}$. For simplicity and to be consistent with [CF23b], we write $\Xi_{\text{mf}} \equiv \Xi_0 = {}^t(\Xi_{0r}, \Xi_{0\theta}, \Xi_{0\nu}, \Xi_{0\zeta})$.

Lemma 2.4.8 (Finding \bar{A}_1). *We have*

$$\bar{A}_1(z_0; \mathfrak{z}; s) = D_{\mathfrak{z}} \Xi_0(z_0; \mathfrak{z}; s)^{-1} \partial_{ts} \left(z_0; t(z_0, s) \right)^{-1} \mathcal{M} \left(\Xi_0(z_0; \mathfrak{z}; s) \right), \quad (2.4.66)$$

where $\mathcal{M} = {}^t(\mathcal{M}_r, \mathcal{M}_\theta, \mathcal{M}_v, \mathcal{M}_\zeta)$ is as follows

$$\mathcal{M}_r(r, \theta, v, \zeta) := -\frac{I(r)}{1 + I(r)^2} \left[\frac{R(r, \theta)}{r} \partial_\theta \Phi(r, \theta) + v^2 (1 - v^2)^{-1/2} \sin \theta \left(\frac{\cos^2 \zeta}{2} + \sin^2 \zeta \right) \right],$$

$$\begin{aligned} \mathcal{M}_\theta(r, \theta, v, \zeta) := & \frac{1}{r} \cos \omega b^{-1} E_r - \frac{1}{2rR} v^2 (1 - v^2)^{-1/2} b^{-1} \sin^2 \omega \cos \omega \cos^2 \zeta \cos \theta \\ & - \frac{1}{2r} v^2 (1 - v^2)^{-1/2} \partial_r(b^{-1}) \cos \omega \cos^2 \zeta \\ & + v^2 (1 - v^2)^{-1/2} b^{-1} \left[-\frac{1}{rR} \cos^3 \omega \sin^2 \zeta \cos \theta - \frac{1}{r^2} \cos \omega \sin^2 \omega \sin^2 \zeta \right. \\ & \left. + \frac{1}{2r} \omega' \sin \omega \cos^2 \zeta + \frac{1}{2r^2} (\cos \omega \cos^2 \zeta - \cos^3 \omega \cos^2 \zeta) \right], \end{aligned}$$

$$\begin{aligned} \mathcal{M}_v(r, \theta, v, \zeta) := & \frac{v}{2} (1 - v^2) \cos^2 \zeta \left[\partial_r(b^{-1}) E_\perp - \partial_\theta(b^{-1}) E_r \frac{\cos \omega}{r} \right] \\ & + \frac{1}{2} v (1 - v^2) b^{-1} \left[\partial_r E_\perp \cos^2 \zeta - \partial_\theta E_r \cos^2 \zeta \frac{\cos \omega}{r} + \frac{2}{R} E_r \sin^2 \zeta \cos \omega \sin \theta \right. \\ & \left. + E_\perp \left(\frac{2 \sin^2 \omega \sin^2 \zeta + \cos^2 \omega \cos^2 \zeta}{r} + \frac{2 \sin^2 \zeta \cos^2 \omega + \sin^2 \omega \cos^2 \zeta}{R} \cos \theta \right) \right], \end{aligned}$$

$$\begin{aligned} \mathcal{M}_\zeta(r, \theta, v, \zeta) := & \cos \zeta \sin \zeta \left\{ \frac{b^{-1}}{2} \left(\frac{\cos \omega}{r} \partial_\theta E_r - \partial_r E_\perp \right) \right. \\ & + \frac{1}{2} \left(\frac{\cos \omega}{r} \partial_\theta(b^{-1}) E_r - \partial_r(b^{-1}) E_\perp \right) + \frac{1}{R} E_r b^{-1} \sin \theta \cos \omega \\ & + \frac{v^2}{2} (1 - v^2)^{-1/2} \left[\frac{1}{R} \sin \theta (\omega' \sin \omega b^{-1} - \partial_r(b^{-1}) \cos \omega) \right. \\ & \quad \left. - \frac{1}{r^2} \sin^2 \omega \cos \omega \partial_\theta(b^{-1}) - \frac{1}{rR} \cos^3 \omega \cos \theta \partial_\theta(b^{-1}) \right] \\ & \left. + E_\perp b^{-1} \left(\frac{\sin^2 \omega}{r} + \cos \theta \frac{\cos^2 \omega}{R} - \frac{\cos^2 \omega}{2r} - \frac{\sin^2 \omega \cos \theta}{2R} \right) \right\}. \end{aligned}$$

Proof. We are interested in finding $\bar{\mathbf{A}}_1(z_0, \mathfrak{z}; s)$. For this reason, we do not have to take into account the contribution $\partial_s(\mathbf{V}_0^{-1} \partial_\nu^{-1} \mathbf{A}_0^*)$ inside (2.4.65). Indeed, its mean with respect to $\nu \in \mathbb{T}$ is just equal to zero since the term \mathbf{V}_0 is independent of $\nu \in \mathbb{T}$. Now, in view of (2.4.5) as well as since $D_{\mathfrak{z}} \Xi_{\text{mf}}^{-1}$ and $\partial_t s^{-1}$ are independent of $\nu \in \mathbb{T}$, we can average (2.4.65) to get

$$\bar{\mathbf{A}}_1 \equiv \begin{pmatrix} \bar{\mathbf{A}}_{1r} \\ \bar{\mathbf{A}}_{1\theta} \\ \bar{\mathbf{A}}_{1v} \\ \bar{\mathbf{A}}_{1\zeta} \end{pmatrix} (z_0; \mathfrak{z}; s) = D_{\mathfrak{z}} \Xi_0(z_0; \mathfrak{z}; s)^{-1} \partial_t s(z_0; t(z_0, s))^{-1} \begin{pmatrix} \mathcal{M}_r \\ \mathcal{M}_\theta \\ \mathcal{M}_v \\ \mathcal{M}_\zeta \end{pmatrix} (\Xi_0(z_0; \mathfrak{z}; s)),$$

where we have introduced the terms \mathcal{M}_\star with $\star \in \{r, \theta, v, \zeta\}$ as follows

$$\mathcal{M}_\star = \mathcal{V}_0^{-1} \overline{(\partial_\nu^{-1} \mathcal{A}^\star \cdot \nabla_z) \mathcal{A}_\star} - \mathcal{V}_0^{-1} \mathcal{V}_1 \mathcal{A}_\star^\star - \mathcal{V}_0^{-2} \overline{(\partial_\nu^{-1} \mathcal{A}^\star \cdot \nabla_z) (\mathcal{V}_0) \mathcal{A}_\star^\star}. \quad (2.4.67)$$

The passage from the above line to the explicit expressions for the \mathcal{M}_\star enumerated in Lemma 2.4.8 is detailed in Paragraph 2.4.3.4 \square

Let us now examine the effect of the mean operator along $s \in \mathbb{T}$ to the function $\overline{\mathbb{A}}_1(z_0, \mathfrak{z}, s)$ through the following lemma.

Lemma 2.4.9 (Finding $\langle \overline{\mathbb{A}}_1 \rangle$). *We have*

$$\langle \overline{\mathbb{A}}_1 \rangle(z_0; \mathfrak{z}) = \frac{1}{2\pi} \int_0^{\mathbb{P}(z_0)} D_{z_0} Z(\mathfrak{z}; \tilde{t})^{-1} \mathcal{M}(Z(\mathfrak{z}; \tilde{t})) d\tilde{t}, \quad (2.4.68)$$

where \mathcal{M} is identified in Lemma 2.4.8 and $\mathbb{P}(z_0)$ is the period provided by (2.3.31).

Proof. In view of (2.4.9), we can see that

$$D_{\mathfrak{z}} \Xi_0(z_0; \mathfrak{z}; s) = D_{\mathfrak{z}} Z(\mathfrak{z}; t(z_0; s)). \quad (2.4.69)$$

With the aid of (2.4.9), (2.4.69) and Lemma 2.4.8, we get

$$\langle \overline{\mathbb{A}}_1 \rangle(z_0; \mathfrak{z}) = \frac{1}{2\pi} \int_0^{2\pi} D_{\mathfrak{z}} Z(\mathfrak{z}; t(z_0; s))^{-1} \partial_t s(z_0; t(z_0, s))^{-1} \mathcal{M}(Z(\mathfrak{z}; t(z_0; s))) ds. \quad (2.4.70)$$

Now, inside (2.4.70), we perform the change of variable $\tilde{t} = t(z_0; s)$, and since $t(z_0; \cdot)$ is the inverse of $s(z_0; \cdot)$, we get

$$d\tilde{t} = \partial_s t(z_0; s) ds = (\partial_t s(z_0; t(z_0, s)))^{-1} ds.$$

From (2.3.15) and (2.3.28), it is obvious that $s(z_0; 0) = 0$ and thus $t(z_0; 0) = 0$, we exploit then (2.4.2) with $s = 0$ to see that $t(z_0; 2\pi) = \mathbb{P}(z_0)$. We get at the end

$$\begin{aligned} \langle \overline{\mathbb{A}}_1 \rangle(z_0; \mathfrak{z}) &= \frac{1}{2\pi} \int_0^{2\pi} D_{z_0} Z(\mathfrak{z}; \underbrace{t(z_0; s)}_{=\tilde{t}})^{-1} \mathcal{M}(Z(\mathfrak{z}; \underbrace{t(z_0; s)}_{=\tilde{t}})) \underbrace{(\partial_t s(z_0; t(z_0, s)))^{-1} ds}_{=d\tilde{t}} \\ &= \frac{1}{2\pi} \int_0^{\mathbb{P}(z_0)} D_{z_0} Z(\mathfrak{z}; \tilde{t})^{-1} \mathcal{M}(Z(\mathfrak{z}; \tilde{t})) d\tilde{t}. \end{aligned}$$

\square

2.4.3.2 Confinement properties

Of particular interest is the radius \tilde{r} which appears as the first component of $\tilde{\mathbf{z}} = {}^t(\tilde{r}, \tilde{\theta}, \tilde{v}, \tilde{\zeta})$. It owns an asymptotic expansion as indicated in (2.4.19) with the main term \tilde{Z}_{0r} as in (2.4.30). In other words, we have

$$\tilde{r}(x_0, v_0; \tau) = \langle \bar{\mathfrak{Z}}_{0r} \rangle(z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)) + \mathcal{O}(\varepsilon), \quad (2.4.71)$$

where $\langle \bar{\mathfrak{Z}}_{0r} \rangle$ solves

$$\partial_\tau \langle \bar{\mathfrak{Z}}_{0r} \rangle = \langle \bar{A}_{1r} \rangle(z_0; \langle \bar{\mathfrak{Z}}_0 \rangle(z_0; \tau)), \quad \langle \bar{\mathfrak{Z}}_{0r} \rangle(z_0; 0) = r_0. \quad (2.4.72)$$

It is interesting to see that we have a stability property related to $\langle \bar{\mathfrak{Z}}_{0r} \rangle$. This is illustrated in the two following lemmas.

Lemma 2.4.10 (Conservative form). *We have*

$$\mathcal{M}_r(Z(\mathfrak{z}; t)) = \frac{d}{dt} \left[F_r(Z(\mathfrak{z}; t)) \right], \quad (2.4.73)$$

where $Z(\mathfrak{z}; t) = {}^t(\mathfrak{z}_r, Z_\theta, Z_v, Z_\zeta)(\mathfrak{z}, t)$ is the average flow solving the system (2.3.2) with initial data \mathfrak{z} (instead of z_0) and where we have introduced $F_r(\cdot)$ as follows

$$F_r(r, \theta, v, \zeta) := \frac{I(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} v (1 - v^2)^{-1/2} R(r, \theta) \sin \zeta.$$

Proof. Note first that since $Z_r(\mathfrak{z}; t) \equiv \mathfrak{z}_r$, we have

$$\frac{d}{dt} \left[F_r(Z(\mathfrak{z}; t)) \right] = \partial_\theta F_r(Z(\mathfrak{z}; t)) \frac{dZ_\theta}{dt} + \partial_v F_r(Z(\mathfrak{z}; t)) \frac{dZ_v}{dt} + \partial_\zeta F_r(Z(\mathfrak{z}; t)) \frac{dZ_\zeta}{dt}. \quad (2.4.74)$$

The partial derivatives of F_r are given by

$$\begin{aligned} \partial_\theta F_r(r, \theta, v, \zeta) &= -\frac{I(r)}{1 + I(r)^2} \frac{r}{\sin \omega(r)} v (1 - v^2)^{-1/2} \sin \theta \sin \zeta, \\ \partial_v F_r(r, \theta, v, \zeta) &= +\frac{I(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} (1 - v^2)^{-3/2} R(r, \theta) \sin \zeta, \\ \partial_\zeta F_r(r, \theta, v, \zeta) &= +\frac{I(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} v (1 - v^2)^{-1/2} R(r, \theta) \cos \zeta. \end{aligned} \quad (2.4.75)$$

Exploit (2.4.75) and (2.3.2) inside (2.4.74) to obtain that

$$\begin{aligned} \frac{d}{dt} \left[F_r(Z(\mathfrak{z}; t)) \right] &= -\frac{I(\mathfrak{z}_r)}{1 + I(\mathfrak{z}_r)^2} \left[Z_v^2 (1 - Z_v^2)^{-1/2} \sin Z_\theta (\sin Z_\zeta)^2 \right. \\ &\quad + \frac{1}{\sin \omega(\mathfrak{z}_r)} R(\mathfrak{z}_r, Z_\theta) E_{\parallel}(\mathfrak{z}_r, Z_\theta) (\sin Z_\zeta)^2 \\ &\quad + \frac{Z_v^2}{2} (1 - Z_v^2)^{-1/2} \sin Z_\theta (\cos Z_\zeta)^2 \\ &\quad \left. + \frac{1}{\sin \omega(\mathfrak{z}_r)} R(\mathfrak{z}_r, Z_\theta) E_{\parallel}(\mathfrak{z}_r, Z_\theta) (\cos Z_\zeta)^2 \right]. \end{aligned} \quad (2.4.76)$$

From (2.2.7), we have $E_{\parallel}(\mathfrak{z}_r, Z_\theta) = 1/\mathfrak{z}_r \sin \omega(\mathfrak{z}_r) \partial_\theta \Phi(\mathfrak{z}_r, Z_\theta)$. Insert the latter into (2.4.76) to display the term \mathcal{M}_r of Lemma 2.4.8. \square

Lemma 2.4.11 (Cancellation property). *We have $\langle \bar{A}_{1r} \rangle(z_0; \mathfrak{z}) = 0$ implying that $\langle \bar{\mathfrak{z}}_{0r} \rangle(z_0; \tau) = r_0$ for all τ .*

Proof. Note that since $Z(\mathfrak{z}; t) = \mathfrak{z}_r$, the first row of the matrix $(D_{z_0} Z(\mathfrak{z}; t))^{-1}$ is $(1 \ 0 \ 0 \ 0)$. Exploit (2.4.73) inside (2.4.68). This gives

$$\langle \bar{A}_{1r} \rangle(z_0; \mathfrak{z}) = \frac{1}{2\pi} \int_0^{\mathbb{P}(z_0)} \mathcal{M}_r(Z(\mathfrak{z}; \tilde{t})) d\tilde{t} = \frac{1}{2\pi} \int_0^{\mathbb{P}(z_0)} \frac{d}{d\tilde{t}} \left[F_r(Z(\mathfrak{z}; \tilde{t})) \right] d\tilde{t},$$

which must be zero since the average flow $Z(\mathfrak{z}; \cdot)$ is periodic of period $\mathbb{P}(z_0)$ by Lemma 2.3.7. \square

From (2.4.72), it follows that $\langle \bar{\mathfrak{z}}_{0r} \rangle(z_0; \tau) = r_0$ for all τ . Coming back to (2.4.59) together with (2.3.1), and then using (2.2.4), we can see that

$$|\tilde{x}(x_0, v_0; \tau) - R_0| = r_0 + \mathcal{O}(\varepsilon), \quad \forall \tau \in [0, \tilde{\mathcal{T}}(x_0, v_0)].$$

This implies that the portion of the plasma that is issued from $z_0 \in \mathcal{U}^{-1}(\mathcal{D})$ remains trapped during long times inside a toroidal chamber (in particular, it does not collide with the walls). Thus, for such positions z_0 , the conclusion is that the introduction of an electric field satisfying Assumption 2.1.7 is not a factor of deconfinement.

Still, looking at (2.4.59) together with (2.4.57)-(2.4.58), the leading long time behavior of \tilde{x} and \tilde{v} involves large amplitude oscillations at frequencies ε^{-1} and ε^{-2} . Let us now look more closely at what happens concerning \tilde{v} .

2.4.3.3 The case of a radial electric field

We suppose in this paragraph that the electric field is radial. In other words, we work with $\Phi \equiv \Phi(r)$ in Assumption [2.1.7](#). This implies that the two components of the electric field: E_\perp in [\(2.2.6\)](#) and E_\parallel in [\(2.2.7\)](#) are such that $E_\parallel = E_\perp = 0$. In this case, the system for the average flow [\(2.3.2\)](#) with initial data \mathfrak{z} instead of z_0 is reduced to

$$Z_r(\mathfrak{z}; t) = \mathfrak{z}_r, \quad (2.4.77)$$

$$dZ_\theta/dt = +\mathfrak{z}_v \mathfrak{z}_r^{-1} \sin \omega(\mathfrak{z}_r) \sin Z_\zeta, \quad Z_\theta(\mathfrak{z}; 0) = \mathfrak{z}_\theta, \quad (2.4.78)$$

$$Z_v(\mathfrak{z}; t) = \mathfrak{z}_v, \quad (2.4.79)$$

$$dZ_\zeta/dt = -\mathfrak{z}_v \left(2 R(\mathfrak{z}_r, Z_\theta)\right)^{-1} \sin \omega(\mathfrak{z}_r) \sin Z_\theta \cos Z_\zeta, \quad Z_\zeta(\mathfrak{z}; 0) = \mathfrak{z}_\zeta. \quad (2.4.80)$$

On the other hand, the expression of the term \mathcal{M}_v in Lemma [2.4.8](#) is also simplified to

$$\mathcal{M}_v(r, \theta, v, \zeta) = v (1 - v^2) \frac{I(r)}{1 + I(r)^2} \partial_r \Phi(r) \sin \theta \left(\frac{\cos^2 \zeta}{2} + \sin^2 \zeta \right). \quad (2.4.81)$$

Then, in view of [\(2.4.19\)](#), [\(2.4.29\)](#) and [\(2.4.79\)](#), we can assert that the momentum \tilde{v} which appears as the third component of $\tilde{\mathfrak{z}} = {}^t(\tilde{r}, \tilde{\theta}, \tilde{v}, \tilde{\zeta})$ owns the following expansion

$$\tilde{v}(x_0, v_0; \tau) = \langle \bar{\mathfrak{z}}_{0v} \rangle (z_0(x_0, v_0); \Psi^1(x_0, v_0; \tau)) + \mathcal{O}(\varepsilon), \quad (2.4.82)$$

where $\langle \bar{\mathfrak{z}}_{0v} \rangle$ solves

$$\partial_\tau \langle \bar{\mathfrak{z}}_{0v} \rangle = \langle \bar{A}_{1v} \rangle (z_0; \langle \bar{\mathfrak{z}}_0 \rangle (z_0; \tau)), \quad \langle \bar{\mathfrak{z}}_{0v} \rangle (z_0; 0) = v_0. \quad (2.4.83)$$

Since the average flow $Z(\mathfrak{z}; t) = {}^t(\mathfrak{z}_r, Z_\theta(\mathfrak{z}; t), \mathfrak{z}_v, Z_\zeta(\mathfrak{z}; t))$ is as in [\(2.4.77\)](#), [\(2.4.78\)](#), [\(2.4.79\)](#) and [\(2.4.80\)](#), we get from [\(2.4.68\)](#) that

$$\langle \bar{A}_{1v} \rangle (z_0; \mathfrak{z}) = \frac{1}{2\pi} \int_0^{\mathbb{P}(z_0)} \mathcal{M}_v(Z(\mathfrak{z}; \tilde{t})) d\tilde{t}. \quad (2.4.84)$$

Below, we show that we have a stability property related to $\langle \bar{\mathfrak{z}}_{0v} \rangle$.

Lemma 2.4.12 (Conservative form for the momentum term). *For all \mathfrak{z} in a compact set, we have that \mathcal{M}_v is conservative in the sense*

$$\mathcal{M}_v(Z(\mathfrak{z}; t)) = \frac{d}{dt} \left[F_v(Z(\mathfrak{z}; t)) \right], \quad (2.4.85)$$

where $Z(\mathfrak{z}; t) = {}^t(\mathfrak{z}_r, Z_\theta, \mathfrak{z}_v, Z_\zeta)(\mathfrak{z}, t)$ is the average flow as in (2.4.77), (2.4.78), (2.4.79) and (2.4.80) and where we have introduced $F_v(\cdot)$ as follows

$$F_v(r, \theta, v, \zeta) := -(1 - v^2) \frac{I(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} \partial_r \Phi(r) R(r, \theta) \sin \zeta.$$

Proof. Note first that since $Z_r(\mathfrak{z}; t) \equiv \mathfrak{z}_r$ and $Z_v(\mathfrak{z}; t) \equiv \mathfrak{z}_v$, we have

$$\frac{d}{dt} \left[F_v(Z(\mathfrak{z}; t)) \right] = \partial_\theta F_v(Z(\mathfrak{z}; t)) \frac{dZ_\theta}{dt} + \partial_\zeta F_v(Z(\mathfrak{z}; t)) \frac{dZ_\zeta}{dt}. \quad (2.4.86)$$

With $R(\cdot)$ as in (2.1.12), the partial derivatives of F_v are computed as follows

$$\begin{aligned} \partial_\theta F_v(r, \theta, v, \zeta) &= \frac{rI(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} \partial_r \Phi(r) \sin \theta (1 - v^2) \sin \zeta, \\ \partial_\zeta F_v(r, \theta, v, \zeta) &= -\frac{I(r)}{1 + I(r)^2} \frac{1}{\sin \omega(r)} \partial_r \Phi(r) R(r, \theta) (1 - v^2) \cos \zeta. \end{aligned} \quad (2.4.87)$$

Exploit (2.4.87), and then substitute dZ_θ/dt , dZ_ζ/dt as indicated in (2.4.78) and (2.4.80) inside (2.4.86) to find that

$$\begin{aligned} \frac{d}{dt} \left[F_v(Z(\mathfrak{z}; t)) \right] &= \frac{I(\mathfrak{z}_r)}{1 + I(\mathfrak{z}_r)^2} \frac{\mathfrak{z}_r}{\sin \omega} \partial_r \Phi(\mathfrak{z}_r) \sin Z_\theta (1 - \mathfrak{z}_v^2) \sin Z_\zeta \frac{\mathfrak{z}_v}{\mathfrak{z}_r} \sin \omega \sin Z_\zeta \\ &+ \frac{I(\mathfrak{z}_r)}{1 + I(\mathfrak{z}_r)^2} \frac{1}{\sin \omega} \partial_r \Phi(\mathfrak{z}_r) R(\mathfrak{z}_r, Z_\theta) (1 - \mathfrak{z}_v^2) \cos Z_\zeta \mathfrak{z}_v \left(2R(\mathfrak{z}_r, Z_\theta) \right)^{-1} \sin \omega \sin Z_\theta \cos Z_\zeta \\ &= \mathcal{M}_v(Z(\mathfrak{z}; t)). \end{aligned}$$

□

Now, we exploit (2.4.85) inside (2.4.84), we find that $\langle \bar{A}_{1v} \rangle(z_0; \mathfrak{z}) = 0$ since the average flow $Z(\mathfrak{z}; \cdot)$ is periodic of period $P(z_0)$ by Lemma 2.3.7. This implies that $\langle \bar{\mathfrak{z}}_{0v} \rangle(z_0; \tau) = v_0$ for all τ . And thus, the equation (2.4.82) becomes

$$\tilde{v}(x_0, v_0; \tau) = v_0(x_0, v_0) + \mathcal{O}(\varepsilon). \quad (2.4.88)$$

This means that the kinetic energy is, at least at the leading order, a conserved quantity.

2.4.3.4 Appendix

To proceed, we need to consider the Fourier series expansions with respect to the variable $\nu \in \mathbb{T}$ of the functions $\mathcal{A}(z, \cdot)$ and $\mathcal{V}_j(z, \cdot)$ for $j \in \{0, 1\}$, which can be written as

$$\begin{aligned}\mathcal{A}_\star(z, \nu) &= \bar{\mathcal{A}}_\star(z) + \sum_{n=1}^2 \frac{1}{n} \mathcal{C}\mathcal{A}_{\star n}(z) \cos(n\nu) + \sum_{n=1}^2 \frac{1}{n} \mathcal{S}\mathcal{A}_{\star n}(z) \sin(n\nu), \quad \star \in \{r, \theta, v, \zeta\}, \\ \mathcal{V}_j(z, \nu) &= \bar{\mathcal{V}}_j(z) + \sum_{n=1}^2 \frac{1}{n} \mathcal{C}\mathcal{V}_{jn}(z) \cos(n\nu) + \sum_{n=1}^2 \frac{1}{n} \mathcal{S}\mathcal{V}_{jn}(z) \sin(n\nu), \quad j \in \{0, 1\}.\end{aligned}$$

From (2.2.9), we know that

$$\begin{aligned}\mathcal{C}\mathcal{A}_{r1} &= v \cos \zeta, \quad \mathcal{S}\mathcal{A}_{r1} = 0, & \mathcal{C}\mathcal{A}_{rn} &= \mathcal{S}\mathcal{A}_{rn} = 0, & \forall n \geq 2, \\ \mathcal{C}\mathcal{A}_{\theta 1} &= 0, \quad \mathcal{S}\mathcal{A}_{\theta 1} = \frac{v}{r} \cos \omega \cos \zeta, & \mathcal{C}\mathcal{A}_{\theta n} &= \mathcal{S}\mathcal{A}_{\theta n} = 0, & \forall n \geq 2.\end{aligned}$$

From (2.2.12), we can deduce in particular that

$$\begin{aligned}\mathcal{C}\mathcal{A}_{v1} &= -(1 - v^2)^{3/2} \cos \zeta E_r, & \mathcal{S}\mathcal{A}_{v1} &= -(1 - v^2)^{3/2} \cos \zeta E_\perp, \\ \mathcal{S}\mathcal{A}_{\zeta 1} &= v^{-1} (1 - v^2)^{1/2} \sin \zeta E_\perp + \frac{v}{R} \sin \theta \cos \omega \sin \zeta, \\ \mathcal{C}\mathcal{A}_{\zeta 1} &= v^{-1} (1 - v^2)^{1/2} \sin \zeta E_r - \frac{v}{r} \sin^2 \omega \sin \zeta - \frac{v}{R} \cos \theta \cos^2 \omega \sin \zeta, \\ \mathcal{C}\mathcal{A}_{\zeta 2} &= \frac{v}{R} \sin \theta \sin \omega \cos \zeta, \\ \mathcal{S}\mathcal{A}_{\zeta 2} &= v \omega' \cos \zeta - \frac{v}{r} \sin \omega \cos \omega \cos \zeta + \frac{v}{R} \cos \theta \cos \omega \sin \omega \cos \zeta,\end{aligned}$$

as well as $\mathcal{C}\mathcal{A}_{vn} = \mathcal{S}\mathcal{A}_{vn} = \mathcal{C}\mathcal{A}_{\zeta n} = \mathcal{S}\mathcal{A}_{\zeta n} = 0$ for all $n \geq 2$. On the other hand, from (2.2.15), it is obvious that $\mathcal{C}\mathcal{V}_{0n} = 0$ and $\mathcal{S}\mathcal{V}_{0n} = 0$ for all $n \in \mathbb{N}$. Looking at (2.2.16), we can assert that

$$\begin{aligned}\mathcal{C}\mathcal{V}_{11} &= -\frac{v}{R} \sin^2 \zeta \cos \omega \cos^{-1} \zeta \sin \theta - v^{-1} \cos^{-1} \zeta (1 - v^2)^{1/2} E_\perp, \\ \mathcal{S}\mathcal{V}_{11} &= -\frac{v}{r} \cos^2 \omega \cos \zeta - \frac{v}{r} \sin^2 \omega \sin^2 \zeta \cos^{-1} \zeta - \frac{v}{R} \sin^2 \zeta \cos^{-1} \zeta \cos^2 \omega \cos \theta \\ &\quad - \frac{v}{R} \sin^2 \omega \cos \zeta \cos \theta + v^{-1} \cos^{-1} \zeta (1 - v^2)^{1/2} E_r, \\ \mathcal{C}\mathcal{V}_{12} &= -v \omega' \sin \zeta + \frac{v}{r} \cos \omega \sin \omega \sin \zeta - \frac{v}{R} \sin \zeta \sin \omega \cos \omega \cos \theta, \\ \mathcal{S}\mathcal{V}_{12} &= \frac{v}{R} \sin \zeta \sin \omega \sin \theta.\end{aligned}$$

We now come back to (2.4.67) in order to compute the different terms occurring in \mathcal{S}_* . We have

$$\begin{aligned} (\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) &= (\partial_\nu^{-1} \mathcal{A}_r^* \partial_r) + (\partial_\nu^{-1} \mathcal{A}_\theta^* \partial_\theta) + (\partial_\nu^{-1} \mathcal{A}_v^* \partial_v) + (\partial_\nu^{-1} \mathcal{A}_\zeta^* \partial_\zeta) \\ &= \mathcal{C}\mathcal{A}_{r1} \sin \nu \partial_r - \mathcal{S}\mathcal{A}_{\theta1} \cos \nu \partial_\theta + [\mathcal{C}\mathcal{A}_{v1} \sin \nu - \mathcal{S}\mathcal{A}_{v1} \cos \nu] \partial_v \\ &\quad + \left[\mathcal{C}\mathcal{A}_{\zeta1} \sin \nu - \mathcal{S}\mathcal{A}_{\zeta1} \cos \nu + \frac{1}{4} \mathcal{C}\mathcal{A}_{\zeta2} \sin 2\nu - \frac{1}{4} \mathcal{S}\mathcal{A}_{\zeta2} \cos 2\nu \right] \partial_\zeta. \end{aligned}$$

It follows that

$$\begin{aligned} \overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) \mathcal{A}_r} &= -\frac{1}{2} \left[\mathcal{S}\mathcal{A}_{v1} \partial_v (\mathcal{C}\mathcal{A}_{r1}) + \mathcal{S}\mathcal{A}_{\zeta1} \partial_\zeta (\mathcal{C}\mathcal{A}_{r1}) \right] \\ &= \frac{(1-v^2)^{3/2}}{2} \cos^2 \zeta E_\perp + \frac{v^2}{2R} \sin^2 \zeta \cos \omega \sin \theta + \frac{(1-v^2)^{1/2}}{2} \sin^2 \zeta E_\perp. \\ \overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) \mathcal{A}_\theta} &= \frac{1}{2} \left[\mathcal{C}\mathcal{A}_{r1} \partial_r (\mathcal{S}\mathcal{A}_{\theta1}) + \mathcal{C}\mathcal{A}_{v1} \partial_v (\mathcal{S}\mathcal{A}_{\theta1}) + \mathcal{C}\mathcal{A}_{\zeta1} \partial_\zeta (\mathcal{S}\mathcal{A}_{\theta1}) \right] \\ &= \frac{1}{2} v^2 \cos^2 \zeta \frac{\partial}{\partial r} \left(\frac{\cos \omega}{r} \right) - \frac{(1-v^2)^{3/2}}{2r} \cos^2 \zeta \cos \omega E_r \\ &\quad + \frac{v^2}{2r} \sin^2 \zeta \cos \omega \left(\frac{\sin^2 \omega}{r} + \frac{\cos \theta \cos^2 \omega}{R} \right) - \frac{(1-v^2)^{1/2}}{2r} \sin^2 \zeta \cos \omega E_r. \\ \overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) \mathcal{A}_v} &= \frac{1}{2} \left[-\mathcal{S}\mathcal{A}_{\theta1} \partial_\theta (\mathcal{C}\mathcal{A}_{v1}) - \mathcal{S}\mathcal{A}_{v1} \partial_v (\mathcal{C}\mathcal{A}_{v1}) - \mathcal{S}\mathcal{A}_{\zeta1} \partial_\zeta (\mathcal{C}\mathcal{A}_{v1}) \right. \\ &\quad \left. + \mathcal{C}\mathcal{A}_{r1} \partial_r (\mathcal{S}\mathcal{A}_{v1}) + \mathcal{C}\mathcal{A}_{v1} \partial_v (\mathcal{S}\mathcal{A}_{v1}) + \mathcal{C}\mathcal{A}_{\zeta1} \partial_\zeta (\mathcal{S}\mathcal{A}_{v1}) \right] \\ &= -\frac{v}{2} (1-v^2)^{3/2} \cos^2 \zeta \partial_r E_\perp + \frac{v}{2r} (1-v^2)^{3/2} \cos^2 \zeta \cos \omega \partial_\theta E_r \\ &\quad - \frac{v}{2R} (1-v^2)^{3/2} \sin^2 \zeta \cos \omega \sin \theta E_r - \frac{v}{2} (1-v^2)^{3/2} \sin^2 \zeta E_\perp \left(\frac{\sin^2 \omega}{r} + \frac{\cos \theta \cos^2 \omega}{R} \right). \\ \overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) \mathcal{A}_\zeta} &= \frac{1}{2} \left[-\mathcal{S}\mathcal{A}_{\theta1} \partial_\theta (\mathcal{C}\mathcal{A}_{\zeta1}) - \mathcal{S}\mathcal{A}_{v1} \partial_v (\mathcal{C}\mathcal{A}_{\zeta1}) - \mathcal{S}\mathcal{A}_{\zeta1} \partial_\zeta (\mathcal{C}\mathcal{A}_{\zeta1}) \right. \\ &\quad \left. + \mathcal{C}\mathcal{A}_{r1} \partial_r (\mathcal{S}\mathcal{A}_{\zeta1}) + \mathcal{C}\mathcal{A}_{v1} \partial_v (\mathcal{S}\mathcal{A}_{\zeta1}) + \mathcal{C}\mathcal{A}_{\zeta1} \partial_\zeta (\mathcal{S}\mathcal{A}_{\zeta1}) \right] \\ &\quad + \frac{1}{16} \left[\mathcal{C}\mathcal{A}_{\zeta2} \partial_\zeta (\mathcal{S}\mathcal{A}_{\zeta2}) - \mathcal{S}\mathcal{A}_{\zeta2} \partial_\zeta (\mathcal{C}\mathcal{A}_{\zeta2}) \right] \\ &= \frac{v^2}{2R^2} \cos \zeta \sin \zeta \sin \theta \cos \theta \cos \omega (\cos^2 \omega - 1) + \frac{(1-v^2)^{1/2}}{2} \sin \zeta \cos \zeta \left(\partial_r E_\perp - \frac{\cos \omega}{r} \partial_\theta E_r \right) \\ &\quad - \frac{v^2}{2R} \omega' \cos \zeta \sin \zeta \sin \theta \sin \omega - \frac{v^2}{2rR} \cos \zeta \sin \zeta \sin \theta \cos^3 \omega \\ &\quad - \frac{(1-v^2)^{3/2}}{2} \sin \zeta \cos \zeta \left(\frac{\sin^2 \omega}{r} E_\perp + \sin \theta \frac{\cos \omega}{R} E_r + \cos \theta \frac{\cos^2 \omega}{R} E_\perp \right). \end{aligned}$$

We find also

$$\overline{\mathcal{V}_1 \mathcal{A}_r^*} = \frac{1}{2} \mathcal{C} \mathcal{A}_{r1} \mathcal{C} \mathcal{V}_{11} = -\frac{v^2}{2R} \sin^2 \zeta \cos \omega \sin \theta - \frac{(1-v^2)^{1/2}}{2} E_\perp,$$

as well as

$$\begin{aligned} \overline{\mathcal{V}_1 \mathcal{A}_\theta^*} &= \frac{1}{2} \mathcal{S} \mathcal{A}_{\theta 1} \mathcal{S} \mathcal{V}_{11} = -\frac{v^2}{2r^2} \cos^3 \omega \cos^2 \zeta - \frac{v^2}{2r^2} \cos \omega \sin^2 \omega \sin^2 \zeta \\ &+ \frac{(1-v^2)^{1/2}}{2r} \cos \omega E_r - \frac{v^2}{2rR} \sin^2 \zeta \cos^3 \omega \cos \theta - \frac{v^2}{2rR} \sin^2 \omega \cos \omega \cos^2 \zeta \cos \theta. \end{aligned}$$

$$\begin{aligned} \overline{\mathcal{V}_1 \mathcal{A}_v^*} &= \frac{1}{2} [\mathcal{C} \mathcal{A}_{v1} \mathcal{C} \mathcal{V}_{11} + \mathcal{S} \mathcal{V}_{11} \mathcal{S} \mathcal{A}_{v1}] = \frac{v}{2r} (1-v^2)^{3/2} E_\perp [\cos^2 \omega \cos^2 \zeta + \sin^2 \omega \sin^2 \zeta] \\ &+ \frac{v(1-v^2)^{3/2}}{2R} E_r \sin^2 \zeta \cos \omega \sin \theta + \frac{v}{2R} (1-v^2)^{3/2} E_\perp \cos \theta [\sin^2 \zeta \cos^2 \omega + \sin^2 \omega \cos^2 \zeta]. \end{aligned}$$

$$\begin{aligned} \overline{\mathcal{V}_1 \mathcal{A}_\zeta^*} &= \frac{1}{2} [\mathcal{C} \mathcal{A}_{\zeta 1} \mathcal{C} \mathcal{V}_{11} + \mathcal{S} \mathcal{A}_{\zeta 1} \mathcal{S} \mathcal{V}_{11}] + \frac{1}{8} [\mathcal{C} \mathcal{A}_{\zeta 2} \mathcal{C} \mathcal{V}_{12} + \mathcal{S} \mathcal{A}_{\zeta 1} \mathcal{S} \mathcal{V}_{12}] \\ &= -\frac{v^2}{2rR} \sin \theta \cos^3 \omega \sin \zeta \cos \zeta - \frac{v^2}{2R^2} \sin \zeta \cos \zeta \sin \theta \cos \theta \sin^2 \omega \cos \omega \\ &+ \frac{(1-v^2)^{1/2}}{2R} \sin \theta \cos \omega \sin \zeta \cos \zeta E_r \\ &+ \frac{(1-v^2)^{1/2}}{2} \sin \zeta \cos \zeta E_\perp (\sin^2 \omega - \cos^2 \omega) \left(\frac{1}{r} - \frac{\cos \theta}{R} \right). \end{aligned}$$

There remains to compute

$$\begin{aligned} \overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) (\mathcal{V}_0) \mathcal{A}_r^*} &= -\frac{1}{2} \mathcal{C} \mathcal{A}_{r1} [\mathcal{S} \mathcal{A}_{\theta 1} \partial_\theta \mathcal{V}_0 + \mathcal{S} \mathcal{A}_{v1} \partial_v \mathcal{V}_0] \\ &= \frac{v^2(1-v^2)^{1/2}}{2r} \partial_\theta b \cos \omega \cos^2 \zeta + \frac{v^2(1-v^2)}{2} b E_\perp \cos^2 \zeta, \end{aligned}$$

$$\begin{aligned} \overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) (\mathcal{V}_0) \mathcal{A}_\theta^*} &= \frac{1}{2} \mathcal{S} \mathcal{A}_{\theta 1} [\mathcal{C} \mathcal{A}_{r1} \partial_r \mathcal{V}_0 + \mathcal{C} \mathcal{A}_{v1} \partial_v \mathcal{V}_0] \\ &= \frac{v}{2r} \cos \omega \cos \zeta [-v(1-v^2)^{1/2} \partial_r b \cos \zeta - v(1-v^2) b \cos \zeta E_r], \\ \overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) (\mathcal{V}_0) \mathcal{A}_v^*} &= \frac{1}{2} [\mathcal{C} \mathcal{A}_{r1} \partial_r \mathcal{V}_0 \mathcal{S} \mathcal{A}_{v1} - \mathcal{S} \mathcal{A}_{\theta 1} \partial_\theta \mathcal{V}_0 \mathcal{C} \mathcal{A}_{v1}] \\ &= \frac{v}{2} (1-v^2)^2 \cos^2 \zeta \left(\partial_r b E_\perp - \frac{\cos \omega}{r} \partial_\theta b E_r \right), \end{aligned}$$

as well as

$$\begin{aligned}
\overline{(\partial_\nu^{-1} \mathcal{A}^* \cdot \nabla_z) (\mathcal{V}_0) \mathcal{A}_\zeta^*} &= \frac{1}{2} \left[\mathcal{C} \mathcal{A}_{r1} \mathcal{S} \mathcal{A}_{\zeta 1} \partial_r \mathcal{V}_0 - \mathcal{S} \mathcal{A}_{\theta 1} \mathcal{C} \mathcal{A}_{\zeta 1} \partial_\theta \mathcal{V}_0 \right. \\
&\quad \left. + (\mathcal{C} \mathcal{A}_{v1} \mathcal{S} \mathcal{A}_{\zeta 1} - \mathcal{S} \mathcal{A}_{v1} \mathcal{C} \mathcal{A}_{\zeta 1}) \partial_v \mathcal{V}_0 \right] \\
&= -\frac{v^2}{2r} (1-v^2)^{1/2} \partial_\theta b \cos \omega \cos \zeta \sin \zeta \left(\frac{\sin^2 \omega}{r} + \frac{\cos \theta \cos^2 \omega}{R} \right) \\
&\quad - \frac{1}{2} (1-v^2) E_\perp \partial_r b \sin \zeta \cos \zeta + \frac{1}{2r} (1-v^2) \partial_\theta b \cos \omega \cos \zeta \sin \zeta E_r \\
&\quad - \frac{v^2}{2} (1-v^2) \cos \zeta \sin \zeta E_\perp b \left(\frac{\sin^2 \omega}{r} + \frac{\cos \theta \cos^2 \omega}{R} \right) \\
&\quad - \frac{v^2}{2R} (1-v^2)^{1/2} \partial_r b \cos \zeta \cos \omega \sin \theta \sin \zeta - \frac{v^2}{2R} (1-v^2) \sin \theta \cos \omega \sin \zeta \cos \zeta b E_r.
\end{aligned}$$

We can compile the above results at the level of (2.4.67) to recover the content of the \mathcal{M}_* inside Lemma 2.4.8.

PART II

The statistical approach

QUANTUM-CLASSICAL MOTION OF CHARGED PARTICLES INTERACTING WITH SCALAR FIELDS

Abstract. *The goal of this chapter is to investigate the dynamics of semi-relativistic or non-relativistic charged particles in interaction with a scalar meson field. Our main contribution is the derivation of the classical dynamics of a particle-field system as an effective equation of the quantum microscopic Nelson model, in the classical limit where the value of the Planck constant approaches zero ($\hbar \rightarrow 0$). Thus, firstly we prove the validity of Bohr's correspondence principle, that is to establish the transition from quantum to classical dynamics; and secondly we exhibit the global well-posedness of the related particle-field equation. We use a Wigner measure approach to study such transition. Then, as a consequence of this interplay between classical and quantum dynamics, we establish the global well-posedness of the classical interacting system, despite the low regularity of the related vector field, which prevents the use of a fixed point argument.*

3.1 Introduction

Classical and quantum mechanics may initially appear to be fundamentally different, as classical mechanics deals with the trajectories of particles while quantum mechanics focuses on wave functions evolution. Furthermore, quantum mechanics are successful in describing microscopic objects, whereas macroscopic systems are better described by classical theories like classical mechanics and classical electrodynamics. The point at which quantum and classical physics are in accordance is known as the correspondence limit, or the classical limit. The *Correspondence Principle* has been introduced to quantum theory in 1920 by Niels Bohr. Such principle emphasizes the importance of understanding the relationship between the two theories and how they converge in specific scaling limits.

Bohr proposed that as the quantum numbers increase, the system behaves more classically and the predictions of quantum mechanics align with those of classical mechanics. In the mathematical physics literature, the Bohr's principle is discussed in different frameworks (quantum mechanics, many-body theory, quantum field theory); and rigorously proved using mainly the Hepp's method [Hep74]. However, this method applies only to a specific selection of density matrices, namely coherent states. In this chapter, we explore this principle by studying the convergence from quantum to classical dynamics in a particle-field interaction model as the value of the Planck constant, denoted by \hbar , approaches zero (see also [Fal12, GNV06]).

On the other hand, the interaction between matter and fields has been a subject of great interest in recent decades. Here, the focus is on exploring the dynamics of charged particles and a scalar meson field interacting according to the Yukawa theory. Recall that the Yukawa theory models the strong nuclear force as an interaction between nucleons (non-relativistic or semi-relativistic particles) and mesons (fields). It is known that, despite the ability of classical mechanics in resolving many physical problems, there are still some phenomena that can not be explained by classical laws alone. Here, the low regularity of the vector field associated to the interacting system makes it difficult to construct global solutions using standard arguments. To overcome this issue, we use the quantum-classical transition of the Yukawa theory. Then, by employing transition, it becomes possible to construct global solutions for the classical interacting system. Another recent study [AFH22] has also explored this type of convergence for non-relativistic particles interacting with the electro-magnetic field, considering the Pauli-Fierz Hamiltonian which converges to the Newton-Maxwell equation.

From a classical standpoint, in our case the dynamics are governed by a particle-field equation (3.1.1), also known as Newton-Klein-Gordon equation, which is a nonlinear system of coupled PDE-ODEs. Previous studies have examined this type of equation, as demonstrated in the articles [KKS99, KS98, KSK97]. These works focus on analyzing the long-term behavior of the solutions to a particle-field equation. Specifically, the authors of these articles introduce a form factor within the interacting system to ensure that the Hamiltonian remains bounded from below, and they assume that this form factor is compactly supported. In our investigation, we adopt a more general framework by imposing less restrictions on this form factor.

From a quantum standpoint, the time evolution is generated by the so-called Nelson Hamiltonian (3.1.5). The Nelson model was first introduced by Edward Nelson in [Nel64a].

[Nel64b] to describe the interaction between particles (nucleons) and meson field (strong nuclear force). The Nelson model has been widely studied by many researchers, and a selection of relevant articles includes [AH12, Alb73, Amm00, Ara01, BFS99, CHPS15, CP14, FGS00, Ger02, GHPS11, Hir96, Hø69, Teu02].

Our objectives are then:

- Proving the validity of Bohr’s correspondence principle. More precisely, we want to establish a relationship between quantum and classical dynamics by showing that Nelson model reduces to the classical particle-field equation in the classical limit $\hbar \rightarrow 0$;
- Establishing the global well-posedness of a particle-field equation under weak assumptions on the form factor χ (see (3.1.6)) and on the potential V (see (3.1.7)).

The quantum dynamics have a well-defined global behavior. Our method involves transferring certain quantum regularization effects to the classical dynamics. This leads to the derivation of the classical dynamics of the particle-field system as an effective equation of a quantum microscopic dynamical system in the limit $\hbar \rightarrow 0$.

To achieve this scenario, we investigate the transition using Wigner measures approach in infinite dimensional bosonic quantum field theory. In recent years, these Wigner measure method have been used in many-body theory [AN08] and in quantum field theory [AF17] with an a priori knowledge of global well-posedness (GWP) for effective equations. Whereas in this work, our strategy furnishes global well-posedness and convergence at the same time. Usually, this convergence is non-trivial, and there is no prior guarantee of obtaining unique limits. However, we overcome this difficulty by relying on our assumptions.

The main results are the classical limit (Theorem 3.1.1) and the global well-posedness of a particle field equation (Theorem 3.1.3). To prove these outcomes, our strategy is summarized in the steps below:

- We first extract the quantum dynamical system using the family of density matrices $(\varrho_{\hbar})_{\hbar}$ satisfying (3.1.8) and (3.1.9);
- Then, after proving the propagation (uniformly in any compact time interval) of the two uniform estimates (3.1.8) and (3.1.9), we take the limit to obtain the classical dynamics on the inverse Fourier transform of the Wigner measure. This results in a specific classical equation which is equivalent to a statistical Liouville equation, thanks to the regularities associated with the Wigner measure and vector field in

this context;

- We employ then measure-theoretical methods [AL18, ALR20, AFH22] which provides us with the almost sure existence of global solutions. This requires us to prove the uniqueness of the solutions to a particle-field equation by using classical tools;
- Finally, we extend the existence result to all initial data. It is important to note, however, that the associated flow is Borel measurable with respect to initial data and may not be continuous.

3.1.1 General framework

This section provides a concrete mathematical description of the previous introduction. From a classical perspective, the dynamics are governed by a particle-field equation, as detailed in Paragraph 3.1.1.1. From a quantum perspective, the dynamics are governed by the Nelson Hamiltonian, which is explained in Paragraph 3.1.1.2.

3.1.1.1 The particle-field equation

Consider n fixed number of classical particles in the configuration space \mathbb{R}^d with $d \in \mathbb{N}^*$, interacting with field. Let M_j be the mass of the j th particle. The dynamic of the particles is characterized by their momenta $p_j \in \mathbb{R}^d$ and their positions $q_j \in \mathbb{R}^d$. Whereas, the field is described by $\alpha : \mathbb{R}^d \rightarrow \mathbb{C}$. Let $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$ and $f_j : \mathbb{R}^d \rightarrow \mathbb{R}$, the Hamiltonian of the particle-field system is

$$H(p, q, \alpha) = \sum_{j=1}^n f_j(p_j) + V(q_1, \dots, q_n) + \int_{\mathbb{R}^d} \overline{\alpha(k)} \omega(k) \alpha(k) dk \\ + \sum_{j=1}^n \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} [\alpha(k) e^{2\pi i k \cdot q_j} + \overline{\alpha(k)} e^{-2\pi i k \cdot q_j}] dk.$$

We consider two cases:

- Choosing $f_j(p_j) = \sqrt{p_j^2 + M_j^2}$ corresponds to the semi-relativistic case.
- Choosing $f_j(p_j) = p_j^2/2M_j$ corresponds to the non-relativistic case.

