



**HAL**  
open science

# Effective methods for multilinear birational transformations and contributions to polynomial data analysis

Pablo González-Mazón

► **To cite this version:**

Pablo González-Mazón. Effective methods for multilinear birational transformations and contributions to polynomial data analysis. Commutative Algebra [math.AC]. Université Côte d'Azur, 2023. English. NNT : 2023COAZ4138 . tel-04597763

**HAL Id: tel-04597763**

**<https://theses.hal.science/tel-04597763v1>**

Submitted on 3 Jun 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

$$\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + \nabla \cdot T + f$$

$$e^{i\pi} + 1 = 0$$

# THÈSE DE DOCTORAT

**Méthodes effectives pour les transformations birationnelles  
multilinéaires et contributions à l'analyse polynomial de données**

**Pablo GONZÁLEZ-MAZÓN**  
Université Côte d'Azur, Inria AROMATH

Présentée en vue de l'obtention du grade de  
docteur en **Mathématiques** d'Université Côte  
d'Azur

Dirigée par:

**Laurent BUSÉ**, Directeur de Recherche,  
Centre Inria d'Université Côte d'Azur, France

Soutenue le : 19 Décembre 2023

Devant le jury, composé de :

**Laurent BUSÉ**, Directeur de Recherche,  
Centre Inria d'Université Côte d'Azur, France

**Alicia DICKENSTEIN**, Professeure Émérite,  
Universidad de Buenos Aires, Argentine

**Daniele FAENZI**, Professeur,  
Université de Bourgogne, France

**Andreas HÖRING**, Professeur,  
Université Côte d'Azur, France

**Sonia PÉREZ-DÍAZ**, Professeure,  
Universidad de Alcalá, Spain

**Hal SCHENCK**, Professeur,  
Auburn University, Alabama, USA



# Méthodes effectives pour les transformations birationnelles multilinéaires et contributions à l'analyse polynomial de données

---

## Effective methods for multilinear birational transformations and contributions to polynomial data analysis

Pablo GONZÁLEZ-MAZÓN

### Jury

#### Rapporteurs:

- Alicia DICKENSTEIN, Professeure Émérite, Universidad de Buenos Aires, Argentine
- Daniele FAENZI, Professeur, Université de Bourgogne, France
- Hal SCHENCK, Professeur, Auburn University, Alabama, USA

#### Examineurs:

- Andreas HÖRING, Professeur, Université Côte d'Azur, France
- Sonia PÉREZ-DÍAZ, Professeure, Universidad de Alcalá, Spain

#### Directeur:

- Laurent BUSÉ, Directeur de Recherche, Centre Inria d'Université Côte d'Azur, France



*To mom and dad,  
to my brother, to my aunt,  
and to my best friend Lucía*

*A mon père et à ma mère,  
à mon frère, à ma tante,  
et à ma meilleure amie Lucía*



# Abstract

This thesis explores two distinct subjects at the intersection of commutative algebra, algebraic geometry, multilinear algebra, and computer-aided geometric design:

- (i) The study and effective construction of multilinear birational maps
- (ii) The extraction of information from measures and data using polynomials

The primary and most extensive part of this work is devoted to multilinear birational maps. A multilinear birational map is a rational map  $\phi : (\mathbb{P}^1)^n \dashrightarrow \mathbb{P}^n$ , defined by multilinear polynomials, which admits an inverse rational map. Birational transformations between projective spaces have been a central theme in algebraic geometry, tracing back to the seminal works of Cremona, which has witnessed significant advancement in the last decades. Additionally, there has been a recent surge of interest in tensor-product birational maps, driven by the study of multiprojective spaces in commutative algebra and their practical application in computer-aided geometric design.

In the first part, we address algebraic and geometric aspects of multilinear birational maps. We primarily focus on trilinear birational maps  $\phi : (\mathbb{P}^1)^3 \dashrightarrow \mathbb{P}^3$ , that we classify according to the algebraic structure of their space, base loci, and the minimal graded free resolutions of the ideal generated by the defining polynomials. Furthermore, we develop the first methods for constructing and manipulating nonlinear birational maps in 3D with sufficient flexibility for geometric modeling and design. Interestingly, we discover a characterization of birationality based on tensor rank, which yields effective constructions and opens the door to the application of tools from tensors to birationality. We also extend our results to multilinear birational maps in arbitrary dimension, in the case that there is a multilinear inverse.

In the second part, our focus shifts to the application of polynomials in analyzing data and measures. We tackle two distinct problems. Firstly, we derive bounds for the size of  $(1 - \epsilon)$ -nets for superlevel sets of real polynomials. Our results allow us to extend the classical centerpoint theorem to polynomial inequalities of higher degree. Secondly, we address the classification of real cylinders through five-point configurations where four points are cocyclic, i.e. they lie on a circumference. This is an instance of the more general problems of real root classification of systems of real polynomials and the extraction of algebraic primitives from raw data.

**Key words:** birational map, multiprojective space, multilinear, syzygy, tensor, geometric modeling





# Résumé

Cette thèse explore deux sujets distincts à l'intersection de l'algèbre commutative, de la géométrie algébrique, de l'algèbre multilinéaire et de la modélisation géométrique:

- (i) L'étude et la construction effective des transformations birationnelles multilinéaires
- (ii) L'extraction d'informations à partir de données discrètes à l'aide de modèles polynomiaux

La partie principale de ce travail est consacrée aux transformations birationnelles multilinéaires. Une transformation birationnelle multilinéaire est une transformation rationnelle  $\phi : (\mathbb{P}^1)^n \dashrightarrow \mathbb{P}^n$ , définie par des polynômes multilinéaires, qui admet une transformation rationnelle inverse. Les transformations birationnelles entre espaces projectifs constituent un sujet d'étude important de la géométrie algébrique, initié par les travaux fondateurs de Cremona, qui a connu des avancées significatives au cours des dernières décennies. Plus récemment, les transformations birationnelles multiprojectives, c'est-à-dire définies par des polynômes multi-homogènes, ont récemment suscité un regain d'intérêt, motivé notamment par l'étude des structures multigraduées en algèbre commutative et leurs applications pratique en modélisation géométrique.

Dans la première partie, nous étudions les aspects algébriques et géométriques des transformations birationnelles multilinéaires. Nous nous concentrons principalement sur les transformations birationnelles trilinéaires  $\phi : (\mathbb{P}^1)^3 \dashrightarrow \mathbb{P}^3$  dont nous établissons une classification en fonction de la structure algébrique de leur espace, du lieu base, et des résolutions libres graduées minimales de l'idéal engendré par les polynômes de définition. En outre, nous développons les premières méthodes qui permettent de construire et de manipuler des transformations birationnelles non linéaires en dimension 3 avec une flexibilité suffisante pour les applications visées en modélisation géométrique. De plus, nous établissons une caractérisation de la birationalité basée sur le rang de tenseurs, qui permet de construire efficacement et ouvre la voie à l'application des outils de l'algèbre tensorielle à la birationalité. Nous étendons également nos résultats aux transformations birationnelles multilinéaires en dimension arbitraire, dans le cas où il existe un inverse multilinéaire.

Dans la deuxième partie, nous nous concentrons sur l'application des polynômes à l'analyse des données discrètes. Nous nous attaquons à deux problèmes distincts. Tout d'abord, nous dérivons des bornes pour la taille des  $(1 - \epsilon)$ -nets pour les ensembles de non-négativité de polynômes réels. Nos résultats nous permettent d'étendre le théorème classique du point central aux inégalités polynomiales de degré supérieur. Ensuite, nous abordons la classification des cylindres réels qui passent par cinq points qui sont tels que quatre d'entre eux sont cocycliques, c'est-à-dire qu'ils se trouvent sur un cercle. Il s'agit d'un cas particulier de problèmes plus généraux que sont la classification des racines réelles des systèmes de polynômes réels et l'extraction de primitives algébriques à partir de données brutes.

**Mots-clés:** transformation birationnelle, espace multiprojectif, multilinéaire, syzygie, tenseur, modélisation géométrique



# Acknowledgments

First, I express my sincere gratitude to my PhD advisor Laurent Busé. Thank you for guiding me towards a research path that I find truly fulfilling, where the learning is endless, and where I feel that I contribute with my work. I am also appreciative of the considerable time you devoted to me and for the exceptional treatment I always received from you.

My sincere thanks to Professors Alicia Dickenstein, Daniele Faenzi, and Hal Scheck for agreeing to be the reviewers of this thesis. Thank you very much for your time and the analysis of my work. Additionally, I would like to express my sincere thanks to Professors Andreas Höring and Sonia Pérez-Díaz for agreeing to be the examiners of this thesis.

Special thanks to Professor Carlos D'Andrea. Thanks for the support, the scientific discussions, and all the mathematics I have learned from you. Thanks also to Professor Joan Carles Naranjo for teaching me so much algebraic geometry. I am truly thankful to both for supervising my research during the three months I spent at the Universitat de Barcelona.

Thanks to Alfredo Hubard for his supervision and guiding during my first steps as a researcher. I have learned, and continue to learn, a lot from you. Thanks also to Roman Karasev for teaching me new mathematics. It has been an absolute pleasure to work with both of you.

Thanks to Alexander Leutgeb and Christoph Hofer for your supervision during my stay at RISC-Software. Thanks to Professor Josef Schicho for the scientific discussions and the introduction to new problems and scientific perspectives. It has been a pleasure to have worked with all of you.

Thanks again to Professors Joan Carles Naranjo and Andreas Höring for having been part of the two monitoring committees of my PhD.

Thanks to all the members of the AROMATH research team, which I have had the pleasure to be part of during the last three years. Thank you for your support, guidance, and advice. In particular, thanks to the team members at Inria, Bernard Mourrain, Angelos Mantzaflaris, Evelyne Hubert, Sophie Honnorat, Ayoub Belhachmi, Mehran Hatamzadeh, Lorenzo Baldi, Tobias Metzloff, Martin Jalard, and Rima Khouja. Special thanks to Michelangelo Marsala, who has lived the PhD experience with me at the front desk.

Thanks to the committee, directors, partners, and fellows of the GRAPES PhD network. Thank you for designing an international project that has allowed me to work with excellent researchers and learn about multiple research fields. Special thanks to the fellows Carles Checa and Amrutha Balachandran, with whom I became specially close. Thank you Carles for the patience and so many good moments. It has been a pleasure to live this experience with all of you.

Thanks also to the professors who, during these three years, have invited me to seminars and conferences to talk about my work. Thanks to Tomás Recio, Carlos D'Andrea, Joan Carles Naranjo, Philippe Gimenez, Andreas Höring, Josef Schicho, Josué Tonelli-Cueto, Angélica Torres, Sonia Pérez-Díaz, Krunal Raval, and Hendrik Speleers. Thanks also to John Cobb, Philippe Gimenez, Mario González-Sánchez, Mahrud Sayrafi, Lisa Seccia, Junho Choe, and Martí Salat-Moltó for participating in the session "Castelnuovo-Mumford Regularity and Applications" of the SIAM (AG23) conference that I had the pleasure of co-organizing with Carles Checa and Laurent Busé.

Thanks to the Institut de Matemàtica de la Universitat de Barcelona (IMUB) for hosting me during my three-month stay in Barcelona. Also, thanks to RISC-Software and the Johann Radon Institute for Computational and Applied Mathematics (RICAM) for hosting me during my three-month stay in Linz. Thanks to the Università di Genova for paying my accommodation during the Nice-Genova-Torino Algebraic Geometry Seminar 2023. Thanks to the Centre National de la Recherche Scientifique (CNRS) for paying for my stays at the Centre International de Rencontres Mathématiques (CIRM) during the JNCF conferences in 2022 and 2023.

Thanks to the professors of the Universidad de Cantabria who guided me to become a researcher in mathematics. Thanks to Tomás Recio, Daniel Sadornil, Javier Junquera, Francisco Santos, and Luis Miguel Pardo. Special thanks to my classmates of the bachelor degrees in mathematics and physics, with whom I have cultivated a friendship that I hope will last forever.

Thanks to the PhD students at the Laboratoire Jean Alexandre Dieudonné (LJAD), with whom I have had the pleasure to have fruitful conversations, to solve mathematical doubts, and to enjoy good times.

Thanks to all my friends for being the people in the world that make me happiest. Special thanks to Gonzalo Gutiérrez for the long phone calls and for being there when I needed you. Thanks to my best friend Lucía Fernández, with whom I have had a parallel life for the last two years. You are a synonym of happiness to me. I love you two very much. Thanks also to my therapist Ángela, for the positive impact you have had on my life over the past year.

Lastly, and most importantly, thanks to my family. You are the most important people in my life, and I owe you everything. Thanks to my parents, Yolanda and Santiago, for having done everything you could for me. Thanks to my aunt Maru, for taking care of me and for all your love. Thanks also to my uncle Gelo, for all the faith you have always shown in me. Your presence at both my undergraduate defense and my PhD defense means a great deal to me. I love you all.

# Funding

This thesis is part of the European project GRAPES, that has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant, agreement n°860843.



Marie Skłodowska-Curie  
Actions



# Contents

|  |           |
|--|-----------|
| <b>1. Introduction</b>   | <b>1</b>  |
| 1.1. Motivation and state-of-the-art   | 3         |
| 1.1.1. Construction and manipulation of birational transformations             | 3         |
| 1.1.2. Criteria for birationality  | 6         |
| 1.1.3. Classification of birational maps                                       | 9         |
| 1.1.4. Weak $\epsilon$ -nets for geometric range systems                       | 11        |
| 1.1.5. Extraction and count of real cylinders                                  | 12        |
| 1.2. Contributions   | 13        |
| 1.2.1. Classification of trilinear birational maps                             | 13        |
| 1.2.2. Birationality criteria for multilinear rational maps                    | 16        |
| 1.2.3. Construction and manipulation of multilinear birational transformations | 18        |
| 1.2.4. Weak $(1 - \epsilon)$ -nets for polynomial superlevel sets              | 28        |
| 1.2.5. Classification of real cylinders through five points with four cocyclic | 30        |
| 1.3. Organization of the thesis  | 31        |
| <b>2. Preliminaries</b>  | <b>35</b> |
| 2.1. Birational maps between multiprojective spaces                            | 35        |
| 2.1.1. Multiprojective schemes   | 35        |
| 2.1.2. Syzygies and free resolutions   | 39        |
| 2.1.3. Birational maps between multiprojective spaces                          | 43        |
| 2.1.4. The Rees algebra and blow-ups   | 45        |
| 2.2. Tensors   | 47        |
| 2.2.1. Elementary definitions  | 47        |
| 2.2.2. Tensor rank and CP decompositions                                       | 49        |
| 2.2.3. Some spaces of tensors  | 52        |
| 2.2.4. Bilinear birational maps  | 55        |
| <b>I. MULTILINEAR BIRATIONAL MAPS</b>  | <b>59</b> |
| <b>3. Trilinear birational maps in dimension three</b>                         | <b>61</b> |
| 3.1. Preliminaries   | 61        |
| 3.1.1. Basic definitions and properties  | 61        |
| 3.1.2. The Jacobian dual criterion for trilinear rational maps                 | 63        |
| 3.1.3. Automorphisms of $(\mathbb{P}_{\mathbb{C}}^1)^3$                        | 63        |
| 3.1.4. Curves and surfaces in $(\mathbb{P}_{\mathbb{C}}^1)^3$                  | 64        |
| 3.1.5. Pairing between the type and the tridegree of the base locus            | 66        |
| 3.2. Factorization of trilinear birational maps                                | 67        |
| 3.2.1. Commutative diagram for the factorization                               | 67        |
| 3.2.2. Factorization according to the type of $\phi$                           | 68        |



|  |            |
|--|------------|
| 3.3. The algebraic set of trilinear birational maps . . . . .  | 71         |
| 3.3.1. The sets $\mathfrak{Bir}_{(1,1,1)}$ and $\text{Bir}_{(d_1, d_2, d_3)}$ . . . . .                        | 71         |
| 3.3.2. The group action of $\text{Aut}((\mathbb{P}^1_{\mathbb{C}})^3)$ on $\mathfrak{Bir}_{(1,1,1)}$ . . . . . | 72         |
| 3.3.3. The algebraic set $\mathfrak{Bir}_{(1,1,1)}$ . . . . .  | 72         |
| 3.4. Classification of the base loci . . . . .   | 74         |
| 3.4.1. Orbits of birational maps of type $(1, 1, 1)$ . . . . .   | 74         |
| 3.4.2. Orbits of birational maps of type $(1, 1, 2)$ , $(1, 2, 1)$ and $(2, 1, 1)$ . . . . .                   | 76         |
| 3.4.3. Orbits of birational maps of type $(1, 2, 2)$ , $(2, 1, 2)$ and $(2, 2, 1)$ . . . . .                   | 78         |
| 3.4.4. Orbits of birational maps of type $(2, 2, 2)$ . . . . .   | 81         |
| 3.4.5. The group actions of the stabilizers on $Y_V$ . . . . .   | 83         |
| 3.4.6. Degenerations of the base loci . . . . .  | 86         |
| 3.5. Syzygies and minimal graded free resolutions . . . . .  | 87         |
| 3.5.1. Birational maps of type $(1, 1, 1)$ . . . . .   | 88         |
| 3.5.2. Birational maps of type $(1, 1, 2)$ . . . . .   | 89         |
| 3.5.3. Birational maps of type $(1, 2, 2)$ . . . . .   | 90         |
| 3.5.4. Birational maps of type $(2, 2, 2)$ . . . . .   | 92         |
| <b>4. Construction and manipulation of birational trilinear volumes . . . . .</b>                              | <b>95</b>  |
| 4.1. Preliminaries . . . . .   | 95         |
| 4.1.1. Motivating by applications . . . . .  | 95         |
| 4.1.2. Notation for this chapter . . . . .   | 98         |
| 4.2. Linear syzygies . . . . .   | 100        |
| 4.2.1. Rank characterization . . . . .   | 100        |
| 4.2.2. Geometric characterizations . . . . .   | 101        |
| 4.3. Hexahedral birational maps . . . . .  | 103        |
| 4.3.1. Construction of control points . . . . .  | 103        |
| 4.3.2. Effective computation of weights . . . . .  | 104        |
| 4.3.3. Distance to birationality . . . . .   | 106        |
| 4.3.4. Inverse rational map and base locus . . . . .   | 108        |
| 4.3.5. Deformation of birational maps . . . . .  | 111        |
| 4.4. Pyramidal birational maps . . . . .   | 112        |
| 4.4.1. Construction of control points . . . . .  | 112        |
| 4.4.2. Effective computation of weights . . . . .  | 114        |
| 4.4.3. Distance to birationality . . . . .   | 119        |
| 4.4.4. Inverse rational map and base locus . . . . .   | 119        |
| 4.5. Scaffold birational maps . . . . .  | 122        |
| 4.5.1. Construction of control points . . . . .  | 123        |
| 4.5.2. Effective computation of weights . . . . .  | 124        |
| 4.5.3. Distance to birationality . . . . .   | 130        |
| 4.5.4. Inverse rational map and base locus . . . . .   | 132        |
| <b>5. Construction of multilinear birational maps with multilinear inverse . . . . .</b>                       | <b>137</b> |
| 5.1. Rank-based characterization of birationality . . . . .  | 137        |
| 5.2. Inverse rational map and base locus . . . . .   | 140        |

---

|  |            |
|--|------------|
| <b>II. POLYNOMIAL DATA ANALYSIS</b>  | <b>145</b> |
| <b>6. Weak <math>(1 - \epsilon)</math>-nets for polynomial superlevel sets</b> | <b>147</b> |
| 6.1. Preliminaries . . . . .   | 147        |
| 6.1.1. Motivating example: disks and annuli . . . . .                          | 147        |
| 6.1.2. Weak $(1 - \epsilon)$ nets for polynomial superlevel sets . . . . .     | 149        |
| 6.1.3. The Carathéodory number of Veronese varieties . . . . .                 | 150        |
| 6.2. Contributions and related work . . . . .                                  | 151        |
| 6.2.1. Statement of our results . . . . .                                      | 151        |
| 6.2.2. Related work and discussion . . . . .                                   | 152        |
| 6.3. $(1 - \epsilon)$ -nets for superlevel sets of polynomials . . . . .       | 152        |
| 6.4. The Carathéodory number of Veronese varieties . . . . .                   | 155        |
| 6.4.1. Even degree . . . . .   | 155        |
| 6.4.2. Odd degree . . . . .  | 160        |
| <b>7. Cylinders through four cocyclic points</b>                               | <b>161</b> |
| 7.1. Projective definition of circular cylinders . . . . .                     | 161        |
| 7.2. The conic of admissible vertices . . . . .                                | 162        |
| 7.3. Classification of real circular cylinders . . . . .                       | 165        |
| <b>Bibliography</b>  | <b>184</b> |



# Chapter 1

## Introduction

This thesis focuses on two separated subjects that lie in the intersection of commutative algebra, algebraic geometry, multilinear algebra, and computer-aided geometric design:

- (i) The study and effective construction of multilinear birational maps
- (ii) The extraction of information from measures and data using polynomials

The thesis is divided into two parts, each addressing one of the aforementioned topics.

The first part, spanning Chapters 3 to 5, receives a more comprehensive treatment owing to its inclusion of a greater number of contributions. It is devoted to the study of multilinear birational maps. A multilinear rational map  $\phi : (\mathbb{P}^1)^n \dashrightarrow \mathbb{P}^n$  is a rational map from a product of projective lines to a projective space, which is defined by a tuple of multilinear polynomials. If it admits an inverse rational map, we say that it is birational. Birational maps pose multiple challenges and have historically occupied a central role in algebraic geometry (e.g. [72, 73, 119, 50, 2, 53, 85, 111, 109, 86]). However, much of the classic literature has primarily concentrated on birational automorphisms of projective spaces, also known as Cremona maps, and did not give equal attention to birational maps between multiprojective spaces. The study of multiprojective spaces is a very active topic in commutative algebra, driven by multiple theoretical inquiries [146, 104, 16, 107, 107, 145] and its real-world applications due to the connection with tensor product maps [41, 59, 51, 28, 43]. Additionally, during the last decade interest in birational transformations within geometric modeling and computer-aided geometric design (CAGD) has surged, and several works dedicated to their construction and manipulation have appeared [185, 183, 203, 202, 99].

The formal statements of the principal results of the first part of the thesis are collected in §1.2, along with examples and applications. We outline here the main objectives:

- (i) **Construction and manipulation of trilinear birational volumes:** the ultimate goal is to develop effective methods for the construction and manipulation of trilinear birational maps, i.e. birational maps  $\phi : (\mathbb{P}^1)^3 \dashrightarrow \mathbb{P}^3$  defined by trilinear polynomials. We develop tools to operate with these birational maps in a flexible way. In particular, we address several problems: the description of the necessary geometric constraints for birationality, the explicit computation of the inverse rational map, the computation of a birational approximation for a rational map, and the deformation of birational maps.
- (ii) **Effective birationality criteria for trilinear rational maps:** in order to achieve the first goal, we require results that characterize the birationality of trilinear rational maps. Our second objective is the derivation of new birationality criteria. We provide two main contributions in this direction. The first is a syzygy-based characterization of birationality, which is effective

for testing birationality but not as immediately applicable to construction. The second contribution is a characterization of birationality involving tensor rank. More specifically, under some necessary conditions, birationality holds if and only if a specific tensor of format  $2 \times 2 \times 2$  has rank one. This result establishes a bridge between birationality and tensors that, as far as we know, had remained unnoticed. From this perspective, we can apply tools from a vast literature on tensor rank to the construction of multilinear birational mappings.

- (iii) **Classification of the space and base loci of trilinear birational maps:** the derivation of our birationality criteria still requires an in-depth study of trilinear birational maps. We describe the geometry of the space of trilinear birational maps, and find that it consists of eight irreducible components of varying dimension. Roughly, each of these components consists of the birational maps that exhibit the same algebraic behavior. Additionally, we provide a complete list of the possible base loci of these transformations. In particular, we find exactly 19 isomorphism classes for the base loci.
- (iv) **Construction of multilinear birational maps with multilinear inverse:** we also address the construction of multilinear birational maps  $\phi : (\mathbb{P}^1)^n \dashrightarrow \mathbb{P}^n$  with a multilinear inverse. Algebraically, the defining polynomials of these birational maps admit  $n$  independent linear syzygies and have a Hilbert-Burch minimal graded free resolution. We extend the approach established for trilinear rational maps to this context. Namely, we prove that birationality is achieved if and only if a tensor of format  $2 \times n$  has rank one, and present effective methods for their construction.

In the second part, consisting of Chapters 6 and 7, we delve into inquiries related to using polynomials in analyzing data and measures. We focus on two distinct problems. The first one involves the investigation of  $(1 - \epsilon)$ -nets for superlevel sets of real polynomials, and relates to tensor rank [26, 17] and convex algebraic geometry [83, 84]. The second problem is the classification of the real cylinders through four cocyclic points. It can be regarded as an instance of a more general problem in real algebraic geometry, known as real root classification (e.g. [137, 142, 209, 140]). Moreover, it resides within the extraction of algebraic primitives from raw data (e.g. [165, 81, 42, 143]). The main objectives of the second part are summarized here:

- (i)  **$(1 - \epsilon)$ -nets for polynomial superlevel sets and the Carathéodory number of Veronese varieties:** we prove bounds on the size of a  $(1 - \epsilon)$ -net for the range space defined by the superlevel sets of real polynomials with bounded degree, i.e. the loci defined by a single polynomial inequality of bounded degree. These bounds allow us to generalize the classical centerpoint theorem, replacing half-spaces by more general polynomial superlevel sets. Our results build upon novel estimates of the Carathéodory number of real Veronese varieties.
- (ii) **Classification of real cylinders through five points with four of them cocyclic:** we provide algebraic certificates to ascertain the number of cylinders through a given configuration of five points with four cocyclic points (i.e. lying on a common circumference), and present closed formulas for their defining equations. These certificates are the equations that, in the space of five-point configurations with four cocyclic points, determine the regions with a fixed count of real cylinders.

## 1.1. Motivation and state-of-the-art

In this section, we provide the motivation that supports our research, and establish its context within the existing body of literature and contemporary research trajectories. Furthermore, we introduce the central ideas underlying the problems that we address. To ensure coherence, we introduce elementary notation as it becomes necessary. The section is thematically divided based on the research domains where our contributions lie.

### 1.1.1. Construction and manipulation of birational transformations

Surprisingly, although birational geometry is a classical topic in algebraic geometry with a trajectory of over 150 years, it wasn't until 2015 that the practical application of (nonlinear) birational transformations to design emerged, thanks to the work of researchers in computer-aided geometric design [185, 183].

In the fields of geometric modeling and computer-aided geometric design (CAGD), rational maps play a pivotal role. They offer an intuitive means of representing curves, surfaces, and volumes [93, 166, 69], and have been instrumental for the modern development of these areas since the seminal works of Pierre Bézier and Paul de Casteljau in the 1950s and 1960s [18, 74, 75].

A rational map is defined by ratios of polynomials. The primary representation of rational parametrizations in practical scenarios involve control points, nonnegative weights, and blending functions. The most frequent parametrizations of rational surfaces and volumes in CAGD rely on tensor-product polynomials. Nevertheless, it is worth noting that other useful parametrizations exist in geometric design, such as simplicial and toric Bézier patches [129]. For rational curves and surfaces, the typical parametrizations take the following form

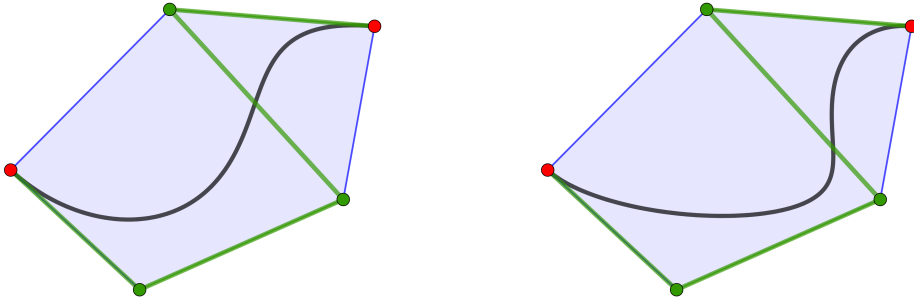
$$\begin{aligned} \phi : [0, 1] &\rightarrow \mathbb{A}_{\mathbb{R}}^2 & \phi : [0, 1]^2 &\rightarrow \mathbb{A}_{\mathbb{R}}^3 & (1.1) \\ s &\mapsto \frac{\sum_{i=0}^n w_i \mathbf{P}_i b_i^n(s)}{\sum_{i=0}^n w_i b_i^n(s)} & (s, t) &\mapsto \frac{\sum_{i=0}^n \sum_{j=0}^m w_{ij} \mathbf{P}_{ij} b_i^n(s) b_j^m(t)}{\sum_{i=0}^n \sum_{j=0}^m w_{ij} b_i^n(s) b_j^m(t)}, \end{aligned}$$

where  $\mathbf{P}_i = (x_i, y_i)$  in  $\mathbb{R}^2$  (resp.  $\mathbf{P}_{ij} = (x_{ij}, y_{ij}, z_{ij})$  in  $\mathbb{R}^3$ ) are the control points,  $w_i \geq 0$  (resp.  $w_{ij} \geq 0$ ) are the weights, and

$$b_i^n(s) = \binom{n}{i} (1-s)^{n-i} s^i$$

are the polynomials in the Bernstein basis of degree  $n$  in the variable  $s$ .

This formulation might seem unusual, or unnecessary, for readers with a background in algebra and algebraic geometry. However, there are substantial arguments in its favour. First and foremost, the control points and weights offer intuitive insights into the geometry of the rational map. The parametrized shape mimics the net of control points, and the weights have a pull-push effect towards the control points. Furthermore, the control points provide useful differential information of the rational map, and the shape always lies in the convex hull of the control points. All these properties are visually depicted in **Figure 1.1**, with the parametrization of a rational curve by means of the representation in (1.1). We refer the reader to [93] and [69, Chapter 3] for a more detailed description of the properties of these representations. In this thesis, we provide new arguments (of algebraic nature) supporting this representation.



**Figure 1.1.:** a rational curve parametrized using control points (green and red points) and their associated weights. The curve approximates the geometry of the sequence of segments connecting adjacent control points (green segments). In the image on the left, we use uniform weights equal to one. The right image showcases the effect of increasing the weight of the third control point to three, resulting in a “pull” effect that adjusts the curve towards the control point. Moreover, since all the weights are positive, the curve lies in the convex hull of the control points.

A rational parametrization is birational if it admits an inverse map which is also rational [109, 111]. Birational maps have several advantages in applications. One key benefit is that they ensure global injectivity (on a Zariski open set). Often, manipulating control points can lead to singularities and self-intersections of shapes due to noninjective parametrizations. More importantly, the inverse can be exploited for computing preimages. Some applications require the computation of preimages for various purposes [64, 97], and it is convenient for others such as image and volume warping [177, 194], morphing [193, 144], texturing [36, 150], or the generation of 3D curved meshes for geometric analysis [173, 138]. Birational maps offer computational advantages since the inverse yields formulas for these preimages without invoking numerical solving methods.

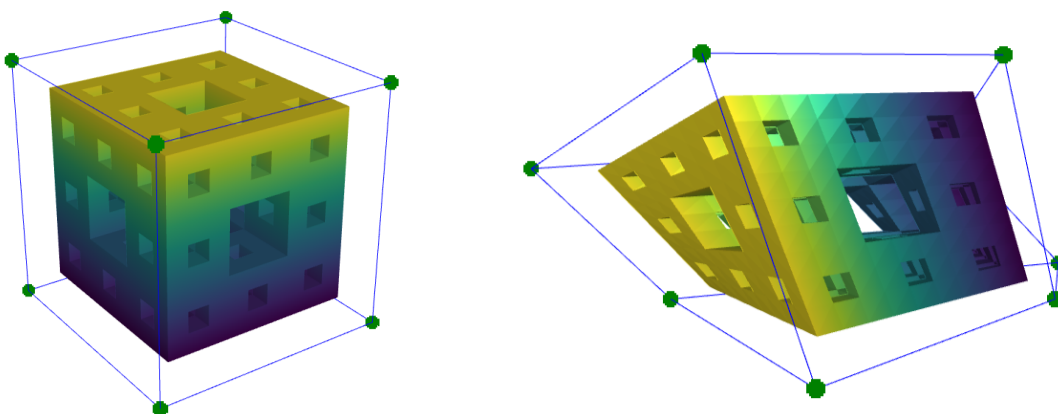
By a dimensional argument, the curve and surface parametrizations in (1.1) are birational for a general choice of control points and weights. However, if the dimension of the source and target spaces is the same, the situation is completely different. Such rational transformations are ubiquitous in CAGD. For example, free-form deformations (FFDs) have revealed a powerful technique with multiple applications (e.g. [184, 200, 158]), where the ambient space is transformed by means of a rational map as shown in Figure 1.2.

More specifically, given control points  $\mathbf{P}_{ijk} = (x_{ijk}, y_{ijk}, z_{ijk})$  and weights  $w_{ijk} \geq 0$ , the volumetric parametrization

$$\begin{aligned} \phi : \mathbb{A}_{\mathbb{R}}^3 &\dashrightarrow \mathbb{A}_{\mathbb{R}}^3 & (1.2) \\ (s, t, u) &\mapsto \frac{\sum_{i=0}^n \sum_{j=0}^m \sum_{k=0}^l w_{ijk} \mathbf{P}_{ijk} b_i^n(s) b_j^m(t) b_k^l(u)}{\sum_{i=0}^n \sum_{j=0}^m \sum_{k=0}^l w_{ijk} b_i^n(s) b_j^m(t) b_k^l(u)}, \end{aligned}$$

is in general not birational. Furthermore, the locus of birational transformations in the space of such parametrizations generally has a large codimension. Therefore, birationality represents a notably restrictive condition, making the construction of birational maps a challenging algebraic problem. Hence, it must be treated using tools from commutative algebra.

In order to address the problem of constructing tensor-product birational maps with the eyes of an algebraist, it is more convenient to adopt a multiprojective formulation. Let us consider the



**Figure 1.2.:** free-form deformation of a 3D complex model by means of a trilinear rational map, defined by a net of  $2 \times 2 \times 2 = 8$  control points.

homogeneous variables  $s_0, s_1$  on a real projective line  $\mathbb{P}_{\mathbb{R}}^1$ , and redefine the homogeneous Bernstein polynomials of degree  $n$  in the variables  $s_0, s_1$  as

$$b_i^n(s_0, s_1) = \binom{n}{i} (s_0 - s_1)^{n-i} s_1^i$$

for every  $0 \leq i \leq n$ . Moreover, we define the ring  $R = \mathbb{R}[s_0, s_1] \otimes \mathbb{R}[t_0, t_1] \otimes \mathbb{R}[u_0, u_1]$ , which is canonically graded by three integers as it inherits the degree from the three pairs of homogeneous variables. If we redefine  $\mathbf{P}_{ijk} = (1, x_{ijk}, y_{ijk}, z_{ijk})$ , the previous volume parametrization can be equivalently written as

$$\begin{aligned} \phi : \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{R}}^3 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (f_0 : f_1 : f_2 : f_3) \end{aligned} \quad (1.3)$$

where

$$\mathbf{f} = (f_0, f_1, f_2, f_3) = \sum_{i=0}^n \sum_{j=0}^m \sum_{k=0}^l w_{ijk} \mathbf{P}_{ijk} b_i^n(s_0, s_1) b_j^m(t_0, t_1) b_k^l(u_0, u_1) \in R^4$$

In particular, the projective point  $(1 : x_{ijk} : y_{ijk} : z_{ijk})$  lies in the affine chart  $\mathbb{A}_{\mathbb{R}}^3 \subset \mathbb{P}_{\mathbb{R}}^3$  defined by  $y_0 \neq 0$ . Since  $\mathbb{A}_{\mathbb{R}}^3 \cong \text{Spec}(\mathbb{R}[Y_1, Y_2, Y_3])$  for the variables  $Y_n = y_n/y_0$  for each  $n = 1, 2, 3$ , the point  $(1 : x_{ijk} : y_{ijk} : z_{ijk})$  is identified with  $(x_{ijk}, y_{ijk}, z_{ijk})$  in  $\mathbb{A}_{\mathbb{R}}^3$ . Consequently, both rational maps coincide (see Definition 2.1.30) and the polynomial  $f_0$  can be thought of as the common denominator in (1.2). The advantage of this multiprojective setting is that we can transfer the study of the rational map  $\phi$  to the study of the ideal  $B = (f_0, f_1, f_2, f_3)$  in  $R$ , called the base ideal (Definition 2.1.32), and leverage homological methods for treating  $B$  as a graded  $R$ -module.

One of the central tools in commutative algebra are syzygies. In our context, a syzygy of  $\mathbf{f}$  is a tuple  $\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$  of homogeneous polynomials in  $R$  satisfying the relation

$$\langle \boldsymbol{\sigma}, \mathbf{f} \rangle = \sigma_0 f_0 + \sigma_1 f_1 + \sigma_2 f_2 + \sigma_3 f_3 = 0,$$

where  $\langle -, - \rangle$  stands for the usual scalar product. To put it simply, a syzygy of  $\mathbf{f}$  represents a linear relation among the polynomials  $f_0, f_1, f_2$ , and  $f_3$  with coefficients belonging to  $R$ . Approximately



thirty years ago syzygies became involved in CAGD, primarily motivated by the problem of finding the implicit equation of parametric rational curves and surfaces by the method of moving curves and surfaces [182]. They have proven to be an instrumental tool in the modern theory of implicitization since then (e.g. [123, 30, 44, 40, 39]). In the last decade, syzygies have also become involved in the construction of birational maps.

The literature concerning the construction of birational maps for CAGD is still in its infancy. Much of the research has focused on developing sufficient criteria for local injectivity, aiming to determine conditions under which a rational parametrization is injective within a specific domain (e.g. [206, 207, 95, 168]). However, this approach falls short when it comes to efficiently computing preimages. The earliest works on constructing birational maps dealt with 2D tensor-product parametrizations  $\phi : \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \dashrightarrow \mathbb{P}_{\mathbb{R}}^2$  defined by polynomials of low degree. In [185], the authors give a simple characterization of birationality for rational maps with bilinear entries, and derive an effective method to construct them with much flexibility. Notably, their approach does not impose geometric constraints on control points and provides explicit formulas for defining the inverse rational map. More general (nonrational) inversion formulas for nonbirational bilinear maps are studied in [94]. Research has also explored birational maps  $\phi : \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \dashrightarrow \mathbb{P}_{\mathbb{R}}^2$  with entries of degree  $(1, n)$ , i.e. linear with respect to one of the parameters. In [183], the authors propose sufficient conditions for birationality and methods for construction. Remarkably, the control points defining the rational map remain unconstrained, and nonnegative weights can be computed to ensure birationality. Recently, novel conditions for birationality have been proposed for parametrizations  $\phi : \mathbb{P}_{\mathbb{R}}^2 \dashrightarrow \mathbb{P}_{\mathbb{R}}^2$  with quadratic entries, relying on the complex rational representation of a rational map [203, 202]. These works introduce methods for the construction of birational triangular patches, albeit with some constraints on the geometry of the control points.

It is important to highlight three characteristics common to all the works addressing the construction of birational maps in CAGD published to date:

- (i) They only treat 2D parameterizations. Methods for constructing and manipulating 3D nonlinear birational maps are yet to be developed
- (ii) They rely on the imposition of specific syzygies to the entries of the rational map in order to achieve birationality
- (iii) They provide strategies for constructing (possibly constrained) nets of control points with sufficient flexibility, followed by the computation of weights that ensure birationality

### 1.1.2. Criteria for birationality

The computation of the degrees and birationality of rational maps between multiprojective and projective spaces has received attention recently [41, 59, 43, 28]. Approximately thirty years ago, modern techniques from commutative algebra were incorporated to the study of birational maps, primarily motivated by computational purposes. Syzygy-based birationality criteria date back to [181], and have been further developed ever since (e.g. [178, 188]). In particular, they are necessary in all the existing methods for the construction of birational transformations in CAGD.

The Jacobian Dual Criterion (JDC) is a general method to decide if a rational map between projective varieties, or from a multiprojective variety to a projective variety, is birational [87, 178, 41, 188]. The JDC relies on the computation of the defining equations of the Rees algebra associated to the rational map. The Rees algebra is a central object in commutative algebra (e.g. [57, 170, 198, 199]) that has proven useful in a number of scenarios (e.g. [66, 65, 67, 135, 134, 133]).

More specifically, the defining equations of the Rees algebra associated to an ideal encompass all the polynomial relations satisfied by the generators of the ideal. Among these relations we have the syzygies, which are precisely the linear relations. The algebra induced by the syzygies is the symmetric algebra, which is a rough approximation to the Rees algebra. However, in some particular cases the symmetric algebra provides useful information about the Rees algebra [89, 151, 41, 132].

In the context of rational maps, the Rees algebra can be understood as the bihomogeneous coordinate ring of the graph of the rational map [111, §II.7]. More specifically, let  $R = \mathbb{C}[\mathbf{x}_1] \otimes \dots \otimes \mathbb{C}[\mathbf{x}_n]$  be the homogeneous coordinate ring of  $\mathbb{P}_{\mathbb{C}}^{m_1} \times \dots \times \mathbb{P}_{\mathbb{C}}^{m_n}$ , where  $\mathbf{x}_i = (x_{0i}, \dots, x_{m_i i})$  are standard  $\mathbb{Z}$ -graded variables (see §2.1.1). In particular, the tensor product ring  $R$  is standard  $\mathbb{Z}^n$ -graded. Additionally, set  $m_0 = m_1 + \dots + m_n$  and let  $\mathbf{y} = (y_0, \dots, y_{m_0})$  be standard  $\mathbb{Z}$ -graded variables. Furthermore, let  $\mathbf{f} = (f_0, \dots, f_{m_0})$  be homogeneous polynomials in  $R$ . Clearly,  $\mathbf{f}$  induces a rational map  $\phi : \mathbb{P}_{\mathbb{C}}^{m_1} \times \dots \times \mathbb{P}_{\mathbb{C}}^{m_n} \dashrightarrow \mathbb{P}_{\mathbb{C}}^{m_0}$ . The Rees ideal  $J$  associated to  $\mathbf{f}$ , or  $\phi$ , is the kernel of the homomorphism of graded rings (see §2.1.4 for more details)

$$\begin{aligned} R[\mathbf{y}] &\rightarrow R \\ y_j &\mapsto f_j, \end{aligned}$$

where  $R[\mathbf{y}]$  is  $\mathbb{Z}^{n+1}$ -graded since it inherits the degree from  $R$  and the variables in  $\mathbf{y}$ . The Rees algebra associated to  $\mathbf{f}$ , or  $\phi$ , is the quotient  $\mathcal{R} := R[\mathbf{y}]/J$ .

Let us denote by  $(d_1, \dots, d_n; e)$  the  $\mathbb{Z}^{n+1}$ -degree in  $R[\mathbf{y}]$ , which is the concatenation of the  $\mathbb{Z}^n$ -degree in  $R$  and the  $\mathbb{Z}$ -degree of  $\mathbf{y}$ . Namely, a polynomial in  $R[\mathbf{y}]$  has degree  $(d_1, \dots, d_n; e)$  if it has degree  $d_i$  with respect to the variables in  $\mathbf{x}_i = (x_{0i}, \dots, x_{m_i i})$  and degree  $e$  with respect to the variables in  $\mathbf{y} = (y_0, \dots, y_{m_0})$ . For each  $1 \leq i \leq n$ , let  $\mathbf{e}_i$  be the  $i$ -th canonical vector in  $\mathbb{Z}^n$ , i.e.

$$\mathbf{e}_i = (0, \dots, 0, \underbrace{1}_{i\text{-th coordinate}}, 0, \dots, 0).$$

Additionally, we define the  $\mathbb{C}[\mathbf{y}]$ -module

$$J_{(\mathbf{e}_i; *)} = \bigoplus_{j=1}^{\infty} J_{(\mathbf{e}_i; j)}$$

where  $J_{(\mathbf{e}_i; j)}$  stands for the graded component of  $J$  in degree  $(\mathbf{e}_i; j)$  (see Definition 2.1.3). If we denote by  $\mathbb{C}(\mathbf{y})$  the field of fractions of  $\mathbb{C}[\mathbf{y}]$ , the multigraded JDC (Theorem 2.1.40) states that  $\phi$  is birational if and only if the  $\mathbb{C}(\mathbf{y})$ -vector space  $\mathbb{C}(\mathbf{y}) \otimes_{\mathbb{C}[\mathbf{y}]} J_{(\mathbf{e}_i; *)}$  has dimension one for every  $1 \leq i \leq n$ .