The parameter ω represents the dispersion relation defined by $\omega(k) = \sqrt{k^2 + m_f^2} \geq m_f > 0$, where m_f is the mass of the meson field. The function $V : \mathbb{R}^{dn} \rightarrow \mathbb{R}$ represents the external potential and $\chi : \mathbb{R}^d \rightarrow \mathbb{R}$ is the form factor. The equation of motion for the

particle-field system is given by

$$\begin{cases} \partial_t p_j = -\frac{\partial H}{\partial q_j} = -\nabla_{q_j} V(q) - \int_{\mathbb{R}^d} 2\pi i k \frac{\chi(k)}{\sqrt{\omega(k)}} [\alpha(k) e^{2\pi i k \cdot q_j} - \overline{\alpha(k)} e^{-2\pi i k \cdot q_j}] dk; \\ \partial_t q_j = \frac{\partial H}{\partial p_j} = \nabla f_j(p_j); \\ i\partial_t \alpha = \frac{\partial H}{\partial \alpha} = \omega(k) \alpha(k) + \sum_{j=1}^n \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot q_j}. \end{cases} \quad (3.1.1)$$

To clarify, the interaction term between particles and the scalar field has a specific form which is: linear in the field (for both semi and non-relativistic case); and in the momentum (only in the non-relativistic case). The solution $u = (p, q, \alpha)$ to (3.1.1) belongs to the following classical space

$$X^\sigma := \mathbb{R}^{dn} \times \mathbb{R}^{dn} \times \mathcal{G}^\sigma,$$

where \mathcal{G}^σ with $\sigma \geq 0$ is the weighted L^2 lebesgue space endowed with the following norm

$$\|\alpha\|_{\mathcal{G}^\sigma}^2 := \langle \alpha, \omega(\cdot)^{2\sigma} \alpha \rangle_{L^2} = \int_{\mathbb{R}^d} \omega(k)^{2\sigma} |\alpha(k)|^2 dk = \|\omega^\sigma \alpha\|_{L^2}^2.$$

We have then for $u = (p, q, \alpha) \in X^\sigma$ the following norm

$$\|u\|_{X^\sigma}^2 := \sum_{j=1}^n (|q_j|^2 + |p_j|^2) + \|\alpha\|_{\mathcal{G}^\sigma}^2.$$

The form factor serves as a way to term the interaction between particles and the field, by smoothing out the Hamiltonian and ensuring that it is bounded from below under certain assumptions. The magnitude of the coupling between the particles and the field is controlled by the form factor. We consider the energy space where the Hamiltonian is well-defined, namely $X^{1/2}$, but our main results are stated in the spaces X^σ with $\sigma \in [\frac{1}{2}, 1]$.

3.1.1.2 The Nelson model

The particle-field equation can be formally quantized to obtain the Nelson model. The Hilbert space of the quantized particle-field system is

$$\mathcal{H} := L^2(\mathbb{R}_x^{dn}, \mathbb{C}) \otimes \Gamma_s(L^2(\mathbb{R}_k^d, \mathbb{C})),$$

where $\Gamma_s(L^2(\mathbb{R}_k^d, \mathbb{C}))$ is the symmetric Fock space which could be identified with

$$\Gamma_s(L^2(\mathbb{R}_k^d, \mathbb{C})) := \bigoplus_{m=0}^{+\infty} L^2(\mathbb{R}^d, \mathbb{C})^{\otimes_s m} \simeq \bigoplus_{m=0}^{+\infty} L^2(\mathbb{R}^{dm}, \mathbb{C}).$$

We denote by $\mathcal{F}^m := L^2(\mathbb{R}^{dm}, \mathbb{C})$ the symmetric L^2 space over \mathbb{R}^{dm} . Let

$$\begin{aligned} X_n &= (x_1, \dots, x_n), & dX_n &= dx_1 \cdots dx_n, \\ K_m &= (k_1, \dots, k_m), & dK_m &= dk_1 \cdots dk_m. \end{aligned} \quad (3.1.2)$$

Then, the Hilbert space \mathcal{H} is endowed with the following norm for all $\psi = \{\psi^m\}_{m \geq 0}$

$$\|\psi\|_{\mathcal{H}} := \left[\sum_{m=0}^{\infty} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} |\psi^m(X_n, K_m)|^2 dX_n dK_m \right]^{1/2}.$$

Let \hat{p}_j and \hat{q}_j be the quantized momentum and position operators such that for all $j \in \{1, \dots, n\}$

$$\hat{p}_j = -i\hbar \nabla_{x_j}, \quad \hat{q}_j = x_j.$$

The \hbar scaled creation-annihilation operators for the field are defined on Γ_s for any $f \in L^2(\mathbb{R}^d, \mathbb{C})$ as

$$\hat{a}_{\hbar}(f) = \int_{\mathbb{R}^d} \overline{f(k)} \hat{a}_{\hbar}(k) dk, \quad \hat{a}_{\hbar}^*(f) = \int_{\mathbb{R}^d} f(k) \hat{a}_{\hbar}^*(k) dk,$$

where $\hat{a}_{\hbar}(k)$ and $\hat{a}_{\hbar}^*(k)$ are the creation-annihilation operator-valued distributions defined as follows

$$\begin{aligned} [\hat{a}_{\hbar}(k) \psi]^m(k_1, \dots, k_m) &= \sqrt{\hbar(m+1)} \psi^{m+1}(k, k_1, \dots, k_m); \\ [\hat{a}_{\hbar}^*(k) \psi]^m(k_1, \dots, k_m) &= \frac{\sqrt{\hbar}}{\sqrt{m}} \sum_{j=1}^m \delta(k - k_j) \psi^{m-1}(k_1, \dots, \hat{k}_j, \dots, k_m). \end{aligned}$$

In our case, we will work with the generalized \hbar scaled creation-annihilation operators. The two operators $\hat{a}_{\hbar}^{\sharp}(G) : \mathcal{H} \rightarrow \mathcal{H}$ are defined for

$$\begin{aligned} G : L^2(\mathbb{R}_x^{dn}, \mathbb{C}) &\longrightarrow L^2(\mathbb{R}_x^{dn}, \mathbb{C}) \otimes L^2(\mathbb{R}_k^d, \mathbb{C}) \\ \psi &\longmapsto G \psi. \end{aligned}$$

with

$$(G \psi)(X_n, k) = \sum_{j=1}^n \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot \hat{q}_j} \psi(X_n).$$

In general, we have

$$\begin{aligned} [\hat{a}_\hbar(G) \psi(X_n)]^m(K_m) &= \sqrt{\hbar(m+1)} \sum_{j=1}^n \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{2\pi i k \cdot \hat{q}_j} \psi^{m+1}(X_n; K_m, k) dk; \quad (3.1.3) \\ [\hat{a}_\hbar^*(G) \psi(X_n)]^m(K_m) &= \frac{\sqrt{\hbar}}{\sqrt{m}} \sum_{j=1}^m \sum_{\ell=1}^n \frac{\chi(k_j)}{\sqrt{\omega(k_j)}} e^{-2\pi i k_j \cdot \hat{q}_\ell} \psi^{m-1}(X_n; k_1, \dots, \hat{k}_j, \dots, k_m). \end{aligned} \quad (3.1.4)$$

Introduce the second quantization $d\Gamma(A) : \mathcal{H} \rightarrow \mathcal{H}$ for the self-adjoint operator A with $d\Gamma(A) \psi = \{[d\Gamma(A) \psi]^m\}_{m>0}$ and where

$$[d\Gamma(A) \psi]^m = \hbar \sum_{j=1}^m \psi \otimes \dots \otimes \underbrace{A\psi}_{j\text{th position}} \otimes \dots \otimes \psi.$$

The \hbar scaled number operator $\hat{N}_\hbar = d\Gamma(\text{Id})$ and the number operator \hat{N} are defined as follows

$$[\hat{N}_\hbar \psi]^m = \hbar m \psi^m, \quad [\hat{N} \psi]^m = m \psi^m.$$

The free field Hamiltonian $d\Gamma(\omega) : \mathcal{H} \rightarrow \mathcal{H}$ is defined as follows

$$[d\Gamma(\omega) \psi(X_n)]^m = \hbar \sum_{j=1}^m \omega(k_j) \psi^m(X_n; K_m).$$

Formally, one can express this as:

$$d\Gamma(\omega) = \int_{\mathbb{R}^d} \hat{a}_\hbar^*(k) \omega(k) \hat{a}_\hbar(k) dk.$$

The non-interacting Hamiltonian is defined as follows

$$\hat{H}_0 := \hat{H}_{01} + \hat{H}_{02},$$

where we have introduced the two terms \hat{H}_{01} and \hat{H}_{02} as follows

$$\hat{H}_{01} = \sum_{j=1}^n f_j(\hat{p}_j), \quad \hat{H}_{02} = d\Gamma(\omega).$$

The interaction Hamiltonian $\hat{H}_1 : \mathcal{H} \rightarrow \mathcal{H}$ is defined in terms of $\hat{a}_\hbar, \hat{a}_\hbar^*$ as in (3.1.3)-(3.1.4) as follows

$$\hat{H}_1 = \hat{a}_\hbar(G) + \hat{a}_\hbar^*(G).$$

The Nelson-Hamiltonian takes then the following form

$$\hat{H}_\hbar \equiv \hat{H} = \hat{H}_0 + V(\hat{q}) + \hat{H}_1. \quad (3.1.5)$$

The inclusion of a form factor χ in the interaction term of the particle-field equation ensures the well-definedness of the corresponding quantum dynamics and leads to a self-adjoint Nelson Hamiltonian. It has been demonstrated that, under certain mild assumptions on χ and the potential V , the unbounded operator \hat{H}_\hbar is indeed self-adjoint (as discussed in [AF17] and references therein). In the following, we aim to identify the minimal conditions on χ and V that enable further analysis.

3.1.2 Assumptions and main results

We have to impose the following assumptions on the external potential $V : \mathbb{R}^{dn} \mapsto \mathbb{R}$ and the form factor $\chi : \mathbb{R}^d \mapsto \mathbb{R}$ with $\sigma \geq 0$:

$$V \in \mathcal{C}_b^2(\mathbb{R}^{dn}; \mathbb{R}), \quad (3.1.6)$$

$$\omega(\cdot)^{\frac{3}{2}-\sigma} \chi(\cdot) \in L^2(\mathbb{R}^d; \mathbb{R}). \quad (3.1.7)$$

Note that the following identities hold true:

- If $\omega(\cdot)^{\frac{3}{2}-\sigma} \chi(\cdot) \in L^2(\mathbb{R}^d; \mathbb{R})$ then $\chi(\cdot) \in L^2(\mathbb{R}^d; \mathbb{R})$;
- If $\chi(\cdot) \in L^2(\mathbb{R}^d; \mathbb{R})$ then for any $\gamma > 0$, we have $\omega(\cdot)^{-\gamma} \chi(\cdot) \in L^2(\mathbb{R}^d; \mathbb{R})$.

Let $(\varrho_\hbar)_{\hbar \in (0,1)}$ be a family of density matrices on \mathcal{H} of the particle-field quantum system. The main assumptions on the family of states $(\varrho_\hbar)_{\hbar \in (0,1)}$ are:

$$\exists C_0 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_\hbar d\Gamma(\omega^{2\sigma})] \leq C_0, \quad (3.1.8)$$

$$\exists C_1 > 0, \forall \hbar \in (0, 1), \quad \text{Tr}[\varrho_\hbar (\hat{q}^2 + \hat{p}^2)] \leq C_1. \quad (3.1.9)$$

Remark that the following identities hold true:

- If $\text{Tr}[\varrho_\hbar d\Gamma(\omega^{2\sigma})] \leq c_0$, then $\text{Tr}[\varrho_\hbar d\Gamma(\omega)] \leq c'_0$ for some $c_0, c'_0 \in \mathbb{R}_+^*$;
- If $\text{Tr}[\varrho_\hbar d\Gamma(\omega)] \leq c_1$, then $\text{Tr}[\varrho_\hbar \hat{N}_\hbar] \leq c'_1$ for some $c_1, c'_1 \in \mathbb{R}_+^*$;

- If $\text{Tr}[\varrho_{\hbar} (\hat{q}^2 + \hat{p}^2)] \leq c_2$, then $\text{Tr}[\varrho_{\hbar} (\hat{H}_0 + 1)] \leq c'_2$ for some $c_2, c'_2 \in \mathbb{R}_+^*$.

The first result presented in this section concerns the flow of the particle-field equation. Notably, this result does not require any assumptions from the quantum perspective.

Theorem 3.1.1 (Global well-posedness of the particle-field equation). *Let $\sigma \in [\frac{1}{2}, 1]$. Assume (3.1.6) and (3.1.7) hold. Then for any initial condition $u_0 \in X^\sigma$ there exists a unique global strong solution $u(\cdot) \in \mathcal{C}(\mathbb{R}, X^\sigma) \cap \mathcal{C}^1(\mathbb{R}, X^{\sigma-1})$ of the particle-field equation (3.1.1). Moreover, the global flow map $u_0 \rightarrow \Phi_t(u_0) = u(t)$ associated to the particle-field equation (3.1.1) is Borel measurable.*

The above global flow is not constructed from a fixed point argument, whereas it is constructed by means of statistical arguments. More precisely, we use measure theoretical techniques to construct this flow. And thus, it is only Borel measurable and not necessarily continuous.

Denote by $\mathcal{P}(X^0)$ the set of all Borel probability measure over the space X^0 .

Definition 3.1.2 (Wigner measures). A Borel probability measure $\mu \in \mathcal{P}(X^0)$ is a Wigner measure of a family of density matrices $(\varrho_{\hbar})_{\hbar \in (0,1)}$ on the Hilbert space \mathcal{H} if and only if there exists a countable subset $\mathcal{A} \subset (0, 1)$ with $0 \in \overline{\mathcal{A}}$ such that for any $\xi = (p_0, q_0, \alpha_0) \in X^0$:

$$\lim_{\hbar \rightarrow 0, \hbar \in \mathcal{A}} \text{Tr} \left[\mathcal{W}(2\pi q_0, -2\pi p_0, \sqrt{2\pi} \alpha_0) \varrho_{\hbar} \right] = \int_{X^0} e^{2\pi i \Re \langle \xi, u \rangle_{X^0}} d\mu(u).$$

The next result concerns the classical limit which relies on the construction of a Wigner measure in the context of infinite-dimensional bosonic quantum field theory. This allows us to establish convergence from the quantum to the classical dynamics. Denote by

$$\mathcal{M}(\varrho_{\hbar}, \hbar \in \mathcal{A}),$$

the set of all Wigner measure associated to the density matrices $(\varrho_{\hbar})_{\hbar \in \mathcal{A}}$.

Theorem 3.1.3 (Validity of Bohr's correspondence principle). *Let $\sigma \in [\frac{1}{2}, 1]$ and assume (3.1.6) and (3.1.7) hold true. Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices on \mathcal{H} satisfying (3.1.8) and (3.1.9). Assume that $\mathcal{M}(\varrho_{\hbar}, \hbar \in (0, 1)) = \{\mu_0\}$. Then for all times $t \in \mathbb{R}$, we have*

$$\mathcal{M}(e^{-i\frac{t}{\hbar}\hat{H}} \varrho_{\hbar} e^{i\frac{t}{\hbar}\hat{H}}, \hbar \in (0, 1)) = \{\mu_t\},$$

where $\mu_t \in \mathcal{P}(X^0)$ satisfying

- (i) μ_t is concentrated on X^σ i.e. $\mu_t(X^\sigma) = 1$;
- (ii) $\mu_t = (\Phi_t)_\# \mu_0$, where $u_0 \mapsto \Phi_t(u_0) = u(t)$ is the Borel measurable global flow of the particle-field equation (3.1.1).

The result above indicates that when ϱ_\hbar are density matrices on \mathcal{H} that approach the Wigner probability measure μ_0 as \hbar approaches zero, the evolved density matrices $\varrho_\hbar(t)$ will converge to $\mu_t = (\Phi_t)_\# \mu_0$ for all times t . Here, Φ_t is the flow that solves (3.1.1).

To demonstrate the aforementioned results, we adopt the following approach: Firstly, we employ classical techniques to establish the uniqueness property of the particle-field solutions. Subsequently, we establish crucial uniform propagation estimates on the quantum dynamics. Then, we present a probabilistic representation of measure-valued solutions for the Liouville's equation (see [AL18, ALR20]). This representation is used to construct a generalized global flow for a particle-field equation. As a conclusion, we establish by means of Wigner measures the global well-posedness for the particle-field equation and the Bohr's correspondence principle for the Nelson model.

3.2 The classical system

This section is dedicated to examining various classical properties of the particle-field equation. Firstly, in Subsection 3.2.1, we introduce the particle-field equation as a semi-linear partial differential equation and establish its interaction representation. In Subsection 3.2.2, we prove the uniqueness of the particle-field equation using this representation. Finally, in Subsection 3.2.3, we demonstrate the local well-posedness of the particle-field equation in the special subspace $X^{1/2}$ of X^0 .

3.2.1 The interaction representation

The particle-field equation (3.1.1) takes the following form

$$\begin{cases} \frac{du(t)}{dt} = w(u(t)) = \mathcal{L}(u(t)) + \mathcal{N}(u(t)), \\ u(0) = u_0 \in X^\sigma, \end{cases} \quad (\text{PFE})$$

where $t \rightarrow u(t) = (p(t), q(t), \alpha(t))$ is a solution, $\mathcal{L}(u) = (0, 0, -i\omega\alpha)$ is a linear operator such that $\mathcal{L} : X^\sigma \rightarrow X^{\sigma-1}$ and \mathcal{N} is the nonlinearity given by

$$\begin{aligned} (\mathcal{N}(u))_{p_j} &:= -\nabla_{q_j} V(q) - \nabla_{q_j} I_j(q, \alpha), \\ (\mathcal{N}(u))_{q_j} &:= \nabla f_j(p_j), \\ (\mathcal{N}(u))_\alpha(k) &:= -i \sum_{j=1}^n \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot q_j}, \end{aligned} \tag{3.2.1}$$

where we have introduced

$$I_j(q, \alpha) := \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} \left[\alpha(k) e^{2\pi i k \cdot q_j} + \overline{\alpha(k)} e^{-2\pi i k \cdot q_j} \right] dk \tag{3.2.2}$$

with

$$\nabla_{q_j} I_j(q, \alpha) = \int_{\mathbb{R}^d} 2\pi i k \frac{\chi(k)}{\sqrt{\omega(k)}} \left[\alpha(k) e^{2\pi i k \cdot q_j} - \overline{\alpha(k)} e^{-2\pi i k \cdot q_j} \right] dk. \tag{3.2.3}$$

We consider now the particle-field equation as a non-autonomous initial value problem over the Hilbert space X^σ with

$$\begin{cases} \frac{du(t)}{dt} = v(t, u(t)), \\ u(0) = u_0 \in X^\sigma. \end{cases} \tag{IVP}$$

The non-autonomous vector field v is defined in terms of the non-linearity $\mathcal{N} : X^\sigma \rightarrow X^\sigma$ of the particle-field equation as well as the free field flow $\Phi_t^f : X^\sigma \rightarrow X^\sigma$ as follows:

$$v(t, u) = \Phi_{-t}^f \circ \mathcal{N} \circ \Phi_t^f(u), \tag{3.2.4}$$

where we have introduced the free field flow Φ_t^f as follows

$$\Phi_t^f(p, q, \alpha) = (p, q, e^{-it\omega(k)}\alpha). \tag{3.2.5}$$

Lemma 3.2.1 (Explicit expression for the vector field v). *The vector field $v : \mathbb{R} \times X^\sigma \rightarrow$*

X^σ takes the following explicit form:

$$\begin{aligned} (v(t, u))_{p_j} &= \left(\mathcal{N} \circ \Phi_t^f(u) \right)_{p_j}, \\ (v(t, u))_{q_j} &= \left(\mathcal{N} \circ \Phi_t^f(u) \right)_{q_j}, \\ (v(t, u))_\alpha(k) &= e^{it\omega(k)} \left(\mathcal{N} \circ \Phi_t^f(u) \right)_\alpha(k), \end{aligned} \quad (3.2.6)$$

where $v(t, u) = {}^t((v(t, u))_{p_1}, \dots, (v(t, u))_{p_n}, (v(t, u))_{q_1}, \dots, (v(t, u))_{q_n}, v(t, u)_\alpha)$.

Proof. The result follows from direct computations of v using the relation (3.2.4). \square

Proposition 3.2.2 (Equivalence between (PFE) and (IVP)). *Assume (3.1.6) and (3.1.7) are satisfied. Let I be a bounded open interval containing the origin. Then, the statements below are equivalent:*

1. $u(\cdot) \in \mathcal{C}^1(I, X^\sigma)$ is a strong solution of (IVP);
2. $u(\cdot) \in \mathcal{C}(I, X^\sigma)$ solves the following Duhamel formula

$$u(t) = u_0 + \int_0^t v(s, u(s)) ds, \quad \forall t \in I.$$

3. The curve $t \mapsto \Phi_t^f(u(t)) \in \mathcal{C}(I, X^\sigma) \cap \mathcal{C}^1(I, X^{\sigma-1})$ is a strong solution to the particle-field equation (3.1.1).

Proof. The first two assertions can be proved easily since v is continuous vector field (by Lemma 3.2.5). Let us now prove the equivalence between (1) and (3). Suppose that $u(t) = (p(t), q(t), \alpha(t))$ is a solution to (IVP). Require to prove that

$$\tilde{u}(t) = (\tilde{p}(t), \tilde{q}(t), \tilde{\alpha}(t)) = \Phi_t^f(u(t)) = (p(t), q(t), e^{-it\omega(k)}\alpha(t)),$$

is a solution to (PFE). The first term

$$\partial_t \tilde{p}_j = \partial_t p_j = (v(t, u))_{p_j} = \left(\mathcal{N} \circ \Phi_t^f(u) \right)_{p_j} = (w(\tilde{u}))_{p_j}.$$

The second term

$$\partial_t \tilde{q}_j = \partial_t q_j = \nabla f_j(p_j) = (w(\tilde{u}))_{q_j}.$$

The third term

$$\begin{aligned}
 \partial_t \tilde{\alpha} &= -i\omega(k) e^{-it\omega(k)} \alpha(k) + e^{-it\omega(k)} \partial_t \alpha \\
 &= -i\omega(k) \tilde{\alpha}(k) + e^{-it\omega(k)} e^{it\omega(k)} \left(\mathcal{N} \circ \Phi_t^f(u) \right)_\alpha(k) \\
 &= -i\omega(k) \tilde{\alpha}(k) + (\mathcal{N}(\tilde{u}))_\alpha(k) \\
 &= (\mathfrak{w}(\tilde{u}))_\alpha(k).
 \end{aligned}$$

We conclude that \tilde{u} is a solution to (PFE). Similarly, we can prove the reverse sense. \square

Let I be an open interval containing the origin. We are interested in strong solution to the particle-field equation (PFE) such that

$$u(\cdot) \in \mathcal{C}(I, X^\sigma) \cap \mathcal{C}^1(I, X^{\sigma-1}),$$

and (PFE) is satisfied for all $t \in I$. In particular, from the second assertion of Proposition 3.2.2, these solutions satisfy the following Duhamel formula for all $t \in I$

$$u(t) = \Phi_t^f(u(0)) + \int_0^t \Phi_{t-s}^f \circ \mathcal{N}(u(s)) ds, \quad (3.2.7)$$

where $\Phi_t^f(\cdot)$ is the free field flow (associated to the linear operator \mathcal{L}) defined above in (3.2.5).

3.2.2 Properties of the particle-field equation

In this section, we establish various properties related to the particle-field equation and its time interaction representation. Of most significance is the recovery of the uniqueness property of solutions to the particle-field equation (PFE) on the energy space X^σ . Our approach starts with deriving estimates for $\nabla_{q_j} I_j(\cdot)$.

Lemma 3.2.3 (Estimates for $\nabla_{q_j} I_j$). *We have the following two estimates.*

- (i) *Assume $\omega^{1/2}\chi \in L^2(\mathbb{R}^d, dk)$. Then, for all $(q, \alpha) \in \mathbb{R}^{dn} \times L^2(\mathbb{R}^d, \mathbb{C})$, for all $j \in \{1, \dots, n\}$, we have the following estimate*

$$|\nabla_{q_j} I_j(q, \alpha)| \leq 4\pi \|\omega^{1/2}\chi\|_{L^2} \|\alpha\|_{L^2}. \quad (3.2.8)$$

(ii) Assume (3.1.7) is satisfied. Then, for all $j \in \{1, \dots, n\}$, for all $q_1, q_2 \in \mathbb{R}^{dn}$ with $q_1 = (q_{1j})_j$ and $q_2 = (q_{2j})_j$, for all $\alpha_1, \alpha_2 \in \mathcal{G}^\sigma$, we have

$$\begin{aligned} & |\nabla_{q_j} I_j(q_1, \alpha_1) - \nabla_{q_j} I_j(q_2, \alpha_2)| \\ & \leq 4\pi \|\omega^{1/2} \chi\|_{L^2} \|\alpha_1 - \alpha_2\|_{L^2} + 8\sqrt{2}\pi^2 \|\omega^{\frac{3}{2}-\sigma} \chi\|_{L^2} |q_{1j} - q_{2j}| \|\alpha_2\|_{\mathcal{G}^\sigma}. \end{aligned}$$

Proof. For (i), by Cauchy-Schwartz inequality, we have $\forall (q, \alpha) \in \mathbb{R}^{dn} \times L^2(\mathbb{R}^d, \mathbb{C})$

$$\begin{aligned} |\nabla_{q_j} I_j(q, \alpha)| &= \left| \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} 2\pi i k \left[\alpha(k) e^{2\pi i k \cdot q_j} - \overline{\alpha(k)} e^{-2\pi i k \cdot q_j} \right] dk \right| \\ &\leq 4\pi \int_{\mathbb{R}^d} \left| \frac{\chi(k)}{\sqrt{\omega(k)}} \omega(k) \right| |\alpha(k)| dk \\ &\stackrel{c-s}{\leq} 4\pi \|\omega^{1/2} \chi\|_{L^2} \|\alpha\|_{L^2}. \end{aligned}$$

For (ii), by Cauchy-Schwartz inequality and using the estimate $|e^{iy} - 1| \leq \sqrt{2} |y|$, we have $\forall (q_1, \alpha_1), (q_2, \alpha_2) \in \mathbb{R}^{dn} \times \mathcal{G}^\sigma, \forall j \in \{1, \dots, n\}$ the following estimates

$$\begin{aligned} & |\nabla_{q_j} I_j(q_1, \alpha_1) - \nabla_{q_j} I_j(q_2, \alpha_2)| \\ &= \left| \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} 2\pi i k \left[\alpha_1(k) e^{2\pi i k \cdot q_{1j}} - \overline{\alpha_1(k)} e^{-2\pi i k \cdot q_{1j}} - \alpha_2(k) e^{2\pi i k \cdot q_{2j}} + \overline{\alpha_2(k)} e^{-2\pi i k \cdot q_{2j}} \right] dk \right| \\ &= \left| \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} 2\pi i k \left[(\alpha_1(k) - \alpha_2(k)) e^{2\pi i k \cdot q_{1j}} + \alpha_2(k) (e^{2\pi i k \cdot q_{1j}} - e^{2\pi i k \cdot q_{2j}}) \right. \right. \\ &\quad \left. \left. + (\overline{\alpha_2(k)} - \overline{\alpha_1(k)}) e^{-2\pi i k \cdot q_{1j}} + \overline{\alpha_2(k)} (e^{-2\pi i k \cdot q_{2j}} - e^{-2\pi i k \cdot q_{1j}}) \right] \right| \\ &\leq 4\pi \int_{\mathbb{R}^d} \left| \frac{\chi(k)}{\sqrt{\omega(k)}} \omega(k) \right| \left[|\alpha_1(k) - \alpha_2(k)| + |\alpha_2(k)| (e^{2\pi i k \cdot (q_{1j} - q_{2j})} - 1) \right] \\ &\leq 4\pi \int_{\mathbb{R}^d} \left| \sqrt{\omega(k)} \chi(k) \right| \left[|\alpha_1(k) - \alpha_2(k)| + |\alpha_2(k)| \sqrt{2} 2\pi k \cdot (q_{1j} - q_{2j}) \right] \\ &\leq 4\pi \int_{\mathbb{R}^d} \left| \sqrt{\omega(k)} \chi(k) \right| \left[|\alpha_1(k) - \alpha_2(k)| + 2\sqrt{2} \pi |\alpha_2(k)| \omega(k) |q_{1j} - q_{2j}| \right] \\ &\stackrel{c-s}{\leq} 4\pi \|\omega^{1/2} \chi\|_{L^2} \|\alpha_1 - \alpha_2\|_{L^2} + 8\sqrt{2}\pi^2 \|\omega^{\frac{3}{2}-\sigma} \chi\|_{L^2} |q_{1j} - q_{2j}| \|\alpha_2\|_{\mathcal{G}^\sigma}. \end{aligned}$$

□

The vector field \mathcal{N} , which characterizes the nonlinearity of the particle-field equation, possesses the following properties.

Proposition 3.2.4 (Continuity and boundedness of \mathcal{N}). *Assume (3.1.6) and (3.1.7) are satisfied. Then, the nonlinearity $\mathcal{N} : X^\sigma \rightarrow X^\sigma$ is a continuous, and bounded on bounded sets, vector field.*

Proof. Let us prove first that $\mathcal{N} : X^\sigma \rightarrow X^\sigma$ is bounded on bounded sets. Let $u \in X^\sigma$ be a bounded such that $\|u\|_{X^\sigma} \leq c_0$, for some $c_0 > 0$. Require to prove $\|\mathcal{N}(u)\|_{X^\sigma}^2 \leq c_1$ for some $c_1 > 0$. We have first with some $c_2 > 0$

$$\begin{aligned} \left| (\mathcal{N}(u))_{p_j} \right| &= \left| -\nabla_{q_j} V(q) - \nabla_{q_j} I_j(q, \alpha) \right| \\ &\leq \left| \nabla_{q_j} V(q) \right| + \left| \nabla_{q_j} I_j(q, \alpha) \right| \\ &\stackrel{\text{Lemma 3.2.3 (i)}}{\leq} \left\| \nabla_{q_j} V \right\|_{L^\infty} + 4\pi \|\omega^{1/2} \chi\|_{L^2} \|\alpha\|_{L^2} \\ &\leq \sup_{j=1}^n \left\| \nabla_{q_j} V \right\|_{L^\infty} + \frac{4\pi}{m_f^\sigma} \|\omega^{1/2} \chi\|_{L^2} \|\alpha\|_{\mathcal{G}^\sigma} \\ &\leq \sup_{j=1}^n \left\| \nabla_{q_j} V \right\|_{L^\infty} + \frac{4\pi}{m_f^\sigma} \|\omega^{1/2} \chi\|_{L^2} c_0 := c_2. \end{aligned}$$

We also have with some $c_3, c_4 > 0$

$$\left| (\mathcal{N}(u))_{q_j} \right| = \left| \nabla f_j(p_j) \right| \leq c_3 |p_j| \leq c_3 c_0 := c_4.$$

Finally, we have with some $c_5 > 0$

$$\begin{aligned} \left\| (\mathcal{N}(u))_\alpha \right\|_{\mathcal{G}^\sigma}^2 &= \left\| \omega^\sigma (\mathcal{N}(u))_\alpha \right\|_{L^2}^2 \\ &= \int_{\mathbb{R}^d} \left| -i \sum_{j=1}^n \omega^\sigma(k) \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot q_j} \right|^2 dk \\ &\lesssim n^2 \|\omega^{\sigma-\frac{1}{2}} \chi\|_{L^2}^2 := c_5, \end{aligned}$$

where $\|\omega^{\sigma-\frac{1}{2}} \chi\|_{L^2}$ is finite since $\sigma - \frac{1}{2} < \frac{3}{2} - \sigma$ for $\sigma \in [\frac{1}{2}, 1]$. This implies that there exists $c_1 > 0$ such that

$$\|\mathcal{N}(u)\|_{X^\sigma}^2 \leq n (c_2^2 + c_4^2) + c_5 := c_1.$$

It remains to prove the continuity of the nonlinear term $\mathcal{N} : X^\sigma \rightarrow X^\sigma$. Suppose that

$$u_\ell = (p_\ell, q_\ell, \alpha_\ell) \xrightarrow{\ell \rightarrow +\infty} u = (p, q, \alpha), \quad \text{in } X^\sigma \text{ i.e. } \|u_\ell - u\|_{X^\sigma} \xrightarrow{\ell \rightarrow +\infty} 0.$$

Require to prove

$$\mathcal{N}(u_\ell) \xrightarrow{\ell \rightarrow +\infty} \mathcal{N}(u) \text{ in } X^\sigma \text{ i.e. } \|\mathcal{N}(u_\ell) - \mathcal{N}(u)\|_{X^\sigma} \xrightarrow{\ell \rightarrow +\infty} 0.$$

Indeed, we have

$$\|\mathcal{N}(u_\ell) - \mathcal{N}(u)\|_{X^\sigma}^2 = \sum_{j=1}^n \left[\left| (\mathcal{N}(u_\ell) - \mathcal{N}(u))_{p_j} \right|^2 + \left| (\mathcal{N}(u_\ell) - \mathcal{N}(u))_{q_j} \right|^2 \right] + \|(\mathcal{N}(u_\ell) - \mathcal{N}(u))_\alpha\|_{\mathcal{G}^\sigma}^2.$$

By Lemma [3.2.3](#)(ii), we can assert that

$$\begin{aligned} & \left| (\mathcal{N}(u_\ell) - \mathcal{N}(u))_{p_j} \right| \\ & \leq \left| \nabla_{q_j} V(q_\ell) - \nabla_{q_j} V(q) \right| + \frac{4\pi}{m_f^\sigma} \|\omega^{1/2} \chi\|_{L^2} \|\alpha_\ell - \alpha\|_{\mathcal{G}^\sigma} + 8\sqrt{2}\pi^2 \|\omega^{\frac{3}{2}-\sigma} \chi\|_{L^2} |q_{\ell j} - q_j| \|\alpha\|_{\mathcal{G}^\sigma} \\ & \leq \sum_{j'=1}^n \|\nabla_{q_{j'}} \nabla_{q_j} V\|_{L^\infty} |q_{\ell j'} - q_{j'}| + \frac{4\pi}{m_f^\sigma} \|\omega^{1/2} \chi\|_{L^2} \|\alpha_\ell - \alpha\|_{\mathcal{G}^\sigma} + 8\sqrt{2}\pi^2 \|\omega^{\frac{3}{2}-\sigma} \chi\|_{L^2} |q_{\ell j} - q_j| \|\alpha\|_{\mathcal{G}^\sigma} \\ & \leq \left[n \sup_{j', j=1}^n \|\nabla_{q_{j'}} \nabla_{q_j} V\|_{L^\infty} + \frac{4\pi}{m_f^\sigma} \|\omega^{1/2} \chi\|_{L^2} + 8\sqrt{2}\pi^2 \|\omega^{\frac{3}{2}-\sigma} \chi\|_{L^2} \|\alpha\|_{\mathcal{G}^\sigma} \right] \|u_\ell - u\|_{X^\sigma} \xrightarrow{\ell \rightarrow +\infty} 0. \end{aligned}$$

$$\left| (\mathcal{N}(u_\ell) - \mathcal{N}(u))_{q_j} \right| \leq c |p_{\ell j} - p_j| \leq c \|u_\ell - u\|_{X^\sigma} \xrightarrow{\ell \rightarrow +\infty} 0.$$

For the last term we have

$$\begin{aligned} \|(\mathcal{N}(u_\ell) - \mathcal{N}(u))_\alpha\|_{\mathcal{G}^\sigma}^2 &= \int_{\mathbb{R}^d} \left| -i \sum_{j=1}^n \omega^\sigma(k) \frac{\chi(k)}{\sqrt{\omega(k)}} \left[e^{-2\pi i k \cdot q_{\ell j}} - e^{-2\pi i k \cdot q_j} \right] \right|^2 dk \\ &= \int_{\mathbb{R}^d} \left| -i \sum_{j=1}^n \omega^{\sigma-\frac{1}{2}}(k) \chi(k) \left[e^{-2\pi i k \cdot q_{\ell j}} - e^{-2\pi i k \cdot q_j} \right] \right|^2 dk, \end{aligned}$$

where we have

- $\int_{\mathbb{R}^d} \left| -i \sum_{j=1}^n \omega^{\sigma-\frac{1}{2}}(k) \chi(k) \left[e^{-2\pi i k \cdot q_{\ell j}} - e^{-2\pi i k \cdot q_j} \right] \right|^2 dk \leq 4n^2 \|\omega^{\sigma-\frac{1}{2}} \chi\|_{L^2}^2 < +\infty;$
- $\left| e^{-2\pi i k \cdot q_{\ell j}} - e^{-2\pi i k \cdot q_j} \right| \xrightarrow{\ell \rightarrow +\infty} 0.$

Hence by lebesgue dominated convergence theorem, we get

$$\|(\mathcal{N}(u_\ell) - \mathcal{N}(u))_\alpha\|_{\mathcal{G}^\sigma}^2 \xrightarrow{\ell \rightarrow +\infty} 0.$$

And thus,

$$\|\mathcal{N}(u_\ell) - \mathcal{N}(u)\|_{X^\sigma}^2 \xrightarrow{\ell \rightarrow +\infty} 0.$$

□

The above theorem implies the following results on the vector field v .

Lemma 3.2.5 (Continuity and boundedness properties of the vector field v). *Assume (3.1.6) and (3.1.7) are satisfied. Then, the vector field $v : \mathbb{R} \times X^\sigma \rightarrow X^\sigma$ is continuous and bounded on bounded subsets of $\mathbb{R} \times X^\sigma$.*

Proof. This is a consequence of the continuity and boundedness properties of the nonlinear term $\mathcal{N} : X^\sigma \rightarrow X^\sigma$ in Proposition 3.2.4. □

As a consequence of the above properties, we have the following uniqueness property.

Proposition 3.2.6 (Uniqueness property). *Assume (3.1.6) and (3.1.7) are satisfied. Let I be an open interval containing the origin and let $u_1, u_2 \in \mathcal{C}(I, X^\sigma)$ be two strong solutions of the particle-field equation (PFE) such that $u_1(0) = u_2(0)$. Then $u_1(t) = u_2(t)$ for all $t \in I$.*

Proof. Note first that using Duhamel formula (3.2.7) as well as $u_1(0) = u_2(0)$, we have for all $t \geq 0$

$$\|u_1(t) - u_2(t)\|_{X^0} \leq \int_0^t \|\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s))\|_{X^0} ds.$$

We claim that for all $s \in [0, t]$, there exists $C > 0$ such that

$$\|\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s))\|_{X^0} \leq C \|u_1(s) - u_2(s)\|_{X^0}.$$

Indeed, by using the Mean Value Theorem for multivariate vector-valued function $\nabla_{q_j} V$, the first component yields to

$$\begin{aligned} & \left| \left(\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s)) \right)_{p_j} \right| \\ & \leq \left| \nabla_{q_j} V(q_1(s)) - \nabla_{q_j} V(q_2(s)) \right| + \left| \nabla_{q_j} I_j(q_1(s), \alpha_1(s)) - \nabla_{q_j} I_j(q_2(s), \alpha_2(s)) \right| \\ & \stackrel{\text{Lemma 3.2.3-(ii)}}{\lesssim} \|\nabla_q \nabla_{q_j} V\|_{L^\infty} |q_1(s) - q_2(s)| + \|\alpha_1(s) - \alpha_2(s)\|_{L^2} + \|\alpha_2(s)\|_{\mathcal{G}^\sigma} |q_1(s) - q_2(s)| \\ & \lesssim \|u_1(s) - u_2(s)\|_{X^0}, \end{aligned}$$

where we have used for some bounded interval $J \subset I$

$$\|\alpha_2(s)\|_{\mathcal{G}^\sigma} \leq \sup_{s \in J} \|u_2(s)\|_{X^\sigma} < +\infty.$$

The second component yields to

$$\left| \left(\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s)) \right)_{q_j} \right| \lesssim |p_1(s) - p_2(s)| \lesssim \|u_1(s) - u_2(s)\|_{X^0}.$$

The third component yields to

$$\left\| \left(\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s)) \right)_\alpha \right\|_{L^2} \lesssim |q_1(s) - q_2(s)| \lesssim \|u_1(s) - u_2(s)\|_{X^0}.$$

Therefore, we get by combining the above three components the following estimate

$$\begin{aligned} \|u_1(t) - u_2(t)\|_{X^0} &\leq \int_0^t \|\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s))\|_{X^0} ds \\ &\leq C \int_0^t \|u_1(s) - u_2(s)\|_{X^0} ds. \end{aligned}$$

Then, by Gronwall's Lemma, we get $\|u_1(t) - u_2(t)\|_{X^0} = 0$. Thus, we get the desired result. \square

3.2.3 Local well-posedness in special subspaces

We prove here the local well-posedness of the particle-field equation in the energy space $X^{1/2}$ by means of standard fixed point argument. Let $u_0 = (p_0, q_0, \alpha_0) \in X^{1/2}$ fixed initial data, $T > 0$ a given time and R a positive real number. Consider the natural complete metric space $Y = \mathcal{C}([0, T], X^{1/2})$ associated with the norm $\|u\|_Y = \sup_{[0, T]} \|u(t)\|_{X^{1/2}}$. Define $\tilde{u}_0(t) := \Phi_t^f(u_0)$. Consider the closed ball E of Y defined as follows

$$E := B_Y(\tilde{u}_0, R) = \{u(\cdot) \in Y; \|u - \tilde{u}_0\|_Y = \sup_{[0, T]} \|u(t) - \tilde{u}_0(t)\|_{X^{1/2}} \leq R\}.$$

In other words, that mean we consider all the curves that live in the ball of radius R starting from the initial data \tilde{u}_0 . Define the map

$$\begin{aligned} \Gamma : E &\longrightarrow E \\ u(\cdot) &\longmapsto \Gamma(u)(\cdot), \end{aligned}$$

where

$$\Gamma(u)(t) := \tilde{u}_0(t) + \int_0^t \Phi_{t-s}^f \circ \mathcal{N}(u(s)) \, ds.$$

Lemma 3.2.7 (Properties of the function Γ). *Let $\sigma = \frac{1}{2}$ and assume (3.1.6) and (3.1.7) hold true. Then for $T \in \mathbb{R}_+^*$ small enough and $R \in \mathbb{R}_+^*$ large enough, the function Γ maps E into itself and owns the contraction property, i.e. there exists $0 < K < 1$ such that for all $u_1, u_2 \in E$*

$$\|\Gamma(u_1) - \Gamma(u_2)\|_Y \leq K \|u_1 - u_2\|_Y.$$

Proof. Let us first prove that Γ maps E into itself. To do that, it is sufficient to prove

$\|\Gamma(u) - \tilde{u}_0\|_Y \leq R$ whenever $\|u - \tilde{u}_0\|_Y \leq R$. Indeed, we have

$$\begin{aligned} \|\Gamma(u) - \tilde{u}_0\|_Y &= \sup_{[0,T]} \|\Gamma(u)(t) - \tilde{u}_0(t)\|_{X^{1/2}} \\ &= \sup_{[0,T]} \left\| \int_0^t \Phi_{t-s}^f \circ \mathcal{N}(u(s)) \, ds \right\|_{X^{1/2}} \\ &\leq \sup_{[0,T]} \int_0^T \|\mathcal{N}(u(s))\|_{X^{1/2}} \, ds. \end{aligned} \quad (3.2.9)$$

We claim now that $\|\mathcal{N}(u(s))\|_{X^{1/2}} \leq C_1(R)$ where $C_1(R) \in \mathbb{R}_+^*$. Indeed, we have

$$\begin{aligned} |(\mathcal{N}(u(s)))_{p_j}| &= |\nabla_{q_j} V(q(s)) + \nabla_{q_j} I(q(s), \alpha(s))| \\ &\leq \sup_{j=1}^n \|\nabla_{q_j} V\|_{L^\infty} + \frac{4\pi}{\sqrt{m_f}} \|\chi\|_{\mathcal{G}^{1/2}} \|\alpha(s)\|_{\mathcal{G}^{1/2}} \\ &\leq \sup_{j=1}^n \|\nabla_{q_j} V\|_{L^\infty} + \frac{4\pi}{\sqrt{m_f}} \|\chi\|_{\mathcal{G}^{1/2}} \|u(s)\|_{X^{1/2}} \\ &\leq \sup_{j=1}^n \|\nabla_{q_j} V\|_{L^\infty} + \frac{4\pi}{\sqrt{m_f}} \|\chi\|_{\mathcal{G}^{1/2}} (R + \|u_0\|_{X^{1/2}}); \end{aligned} \quad (3.2.10)$$

$$|(\mathcal{N}(u(s)))_{q_j}| = |\nabla f_j(p_j(s))| \lesssim |p_j(s)| \leq \|u(s)\|_{X^{1/2}} \leq c (R + \|u_0\|_{X^{1/2}}); \quad (3.2.11)$$

$$\|(\mathcal{N}(u(s)))_\alpha\|_{\mathcal{G}^{1/2}}^2 = \int_{\mathbb{R}^d} \left| -i \sum_{j=1}^n \sqrt{\omega(k)} \frac{\chi}{\sqrt{\omega(k)}} e^{-2\pi k i \cdot q_j(s)} \right|^2 dk \leq n^2 \|\chi\|_{L^2}^2. \quad (3.2.12)$$

Combining (3.2.10), (3.2.11) and (3.2.12), one obtains

$$\begin{aligned} \|\mathcal{N}(u(s))\|_{X^{1/2}} &= \left[\sum_{j=1}^n |(\mathcal{N}(u(s)))_{p_j}|^2 + |(\mathcal{N}(u(s)))_{q_j}|^2 + \|(\mathcal{N}(u(s)))_{\alpha}\|_{\mathcal{G}^{1/2}}^2 \right]^{1/2} \\ &\leq \underbrace{\sqrt{n} \left[\left(\sup_{j=1}^n \|\nabla_{q_j} V\|_{L^\infty} + \frac{4\pi}{\sqrt{m_f}} \|\chi\|_{\mathcal{G}^{1/2}} (R + \|u_0\|_{X^{1/2}}) \right)^2 + c^2 (R + \|u_0\|_{X^{1/2}})^2 + n \|\chi\|_{L^2}^2 \right]^{1/2}}_{:=C_1(R)}. \end{aligned}$$

Plugging the above equality in (3.2.9), we find

$$\|\Gamma(u) - \tilde{u}_0\|_Y \leq T C_1(R).$$

Then choose T small enough such that $T C_1(R) \leq R$.

We now prove that Γ verifies the contraction property. Let $u_1, u_2 \in E$. We have

$$\begin{aligned} \|\Gamma(u_1) - \Gamma(u_2)\|_Y &= \sup_{[0,T]} \|\Gamma(u_1)(t) - \Gamma(u_2)(t)\|_{X^{1/2}} \\ &= \sup_{[0,T]} \left\| \int_0^t \Phi_{t-s}^f (\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s))) ds \right\|_{X^{1/2}} \\ &\leq \sup_{[0,T]} \int_0^T \|\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s))\|_{X^{1/2}} ds \end{aligned}$$

Claim that there exists a constant $C_2(R) \in \mathbb{R}_+^*$ such that

$$\|\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s))\|_{X^{1/2}} \leq C_2(R) \|u_1(s) - u_2(s)\|_{X^{1/2}} \quad (3.2.13)$$

If so, we get

$$\|\Gamma(u_1) - \Gamma(u_2)\|_Y \leq T C_2(R) \|u_1 - u_2\|_Y.$$

By this way, we choose T small enough such that $T C_2(R) < 1$. Come back now to prove

the claim [\(3.2.13\)](#). Indeed, exploit the proof of Lemma [3.2.3](#)(ii), we get

$$\begin{aligned} & \left| \left(\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s)) \right)_{p_j} \right| \\ & \lesssim \left[\|\nabla_q \nabla_{q_j} V\|_{L^\infty} + (R + \|u_0\|_{X^{1/2}}) \right] |q_1(s) - q_2(s)| + \|\alpha_1(s) - \alpha_2(s)\|_{\mathcal{G}^{1/2}} \end{aligned} \quad (3.2.14)$$

$$\lesssim C_3(R) \|u_1(s) - u_2(s)\|_{X^{1/2}};$$

$$\left| \left(\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s)) \right)_{q_j} \right| \lesssim \|u_1(s) - u_2(s)\|_{X^{1/2}}; \quad (3.2.15)$$

$$\begin{aligned} & \left\| \left(\mathcal{N}(u_1(s)) - \mathcal{N}(u_2(s)) \right)_\alpha \right\|_{\mathcal{G}^{1/2}}^2 \\ & = \int_{\mathbb{R}^d} \left| -i \sum_{j=1}^n \sqrt{\omega(k)} \frac{\chi(k)}{\sqrt{\omega(k)}} \left(e^{-2\pi k i \cdot q_{1j}(s)} - e^{-2\pi k i \cdot q_{2j}(s)} \right) \right|^2 dk \end{aligned} \quad (3.2.16)$$

$$\leq \int_{\mathbb{R}^d} \left| -i \sum_{j=1}^n \chi(k) (2\pi \omega(k)) |q_{1j}(s) - q_{2j}(s)| \right|^2 dk$$

$$\lesssim \|\chi\|_{\mathcal{G}^1} \|u_1(s) - u_2(s)\|_{X^{1/2}}^2.$$

□

3.3 The quantum system

In Subsection [3.3.1](#), we prove some quantum estimates, which we then apply in Subsection [3.3.2](#) to establish the self-adjointness of the Nelson Hamiltonian using the Kato-Rellich theorem. Lastly, in Subsection [3.3.3](#), we discuss a dynamical equation for the quantum system.

3.3.1 Quantum estimates

Our initial focus here is on providing the reader with estimates that are necessary to establish the self-adjointness of the Nelson Hamiltonian. Denote by $\mathcal{L}(\mathcal{H})$ the set of all bounded operator and by $\mathcal{L}^1(\mathcal{H})$ the class of trace-class operators on \mathcal{H} .