The following example shows how the multigraded JDC can be used to test birationality and compute the inverse rational map explicitly.

**Example 1.1.1.** Consider the rational map

$$\begin{aligned} \phi : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^3 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (s_1 t_0 u_1 : s_0 t_0 u_1 - s_0 t_1 u_1 : s_1 t_1 u_0 : s_0 t_1 u_0 - s_0 t_1 u_1) \end{aligned}$$

and let  $R = \mathbb{C}[s_0, s_1] \otimes \mathbb{C}[t_0, t_1] \otimes \mathbb{C}[u_0, u_1]$  be standard  $\mathbb{Z}^3$ -graded, and  $\mathbf{y} = (y_0, y_1, y_2, y_3)$  be standard  $\mathbb{Z}$ -graded. Using *Macaulay2* [101], we derive that the Rees ideal associated to  $\phi$  is

generated by the seven relations

$$\begin{aligned}
& s_0 (y_0 - y_2) - s_1 (y_1 - y_3) , \\
& (s_0 u_0 - s_0 u_1) y_2 - s_1 u_0 y_3 , \\
& s_1 t_1 y_1 + (s_0 t_1 - s_0 t_0) y_2 + (s_1 t_0 - s_1 t_1) y_3 , \\
& (t_1 u_0 - t_1 u_1) y_1 + (t_1 u_1 - t_0 u_1) y_3 , \\
& t_1 u_0 y_0 - t_0 u_1 y_2 , \\
& t_0 (y_0 y_3 - y_1 y_2) - t_1 (y_0 y_3 - y_0 y_1) , \\
& u_0 (y_1 y_2 - y_0 y_3) - u_1 (y_1 y_2 - y_2 y_3) .
\end{aligned}$$

In particular, we find

$$\begin{aligned}
J_{(1,0,0;1)} &= \mathbb{C}\langle s_0 (y_0 - y_2) - s_1 (y_1 - y_3) \rangle , \\
J_{(0,1,0;2)} &= \mathbb{C}\langle t_0 (y_0 y_3 - y_1 y_2) - t_1 (y_0 y_3 - y_0 y_1) \rangle , \\
J_{(0,0,1;2)} &= \mathbb{C}\langle u_0 (y_1 y_2 - y_0 y_3) - u_1 (y_1 y_2 - y_2 y_3) \rangle ,
\end{aligned}$$

where  $\mathbb{C}\langle - \rangle$  stands for the  $\mathbb{C}$ -vector space spanned by  $-$ . Furthermore, the  $\mathbb{C}(\mathbf{y})$ -vector spaces  $\mathbb{C}(\mathbf{y}) \otimes_{\mathbb{C}[\mathbf{y}]} J_{(e_i;*)}$  are spanned by the generators of the vector spaces above. By the multigraded JDC, it follows that  $\phi$  is birational.

Additionally, the defining relations of the Rees algebra can be used to efficiently compute preimages. Specifically, the Rees ideal  $J$  is the defining ideal of the graph of  $\phi$  in the product space  $(\mathbb{P}_{\mathbb{C}}^1)^3 \times \mathbb{P}_{\mathbb{C}}^3$  (see §2.1.4), or equivalently, the graph of  $\phi$  is the scheme in  $(\mathbb{P}_{\mathbb{C}}^1)^3 \times \mathbb{P}_{\mathbb{C}}^3$  defined by the Rees algebra  $\mathcal{R} = R[\mathbf{y}]/J$  via the MultiProj construction (see §2.1.1). Namely, a point

$$(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1) \times (\delta_0 : \delta_1 : \delta_2 : \delta_3) \in (\mathbb{P}_{\mathbb{C}}^1)^3 \times \mathbb{P}_{\mathbb{C}}^3$$

lies in the graph of  $\phi$  if and only if

$$g(\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1, \delta_0, \delta_1, \delta_2, \delta_3) = 0$$

for every  $g$  in  $J$ . In particular, the pullback  $\phi^{-1}$  of the point  $(y_0 : y_1 : y_2 : y_3) = (1 : 1 : 2 : -3)$  can be retrieved from the specializations of the generators of  $J_{(e_i;*)}$ . More explicitly,

$$\begin{aligned}
& s_0 (1 - 2) - s_1 (1 + 3) = s_0 (-1) - s_1 (4) , \\
& t_0 (1 \cdot (-3) - 1 \cdot 2) - t_1 (1 \cdot (-3) - 1 \cdot 1) = t_0 (-5) - t_1 (-4) , \\
& u_0 (1 \cdot 2 - 1 \cdot (-3)) - u_1 (1 \cdot 2 - 2 \cdot (-3)) = u_0 (5) - u_1 (8)
\end{aligned}$$

implying that

$$\phi^{-1}(1 : 1 : 2 : -3) = (4 : -1) \times (4 : 5) \times (8 : 5) .$$

More generally, the inverse rational map is given by

$$\begin{aligned}
\phi : \mathbb{P}_{\mathbb{C}}^3 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \\
(y_0 : y_1 : y_2 : y_3) &\mapsto (y_1 - y_3 : y_0 - y_2) \times (y_0 y_3 - y_0 y_1 : y_0 y_3 - y_1 y_2) \times (y_1 y_2 - y_2 y_3 : y_1 y_2 - y_0 y_3) .
\end{aligned}$$

Unfortunately, there are two important drawbacks inherent to the JDC:

- (i) The derivation of the defining equations of the Rees algebra is a difficult problem [27, 38, 70, 68] that entails a high computational cost.
- (ii) The JDC is suitable to test birationality and computing the inverse, but not so easily applicable for the construction of birational transformations. More specifically, how can we effectively ensure that the entries of a rational map  $\phi : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \dashrightarrow \mathbb{P}_{\mathbb{C}}^3$ , defined by control points and weights, admit relations of degrees

$$(1, 0, 0; d_1) \quad , \quad (0, 1, 0; d_2) \quad , \quad (0, 0, 1; d_3)$$

for some positive integers  $d_1, d_2$ , and  $d_3$ ?

For applications to CAGD, specialized methods are more frequent. In particular, syzygy-based criteria for birationality are preferred due to efficiency and applicability reasons. In dimension two, there are birationality criteria that rely on the existence of particular syzygies for specific families of birational maps. Namely, dominant rational parametrizations  $\phi : (\mathbb{P}_{\mathbb{C}}^1)^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$  with bilinear entries, i.e. of  $\mathbb{Z}^2$ -degree  $(1, 1)$ , are known to be birational if and only if the entries admit a syzygy of degree either  $(1, 0)$  or  $(0, 1)$  [28]. Moreover, the minimal  $\mathbb{Z}^2$ -graded free resolution of the ideal generated by the entries, called the base ideal, is Hilbert-Burch (see §2.2.4). This relates with the recent interest on the study of resolutions of the base ideals of plane Cremona maps [112], and more generally with the very active research on resolutions in products of projective spaces [112, 31, 16, 107, 145]. Larger degrees for parametrizations  $\phi : (\mathbb{P}_{\mathbb{C}}^1)^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$  have also been considered, but only partially. More specifically, sufficient conditions under which birationality holds are known for maps with entries of degree  $(2, 2)$  [28] and  $(1, n)$  [183].

### 1.1.3. Classification of birational maps

The classification of birational maps has a rich history within algebraic geometry, with a vast body of literature devoted to this topic (e.g. [119, 2, 186, 163, 79, 46, 23, 21, 14, 22, 25, 77]). Like any other objects, birational maps can be examined from multiple perspectives, giving rise to various notions for classification. The most relevant approaches can be summarized as follows:

- (i) The study of groups of birational maps
- (ii) The study of spaces of birational maps
- (iii) The study of the base loci of birational maps
- (iv) The study of the algebraic relations between the defining polynomials of birational maps

Naturally, the appropriateness of each point of view depends on the context, and the tools employed to address them usually vary. Furthermore, these perspectives are interconnected. In this thesis, we focus on (ii), (iii), and (iv).

The  $n$ -dimensional Cremona group over  $\mathbb{C}$ , denoted by either  $Cr_n(\mathbb{C})$  or  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$ , is the group of birational endomorphisms of the complex projective space  $\mathbb{P}_{\mathbb{C}}^n$ . Equivalently, it is the group of  $\mathbb{C}$ -automorphisms of the field  $\mathbb{C}(x_0, \dots, x_n)$  where  $x_0, \dots, x_n$  are the homogeneous variables in  $\mathbb{P}_{\mathbb{C}}^n$ . Given an integer  $d$ , the subset  $\text{Bir}_d(\mathbb{P}_{\mathbb{C}}^n) \subset Cr_n(\mathbb{C})$  consists of all the birational endomorphisms that are determined by  $n + 1$  homogeneous polynomials of degree  $d$ . In particular, we have an increasing sequence

$$\text{PGL}(n + 1, \mathbb{C}) \cong \text{Aut}(\mathbb{P}_{\mathbb{C}}^n) = \text{Bir}_1(\mathbb{P}_{\mathbb{C}}^n) \subset \text{Bir}_2(\mathbb{P}_{\mathbb{C}}^n) \subset \dots \subset \text{Bir}_d(\mathbb{P}_{\mathbb{C}}^n) \subset \dots \subset Cr_n(\mathbb{C}) .$$

A natural question is whether  $\text{Cr}_n(\mathbb{C})$  has the structure of an algebraic group (of infinite dimension), i.e. if the group  $\text{Cr}_n(\mathbb{C})$  is compatible with the structure of an algebraic variety [111, 109]. The answer is negative for  $n \geq 2$  [24]. However, the subset  $\text{Bir}_d(\mathbb{P}_{\mathbb{C}}^n)$  is always a locally closed projective variety [24, Lemma 2.4]. Consequently, the investigation of spaces of Cremona maps primarily hinges on the study of  $\text{Bir}_d(\mathbb{P}_{\mathbb{C}}^n)$  as an algebraic set. Regrettably, the algebraic structure of these sets is generally intricate. The case  $n = 2$  of birational endomorphisms of the projective plane, or plane Cremona maps, is the simplest and best understood. We have a good understanding of  $\text{Bir}_d(\mathbb{P}_{\mathbb{C}}^2)$  for any  $d \geq 2$  [20, 53], and a complete classification of the base loci is known for  $d = 2, 3$  [53].

In higher dimensions the situation quickly becomes more complicated. A very nice property of plane Cremona maps is that the inverse of a birational map in  $\text{Bir}_d(\mathbb{P}_{\mathbb{C}}^2)$  also lies in  $\text{Bir}_d(\mathbb{P}_{\mathbb{C}}^2)$  [45, 78]. However, this property no longer holds true for  $n \geq 3$ . More specifically, let  $\phi$  be a birational automorphism of  $\mathbb{P}_{\mathbb{C}}^3$  defined by polynomials of degree  $d$  without a common factor, and let  $\phi^{-1}$  be defined by polynomials of degree  $d'$  without a common factor. Then, we have (see [45, 78, 80])

$$d^2 = d' + \deg C, \quad (1.4)$$

where  $C \subset \mathbb{P}_{\mathbb{C}}^n$  is the curve component of the base locus of  $\phi$ . Hence, the degree of the defining polynomials of a birational automorphism of  $\mathbb{P}_{\mathbb{C}}^n$  is in general different from the degree of the defining polynomials of its inverse. In §3.1.5, we establish the analogue of (1.4) transposed to our multiprojective setting. This observation is crucial in this thesis, as it delineates the structure of Chapters 3-5 since each type of inverse (see Definition 1.2.3) is studied separately.

**Example 1.1.2.** The trilinear birational map

$$\begin{aligned} \phi_1 : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^3 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (s_1 t_1 u_1 : s_0 t_1 u_1 : s_1 t_0 u_1 : s_1 t_1 u_0) \end{aligned}$$

admits the inverse

$$\begin{aligned} \phi_1^{-1} : \mathbb{P}_{\mathbb{C}}^3 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \\ (y_0 : y_1 : y_2 : y_3) &\mapsto (y_1 : y_0) \times (y_2 : y_0) \times (y_3 : y_0), \end{aligned}$$

where all the defining polynomials are linear. On the other hand, the trilinear birational map

$$\begin{aligned} \phi_2 : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^3 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (s_1 t_0 u_1 - s_0 t_1 u_1 : s_0 t_0 u_1 : s_1 t_1 u_0 : s_0 t_1 u_0) \end{aligned}$$

admits the inverse

$$\begin{aligned} \phi_2^{-1} : \mathbb{P}_{\mathbb{C}}^3 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \\ (y_0 : y_1 : y_2 : y_3) &\mapsto (y_3 : y_2) \times (y_1 y_3 : y_1 y_2 - y_0 y_3) \times (y_3^2 : y_1 y_2 - y_0 y_3), \end{aligned}$$

defined by a pair of linear polynomials and two pairs of quadratic polynomials. As we shall see, the algebraic behavior of these two birational maps is different. Moreover, they belong to different irreducible components in the space of trilinear birational maps (see Theorem 1.2.4).

The algebraic structure of  $\text{Bir}_2(\mathbb{P}_{\mathbb{C}}^3)$  has been described in [161], together with a complete classification of the base loci. In particular,  $\text{Bir}_2(\mathbb{P}_{\mathbb{C}}^3)$  has three irreducible components, each

consisting of the birational maps with inverse defined by polynomials of degree 2, 3, and 4 respectively. Additionally, some of the irreducible components of  $\text{Bir}_3(\mathbb{P}_{\mathbb{C}}^3)$  have been properly described [80], and a variety (75 classes) of the possible base loci is understood and classified [80, Appendix A], [119, Table VI]. Unfortunately, we still lack analogous results for either  $n \geq 4$  or  $d \geq 4$ . In the context of multiprojective birational maps, the literature has only treated a classification for the family of bilinear birational maps  $(\mathbb{P}_{\mathbb{C}}^1)^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2$  to this date [185, 28]. Specifically, the base locus is always a closed point. Furthermore, the space of such birational transformations is an irreducible locally closed hypersurface in  $\mathbb{P}_{\mathbb{C}}^{11}$ . Interestingly, Corollary 2.2.26 establishes that the space of bilinear birational maps is identified with the hyperdeterminant of the tensors of format  $3 \times 2 \times 2$ .

Cremona maps have also been investigated putting emphasis on the ideal-theoretic properties of the base ideal [60, 162, 164, 188, 190, 191, 189]. In this direction, an in-depth homological classification of plane Cremona maps has been undertaken for degrees  $5 \leq d \leq 7$  [112]. This study is complemented by novel homological results concerning the base ideal of de Jonquières transformations and their associated Rees algebra. Regarding multiprojective birational maps, such homological classification is limited to the case of bilinear birational maps. Namely, the minimal graded free resolution is Hilbert-Burch, with two linear syzygies [28].

#### 1.1.4. Weak $\epsilon$ -nets for geometric range systems

In the fields of computational geometry and machine learning,  $\epsilon$ -nets have proven a useful tool with applications to statistical learning theory [197, 5, 113], efficient search algorithms [187, 147, 114] approximate solutions [105, 61, 7, 8], randomized algorithms [62, 33, 154], dimensionality reduction [106], and data compression [1]. A geometric range system is a pair  $(\mu, F)$  consisting of a probability measure  $\mu$  on  $\mathbb{A}_{\mathbb{R}}^n$  and a family  $F$  of subsets of  $\mathbb{A}_{\mathbb{R}}^n$ , also called ranges. Given  $0 \leq \epsilon \leq 1$ , a weak  $\epsilon$ -net for  $(\mu, F)$  is a subset  $Y \subset \mathbb{A}_{\mathbb{R}}^n$  with the property that every range  $R$  in  $F$  satisfying  $\mu(R) \geq \epsilon$  has a nonempty intersection with  $Y$ .

One of the primary focuses in  $\epsilon$ -nets research is the determination of their minimal size. In general, this is a difficult task. Therefore, a substantial body of work has been dedicated to establishing bounds on their cardinality (e.g. [176, 131, 56, 35, 155]). Naturally, these bounds vary significantly depending on the choice of range systems. The most frequent range systems involve semialgebraic sets, including lines [114], disks and balls [128, 37], triangles [6], axis-parallel rectangles [114, 6]. It is noteworthy, however, that the exploration of  $\epsilon$ -nets for range systems defined by general polynomial inequalities has been largely overlooked, except for the linear case where the ranges are half-spaces in  $\mathbb{A}_{\mathbb{R}}^n$  [148, 128]. More explicitly, let  $d \geq 2$  and let  $\mathbb{R}[x_1, \dots, x_n]_{\leq d}$  be the  $\mathbb{R}$ -vector space of real polynomials of degree at most  $d$ . We still lack results bounding the size of  $\epsilon$ -nets involving the range space

$$F = \{(f \geq 0) : f \in \mathbb{R}[x_1, \dots, x_n]_{\leq d}\},$$

where

$$(f \geq 0) := \{(x_1, \dots, x_n) \in \mathbb{A}_{\mathbb{R}}^n : f(x_1, \dots, x_n) \geq 0\}.$$

Regarding linear inequalities, the classical centerpoint theorem (Theorem 1.2.28) of Rado and Birch ([169, 19], see [160] as well) states that for any probability measure  $\mu$  on  $\mathbb{R}^n$  there exists a point  $c$  in  $\mathbb{R}^n$  such that every closed half-space  $H \subset \mathbb{R}^n$  containing  $c$  satisfies  $\mu(H) \geq \frac{1}{n+1}$ . While there have been several extensions of the centerpoint theorem (e.g. [122, 167, 153, 121]), none have addressed ranges defined by nonlinear polynomial inequalities to date.

### 1.1.5. Extraction and count of real cylinders

The extraction of algebraic primitives from raw data involves computing algebraic shapes that interpolate or approximate a given set of data points. The applications of the extraction process are diverse, spanning various fields such as robotics [205, 54, 15], CAGD [55], computer-aided surgery [205, 174], and data analysis [204]. In practice, the preference often leans towards elementary shapes. In the context of 3D data, these shapes can include spheres, cylinders, cones, and tori.

In this thesis, our primary focus is on the extraction of primitive cylinders. While this problem has already garnered attention [165, 81, 42, 143], there are important open questions regarding classification. Specifically, there are exactly six cylinders over the complex numbers through five general points in  $\mathbb{A}_{\mathbb{R}}^3$  (e.g. [81, 42, 143]). However, the understanding of the number of such real cylinders is quite limited. The problem of computing the number of real cylinders through five general points boils down to computing the number of real solutions of a system of real polynomial equations. Therefore, the generic count of real cylinders is either zero, two, four, or six. The following questions arise naturally:

- (i) Given five distinct points in  $\mathbb{A}_{\mathbb{R}}^3$ , how many real cylinders interpolate them?
- (ii) Can we describe the configurations of five points that yield a fixed number (0, 2, 4, or 6) of real cylinders?
- (iii) Can we compute the defining equations of the cylinders through five points?

In full generality, the first two questions above are very complicated. Formally, the first question seeks a list of certificates for the five points to discriminate the number of real cylinders through them. As real cylinders correspond to the real roots of a system of polynomial equations (e.g. [81, 42, 143]), these certificates manifest as algebraic equations in the coordinates of the points [108].

The second question delves into a considerably broader topic in the field of real algebraic geometry: the real root classification. It entails identifying all semi-algebraic conditions within the space  $(\mathbb{A}_{\mathbb{R}}^3)^5$  of five-points configurations, which discriminate all possible counts of real solutions to the initial polynomial system (e.g. [139, §1.2.3]). Although some configurations of five general points that yield zero real cylinders are known, such as when one point lies within the convex hull of the others, and other conjectures of configurations with a fixed count of real cylinders are established [143], a real root classification remains elusive. Furthermore, existing algorithms for the real root classification of polynomial systems (e.g. [137, 142, 141, 208, 209, 140]) are inadequate for this task, primarily due to the complexity of system, which after a suitable choice of coordinates involves 8 parameters. More specifically, by means of a change of coordinates the five points take the form

$$Q_1 = (0, 0, 0) \quad , \quad Q_2 = (1, 0, 0) \quad , \quad Q_3 = (x_3, y_3, 0) \quad , \quad Q_4 = (x_4, y_4, z_4) \quad , \quad Q_5 = (x_5, y_5, z_5)$$

and the space of five-points configurations reduces to  $\mathbb{A}_{\mathbb{R}}^2 \times \mathbb{A}_{\mathbb{R}}^3 \times \mathbb{A}_{\mathbb{R}}^3$ . Even if we were able to obtain a complete list of certificates, interpreting the intricate semi-algebraic conditions involved would pose a significant challenge.

In this thesis, we take a first step in this direction by providing a complete real root classification for a specific configuration of the five points: four of them are cocyclic, meaning they lie in a common circumference. Although this configuration might initially appear restrictive (it is not Zariski dense in the space of five-point configurations  $(\mathbb{A}_{\mathbb{R}}^3)^5$ ), our findings demonstrate that achieving a real root classification is, in fact, far from trivial.

Regarding the third question, there are several methods available for computing the implicit equations of the cylinders. Some of these methods involve minimization algorithms aimed at approximating optimal cylinders that fit a given point cloud (e.g. [165, 175, 180]). In our five-points context, there are methods relying on Gröbner basis [143], eigenvalue computations [42, 81], and resultants [81]. However, it is worth noting that none of these algebraic approaches provides closed-form formulas for the defining equations of cylinders, which are often preferred.

## 1.2. Contributions

In this section we present the main results of the thesis. Our contributions span both theoretical advancements and practical applications. For the latter, we provide illustrative examples to showcase their usefulness. Semantically, the contributions are organized within the same topics discussed in §1.1.

### 1.2.1. Classification of trilinear birational maps

Let  $A_1 = \mathbb{C}[s_0, s_1]$ ,  $A_2 = \mathbb{C}[t_0, t_1]$ , and  $A_3 = \mathbb{C}[u_0, u_1]$  be standard  $\mathbb{Z}$ -graded rings, each graded by the corresponding pair of variables. The tensor product  $R = A_1 \otimes A_2 \otimes A_3$  is naturally endowed with a standard  $\mathbb{Z}^3$ -grading, as it inherits the degrees from each factor. The multiprojective space defined by  $R$  is the product  $(\mathbb{P}_{\mathbb{C}}^1)^3$  (see §2.1.1, in particular Example 2.1.10). Moreover, we let  $\mathbb{C}[\mathbf{y}] = \mathbb{C}[y_0, y_1, y_2, y_3]$  be the standard  $\mathbb{Z}$ -graded homogeneous coordinate ring of  $\mathbb{P}_{\mathbb{C}}^3$ .

**Definition 1.2.1.** A *trilinear rational map* in dimension three is a rational map

$$\begin{aligned} \phi : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^3 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (f_0 : f_1 : f_2 : f_3) \end{aligned} \quad (1.5)$$

where  $f_n = f_n(s_0, s_1, t_0, t_1, u_0, u_1)$  is homogeneous of degree  $(1, 1, 1)$  for each  $0 \leq n \leq 3$ , i.e. linear and homogeneous with respect to the three pairs of variables, and  $\gcd(f_0, f_1, f_2, f_3) = 1$ . Additionally:

- The *base ideal* of  $\phi$  is the homogeneous ideal  $B = (f_0, f_1, f_2, f_3)$  in  $R$
- The subscheme  $Z$  of  $X$  defined by  $B$  is the *base locus* of  $\phi$ .

If  $\phi$  is birational, the inverse rational map has the form

$$\begin{aligned} \phi^{-1} : \mathbb{P}_{\mathbb{C}}^3 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \\ (y_0 : y_1 : y_2 : y_3) &\mapsto (a_0 : a_1) \times (b_0 : b_1) \times (c_0 : c_1), \end{aligned} \quad (1.6)$$

where the  $a_i = a_i(\mathbf{y})$  (resp.  $b_j = b_j(\mathbf{y})$  and  $c_k = c_k(\mathbf{y})$ ) are homogeneous of the same degree for  $i = 0, 1$  (resp.  $j = 0, 1$  and  $k = 0, 1$ ). Furthermore, without loss of generality we can assume that  $\gcd(a_0, a_1) = \gcd(b_0, b_1) = \gcd(c_0, c_1) = 1$ . Additionally, the degree of the defining polynomials of  $\phi^{-1}$  is either one or two.

**Remark 1.2.2.** The last sentence in the previous paragraph follows from the fact that the inverse rational map on each parameter yields a linear parametrization of the parametric surfaces (see Definition 1.2.10), which are either planes or quadrics (see Remark 3.1.4).

**Definition 1.2.3** (Type of  $\phi$ ). If  $\phi$  is birational, the *type* of  $\phi$  is the triple  $(\deg a_i, \deg b_j, \deg c_k)$ .



We classify trilinear birational maps according to perspectives (ii), (iii), and (iv) as discussed in §1.1.3. Specifically, our contributions focus on three aspects: studying the space of trilinear birational maps as an algebraic set, computing the complete list of base loci up to isomorphism of schemes, and classifying all the possible minimal  $\mathbb{Z}^3$ -graded free resolutions of the base ideal  $B$  as an  $R$ -module.

Our first main result describes the space of trilinear birational maps and its irreducible components. Given a vector space  $V$ , we denote by  $\text{Gr}(n, V)$  the Grassmannian of  $n$ -dimensional vector subspaces of  $V$ .

**Theorem 1.2.4** (Theorem 3.3.6, Corollary 3.4.12). *The set of trilinear birational maps is an algebraic subset of  $\text{Gr}(4, R_{(1,1,1)}) \times \mathbb{P}_{\mathbb{C}}^{15}$  with eight irreducible components, and each component consists of the birational maps of a fixed type. Moreover, all the components are unirational and:*

- the component of type  $(1, 1, 1)$  has dimension 21
- the components of type  $(1, 1, 2)$ ,  $(1, 2, 1)$ , and  $(2, 1, 1)$  have dimension 22
- the components of type  $(1, 2, 2)$ ,  $(2, 1, 2)$ , and  $(2, 2, 1)$  have dimension 23
- the component of type  $(2, 2, 2)$  has dimension 23

On the other hand, let  $\text{Aut}(X)$  stand for the group of automorphisms of a variety  $X$ . The composition of trilinear birational maps with automorphisms of the source  $(\mathbb{P}_{\mathbb{C}}^1)^3$  and target  $\mathbb{P}_{\mathbb{C}}^3$  spaces induces a group action of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) \times \text{Aut}(\mathbb{P}_{\mathbb{C}}^3)$  on the space of trilinear birational maps. In particular, we have the following.

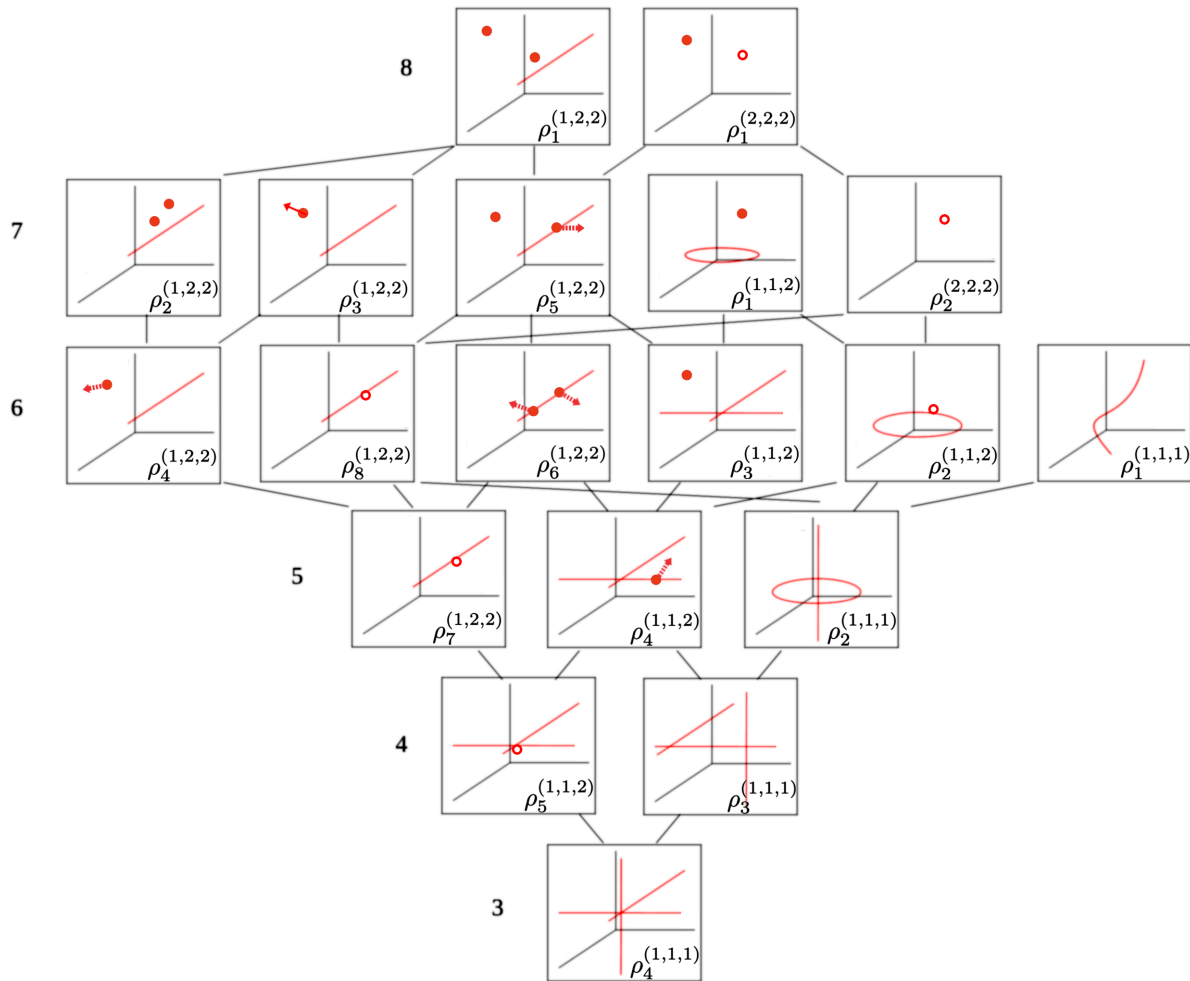
**Theorem 1.2.5** (Theorems 3.4.3, 3.4.6, 3.4.8, 3.4.10). *There are exactly 19 isomorphism classes of the base loci of trilinear birational maps, depicted in Figure 1.3. Furthermore, there is a one-to-one correspondence between these classes and the orbits of the group action of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) \times \text{Aut}(\mathbb{P}_{\mathbb{C}}^3)$  on the space of trilinear birational maps.*

Regarding the algebraic relations of the defining polynomials, the structure of the syzygy modules of the base ideal  $B$  is determined by the type of  $\phi$ . More explicitly, there is a one-to-one correspondence between the type of  $\phi$  and the shape of its minimal  $\mathbb{Z}^3$ -graded free resolution.

**Theorem 1.2.6** (Theorems 3.4.3, 3.4.6, 3.4.8, 3.4.10). *Let  $\phi$  be a dominant trilinear rational map. We have the following:*

- $\phi$  is birational of type  $(1, 1, 1)$  if and only if its minimal  $\mathbb{Z}^3$ -graded free resolution has the Hilbert-Burch shape

$$\begin{array}{c}
 R(-2, -1, -1) \\
 \oplus \\
 0 \rightarrow R(-1, -2, -1) \rightarrow R(-1, -1, -1)^4 \xrightarrow{(f_0 \ f_1 \ f_2 \ f_3)} B \rightarrow 0 \\
 \oplus \\
 R(-1, -1, -2)
 \end{array}$$



**Figure 1.3.:** the 19 isomorphism classes of the base loci of trilinear birational maps, and their degenerations. Equivalently, these can be regarded as the orbits of the group action of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3) \times \text{Aut}(\mathbb{P}_{\mathbb{C}}^3)$ . The number on the left-hand side is the dimension of the corresponding orbit. An orbit degenerates into another if it has larger dimension and the two are connected by a segment. The labels for each class are specified in §3.4, and the meaning of the symbols in §3.4.6.

- $\phi$  is birational of type  $(1, 1, 2)$  if and only its minimal  $\mathbb{Z}^3$ -graded free resolution has the shape

$$\begin{array}{ccccccc}
& & & R(-2, -1, -1) & & & \\
& & & \oplus & & & \\
& & & R(-1, -2, -1) & & & \\
0 \rightarrow & R(-2, -2, -2) & \rightarrow & \oplus & \rightarrow & R(-1, -1, -1)^4 \xrightarrow{(f_0 \ f_1 \ f_2 \ f_3)} & B \rightarrow 0 \\
& & & R(-2, -1, -2) & & & \\
& & & \oplus & & & \\
& & & R(-1, -2, -2) & & & 
\end{array}$$

- $\phi$  is birational of type  $(1, 2, 2)$  if and only if its minimal  $\mathbb{Z}^3$ -graded free resolution has the shape

$$\begin{array}{ccccccc}
& & & R(-2, -1, -1) & & & \\
& & & \oplus & & & \\
& & & R(-2, -2, -1) & & & \\
0 \rightarrow & R(-2, -2, -2)^2 & \rightarrow & \oplus & \rightarrow & R(-1, -1, -1)^4 \xrightarrow{(f_0 \ f_1 \ f_2 \ f_3)} & B \rightarrow 0 \\
& & & R(-2, -1, -2) & & & \\
& & & \oplus & & & \\
& & & R(-1, -2, -2)^2 & & & 
\end{array}$$

- $\phi$  is birational of type  $(2, 2, 2)$  if and only if its minimal  $\mathbb{Z}^3$ -graded free resolution has the shape

$$\begin{array}{ccccccc}
& & & R(-2, -2, -1)^2 & & & \\
& & & \oplus & & & \\
0 \rightarrow & R(-2, -2, -2)^3 & \rightarrow & R(-2, -1, -2)^2 & \rightarrow & R(-1, -1, -1)^4 \xrightarrow{(f_0 \ f_1 \ f_2 \ f_3)} & B \rightarrow 0 \\
& & & \oplus & & & \\
& & & R(-1, -2, -2)^2 & & & 
\end{array}$$

and some extra splitting conditions are satisfied (see Condition 3.5.7).

### 1.2.2. Birationality criteria for multilinear rational maps

From our classification, we derive syzygy-based criteria for determining the birationality of trilinear rational maps, forming the foundational basis of our constructive results. In contrast to 2D parametrizations, 3D parametrizations introduce a greater degree of complexity due to the potential variation in the degrees of the defining polynomials of the inverse. Consequently, we present four distinct birationality criteria, each corresponding to the types  $(1, 1, 1)$ ,  $(1, 1, 2)$  (and permutations),  $(1, 2, 2)$  (and permutations), and  $(2, 2, 2)$ . In essence, our criteria rely exclusively on assessing the existence of certain first syzygies at some particular degrees. In order to simplify the statement, in  $\mathbb{Z}^3$  we set

$$\mathbf{e}_1 = (1, 0, 0), \quad \mathbf{e}_2 = (0, 1, 0), \quad \mathbf{e}_3 = (0, 0, 1).$$

**Theorem 1.2.7** (Theorem 3.5.1). *Let  $\phi$  be dominant, and let  $i, j, k$  be indices such that  $\{i, j, k\} = \{1, 2, 3\}$ . Then,  $\phi$  is birational if and only if one of the following conditions holds:*

- (i)  $\mathbf{f}$  has syzygies of degrees  $\mathbf{e}_1, \mathbf{e}_2$ , and  $\mathbf{e}_3$ . In this case,  $\phi$  has type  $(1, 1, 1)$ .

- (ii)  $\mathbf{f}$  has syzygies of degrees  $\mathbf{e}_i$  and  $\mathbf{e}_j$ , but not  $\mathbf{e}_k$ . In this case,  $\phi$  has type  $(1, 1, 1) + \mathbf{e}_k$ .
- (iii)  $\mathbf{f}$  has a syzygy of degree  $\mathbf{e}_i$ , but neither  $\mathbf{e}_j$  nor  $\mathbf{e}_k$ . Moreover,  $\mathbf{f}$  has a syzygy of degree either  $\mathbf{e}_i + \mathbf{e}_j$  or  $\mathbf{e}_i + \mathbf{e}_k$ , independent from the first one. In this case,  $\phi$  has type  $(1, 1, 1) + \mathbf{e}_j + \mathbf{e}_k$ .
- (iv)  $\mathbf{f}$  does not have syzygies of degree  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , or  $\mathbf{e}_3$ . Moreover,  $\mathbf{f}$  has two independent syzygies in each of the degrees  $\mathbf{e}_1 + \mathbf{e}_2$ ,  $\mathbf{e}_1 + \mathbf{e}_3$ , and  $\mathbf{e}_2 + \mathbf{e}_3$  satisfying the splitting property in Condition 3.5.7. In this case,  $\phi$  has type  $(2, 2, 2)$ .

We illustrate the usefulness of our criteria to test the birationality of a trilinear rational map.

**Example 1.2.8.** Consider the trilinear rational map

$$\begin{aligned} \phi : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^3 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (f_0 : f_1 : f_2 : f_3) = (s_0 t_1 u_1 - s_1 t_0 u_1 + 2 s_0 t_0 u_1 + 3 s_1 t_1 u_0 : \\ &3 s_1 t_0 u_0 - s_0 t_0 u_1 + s_1 t_1 u_1 : \\ &s_1 t_1 u_0 - s_0 t_0 u_1 - 2 s_1 t_0 u_0 : s_1 t_0 u_0 + s_0 t_1 u_1) \end{aligned}$$

We can compute the degrees of a set of generators of the module of first syzygies of  $\mathbf{f} = (f_0, f_1, f_2, f_3)$  with the help of *Macaulay2* [101].

```
i1 : R = QQ[s_0,s_1,t_0,t_1,u_0,u_1,
      Degrees=>{{1,0,0},{1,0,0},{0,1,0},{0,1,0},{0,0,1},{0,0,1}}]
o1 = R
o1 : PolynomialRing
i2 : f_0 = s_0*t_1*u_1 - s_1*t_0*u_1 + 2*s_0*t_0*u_1 + 3*s_1*t_1*u_0;
      -- ...initialization of entries
i5 : f_3 = s_1*t_0*u_0 + s_0*t_1*u_1;
i6 : f = matrix{{f_0,f_1,f_2,f_3}};
      1      4
o6 : Matrix R <--- R
i7 : M = syz f;
      4      17
o7 : Matrix R <--- R
i8 : source M
      17
o8 = R
o8 : R-module, free, degrees {{1, 2, 2}, {2, 2, 1}, 3:{2, 2, 2},
      {2, 3, 1}, {2, 1, 3}, {1, 2, 3}, {2, 1, 3},
      2:{1, 3, 2}, {2, 2, 2}, 2:{3, 1, 2},
      {1, 4, 1}, {2, 3, 1}, {3, 2, 1}}
```

We obtain a complex of graded  $R$ -modules

$$R^{17} \rightarrow R(-1, -1, -1) \xrightarrow{\begin{pmatrix} f_0 & f_1 & f_2 & f_3 \end{pmatrix}} R \rightarrow 0$$

where the Betti numbers of  $R^{17}$  are indicated in the output o8. In particular, we find no linear syzygies and only two bilinear syzygies. Hence,  $\phi$  is not birational by Theorem 1.2.7. Now, consider

the trilinear rational map

$$\begin{aligned} \phi : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^3 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (f_0 : f_1 : f_2 : f_3) = (s_0 t_1 u_1 - s_1 t_0 u_1 + 2 s_0 t_0 u_1 + 3 s_1 t_1 u_0 : \\ &3 s_1 t_0 u_0 - s_0 t_0 u_1 : s_1 t_1 u_0 - s_0 t_0 u_1 - 2 s_1 t_0 u_0 : \\ &s_1 t_0 u_0 + s_0 t_1 u_1 - s_1 t_0 u_1) \end{aligned}$$

where we have updated the entries  $f_1$  and  $f_3$ . Now, *Macaulay2* outputs the following.

```
i9 : f_1 = 3*s_1*t_0*u_0 - s_0*t_0*u_1;
i10 : f_3 = s_1*t_0*u_0 + s_0*t_1*u_1 - s_1*t_0*u_1;
i11 : f = matrix{{f_0,f_1,f_2,f_3}};
      1      4
o11 : Matrix R <--- R
i12 : M = syz f;
      4      5
o12 : Matrix R <--- R
i13 : source M
      5
o13 = R
o13 : R-module, free, degrees {{1, 2, 1}, {2, 1, 2}, {1, 2, 2},
      {2, 1, 2}, {2, 2, 1}}
```

This time, we obtain a complex of graded  $R$ -modules

$$R^5 \rightarrow R(-1, -1, -1) \xrightarrow{\begin{pmatrix} f_0 & f_1 & f_2 & f_3 \end{pmatrix}} R \rightarrow 0$$

where the Betti numbers of  $R^5$  are indicated in the output o13. In particular, we find a linear syzygy of degree  $(0, 1, 0)$  and bilinear syzygies of degrees  $(1, 1, 0)$  and  $(0, 1, 1)$ . By Theorem 1.2.7, it follows that  $\phi$  is birational of type  $(2, 1, 2)$ .

### 1.2.3. Construction and manipulation of multilinear birational transformations

In this thesis, we present effective methods for constructing and manipulating birational tensor-product maps  $\phi : \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \dashrightarrow \mathbb{P}_{\mathbb{R}}^3$  with trilinear entries, i.e. defined by polynomials of degree  $(1, 1, 1)$ . These are the first methods for the construction of nonlinear birational maps in 3D with sufficient flexibility for geometric modeling and design [183, see §7]. Our approach aligns with the principles of (ii) and (iii) stated in §1.1.1. Specifically, we rely on our birationality criteria and compute suitable weights for an adequately constrained net of control points.

**Notation 1.2.9.** To simplify the notation, we write

$$b_0^1(s_0, s_1) = b_0(s_0, s_1) = s_0 - s_1 \quad , \quad b_1^1(s_0, s_1) = b_1(s_0, s_1) = s_1 \quad ,$$

i.e. we drop the superindex for the linear Bernstein polynomials.

We adopt the conventional CAGD formulation, and define  $\phi$  by means of control points and weights. Given  $2 \times 2 \times 2 = 8$  control points  $\mathbf{P}_{ijk} = (1, x_{ijk}, y_{ijk}, z_{ijk})$  in  $\mathbb{R}^4$  and nonnegative weights

$w_{ijk}$  for each  $0 \leq i, j, k \leq 1$ , we define

$$\mathbf{f} = (f_0, f_1, f_2, f_3) := \sum_{0 \leq i, j, k \leq 1} w_{ijk} \mathbf{P}_{ijk} b_i(s_0, s_1) b_j(t_0, t_1) b_k(u_0, u_1). \quad (1.7)$$

Here,  $\mathbf{P}_{ijk}$  is identified with the projective point  $(1 : x_{ijk} : y_{ijk} : z_{ijk})$  in  $\mathbb{P}_{\mathbb{R}}^3 = \text{Proj}(\mathbb{R}[\mathbf{y}]) = \text{Proj}(\mathbb{R}[y_0, y_1, y_2, y_3])$ . Specifically, in the affine chart  $\mathbb{A}_{\mathbb{R}}^3 \subset \mathbb{P}_{\mathbb{R}}^3$  defined by  $y_0 \neq 0$ ,  $\mathbf{P}_{ijk}$  coincides with  $(x_{ijk}, y_{ijk}, z_{ijk})$ . The control points are the images of the vertices of the unit cube  $[0, 1]^3$  in  $\mathbb{A}_{\mathbb{R}}^3$ . More explicitly, the supporting planes of the unit cube in the affine chart  $\mathbb{A}_{\mathbb{R}}^3 \subset (\mathbb{P}_{\mathbb{R}}^1)^3$  defined by  $s_0 \neq 0$ ,  $t_0 \neq 0$ , and  $u_0 \neq 0$  are

$$b_0(s_0, s_1) = 0, \quad b_1(s_0, s_1) = 0, \quad b_0(t_0, t_1) = 0, \quad b_1(t_0, t_1) = 0, \quad b_0(u_0, u_1) = 0, \quad b_1(u_0, u_1) = 0.$$

Therefore, the vertices of the unit cube in  $(\mathbb{P}_{\mathbb{R}}^1)^3$  correspond to the points

$$(1 : i) \times (1 : j) \times (1 : k)$$

for each  $0 \leq i, j, k \leq 1$ . Furthermore, if all the weights are nonnegative, the image of the unit cube lies within the convex hull of the affine control points. This allows us to interpret  $\phi$  as a deformation of the unit cube, as illustrated in **Figure 1.2**.

The concepts of parametric lines and surfaces help to describe the geometry of the rational map. Similarly, the boundary lines and surfaces are very useful. Unless explicitly mentioned differently, we always work with the Zariski topology.

**Definition 1.2.10** (Parametric surfaces and lines).

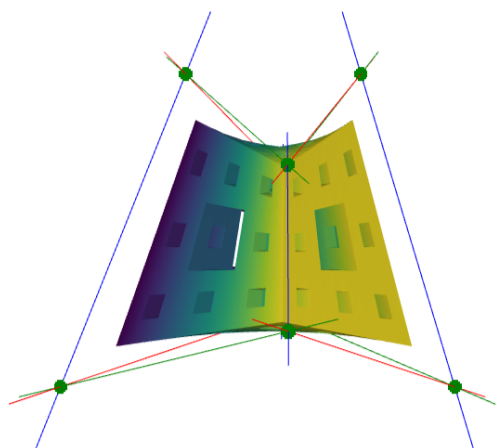
- (i) A *parametric surface* is the closure of  $\phi(\mathbb{P}^1 \times \mathbb{P}^1)$  for the specialization of one parameter to a general point in  $\mathbb{P}^1$ . If the parameter specialized is  $(s_0 : s_1)$  (resp.  $(t_0 : t_1)$  and  $(u_0 : u_1)$ ), we refer to the parametric surface as *s-surface* (resp. *t-* and *u-surface*).
- (ii) A *parametric line* is  $\phi(\mathbb{P}^1)$  for the specialization of two parameters to general points in  $\mathbb{P}^1$ . If the parameter that is not specialized is  $(s_0 : s_1)$  (resp.  $(t_0 : t_1)$  and  $(u_0 : u_1)$ ), we refer to the parametric line as *s-line* (resp. *t-* and *u-line*).

**Definition 1.2.11** (Boundary surfaces and lines). The *boundary surfaces* are the parametric surfaces defined by the supporting planes of the unit cube. Specifically, for each  $0 \leq i, j, k \leq 1$  they are:

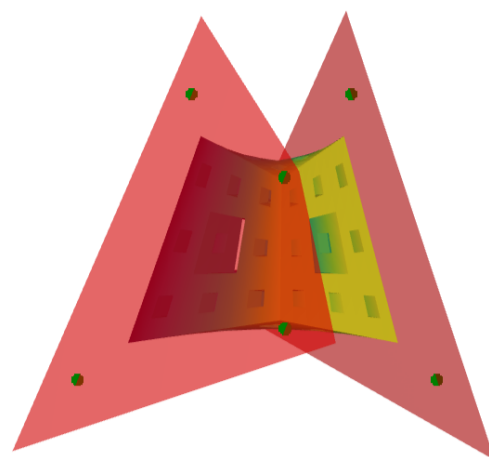
- (i)  $\Sigma_i$  defined by  $(s_0 : s_1) = (1 : i)$
- (ii)  $T_j$  defined by  $(t_0 : t_1) = (1 : j)$
- (iii)  $Y_k$  defined by  $(u_0 : u_1) = (1 : k)$

The *boundary lines* are the parametric lines defined by the supporting lines of the edges of the unit cube. Specifically, they are:

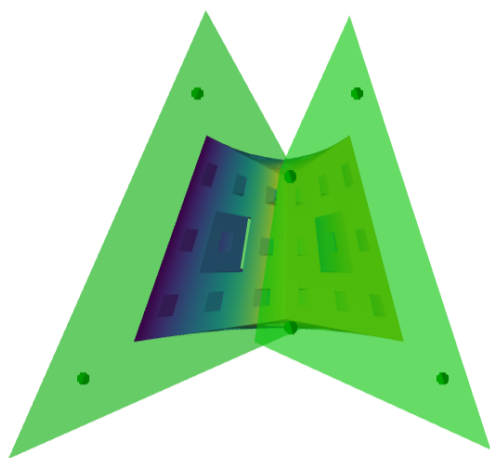
- (i)  $s_{jk}$  defined by  $(t_0 : t_1) = (1 : j)$  and  $(u_0 : u_1) = (1 : k)$ , or equivalently  $s_{jk} = \overline{\mathbf{P}_{0jk} \mathbf{P}_{1jk}}$
- (ii)  $t_{ik}$  defined by  $(s_0 : s_1) = (1 : i)$  and  $(u_0 : u_1) = (1 : k)$ , or equivalently  $t_{ik} = \overline{\mathbf{P}_{i0k} \mathbf{P}_{i1k}}$
- (iii)  $u_{ij}$  defined by  $(s_0 : s_1) = (1 : i)$  and  $(t_0 : t_1) = (1 : j)$ , or equivalently  $u_{ij} = \overline{\mathbf{P}_{ij0} \mathbf{P}_{ij1}}$



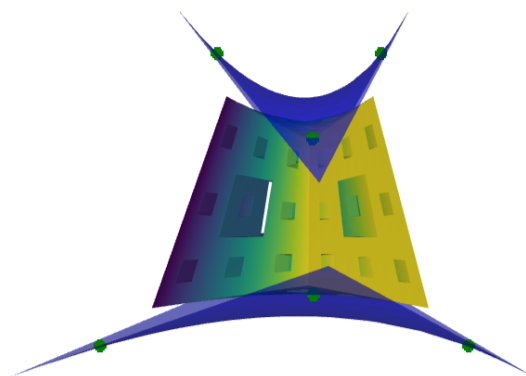
(a) The boundary lines  $s_{jk}$  (red),  $t_{ik}$  (green), and  $u_{ij}$  (blue), for each  $0 \leq i, j, k \leq 1$



(b) The boundary surfaces  $\Sigma_0$  and  $\Sigma_1$



(c) The boundary surfaces  $T_0$  and  $T_1$



(d) The boundary surfaces  $Y_0$  and  $Y_1$

Figure 1.4.: The boundary lines and surfaces associated to a trilinear rational map.

To gain some geometric intuition, **Figure 1.4** illustrates the boundary lines and the boundary surfaces associated to a trilinear rational map.

In the context of CAGD, the challenge lies in reconciling our syzygy-based birationality criteria with the flexible manipulation of the control points and weights required by applications. As we shall prove, the existence of the necessary syzygies for birationality imposes geometric constraints on the control points, that are formalized in the following definition.

**Definition 1.2.12** (Definitions 4.3.1, 4.4.1, 4.5.1). Let  $\phi$  be a dominant trilinear rational map, satisfying a nondegeneracy assumption (see Property 1). We define the following classes:

- (i)  $\phi$  is *hexahedral* if all the boundary surfaces are planes
- (ii)  $\phi$  is *pyramidal* if all the boundary lines in one parameter meet at a point, and the boundary surfaces in the same parameter are smooth quadrics
- (iii)  $\phi$  is *scaffold* if the boundary surfaces in one parameter are planes that intersect at a line  $\ell$ , the boundary surfaces in the other two parameters are smooth quadrics, and there are two other lines that intersect  $\ell$  and all the parametric lines in the same parameter

**Remark 1.2.13.** The properties of being hexahedral, pyramidal, or ladder depend exclusively on the control points.

Each of the previous classes are depicted in **Figure 1.5**. Specifically, by Theorem 1.2.4 the space of trilinear birational maps has multiple irreducible components. The components of type  $(1, 1, 1)$ ,  $(1, 1, 2)$  with permutations, and  $(1, 2, 2)$  with permutations, respectively correspond to the previous geometric constraints, as explained by the following result.

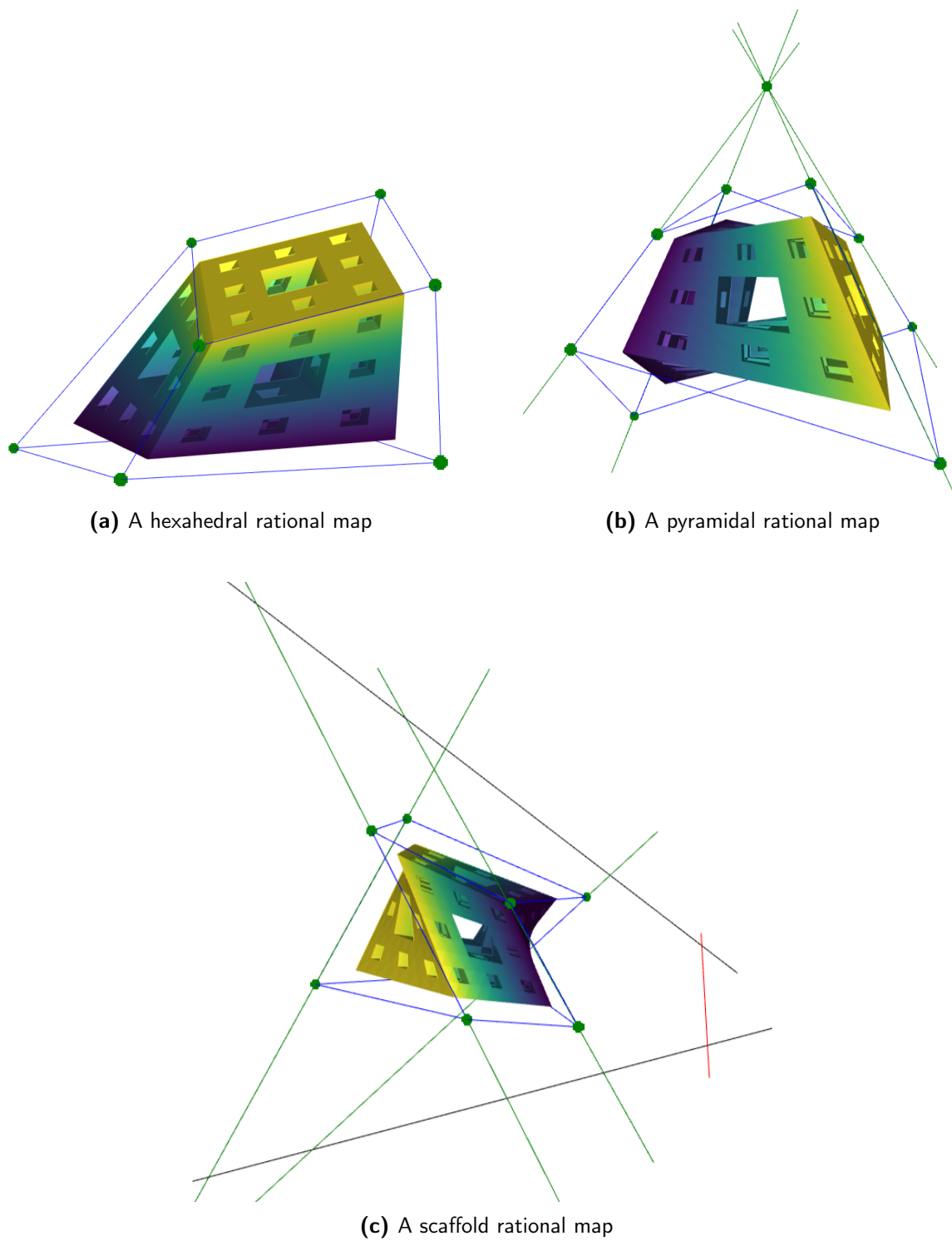
**Theorem 1.2.14.** Let  $\phi$  be a general trilinear birational map, satisfying a nondegeneracy assumption (see Property 1). We have the following:

- (i) If  $\phi$  has type  $(1, 1, 1)$  then  $\phi$  is hexahedral
- (ii) If  $\phi$  has type either  $(1, 1, 2)$ ,  $(1, 2, 1)$ , or  $(2, 1, 1)$  then  $\phi$  is pyramidal
- (iii) If  $\phi$  has type either  $(1, 2, 2)$ ,  $(2, 1, 2)$ , or  $(2, 2, 1)$  then  $\phi$  is scaffold

In this thesis, we do not discuss the construction of the birational maps of type  $(2, 2, 2)$ . Namely, Theorem 1.2.7 establishes that the syzygies of the defining polynomials of a birational map of type  $(2, 2, 2)$  require to satisfy additional splitting conditions. Since our methods are syzygy-based, these extra conditions make their application more complicated. Additionally, the construction of birational maps of type  $(1, 2, 2)$  is technically challenging (see §4.5), and the geometric constraints on the control points are complicated due to the presence of quadric parametric surfaces for two of the parameters. Since all the parametric surfaces of  $(2, 2, 2)$  birational maps are quadrics, we expect that they are not very suitable for CAGD.

The syzygy-based birationality criteria of Theorem 1.2.7 offers an effective method to test birationality. However, its application in the context of constructing trilinear birational maps in CAGD is not straightforward, and introduces further challenges. To address them, we establish a connection between birationality and tensor rank. Specifically, we prove that the simultaneous existence of the necessary syzygies for birationality occurs if and only if a certain tensor with a  $2 \times 2 \times 2$  format has rank one. This connection opens the door to the application of an extensive





**Figure 1.5.:** the distinct geometric constraints on the control points necessary for the birationality of type  $(1, 1, 1)$ ,  $(1, 1, 2)$ ,  $(1, 2, 2)$  and permutations.

body of work related to tensor rank and fixed-rank approximations to the study of birational maps. An important tool is the Canonical Polyadic Decomposition (CPD) (e.g. [76, 126, 118, 136]), which enables us to efficiently compute rank-one approximations. Furthermore, it provides useful formulas for computing weights that ensure birationality, explicitly revealing the underlying geometric structure of the Segre variety associated with  $(\mathbb{P}_{\mathbb{R}}^1)^3$ .

In particular, we have the following birationality criterion.

**Theorem 1.2.15.** (Theorems 4.3.6, 4.4.7, 4.5.7) *Let  $\phi$  be either hexahedral, pyramidal, or scaffold. Then,  $\phi$  is birational of the corresponding type (see Theorem 1.2.14) if and only if the  $2 \times 2 \times 2$  tensor*

$$W = \left( \frac{w_{ijk}}{\Delta_{ijk}} \right)_{0 \leq i, j, k \leq 1} \quad (1.8)$$

has rank one, where  $\Delta_{ijk}$  is a rational function on the control points (see Chapter 4).

**Corollary 1.2.16.** *Let  $\phi$  be either hexahedral, pyramidal, or scaffold. Then,  $\phi$  is birational of the corresponding type if and only if*

$$w_{ijk} = \alpha_i \beta_j \gamma_k \Delta_{ijk} \quad (1.9)$$

for some  $(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1)$  in  $(\mathbb{P}_{\mathbb{C}}^1)^3$ .

Theorem 1.2.15 is yet another birationality criterion, which is now based on tensor rank. Notably, it stands out for its effectiveness in the construction of birational maps, since Corollary 1.2.16 provides formulas for the weights that yield birationality.

Furthermore, Theorem 1.2.15 offers a means to measure the distance of  $\phi$  to a component of birational maps, assuming one of the geometric configurations in Definition 1.2.12. Specifically, the distance to birationality can be quantified as the distance of  $W$  to the locus of  $2 \times 2 \times 2$  tensors of rank one. Additionally, a closest birational map to  $\phi$  can be computed by finding an optimal rank-one approximation of  $W$ , and retrieving the weights from (1.9)

The following example illustrates how our results can be applied in practice.

**Example 1.2.17** (Birational approximation of a pyramidal rational map). Consider the trilinear rational map  $\phi$  defined by the control points

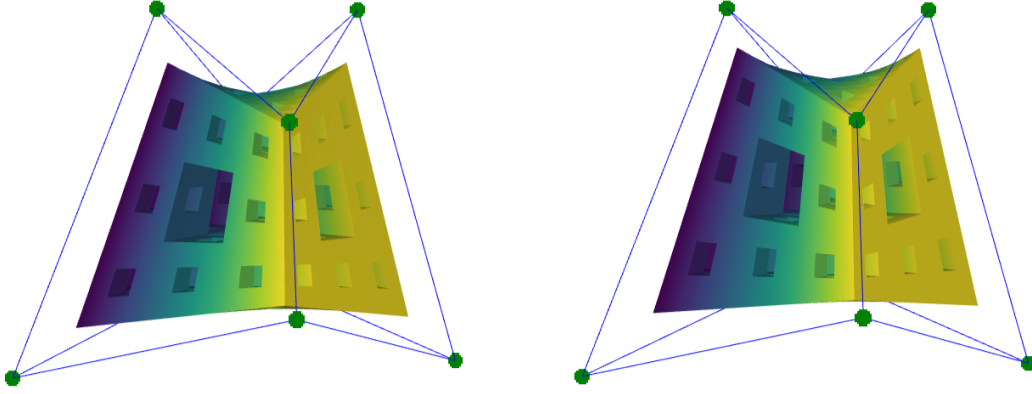
$$\begin{aligned} \mathbf{P}_{000} &= \left(1 \ \frac{-8}{5} \ 0 \ 1\right), \quad \mathbf{P}_{100} = \left(1 \ 0 \ -\frac{9}{5} \ \frac{1}{2}\right), \quad \mathbf{P}_{010} = \left(1 \ 0 \ \frac{27}{20} \ \frac{1}{2}\right), \quad \mathbf{P}_{110} = \left(1 \ \frac{4}{5} \ 0 \ 1\right), \\ \mathbf{P}_{001} &= \left(1 \ -\frac{11}{10} \ 0 \ \frac{9}{4}\right), \quad \mathbf{P}_{101} = \left(1 \ 0 \ -\frac{4}{5} \ 3\right), \quad \mathbf{P}_{011} = \left(1 \ 0 \ \frac{3}{5} \ 3\right), \quad \mathbf{P}_{111} = \left(1 \ \frac{11}{20} \ 0 \ \frac{9}{4}\right), \end{aligned}$$

and  $w_{ijk} = 1$  for every  $0 \leq i, j, k \leq 1$ . Since the boundary  $u$ -lines, i.e.  $u_{ij} = \overline{\mathbf{P}_{ij0} \mathbf{P}_{ij1}}$  for each  $0 \leq i, j \leq 1$ , meet at the point  $\mathbf{V} = (1 \ 0 \ 0 \ 5)$  it follows that  $\phi$  is pyramidal. While the rational map  $\phi$  is not birational, we can render it birational by computing new weights according to Corollary 1.2.16. In this case, the computed values for the  $\Delta_{ijk}$ 's (see §4.4) are

$$\begin{aligned} \Delta_{000} &= \frac{5}{6}, \quad \Delta_{100} = \frac{20}{21}, \quad \Delta_{010} = \frac{80}{63}, \quad \Delta_{110} = \frac{5}{3}, \\ \Delta_{001} &= \frac{40}{33}, \quad \Delta_{101} = \frac{15}{7}, \quad \Delta_{011} = \frac{20}{7}, \quad \Delta_{111} = \frac{80}{33}. \end{aligned}$$

Subsequently, for any choice of  $(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1) \in (\mathbb{P}_{\mathbb{R}}^1)^3$  the computation

$$w_{ijk} = \alpha_i \beta_j \gamma_k \Delta_{ijk}$$



**Figure 1.6.:** Deformations of a (truncated) Menger sponge using the trilinear birational map presented in Example 1.2.17. In the left image, we use uniform weights, specifically  $w_{ijk} = 1$  for every  $0 \leq i, j, k \leq 1$ . The right image showcases the effect of utilizing the computed birational weights. These weight adjustments subtly influence the deformation, while simultaneously ensuring the existence of an inverse transformation.

ensures that  $\phi$  is birational. However, the new birational map  $\phi$  might differ significantly from the original  $\phi$  if we make a bad choice of  $(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1)$ . Hence, we aim to compute birational weights that are as close to 1 as possible. To achieve this, we can compute a rank-one CP decomposition (see §2.2.2) of the tensor  $W$  in (1.8). Using the *Python* library *TensorFox* [32], we compute the rational approximation

$$W_{\text{approx}} = (\alpha_0 \ \alpha_1) \otimes (\beta_0 \ \beta_1) \otimes (\gamma_0 \ \gamma_1) = (0.95 \ 0.91) \otimes (1.06 \ 0.78) \otimes (1.08 \ 0.75) .$$

Specifically, this leads to the exact rational weights

$$w_{ijk} = \alpha_i \beta_j \gamma_k \Delta_{ijk} \tag{1.10}$$

for each  $0 \leq i, j, k \leq 1$ . Additionally, if we introduce the Frobenius norm (see Definition 2.2.20) of a real tensor  $A = (a_{ijk})_{0 \leq i, j, k \leq 1}$  with  $2 \times 2 \times 2$  format as

$$\|A\|^2 = \sum_{0 \leq i, j, k \leq 1} a_{ijk}^2 ,$$

we can measure the relative distance of  $W$  to the locus of birational maps as

$$\frac{\|W - W_{\text{approx}}\|}{\|W\|} \sim 0.2594 .$$

In **Figure 1.6**, we present a comparison between the deformations resulting from the original rational map, with uniform weights  $w_{ijk} = 1$  for every  $0 \leq i, j, k \leq 1$ , and its birational approximation, utilizing the weights specified in (1.10).

While birational maps indeed ensure injectivity on an open set, one of the most compelling applications is the exact and fast computation of preimages avoiding numerical solving methods. This computation relies on the inverse rational map. Therefore, having explicit formulas for  $\phi^{-1}$  is

imperative. Birational maps of the same type are algebraically equivalent, and the inverses obey the same formulas. However, the behavior between distinct types presents significant differences, and different formulas are required. To simplify our statements, we establish the following notation for the defining equations of the boundary surfaces, that is maintained throughout Part I.

**Notation 1.2.18.** Let  $0 \leq i, j, k \leq 1$ . If the boundary surface  $\Sigma_i$  is a plane, it is defined by the equation (we consider row vectors)

$$\boldsymbol{\sigma}_i(\mathbf{y}) := \boldsymbol{\sigma}_i \cdot \mathbf{y}^T = 0 \quad (1.11)$$

for some vector  $\boldsymbol{\sigma}_i$  in  $\mathbb{R}^4$ . On the other hand, if  $\Sigma_i$  is a quadric, it is defined by

$$\boldsymbol{\sigma}_i(\mathbf{y}) := \mathbf{y} \cdot \boldsymbol{\sigma}_i \cdot \mathbf{y}^T = 0 \quad (1.12)$$

for some symmetric  $4 \times 4$  matrix  $\boldsymbol{\sigma}_i$  in  $\mathbb{R}^{4 \times 4}$ . Similarly, we denote by  $\boldsymbol{\tau}_j$  and  $\mathbf{v}_k$  the vectors or matrices defining the equations of the boundary surfaces  $T_j$  and  $Y_k$ .

**Remark 1.2.19.** Clearly, the vectors and matrices  $\boldsymbol{\sigma}_i$ ,  $\boldsymbol{\tau}_j$ , and  $\mathbf{v}_k$  are defined up to nonzero scalar.

The following result provides explicit formulas for the inverse of a trilinear birational map. By definition, the degrees of the defining polynomials for  $\phi^{-1}$  are determined by the type of  $\phi$ . This statement is a general version, encompassing hexahedral, pyramidal, and scaffold rational maps. For more specific statements, tailored to each of the distinct classes, we refer the reader to Chapter 4.

**Theorem 1.2.20** (Theorems 4.3.13, 4.4.19, 4.5.15). *Let  $\phi$  be either a hexahedral, pyramidal, or scaffold birational map, with weights as in Corollary 1.2.16. Then, maintaining the notation of (1.6), the inverse  $\phi^{-1}$  is given by*

$$(a_0 : a_1) = (\alpha_0 \lambda_0 \boldsymbol{\sigma}_0(\mathbf{y}) - \alpha_1 \lambda_1 \boldsymbol{\sigma}_1(\mathbf{y}) : \alpha_0 \lambda_0 \boldsymbol{\sigma}_0(\mathbf{y})) ,$$

$$(b_0 : b_1) = (\beta_0 \mu_0 \boldsymbol{\tau}_0(\mathbf{y}) - \beta_1 \mu_1 \boldsymbol{\tau}_1(\mathbf{y}) : \beta_0 \mu_0 \boldsymbol{\tau}_0(\mathbf{y})) ,$$

$$(c_0 : c_1) = (\gamma_0 \nu_0 \mathbf{v}_0(\mathbf{y}) - \gamma_1 \nu_1 \mathbf{v}_1(\mathbf{y}) : \gamma_0 \nu_0 \mathbf{v}_0(\mathbf{y}))$$

for some  $(\lambda_0 : \lambda_1)$ ,  $(\mu_0 : \mu_1)$ , and  $(\nu_0 : \nu_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$  which are rational functions on the coordinates of the control points (see Chapter 4). Furthermore, we also provide explicitly the defining equations of the irreducible components of the base loci of  $\phi$  and  $\phi^{-1}$  in Chapter 4, and we describe the blow-ups and contractions.

The following example shows the explicit computation of the inverse of the trilinear birational map of Example 1.2.17.

**Example 1.2.21.** We continue with Example 1.2.17, using the computed birational weights. In this case, the vectors and matrices defining the boundary surfaces are

$$\boldsymbol{\sigma}_0 = \left( 5 \quad -\frac{5}{2} \quad \frac{10}{3} \quad 1 \right) , \quad \boldsymbol{\sigma}_1 = \left( 4 \quad -4 \quad 2 \quad -\frac{4}{5} \right) ,$$

$$\boldsymbol{\tau}_0 = \left( 5 \quad \frac{5}{2} \quad \frac{5}{2} \quad -1 \right) , \quad \boldsymbol{\tau}_1 = \left( -3 \quad 3 \quad 2 \quad \frac{3}{5} \right) ,$$

$$\mathbf{v}_0 = \begin{pmatrix} 390 & -30 & -\frac{45}{2} & -294 \\ -30 & 75 & 0 & 60 \\ -\frac{45}{2} & 0 & -50 & \frac{45}{2} \\ -294 & 60 & \frac{45}{2} & 102 \end{pmatrix} , \quad \mathbf{v}_1 = \begin{pmatrix} 1530 & -\frac{495}{2} & -45 & -438 \\ -\frac{495}{2} & -225 & 0 & \frac{165}{2} \\ -45 & 0 & 150 & 20 \\ -438 & \frac{165}{2} & 20 & 114 \end{pmatrix} .$$

Furthermore, using the formulas of Chapter 4, we find

$$(\lambda_0 : \lambda_1) \times (\mu_0 : \mu_1) \times (\nu_0 : \nu_1) = (4 : -5) \times (3 : -5) \times (11 : -18) .$$

In particular, the defining polynomials of  $\phi^{-1}$  yield linear parametrizations of the pencils of  $s$ -,  $t$ -, and  $u$ -surfaces. More explicitly, we find the relations

$$\begin{aligned} \boldsymbol{\sigma}(s_0, s_1) \cdot \mathbf{f}^T &= (\alpha_0 \lambda_0 \boldsymbol{\sigma}_0 b_0(s_0, s_1) + \alpha_1 \lambda_1 \boldsymbol{\sigma}_1 b_1(s_0, s_1)) \cdot \mathbf{f}^T = 0 , \\ \boldsymbol{\tau}(t_0, t_1) \cdot \mathbf{f}^T &= (\beta_0 \mu_0 \boldsymbol{\tau}_0 b_0(t_0, t_1) + \beta_1 \mu_1 \boldsymbol{\tau}_1 b_1(t_0, t_1)) \cdot \mathbf{f}^T = 0 , \\ \mathbf{f} \cdot \mathbf{u}(u_0, u_1) \cdot \mathbf{f}^T &= \mathbf{f} \cdot (\gamma_0 \nu_0 \mathbf{u}_0 b_0(u_0, u_1) + \gamma_1 \nu_1 \mathbf{u}_1 b_1(u_0, u_1)) \cdot \mathbf{f}^T = 0 . \end{aligned}$$

In the context of higher-dimensional rational maps, we are able to extend the tensor rank criterion for birationality of Theorem 1.2.15 to the simplest class within the family of multilinear rational maps in arbitrary dimension: multilinear birational maps with a multilinear inverse.

**Definition 1.2.22** (Multilinear rational map and multilinear inverse). A *multilinear rational map* in dimension  $n$  is a rational map

$$\begin{aligned} \phi : (\mathbb{P}_{\mathbb{C}}^1)^n &\dashrightarrow \mathbb{P}_{\mathbb{C}}^n \\ (x_{01} : x_{11}) \times \dots \times (x_{0n} : x_{1n}) &\mapsto (f_0, \dots, f_n) \end{aligned} \quad (1.13)$$

where  $f_k = f_k(x_{01}, x_{11}, \dots, x_{0n}, x_{1n})$  is multilinear for each  $0 \leq k \leq n$  and  $\gcd(f_0, f_1, \dots, f_n) = 1$ . If  $\phi$  is birational, the inverse rational map has the form

$$\begin{aligned} \phi^{-1} : \mathbb{P}_{\mathbb{C}}^n &\dashrightarrow (\mathbb{P}_{\mathbb{C}}^1)^n \\ (y_0, \dots, y_n) &\mapsto (a_{01} : a_{11}) \times \dots \times (a_{0n} : a_{1n}) \end{aligned} \quad (1.14)$$

where  $a_{0k} = a_{0k}(\mathbf{y})$  and  $a_{1k} = a_{1k}(\mathbf{y})$  are homogeneous of the same degree and coprime, for each  $1 \leq k \leq n$ . If  $a_{0k}, a_{1k}$  are linear for every  $1 \leq k \leq n$ , we say that  $\phi$  admits a *multilinear inverse*.

Once more, we define rational maps by means of control points and their associated weights. Specifically, we introduce

$$\mathbf{f} = (f_0, \dots, f_n)^T := \sum_{0 \leq i_1, \dots, i_n \leq 1} w_{i_1 \dots i_n} \mathbf{P}_{i_1 \dots i_n} b_{i_1}(x_{10}, x_{11}) \dots b_{i_n}(x_{n0}, x_{n1}) \quad (1.15)$$

where we allow  $\mathbf{P}_{i_1 \dots i_n} = (1, y_{1i_1 \dots i_n}, \dots, y_{ni_1 \dots i_n})$  to lie in  $\mathbb{C}^{n+1}$ , and the weights  $w_{i_1 \dots i_n}$  to be nonzero complex numbers, for each  $0 \leq i_1, \dots, i_n \leq 1$ . At this point, it comes as no surprise that, for  $\phi$  to admit a multilinear inverse, some geometric constraints must be imposed on the control points. Namely, we have the following definition.

**Definition 1.2.23** (Hypercubic rational map).

- For each  $1 \leq k \leq n$  and  $i_k = 0, 1$ , the *boundary hypersurface*  $X_{i_k k}$  of  $\phi$  is the closure of the image of  $\phi$  after the specialization of  $(x_{0k} : x_{1k}) = (1 : i_k)$
- A multilinear rational map  $\phi : (\mathbb{P}_{\mathbb{C}}^1)^n \dashrightarrow \mathbb{P}_{\mathbb{C}}^n$  is *n-cubic*, *hypercubic*, or simply *cubic*, if it is dominant and all the boundary hypersurfaces are hyperplanes

The following result is the extension of Theorem 1.2.15 to the class of hypercubic rational maps.

**Theorem 1.2.24.** (Theorem 5.1.5) *Let  $\phi$  be  $n$ -cubic. Then,  $\phi$  is birational with multilinear inverse if and only if the tensor with  $2^{\times n}$  format*

$$W = \left( \frac{w_{i_1 \dots i_n}}{\Delta_{i_1 \dots i_n}} \right)_{0 \leq i_1, \dots, i_n \leq 1} \quad (1.16)$$

has rank one, where  $\Delta_{i_1 \dots i_n}$  is a rational function on the coordinates of the control points.

**Corollary 1.2.25.** *Let  $\phi$  be  $n$ -cubic. Then,  $\phi$  is birational with multilinear inverse if and only if*

$$w_{i_1 \dots i_n} = \alpha_{i_1 1} \dots \alpha_{i_n n} \Delta_{i_1 \dots i_n} \quad (1.17)$$

for some  $(\alpha_{01} : \alpha_{11}) \times \dots \times (\alpha_{0n} : \alpha_{1n})$  in  $(\mathbb{P}_{\mathbb{C}}^1)^n$ .

The minimal graded free resolution of a hypercubic birational map is always Hilbert-Burch, and  $\mathbf{f}$  admits  $n$  independent linear syzygies. To elaborate, let  $n \geq 4$ . For each  $1 \leq k \leq n$ , we define the standard  $\mathbb{Z}$ -graded polynomial ring  $A_k = \mathbb{C}[\mathbf{x}_k] = \mathbb{C}[x_{0k}, x_{1k}]$ . In particular, the tensor product ring  $R := A_1 \otimes \dots \otimes A_n$  is hence standard  $\mathbb{Z}^n$ -graded by

$$R = \bigoplus_{(i_1 \dots i_n) \in \mathbb{Z}^n} R_{(i_1 \dots i_n)} = \bigoplus_{(i_1 \dots i_n) \in \mathbb{Z}^n} (A_1)_{i_1} \otimes \dots \otimes (A_n)_{i_n}.$$

The following result provides the explicit expression of the multilinear inverse of a birational  $n$ -cubic map. Furthermore, it describes the base loci of the birational map and its inverse. For simplicity in the statement, we write  $\mathbf{1} = (1, \dots, 1)$  in  $\mathbb{Z}^n$  and denote by  $\mathbf{e}_k$  the  $k$ -th canonical vector.

**Notation 1.2.26.** If  $\phi$  is  $n$ -cubic, the boundary hyperplane  $X_{ik}$  is defined by

$$\mathbf{x}_{ik}(\mathbf{y}) := \mathbf{x}_{ik} \cdot \mathbf{y}^T = 0$$

for some vector  $\mathbf{x}_{ik} = (x_{0ik}, \dots, x_{nik})$  in  $\mathbb{C}^{n+1}$ .

**Theorem 1.2.27** (Theorem 5.2.1). *Let  $\phi$  be  $n$ -cubic. Then, maintaining the notation of (1.14) for the inverse  $\phi^{-1}$ , we have*

$$(a_{0k} : a_{1k}) = (\alpha_{0k} \langle \mathbf{x}_{0k}, \mathbf{y} \rangle - \alpha_{1k} \langle \mathbf{x}_{1k}, \mathbf{y} \rangle : \alpha_{0k} \langle \mathbf{x}_{0k}, \mathbf{y} \rangle)$$

for each  $1 \leq k \leq n$ . In particular,  $\mathbf{f}$  admits the syzygy

$$\alpha_{0k} \mathbf{x}_{0k} b_0(x_{0k}, x_{1k}) + \alpha_{1k} \mathbf{x}_{1k} b_1(x_{0k}, x_{1k})$$

for each  $1 \leq k \leq n$ , and the base ideal  $B = (f_0, \dots, f_n)$  in  $R$  has a Hilbert-Burch minimal  $\mathbb{Z}^n$ -graded free resolution

$$0 \rightarrow \bigoplus_{k=1}^n R(-\mathbf{1} - \mathbf{e}_k) \rightarrow R(-\mathbf{1})^{n+1} \xrightarrow{(f_0 \dots f_n)} B \rightarrow 0.$$

Therefore,  $\mathbf{f}$  coincides with the signed minors of maximal size of the  $(n+1) \times n$  syzygy matrix. Additionally, the base locus of  $\phi^{-1}$  is the union of the  $(n-2)$ -planes  $X_{0k} \cap X_{1k}$  for each  $1 \leq k \leq n$ .

### 1.2.4. Weak $(1 - \epsilon)$ -nets for polynomial superlevel sets

In Chapter 6, we extend the classical centerpoint theorem by replacing half-spaces with superlevel sets of polynomials, i.e. the loci in  $\mathbb{A}_{\mathbb{R}}^n$  defined by a polynomial inequality (of arbitrary degree). More specifically, the centerpoint theorem can be stated as follows.

**Theorem 1.2.28** (Centerpoint theorem). *Let  $\mu$  be a probability measure on  $\mathbb{A}_{\mathbb{R}}^n$ . There exists a point  $c$  in  $\mathbb{A}_{\mathbb{R}}^n$ , called centerpoint, such that every half-space  $H$  containing  $c$  satisfies*

$$\mu(H) \geq \frac{1}{n+1}.$$

For our generalization to hold true, we must replace the centerpoint  $c$  by a finite set  $X$  in  $\mathbb{A}_{\mathbb{R}}^n$ . Namely,  $X$  is a weak  $(1 - \epsilon)$ -net for the range space of superlevel sets of polynomials of bounded degree (see §6.1.2) and some small  $\epsilon > 0$ .

We now present our first generalization of the centerpoint theorem, which deals with quadratic polynomial inequalities.

**Theorem 1.2.29.** (Theorem 6.2.1) *Let  $\mu$  be a probability measure on  $\mathbb{A}_{\mathbb{R}}^n$ . There exists a set  $X$  of  $n+1$  points in  $\mathbb{A}_{\mathbb{R}}^n$  such that any quadratic  $f$  in  $\mathbb{R}[x_1, \dots, x_n]$ , that is nonnegative on every point of  $X$ , satisfies*

$$\mu\{\mathbf{x} \in \mathbb{A}_{\mathbb{R}}^n : f(\mathbf{x}) \geq 0\} \geq \frac{2}{(n+2)(n+1)}.$$

Notably, in the previous statement  $n+1$  is the least number of points for such a result to hold true, for any other positive fraction (see Lemma 6.3.2). Specifically, given  $n$  (or less) points in  $\mathbb{A}_{\mathbb{R}}^n$  we can always find a linear polynomial  $L$  in  $\mathbb{R}[x_1, \dots, x_n]$  that vanishes at all these points. In particular, the polynomial  $f = -L^2$  is nonpositive on  $\mathbb{A}_{\mathbb{R}}^n$ , and zero (thus nonnegative) at every point. Therefore,

$$\mu\{\mathbf{x} \in \mathbb{A}_{\mathbb{R}}^n : f(\mathbf{x}) \geq 0\} = \mu\{\mathbf{x} \in \mathbb{A}_{\mathbb{R}}^n : L(\mathbf{x}) = 0\}$$

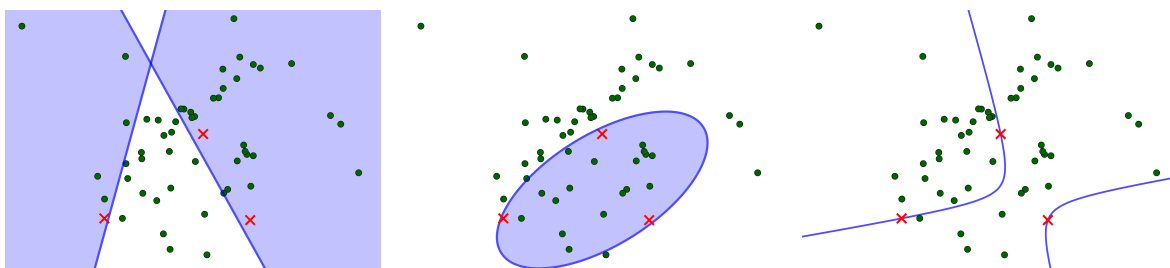
which has zero measure for multiple probability measures (for instance, if they are absolutely continuous). Additionally, we are able to extend the centerpoint theorem to encompass polynomial inequalities of arbitrary degree. Nevertheless, in this case we require the probability measure to be absolutely continuous with respect to the Lebesgue measure (see Definition 6.1.5).

**Theorem 1.2.30.** (Theorem 6.2.2) *Let  $\mu$  be a probability measure on  $\mathbb{A}_{\mathbb{R}}^n$  which is absolutely continuous with respect to the Lebesgue measure. For any  $\delta > 0$ , there exists a set  $X$  of at most  $\binom{n+2d}{n} - n - 1$  points in  $\mathbb{A}_{\mathbb{R}}^n$  such that any  $f$  in  $\mathbb{R}[x_1, \dots, x_n]_{\leq d}$ , that is nonnegative on every point of  $X$ , satisfies*

$$\mu\{\mathbf{x} \in \mathbb{A}_{\mathbb{R}}^n : f(\mathbf{x}) \geq 0\} \geq \left( \binom{n+2d}{2d} + 1 \right)^{-1} - \delta.$$

Our results rely on novel estimations of the Carathéodory number of Veronese varieties (see Definition 6.1.13). Since the Carathéodory number is inherently a property of real affine sets, and the Veronese embedding typically refers to a projective space, we recall the definitions of the varieties that we are interested at.

**Definition 1.2.31** (Real affine Veronese variety and Veronese cone).



**Figure 1.7.:** A sample comprising  $N = 50$  (green) points in  $\mathbb{A}_{\mathbb{R}}^2$ . Let  $\mu$  denote the probability measure derived from the normalized sum of  $\delta$  masses associated with each of these points. According to Theorem 1.2.29, there exist three points (red crosses) such that the superlevel sets of every quadratic polynomial (blue regions) containing these three points have measure at least  $1/6$ . In the right image, there is a plane conic passing through these three points. In particular, it divides  $\mathbb{A}_{\mathbb{R}}^2$  into two regions of measure of at least  $1/6$ .

- (i) The  $n$ -dimensional *real affine Veronese variety* in degree  $d$ , denoted by  $V(n, d)$ , is the image of

$$\begin{aligned} v : \mathbb{A}_{\mathbb{R}}^n &\rightarrow \mathbb{A}_{\mathbb{R}}^{\binom{n+d}{n}-1} \\ \mathbf{x} = (1, x_1, \dots, x_n) \equiv (x_1, \dots, x_n) &\mapsto \mathbf{x}^{\otimes d} \end{aligned}$$

Specifically, it is the restriction of the Veronese embedding to an affine chart in  $\mathbb{P}_{\mathbb{R}}^n$  (see Definition 2.2.13).

- (ii) The  $(n + 1)$ -dimensional *real Veronese cone in degree  $d$* , denoted by  $\hat{V}(n, d)$ , is the affine cone in  $\mathbb{A}_{\mathbb{R}}^{n+1}$  over the (projective) real Veronese variety (see Definition 2.2.13).

Remarkably, the Carathéodory number of the Veronese cone can be construed as the maximum nonnegative symmetric rank of a real symmetric tensor (see Definition 2.2.19 and §6.1.3). This relates with the literature on tensors, where the estimation of the maximum, typical and generic (symmetric) ranks of real and complex (symmetric) tensors has garnered substantial attention previously (e.g. [26, 171, 17, 136]).

The following result underlies our generalizations of the centerpoint theorem. Namely, it provides bounds for the Carathéodory number of the real Veronese varieties.

**Theorem 1.2.32** (Theorem 6.2.3). *The Carathéodory numbers of the Veronese varieties, denoted by  $\kappa(V(n, d))$  and  $\kappa(\hat{V}(n, d))$ , satisfy the following:*

- (i)  $\kappa(V(n, 2)) = n + 1$
- (ii) (di Dio and Kummer, [84])  $\kappa(V(n, 2d)) \geq \binom{n+2d}{n} - n\binom{n+d}{n} + \binom{n}{2}$
- (iii)  $\kappa(\hat{V}(n+1, 2d)) \leq \binom{n+2d}{n} - n - 1$
- (iv)  $\kappa(V(n, 2d)) \leq \kappa(V(n, 2d+1)) \leq \kappa(V(n, 2d+2))$



### 1.2.5. Classification of real cylinders through five points with four cocyclic

In Chapter 7, we address a very specific problem: the classification and computation of the real cylinders through five points in  $\mathbb{A}_{\mathbb{R}}^3$ , where four of them are cocyclic.