Lemma 3.3.1 (Creation-Annihilation estimates). *Let*

$$F \in \mathcal{L}\left(L^2(\mathbb{R}^{dn}, dX_n), L^2(\mathbb{R}^{dn}, dX_n) \otimes L^2(\mathbb{R}^d, dk)\right).$$

(i) For every $\psi \in D(\hat{N}_\hbar^{1/2})$, we have

$$\|\hat{a}_\hbar(F)\psi\|_{\mathcal{H}} \leq \|\hat{N}_\hbar^{1/2}\psi\|_{\mathcal{H}} \|F\|_{\mathcal{L}(L^2, L^2 \otimes L^2)}; \quad (3.3.1)$$

$$\|\hat{a}_\hbar^*(F)\psi\|_{\mathcal{H}} \leq \|(\hat{N}_\hbar + 1)^{1/2}\psi\|_{\mathcal{H}} \|F\|_{\mathcal{L}(L^2, L^2 \otimes L^2)}. \quad (3.3.2)$$

(ii) For all $\psi \in D((\hat{H}_{02})^{1/2})$, we have

$$\|\hat{a}_\hbar(F)\psi\|_{\mathcal{H}} \leq \left\| \frac{F}{\sqrt{\omega}} \right\|_{\mathcal{L}} \|(\hat{H}_{02} + 1)^{1/2}\psi\|_{\mathcal{H}}; \quad (3.3.3)$$

$$\|\hat{a}_\hbar^*(F)\psi\|_{\mathcal{H}}^2 \leq \left\| \frac{F}{\sqrt{\omega}} \right\|_{\mathcal{L}}^2 \|(\hat{H}_{02} + 1)^{1/2}\psi\|_{\mathcal{H}}^2 + \hbar \|F\|_{\mathcal{L}}^2 \|\psi\|^2. \quad (3.3.4)$$

Proof. Let K_m and X_n as indicated in (3.1.2). For (i)-(3.3.1), we have with $\|\cdot\|_{\mathcal{L}} \equiv \|\cdot\|_{\mathcal{L}(L^2, L^2 \otimes L^2)}$

$$\begin{aligned} \|\hat{a}_\hbar(F)\psi\|_{\mathcal{H}}^2 &= \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| [\hat{a}_\hbar(F)\psi]^m(X_n, K_m) \right|^2 dX_n dK_m \\ &= \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| \int_{\mathbb{R}^d} \sqrt{\hbar(m+1)} \overline{F(k)} \psi^{m+1}(X_n, K_m, k) dk \right|^2 dX_n dK_m \\ &\leq_{c-s} \|F\|_{\mathcal{L}}^2 \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left[\int_{\mathbb{R}^d} \left| \sqrt{\hbar(m+1)} \psi^{m+1}(X_n, K_m, k) \right|^2 dk \right] dX_n dK_m \\ &\leq \|F\|_{\mathcal{L}}^2 \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{d(m+1)}} \hbar(m+1) \left| \psi^{m+1}(X_n, K_{m+1}) \right|^2 dX_n dK_{m+1} \\ &\leq \|F\|_{\mathcal{L}}^2 \|\hat{N}_\hbar^{1/2}\psi\|_{\mathcal{H}}^2. \end{aligned}$$

For (i)-(3.3.2), we have

$$\begin{aligned} \|\hat{a}_\hbar^*(F)\psi\|_{\mathcal{H}}^2 &= \langle \psi, \hat{a}_\hbar(F)\hat{a}_\hbar^*(F)\psi \rangle \\ &= \langle \psi, [\hat{a}_\hbar(F), \hat{a}_\hbar^*(F)] + \hat{a}_\hbar^*(F)\hat{a}_\hbar(F) \rangle \psi \\ &\leq \hbar \|F\|_{\mathcal{L}}^2 \|\psi\|^2 + \|\hat{a}_\hbar(F)\psi\|^2 \\ &\lesssim \|F\|_{\mathcal{L}}^2 \|(\hat{N}_\hbar + 1)^{1/2}\psi\|^2. \end{aligned}$$

For (ii)-(3.3.3), we have

$$\begin{aligned}
\|\hat{a}_\hbar(F) \psi\|_{\mathcal{H}}^2 &= \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| [\hat{a}_\hbar(F) \psi]^m(X_n, K_m) \right|^2 dK_m dX_n \\
&= \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| \int_{\mathbb{R}^d} \sqrt{\hbar(m+1)} \overline{F(k)} \psi^{m+1}(X_n, K_m, k) dk \right|^2 dK_m dX_n \\
&= \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left[\int_{\mathbb{R}^d} \left| \sqrt{\hbar(m+1)} \frac{\overline{F(k)}}{\sqrt{\omega(k)}} \sqrt{\omega(k)} \psi^{m+1}(X_n, K_m, k) \right| dk \right]^2 dK_m dX_n \\
&\stackrel{c-s}{\leq} \left\| \frac{F}{\sqrt{\omega}} \right\|_{\mathcal{L}}^2 \sum_{m \geq 0} \int_{\mathbb{R}^{dm}} \left[\int_{\mathbb{R}^d} \left| \sqrt{\hbar(m+1)} \sqrt{\omega(k)} \psi^{m+1}(X_n, K_m, k) \right| dk \right]^2 dk dK_m dX_n \\
&\leq \left\| \frac{F}{\sqrt{\omega}} \right\|_{\mathcal{L}}^2 \sum_{m \geq 0} \int_{\mathbb{R}^{dm}} \int_{\mathbb{R}^d} \hbar(m+1) \omega(k) \left| \psi^{m+1}(X_n, K_m, k) \right|^2 dk dK_m dX_n \\
&\leq \left\| \frac{F}{\sqrt{\omega}} \right\|_{\mathcal{L}}^2 \|(\hat{H}_{02})^{1/2} \psi\|_{\mathcal{H}}^2.
\end{aligned}$$

Similar discussion as for (i)-(3.3.2) works perfectly to prove (ii)-(3.3.4). \square

Lemma 3.3.2 (Field and number estimates). *For all $\psi \in D(\hat{H}_{02})$, we have*

1. $\|\hat{H}_{02} \psi\|_{\mathcal{H}} \geq m_f \|\hat{N}_\hbar \psi\|_{\mathcal{H}}$;
2. $\|(\hat{N}_\hbar + 1)^{1/2} \psi\|_{\mathcal{H}} \leq \varepsilon \|\hat{N}_\hbar \psi\|_{\mathcal{H}} + b(\varepsilon) \|\psi\|_{\mathcal{H}}$ for some $\varepsilon < 1$ and $b(\varepsilon) < +\infty$.

Proof. For (1), we have

$$\begin{aligned}
\|\hat{H}_{02} \psi\|_{\mathcal{H}}^2 &= \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| [\hat{H}_{02} \psi]^m(X_n, k_1, \dots, k_m) \right|^2 dX_n dk_1 \cdots dk_m \\
&= \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| \hbar \sum_{l=1}^m \underbrace{w(k_l)}_{\geq m_f} \psi^m(X_n, k_1, \dots, k_m) \right|^2 dX_n dk_1 \cdots dk_m \\
&\geq \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| \hbar m m_f \psi^m(X_n, k_1, \dots, k_m) \right|^2 dX_n dk_1 \cdots dk_m \\
&\geq m_f^2 \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \hbar^2 m^2 \left| \psi^m(X_n, k_1, \dots, k_m) \right|^2 dX_n dk_1 \cdots dk_m \\
&\geq m_f^2 \|\hat{N}_\hbar \psi\|_{\mathcal{H}}^2.
\end{aligned}$$

For (2), we have

$$\begin{aligned} \|(\hat{N}_{\hbar} + 1)^{1/2}\psi\|_{\mathcal{H}}^2 &= \langle \psi, (\hat{N}_{\hbar} + 1)\psi \rangle = \langle \psi, \hat{N}_{\hbar}\psi \rangle + \|\psi\|_{\mathcal{H}}^2 \\ &\leq \|\psi\|_{\mathcal{H}} \|\hat{N}_{\hbar}\psi\|_{\mathcal{H}} + \|\psi\|_{\mathcal{H}}^2 \\ &\leq \frac{\tilde{\varepsilon}}{2} \|\hat{N}_{\hbar}\psi\|_{\mathcal{H}}^2 + \left(1 + \frac{1}{2\tilde{\varepsilon}}\right) \|\psi\|_{\mathcal{H}}^2. \end{aligned}$$

This implies that, by choosing appropriate $\tilde{\varepsilon}$, there exists $\varepsilon < 1$ and $b(\varepsilon) < +\infty$ such that

$$\|(\hat{N}_{\hbar} + 1)^{1/2}\psi\|_{\mathcal{H}} \leq \varepsilon \|\hat{N}_{\hbar}\psi\|_{\mathcal{H}} + b(\varepsilon) \|\psi\|_{\mathcal{H}}.$$

□

Below, we give an important inequality between \hat{H} and \hat{H}_0 which is useful for the coming discussions. Suppose $\hat{H} + a > 0$ and $\hat{H}_0 + b > 0$ with some $a, b \in \mathbb{R}$.

Lemma 3.3.3 (Equivalence between \hat{H} and \hat{H}_0). *Assume (3.1.6) and $\omega^{-\frac{1}{2}}\chi \in L^2(\mathbb{R}^d, dk)$ are satisfied. Then there exists $c, C > 0$ independent of \hbar such that for all $\hbar \in (0, 1)$ and for all $\psi \in D(\hat{H}_0)$*

$$c \langle \psi, (\hat{H} + a)\psi \rangle \leq \langle \psi, (\hat{H}_0 + b)\psi \rangle \leq C \langle \psi, (\hat{H} + a)\psi \rangle. \quad (3.3.5)$$

In particular, for all $\psi \in D((\hat{H}_0)^{1/2})$, we have

$$c \|(\hat{H} + a)^{1/2}\psi\| \leq \|(\hat{H}_0 + b)^{1/2}\psi\| \leq C \|(\hat{H} + a)^{1/2}\psi\|. \quad (3.3.6)$$

Proof. Note that first we have the following estimates on V below

$$-\|V\|_{L^\infty} \leq V \leq \|V\|_{L^\infty} \Rightarrow \|V\|_{L^\infty} + V \geq 0.$$

We have

$$\begin{aligned}
 \langle \psi, (\hat{H} + a) \psi \rangle &= \langle \psi, \hat{H}_0 \psi \rangle + \langle \psi, V \psi \rangle + \langle \psi, \hat{H}_1 \psi \rangle + \langle \psi, a \psi \rangle \\
 &\leq \langle \psi, \hat{H}_0 \psi \rangle + \|V\|_{L^\infty} \langle \psi, \psi \rangle + \|\psi\| \|\hat{H}_1 \psi\| + \langle \psi, a \psi \rangle \\
 &\leq \langle \psi, \hat{H}_0 \psi \rangle + \|V\|_{L^\infty} \langle \psi, \psi \rangle + 2\|\omega^{-1/2} \chi\|_{L^2} \|\psi\| \|(\hat{H}_0 + 1)^{1/2} \psi\| + \langle \psi, a \psi \rangle \\
 &\leq \langle \psi, \hat{H}_0 \psi \rangle + \|V\|_{L^\infty} \langle \psi, \psi \rangle + \|\omega^{-1/2} \chi\|_{L^2}^2 \|\psi\|^2 + \|(\hat{H}_0 + 1)^{1/2} \psi\|^2 + \langle \psi, a \psi \rangle \\
 &\leq \langle \psi, \left(2\hat{H}_0 + \|V\|_{L^\infty} + \|\omega^{-1/2} \chi\|_{L^2}^2 + 1 + a \right) \psi \rangle \\
 &\leq c \langle \psi, (\hat{H}_0 + b) \psi \rangle,
 \end{aligned}$$

where $c \in \mathbb{R}_*^+$ depends on $\|V\|_{L^\infty}$, $\|\omega^{-1/2} \chi\|_{L^2}^2$ and independent on \hbar .

On the reverse side, we have

$$\begin{aligned}
 \langle \psi, \hat{H}_0 \psi \rangle &= \langle \psi, \hat{H}_{01} \psi \rangle + \langle \psi, \hat{H}_{02} \psi \rangle \\
 &\leq \langle \psi, (\hat{H}_{01} + V + \|V\|_{L^\infty}) \psi \rangle + \langle \psi, \hat{H}_{02} \psi \rangle.
 \end{aligned}$$

Then by using Lemma [3.3.1](#), we can also assert that for $\varepsilon > 0$, there exists $c_\varepsilon > 0$ (depends on the norm $\|\omega^{-1} \chi\|_{L^2}$) such that

$$|\langle \psi, \hat{H}_1 \psi \rangle| \leq \|\omega^{-1} \chi\|_{L^2} \left[\frac{1}{\varepsilon} \langle \psi, \psi \rangle + \varepsilon \langle \psi, (\hat{H}_{02} + 1) \psi \rangle \right].$$

This means as quadratic form there exists a constant $c_\varepsilon \in \mathbb{R}_+^*$ which depends on ε such that

$$\begin{aligned}
 \hat{H}_{02} + \hat{H}_1 &= (1 - \varepsilon) \hat{H}_{02} + \varepsilon \hat{H}_{02} + \hat{H}_1 \\
 &\geq (1 - \varepsilon) \hat{H}_{02} - c_\varepsilon \mathbb{1}
 \end{aligned}$$

This implies

$$\hat{H}_{02} \leq \frac{1}{1 - \varepsilon} \left[\hat{H}_{02} + \hat{H}_1 + c_\varepsilon \right].$$

We conclude that there exists $C > 0$ such that

$$\begin{aligned}
 \langle \psi, (\hat{H}_0 + b) \psi \rangle &\leq \langle \psi, (\hat{H}_{01} + V + \|V\|_{L^\infty}) \psi \rangle + \langle \psi, \frac{1}{1 - \varepsilon} \left[\hat{H}_{02} + \hat{H}_1 + c_\varepsilon \right] \psi \rangle \\
 &\leq C \langle \psi, (\hat{H} + a) \psi \rangle.
 \end{aligned}$$

□

3.3.2 Self-adjointness of Nelson Hamiltonian

We prove here the self-adjointness of the Nelson Hamiltonian using the estimates provided in the previous section.

Proposition 3.3.4 (Self-adjointness of the Nelson Hamiltonian). *Assume (3.1.6) and $\omega^{-\frac{1}{2}}\chi \in L^2(\mathbb{R}^d, dk)$. Then, the operator $\hat{H} : \mathcal{H} \rightarrow \mathcal{H}$ is self-adjoint operator on $D(\hat{H}_0) = D(\hat{H})$.*

Proof. We have first with some $C_{\tilde{\varepsilon}} \in \mathbb{R}_+^*$

$$\begin{aligned} \|\hat{H}_1\psi\|_{\mathcal{H}} &= \|\hat{a}_h(G)\psi + \hat{a}_h^*(G)\psi\|_{\mathcal{H}} \\ &\leq \|\hat{a}_h(G)\psi\|_{\mathcal{H}} + \|\hat{a}_h^*(G)\psi\|_{\mathcal{H}} \\ &\leq 2\|G\| \|(\hat{N}_h + 1)^{1/2}\psi\|_{\mathcal{H}} \\ &\leq 2n \left\| \frac{\chi}{\sqrt{\omega}} \right\|_{L^2} \left[\tilde{\varepsilon} \|\hat{N}_h\psi\|_{\mathcal{H}} + b(\tilde{\varepsilon}) \|\psi\|_{\mathcal{H}} \right] \\ &\leq \frac{2n}{m_f} \left\| \frac{\chi}{\sqrt{\omega}} \right\|_{L^2} \tilde{\varepsilon} \|\hat{H}_{02}\psi\|_{\mathcal{H}} + C_{\tilde{\varepsilon}} \|\psi\|_{\mathcal{H}}, \end{aligned}$$

where we have used the estimates in Lemmas 3.3.1 and 3.3.2. Choose $\tilde{\varepsilon}$ small enough such that $2n \left\| \frac{\chi}{\sqrt{\omega}} \right\|_{L^2} \tilde{\varepsilon} < m_f$. We conclude that there exists $\varepsilon < 1$ and $C_{\varepsilon} \in \mathbb{R}_+^*$ such that

$$\|\hat{H}_1\psi\|_{\mathcal{H}} \leq \varepsilon \|\hat{H}_{02}\psi\|_{\mathcal{H}} + C_{\varepsilon} \|\psi\|_{\mathcal{H}}.$$

The operator \hat{H}_{02} is self-adjoint operator and \hat{H}_1 is symmetric operator. Thus by Kato-Rellich theorem, $\hat{H}_{02} + \hat{H}_1$ is self-adjoint on $D(\hat{H}_{02})$. Remark also that by (3.1.6), we have

$$\|V\psi\|_{L^2} \leq \varepsilon' \|\hat{H}_{01}\psi\|_{L^2} + b \|\psi\|_{L^2}.$$

Then, again by Kato-Rellich theorem, $\hat{H}_{01} + V$ is self-adjoint on $D(\hat{H}_{01}) \subseteq D(V)$. We also have as a consequence of Kato-Rellich theorem that $\hat{H}_{01} + V \geq -c\mathbb{1}$. This means $\hat{H}_{01} + V + c\mathbb{1} \geq 0$. This gives

$$\begin{aligned} \|\hat{H}_{02}\psi\|_{\mathcal{H}}^2 &= \langle \psi, \hat{H}_{02}^2 \psi \rangle \\ &\leq \langle \psi, (\hat{H}_{02} + \hat{H}_{01} + V + c\mathbb{1})^2 \psi \rangle \\ &\leq \|(\hat{H}_{02} + \hat{H}_{01} + V + c\mathbb{1})\psi\|_{\mathcal{H}}^2. \end{aligned}$$

We conclude

$$\|\hat{H}_1\psi\|_{\mathcal{H}} \leq \varepsilon \left\| \left(\hat{H}_{02} + \hat{H}_{01} + V + c\mathbb{1} \right) \psi \right\|_{\mathcal{H}} + C(\varepsilon) \|\psi\|_{\mathcal{H}}.$$

Remark that $\hat{H}_{01} + V + c\mathbb{1}$ commutes with \hat{H}_{02} . This means $\hat{H}_{01} + V + c\mathbb{1} + \hat{H}_{02}$ is self adjoint in $D(\hat{H}_{01} + V + c\mathbb{1} + \hat{H}_{02}) = D(\hat{H}_{01} + \hat{H}_{02})$. By Kato-Rellich theorem, \hat{H} is self adjoint on $D(\hat{H}_0) = D(\hat{H}_{01} + \hat{H}_{02}) = D(\hat{H}_{01}) \cap D(\hat{H}_{02})$. \square

3.3.3 The dynamical equation

The primary objective of this section is to determine the dynamical equation of the quantum system. This equation should converge, as \hbar approaches zero, to a classical dynamical equation that involves the inverse Fourier transform of a specific Wigner measure. To achieve this, in Paragraph [3.3.3.1](#), we derive the Duhamel formula for the quantum system. Then, in Paragraph [3.3.3.2](#), we expand the commutator within this Duhamel formula.

3.3.3.1 Duhamel formula

We begin by introducing the Weyl Heisenberg operator, which acts on the entire interacting Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{dn}, \mathbb{C}) \otimes \Gamma_s(\mathcal{G}^0)$, as the following map

$$\xi = (z, \alpha) \in X^0 \equiv \mathbb{C}^{dn} \oplus \mathcal{G}^0 \longmapsto \mathcal{W}(\xi) \equiv \mathcal{W}(z, \alpha) := W_1(z) \otimes W_2(\alpha) \quad (3.3.7)$$

where we have introduced with $\Im m\langle z, z' \rangle = q \cdot p' - p \cdot q'$, $\forall (p, q), (p', q') \in \mathbb{R}^{dn} \times \mathbb{R}^{dn}$:

- the Weyl operator on the particle variable which is defined, for all $(p, q) \in \mathbb{R}^{dn} \times \mathbb{R}^{dn}$ and for $z = q + ip \in \mathbb{C}^{dn}$, as follows:

$$W_1(z) = e^{i \Im m\langle \hat{q} + i\hat{p}, z \rangle} = e^{i(p \cdot \hat{q} - q \cdot \hat{p})}; \quad (3.3.8)$$

- the Weyl operator on the Fock space $\Gamma_s(L^2(\mathbb{R}^d, \mathbb{C}))$ which is defined for any $\alpha \in L^2(\mathbb{R}^d, \mathbb{C})$ as follows:

$$W_2(\alpha) = e^{\frac{i}{\sqrt{2}}(\hat{a}_\hbar(\alpha) + \hat{a}_\hbar^*(\alpha))}. \quad (3.3.9)$$

The above operators satisfy the following commutation relations

$$W_1(z)W_1(z') = e^{-i\frac{\hbar}{2}\Im m\langle z, z' \rangle} W_1(z + z'), \quad \forall z, z' \in \mathbb{C}^{dn}, \quad (3.3.10)$$

$$W_2(\alpha)W_2(\beta) = e^{-i\frac{\hbar}{2}\Im m\langle \alpha, \beta \rangle_{L^2}} W_2(\alpha + \beta), \quad \forall \alpha, \beta \in \mathcal{G}^0. \quad (3.3.11)$$

Below, we mention several crucial estimates that are necessary to establish a Duhamel formula for the evolved states of a quantum system. The prove of the following identities requires the estimates derived in Lemma [3.3.1](#), we refer the reader to [\[AFH22\]](#) for more details on the proof.

Lemma 3.3.5 (Weyl Heisenberg estimates). *There exists a constant $C > 0$ such that for any $\hbar \in (0, 1)$*

(i) *for any $\alpha \in L^2(\mathbb{R}^d, \mathbb{C})$ and any $\psi \in D(\hat{N}_\hbar)$*

$$\|(\hat{N}_\hbar)^{1/2} W_2(\alpha) \psi\|_{\Gamma_s} \leq C \|(\hat{N}_\hbar + 1)^{1/2} \psi\|_{\Gamma_s};$$

(ii) *for any $\alpha \in \mathcal{G}^{1/2}$ and any $\psi \in D(\hat{H}_0)$*

$$\|(\hat{H}_0)^{1/2} W_2(\alpha) \psi\|_{\Gamma_s} \leq C \|(\hat{H}_0 + 1)^{1/2} \psi\|_{\Gamma_s};$$

(iii) *for any $z \in \mathbb{C}^{dn}$ and any $\psi \in D((\hat{p}^2 + \hat{q}^2)^{1/2})$*

$$\|(\hat{p}^2 + \hat{q}^2)^{1/2} W_1(z) \psi\|_{L^2(\mathbb{R}^{dn})} \leq C \|(\hat{p}^2 + \hat{q}^2 + 1)^{1/2} \psi\|_{L^2(\mathbb{R}^{dn})}.$$

The matter here is to understand the propagation of the density matrices ϱ_\hbar on the Hilbert space \mathcal{H} . To this end, we define

$$\varrho_\hbar(t) = e^{-i\frac{t}{\hbar}\hat{H}} \varrho_\hbar e^{i\frac{t}{\hbar}\hat{H}} \quad \text{and} \quad \tilde{\varrho}_\hbar(t) = e^{i\frac{t}{\hbar}\hat{H}_{02}} \varrho_\hbar(t) e^{-i\frac{t}{\hbar}\hat{H}_{02}}. \quad (3.3.12)$$

In order to prove the main results Theorems [3.1.1](#) and [3.1.3](#), it is necessary to identify the Wigner measures of the evolved state $\varrho_\hbar(t)$. However, the complexity inherited from the interaction between particles and field makes direct identification unfeasible. Instead, we use the interaction representation $\tilde{\varrho}_\hbar(t)$, which helps us overcome several nonlinearities that could lead to imprecise formulas. Furthermore, recovering the Wigner measures of $\varrho_\hbar(t)$ from those of $\tilde{\varrho}_\hbar(t)$ is not difficult. To this end, we start below derivation of the quantum dynamical system.

Proposition 3.3.6. Assume that (3.1.6) and $\omega^{1/2}\chi \in L^2(\mathbb{R}^d, dk)$. Let $(\varrho_h)_{h \in (0,1)}$ be a family of density matrices satisfying (3.1.8) and (3.1.9). Then for all $\xi \in X^{1/2}$, for all $h \in (0, 1)$ and for all $t, t_0 \in \mathbb{R}$, we have

$$\mathrm{Tr}\left[\mathcal{W}(\xi)\tilde{\varrho}_h(t)\right] = \mathrm{Tr}\left[\mathcal{W}(\xi)\tilde{\varrho}_h(t_0)\right] - \frac{i}{\hbar} \int_{t_0}^t \mathrm{Tr}\left(\left[\mathcal{W}(\xi), \hat{H}_I(s)\right] \tilde{\varrho}_h(s)\right) ds, \quad (3.3.13)$$

where

$$\hat{H}_I(s) := e^{i\frac{s}{\hbar}\hat{H}_{02}}(\hat{H} - \hat{H}_{02})e^{-i\frac{s}{\hbar}\hat{H}_{02}}. \quad (3.3.14)$$

Proof. By Duhamel's formula, we have

$$\mathrm{Tr}\left[\mathcal{W}(\xi)\tilde{\varrho}_h(t)\right] = \mathrm{Tr}\left[\mathcal{W}(\xi)\tilde{\varrho}_h(t_0)\right] + \int_{t_0}^t \frac{d}{ds} \mathrm{Tr}\left[\mathcal{W}(\xi)\tilde{\varrho}_h(s)\right] ds.$$

We have also

$$\frac{d}{dt} \mathrm{Tr}\left[\mathcal{W}(\xi)\tilde{\varrho}_h(t)\right] = \lim_{s \rightarrow t} \frac{\mathrm{Tr}\left[\mathcal{W}(\xi)\left(\tilde{\varrho}_h(t) - \tilde{\varrho}_h(s)\right)\right]}{t - s}.$$

Let $S = (\hat{H}_0 + 1)^{1/2}$. We start by

$$\begin{aligned} & \mathrm{Tr}\left[\mathcal{W}(\xi)\left(\tilde{\varrho}_h(t) - \tilde{\varrho}_h(s)\right)\right] \\ &= \mathrm{Tr}\left[\mathcal{W}(\xi)\left(e^{i\frac{t}{\hbar}\hat{H}_{02}}e^{-i\frac{t}{\hbar}\hat{H}}\varrho_h e^{i\frac{t}{\hbar}\hat{H}}e^{-i\frac{t}{\hbar}\hat{H}_{02}} - e^{i\frac{s}{\hbar}\hat{H}_{02}}e^{-i\frac{s}{\hbar}\hat{H}}\varrho_h e^{i\frac{s}{\hbar}\hat{H}}e^{-i\frac{s}{\hbar}\hat{H}_{02}}\right)\right] \\ &= \mathrm{Tr}\left[\mathcal{W}(\xi)\left(e^{i\frac{t}{\hbar}\hat{H}_{02}}e^{-i\frac{t}{\hbar}\hat{H}} - e^{i\frac{s}{\hbar}\hat{H}_{02}}e^{-i\frac{s}{\hbar}\hat{H}}\right)\varrho_h e^{i\frac{t}{\hbar}\hat{H}}e^{-i\frac{t}{\hbar}\hat{H}_{02}}\right] \\ & \quad + \mathrm{Tr}\left[\mathcal{W}(\xi)e^{i\frac{s}{\hbar}\hat{H}_{02}}e^{-i\frac{s}{\hbar}\hat{H}}\tilde{\varrho}_h\left(e^{i\frac{t}{\hbar}\hat{H}}e^{-i\frac{t}{\hbar}\hat{H}_{02}} - e^{i\frac{s}{\hbar}\hat{H}}e^{-i\frac{s}{\hbar}\hat{H}_{02}}\right)\right] \\ &= \mathrm{Tr}\left[S^{-1}\mathcal{W}(\xi)S S^{-1}\left(e^{i\frac{t}{\hbar}\hat{H}_{02}}e^{-i\frac{t}{\hbar}\hat{H}} - e^{i\frac{s}{\hbar}\hat{H}_{02}}e^{-i\frac{s}{\hbar}\hat{H}}\right)\varrho_h S S^{-1}e^{i\frac{t}{\hbar}\hat{H}}S S^{-1}e^{-i\frac{t}{\hbar}\hat{H}_{02}}S\right] \\ & \quad + \mathrm{Tr}\left[\mathcal{W}(\xi)e^{i\frac{s}{\hbar}\hat{H}_{02}}e^{-i\frac{s}{\hbar}\hat{H}}\tilde{\varrho}_h S S^{-1}\left(e^{i\frac{t}{\hbar}\hat{H}}e^{-i\frac{t}{\hbar}\hat{H}_{02}} - e^{i\frac{s}{\hbar}\hat{H}}e^{-i\frac{s}{\hbar}\hat{H}_{02}}\right)\right] \end{aligned}$$

Remark that each step makes sense. Indeed, we have that

$$\mathcal{W}(\xi), e^{i\frac{t}{\hbar}\hat{H}_{02}}, e^{i\frac{t}{\hbar}\hat{H}}, S^{-1}e^{i\frac{t}{\hbar}\hat{H}}S, S^{-1}e^{-i\frac{t}{\hbar}\hat{H}_{02}}S \in \mathcal{L}(\mathcal{H}), \quad \varrho_h, \varrho_h(\hat{H}_0 + 1) \in \mathcal{L}^1(\mathcal{H}).$$

We have also

$$\begin{aligned} \lim_{s \rightarrow t} S^{-1} \frac{\left(e^{i\frac{t}{\hbar}\hat{H}} e^{-i\frac{t}{\hbar}\hat{H}_{02}} - e^{i\frac{s}{\hbar}\hat{H}} e^{-i\frac{s}{\hbar}\hat{H}_{02}} \right)}{t-s} &= \frac{i}{\hbar} S^{-1} e^{i\frac{t}{\hbar}\hat{H}} (\hat{H} - \hat{H}_{02}) e^{-i\frac{t}{\hbar}\hat{H}_{02}}; \\ \lim_{s \rightarrow t} S^{-1} \frac{\left(e^{i\frac{t}{\hbar}\hat{H}_{02}} e^{-i\frac{t}{\hbar}\hat{H}} - e^{i\frac{s}{\hbar}\hat{H}_{02}} e^{-i\frac{s}{\hbar}\hat{H}} \right)}{t-s} &= -\frac{i}{\hbar} S^{-1} e^{i\frac{t}{\hbar}\hat{H}_{02}} (\hat{H} - \hat{H}_{02}) e^{-i\frac{t}{\hbar}\hat{H}}. \end{aligned}$$

Plugging these limits in the Duhamel's formula, we get the desired result. \square

3.3.3.2 The commutator expansion

The aim of this subsection is to expand the commutator $[\mathcal{W}(\xi), \hat{H}_I(s)]$ in the above Duhamel formula (3.3.13) in terms of the parameter $\hbar \in (0, 1)$.

Lemma 3.3.7 (Time evolved equation of $\hat{H}_I(s)$). *For any $s \in \mathbb{R}$, the time evolved interaction term $\hat{H}_I(s)$ takes the following form*

$$\hat{H}_I(s) = \sum_{j=1}^n f_j(\hat{p}_j) + V(\hat{q}) + \sum_{j=1}^n \hat{a}_\hbar(g_j(s)) + \hat{a}_\hbar^*(g_j(s)), \quad (3.3.15)$$

where we have introduced

$$g_j(s) \equiv g_j(s)(\hat{q}) := \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot \hat{q}_j + i s \omega(k)}. \quad (3.3.16)$$

Proof. We have

$$\hat{H} - \hat{H}_{02} = \sum_{j=1}^n f_j(\hat{p}_j) + V(\hat{q}) + \sum_{j=1}^n \hat{a}_\hbar(g_j) + \hat{a}_\hbar^*(g_j),$$

where the function g_j is given by

$$g_j \equiv g_j(\hat{q}) := \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot \hat{q}_j}. \quad (3.3.17)$$

Then, we have with $\hat{q} = (\hat{q}_1, \dots, \hat{q}_n) \in \mathbb{R}^{dn}$

$$\hat{H}_I(s) = e^{i\frac{s}{\hbar}\hat{H}_{02}} \left(\sum_{j=1}^n f_j(\hat{p}_j) + V(\hat{q}) + \sum_{j=1}^n \hat{a}_\hbar(g_j) + \hat{a}_\hbar^*(g_j) \right) e^{-i\frac{s}{\hbar}\hat{H}_{02}}.$$

It is sufficient then to look at the following identity

$$e^{i\frac{s}{\hbar}\hat{H}_{02}} \hat{a}^\#(g_j) e^{-i\frac{s}{\hbar}\hat{H}_{02}} = \hat{a}^\#(g_j(s)).$$

□

Now, since the Weyl operator $\mathcal{W}(\xi)$ is a unitary operator, we have

$$\frac{1}{\hbar} [\mathcal{W}(\xi), \hat{H}_I(s)] = \frac{1}{\hbar} \left(\mathcal{W}(\xi) \hat{H}_I(s) \mathcal{W}(\xi)^* - \hat{H}_I(s) \right) \mathcal{W}(\xi). \quad (3.3.18)$$

Lemma 3.3.8 (Expression for the commutators). *For any $s \in \mathbb{R}$ and $\xi = (p_0, q_0, \alpha_0) \in X^{1/2}$, the following holds true with $q_0 = (q_{01}, \dots, q_{0n})$ and $p_0 = (p_{01}, \dots, p_{0n})$*

$$\begin{aligned} \mathcal{W}(\xi) \hat{H}_I(s) \mathcal{W}(\xi)^* &= \sum_{j=1}^n f_j(\hat{p}_j - \hbar p_{0j}) + V(\hat{q} - \hbar q_0) \\ &+ \sum_{j=1}^n \hat{a}_\hbar(\tilde{g}_j(s)) + \hat{a}_\hbar^*(\tilde{g}_j(s)) + \frac{i\hbar}{\sqrt{2}} \left(\langle \alpha_0, \tilde{g}_j(s) \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} - \langle \tilde{g}_j(s), \alpha_0 \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} \right), \end{aligned}$$

where we have introduced

$$\tilde{g}_j(s) := \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot (\hat{q}_j - \hbar q_{0j}) + i s \omega(k)} = e^{2\pi i k \cdot q_{0j} \hbar} g_j(s). \quad (3.3.19)$$

Proof. The results follow from the following identities

$$W_1(z_0) \hat{q}_j^\nu W_1(z_0)^* = \hat{q}_j^\nu - \hbar q_{0j}^\nu, \quad (3.3.20)$$

$$W_1(z_0) \hat{p}_j^\nu W_1(z_0)^* = \hat{p}_j^\nu - \hbar p_{0j}^\nu, \quad (3.3.21)$$

$$W_2(\alpha_0) \hat{a}_\hbar^*(f) W_2(\alpha)^* = \hat{a}_\hbar^*(f) + \frac{i\hbar}{\sqrt{2}} \langle \alpha_0, f \rangle_{L^2}, \quad (3.3.22)$$

$$W_2(\alpha_0) \hat{a}_\hbar(f) W_2(\alpha) = \hat{a}_\hbar(f) - \frac{i\hbar}{\sqrt{2}} \langle f, \alpha_0 \rangle_{L^2}. \quad (3.3.23)$$

We start proving the first identity (3.3.20). Recall from (3.3.10) that we have

$$W_1(z_0) = e^{i(p_0 \cdot \hat{q} - q_0 \cdot \hat{p})}, \quad W_1(z_0)^* = e^{-i(p_0 \cdot \hat{q} - q_0 \cdot \hat{p})}$$

Define

$$K(t) := e^{it(p_0 \cdot \hat{q} - q_0 \cdot \hat{p})} \hat{q}_j^\nu e^{-it(p_0 \cdot \hat{q} - q_0 \cdot \hat{p})}$$

Since \hat{q} and \hat{p} are self adjoint operators, we claim using Taylor expansions that

$$K(t) = K(0) + tK'(0). \quad (3.3.24)$$

Indeed, we have, using the commutation relation $[\hat{q}_j^\nu, \hat{p}_j^\nu] = i\hbar$, that

$$K'(0) = \frac{d}{dt}K(t)|_{t=0} = e^{it(p_0 \cdot \hat{q} - q_0 \cdot \hat{p})} i[(p_0 \cdot \hat{q} - q_0 \cdot \hat{p}), \hat{q}_j^\nu] e^{-it(p_0 \cdot \hat{q} - q_0 \cdot \hat{p})}|_{t=0} = -\hbar q_{0j}^\nu.$$

This implies that $K^r(0) = 0$, for all $r \geq 2$. Take $t = 1$ in (3.3.24) and since $K(0) = \hat{q}_j^\nu$, we get (3.3.20). Similarly, we can prove the identity (3.3.21). Also the two identities (3.3.22) and (3.3.23) can be proved by simiar way using the commutation relations on the Fock space.

In particular, the identity (3.3.20) gives

$$W_1(z_0) g_j(s) W_1(z_0)^* = \tilde{g}_j(s).$$

□

Lemma 3.3.9 (The expansion of the commutator). *For any $s \in \mathbb{R}$ and $\xi = (p_0, q_0, \alpha_0) \in X^{1/2}$, we have the following expansion of the commutator in terms of the semiclassical parameter $\hbar \in (0, 1)$*

$$\frac{1}{\hbar} [\mathcal{W}(\xi), \hat{H}_I(s)] = \left(B_0(s, \hbar, \xi) + \hbar B_1(s, \hbar, \xi) \right) \mathcal{W}(\xi). \quad (3.3.25)$$

The two terms B_0 and B_1 are identified as follows

$$\begin{aligned} B_0(s, \hbar, \xi) := & - \sum_{j=1}^n \nabla f_j(\hat{p}_j) \cdot p_{0j} - \nabla V(\hat{q}) \cdot q_0 \\ & + \sum_{j=1}^n \hat{a}_\hbar \left(\frac{\tilde{g}_j(s) - g_j(s)}{\hbar} \right) + \hat{a}_\hbar^* \left(\frac{\tilde{g}_j(s) - g_j(s)}{\hbar} \right) \\ & + \sum_{j=1}^n \frac{i}{\sqrt{2}} \left(\langle \alpha_0, \tilde{g}_j(s) \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} - \langle \tilde{g}_j(s), \alpha_0 \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} \right), \end{aligned} \quad (3.3.26)$$

$$B_1(s, \hbar, \xi) := \Theta_1(\hbar, \xi) + \Theta_2(\hbar, \xi), \quad (3.3.27)$$

where Θ_1 and Θ_2 are identified below in the proof. Moreover, we have also the following

estimates

$$\|(\hat{H}_0 + 1)^{-1/2} \mathbf{B}_0(s, \hbar, \xi) (\hat{H}_0 + 1)^{-1/2}\|_{\mathcal{L}(\mathcal{H})} \lesssim \left(\|\chi\|_{L^2} + \left\| \sqrt{\omega} \chi \right\|_{L^2} \right) \|\xi\|_{X^0}, \quad (3.3.28)$$

$$\|(\hat{H}_0 + 1)^{-1/2} \mathbf{B}_1(s, \hbar, \xi) (\hat{H}_0 + 1)^{-1/2}\|_{\mathcal{L}(\mathcal{H})} \lesssim \|\xi\|_{X^0}^2. \quad (3.3.29)$$

Proof. Exploiting Lemma 3.3.7 and Lemma 3.3.8 inside (3.3.18), the commutator expansion becomes

$$\begin{aligned} \frac{1}{\hbar} [\mathcal{W}(\xi), \hat{H}_I(s)] &= \frac{1}{\hbar} \left[\sum_{j=1}^n \left(f_j(\hat{p}_j - \hbar p_{0j}) - f_j(\hat{p}_j) \right) + V(\hat{q} - \hbar q_0) - V(\hat{q}) \right. \\ &\quad \left. + \sum_{j=1}^n \left(\hat{a}_{\hbar}(\tilde{g}_j(s) - g_j(s)) + \hat{a}_{\hbar}^*(\tilde{g}_j(s) - g_j(s)) \right) \right. \\ &\quad \left. + \frac{i\hbar}{\sqrt{2}} \sum_{j=1}^n \left(\langle \alpha_0, \tilde{g}_j(s) \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} - \langle \tilde{g}_j(s), \alpha_0 \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} \right) \right] \mathcal{W}(\xi). \end{aligned} \quad (3.3.30)$$

We start first by expanding the first line and then proving some estimates for the remaining terms. Let $X \in \mathbb{R}^d$ and $Y = (Y_1, \dots, Y_n) \in \mathbb{R}^{dn}$. We apply Taylor series to the two functions

$$\begin{aligned} t &\longrightarrow A(t) := f_j(X - t \hbar p_{0j}), \\ t &\longrightarrow B(t) := V(Y - t \hbar q_0). \end{aligned}$$

We get

$$A(t) = A(0) + tA'(0) + \int_0^t A''(s)(t-s) ds,$$

and

$$B(t) = B(0) + tB'(0) + \int_0^t B''(s)(t-s) ds.$$

Let $t = 1$ in the above formulas and since \hat{p} and \hat{q} are self adjoint operators, we get

$$f_j(\hat{p}_j - \hbar p_{0j}) = f_j(\hat{p}_j) - \hbar \nabla f_j(\hat{p}_j) \cdot p_{0j} + \underbrace{\hbar^2 \int_0^1 p_{0j}^T H_{f_j}(\hat{p}_j - \hbar p_{0j}s) p_{0j} (1-s) ds}_{:=\Theta_1(\hbar, \xi)}; \quad (3.3.31)$$

$$V(\hat{q} - \hbar q_0) = V(\hat{q}) - \hbar \nabla V(\hat{q}) \cdot q_0 + \underbrace{\hbar^2 \int_0^1 q_0^T H_V(\hat{q} - \hbar q_0s) q_0 (1-s) ds}_{:=\Theta_2(\hbar, \xi)}, \quad (3.3.32)$$

where the notation \cdot^T represents the transpose. Moreover, the two terms H_{f_j} and H_V are respectively the Hessian matrices related to f_j and V . This implies

$$f_j(\hat{p}_j - \hbar p_{0j}) - f_j(\hat{p}_j) = -\hbar \nabla f_j(\hat{p}_j) \cdot p_{0j} + \hbar^2 \Theta_1(\hbar, \xi), \quad (3.3.33)$$

$$V(\hat{q} - \hbar q_0) - V(\hat{q}) = -\hbar \nabla V(\hat{q}) \cdot q_0 + \hbar^2 \Theta_2(\hbar, \xi). \quad (3.3.34)$$

And thus, using (3.3.33)-(3.3.34), the commutator is expanded as indicated in (3.3.25). Now, to obtain the two estimates (3.3.28) and (3.3.29), we need first to prove that the function

$$F_j(\hbar, s) := (\tilde{g}_j(s) - g_j(s))/\hbar : L^2(\mathbb{R}^d, dx_j) \longrightarrow L^2(\mathbb{R}^d, dx_j) \otimes L^2(\mathbb{R}^d, dk)$$

is bounded uniformly in $\hbar \in (0, 1)$. Indeed, we have for all $\psi \in L^2(\mathbb{R}^d, dx_j)$

$$\begin{aligned} \|F_j(\hbar, s) \psi\|_{L^2_{x_j} \otimes L^2_k}^2 &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| (F_j(\hbar, s) \psi)(x_j, k) \right|^2 dx_j dk \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| g_j(s) \left(\frac{e^{2\pi i k \cdot q_{0j} \hbar} - 1}{\hbar} \right) \psi \right|^2 dx_j dk \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot \hat{q}_j + i s \omega(k)} \left(\frac{e^{2\pi i k \cdot q_{0j} \hbar} - 1}{\hbar} \right) \psi(x_j) \right|^2 dx_j dk. \end{aligned}$$

Now, with the aid of Fubini and the estimate $|e^{iy} - 1| \leq \sqrt{2}|y|$, we find that

$$\begin{aligned} \|F_j(\hbar, s) \psi\|_{L^2_{x_j} \otimes L^2_k}^2 &\leq 8\pi^2 \|\xi\|_{X^0}^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \sqrt{\omega(k)} \chi(k) \right|^2 \left| \psi(x_j) \right|^2 dx_j dk \\ &= 8\pi^2 \|\chi\|_{\mathcal{G}^{1/2}}^2 \|\xi\|_{X^0}^2 \|\psi\|_{L^2_{x_j}}^2. \end{aligned}$$

We get finally, with some $C > 0$, that

$$\|F_j(\hbar, s)\|_{\mathcal{L}(L^2_{x_j}, L^2_{x_j} \otimes L^2_k)} \leq C \|\chi\|_{\mathcal{G}^{1/2}} \|\xi\|_{X^0}. \quad (3.3.35)$$

Now, using the estimates in Lemma 3.3.1 on the creation-annihilation operators together with the above estimate for $F_j(\hbar, s)$, we can easily prove (3.3.28). It is also not hard to see that (3.3.29) hold true as a consequence of the fact that the Hessian matrices of f_j and V are bounded. \square

Our focus is on taking the classical limit $\hbar \rightarrow 0$. To accomplish this, it is crucial to establish a uniform bound on the expansion derived in Lemma 3.3.9, particularly for the

remainder term. Let $S = (\hat{H}_0 + 1)^{1/2}$, we have

$$\begin{aligned} \mathrm{Tr}\left(\frac{1}{\hbar}[\mathcal{W}(\xi), \hat{H}_I(s)] \tilde{\varrho}_\hbar(s)\right) &= \mathrm{Tr}\left[\underbrace{S^{-1} B_0(s, \hbar, \xi) S^{-1}}_{\in \mathcal{L}(\mathcal{H})} \underbrace{S \mathcal{W}(\xi) S^{-1}}_{\in \mathcal{L}(\mathcal{H})} \underbrace{S \tilde{\varrho}_\hbar(s) S}_{\in \mathcal{L}^1(\mathcal{H})}\right] \\ &\quad + \hbar \mathrm{Tr}\left[\underbrace{S^{-1} B_1(\hbar, s, \xi) S^{-1}}_{\in \mathcal{L}(\mathcal{H})} \underbrace{S \mathcal{W}(\xi) S^{-1}}_{\in \mathcal{L}(\mathcal{H})} \underbrace{S \tilde{\varrho}_\hbar(s) S}_{\in \mathcal{L}^1(\mathcal{H})}\right] \end{aligned} \quad (3.3.36)$$

- ◁ Lemma 3.3.9 assures that the first term in each of the above two lines in (3.3.36) is bounded.
- ◁ The Weyl-Heisenberg operator estimates presented in Lemma 3.3.5 guarantee that the bound of the second term in the above two lines in (3.3.36) holds.
- ◁ The bound of the last term in each of the above two lines in (3.3.36) follows from Assumption (3.1.8) and (3.1.9) in conjunction with the equivalent relation between \hat{H} and \hat{H}_0 outlined in Lemma 3.3.3.

Our next step is to take the limit in the Duhamel formula (3.3.13) as \hbar approaches zero. Using the above arguments, we can disregard the remainder term when passing to the limit $\hbar \rightarrow 0$ in the Duhamel formula (3.3.13). We achieve this in the next section by extracting a subsequence.

3.4 Existence of Wigner measure

According to Definition 3.1.2, the Wigner measures of $\tilde{\varrho}_\hbar(t)$ is obtained by taking limits of the following map:

$$\xi \rightarrow \mathrm{Tr}\left[\mathcal{W}(\xi) \tilde{\varrho}_\hbar(t)\right]. \quad (3.4.1)$$

Thus, the first task is to verify that the Wigner measure associated to the above map is unique for all times. It is worth noting that, given our assumptions on the initial states $(\varrho_\hbar)_{\hbar \in (0,1)}$, the associated set of Wigner measures

$$\mathcal{M}(\varrho_\hbar, \hbar \in (0, 1))$$

is non-empty. To ensure that the sets of Wigner measures

$$\mathcal{M}(\varrho_\hbar(t), \hbar \in (0, 1)) \quad \text{and} \quad \mathcal{M}(\tilde{\varrho}_\hbar(t), \hbar \in (0, 1))$$

are also non-empty, it is crucial to demonstrate that assumptions (3.1.8) and (3.1.9) can be uniformly propagated in time by both families of states $(\varrho_h(t))_{h \in (0,1)}$ and $(\tilde{\varrho}_h(t))_{h \in (0,1)}$. This is established in Subsection 3.4.1. Subsequently, in Subsection 3.4.2, we prove that the map (3.4.1) has a unique limit that holds for all times in compact interval.

3.4.1 Propagation of assumptions

In order to establish the existence of a unique Wigner measure that holds for all times, we require certain localization estimates for particles along the Nelson dynamics. We demonstrate that if an initial state ϱ_h is localized uniformly in \hbar , then it will remain localized uniformly with respect to the semiclassical parameter $\hbar \in (0, 1)$ for all times in compact interval. We prove this result separately for particle operators in Paragraph 3.4.1.1 and for field operators in Paragraph 3.4.1.2. Finally, in Paragraph 3.4.1.3, we establish that both families of states $(\varrho_h(t))_{h \in (0,1)}$ and $(\tilde{\varrho}_h(t))_{h \in (0,1)}$ uniformly satisfy (3.1.8) and (3.1.9) for all times.

3.4.1.1 Position and Momentum operator estimates

In this part, we prove some uniform estimates (in \hbar) related to the two operators \hat{p}^2 and \hat{q}^2 .

Lemma 3.4.1 (Position operator's estimate). *Assume that (3.1.6) and $\omega^{-1/2}\chi \in L^2(\mathbb{R}^d, dk)$. Then, there exists constants $C_1, C_2 > 0$ such that for all $\psi \in D(\hat{H}_0^{1/2}) \cap D(\hat{q})$, all $t \in \mathbb{R}$ and all $\hbar \in (0, 1)$:*

$$\langle e^{-i\frac{t}{\hbar}\hat{H}}\psi, \hat{q}^2 e^{-i\frac{t}{\hbar}\hat{H}}\psi \rangle \leq C_1 \langle \psi, (\hat{H}_0 + \hat{q}^2 + 1) \psi \rangle e^{C_2|t|}. \quad (3.4.2)$$

Proof. Let $\Theta_1(t) := \langle e^{-i\frac{t}{\hbar}\hat{H}}\psi, \hat{q}^2 e^{-i\frac{t}{\hbar}\hat{H}}\psi \rangle$. We have

$$\Theta_1(t) = \Theta_1(0) + \int_0^t \dot{\Theta}_1(s) ds.$$

Then Stone's Theorem implies that

$$\dot{\Theta}_1(t) = \frac{1}{\hbar} \langle e^{-i\frac{t}{\hbar}\hat{H}}\psi, i[\hat{H}, \hat{q}^2] e^{-i\frac{t}{\hbar}\hat{H}}\psi \rangle.$$

Now, using some commutation relations, we get

$$i[\hat{H}, \hat{q}^2] = i \sum_{j=1}^n [f_j(\hat{p}_j), \hat{q}_j^2] = \hbar \sum_{j=1}^n [\nabla f_j(\hat{p}_j) \cdot \hat{q}_j + \hat{q}_j \cdot \nabla f_j(\hat{p}_j)].$$

Define $\psi(t) := e^{-i\frac{t}{\hbar}\hat{H}}\psi$. Since \hat{q}_j and $\nabla f_j(\hat{p}_j)$ are self-adjoint operators, we have the following estimates

$$\begin{aligned} \langle \psi(t), \nabla f_j(\hat{p}_j) \cdot \hat{q}_j \psi(t) \rangle &\leq \|\nabla f_j(\hat{p}_j) \psi(t)\| \|\hat{q}_j \psi(t)\| \\ &\leq \frac{1}{2} \left[\|\nabla f_j(\hat{p}_j) \psi(t)\|^2 + \|\hat{q}_j \psi(t)\|^2 \right] \\ &= \frac{1}{2} \left[\|\nabla f_j(\hat{p}_j) \psi(t)\|^2 + \Theta_1(t) \right]. \end{aligned} \quad (3.4.3)$$

$$\begin{aligned} \langle \psi(t), \hat{q}_j \cdot \nabla f_j(\hat{p}_j) \psi(t) \rangle &\leq \|\hat{q}_j \psi(t)\| \|\nabla f_j(\hat{p}_j) \psi(t)\| \\ &\leq \frac{1}{2} \left[\|\nabla f_j(\hat{p}_j) \psi(t)\|^2 + \|\hat{q}_j \psi(t)\|^2 \right] \\ &= \frac{1}{2} \left[\|\nabla f_j(\hat{p}_j) \psi(t)\|^2 + \Theta_1(t) \right], \end{aligned} \quad (3.4.4)$$

where we have used the identity $2a \cdot b \leq a^2 + b^2$. At this stage, we have to consider separately the two cases: the semi-relativistic and the non-relativistic case since the function ∇f_j is bounded in the first case and not in the second one.