**Definition 1.2.33** (Cocyclic points). A family of points in  $\mathbb{A}_{\mathbb{R}}^2$  is *cocyclic* if they lie in a circumference.

To elaborate, by “classification” we refer to the real root classification of the system of two polynomial equations defining the (possibly complex) cylinders through a five-point configuration. Specifically, our objective is to establish semialgebraic equations that determine the count of real roots for this polynomial system. Thanks to the inherent constraints of our configuration, we can simplify the system to just two cubic equations in two homogeneous variables. Remarkably, our semialgebraic certificates reduce to the classical discriminant of an homogeneous cubic equation in two variables.

The following is the primary result in this direction. It permits the count of the real cylinders through a configuration of five affine points, with four of them cocyclic. Additionally, it provides closed formulas for the implicit equations of such cylinders, with rational coefficients in the coordinates of the points.

**Theorem 1.2.34** (Theorem 7.3.5, Corollary 7.3.7). *Let  $Q_1, Q_2, Q_3, Q_4$ , and  $Q_5 = (x_5, y_5, z_5)$  be points in  $\mathbb{A}_{\mathbb{R}}^3$ . Additionally, suppose that  $Q_1, Q_2, Q_3, Q_4$  are cocyclic and do not lie on a rectangle. Then, for each  $i = 0, 1$  there exists a bihomogeneous polynomial (see Notation 7.3.4)*

$$H_i(s_0, s_1; w, x, y, z)$$

*such that there is a one-to-one correspondence between the real cylinders through the five points and the real roots of the cubic equations*

$$h_0(s_0, s_1) = H_0(s_0, s_1; 1, x_5, y_5, z_5) = 0 \quad , \quad h_1(s_0, s_1) = H_1(s_0, s_1; 1, x_5, y_5, z_5) = 0 .$$

*More explicitly, if  $(s_0 : s_1) = (\alpha_0 : \alpha_1)$  in  $\mathbb{P}_{\mathbb{R}}^1$  is a root of  $h_i(s_0, s_1)$  for some  $i = 0, 1$ , then  $H_i(\alpha_0, \alpha_1; w, x, y, z) = 0$  is the defining equation in  $\mathbb{P}_{\mathbb{R}}^3$  of a real cylinder through the five points. In particular, if  $D_i = D_i(w, x, y, z)$  is the discriminant of  $h_i(s_0, s_1)$  and we let  $Q_5 \equiv (1, x_5, y_5, z_5)$ , the number of real cylinders through the five points is. . .*

- (i) . . . six if and only if  $D_0(Q_5), D_1(Q_5) > 0$
- (ii) . . . four if and only if  $D_0(Q_5) \cdot D_1(Q_5) < 0$
- (iii) . . . two if and only if  $D_0(Q_5), D_1(Q_5) < 0$

We illustrate the use of Theorem 1.2.34 with an example.

**Example 1.2.35** (Count and computation of the real cylinders through five points). Let

$$Q_1 = (0, 0, 0) \quad , \quad Q_2 = (1, 0, 0) \quad , \quad Q_3 = (-2.2336, 2.3888, 0) \quad , \quad Q_4 = (3.16, 3.41, 0) \quad ,$$

which are cocyclic points in the plane  $z = 0$ . Furthermore, we compute the constants (see Chapter 7)

$$\rho_0 = -10.5633 \quad , \quad \mu_0 = 14.8504 \quad , \quad \mu_1 = 5.4139 .$$

Now, let  $Q_5 = (1.6667, 0.5, 4)$ . Computing the discriminants of Theorem 1.2.34, we find

$$D_0(Q_5) < 0 \quad , \quad D_1(Q_5) > 0 \quad ,$$

and therefore there are exactly four circular cylinders through the five points. Moreover, the unique root of  $H_0(s_0, s_1; Q_5)$  in  $\mathbb{P}_{\mathbb{R}}^1$  is

$$(\alpha_0 : \alpha_1) = (0.0342 : -0.9994) \quad ,$$

and the three roots of  $H_1(s_0, s_1; Q_5)$  in  $\mathbb{P}_{\mathbb{R}}^1$  are

$$(\beta_0 : \beta_1) = (0.5485 : 0.8361) \quad , \quad (\gamma_0 : \gamma_1) = (0.0296 : -0.9996) \quad , \quad (\delta_0 : \delta_1) = (0.0065 : 1.0000) \quad .$$

Therefore, the defining polynomials in  $\mathbb{A}_{\mathbb{R}}^3$  of the four cylinders through the five points are

$$C_1(x, y, z) = H_0(\alpha_0, \alpha_1; x, y, z) = -1.00292 x^2 - 0.000876155 x y - 0.99829 y^2 - 0.0128117 x z + 0.136535 y z - 0.00470974 z^2 + 1.00292 x + 5.41667 y - 0.363995 z \quad ,$$

$$C_2(x, y, z) = H_1(\beta_0, \beta_1; x, y, z) = 1.59086 x^2 - 21.4486 x y + 114.874 y^2 + 16.3473 x z + 1.53394 y z + 115.296 z^2 - 1.59086 x - 327.132 y - 451.458 z \quad ,$$

$$C_3(x, y, z) = H_1(\gamma_0, \gamma_1; x, y, z) = -1.00219 x^2 + 0.0748174 x y - 1.39735 y^2 + 1.26197 x z + 0.118416 y z - 0.402178 z^2 + 1.00219 x + 6.53683 y - 1.02075 z \quad ,$$

$$C_4(x, y, z) = H_1(\delta_0, \delta_1; x, y, z) = 1.00011 x^2 - 0.00360464 x y + 1.01914 y^2 + 0.277173 x z + 0.0260084 y z + 0.0193766 z^2 - 1.00011 x - 5.46799 y - .209721 z \quad .$$

### 1.3. Organization of the thesis

The thesis is divided into two parts, each organized thematically as outlined at the beginning of **Chapter 1 Introduction**. More specifically, it respects the following structure:

- **Chapter 2 Preliminaries** introduces the elementary concepts and results in multiprojective geometry and tensors that are necessary for the thesis.
- **Part I MULTILINEAR BIRATIONAL MAPS** deals with the main research direction of this thesis, which is the study and effective application of multilinear birational maps. It comprises three chapters:
  - **Chapter 3 Trilinear birational maps in dimension three** delves into the algebro-geometric classification of trilinear birational maps, and computational aspects related to syzygies. The content of this chapter appears in the research article [43] Laurent Busé, Pablo González-Mazón, and Josef Schicho. “Tri-linear birational maps in dimension three”. In: *Math. Comp.* 92.342 (2023), pp. 1837–1866.
  - In **Chapter 4 Construction and manipulation of birational trilinear volumes** we develop effective methods for constructing and manipulating trilinear birational transformations. Some of the results of this chapter are available in the preprint

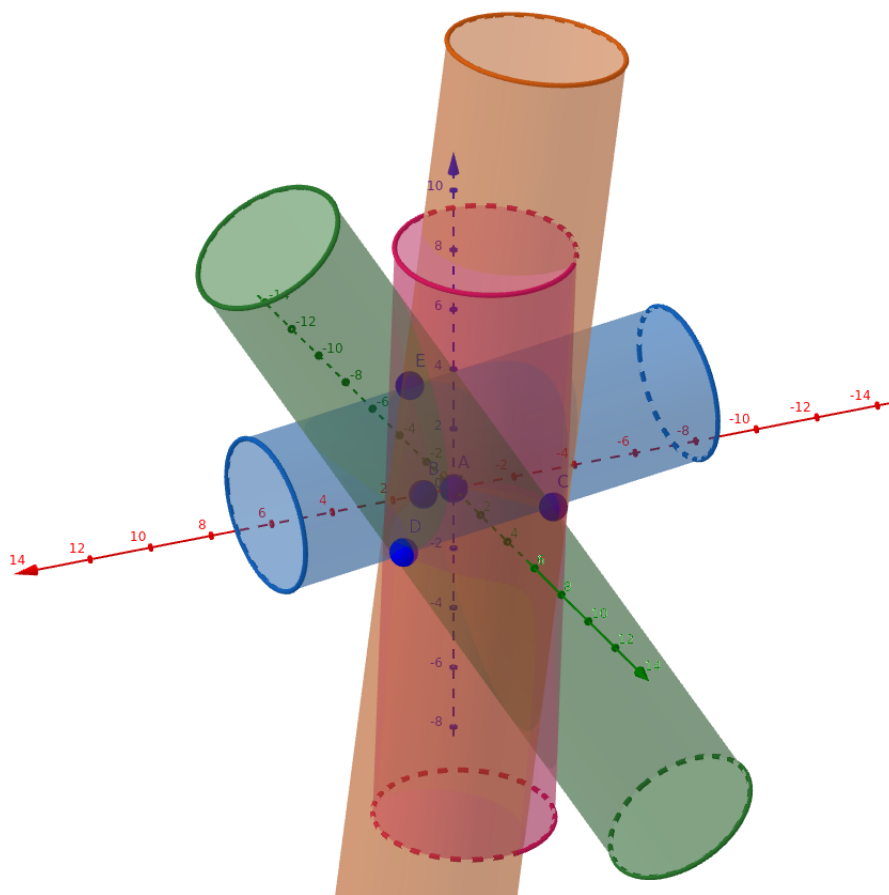


Figure 1.8.: The four circular cylinders through the five (blue) points in Example 1.2.35.

[99] Pablo González-Mazón and Laurent Busé. “Construction of birational trilinear volumes via tensor rank criteria” (2024).

- In **Chapter 5 Construction of multilinear birational maps with multilinear inverse** we extend the strategy developed for the construction of hexahedral birational maps in **Chapter 4** to the class of hypercubic birational maps in arbitrary dimension.
- **Part II POLYNOMIAL DATA ANALYSIS** addresses two lateral problems, involving the use of polynomials to extract information from data. It comprises two chapters:
  - **Chapter 6 Weak  $(1 - \epsilon)$  nets for polynomial superlevel sets** focuses on the derivation of bounds for the size of  $(1 - \epsilon)$ -nets for range systems defined by general polynomial inequalities, the generalization of the centerpoint theorem, and the estimation of the Carathéodory number of real affine Veronese varieties. The content of this chapter is the conclusion of a research that originated during Pablo González Mazón’s master’s thesis, under the supervision of Professor Alfredo Hubard, and is available in the preprint [100] Pablo González-Mazón, Alfredo Hubard, and Roman Karasev. “Weak  $(1 - \epsilon)$ -nets for polynomial superlevel sets” (2023).
  - **Chapter 7 Cylinders through four cocyclic points** addresses the problem of counting

and computing the real circular cylinders passing through a configuration of five points where four are cocyclic. This work emerged during a three-month secondment at the Universitat de Barcelona, in collaboration with the Ph.D. students Carles Checa and Amrutha B. Nair, and supervised by the Professors Carlos D'Andrea and Joan Carles Naranjo.



# Chapter 2

## Preliminaries

In this chapter, we introduce the fundamental objects studied in this thesis, along with some key results that provide the necessary background for our work. We also establish some notation used throughout the thesis.

The chapter comprises two sections, each dedicated to distinct aspects: multiprojective geometry and tensors. The first section is oriented towards Part I, and presents the concepts of multiprojective scheme, rational and birational maps between multiprojective spaces, base loci and base ideal, the Rees algebra, and the Jacobian dual criterion. The second section serves as valuable background for both Part I and Part II. It presents tensors and their more relevant attributes, the notion of rank with respect to a variety, and CP decompositions. Moreover, we discuss the geometry of some spaces of tensors that are relevant in this work, and in §2.2.4 we prepare for our exploration of trilinear birational maps by discussing their 2D counterparts, namely bilinear birational maps.

### 2.1. Birational maps between multiprojective spaces

The primary focus in the first part of this thesis centers on multilinear birational maps  $\phi : (\mathbb{P}_{\mathbb{C}}^1)^n \dashrightarrow \mathbb{P}_{\mathbb{C}}^n$ . To this purpose, we delve into various aspects, including  $\mathbb{Z}^n$ -graded free resolutions and syzygies, the classification of the base loci of trilinear birational maps, and birationality criteria. Thus, our approach requires familiarity with commutative algebra and algebraic geometry.

In this section, we survey the main tools and concepts that we address. For a more comprehensive treatment of the ideas discussed in this section, we refer the reader to [90, 89, 179, 71, 111].

#### 2.1.1. Multiprojective schemes

We begin by introducing multiprojective schemes using the MultiProj construction, which parallels the Proj construction for projective schemes but employs a  $\mathbb{Z}^n$ -grading rather than a  $\mathbb{Z}$ -grading.

**Definition 2.1.1** ( $\mathbb{Z}^n$ -graded ring and modules). A ring  $R$  is  $\mathbb{Z}^n$ -graded if it admits a direct sum decomposition of abelian groups

$$R = \bigoplus_{\mathbf{d} \in \mathbb{Z}^n} R_{\mathbf{d}}$$

such that

$$R_{\mathbf{d}_1} \cdot R_{\mathbf{d}_2} \subset R_{\mathbf{d}_1 + \mathbf{d}_2} .$$

An  $R$ -module  $M$  is  $\mathbb{Z}^n$ -graded if it admits a direct sum decomposition of abelian groups

$$M = \bigoplus_{\mathbf{d} \in \mathbb{Z}^n} M_{\mathbf{d}}$$

such that

$$R_{\mathbf{d}_1} \cdot M_{\mathbf{d}_2} \subset M_{\mathbf{d}_1 + \mathbf{d}_2} .$$

**Notation 2.1.2.** Let  $n \geq 1$ . We write

$$\mathbf{e}_i = (0, \dots, 0, \underbrace{1}_{i\text{-th entry}}, 0, \dots, 0)$$

for the  $i$ -th canonical vector in  $\mathbb{Z}^n$ .

The following are elementary definitions associated to graded rings and modules.

**Definition 2.1.3.** Let  $R$  be a ring and  $M$  be an  $R$ -module, both  $\mathbb{Z}^n$ -graded.

- The group  $R_{\mathbf{d}}$  (resp.  $M_{\mathbf{d}}$ ) is the *graded component in degree  $\mathbf{d}$*  of  $R$  (resp.  $M$ )
- An element is *homogeneous of degree  $\mathbf{d}$*  if it lies in  $R_{\mathbf{d}}$  (resp.  $M_{\mathbf{d}}$ )
- An ideal  $I \subset R$  is *homogeneous* if it is generated by homogeneous elements

**Notation 2.1.4.** If  $n \geq 2$ , it is common to say that  $R$  and  $M$  are *multigraded*. Likewise, we often refer to homogeneous elements and ideals as *multihomogeneous*. We adhere to these conventions in this work.

In this thesis, we exclusively focus on standard gradings. From a geometric perspective, these gradings are very important as they give rise to projective and multiprojective spaces, along with their associated subschemes. We remark that other gradings exist, leading to intriguing geometric objects such as weighted projective spaces and more general toric varieties (see [71]).

**Notation 2.1.5.** Whenever we work with  $\mathbb{Z}^n$ -gradings, we denote by  $\mathbf{0} = (0, \dots, 0)$  the identically zero vector in  $\mathbb{Z}^n$ .

**Definition 2.1.6** (Standard grading). Let  $k$  be a field, and  $R$  be a finitely generated  $\mathbb{Z}^n$ -graded  $k$ -algebra. We say that  $R$  is *standard graded* if  $R_{\mathbf{0}} = k$  and all the generators have degree  $\mathbf{e}_i$  for  $1 \leq i \leq n$ .

**Definition 2.1.7.** Let  $k$  be a field, and  $R$  be a finitely generated standard  $\mathbb{Z}^n$ -graded  $k$ -algebra. For each  $1 \leq i \leq n$ , we can recover the standard  $\mathbb{Z}$ -graded  $k$ -algebra

$$R_{(i)} = \bigoplus_{d \in \mathbb{Z}} R_{d \cdot \mathbf{e}_i},$$

and we denote  $\mathfrak{N}_i = (R_{\mathbf{e}_i})$ . We establish the following definitions:

- (i) The *multigraded irrelevant ideal* of  $R$  is the ideal

$$\mathfrak{N} = \bigcap_{i=1}^n \mathfrak{N}_i$$

- (ii) A multihomogeneous ideal  $I \subset R$  is *relevant* if  $\mathfrak{N} \not\subset I$

Projective and multiprojective schemes can be introduced using, respectively, homogeneous and multihomogeneous prime ideals. While schemes can be introduced in a more general context (see e.g. [111, 92]), such generality is not required for this thesis.

**Definition 2.1.8** (Multiprojective scheme). Let  $k$  be a field, and  $R$  be a finitely generated standard  $\mathbb{Z}^n$ -graded  $k$ -algebra. The *multiprojective scheme* associated to  $R$  is

$$\text{MultiProj}(R) = \{P \in R : P \text{ is prime, multihomogeneous, and relevant}\}$$

with the scheme structure induced by localizations at multihomogeneous relevant prime ideals (see e.g. [120, §1]). In particular, the topology on  $\text{MultiProj}(R)$  is defined by taking the closed sets of the form

$$V(I) = \{P \in \text{MultiProj}(R) : P \subset I\} .$$

The  $\text{MultiProj}$  construction establishes an identification between multiprojective schemes and multihomogeneous ideals, with the caveat that multihomogeneous ideals define the same multiprojective scheme up to saturation with respect to the irrelevant ideal  $\mathfrak{N}$ . The initial examples are projective and multiprojective spaces.

**Notation 2.1.9.** When working over a field  $k$ , we denote  $\otimes_k$  as  $\otimes$  by default.

**Example 2.1.10** (Multiprojective spaces). Let  $n \geq 1$ , and let  $1 \leq i \leq n$ .

- (i) If  $R_{(i)} = k[\mathbf{x}_i]$  for the variables  $\mathbf{x}_i = (x_{0i}, \dots, x_{m_i i})$  is standard  $\mathbb{Z}$ -graded, with irrelevant ideal  $\mathfrak{N}_i = (\mathbf{x}_i)$ , then  $\text{MultiProj}(R_{(i)}) = \text{Proj}(R_{(i)}) = \mathbb{P}_k^{m_i}$ .
- (ii) Similarly, if  $R = k[\mathbf{x}_1] \otimes \dots \otimes k[\mathbf{x}_n]$  is standard  $\mathbb{Z}^n$ -graded by

$$R = \bigotimes_{i=1}^n R_{(i)} = \bigoplus_{(d_1, \dots, d_n) \in \mathbb{Z}^n} (R_{(1)})_{d_1} \oplus \dots \oplus (R_{(n)})_{d_n} ,$$

with irrelevant ideal

$$\mathfrak{N} = \bigcap_{i=1}^n \mathfrak{N}_i = \bigcap_{i=1}^n (\mathbf{x}_i) ,$$

then  $\text{MultiProj}(R) = \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$ . Equivalently,  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  is just the product of the projective spaces  $\mathbb{P}_k^{m_i}$  for each  $1 \leq i \leq n$ . The points in  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  are thus represented by means of multihomogeneous coordinates

$$(\alpha_{01} : \dots : \alpha_{m_1 1}) \times \dots \times (\alpha_{0n} : \dots : \alpha_{m_n n})$$

where  $(\alpha_{0i} : \dots : \alpha_{m_i i})$  lies in  $\mathbb{P}_k^{m_i}$  for each  $1 \leq i \leq n$ . In particular, this point is identified with the ideal

$$\mathfrak{m}_1 + \dots + \mathfrak{m}_n$$

where

$$\mathfrak{m}_i = (\alpha_{0i} x_{1i} - \alpha_{1i} x_{0i}, \dots, \alpha_{0i} x_{n_i i} - \alpha_{n_i i} x_{0i}) .$$

**Notation 2.1.11.** If  $Y \subset \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  is a subscheme, we denote by  $I_Y \subset R$  the defining ideal of  $Y$ .

Of course, multiprojective schemes can exhibit multiple properties and behaviors. Here, we introduce some particular geometric properties that we address in Chapter 3. We refer the reader to [90, Appendix 2] and [89, 111, 92] for more details about these definitions.



**Definition 2.1.12.** Let  $R = \mathbb{C}[\mathbf{x}_1] \otimes \dots \otimes \mathbb{C}[\mathbf{x}_n]$  be standard  $\mathbb{Z}^n$ -graded,  $I \subset R$  be a relevant multihomogeneous ideal, and  $S = \text{MultiProj}(R/I)$ . We have the following definitions:

- (i)  $S$  is *equidimensional* if all its irreducible components have the same dimension
- (ii)  $S$  has an *embedded component* at  $Y$  if  $I_Y$  is an associated prime to  $R/I$  that is not minimal
- (iii)  $S$  is *Cohen-Macaulay* if the localization of  $R/I$  at  $P$  is a Cohen-Macaulay ring for every relevant multihomogeneous prime ideal  $P$

For schemes of dimension zero and one, the property of being Cohen-Macaulay can be characterized as follows.

**Lemma 2.1.13.** ([195, Lemma 31.4.4]) Let  $S \subset \mathbb{P}_{\mathbb{C}}^{m_1} \times \dots \times \mathbb{P}_{\mathbb{C}}^{m_n}$  be a subscheme with  $\dim(S) \leq 1$ . The following are equivalent:

- (i)  $S$  is Cohen-Macaulay
- (ii)  $S$  has no embedded points

In the following example, we illustrate the properties of Definition 2.1.12 on the subscheme of  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  obtained as the base locus (see Definition 2.1.32) of one of the trilinear birational maps that we classify in Chapter 3.

**Example 2.1.14** (Base locus of a trilinear birational map of type  $(1, 2, 2)$ ). Let  $R = \mathbb{C}[s_0, s_1] \otimes \mathbb{C}[t_0, t_1] \otimes \mathbb{C}[u_0, u_1]$  be standard  $\mathbb{Z}^3$ -graded, and consider the multihomogeneous ideal

$$I = (s_0 t_1 u_1, s_1 t_0 u_1, s_1 t_1 u_0, s_0 t_1 u_0 - s_0 t_0 u_1) .$$

The multiprojective scheme  $S \subset \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  defined by  $I$  is  $S = \text{MultiProj}(R/I)$ . Specifically, since  $I$  admits the primary decomposition

$$I = (t_1, u_1) \cap (s_0, t_0, u_0) \cap (s_1, t_0 u_1 - t_1 u_0, t_1^2, t_1 u_1, u_1^2) \cap (s_0, s_1) \cap (t_0, t_1) \cap (u_0, u_1) .$$

We conclude the following:

- (i)  $S$  is not equidimensional, since  $(t_1, u_1)$  defines an irreducible component of dimension one, but  $(s_0, t_0, u_0)$  defines an irreducible component of dimension zero, i.e. a (closed) point
- (ii)  $P = (s_1, t_1, u_1)$  is associated to  $R/I$ , since it is the annihilator of the element  $t_1 \in R/I$  (see [90, Appendix 2]). However, it is not a minimal prime, as we find

$$(t_1, u_1) \subset P .$$

Therefore,  $S$  has an embedded point at

$$(1 : 0) \times (1 : 0) \times (1 : 0) = \text{MultiProj} \left( \frac{R}{P} \right) = \text{MultiProj}(\mathbb{C}[s_0, t_0, u_0]) .$$

- (iii) By Lemma 2.1.13,  $S$  is not Cohen-Macaulay since it has an embedded point and  $\dim S = 1$

### 2.1.2. Syzygies and free resolutions

The MultiProj construction yields an equivalence between multihomogeneous ideals, saturated with respect to the irrelevant ideal, and multiprojective schemes. Therefore, all the geometric properties of a multiprojective scheme can be extracted from either the ideal  $I$  or the ring  $R/I$ . Among these properties, two of the most relevant are the “dimension” and “degree” (or “multidegree”), which are closely related to the graded structure of  $R/I$ . The formalization of these two important properties requires the introduction of minimal graded free resolutions. To this goal, the first step is to introduce the Hilbert function.

**Definition 2.1.15** (Hilbert function for  $\mathbb{Z}^n$ -graded modules). Let  $k$  be a field, let  $R = k[\mathbf{x}_1] \otimes \dots \otimes k[\mathbf{x}_n]$  be standard  $\mathbb{Z}^n$ -graded, and let  $M$  be a finitely generated graded  $R$ -module. The *Hilbert function* of  $M$  is

$$\begin{aligned} HF_M : \mathbb{Z}^n &\rightarrow \mathbb{N} \\ \mathbf{d} &\mapsto \dim_k M_{\mathbf{d}} \end{aligned}$$

The most common method for computing the Hilbert function of a module involves the computation of a graded free resolution. This approach traces its roots back to David Hilbert [116], who proposed the idea of recursively comparing a module  $M$  with free modules, which are much simpler. Minimal graded free resolutions are fundamental objects in commutative algebra, and much research is devoted to their study (e.g. [31, 112, 16, 107, 145]). In Chapter 3, we study the graded free resolutions that arise from the base loci of trilinear birational maps.

In the following lines, we introduce some elementary definitions related to free resolutions.

**Definition 2.1.16.** Let  $M, N$  be  $\mathbb{Z}^n$ -graded  $R$ -modules. An homomorphism  $\varphi : M \rightarrow N$  is *graded* of degree  $\mathbf{d}$  if  $\varphi(M_{\mathbf{p}}) \subset N_{\mathbf{d}+\mathbf{p}}$  for every  $\mathbf{p}$  in  $\mathbb{Z}^n$ .

**Definition 2.1.17.** Let

$$\mathbf{F}_{\bullet} : \dots \rightarrow F_n \xrightarrow{\delta_n} \dots \rightarrow F_1 \xrightarrow{\delta_1} F_0$$

be a chain complex of  $R$ -modules, and let  $M = \operatorname{coker} \delta_1 \subset F_0$ .

- The *homology* of  $\mathbf{F}_{\bullet}$  at  $F_i$  is  $H_i = \ker \delta_i / \operatorname{coker} \delta_{i+1}$
- $\mathbf{F}_{\bullet}$  is *exact* if  $H_i = 0$  for every  $i \geq 1$
- $\mathbf{F}_{\bullet}$  is *finite* if  $F_i = 0$  for every  $i \geq n$ , for some  $n \geq 1$ . The minimal value  $n$  is the *length* of  $\mathbf{F}_{\bullet}$ .
- $\mathbf{F}_{\bullet}$  is *free* if  $F_i$  is a free  $R$ -module for every  $i \geq 0$
- $\mathbf{F}_{\bullet}$  is  $\mathbb{Z}^n$ -*graded*, or simply *graded*, if  $F_i$  is  $\mathbb{Z}^n$ -graded and  $\delta_i : F_i \rightarrow F_{i-1}$  is graded of degree  $\mathbf{0}$  for every  $i \geq 1$
- $\mathbf{F}_{\bullet}$  is a *graded free resolution* of  $M$  if it is free, graded, and exact
- A *graded free resolution* is *minimal* if  $\delta_i$  sends a basis of  $F_i$  to a minimal set of generators of the image of  $\delta_i$

**Remark 2.1.18.** Let

$$\mathbf{F}_{\bullet} : \dots \rightarrow F_n \xrightarrow{\delta_n} \dots \rightarrow F_1 \xrightarrow{\delta_1} F_0 \tag{2.1}$$

be a minimal graded free resolution of  $M = \operatorname{coker} \delta_1$ . With a small abuse of notation, we say that the complex

$$\mathbf{F}_\bullet : \dots \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

is a minimal graded free resolution of  $M$ , although  $M$  might not be free.

The classical Hilbert's syzygy theorem guarantees the existence of finite free resolutions for finitely generated modules over standard  $\mathbb{Z}$ -graded polynomial rings.

**Theorem 2.1.19.** (Hilbert's syzygy theorem, [90, Theorem 1.1]) *Let  $M$  be a finitely generated graded module over the standard  $\mathbb{Z}$ -graded polynomial ring  $k[\mathbf{x}] = k[x_0, \dots, x_m]$ . Then,  $M$  admits a minimal graded free resolution of the form*

$$0 \rightarrow F_r \rightarrow \dots \rightarrow F_1 \rightarrow F_0$$

for some  $r \leq m + 1$ .

Hilbert's syzygy theorem readily generalizes to  $\mathbb{Z}^n$ -gradings with the following straightforward extension.

**Corollary 2.1.20** (Multigraded Hilbert's syzygy theorem). *Let  $M$  be a finitely generated graded module over the standard  $\mathbb{Z}^n$ -graded polynomial ring  $k[\mathbf{x}_1, \dots, \mathbf{x}_n]$ , where  $\mathbf{x}_i = (x_{0i}, \dots, x_{m_i})$  for each  $1 \leq i \leq n$ . Then,  $M$  admits a minimal graded free resolution of length  $r \leq \sum_{i=1}^n (m_i + 1)$ .*

Some of the most relevant attributes of a minimal graded free resolution are the Betti numbers, that we introduce in the following definition. Additionally, the concept of "syzygy" plays a pivotal role in this thesis. In particular, we rely on syzygies to establish our birationality criteria for multilinear rational maps. Syzygies, in essence, represent  $R$ -linear relations within an  $R$ -module. Specifically, for an  $R$ -module  $M$  equipped with a free resolution as (2.1), the (first) syzygies of its generators are elements belonging to  $\ker \delta_1$ . Recursively, the  $n$ -th syzygies of  $M$  are the elements in  $\ker \delta_n$ .

**Notation 2.1.21.** Let  $M$  be a  $\mathbb{Z}^n$ -graded module, and let  $\mathbf{d}$  in  $\mathbb{Z}^n$ . We write  $M(-\mathbf{d})$  for the *shift*, or *twist*, of the module  $M$  by  $\mathbf{d}$ , namely

$$M(-\mathbf{d})_{\mathbf{e}} = M_{\mathbf{e}+\mathbf{d}}$$

for every  $\mathbf{e}$  in  $\mathbb{Z}^n$ . In particular, we have

$$\dim_k R(-\mathbf{d})_{\mathbf{e}} = \binom{m_1 + e_1 - d_1}{m_1} \dots \binom{m_n + e_n - d_n}{m_n}$$

where  $\mathbf{e} = (e_1, \dots, e_n)$  and  $\mathbf{d} = (d_1, \dots, d_n)$ .

**Definition 2.1.22.** Let  $M$  be a finitely generated  $\mathbb{Z}^n$ -graded  $R$ -module, with a minimal graded free resolution

$$0 \rightarrow F_r \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow 0.$$

Then, for each  $1 \leq i \leq r$ , we can write

$$F_i = \bigoplus_{\mathbf{d} \in \mathbb{Z}^n} F_i(-\mathbf{d})^{\beta_{i\mathbf{d}}},$$

where  $\beta_{i\mathbf{d}} \neq 0$  only for finitely many  $\mathbf{d}$  in  $\mathbb{Z}^n$ . For each  $1 \leq i \leq r$ , we have the following definitions:

- (i) The exponents  $\beta_{id}$  are the *Betti numbers* of  $M$
- (ii) The *shape*, also called the *Betti table*, of the minimal graded free resolution is the collection of all the Betti numbers
- (iii)  $\ker \delta_i$  is the module of  *$i$ -th syzygies* of  $M$
- (iv) The elements in  $\ker \delta_i$  are called the  *$i$ -th syzygies* of  $M$

**Remark 2.1.23.** The Betti numbers and the shape are intrinsic properties of a module and are independent of its minimal free resolution. Furthermore, two distinct graded free resolutions of the same module are isomorphic (see e.g. [90, Theorem 1.6]).

When computing minimal graded free resolutions, certain shapes often recur. One of the most significant and well-studied shapes is the Hilbert-Burch resolution. If a module  $M$  admits a Hilbert-Burch resolution, the syzygy modules exhibit a remarkable simplicity: there are only first syzygies, and the module of first syzygies is free of rank  $r$ , where  $r + 1$  is the size of a minimal set of generators of  $M$ .

More precisely, the following is the statement of the classical Hilbert-Burch theorem.

**Theorem 2.1.24.** (*Hilbert-Burch theorem, [90, Theorem 3.2]*) *Let  $I$  be an ideal in a Noetherian ring  $R$  admitting a free resolution*

$$0 \rightarrow F \xrightarrow{M} G \rightarrow I \rightarrow 0 .$$

*If the rank of  $F$  is  $r$ , then the rank of  $G$  is  $r + 1$ , and there exists a nonzerodivisor  $a \in R$  such that  $I = aI_r(M)$ , where  $I_r$  is the ideal of  $r \times r$  minors of the syzygy matrix  $M$ . Specifically,  $I$  is generated by the signed maximal minors of  $M$  multiplied by  $a$ . Furthermore,  $\text{grade}(I_r(M))$  is two.*

*Conversely, given a nonzerodivisor  $a \in R$  and a  $(r + 1) \times r$  matrix  $M$  with entries in  $R$ , such that  $\text{grade}(I_r(M)) \geq 2$ , the ideal  $I = aI_r(M)$  admits a free resolution as above. Additionally,  $\text{grade}(I) = 2$  if and only if  $a$  is a unit.*

Hilbert-Burch resolutions appear frequently throughout Part I of this thesis. Specifically, they define the resolutions of the base ideals of multilinear birational maps with multilinear inverse (see §3.5.1 and Chapter 5). The next example illustrates a Hilbert-Burch minimal  $\mathbb{Z}^5$ -graded free resolution.

**Example 2.1.25** (Hilbert-Burch resolution of the base ideal of a multilinear birational map). Let  $\mathbf{1}$  be the identically one vector in  $\mathbb{Z}^5$ . The base ideal  $B = (f_0, \dots, f_5)$  of the standard  $\mathbb{Z}^5$ -graded ring  $R = \mathbb{C}[\mathbf{x}_1, \dots, \mathbf{x}_5]$ , where  $\mathbf{x}_i = (x_{0i}, x_{1i})$  for each  $1 \leq i \leq 5$ , that we have taken from the multilinear birational map  $\phi : (\mathbb{P}_{\mathbb{C}}^1)^5 \dashrightarrow \mathbb{P}_{\mathbb{C}}^5$  constructed in Example 5.2.4, admits a Hilbert-Burch minimal graded free resolution. Specifically, it has the form

$$0 \rightarrow \bigoplus_{i=1}^5 R(-\mathbf{1} - \mathbf{e}_i) \rightarrow R(-\mathbf{1})^6 \xrightarrow{(f_0 \dots f_5)} B \rightarrow 0 .$$

From the Betti numbers of the resolution, we deduce that  $B$  admits 5 independent syzygies, of degrees  $\mathbf{e}_i$  for each  $1 \leq i \leq 5$  respectively. In particular, by Theorem 2.1.24 the generators of  $B$  can be retrieved as the wedge of these syzygies, up to a nonzero scalar.

Returning to geometry, if  $I$  is a relevant multihomogeneous ideal within a standard  $\mathbb{Z}^n$ -graded ring  $R = k[\mathbf{x}_1, \dots, \mathbf{x}_n]$ , the minimal graded free resolution of  $M = R/I$  provides valuable information for the associated scheme. In particular, it permits the computation of the dimension, as well as the degree for projective schemes, and the multidegrees for multiprojective schemes. These attributes are all derived from the Hilbert polynomial of  $M$ , which stands as a central tool in both commutative algebra and algebraic geometry. More precisely, if  $M$  admits the minimal  $\mathbb{Z}^n$ -graded free resolution

$$\mathbf{F}_\bullet : 0 \rightarrow F_r \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow 0 ,$$

then for each multidegree  $\mathbf{d}$  in  $\mathbb{Z}^n$ , we can take graded components to derive the resolution of vector spaces

$$0 \rightarrow (F_r)_{\mathbf{d}} \rightarrow \dots \rightarrow (F_1)_{\mathbf{d}} \rightarrow (F_0)_{\mathbf{d}} \rightarrow 0 .$$

By exactness, the value of the Hilbert function is thus

$$HF_M(\mathbf{d}) = \sum_{i=0}^r (-1)^i HF_{F_i}(\mathbf{d}) .$$

It is a classical result that, for  $\mathbb{Z}$ -graded modules, the Hilbert function becomes a polynomial for sufficiently large degrees. The following represents the multigraded analog of this result.

**Theorem 2.1.26.** ([115, Theorem 4.1], [146, Lemma 2.8], [29, Proposition 4.27]) *Let  $k$  be a field,  $R = k[\mathbf{x}_1, \dots, \mathbf{x}_n]$  be standard  $\mathbb{Z}^n$ -graded, and let  $M$  be a finitely generated graded  $R$ -module. Then, there is a polynomial  $HP_M$  in  $\mathbb{Q}[\mathbf{t}] = \mathbb{Q}[t_1, \dots, t_n]$ , called the multigraded Hilbert polynomial of  $M$ ,*

$$HP_M(\mathbf{t}) = \sum_{(d_1, \dots, d_n) \in \mathbb{Z}^n} m(d_1, \dots, d_n) \binom{t_1 + d_1}{d_1} \dots \binom{t_n + d_n}{d_n}$$

such that  $HF_M(\mathbf{e}) = HP_M(\mathbf{e})$  for every  $\mathbf{e} = (e_1, \dots, e_n)$  with  $e_i$  sufficiently large for every  $1 \leq i \leq n$ .

As for projective schemes, the dimension and the degree of a multiprojective scheme can be read from the Hilbert polynomial.

**Definition 2.1.27.** ([51, Definition 2.7]) *Let  $k$  be a field,  $R = k[\mathbf{x}_1, \dots, \mathbf{x}_n]$  be standard  $\mathbb{Z}^n$ -graded,  $I \subset R$  be a relevant multihomogeneous ideal,  $M = R/I$ , and  $S = \text{MultiProj}(M)$ . With the notation of Theorem 2.1.26:*

- The *dimension* of  $S$ , denoted by  $\dim S$ , is the degree of  $HP_M(\mathbf{t})$
- For each  $\mathbf{d} = (d_1, \dots, d_n)$  in  $\mathbb{Z}^n$  such that  $d_1 + \dots + d_n = \dim S$ , the *multidegree* of  $S$  of type  $\mathbf{d}$  (with respect to the embedding space  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$ ) is  $m(\mathbf{d})$

**Remark 2.1.28.** In classical algebro-geometric terms, the multidegree  $m(\mathbf{d})$  of type  $\mathbf{d} = (d_1, \dots, d_n)$  is the number of points (counting multiplicity) in the intersection of  $S$  with the product  $L_1 \times \dots \times L_n \subset \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  where  $L_i \subset \mathbb{P}_k^{m_i}$  is a general  $(m_i - d_i)$ -dimensional subspace, for each  $1 \leq i \leq n$  (e.g. [59, §2.1], [58, Theorem 4.7], [51, §2]).

In the following example, we compute the Hilbert polynomial of one of the subschemes of interest in this thesis, namely the base locus of a trilinear birational map.

**Example 2.1.29.** Continuing with Example 2.1.14, the minimal  $\mathbb{Z}^3$ -graded free resolution of the ideal  $I$  is the one corresponding to trilinear birational maps of type  $(1, 2, 2)$  according to Theorem 1.2.6. Using this resolution, we can compute the Hilbert polynomial of the module  $M = R/I$ , or equivalently, of the subscheme  $S$ . Namely,

$$HP_M(t_1, t_2, t_3) = t_1 + 3 = \binom{t_1 + 1}{1} \binom{t_2}{0} \binom{t_3}{0} + 2 \binom{t_1}{0} \binom{t_2}{0} \binom{t_3}{0}.$$

Therefore:

- (i)  $S$  has dimension one
- (ii) The multidegrees of  $S$  of types

$$(1, 0, 0), (0, 1, 0), (0, 0, 1)$$

are respectively  $1, 0, 0$ . In particular, we can group all these multidegrees in the triple  $(1, 0, 0)$  in  $\mathbb{Z}^3$ . This will be our notion of tridegree for curves in  $(\mathbb{P}_{\mathbb{C}}^1)^3$  (see Definition 3.1.12).

### 2.1.3. Birational maps between multiprojective spaces

We now introduce rational and birational maps from a multiprojective space to a projective space, along with the concepts of base ideal and base locus. We recall that we are working over the Zariski topology.

**Definition 2.1.30** (Rational map from a multiprojective to a projective space). Let  $k$  be a field. A *rational map* from  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  to  $\mathbb{P}_k^{m_0}$  is an equivalence class of pairs  $(U, \phi)$  where:

- (i)  $U \subset \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  is an open set
- (ii)  $\phi : U \rightarrow \mathbb{P}_k^{m_0}$  is a morphism
- (iii)  $(U_1, \phi_1) \sim (U_2, \phi_2)$  if  $\phi_1|_{U_1 \cap U_2} = \phi_2|_{U_1 \cap U_2}$

If  $\phi(U)$  is dense in the target space, we say that  $\phi$  is *dominant*. More concretely, if  $R = k[\mathbf{x}_1, \dots, \mathbf{x}_n]$  in standard  $\mathbb{Z}^n$ -graded, where  $\mathbf{x}_i = (x_{0i}, \dots, x_{m_i i})$  for each  $1 \leq i \leq n$ , a rational map from  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  to  $\mathbb{P}_k^{m_0}$  can equivalently be defined as

$$\begin{aligned} \phi : \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n} &\dashrightarrow \mathbb{P}_k^{m_0} \\ \mathbf{x}_1 \times \dots \times \mathbf{x}_n &\mapsto (f_0 : \dots : f_{m_0}) \end{aligned} \quad (2.2)$$

where  $f_j = f_j(\mathbf{x}_1, \dots, \mathbf{x}_n)$  lies in  $R_{\mathbf{d}}$  for each  $1 \leq j \leq m_0$ , for some  $\mathbf{d}$  in  $\mathbb{Z}^n$ . Since  $\phi$  is defined up to an open set, we can assume  $\gcd(f_0, \dots, f_{m_0}) = 1$ .

**Remark 2.1.31.** The following are two straightforward observations:

- (i) Rational maps are denoted by  $\dashrightarrow$  instead of  $\longrightarrow$  because, despite their name, they might not be maps. More precisely, they are not defined at points where all the defining polynomials vanish.
- (ii) The definition of rational maps to a multiprojective target follows immediately from Definition 2.1.30, by considering multiple tuples of multihomogeneous polynomials.

Our approach to study multilinear birational maps relies on the analysis of two intimately related objects: the base ideal and the base locus. Specifically, we are interested on the multiprojective schemes that arise as the base loci of these birational transformations. Therefore, all the generators of their defining ideals have the same degree.

**Definition 2.1.32** (Base ideal and base locus). Let  $k$  be a field, and  $R = k[\mathbf{x}_1, \dots, \mathbf{x}_n]$  be standard  $\mathbb{Z}^n$ -graded. Consider the rational map

$$\begin{aligned} \phi : \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n} &\dashrightarrow \mathbb{P}_k^{r_1} \times \dots \times \mathbb{P}_k^{r_s} \\ \mathbf{x}_1 \times \dots \times \mathbf{x}_n &\mapsto (f_{01} : \dots : f_{r_1 1}) \times \dots \times (f_{0s} : \dots : f_{r_s s}) \end{aligned} \quad (2.3)$$

(i) The *base ideal* of  $\phi$  is the multihomogeneous ideal  $B \subset R$  defined by

$$B = \bigcap_{i=1}^s (f_{0i}, \dots, f_{r_i i})$$

(ii) The *base locus* of  $\phi$  is the multiprojective subscheme  $Z \subset \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  defined by  $B$

**Remark 2.1.33.** By our hypothesis  $\gcd(f_{0i}, \dots, f_{r_i i}) = 1$  for each  $1 \leq i \leq s$ , it follows that  $\text{codim } Z \geq 2$ . Furthermore,  $Z$  is precisely the locus where  $\phi$  is not defined.

Now, we introduce birational maps, which are the central objects of study in this thesis.

**Definition 2.1.34** (Birational map). Maintaining the notation of Definition 2.1.30,  $\phi$  is birational if there exists a rational map  $(V, \varphi)$  from  $\mathbb{P}_k^{m_0}$  to  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  such that the composition  $(U', \varphi \circ \phi)$  is the identity, for some open  $U' \subset \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$ .

In particular, if  $\phi$  is birational there exists an inverse rational map

$$\begin{aligned} \phi^{-1} : \mathbb{P}_k^{m_0} &\dashrightarrow \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n} \\ \mathbf{y} &\mapsto (g_{01} : \dots : g_{m_1 1}) \times \dots \times (g_{0n} : \dots : g_{m_n n}) \end{aligned} \quad (2.4)$$

where for each  $1 \leq i \leq n$  and  $1 \leq j \leq m_i$ ,  $\mathbf{y} = (y_0, \dots, y_{m_0})$  and  $g_{j i} = g_{j i}(\mathbf{y})$  lies in  $R[\mathbf{y}]_{d_i}$ , for some  $d_i$  in  $\mathbb{Z}$ . Again, we can assume that  $\gcd(g_{0i}, \dots, g_{m_i i}) = 1$ .

In particular, birational maps can be seen as isomorphisms between the fraction fields of the underlying rings of the source and target. In our multiprojective setting, we encounter the following.

**Remark 2.1.35** (Isomorphisms of fraction fields). Maintaining the notation of Definitions 2.1.30 and 2.1.34, let  $\mathbf{f} = (f_0, \dots, f_{m_0})$ . Since the composition  $\phi^{-1} \circ \phi$  yields the identity on  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$ , it induces the automorphism between the fields of fractions  $\Phi : \text{Frac}(R) \rightarrow \text{Frac}(R)$  given by

$$(x_{0i}, \dots, x_{m_i i}) \mapsto (g_{0i}(\mathbf{f}), \dots, g_{m_i i}(\mathbf{f})) = h_i \cdot (x_{0i}, \dots, x_{m_i i})$$

for each  $1 \leq i \leq n$ , where  $h_i$  is a multihomogeneous polynomial in  $R$ . Thus, birational maps induce field isomorphisms between the fraction fields  $\text{Frac}(R)$  and  $\text{Frac}(k[\mathbf{y}]) = k(y_0, y_1, y_2, y_3)$ .

### 2.1.4. The Rees algebra and blow-ups

In this subsection, we introduce the Rees algebra, a fundamental concept in commutative algebra widely studied in the literature (e.g., [57, 170, 198, 199]).

From a geometrical point of view, the Rees algebra can be used to define the coordinate ring of the graph of a rational map, i.e. the MultiProj construction applied on the Rees algebra of the base ideal of a rational map  $\phi : \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n} \dashrightarrow \mathbb{P}_k^{m_0}$  yields the graph of  $\phi$ . Additionally, it serves as the coordinate ring for the blow-up of a multiprojective variety along a multiprojective subscheme [111, §II.7]. Moreover, the Jacobian Dual Criterion (JDC) is a characteristic-free criterion for birationality that relies on the defining equations of the Rees algebra [178, 87, 41].

**Definition 2.1.36.** Let  $k$  be a field,  $R$  be a standard  $\mathbb{Z}^n$ -graded ring,  $I \subset R$  be a multihomogeneous ideal, and  $t$  be a variable. The *Rees algebra* of  $I$  is

$$\mathcal{R}_R(I) = R[t] = \bigoplus_{i \geq 0} I^i t^i = R \oplus It \oplus I^2 t^2 \oplus I^3 t^3 \oplus \dots$$

Given generators of  $I$ , say  $I = (f_0, \dots, f_r)$ , then  $\mathcal{R}_R(I) = R[f_0 t, \dots, f_r t]$ . In particular,  $\mathcal{R}_R(I)$  is the image of the ring homomorphism

$$\begin{aligned} \Phi : R[\mathbf{y}] &\rightarrow R[t] \\ y_i &\mapsto f_i t \end{aligned}$$

where  $\mathbf{y} = (y_0, \dots, y_r)$ . The kernel  $J = \ker \Phi \subset R[\mathbf{y}] = k[\mathbf{x}_1, \dots, \mathbf{x}_n] \otimes k[\mathbf{y}]$  is the *Rees ideal* associated to  $I$ , and the generators of  $J$  are the *defining equations of the Rees algebra*  $\mathcal{R}_R(I)$ .

The Rees ideal  $J$  is thus the ideal consisting of all the algebraic relations of the generators of  $I = (f_0, \dots, f_r)$ , i.e. a multihomogeneous polynomial  $g = g(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y})$  in  $R[\mathbf{y}] = k[\mathbf{x}_1, \dots, \mathbf{x}_n] \otimes k[\mathbf{y}]$  belongs to  $J$  if and only if

$$g(f_0 t, \dots, f_r t) = 0$$

or equivalently,

$$g(f_0, \dots, f_r) = 0 .$$

**Remark 2.1.37** (Syzygies are linear relations in the Rees ideal). In particular, the linear relations in  $\mathbf{y}$ , i.e. those of the form

$$g = a_0 y_0 + \dots + a_r y_r$$

with  $a_i = a_i(\mathbf{x}_1, \dots, \mathbf{x}_n)$  in  $R$ , represent the first syzygies of the generators of  $I = (f_0, \dots, f_r)$ . More precisely, the vector of polynomials  $\mathbf{a} = (a_0 \dots a_r)$  is a syzygy of  $\mathbf{f} = (f_0 \dots f_r)$  if and only if

$$\langle \mathbf{a}, \mathbf{f} \rangle = \mathbf{a} \cdot \mathbf{f}^T = a_0 f_0 + \dots + a_r f_r = 0 .$$

Therefore, we talk indistinctly between syzygies, understood as vectors of polynomials defining relations between  $f_0, \dots, f_r$ , and the linear polynomials in  $\mathbf{y}$  defining the same relations.

In our multiprojective setting, we introduce blow-ups of multiprojective spaces using the Rees algebra as follows.



**Definition 2.1.38** (Blow-up of a multiprojective space along a subscheme). Let  $k$  be a field,  $R = k[\mathbf{x}_1, \dots, \mathbf{x}_n]$  be a standard  $\mathbb{Z}^n$ -graded ring,  $I \subset R$  be a multihomogeneous relevant ideal, and  $S = \text{MultiProj}(R/I)$ . The *blow-up* of  $X = \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  along  $S$  is the multiprojective scheme

$$\text{Bl}_S(X) = \text{MultiProj}(\mathcal{R}_R(I)) .$$

If additionally  $I = (f_0, \dots, f_{m_0})$ , where all the  $f_i$  have the same multidegree, then  $\text{Bl}_S X$  defines the *graph* of the rational map

$$\begin{aligned} \phi : X &\dashrightarrow \mathbb{P}_k^{m_0} \\ \mathbf{x}_1 \times \dots \times \mathbf{x}_n &\mapsto (f_0 : \dots : f_{m_0}) , \end{aligned}$$

which can be regarded as a subscheme of  $X \times \mathbb{P}_k^{m_0}$ , since  $\mathcal{R}_R(I) \cong R[\mathbf{y}]/J = (k[\mathbf{x}_1, \dots, \mathbf{x}_n] \otimes k[\mathbf{y}])/J$ .

In the following paragraphs, we present the Jacobian dual criterion for multiprojective rational maps. Actually, we state a weaker version of the general result, which can be formulated for rational maps defined over multiprojective varieties (see [41, Theorem 4.4]). Nevertheless, in this thesis we restrict to multiprojective spaces for the source.

**Notation 2.1.39.** Let  $I = (f_0, \dots, f_{m_0})$  where  $f_j = f_j(\mathbf{x}_1, \dots, \mathbf{x}_n)$  is homogeneous of degree  $\mathbf{d}$  for every  $0 \leq j \leq m_0$ , and let  $J \subset k[\mathbf{x}_1, \dots, \mathbf{x}_n] \otimes k[\mathbf{y}]$  be the Rees ideal associated to  $I$ . For each  $1 \leq i \leq n$ , define the  $k[\mathbf{y}]$ -module

$$J_{(\mathbf{e}_i; *)} = \bigoplus_{j=1}^{\infty} J_{(\mathbf{e}_i; j)} ,$$

where  $(\mathbf{d}; \mathbf{e})$  stands for the  $(\mathbb{Z}^n \times \mathbb{Z})$ -degree in  $k[\mathbf{x}_1, \dots, \mathbf{x}_n] \otimes k[\mathbf{y}]$ , and let  $\{h_{1i}, \dots, h_{r_i i}\}$  be a minimal set of generators of  $J_{(\mathbf{e}_i; *)}$ . Let  $\psi_i$  be the Jacobian matrix with respect to the variables  $\mathbf{x}_i = (x_{0i}, \dots, x_{m_i i})$  associated to this set of generators, i.e.

$$\psi_i = \begin{pmatrix} \frac{\partial h_{1i}}{\partial x_{0i}} & \dots & \frac{\partial h_{r_i i}}{\partial x_{0i}} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_{1i}}{\partial x_{m_i i}} & \dots & \frac{\partial h_{r_i i}}{\partial x_{m_i i}} \end{pmatrix} ,$$

which has size  $(m_i + 1) \times r_i$ . Similarly, denote by  $\psi$  the diagonal concatenation

$$\psi = \begin{pmatrix} \psi_1 & 0 & \dots & 0 \\ 0 & \psi_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \psi_n \end{pmatrix} .$$

**Theorem 2.1.40.** (Multigraded Jacobian dual criterion, [41, Theorem 4.4]) *Let the rational map*

$$\begin{aligned} \phi : \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n} &\dashrightarrow \mathbb{P}_k^{m_0} \\ \mathbf{x}_1 \times \dots \times \mathbf{x}_n &\mapsto (f_0 : \dots : f_{m_0}) \end{aligned} \quad (2.5)$$

be dominant. With Notation 2.1.39, the following are equivalent:

- (i)  $\phi$  is birational

(ii)  $\text{rank}_{k(\mathbf{y})}(\psi_i) = m_i$  for each  $1 \leq i \leq n$

(iii)  $\text{rank}_{k(\mathbf{y})}(\psi) = m_1 + \dots + m_n$

Unfortunately, deriving the defining equations of the Rees algebra is a challenging task requiring significant computational resources [27, 38, 70, 68], which makes the JDC difficult to apply in practice. Moreover, while suitable for testing birationality, it is less effective for constructing birational transformations.

In applications, specialized methods tailored to specific families of birational transformations are often preferred. In Chapter 3, we develop birationality criteria exclusively based on the first syzygies of the defining polynomials, specifically designed for the family of trilinear birational maps.

## 2.2. Tensors

Tensors appear frequently in this thesis. This comes as no surprise, since general tensors are, fundamentally, multilinear polynomials. They also serve to encode multilinear rational maps, which constitute the primary object of study of Part I. In Part II, our exploration of  $(1 - \epsilon)$ -nets hinges upon (real, affine) Veronese varieties, which define the loci of symmetric tensors of rank one.

In this section, we provide an overview of the fundamental concepts and results related to tensors that are used in this thesis. We follow the references [136, 98, 152, 159, 159, 126], and refer the reader to them for further details.

### 2.2.1. Elementary definitions

The tensor product is a fundamental algebraic construction applicable to modules and other abstract structures, such as sheaves. Nevertheless, the term “tensors” predominantly refers to elements within a tensor product of vector spaces

**Definition 2.2.1** (Tensor). Let  $V_0, V_1, \dots, V_n$  be (finite-dimensional) vector spaces over a field  $k$ . A *multilinear map* or *tensor* is a function

$$V_1 \times \dots \times V_n \rightarrow V_0$$

which is linear in  $V_i$  for each  $1 \leq i \leq n$ . The space of such multilinear maps is the tensor product  $V_0 \otimes V_1^* \otimes \dots \otimes V_n^*$ , where  $V^*$  stands for the dual vector space of  $V$ . With this notation, a tensor has the following attributes:

- The *order*, or *dimension*, is  $n + 1$
- If  $\dim_k V_i = m_i$  for each  $0 \leq i \leq n$ , tensors have *format*  $m_0 \times m_1 \times \dots \times m_n$

Clearly,  $V_1^* \otimes \dots \otimes V_n^*$  is isomorphic to  $V_1 \otimes \dots \otimes V_n$ , so we drop the superindex  $*$ .

**Remark 2.2.2** (Alternative definitions of tensors). The following alternative ways to define tensors are also very frequent in the literature, and might be more adequate depending on the context.

- (i) A tensor is just an element in a finite tensor product of vector spaces.
- (ii) (Linear maps over a tensor product of vector spaces) A tensor of order  $n + 1$  can be regarded simply as a linear map

$$V_1 \otimes \dots \otimes V_n \rightarrow V_0$$

- (iii) (Multilinear polynomials, our favorite) Let  $B_i = \{e_{0i}, \dots, e_{m_i i}\}$  be a basis of  $V_i$  for each  $1 \leq i \leq n$ . Then, a tensor of format  $(m_1 + 1) \times \dots \times (m_n + 1)$  can be expressed as the sum

$$\sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} \lambda_{i_1 \dots i_n} e_{i_1 1} \otimes \dots \otimes e_{i_n n} ,$$

which is identified with the multilinear polynomial

$$\sum_{i_1=1}^{m_1} \dots \sum_{i_n=1}^{m_n} \lambda_{i_1 \dots i_n} x_{i_1 1} \dots x_{i_n n} ,$$

in the variables of  $\mathbf{x}_i = (x_{0i}, \dots, x_{m_i i})$  for each  $1 \leq i \leq n$ . Hence, tensors can be defined as multilinear polynomials in  $k[\mathbf{x}_1] \otimes \dots \otimes k[\mathbf{x}_n]$  (with the standard  $\mathbb{Z}^n$ -grading!).

- (iv) (Multidimensional array) Tensors are often depicted as  $n$ -dimensional arrays with entries in  $k$

$$M = (\lambda_{i_1 \dots i_n})_{0 \leq i_1 \leq m_1, \dots, 0 \leq i_n \leq m_n} .$$

In particular, matrices are tensors of order two, or two-dimensional.

The equivalent concepts of *flattening*, *matricization*, *contraction*, or *unfolding*, [126, 136, 159, 127] refer to the process of reordering the entries of a higher-order tensor into a matrix.

**Definition 2.2.3** (Flattening). Let  $V = \bigoplus_{i=1}^n V_i$ , and let  $I, J$  be a bipartition of  $\{1, \dots, n\}$ . The *flattening* associated to  $(I, J)$  of a tensor in  $M$  is the matrix induced by the canonical isomorphism from  $V$  to

$$W_1 = \bigotimes_{i \in I} V_i^* \rightarrow W_0 = \bigotimes_{j \in J} V_j .$$

When we mention the flattenings of a tensor without providing specific details, we are referring to the flattenings associated to all possible bipartitions consisting of 1 and  $n - 1$  elements.

**Notation 2.2.4.** Given a vector  $v$  in  $V$ , we denote

$$v^{\otimes n} := \underbrace{v \otimes \dots \otimes v}_{n \text{ times}}$$

and call  $v^{\otimes n}$  the  $n$ -th tensor power of  $v$ . Similarly, we denote  $V^{\otimes n} := \underbrace{V \otimes \dots \otimes V}_{n \text{ times}}$  and

$$(m+1)^{\times n} := \underbrace{(m+1) \times \dots \times (m+1)}_{n \text{ times}}$$

for specifying this tensor format.

**Example 2.2.5.** In this thesis, we work with the flattenings of  $2 \times 2 \times 2$  tensors  $M = (\lambda_{ijk})$  in  $(\mathbb{R}^2)^{\otimes 3}$  in Chapter 4, since these tensors represent trilinear polynomials. These flattenings are the  $2 \times 4$  matrices that rearrange the elements of  $M$ . Specifically, if  $I = \{1\}$  we have

$$M_{(1)} = \begin{pmatrix} \lambda_{000} & \lambda_{010} & \lambda_{001} & \lambda_{011} \\ \lambda_{100} & \lambda_{110} & \lambda_{101} & \lambda_{111} \end{pmatrix} .$$

Similarly, for  $I = \{2\}$  and  $I = \{3\}$  we respectively find

$$M_{(2)} = \begin{pmatrix} \lambda_{000} & \lambda_{100} & \lambda_{001} & \lambda_{101} \\ \lambda_{010} & \lambda_{110} & \lambda_{011} & \lambda_{111} \end{pmatrix} , \quad M_{(3)} = \begin{pmatrix} \lambda_{000} & \lambda_{100} & \lambda_{010} & \lambda_{110} \\ \lambda_{001} & \lambda_{101} & \lambda_{011} & \lambda_{111} \end{pmatrix} .$$

The family of symmetric tensors has special relevance. In particular, they serve to encode homogeneous polynomials in some (standard  $\mathbb{Z}$ -graded) variables  $\mathbf{x} = (x_0, \dots, x_n)$ .

**Definition 2.2.6** (Symmetric tensor). Let  $M$  be a tensor of format  $(m+1)^{\times n}$ . We say that  $M = (\lambda_{i_1 \dots i_n})_{0 \leq i_1, \dots, i_n \leq m}$  is *symmetric* if

$$\lambda_{i_1 \dots i_n} = \lambda_{\sigma(i_1) \dots \sigma(i_n)}$$

for every permutation  $\sigma$  of  $\{1, \dots, n\}$ .

**Remark 2.2.7.** (Alternative definition of symmetric tensors) Symmetric tensors can be equivalently defined as homogeneous polynomials. More explicitly, let  $\{e_0, \dots, e_m\}$  be a basis of  $V$ . Similarly, for an integer  $d$ , let  $B$  be the monomial basis of the  $k$ -vector space  $k[x_0, \dots, x_m]_d$ , which has dimension  $\binom{m+d}{m}$ . In particular, elements in  $B$  can be regarded as multi-indices  $\mathbf{d} = (d_0, \dots, d_m)$  in  $\mathbb{Z}^{m+1}$  such that  $\sum d_j = d$ , where  $x^{\mathbf{d}} \equiv x_0^{d_0} \dots x_m^{d_m}$ . Then, a symmetric tensor of format  $(m+1)^{\times d}$  can be written as

$$\sum_{d_1=0}^m \dots \sum_{d_n=0}^m \lambda_{d_1 \dots d_n} e_{d_1} \otimes \dots \otimes e_{d_n}$$

with  $\lambda_{d_1 \dots d_n} = \lambda_{\sigma(d_1) \dots \sigma(d_n)}$  for every  $\sigma$  in the group  $S_n$  of permutations of  $\{1, \dots, n\}$ , or equivalently

$$\sum_{\mathbf{d} \in B} \frac{\lambda_{\mathbf{d}}}{d!} \sum_{\sigma \in S_n} e_{\sigma(d_1)} \otimes \dots \otimes e_{\sigma(d_n)} .$$

In particular, we can identify the former symmetric tensor with the homogeneous polynomial

$$\sum_{\mathbf{d} \in B} \frac{\lambda_{\mathbf{d}}}{d!} x^{\mathbf{d}} .$$

This yields a one-to-one identification between homogeneous polynomial in  $k[x_0, \dots, x_m]_d$  and symmetric tensors of format  $(m+1)^{\times d}$  with entries in  $k$ . Hence, symmetric tensors can be defined simply as homogeneous polynomials of degree  $d$  in  $k[x_0, \dots, x_m]$  (with the standard  $\mathbb{Z}$ -grading!).

## 2.2.2. Tensor rank and CP decompositions

The notions of tensor rank and tensor decomposition, originally introduced by Hitchcock [117] and later by Kruskal [130], is probably the central feature in the study of tensors, driven by both its theoretical significance and its multiple practical applications.

### 2.2.2.1. Tensor rank

The literature on tensor rank is vast, encompassing numerous aspects that extend beyond the scope of our discussion. Complexity issues (determining the rank of a tensor is an NP-hard problem), uniqueness of decompositions, and the existence of optimal approximations, among others, offer a wide variety of challenges. Interested readers are encouraged to delve into these subjects through [126] and the references therein.

We introduce tensor rank adopting an algebro-geometric point of view. First, we describe a very general definition of rank with respect to a nondegenerate projective variety. Later, we particularize this definition to the more usual notions of general and symmetric rank.

**Definition 2.2.8** (Nondegenerate variety). A variety  $V$  in  $\mathbb{P}_k^n$  is nondegenerate if it is not contained in a hyperplane.

**Definition 2.2.9** (Rank with respect to a variety, [26]). Let  $V$  be a nondegenerate variety in  $\mathbb{P}_k^n$ , and denote by  $\hat{V}$  its affine cone in  $k^{n+1}$ . The *rank of a point  $P$  in  $k^{n+1}$  with respect to  $V$* , denoted by  $\text{rank}_V(P)$ , is

$$\text{rank}_V(P) = \min\{r : P = \sum_{i=1}^r P_i \text{ where } P_i \text{ lies in } \hat{V} \text{ for every } 1 \leq i \leq r\}$$

**Remark 2.2.10.** The projective span of a nondegenerate variety is the projective space where it is embedded. This is the reason why we ask for nondegeneracy, to ensure that every point has a “rank” with respect to the variety.

What is the geometry of the spaces of general and symmetric tensors? We can identify all proportional tensors and dismiss the zero tensor, so that we can regard tensors as points in projective spaces. Thus, a general tensor of order  $(m_1 + 1) \times \dots \times (m_n + 1)$  is a point in  $\mathbb{P}_k^{(m_1+1)\dots(m_n+1)-1}$ , whereas a symmetric tensor of order  $(m + 1)^{\times n}$  is a point in  $\mathbb{P}_k^{\binom{m+n}{m}-1}$ .

**Notation 2.2.11.** For each  $1 \leq i \leq n$ , let  $\mathbf{x}_i = (x_{0i} : \dots : x_{m_i i})$  in  $\mathbb{P}_k^{m_i}$ . With a small abuse of notation, we write

$$\mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_n$$

for the tensor product  $v_1 \otimes \dots \otimes v_n$  of any vectors  $v_i$  in  $k^{m_i+1}$  such that  $v_i$  projectivizes to  $\mathbf{x}_i$ .

**Definition 2.2.12** (Segre embedding). The *Segre embedding* of  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$  is

$$\begin{aligned} \sigma : \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n} &\rightarrow \mathbb{P}_k^{(m_1+1)\dots(m_n+1)-1} \\ \mathbf{x}_1 \times \dots \times \mathbf{x}_n &\mapsto \mathbf{x}_1 \otimes \dots \otimes \mathbf{x}_n, \end{aligned} \quad (2.6)$$

i.e. the embedding defined by all multilinear monomials in  $k[\mathbf{x}_1] \otimes \dots \otimes k[\mathbf{x}_n]$ . The image of this embedding is the *Segre variety*.

**Definition 2.2.13** (Veronese embedding). The *Veronese embedding* of degree  $d$  (in  $\mathbb{Z}$ ) of  $\mathbb{P}_k^m$  is

$$\begin{aligned} v_d : \mathbb{P}_k^m &\rightarrow \mathbb{P}_k^{\binom{m+d}{m}-1} \\ \mathbf{x} &\mapsto \mathbf{x}^{\otimes d}, \end{aligned} \quad (2.7)$$

i.e. the embedding defined by all homogeneous monomials in  $k[\mathbf{x}]_d$ . The image of this embedding is the *Veronese variety*, and its affine cone  $\hat{V}(n, d)$  in  $k^{m+1}$  is the *Veronese cone*. The *affine Veronese variety*, denoted by  $V(n, d)$ , is the embedding of the affine chart  $\mathbb{A}_k^m$  defined by  $x_0 \neq 0$ . More explicitly, it is the image (contained in the hyperplane  $y_0 = 1$ ) of

$$\begin{aligned} v_d : \mathbb{A}_k^m &\rightarrow \mathbb{P}_k^{\binom{m+d}{m}-1} \\ \mathbf{x} = (1, x_1, \dots, x_m) &\mapsto \mathbf{x}^{\otimes d}. \end{aligned} \quad (2.8)$$

**Definition 2.2.14** (Rank and symmetric rank). Let  $M$  be a general tensor of format  $(m_1 + 1) \times \dots \times (m_n + 1)$ , and let  $N$  be a symmetric tensor of format  $(m + 1)^d$ .

- (i) The *rank* of  $M$ , denoted by  $\text{rank}(M)$ , is the rank of  $M$  as a point in  $k^{(m_1+1)\dots(m_n+1)}$  with respect to Segre variety
- (ii) The *symmetric rank* of  $N$ , denoted by  $\text{rank}_S(N)$ , is the rank of  $N$  as a point in  $k^{\binom{m+d}{m}}$  with respect to the Veronese variety

In particular, the affine cones over the Segre and Veronese varieties are the loci of rank-one tensors.

**Remark 2.2.15.** The rank of  $M$  can be recovered as

$$\text{rank}(M) = \min\{r : M = \sum_{i=1}^r \lambda_i \mathbf{x}_{1i} \otimes \dots \otimes \mathbf{x}_{ni} \text{ where } \mathbf{x}_{ji} \in k^{m_j+1} \text{ and } \lambda_i \in k \text{ for every } 1 \leq i \leq r\}.$$

Similarly, the symmetric rank of  $N$  can be expressed as

$$\text{rank}_S(N) = \min\{r : N = \sum_{i=1}^r \lambda_i \mathbf{x}_i^{\otimes d} \text{ where } \mathbf{x}_i \in k^{m+1} \text{ and } \lambda_i \in k \text{ for every } 1 \leq i \leq r\}.$$

The rank of a three-dimensional tensors can be characterized in terms of its flattenings.

**Lemma 2.2.16.** ([159, Proposition 3.11]) *A tensor of format  $a \times b \times c$  has rank one if and only if two of its flattenings have rank one.*

A natural query is whether the rank and the symmetric rank coincide for symmetric tensors. The answer is negative, as asserted by the following result.

**Theorem 2.2.17.** ([201, Theorem 1.9]) *There is an order six real tensor whose rank and symmetric rank differ.*

An important feature is that the rank of a tensor is sensitive to the choice of the base field. In general, the complex rank is different from the real rank for real tensors, as illustrated by the following example. For different examples, we refer the reader to [171, 136].

**Example 2.2.18.** ([126, §3.1]) The  $2 \times 2 \times 2$  real tensor  $M$  defined by

$$M_{\{k=0\}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{\{k=1\}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

admits the minimal decomposition over  $\mathbb{R}$

$$M = (1 \ 0) \otimes (1 \ 0) \otimes (1 \ -1) + (0 \ 1) \otimes (0 \ 1) \otimes (1 \ 1) + (1 \ -1) \otimes (1 \ 1) \otimes (0 \ 1),$$

whereas it can also be written as

$$M = \frac{1}{2} (1 \ -i) \otimes (1 \ i) \otimes (1 \ i) + \frac{1}{2} (1 \ i) \otimes (1 \ -i) \otimes (1 \ -i).$$

In some contexts, the coefficients in the linear combinations of the definition of the rank are constrained. This is the case for the nonnegative symmetric rank of real tensors.

**Definition 2.2.19.** The *nonnegative symmetric rank* of a real symmetric tensor  $N$  is

$$\min\{r : N = \sum_{i=1}^r \lambda_i \mathbf{x}_i^{\otimes d} \text{ where } \mathbf{x}_i \in \mathbb{R}^{m+1} \text{ and } \lambda_i \geq 0 \text{ for every } 1 \leq i \leq r\}$$

in the case that it exists, or  $\infty$  otherwise.

Similarly to the choice of field, it is interesting to see how the rank is affected when restricting to nonnegative combinations. In Chapter 6, we are interested in nonnegative combinations over the Veronese cone  $\hat{V}(n, d)$ , since the maximum nonnegative symmetric rank can be regarded as the Carethéodory number of  $\hat{V}(n, d)$ .

### 2.2.2.2. CP decompositions

The *CP decomposition* of a tensor is its expression as a finite sum of rank-one tensors. Over time, different authors have introduced various names for this decomposition, including polyadic decomposition [117], PARAFAC [110], CANDECOMP [48], CP (CANDECOMP/PARAFAC) [48]. When the number of rank-one tensors in the decomposition equals the rank of the tensor, we refer to *canonical* CP decompositions.

In practice, the computation of an exact CP decomposition is often not realistic or very challenging. Therefore, the problem frequently involves finding an approximation.

**Definition 2.2.20** (Frobenius norm). Let  $M = (\lambda_{i_1 \dots i_n})_{i_1, \dots, i_n}$  be a complex or real tensor of format  $(m_1 + 1) \times \dots \times (m_n + 1)$ . We define its *Frobenius norm* as

$$\|M\|^2 = \sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} |\lambda_{i_1 \dots i_n}|^2.$$

The task of computing a rank- $r$  CP decomposition  $N$  for a tensor  $M$  of order  $n$  can be formally expressed as the optimization problem

$$\min_N \|M - N\| \text{ where } N = \sum_{i=1}^r \mathbf{x}_{1i} \otimes \dots \otimes \mathbf{x}_{ni} \text{ and } \mathbf{x}_{ij} \text{ lies in either } \mathbb{R}^n \text{ or } \mathbb{C}^n.$$

In particular, this formulation necessitates an appropriate rank  $r$  as an input. In most practical scenarios, this rank is unknown (and NP-hard to compute). Consequently, algorithms typically proceed by employing various rank until an acceptable approximation of  $M$  is achieved. Low-rank approximations are often preferred in such cases. Several strategies for computing CP approximations exist, though we do not delve into their technicalities here. Some notable approaches include Alternating Least Squares (ALS) [48, 110], nonlinear least squares [32, §3.4], and limited-memory BFGS method [192]. Furthermore, several software packages are available, such as *TensorFox* [32] in *Python* and the *tensor toolbox* [125] in *Matlab*. In this thesis, we adopt a user perspective by employing *TensorFox* to compute rank-one approximations of  $2 \times 2 \times 2$  tensors.

### 2.2.3. Some spaces of tensors

Since the locus of rank-one tensors of format  $(m_1 + 1) \times \dots \times (m_n + 1)$  is  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$ , it is natural to study the orbits of the natural group action of  $\text{PGL}(m_1 + 1, k) \times \text{PGL}(m_n + 1, k)$  on the space of tensors. More explicitly, this action is given by

$$(\varphi_1 \times \dots \times \varphi_n) \times \left( \sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} \lambda_{i_1 \dots i_n} x_{i_1 1} \dots x_{i_n n} \right) \mapsto \left( \sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} \lambda_{i_1 \dots i_n} \varphi_1(x_{i_1 1}) \dots \varphi_n(x_{i_n n}) \right).$$

In the case of matrices, the orbits of this group action are well-known: the orbits are the loci of matrices of fixed rank. For higher-dimensional tensors, the study of these orbits is in general very

difficult. In the following subsection, we explain the first nonmatricial example: general tensors of rank  $2 \times 2 \times 2$  over  $\mathbb{C}$ . Tensors of this format are identified to trilinear polynomials, which define the rational maps studied in Chapters 3 and 4.

### 2.2.3.1. The space of trilinear polynomials

In this subsection, we describe the geometry of the space  $\mathbb{P}_{\mathbb{C}}^7$  of  $2 \times 2 \times 2$  tensors, or equivalently trilinear polynomials in  $\mathbb{C}[s_0, s_1] \otimes \mathbb{C}[t_0, t_1] \otimes \mathbb{C}[u_0, u_1]$ . There are six orbits of the action of  $G = \text{PGL}(2, \mathbb{C})^3$ , represented in Figure 2.1. We describe the most important features of these orbits in the following lines:

- (i) The Segre variety  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  is the orbit of rank-one tensors, or trilinear polynomials with three factors. This orbit is straightforward, since elements in  $G$  are automorphisms of  $(\mathbb{P}_{\mathbb{C}}^1)^3$ . It is a closed orbit, generated by the action of  $G$  on  $s_0 t_0 u_0$ .
- (ii) Above  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  we find three Segre varieties  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3)$ , corresponding to the trilinear polynomials with one linear factor. Each of the three embeddings represent the polynomials with a factor in  $\{s_0, s_1\}$ ,  $\{t_0, t_1\}$ , and  $\{u_0, u_1\}$ , respectively. The corresponding orbits of  $G$  are the complements of  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  in each of the  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3)$ . Tensors in these orbits have rank two, and are respectively generated by the action of  $G$  on  $s_0(t_0 u_0 + t_1 u_1)$ ,  $t_0(s_0 u_0 + s_1 u_1)$ , and  $u_0(s_0 t_0 + s_1 t_1)$ .
- (iii) The closure of the next orbit is the hyperdeterminant  $\text{Det}$  (Definition 2.2.27) of format  $2 \times 2 \times 2$ , defined by the polynomial

$$\begin{aligned}
 H = H(\lambda) &= \lambda_{000}^2 \lambda_{111}^2 + \lambda_{001}^2 \lambda_{110}^2 + \lambda_{010}^2 \lambda_{101}^2 + \lambda_{100}^2 \lambda_{011}^2 - 2 \lambda_{000} \lambda_{001} \lambda_{110} \lambda_{111} \quad (2.9) \\
 &- 2 \lambda_{000} \lambda_{010} \lambda_{101} \lambda_{111} - 2 \lambda_{000} \lambda_{011} \lambda_{100} \lambda_{111} - 2 \lambda_{001} \lambda_{010} \lambda_{101} \lambda_{110} - 2 \lambda_{001} \lambda_{011} \lambda_{110} \lambda_{100} \\
 &- 2 \lambda_{010} \lambda_{011} \lambda_{101} \lambda_{100} + 4 \lambda_{000} \lambda_{011} \lambda_{0101} \lambda_{110} + 4 \lambda_{001} \lambda_{010} \lambda_{100} \lambda_{111} .
 \end{aligned}$$

Interestingly, we have the following result.

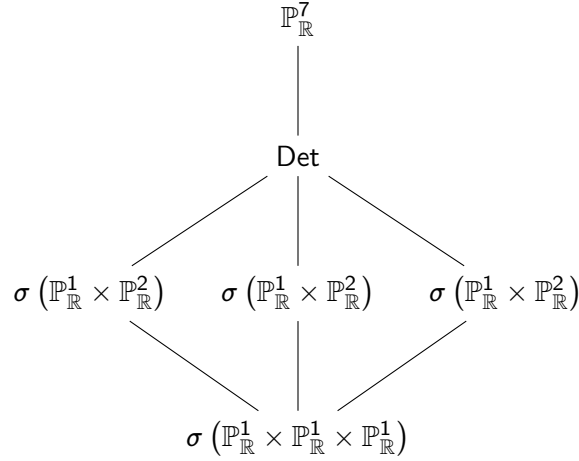
**Lemma 2.2.21.** ([98, §14.1.A]) *Let  $\varphi = \varphi_1 \times \varphi_2 \times \varphi_3$  in  $\text{GL}(2, \mathbb{C})^3$ , and let  $M = (\lambda_{ijk})_{0 \leq i, j, k \leq 1}$  be a  $2 \times 2 \times 2$  tensor. Then,*

$$H(\varphi \cdot M) = \det(\varphi_1) \det(\varphi_2) \det(\varphi_3) H(M) .$$

In particular,  $\text{Det}$  is invariant by the action of  $G$ . Specifically, the complement in  $\text{Det}$  of the union of the three Segre varieties  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^3)$  defines an orbit, which is generated by the action of  $\text{PGL}(2, \mathbb{C})^3$  on  $s_1 t_0 u_0 + s_0 t_1 u_0 + s_0 t_0 u_1$ . This orbit consists of the trilinear polynomials with exactly one singular point. Remarkably, tensors in this orbit have rank three.

- (iv) Finally, the complement of  $\text{Det}$  in  $\mathbb{P}_{\mathbb{C}}^7$  defines an open orbit of tensors of rank two. This orbit consists of the trilinear polynomials without a singular point.





**Figure 2.1.:** The closure of the six orbits of the action of  $\mathrm{PGL}(2, \mathbb{C})^3$  in  $\mathbb{P}_{\mathbb{C}}^7$ . An orbit is contained in all the orbits above it.

### 2.2.3.2. Quadrics in $\mathbb{P}_{\mathbb{C}}^3$

Quadrics in  $\mathbb{P}_{\mathbb{C}}^3$  are defined by homogeneous quadratic polynomials in  $\mathbb{C}[\mathbf{y}] = \mathbb{C}[y_0, y_1, y_2, y_3]$ , or equivalently, symmetric  $4 \times 4$  matrices. In particular, the space of quadrics is stratified by the matrix rank (see e.g. [159, Theorem 1.1]). There are four orbits of the natural group action given by the automorphisms of  $\mathbb{P}_{\mathbb{C}}^3$ : nonsingular (rank 4), cones (3), union of planes (2), and double planes (1).

Quadrics are important in these thesis, since the inverse for at least one of the parameters is quadratic in all classes of trilinear birational maps, except for birational maps with a multilinear inverse, and thus the parametric surfaces are quadrics. Additionally, the geometric constraints on the control points for some of these classes require to be explained using properties of quadric surfaces.

We gather the following elementary results, which we will utilize in Chapter 4 for the description of the admissible configurations of control points for birationality.

**Lemma 2.2.22.** ([111, Exercise 2.15] or [109, Exercise 6.6]) *A smooth quadric in  $\mathbb{P}_{\mathbb{C}}^3$  is isomorphic to  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ .*

**Lemma 2.2.23.** ([49, Corollary 8.3.19] or [109, Exercise 2.12]) *Let  $a, b, c$  be three mutually nonintersecting lines in  $\mathbb{P}_{\mathbb{C}}^3$ . There exists a unique quadric  $Q \subset \mathbb{P}_{\mathbb{C}}^3$  through  $a, b, c$ , which is smooth.*

**Lemma 2.2.24.** *Let  $l_{00}, l_{10}, l_{01}, l_{11}$  be four mutually nonintersecting lines in  $\mathbb{P}_{\mathbb{C}}^3$ . If the four lines do not lie on a common quadric, then there are exactly two (counting multiplicity) lines  $r_1, r_2$  that meet  $l_{ij}$  for every  $0 \leq i, j \leq 1$ .*

*Proof.* By Lemma 2.2.23, the lines  $l_{00}, l_{10}, l_{01}$  determine a unique smooth quadric  $Q \subset \mathbb{P}_{\mathbb{C}}^3$ . Since by hypothesis  $l_{11}$  does not lie in  $Q$ ,  $l_{11}$  meets  $Q$  at two points  $P_1, P_2$  (maybe one with multiplicity two). Thus, the only lines that meet the four lines are the two lines in the ruling of  $Q$ , distinct from the ruling of  $l_{00}, l_{10}, l_{01}$ , through the points  $P_1, P_2$  respectively.  $\square$

### 2.2.4. Bilinear birational maps

This subsection is a warm-up for our study of trilinear birational maps of Chapters 3, and we study bilinear birational maps. In particular, we establish a connection between birationality, hyperdeterminants, and syzygies.

Bilinear birational maps are the two-dimensional analogs of the multilinear birational maps with multilinear inverse in arbitrary dimension studied in Chapter 5. Specifically, the resolution of the base ideal of these transformations is always Hilbert-Burch.

A bilinear rational map is a rational map

$$\begin{aligned} \phi : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^2 \\ (s_0 : s_1) \times (t_0 : t_1) &\mapsto (f_0 : f_1 : f_2) = \left( \sum_{0 \leq i, j \leq 1} \lambda_{0ij} s_i t_j : \sum_{0 \leq i, j \leq 1} \lambda_{1ij} s_i t_j : \sum_{0 \leq i, j \leq 1} \lambda_{2ij} s_i t_j \right) \end{aligned}$$

In particular,  $\phi$  can be encoded as the tensor  $\phi \equiv (\lambda_{ijk})_{ijk}$  of format  $3 \times 2 \times 2$ . Thus, the space of bilinear rational maps is  $\mathbb{P}_{\mathbb{C}}^{3 \cdot 2 \cdot 2 - 1} = \mathbb{P}_{\mathbb{C}}^{11}$ . We have the following characterization of their birationality.

**Proposition 2.2.25.** ([28, Proposition 8]) *Let  $\phi$  be dominant. The following are equivalent:*

- (i)  $\phi$  is birational
- (ii) The base ideal  $B = (f_0, f_1, f_2)$  defines a point in  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$
- (iii)  $(f_0, f_1, f_2)$  admits a syzygy of degree  $(1, 0)$
- (iv)  $(f_0, f_1, f_2)$  admits a syzygy of degree  $(0, 1)$

We denote by  $\boldsymbol{\lambda} = (\dots, \lambda_{ijk}, \dots)$  the vector (or 3D-array) of the variables  $\lambda_{ijk}$ , for each  $0 \leq i, j \leq 1$  and  $0 \leq k \leq 2$ . We have the following characterization of birationality.

**Corollary 2.2.26.** *The space of bilinear birational maps is the hyperdeterminant of format  $3 \times 2 \times 2$  in  $\mathbb{P}_{\mathbb{C}}^{11}$ . More explicitly,  $\phi$  is birational if and only if  $H(\phi) = 0$ , where  $H = H(\boldsymbol{\lambda})$  is the defining polynomial of this hyperdeterminant.*

In this section, we introduce the minimal concepts for Corollary 2.2.26 to make sense. As the name suggests, hyperdeterminants are higher-dimensional analogs of matrix determinants, that were first studied by Cayley [52].

**Definition 2.2.27** (Hyperdeterminant). The *hyperdeterminant*, or simply *determinant*, of format  $(m_1 + 1) \times \dots \times (m_n + 1)$  is the projective dual of the Segre variety  $\sigma(\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n})$ . Explicitly, it is discriminant of multilinear polynomials in  $k[\mathbf{x}_1] \otimes \dots \otimes k[\mathbf{x}_n]$ , i.e. the subscheme of the space of coefficients  $\mathbb{P}_{\mathbb{C}}^{(m_1+1)\dots(m_n+1)-1}$  for which the generic multilinear polynomial with coefficients in  $\mathbb{Z}[\boldsymbol{\lambda}]$ , where  $\boldsymbol{\lambda}$  is the vector of all the variables  $\lambda_{i_1 \dots i_n}$  where  $0 \leq i_j \leq m_j$  and  $1 \leq j \leq n$ , given by

$$F = \sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} \lambda_{i_1 \dots i_n} x_{i_1 1} \dots x_{i_n n}$$

has a singular point, i.e. the ideal generated all the partial derivatives

$$\frac{\partial F}{\partial x_{ij}}$$

defines (at least) a point in  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$ .

**Example 2.2.28.** We show that the hyperdeterminant of square matrices coincides with the classical determinant. Namely, a matrix of size  $(m+1) \times (m+1)$  is a bilinear polynomial

$$M = \sum_{i=0}^m \sum_{j=0}^m \lambda_{ij} s_i t_j$$

in  $k[s_0, \dots, s_m] \otimes k[t_0, \dots, t_m]$ . In particular, the partial derivatives can be identified with the rows and columns of  $M$ , as

$$\frac{\partial M}{\partial s_i} = \sum_{j=0}^m \lambda_{ij} t_j, \quad \frac{\partial M}{\partial t_j} = \sum_{i=0}^m \lambda_{ij} s_i$$

for each  $0 \leq i, j \leq m$ . Therefore,

$$\left( \frac{\partial M}{\partial t_0} \quad \dots \quad \frac{\partial M}{\partial t_m} \right) = (s_0 \quad \dots \quad s_m) \begin{pmatrix} \lambda_{00} & \dots & \lambda_{0m} \\ \vdots & \ddots & \vdots \\ \lambda_{m0} & \dots & \lambda_{mm} \end{pmatrix} = (0 \quad \dots \quad 0)$$

has a solution in  $\mathbb{P}_k^m$  if and only if

$$\begin{pmatrix} \frac{\partial M}{\partial s_0} \\ \vdots \\ \frac{\partial M}{\partial s_m} \end{pmatrix} = \begin{pmatrix} \lambda_{00} & \dots & \lambda_{0m} \\ \vdots & \ddots & \vdots \\ \lambda_{m0} & \dots & \lambda_{mm} \end{pmatrix} \begin{pmatrix} t_0 \\ \vdots \\ t_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a solution in  $\mathbb{P}_k^m$  if and only if  $\det(\lambda_{ij})_{0 \leq i, j \leq m} = 0$ .

The following definition encompasses the tensors of interest in the study of multilinear rational maps between spaces of equal dimension (recall Definition 2.1.30).

**Definition 2.2.29** (Boundary format). A  $(m_0+1) \times \dots \times (m_n+1)$  tensor has *boundary format* if  $m_0 = \sum_{i=1}^n m_i$ .