For semi-relativistic case:

Note that $\nabla f_j(\hat{p}_j)$ is a bounded operator. This implies that for some $c_1 > 0$, we have

$$\|\nabla f_j(\hat{p}_j) \psi(t)\|^2 \leq c_1 \|\psi\|^2.$$

This gives

$$\Theta_1(t) \leq \Theta_1(0) + c_1 \langle \psi, \psi \rangle t + \int_0^t \Theta_1(s) ds.$$

Now using Gronwall's lemma and the estimate $te^t \leq e^{ct}$ for some $c > 0$, we find with some $C_1, C_2 > 0$

$$\Theta_1(t) \leq \left[\Theta_1(0) + c_1 \langle \psi, \psi \rangle t \right] e^t \leq C_1 \langle \psi, (\hat{q}^2 + 1) \psi \rangle e^{C_2|t|} \leq C_1 \langle \psi, (\hat{H}_0 + \hat{q}^2 + 1) \psi \rangle e^{C_2|t|}.$$

For non-relativistic case:

We have $\nabla f_j(\hat{p}_j) = \hat{p}_j/M_j$. This implies (3.4.3) and (3.4.4) become

$$\begin{aligned} \langle \psi(t), \nabla f_j(\hat{p}_j) \cdot \hat{q}_j \psi(t) \rangle &\leq \frac{1}{2} \left[\langle \psi(t), \frac{\hat{p}_j^2}{M_j^2} \psi(t) \rangle + \Theta_1(t) \right] \\ &\leq \frac{1}{M_j} \langle \psi(t), \hat{H}_0 \psi(t) \rangle + \frac{1}{2} \Theta_1(t) \\ \langle \psi(t), \hat{q}_j \cdot \nabla f_j(\hat{p}_j) \psi(t) \rangle &\leq \frac{1}{2} \left[\langle \psi(t), \frac{\hat{p}_j^2}{M_j^2} \psi(t) \rangle + \Theta_1(t) \right] \\ &\leq \frac{1}{M_j} \langle \psi(t), \hat{H}_0 \psi(t) \rangle + \frac{1}{2} \Theta_1(t). \end{aligned}$$

By Lemma 3.3.3, we have

$$\langle e^{-i\frac{t}{\hbar}\hat{H}}\psi, \hat{H}_0 e^{-i\frac{t}{\hbar}\hat{H}}\psi \rangle \lesssim \langle e^{-i\frac{t}{\hbar}\hat{H}}\psi, (\hat{H} + 1) e^{-i\frac{t}{\hbar}\hat{H}}\psi \rangle = \langle \psi, (\hat{H} + 1) \psi \rangle \lesssim \langle \psi, (\hat{H}_0 + 1) \psi \rangle.$$

This leads with some $c_2 \in \mathbb{R}_+^*$ to the following inequality

$$\Theta_1(t) \leq \Theta_1(0) + c_2 \langle \psi, (\hat{H}_0 + 1) \psi \rangle t + \int_0^t \Theta_1(s) ds.$$

Now using Gronwall's Lemma and the estimates $te^{ct} \leq e^{c't}$, we find with some $C_1, C_2 > 0$

$$\begin{aligned} \Theta_1(t) &\leq \left[\Theta_1(0) + c_2 \langle \psi, (\hat{H}_0 + 1) \psi \rangle t \right] e^{\int_0^t 1 ds} \leq C_1 \langle \psi, (\hat{H}_0 + \hat{q}^2 + 1) \psi \rangle e^{C_2|t|} \\ &\leq C_1 \langle \psi, (\hat{H}_0 + \hat{q}^2 + 1) \psi \rangle e^{C_2|t|}. \end{aligned}$$

□

Now, we give some uniform estimates for the momentum operator just in the semi-relativistic case: $f_j(\hat{p}_j) = \sqrt{\hat{p}_j^2 + M_j^2}$.

Lemma 3.4.2 (Momentum operator's estimate). *Assume that (3.1.6) and $\omega^{1/2}\chi \in L^2(\mathbb{R}^d, dk)$. Then, there exists constants $C_1, C_2 > 0$ such that for all $\psi \in D(\hat{H}_0^{1/2}) \cap D(\hat{p})$, all $t \in \mathbb{R}$ and all $\hbar \in (0, 1)$:*

$$\langle e^{-i\frac{t}{\hbar}\hat{H}}\psi, \hat{p}^2 e^{-i\frac{t}{\hbar}\hat{H}}\psi \rangle \leq C_1 \langle \psi, (\hat{H}_0 + \hat{p}^2 + 1) \psi \rangle e^{C_2|t|}. \quad (3.4.5)$$

Proof. Define

$$\Theta_2(t) := \langle \psi(t), \hat{p}^2 \psi(t) \rangle, \quad \psi(t) := e^{-i\frac{t}{\hbar}\hat{H}} \psi.$$

We have that the map $t \rightarrow \Theta_2(t)$ is differentiable with

$$\dot{\Theta}_2(t) = \frac{i}{\hbar} \langle \psi(t), [\hat{H}, \hat{p}^2] \psi(t) \rangle.$$

Then, Duhamel formula implies that

$$\Theta_2(t) = \Theta_2(0) + \int_0^t \dot{\Theta}_2(s) ds.$$

Let us compute first the explicit expression for the function $\dot{\Theta}_2(t)$. To do that, we need first to deal with the commutator $[\hat{H}, \hat{p}^2]$. Indeed, we have

$$\begin{aligned} [\hat{H}, \hat{p}^2] &= [d\Gamma(\omega) + \sum_{j=1}^n \sqrt{\hat{p}_j^2 + M_j^2} + V(\hat{q}) + \hat{H}_1, \hat{p}^2] \\ &= [V(\hat{q}), \hat{p}] \hat{p} + \hat{p} [V(\hat{q}), \hat{p}] + [\hat{H}_1, \hat{p}] \hat{p} + \hat{p} [\hat{H}_1, \hat{p}]. \end{aligned}$$

Recall that

$$\hat{a}_\hbar^\sharp(G) = \sum_{j=1}^n \hat{a}_\hbar^\sharp(g_j(\hat{q})), \quad g_j(\hat{q})(k) := \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot \hat{q}_j}.$$

We can then assert that

$$\begin{aligned} \hat{a}_\hbar(G) &= \sum_{j=1}^n \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{2\pi i k \cdot \hat{q}_j} \hat{a}_\hbar(k) dk =: \sum_{j=1}^n B_j(\hat{q}_j), \\ \hat{a}_\hbar^*(G) &= \sum_{j=1}^n \int_{\mathbb{R}^d} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot \hat{q}_j} \hat{a}_\hbar^*(k) dk =: \sum_{j=1}^n B_j^*(\hat{q}_j), \end{aligned}$$

where $q_j \rightarrow B_j^\sharp(q_j)$ is analytic function. We know that for any analytic function F

$$[F(\hat{q}_j), \hat{p}_j] = i\hbar \frac{\partial F(\hat{q}_j)}{\partial q_j},$$

This gives

$$\begin{aligned}
 [V(\hat{q}), \hat{p}] \hat{p} &= \sum_{j=1}^n i\hbar \nabla_{q_j} V(\hat{q}) \cdot \hat{p}_j, \\
 \hat{p} [V(\hat{q}), \hat{p}] &= \sum_{j=1}^n i\hbar \hat{p}_j \cdot \nabla_{q_j} V(\hat{q}), \\
 [\hat{a}_\hbar(G), \hat{p}] \hat{p} &= \left[\sum_{j=1}^n B_j(\hat{q}_j), \hat{p} \right] \hat{p} = \sum_{j=1}^n i\hbar \frac{\partial B_j(\hat{q}_j)}{\partial q_j} \cdot \hat{p}_j = \sum_{j=1}^n i\hbar \hat{a}_\hbar(\tilde{g}_j) \cdot \hat{p}_j, \\
 \hat{p} [\hat{a}_\hbar(G), \hat{p}] &= \sum_{j=1}^n i\hbar \hat{p}_j \cdot \hat{a}_\hbar(\tilde{g}_j), \\
 [\hat{a}_\hbar^*(G), \hat{p}] \hat{p} &= \left[\sum_{j=1}^n B_j^*(\hat{q}_j), \hat{p} \right] \hat{p} = \sum_{j=1}^n i\hbar \frac{\partial B_j^*(\hat{q}_j)}{\partial q_j} \cdot \hat{p}_j = \sum_{j=1}^n i\hbar \hat{a}_\hbar^*(\tilde{g}_j) \cdot \hat{p}_j, \\
 \hat{p} [\hat{a}_\hbar^*(G), \hat{p}] &= \sum_{j=1}^n i\hbar \hat{p}_j \cdot \hat{a}_\hbar^*(\tilde{g}_j),
 \end{aligned}$$

where we have introduced the term \tilde{g}_j as follows

$$\tilde{g}_j := -2\pi i k \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot \hat{q}_j}.$$

This implies that

$$[\hat{H}, \hat{p}^2] = i\hbar \sum_{j=1}^n \left[\hat{p}_j \cdot \left(\nabla_{q_j} V(\hat{q}) + \hat{a}_\hbar(\tilde{g}_j) + \hat{a}_\hbar^*(\tilde{g}_j) \right) + \left(\nabla_{q_j} V(\hat{q}) + \hat{a}_\hbar(\tilde{g}_j) + \hat{a}_\hbar^*(\tilde{g}_j) \right) \cdot \hat{p}_j \right].$$

We conclude that

$$\begin{aligned}
 \dot{\Theta}_2(t) &= - \sum_{j=1}^n \left[\langle \psi(t), \hat{p}_j \cdot \left(\nabla_{q_j} V(\hat{q}) + \hat{a}_\hbar(\tilde{g}_j) + \hat{a}_\hbar^*(\tilde{g}_j) \right) \psi(t) \rangle \right. \\
 &\quad \left. + \langle \psi(t), \left(\nabla_{q_j} V(\hat{q}) + \hat{a}_\hbar(\tilde{g}_j) + \hat{a}_\hbar^*(\tilde{g}_j) \right) \cdot \hat{p}_j \psi(t) \rangle \right].
 \end{aligned}$$

We estimate now each part. Indeed, we have

$$\begin{aligned}
 \langle \psi(t), \hat{p}_j \cdot \nabla_{q_j} V(\hat{q}) \psi(t) \rangle &= \langle \hat{p}_j \psi(t), \nabla_{q_j} V(\hat{q}) \psi(t) \rangle \\
 &\leq \|\hat{p}_j \psi(t)\| \|\nabla_{q_j} V(\hat{q}) \psi(t)\| \\
 &\leq \|\hat{p}_j \psi(t)\| \|\nabla_{q_j} V(\hat{q})\|_{L^\infty} \|\psi\| \\
 &\leq \frac{1}{2} \|\nabla_{q_j} V(\hat{q})\|_{L^\infty} \left[\|\hat{p}_j \psi(t)\|^2 + \|\psi\|^2 \right].
 \end{aligned}$$

Similarly, we have

$$\langle \psi(t), \nabla_{q_j} V(\hat{q}) \cdot \hat{p}_j \psi(t) \rangle \leq \frac{1}{2} \|\nabla_{q_j} V(\hat{q})\|_{L^\infty} \left[\|\hat{p}_j \psi(t)\|^2 + \|\psi\|^2 \right].$$

We have also using Lemmas [3.3.1](#) and [3.3.3](#)

$$\begin{aligned}
 \langle \psi(t), \hat{a}_h^*(\tilde{g}_j) \cdot \hat{p}_j \psi(t) \rangle &= \langle \hat{a}_h^*(\tilde{g}_j) \psi(t), \hat{p}_j \psi(t) \rangle \\
 &\leq \|\hat{p}_j \psi(t)\| \|\hat{a}_h^*(\tilde{g}_j) \psi(t)\| \\
 &\lesssim \left[\|\chi\|_{L^2} + \|\sqrt{\omega} \chi\|_{L^2} \right] \|\hat{p}_j \psi(t)\| \|(\hat{H}_{02} + 1)^{1/2} \psi(t)\| \\
 &\lesssim \left[\|\chi\|_{L^2} + \|\sqrt{\omega} \chi\|_{L^2} \right] \|\hat{p}_j \psi(t)\| \|(\hat{H}_0 + 1)^{1/2} \psi(t)\| \\
 &\lesssim \left[\|\chi\|_{L^2} + \|\sqrt{\omega} \chi\|_{L^2} \right] \|\hat{p}_j \psi(t)\| \|(\hat{H} + a)^{1/2} \psi(t)\| \\
 &\lesssim \left[\|\chi\|_{L^2} + \|\sqrt{\omega} \chi\|_{L^2} \right] \|\hat{p}_j \psi(t)\| \|(\hat{H} + a)^{1/2} \psi\| \\
 &\lesssim \left[\|\chi\|_{L^2} + \|\sqrt{\omega} \chi\|_{L^2} \right] \|\hat{p}_j \psi(t)\| \|(\hat{H}_0 + 1)^{1/2} \psi\| \\
 &\lesssim \left[\|\chi\|_{L^2} + \|\sqrt{\omega} \chi\|_{L^2} \right] \left[\|\hat{p}_j \psi(t)\|^2 + \|(\hat{H}_0 + 1)^{1/2} \psi\|^2 \right].
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 \langle \psi(t), \hat{a}_h^*(\tilde{g}_j) \cdot \hat{p}_j \psi(t) \rangle &\lesssim \|\chi\|_{L^2} \left[\|\hat{p}_j \psi(t)\|^2 + \|(\hat{H}_0 + 1)^{1/2} \psi\|^2 \right], \\
 \langle \psi(t), \hat{p}_j \cdot \hat{a}_h(\tilde{g}_j) \psi(t) \rangle &\lesssim \|\chi\|_{L^2} \left[\|\hat{p}_j \psi(t)\|^2 + \|(\hat{H}_0 + 1)^{1/2} \psi\|^2 \right], \\
 \langle \psi(t), \hat{p}_j \cdot \hat{a}_h^*(\tilde{g}_j) \psi(t) \rangle &\lesssim \left[\|\chi\|_{L^2} + \|\sqrt{\omega} \chi\|_{L^2} \right] \left[\|\hat{p}_j \psi(t)\|^2 + \|(\hat{H}_0 + 1)^{1/2} \psi\|^2 \right].
 \end{aligned}$$

We get at the end that there exists some $C \in \mathbb{R}_+^*$ depending on the quantities $\|\chi\|_{L^2}$, $\|\sqrt{\omega} \chi\|_{L^2}$ and $\|\nabla_{\hat{q}_j} V(\hat{q})\|_{L^\infty}$ such that

$$\dot{\Theta}_2(t) \leq C \left[\langle \psi, (\hat{H}_0 + 1) \psi \rangle + \Theta_2(t) \right].$$

We find then

$$\Theta_2(t) \leq \Theta_2(0) + C \langle \psi, (\hat{H}_0 + 1) \psi \rangle t + \int_0^t \Theta_2(s) ds.$$

This implies using Gronwall's Lemma that there exists $C_1, C_2 > 0$ depend on the quantities $\|\chi\|_{L^2}$, $\|\sqrt{\omega} \chi\|_{L^2}$ and $\|\nabla_{q_j} V(\hat{q})\|_{L^\infty}$ such that

$$\Theta_2(t) \leq C_1 \langle \psi, (\hat{H}_0 + \hat{p}^2 + 1) \psi \rangle e^{C_2|t|}.$$

And thus the result follows. \square

3.4.1.2 Field operator's estimates

Below, we give some estimates for the field operator $d\Gamma(\omega^{2\sigma})$.

Let Γ_{fin} be a dense subspace in the Fock space. Let $\psi \in D(\hat{H}_0)$ and define

$$\Theta_3(t) := \langle \psi(t), d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})} \psi(t) \rangle, \quad \psi(t) := e^{-i\frac{t}{\hbar} \hat{H}} \psi.$$

Note that $d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}$ is a bounded and positive approximation of $d\Gamma(\omega^{2\sigma})$ that strongly converges monotonically to it. The quantity $\Theta_3(t)$ is well-defined for each $t \in \mathbb{R}$ and $\delta > 0$. In addition, the map $t \rightarrow \Theta_3(t)$ is differentiable with

$$\dot{\Theta}_3(t) = \frac{i}{\hbar} \langle \psi(t), [\hat{H}, d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}] \psi(t) \rangle.$$

Lemma 3.4.3. Assume [\(3.1.6\)](#) and $\omega^{\sigma-\frac{1}{2}} \chi \in L^2(\mathbb{R}^d, dk)$. For $\sigma \in [1/2, 1]$, there exists $C > 0$ such that for all $\delta > 0$, for all $\hbar \in (0, 1)$ and for all $\phi, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^{dn}) \otimes \Gamma_{fin}$:

$$\left| \frac{i}{\hbar} \langle \phi, [\hat{H}, d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}] \psi \rangle \right| \leq C \|\omega^{\sigma-\frac{1}{2}} \chi\|_{L^2} \left[\|\phi\| \|d\Gamma(\omega^{2\sigma})^{\frac{1}{2}} \psi\| + \|\psi\| \|d\Gamma(\omega^{2\sigma})^{\frac{1}{2}} \phi\| \right].$$

Proof. Let us deal first with the term

$$\begin{aligned} [d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}, \hat{H}] &= [d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}, \hat{H}_1] \\ &= [d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}, \hat{a}_\hbar(G) + \hat{a}_\hbar^*(G)] \\ &= d\Gamma(\omega^{2\sigma}) [e^{-\delta d\Gamma(\omega^{2\sigma})}, \hat{a}_\hbar(G)] + [d\Gamma(\omega^{2\sigma}), \hat{a}_\hbar(G)] e^{-\delta d\Gamma(\omega^{2\sigma})} \\ &\quad + d\Gamma(\omega^{2\sigma}) [e^{-\delta d\Gamma(\omega^{2\sigma})}, \hat{a}_\hbar^*(G)] + [d\Gamma(\omega^{2\sigma}), \hat{a}_\hbar^*(G)] e^{-\delta d\Gamma(\omega^{2\sigma})}. \end{aligned}$$

We also have

$$(i) \quad [d\Gamma(\omega^{2\sigma}), \hat{a}_\hbar(G)] = -\hbar \hat{a}_\hbar(\omega^{2\sigma} G),$$

$$(ii) \quad [d\Gamma(\omega^{2\sigma}), \hat{a}_\hbar^*(G)] = \hbar \hat{a}_\hbar^*(\omega^{2\sigma} G),$$

$$(iii) \quad [e^{-\delta d\Gamma(\omega^{2\sigma})}, \hat{a}_\hbar(G)] = e^{-\delta d\Gamma(\omega^{2\sigma})} \hat{a}_\hbar(\beta G), \quad \beta = 1 - e^{-\delta \hbar \omega^{2\sigma}},$$

$$(iv) \quad [e^{-\delta d\Gamma(\omega^{2\sigma})}, \hat{a}_\hbar^*(G)] = \hat{a}_\hbar^*(-\beta G) e^{-\delta d\Gamma(\omega^{2\sigma})}, \quad \beta = 1 - e^{-\delta \hbar \omega^{2\sigma}}.$$

Using (i)-(ii)-(iii) and (iv), we get

$$[d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}, \hat{H}] = \hbar (B_1 + B_2 + B_3),$$

where we have introduced the three terms B_1 , B_2 and B_3 as follows

$$B_1 := [\hat{a}_\hbar^*(\omega^{2\sigma} G) - \hat{a}_\hbar(\omega^{2\sigma} G)] e^{-\delta d\Gamma(\omega^{2\sigma})},$$

$$B_2 := d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})} \hat{a}_\hbar\left(\frac{\beta G}{\hbar}\right),$$

$$B_3 := d\Gamma(\omega^{2\sigma}) \hat{a}_\hbar^*\left(\frac{-\beta G}{\hbar}\right) e^{-\delta d\Gamma(\omega^{2\sigma})}.$$

We get then

$$\begin{aligned} \frac{i}{\hbar} \langle \phi, [\hat{H}, d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}] \psi \rangle &= -i \langle \phi, (B_1 + B_2 + B_3) \psi \rangle \\ &= \underbrace{-i \langle \phi, B_1 \psi \rangle}_{(a)} + \underbrace{-i \langle \phi, B_2 \psi \rangle}_{(b)} + \underbrace{-i \langle \phi, B_3 \psi \rangle}_{(c)}. \end{aligned}$$

For (b), we have

$$\begin{aligned} |\langle \phi, B_2 \psi \rangle| &= \left| \langle \phi, d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})} \hat{a}_\hbar\left(\frac{\beta G}{\hbar}\right) \psi \rangle \right| \\ &= \left| \langle \phi, \delta d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})} \hat{a}_\hbar\left(\frac{\beta G}{\delta \hbar}\right) \psi \rangle \right| \\ &\leq \|\phi\| \|\delta d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})} \hat{a}_\hbar\left(\frac{\beta G}{\delta \hbar}\right) \psi\| \\ &\leq \frac{1}{e} \|\phi\| \|\hat{a}_\hbar\left(\frac{\beta G}{\delta \hbar}\right) \psi\|, \end{aligned}$$

where in the last line we have used the fact that $\sup_{\delta>0} \|\delta d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}\| \leq 1/e$.

Remark also that we have with K_m and X_n as in (3.1.2)

$$\begin{aligned}
 \|\hat{a}_h\left(\frac{\beta G}{\delta \hbar}\right) \psi\|_{\mathcal{H}}^2 &= \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| \left[\hat{a}_h\left(\frac{\beta G}{\delta \hbar}\right) \psi \right]^m(X_n, K_m) \right|^2 dK_m dX_n \\
 &= \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| \int_{\mathbb{R}^d} \sum_{j=1}^n \sqrt{\hbar(m+1)} \frac{1 - e^{-\delta \hbar \omega^{2\sigma}}}{\hbar \delta} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{2\pi i k \cdot \hat{q}_j} \psi^{m+1}(X_n, K_m, k) dk \right|^2 dK_m dX_n \\
 &\leq \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| \int_{\mathbb{R}^d} \sum_{j=1}^n \sqrt{\hbar(m+1)} \omega^{2\sigma}(k) \frac{\chi(k)}{\sqrt{\omega(k)}} e^{2\pi i k \cdot \hat{q}_j} \psi^{m+1}(X_n, K_m, k) dk \right|^2 dK_m dX_n \\
 &= \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left| \int_{\mathbb{R}^d} \sum_{j=1}^n \omega^{\sigma - \frac{1}{2}}(k) \chi(k) \sqrt{\hbar(m+1)} \omega^{\sigma}(k) e^{2\pi i k \cdot \hat{q}_j} \psi^{m+1}(X_n, K_m, k) dk \right|^2 dK_m dX_n \\
 &\stackrel{c-s}{\leq} n^2 \|\omega^{\sigma - \frac{1}{2}} \chi\|_2^2 \sum_{m \geq 0} \int_{\mathbb{R}^{dn}} \int_{\mathbb{R}^{dm}} \left[\int_{\mathbb{R}^d} \hbar(m+1) \omega^{2\sigma}(k) \left| \psi^{m+1}(X_n, K_m, k) \right|^2 dk \right] dK_m dX_n \\
 &\leq n^2 \|\omega^{\sigma - \frac{1}{2}} \chi\|_2^2 \|d\Gamma(\omega^{2\sigma})^{1/2} \psi\|_{\mathcal{H}}^2.
 \end{aligned}$$

Using the above estimates, we find that

$$|\langle \phi, B_2 \psi \rangle| \leq \frac{n}{e} \|\omega^{\sigma - \frac{1}{2}} \chi\|_2 \|\phi\| \|d\Gamma(\omega^{2\sigma})^{1/2} \psi\|_{\mathcal{H}}.$$

For (c), remark first that we have

$$[d\Gamma(\omega^{2\sigma}), \hat{a}_h^*\left(\frac{-\beta G}{\hbar}\right)] = \hbar \hat{a}_h^*\left(\frac{-\omega^{2\sigma} \beta G}{\hbar}\right),$$

and

$$[d\Gamma(\omega^{2\sigma}), \hat{a}_h^*\left(\frac{-\beta G}{\hbar}\right)] = d\Gamma(\omega^{2\sigma}) \hat{a}_h^*\left(\frac{-\beta G}{\hbar}\right) - \hat{a}_h^*\left(\frac{-\beta G}{\hbar}\right) d\Gamma(\omega^{2\sigma})$$

This implies that

$$d\Gamma(\omega^{2\sigma}) \hat{a}_h^*\left(\frac{-\beta G}{\hbar}\right) = \hat{a}_h^*\left(-\omega^{2\sigma} \beta G\right) + \hat{a}_h^*\left(\frac{-\beta G}{\hbar}\right) d\Gamma(\omega^{2\sigma}).$$

Then, we have

$$\begin{aligned}
 |\langle \phi, B_3 \psi \rangle| &\leq |\langle \phi, \hat{a}_h^*\left(-\omega^{2\sigma} \beta G\right) e^{-\delta d\Gamma(\omega^{2\sigma})} \psi \rangle| + |\langle \phi, \hat{a}_h^*\left(\frac{-\beta G}{\hbar}\right) d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})} \psi \rangle| \\
 &\lesssim \|\omega^{\sigma - \frac{1}{2}} \chi\|_2 \|\psi\| \|d\Gamma(\omega^{2\sigma})^{1/2} \phi\|_{\mathcal{H}},
 \end{aligned}$$

where in the last line, we have used the same tricks as before as well as the fact that $|\beta| \leq 2$ and $e^{-\delta d\Gamma(\omega^{2\sigma})} \in \mathcal{L}(\mathcal{H})$ with $\|e^{-\delta d\Gamma(\omega^{2\sigma})}\| \leq 1$. Similarly for (a), we can have by same techniques that

$$\begin{aligned} |\langle \phi, \hat{a}_h^*(\omega^{2\sigma} G) e^{-\delta d\Gamma(\omega^{2\sigma})} \psi \rangle| &= |\langle \hat{a}_h(\omega^{2\sigma} G) \phi, e^{-\delta d\Gamma(\omega^{2\sigma})} \psi \rangle| \\ &\lesssim \|\omega^{\sigma-\frac{1}{2}} \chi\|_2 \|\psi\| \|d\Gamma(\omega^{2\sigma})^{1/2} \phi\|_{\mathcal{H}}. \end{aligned}$$

For the other term in (a), note that

$$[e^{-\delta d\Gamma(\omega^{2\sigma})}, \hat{a}_h(\omega^{2\sigma} G)] = e^{-\delta d\Gamma(\omega^{2\sigma})} \hat{a}_h(\beta \omega^{2\sigma} G).$$

This implies that

$$\hat{a}_h(\omega^{2\sigma} G) e^{-\delta d\Gamma(\omega^{2\sigma})} = e^{-\delta d\Gamma(\omega^{2\sigma})} \hat{a}_h((1-\beta) \omega^{2\sigma} G).$$

Then, using the above equality, we get

$$|\langle \phi, \hat{a}_h(\omega^{2\sigma} G) e^{-\delta d\Gamma(\omega^{2\sigma})} \psi \rangle| \lesssim \|\omega^{\sigma-\frac{1}{2}} \chi\|_2 \|\phi\| \|d\Gamma(\omega^{2\sigma})^{1/2} \psi\|_{\mathcal{H}}.$$

We conclude that

$$|\langle \phi, B_1 \psi \rangle| \lesssim \|\omega^{\sigma-\frac{1}{2}} \chi\|_2 \left[\|\psi\| \|d\Gamma(\omega^{2\sigma})^{1/2} \phi\|_{\mathcal{H}} + \|\phi\| \|d\Gamma(\omega^{2\sigma})^{1/2} \psi\|_{\mathcal{H}} \right].$$

And thus, the final result follows. □

Lemma 3.4.4. *There exists $C_1, C_2 \in \mathbb{R}_+^*$ such that*

$$\Theta_3(t) \leq C_1 \langle \psi, (d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})} + 1) \psi \rangle e^{C_2|t|}.$$

Proof. Use the previous Lemma [3.4.3](#) with $\phi = \psi = \psi(t)$, we get that

$$\begin{aligned}
 \dot{\Theta}_3(t) &= \frac{i}{\hbar} \langle \psi(t), [\hat{H}, d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}] \psi(t) \rangle \\
 &\lesssim \|\omega^{\sigma-\frac{1}{2}} \chi\|_2 \left[\|\psi\| \|d\Gamma(\omega^{2\sigma})^{1/2} \psi(t)\|_{\mathcal{H}} \right] \\
 &\lesssim \|\omega^{\sigma-\frac{1}{2}} \chi\|_2 \left[\|\psi\| \| [d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}]^{1/2} \psi(t) \|_{\mathcal{H}} \right] \\
 &\lesssim \|\omega^{\sigma-\frac{1}{2}} \chi\|_2 \left[\|\psi\|^2 + \| [d\Gamma(\omega^{2\sigma}) e^{-\delta d\Gamma(\omega^{2\sigma})}]^{1/2} \psi(t) \|_{\mathcal{H}}^2 \right] \\
 &\lesssim c \langle \psi, \psi \rangle + c \Theta_3(t).
 \end{aligned}$$

And thus the result follows by applying the Gronwall's Lemma. \square

Lemma 3.4.5 (Field estimate). *Assume that [\(3.1.6\)](#) and $\omega^{\sigma-\frac{1}{2}} \chi \in L^2(\mathbb{R}^d, dk)$. Then, there exists constants $C_1, C_2 > 0$ such that for all $\psi \in D(\hat{H}_0^{1/2}) \cap D(d\Gamma(\omega^{2\sigma})^{1/2})$, all $t \in \mathbb{R}$ and all $\hbar \in (0, 1)$:*

$$\langle e^{-i\frac{t}{\hbar} \hat{H}} \psi, d\Gamma(\omega^{2\sigma}) e^{-i\frac{t}{\hbar} \hat{H}} \psi \rangle \leq C_1 \langle \psi, (d\Gamma(\omega^{2\sigma}) + 1) \psi \rangle e^{C_2|t|}. \quad (3.4.6)$$

Proof. It is a consequence of the previous Lemma [3.4.4](#). Indeed, the approximation map $e^{-\delta d\Gamma(\omega^{2\sigma})} d\Gamma(\omega^{2\sigma})$ converges strongly to $d\Gamma(\omega^{2\sigma})$. This leads to

$$\begin{aligned}
 \|(d\Gamma(\omega^{2\sigma}))^{1/2} \psi(t)\|^2 &= \lim_{\delta \rightarrow 0} \|(e^{-\delta d\Gamma(\omega^{2\sigma})} d\Gamma(\omega^{2\sigma}))^{1/2} \psi(t)\|^2 \\
 &\lesssim \lim_{\delta \rightarrow 0} C_1 \|(e^{-\delta d\Gamma(\omega^{2\sigma})} d\Gamma(\omega^{2\sigma}) + 1)^{1/2} \psi\|^2 e^{C_2|t|} \\
 &= C_1 \|(d\Gamma(\omega^{2\sigma}) + 1)^{1/2} \psi\|^2 e^{C_2|t|}.
 \end{aligned}$$

And thus, we achieve the desired result. \square

3.4.1.3 Propagation of estimates uniformly for all times

As a consequence of the previous estimates, the uniform bound on the initial states $(\varrho_{\hbar})_{\hbar \in (0,1)}$ propagates in time.

Lemma 3.4.6 (Propagation of the assumptions [\(3.1.8\)](#) and [\(3.1.9\)](#) in time). *Assume [\(3.1.6\)](#) and $\omega^{1/2} \chi \in L^2(\mathbb{R}^d, dk)$. Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices satisfying [\(3.1.8\)](#) and [\(3.1.9\)](#). Then, the family of states $(\varrho_{\hbar}(t))_{\hbar \in (0,1)}$ and $(\tilde{\varrho}_{\hbar}(t))_{\hbar \in (0,1)}$ satisfy the same assumptions [\(3.1.8\)](#) and [\(3.1.9\)](#) uniformly for any $t \in \mathbb{R}$ in arbitrary compact interval.*

Proof. Before we begin the proof, remark that by spectral decomposition, we have

$$\varrho_h = \sum_{m \in \mathbb{N}} \lambda_h(m) |e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m)\rangle \langle e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m)|,$$

where $\lambda_h(m)$ are the eigenvalues and $\psi_h(m)$ are their related eigenfunctions. Let J be a compact interval. Then for all $t \in J$:

- We have with some $c \in \mathbb{R}_+^*$ the following uniform estimate

$$\begin{aligned} \mathrm{Tr}[\tilde{\varrho}_h(t) \hat{p}^2] &= \mathrm{Tr}\left[e^{i\frac{t}{\hbar}\hat{H}_{02}} \varrho_h(t) e^{-i\frac{t}{\hbar}\hat{H}_{02}} \hat{p}^2\right] \\ &= \mathrm{Tr}[\varrho_h(t) \hat{p}^2] \\ &= \sum_{m \in \mathbb{N}} \lambda_h(m) \|\hat{p} e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m)\|^2 \\ &= \sum_{m \in \mathbb{N}} \lambda_h(m) \langle e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m), \hat{p}^2 e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m) \rangle \\ &\leq C_1 \sum_{m \in \mathbb{N}} \lambda_h(m) \langle \psi_h(m), (\hat{H}_0 + \hat{p}^2 + 1) \psi_h(m) \rangle e^{C_2|t|} \leq c, \end{aligned}$$

where we have used Lemma [3.4.2](#) as well as assumptions [\(3.1.8\)](#) and [\(3.1.9\)](#).

- We have with some $c' \in \mathbb{R}_+^*$ the following estimate

$$\begin{aligned} \mathrm{Tr}[\tilde{\varrho}_h(t) \hat{q}^2] &= \mathrm{Tr}\left[e^{i\frac{t}{\hbar}\hat{H}_{02}} \varrho_h(t) e^{-i\frac{t}{\hbar}\hat{H}_{02}} \hat{q}^2\right] \\ &= \mathrm{Tr}[\varrho_h(t) \hat{q}^2] \\ &= \sum_{m \in \mathbb{N}} \lambda_h(m) \|\hat{q} e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m)\|^2 \\ &= \sum_{m \in \mathbb{N}} \lambda_h(m) \langle e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m), \hat{q}^2 e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m) \rangle \\ &\leq C_1 \sum_{m \in \mathbb{N}} \lambda_h(m) \langle \psi_h(m), (\hat{H}_0 + \hat{q}^2 + 1) \psi_h(m) \rangle e^{C_2|t|} \leq c', \end{aligned}$$

where we have used Lemma [3.4.1](#) as well as assumptions [\(3.1.8\)](#) and [\(3.1.9\)](#).

- We have for some c'' the following uniform estimate

$$\begin{aligned}
 \mathrm{Tr}[\tilde{\varrho}_h(t) d\Gamma(\omega^{2\sigma})] &= \mathrm{Tr}\left[e^{i\frac{t}{\hbar}\hat{H}_{02}} \varrho_h(t) e^{-i\frac{t}{\hbar}\hat{H}_{02}} d\Gamma(\omega^{2\sigma})\right] \\
 &= \mathrm{Tr}\left[\varrho_h(t) d\Gamma(\omega^{2\sigma})\right] \\
 &= \sum_{m \in \mathbb{N}} \lambda_h(m) \|d\Gamma(\omega^{2\sigma})^{1/2} e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m)\|^2 \\
 &= \sum_{m \in \mathbb{N}} \lambda_h(m) \langle e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m), d\Gamma(\omega^{2\sigma}) e^{-i\frac{t}{\hbar}\hat{H}} \psi_h(m) \rangle \\
 &\leq C_1 \sum_{m \in \mathbb{N}} \lambda_h(m) \langle \psi_h(m), (d\Gamma(\omega^{2\sigma}) + 1) \psi_h(m) \rangle e^{C_2|t|} \leq c''.
 \end{aligned}$$

where we have used Lemma [3.4.5](#) as well as assumptions [\(3.1.8\)](#).

□

3.4.2 Existence of unique Wigner measure

In this section, we prove that for any family of states $(\varrho_h)_{h \in (0,1)}$ which satisfies [\(3.1.8\)](#) and [\(3.1.9\)](#) and for any sequence $\hbar_n \rightarrow 0$, we can extract a subsequence $\hbar_{n_\ell} \rightarrow 0$ such that the set of Wigner measure

$$\mathcal{M}(\tilde{\varrho}_{\hbar_{n_\ell}}(t), \ell \in \mathbb{N})$$

is singleton. The main results are stated below:

Proposition 3.4.7 (Existence of unique Wigner measure $\tilde{\mu}_t$ for all times). *Assume that [\(3.1.6\)](#) and [\(3.1.7\)](#) hold true. Let $(\varrho_h)_{h \in (0,1)}$ be a family of density matrices satisfying [\(3.1.8\)](#) and [\(3.1.9\)](#). For any sequence $(\hbar_n)_{n \in \mathbb{N}}$ in $(0,1)$ such that $\hbar_n \rightarrow 0$, there exists a subsequence $(\hbar_{n_\ell})_{\ell \in \mathbb{N}}$ and a family of probability measures $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ such that for all $t \in \mathbb{R}$,*

$$\mathcal{M}(\tilde{\varrho}_{\hbar_{n_\ell}}(t), \ell \in \mathbb{N}) = \{\tilde{\mu}_t\}.$$

Moreover, for every compact time interval J there exists a constant $C > 0$ such that for all times $t \in J$,

$$\int_{X^0} \|u\|_{X^\sigma}^2 d\tilde{\mu}_t < C. \quad (3.4.7)$$

Proof. We prove the above proposition in two steps. Step 1 is dedicated to the extraction of a unique Wigner measure at fixed times. Step 2 generalizes for all times. To establish Step 1, it is necessary to recall the following result from [\[AN08, Theorem 6.2\]](#).

Proposition 3.4.8 (The set of Wigner measure is not empty). *Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices satisfying (3.1.8) and (3.1.9). Then for all sequences $(\hbar_n)_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \hbar_n = 0$, there exists a subsequence $(\hbar_{n_\ell})_{\ell \in \mathbb{N}}$ with $\lim_{\ell \rightarrow \infty} \hbar_{n_\ell} = 0$ such that*

$$\mathcal{M}(\varrho_{\hbar_{n_\ell}}, \ell \in \mathbb{N}) = \{\mu\}.$$

Moreover, we have

$$\int_{X^0} \|u\|_{X^0}^2 d\mu < +\infty, \quad \int_{X^0} \|u\|_{X^\sigma}^2 d\mu < +\infty. \quad (3.4.8)$$

Step 1: *Extraction of a unique Wigner measure at fixed times.*

Let $s \in \mathbb{R}$ be a fixed time. Let $\hbar_n \rightarrow 0$. Then, by Proposition 3.4.8, there exists a subsequence $(\hbar_\ell)_{\ell \in \mathbb{N}} \equiv (\hbar_{n_\ell})_{\ell \in \mathbb{N}}$ such that $\hbar_\ell \rightarrow 0$ and a probability measure $\tilde{\mu}_s \in \mathcal{P}(X^0)$ such that

$$\mathcal{M}(\tilde{\varrho}_{\hbar_{n_\ell}}(s), \ell \in \mathbb{N}) = \{\tilde{\mu}_s\}.$$

Moreover, we have the following integrability formula

$$\int_{X^0} \|u\|_{X^\sigma}^2 d\tilde{\mu}_s = \int_{X^0} (p^2 + q^2 + \|\alpha\|_{\mathcal{G}^\sigma}^2) d\tilde{\mu}_s < +\infty. \quad (3.4.9)$$

The above integrability formula is a consequence of the following implications proved in [AN15, Lemma 3.12] for some $C > 0$:

- (i) If $\text{Tr}[\varrho_{\hbar} \hat{N}_{\hbar}] \leq C \implies \forall \mu \in \mathcal{M}(\varrho_{\hbar}; \hbar \in (0, 1)), \int_{X^0} \|\alpha\|_{\mathcal{G}^0}^2 d\mu \leq C$;
- (ii) If $\text{Tr}[\varrho_{\hbar} d\Gamma(\omega^{2\sigma})] \leq C \implies \forall \mu \in \mathcal{M}(\varrho_{\hbar}; \hbar \in (0, 1)), \int_{X^0} \|\alpha\|_{\mathcal{G}^\sigma}^2 d\mu \leq C$;
- (iii) If $\text{Tr}[\varrho_{\hbar} (\hat{q}^2 + \hat{p}^2)] \leq C \implies \forall \mu \in \mathcal{M}(\varrho_{\hbar}; \hbar \in (0, 1)), \int_{X^0} (q^2 + p^2) d\mu \leq C$.

Now, by the help of uniform estimate in Lemma 3.4.6, we have the two family of states $(\varrho_{\hbar}(t))_{\hbar \in (0,1)}$ and $(\tilde{\varrho}_{\hbar}(t))_{\hbar \in (0,1)}$ satisfy uniformly the bounds of (i)-(ii)-(iii), one obtains then that (3.4.9) holds true as a consequence of (3.4.7) in Proposition 3.4.8.

Step 2: *Generalization for all times.*

Claim first that we have for all times $t \in \mathbb{R}$

$$\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t), \ell \in \mathbb{N}) = \{\tilde{\mu}_t\}. \quad (3.4.10)$$

Let us now prove the integrability formula in Proposition 3.4.7. Recall that our density matrices $(\varrho_{\hbar})_{\hbar \in (0,1)}$ satisfies the assumptions (3.1.8) and (3.1.9). And, by Lemma 3.4.6,

the family of states $(\tilde{\varrho}_h(t))_{h \in (0,1)}$ satisfies the same assumptions uniformly in any compact time interval J . Then, using (3.4.8), for all $t \in J$, we have (3.4.7). we come back now to prove the claim (3.4.10). Let $(t_j)_{j \in \mathbb{N}}$ be a countable dense set in \mathbb{R} . We have by Step 1 that

- for t_1 , for $\hbar_n \xrightarrow{n \rightarrow \infty} 0$, there exists a subsequence $\hbar_\ell \xrightarrow{\ell \rightarrow \infty} 0$ such that

$$\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t_1), \ell \in \mathbb{N}) = \{\tilde{\mu}_{t_1}\}.$$

- for t_2 , for $(\hbar_\ell)_{\ell \in \mathbb{N}}$, there exists a subsequence $(\hbar_{\phi_2(\ell)})_{\ell \in \mathbb{N}} \subset (\hbar_\ell)_{\ell \in \mathbb{N}}$ such that

$$\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t_2), \ell \in \mathbb{N}) = \{\tilde{\mu}_{t_2}\}.$$

- for t_3 , for $(\hbar_{\phi_2(\ell)})_{\ell \in \mathbb{N}}$, there exists a subsequence $(\hbar_{\phi_3(\ell)})_{\ell \in \mathbb{N}} \subset (\hbar_{\phi_2(\ell)})_{\ell \in \mathbb{N}}$ such that

$$\mathcal{M}(\tilde{\varrho}_{\hbar_{\phi_3(\ell)}}(t_3), \ell \in \mathbb{N}) = \{\tilde{\mu}_{t_3}\}.$$

- And so on, for t_j , for $(\hbar_{\phi_{j-1}(\ell)})_{\ell \in \mathbb{N}}$, there exists a subsequence $(\hbar_{\phi_j(\ell)})_{\ell \in \mathbb{N}} \subset (\hbar_{\phi_{j-1}(\ell)})_{\ell \in \mathbb{N}}$ such that

$$\mathcal{M}(\tilde{\varrho}_{\hbar_{\phi_j(\ell)}}(t_j), \ell \in \mathbb{N}) = \{\tilde{\mu}_{t_j}\}.$$

By diagonal arguments, we extract the subsequence $(\hbar_{\phi_\ell(\ell)})_\ell$ denoted by $(\hbar_\ell)_\ell$ for simplicity such that for all $j \in \mathbb{N}$, we have

$$\mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t_j), \ell \in \mathbb{N}) = \{\tilde{\mu}_{t_j}\}.$$

The above formula implies that for all $\xi = (p_0, q_0, \alpha_0) \in X^0$ and $\tilde{\xi} = (-2\pi q_0, 2\pi p_0, \sqrt{2}\pi \alpha_0)$

$$\lim_{\ell \rightarrow \infty} \text{Tr} \left[\mathcal{W}(\tilde{\xi}) \tilde{\varrho}_{\hbar_\ell}(t_j) \right] = \int_{X^0} e^{2\pi i \Re \langle \xi, u \rangle_{X^0}} d\tilde{\mu}_{t_j} \quad (3.4.11)$$

We have

$$\int_{X^0} \|u\|_{X^\sigma}^2 d\tilde{\mu}_{t_j} < +\infty.$$

The above formula implies that the set of Wigner measure $\{\tilde{\mu}_{t_j}\}_{j \in \mathbb{N}}$ is tight in $\mathcal{P}(X^0)$. This implies that according to Prokhorov's theorem in Lemma 3.7.1, for all $\xi \in X^0$, there exists a subsequence still denoted by t_j and a probability measure $\tilde{\mu}_t \in \mathcal{P}(X^0)$ such that $\tilde{\mu}_{t_j}$ converges weakly narrowly to $\tilde{\mu}_t$. This gives since the function $e^{2\pi i \Re \langle \xi, u \rangle_{X^0}}$ is bounded

that

$$\int_{X^0} e^{2\pi i \Re \langle \xi, u \rangle_{X^0}} d\tilde{\mu}_{t_j} \xrightarrow{t_j \rightarrow t} \int_{X^0} e^{2\pi i \Re \langle \xi, u \rangle_{X^0}} d\tilde{\mu}_t$$

Now, we need to prove that

$$\int_{X^0} e^{2\pi i \Re \langle \xi, u \rangle_{X^0}} d\tilde{\mu}_{t_j} \xrightarrow{t_j \rightarrow t} \lim_{\ell \rightarrow \infty} \text{Tr} \left[\mathcal{W}(\tilde{\xi}) \tilde{\varrho}_{\tilde{h}_\ell}(t) \right] \quad (3.4.12)$$

We start by

$$\begin{aligned} & \left| \int_{X^0} e^{2\pi i \Re \langle \xi, u \rangle_{X^0}} d\tilde{\mu}_{t_j} - \lim_{\ell \rightarrow \infty} \text{Tr} \left[\mathcal{W}(\tilde{\xi}) \tilde{\varrho}_{\tilde{h}_\ell}(t) \right] \right| \\ & \leq \left| \int_{X^0} e^{2\pi i \Re \langle \xi, u \rangle_{X^0}} d\tilde{\mu}_{t_j} - \lim_{\ell \rightarrow \infty} \text{Tr} \left[\mathcal{W}(\tilde{\xi}) \tilde{\varrho}_{\tilde{h}_\ell}(t_j) \right] \right| \quad \dots \quad (1) \\ & \quad + \left| \lim_{\ell \rightarrow \infty} \text{Tr} \left[\mathcal{W}(\tilde{\xi}) \tilde{\varrho}_{\tilde{h}_\ell}(t_j) \right] - \lim_{\ell \rightarrow \infty} \text{Tr} \left[\mathcal{W}(\tilde{\xi}) \tilde{\varrho}_{\tilde{h}_\ell}(t) \right] \right| \quad \dots \quad (2) \end{aligned} \quad (3.4.13)$$

The quantity (1) is zero by (3.4.11). The quantity (2) is zero by using the following estimates:

(i) For $\xi \in X^0$, for all $t, t_0 \in J$ where J is compact interval, we have

$$\left| \text{Tr} \left[\mathcal{W}(\xi) \left(\tilde{\varrho}_h(t) - \tilde{\varrho}_h(t_0) \right) \right] \right| \lesssim |t - t_0| \|\xi\|_{X^0} \left[\|\xi\|_{X^0} + \|\chi\|_{L^2} + \|\chi\|_{\mathcal{G}^0} \right];$$

(ii) For all $\xi_1, \xi_2 \in X^0$, for all $t \in J$ where J is compact interval, we have

$$\left| \text{Tr} \left[\left(\mathcal{W}(\xi_1) - \mathcal{W}(\xi_2) \right) \tilde{\varrho}_h(t) \right] \right| \lesssim \|\xi_1 - \xi_2\|_{X^0} \left[\|\xi_1\|_{X^0} + \|\xi_2\|_{X^0} + 1 \right].$$

For (i), we exploit (3.3.13), we have with $S = (\hat{H}_0 + 1)^{1/2}$

$$\left| \text{Tr} \left[\mathcal{W}(\xi) \left(\tilde{\varrho}_h(t) - \tilde{\varrho}_h(t_0) \right) \right] \right| \leq |t - t_0| \left\| (B_0 + \hbar B_1) S^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \left\| S \mathcal{W}(\xi) S^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \left\| S \tilde{\varrho}_h(t) \right\|_{\mathcal{L}^1(\mathcal{H})}.$$

Now using Lemma 3.3.5, the two estimates (3.3.28) and (3.3.29) and the two assumptions (3.1.8) and (3.1.9), we get the desired result.

For (ii), we have

$$\left| \text{Tr} \left[\left(\mathcal{W}(\xi_1) - \mathcal{W}(\xi_2) \right) \tilde{\varrho}_h(t) \right] \right| \leq \left\| \left(\mathcal{W}(\xi_1) - \mathcal{W}(\xi_2) \right) (\hat{N}_h + 1)^{-1/2} \right\|_{\mathcal{L}(\mathcal{H})} \underbrace{\left\| (\hat{N}_h + 1)^{1/2} \tilde{\varrho}_h(t) \right\|_{\mathcal{L}^1(\mathcal{H})}}_{< \infty \text{ by (3.1.8)}}.$$

And thus, using following the same computations as in [AN08, Lemma 3.1], the result follows. \square

3.5 Derivation of the characteristic equations

Subsection 3.5.1 focuses on investigating the convergence of the quantum dynamics towards the evolution of the particle-field equation. In Subsection 3.5.2, we derive the characteristic equation that the Wigner measure satisfies. Finally, in Subsection 3.5.3, we demonstrate that this characteristic equation is equivalent to a Liouville equation.

3.5.1 Convergence

In this section, we take the classical limit $\hbar_{n_\ell} \rightarrow 0$ as $\ell \rightarrow \infty$ in the Duhamel formula (3.3.13) to derive the characteristics equation satisfied by the Wigner measure $(\tilde{\mu}_t)_{t \in \mathbb{R}}$.