**Remark 2.2.30.** Rational maps from a multiprojective space to a projective space of the same dimension are encoded by tensors of boundary format. Specifically, the rational map

$$\begin{aligned} \phi : \mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n} &\dashrightarrow \mathbb{P}_k^{m_0} \\ \mathbf{x}_1 \times \dots \times \mathbf{x}_n &\mapsto (f_0 : \dots : f_{m_0}) \end{aligned}$$

where

$$f_{i_0} = f_{i_0}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} \lambda_{i_0 i_1 \dots i_n} x_{i_1 1} \dots x_{i_n n} \quad (2.10)$$

for each  $0 \leq j \leq n$  and  $0 \leq i_j \leq m_j$ , can be represented by the  $(m_0+1) \times \dots \times (m_n+1)$  tensor

$$\phi \equiv \sum_{i_0=0}^{m_0} \sum_{i_1=0}^{m_1} \dots \sum_{i_n=0}^{m_n} \lambda_{i_0 i_1 \dots i_n} x_{i_0 0} x_{i_1 1} \dots x_{i_n n}. \quad (2.11)$$

The following result asserts that the hyperdeterminant of a tensor of boundary format  $(m_0 + 1) \times \dots \times (m_n + 1)$  is a resultant of multilinear polynomials.

**Theorem 2.2.31.** ([98, Theorem 3.1]) *The hyperdeterminant of a tensor of boundary format as (2.11) is the resultant of the polynomials  $f_0, \dots, f_{m_0}$ . Explicitly, a point  $\lambda = (\dots : \lambda_{i_0 i_1 \dots i_n} : \dots)$  lies in the hyperdeterminant if and only if the system of multilinear polynomials*

$$f_0 = \dots = f_{m_0} = 0$$

has a solution in  $\mathbb{P}_k^{m_1} \times \dots \times \mathbb{P}_k^{m_n}$ .

*Proof of Corollary 2.2.26.* By Proposition 2.2.25, the bilinear rational map  $\phi$  is birational if and only if the base ideal  $B = (f_0, f_1, f_2)$  defines a point in  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ , or equivalently, if  $\phi$  lies in the resultant of  $f_0, f_1, f_2$ . By Theorem 2.2.31, this is precisely the hyperdeterminant of format  $3 \times 2 \times 2$ .  $\square$

Now, we describe the geometry of the syzygies of a bilinear birational map within the space  $\mathbb{P}_{\mathbb{C}}^3$  of bilinear polynomials.

**Corollary 2.2.32.** *A bilinear rational map  $\phi$  is birational if and only if the  $\mathbb{C}$ -vector space  $\mathbb{C}\langle f_0, f_1, f_2 \rangle$  defines a tangent plane to the Segre variety  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) \subset \mathbb{P}_{\mathbb{C}}^3$ , or equivalently, the projectivization of  $\mathbb{C}\langle f_0, f_1, f_2 \rangle$  determines a point in the dual of  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  (i.e. the determinant of  $2 \times 2$  matrices).*

*Proof.* Since bilinear polynomials are  $2 \times 2$  matrices, we can regard the linear system  $\mathbb{C}\langle f_0, f_1, f_2 \rangle$  as a plane in the space  $\mathbb{P}_{\mathbb{C}}^3$  of  $2 \times 2$  matrices, or equivalently, as a point in the dual  $(\mathbb{P}_{\mathbb{C}}^3)^\vee$ . Furthermore, we can write

$$\mathbf{f} = (f_0, f_1, f_2) = s_0 \mathbf{f}_0 + s_1 \mathbf{f}_1 = s_0 (t_0 \mathbf{P}_{00} + t_1 \mathbf{P}_{01}) + s_1 (t_0 \mathbf{P}_{10} + t_1 \mathbf{P}_{11})$$

for some  $\mathbf{P}_{ij}$  in  $\mathbb{C}^3$ . By Proposition 2.2.25,  $\phi$  is birational if and only if it admits a syzygy of degree  $(1, 0)$ . In particular, we find

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(s_0, s_1) = \boldsymbol{\sigma}_0 s_0 + \boldsymbol{\sigma}_1 s_1$$

for some  $\boldsymbol{\sigma}_i$  in  $k^3$ , such that

$$\langle \boldsymbol{\sigma}, \mathbf{f} \rangle = s_0 \langle \boldsymbol{\sigma}_0, \mathbf{f}_0 \rangle + s_1 \langle \boldsymbol{\sigma}_1, \mathbf{f}_1 \rangle = s_0^2 \langle \boldsymbol{\sigma}_0, \mathbf{f}_0 \rangle + s_0 s_1 (\langle \boldsymbol{\sigma}_0, \mathbf{f}_1 \rangle + \langle \boldsymbol{\sigma}_1, \mathbf{f}_0 \rangle) + s_1^2 \langle \boldsymbol{\sigma}_1, \mathbf{f}_1 \rangle = 0.$$

Hence, we have  $\langle \boldsymbol{\sigma}_i, \mathbf{f}_i \rangle = 0$  and

$$\langle \boldsymbol{\sigma}_0, \mathbf{f}_1 \rangle = t_0 \langle \boldsymbol{\sigma}_0, \mathbf{P}_{10} \rangle + t_1 \langle \boldsymbol{\sigma}_0, \mathbf{P}_{11} \rangle = g(t_0, t_1) = -t_0 \langle \boldsymbol{\sigma}_1, \mathbf{P}_{00} \rangle - t_1 \langle \boldsymbol{\sigma}_1, \mathbf{P}_{01} \rangle = -\langle \boldsymbol{\sigma}_1, \mathbf{f}_0 \rangle.$$

In particular,

$$\langle \boldsymbol{\sigma}_0, \mathbf{f} \rangle = s_1 g(t_0, t_1), \quad \langle \boldsymbol{\sigma}_1, \mathbf{f} \rangle = -s_0 g(t_0, t_1)$$

and the image of the parametrization  $\mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{C}\langle f_0, f_1, f_2 \rangle$  defined by

$$(s_0 : s_1) \mapsto \langle \boldsymbol{\sigma}(s_0, s_1), \mathbf{f} \rangle$$

is a line  $\ell$  of degenerate bilinear polynomials, i.e. they have two linear factors. Thus,  $\ell$  lies in  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$ , which is the locus of degenerate or rank-one bilinear polynomials. Since  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  is a smooth quadric and  $\mathbb{C}\langle f_0, f_1, f_2 \rangle$  is a plane, their intersection is a singular conic. Therefore,  $\mathbb{C}\langle f_0, f_1, f_2 \rangle$  must be a tangent plane to  $\sigma(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$ , and hence a point in the  $2 \times 2$  determinant.  $\square$

In Chapter 4, we try to extend some of the ideas briefly exposed in this subsection to trilinear birational maps. As already illustrated in §2.2.3.1, the space of trilinear polynomials is much more complicated than the space of bilinear polynomials. This makes the study of trilinear birational maps more complicated. In particular, there is no hope for a characterization of the space of trilinear birational maps as a hyperdeterminant, since the base loci of trilinear birational maps are not mere simple points.

**Part I**

**MULTILINEAR BIRATIONAL  
MAPS**



# Chapter 3

## Trilinear birational maps in dimension three

In this chapter, we address algebraic and geometric aspects about trilinear birational maps in dimension three. In all the chapter, we assume that  $\phi$  is as in Definition 1.2.1.

**Commutative algebra:** we provide a characterization of birationality based on the first syzygies of the entries (Theorem 3.5.1). More generally, we describe all the possible minimal graded free resolutions of the ideal  $B = (f_0, f_1, f_2, f_3)$  in the tensor product ring  $R = \mathbb{C}[s_0, s_1] \otimes \mathbb{C}[t_0, t_1] \otimes \mathbb{C}[u_0, u_1]$  (Propositions 3.5.2, 3.5.4, 3.5.5, and 3.5.8).

**Algebraic geometry:** we show that quotient  $\mathfrak{Bir}_{(1,1,1)}$  of the set of trilinear birational maps by  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^3)$  is a locally closed algebraic subset of the Grassmannian  $\text{Gr}(4, R_{(1,1,1)})$ , and has eight irreducible components (Theorem 3.3.6). We also prove that the group action on  $\mathfrak{Bir}_{(1,1,1)}$  given by  $\text{Aut}((\mathbb{P}_{\mathbb{C}}^1)^3)$  defines 19 orbits, and each of these orbits determines an isomorphism class of the base loci of these transformations (Theorems 3.4.3, 3.4.6, 3.4.8, and 3.4.10).

### 3.1. Preliminaries

In this section, we present the fundamental concepts and notation specific to this chapter. While some of these concepts were introduced more generally in Chapter 2, here we provide brief, specialized definitions tailored to the context of trilinear birational maps.

#### 3.1.1. Basic definitions and properties

Let  $A_1 = \mathbb{C}[s_0, s_1]$ ,  $A_2 = \mathbb{C}[t_0, t_1]$ , and  $A_3 = \mathbb{C}[u_0, u_1]$  be the homogeneous coordinate rings of three projective lines, with the standard grading in each ring given by the corresponding pair of variables.

**Notation 3.1.1.** We set  $R := A_1 \otimes A_2 \otimes A_3$ , which is a standard  $\mathbb{Z}^3$ -graded ring. The multiprojective scheme associated to  $R$  is the product  $X := (\mathbb{P}_{\mathbb{C}}^1)^3$ . The irrelevant ideal of  $R$  is

$$\mathfrak{R} = \bigoplus_{i>0, j>0, k>0} (A_1)_i \otimes (A_2)_j \otimes (A_3)_k = (s_0, s_1) \cap (t_0, t_1) \cap (u_0, u_1).$$

**Definition 3.1.2** (Base ideal and base locus of  $\phi$ ). The *base ideal* of  $\phi$  is the homogeneous ideal  $B = (f_0, f_1, f_2, f_3)$  in  $R$ . The subscheme  $Z$  of  $X$  defined by  $B$  is the *base locus* of  $\phi$ .

**Remark 3.1.3.** If  $\phi$  is dominant, then  $\text{codim}(Z) \geq 2$  since otherwise we find a common factor to the entries.



Let  $(\lambda_0 : \lambda_1) \in \mathbb{P}_{\mathbb{C}}^1$  be a general point. If  $\phi$  is dominant, then the image of the restriction

$$\begin{aligned} \phi_s : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^3 \\ (t_0 : t_1) \times (u_0 : u_1) &\mapsto (f'_0 : f'_1 : f'_2 : f'_3), \end{aligned} \quad (3.1)$$

where  $f'_i = f'_i(t_0, t_1, u_0, u_1) = f_i(\lambda_0, \lambda_1, t_0, t_1, u_0, u_1)$ , is a parametric  $s$ -surface, described in Definition 1.2.10, and is either a plane or a smooth quadric. The same observation holds true for the restrictions  $\phi_t$  and  $\phi_u$  to the other two parameters, which respectively yield parametric  $t$ - and  $u$ -surfaces.

**Remark 3.1.4.** More precisely,  $\phi_s$  is a bilinear parametrization to  $\mathbb{P}_{\mathbb{C}}^3$ . In particular, for each  $0 \leq i \leq 3$  we can write

$$f'_i(t_0, t_1, u_0, u_1) = \alpha_{0i} t_0 u_0 + \alpha_{1i} t_1 u_0 + \alpha_{2i} t_0 u_1 + \alpha_{3i} t_1 u_1$$

for some  $\alpha_{ji}$  not all zero. If the image of  $\phi_s$  does not lie on a plane, then the  $4 \times 4$  coefficient matrix  $(\alpha_{ji})_{ji}$  has full rank. Therefore,  $\phi_s$  coincides with the Segre embedding of  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  up to an automorphism of  $\mathbb{P}_{\mathbb{C}}^3$ , and is thus a smooth quadric.

The following definition extends the notion of type, introduced in Definition 1.2.3 for trilinear birational maps, to trilinear rational maps without the requirement of being birational.

**Definition 3.1.5** (Type of  $\phi$ ). The *type of  $\phi$*  is the triple  $(d_1, d_2, d_3)$  in  $\mathbb{Z}^3$ , where  $d_1$  (resp.  $d_2$  and  $d_3$ ) is the degree of the  $s$ -surfaces (resp.  $t$ - and  $u$ -surfaces).

It follows immediately that there are only eight possible types for  $\phi$ , namely

$$(1, 1, 1), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 2, 2), (2, 1, 2), (2, 2, 1), (2, 2, 2), \quad (3.2)$$

which up to permutation, reduce to the four possibilities

$$(1, 1, 1), (1, 1, 2), (1, 2, 2), (2, 2, 2).$$

Let  $\phi$  be birational with an inverse rational map as in (1.6), where  $\mathbf{y} = (y_0, y_1, y_2, y_3)$  are standard  $\mathbb{Z}$ -graded variables. The composition  $\phi^{-1} \circ \phi : (\mathbb{P}_{\mathbb{R}}^1)^3 \dashrightarrow (\mathbb{P}_{\mathbb{R}}^1)^3$  yields the identity on some open set of  $X$ . More explicitly,  $\phi^{-1} \circ \phi$  is generically given by

$$\begin{aligned} (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (a_0(\mathbf{f}) : a_1(\mathbf{f})) \times (b_0(\mathbf{f}) : b_1(\mathbf{f})) \times (c_0(\mathbf{f}) : c_1(\mathbf{f})) \\ &= (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1), \end{aligned}$$

implying that the  $2 \times 2$  determinants in  $R[\mathbf{y}]$

$$\begin{vmatrix} s_0 & s_1 \\ a_0(\mathbf{y}) & a_1(\mathbf{y}) \end{vmatrix}, \quad \begin{vmatrix} t_0 & t_1 \\ b_0(\mathbf{y}) & b_1(\mathbf{y}) \end{vmatrix}, \quad \begin{vmatrix} u_0 & u_1 \\ c_0(\mathbf{y}) & c_1(\mathbf{y}) \end{vmatrix} \quad (3.3)$$

vanish when we specialize  $y_n \mapsto f_n$  for each  $0 \leq n \leq 3$ . Therefore, these determinants represent algebraic relations satisfied by the defining polynomials of  $\phi$ . Geometrically, they define the implicit equations of the parametric surfaces. More specifically, given a general  $(s_0 : s_1) = (\lambda_0 : \lambda_1)$  in  $\mathbb{P}_{\mathbb{R}}^1$ , the equation of the corresponding  $s$ -surface in  $\mathbb{P}_{\mathbb{R}}^3$  is

$$\begin{vmatrix} \lambda_0 & \lambda_1 \\ a_0(\mathbf{y}) & a_1(\mathbf{y}) \end{vmatrix} = 0,$$

and similarly for the  $t$ - and  $u$ -surfaces. In particular, if  $a_0(\mathbf{y}), a_1(\mathbf{y})$  are linear, then the  $s$ -surfaces form a pencil of planes. If they are quadratic, then the  $s$ -surfaces form a pencil of quadric surfaces. The obvious observations hold for the  $t$ - and  $u$ -surfaces, as well as the following remark.

**Remark 3.1.6.** If  $\phi$  is birational with an inverse as in (1.6), then the type of  $\phi$  is  $(\deg a_i, \deg b_j, \deg c_k)$ . In particular, the notions of type of Definitions 1.2.3 and 3.1.5 coincide on birational maps.

### 3.1.2. The Jacobian dual criterion for trilinear rational maps

Let  $R[\mathbf{y}] = R[y_0, y_1, y_2, y_3]$  be the  $(\mathbb{Z}^3 \times \mathbb{Z})$ -graded ring of the product  $X \times \mathbb{P}_{\mathbb{R}}^3$ , since  $R[\mathbf{y}]$  inherits the  $\mathbb{Z}^3$ -grading from  $R$  and the standard  $\mathbb{Z}$ -grading of  $\mathbb{C}[\mathbf{y}]$ . Consider the ring homomorphism

$$\begin{aligned} R[\mathbf{y}] &\rightarrow R[u] \\ y_i &\mapsto f_i u . \end{aligned} \quad (3.4)$$

**Definition 3.1.7** (Rees ideal of  $\phi$ ). The kernel  $J$  of (3.4) is the *Rees ideal* of  $\phi$ .

The Rees ideal  $J$  is prime, and hence saturated with respect to the irrelevant  $\mathfrak{N}$ . Specifically,  $J$  defines the scheme-theoretic graph of  $\phi$  in  $X \times \mathbb{P}_{\mathbb{C}}^3$  (recall §2.1.4), which is irreducible. In particular, if  $\phi$  is birational, the polynomials in (3.3) belong to  $J$ .

**Notation 3.1.8.** Given a vector of polynomials  $\mathbf{g} = (g_0, g_1, g_2, g_3)$  in  $R$ , we denote by  $\mathbf{y} \mapsto \mathbf{g}$  the specialization of the ring  $R[\mathbf{y}]$  to  $R$  given by  $y_n \mapsto g_n$  for every  $0 \leq n \leq 3$ . Moreover, we use the notation  $(d_1, d_2, d_3; d_4) \in \mathbb{Z}^3 \times \mathbb{Z}$  for the degree in  $R[\mathbf{y}]$ . In particular, we have

$$g = g(s_0, s_1, t_0, t_1, u_0, u_1, y_0, y_1, y_2, y_3) \in J_{(d_1, d_2, d_3; d_4)}$$

if  $g$  is homogeneous of degree  $(d_1, d_2, d_3; d_4)$  and additionally

$$g(s_0, s_1, t_0, t_1, u_0, u_1, f_0, f_1, f_2, f_3) = 0 ,$$

i.e.  $g$  vanishes after the specialization  $\mathbf{y} \mapsto \mathbf{f}$ .

For any  $(d_1, d_2, d_3) \in \mathbb{Z}^3$ , we define the  $\mathbb{C}[\mathbf{y}]$ -module

$$J_{(d_1, d_2, d_3; *)} := \bigoplus_{d_4 \in \mathbb{Z}} J_{(d_1, d_2, d_3; d_4)} .$$

The multigraded Jacobian dual criterion (Theorem 2.1.40) characterizes birationality in terms of the ideal  $J$ . In particular, we find the following corollary.

**Corollary 3.1.9** (of Theorem 2.1.40). *The trilinear rational map  $\phi$  is birational if and only if all*

$$J_{(1,0,0; *)} , J_{(0,1,0; *)} , J_{(0,0,1; *)} \quad (3.5)$$

*have rank one as vector spaces over the field  $\mathbb{C}(\mathbf{y}) = \mathbb{C}(y_0, y_1, y_2, y_3)$ .*

### 3.1.3. Automorphisms of $(\mathbb{P}_{\mathbb{C}}^1)^3$

The automorphisms of  $X = (\mathbb{P}_{\mathbb{C}}^1)^3$  are in one-to-one correspondence with the automorphisms of  $R$  that leave the irrelevant ideal  $\mathfrak{N}$  invariant. Let  $H \trianglelefteq \text{Aut}(X)$  be the normal subgroup of all the automorphisms  $\varphi : X \rightarrow X$  such that the induced ring automorphism  $\varphi^\# : R \rightarrow R$  is graded of degree  $\mathbf{0} = (0, 0, 0)$ , i.e. it preserves the degree of the elements in  $R$ . Namely, these have the form

$$(s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) \mapsto \eta_1(s_0 : s_1) \times \eta_2(t_0 : t_1) \times \eta_3(u_0 : u_1) ,$$

for some  $\eta_1, \eta_2, \eta_3$  in  $\text{PGL}(2, \mathbb{C})$ . Clearly, we have an isomorphism  $H \cong \text{PGL}(2, \mathbb{C})^3$ .

**Example 3.1.10.** Not all the automorphisms of  $R$  are graded. In particular,

$$\begin{aligned} \varphi : X &\rightarrow X \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (t_0 : t_1) \times (u_0 : u_1) \times (s_0 : s_1) \end{aligned}$$

induces

$$\begin{aligned} \varphi^\# : R &\rightarrow R \\ s_i &\mapsto t_i \quad t_j \mapsto u_j \quad u_k \mapsto s_k \end{aligned}$$

which is not graded for any  $\mathbf{d} = (d_1, d_2, d_3)$  in  $\mathbb{Z}^3$ . In general, an automorphism of  $R$  always applies a permutation to the integers in the  $\mathbb{Z}^3$ -degree of  $R$ .

Let  $K \leq \text{Aut}(X)$  be the subgroup consisting of the automorphisms that simply permute the factors of  $X$ , which is isomorphic to the symmetric group  $\mathfrak{S}_3$  acting on three elements. Interestingly,  $H$  and  $K$  satisfy the following properties:

- $H \cap K = \{\text{id}_X\}$
- The map  $K \times H \rightarrow \text{Aut}(X) : (\sigma, \eta) \mapsto \sigma \circ \eta$  is a bijection of sets

Therefore,  $\text{Aut}(X)$  is isomorphic to the semidirect product of groups  $K \rtimes H \cong \mathfrak{S}_3 \rtimes \text{PGL}(2, \mathbb{C})^3$ .

### 3.1.4. Curves and surfaces in $(\mathbb{P}_{\mathbb{C}}^1)^3$

**Definition 3.1.11** (Surfaces and curves in  $(\mathbb{P}_{\mathbb{C}}^1)^3$ ). A subscheme  $Y$  of  $X$  is a *surface* (resp. *curve*) if it is equidimensional of dimension two (resp. one).

In particular, a surface in  $X$  is the vanishing locus of a homogeneous polynomial in  $R$ .

**Definition 3.1.12** (Tridegree of a subscheme in  $(\mathbb{P}_{\mathbb{C}}^1)^3$ ). Let  $Y$  be an equidimensional subscheme of  $X$  of dimension either one or two. The *tridegree* of  $Y$ , denoted by  $\text{trideg}(Y)$ , is defined as follows:

- If  $\dim(Y) = 2$ ,  $\text{trideg}(Y)$  is the degree of its defining polynomial
- If  $\dim(Y) = 1$ ,  $\text{trideg}(Y) = (d_1, d_2, d_3)$  where  $d_i$  is the number of points, counted with multiplicity, in the intersection of  $Y$  with the subspace of  $X$  determined by a general linear form in  $A_i$ . Equivalently,  $d_i$  is the degree of the projection of  $Y$  onto the  $i$ -th  $\mathbb{P}_{\mathbb{C}}^1$  factor of  $X$

If  $Y$  is not equidimensional, we define its tridegree as the tridegree of its top-dimensional scheme, i.e. the union of all the irreducible components of highest dimension.

**Remark 3.1.13.** Let  $Y$  be a subscheme of  $X$  with  $\dim(Y) \geq 1$ . Definition 3.1.12 gives the correct notion of  $\mathbb{Z}^3$ -degree of  $Y$ . Namely, the Chow ring of  $X$  is

$$CR(X) = \frac{\mathbb{Z}[a, b, c]}{(a^2, b^2, c^2)},$$

where  $a$ ,  $b$ , and  $c$  are respectively the classes of surfaces of tridegree  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ . This Chow ring is  $\mathbb{Z}$ -graded by the dimension of the classes. Specifically,

$$CR(X) = CR(X)_2 \oplus CR(X)_1 \oplus CR(X)_0,$$

where

$$CR(X)_2 = \mathbb{Z}\langle a, b, c \rangle, \quad CR(X)_1 = \mathbb{Z}\langle bc, ac, ab \rangle, \quad CR(X)_0 = \mathbb{Z}\langle abc \rangle.$$

Therefore, if  $\text{trideg}(Y) = (d_1, d_2, d_3)$  then the class  $[Y]$  of  $Y$  in  $CR(X)$  can be written as

$$[Y] = d_1 a + d_2 b + d_3 c$$

if  $\dim(Y) = 2$ , or

$$[Y] = d_1 bc + d_2 ac + d_3 ab$$

if  $\dim(Y) = 1$ . Additionally, the tridegree of  $Y$  can be regarded as the concatenation of all the multidegrees extracted by the Hilbert polynomial of  $Y$ , as explained in Definition 2.1.27 and Example 2.1.29.

**Notation 3.1.14.** If  $Y \subset X$  is zero-dimensional, the class  $[Y]$  in the Chow ring is determined by a single integer. For simplicity in upcoming notation, we denote  $\text{trideg}(Y) = (0, 0, 0)$  whenever  $Y$  is zero-dimensional.

### 3.1.4.1. Cohen-Macaulay curves in $(\mathbb{P}_{\mathbb{C}}^1)^3$

Recall that, by Lemma 2.1.13, a curve is Cohen-Macaulay if and only if it has no embedded points. The following is a useful description of the Cohen-Macaulay curves in  $X$  of the tridegrees of interest for the following sections.

**Lemma 3.1.15.** *Let  $Y$  be a Cohen-Macaulay curve in  $X$ . Then, we have the following:*

- If  $\text{trideg}(Y) = (1, 1, 1)$ , there is an automorphism of  $X$  that sends  $Y$  to the curve defined by one of the following ideals:

$$(i) (s_0 t_1 - s_1 t_0, s_0 u_1 - s_1 u_0, t_0 u_1 - t_1 u_0) \quad (v) (s_1, t_1) \cap (s_1, u_1) \cap (t_0, u_0)$$

$$(ii) (s_0 t_1 - s_1 t_0, u_1) \cap (s_1, t_0) \quad (vi) (s_1, t_1) \cap (s_1, u_1) \cap (t_1, u_0)$$

$$(iii) (s_0 t_1 - s_1 t_0, u_1) \cap (s_1, t_1) \quad (vii) (s_1, t_1) \cap (s_1, u_1) \cap (t_1, u_1)$$

$$(iv) (s_1, t_1) \cap (s_0, u_1) \cap (t_0, u_0)$$

- If  $\text{trideg}(Y)$  is either  $(1, 1, 0)$ ,  $(1, 0, 1)$ , or  $(0, 1, 1)$ , there is an automorphism of  $X$  that sends  $Y$  to the curve defined by one of the following ideals:

$$(viii) (s_0 t_1 - s_1 t_0, u_1) \quad (x) (s_1, u_1) \cap (t_1, u_1)$$

$$(ix) (s_1, u_0) \cap (t_1, u_1)$$

- If  $\text{trideg}(Y)$  is either  $(1, 0, 0)$ ,  $(0, 1, 0)$ , or  $(0, 0, 1)$ , then  $Y$  is a projective line. In particular, it can be transformed by an automorphism of  $X$  into the line defined by  $(t_1, u_1)$ .

*Proof.* Up to an automorphism in  $K \leq \text{Aut}(X)$ , that just permutes the factors (see §3.1.3), we can assume that  $Y$  has tridegree either  $(1, 1, 1)$ ,  $(1, 1, 0)$ , or  $(1, 0, 0)$ . In the latter case, the projection  $\pi_1 : Y \rightarrow \mathbb{P}_{\mathbb{C}}^1$  onto the first factor of  $X$  is an isomorphism, so  $Y$  is a projective line. Clearly, there is an automorphism of  $X$  sending  $Y$  to the line  $(t_1, u_1)$ .

Now, assume that  $Y$  is irreducible of tridegree either  $(1, 1, 1)$  or  $(1, 1, 0)$ . The projections  $\pi_i : Y \rightarrow \mathbb{P}_{\mathbb{C}}^1$  onto the first two factors of  $X$  have degree one, so  $Y$  is rational. Hence,  $Y$  is the image of a regular map  $\varphi : \mathbb{P}_{\mathbb{C}}^1 \rightarrow Y$  given by

$$(x_0 : x_1) \mapsto (a_0(x_0, x_1) : a_1(x_0, x_1)) \times (b_0(x_0, x_1) : b_1(x_0, x_1)) \times (c_0(x_0, x_1) : c_1(x_0, x_1)), \quad (3.6)$$

for some linear forms  $a_i, b_j, c_k$  in  $\mathbb{C}[x_0, x_1]$ . If  $Y$  has tridegree  $(1, 1, 0)$ , then  $c_0, c_1$  are proportional and we have  $(c_0(x_0, x_1) : c_1(x_0, x_1)) = (\gamma_0 : \gamma_1)$  for some  $(\gamma_0 : \gamma_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$ . In particular, any two parametrizations of curves of the same tridegree coincide up to the action of  $\mathrm{PGL}(2, \mathbb{C})^3 \cong H \triangleleft \mathrm{Aut}(X)$  (see §3.1.3). Hence, if  $Y$  has tridegree  $(1, 1, 1)$  (resp.  $(1, 1, 0)$ ) there is an automorphism of  $X$  sending it to the curve defined by the ideal (i) (resp. (viii)) in the statement.

Now, assume that  $Y$  has exactly two irreducible components of tridegrees  $(1, 1, 0)$  and  $(0, 0, 1)$ . By the argument above, the component of tridegree  $(1, 1, 0)$  can be transformed into the curve  $D \subset X$  defined by the ideal (viii). Moreover, the component of tridegree  $(0, 0, 1)$  can be parametrized as

$$(x_0 : x_1) \mapsto (\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (c'_0(x_0, x_1) : c'_1(x_0, x_1)),$$

for some  $(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \in \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  and linear forms  $c'_0, c'_1$  in  $\mathbb{C}[x_0, x_1]$ . In particular, if  $(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \in \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  satisfies  $\alpha_0\beta_1 - \alpha_1\beta_0 = 0$  we find an automorphism of  $X$  that stabilizes  $D$  and sends

$$(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (c'_0(t_0, t_1) : c'_1(t_0, t_1)) \mapsto (1 : 0) \times (1 : 0) \times (c'_0(t_0, t_1) : c'_1(t_0, t_1)).$$

On the other hand, if  $\alpha_0\beta_1 - \alpha_1\beta_0 \neq 0$  then we find an automorphism sending

$$(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (c'_0(t_0, t_1) : c'_1(t_0, t_1)) \mapsto (1 : 0) \times (0 : 1) \times (c'_0(t_0, t_1) : c'_1(t_0, t_1))$$

that stabilizes  $D$  as well. The images of the two last parametrizations are the projective lines in  $X$  respectively defined by  $(s_1, t_1)$  and  $(s_1, t_0)$ . Therefore,  $Y$  can be transformed into the curve defined by either the ideal (iii) or (ii) in the statement by means of an automorphism of  $X$ .

Proceeding similarly with curves of tridegree  $(1, 1, 1)$  with three irreducible components, one derives the curves from (iv) to (vii). On the other hand, if  $Y$  is reducible of tridegree  $(1, 1, 0)$  it must have exactly two irreducible components of tridegrees  $(1, 0, 0)$  and  $(0, 1, 0)$ . Depending on whether these components intersect,  $Y$  can be transformed into the curve defined by either (ix) or (x).  $\square$

### 3.1.5. Pairing between the type and the tridegree of the base locus

Let  $\phi$  be birational, with inverse as in (1.6). Given a general line  $\ell \subset \mathbb{P}_{\mathbb{C}}^3$ , the image  $\phi^{-1}(\ell)$  is a closed rational curve in  $X$ . Additionally, writing  $\pi_i : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$  for the projection of  $X$  onto the  $i$ -th factor, the composition  $\pi_1 \circ \phi^{-1}|_{\ell}$

$$\begin{aligned} \mathbb{P}_{\mathbb{C}}^1 \cong \ell \subset \mathbb{P}_{\mathbb{C}}^3 &\xrightarrow{\phi^{-1}|_{\ell}} \phi^{-1}(\ell) \subset X = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \xrightarrow{\pi_1} \mathbb{P}_{\mathbb{C}}^1 \\ (y_0 : y_1 : y_2 : y_3) &\mapsto (a_0 : a_1) \times (b_0 : b_1) \times (c_0 : c_1) \mapsto (a_0 : a_1), \end{aligned}$$

yields an endomorphism of  $\mathbb{P}_{\mathbb{C}}^1$  of degree  $\deg a_i$ . Similarly, the projections onto the other factors yield endomorphisms of degree  $\deg b_j$  and  $\deg c_k$ , implying that the tridegree of  $\phi^{-1}(\ell)$  is  $(\deg a_i, \deg b_j, \deg c_k)$  (recall Definition 3.1.12). On the other hand, let

$$L = \left( \sum_{n=0}^3 \delta_n y_n, \sum_{n=0}^3 \epsilon_n y_n \right)$$

be the defining ideal of  $\ell$  in  $\mathbb{C}[\mathbf{y}]$ . Then, the homogeneous ideal

$$I = \left( \sum_{n=0}^3 \delta_n f_n, \sum_{i=0}^3 \epsilon_n f_n \right)$$

defines the scheme-theoretic union of  $\phi^{-1}(\ell)$  and the base locus  $Z$ , and has tridegree  $(2, 2, 2)$ . Namely, the class of each generator of  $I$  in the Chow ring  $CR(X)$  is  $(a + b + c)$ , and hence (see Remark 3.1.13)

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = 2ab + 2ac + 2bc \pmod{(a^2, b^2, c^2)}.$$

Therefore, it follows the identity

$$\text{trideg}(\phi^{-1}(\ell)) + \text{trideg}(Z) = (\deg a_i, \deg b_j, \deg c_k) + \text{trideg}(Z) = (2, 2, 2), \quad (3.7)$$

where we set  $\text{trideg}(Z) = (0, 0, 0)$  for zero-dimensional schemes (recall Notation 3.1.14).

**Remark 3.1.16.** A projective version of the identity (3.7) can be found in [80, §1].

We thus obtain the pairing between the possible types of  $\phi$  and the tridegree of the base locus  $Z$  in Table 3.1. In particular, it follows immediately that if  $\phi$  has type  $(2, 2, 2)$  the base locus is 0-dimensional.

| type of $\phi$             | trideg( $Z$ )              |
|----------------------------|----------------------------|
| (1,1,1)                    | (1,1,1)                    |
| (1,1,2) (and permutations) | (1,1,0) (and permutations) |
| (1,2,2) (and permutations) | (1,0,0) (and permutations) |
| (2,2,2)                    | (0,0,0)                    |

**Table 3.1.:** Pairing between the type of a birational map and the tridegree of the base locus.

## 3.2. Factorization of trilinear birational maps

In this section, we explain the strategy followed in our geometric analysis. It is inspired by [161], where the classification of birational endomorphisms of  $\mathbb{P}_{\mathbb{C}}^3$  with quadratic entries is addressed. The key idea is the factorization of trilinear rational maps through a linear system and a projection.

### 3.2.1. Commutative diagram for the factorization

Maintaining the notation of §3.1.1, let  $V$  be a  $\mathbb{C}$ -vector space satisfying

$$B_{(1,1,1)} \subset V \subset R_{(1,1,1)},$$

and let  $N + 1 = \dim V$ . Let  $\zeta_V : X \dashrightarrow \mathbb{P}(V^\vee)$  be the rational map whose entries are the vectors in a basis of the dual vector space  $V^\vee$ , and write  $Y_V$  for the closure in  $\mathbb{P}(V^\vee)$  of the image of  $\zeta_V$ . In particular, we have the commutative diagram

$$\begin{array}{ccc}
 & Y_V \hookrightarrow \mathbb{P}(V^\vee) \cong \mathbb{P}_{\mathbb{C}}^N & \\
 \nearrow \pi_V & \uparrow \zeta_V & \downarrow \pi_L \\
 \text{Bl}_V X \subset X \times \mathbb{P}_{\mathbb{C}}^N & \xrightarrow{\pi_V} X & \dashrightarrow \phi \dashrightarrow \mathbb{P}_{\mathbb{C}}^3
 \end{array} \tag{3.8}$$

where  $\pi_V : \text{Bl}_V X \rightarrow X$  is the blow-up of  $X$  along the subscheme defined by the entries of  $\zeta_V$ ,  $\pi_V : \text{Bl}_V X \rightarrow Y_V$  is the projection onto the second factor, and  $\pi_L : \mathbb{P}(V^\vee) \dashrightarrow \mathbb{P}_{\mathbb{C}}^3$  is the linear projection from some suitable subspace  $L$  in  $\mathbb{P}(V^\vee)$ . If  $N = 3$ , then  $\mathbb{P}(V^\vee) \cong \mathbb{P}_{\mathbb{C}}^3$  and  $\pi_L$  is just an automorphism of  $\mathbb{P}_{\mathbb{C}}^3$ . It follows from the diagram of (3.8) that  $\phi$  is birational if and only if both  $\zeta_V$  and the restriction of  $\pi_L$  to  $Y_V$  are birational.

### 3.2.2. Factorization according to the type of $\phi$

**Notation 3.2.1.** Let  $Y$  be a subscheme in  $X$ .

- We denote by  $I_Y \subset R$  the (saturated) defining ideal of  $Y$
- If  $\dim(Y) \geq 1$ , we write  $C_Y$  for the curve defined by the intersection of all the associated primes to  $I_Y$  of codimension exactly two. In this case  $C_Y$  does not have embedded points, and by Lemma 2.1.13 it is a Cohen-Macaulay curve
- With a small abuse of notation, when  $Y = Z$  (the base locus of  $\phi$ ) we denote  $C_Z$  just by  $C$

By definition, if  $\dim(Z) = 1$  then we have  $B \subset I_C$ . In particular, we can set  $V = (I_C)_{(1,1,1)}$  in the diagram of (3.8). The situation is a little bit different if  $\dim(Z) = 0$ , i.e. if  $\phi$  has type  $(2, 2, 2)$  (see §3.1.1). Namely, given a point

$$Q = (\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1) \in X \tag{3.9}$$

we define the linear polynomials

$$\Delta_1^Q := \begin{vmatrix} x_0 & x_1 \\ \alpha_0 & \alpha_1 \end{vmatrix}, \quad \Delta_2^Q := \begin{vmatrix} y_0 & y_1 \\ \beta_0 & \beta_1 \end{vmatrix}, \quad \Delta_3^Q := \begin{vmatrix} z_0 & z_1 \\ \gamma_0 & \gamma_1 \end{vmatrix}.$$

The following result allows us to use the diagram of (3.8) with birational maps of type  $(2, 2, 2)$ .

**Lemma 3.2.2.** *Let  $\phi$  be birational of type  $(2, 2, 2)$ . Then, there exists a unique point  $Q$  as (3.9) such that, for every  $0 \leq n \leq 3$ , the entries of  $\phi$  satisfy*

$$f_n \in \left( \lambda \Delta_1^Q y_j z_k + \mu x_i \Delta_2^Q z_k + \nu x_i y_j \Delta_3^Q \right) + \left( \Delta_1^Q, \Delta_2^Q, \Delta_3^Q \right)^2 \tag{3.10}$$

for some  $\lambda, \mu, \nu \in \mathbb{C}^*$  and indices  $0 \leq i, j, k \leq 1$  where  $\alpha_i, \beta_j, \gamma_k$  are all nonzero.

*Proof.* Consider the Segre embedding  $\sigma : X \hookrightarrow \Sigma \subset \mathbb{P}_{\mathbb{C}}^7$  of  $X$ , where the Segre variety  $\Sigma$  is smooth of degree 6. From this isomorphism, it follows that the Hilbert series of  $\Sigma$  is

$$\text{HS}_{\Sigma}(t) = \sum_{d \geq 0} \dim_{\mathbb{C}} \Gamma(X, \mathcal{O}_X(d, d, d)) t^d = \sum_{d \geq 0} (d+1)^3 t^d = \frac{1+4t+t^2}{(1-t)^4},$$

where  $\mathcal{O}_X$  is the structure sheaf of  $X$  and  $(1-t)^4 \text{HS}_{\Sigma}(t)$  is a polynomial because  $\dim \Sigma = 3$ .

On the other hand, given a general line  $\ell \subset \mathbb{P}_{\mathbb{C}}^3$  the image  $\phi^{-1}(\ell)$  is a closed, irreducible curve in  $X$ . Moreover, if

$$L = \left( \sum_{n=0}^3 \delta_n y_n, \sum_{n=0}^3 \epsilon_n y_n \right)$$

is the defining ideal of  $\ell$ , then

$$I = \left( \sum_{n=0}^3 \delta_n f_n, \sum_{i=0}^3 \epsilon_n f_n \right)$$

defines  $\phi^{-1}(\ell)$ . More specifically,  $I$  is the ideal of the scheme-theoretic union of  $\phi^{-1}(\ell)$  and the base locus  $Z$ . Since  $\text{codim } I = 2$  and  $R$  is a Cohen-Macaulay ring, all the associated primes to  $I$  have codimension two. However, by (3.7) we have  $\dim Z = 0$ , so  $\text{codim } B_{\phi} = 3$ . Therefore,  $I$  must be the defining ideal of  $\phi^{-1}(\ell)$ . In particular, the curve  $\sigma(\phi^{-1}(\ell))$  is determined by two hyperplane sections in  $\mathbb{P}_{\mathbb{C}}^7$ , and its Hilbert series [102, Chapter 5] is

$$\text{HS}_{\sigma(\phi^{-1}(\ell))}(t) = \text{HS}_{\phi^{-1}(\ell)}(t) = (1-t)^2 \text{HS}_{\Sigma}(t) = \frac{1+4t+t^2}{(1-t)^2},$$

implying that the Hilbert polynomial of  $\phi^{-1}(\ell)$  is  $\text{HP}_{\phi^{-1}(\ell)}(t) = 6t$ . In particular, the arithmetic genus of  $\phi^{-1}(\ell)$  is one. By definition,  $\phi^{-1}(\ell)$  is a rational curve and its normalization is  $\mathbb{P}_{\mathbb{C}}^1$ . Then, from [111, Chapter 4, Exercise 1.8]  $\phi^{-1}(\ell)$  has a unique singular point  $Q$ . Since  $\Sigma$  is smooth, by the strong Bertini's theorem [91, Theorem 0.5] the curve  $\phi^{-1}(\ell)$  is smooth outside  $Z$ . Hence,  $Q$  is supported in  $Z$ .

Without loss of generality, we can assume that  $Q$  belongs to the affine subset  $U \subset X$  given by  $s_0 \neq 0$ ,  $t_0 \neq 0$ , and  $u_0 \neq 0$ , i.e. we can assume indices  $(i, j, k) = (0, 0, 0)$  in the statement. From the Jacobian criterion [111, §1 Exercise 5.8] in  $U$ , since  $Q$  is a singular point of  $\phi^{-1}(\ell)$  the rank of the Jacobian matrix

$$\text{Jac}_{\phi^{-1}(\ell)} = \begin{pmatrix} \sum_{n=0}^3 \delta_n \frac{\partial f_n}{\partial s_1} & \sum_{n=0}^3 \delta_n \frac{\partial f_n}{\partial t_1} & \sum_{n=0}^3 \delta_n \frac{\partial f_n}{\partial u_1} \\ \sum_{n=0}^3 \epsilon_n \frac{\partial f_n}{\partial s_1} & \sum_{n=0}^3 \epsilon_n \frac{\partial f_n}{\partial t_1} & \sum_{n=0}^3 \epsilon_n \frac{\partial f_n}{\partial u_1} \end{pmatrix}$$

specialized at  $Q$  is one. Equivalently, writing

$$\nabla f_n = \left( \frac{\partial f_n}{\partial s_1}, \frac{\partial f_n}{\partial t_1}, \frac{\partial f_n}{\partial u_1} \right),$$

then the vectors

$$\sum_{n=0}^3 \delta_n \nabla f_n(Q), \quad \sum_{n=0}^3 \epsilon_n \nabla f_n(Q)$$

are proportional. The previous observations hold true for any choice of a general  $\ell \subset \mathbb{P}_{\mathbb{C}}^3$ , i.e. for general values of the  $\delta_n$ 's and  $\epsilon_n$ 's. Therefore, it can only occur if the  $\nabla f_n(Q)$ 's are all proportional to a vector  $(\lambda, \mu, \nu)$  in  $\mathbb{C}^3$ . In particular, from the Taylor expansion of  $f_n$  centered at  $Q$  we derive that

$$f_n = \theta_n (\lambda \Delta_1^Q y_0 z_0 + \mu x_0 \Delta_2^Q z_0 + \nu x_0 y_0 \Delta_3^Q) + h_n,$$

for some  $\theta_n \in \mathbb{C}$  and  $h_n \in (\Delta_1^Q, \Delta_2^Q, \Delta_3^Q)^2$ . Moreover, if  $\lambda = 0$  (resp.  $\mu = 0$  or  $\nu = 0$ ) we find a line in the base locus  $Z$ , against  $\dim Z = 0$ . Hence,  $\lambda, \mu, \nu$  are all nonzero.  $\square$



**Notation 3.2.3.** Given a point  $Q$  as in (3.9) and  $\lambda, \mu, \nu \in \mathbb{C}^*$ , we define the ideal

$$I_{Q(\lambda, \mu, \nu)} := \left( \lambda \Delta_1^Q y_j z_k + \mu x_i \Delta_2^Q z_k + \nu x_i y_j \Delta_3^Q \right) + \left( \Delta_1^Q, \Delta_2^Q, \Delta_3^Q \right)^2 \subset R$$

for the smallest indices  $0 \leq i, j, k \leq 1$  such that  $\alpha_i, \beta_j, \gamma_k$  are all nonzero. The graded component of  $I_{Q(\lambda, \mu, \nu)}$  in degree  $(1, 1, 1)$  is spanned by the five independent polynomials

$$\lambda \Delta_1^Q t_j u_k + \mu s_i \Delta_2^Q u_k + \nu s_i t_j \Delta_3^Q, \Delta_1^Q \Delta_2^Q u_k, \Delta_1^Q t_j \Delta_3^Q, s_i \Delta_2^Q \Delta_3^Q, \Delta_1^Q \Delta_2^Q \Delta_3^Q.$$

**Notation 3.2.4.** Let  $\phi$  be birational.

- If  $\dim(Z) = 1$ , we define  $V_\phi = (I_C)_{(1,1,1)}$  (see Notation 3.2.1)
- If  $\dim(Z) = 0$ , we define  $V_\phi = (I_{Q(\lambda, \mu, \nu)})_{(1,1,1)}$  for the  $Q$  and  $\lambda, \mu, \nu$  in the statement of Lemma 3.2.2 (see Notation 3.2.3)
- By definition, in both cases we have  $B_{(1,1,1)} \subset V_\phi$

**Definition 3.2.5** (Variety of minimal degree). A projective variety  $Y \subset \mathbb{P}_{\mathbb{C}}^N$  has *minimal degree* if it is nondegenerate (recall Definition 2.2.8) and  $\deg V = 1 + \text{codim } V$ .