Lemma 3.5.1 (Convergence). *Assume (3.1.6) and $\omega^{1/2}\chi \in L^2(\mathbb{R}^d, dk)$. Let $(\varrho_\hbar)_{\hbar \in (0,1)}$ be a family of density matrices satisfying (3.1.8) and (3.1.9). Then for all $\xi = (z_0, \alpha_0) \in X^0$ and all $t, t_0 \in \mathbb{R}$, the Duhamel formula (3.3.13) converges to the following characteristics equation*

$$\int_{X^0} e^{Q(\xi, u)} d\tilde{\mu}_t(u) = \int_{X^0} e^{Q(\xi, u)} d\tilde{\mu}_{t_0}(u) - i \int_{t_0}^t \int_{X^0} b(s, \xi) e^{Q(\xi, u)} d\tilde{\mu}_s(u) ds \quad (3.5.1)$$

with

$$Q(\xi, u) = i\Im m\langle z, z_0 \rangle + \sqrt{2}i\Re e\langle \alpha_0, \alpha \rangle, \quad \xi = (z_0, \alpha_0), \quad u = (z, \alpha), \quad (3.5.2)$$

and where we have introduced

$$\begin{aligned} b(s, \xi) := & - \sum_{j=1}^n \nabla f_j(p_j) \cdot p_{0j} - \sum_{j=1}^n \nabla_{q_j} V(q) \cdot q_{0j} \\ & + \sum_{j=1}^n \left(\langle \alpha, b_j^0(s) \rangle_{L^2} + \langle b_j^0(s), \alpha \rangle_{L^2} \right) + \frac{i}{\sqrt{2}} \left(\langle \alpha_0, g_j(s) \rangle_{L^2} - \langle g_j(s), \alpha_0 \rangle_{L^2} \right). \end{aligned}$$

The function $b_j^0(s)$ is such that $b_j^0(s) \equiv b_j^0(s)(p_j, q_j)$ is defined as follows

$$b_j^0(s) := 2\pi i k \cdot q_{0j} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot q_j + i s \omega(k)} \quad (3.5.3)$$

Proof. From the Definition [3.1.2](#) of Wigner measure, we have

$$\lim_{\ell \rightarrow +\infty} \text{Tr}[\mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] = \int_{X^0} e^{Q(\xi, u)} d\tilde{\mu}_s(u).$$

We plug [\(3.3.25\)](#) in the Duhamel's formula [\(3.3.13\)](#), we get

$$\int_{X^0} e^{Q(\xi, u)} d\tilde{\mu}_t(u) = \int_{X^0} e^{Q(\xi, u)} d\tilde{\mu}_{t_0}(u) - i \int_{t_0}^t \lim_{\ell \rightarrow \infty} \text{Tr}[\mathbb{B}_0(s, \hbar_{n_\ell}, \xi) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] ds.$$

We have to prove then

$$\lim_{\ell \rightarrow +\infty} \text{Tr}[\mathbb{B}_0(s, \hbar_{n_\ell}, \xi) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] = \int_{X^0} b(s, \xi) e^{Q(\xi, u)} d\tilde{\mu}_s(u).$$

We start with

$$\begin{aligned} & \text{Tr}[\mathbb{B}_0(s, \hbar_{n_\ell}, \xi) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] \\ &= - \sum_{j=1}^n \text{Tr}[\nabla f_j(\hat{p}_j) \cdot p_{0j} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] - \sum_{j=1}^n \text{Tr}[\nabla_{q_j} V(\hat{q}) \cdot q_{0j} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] \\ &+ \sum_{j=1}^n \text{Tr}[\hat{a}_{\hbar_{n_\ell}} \left(\frac{\tilde{g}_j(s) - g_j(s)}{\hbar_{n_\ell}} \right) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] + \text{Tr}[\hat{a}_{\hbar_{n_\ell}}^* \left(\frac{\tilde{g}_j(s) - g_j(s)}{\hbar_{n_\ell}} \right) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] \\ &+ \sum_{j=1}^n \frac{i}{\sqrt{2}} \left(\text{Tr}[\langle \alpha_0, \tilde{g}_j(s) \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] - \text{Tr}[\langle \tilde{g}_j(s), \alpha_0 \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] \right). \end{aligned}$$

Let us start with the first two terms. We have

$$\lim_{\ell \rightarrow \infty} \text{Tr}[\nabla f_j(\hat{p}_j) \cdot p_{0j} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] = \int_{X^0} e^{Q(\xi, u)} \nabla f_j(p_j) \cdot p_{0j} d\tilde{\mu}_s(u), \quad (3.5.4)$$

$$\lim_{\ell \rightarrow \infty} \text{Tr}[\nabla_{q_j} V(\hat{q}) \cdot q_{0j} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] = \int_{X^0} e^{Q(\xi, u)} \nabla_{q_j} V(q) \cdot q_{0j} d\tilde{\mu}_s(u), \quad (3.5.5)$$

where we have used in the above two lines the convergent results in [\[AFH22\]](#) Lemma B.1] since $\langle p_j \rangle^{-1} \nabla f_j(p_j) \cdot p_{0j} \in L^\infty$ and $\langle q \rangle^{-1} \nabla_{q_j} V(q) \cdot q_{0j} \in L^\infty$. Let us deal now with the second line. The goal is to prove the following limit:

$$\lim_{\ell} \text{Tr} \left[\hat{a}_{\hbar_{n_\ell}}^* \left(\frac{\tilde{g}_j(s) - g_j(s)}{\hbar_{n_\ell}} \right) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s) \right] = \int_{X^0} e^{Q(\xi, u)} \langle \alpha, b_j^0(s) \rangle_{L^2} d\tilde{\mu}_s(u).$$

We start then with

$$\begin{aligned}
 & \left| \text{Tr} \left[\hat{a}_{\hbar_{n_\ell}}^* \left(\frac{\tilde{g}_j(s) - g_j(s)}{\hbar_{n_\ell}} \right) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s) \right] - \int_{X^0} e^{Q(\xi, u)} \langle \alpha, b_j^0(s) \rangle_{L^2} d\tilde{\mu}_s(u) \right| \\
 & \leq \underbrace{\left| \text{Tr} \left[\hat{a}_{\hbar_{n_\ell}}^* \left(\frac{\tilde{g}_j(s) - g_j(s)}{\hbar_{n_\ell}} - \tilde{b}_j^0(s) \right) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s) \right] \right|}_{(1)} \\
 & + \underbrace{\left| \text{Tr} \left[\hat{a}_{\hbar_{n_\ell}}^* \left(\tilde{b}_j^0(s) \right) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s) \right] - \int_{X^0} e^{Q(\xi, u)} \langle \alpha, b_j^0(s) \rangle_{L^2} d\tilde{\mu}_s(u) \right|}_{(2)},
 \end{aligned}$$

where

$$\tilde{b}_j^0(s) := 2\pi i k \cdot q_{0j} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot \hat{q}_j + i s \omega(k)}. \quad (3.5.6)$$

For (1), let $S = (\hat{H}_0 + 1)^{1/2}$, we have

$$\begin{aligned}
 & \left| \text{Tr} \left[\hat{a}_{\hbar_{n_\ell}}^* \left(\frac{\tilde{g}_j(s) - g_j(s)}{\hbar_{n_\ell}} - \tilde{b}_j^0(s) \right) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s) \right] \right| \\
 & \leq \underbrace{\left\| \frac{1}{\sqrt{\omega}} \left[\frac{\tilde{g}_j(s) - g_j(s)}{\hbar_{n_\ell}} - \tilde{b}_j^0(s) \right] \right\|_{\mathcal{L}(L^2, L^2 \otimes L^2)}}_{\xrightarrow{\ell \rightarrow \infty} 0} \|S \mathcal{W}(\xi) S^{-1}\|_{\mathcal{L}(\mathcal{H})} \|S \tilde{\varrho}_{\hbar_{n_\ell}}(s)\|_{\mathcal{L}^1(\mathcal{H})}.
 \end{aligned}$$

The above convergence follows from dominated convergence theorem and our assumptions.

For (2), according to the expression (3.5.6), we have

$$\tilde{b}_j^0(s) = e^{-2\pi i k \cdot \hat{q}_j} \varphi_j(k),$$

for some $\varphi_j \in L^2(\mathbb{R}^d)$. Hence, applying Lemma 3.7.6 in the appendix, we conclude that (2) converges to zero as $\ell \rightarrow \infty$. Similar discussions lead to

$$\lim_{\ell \rightarrow \infty} \text{Tr} \left[\hat{a}_{\hbar_{n_\ell}}^* \left(\frac{\tilde{g}_j(s) - g_j(s)}{\hbar_{n_\ell}} \right) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s) \right] = \int_{X^0} e^{Q(\xi, u)} \langle b_j^0(s), \alpha \rangle_{L^2} d\tilde{\mu}_s(u)$$

We deal now with the last line

$$\begin{aligned}
& \text{Tr}[\langle \alpha_0, \tilde{g}_j(s) \rangle_{L^2} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] \\
&= \int_{\mathbb{R}^d} \overline{\alpha_0(k)} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{is\omega(k)} e^{2\pi i k \cdot q_0 j \hbar_{n_\ell}} \text{Tr}[e^{-2\pi i k \cdot \hat{q}_j} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] dk \\
&= \int_{\mathbb{R}^d} \overline{\alpha_0(k)} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{is\omega(k)} e^{2\pi i k \cdot q_0 j \hbar_{n_\ell}} \text{Tr}[W_1(-2\pi k, 0) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] dk \\
&= \int_{\mathbb{R}^d} \overline{\alpha_0(k)} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{is\omega(k)} e^{2\pi i k \cdot q_0 j \hbar_{n_\ell}} \text{Tr}[\mathcal{W}(-2\pi k, 0, 0) \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] dk \\
&= \int_{\mathbb{R}^d} \overline{\alpha_0(k)} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{is\omega(k)} e^{2\pi i k \cdot q_0 j \hbar_{n_\ell}} \text{Tr}[e^{\frac{-i\hbar_{n_\ell}}{2} \mathfrak{Im}\langle i2\pi k, q_0 + ip_0 \rangle} \mathcal{W}(p_0 - 2\pi k e_j, q_0, \alpha_0) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] dk.
\end{aligned}$$

We conclude

$$\begin{aligned}
& \lim_{\ell \rightarrow +\infty} \text{Tr}[\langle \alpha_0, \tilde{g}_j(s) \rangle_{L^2} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] \\
&= \int_{\mathbb{R}^d} \overline{\alpha_0(k)} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{is\omega(k)} \lim_{\ell \rightarrow +\infty} \text{Tr}[\mathcal{W}(p_0 - 2\pi k, q_0, \alpha_0) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] dk \\
&= \int_{\mathbb{R}^d} \overline{\alpha_0(k)} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{is\omega(k)} \int_{X^0} e^{-2\pi i k \cdot q_j} e^{Q(\xi, u)} d\tilde{\mu}_s(u) dk
\end{aligned}$$

By Fubini, we get

$$\begin{aligned}
& \lim_{\ell \rightarrow +\infty} \text{Tr}[\langle \alpha_0, \tilde{g}_j(s) \rangle_{L^2} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] \\
&= \int_{X^0} e^{Q(\xi, u)} \int_{\mathbb{R}^d} \overline{\alpha_0(k)} \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot q_j + is\omega(k)} dk d\tilde{\mu}_s(u) \\
&= \int_{X^0} e^{Q(\xi, u)} \langle \alpha_0, g_j(s) \rangle_{L^2} d\tilde{\mu}_s(u).
\end{aligned}$$

Similar discussions also work to prove

$$\lim_{\ell \rightarrow +\infty} \text{Tr}[\langle \tilde{g}_j(s), \alpha_0 \rangle_{L^2} \mathcal{W}(\xi) \tilde{\varrho}_{\hbar_{n_\ell}}(s)] = \int_{X^0} \langle g_j(s), \alpha_0 \rangle_{L^2} e^{Q(\xi, u)} d\tilde{\mu}_s(u).$$

□

3.5.2 The characteristic equation

Below, we derive the final form of the time-evolution equation satisfied by the Wigner measure $\tilde{\mu}_t$.

Corollary 3.5.2 (Characteristic equation). *Assume (3.1.6) and $\omega^{1/2}\chi \in L^2(\mathbb{R}^d, dk)$. Then, the characteristic equation (3.5.1) can be further reduced to the following form*

$$\begin{aligned} \int_{X^0} e^{2i\pi\Re\langle y, u \rangle_{X^\sigma}} d\tilde{\mu}_t(u) &= \int_{X^0} e^{2i\pi\Re\langle y, u \rangle_{X^\sigma}} d\tilde{\mu}_{t_0}(u) \\ &\quad + 2\pi i \int_{t_0}^t \int_{X^0} e^{2i\pi\Re\langle y, u \rangle_{X^\sigma}} \Re\langle v(s, u), y \rangle_{X^\sigma} d\tilde{\mu}_s(u) ds, \end{aligned} \quad (3.5.7)$$

for all $t, t_0 \in \mathbb{R}$ and $y \in X^\sigma$.

Proof. Define

$$\tilde{\xi} := \left(\frac{z_0}{2i\pi}, \frac{\alpha_0}{\sqrt{2\pi}} \right) \in X^0, \quad \text{with } \xi = (z_0, \alpha_0) \in X^0.$$

We claim that

$$b(s, \xi) = -2\pi \Re\langle v(s, u), \tilde{\xi} \rangle_{X^0}.$$

Indeed, we first remark that

$$-2\pi \Re\langle v(s, u), \tilde{\xi} \rangle_{X^0} = \underbrace{-2\pi \Re\langle (v(s, u))_z, \frac{z_0}{2i\pi} \rangle}_{(1)} - \underbrace{2\pi \Re\langle (v(s, u))_\alpha, \frac{\alpha_0}{\sqrt{2\pi}} \rangle_{L^2}}_{(2)}.$$

For (1), we have

$$\begin{aligned} -2\pi \Re\langle (v(s, u))_z, \frac{z_0}{2i\pi} \rangle &= -\Im\langle (v(s, u))_z, z_0 \rangle \\ &= -\sum_{j=1}^n \left((v(s, u))_{q_j} \cdot p_{0j} - (v(s, u))_{p_j} \cdot q_{0j} \right) \\ &= -\sum_{j=1}^n \nabla_{q_j} V(q) \cdot q_{0j} + \sum_{j=1}^n \left(\langle \alpha, b_j^0(s) \rangle_{L^2} + \langle b_j^0(s), \alpha \rangle_{L^2} \right), \end{aligned}$$

where recall that $v(s, u)$ is as in (3.2.6) and $b_j^0(s)$ is as in (3.5.3).

For (2), we have

$$\begin{aligned}
 & -2\pi \Re e \langle (v(s, u))_\alpha, \frac{\alpha_0}{\sqrt{2\pi}} \rangle_{L^2} = -\sqrt{2} \Re e \langle (v(s, u))_\alpha, \alpha_0 \rangle_{L^2} \\
 & = -\sqrt{2} \Re e \left\langle -i \sum_{j=1}^n \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-2\pi i k \cdot q_j + i s \omega(k)}, \alpha_0 \right\rangle_{L^2} \\
 & = \sqrt{2} \sum_{j=1}^n \Im m \langle g_j(s), \alpha_0 \rangle_{L^2}.
 \end{aligned}$$

where $g_j(s)$ is as in (3.3.16). On the other hand, we have

$$\begin{aligned}
 & \frac{i}{\sqrt{2}} \left(\langle \alpha_0, g_j(s) \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} - \langle g_j(s), \alpha_0 \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} \right) \\
 & = -\sqrt{2} \Im m \langle \alpha_0, g_j(s) \rangle_{L^2(\mathbb{R}^d, \mathbb{C})} = \sqrt{2} \Im m \langle g_j(s), \alpha_0 \rangle_{L^2(\mathbb{R}^d, \mathbb{C})}.
 \end{aligned}$$

And, thus combining the above arguments, we prove the claimed results. The Characteristic equation (3.5.1) becomes then

$$\int_{X^0} e^{Q(\xi, u)} d\tilde{\mu}_t(u) = \int_{X^0} e^{Q(\xi, u)} d\tilde{\mu}_{t_0}(u) + 2\pi i \int_{t_0}^t \int_{X^0} e^{Q(\xi, u)} \Re e \langle v(s, u), \tilde{\xi} \rangle_{X^0} d\tilde{\mu}_s(u) ds. \quad (3.5.8)$$

We have, with $Q(\xi, u)$ as in (3.5.2), that

$$Q(\xi, u) = 2\pi i \Re e \langle \tilde{\xi}, u \rangle_{X^0}. \quad (3.5.9)$$

We have also for all $y = (p, q, \alpha) \in X^{2\sigma}$ and all $\tilde{\xi} = (p, q, \omega^{2\sigma} \alpha) \in X^0$ that

$$\begin{aligned}
 \Re e \langle y, u \rangle_{X^\sigma} &= \Re e \langle \tilde{\xi}, u \rangle_{X^0}, \\
 \Re e \langle v(s, u), y \rangle_{X^\sigma} &= \Re e \langle v(s, u), \tilde{\xi} \rangle_{X^0}
 \end{aligned} \quad (3.5.10)$$

By this way, plugging (3.5.9)-(3.5.10) in (3.5.8) gives that (3.5.7) is valid for all $y \in X^{2\sigma}$. The latter could be extended to all $y \in X^\sigma$ by dominated convergence theorem and the bound (3.5.12). \square

3.5.3 The Liouville equation

In this part, we relate the characteristic equation (3.5.7) satisfied by the set of Wigner measures $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ to a special Liouville equation. To do that, we need to have some inte-

grability condition of the vector field v of (IVP) with respect to this Wigner measure and some regularities of the latter measure.

Lemma 3.5.3 (Integrability of the vector field v). *Assume (3.1.6) and (3.1.7) hold true. Then, there exists a constant $C > 0$ such that for all $u = (p, q, \alpha) \in X^\sigma$,*

$$\|v(t, u)\|_{X^\sigma} \leq C(\|u\|_{X^0}^2 + 1). \quad (3.5.11)$$

Moreover, for any bounded open interval I ,

$$\int_I \int_{X^\sigma} \|v(t, u)\|_{X^\sigma} d\tilde{\mu}_t(u) dt < +\infty. \quad (3.5.12)$$

Proof. The non-autonomous vector field v is defined in terms of the nonlinearity \mathcal{N} as indicated in (3.2.6). Then it is not hard to see by looking at the proof of Proposition 3.2.4 that

- in the semi-relativistic case, since the function $\nabla f_j(\hat{p}_j)$ is bounded, we get

$$\|v(t, u)\|_{X^\sigma} \leq C(\|\alpha\|_{L^2}^2 + 1). \quad (3.5.13)$$

- in the non-relativistic case, we get

$$\|v(t, u)\|_{X^\sigma} \leq C(\|\alpha\|_{L^2}^2 + |p|^2 + 1). \quad (3.5.14)$$

Thus, both inequalities (3.5.13) and (3.5.14) lead to (3.5.11). Now, the integrability condition (3.5.12) is a consequence of (3.4.7) in Proposition 3.4.7 \square

We establish now some regularity of the Wigner measures $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ with respect to time.

Lemma 3.5.4 (Regular properties of the Wigner Measure $\tilde{\mu}_t$). *The Wigner measures $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ extracted in Proposition 3.4.7 satisfy*

- (i) $\tilde{\mu}_t$ concentrates on X^σ i.e. $\tilde{\mu}_t(X^\sigma) = 1$;
- (ii) $\mathbb{R} \ni t \mapsto \tilde{\mu}_t \in \mathcal{P}(X^\sigma)$ is weakly narrowly continuous.

Proof. For the first assertion (i), we have from Proposition 3.4.7 that

$$\int_{X^0} \|u\|_{X^\sigma}^2 d\tilde{\mu}_t < C.$$

And, from the Markov's inequality, we have

$$\tilde{\mu}_t(\{u \in X^0 : \|u\|_{X^\sigma} \geq \varepsilon\}) \leq \frac{1}{\varepsilon^2} \tilde{\mu}_t(\|u\|_{X^\sigma}).$$

Let $\varepsilon \rightarrow \infty$, we get

$$\tilde{\mu}_t(\{u \in X^0; u \notin X^\sigma\}) = 0.$$

Hence, we get that the measure $\tilde{\mu}_t$ is concentrated in X^σ . The second assertion (ii) is proved in a similar fashion as in [AFH22, Lemma 5.5] using Prokhorov's Theorem. \square

In the coming discussions, for more details, we refer the reader to Appendix A in [AFH22]. Let I be an open bounded interval. Define the space of smooth cylindrical functions on $I \times X^\sigma$, denoted by $\mathcal{C}_{0,cyl}^\infty(I \times X^\sigma)$, as follows

$$\mathcal{C}_{0,cyl}^\infty(I \times X^\sigma) := \left\{ \phi : I \times X^\sigma \rightarrow \mathbb{R}; \phi(t, u) = \psi(t, \pi(u)), \psi \in \mathcal{C}_0^\infty(I \times \mathbb{R}^{d'}), \right. \\ \left. \pi : X^\sigma \rightarrow \mathbb{R}^{d'}, d' \in \mathbb{N} \right\},$$

where $\pi : X^\sigma \rightarrow \mathbb{R}^{d'}$ is a projection of the form $\pi : u \rightarrow \pi(u) = (\Re\langle u, e_1 \rangle_{X^\sigma}, \dots, \Re\langle u, e_{d'} \rangle_{X^\sigma})$, with $(e_1, \dots, e_{d'})$ is an arbitrary orthonormal family of X^σ .

Proposition 3.5.5. *The family of Wigner measures $(\tilde{\mu}_t)_{t \in \mathbb{R}}$ defined in Proposition 3.4.7 is a weakly narrowly continuous solution to the following Liouville equation*

$$\int_I \int_{X^\sigma} \{ \partial_t \phi(t, u) + \Re\langle v(t, u), \nabla \phi(t, u) \rangle_{X^\sigma} \} d\tilde{\mu}_t(u) dt = 0, \quad (\text{LE})$$

for any bounded open interval I containing the origin with $\phi \in \mathcal{C}_{0,cyl}^\infty(I \times X^\sigma)$.

Proof. It is a direct consequence of Lemma 3.7.2 by selecting $H \equiv X^\sigma$ which is a Hilbert space. More precisely, all the prerequisites of Lemma 3.7.2 are satisfied. Indeed, we have

- from Corollary 3.5.2 that the set of Wigner measures $\{\tilde{\mu}_t\}_{t \in I}$ solves the characteristic equation (3.5.7);
- from Lemma 3.5.4, we have checked that $\tilde{\mu}_t \in \mathcal{P}(X^\sigma)$ is a weakly narrowly continuous;
- from Lemma 3.5.3, we have checked the integrability condition of v with respect to $\tilde{\mu}_t$.

And thus the result follows. \square

3.6 Proof of the main result

In order to prove the main Theorems [3.1.1](#) and [3.1.3](#), we must establish some identities. It is important to note that the statement of Theorem [3.1.1](#) is not related to the quantum dynamics and does not require any restrictions on it. Therefore, our plan is to ensure that the assumptions [\(3.1.8\)](#) and [\(3.1.9\)](#) are applied to a specific class of density matrices, namely the coherent states. To achieve this, we must first define the coherent states for the particle and field components separately, and then generalize to the entire interacting space since we are dealing with an interaction between particles and field. Let $u_0 = (z_0, \alpha_0) \in X^0$ and consider the family of coherent states

$$\mathcal{C}_h(u_0) = \left| W_1\left(\frac{\sqrt{2}}{i\hbar}z_0\right)\psi \otimes W_2\left(\frac{\sqrt{2}}{i\hbar}\alpha_0\right)\Omega \right\rangle \left\langle W_1\left(\frac{\sqrt{2}}{i\hbar}z_0\right)\psi \otimes W_2\left(\frac{\sqrt{2}}{i\hbar}\alpha_0\right)\Omega \right|$$

where we have introduced

- the coherent vector: $W_1\left(\frac{\sqrt{2}}{i\hbar}z_0\right)\psi$, centered on $z_0 \in \mathbb{C}^{dn}$ where $\psi(x) = (\pi\hbar)^{-dn/4} e^{-x^2/2\hbar} \in L^2(\mathbb{R}^{dn}, dx)$ is the normalized gaussian function on the particles related to the particle space $L^2(\mathbb{R}^{dn}, \mathbb{C})$.
- the coherent vector: $W_2\left(\frac{\sqrt{2}}{i\hbar}\alpha_0\right)\Omega$ in the Fock space, for $\alpha \in \mathcal{G}^0$ and Ω is the vacuum vector on the fock space.

It bears noting that these family of coherent states gives rise to a family of density matrices satisfying the assumptions [\(3.1.8\)](#) and [\(3.1.9\)](#).

Lemma 3.6.1 (The family of coherent states). *The family of coherent states $(\mathcal{C}_h(u_0))_{h \in (0,1)}$ satisfies*

$$\mathcal{M}(\mathcal{C}_h(u_0), \hbar \in (0, 1)) = \{\delta_{u_0}\},$$

where δ_{u_0} is the Dirac measure centered on u_0 . Moreover, if $u_0 = (z_0, \alpha_0) \in X^\sigma$, then $(\mathcal{C}_h(u_0))_{h \in (0,1)}$ satisfies [\(3.1.8\)](#) and [\(3.1.9\)](#).

Proof. We find also

$$\begin{aligned}\mathrm{Tr}(\mathcal{C}_{\hbar}(u_0) d\Gamma(\omega^{2\sigma})) &= \|\alpha_0\|_{\mathcal{G}^\sigma}^2 \\ \mathrm{Tr}(\mathcal{C}_{\hbar}(u_0) \hat{p}^2) &= \langle \psi, \hat{p}^2 \psi \rangle - 2p_0^2 \\ \mathrm{Tr}(\mathcal{C}_{\hbar}(u_0) \hat{q}^2) &= \langle \psi, \hat{q}^2 \psi \rangle - 2q_0^2.\end{aligned}$$

□

Below, we give useful lemma which relates the Wigner measure $\tilde{\mu}_t$ to μ_t in terms of the free field flow Φ_t^f .

Lemma 3.6.2 (Relations between the sets of Wigner measure). *Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices satisfying (3.1.8) and (3.1.9). Define*

$$\tilde{\varrho}_{\hbar}(t) := e^{i\frac{t}{\hbar}d\Gamma(\omega)} \varrho_{\hbar} e^{-i\frac{t}{\hbar}d\Gamma(\omega)}.$$

Then, we can assert that

1. the family of states $(\tilde{\varrho}_{\hbar}(t))_{\hbar \in (0,1)}$ satisfies (3.1.8) and (3.1.9);
2. for all sequences $(\hbar_n)_{n \in \mathbb{N}}$ with $\hbar_n \rightarrow 0$, there exists a subsequence \hbar_{n_ℓ} with $\hbar_{n_\ell} \rightarrow 0$ such that

$$\mathcal{M}(\tilde{\varrho}_{\hbar_{n_\ell}}(t), \ell \in \mathbb{N}) = \{(\Phi_{-t}^f)_\# \mu; \mu \in \mathcal{M}(\tilde{\varrho}_{\hbar_{n_\ell}}, \ell \in \mathbb{N})\},$$

where Φ_t^f is the free field flow as in (3.2.5) .

Proof. The first assertion is a consequence of Lemma 3.4.6. Let $\mu \in \mathcal{M}(\varrho_{\hbar_\ell}, \ell \in \mathbb{N})$ and $\tilde{\mu}_t \in \mathcal{M}(\tilde{\varrho}_{\hbar_\ell}(t), \ell \in \mathbb{N})$. On one hand, we have

$$\begin{aligned}\lim_{\ell} \mathrm{Tr} \left[\tilde{\varrho}_{\hbar_\ell}(t) \mathcal{W}(\xi) \right] &= \lim_{\ell} \mathrm{Tr} \left[\varrho_{\hbar_\ell} e^{-i\frac{t}{\hbar}d\Gamma(\omega)} \mathcal{W}(\xi) e^{i\frac{t}{\hbar}d\Gamma(\omega)} \right] = \lim_{\ell} \mathrm{Tr} \left[\varrho_{\hbar_\ell} \mathcal{W}(\Phi_t^f(\xi)) \right] \\ &= \int_{X^0} e^{Q(\Phi_t^f(\xi), u)} d\mu(u) = \int_{X^0} e^{Q(\xi, \Phi_{-t}^f(u))} d\mu(u) = \int_{X^0} e^{Q(\xi, u)} d(\Phi_{-t}^f)_\# \mu(u).\end{aligned}$$

On the other hand, we have

$$\lim_{\ell} \mathrm{Tr} \left[\tilde{\varrho}_{\hbar_\ell}(t) \mathcal{W}(\xi) \right] = \int_{X^0} e^{Q(\xi, u)} d\tilde{\mu}(u).$$

We conclude then that

$$\tilde{\mu}_t = (\Phi_{-t}^f)_\# \mu.$$

□

Below, we start the proof Theorems [3.1.1](#) and [3.1.3](#).

Proof of Theorem [3.1.1](#). Let $u_0 \in X^\sigma$ and defines the density matrices for all $\hbar \in (0, 1)$ as follows

$$\varrho_\hbar := \mathcal{C}_\hbar(u_0).$$

Then since $u_0 \in X^\sigma$, we can assert by Lemma [3.6.1](#) that the family of density matrices $(\varrho_\hbar)_{\hbar \in (0,1)}$ satisfies [\(3.1.8\)](#) and [\(3.1.9\)](#). Thus, with this choice of density matrices and using the arguments of Proposition [3.4.7](#), we can assert that for each sequence $(\hbar_n)_{n \in \mathbb{N}}$ with $\hbar_n \xrightarrow{n \rightarrow \infty} 0$, there exists a subsequence $(\hbar_{n_\ell})_{\ell \in \mathbb{N}}$ with $\hbar_{n_\ell} \xrightarrow{\ell \rightarrow \infty} 0$ and a family of Borel probability measure $\{\tilde{\mu}_t\}_{t \in \mathbb{R}}$ in X^0 such that

$$\mathcal{M}(e^{i\frac{t}{\hbar_{n_\ell}}d\Gamma(\omega)} e^{-i\frac{t}{\hbar_{n_\ell}}\hat{H}} \mathcal{C}_{\hbar_{n_\ell}}(u_0) e^{i\frac{t}{\hbar_{n_\ell}}\hat{H}} e^{-i\frac{t}{\hbar_{n_\ell}}d\Gamma(\omega)}, \ell \in \mathbb{N}) = \{\tilde{\mu}_t\}.$$

Now, on one hand, we do have from Proposition [3.5.5](#) that $\{\tilde{\mu}_t\}_{t \in \mathbb{R}}$ is weakly narrowly continuous solution to the Liouville equation [\(LE\)](#); from the other hand, from Lemma [3.5.3](#), we can assert that all the prerequisites to apply Theorem [3.7.3](#) are in our hand. To recover the proof of Theorem [3.1.1](#), we follow the steps below.

- ◁ We apply Theorem [3.7.3](#) with the measure $\tilde{\mu}_t$ obtained above, we get the global well posedness of the initial value problem [\(IVP\)](#) $\tilde{\mu}_0$ -almost all initial data in X^σ as well as the existence of a generalized Borel measurable global flow $\tilde{\Phi}_t$ as follows

$$\begin{aligned} \tilde{\Phi}_t : \mathfrak{G} &\longrightarrow X^\sigma \\ u_0 &\longmapsto u(t), \end{aligned}$$

where \mathfrak{G} is the ensemble of initial data obtained from Theorem [3.7.3](#).

- ◁ Let $u_0 \in X^\sigma$. From Lemma [3.6.1](#), we have $\tilde{\mu}_0(\mathfrak{G}) = \delta_{u_0}(\mathfrak{G}) = 1$. This implies $u_0 \in \mathfrak{G}$;
- ◁ Use the equivalence between the solution to [\(IVP\)](#) and [\(3.1.1\)](#), we can show the existence and uniqueness of the solution of [\(3.1.1\)](#) with a generalized global flow

$$\Phi_t(u_0) = \Phi_t^f \circ \tilde{\Phi}_t(u_0),$$

where Φ_t^f is the free flow and $\tilde{\Phi}_t$ is the generalized flow of [\(IVP\)](#);

Proof of Theorem 3.1.3 We have here to prove the validity of Bohr’s correspondence principle. Assume $(\varrho_{\hbar})_{\hbar}$ is a family of density matrices satisfying the Assumptions (3.1.8) and (3.1.9). Then, using Proposition 3.4.7, we can assert that for each sequence $(\hbar_n)_{n \in \mathbb{N}}$ with $\hbar_n \rightarrow 0$, there exists a subsequence $(\hbar_{n_\ell})_{\ell \in \mathbb{N}}$ with $\hbar_{n_\ell} \rightarrow 0$ and a family of Borel probability measures $\{\tilde{\mu}_t\}_{t \in \mathbb{R}}$ in X^0 such that

$$\mathcal{M}(\tilde{\varrho}_{\hbar_{n_\ell}}(t), \ell \in \mathbb{N}) = \{\tilde{\mu}_t\}.$$

By Lemma 3.6.2, we have

$$\mathcal{M}(\varrho_{\hbar_{n_\ell}}(t), \ell \in \mathbb{N}) = \{(\Phi_t^f)_\# \tilde{\mu}_t; \tilde{\mu}_t \in \mathcal{M}(\tilde{\varrho}_{\hbar_{n_\ell}}(t), \ell \in \mathbb{N})\}.$$

This implies that

$$\mathcal{M}(\varrho_{\hbar_{n_\ell}}(t), \ell \in \mathbb{N}) = \{\mu_t\} = \{(\Phi_t^f)_\# \tilde{\mu}_t\}$$

From (ii) in Probabilistic representation, we can assert that for any bounded Borel functions $\psi : X^\sigma \rightarrow \mathbb{R}$

$$\int_{X^\sigma} \psi(u) d\tilde{\mu}_t = \int_{\mathcal{F}_I} \psi(e_t(u_0, u(\cdot))) d\eta(u_0, u(\cdot)).$$

Since, we have the generalized global flow $\tilde{\Phi}_t$ to (IVP), we get

$$e_t(u_0, u(\cdot)) = \tilde{\Phi}_t(e_0(u_0, u(\cdot))) = \tilde{\Phi}_t(u_0).$$

This gives

$$\int_{X^\sigma} \psi(u) d\tilde{\mu}_t = \int_{X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma)} \psi \circ \tilde{\Phi}_t(e_0(u_0, u(\cdot))) d\eta(u_0, u(\cdot)) = \int_{X^\sigma} \psi \circ \tilde{\Phi}_t(u) d\tilde{\mu}_0.$$

We conclude that $\tilde{\mu}_t = (\tilde{\Phi}_t)_\# \tilde{\mu}_0$. This implies that:

$$\mu_t = (\Phi_t^f)_\# \tilde{\mu}_t = (\Phi_t^f \circ \tilde{\Phi}_t)_\# \tilde{\mu}_0 = (\Phi_t)_\# \tilde{\mu}_0 = (\Phi_t)_\# \mu_0$$

and where we have used $\tilde{\mu}_0 = \mu_0$ as a consequence of

$$\tilde{\varrho}_{\hbar}(0) = \varrho_{\hbar}(0) = \varrho_{\hbar}.$$

□

3.7 Appendix

3.7.1 Prokhorov theorem

Let X be separable metric space. The proof of the following result is proved in [AGS08, Theorem 5.1.3].

Theorem 3.7.1 (Prokhorov Theorem). *If a set $\mathcal{K} \subset \mathcal{P}(X)$ is tight i.e.*

$$\forall \varepsilon > 0, \exists K_\varepsilon \text{ compact in } X \text{ such that } \mu(X \setminus K_\varepsilon) \leq \varepsilon, \forall \mu \in \mathcal{K},$$

then \mathcal{K} is relatively compact in $\mathcal{P}(X)$.

3.7.2 Useful results

The following results relate the Liouville equations and the Characteristic equations satisfied by a family of Wigner measures. For more details, we refer the reader to [Rou18a, Proposition 4.2]. Let \mathbb{H} be a Hilbert space.

Lemma 3.7.2 (Equivalence). *Let $v : \mathbb{R} \times \mathbb{H} \rightarrow \mathbb{H}$ be a continuous vector field such that it is bounded on bounded sets. Let $I \ni t \rightarrow \mu_t$ a weakly narrowly continuous curve in $\mathcal{P}(\mathbb{H})$ such that we have the following integrability condition*

$$\int_I \int_{\mathbb{H}} \|v(t, u)\|_{\mathbb{H}} d\mu_t dt < +\infty.$$

Then, the following statements are equivalent:

- (i) $\{\mu_t\}_{t \in I}$ is a solution of Liouville equation (LE);
- (ii) $\{\mu_t\}_{t \in I}$ solves the characteristic equation (3.5.7) for all $t \in I$ and for all $y \in \mathbb{H}$.

The subsequent outcomes illustrate how to build the global solution to the (IVP) utilizing measure-theoretical approaches and certain probabilistic representations of the measure-valued solutions for the Liouville equation. Additional information on the topic can be found in the Appendices of [AFH22].

Theorem 3.7.3 (Global flow of the initial value problem). *Let $v : \mathbb{R} \times X^\sigma \rightarrow X^\sigma$ be a continuous vector field bounded on bounded sets. Assume*

- $\exists t \in \mathbb{R} \rightarrow \tilde{\mu}_t \in \mathcal{P}(X^\sigma)$ a weakly narrowly continuous solution to (LE) satisfying the integrability condition (3.5.12) on I ;

- There is at most one solution of the initial value problem (IVP) over any bounded open interval I containing the origin .

Then for $\tilde{\mu}_0$ -almost all initial conditions u_0 in X^σ , there exists a unique global strong solution to (IVP) . In addition, the set

$$\mathfrak{G} := \{u_0 \in X^\sigma : \exists u(\cdot) \text{ a global strong solution of } (\text{IVP}) \\ \text{with the initial condition } u_0\},$$

is Borel subset of X^σ with $\tilde{\mu}_0(\mathfrak{G}) = 1$ and for any time $t \in \mathbb{R}$ the map $u_0 \in \mathfrak{G} \rightarrow \tilde{\Phi}_t(u_0) = u(t)$ is Borel measurable.

Proposition 3.7.4 (Superposition principle). *There exists $\eta \in \mathcal{P}(X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma))$ satisfying:*

(i) $\eta(\mathcal{F}_I) = 1$ where

$$\mathcal{F}_I := \left\{ (u_0, u(\cdot)) \in X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma) : u(\cdot) \text{ satisfies } (\text{IVP}) \text{ on } I \text{ with } u_0 \right\}$$

(ii) $\tilde{\mu}_t = (e_t)_\# \eta$, $\forall t \in I$, where the map

$$e_t : (u_0, u(\cdot)) \in X^\sigma \times \mathcal{C}(\bar{I}, X^\sigma) \rightarrow u(t) \in X^\sigma$$

is the evaluation map.

3.7.3 Technical results about convergence

Finally, we prove two technical lemmas which are useful for the study of the quantum-classical convergence in Subsection $(3.5.1)$. We denote by \mathcal{F} the Fourier transform on \mathbb{R}^d .

Lemma 3.7.5. *Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices on the Hilbert space \mathcal{H} satisfying $(3.1.8)$ - $(3.1.9)$ for $\sigma = \frac{1}{2}$. Assume that for some sequence $(\hbar_\ell)_{\ell \in \mathbb{N}} \subset (0, 1)$, $\hbar_\ell \rightarrow 0$, there exists a (unique) Borel measure $\mu \in \mathcal{P}(X^0)$ such that*

$$\mathcal{M}(\varrho_{\hbar_\ell}, \ell \in \mathbb{N}) = \{\mu\}.$$

Then for any $\varphi \in L^2(\mathbb{R}^d)$, $\beta \in \mathcal{F}(L^1(\mathbb{R}^d))$ and $\xi \in X^0$, $j = 1, \dots, n$,

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \operatorname{Tr} \left[\beta(\hat{q}_j) \hat{a}_{\hbar_\ell}(\varphi) \mathcal{W}(\xi) \varrho_{\hbar_\ell} \right] &= \int_{X^0} \beta(q_j) \langle \varphi, \alpha \rangle_{L^2(\mathbb{R}^d)} e^{Q(\xi, u)} d\mu(u), \\ \lim_{\ell \rightarrow \infty} \operatorname{Tr} \left[\beta(\hat{q}_j) \hat{a}_{\hbar_\ell}^*(\varphi) \mathcal{W}(\xi) \varrho_{\hbar_\ell} \right] &= \int_{X^0} \beta(q_j) \langle \alpha, \varphi \rangle_{L^2(\mathbb{R}^d)} e^{Q(\xi, u)} d\mu(u), \end{aligned}$$

with $u = (p, q, \alpha) \in X^0$ and $Q(\cdot, \cdot)$ is the phase given in (3.5.2).

Proof. The two limits are similar. By linear combinations one can use instead the fields operators

$$\hat{\phi}_{\hbar}(\varphi) = \frac{\hat{a}_{\hbar}^*(\varphi) + \hat{a}_{\hbar}(\varphi)}{\sqrt{2}} \quad \text{and} \quad \hat{\pi}_{\hbar}(\varphi) = \frac{i\hat{a}_{\hbar}^*(\varphi) - i\hat{a}_{\hbar}(\varphi)}{\sqrt{2}}.$$

So, it is enough to show

$$\lim_{\ell \rightarrow \infty} \operatorname{Tr} \left[\beta(\hat{q}_j) \hat{\phi}_{\hbar_\ell}(\varphi) \mathcal{W}(\xi) \varrho_{\hbar_\ell} \right] = \sqrt{2} \int_{X^0} \beta(q_j) \Re \langle \alpha, \varphi \rangle_{L^2(\mathbb{R}^d)} e^{Q(\xi, u)} d\mu(u). \quad (3.7.1)$$

Our goal is to prove (3.7.1). Since $\beta = \mathcal{F}(g)$ for some $g \in L^1(\mathbb{R}^d)$, one can write

$$\operatorname{Tr} \left[\beta(\hat{q}_j) \hat{\phi}_{\hbar_\ell}(\varphi) \mathcal{W}(\xi) \varrho_{\hbar_\ell} \right] = \int_{\mathbb{R}^d} g(y) \operatorname{Tr} \left[e^{-2\pi i y \cdot \hat{q}_j} \hat{\phi}_{\hbar_\ell}(\varphi) \mathcal{W}(\xi) \varrho_{\hbar_\ell} \right] dy. \quad (3.7.2)$$

Furthermore, dominated convergence applies to the right hand side of (3.7.2) thanks to the assumptions (3.1.8)-(3.1.9) and the estimates in Lemma 3.3.1. Thus, the limit (3.7.1) reduces to

$$\lim_{\ell \rightarrow \infty} \operatorname{Tr} \left[e^{-2\pi i y \cdot \hat{q}_j} \hat{\phi}_{\hbar_\ell}(\varphi) \mathcal{W}(\xi) \varrho_{\hbar_\ell} \right] = \sqrt{2} \int_{X^0} e^{-2\pi i y \cdot q_j} \Re \langle \alpha, \varphi \rangle_{L^2(\mathbb{R}^d)} e^{Q(\xi, u)} d\mu(u). \quad (3.7.3)$$

Now, applying [AFH22, Lemma B.2], we obtain (3.7.3) for all $y \in \mathbb{R}^d$ since

$$e^{-2\pi i y \cdot \hat{q}_j} \mathcal{W}(\xi) = W_1(-2\pi y, 0) \mathcal{W}(\xi) = \mathcal{W}(-2\pi y, 0, 0) \mathcal{W}(\xi) = e^{i\hbar_\ell \pi y \cdot p_0} \mathcal{W}(\tilde{\xi}),$$

with $\tilde{\xi} = (-2\pi y, 0, 0) + \xi$ and $\xi = (p_0, q_0, \alpha)$. Recall that the Weyl-Heisenberg operator $W_1(\cdot)$ is given in (3.3.10) while $\mathcal{W}(\cdot)$ is defined by (3.3.9)-(3.3.7). \square

Lemma 3.7.6. *Let $(\varrho_{\hbar})_{\hbar \in (0,1)}$ be a family of density matrices on the Hilbert space \mathcal{H} satisfying (3.1.8)-(3.1.9) for $\sigma = \frac{1}{2}$. Assume that for some sequence $(\hbar_\ell)_{\ell \in \mathbb{N}} \subset (0, 1)$,*

$\hbar_\ell \rightarrow 0$, there exists a (unique) Borel measure $\mu \in \mathcal{P}(X^0)$ such that

$$\mathcal{M}(\varrho_{\hbar_\ell}, \ell \in \mathbb{N}) = \{\mu\}.$$

Then for any $\varphi \in L^2(\mathbb{R}_k^d)$, $\xi \in X^0$ and $j = 1, \dots, n$,

$$\begin{aligned} \lim_{\ell \rightarrow \infty} \operatorname{Tr} \left[\hat{a}_{\hbar_\ell} (e^{-2\pi i k \cdot \hat{q}_j} \varphi) \mathcal{W}(\xi) \varrho_{\hbar_\ell} \right] &= \int_{X^0} \langle e^{-2\pi i k \cdot q_j} \varphi, \alpha \rangle_{L^2(\mathbb{R}_k^d)} e^{Q(\xi, u)} d\mu(u), \\ \lim_{\ell \rightarrow \infty} \operatorname{Tr} \left[\hat{a}_{\hbar_\ell}^* (e^{-2\pi i k \cdot \hat{q}_j} \varphi) \mathcal{W}(\xi) \varrho_{\hbar_\ell} \right] &= \int_{X^0} \langle \alpha, e^{-2\pi i k \cdot q_j} \varphi \rangle_{L^2(\mathbb{R}_k^d)} e^{Q(\xi, u)} d\mu(u), \end{aligned}$$

with $u = (p, q, \alpha) \in X^0$ and $Q(\cdot, \cdot)$ is the phase given in [\(3.5.2\)](#).

Proof. According to Definition [3.1.2](#) of Wigner measures and [\[AN08\]](#), Theorem 6.2 and Proposition 6.4], we deduce that

$$\mathcal{M}(\mathcal{W}(\xi) \varrho_{\hbar_\ell}, \ell \in \mathbb{N}) = \{e^{Q(\xi, \cdot)} \mu\}.$$

Here, we have used the extension of the notion of Wigner measures to trace-class operators which are not necessary non-negative nor trace normalized (see [\[AN08\]](#), Proposition 6.4). Let $\{e_m\}_{m \in \mathbb{N}}$ be O.N.B of the Hilbert space $L^2(\mathbb{R}_k^d)$. The two limits are similar (almost conjugate) and it is enough to explain the argument for the second one. We denote $\varrho_{\hbar_\ell}(\xi) := \mathcal{W}(\xi) \varrho_{\hbar_\ell}$ and $b(\hat{q}_j) = e^{-2\pi i k \cdot \hat{q}_j} \varphi$. We have

$$\begin{aligned} & \left| \operatorname{Tr} \left[\hat{a}_{\hbar_\ell}^* (b(\hat{q}_j)) \varrho_{\hbar_\ell}(\xi) \right] - \int_{X^0} e^{Q(\xi, u)} \langle \alpha, b(q_j) \rangle_{L^2} d\mu(u) \right| \\ & \leq \underbrace{\left| \operatorname{Tr} \left[\left(\hat{a}_{\hbar_\ell}^* (b(\hat{q}_j)) - \sum_{m=1}^R \hat{a}_{\hbar_\ell}^* (e_m) \mathcal{F}[\varphi \bar{e}_m](\hat{q}_j) \right) \varrho_{\hbar_\ell}(\xi) \right] \right|}_{(1)} \\ & + \underbrace{\sum_{m=1}^R \left| \operatorname{Tr} \left[\hat{a}_{\hbar_\ell}^* (e_m) \mathcal{F}[\varphi \bar{e}_m](\hat{q}_j) \varrho_{\hbar_\ell}(\xi) \right] - \int_{X^0} e^{Q(\xi, u)} \langle \alpha, e_m \rangle_{L^2} \mathcal{F}[\varphi \bar{e}_m](q_j) d\mu(u) \right|}_{(2)}, \\ & + \underbrace{\left| \int_{X^0} e^{Q(\xi, u)} \left(\sum_{m=1}^R \langle \alpha, e_m \rangle_{L^2} \mathcal{F}[\varphi \bar{e}_m](q_j) - \langle \alpha, b(q_j) \rangle_{L^2} \right) d\mu(u) \right|}_{(3)}. \end{aligned}$$

Using estimates as in Lemma 3.3.1 and assumptions (3.1.8)-(3.1.9), one proves

$$\begin{aligned} (1)^2 &\lesssim \left\| \langle x \rangle^{-1} \left(e^{-2\pi i k \cdot x} \varphi - \sum_{m=1}^R e_m \langle e_m, e^{-2\pi i k \cdot x} \varphi \rangle_{L^2(\mathbb{R}_k^d)} \right) \right\|_{L^\infty(\mathbb{R}_x^d, L^2(\mathbb{R}_k^d))}^2 \\ &\lesssim \sup_{x \in \mathbb{R}^d} \sum_{m=R+1}^{\infty} \langle x \rangle^{-1} \left| \langle e_m, e^{-2\pi i k \cdot x} \varphi \rangle_{L^2(\mathbb{R}_k^d)} \right|^2. \end{aligned}$$

So, thanks to a further localization argument in the variable x combined to Dini's theorem, one concludes that (1) converges to zero uniformly in $\hbar \in (0, 1)$ as $R \rightarrow \infty$. Similarly, using the pointwise convergence for any $x \in \mathbb{R}^d$,

$$\sum_{m=1}^R \langle \alpha, e_m \rangle_{L^2} \mathcal{F}[\varphi \bar{e}_m](x) - \langle \alpha, e^{-2\pi i k \cdot x} \varphi \rangle_{L^2} \xrightarrow{R \rightarrow \infty} 0,$$

and dominated convergence, one concludes that (3) converges to zero as $R \rightarrow \infty$. Now, applying Lemma 3.7.5 with $\beta := \mathcal{F}[\varphi \bar{e}_m] \in \mathcal{F}(L^1(\mathbb{R}^d))$, we obtain that (2) converges also to zero for any fixed $R \in \mathbb{N}$ as $\hbar_\ell \rightarrow 0$. Hence, using an $\varepsilon/3$ -argument we prove the claimed statement. \square

ALMOST SURE EXISTENCE OF GLOBAL SOLUTIONS FOR GENERAL INITIAL VALUE PROBLEMS

Abstract. *This chapter is concerned with the almost sure existence of global solutions for initial value problems of the form $\dot{\gamma}(t) = v(t, \gamma(t))$ on separable dual Banach spaces. We prove a general result stating that whenever there exists $(\mu_t)_{t \in \mathbb{R}}$ a family of probability measures satisfying a related statistical Liouville equation, there exist global solutions to the initial value problem for μ_0 -almost all initial data, possibly without uniqueness. The main assumption is a mild integrability condition of the vector field v with respect to $(\mu_t)_{t \in \mathbb{R}}$. As a notable application, we obtain from the above principle that Gibbs and Gaussian measures yield low regularity global solutions for several nonlinear dispersive PDEs as well as fluid mechanics equations including Hartree, Klein-Gordon, NLS, Euler and modified surface quasi-geostrophic equations. In this regard, our contribution generalizes Bourgain's method [Bou96] as well as Albeverio & Cruzeiro's method [AC90] of constructing low regularity global solutions, without the need for local well-posedness analysis.*

4.1 Introduction and main results

Initial value problems, including ODEs, PDEs and stochastic PDEs, are of major interest to both applied and fundamental Mathematics. There is an abundant literature for this broad field of research, covering often important evolution equations in science, see *e.g.* [Bar76, CL55, MB02]. From a theoretical point of view, one can mainly recognize two qualitative approaches:

- A specific analysis that relies on the exact or almost exact form of initial value problems using particular features of given equations (*e.g.* exactly solvable equations, dispersive, hyperbolic, parabolic and to some extent semilinear equations).

- A more general analysis that ignores the exact form of initial value problems and instead focuses on finding general criteria that ensure uniqueness, local and global existence of solutions (*e.g.* Carathéodory, Cauchy-Lipschitz, Peano theorems and to some extent fixed-point theorems).

Of course, these two perspectives complement each other. On the other hand, there is a sharp distinction between initial value problems over finite and infinite dimensional spaces. For instance, in infinite dimensions it is known that the Peano theorem is in general not true and that there exist finite lifespan solutions to initial value problems without blowup. However, the Cauchy-Lipschitz theorem still holds true on Banach spaces, thus indicating that certain results could indeed survive in infinite dimensions. The present article is concerned with the second approach.

In the past few decades, there have been significant advances in the field of dispersive PDEs, stemming from the combination of probability theory, harmonic analysis and quantum field theory methods to construct global solutions with low regularity. These advances were inspired by the pioneering work of Bourgain [Bou94, Bou96, Bou97]. The latter papers continued the line of research initiated by Lebowitz–Rose–Speer [LRS88] and Zhidkov [Zhi91], as well as McKean–Vaninsky [MKV97, MV97, MKV94]. There have been many subsequent contributions on this subject, see [Bri22b, Bri22a, BDNY22, BT08b, CdS20, CFL16, DNY21, DNY20, DNY22, FOSW21, GLV16, GOTW22, KM21, Nah12, NRBSS11, Nah15, OST21, OT18, OT20, ST20] and the works quoted there. For an overview, we also refer the reader to the expository works [BTT18, NS19, NSW23] and the references therein. One of the main ideas of the aforementioned works is that Gibbs measures for appropriate Hamiltonian PDEs are well-defined over Sobolev spaces with non-positive regularity exponent and that they are formally invariant under the flow. However, the other known conservation laws are available only at higher regularity. Consequently, the Gibbs measures can be used as a substitute for a conservation law. When combined with local well-posedness theory, they can be used to construct low regularity global solutions for almost every initial data. On the other hand, in the field of fluid mechanics, there is a long-standing interest in constructing invariant measures and global solutions using probabilistic methods (see *e.g.* [ARdFHK79, AHK89, Cip99]). In particular, in the nineties, Albeverio and Cruzeiro proved the existence almost surely of global solutions to the Euler equation on the two dimensional torus [AC90]. More recently, there has been a renewed interest for such invariant measures and stochastic flows in fluid mechanics (see *e.g.* [Fer23, Fla18, FS19]). In the work of Nahmod–Pavlović–Staffilani–Totz

[NPST18], the authors extended the result of Albeverio–Cruzeiro [AC90] to the modified quasi-geostrophic (mSQG) equations which interpolate between the Euler and the SQG equation. Again the idea behind the construction of these global flows is the invariance of a given well-understood (Gaussian) measure combined with some probabilistic compactness arguments in the spirit of Prokhorov’s or Skorokhod’s theorems. Our claim here is that the previous circle of ideas is quite general and robust and could be formulated as a general principle for abstract initial value problems.

Our aim in this article is to address the question of almost sure existence of global solutions from a more general perspective. Precisely, we consider an abstract initial value problem of the form

$$\dot{\gamma}(t) = v(t, \gamma(t)). \tag{4.1.1}$$

over a separable dual Banach space \mathbb{B} $(B, \|\cdot\|)$ with $v : \mathbb{R} \times B \rightarrow B$ a Borel vector field. We assume that there exists a narrowly continuous \mathbb{P} probability measure solution $(\mu_t)_{t \in \mathbb{R}}$ to the statistical Liouville equation

$$\frac{d}{dt} \int_B F(u) \mu_t(du) = \int_B \langle v(t, u), \nabla F(u) \rangle \mu_t(du), \tag{4.1.2}$$

with the same vector field. Here $\langle \cdot, \cdot \rangle$ denotes the duality bracket between the topological dual space $B = E^*$ and its predual E while F are smooth test functions with ∇F denoting their Fréchet differentials. The statistical Liouville equation (4.1.2) is explained in detail in Paragraph 4.1.1, particularly the considered class of smooth test functions is given in Definition 4.1.3. Under such assumptions, we show that for μ_0 -almost all initial data in B there exist global solutions to the initial value problem (4.1.1), if the vector field satisfies the integrability condition

$$\int_{\mathbb{R}} \int_B \|v(t, u)\| \mu_t(du) \frac{dt}{\omega(|t|)} < +\infty, \tag{4.1.3}$$

for some non-decreasing positive function $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$. Remarkably, such result does not depend on the shape of the vector field nor on a local well-posedness theory. Moreover, the method applies equally to finite or infinite dimensional spaces while the vector field v may not be continuous. In practice, our result reduces the problem of constructing global

1. *i.e.* this means that there exists E a Banach space such that B is a topological dual of E and B is separable.

2. see Definition 4.1.4

solutions of ODEs or PDEs to finding out measure solutions for the statistical Liouville equation (4.1.2). So, one may wonder what is the gain so far? In fact in many cases, as we will see later, global measure solutions $(\mu_t)_{t \in \mathbb{R}}$ can be constructed which are either stationary or stationary modulo a pushforward. Generally speaking, recall that for Hamiltonian systems the Liouville theorem ensures the existence of invariant measures while for dynamical systems the Krylov-Bogolyubov theorem ([Sin76, Lecture 2] and [DPZ96]) is an efficient tool for constructing invariant measures. Note that our understanding of invariance here is in terms of stationary solutions for the statistical Liouville equation (4.1.2) instead of the invariance with respect to the flow, since the latter may not exist in general.

On the other hand, in the recent work [AS22], the Kubo-Martin-Schwinger (KMS) equilibrium states were introduced for Hamiltonian PDEs (like Hartree, NLS, Wave equations). In particular, it is proved in this context that Gibbs measures are KMS equilibrium states satisfying the stationary statistical Liouville equation (4.1.2) with an appropriate choice of the vector field v in accordance with the given PDE. Hence, as a consequence of the above principle and the stationarity of Gibbs measures one deduces straightforwardly the existence of low regularity global solutions for almost all initial data for several Hamiltonian PDEs. It is also worth highlighting here the two features of our approach:

- No dispersive properties are needed.
- More general nonlinearities can be included by relying on Malliavin calculus (see [Nua06, Chapter 1]).

We refer the reader to Section 4.1.4 for more details on the examples of nonlinear PDEs considered here (including Hartree, NLS and Wave equations on the flat torus \mathbb{T}^d , $d = 1, 2, 3$ (see [BTT18]) and Euler, modified surface quasi-geostrophic (mSQG) equations on the 2-dimensional torus (see [AC90, NPST18])). Of course, we do not prove here global well-posedness for such PDEs. Instead, we emphasize the fact that the solutions of such PDEs are almost surely global in time. Potentially, one can try to combine our method with a local posedness theory in order to prove global well-posedness (see [BT08a, BT08b]). In contrast, uniqueness generally speaking depends more on the particular properties of the considered initial value problem. Another aspect that we did not address is global solutions for random initial value problems, instead we focused here on deterministic equations. Such random systems, which will be addressed elsewhere, are very interesting as it includes stochastic PDEs and random dynamical systems with widespread

applications in fluid mechanics and stochastic quantization for instance (see [VF88] and [DPD03]).

Techniques: Our approach is quite related to statistical physics in spirit and consists of studying the evolution of ensembles of initial data through statistical Liouville equations. However, the key argument comes from transport theory via the superposition principle (or probabilistic representation) proved for instance in [AGS08, Ber08, Man07] (see also [Ber10]). The superposition principle shows in particular that if one has a (possibly stationary) probability measure solution $(\mu_t)_{t \in [0, T]}$ to the statistical Liouville equation (4.1.2) then there exists a probability path measure η concentrated on the set of local solutions of the initial value problem (4.1.1) and such that the projection of η at each fixed time is equal to μ_t (see Proposition 4.2.1). Such a result is extended to infinite dimension and adapted to PDE analysis in [AL18]. Here, we extend such principle to separable dual Banach spaces and more importantly to global solutions of initial value problems and Liouville equations in such a way that this tool yields a powerful globalisation argument. In particular, by using the measurable projection theorem and the properties of the path measure η one is able to find a universally measurable subset S of the Banach space B such that $\mu_0(S) = 1$ and for each $x \in S$ there exists a global solution to the initial value problem (4.1.1).

In conclusion, the result obtained here by measure theoretical techniques is quite general and to the best of our knowledge is new. It formalizes some of the deep ideas in the topic of constructing almost surely low regularity global solutions to dispersive PDEs and to fluid mechanics equations. It also bridges the problem of constructing global flows for PDEs to the topic of continuity and transport equations ([AGS08, AF09a]). We hope that it will be of interest for the ODE & PDE communities.

4.1.1 General framework

Let $(B, \|\cdot\|)$ be a real separable dual Banach space. This means that there exists $(E, \|\cdot\|_E)$ a real Banach space such that B is the topological dual of E (i.e. $B = E^*$, $\|\cdot\| = \|\cdot\|_{E^*}$) and $(B, \|\cdot\|)$ is separable. Recall that since $(B, \|\cdot\|)$ is separable then $(E, \|\cdot\|_E)$ is also separable (see [Bre, Theorem III.23]). When there is no possible confusion, we will denote the duality bracket $\langle \cdot, \cdot \rangle_{E^*, E}$ simply by $\langle \cdot, \cdot \rangle$.

In all the sequel, I denotes a closed unbounded time interval (e.g. $I = \mathbb{R}$ or $I = \mathbb{R}_+$) with $t_0 \in I$ is any initial time if $I = \mathbb{R}$ or if I is half-closed then $t_0 \in I$ is its endpoint. Our

main purpose is the study of the *initial value problem*

$$\begin{cases} \dot{\gamma}(t) = v(t, \gamma(t)), \\ \gamma(t_0) = x \in B, \end{cases} \quad (4.1.4)$$

where $v : I \times B \rightarrow B$ is a Borel vector field. Generally speaking, there are several notions of solutions to (4.1.4). A *strong solution* is a curve γ belonging to $\mathcal{C}^1(I, B)$ and satisfying (4.1.4) for all times $t \in I$. However, to study such curves one usually requires v to be at least continuous in order to have a consistent equation. Instead, we focus on *mild solutions* of (4.1.4) which are continuous curves $\gamma \in \mathcal{C}(I; B)$ such that $v(\cdot, \gamma(\cdot)) \in L^1_{loc}(I, dt; B)$ and for all $t \in I$ the following integral formula is satisfied:

$$\gamma(t) = x + \int_{t_0}^t v(s, \gamma(s)) ds. \quad (4.1.5)$$

Here, the integration in the right hand side is a Bochner integral and the function $s \in I \mapsto v(s, \gamma(s))$ is strongly measurable and satisfies for all $a, b \in I, a < b$,

$$\int_a^b \|v(s, \gamma(s))\| ds < +\infty.$$

Equivalently, we define the space of locally absolutely continuous curves $AC^1_{loc}(I; B)$ to be the space of all functions $u : I \rightarrow B$ such that there exists $m \in L^1_{loc}(I, dt)$ satisfying

$$\forall s, r \in I, s < r : \quad \|u(s) - u(r)\| \leq \int_s^r m(t) dt.$$

Since separable dual Banach spaces satisfy the Radon-Nikodym property (see [Rya]), the functions in the space $AC^1_{loc}(I; B)$ are continuous and almost everywhere differentiable on I with a derivative $\dot{u}(\cdot) \in L^1_{loc}(I, dt; B)$. Hence, a curve $\gamma : I \rightarrow B$ is a mild solution of (4.1.4) if and only if $\gamma \in AC^1_{loc}(I; B)$, $\gamma(t_0) = x$ and for almost all $t \in I$

$$\dot{\gamma}(t) = v(t, \gamma(t)).$$

Statistical Liouville equation: When studying the statistical Liouville equation (4.1.2), the following notion will be useful.

Definition 4.1.1 (Fundamental strongly total biorthogonal system). We say that the families $\{e_k\}_{k \in \mathbb{N}}$ and $\{e_k^*\}_{k \in \mathbb{N}}$ in E, E^* respectively form a *fundamental strongly total*

biorthogonal system if the following properties hold.

- (a) $\text{Span}\{e_k, k \in \mathbb{N}\}$ is dense in E (fundamental),
- (b) $\text{Span}\{e_k^*, k \in \mathbb{N}\}$ is dense in $B = E^*$ (strongly total),
- (c) $\langle e_{k'}^*, e_k \rangle = \delta_{k',k}, \forall k, k' \in \mathbb{N}$ (biorthogonal).

We note that such an object exists in our framework.

Lemma 4.1.2. *Let $(B, \|\cdot\|)$ be a separable dual Banach space. Then a fundamental strongly total biorthogonal system $\{e_k\}_{k \in \mathbb{N}}, \{e_k^*\}_{k \in \mathbb{N}}$ as in Definition 4.1.1 exists.*

Lemma 4.1.2 is proved in [LT77, Proposition 1.f.3]. We henceforth fix a system $\{e_k\}_{k \in \mathbb{N}}, \{e_k^*\}_{k \in \mathbb{N}}$ as in Definition 4.1.1, whose existence is guaranteed by Lemma 4.1.2 above. This allows us to define a convenient class of cylindrical test functions.

Definition 4.1.3 (Cylindrical test functions). A function $F : B = E^* \rightarrow \mathbb{R}$ belongs to $\mathcal{C}_{c,cyl}^\infty(B)$ (resp. $\mathcal{S}_{cyl}(B)$ or $\mathcal{C}_{b,cyl}^\infty(B)$) if there exists $n \in \mathbb{N}$ and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ (resp. $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{C}_b^\infty(\mathbb{R}^n)$) such that

$$F(u) = \varphi(\langle u, e_1 \rangle, \dots, \langle u, e_n \rangle), \quad \forall u \in B = E^*. \quad (4.1.6)$$

Here, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space and we have the inclusions

$$\mathcal{C}_{c,cyl}^\infty(B) \subset \mathcal{S}_{cyl}(B) \subset \mathcal{C}_{b,cyl}^\infty(B).$$

Let us note that any $F \in \mathcal{C}_{b,cyl}^\infty(B)$ is Fréchet differentiable with a differential $DF(u) \in B^* = E^{**}$ identified with an element of E . Hence, we denote simply ∇F for the differential of F so that $\nabla F(u) \in E$ for all $u \in B$. In particular, we have

$$\nabla F(u) = \sum_{k=1}^n \partial_k \varphi(\langle u, e_1 \rangle, \dots, \langle u, e_n \rangle) e_k \in E. \quad (4.1.7)$$

Let $\mathcal{B}(B)$ and $\mathcal{P}(B)$ denote respectively the Borel σ -algebra and the space of Borel probability measures on the Banach space $(B, \|\cdot\|)$. We endow $\mathcal{P}(B)$ with the narrow topology.

Definition 4.1.4 (Narrow continuity). We say a curve $(\mu_t)_{t \in I}$ in $\mathcal{P}(B)$ is *narrowly continuous* if for any bounded continuous real-valued function $F \in \mathcal{C}_b(B, \mathbb{R})$, the map

$$t \in I \mapsto \int_B F(u) d\mu_t \in \mathcal{C}(I, \mathbb{R}). \quad (4.1.8)$$

is continuous.

We say that a narrowly continuous curve $(\mu_t)_{t \in I}$ in $\mathcal{P}(B)$ satisfies the statistical Liouville equation with respect to the Borel vector field $v : I \times B \rightarrow B$ if

$$\frac{d}{dt} \int_B F(u) \mu_t(du) = \int_B \langle v(t, u), \nabla F(u) \rangle \mu_t(du), \quad \forall F \in \mathcal{C}_{c, \text{cyl}}^\infty(B), \quad (4.1.9)$$

in the sense of distributions on the interior of I . In the right hand side of (4.1.9), the quantity $\langle v(t, u), \nabla F(u) \rangle$ refers to the duality bracket E^*, E . In order that the above statistical Liouville equation make sense, one needs a further assumption on the vector field v which ensures the integrability of the right hand side of (4.1.9).

Assumption 4.1.5 (Assumption on the vector field v). We assume that $v : I \times B \rightarrow B$ is a Borel vector field such that there exists a non-decreasing positive function $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\int_I \int_B \|v(t, u)\| \mu_t(du) \frac{dt}{\omega(|t|)} < +\infty. \quad (4.1.10)$$

In particular, it follows that the duality pairing $\langle v(t, u), \nabla F(u) \rangle \equiv \langle v(t, u), \nabla F(u) \rangle_{E^*, E}$ makes sense and satisfies

$$|\langle v(t, u), \nabla F(u) \rangle| \leq \|v(t, u)\| \|\nabla F(u)\|_E \leq C \|v(t, u)\|, \quad (4.1.11)$$

for some constant $C > 0$ depending on F . Using (4.1.11) and Assumption 4.1.5, it follows that the right-hand side of (4.1.9) is finite for almost every $t \in I$.

Remark 4.1.6. The above condition (4.1.10) implies

$$t \in I \mapsto \int_B \|v(t, u)\| \mu_t(du) \in L_{loc}^1(I, dt). \quad (4.1.12)$$

4.1.2 Main result

We give in this subsection our main abstract result that holds true on any separable dual Banach space. For instance, one can consider initial value problems on finite dimensional normed spaces or on separable reflexive Banach spaces like Lebesgue spaces L^p for $p \in (1, +\infty)$; but one can also consider non reflexive Banach spaces like the sequence space $\ell^1(\mathbb{N})$ which is a separable dual space.

Theorem 4.1.7. *Let B be a separable dual Banach space and $v : \mathbb{R} \times B \rightarrow B$ a Borel vector field. Suppose that there exists $(\mu_t)_{t \in I}$ a narrowly continuous curve in $\mathcal{P}(B)$ such that*

the Assumption [4.1.5](#) holds true and assume that $(\mu_t)_{t \in I}$ satisfies the statistical Liouville equation [\(4.1.9\)](#). Then there exists a universally measurable subset \mathcal{G} of B of total measure $\mu_{t_0}(\mathcal{G}) = 1$ such that for any $x \in \mathcal{G}$ there exists a global mild solution to the initial value problem [\(4.1.4\)](#).

Remark 4.1.8. The following comments are useful:

- The above theorem extends straightforwardly to any Banach space that is isometrically isomorphic to a separable dual Banach space.
- The assumption on narrow continuity of $(\mu_t)_{t \in I}$ can be relaxed to weak narrow continuity given in Definition [4.2.3](#).
- The notion of mild solutions and the statistical Liouville equation are explained in Subsection [4.1.1](#).
- A universally measurable set is a subset of a Polish space (here B) that is measurable with respect to every complete probability measure. In particular, \mathcal{G} is μ_{t_0} -measurable.
- The above theorem provide no information about uniqueness of mild solutions.

4.1.3 Application to ODEs

Consider the euclidean space \mathbb{R}^{2d} endowed with a symplectic structure given by a skew-symmetric matrix J satisfying $J^2 = -I_{2d}$. Let $h : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ be a Borel function in the local Sobolev space $W_{loc}^{2,2}(\mathbb{R}^{2d}, \mathbb{R})$. Furthermore, suppose that there exists a non-negative \mathcal{C}_b^1 -function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$0 < \int_{\mathbb{R}^{2d}} (1 + \|\nabla h(u)\|_{\mathbb{R}^{2d}}) F(h(u)) L(du) < +\infty, \quad (4.1.13)$$

where L denotes the Lebesgue measure over \mathbb{R}^{2d} . Consider the initial value problem given by the Hamiltonian system:

$$\begin{cases} \dot{\gamma}(t) = J\nabla h(\gamma(t)), \\ \gamma(0) = u_0 \in \mathbb{R}^{2d}. \end{cases} \quad (4.1.14)$$

Observing that $\mu = \frac{F(h(\cdot))L}{\int F(h(u))L(du)} \in \mathcal{P}(\mathbb{R}^{2d})$ is a stationary solution to the statistical Liouville equation [\(4.1.9\)](#) with the vector field $v = J\nabla h$, yields the following consequence of Theorem [4.1.7](#)

Corollary 4.1.9. *Assume that $h \in W_{loc}^{2,2}(\mathbb{R}^{2d}, \mathbb{R})$ and the condition (4.1.13) is satisfied. Then the Hamiltonian system (4.1.14) admits a mild global solution for Lebesgue-almost any initial condition $u_0 \in \mathbb{R}^{2d}$.*

As an illustrative example for $d \geq 5$, one can take $F(t) = e^{-\beta t}$, $t \in \mathbb{R}_+$, for some $\beta > 0$ and

$$h(u) = \phi(p) + \|q\|_{\mathbb{R}^d}^2 + \frac{1}{\|q\|_{\mathbb{R}^d}^\alpha},$$

with $\phi \in W_{loc}^{2,2}(\mathbb{R}^d, \mathbb{R}_+)$ such that $F(\phi)$ and $\|\nabla\phi\|_{\mathbb{R}^d} F(\phi)$ belong to $L^1(\mathbb{R}^d)$ and $\alpha < \frac{d}{2} - 2$ (here $u = (q, p) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\|\cdot\|_{\mathbb{R}^d}$ is the euclidean norm).

4.1.4 Application to PDEs

Consider a complex Hilbert space $(H, \|\cdot\|_H)$ and a self-adjoint operator $A : D(A) \subset H \rightarrow H$ such that there exists a constant $c > 0$,

$$A \geq c \mathbb{1},$$

and A having a compact resolvent. So, there exist a sequence of eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$ and an O.N.B of eigenvectors

$$Ae_k = \lambda_k e_k \tag{4.1.15}$$

such that $Ae_k = \lambda_k e_k$ for all $k \in \mathbb{N}$. Furthermore, assume that there exists $s \geq 0$ such that

$$\sum_{k \in \mathbb{N}} \lambda_k^{-(s+1)} < +\infty. \tag{4.1.16}$$

Then, one can define Sobolev spaces with positive exponent $r > 0$ as

$$H^r = (D(A^{r/2}), \|A^{r/2} \cdot\|_H)$$

and Sobolev spaces with negative exponent as

$$H^{-r} = \overline{(H, \|A^{-r/2} \cdot\|_H)}^{\text{completion}}.$$

From now on, we regard H, H^r, H^{-r} as real Hilbert spaces endowed respectively with the scalar products

$$\langle \cdot, \cdot \rangle_{H_{\mathbb{R}}} = \text{Re} \langle \cdot, \cdot \rangle_H, \quad \langle \cdot, \cdot \rangle_{H_{\mathbb{R}}^r} = \text{Re} \langle \cdot, A^r \cdot \rangle_H, \quad \langle \cdot, \cdot \rangle_{H_{\mathbb{R}}^{-r}} = \text{Re} \langle \cdot, A^{-r} \cdot \rangle_H,$$

and denote them respectively by $H_{\mathbb{R}}, H_{\mathbb{R}}^r, H_{\mathbb{R}}^{-r}$ (note that $\text{Re}(\cdot)$ refer to the real part). Then it is well-known that there exists a unique centred Gaussian probability measure ν_0 on the Sobolev space H^{-s} with $s \geq 0$ satisfying (4.1.16) and such that for all $\xi \in H^{-s}$,

$$\int_{H^{-s}} \exp(i\langle u, \xi \rangle_{H_{\mathbb{R}}^{-s}}) \nu_0(du) = \exp(-\frac{1}{2}\langle \xi, A^{-(s+1)}\xi \rangle_{H_{\mathbb{R}}^{-s}}). \quad (4.1.17)$$

Once we have such centred Gaussian measure ν_0 , one can define the Gross-Sobolev space

$$\mathbb{D}^{1,2}(\nu_0) = \{F \in L^2(H^{-s}, \nu_0) : \nabla F \in L^2(H^{-s}, \nu_0; H^{-s})\},$$

where here ∇F is the Malliavin derivative of F (see for instance [Nua06] or [AS22] for brief details). In particular, $\mathbb{D}^{1,2}(\nu_0)$ is a Hilbert space when endowed with the inner product

$$\langle F, G \rangle_{\mathbb{D}^{1,2}(\nu_0)} = \langle F, G \rangle_{L^2(H^{-s}, \nu_0)} + \langle \nabla F, \nabla G \rangle_{L^2(H^{-s}, \nu_0; H^{-s})}.$$

Our purpose is to prove that the initial value problem (4.1.4) admit global solutions for ν_0 -almost any initial condition $x \in H^{-s}$ (here $B = H^{-s}$) when the vector field $v : \mathbb{R} \times H^{-s} \rightarrow H^{-s}$ is given explicitly by

$$v(t, u) = -ie^{itA}\nabla h_{NL}(e^{-itA}u), \quad (4.1.18)$$

and h_{NL} is any function in the Gross-Sobolev space $\mathbb{D}^{1,2}(\nu_0)$ satisfying the condition

$$\exp(-h_{NL}) \in L^2(H^{-s}, \nu_0). \quad (4.1.19)$$

Thanks to the above assumptions, the following Gibbs measure is well-defined:

$$d\mu_0 = \frac{e^{-h_{NL}} d\nu_0}{\int_{H^{-s}} e^{-h_{NL}} d\nu_0}. \quad (4.1.20)$$

Then within the above framework, we prove in Section 4.3 the following result.

Proposition 4.1.10. *Consider the time-dependent push-forward Gibbs measures*

$$\mu_t = (e^{itA})_{\#}\mu_0. \quad (4.1.21)$$

Then for all $t \in \mathbb{R}$ and any $F \in \mathcal{C}_{b,cyl}^\infty(H^{-s})$,

$$\frac{d}{dt} \int_{H^{-s}} F(u) \mu_t(du) = \int_{H^{-s}} \langle v(t, u), \nabla F(u) \rangle \mu_t(du). \quad (4.1.22)$$

This implies that there exists a narrowly continuous curve $(\mu_t)_{t \in \mathbb{R}}$ in $\mathcal{P}(H^{-s})$, given in (4.1.21), satisfying the statistical Liouville equation (4.1.22) and the integrability condition

$$\int_{\mathbb{R}} \int_{H^{-s}} \|v(t, u)\|_{H^{-s}} \mu_t(du) \frac{dt}{\langle t \rangle^2} = \int_{\mathbb{R}} \frac{dt}{\langle t \rangle^2} \int_{H^{-s}} \|\nabla h_{NL}(u)\|_{H^{-s}} \mu_0(du) < +\infty,$$

since $e^{-h_{NL}} \in L^2(H^{-s}, \nu_0)$ and $\nabla h_{NL} \in L^2(H^{-s}, \nu_0; H^{-s})$ (recall that $h_{NL} \in \mathbb{D}^{1,2}(\nu_0)$), which corresponds to Assumption 4.1.5 with $\omega(t) = \langle t \rangle^2$. Thus, Theorem 4.1.7 yields the following statement.

Corollary 4.1.11. *For any nonlinear functional $h_{NL} : H^{-s} \rightarrow \mathbb{R}$ belonging to the Gross-Sobolev space $\mathbb{D}^{1,2}(\nu_0)$ and satisfying*

$$\exp(-h_{NL}) \in L^2(H^{-s}, \nu_0),$$

the initial value problem

$$\begin{cases} i\partial_t u(t) = Au(t) + \nabla h_{NL}(u(t)), \\ u(0) = x \in H^{-s}, \end{cases} \quad (4.1.23)$$

admit global mild solutions for ν_0 -almost any initial data in H^{-s} .

Such result is new to the best of our knowledge. It is a straightforward consequence of the Theorem 4.1.7 and Proposition 4.1.10. The main point in Corollary 4.1.11 is that the existence of well-defined Gibbs measure provides a global solution to the statistical Liouville equation and hence by Theorem 4.1.7, one deduces the almost sure existence of global solutions. It is clear therefore that there is a tight connection between Theorem 4.1.7 and the subject of Gibbs measures and low regularity solutions of dispersive PDEs. In fact, several examples of concrete PDEs like NLS, Hartree and Wave equations can be recast as the above initial value problem (4.1.23). One needs only to specify the Hilbert space H , the operator A and the nonlinear functional h_{NL} .

Gibbs measures for nonlinear dispersive equations are well-studied and the literature on

the subject is quite large as was summarised above. To highlight the connection of our results with this topic, we provide here some applications of Corollary 4.1.11 to concrete examples. Our aim is not to give all the possible applications, but rather to illustrate our method.

Hartree and NLS equations: Let H be the Hilbert space $L^2(\mathbb{T}^d)$ with $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$ is the flat d -dimensional torus. Take the operator A as

$$A = -\Delta + 1, \tag{4.1.24}$$

where Δ is the Laplacian on \mathbb{T}^d . So, the family $\{e_k = e^{ikx}\}_{k \in \mathbb{Z}^d}$ forms an O.N.B of eigenvectors for the operator A which admits a compact resolvent. Now, consider an exponent $s \geq 0$ such that

$$\boxed{s > \frac{d}{2} - 1}, \tag{4.1.25}$$

and define the Sobolev space H^{-s} accordingly. Denote ν_0 to be the well-defined centred Gaussian measure on H^{-s} given by (4.1.17). Then, we list some nonlinear functionals, $h_{NL} : H^{-s} \rightarrow \mathbb{R}$, for which Corollary 4.1.11 applies; specified according to the dimension d and the type equation.

- *The Hartree equation on \mathbb{T} :* Let $V : \mathbb{T} \rightarrow \mathbb{R}$ be a nonnegative even L^1 function and

$$h_{NL}(u) = \frac{1}{4} \int_{\mathbb{T}} \int_{\mathbb{T}} |u(x)|^2 V(x - y) |u(y)|^2 dx dy. \tag{4.1.26}$$

- *The Hartree equation on \mathbb{T}^d , $d = 2, 3$:* Let $V \in L^1(\mathbb{T}^d)$ even and of positive type such that there exist $\epsilon > 0$ and $C > 0$ so that for all $k \in \mathbb{Z}^d$,

$$\begin{cases} \hat{V}(k) \leq \frac{C}{\langle k \rangle^\epsilon} & \text{if } d = 2 \\ \hat{V}(k) \leq \frac{C}{\langle k \rangle^{2+\epsilon}} & \text{if } d = 3. \end{cases} \tag{4.1.27}$$

Then take

$$h_{NL}(u) = \frac{1}{4} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} : |u(x)|^2 : V(x - y) : |u(y)|^2 : dx dy. \tag{4.1.28}$$

- *The NLS equation on \mathbb{T}* : Let $r \in \mathbb{N}$ and

$$h_{NL}(u) = \frac{1}{2r} \int_{\mathbb{T}} |u(x)|^{2r} dx \geq 0. \quad (4.1.29)$$

- *The NLS equation on \mathbb{T}^2* : Let $r \in \mathbb{N}$ with

$$h_{NL}(u) = \frac{1}{2r} \int_{\mathbb{T}^2} : |u|^{2r} : dx. \quad (4.1.30)$$

Here, the notation $: \cdot :$ refers to Wick ordering with respect to the Gaussian measure ν_0 . See for instance [OT18] for a self-contained construction of these Wick ordered nonlinearities (4.1.28)-(4.1.30).

Wave equations: Consider the Hilbert space $H = L_{\mathbb{R}}^2(\mathbb{T}^d) \oplus L_{\mathbb{R}}^2(\mathbb{T}^d)$ with $L_{\mathbb{R}}^2$ is the space of real-valued square integrable functions. For $s \in \mathbb{R}$ satisfying (4.1.25), define the Sobolev space

$$H^{-s} = H_{\mathbb{R}}^{-s}(\mathbb{T}^d) \oplus H_{\mathbb{R}}^{-s-1}(\mathbb{T}^d). \quad (4.1.31)$$

The nonlinear wave equation takes the form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{1} \\ \Delta - \mathbf{1} & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -\nabla h_{NL}(u), \end{pmatrix}$$

with h_{NL} a nonlinear functional on $H_{\mathbb{R}}^{-s}(\mathbb{T}^d)$ given by

$$h_{NL} = \begin{cases} (4.1.26) \text{ or } (4.1.29) & \text{if } d = 1, \\ (4.1.28) \text{ or } (4.1.30) & \text{if } d = 2, \\ (4.1.28) & \text{if } d = 3. \end{cases} \quad (4.1.32)$$

The Gaussian measure ν_0 in the case of the wave equation is defined as the product measure

$$\nu_0 = \nu_0^1 \otimes \nu_0^2,$$

with ν_0^1 and ν_0^2 are Gaussian measures on the distribution space $\mathcal{D}'(\mathbb{T}^d)$ with covariance operators $(-\Delta + \mathbf{1})^{-1}$ and $\mathbf{1}$ respectively. Moreover, one can define rigorously the Gibbs

measure for the nonlinear wave equation as

$$d\mu_0 = \frac{e^{-h_{NL}} d\nu_0^1 \otimes \nu_0^2}{\int e^{-h_{NL}} d\nu_0^1}.$$

We recall the following result, proved in [AS22].

Proposition 4.1.12. *The nonlinear Borel functionals $h_{NL}(\cdot)$ given by (4.1.26), (4.1.28), (4.1.29) and (4.1.30) belong to the Gross-Sobolev space $\mathbb{D}^{1,2}(\nu_0)$.*

Hence, we recover the following statement as a consequence of Corollary 4.1.11.

Corollary 4.1.13. *The above Hartree, NLS and Wave equations with the nonlinearities (4.1.26), (4.1.28), (4.1.29) and (4.1.30) admit global mild solutions almost surely with respect to the Gaussian measure ν_0 .*

Fluid mechanics equations: In this paragraph, we follow the work of [NPST18] and refer the reader to the references therein for more details on the Euler and modified SQG equations. Indeed, the mSQG equation takes the form:

$$\begin{cases} \partial_t \theta = -(\varphi \cdot \nabla) \theta, & x \in \mathbb{T}^2, t > 0, \\ \varphi = R^\perp |D|^{-\delta} \theta, \end{cases} \quad (4.1.33)$$

for $\delta > 0$. Here $\theta : \mathbb{T}^2 \rightarrow \mathbb{R}$, $\varphi : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ are functions, $|D| := (-\Delta)^{1/2}$ and $R^\perp := \nabla^\perp |D|^{-1}$ denotes the Riesz transform with $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. The case $\delta = 1$ in the above equation corresponds to the 2D Euler equation. The streamline formulation ($u = |D|^{-1} \theta$) of the above mSQG equation yields

$$\begin{cases} \partial_t u = -|D|^{-1} (\varphi \cdot \nabla) |D| u, \\ \varphi = \nabla^\perp |D|^{-\delta} u, \end{cases} \quad (4.1.34)$$

so that the original mSQG equation is rephrased as an initial value problem with an autonomous vector field given by

$$v(u) \equiv v(t, u) = -|D|^{-1} (\nabla^\perp |D|^{-\delta} u \cdot \nabla) |D| u. \quad (4.1.35)$$

Consider now the centred Gaussian measure ν_0 defined on the negative the Sobolev space H^{-s} , $s > 0$, as in the previous section (with $A = -\Delta$ and the Hilbert space $L_0^2(\mathbb{T}^2)$ of

mean zero square integrable functions). There are nice results on one hand by Albeverio and Cruzeiro [AC90] for the Euler equation and on the other by Nahmod–Pavlović–Staffilani–Totz [NPST18] for the mSQG equation ($0 < \delta \leq 1$), establishing the existence of solutions for arbitrarily large lifespan and for almost initial data in the spaces H^s , $s < -2$, with respect to the Gaussian measure ν_0 . Actually, thanks to preliminary results in [NPST18], one can apply our Theorem 4.1.7 to these equations too. Indeed, take $\mu_t = \nu_0$, $B = H^{-s}$, $s > 2$, and the vector field v as in (4.1.35), then in [NPST18, Proposition 4.1] it is proved that $v \in L^2(H^{-s}, \nu_0; H^{-s})$ for all $s > 2$. This implies that the integrability condition (4.1.10) is satisfied with any ω such that $\omega^{-1} \in L^1(\mathbb{R}_+, dt)$. Moreover, the stationary Liouville equation (4.1.9) is satisfied by the Gaussian measure ν_0 thanks to the proof of [NPST18, Lemma 5.1]. Hence, as a consequence of Theorem 4.1.7, we obtain ν_0 -almost surely the existence of global solutions to the mSQG equation (4.1.33)–(4.1.34) in $\mathcal{C}(\mathbb{R}_+, H^{-s})$ for all $s > 2$. In particular, our application of Theorem 4.1.7 yields an improvement of the result [NPST18] as it gives directly almost sure global solutions instead of arbitrarily large lifespan solutions.

4.2 Global superposition principle

Our purpose in this part is to state and prove the global superposition principle (Proposition 4.2.1). For that, we need to introduce the path space

$$\mathfrak{X} := B \times \mathcal{C}(I; B), \quad (4.2.1)$$

composed of pairs (x, γ) where $x \in B$ and γ is a continuous curve in $(B, \|\cdot\|)$. Denote, for any $\gamma \in \mathcal{C}(I; B)$ and $m \in \mathbb{N}$,

$$\|\gamma\|_m = \sup_{[-m, m] \cap I} \|\gamma(t)\|, \quad (4.2.2)$$

with the convention $\|\gamma\|_m = 0$ if $[-m, m] \cap I$ is the empty set. Since the interval I is unbounded and closed, it is convenient to equip the space $\mathcal{C}(I; B)$ with the compact-open topology which is metrizable in our case with the metric:

$$d_0(\gamma_1, \gamma_2) = \sum_{m \in \mathbb{N}} \frac{1}{2^m} \frac{\|\gamma_1 - \gamma_2\|_m}{1 + \|\gamma_1 - \gamma_2\|_m}. \quad (4.2.3)$$

Then, we accordingly define a distance d over the product space $\mathfrak{X} = B \times \mathcal{C}(I; B)$,

$$d((x_1, \gamma_1); (x_2, \gamma_2)) = \|x_1 - x_2\| + d_0(\gamma_1, \gamma_2). \quad (4.2.4)$$

Furthermore, we define, for each $t \in I$, the evaluation map

$$\begin{aligned} \Xi_t : \quad \mathfrak{X} &\longrightarrow B \\ (x, \gamma) &\longmapsto \gamma(t). \end{aligned}$$

Now, we are in position to state the global superposition principle. Recall that I is an unbounded closed interval and B is a separable dual Banach space equipped with a biorthogonal system satisfying (a)-(c) given in Definition 4.1.1.

Proposition 4.2.1 (Global superposition principle). *Let $v : I \times B \rightarrow B$ be a Borel vector field. Let $(\mu_t)_{t \in I}$ be a narrowly continuous curve in $\mathcal{P}(B)$ satisfying the integrability condition (4.1.10) and the statistical Liouville equation (4.1.9). Then, there exists a Borel probability measure $\eta \in \mathcal{P}(\mathfrak{X})$ such that:*

- (i) η concentrates on the set of pairs (x, γ) such that $\gamma \in AC_{loc}^1(I; B)$ is a mild solution of the initial value problem (4.1.4) for a.e. $t \in I$ with $\gamma(t_0) = x$.
- (ii) μ_t is equal to the image measure $(\Xi_t)_\# \eta$, for all $t \in I$: i.e., $\forall \mathcal{O} \in \mathcal{B}(B)$, $\mu_t(\mathcal{O}) = \eta(\Xi_t^{-1}(\mathcal{O}))$.

The remaining part of this section is dedicated to the proof the above proposition. In subsection 4.2.1, we introduce convenient weaker topologies on the space B and the path space \mathfrak{X} . In Subsection 4.2.2, we set up a finite dimensional projection argument. In Subsection 4.2.3, we prove Proposition 4.2.1 when $B = \mathbb{R}^d$. Then, we extend such result to infinite-dimensional separable dual Banach spaces in Subsection 4.2.4.

4.2.1 Weak topologies

The following topological and measure theoretical aspects will be very useful in the proofs of our main Theorem 4.1.7 and the global superposition principle (Proposition 4.2.1).

It is useful to introduce a norm over B that ensures relative compactness of bounded sets,

$$\|x\|_* := \sum_{k \in \mathbb{N}} \frac{1}{2^k \|e_k\|_E} |\langle x, e_k \rangle|, \quad x \in B. \quad (4.2.5)$$

Here $\langle \cdot, \cdot \rangle$ is the duality pairing of E^* , E and $\{e_k\}_{k \in \mathbb{N}}$, $\{e_k^*\}_{k \in \mathbb{N}}$ is the fixed biorthogonal system in Definition 4.1.1. Actually, the norm $\|\cdot\|_*$ yields a distance on $B = E^*$ which metrizes the weak-* topology $\sigma(E^*, E)$ on bounded sets. For convenience, we denote by B_w the space B endowed with the above norm (4.2.5) and remark that $B_w = (B, \|\cdot\|_*)$ is separable.

Recall that $\mathcal{P}(B)$ denotes the space of Borel probability measures on $(B, \|\cdot\|)$. The following lemma shows that $\mathcal{P}(B)$ is unchanged as a set, if we equip the space B with the norms $\|\cdot\|$ or $\|\cdot\|_*$.

Lemma 4.2.2. *The σ -algebras of Borel sets of $(B, \|\cdot\|)$ and $(B, \|\cdot\|_*)$ coincide.*

Proof. See [ALR20, Lemma C1]. □

It is useful to distinguish two narrow topologies over $\mathcal{P}(B)$. Namely, the (strong) narrow topology when B is equipped with the original norm $\|\cdot\|$ and the "weak" narrow topology when B is endowed with the norm $\|\cdot\|_*$.

Definition 4.2.3 (Weak narrow topology). We say that a sequence $(\mu_n)_{n \in \mathbb{N}}$ of Borel probability measures in $\mathcal{P}(B)$ converges *weakly narrowly* to $\mu \in \mathcal{P}(B)$ if for every bounded continuous function $F \in \mathcal{C}_b(B_w, \mathbb{R})$,

$$\lim_n \int_B F(u) \mu_n(du) = \int_B F(u) \mu(du).$$

In such a case, we denote

$$\mu_n \xrightarrow[n \rightarrow \infty]{} \mu.$$

Accordingly, a curve $(\mu_t)_{t \in I}$ in $\mathcal{P}(B)$ is said to be *weakly narrowly continuous* if the real-valued map

$$t \in I \longmapsto \int_B F(u) \mu_t(du),$$

is continuous for every $F \in \mathcal{C}_b(B_w, \mathbb{R})$.

Remark 4.2.4. Note that in finite dimensions the weak narrow and (strong) narrow topologies coincide.

On the other hand, we define similarly new distances on $\mathcal{C}(I; B)$ and the path space $\mathfrak{X} = B \times \mathcal{C}(I; B)$ given respectively by

$$d_{0,*}(\gamma_1; \gamma_2) = \sum_{m \in \mathbb{N}} \frac{1}{2^m} \frac{\|\gamma_1 - \gamma_2\|_{*,m}}{1 + \|\gamma_1 - \gamma_2\|_{*,m}}, \quad (4.2.6)$$

and

$$d_*((x_1, \gamma_1); (x_2, \gamma_2)) = \|x_1 - x_2\|_* + d_{0,*}(\gamma_1; \gamma_2), \quad (4.2.7)$$

where

$$\|\gamma\|_{*,m} = \sup_{[-m,m] \cap I} \|\gamma(t)\|_*, \quad (4.2.8)$$

with the convention $\|\gamma\|_{*,m} = 0$ if $[-m, m] \cap I$ is the empty set. It is clear that the topology induced by d_* is coarser than the one induced by the distance d given in (4.2.4). Moreover, d_* is the induced metric on \mathfrak{X} corresponding to the product topology between $(B, \|\cdot\|_*)$ and the space $\mathcal{C}(I; B_w)$ endowed with the compact-open topology.

Recall that a Polish space is a Hausdorff topological space homeomorphic to a separable complete metric space while a Suslin space is a Hausdorff topological space which is the image of a Polish space under a continuous map. In particular, in our case, $(B, \|\cdot\|_*)$ is a Suslin space.

Lemma 4.2.5. *With the distances d and d_* given respectively in (4.2.4) and (4.2.7), we have:*

- $(\mathcal{C}(I; B), d_0)$ and (\mathfrak{X}, d) are Polish spaces.
- $(\mathcal{C}(I; B), d_{0,*})$ and (\mathfrak{X}, d_*) are Suslin spaces.

Proof. In fact, (\mathfrak{X}, d) is a metric space which is the product of two separable complete metric spaces. Note that $\mathcal{C}(I; B)$ is separable and complete with respect to the compact-open topology because I is a hemicompact space (see e.g. [Mun75, Sim63]). On the other hand, the identity map

$$Id : (\mathfrak{X}, d) \rightarrow (\mathfrak{X}, d_*),$$

is continuous with (\mathfrak{X}, d) a Polish space and hence its image (\mathfrak{X}, d_*) is a Suslin space. The spaces $(\mathcal{C}(I; B), d_0)$ and $(\mathcal{C}(I; B), d_{0,*})$ are treated similarly. \square

It is also useful to stress the following result.

Lemma 4.2.6. *The σ -algebras of (\mathfrak{X}, d) and (\mathfrak{X}, d_*) coincide.*

Proof. See [ALR20, Lemma C.2]. \square

Hence, as a consequence, the set of Borel probability measures on (\mathfrak{X}, d) and on (\mathfrak{X}, d_*) coincide.

4.2.2 Projective argument

We introduce the finite rank linear operators:

$$\begin{aligned} T_n : B &\longrightarrow B \\ x &\longmapsto T_n(x) = \sum_{k=1}^n \langle x, e_k \rangle e_k^*. \end{aligned} \quad (4.2.9)$$

Lemma 4.2.7. *The operators $T_n : B \rightarrow B$ satisfy the following properties:*

- (i) $\|T_n(x)\|_* \leq \|x\|_*, \quad \forall x \in B.$
- (ii) $\lim_{n \rightarrow +\infty} \|T_n(x) - x\|_* = 0, \quad \forall x \in B.$

Proof. Recall that the families $\{e_k\}_{k \in \mathbb{N}}$ and $\{e_k^*\}_{k \in \mathbb{N}}$ define a fundamental and strongly total biorthogonal system satisfying (a)(c) in Definition 4.1.1.

(i) For all $x \in B$,

$$\begin{aligned} \|T_n x\|_* &= \sum_{k' \in \mathbb{N}} \frac{1}{2^{k'} \|e_{k'}\|_E} \left| \left\langle \sum_{k=1}^n \langle x, e_k \rangle e_k^*, e_{k'} \right\rangle \right| \\ &= \sum_{k=1}^n \frac{1}{2^k \|e_k\|_E} |\langle x, e_k \rangle| \\ &\leq \|x\|_*. \end{aligned}$$

(ii) For all $x \in \text{Span}\{e_k^*, k \in \mathbb{N}\}$, one checks

$$\lim_n \|T_n x - x\|_* = 0.$$

Then using (i) and the density of $\text{Span}\{e_k^*, k \in \mathbb{N}\}$ in $(B, \|\cdot\|_*)$, we prove (ii) by an approximation argument. □

For $n \in \mathbb{N}$, let $B_n = T_n(B) = \text{Span}(e_1^*, \dots, e_n^*) \subset B$. Denote by

$$\begin{aligned} P_n : B_n &\longrightarrow \mathbb{R}^n \\ \sum_{k=1}^n a_k e_k^* &\longmapsto (a_1, \dots, a_n). \end{aligned}$$

Define also the maps

$$\pi_n : B \longmapsto \mathbb{R}^n, \quad \pi_n = P_n \circ T_n,$$

and

$$\begin{aligned} \tilde{\pi}_n : \mathbb{R}^n &\longrightarrow B_n \subset B \\ (a_1, \dots, a_n) &\longmapsto \tilde{\pi}_n(a_1, \dots, a_n) = \sum_{k=1}^n a_k e_k^*. \end{aligned} \quad (4.2.10)$$

Remark that we have the following relations

$$P_n \circ \tilde{\pi}_n = \text{id}_{\mathbb{R}^n}, \quad \tilde{\pi}_n \circ \pi_n = T_n. \quad (4.2.11)$$

Let $(\mu_t)_{t \in I}$ be a family of Borel probability measures on B satisfying the assumptions in Proposition [4.2.1](#). Then consider the following image measures

$$\boxed{\mu_t^n := (\pi_n)_\# \mu_t \in \mathcal{P}(\mathbb{R}^n)}, \quad \boxed{\tilde{\mu}_t^n := (\tilde{\pi}_n)_\# \mu_t^n = (T_n)_\# \mu_t \in \mathcal{P}(B_n)}. \quad (4.2.12)$$

According to Definition [4.2.3](#), Lemma [4.2.7](#) implies

$$\tilde{\mu}_t^n \xrightarrow{n \rightarrow \infty} \mu_t \quad (4.2.13)$$

for all $t \in I$. In our case, it is useful to consider \mathbb{R}^n with the norm

$$\|y\|_{\mathbb{R}^n} := \|\tilde{\pi}_n y\|_*. \quad (4.2.14)$$

Lemma 4.2.8 (Projection to finite dimensions). *For each $n \in \mathbb{N}$, the curve $(\mu_t^n)_{t \in I}$ given by [\(4.2.12\)](#) is narrowly continuous and satisfies the statistical Liouville equation,*

$$\frac{d}{dt} \int_{\mathbb{R}^n} \varphi(x) \mu_t^n(dx) = \int_{\mathbb{R}^n} \langle v^n(t, x), \nabla_x \varphi(x) \rangle_{\mathbb{R}^n} \mu_t^n(dx), \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n), \quad (4.2.15)$$

in the distribution sense over the interior of I and for some Borel vector field $v^n : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying

$$\int_I \int_{\mathbb{R}^n} \|v^n(t, x)\|_{\mathbb{R}^n} d\mu_t^n(x) \frac{dt}{\omega(|t|)} < +\infty. \quad (4.2.16)$$

Proof. Notice first that the curve $(\mu_t^n)_{t \in I}$ is narrowly continuous. Indeed, let $\varphi \in \mathcal{C}_b(\mathbb{R}^n; \mathbb{R})$ implies $\varphi \circ \pi_n \in \mathcal{C}_b(B_w; \mathbb{R})$. Hence, the curve $t \mapsto \int_{\mathbb{R}^n} \varphi(x) d\mu_t^n(x) = \int_B \varphi \circ \pi_n(x) d\mu_t(x)$ is continuous since $(\mu_t)_{t \in I}$ is weakly narrowly continuous as it is (strongly) narrowly continuous (see Definitions [4.1.4](#) and [4.2.3](#)).