The following result gives useful information about diagram in (3.8), according to the type of  $\phi$ .

**Proposition 3.2.6.** *Let  $\phi$  be birational, and set  $V = V_\phi$ . We have the following:*

- If  $\phi$  has type  $(1, 1, 1)$ , then  $N = 3$  and  $Y_V \cong \mathbb{P}_{\mathbb{C}}^3$ .
- If  $\phi$  has type  $(1, 1, 2)$ , then  $N = 4$  and  $\deg Y_V = 2$ .
- If  $\phi$  has type  $(1, 2, 2)$ , then  $N = 5$  and  $\deg Y_V = 3$ .
- If  $\phi$  has type  $(2, 2, 2)$ , then  $N = 4$  and  $\deg Y_V = 2$ .

In all the cases,  $Y_V$  has minimal degree and  $\zeta_V$  is birational for any choice of basis in  $V^\vee$ . Moreover, if  $\dim Z = 1$  the curve  $C$  is connected.

*Proof.* We prove each statement separately.

- If  $\phi$  has type  $(1, 1, 1)$ , by (3.7) the base locus  $Z$  has tridegree  $(1, 1, 1)$ . In particular, up to an automorphism of  $X$  the curve  $C$  is one of the seven listed in Lemma 3.1.15. The ideals (i), (iii), (vi), and (vii) determine connected curves, and satisfy  $\dim V = 4$ . On the other hand, the ideals (ii), (iv), (v) yield  $\dim V = 3$ . Since the entries of  $\phi$  are independent and lie in  $V$ , we must have  $\dim V \geq 4$ . Therefore,  $\dim V = 4$  and  $C$  is connected.
- If  $\phi$  has type  $(1, 1, 2)$ ,  $Z$  has tridegree  $(1, 1, 0)$ . In particular, up to an automorphism of  $X$  the curve  $C$  is one of the three listed in Lemma 3.1.15. The ideals (viii) and (x) determine connected curves and satisfy  $\dim V = 5$ . On the other hand, the ideal (ix) defines two skew lines. Let  $I$  be the ideal in (ix). An easy computation (performed with the help of *Macaulay2*) yields  $I \neq (I_{(1,1,1)})$ , since the ideal generated by the polynomials of degree  $(1, 1, 1)$  in  $I$  defines a curve of tridegree  $(1, 1, 1)$ . Therefore, we must have  $\dim V = 5$  and  $C$  is again connected. The degree of  $Y_V$  follows from a direct computation (performed with *Macaulay2*) with the ideals (viii) and (x).

- If  $\phi$  has type  $(1, 2, 2)$ ,  $Z$  has tridegree  $(1, 0, 0)$ . By Lemma 3.1.15  $C$  is a projective line, so it is connected and  $\dim V = 6$ . Similarly, the degree of  $Y_V$  follows from a direct computation on the ideal  $(t_0, u_0)$ .
- If  $\phi$  has type  $(2, 2, 2)$ , by Notation 3.2.3 we have  $\dim V = 5$ . Similarly, it follows from a direct computation that  $\deg Y_V = 2$ .

On the other hand, two distinct bases in  $V^\vee$  yield rational maps  $\zeta_V$  that coincide up to an automorphism of  $\mathbb{P}_\mathbb{C}^N$ . The birationality of  $\zeta_V$  then follows from the explicit construction of this rational map when  $V = I_{(1,1,1)}$  for the ideals  $I$  listed above.  $\square$

### 3.3. The algebraic set of trilinear birational maps

In this section we describe the algebraic set  $\mathfrak{Bir}_{(1,1,1)}$ , which is the quotient of the set of trilinear birational maps by the group action of  $\text{Aut}(\mathbb{P}_\mathbb{C}^3)$  given by composition. Additionally, we introduce the group action of  $\text{Aut}(X)$ .

#### 3.3.1. The sets $\mathfrak{Bir}_{(1,1,1)}$ and $\text{Bir}_{(d_1, d_2, d_3)}$

Let

$$\mathfrak{Rat}_{(1,1,1)} = \mathbb{P} \left( (R_{(1,1,1)})^4 \right) ,$$

i.e. the projective space where (closed) points are 4-tuples of trilinear polynomials in  $R$ , up to nonzero scalar. Let  $U$  be the open set in  $\mathfrak{Rat}_{(1,1,1)}$  of points  $\mathbf{f} = (f_0, f_1, f_2, f_3)$  such that  $\gcd(f_0, f_1, f_2, f_3) = 1$ .

Clearly, there is a one-to-one correspondence between  $U$  and the set of trilinear rational maps. In  $U$ , we define the equivalence relation

$$\phi \sim \phi' \text{ if } V_\phi = V_{\phi'} ,$$

or equivalently,

$$\phi \sim \phi' \text{ if there is a } \rho \in \text{Aut}(\mathbb{P}_\mathbb{C}^3) \text{ such that } V_{\rho \circ \phi} = V_{\phi'} .$$

**Definition 3.3.1.**  $\mathfrak{Bir}_{(1,1,1)}$  is the quotient of  $U$  by the equivalence relation  $\sim$ .

In particular, there is an injective map of sets

$$\begin{aligned} \mathfrak{Bir}_{(1,1,1)} &\rightarrow \text{Gr}(4, R_{(1,1,1)}) \\ [\phi] &\mapsto V_\phi , \end{aligned}$$

so we can write  $V_\phi$  instead of  $[\phi]$ . On the other hand, two birational maps  $\phi$  and  $\phi'$  in the same class of  $\mathfrak{Bir}_{(1,1,1)}$  have the same type. Therefore, the following definition makes sense.

**Definition 3.3.2.** Given  $(d_1, d_2, d_3) \in \mathbb{Z}^3$ ,  $\text{Bir}_{(d_1, d_2, d_3)} = \{V_\phi \in \mathfrak{Bir}_{(1,1,1)} : \phi \text{ has type } (d_1, d_2, d_3)\}$ .

By §3.1.1, the sets  $\text{Bir}_{(d_1, d_2, d_3)}$  indexed by the types listed in (3.2) form a partition of  $\mathfrak{Bir}_{(1,1,1)}$ .

### 3.3.2. The group action of $\text{Aut}((\mathbb{P}_{\mathbb{C}}^1)^3)$ on $\mathfrak{Bir}_{(1,1,1)}$

The map of sets

$$\begin{aligned} \text{Aut}(X) \times \mathfrak{Bir}_{(1,1,1)} &\rightarrow \mathfrak{Bir}_{(1,1,1)} \\ \xi \times V_{\phi} &\mapsto V_{\phi \circ \xi} \end{aligned} \quad (3.11)$$

is a well-defined group action of  $\text{Aut}(X)$  on  $\mathfrak{Bir}_{(1,1,1)}$ .

**Definition 3.3.3.** The group action in (3.11) is the *right-action* on  $\mathfrak{Bir}_{(1,1,1)}$ , which induces an equivalence relation  $\sim_r$ .

**Notation 3.3.4.** If  $V_{\phi} \sim_r V_{\phi'}$ , we say that  $\phi$  and  $\phi'$  lie in the same orbit. More explicitly, this means  $\phi$  and  $\phi'$  lie in the same orbit in  $\mathfrak{Aut}_{(1,1,1)}$  of the group action of  $\text{Aut}(X) \times \text{Aut}(\mathbb{P}_{\mathbb{C}}^3)$  given by composition, since

$$V_{\phi} \sim_r V_{\phi'} \text{ if and only if there are } \rho \in \text{Aut}(\mathbb{P}_{\mathbb{C}}^3) \text{ and } \xi \in \text{Aut}(X) \text{ such that } \phi' = \rho \circ \phi \circ \xi .$$

**Remark 3.3.5.** The base loci of two birational maps in the same orbit coincide up to automorphism of  $X$ .

### 3.3.3. The algebraic set $\mathfrak{Bir}_{(1,1,1)}$

We write  $\mathcal{S}(Y_V)$  for the variety of  $(N - 4)$ -dimensional subspaces secant to  $Y_V$ . Now, we prove the main result of this section. In our proof, we rely on a result of [161], that establishes that birational endomorphisms of a projective space, defined by homogeneous polynomials of a given degree, determine a locally closed algebraic set of a certain Grassmannian.

**Theorem 3.3.6.** *The set  $\mathfrak{Bir}_{(1,1,1)}$  has the structure of a locally closed algebraic subset of the Grassmannian  $\text{Gr}(4, R_{(1,1,1)})$ . Moreover, its irreducible components are the following:*

- $\text{Bir}_{(1,1,1)}$ , of dimension 6.
- $\text{Bir}_{(1,1,2)}$ ,  $\text{Bir}_{(1,2,1)}$ , and  $\text{Bir}_{(2,1,1)}$ , all of dimension 7.
- $\text{Bir}_{(1,2,2)}$ ,  $\text{Bir}_{(2,1,2)}$ , and  $\text{Bir}_{(2,2,1)}$ , all of dimension 8.
- $\text{Bir}_{(2,2,2)}$ , of dimension 8.

**Remark 3.3.7.** In Theorem 1.2.4, the increase in dimensionality of the components arises because we do not consider classes with respect to the group action of  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^3)$ . Therefore, the irreducible components can be viewed as locally closed subvarieties of  $\text{Gr}(4, R_{(1,1,1)}) \times \text{Aut}(\mathbb{P}_{\mathbb{C}}^3)$ , where  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^3) = \text{PGL}(4, \mathbb{C}) \cong \mathbb{P}_{\mathbb{C}}^{15}$ .

*Proof.* The birational map

$$\begin{aligned} \Phi : X &\dashrightarrow \mathbb{P}_{\mathbb{C}}^3 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (s_0 t_0 u_0 : s_1 t_0 u_0 : s_0 t_1 u_0 : s_0 t_0 u_1) \end{aligned} \quad (3.12)$$

is an isomorphism between the affine subset  $U \subset X$  determined by  $s_0 \neq 0, t_0 \neq 0, u_0 \neq 0$  and the affine subset of  $\mathbb{P}_{\mathbb{C}}^3 = \text{Proj}(\mathbb{C}[\mathbf{y}])$  given by  $y_0 \neq 0$ . Write  $V_3$  for the  $\mathbb{C}$ -vector space of cubic forms in  $y_0, y_1, y_2$ , and  $y_3$ . We have the injective linear map  $\chi : R_{(1,1,1)} \rightarrow V_3$  given by

$$\begin{aligned} s_0 t_0 u_0 &\mapsto y_0^3, \quad s_1 t_0 u_0 \mapsto y_0^2 y_1, \quad s_0 t_1 u_0 \mapsto y_0^2 y_2, \quad s_0 t_0 u_1 \mapsto y_0^2 y_3, \\ s_1 t_1 u_0 &\mapsto y_0 y_1 y_2, \quad s_1 t_0 u_1 \mapsto y_0 y_1 y_3, \quad s_0 t_1 u_1 \mapsto y_0 y_2 y_3, \quad s_1 t_1 u_1 \mapsto y_1 y_2 y_3 . \end{aligned}$$

In particular, the rational endomorphism  $\psi : \mathbb{P}_{\mathbb{C}}^3 \dashrightarrow \mathbb{P}_{\mathbb{C}}^3$  that sends

$$(y_0 : y_1 : y_2 : y_3) \mapsto (\chi(f_0) : \chi(f_1) : \chi(f_2) : \chi(f_3))$$

coincides with  $\phi \circ \Phi^{-1}$  on the open set  $\Phi(U)$ , i.e. we have  $\psi = \phi \circ \Phi^{-1}$ . Therefore, the map

$$\phi \mapsto \psi = \phi \circ \Phi^{-1}$$

is a bijection between trilinear rational maps and rational endomorphisms of  $\mathbb{P}_{\mathbb{C}}^3$  with entries in the vector space

$$V' = \langle y_0^3, y_0^2 y_1, y_0^2 y_2, y_0^2 y_3, y_0 y_1 y_2, y_0 y_1 y_3, y_0 y_2 y_3, y_1 y_2 y_3 \rangle \subset V_3.$$

By [161, Proposition B], the classes (up to composition with an automorphism of  $\mathbb{P}_{\mathbb{C}}^3$ ) of birational automorphisms of  $\mathbb{P}_{\mathbb{C}}^3$  with cubic entries form a locally closed algebraic set  $Y' \subset \text{Gr}(4, V_3)$ . On the other hand, the Grassmannian  $\text{Gr}(4, V')$  is a subvariety of  $\text{Gr}(4, V_3)$ . Hence, the intersection

$$\mathfrak{Bir}_{(1,1,1)} = Y' \cap \text{Gr}(4, V')$$

is also a locally closed algebraic subset of the Grassmannian  $\text{Gr}(4, V') \cong \text{Gr}(4, R_{(1,1,1)})$ .

Now, we prove the statements about the irreducible components of  $\mathfrak{Bir}_{(1,1,1)}$ . Recall from Proposition 3.2.6 that if  $\dim Z = 1$  the Cohen-Macaulay curve  $C_\phi$  is connected. Let  $\mathcal{C}_{(1,1,1)}$  be the variety of connected Cohen-Macaulay curves of tridegree  $(1, 1, 1)$ , which is irreducible of dimension 6 [9, Proposition 4.9]. Additionally, consider the algebraic set

$$\mathcal{V}_{(1,1,1)} = \{V_\phi \times C \in \mathfrak{Bir}_{(1,1,1)} \times \mathcal{C}_{(1,1,1)} : V_\phi \subset (I_C)_{(1,1,1)}\},$$

together with the canonical projections  $\Pi_1 : \mathcal{V}_{(1,1,1)} \rightarrow \mathfrak{Bir}_{(1,1,1)}$  and  $\Pi_2 : \mathcal{V}_{(1,1,1)} \rightarrow \mathcal{C}_{(1,1,1)}$ . Given a curve  $C$  in  $\mathcal{C}_{(1,1,1)}$  we have  $\Pi_2^{-1}(C) = V \times C$ , where  $V = (I_C)_{(1,1,1)}$  since  $\dim V = 4$  by Proposition 3.2.6. Hence,  $\mathcal{V}_{(1,1,1)}$  is irreducible of dimension 6. On the other hand, given a  $V_\phi$  in  $\mathfrak{Bir}_{(1,1,1)}$  we have  $\Pi_1^{-1}(V_\phi) = V_\phi \times C_\phi$  if  $\phi$  has type  $(1, 1, 1)$  and  $\Pi_1^{-1}(V_\phi) = \emptyset$  otherwise. Therefore, it follows that  $\text{Bir}_{(1,1,1)} \cong \mathcal{C}_{(1,1,1)}$ .

The study of the irreducible components  $\text{Bir}_{(1,1,2)}$ ,  $\text{Bir}_{(1,2,1)}$ , and  $\text{Bir}_{(2,1,1)}$  is identical, since the associated birational maps differ by a permutation of the factors of  $X$ . Therefore, we can focus on  $\text{Bir}_{(1,1,2)}$ . By Lemma 3.1.15, the ideal of a connected Cohen-Macaulay curve of tridegree  $(1, 1, 0)$  has the form

$$(\alpha_0 s_0 t_0 + \alpha_1 s_1 t_0 + \alpha_2 s_0 t_1 + \alpha_3 s_1 t_1, \gamma_0 u_1 - \gamma_1 u_0),$$

for some  $(\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3) \times (\gamma_0 : \gamma_1) \in \mathbb{P}_{\mathbb{C}}^3 \times \mathbb{P}_{\mathbb{C}}^1$ , where the first generator might be reducible. Thus, we can identify the variety  $\mathcal{C}_{(1,1,0)}$  of such curves with  $\mathbb{P}_{\mathbb{C}}^3 \times \mathbb{P}_{\mathbb{C}}^1$ . Consider now the algebraic set

$$\mathcal{V}_{(1,1,2)} = \{V_\phi \times C \in \mathfrak{Bir}_{(1,1,1)} \times \mathcal{C}_{(1,1,0)} : V_\phi \subset (I_C)_{(1,1,1)}\},$$

together with the projections as before. Given a curve  $C$  in  $\mathcal{C}_{(1,1,0)}$ , by [161, Proposition A] the fiber  $\Pi_2^{-1}(C)$  is isomorphic to an open subset of  $\mathcal{S}(Y_V)$ , where  $V = (I_C)_{(1,1,1)}$ . Hence,  $\mathcal{V}_{(1,1,2)}$  is irreducible of dimension  $\dim \mathcal{C}_{(1,1,0)} + \dim \mathcal{S}(Y_V) = 4 + 3 = 7$ . On the other hand, we have the identity of sets

$$\text{Bir}_{(1,1,2)} = \Pi_1(\mathcal{V}_{(1,1,2)}) \setminus \text{Bir}_{(1,1,1)},$$

so given a general  $V_\phi$  in  $\Pi_1(\mathcal{V}_{(1,1,2)})$  the base locus  $Z$  has tridegree  $(1, 1, 0)$ . Thus, we have  $\Pi_1^{-1}(V_\phi) = (V_\phi, C_\phi)$  and  $\Pi_1$  is birational, implying that  $\text{Bir}_{(1,1,2)}$  is irreducible of dimension 7.

Similarly, one proves that  $\text{Bir}_{(1,2,2)}$ ,  $\text{Bir}_{(2,1,2)}$ , and  $\text{Bir}_{(2,2,1)}$  are all irreducible of dimension 8.

For the irreducible component  $\text{Bir}_{(2,2,2)}$ , we consider the algebraic set

$$\mathcal{V}_{(2,2,2)} = \{V_\phi \times Q \times (\lambda, \mu) \in \mathfrak{Bir}_{(1,1,1)} \times X \times (\mathbb{C}^*)^2 : V_\phi \subset (I_{Q(\lambda, \mu, 1)})_{(1,1,1)}\},$$

with the projections  $\Pi_1 : \mathcal{V}_{(2,2,2)} \rightarrow \mathfrak{Bir}_{(1,1,1)}$  and  $\Pi_2 : \mathcal{V}_{(2,2,2)} \rightarrow X \times (\mathbb{C}^*)^2$ . By [161, Proposition A], given  $Q \times (\lambda, \mu) \in X \times (\mathbb{C}^*)^2$  the fiber  $\Pi_2^{-1}(Q \times (\lambda, \mu))$  is isomorphic to an open subset of  $\mathcal{S}(Y_V)$ , where  $V = (I_{Q(\mu, \nu, 1)})_{(1,1,1)}$ . Hence,  $\mathcal{V}_{(2,2,2)}$  is irreducible of dimension

$$\dim X + \dim(\mathbb{C}^*)^2 + \dim \mathcal{S}(Y_V) = 3 + 2 + 3 = 8.$$

On the other hand, given a general  $V_\phi$  in  $\Pi_1(\mathcal{V}_{(2,2,2)})$  by Lemma 3.2.2 we have  $\Pi_1^{-1}(V_\phi) = V_\phi \times Q \times (\lambda, \mu)$  for the unique singular point  $Q \in Z$  and some  $(\lambda, \mu) \in (\mathbb{C}^*)^2$ . In particular,  $\Pi_1$  is birational. Moreover, since a general point in  $\Pi_1(\mathcal{V}_{(2,2,2)})$  belongs to  $\text{Bir}_{(2,2,2)}$  the latter is irreducible of dimension 8.  $\square$

### 3.4. Classification of the base loci

In this section, we provide the complete list of the orbits of the right-action on  $\mathfrak{Bir}_{(1,1,1)}$ . More specifically, there are 19 orbits. Each of these orbits determines an isomorphism class of the base loci of a trilinear birational map. In the entire section, we maintain the notation of the diagram in (3.8).

One way to understand the different classes of the base loci is through the linear system of surfaces spanned by the entries of  $\phi$ , i.e. the surfaces in  $X$  defined by the equations in the vector subspace  $\mathbb{C}\langle f_0, f_1, f_2, f_3 \rangle$  of  $R_{(1,1,1)}$ . In order to describe this point of view, we recall the following definition.

**Definition 3.4.1** (Points of contact and tangency in  $X$ ). Let  $Y$  be a surface in  $X$ .

- $Q$  is a *point of contact* of  $Y$  to a surface  $S \subset X$  if  $I_Y \subset I_S + I_Q^2$
- $Q$  is a *point of tangency* of  $Y$  to a curve  $C \subset X$  if  $I_Y \subset I_C + I_Q^2$

**Remark 3.4.2.** Let  $\phi$  be birational of type  $(2, 2, 2)$ . By Lemma 3.2.2, and maintaining the same notation, every surface in the linear system  $\mathbb{C}\langle f_0, f_1, f_2, f_3 \rangle$  has  $Q$  as a point of contact to the surface

$$\lambda \Delta_1^Q t_j u_k + \mu s_i \Delta_2^Q u_k + \nu s_i t_j \Delta_3^Q = 0.$$

#### 3.4.1. Orbits of birational maps of type $(1, 1, 1)$

The classification of the orbits in  $\text{Bir}_{(1,1,1)}$  is the most straightforward, since by Proposition 3.5.2 (that we postpone until §3.5) the base locus is a Cohen-Macaulay curve.

**Theorem 3.4.3.** *Let  $\phi$  be birational of type  $(1, 1, 1)$ . Then,  $\phi$  belongs to the orbit of one of the following birational maps:*

- $\rho_1^{(1,1,1)} \equiv (s_1 t_0 u_1 - s_0 t_1 u_1 : s_1 t_1 u_0 - s_0 t_1 u_1 : s_0 t_1 u_0 - s_0 t_0 u_1 : s_1 t_0 u_0 - s_0 t_0 u_1)$

The entries of  $\rho_1^{(1,1,1)}$  are given by the  $3 \times 3$  minors of the matrix

$$\begin{pmatrix} 0 & -t_1 & -u_1 \\ -s_1 & 0 & u_1 \\ s_0 & t_0 & 0 \\ -s_0 & 0 & u_0 \end{pmatrix}.$$

A surface in  $V_{\rho_1}$  contains an irreducible curve of tridegree  $(1, 1, 1)$ . The base ideal is

$$(s_0 t_1 - s_1 t_0, s_0 u_1 - s_1 u_0, t_0 u_1 - t_1 u_0).$$

- $\rho_2^{(1,1,1)} \equiv (s_1 t_1 u_1 : s_0 t_1 u_1 : s_1 t_0 u_1 : s_1 t_0 u_0 - s_0 t_1 u_0)$

The entries of  $\rho_2^{(1,1,1)}$  are given by the  $3 \times 3$  minors of the matrix

$$\begin{pmatrix} 0 & 0 & -u_1 \\ 0 & -t_1 & u_0 \\ -s_1 & 0 & -u_0 \\ s_0 & t_0 & 0 \end{pmatrix},$$

A surface in  $V_{\rho_2}$  contains an irreducible curve of tridegree  $(1, 1, 0)$  and a line of tridegree  $(0, 0, 1)$  that intersect. The base ideal is

$$(s_1 t_0 - s_0 t_1, u_1) \cap (s_1, t_1).$$

- $\rho_3^{(1,1,1)} \equiv (s_1 t_1 u_1 : s_0 t_1 u_1 : s_1 t_1 u_0 : s_1 t_0 u_0)$

The entries of  $\rho_3^{(1,1,1)}$  are given by the  $3 \times 3$  minors of the matrix

$$\begin{pmatrix} 0 & -t_1 & 0 \\ 0 & t_0 & -u_1 \\ -s_1 & 0 & 0 \\ s_0 & 0 & u_0 \end{pmatrix}.$$

A surface in  $V_{\rho_3}$  contains three lines of tridegrees  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ , one of them intersecting the other two at distinct points. The base ideal is

$$(s_1, t_1) \cap (s_1, u_1) \cap (t_1, u_0).$$

- $\rho_4^{(1,1,1)} \equiv (s_1 t_1 u_1 : s_0 t_1 u_1 : s_1 t_0 u_1 : s_1 t_1 u_0)$

The entries of  $\rho_4^{(1,1,1)}$  are given by the  $3 \times 3$  minors of the matrix

$$\begin{pmatrix} 0 & 0 & -u_1 \\ 0 & -t_1 & 0 \\ -s_1 & 0 & 0 \\ s_0 & t_0 & u_0 \end{pmatrix}.$$

A surface in  $V_{\rho_4}$  contains three lines of tridegrees  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$  that intersect at a common point. The base ideal is

$$(s_1, t_1) \cap (s_1, u_1) \cap (t_1, u_1).$$

*Proof.* From Proposition 3.5.2, the base locus of  $\phi$  is a connected Cohen-Macaulay curve of tridegree  $(1, 1, 1)$ , i.e. we have  $Z = C$  (see Notation 3.2.1). Additionally, by the proof of Proposition 3.2.6 given a connected Cohen-Macaulay curve  $C$  of tridegree  $(1, 1, 1)$  we have  $\dim(I_C)_{(1,1,1)} = 4$ . Hence, any such curve determines a unique  $V_\phi$  in  $\mathfrak{Bir}_{(1,1,1)}$ . On the other hand, by Lemma 3.1.15 any connected Cohen-Macaulay curve of tridegree  $(1, 1, 1)$  is equivalent, by means of an automorphism of  $X$ , to the base loci of one of the birational maps in the statement.  $\square$

### 3.4.2. Orbits of birational maps of type $(1, 1, 2)$ , $(1, 2, 1)$ and $(2, 1, 1)$

Let  $C$  be a connected Cohen-Macaulay curve in  $X$  of tridegree either  $(1, 1, 0)$ ,  $(1, 0, 1)$ , or  $(0, 1, 1)$ . From Lemma 3.1.15, by means of an automorphism of  $X$  the curve  $C$  can be transformed into one of the curves  $C_o$  or  $C_\times$  respectively defined by the ideals

$$I_{C_o} = (s_0 t_1 - s_1 t_0, u_1) \quad , \quad I_{C_\times} = (s_1 t_1, u_1) .$$

In particular, any birational map of type either  $(1, 1, 2)$ ,  $(1, 2, 1)$ , or  $(2, 1, 1)$  lies in the orbit of a birational map with either  $C_o$  or  $C_\times$  in its base locus. Therefore, by the diagram of (3.8) in order to study the orbits of the right-action in  $\text{Bir}_{(1,1,2)}$ ,  $\text{Bir}_{(1,2,1)}$  and  $\text{Bir}_{(2,1,1)}$  it is enough to:

- (i) Compute the orbits of the group action given by the stabilizer  $\text{Stab}(C_o) \leq \text{Aut}(X)$  of the curve  $C_o$  acting on  $Y_{V_o}$ , where  $V_o := (I_{C_o})_{(1,1,1)}$
- (ii) Compute the orbits of the group action given by the stabilizer  $\text{Stab}(C_\times) \leq \text{Aut}(X)$  of the curve  $C_\times$  acting on  $Y_{V_\times}$ , where  $V_\times := (I_{C_\times})_{(1,1,1)}$

#### 3.4.2.1. Orbits of $\text{Stab}(C_o)$ on $Y_{V_o}$

In this subsection, we set  $V = V_o$ . The birational map  $\zeta_V$  appearing in the diagram of (3.8) is

$$\zeta_V : (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) \mapsto (s_0 t_0 u_1 : s_1 t_0 u_1 : s_0 t_1 u_1 : s_1 t_1 u_1 : (s_0 t_1 - s_1 t_0) u_0) , \quad (3.13)$$

and  $Y_V \subset \mathbb{P}_{\mathbb{C}}^4$  is the cone determined by  $y_0 y_3 - y_1 y_2 = 0$ . Equivalently,  $Y_V$  is the image of the projection  $\Pi_V : \text{Bl}_V X \subset X \times \mathbb{P}_{\mathbb{C}}^4 \rightarrow \mathbb{P}_{\mathbb{C}}^4$ , where  $\pi_V : \text{Bl}_V X \rightarrow X$  is the blow-up of  $X$  along  $C_o$ . The exceptional divisor  $E$  of  $\text{Bl}_V X$  projects onto a 2-dimensional cone in  $Y_V$ . Namely, the defining equation of  $\Pi_V(E)$  in  $Y_V$  is  $y_1 - y_2 = 0$ , and its singular point is  $P = (0 : 0 : 0 : 0 : 1)$ . We write  $H_{1-2}$  and  $H_4$  for the divisors in  $Y_V$  determined by  $y_1 - y_2 = 0$  and  $y_4 = 0$ , respectively.

**Lemma 3.4.4.** *The group action of  $\text{Stab}(C_o)$  on  $Y_V$  determines the four orbits:*

$$Y_V \setminus H_{1-2} \quad , \quad H_{1-2} \setminus (H_4 \cup P) \quad , \quad H_{1-2} \cap H_4 \quad , \quad P$$

*Proof.* Write  $D_{st}$  and  $D_u$  for the divisors in  $X$  defined by  $s_0 t_1 - s_1 t_0 = 0$  and  $u_1 = 0$ , respectively. By §3.4.5.3, the group action of  $\text{Stab}(C_o)$  on  $X$  determines the four orbits

$$C_o = D_{st} \cap D_u \quad , \quad D_u \setminus C_o \quad , \quad D_{st} \setminus C_o \quad , \quad X \setminus (D_u \cup D_{st}) . \quad (3.14)$$

The last three orbits are respectively transformed by  $\zeta_V$  into  $P$ ,  $H_{1-2} \cap H_4$ , and  $Y_V \setminus H_{1-2}$ . Thus, we find three of the orbits in the statement. The remaining orbit  $H_{1-2} \setminus (H_4 \cup P)$  follows from the extension of the action of  $\text{Stab}(C_o)$  to  $Y_V$ , by the action of the group  $\mathcal{M}_V$  of matrices of the form (3.21). Equivalently, this last orbit corresponds to the projection by  $\Pi_V$  of general orbit on the exceptional divisor  $E$  of the blow-up  $\text{Bl}_V X \rightarrow X$ , when we let  $\text{Stab}(C_o)$  act on  $\text{Bl}_V X$ .  $\square$

### 3.4.2.2. Orbits of $\text{Stab}(C_\times)$ on $Y_{V_\times}$

In this subsection, we set  $V = V_\times$ . The birational map  $\zeta_V$  in the diagram of (3.8) is

$$\zeta_V : (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) \mapsto (s_0 t_0 u_1 : s_1 t_0 u_1 : s_0 t_1 u_1 : s_1 t_1 u_1 : s_1 t_1 u_0) . \quad (3.15)$$

As in §3.4.2.1,  $Y_V$  is determined by  $y_0 y_3 - y_1 y_2 = 0$ . However, the projection  $\Pi_V(E)$  of the exceptional divisor  $E$  is now the union of the loci  $y_1 = y_3 = 0$  and  $y_2 = y_3 = 0$  in  $\mathbb{P}_\mathbb{C}^4$ . We maintain the notation of §3.4.2.1, and write  $H_i$  for the divisor in  $Y_V$  defined by  $y_i = 0$ . Additionally, we set  $O = (1 : 0 : 0 : 0 : 0)$  and write  $\overline{OP}$  for the projective line  $\overline{OP} = H_1 \cap H_2 \cap H_3$ .

**Lemma 3.4.5.** *The group action of  $\text{Stab}(C_\times)$  on  $Y_V$  determines the six orbits:*

$$\begin{aligned} & Y_V \setminus ((H_1 \cup H_2) \cap H_3) , ((H_1 \cup H_2) \cap H_3) \setminus (H_4 \cup \overline{OP}) , \\ & ((H_1 \cup H_2) \cap H_3 \cap H_4) \setminus O , \overline{OP} \setminus (O \cup P) , O , P \end{aligned}$$

*Proof.* Write  $D_s, D_t,$  and  $D_u$  for the divisors in  $X$  respectively determined by  $s_1 = 0, t_1 = 0,$  and  $u_1 = 0,$  and moreover let  $Q' = (1 : 0)^3 \in X$ . By §3.4.5.4, the group action of  $\text{Stab}(C_\times)$  on  $X$  determines the five orbits

$$Q' , C_\times \setminus Q' , D_u \setminus C_\times , (D_s \cup D_t) \setminus C_\times , X \setminus (D_s \cup D_t \cup D_u) .$$

The last three orbits are respectively transformed by  $\zeta_V$  into  $P, ((H_1 \cup H_2) \cap H_3 \cap H_4) \setminus O,$  and  $Y_V \setminus ((H_1 \cup H_2) \cap H_3)$ . Therefore, we find three of the orbits in the statement. The orbits  $((H_1 \cup H_2) \cap H_3) \setminus (H_4 \cup \overline{OP}), \overline{OP} \setminus (O \cup P),$  and  $O$  follow from the extension of the action of  $\text{Stab}(C_\times)$  to  $Y_V,$  by the action of the group  $\mathcal{M}_V$  of matrices of the form (3.22). Equivalently, they correspond to the projection by  $\Pi_V$  of the orbits in  $E$  of the blow-up  $\text{Bl}_V X \rightarrow X,$  when we let  $\text{Stab}(C_\times)$  act on  $\text{Bl}_V X$ .  $\square$

The following is a classification theorem for the trilinear birational maps of type  $(1, 1, 2), (1, 2, 1),$  and  $(2, 1, 1)$ . Namely, it provides the orbits of the right-action in  $\text{Bir}_{(1,1,2)}, \text{Bir}_{(1,2,1)},$  and  $\text{Bir}_{(2,1,1)}$ .

**Theorem 3.4.6.** *Let  $\phi$  be birational of type either  $(1, 1, 2), (1, 2, 1)$  or  $(2, 1, 1)$ . Then,  $\phi$  belongs to the orbit of one of the following birational maps:*

- $\rho_1^{(1,1,2)} \equiv (s_1 t_1 u_1 : s_0 t_1 u_1 : s_0 t_0 u_1 : s_1 t_0 u_0 - s_0 t_1 u_0)$

*A surface in  $V_{\rho_1}$  contains an irreducible curve of tridegree  $(1, 1, 0)$  and an isolated point. The base ideal is*

$$(s_0 t_1 - s_1 t_0, u_1) \cap (s_0, t_1, u_0) .$$

- $\rho_2^{(1,1,2)} \equiv (s_0 t_1 u_1 : s_1 t_0 u_1 : s_0 t_0 u_1 : s_1 t_0 u_0 - s_0 t_1 u_0 - s_1 t_1 u_1)$

*A surface in  $V_{\rho_2}$  contains an irreducible curve of tridegree  $(1, 1, 0)$  and has contact to a surface of tridegree  $(1, 1, 1)$  at a point of the curve. The base ideal is*

$$(s_0 t_1 - s_1 t_0, u_1) \cap (u_1^2, t_0 u_1, s_0 u_1, t_0^2, s_0 t_0, s_0^2, s_1 t_0 u_0 - s_0 t_1 u_0 - s_1 t_1 u_1) .$$

- $\rho_3^{(1,1,2)} \equiv (s_0 t_1 u_1 : s_1 t_0 u_1 : s_0 t_0 u_1 : s_1 t_1 u_0)$

*A surface in  $V_{\rho_3}$  contains a pair of intersecting lines, of tridegrees  $(1, 0, 0)$  and  $(0, 1, 0),$  and an isolated point. The base ideal is*

$$(s_1, u_1) \cap (t_1, u_1) \cap (s_0, t_0, u_0) .$$



- $\rho_4^{(1,1,2)} \equiv (s_1 t_1 u_1 : s_0 t_1 u_1 : s_0 t_0 u_1 : s_1 t_1 u_0 - s_1 t_0 u_1)$

A surface in  $V_{\rho_4}$  contains a pair of intersecting lines, of tridegrees  $(1, 0, 0)$  and  $(0, 1, 0)$ , and is tangent to a curve of tridegree  $(0, 1, 1)$  at one of the points of the line of tridegree  $(1, 0, 0)$ . The base ideal is

$$(s_1, u_1) \cap (t_1, u_1) \cap (s_0, t_0 u_1 - t_1 u_0, t_1^2, t_1 u_1, t_1^2).$$

- $\rho_5^{(1,1,2)} \equiv (s_1 t_1 u_1 : s_0 t_1 u_1 : s_1 t_0 u_1 : s_1 t_1 u_0 + s_0 t_0 u_1)$

A surface in  $V_{\rho_5}$  contains a pair of intersecting lines, of tridegrees  $(1, 0, 0)$  and  $(0, 1, 0)$ , and has contact to a surface of tridegree  $(1, 1, 1)$  at the point of intersection of the lines. The base ideal is

$$(s_1, u_1) \cap (t_1, u_1) \cap (s_1 t_1 u_0 + s_0 t_0 u_1, s_1^2, s_1 u_1, t_1^2, t_1 u_1, u_1^2).$$

*Proof.* By the diagram of (3.8),  $\phi$  lies in the orbit of a birational map that factors as  $\phi' = \zeta_V \circ \pi_L$  for either  $V = V_o$  or  $V = V_x$  and some point  $L$  in  $Y_V$ . In particular, the orbit of  $\phi$  is determined by one of the orbits listed in Lemmas 3.4.2.1 and 3.4.2.2, depending the curve component of  $Z$  and where the point  $L$  belongs. In the case that  $L = P$ , we find a common factor to the entries of  $\phi$ . Excluding this case, the possibilities when  $V = V_o$  are the following:

- If  $L \in Y_V \setminus H_{1-2}$ ,  $\phi$  lies in the orbit of  $\rho_1^{(1,1,2)}$
- If  $L \in H_{1-2} \setminus (H_4 \cup P)$ ,  $\phi$  lies in the orbit of  $\rho_2^{(1,1,2)}$
- If  $L \in H_{1-2} \cap H_4$ ,  $\phi$  lies in the orbit of  $\rho_2^{(1,1,1)}$

Secondly, if  $V = V_x$  the possibilities are the following:

- If  $L \in Y_V \setminus ((H_1 \cup H_2) \cap H_3)$ ,  $\phi$  lies in the orbit of  $\rho_3^{(1,1,2)}$
- If  $L \in ((H_1 \cup H_2) \cap H_3) \setminus (H_4 \cup \overline{OP})$ ,  $\phi$  lies in the orbit of  $\rho_4^{(1,1,2)}$
- If  $L \in ((H_1 \cup H_2) \cap H_3 \cap H_4) \setminus O$ ,  $\phi$  lies in the orbit of  $\rho_3^{(1,1,1)}$
- If  $L \in \overline{OP} \setminus (O \cup P)$ ,  $\phi$  lies in the orbit of  $\rho_5^{(1,1,2)}$
- If  $L = O$ ,  $\phi$  lies in the orbit of  $\rho_4^{(1,1,1)}$

□

### 3.4.3. Orbits of birational maps of type $(1, 2, 2)$ , $(2, 1, 2)$ and $(2, 2, 1)$

By Lemma 3.1.15, any Cohen-Macaulay curve in  $X$  of tridegree either  $(1, 0, 0)$ ,  $(0, 1, 0)$ , or  $(0, 0, 1)$  is a projective line. Therefore, by means of an automorphism of  $X$  it can be transformed into the line  $\ell$  of ideal  $I_\ell = (t_1, u_1)$ . In particular, any birational map of type  $(1, 2, 2)$ ,  $(2, 1, 2)$ , or  $(2, 2, 1)$  lies in the orbit of a birational map with  $\ell$  in its base locus.

By the diagram (3.8), in order to study the orbits of the right-action in  $\text{Bir}_{(1,2,2)}$ ,  $\text{Bir}_{(2,1,2)}$ , and  $\text{Bir}_{(2,2,1)}$  we can compute the orbits of the group action given by the stabilizer  $\text{Stab}(\ell) \leq \text{Aut}(X)$

of  $\ell$  acting on  $Y_V$ , where  $V = (\ell)_{(1,1,1)}$ . Namely, the birational map in the diagram  $\zeta_V : X \dashrightarrow Y_V \subset \mathbb{P}_{\mathbb{C}}^5$  is

$$\zeta_V : (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) \mapsto (s_0 t_0 u_1 : s_1 t_0 u_1 : s_0 t_1 u_1 : s_1 t_1 u_1 : s_0 t_1 u_0 : s_1 t_1 u_0) , \quad (3.16)$$

and  $Y_V$  is determined by the ideal  $(y_0 y_5 - y_1 y_4, y_2 y_5 - y_3 y_4, y_1 y_2 - y_0 y_3)$ . The defining equations of the projection  $\Pi_W(E)$  of the exceptional divisor  $E$  in  $Y_V$  are  $y_2 = y_3 = 0$ . We write  $H_{01}$ ,  $H_{23}$ , and  $H_{45}$  for the subvarieties of  $Y_W$  given respectively by  $y_0 = y_1 = 0$ ,  $y_2 = y_3 = 0$ , and  $y_4 = y_5 = 0$ .