We then consider the statistical Liouville equation [\(4.1.9\)](#) and select $F = \psi \circ \pi_n$ where

$\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. The left part of (4.1.9) is transformed to:

$$\frac{d}{dt} \int_B F(x) d\mu_t(x) = \frac{d}{dt} \int_B \psi \circ \pi_n(x) d\mu_t(x) = \frac{d}{dt} \int_{\mathbb{R}^n} \psi(y) d\mu_t^n(y). \quad (4.2.17)$$

Since the differential $D(\psi \circ \pi_n)(x) = D\psi(\pi_n(x)) \circ \pi_n \in \mathcal{L}(B, \mathbb{R}) = E^{**} \supset E$, identities with the element $\sum_{j=1}^n \partial_j \psi(\pi_n(x)) e_j \in E$, the right part of (4.1.9) is transformed to:

$$\int_B \langle \nabla F(x), v(t, x) \rangle_{E, E^*} d\mu_t(x) = \int_B \langle \nabla \psi(\pi_n(x)), \pi_n \circ v(t, x) \rangle_{\mathbb{R}^n} d\mu_t(x).$$

Remark that since $(B, \|\cdot\|_*)$ is a separable Radon space, we can apply the disintegration Theorem 4.4.13 (see Appendices 4.4.2 and 4.4.5). In particular, there exists a μ_t^n -a.e. determined family of measures $\{\mu_{t,y}^n, y \in \mathbb{R}^n\} \subset \mathcal{P}(B)$ such that $\mu_{t,y}^n(B \setminus (\pi_n)^{-1}(y)) = 0$ and applying formula (4.4.9) with $f(x) = \langle \nabla \psi(\pi_n(x)), \pi_n \circ v(t, x) \rangle_{\mathbb{R}^n}$ yields

$$\int_B \langle \nabla \psi(\pi_n(x)), \pi_n \circ v(t, x) \rangle_{\mathbb{R}^n} d\mu_t(x) = \int_{\mathbb{R}^n} \int_{(\pi_n)^{-1}(y)} \langle \nabla \psi(\pi_n(x)), \pi_n \circ v(t, x) \rangle_{\mathbb{R}^n} d\mu_{t,y}^n(x) d\mu_t^n(y).$$

So, we get

$$\begin{aligned} \int_B \langle \nabla F(x), v(t, x) \rangle_{E, E^*} d\mu_t(x) &= \int_{\mathbb{R}^n} \int_{(\pi_n)^{-1}(y)} \langle \nabla \psi(\pi_n(x)), \pi_n \circ v(t, x) \rangle_{\mathbb{R}^n} d\mu_{t,y}^n(x) d\mu_t^n(y) \\ &= \int_{\mathbb{R}^n} \langle \nabla \psi(y), \int_{(\pi_n)^{-1}(y)} \pi_n \circ v(t, x) d\mu_{t,y}^n(x) \rangle_{\mathbb{R}^n} d\mu_t^n(y) \\ &= \int_{\mathbb{R}^n} \langle \nabla \psi(y), v^n(t, y) \rangle_{\mathbb{R}^n} d\mu_t^n(y). \end{aligned}$$

where we have introduced the vector field v^n as follows:

$$v^n(t, y) := \int_{(\pi_n)^{-1}(y)} \pi_n \circ v(t, x) d\mu_{t,y}^n(x), \text{ for } t \in I \text{ and } \mu_t^n\text{-a.e. } y \in \mathbb{R}^n. \quad (4.2.18)$$

Furthermore, by gathering the above equations, we get

$$\frac{d}{dt} \int_{\mathbb{R}^n} \psi(y) d\mu_t^n(y) = \int_{\mathbb{R}^n} \langle \nabla \psi(y), v^n(t, y) \rangle_{\mathbb{R}^n} d\mu_t^n(y).$$

So, we obtain the statistical Liouville equation (4.2.15). We also obtain that

$$\begin{aligned}
 & \int_I \int_{\mathbb{R}^n} \|v^n(t, y)\|_{\mathbb{R}^n} d\mu_t^n(y) \frac{dt}{\omega(|t|)} \\
 &= \int_I \int_{\mathbb{R}^n} \left\| \int_{(\pi_n)^{-1}(y)} \pi_n \circ v(t, x) d\mu_{t,y}^n(x) \right\|_{\mathbb{R}^n} d\mu_t^n(y) \frac{dt}{\omega(|t|)} \\
 &\leq \int_I \int_B \|\tilde{\pi}_n \circ \pi_n \circ v(t, x)\|_* d\mu_t(x) \frac{dt}{\omega(|t|)} \\
 &\leq \int_I \int_B \|v(t, x)\| d\mu_t(x) \frac{dt}{\omega(|t|)} < +\infty,
 \end{aligned} \tag{4.2.19}$$

where in the above lines, we have used (4.2.18), (4.2.14), the Disintegration Theorem 4.4.13, the second equality in (4.2.11), Lemma 4.2.7-(i) and (4.1.10). \square

4.2.3 Analysis on finite dimensions

In this part, we restrict our selves to the case $B = \mathbb{R}^d$. So, we aim to prove the *global superposition principle* (Proposition 4.2.1) when $B = \mathbb{R}^d$. In fact, similar results are already known in finite dimensions and proved in the book of Ambrosio et al. [AGS08] and in the work of Maniglia [Man07]. The main difference here is that we consider times in unbounded intervals like the half-line \mathbb{R}_+ while in the latter references it is restricted to $[0, T]$. Unfortunately, we could not deduce directly Proposition 4.2.1 in the case $B = \mathbb{R}^d$ from the result of Maniglia [Man07] or even from [AGS08]. In fact, one needs to go through the main ideas and adjust some topological arguments which are behind the compactness properties that lead to the construction of the probability measure η on the path space \mathfrak{X} . We first discuss in Subsection 4.2.3.1 the case where the vector field is locally Lipschitz in the second variable, then we consider the Borel case in Subsection 4.2.3.2.

4.2.3.1 The Lipschitz case

Let $\|\cdot\|_{\mathbb{R}^d}$ be any norm on \mathbb{R}^d . We denote often $v_t = v(t, \cdot)$. In this paragraph, we impose the following local Lipschitz condition.

Assumption 4.2.9 (Lipschitz condition). For every compact set $K \subset \mathbb{R}^d$,

$$t \in I \mapsto \sup_K \|v_t\|_{\mathbb{R}^d} + \text{lip}(v_t, K) \in L^1_{loc}(I, dt), \tag{4.2.20}$$

where $\text{lip}(\cdot, K)$ denotes the Lipschitz constant on K .

Proposition 4.2.10 (The global superposition principle in the *Lipschitz case*). *Consider $B = \mathbb{R}^d$ and $v, (\mu_t)_{t \in I}$ as in Proposition 4.2.1 satisfying the same hypotheses. Additionally, assume that $v : I \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies the local Lipschitz Assumption 4.2.9. Then the conclusion of Proposition 4.2.1 holds true with the Borel probability measure η defined as in (4.2.21).*

Proof. To prove the above result, we recall in Appendix 4.4.1 several auxiliary results from the literature. In particular, the proof is based on Lemmas 4.4.1–4.4.2. By Remark 4.1.6, the assumption (4.1.10) implies that (4.1.12) holds true. Thus, we can apply Lemmas 4.4.1 and 4.4.2 to show that there exists a Borel set $\mathcal{G}^{d,T} \subset \mathbb{R}^d$ such that $\mu_{t_0}(\mathcal{G}^{d,T}) = 1$ and for all $x \in \mathcal{G}^{d,T}$, there is a unique solution $\gamma \in AC^1(I \cap [-T, T]; \mathbb{R}^d)$ to the initial value problem (4.1.4) on $I \cap [-T, T]$ while we take $T > \pm t_0$ so that $I \cap [-T, T]$ has non empty interior.

Let

$$\mathcal{G}^d = \bigcap_{T \in \mathbb{N}, T > \pm t_0} \mathcal{G}^{d,T},$$

which is a Borel subset of \mathbb{R}^d . Remark that by construction $\{\mathcal{G}^{d,T}\}_{T \in \mathbb{N}}$ is a decreasing sequence of Borel sets. Then, by monotone convergence theorem, we have

$$\mu_{t_0}(\mathcal{G}^d) = \mu_{t_0}\left(\bigcap_{T \in \mathbb{N}, T > \pm t_0} \mathcal{G}^{d,T}\right) = \lim_{T \rightarrow +\infty} \mu_{t_0}(\mathcal{G}^{d,T}) = 1.$$

Recall that in the Lipschitz case, we have uniqueness of solutions for the initial value problem (4.1.4). So, we conclude that for each $x \in \mathcal{G}^d$ there exists a global unique solution of (4.1.4). Thus, we construct a well-defined global flow

$$\begin{aligned} \Phi : I \times \mathcal{G}^d &\longrightarrow \mathbb{R}^d \\ (t, x) &\longmapsto \Phi_t(x) = \gamma_x(t), \end{aligned}$$

where γ_x is the global solution of (4.1.4) with the initial condition $\gamma(t_0) = x$. Note that Φ is the flow with prescribed initial conditions at time t_0 . Moreover, using the identity (4.4.1), we have for all $t \in I$,

$$\mu_t = (\Phi_t)_\# \mu_{t_0}.$$

Now, we construct the measure η as

$$\eta = (\text{Id} \times \Phi)_\# \mu_{t_0} \in \mathcal{P}(\mathfrak{X}) \tag{4.2.21}$$

where $\text{Id} \times \Phi_\cdot$ is the map given by

$$\begin{aligned} \text{Id} \times \Phi_\cdot : \mathcal{G}^d &\longrightarrow \mathfrak{X} = \mathbb{R}^d \times \mathcal{C}(I; \mathbb{R}^d) \\ x &\longmapsto (x, \Phi_\cdot(x)) \equiv (x, \gamma_x(\cdot)). \end{aligned}$$

Next, we want to prove that η satisfies the conditions (i) and (ii) of Proposition 4.2.1.

For (i): We have to prove that $\eta(\mathcal{F}^d) = 1$ where

$$\mathcal{F}^d = \left\{ (x, \gamma) \in \mathfrak{X}; \quad \gamma \in AC_{loc}^1(I; \mathbb{R}^d), \quad \gamma(t) = x + \int_{t_0}^t v(\tau, \gamma(\tau)) d\tau, \quad t \in I \right\}.$$

Indeed, we have for $t \in I$

$$\begin{aligned} &\int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t v(\tau, \gamma(\tau)) d\tau \right\|_{\mathbb{R}^d} \eta(dx, d\gamma) \\ &= \int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t v(\tau, \gamma(\tau)) d\tau \right\|_{\mathbb{R}^d} (\text{Id} \times \Phi_\cdot)_\# \mu_{t_0}(dx, d\gamma) \\ &= \int_{\mathbb{R}^d} \left\| \Phi_t(x) - x - \int_{t_0}^t v(\tau, \Phi_\tau(x)) d\tau \right\|_{\mathbb{R}^d} \mu_{t_0}(dx) \\ &= 0, \end{aligned}$$

which implies that we have

$$\gamma(t) = x + \int_{t_0}^t v(\tau, \gamma(\tau)) d\tau, \quad \eta - \text{a.e. } (x, \gamma) \in \mathfrak{X}. \quad (4.2.22)$$

For each $t_j \in \mathbb{Q} \cap I$, a rational number, there exists a null set \mathcal{N}^{t_j} with $\eta(\mathcal{N}^{t_j}) = 0$ and such that (4.2.22) holds true on $\mathfrak{X} \setminus \mathcal{N}^{t_j}$. Then taking

$$\mathcal{N} = \bigcup_{j \in \mathbb{N}} \mathcal{N}^{t_j}$$

with $\eta(\mathcal{N}) = 0$ such that for all $(x, \gamma) \notin \mathcal{N}$ and for all rational numbers in $\{t_j\}_j = \mathbb{Q} \cap I$, we have

$$\gamma(t_j) = x + \int_{t_0}^{t_j} v(\tau, \gamma(\tau)) d\tau. \quad (4.2.23)$$

Now, using the continuity of the curves γ and since $v(\cdot, \gamma(\cdot)) \in L_{loc}^1(I, dt)$ η -almost surely (see condition (4.1.10)), the identity (4.2.23) is well-defined and moreover we get (4.2.22) for all times $t \in I$ and for all $(x, \gamma) \notin \mathcal{N}$.

For (ii): Let $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) ((\Xi_t)_\# \eta)(dx) &= \int_{\mathfrak{X}} \varphi(\Xi_t(x, \gamma)) \eta(dx, d\gamma) = \int_{\mathfrak{X}} \varphi \circ \Xi_t(x, \gamma) (\text{Id} \times \Phi)_\# \mu_{t_0}(dx, d\gamma) \\ &= \int_{\mathcal{G}^d} \varphi \circ \Xi_t \circ (\text{Id} \times \Phi)(x) \mu_{t_0}(dx) = \int_{\mathcal{G}^d} \varphi(\Phi_t(x)) \mu_{t_0}(dx) \\ &= \int_{\mathbb{R}^d} \varphi(x) (\Phi_t)_\# \mu_{t_0}(dx) = \int_{\mathbb{R}^d} \varphi(x) \mu_t(dx). \end{aligned}$$

And thus $(\Xi_t)_\# \eta = \mu_t$, for all $t \in I$. \square

4.2.3.2 The Borel case

In this section, we prove Proposition [4.2.1](#) when $B = \mathbb{R}^d$ and v is a Borel vector field. Now since v is no more assumed to be Lipschitz in the second variable, we have to take into account that the characteristics may not be unique. Indeed, the potential lack of uniqueness of solutions to [\(4.1.4\)](#) on finite intervals makes it impossible to follow the same strategy as before.

Proposition 4.2.11 (The global superposition principle in the Borel case). *Consider $B = \mathbb{R}^d$ and $v, (\mu_t)_{t \in I}$ as in Proposition [4.2.1](#) satisfying the same hypotheses. Then the conclusion of Proposition [4.2.1](#) holds true.*

Proof. The proof of Proposition [4.2.11](#) is based on the three lemmas stated below. The scheme goes as follows. We apply first the regularization Lemma [4.2.12](#) to get an approximating family of probability measures $(\mu_t^\varepsilon)_{t \in I}$ which satisfies a statistical Liouville equation similar to [\(4.1.9\)](#) with a locally Lipschitz vector field v^ε satisfying Assumption [4.2.9](#). Then, we can apply Proposition [4.2.10](#) to the couple $(v^\varepsilon, (\mu_t^\varepsilon)_{t \in I})$ and get a corresponding probability measure $\eta_\varepsilon \in \mathcal{P}(\mathfrak{X})$. Hence, we apply Lemma [4.2.13](#) to prove that the family $\{\eta_\varepsilon\}_\varepsilon$ is tight in $\mathcal{P}(\mathfrak{X})$. Therefore, there exists $\eta \in \mathcal{P}(\mathfrak{X})$ such that $\eta_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \eta$ weakly narrowly (at least for a subsequence). Finally, by Lemma [4.2.14](#), we check that the constructed measure η satisfies [\(i\)](#) and [\(ii\)](#) in Proposition [4.2.1](#). \square

We provide here the above mentioned technical Lemmas [4.2.12](#), [4.2.13](#) and [4.2.14](#).

Lemma 4.2.12 (Regularization). *Consider $B = \mathbb{R}^d$ and $v, (\mu_t)_{t \in I}$ as in Proposition [4.2.1](#) satisfying the same hypotheses. Then, the regularized vector field v_t^ε and the measures μ_t^ε given in [\(4.4.2\)](#) satisfy a statistical Liouville equation as in [\(4.1.9\)](#) over the interval I . Moreover, define*

$$\eta_\varepsilon = (\text{Id} \times \Phi^\varepsilon)_\# \mu_{t_0}^\varepsilon \in \mathcal{P}(\mathfrak{X}) \tag{4.2.24}$$

where $\Phi_t^\varepsilon(x) \equiv \gamma_x^\varepsilon(t)$ is the unique global solution to the initial value problem

$$\dot{\gamma}_x^\varepsilon(t) = v^\varepsilon(t, \gamma_x^\varepsilon(t)), \quad \gamma_x^\varepsilon(t_0) = x. \quad (4.2.25)$$

Then for all $t \in I$,

$$\mu_t^\varepsilon = (\Xi_t)_\# \eta_\varepsilon = (\Phi_t^\varepsilon)_\# \mu_{t_0}^\varepsilon. \quad (4.2.26)$$

Proof. We apply the regularization Lemma 4.4.3, finding the approximation μ_t^ε and v_t^ε in (4.4.2) satisfying the Liouville equation (4.1.9). In particular, the vector field v_t^ε is locally Lipschitz as in Assumption 4.2.9 and satisfies (4.1.10) as a consequence of (4.4.3). Thus, we can apply Proposition 4.2.10 to get $\eta_\varepsilon = (\text{Id} \times \Phi^\varepsilon)_\# \mu_{t_0}^\varepsilon \in \mathcal{P}(\mathfrak{X})$, with $\Phi_t^\varepsilon(x) \equiv \gamma_x^\varepsilon(t)$ the unique global solution to (4.2.25) for all $x \in \mathcal{G}^{d,\varepsilon}$, where we have denoted by $\mathcal{G}^{d,\varepsilon}$ the set of all initial data where (4.2.25) admits a unique global solution. More precisely, we define here

$$\mathcal{F}^{d,\varepsilon} = \left\{ (x, \gamma) \in \mathfrak{X}; \gamma \in AC_{loc}^1(I; \mathbb{R}^d), \gamma(t) = x + \int_{t_0}^t v^\varepsilon(\tau, \gamma(\tau)) d\tau, \forall t \in I \right\}. \quad (4.2.27)$$

We have then

$$\mathcal{G}^{d,\varepsilon} = \left\{ x \in \mathbb{R}^d; \exists \gamma \in AC_{loc}^1(I; \mathbb{R}^d) \text{ s.t. } (x, \gamma) \in \mathcal{F}^{d,\varepsilon} \right\} \quad (4.2.28)$$

with $\mu_{t_0}^\varepsilon(\mathcal{G}^{d,\varepsilon}) = 1$ and $\eta_\varepsilon(\mathcal{F}^{d,\varepsilon}) = 1$. The identity (4.2.26) follows from (4.2.24). \square

Lemma 4.2.13 (Tightness). *The family $\{\eta_\varepsilon\}_\varepsilon$, defined in (4.2.24), is tight in $\mathcal{P}(\mathfrak{X})$.*

Proof. We use here Lemma 4.4.6 with $X \equiv \mathfrak{X}$, $X_1 \equiv \mathbb{R}^d$ and $X_2 \equiv \mathcal{C}(I; \mathbb{R}^d)$. The latter spaces are separable metric spaces. Recall that $\mathcal{C}(I; \mathbb{R}^d)$ is endowed with the compact-open topology (see the metric d in (4.2.2)-(4.2.3)). Define the homeomorphism map $r := r^1 \times r^2 : X \rightarrow X$ by

$$\begin{aligned} r^1 : X &\longrightarrow X_1 & r^2 : X &\longrightarrow X_2 \\ (x, \gamma) &\longmapsto r^1(x, \gamma) = x & (x, \gamma) &\longmapsto r^2(x, \gamma) = \gamma - x. \end{aligned}$$

It is obvious that r is proper. To prove the tightness of $\{\eta_\varepsilon\}_\varepsilon$, it suffices to prove:

1. The family of measures $\{(r^1)_\# \eta_\varepsilon\}_\varepsilon$ is tight in $\mathcal{P}(\mathbb{R}^d)$.
2. The family of measures $\{(r^2)_\# \eta_\varepsilon\}_\varepsilon$ is tight in $\mathcal{P}(\mathcal{C}(I; \mathbb{R}^d))$.

For (1), we have $(r^1)_\# \eta_\varepsilon = \mu_{t_0}^\varepsilon$. Indeed, let $\varphi \in \mathcal{C}_b(\mathbb{R}^d)$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x) (r^1)_\# \eta_\varepsilon(dx) &= \int_{\mathcal{F}^{d,\varepsilon}} \varphi(r^1(x, \gamma)) \eta_\varepsilon(dx, d\gamma) = \int_{\mathcal{F}^{d,\varepsilon}} \varphi(x) \eta_\varepsilon(dx, d\gamma) \\ &= \int_{\mathcal{F}^{d,\varepsilon}} \varphi(\Xi_{t_0}(x, \gamma)) \eta_\varepsilon(dx, d\gamma) = \int_{\mathcal{G}^{d,\varepsilon}} \varphi(x) (\Xi_{t_0})_\# \eta_\varepsilon(dx) = \int_{\mathbb{R}^d} \varphi(x) \mu_{t_0}^\varepsilon(dx), \end{aligned}$$

since $(\Xi_{t_0})_\# \eta_\varepsilon = \mu_{t_0}^\varepsilon$ and $\mu_{t_0}^\varepsilon(\mathbb{R}^d \setminus \mathcal{G}^{d,\varepsilon}) = 0$. Remark also $\mu_{t_0}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \mu_{t_0}$. Since \mathbb{R}^d is a separable Radon space, we get by Lemma 4.4.5, that the family $\{(r^1)_\# \eta_\varepsilon\}_\varepsilon$ is tight in $\mathcal{P}(\mathbb{R}^d)$.

The proof of (2) is more complicated to handle. For that, we apply Lemma 4.4.4. In fact, by Lemma 4.4.9, we get the existence of a non-decreasing superlinear function $\theta : \mathbb{R}_+ \rightarrow [0, +\infty]$ satisfying the inequality (4.4.5). Then we introduce $g : \mathcal{C}(I; \mathbb{R}^d) \rightarrow [0, +\infty]$

$$g(\gamma) := \begin{cases} \int_I \theta(\|\dot{\gamma}\|_{\mathbb{R}^d}) \frac{dt}{\omega(|t|)} & \text{if } \gamma(t_0) = 0 \text{ and } \gamma \in AC_{loc}^1(I; \mathbb{R}^d), \\ +\infty & \text{if } \gamma(t_0) \neq 0 \text{ or } \gamma \notin AC_{loc}^1(I; \mathbb{R}^d). \end{cases}$$

In order to obtain the tightness of the family $\{(r^2)_\# \eta_\varepsilon\}_\varepsilon$, it is enough according to Lemma 4.4.4 to prove the following points:

- (a) $\sup_{\varepsilon > 0} \int_{\mathcal{C}(I; \mathbb{R}^d)} g(\gamma) (r^2)_\# \eta_\varepsilon(d\gamma) < +\infty$.
- (b) For all $c \geq 0$, the sublevel sets $\{\gamma \in \mathcal{C}(I; \mathbb{R}^d); g(\gamma) \leq c\}$ are relatively compact in the space $\mathcal{C}(I; \mathbb{R}^d)$ endowed with the compact-open topology.

For (a), let $\varepsilon > 0$. We have for $\mathcal{F}^{d,\varepsilon}$ and $\mathcal{G}^{d,\varepsilon}$ as defined in (4.2.27) and (4.2.28)

$$\begin{aligned} \int_{\mathcal{C}(I; \mathbb{R}^d)} g(\gamma) (r^2)_\# \eta_\varepsilon(d\gamma) &= \int_{\mathcal{C}(I; \mathbb{R}^d)} \int_I \theta(\|\dot{\gamma}(t)\|_{\mathbb{R}^d}) \frac{dt}{\omega(|t|)} (r^2)_\# \eta_\varepsilon(d\gamma) \\ &= \int_I \int_{\mathcal{F}^{d,\varepsilon}} \theta(\|v^\varepsilon(t, \gamma(t))\|_{\mathbb{R}^d}) \eta_\varepsilon(dx, d\gamma) \frac{dt}{\omega(|t|)} \\ &= \int_I \int_{\mathcal{G}^{d,\varepsilon}} \theta(\|v^\varepsilon(t, \Phi_t^\varepsilon(x))\|_{\mathbb{R}^d}) \mu_{t_0}^\varepsilon(dx) \frac{dt}{\omega(|t|)} \\ &\leq \int_I \int_{\mathbb{R}^d} \theta(\|v^\varepsilon(t, x)\|_{\mathbb{R}^d}) \mu_t^\varepsilon(dx) \frac{dt}{\omega(|t|)} \\ &\leq \int_I \int_{\mathbb{R}^d} \theta(\|v(t, x)\|_{\mathbb{R}^d}) \mu_t(dx) \frac{dt}{\omega(|t|)} < +\infty. \end{aligned}$$

Note that for the above inequality, we used Fubini's theorem and in the last step, we used [Man07, Lemma 3.10] which generalizes (4.4.3). The above inequality holds uniformly in $\varepsilon > 0$.

For (b), thanks to Lemma 4.4.11 and Remark 4.4.12, the sublevels

$$\mathcal{A}_c = \{\gamma \in \mathcal{C}(I; \mathbb{R}^d); g(\gamma) \leq c\}$$

are relatively compact in the separable metric space $(\mathcal{C}(I; \mathbb{R}^d), d_0)$ with the distance d_0 inducing the compact-open topology given in 4.2.3. Hence, thanks to Lemma 4.4.6, we conclude that $\{\eta_\varepsilon\}_\varepsilon$ is tight in $\mathcal{P}(\mathfrak{X})$. □

Lemma 4.2.14 (Concentration and lifting properties). *The subsequential limit η (in the sense of narrow convergence) of the family $\{\eta_\varepsilon\}_\varepsilon$ satisfies (i) and (ii) in Proposition 4.2.1.*

Proof. The existence of η is guaranteed by Lemma 4.2.13. We show that η satisfies (i) and (ii) in Proposition 4.2.1. Remark that we do not have the explicit expression for η in this case. And thus, we cannot proceed as before to prove first (i). Then, we start with proving (ii). For $\varphi \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R})$, we have

$$\int_{\mathbb{R}^d} \varphi(x) \mu_t^\varepsilon(dx) = \int_{\mathfrak{X}} \varphi(\gamma(t)) \eta_\varepsilon(dx, d\gamma),$$

where η_ε is as in 4.2.24. Hence, we can let $\varepsilon \rightarrow 0$ in the above equation and deduce

$$\int_{\mathbb{R}^d} \varphi(x) \mu_t(dx) = \int_{\mathfrak{X}} \varphi(\gamma(t)) \eta(dx, d\gamma).$$

The above equality is true for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{R})$. This gives $(\Xi_t)_\# \eta = \mu_t$, for all $t \in I$. And thus, condition (ii) is satisfied. Finally, we check condition (i). Let $w : I \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a bounded continuous vector field. We write $w(t, x) \equiv w_t(x)$ and introduce the regularized vector field $w_\tau^\varepsilon := \frac{(w_\tau \mu_\tau) * \rho_\varepsilon}{\mu_\tau * \rho_\varepsilon}$ (for ρ_ε as in Lemma 4.4.3). For $T > 0$, we have for all $t \in J = I \cap [-T, T]$,

$$\begin{aligned} & \int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t w(\tau, \gamma(\tau)) d\tau \right\|_{\mathbb{R}^d} \eta_\varepsilon(dx, d\gamma) \\ &= \int_{\mathcal{F}^{d,\varepsilon}} \left\| \int_{t_0}^t v^\varepsilon(\tau, \gamma(\tau)) d\tau - \int_{t_0}^t w(\tau, \gamma(\tau)) d\tau \right\|_{\mathbb{R}^d} \eta_\varepsilon(dx, d\gamma) \\ &= \int_{\mathcal{G}^{d,\varepsilon}} \left\| \int_{t_0}^t v^\varepsilon(\tau, \Phi_\tau^\varepsilon(x)) d\tau - \int_{t_0}^t w(\tau, \Phi_\tau^\varepsilon(x)) d\tau \right\|_{\mathbb{R}^d} \mu_{t_0}^\varepsilon(dx) \\ &\leq \int_J \int_{\mathbb{R}^d} \|v^\varepsilon(\tau, x) - w(\tau, x)\|_{\mathbb{R}^d} \mu_\tau^\varepsilon(dx) d\tau \end{aligned}$$

$$\begin{aligned}
 &\leq \int_J \int_{\mathbb{R}^d} \|v^\varepsilon(\tau, x) - w^\varepsilon(\tau, x)\|_{\mathbb{R}^d} \mu_\tau^\varepsilon(dx) d\tau \\
 &\quad + \int_J \int_{\mathbb{R}^d} \|w^\varepsilon(\tau, x) - w(\tau, x)\|_{\mathbb{R}^d} \mu_\tau^\varepsilon(dx) d\tau \\
 &\leq \int_J \int_{\mathbb{R}^d} \|v(\tau, x) - w(\tau, x)\|_{\mathbb{R}^d} d\mu_\tau(x) d\tau \\
 &\quad + \int_J \int_{\mathbb{R}^d} \|(w_\tau \mu_\tau) * \rho_\varepsilon(x) - w(\tau, x) \mu_\tau * \rho_\varepsilon(x)\|_{\mathbb{R}^d} \mu_\tau(dx) d\tau \\
 &\leq \int_J \int_{\mathbb{R}^d} \|v(\tau, x) - w(\tau, x)\|_{\mathbb{R}^d} \mu_\tau(dx) d\tau \\
 &\quad + \int_J \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \|w(\tau, x) - w(\tau, y)\|_{\mathbb{R}^d} \rho_\varepsilon(x - y) dx \right] \mu_\tau(dy) d\tau.
 \end{aligned}$$

For the above inequality, we used Fubini's theorem as well as the inequality (4.4.3). On the other hand,

$$\begin{aligned}
 &\int_J \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \|w(\tau, x) - w(\tau, y)\|_{\mathbb{R}^d} \rho_\varepsilon(x - y) dx \right] \mu_\tau(dy) d\tau \\
 &\leq \int_J \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \|w(\tau, \varepsilon\tilde{x} + y) - w(\tau, y)\|_{\mathbb{R}^d} \rho(\tilde{x}) d\tilde{x} \right] \mu_\tau(dy) d\tau
 \end{aligned}$$

where $\rho_\varepsilon(x) = 1/\varepsilon^d \rho(x/\varepsilon)$. Moreover, by the Lebesgue dominated convergence theorem, the above expression tends to zero as $\varepsilon \rightarrow 0$. We then deduce that

$$\int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t w(\tau, \gamma(\tau)) d\tau \right\|_{\mathbb{R}^d} \eta(dx, d\gamma) \leq \int_J \int_{\mathbb{R}^d} \|v(\tau, x) - w(\tau, x)\|_{\mathbb{R}^d} \mu_\tau(dx) d\tau. \quad (4.2.29)$$

Follow the same argument as in the proof of Lemma 4.4.9 to define a measure ν as in (4.4.6) on the product space $J \times \mathbb{R}^d$. Then using Lemma 4.4.14, the space $\mathcal{C}_b(J \times \mathbb{R}^d; \mathbb{R}^d)$ is dense in $L^1(J \times \mathbb{R}^d, \nu; \mathbb{R}^d)$. Let $(w_m)_m \subset \mathcal{C}_b(J \times \mathbb{R}^d; \mathbb{R}^d)$ be a sequence of continuous bounded functions converging to $v \in L^1(J \times \mathbb{R}^d, \nu; \mathbb{R}^d)$. Using the fact that $\mu_t = (\Xi_t)_\# \eta$ as well as $\lim_m w_m = v$ in $L^1(J \times \mathbb{R}^d, \nu; \mathbb{R}^d)$, we have for all $T > 0$

$$\begin{aligned}
 &\int_{\mathfrak{X}} \int_J \|w_m(\tau, \gamma(t)) - v(\tau, \gamma(t))\|_{\mathbb{R}^d} d\tau \eta(dx, d\gamma) \\
 &= \int_J \int_{\mathbb{R}^d} \|w_m(\tau, x) - v(\tau, x)\|_{\mathbb{R}^d} \mu_\tau(dx) d\tau \xrightarrow{m \rightarrow +\infty} 0.
 \end{aligned} \quad (4.2.30)$$

At the end, we use the triangle inequality, and apply (4.2.29) with $w \equiv w_m$ together with (4.2.30) to obtain

$$\begin{aligned}
0 &\leq \int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t v(\tau, \gamma(\tau)) d\tau \right\|_{\mathbb{R}^d} \eta(dx, d\gamma) \\
&\leq \int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t w_m(\tau, \gamma(\tau)) d\tau \right\|_{\mathbb{R}^d} \eta(dx, d\gamma) + \int_{\mathfrak{X}} \int_J \| (w_m - v)(\tau, \gamma(\tau)) \|_{\mathbb{R}^d} d\tau \eta(dx, d\gamma) \\
&\leq 2 \int_J \int_{\mathbb{R}^d} \| v(\tau, x) - w_m(\tau, x) \|_{\mathbb{R}^d} \mu_\tau(dx) d\tau.
\end{aligned}$$

We take $m \rightarrow +\infty$ to deduce that: $\forall T \in \mathbb{R}_+^*$, $\forall t \in J = I \cap [-T, T]$,

$$\int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t v(\tau, \gamma(\tau)) d\tau \right\|_{\mathbb{R}^d} \eta(dx, d\gamma) = 0. \quad (4.2.31)$$

Hence, for all $t \in I$, we get (4.2.31). This implies that for each $t \in I$, we have

$$\gamma(t) = x + \int_{t_0}^t v(\tau, \gamma(\tau)) d\tau, \text{ for } \eta - a.e. (x, \gamma) \in \mathbb{R}^d \times \mathcal{C}(I; \mathbb{R}^d). \quad (4.2.32)$$

Now, due to the continuity of the curves γ as well as $v(\cdot, \gamma(\cdot)) \in L_{loc}^1(I; dt)$ η -almost surely, using the same arguments as in the proof of Proposition 4.2.10, we can find by density arguments an η -null set \mathcal{N} such that the Duhamel formula (4.2.32) holds true for all times $t \in I$ and for all $(x, \gamma) \notin \mathcal{N}$. \square

4.2.4 Analysis on Banach spaces

We want to complete the proof of the *global superposition principle* of Proposition 4.2.1 by applying the results in the previous Section 4.2.3.

Proof of Proposition 4.2.1: The strategy of the proof is similar to the finite dimensional case in Proposition 4.2.11. Consider $(B, \|\cdot\|)$ to be an infinite dimensional separable dual Banach space. Let v and $(\mu_t)_{t \in I}$ as in Proposition 4.2.1 satisfying the same hypotheses. Recall the image measures $\mu_t^n \in \mathcal{P}(\mathbb{R}^n)$ and $\tilde{\mu}_t^n \in \mathcal{P}(B_n)$ given in (4.2.12) and the subspace $B_n = \text{Span}(e_1^*, \dots, e_n^*) \subset B$ as well as $\|\cdot\|_{\mathbb{R}^d}$ in (4.2.14). We apply then the projection argument in Lemma 4.2.8. Hence, we conclude that there exists a Borel vector field $v^n : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ given in (4.2.18) such that the probability measures $(\mu_t^n)_{t \in I}$ satisfy the estimate (4.2.16) and the statistical Liouville equation (4.2.15). Therefore, we have all the ingredients to apply Proposition 4.2.11 for the couple $(v_t^n, (\mu_t^n)_{t \in I})$ and get the existence of the path measure $\eta^n \in \mathcal{P}(\mathbb{R}^n \times \mathcal{C}(I; \mathbb{R}^n))$ so that η^n satisfies the concentration and lifting properties in Proposition 4.2.11 for each $n \in \mathbb{N}$.

We then define

$$\tilde{\eta}^n := (\tilde{\pi}_n \times \tilde{\pi}_n)_{\#} \eta^n \in \mathcal{P}(\mathfrak{X}_n), \quad (4.2.33)$$

where $\tilde{\pi}_n$ is introduced in (4.2.10) and $\mathfrak{X}_n := B_n \times \mathcal{C}(I; B_n) \subset \mathfrak{X} = B \times \mathcal{C}(I; B)$. Thanks to Lemma 4.2.15 given below, we obtain that the sequence $\{\tilde{\eta}^n\}_n$ is tight in $\mathcal{P}(\mathfrak{X}) \supset \mathcal{P}(\mathfrak{X}_n)$ (since \mathfrak{X}_n is a Borel subset of \mathfrak{X}). So, there exist an $\eta \in \mathcal{P}(\mathfrak{X})$ and a subsequence that we still denote by $(\tilde{\eta}^n)_n$ such that $\tilde{\eta}^n \xrightarrow[n \rightarrow \infty]{} \eta$ weakly narrowly. Finally, by Lemma 4.2.16, we conclude that the constructed path measure η satisfies (i) and (ii) in Proposition 4.2.1. \square

We are now going to state and prove the aforementioned technical Lemmas 4.2.15 and 4.2.16 used in the proof of Proposition 4.2.1.

Lemma 4.2.15 (Tightness in Banach spaces). *The family of path measures $\{\tilde{\eta}^n\}_n$ given in (4.2.33), is tight for the weak narrow topology of $\mathcal{P}(\mathfrak{X})$.*

Proof. We use here Lemma 4.4.6 with $X \equiv (\mathfrak{X}, d_*)$ defined in (4.2.1), $X_1 \equiv B_w = (B, \|\cdot\|_*)$ and $X_2 \equiv (\mathcal{C}(I; B), d_{0,*})$. The latter spaces are separable metric spaces. Define the homeomorphism map $r := r^1 \times r^2 : X \rightarrow X$ by

$$\begin{aligned} r^1 : X &\longrightarrow X_1 & r^2 : X &\longrightarrow X_2 \\ (x, \gamma) &\longmapsto r^1(x, \gamma) = x & (x, \gamma) &\longmapsto r^2(x, \gamma) = \gamma - x. \end{aligned}$$

According to Lemma 4.4.6, to prove the tightness of $\{\tilde{\eta}^n\}_n$, it suffices to show:

1. The family of measures $\{(r^1)_{\#} \tilde{\eta}^n\}_n$ is tight in $\mathcal{P}(X_1)$.
2. The family of measures $\{(r^2)_{\#} \tilde{\eta}^n\}_n$ is tight in $\mathcal{P}(X_2)$.

For (1), we have $(r^1)_{\#} \tilde{\eta}^n = \tilde{\mu}_{t_0}^n$ for all $n \in \mathbb{N}$. Indeed, let $\varphi \in \mathcal{C}_b(B_w; \mathbb{R})$, we have

$$\begin{aligned} \int_B \varphi(x) (r^1)_{\#} \tilde{\eta}^n(dx) &= \int_{\mathfrak{X}} \varphi(r^1(x, \gamma)) \tilde{\eta}^n(dx, d\gamma) = \int_{\mathfrak{X}} \varphi(x) \tilde{\eta}^n(dx, d\gamma) = \int_{\mathfrak{X}_n} \varphi(\tilde{\pi}_n(x)) \eta^n(dx, d\gamma) \\ &= \int_{\mathfrak{X}_n} \varphi \circ \tilde{\pi}_n \circ \Xi_{t_0}(x, \gamma) \eta^n(dx, d\gamma) = \int_{\mathbb{R}^n} \varphi \circ \tilde{\pi}_n(x) \mu_{t_0}^n(dx) = \int_B \varphi(x) \tilde{\mu}_{t_0}^n(dx). \end{aligned}$$

where $(\tilde{\pi}_n)_{\#} \mu_{t_0}^n = \tilde{\mu}_{t_0}^n$ is given in (4.2.12). Remark also $\tilde{\mu}_{t_0}^n \rightharpoonup \mu_{t_0}$ as $n \rightarrow +\infty$. And since B_w is a separable Radon space, we get by Lemma 4.4.5, that the family $\{(r^1)_{\#} \tilde{\eta}^n\}_n$ is tight in $\mathcal{P}(B)$.

The proof of (2) follows the same strategy as in the finite-dimensional case. Using Lemma 4.4.9, there exists a non-decreasing super-linear continuous convex function $\theta : \mathbb{R}_+ \rightarrow$

$[0, +\infty]$ such that

$$\int_I \int_B \theta(\|v(t, x)\|_B) \mu_t(dx) \frac{dt}{\omega(|t|)} \leq 1. \quad (4.2.34)$$

We want to apply Lemma 4.4.4. To this end, we introduce $g : \mathcal{C}(I; B) \rightarrow [0, +\infty]$

$$g(\gamma) := \begin{cases} \int_I \theta(\|\dot{\gamma}\|_*) \frac{dt}{\omega(|t|)} & \text{if } \gamma(t_0) = 0 \text{ and } \gamma \in AC_{loc}^1(I; B), \\ +\infty & \text{if } \gamma(t_0) \neq 0 \text{ or } \gamma \notin AC_{loc}^1(I; B). \end{cases}$$

According to Lemma 4.4.4, we have to prove the following points:

- (a) $\sup_{n \in \mathbb{N}} \int_{\mathcal{C}(I; B)} g(\gamma) (r^2)_{\#} \tilde{\eta}^n(d\gamma) < +\infty$.
- (b) The sublevel sets $\mathcal{A}_c := \{\gamma \in \mathcal{C}(I; B); g(\gamma) \leq c\}$ are relatively compact in $(\mathcal{C}(I; B), d_{0,*})$ for all $c \geq 0$.

For (a), let $n \in \mathbb{N}$. We have

$$\begin{aligned} \int_{\mathcal{C}(I; B)} g(\gamma) (r^2)_{\#} \tilde{\eta}^n(d\gamma) &= \int_{\mathcal{C}(I; B)} \int_I \theta(\|\dot{\gamma}(t)\|_*) \frac{dt}{\omega(|t|)} (r^2)_{\#} \tilde{\eta}^n(d\gamma) \\ &= \int_{\mathfrak{X}} \int_I \theta(\|\dot{\gamma}(t)\|_*) \frac{dt}{\omega(|t|)} \tilde{\eta}^n(dx, d\gamma) \\ &= \int_{\mathbb{R}^n \times \mathcal{C}(I; \mathbb{R}^n)} \int_I \theta(\|\tilde{\pi}_n(\dot{\gamma}(t))\|_*) \frac{dt}{\omega(|t|)} \eta^n(dx, d\gamma) \\ &= \int_{\mathbb{R}^n \times \mathcal{C}(I; \mathbb{R}^n)} \int_I \theta(\|\tilde{\pi}_n \circ v^n(t, \gamma(t))\|_*) \frac{dt}{\omega(|t|)} \eta^n(dx, d\gamma) \\ &= \int_I \int_{\mathbb{R}^n \times \mathcal{C}(I; \mathbb{R}^n)} \theta(\|v^n(t, \gamma(t))\|_{\mathbb{R}^n}) \eta^n(dx, d\gamma) \frac{dt}{\omega(|t|)} \\ &= \int_I \int_{\mathbb{R}^n} \theta\left(\left\| \int_{(\pi_n)^{-1}(y)} \pi_n \circ v(t, x) \mu_{t,y}^n(dx) \right\|_{\mathbb{R}^n}\right) \mu_t^n(dy) \frac{dt}{\omega(|t|)} \\ &\leq \int_I \int_B \theta(\|v(t, x)\|_*) \mu_t(dx) \frac{dt}{\omega(|t|)}, \end{aligned}$$

where we used the definition of v^n as in (4.2.18), Jensen's inequality and the arguments as for (4.2.19). For the last line, we used

$$\| \pi_n \circ v(t, x) \|_{\mathbb{R}^n} = \| \tilde{\pi}_n \circ \pi_n \circ v(t, x) \| = \| T_n v(t, x) \|_* \leq \| v(t, x) \|_*,$$

which follows from (4.2.14), the second equality in (4.2.11), and Lemma 4.2.7 (i). Then by using the estimate (4.2.34), the monotonicity of θ , and $\|v(t, x)\|_* \leq \|v(t, x)\|$ (which

follows from (4.2.5)), we conclude from the above calculation that

$$\sup_{n \in \mathbb{N}} \int_{\mathcal{C}(I; B)} g(\gamma) (r^2)_{\#} \tilde{\eta}^n(d\gamma) < +\infty.$$

For (b), we apply Lemma 4.4.11 and conclude that the sublevels $\mathcal{A}_c := \{\gamma \in \mathcal{C}(I; B); g(\gamma) \leq c\}$ are relatively compact in $(\mathcal{C}(I; B_w), d_{0,*})$ for all $c \geq 0$. However, we still need to check that \mathcal{A}_c is relatively compact in $(\mathcal{C}(I; B), d_{0,*})$. Let $(\gamma_n)_n$ be a sequence in \mathcal{A}_c . Then there is a subsequence $(\gamma_{n_k})_k$ and $\gamma \in \mathcal{C}(I; B_w)$ such that $d_{0,*}(\gamma_{n_k}; \gamma) \xrightarrow[k \rightarrow \infty]{} 0$. Hence, we just need to prove that $\gamma \in \mathcal{C}(I; B)$. First remark that

$$\|\gamma_n(t) - \gamma_n(t_0)\|_B \leq \int_{[t_0, t]} \|\dot{\gamma}_n(s)\|_B ds.$$

Assume $|t| \leq T$ and consider the set

$$\mathcal{F} = \{\|\dot{\gamma}_n(\cdot)\|_B; n \in \mathbb{N}\}.$$

Remark that

$$\mathcal{F} \subset \left\{ f \in L^1\left(I; \frac{dt}{\omega(|t|)}\right) : \int_I \frac{\theta(|f|)}{c} \frac{ds}{\omega(|s|)} \leq 1 \right\}. \quad (4.2.35)$$

Then by Lemma 4.4.7, \mathcal{F} is equi-integrable. And thus by the Dunford-Pettis theorem 4.4.8, \mathcal{F} is relatively sequentially compact in the topology $\sigma(L^1, L^\infty)$. More precisely, this means there exists $m(\cdot) \in L^1\left(I, \frac{dt}{\omega(|t|)}\right)$ such that

$$\|\dot{\gamma}_n(\cdot)\|_B \rightharpoonup m(\cdot) \text{ in } \sigma(L^1, L^\infty).$$

And thus, for all $f \in L^\infty$

$$\int_I \|\dot{\gamma}_n(s)\|_B f(s) \frac{ds}{\omega(|s|)} \longrightarrow \int_I m(s) f(s) \frac{ds}{\omega(|s|)}.$$

Take $f = 1$, we get

$$\int_I \|\dot{\gamma}_n(s)\|_B \frac{ds}{\omega(|s|)} \longrightarrow \int_I m(s) \frac{ds}{\omega(|s|)}.$$

Since $\|\cdot\|_*$ induces the weak-* topology on bounded sets, for $|t|, |t_0| \leq T$, we get

$$\begin{aligned} \|\gamma(t) - \gamma(t_0)\|_B &\leq \liminf_{n \rightarrow +\infty} \|\gamma_n(t) - \gamma_n(t_0)\|_B \\ &\leq \liminf_{n \rightarrow +\infty} \int_{[t_0, t]} \|\dot{\gamma}_n(s)\|_B ds \\ &\lesssim \liminf_{n \rightarrow +\infty} \int_{[t_0, t]} \|\dot{\gamma}_n(s)\|_B \frac{ds}{\omega(|s|)} \\ &\lesssim \int_{[t_0, t]} \frac{m(s)}{\omega(|s|)} ds. \end{aligned}$$

Thus, we conclude that $\gamma \in AC_{loc}^1(I; B) \subset \mathcal{C}(I; B)$. \square

We give now the proof of the concentration and lifting properties (i) and (ii) in Proposition 4.2.1.

Lemma 4.2.16. *Let $\eta \in \mathcal{P}(\mathfrak{X})$ be any cluster point of the tight sequence $\{\tilde{\eta}^n\}_n$ defined in (4.2.33). Then η satisfies the properties (i) and (ii) of Proposition 4.2.1.*

Proof. We start to give the proof of (ii). Then, we address to the proof of (i) which can be achieved using (ii).

For (ii):

We have, for $n \in \mathbb{N}$, for $\varphi \in \mathcal{C}_b(B_w)$, by (4.2.33), Proposition 4.2.11 and (4.2.12)

$$\begin{aligned} \int_{\mathfrak{X}} \varphi(\gamma(t)) \tilde{\eta}^n(dx, d\gamma) &= \int_{\mathbb{R}^n \times \mathcal{C}(I; \mathbb{R}^n)} \varphi \circ \tilde{\pi}_n(\gamma(t)) \eta^n(dx, d\gamma) = \int_{\mathbb{R}^n} \varphi(\tilde{\pi}_n(x)) \mu_t^n(dx) \\ &= \int_B \varphi(x) (\tilde{\pi}_n)_\# \mu_t^n(dx) = \int_B \varphi(x) \tilde{\mu}_t^n(dx). \end{aligned}$$

By (4.2.12), we have $\tilde{\mu}_t^n \rightharpoonup \mu_t$ and $\tilde{\eta}^n \rightharpoonup \eta$ as $n \rightarrow +\infty$. We take limits in the above formula to get

$$\int_{\mathfrak{X}} \varphi(\gamma(t)) \eta(dx, d\gamma) = \int_B \varphi(x) \mu_t(dx).$$

The above equality implies that $\Xi_{t\#} \eta = \mu_t$, for all $t \in I$.

For (i):

Let $h : I \times B \rightarrow B$ be any bounded continuous function. Then, we claim that for all $t \in I$

$$\int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t h(\tau, \gamma(\tau)) d\tau \right\|_* \eta(dx, d\gamma) \leq \int_{[t_0, t]} \int_B \|v(\tau, x) - h(\tau, x)\|_* \mu_\tau(dx) d\tau. \quad (4.2.36)$$

To prove the above inequality, we consider first the following points:

1. Using the disintegration Theorem [4.4.13](#) with the projection $T_n : B \rightarrow B_n$, and since $\tilde{\mu}_t^n = (T_n)_\# \mu_t \in \mathcal{P}(B_n)$, there exists a $\tilde{\mu}_t^n$ -a.e. uniquely determined family of Borel probability measures $\{\tilde{\mu}_{t,y}^n\}_{y \in B_n} \subset \mathcal{P}(B)$ such that $\tilde{\mu}_{t,y}^n(B \setminus T_n^{-1}(y)) = 0$ for $\tilde{\mu}_t^n$ -a.e. $y \in B_n$ and

$$\int_B f(x) \mu_t(dx) = \int_{B_n} \left(\int_{T_n^{-1}(y)} f(x) \tilde{\mu}_{t,y}^n(dx) \right) \tilde{\mu}_t^n(dy)$$

for every Borel map $f : B \rightarrow [0, +\infty]$ with T_n is the map introduced in Section [4.2.2](#) and satisfying the properties in Lemma [4.2.7](#).

2. Recall [\(4.2.33\)](#) and [\(4.2.12\)](#). We have

$$\tilde{\mu}_t^n = (\Xi_t)_\# \tilde{\eta}^n \quad (4.2.37)$$

for all $t \in I$. Moreover, $\tilde{\eta}^n$ is concentrated on the set of pairs (x, γ) such that $\gamma \in AC_{loc}^1(I; B_n)$ is a solution of the $\dot{\gamma}(t) = \tilde{v}^n(t, \gamma(t))$, $\gamma(t_0) = x \in B_n$ for a.e. $t \in I$ where the vector field \tilde{v}^n is the projected one defined as follows

$$\tilde{v}^n(t, y) := \int_{T_n^{-1}(y)} T_n \circ v(t, x) \tilde{\mu}_{t,y}^n(dx) \quad \text{for } \tilde{\mu}_t^n \text{ a.e. } t \in I. \quad (4.2.38)$$

We obtain [\(4.2.38\)](#) by arguing analogously as for [\(4.2.18\)](#) in the proof of Lemma [4.2.8](#) above.

3. We have also

$$T_n \circ h(t, y) = \int_{T_n^{-1}(y)} T_n \circ h(t, T_n(x)) \tilde{\mu}_{t,y}^n(dx), \quad \forall y \in B_n.$$

The above identity is true by the support property of $\tilde{\mu}_{t,y}^n$.

To prove [\(4.2.36\)](#), we start with

$$\begin{aligned} & \int_{\tilde{\mathfrak{X}}} \left\| \gamma(t) - x - \int_{t_0}^t h(\tau, \gamma(\tau)) d\tau \right\|_* \tilde{\eta}^n(dx, d\gamma) \\ & \leq \int_{\tilde{\mathfrak{X}}} \left\| \gamma(t) - x - \int_{t_0}^t T_n \circ h(\tau, \gamma(\tau)) d\tau \right\|_* \tilde{\eta}^n(dx, d\gamma) \\ & + \int_{\tilde{\mathfrak{X}}} \left\| \int_{t_0}^t [T_n \circ h(\tau, \gamma(\tau)) - h(\tau, \gamma(\tau))] d\tau \right\|_* \tilde{\eta}^n(dx, d\gamma). \end{aligned} \quad (4.2.39)$$

Gathering (1), (2) and (3), [\(4.2.37\)](#), Lemma [4.2.7](#) then using the disintegration Theorem

[4.4.13](#) and the Lebesgue dominated convergence theorem, the second line in [\(4.2.39\)](#) gives

$$\begin{aligned}
 & \int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t T_n \circ h(\tau, \gamma(\tau)) d\tau \right\|_* \tilde{\eta}^n(dx, d\gamma) \\
 & \leq \int_{\mathfrak{X}_n} \left\| \int_{[t_0, t]} [\tilde{v}^n(\tau, \gamma(\tau)) - T_n \circ h(\tau, \gamma(\tau))] d\tau \right\|_* \tilde{\eta}^n(dx, d\gamma) \\
 & \leq \int_{[t_0, t]} \int_{\mathfrak{X}_n} \|\tilde{v}^n(\tau, \gamma(\tau)) - T_n \circ h(\tau, \gamma(\tau))\|_* \tilde{\eta}^n(dx, d\gamma) d\tau \\
 & = \int_{[t_0, t]} \int_{B_n} \|\tilde{v}^n(\tau, y) - T_n \circ h(\tau, y)\|_* \tilde{\mu}_\tau^n(dy) d\tau \\
 & = \int_{[t_0, t]} \int_{B_n} \left\| \int_{T_n^{-1}(y)} T_n \circ v(\tau, x) \tilde{\mu}_{\tau, y}^n(dx) - \int_{T_n^{-1}(y)} T_n \circ h(\tau, T_n(x)) \tilde{\mu}_{\tau, y}^n(dx) \right\|_* \tilde{\mu}_\tau^n(dy) d\tau \\
 & \leq \int_{[t_0, t]} \int_B \|T_n \circ v(\tau, x) - T_n \circ h(\tau, T_n(x))\|_* \mu_\tau(dx) d\tau \\
 & \leq \int_{[t_0, t]} \int_B \|v(\tau, x) - h(\tau, T_n(x))\|_* \mu_\tau(dx) d\tau \\
 & \leq \left[\int_{[t_0, t]} \int_B \|v(\tau, x) - h(\tau, x)\|_* \mu_\tau(dx) d\tau + \int_{[t_0, t]} \int_B \|h(\tau, T_n x) - h(\tau, x)\|_* \mu_\tau(dx) d\tau \right] \\
 & = \int_{[t_0, t]} \int_B \|v(\tau, x) - h(\tau, x)\|_* \mu_\tau(dx) d\tau + \varepsilon_1(n),
 \end{aligned}$$

with $\varepsilon_1(n) \rightarrow 0$ as $n \rightarrow +\infty$. Again by the Lebesgue dominated convergence theorem, the third line of [\(4.2.39\)](#) gives

$$\begin{aligned}
 & \int_{\mathfrak{X}} \left\| \int_{t_0}^t [T_n \circ h(\tau, \gamma(\tau)) - h(\tau, \gamma(\tau))] d\tau \right\|_* \tilde{\eta}^n(dx, d\gamma) \\
 & \leq \int_{[t_0, t]} \int_{\mathfrak{X}} \|T_n \circ h(\tau, \gamma(\tau)) - h(\tau, \gamma(\tau))\|_* \tilde{\eta}^n(dx, d\gamma) d\tau \\
 & = \int_{[t_0, t]} \int_{B_n} \|T_n \circ h(\tau, x) - h(\tau, x)\|_* \tilde{\mu}_\tau^n(dx) d\tau \\
 & = \int_{[t_0, t]} \int_{B_n} \|T_n \circ h(\tau, T_n x) - h(\tau, T_n x)\|_* \mu_\tau(dx) d\tau \\
 & \leq \varepsilon_2(n),
 \end{aligned}$$

with $\varepsilon_2(n) \rightarrow 0$ as $n \rightarrow +\infty$. Combining the bounds on the second and third lines in [\(4.2.39\)](#) and the above calculations, we conclude

$$\begin{aligned}
 \int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_0^t h(\tau, \gamma(\tau)) d\tau \right\|_* \tilde{\eta}^n(dx, d\gamma) & \leq \int_{[t_0, t]} \int_B \|v(\tau, x) - h(\tau, x)\|_* \mu_\tau(dx) d\tau \\
 & \quad + \varepsilon_1(n) + \varepsilon_2(n).
 \end{aligned}$$

Letting $n \rightarrow +\infty$ in the above equality, we get (4.2.36). We have then

$$\begin{aligned}
 & \int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t v(\tau, \gamma(\tau)) d\tau \right\|_* \eta(dx, d\gamma) \\
 & \leq \int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t h(\tau, \gamma(\tau)) d\tau \right\|_* \eta(dx, d\gamma) + \int_{\mathfrak{X}} \left\| \int_{t_0}^t [v(\tau, \gamma(\tau)) - h(\tau, \gamma(\tau))] d\tau \right\|_* \eta(dx, d\gamma) \\
 & \leq 2 \int_{[t_0, t]} \int_B \|v(\tau, x) - h(\tau, x)\|_* \mu_\tau(dx) d\tau \\
 & \leq 2 \left[\int_{[t_0, t]} \int_B \|T_n \circ v(\tau, x) - v(\tau, x)\|_* \mu_\tau(dx) d\tau + \int_{[t_0, t]} \int_B \|T_n \circ v(\tau, x) - h(\tau, x)\|_* \mu_\tau(dx) d\tau \right].
 \end{aligned}$$

First, since we have (4.1.10), then by the Lebesgue dominated convergence theorem

$$\int_{[t_0, t]} \int_B \|T_n \circ v(\tau, x) - v(\tau, x)\|_* \mu_\tau(dx) d\tau \xrightarrow{n \rightarrow +\infty} 0.$$

Remember that $\|\cdot\|_* \leq \|\cdot\|$. So, it remains to seek a sequence of continuous bounded functions $h_n : [t_0, t] \times B \rightarrow B$ such that

$$\lim_{n \rightarrow +\infty} \int_{[t_0, t]} \int_B \|T_n \circ v(\tau, x) - h_n(\tau, x)\|_B \mu_\tau(dx) d\tau = 0.$$

To this end, set for $a, b \in [t_0, t]$ with and for every Borel set K in B

$$\tilde{\nu}([a, b] \times K) := \int_{[a, b]} \mu_\tau(K) d\tau.$$

We have

$$\int_{[t_0, t] \times B} \|T_n \circ v(\tau, x)\|_B \tilde{\nu}(d\tau, dx) \leq \int_{[t_0, t]} \int_B \|v(\tau, x)\|_B \mu_\tau(dx) d\tau < +\infty.$$

This implies $T_n \circ v \in L^1([t_0, t] \times B, \tilde{\nu}; B_n)$. Remark that B_n can be identified with \mathbb{R}^n . Then, applying Lemma 4.4.14, one obtains a sequence of continuous bounded functions $(h_n)_{n \in \mathbb{N}}$ from $[t_0, t] \times B$ to $B_n \subset B$ such that

$$\|T_n \circ v - h_n\|_{L^1([t_0, t] \times B, \tilde{\nu}; B_n)} \xrightarrow{n \rightarrow +\infty} 0.$$

Hence, we have

$$\int_{[t_0, t]} \int_B \|T_n \circ v(\tau, x) - h_n(\tau, x)\|_B \mu_\tau(dx) d\tau = \int_{[t_0, t] \times B} \|T_n \circ v(\tau, x) - h_n(\tau, x)\|_B \tilde{\nu}(d\tau, dx) \xrightarrow{n \rightarrow \infty} 0.$$

We conclude

$$\int_{\mathfrak{X}} \left\| \gamma(t) - x - \int_{t_0}^t v(\tau, \gamma(\tau)) d\tau \right\|_* \eta(dx, d\gamma) = 0.$$

This implies that there exists an η null set \mathcal{N} such that

$$\gamma(t) = x + \int_{t_0}^t v(\tau, \gamma(\tau)) d\tau = 0, \quad \forall (x, \gamma) \in \mathfrak{X} \setminus \mathcal{N}.$$

Then using a density argument and the continuity of the curves γ in B as in the finite dimensional case, we obtain the concentration property (i) in Proposition 4.2.1. \square

4.3 Proof of main results

We give the proof of our main results. In particular, Theorem 4.1.7 is proved in the subsection below while the applications to ODEs and PDEs are analyzed in Subsection 4.3.2.

4.3.1 The globalization argument

In order to prove Theorem 4.1.7, we rely in the global superposition principle Proposition 4.2.1 proved in the latter Section 4.2 and the measurable projection theorem recalled below (see [Cra02, Theorem 2.12]).

Theorem 4.3.1. *If (X, \mathcal{T}) is a measurable space and (Y, \mathcal{B}) is a Polish space with \mathcal{B} its Borel σ -algebra. Then for every set S in the product σ -algebra $\mathcal{T} \otimes \mathcal{B}$ the projected set $p(S)$, $p : X \times Y \rightarrow X, p(x, y) = x$, is a universally measurable set of X relatively to \mathcal{T} .*

We will also need a measure theoretical argument provided for instance in [ALR20, Lemma 4.3 and Lemma C.2] and in the PhD thesis of C. Rouffort [Rou18b, Lemma 3.A.1].

Lemma 4.3.2 (see [Rou18b, Lemma 3.A.1]). *Let (M, d_M) be a metric space, let $a, b \in \mathbb{R}$, $a < b$. Then, for any Borel measurable function $f : [a, b] \times M \rightarrow \mathbb{R}$ such that for all*

$u \in M, f(\cdot, u) \in L^1([a, b])$, the mapping given by

$$M \longrightarrow \mathbb{R} \quad (4.3.1)$$

$$x \longmapsto \int_{[a,b]} f(s, x) ds \quad (4.3.2)$$

is Borel measurable.

Define the set

$$\mathcal{F} = \{(x, \gamma) \in \mathfrak{X} : \gamma \text{ is a mild global solution of (4.1.4) s.t. } \gamma(t_0) = x\}. \quad (4.3.3)$$

Lemma 4.3.3. *The set \mathcal{F} is a Borel subset of (\mathfrak{X}, d) satisfying $\eta(\mathcal{F}) = 1$ with η is the Borel probability measure on \mathfrak{X} provided by the global superposition principle in Proposition 4.2.1.*

Proof. According to Proposition 4.2.1, the path measure η constructed there concentrates on the set \mathcal{F} of global solutions with specified initial conditions given in (4.3.3). More precisely, the concentration property (i) says that $\mathfrak{X} \setminus \mathcal{F}$ is a η -null set. So, to prove the lemma it is enough to show that \mathcal{F} is a Borel subset of \mathfrak{X} . Such statement follows from Lemma 4.3.2. We have

$$\mathcal{F} = \{(x, \gamma) \in \mathfrak{X} : v(\cdot, \gamma(\cdot)) \in L^1_{loc}(I) \text{ and } \gamma(t) = x + \int_{t_0}^t v(s, \gamma(s)) ds, \forall t \in I\} \quad (4.3.4)$$

$$= \bigcap_{t_j \in \mathbb{Q} \cap I} \{(x, \gamma) \in \mathfrak{X} : v(\cdot, \gamma(\cdot)) \in L^1_{loc}(I) \text{ and } \gamma(t_j) = x + \int_{t_0}^{t_j} v(s, \gamma(s)) ds\} \quad (4.3.5)$$

On the other hand, for each $j \in \mathbb{N}$

$$\mathcal{F}_j := \{(x, \gamma) : v(\cdot, \gamma(\cdot)) \in L^1_{loc}(I) \text{ and } \gamma(t_j) = x + \int_{t_0}^{t_j} v(s, \gamma(s)) ds\} \quad (4.3.6)$$

$$= \bigcap_{k \in \mathbb{N}} \{(x, \gamma) : v(\cdot, \gamma(\cdot)) \in L^1_{loc}(I) \text{ and } \langle \gamma(t_j) - x - \int_{t_0}^{t_j} v(s, \gamma(s)) ds, e_k \rangle = 0\} \quad (4.3.7)$$

$$=: \bigcap_{k \in \mathbb{N}} E_{j,k}, \quad (4.3.8)$$

where $\{e_k\}_{k \in \mathbb{N}}$ is the elements of the biorthogonal system in B (see Definition 4.1.1). Hence, it is enough to show $E_{j,k}$ are Borel sets. Let $\mathfrak{L}(I, B)$ denotes the set of curves

$\gamma \in \mathcal{C}(I; B)$ such that $v(\cdot, \gamma(\cdot)) \in L^1_{loc}(I, B)$. Taking the functions

$$\begin{aligned} \Lambda_T : \mathcal{C}(I; B) &\longrightarrow \overline{\mathbb{R}} \\ \gamma &\longmapsto \int_{[-T, T] \cap I} \|v(s, \gamma(s))\| ds \\ &= \lim_{N \rightarrow +\infty} \int_{[-T, T] \cap I} \max(N, \|v(s, \gamma(s))\|) ds, \end{aligned}$$

we prove using Lemma 4.3.2 with $M = \mathcal{C}(I; B)$ and monotone convergence theorem that Λ_T are Borel measurable for all $T \in \mathbb{N}$. Hence, we conclude that $\mathfrak{L}(I, B) = \bigcap_{T \in \mathbb{N}} \Lambda_T^{-1}(\mathbb{R})$ is a Borel subset of $\mathcal{C}(I; B)$. In particular, Borel sets of $(\mathfrak{L}(I, B), d_0)$ equipped with the induced metric d_0 in (4.2.2) coincide with Borel sets of $\mathcal{C}(I; B)$ which are in $\mathfrak{L}(I, B)$. Now, using again Lemma 4.3.2 with $M = (\mathfrak{L}(I, B), d_0)$, we show that the map

$$\begin{aligned} \psi_1 : B \times \mathfrak{L}(I, B) &\longrightarrow \mathbb{R} \\ (x, \gamma) &\longmapsto \int_{t_0}^{t_j} \langle v(s, \gamma(s)), e_k \rangle ds \end{aligned}$$

is Borel measurable. While, the map

$$\begin{aligned} \psi_2 : B \times \mathfrak{L}(I, B) &\longrightarrow \mathbb{R} \\ (x, \gamma) &\longmapsto \langle \gamma(t_j) - x, e_k \rangle \end{aligned}$$

is continuous. Hence, we get

$$E_{j,k} = (\psi_1 - \psi_2)^{-1}(\{0\})$$

is Borel. □

Define the set

$$\mathcal{G} = \{x \in B : \exists \gamma \text{ a mild global solution of (4.1.4) s.t. } \gamma(t_0) = x\}. \quad (4.3.9)$$

Lemma 4.3.4. *The set \mathcal{G} is a universally measurable subset of $(B, \|\cdot\|)$.*

Proof. Take the projection map $p : B \times \mathcal{C}(I; B) \rightarrow B$, $p(x, \gamma) = x$. Recall that according to Lemma 4.2.5, $\mathcal{C}(I; B)$ endowed with the metric d_0 of compact-open topology is a Polish space. Then using Lemma 4.3.3 and the measurable projection Theorem 4.3.1, we obtain

that

$$p(\mathcal{F}) = \mathcal{G}$$

is a universally measurable set of $(B, \|\cdot\|)$. \square

Proof of Theorem 4.1.7: The global superposition principle, Proposition 4.2.1, yields the existence of a probability measure $\eta \in \mathcal{P}(\mathfrak{X})$ such that $\mu_t = (\Xi_t)_\# \eta$, for all $t \in I$ where Ξ_t is the evaluation map

$$(x, \gamma) \in \mathfrak{X} \mapsto \gamma(t) \in B.$$

Thanks to Lemma 4.3.4, we know that the set \mathcal{G} is μ_{t_0} -measurable since it is universally measurable. Hence, Proposition 4.2.1 and Lemma 4.3.3 imply

$$\mu_{t_0}(\mathcal{G}) = (\Xi_{t_0})_\# \eta(\mathcal{G}) = \eta(\Xi_{t_0}^{-1}(\mathcal{G})) \geq \eta(\mathcal{F}) = 1.$$

The last inequality is a consequence of the inclusion $\mathcal{F} \subset \Xi_{t_0}^{-1}(\mathcal{G})$. \square

4.3.2 Analysis of ODEs and PDEs

We sketch here the proofs of Corollaries 4.1.9 and 4.1.11 which provide respectively two applications for ODEs and PDEs.

Analysis of ODEs: Recall the initial value problem (4.1.14) and assume that the assumptions of Corollary 4.1.9 are satisfied.

Proof of Corollary 4.1.9: Without loss of generality, we may assume that the symplectic structure J is canonical. Precisely, the skew-symmetric matrix J satisfying $J^2 = -I_{2d}$ is given by $J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$. For any $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$, we have by integration by parts

$$\begin{aligned} & \int_{\mathbb{R}^{2d}} \frac{\partial h}{\partial q}(u) \frac{\partial \varphi}{\partial p}(u) F(h(u)) L(du) = \\ & \int_{\mathbb{R}^{2d}} \varphi(u) \left(-\frac{\partial^2 h}{\partial p \partial q}(u) F(h(u)) - \frac{\partial h}{\partial q}(u) \frac{\partial h}{\partial p}(u) F'(h(u)) \right) L(du), \end{aligned}$$

and similarly

$$\begin{aligned}
 & - \int_{\mathbb{R}^{2d}} \frac{\partial h}{\partial p}(u) \frac{\partial \varphi}{\partial q}(u) F(h(u)) L(du) = \\
 & \int_{\mathbb{R}^{2d}} \varphi(u) \left(\frac{\partial^2 h}{\partial q \partial p}(u) F(h(u)) + \frac{\partial h}{\partial p}(u) \frac{\partial h}{\partial q}(u) F'(h(u)) \right) L(du).
 \end{aligned}$$

Combining the two identities, we obtain

$$\int_{\mathbb{R}^{2d}} \left(\frac{\partial h}{\partial q}(u) \frac{\partial \varphi}{\partial p}(u) - \frac{\partial h}{\partial p}(u) \frac{\partial \varphi}{\partial q}(u) \right) F(h(u)) L(du) = 0. \quad (4.3.10)$$

Hence, using the symplectic structure on \mathbb{R}^{2d} and the Hamiltonian character of the initial value problem, we prove

$$\int_{\mathbb{R}^{2d}} \langle \nabla \varphi, J \nabla h(u) \rangle F(h(u)) L(du) = 0. \quad (4.3.11)$$

This shows that the measure $\frac{F(h(\cdot)) dL}{\int F(h(u)) L(du)}$ satisfies the statistical Liouville equation and so we are within the framework of Theorem [4.1.7](#). The latter grants us the almost sure existence of global solutions to the ODE [\(4.1.14\)](#). \square

Analysis of PDEs: In this paragraph, we provide the proof of Proposition [4.1.10](#) and Corollary [4.1.11](#). Recall that the initial value problem [\(4.1.23\)](#) can be written equivalently in the interaction representation as

$$\dot{\gamma}(t) = v(t, \gamma(t)),$$

with a Borel vector field $v : \mathbb{R} \times H^{-s} \rightarrow H^{-s}$ given by

$$v(t, u) = -ie^{itA} \nabla h_{NL}(e^{-itA} u), \quad x \in H^{-s}. \quad (4.3.12)$$

First, we notice the following invariance:

Proposition 4.3.5. *Consider the Gibbs measure*

$$d\mu_0 = \frac{e^{-h_{NL}} d\nu_0}{\int_{H^{-s}} e^{-h_{NL}} d\nu_0}. \quad (4.3.13)$$

Then for any $F \in \mathcal{C}_{b,cyl}^\infty(H^{-s})$,

$$\int_{H^{-s}} \langle \nabla F(u), -iAu - i\nabla h_{NL}(u) \rangle \mu_0(du) = 0. \quad (4.3.14)$$

Proof. For $k \in \mathbb{N}$, let e_k be as in (4.1.15) above. Note that two copies of $\{e_k, ie_k, k \in \mathbb{N}\}$ forms a fundamental strongly total biorthogonal system on H^{-s} . Using [AS22, Theorem 4.11 and (4.38)], we have for all $F, G \in \mathcal{C}_{c,cyl}^\infty(H^{-s})$

$$\int_{H^{-s}} \{F, G\}(u) \mu_0(du) = \int_{H^{-s}} \langle \nabla F(u), -iAu - i\nabla h_{NL}(u) \rangle G(u) \mu_0(du), \quad (4.3.15)$$

where $\{\cdot, \cdot\}$ refers to the Poisson bracket (see [AS22]). Take a sequence of functions $(G_n)_{n \in \mathbb{N}}$ in $\mathcal{C}_{c,cyl}^\infty(H^{-s})$ with given fixed basis³. Suppose that $G_n \rightarrow 1$ pointwisely with $\partial_j G_n$ are uniformly bounded with respect to n and $\partial_j G_n \rightarrow 0$ pointwise. Then replacing G by the sequence G_n and letting $n \rightarrow \infty$ in (4.3.15), yields the identity (4.3.14). \square

Proof of Proposition 4.1.10: We let $\{f_k, k \in \mathbb{N}\} := \{e_k, ie_k, k \in \mathbb{N}\}$. Let $F \in \mathcal{C}_{b,cyl}^\infty(H^{-s})$ such that

$$F(u) = \varphi(\langle u, f_1 \rangle, \dots, \langle u, f_m \rangle),$$

for some $\varphi \in \mathcal{C}_b^\infty(\mathbb{R}^m)$. Then

$$\frac{d}{dt} \int_{H^{-s}} F(u) \mu_t(du) = \sum_{k=1}^m \int_{H^{-s}} \partial_k \varphi(\langle e^{itA}u, f_1 \rangle, \dots, \langle e^{itA}u, f_m \rangle) \langle f_k, iAe^{itA}u \rangle \mu_0(du). \quad (4.3.16)$$

On the other hand, we check

$$\begin{aligned} \int_{H^{-s}} \langle v(t, u), \nabla F(u) \rangle \mu_t(du) &= \\ &= \sum_{k=1}^m \int_{H^{-s}} \partial_k \varphi(\langle e^{itA}u, f_1 \rangle, \dots, \langle e^{itA}u, f_m \rangle) \langle f_k, -ie^{itA} \nabla h_{NL}(u) \rangle \mu_0(du). \end{aligned}$$

Hence, Proposition 4.3.5 yields

$$\begin{aligned} \int_{H^{-s}} \langle v(t, u), \nabla F(u) \rangle \mu_t(du) - \frac{d}{dt} \int_{H^{-s}} F(u) \mu_t(du) &= \int_{H^{-s}} \langle \nabla \tilde{F}(u), -iAu - i\nabla h_{NL}(u) \rangle \mu_0(du) \\ &= 0, \end{aligned}$$

3. i.e. $G(u) = \varphi(\langle u, e_1 \rangle, \dots, \langle u, e_m \rangle; \langle u, ie_1 \rangle, \dots, \langle u, ie_m \rangle)$, for m fixed

for $\tilde{F} \in \mathcal{C}_{b,cyl}^\infty(H^{-s})$ given by

$$\tilde{F}(u) = \varphi(\langle e^{it\lambda_1}u, f_1 \rangle, \dots, \langle e^{it\lambda_m}u, f_m \rangle),$$

and satisfying

$$\nabla \tilde{F}(u) = \sum_{k=1}^m \partial_k \varphi(\langle e^{it\lambda_1}u, f_1 \rangle, \dots, \langle e^{it\lambda_m}u, f_m \rangle) e^{-it\lambda_k} f_k.$$

□

Proof of Corollary 4.1.11:

In this framework: $B = H^{-s}$ is a separable dual Banach space with predual $E = H^s$. The vector field $v : \mathbb{R} \times H^{-s} \rightarrow H^{-s}$ given by (4.3.12) is Borel measurable. The Gaussian measure ν_0 and the Gibbs measures μ_t are well-defined Borel probabilities on B . Then, one checks thanks to Proposition 4.1.10 that $(\mu_t)_{t \in \mathbb{R}}$ is a narrowly continuous curve in $\mathcal{P}(B)$ satisfying the condition (4.1.10) and the statistical Liouville equation (4.1.9). Hence, applying Theorem 4.1.7, we obtain the μ_0 -almost sure existence of global mild solutions for the initial value problem (4.1.23). Taking now into account the expression (4.3.13) of the measure μ_0 , one deduces the ν_0 -almost sure existence of global solutions as stated in Corollary 4.1.11. □

4.4 Appendices

4.4.1 Results in finite dimensions

In this appendix we recall some useful results on finite dimensions. Here, we consider $B = \mathbb{R}^d$ and I is a closed unbounded interval either equal to \mathbb{R} or half-closed. Take t_0 any initial time if $I = \mathbb{R}$ or t_0 is its end-point if I is half-closed.

The following Cauchy-Lipschitz type theorem holds true.

Theorem 4.4.1 (Cauchy–Lipschitz theorem (AGS08, Lemma 8.1.4)). *Let $v : I \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a Borel vector field satisfying the local Lipschitz Assumption 4.2.9. Then, for every $x \in \mathbb{R}^d$, the initial value problem (4.1.4) admits a unique maximal solution in a relatively open interval $I(x) \subseteq I$ with $t_0 \in I(x)$.*

We recall below the characteristics representation formula for solution of statistical Liouville equations in finite dimensions with locally Lipschitz vector fields (see [AGS08, Proposition 8.1.8]).

Lemma 4.4.2 (Characteristics representation formula). *Let $v : I \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a Borel vector field satisfying the local Lipschitz Assumption [4.2.9]. Let $(\mu_t)_{t \in I}$ be a narrowly continuous curve in $\mathcal{P}(\mathbb{R}^d)$ satisfying [4.1.12] and the statistical Liouville equation [4.1.9]. Then for all $T > 0$ and for μ_{t_0} -a.e $x \in \mathbb{R}^d$, the initial value problem [4.1.4] admits a solution $\Phi_t(x) \equiv \gamma_x(t)$ over $I \cap [-T, T]$ and*

$$\mu_t = (\Phi_t)_\# \mu_{t_0}, \quad \forall t \in I \cap [-T, T]. \quad (4.4.1)$$

We recall the following regularization argument for solutions of Liouville equations in finite dimensions (see [AGS08, Lemma 8.1.9]).

Lemma 4.4.3 (Approximation by regular curves). *Let $v : I \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a Borel vector field and $(\mu_t)_{t \in I}$ be a narrowly continuous curve in $\mathcal{P}(\mathbb{R}^d)$ satisfying [4.1.12] and the statistical Liouville equation [4.1.9]. Let $\rho \in \mathcal{C}^\infty(\mathbb{R}^d)$ strictly positive such that $\int_{\mathbb{R}^d} \rho(x) dx = 1$. Set $\rho_\varepsilon(x) = \frac{1}{\varepsilon^d} \rho(\frac{x}{\varepsilon})$ and*

$$\mu_t^\varepsilon := \mu_t * \rho_\varepsilon, \quad v_t^\varepsilon := \frac{(v_t \mu_t) * \rho_\varepsilon}{\mu_t^\varepsilon}. \quad (4.4.2)$$

Then $(\mu_t^\varepsilon)_{t \in I}$ is a narrowly continuous solution of the Liouville equation [4.1.9] with respect to the vector field v^ε . Moreover, v_t^ε satisfies the local Lipschitz Assumption [4.2.9] and the uniform integrability bound

$$\int_{\mathbb{R}^d} \|v_t^\varepsilon(x)\|_{\mathbb{R}^d} \mu_t^\varepsilon(dx) \leq \int_{\mathbb{R}^d} \|v_t(x)\|_{\mathbb{R}^d} \mu_t(dx), \quad \forall t \in I. \quad (4.4.3)$$

4.4.2 Radon spaces, Tightness

Radon spaces: Let X be a Hausdorff topological space. A Borel measure μ is a Radon measure if it is locally finite and inner regular. A topological space is called a Radon space if every finite Borel measure is a Radon measure (see e.g. [Sch73, chapter 2]). In particular, separable complete metric spaces and Suslin spaces [Sch73, Theorem 10] are Radon spaces.

Tightness: Let X be a separable metric space and $\mathcal{P}(X)$ the set of Borel probability

measures on X . A set $\mathcal{K} \subset \mathcal{P}(X)$ is tight if

$$\forall \varepsilon > 0, \exists K_\varepsilon \text{ compact in } X \text{ such that } \mu(X \setminus K_\varepsilon) \leq \varepsilon, \forall \mu \in \mathcal{K}.$$

Prokhorov's theorem [AGS08, Theorem 5.1.3] says that any tight set $\mathcal{K} \subset \mathcal{P}(X)$ is relatively (sequentially) compact in $\mathcal{P}(X)$ in the narrow topology. A useful characterization is given below.

Lemma 4.4.4 ([AGS08, Remark 5.1.5]). *A set $\mathcal{K} \subset \mathcal{P}(X)$ is tight if and only if there exists a function $\varphi : X \rightarrow [0, +\infty]$ whose sublevel sets $\{x \in X; \varphi(x) \leq c\}$ are relatively compact in X for all $c \geq 0$ and which satisfies*

$$\sup_{\mu \in \mathcal{K}} \int_X \varphi(x) \mu(dx) < +\infty.$$

Lemma 4.4.5 ([AGS08, page 108]). *Let (X, d) be a (metric, separable) Radon space. Then every narrowly converging sequence $(\mu_n)_n$ is tight.*

Lemma 4.4.6 ([AGS08, Lemma 5.2.2]). *Let X, X_1 and X_2 be separable spaces and let $r^i : X \rightarrow X_i$ be continuous maps such that the product map $r := r^1 \times r^2 : X \rightarrow X_1 \times X_2$ is proper. Let $\mathcal{K} \subset \mathcal{P}(X)$ be such that $\mathcal{K}_i := (r^i)_\# \mathcal{K}$ is tight in $\mathcal{P}(X_i)$ for $i = 1, 2$. Then, \mathcal{K} is tight in $\mathcal{P}(X)$.*

4.4.3 Equi-integrability and Dunford-Pettis theorem

Let (X, ν) be a measurable space. Recall that the dual of $L^1 \equiv L^1(X, \nu)$ is $L^\infty \equiv L^\infty(X, \nu)$. More precisely, we have $(L^1)^* = L^\infty$. The weak topology $\sigma(L^1, L^\infty)$ on L^1 is the coarsest topology associated to the collection of linear functions $(\varphi_f)_{f \in L^\infty}$ where

$$\begin{aligned} \varphi_f : L^1 &\longrightarrow \mathbb{R} \\ g &\longmapsto (\varphi_f)g = \langle f, g \rangle_{L^1, L^\infty} = \int_X f(x)g(x) \nu(dx). \end{aligned}$$

The Dunford-Pettis theorem is a standard criterion that allows us to deduce the relative compactness of sets in L^1 with respect to the weak topology $\sigma(L^1, L^\infty)$.

Let (X, Σ) be a measurable space and μ a finite measure on (X, Σ) . We say that a family $\mathcal{F} \subset L^1(X, \mu)$ is equi-integrable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\forall K \in \Sigma, \mu(K) < \delta \Rightarrow \sup_{f \in \mathcal{F}} \int_K |f| d\mu < \varepsilon.$$

A characterization of equi-integrability is given below.

Lemma 4.4.7 ([AFP00, Proposition 1.27]). *A bounded set \mathcal{K} in $L^1(X, \mu)$ is equi-integrable if and only if*

$$\mathcal{K} \subset \{f \in L^1(X, \mu) : \int_X \theta(|f|) d\mu \leq 1\},$$

for some nondecreasing convex continuous function $\theta : \mathbb{R}_+ \mapsto [0, \infty]$ satisfying $\theta(t)/t \rightarrow \infty$ when $t \rightarrow \infty$ (superlinear) or equivalently if and only if

$$\lim_{L \rightarrow +\infty} \sup_{f \in \mathcal{K}} \int_{\{|f| > L\}} |f| d\mu = 0. \quad (4.4.4)$$

Lemma 4.4.8 (Dunford-Pettis [AFP00, Theorem 1.38]). *A bounded set \mathcal{K} in $L^1(X, \mu)$ is relatively sequentially compact for the weak topology $\sigma(L^1, L^\infty)$ if and only if \mathcal{K} is equi-integrable.*

4.4.4 Compactness argument

We discuss in this paragraph the main compactness argument used throughout the text. Let $v : I \times B \rightarrow B$ be Borel vector field and $(\mu_t)_{t \in I}$ a weakly narrowly continuous curve in $\mathcal{P}(B)$.

Lemma 4.4.9. *Assume that v and $(\mu_t)_{t \in I}$ satisfy the integrability condition (4.1.10) for some non-decreasing positive function $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$. Then there exists a non-decreasing super-linear convex function $\theta : \mathbb{R}_+ \rightarrow [0, +\infty]$ such that*

$$\int_I \int_B \theta(\|v(t, u)\|) \mu_t(du) \frac{dt}{\omega(|t|)} \leq 1. \quad (4.4.5)$$

Proof. For every $\alpha, \beta \in I$ and for every Borel subset K of B , define the measure

$$\nu([\alpha, \beta] \times K) := \int_{[\alpha, \beta]} \mu_\tau(K) \frac{d\tau}{\omega(|\tau|)}. \quad (4.4.6)$$

We have

$$\int_{I \times B} \|v(t, x)\| d\nu(t, u) = \int_I \int_{\mathbb{R}^d} \|v(t, u)\|_{\mathbb{R}^d} \mu_t(du) \frac{dt}{\omega(|t|)} < +\infty,$$

which means $\|v(\cdot)\|_{\mathbb{R}^d} \in L^1(I \times B, \nu)$. Since the singleton $\{\|v(\cdot)\|\}$ is a compact set, then by the Dunford-Pettis Theorem [4.4.8], it is equi-integrable. And thus, by using Lemma

4.4.7, there exists a non-decreasing super-linear convex function $\theta : \mathbb{R}_+ \rightarrow [0, +\infty]$ such that

$$\int_I \int_B \theta(\|v(t, u)\|) \mu_t(du) \frac{dt}{\omega(|t|)} \leq 1.$$

□

Remark 4.4.10. The previous result applies for finite dimensions $B = \mathbb{R}^d$ with any norm on \mathbb{R}^d .

We consider the space of continuous functions $\mathcal{C}(I; B)$ endowed with the compact-open topology induced by the metric $d_{0,*}$ defined in **(4.2.6)**.

Lemma 4.4.11 (Compactness). *Let $\theta : \mathbb{R}_+ \rightarrow [0, +\infty]$ be a non-decreasing super-linear convex function. Define the function $g : \mathcal{C}(I; B) \rightarrow [0, +\infty]$ as*

$$g(\gamma) := \begin{cases} \int_I \theta(\|\dot{\gamma}\|_*) \frac{dt}{\omega(|t|)} & \text{if } \gamma(t_0) = 0 \text{ and } \gamma \in AC_{loc}^1(I; B), \\ +\infty & \text{if } \gamma(t_0) \neq 0 \text{ or } \gamma \notin AC_{loc}^1(I; B). \end{cases}$$

Then for all $c \geq 0$, the sublevel set

$$\mathcal{A}_c = \{\gamma \in \mathcal{C}(I; B); g(\gamma) \leq c\}$$

is relatively compact in the space $(\mathcal{C}(I; B_w), d_{0,*})$.

Proof. By the general Arzela-Ascoli theorem (see **[Mun75]**, Theorem 6.1), \mathcal{A}_c is relatively compact in $(\mathcal{C}(I; B_w), d_{0,*})$ provided that we prove the following claims:

- For all $t \in I$, the set $\mathcal{A}_c(t) = \{\gamma(t); \gamma \in \mathcal{A}_c\}$ is relatively compact in B_w .
- The set \mathcal{A}_c is equicontinuous.

• $\mathcal{A}_c(t)$ relatively compact:

In fact, remark that $\mathcal{A}_c(t)$ is bounded. Indeed, by Jensen's inequality

$$\theta(\|\gamma(t)\|_*) \leq \theta\left(\int_{t_0}^t \|\dot{\gamma}(s)\|_* ds\right) \leq \int_{[t_0, t]} \theta(\|\dot{\gamma}(s)\|_*) ds \leq c\omega(\max(|t_0|, |t|)),$$

for all $\gamma \in \mathcal{A}_c$. Since θ is superlinear, we get $\mathcal{A}_c(t)$ is bounded in B_w . Now, since $t \in I$ is fixed and the norm $\|\cdot\|_*$ induces the weak-* topology in B on bounded sets, it follows that $\mathcal{A}_c(t)$ is relatively compact in B_w .

• \mathcal{A}_c is equi-continuous:

The case $c = 0$ is trivial, so we may assume $c > 0$. Let $t_1 \in I$, we have to prove:

$\forall \varepsilon > 0, \exists \delta > 0, \forall t \in I, \forall \gamma \in \mathcal{A}_c$

$$|t - t_1| \leq \delta \Rightarrow \|\gamma(t) - \gamma(t_1)\|_* \leq \varepsilon.$$

Assume $\delta \leq 1$. We have for $\gamma \in \mathcal{A}_c$,

$$\begin{aligned} \|\gamma(t) - \gamma(t_1)\|_* &= \left\| \int_{t_1}^t \dot{\gamma}(s) ds \right\|_* \leq \int_{[t_1, t]} \|\dot{\gamma}(s)\|_* ds \\ &\leq \int_{\{s \in [t_1, t]; \|\dot{\gamma}(s)\|_* \leq L\}} L ds + \int_{\{s \in [t_1, t]; \|\dot{\gamma}(s)\|_* > L\}} \|\dot{\gamma}(s)\|_* ds \\ &\leq L|t - t_1| + \int_{\{s \in [t_1, t]; \|\dot{\gamma}(s)\|_* > L\}} \|\dot{\gamma}(s)\|_* ds \\ &\leq L|t - t_1| + \omega(\max |t|, |t_1|) \int_{\{s \in [t_1, t]; \|\dot{\gamma}(s)\|_* > L\}} \|\dot{\gamma}(s)\|_* \frac{ds}{\omega(|s|)}. \end{aligned}$$

Remark that the set $\mathcal{K} = \{s \in [t_1, t] \mapsto \|\dot{\gamma}(s)\|_*, \gamma \in \mathcal{A}_c\} \subset L^1([t_1, t]; \frac{ds}{\omega(|s|)})$ since the function θ is superlinear and by assumption

$$\int_{[t_1, t]} \theta(\|\dot{\gamma}(s)\|_*) \frac{ds}{\omega(|s|)} \leq g(\gamma) \leq c. \quad (4.4.7)$$

On the other hand, this means that there exists $\tilde{\theta} = \theta/c$ a non-decreasing convex super-linear function such that

$$\int_{[t_1, t]} \tilde{\theta}(\|\dot{\gamma}(s)\|_*) \frac{ds}{\omega(|s|)} \leq 1. \quad (4.4.8)$$

Hence, by Lemma [4.4.7](#), we have that \mathcal{K} is equi-integrable and

$$\lim_{L \rightarrow \infty} \sup_{\gamma \in \mathcal{A}_c} \int_{\{s \in [t_1, t]; \|\dot{\gamma}(s)\|_* > L\}} \|\dot{\gamma}(s)\|_* \frac{ds}{\omega(|s|)} = 0.$$

This means for given $\varepsilon > 0$, one can choose $L > 0$ such that for all $\gamma \in \mathcal{A}_c$

$$\|\gamma(t) - \gamma(t_1)\|_* \leq L\delta + \varepsilon/2.$$

Hence, choosing $0 < \delta \leq \varepsilon/2L$, we show that \mathcal{A}_c is equi-continuous at t_1 . \square

Remark 4.4.12. The above lemma applies, mutatis mutandis, to finite dimensions with any norm on \mathbb{R}^d .

4.4.5 Disintegration theorem

Let E and F be Radon separable metric spaces. We say that a measure-valued map $x \in E \rightarrow \mu_x \in \mathcal{P}(E)$ is Borel if $x \in F \rightarrow \mu_x(B)$ is a Borel map for any Borel set B of E . We recall below the disintegration theorem (see [AGS08, Theorem 5.3.1]).

Theorem 4.4.13. *Let E and F be Radon separable metric spaces and $\mu \in \mathcal{P}(E)$. Let $\pi : E \rightarrow F$ be a Borel map and $\nu = \pi_{\#}\mu \in \mathcal{P}(F)$. Then, there exists a ν -a.e. uniquely determined Borel family of probability measures $\{\mu_y\}_{y \in F} \subset \mathcal{P}(E)$ such that $\mu_y(E \setminus \pi^{-1}(y)) = 0$ for ν -a.e. $y \in F$ and*

$$\int_E f(x) \mu(dx) = \int_F \left(\int_{\pi^{-1}(y)} f(x) \mu_y(dx) \right) \nu(dy) \quad (4.4.9)$$

for every Borel map $f : E \rightarrow [0, +\infty]$.

4.4.6 Density argument

Let (X, μ) be a measurable space. Define the space

$$L^1(X, \mu; \mathbb{R}^n) := \left\{ f : X \rightarrow \mathbb{R}^n; \int_X \|f(x)\|_{\mathbb{R}^n} \mu(dx) < +\infty \right\}$$

Lemma 4.4.14. *Let X be a metric space, μ a finite Radon measure on X . Then the space of bounded Lipschitz functions $Lip_b(X; \mathbb{R}^n)$ is dense in $L^1(X, \mu; \mathbb{R}^n)$. In particular, $\overline{\mathcal{C}_b(X; \mathbb{R}^n)} = L^1(X, \mu; \mathbb{R}^n)$.*

Proof. By the result in [AL18, Corollary C.3, Appendix C], we have $Lip_b(X; \mathbb{R})$ is dense in $L^1(X, \mu; \mathbb{R})$. Let $f \in L^1(X, \mu; \mathbb{R}^n)$, then $f = (f^i)_{i=1}^n$. Now for every $1 \leq i \leq n$, $f^i \in L^1(X, \mu)$. This implies for all $1 \leq i \leq n$ there exists a sequence of Lipschitz bounded functions $(f_k^i)_{k \in \mathbb{N}}$ such that

$$\|f^i - f_k^i\|_{L^1(X, \mu)} \xrightarrow{k \rightarrow +\infty} 0.$$

Let $f_k = (f_k^i)_{i=1}^n$ a bounded Lipschitz function i.e. $f_k \in Lip_b(X; \mathbb{R}^n)$, we have

$$\begin{aligned} \|f - f_k\|_{L^1(X, \mu; \mathbb{R}^n)} &= \int_X \|f(x) - f_k(x)\|_{\mathbb{R}^n} \mu(dx) \\ &\lesssim \int_X \sum_{i=1}^n |f^i(x) - f_k^i(x)| \mu(dx) \\ &= \sum_{i=1}^n \|f^i - f_k^i\|_{L^1(X, \mu)} \xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

And thus $L^1(X, \mu; \mathbb{R}^n) = \overline{Lip_b(X; \mathbb{R}^n)}$. This implies that

$$L^1(X, \mu; \mathbb{R}^n) = \overline{Lip_b(X; \mathbb{R}^n)} \subseteq \overline{\mathcal{C}_b(X; \mathbb{R}^n)} \subseteq L^1(X, \mu; \mathbb{R}^n).$$

□

CONCLUSION AND PERSPECTIVES

This thesis is concerned with the study of systems of nonlinear differential equations in both finite and infinite dimensional spaces. Below, we present the conclusions drawn from Part I (composed of Chapters 1 and 2) and Part II (composed of Chapters 3 and 4), emphasizing the respective perspectives and the correlation between the two parts.

Conclusion and perspectives of Part I.

To sum up, Chapter 1 has addressed the difficulties posed by oscillations of source terms and strong nonlinearities in special nonlinear differential equations within finite dimensional spaces. Under integrability conditions, and using blow-up procedure and a WKB approximation (of super-critical type), our research has resulted in uniform existence (Theorem [1.2.5](#)), asymptotic expansions of solutions (Theorem [1.2.7](#)), with stability results and coherent structures that account for the emergence of new scales and phases. By using averaging procedures, we have also derived long time reduced equations (Theorem [1.2.10](#)) that, despite their strong nonlinearity, are well-posed, enabling us to analyze their qualitative properties. Then, the preceding theoretical study has allowed us to develop two applications. Indeed, using the obtained outcomes, we prove uniform existence results and asymptotic expansions of the solutions to

- first application: a class of oscillating Hamilton-Jacobi equation (Theorem [1.1.5](#)) by exploiting transparency condition (coming from the integrability conditions), by implementing the Hadamard's global inverse function theorem and by involving some specific WKB analysis;
- second application: the characteristics of the Vlasov equation in the presence of a fixed electric field $E \neq 0$ and strong magnetic field $|B| \gg 1$ (Theorem [2.1.8](#)). This application is addressed in Chapter 2 since it requires an extensive preparatory work. Furthermore, our study of the characteristics of Vlasov equation has resulted in the proof of: a spatial confinement in position (for the entire range of electric field) from one hand; a confinement in momentum (for radial electric field only) from the other hand. Both appear as a consequence of Theorem [2.1.9](#).

In our contributions in Chapter 2, the electromagnetic field (E, B) is viewed as a fixed

external field, and the focus is on the Vlasov equation. It would be interesting to complement this progress with a WKB analysis of the full relativistic Vlasov-Maxwell system [CI23]. The aim would be to extract and justify approximate solutions for dense, cold and strongly magnetized plasmas. There are indeed plenty of original effects that could be investigated in this way.

Conclusion and perspectives of Part II.

As a summary, Part II of our thesis addresses two related topics discussed in Chapters 3 and 4. The first topic in Chapter 3 concerns the quantum-to-classical dynamics transition of particle-field equations. The primary contributions of our research are: the global well-posedness of the particle-field equations (Theorem 3.1.1) and the validity of Bohr's correspondence principle (Theorem 3.1.3). To fulfil these results, we have used the approach of Wigner measures to derive the classical dynamics of the particle-field system as an effective equation of a specific microscopic dynamical system in the classical limit of small Planck constant, denoted by $\hbar \rightarrow 0$. The second topic in Chapter 4 concerns with the study of abstract initial value problems associated with a Borel vector field in infinite-dimensional space. We show that the existence of stationary measure-valued solutions to some Liouville equation implies the existence of low regularity global solutions to the related abstract initial value problems. Our work shows that under the low regularity of the vector field, one can provide mild global (but not necessary unique) solutions for almost all initial data (Theorem 4.1.7). The latter result has diverse applications, including nonlinear dispersive PDEs and fluid mechanics equations, such as Hartree, Klein-Gordon, NLS, Euler, and modified surface quasi-geostrophic equations.

In our works concerning quantum-classical transition in Chapter 3, we could try to extend the analysis to the Yukawa interaction without cutoffs (i.e. pointwise charge distribution $\chi = 1$). Another direction would be to focus on additional relevant physical models [FLMP21, AF17]. For instance, in condensed matter physics the Polaron model (which describes the motion of an electron in a crystal lattice) could be investigated in the same spirit through Wigner measures and transport equations. In such case, drawing a link with the well-known Landau-Pekar's theory of Polaron would be challenging [FLST11].

Common perspective of Parts I and II.

There are links between Parts I and II. We can consider the dynamics of many fermions interacting with a scalar quantized field and study the mean field semi-classical effective

limit, see [LP19]. Presumably, this limit could appear in the form of relativistic Vlasov-Maxwell (RVM) system. This could be completed with the insights of the recent papers [CI23, JM15] to better understand the notion of weak solutions, and then uniqueness and stability.

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Titre : Analyse multi-échelle des interactions entre les ondes et les particules

Mot clés : Equations de Hamilton-Jacobi, de Vlasov, de type Liouville, et sur les champs de particules ; Modèle de Nelson ; Problèmes aux valeurs initiales.

Résumé : Cette thèse contient deux parties.

La partie I étudie une classe spécifique d'équations différentielles non linéaires oscillantes dans des espaces de dimension finie. Sous des conditions d'intégrabilité, une procédure de « blow-up » et une approximation WKB (de type sur-critique) conduisent à l'existence en temps longs, à des développements asymptotiques avec une forme forte de stabilité, ainsi qu'à des modèles réduits. En exploitant des conditions de transparence et le théorème d'inversion globale de Hadamard, les résultats sont ensuite appliqués à l'étude d'une classe d'équations de Hamilton-Jacobi oscillantes. Ils sont aussi utilisés pour décrire les caractéristiques de l'équation de Vlasov en présence d'un champ électromagnétique fixe (E, B) tel que $E \neq 0$ et $|B| \gg 1$.

La partie II explore la transition de la dynamique quantique vers la dynamique classique des interactions particule-champ. Nous obtenons : le caractère globalement bien-posé en temps du système particule-champ et la validation du principe de correspondance de Bohr du modèle de Nelson. Ensuite, nous étudions la construction de solutions globales de faible régularité pour des problèmes de valeur initiale abstraits dans des espaces de dimension infinie. Nous utilisons des techniques de la théorie de la mesure, une représentation probabiliste et des arguments projectifs pour montrer l'existence de solutions globales pour presque toutes les données initiales. Ceci est appliqué pour construire des solutions globales pour des EDPs non linéaires comme les équations de Hartree, Klein-Gordon, NLS, Euler et mSQG.

Title: Multi-scale analysis of interactions between waves and particles

Keywords: Hamilton-Jacobi equation; Vlasov equation; Liouville equations; Initial value problems; Particle-field equation; Nelson model.

Abstract: This thesis consists of two parts.

Part I studies a special class of oscillating nonlinear differential equations in finite dimensional spaces. Under integrability conditions, a “blow-up” procedure and a WKB approximation (of super-critical type) lead to long-time existence, asymptotic expansions with a strong form of stability, and reduced models. These outcomes are then applied to study a class of oscillating Hamilton-Jacobi equations by exploiting transparency conditions and Hadamard's global inverse function theorem. Additionally, they are used to study the characteristics of the Vlasov equation in the presence of a fixed electromagnetic field (E, B) such that $E \neq 0$ and $|B| \gg 1$.

Part II explores the transition from quantum to classical dynamics of particle-field interactions. We obtain: the global well-posedness of a particle-field system and the validation of Bohr's correspondence principle of Nelson model. Then, part II focuses on constructing low regularity global solutions to abstract initial value problems in infinite dimensional spaces. We employ measure-theoretical techniques, a probabilistic representation, and projective arguments to show the existence of mild global solutions to initial value problems for almost all initial data. This is applied to construct global solutions for nonlinear PDEs like Hartree, Klein-Gordon, NLS, Euler and mSQG equations.