**Lemma 3.4.7.** *The group action of  $\text{Stab}(\ell)$  on  $Y_V$  determines the three orbits:*

$$Y_V \setminus H_{23} , \quad H_{23} \setminus (H_{01} \cup H_{45}) , \quad H_{23} \cap (H_{01} \cup H_{45})$$

*Proof.* Write  $D_t$  and  $D_u$  for the divisors in  $X$  respectively determined by  $t_1 = 0$  and  $u_1 = 0$ . By §3.4.5.5, the group action of  $\text{Stab}(\ell)$  on  $X$  determines the three orbits

$$\ell , \quad (D_t \cup D_u) \setminus \ell , \quad X \setminus (D_t \cup D_u) .$$

The last two orbits are respectively transformed by  $\zeta_V$  into  $H_{23} \cap (H_{01} \cup H_{45})$  and  $Y_V \setminus H_{23}$ . Thus, we find two of the orbits in the statement. The orbit  $H_{23} \setminus (H_{01} \cup H_{45})$  follows from the extension of the action of  $\text{Stab}(\ell)$  to  $Y_V$ , by the action of the group  $\mathcal{M}_V$  of matrices of the form (3.23). Equivalently, this last orbit corresponds to the projection by  $\Pi_V$  of general orbit on the exceptional divisor  $E$  of the blow-up  $\text{Bl}_V X \rightarrow X$ , when we let  $\text{Stab}(\ell)$  act on  $\text{Bl}_V X$ .  $\square$

**Theorem 3.4.8.** *Let  $\phi$  be birational of type either  $(1, 2, 2)$ ,  $(2, 1, 2)$  or  $(2, 2, 1)$ . Then,  $\phi$  belongs to the orbit of one of the following birational maps:*

- $\rho_1^{(1,2,2)} \equiv (s_1 t_0 u_1 - s_0 t_1 u_1 : s_0 t_0 u_1 - s_0 t_1 u_1 : s_1 t_1 u_0 - s_0 t_1 u_1 : s_0 t_1 u_0 - s_0 t_1 u_1)$

*A surface in  $V_{\rho_1}$  contains a line and two isolated points with all the coordinates different, i.e. the projections onto the factors of  $X = (\mathbb{P}_{\mathbb{C}}^1)^3$  are all different. The base ideal is*

$$(t_1, u_1) \cap (s_0, t_0, u_0) \cap (s_0 - s_1, t_0 - t_1, u_0 - u_1) .$$

- $\rho_2^{(1,2,2)} \equiv (s_1 t_0 u_1 : s_0 t_0 u_1 : s_1 t_1 u_0 - s_0 t_1 u_1 : s_0 t_1 u_0 + s_0 t_1 u_1)$

*A surface in  $V_{\rho_2}$  contains a line and two isolated points with the same coordinate in exactly one of the factors of  $X$ . The base ideal is*

$$(t_1, u_1) \cap (s_0, t_0, u_0) \cap (s_0 + s_1, t_0, u_0 + u_1) .$$

- $\rho_3^{(1,2,2)} \equiv (s_1 t_0 u_1 - s_0 t_1 u_1 : s_0 t_0 u_1 : s_1 t_1 u_0 - s_0 t_1 u_1 : s_0 t_1 u_0)$

*A surface in  $V_{\rho_3}$  contains a line and is tangent to a curve of tridegree  $(1, 1, 1)$  at an isolated point. The base ideal is*

$$(t_1, u_1) \cap (s_0 t_1 - s_1 t_0, s_0 u_1 - s_1 u_0, t_0 u_1 - t_1 u_0, s_0^2, s_0 t_0, s_0 u_0, t_0^2, t_0 u_0, u_0^2) .$$

- $\rho_4^{(1,2,2)} \equiv (s_1 t_0 u_1 - s_0 t_1 u_1 : s_0 t_0 u_1 : s_1 t_1 u_0 : s_0 t_1 u_0)$

*A surface in  $V_{\rho_4}$  contains a line of tridegree  $(1, 0, 0)$  and is tangent to a curve of tridegree  $(1, 1, 0)$  at an isolated point. The base ideal is*

$$(t_1, u_1) \cap (u_0, s_0 t_1 - s_1 t_0, s_0^2, s_0 t_0, t_0^2) .$$

- $\rho_5^{(1,2,2)} \equiv (s_0 t_1 u_1 : s_1 t_0 u_1 : s_1 t_1 u_0 : s_0 t_1 u_0 - s_0 t_0 u_1)$

A surface in  $V_{\rho_5}$  contains a line of tridegree  $(1, 0, 0)$  and an isolated point, and is tangent to a curve of tridegree  $(0, 1, 1)$  at a point of the line. The base ideal is

$$(t_1, u_1) \cap (s_0, t_0, u_0) \cap (s_1, t_0 u_1 - t_1 u_0, t_1^2, t_1 u_1, u_1^2) .$$

- $\rho_6^{(1,2,2)} \equiv (s_1 t_1 u_1 : s_0 t_1 u_1 : s_1 t_1 u_0 + s_1 t_0 u_1 : s_0 t_1 u_0 - s_0 t_0 u_1 - 2s_1 t_0 u_1)$

A surface in  $V_{\rho_6}$  contains a line of tridegree  $(1, 0, 0)$  and is tangent to two curves of tridegree  $(0, 1, 1)$  at two distinct points of the line. The base ideal is

$$(t_1, u_1) \cap (s_1, t_0 u_1 - t_1 u_0, t_1^2, t_1 u_1, u_1^2) \cap (s_0 + s_1, t_0 u_1 + t_1 u_0, t_1^2, t_1 u_1, u_1^2) .$$

- $\rho_7^{(1,2,2)} \equiv (s_1 t_1 u_1 : s_0 t_1 u_1 : s_1 t_1 u_0 - s_1 t_0 u_1 : s_0 t_1 u_0 - s_0 t_0 u_1 + s_1 t_0 u_1)$

A surface in  $V_{\rho_7}$  contains a line and has contact to a surface of tridegree  $(0, 1, 1)$  at a point of the line. The base ideal is

$$(t_1, u_1) \cap (s_1 t_1 u_0 - s_1 t_0 u_1, s_0 t_1 u_0 - s_0 t_0 u_1 + s_1 t_0 u_1, s_1^2, t_1^2, t_1 u_1, u_1^2) .$$

- $\rho_8^{(1,2,2)} \equiv (s_1 t_1 u_1 : s_1 t_0 u_1 - s_0 t_1 u_1 : s_1 t_1 u_0 - s_0 t_1 u_1 : s_0 t_1 u_0 - s_0 t_0 u_1 - s_0 t_1 u_1)$

A surface in  $V_{\rho_8}$  contains a line and has contact to a surface of tridegree  $(0, 1, 1)$  at a point of the line. The base ideal is

$$(t_1, u_1) \cap (t_1 u_0 - t_0 u_1 - t_1 u_1, s_1^2, u_1^3, t_1 u_1^2, s_1 u_1^2, s_1 u_0 u_1 - s_0 u_1^2, \\ t_1^2 u_1, s_1 t_1 u_1, s_1 t_0 u_1 - s_0 t_1 u_1, t_1^3, s_1 t_1^2, s_1 t_0 t_1 - s_0 t_1^2) .$$

*Proof.* By the diagram (3.8),  $\phi$  lies in the orbit of a birational map that factors as  $\phi' = \zeta_V \circ \pi_L$  for some line  $L$  in  $\mathbb{P}_{\mathbb{C}}^5$  such that the restriction  $\pi_L|_{Y_V}$  has degree one, i.e.  $\pi_L|_{Y_V}$  is birational. From Proposition 3.2.6, we have  $\deg Y_V = 3$ . If  $L \not\subset Y_V$ , then  $\deg(L \cap Y_V) \geq 2$  as otherwise  $\deg(\pi_L|_{Y_V}) > 1$ . On the other hand, consider a linear parametrization  $\mathbb{P}_{\mathbb{C}}^1 \rightarrow L$ . Since the ideal of  $Y_V$  is generated by three quadratic forms, the pullback of  $L \cap Y_V$  yields three quadratic forms in  $\mathbb{P}_{\mathbb{C}}^1$ , implying that  $\deg(L \cap Y_V) \leq 2$ . Then, we must have either  $L \subset Y_V$  or  $L \not\subset Y_V$  and  $\deg(L \cap Y_V) = 2$ .

Given a point  $P \in Y_V \setminus H_{23}$ , we find  $Q = (\alpha_0 : \alpha_1) \times (\beta : 1) \times (\gamma : 1)$  in  $X$  such that  $\zeta_V(Q) = P$ . Write  $Y_t(P) \subset Y_V$  for the closure of the image of the restriction of  $\zeta_V$  to  $\mathbb{P}_{\mathbb{C}}^1 \times (\beta : 1) \times \mathbb{P}_{\mathbb{C}}^1$ . Analogously, we define  $Y_u(P) \subset Y_V$  by restricting  $\zeta_V$  to  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times (\gamma : 1)$ . Notice that the restriction of  $\zeta_V$  to  $(\alpha_0 : \alpha_1) \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  is a projective plane. Both  $Y_t(P)$  and  $Y_u(P)$  are smooth subvarieties of  $Y_V$ . Given a subvariety  $S \subset Y_V$  and a smooth point  $P$  in  $S$ , we write  $T_P(S)$  for the tangent space of  $S$  at  $P$ .

Now, we discuss all the possible intersections between  $L$  and  $Y_V$ . In the first place, we assume that  $L \not\subset Y_V$ :

- If  $L$  meets  $Y_V$  transversally at  $P_1, P_2 \in Y_V \setminus H_{23}$ , and moreover we have  $P_2 \notin Y_t(P_1)$  and  $P_2 \notin Y_u(P_1)$ ,  $\phi$  lies in the orbit of  $\rho_1^{(1,2,2)}$
- If  $L$  meets  $Y_V$  transversally at  $P_1, P_2 \in Y_V \setminus H_{23}$ , and either  $P_2 \in Y_t(P_1)$  or  $P_2 \in Y_u(P_1)$ ,  $\phi$  lies in the orbit of  $\rho_2^{(1,2,2)}$

- If  $L \subset T_P(Y_V)$  at a point  $P \in Y_V \setminus H_{23}$ , and moreover we have  $L \not\subset T_P(Y_t(P))$  and  $L \not\subset T_P(Y_u(P))$ ,  $\phi$  lies in the orbit of  $\rho_3^{(1,2,2)}$
- If  $L \subset T_P(Y_V)$  at a point  $P \in Y_V \setminus H_{23}$ , and either  $L \subset T_P(Y_t(P))$  or  $L \subset T_P(Y_u(P))$ ,  $\phi$  lies in the orbit of  $\rho_4^{(1,2,2)}$
- If  $L$  meets  $Y_V$  transversally at  $P_1 \in Y_V \setminus H_{23}$ ,  $P_2 \in H_{23} \setminus (H_{01} \cup H_{45})$ ,  $\phi$  lies in the orbit of  $\rho_5^{(1,2,2)}$
- If  $L$  meets  $Y_V$  transversally at  $P_1, P_2 \in H_{23} \setminus (H_{01} \cup H_{45})$ ,  $\phi$  lies in the orbit of  $\rho_6^{(1,2,2)}$
- If  $L \subset T_P(Y_V)$  at a point  $P \in H_{23} \setminus (H_{01} \cup H_{45})$  and  $L \subset T_P(H_{23})$ ,  $\phi$  lies in the orbit of  $\rho_7^{(1,2,2)}$
- If  $L \subset T_P(Y_V)$  at a point  $P \in H_{23} \setminus (H_{01} \cup H_{45})$  and  $L \not\subset T_P(H_{23})$ ,  $\phi$  lies in the orbit of  $\rho_8^{(1,2,2)}$
- If  $L$  meets  $Y_V$  transversally at  $P_1 \in Y_V \setminus H_{23}$ ,  $P_2 \in H_{23} \cap (H_{01} \cup H_{45})$ ,  $\phi$  lies in the orbit of  $\rho_3^{(1,1,2)}$
- If  $L$  meets  $Y_V$  transversally at  $P_1 \in H_{23} \setminus (H_{01} \cup H_{45})$ ,  $P_2 \in H_{23} \cap (H_{01} \cup H_{45})$ ,  $\phi$  lies in the orbit of  $\rho_4^{(1,1,2)}$
- If  $L \subset T_P(H_{23})$  at a point  $P \in H_{23} \cap (H_{01} \cup H_{45})$ ,  $\phi$  lies in the orbit of  $\rho_5^{(1,1,2)}$

Secondly, we study the cases where  $L \subset Y_V$ :

- If  $L \subset Y_V \setminus H_{23}$ ,  $\phi$  is not dominant
- If  $L \subset Y_V$  and intersects transversally  $H_{23} \setminus (H_{01} \cup H_{45})$ , then  $\phi$  lies in the orbit of  $\rho_2^{(1,1,1)}$
- If  $L \subset Y_V$  and intersects transversally  $H_{23} \cap (H_{01} \cup H_{45})$ ,  $\phi$  lies in the orbit of  $\rho_3^{(1,1,1)}$
- If  $L \subset H_{23} \setminus (H_{01} \cup H_{45})$ , and  $L \not\subset H_{23} \cap (H_{01} \cup H_{45})$ ,  $\phi$  lies in the orbit of  $\rho_4^{(1,1,1)}$

□

#### 3.4.4. Orbits of birational maps of type $(2, 2, 2)$

Let  $Q$  be a point in  $X$  and let  $\lambda, \mu, \nu \in \mathbb{C}^*$ . There is always an automorphism of  $R$  that sends the ideal  $I_{Q(\lambda, \mu, \nu)}$  (see Notation 3.2.3) to

$$I' = I_{Q'(1,1,1)} = (s_1 t_0 u_0 + s_0 t_1 u_0 + s_0 t_0 u_1, s_0 t_1 u_1, s_1 t_0 u_1, s_1 t_1 u_0, s_1 t_1 u_1),$$

where  $Q' = (1 : 0)^3 \in X$ . In particular, any birational map of type  $(2, 2, 2)$  lies in the orbit of a birational map with the singular point defined by  $I'$  in its base locus. Let  $\text{Stab}(Q'(1, 1, 1)) \leq \text{Aut}(X)$  be the stabilizer of this singular. Then, in order to study the orbits of the right-action in  $\text{Bir}_{(2,2,2)}$ , we can compute the orbits of the group action given by the stabilizer  $\text{Stab}(Q'(1, 1, 1))$  acting on  $Y_V$ , where we set  $V = I'_{(1,1,1)}$ . The birational map  $\zeta_V : X \dashrightarrow Y_V \subset \mathbb{P}_{\mathbb{C}}^5$  in the diagram (3.8) is now

$$(s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) \mapsto (s_1 t_0 u_0 + s_0 t_1 u_0 + s_0 t_0 u_1 : s_0 t_1 u_1 : s_1 t_0 u_1 : s_1 t_1 u_0 : s_1 t_1 u_1), \quad (3.17)$$

and the defining equation of  $Y_V$  in  $\mathbb{P}_{\mathbb{C}}^4$  is  $y_1y_2 + y_1y_3 + y_2y_3 - y_0y_4 = 0$ . On the other hand, the projection  $H_4 = \Pi_V(E)$  of the exceptional divisor  $E$  is the hyperplane section in  $Y_V$  given by  $y_4 = 0$ . The divisor  $H_4$  is again a 2-dimensional cone, of singular point  $O = (1:0:0:0:0)$ . Similarly, we define the points

$$P_1 = (0:1:0:0:0), P_2 = (0:0:1:0:0), P_3 = (0:0:0:1:0).$$

**Lemma 3.4.9.** *The group action of  $\text{Stab}(Q'(1, 1, 1))$  on  $Y_V$  determines the four orbits:*

$$Y_W \setminus H_4, H_4 \setminus (\overline{OP_1} \cup \overline{OP_2} \cup \overline{OP_3}), (\overline{OP_1} \cup \overline{OP_2} \cup \overline{OP_3}) \setminus O, O$$

*Proof.* Write  $D_s, D_t,$  and  $D_u$  for the divisors in  $X$  defined respectively by  $s_1 = 0, t_1 = 0,$  and  $u_1 = 0$ . Similarly, write  $D_{st}, D_{su}$  and  $D_{tu}$  for their intersections, with the obvious notation. By §3.4.5.5, the group action of  $\text{Stab}(Q'(1, 1, 1))$  on  $X$  determines the four orbits

$$Q', (D_{st} \cup D_{su} \cup D_{tu}) \setminus Q', (D_s \cup D_t \cup D_u) \setminus (D_{st} \cup D_{su} \cup D_{tu}), X \setminus (D_s \cup D_t \cup D_u).$$

The last three orbits are respectively transformed by  $\zeta_V$  into  $O, (\overline{OP_1} \cup \overline{OP_2} \cup \overline{OP_3}) \setminus O,$  and  $Y_V \setminus H_4$ . Therefore, we find three of the orbits in the statement. The orbit  $H_4 \setminus (\overline{OP_1} \cup \overline{OP_2} \cup \overline{OP_3})$  follows from the extension of the action of  $\text{Stab}(\ell)$  to  $Y_V$ , by the action of the group  $\mathcal{M}_V$  of matrices of the form (3.24). Equivalently, this last orbit corresponds to the projection by  $\Pi_V$  of general orbit on the exceptional divisor  $E$  of the blow-up  $\text{Bl}_V X \rightarrow X$ , when we let  $\text{Stab}(Q'(1, 1, 1))$  act on  $\text{Bl}_V X$ .  $\square$

**Theorem 3.4.10.** *Let  $\phi$  be birational of type  $(2, 2, 2)$ . Then,  $\phi$  belongs to the orbit of one of the following birational maps:*

- $\rho_1^{(2,2,2)} \equiv (s_0 t_1 u_1 : s_1 t_0 u_1 : s_1 t_1 u_0 : s_1 t_0 u_0 + s_0 t_1 u_0 + s_0 t_0 u_1)$

*A surface in  $V_{\rho_1}$  contains a point and has contact to a surface of tridegree  $(1, 1, 1)$  at a distinct point. The base ideal is*

$$(s_0, t_0, u_0) \cap (s_1 t_0 u_0 + s_0 t_1 u_0 + s_0 t_0 u_1, s_1^2, s_1 t_1, s_1 u_1, t_1^2, t_1 u_1, u_1^2). \quad (3.18)$$

- $\rho_2^{(2,2,2)} \equiv (s_1 t_1 u_1 : s_1 t_0 u_1 - s_0 t_1 u_1 : 2s_1 t_1 u_0 + s_0 t_1 u_1 : s_1 t_0 u_0 + s_0 t_1 u_0 + s_0 t_0 u_1)$

*A surface in  $V_{\rho_2}$  has contact to a surface of tridegree  $(1, 1, 1)$  at a point. The base ideal is*

$$(u_1^3, t_1 u_1^2, s_1 u_1^2, 2t_1 u_0 u_1 + t_0 u_1^2, 2s_1 u_0 u_1 + s_0 u_1^2, t_1^2 u_1, s_1 t_1 u_1, s_1 t_0 u_1 - s_0 t_1 u_1, \\ s_1^2 u_1, 2t_1^2 u_0 + t_0 t_1 u_1, 2s_1 t_1 u_0 + s_0 t_1 u_1, s_1 t_0 u_0 + s_0 t_1 u_0 + s_0 t_0 u_1, \\ 2s_1^2 u_0 + s_0 s_1 u_1, t_1^3, s_1 t_1^2, s_1 t_0 t_1 - s_0 t_1^2, s_1^2 t_1, s_1^2 t_0 - s_0 s_1 t_1, s_1^3).$$

*Proof.* By the diagram (3.8),  $\phi$  lies in the orbit of a birational map that factors as  $\phi' = \zeta_V \circ \pi_L$  for some point  $L$  in  $Y_W$ . We have the following four possibilities:

- If  $L \in Y_V \setminus H_4$ ,  $\phi$  lies in the orbit of  $\rho_1^{(2,2,2)}$
- If  $L \in H_4 \setminus (\overline{OP_1} \cup \overline{OP_2} \cup \overline{OP_3})$ ,  $\phi$  lies in the orbit of  $\rho_2^{(2,2,2)}$
- If  $L \in (\overline{OP_1} \cup \overline{OP_2} \cup \overline{OP_3}) \setminus O$ ,  $\phi$  lies in the orbit of  $\rho_2^{(1,1,2)}$

- If  $L = O$ ,  $\phi$  lies in the orbit of  $\rho_4^{(1,1,1)}$

□

**Remark 3.4.11** (Singular point in the base locus of a general  $(2, 2, 2)$  birational map). The singular point in the base locus of a general trilinear birational map of type  $(2, 2, 2)$  is not a local complete intersection. More precisely, let  $Q$  be this singular point. Then,  $Q$  is a local complete intersection if and only if  $\deg(Q) = e(Q)$  (see [149, §14] and [39, §4.1]), where  $e(Q)$  stands for the Hilbert-Samuel multiplicity of  $Q$ . Using *Macaulay2* [101], we find

$$\deg(Q) = \dim_{\mathbb{C}} \frac{R_Q}{B_Q} = 3,$$

where  $R_Q$  and  $B_Q$  respectively stand for the localizations of  $R$  and  $B$  at  $Q$ . On the other hand, since  $\phi$  is birational the Hilbert-Samuel multiplicity of the base locus  $Z$  must be 5, as the intersection of three general trilinear polynomials in  $(\mathbb{P}_{\mathbb{C}}^1)^3$  has multiplicity 6 (see [96, §4.4]). Moreover,  $Z$  is the union of  $Q$  and a simple point  $P$ , satisfying  $\deg(P) = e(P) = 1$ . Therefore, it follows that  $e(Q) = 4 \neq \deg(Q)$ , and  $Q$  is not a complete intesection.

Altogether, Theorems 3.4.3, 3.4.6, 3.4.8 and 3.4.10 provide the complete list of the orbits of the right-action in  $\mathfrak{Bit}_{(1,1,1)}$ . Interestingly, we find the following corollary from this classification.

**Corollary 3.4.12.** *All the irreducible components of  $\mathfrak{Bit}_{(1,1,1)}$  are unirational.*

*Proof.* Let  $Y$  be an irreducible component of  $\mathfrak{Bit}_{(1,1,1)}$ , as listed in Theorem 3.3.6. The action of the subgroup  $H \trianglelefteq \text{Aut}(X)$ , consisting of the automorphisms  $\varphi$  such that the associated ring automorphism  $\varphi^{\#} : R \rightarrow R$  is degree-preserving, on any representative of the general orbit of  $Y$  (of the type that labels the irreducible component  $Y$ !), as listed in either Theorem 3.4.3, 3.4.6, 3.4.8, or 3.4.10, yields a dominant rational map from  $H$  to  $Y$ . Since we find a dense affine set in  $H \cong \text{PGL}(2, \mathbb{C})^3$ , the result follows. □

**Remark 3.4.13** (Deformation of birational maps). The proof of Corollary 3.4.12 provides dominant rational parametrizations for each irreducible component of the space of trilinear birational maps. However, these parametrizations may not be “geometrically intuitive” from the perspective of geometric design.

To clarify, consider a specific domain, such as the unit cube  $[0, 1]^3$  in the affine chart defined by  $s_0 \neq 0, t_0 \neq 0, u_0 \neq 0$ . When we compose  $\phi$  with automorphisms in  $\text{Aut}(X)$  and  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^3)$  to obtain a new birational map  $\phi'$ , it's not immediately clear how the image of  $\phi'([0, 1]^3)$  relates to the image of  $\phi([0, 1]^3)$ .

In Chapter 4, we tackle the question of deforming birational maps in a geometrically meaningful way (see §4.3.5). This involves providing a continuous parametrization of the irreducible components in  $\mathfrak{Bit}_{(1,1,1)}$ , and explaining how to systematically determine new control points for defining trilinear birational maps. This approach allows users to grasp the variation in the geometry of the rational map along with the deformation.

### 3.4.5. The group actions of the stabilizers on $Y_V$

In this section, we describe explicitly the groups of matrices acting on  $\mathbb{P}_{\mathbb{C}}^N$  that extend the group actions of the stabilizers, considered in §3.4.2, §3.4.3, and §3.4.4, from  $X$  to  $Y_V$ . These actions are used in the proofs of Lemmas 3.4.4, 3.4.5, 3.4.7, and 3.4.9, and are thus necessary for the classification of the base loci.

### 3.4.5.1. General strategy for the action extension

Let  $S \subset X$  be one of the stabilized subschemes of the previous subsections, and set  $V = (I_S)_{(1,1,1)}$ . We define a group monomorphism

$$\begin{aligned} \Phi : \text{Stab}(S) &\rightarrow \text{PGL}(N+1, \mathbb{C}) \\ \varphi &\mapsto M_\varphi \end{aligned} \quad (3.19)$$

whose image  $\mathcal{M}_V$  induces an action on  $\mathbb{P}_{\mathbb{C}}^N$  satisfying

$$\zeta_V(\varphi(Q)) = M_\varphi \cdot \zeta_V(Q)$$

for every  $Q$  in  $X$ . In particular, since  $Y_V$  is the closure of the image of  $\zeta_V$ , the action of  $\mathcal{M}_V$  restricts to  $Y_V$ . Therefore, the group of matrices  $\mathcal{M}_V$  extends the action of  $\text{Stab}(S)$  from  $X$  to  $Y_V$ . More explicitly,  $\zeta_V$  has the form

$$\begin{aligned} \zeta_V : X = \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 &\dashrightarrow Y_V \subset \mathbb{P}_{\mathbb{C}}^N = \text{Proj}(\mathbb{C}[y_0, \dots, y_N]) \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (g_0 : \dots : g_N) \end{aligned} \quad (3.20)$$

for some  $g_n = g_n(s_0, s_1, t_0, t_1, u_0, u_1)$  defining a basis of  $V$ . Given an automorphism  $\varphi$  in  $\text{Stab}(S)$ , the induced ring automorphism  $\varphi^\# : R \rightarrow R$  satisfies  $\varphi^\#(I) = I$ . Namely,  $\varphi^\#$  restricts to a  $\mathbb{C}$ -vector space automorphism of  $V$ . Hence, we find an invertible  $(N+1) \times (N+1)$  matrix  $M_\varphi$  such that

$$(\varphi^\#(g_0) \ \dots \ \varphi^\#(g_N))^t = M_\varphi (g_0 \ \dots \ g_N)^t,$$

which yields the group monomorphism  $\Phi : \varphi \mapsto M_\varphi$  in (3.19).

In the following subsections, we derive the group monomorphism  $\Phi$ , or equivalently the group of matrices  $\mathcal{M}_V$ , explicitly.

### 3.4.5.2. Group of matrices in §3.4.2

#### 3.4.5.3. Nonsingular case

If  $S = C_o$ , we find

$$I = (I_{C_o})_{(1,1,1)} = (s_0 t_0 u_1, s_1 t_0 u_1, s_0 t_1 u_1, s_1 t_1 u_1, (s_0 t_1 - s_1 t_0) u_0) = \mathfrak{R} \cap I_{C_o},$$

and  $\zeta_V$  is as in (3.13). Let  $A_o \leq \text{PGL}(2, \mathbb{C})^3$  be the subgroup of elements of the form

$$\varphi = \varphi_1 \times \varphi_2 \times \varphi_3 = \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix} \times \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix} \times \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ 0 & \gamma_{11} \end{pmatrix}.$$

The algebraic group  $\text{Stab}(C_o)$  has two irreducible components  $K_1$  and  $K_2$ , which are isomorphic to  $A_o$ . More specifically, an element in  $K_1$  has the form

$$(s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) \mapsto \varphi_1(s_0, s_1) \times \varphi_2(t_0, t_1) \times \varphi_3(u_0, u_1)$$

whereas elements in  $K_2$  have the form

$$(s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) \mapsto \varphi_1(t_0, t_1) \times \varphi_2(s_0, s_1) \times \varphi_3(u_0, u_1).$$

In particular,  $\Phi$  splits in two monomorphisms  $\Phi_i : K_i \cong A_o \rightarrow \mathrm{PGL}(5, \mathbb{C})$ . The monomorphism  $\Phi_1$  maps  $\varphi$  to

$$\begin{pmatrix} \beta_{00}^2 \gamma_{11} & \beta_{00} \beta_{01} \gamma_{11} & \beta_{00} \beta_{01} \gamma_{11} & \beta_{01}^2 \gamma_{11} & 0 \\ \beta_{00} \beta_{10} \gamma_{11} & \beta_{00} \beta_{11} \gamma_{11} & \beta_{01} \beta_{10} \gamma_{11} & \beta_{01} \beta_{11} \gamma_{11} & 0 \\ \beta_{00} \beta_{10} \gamma_{11} & \beta_{01} \beta_{10} \gamma_{11} & \beta_{00} \beta_{11} \gamma_{11} & \beta_{01} \beta_{11} \gamma_{11} & 0 \\ \beta_{10}^2 \gamma_{11} & \beta_{10} \beta_{11} \gamma_{11} & \beta_{10} \beta_{11} \gamma_{11} & \beta_{11}^2 \gamma_{11} & 0 \\ 0 & (\beta_{01} \beta_{10} - \beta_{00} \beta_{11}) \gamma_{01} & (\beta_{00} \beta_{11} - \beta_{01} \beta_{10}) \gamma_{01} & 0 & (\beta_{00} \beta_{10} - \beta_{00} \beta_{11}) \gamma_{00} \end{pmatrix} \quad (3.21)$$

whereas  $\Phi_2$  maps  $\varphi$  to the same matrix, after the permutation of the columns induced by  $s_i \leftrightarrow t_i$ .

#### 3.4.5.4. Singular case

If  $S = C_\times$ , we find

$$I = (I_{C_\times})_{(1,1,1)} = (s_0 t_0 u_1, s_1 t_0 u_1, s_0 t_1 u_1, s_1 t_1 u_1, s_1 t_1 u_0) = \mathfrak{N} \cap I_{C_\times},$$

and  $\zeta_V$  is as in (3.15). Let  $A_\times \leq \mathrm{PGL}(2, \mathbb{C})^3$  be the subgroup of elements of the form

$$\varphi = \varphi_1 \times \varphi_2 \times \varphi_3 = \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ 0 & \alpha_{11} \end{pmatrix} \times \begin{pmatrix} \beta_{00} & \beta_{01} \\ 0 & \beta_{11} \end{pmatrix} \times \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ 0 & \gamma_{11} \end{pmatrix}.$$

The algebraic group  $\mathrm{Stab}(C_\times)$  has again two irreducible components  $K_1$  and  $K_2$ , which are isomorphic to  $A_\times$ . Similarly, these components encode the permutation of the factors of  $X$ . Once more,  $\Phi$  splits in two monomorphisms  $\Phi_i : K_i \cong A_\times \rightarrow \mathrm{PGL}(5, \mathbb{C})$ , where  $\Phi_1$  maps  $\varphi$  to

$$\begin{pmatrix} \alpha_{00} \beta_{00} \gamma_{11} & \alpha_{01} \beta_{00} \gamma_{11} & \alpha_{00} \beta_{01} \gamma_{11} & \alpha_{01} \beta_{01} \gamma_{11} & 0 \\ 0 & \alpha_{11} \beta_{00} \gamma_{11} & 0 & \alpha_{11} \beta_{01} \gamma_{11} & 0 \\ 0 & 0 & \alpha_{00} \beta_{11} \gamma_{11} & \alpha_{01} \beta_{11} \gamma_{11} & 0 \\ 0 & 0 & 0 & \alpha_{11} \beta_{11} \gamma_{11} & 0 \\ 0 & 0 & 0 & \alpha_{11} \beta_{11} \gamma_{01} & \alpha_{11} \beta_{11} \gamma_{00} \end{pmatrix}, \quad (3.22)$$

and  $\Phi_2$  maps  $\varphi$  to the same matrix after the permutation of the columns induced by  $s_i \leftrightarrow t_i$ .

#### 3.4.5.5. Group of matrices in §3.4.3

If  $S = \ell$ , we find

$$I = (I_\ell)_{(1,1,1)} = (s_0 t_0 u_1, s_1 t_0 u_1, s_0 t_1 u_1, s_1 t_1 u_1, s_0 t_1 u_0, s_1 t_1 u_0) = \mathfrak{N} \cap I_\ell,$$

and  $\zeta_V$  is as in (3.16). Let  $A_\ell \leq \mathrm{PGL}(2, \mathbb{C})^3$  be the subgroup of elements of the form

$$\varphi \equiv \begin{pmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{pmatrix} \times \begin{pmatrix} \beta_{00} & \beta_{01} \\ 0 & \beta_{11} \end{pmatrix} \times \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ 0 & \gamma_{11} \end{pmatrix},$$

The algebraic group  $\mathrm{Stab}(\ell)$  has again two irreducible components  $K_1$  and  $K_2$ , which are isomorphic to  $A_\ell$ . Now, the automorphisms in  $K_1$  fix all the factors of  $X$  while those in  $K_2$  transpose the last





The degenerations of the 0-dimensional base loci require further analysis. Namely, we can derive the degenerations to base loci of tridegree  $(1, 0, 0)$  (resp.  $(0, 1, 0)$ ,  $(0, 0, 1)$ ) by studying the orbits in the closure  $Y'$  of the image of

$$\begin{aligned} \zeta' : X &\dashrightarrow Y' \subset \mathbb{P}_{\mathbb{C}}^4 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (s_1 t_1 u_1 : s_0 t_1 u_1 : s_1 t_0 u_1 : s_1 t_1 u_0 : s_0 t_1 u_0 + s_0 t_0 u_1) . \end{aligned}$$

In this case, the entries of  $\zeta'$  determine a basis for the graded component  $I'_{(1,1,1)}$  of the ideal

$$I' = (x_0 y_1 z_0 + x_0 y_0 z_1) + (x_1, y_1, z_1)^2 .$$

Similarly, the degenerations to base loci of tridegree  $(1, 1, 0)$  (resp.  $(1, 0, 1)$ ,  $(0, 1, 1)$ ) can be deduced from the orbits in the closure  $Y''$  of the image of

$$\begin{aligned} \zeta'' : X &\dashrightarrow Y'' \subset \mathbb{P}_{\mathbb{C}}^4 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (s_1 t_1 u_1 : s_0 t_1 u_1 : s_1 t_0 u_1 : s_1 t_1 u_0 : s_0 t_0 u_1) , \end{aligned}$$

where the entries of  $\zeta''$  now determine a basis of the graded component  $I''_{(1,1,1)}$  of

$$I'' = (x_0 y_0 z_1) + (x_1, y_1, z_1)^2 .$$

**Example 3.4.15.** The orbits on  $Y'$  of the action given by the stabilizer of the subscheme of  $X$  defined by  $I'$  are

$$Y' \setminus H_4 , (H_1 \cap H_4) \setminus H_{2-3} , (H_{2-3} \cap H_4) \setminus (H_1 \cup H_2) , \overline{PQ} \setminus H_1 , H_1 \cap H_{2-3} \cap H_4 , Q \quad (3.25)$$

where  $H_1, H_{2-3}, H_4$  are the divisors in  $Y'$  respectively given by  $t_1 = 0$ ,  $t_2 - t_3 = 0$ , and  $t_4 = 0$ , and

$$Q = (1 : 0 : 0 : 0 : 0) , P = (0 : 1 : 0 : 0 : 0) .$$

Representatives associated to each orbit in (3.25), in the same order, are

$$\rho_5^{(1,2,2)} , \rho_6^{(1,2,2)} , \rho_8^{(1,2,2)} , \rho_2^{(1,1,1)} , \rho_7^{(1,2,2)} , \rho_4^{(1,1,1)} .$$

Since  $\overline{PQ} \subset H_{2-3} \cap H_4$  but  $\overline{PQ} \not\subset H_1 \cap H_4$ , it follows that  $\rho_8^{(1,2,2)}$  degenerates to  $\rho_2^{(1,1,1)}$ , but  $\rho_6^{(1,2,2)}$  does not.

### 3.5. Syzygies and minimal graded free resolutions

In this section we study the syzygies of the entries  $\mathbf{f} = (f_0, f_1, f_2, f_3)$  of trilinear birational maps. Interestingly, novel methods for the construction and manipulation of birational trilinear volumes can be derived from the results in this section. Namely, these methods are developed in Chapter 4, and they are based on the imposition of specific syzygies.

The following theorem yields a birationality criterion that relies on the computation of the first syzygies of  $\mathbf{f}$  in some degrees. In order to simplify the statement, we set

$$\mathbf{e}_1 = (1, 0, 0) , \mathbf{e}_2 = (0, 1, 0) , \mathbf{e}_3 = (0, 0, 1) .$$

**Theorem 3.5.1.** *Let  $\phi$  be dominant, and let  $i, j, k$  be indices such that  $\{i, j, k\} = \{1, 2, 3\}$ . Then,  $\phi$  is birational if and only if one of the following conditions holds:*

- (i)  $\mathbf{f}$  has syzygies of degrees  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$ . In this case,  $\phi$  has type  $(1, 1, 1)$ .
- (ii)  $\mathbf{f}$  has syzygies of degrees  $\mathbf{e}_i$  and  $\mathbf{e}_j$ , but not  $\mathbf{e}_k$ . In this case,  $\phi$  has type  $(1, 1, 1) + \mathbf{e}_k$ .
- (iii)  $\mathbf{f}$  has a syzygy of degree  $\mathbf{e}_i$ , but neither  $\mathbf{e}_j$  nor  $\mathbf{e}_k$ . Moreover,  $\mathbf{f}$  has a syzygy of degree either  $\mathbf{e}_i + \mathbf{e}_j$  or  $\mathbf{e}_i + \mathbf{e}_k$ , independent from the first one. In this case,  $\phi$  has type  $(1, 1, 1) + \mathbf{e}_j + \mathbf{e}_k$ .
- (iv)  $\mathbf{f}$  does not have syzygies of degree  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , or  $\mathbf{e}_3$ . Moreover,  $\mathbf{f}$  has two independent syzygies in each of the degrees  $\mathbf{e}_1 + \mathbf{e}_2$ ,  $\mathbf{e}_1 + \mathbf{e}_3$ , and  $\mathbf{e}_2 + \mathbf{e}_3$  satisfying the splitting property in Condition 3.5.7 (see §3.5.1). In this case,  $\phi$  has type  $(2, 2, 2)$ .

The proof of Theorem 3.5.1 follows immediately from Propositions 3.5.2, 3.5.4, 3.5.5, and 3.5.8, and it relies strongly on the geometric classification of §3.4. More explicitly, the composition of a rational map  $\phi$  with automorphisms of  $(\mathbb{P}_{\mathbb{C}}^1)^3$  and  $\mathbb{P}_{\mathbb{C}}^3$  induces isomorphisms between the syzygy modules of the entries of both rational maps, with maybe a permutation of the Betti numbers (not all the automorphisms of  $(\mathbb{P}_{\mathbb{C}}^1)^3$  are degree-preserving, see §3.1.3).

Therefore, the syzygies of  $\mathbf{f}$  can be computed for the representatives of the finitely many orbits of the right-action in  $\mathfrak{Bir}_{(1,1,1)}$ , listed in Theorems 3.4.3, 3.4.6, 3.4.8, and 3.4.10.

On the other hand, even though Theorem 3.5.1 is suitable for computational purposes, we provide further cohomological information about the base loci. Namely, we prove that there is a one-to-one correspondence between the type of a birational  $\phi$  and the shape of the minimal graded free resolution of its base ideal  $B$ . For the sake of comparison, the minimal graded free resolution of the base ideal of a trilinear rational map with general entries is (we exclude the Betti numbers)

$$0 \rightarrow R \rightarrow R^{21} \rightarrow R^{62} \rightarrow R^{69} \rightarrow R^{30} \rightarrow R^4 \rightarrow B \rightarrow 0. \quad (3.26)$$

As we shall see in this section, the resolution is remarkably simpler if  $\phi$  is birational.

### 3.5.1. Birational maps of type $(1, 1, 1)$

For birational maps of type  $(1, 1, 1)$ , the minimal graded free resolution of  $B$  is Hilbert-Burch.

**Proposition 3.5.2.** *Let  $\phi$  be dominant. The following are equivalent:*

- (i)  $\phi$  is birational of type  $(1, 1, 1)$ .
- (ii) The minimal graded free resolution of the base ideal  $B$  is

$$\begin{array}{c} R(-2, -1, -1) \\ \oplus \\ 0 \rightarrow R(-1, -2, -1) \rightarrow R(-1, -1, -1)^4 \xrightarrow{(f_0 \ f_1 \ f_2 \ f_3)} B \rightarrow 0, \\ \oplus \\ R(-1, -1, -2) \end{array} \quad (3.27)$$

*i.e. it is Hilbert-Burch. In particular, the base locus  $Z$  is a Cohen-Macaulay curve.*

- (iii)  $\mathbf{f}$  has syzygies of degrees  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$ .

*Proof.* We first prove that (i) implies (ii). Let  $\phi$  be birational of type  $(1, 1, 1)$ . Maintaining the notation of (1.6) for  $\phi^{-1}$ , in the Rees ideal  $J$  we find the polynomials

$$g_1 = \begin{vmatrix} s_0 & s_1 \\ a_0 & a_1 \end{vmatrix}, \quad g_2 = \begin{vmatrix} t_0 & t_1 \\ b_0 & b_1 \end{vmatrix}, \quad g_3 = \begin{vmatrix} u_0 & u_1 \\ c_0 & c_1 \end{vmatrix}. \quad (3.28)$$

Consider the  $4 \times 3$  matrix  $M = M(s_i, t_j, u_k)$ , whose  $(m, n)$ -th entry is the coefficient of  $g_n$  in the variable  $y_m$ . By definition, the tuple  $\mathbf{f} = (f_0, f_1, f_2, f_3)$  lies in the cokernel of  $M$ . Now, write  $\Delta_m$  for the  $3 \times 3$  signed minor obtained by deleting the  $m$ -th row of  $M$ . Then, the tuple  $(\Delta_0, \Delta_1, \Delta_2, \Delta_3)$  also lies in the cokernel of  $M$ . Moreover, since  $\phi$  is birational the specialization of the  $g_n$ 's at a general point of  $X$  yields three independent linear forms in  $\mathbf{y}$ , so the matrix  $M$  has generically rank 3. In particular  $M$  has no kernel in  $R^3$ , and its cokernel is a free  $R$ -module of rank one generated by  $(\Delta_0, \Delta_1, \Delta_2, \Delta_3)$ . Since the tuples  $(f_0, f_1, f_2, f_3)$  and  $(\Delta_0, \Delta_1, \Delta_2, \Delta_3)$  have trilinear entries, they must differ by a nonzero constant. Therefore, since  $\text{codim}(B) \geq 2$  the complex

$$0 \rightarrow R^3 \xrightarrow{M} R^4 \xrightarrow{(\Delta_0 \Delta_1 \Delta_2 \Delta_3)} B \rightarrow 0 \quad (3.29)$$

is a minimal graded free resolution of  $B$ . On the other hand, from the Auslander-Buchbaum formula the quotient ring  $R/B$  is Cohen-Macaulay of codimension two (see e.g. [89, §20.4]), which implies that  $Z$  is a Cohen-Macaulay curve.

Clearly (ii) implies (iii). Thus, we now assume (iii) and prove (i). The syzygies of degrees  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , and  $\mathbf{e}_3$  can be written as

$$s_0 a_1 - s_1 a_0 = \begin{vmatrix} s_0 & s_1 \\ a_0 & a_1 \end{vmatrix}, \quad t_0 b_1 - t_1 b_0 = \begin{vmatrix} t_0 & t_1 \\ b_0 & b_1 \end{vmatrix}, \quad u_0 c_1 - u_1 c_0 = \begin{vmatrix} u_0 & u_1 \\ c_0 & c_1 \end{vmatrix},$$

for some linear forms  $a_i = a_i(\mathbf{y})$ ,  $b_j = b_j(\mathbf{y})$ , and  $c_k = c_k(\mathbf{y})$ . Then, the rational map from  $\mathbb{P}_{\mathbb{C}}^3$  to  $X$  given by

$$(y_0 : y_1 : y_2 : y_3) \mapsto (a_0 : a_1) \times (b_0 : b_1) \times (c_0 : c_1)$$

must be the inverse rational map of  $\phi$ . Hence,  $\phi$  is birational of type  $(1, 1, 1)$ .  $\square$

**Remark 3.5.3.** Notice that the proof of Proposition 3.5.2 is independent from the classification of §3.4. In particular, it is valid to use Proposition 3.5.2 in the proof of Theorem 3.4.3. On the other hand, it follows from the proof that the base locus  $Z$  of a birational map  $\phi$  of type  $(1, 1, 1)$  is an arithmetically Cohen-Macaulay curve, i.e. the homogeneous coordinate ring  $R/B$  is Cohen-Macaulay, which is stronger than requiring that  $Z$  is Cohen-Macaulay.

### 3.5.2. Birational maps of type $(1, 1, 2)$

The study of the syzygies of birational maps of type either  $(1, 1, 2)$ ,  $(1, 2, 1)$ , or  $(2, 1, 1)$  is equivalent, by means of an automorphism of  $X$ . In particular, the shape of the minimal graded free resolution of  $B$  is the same up to a permutation of the Betti numbers. Therefore, we restrict to the class of birational maps of type  $(1, 1, 2)$ .

**Proposition 3.5.4.** *Let  $\phi$  be dominant. The following are equivalent:*

- (i)  $\phi$  is birational of type  $(1, 1, 2)$

(ii) The minimal graded free resolution of the base ideal  $B$  is

$$\begin{array}{ccccccc}
& & & R(-2, -1, -1) & & & \\
& & & \oplus & & & \\
& & & R(-1, -2, -1) & & & \\
0 \rightarrow & R(-2, -2, -2) \rightarrow & & \oplus & \rightarrow & R(-1, -1, -1)^4 \xrightarrow{(f_0 \ f_1 \ f_2 \ f_3)} & B \rightarrow 0 \\
& & & R(-2, -1, -2) & & & \\
& & & \oplus & & & \\
& & & R(-1, -2, -2) & & & 
\end{array} \quad (3.30)$$

(iii)  $\mathbf{f}$  has syzygies of degrees  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , but not  $\mathbf{e}_3$

*Proof.* The implication from (i) to (ii) follows from the computation of the minimal graded free resolutions of the base ideals of all the representatives listed in Theorem 3.4.6 (performed with the help of *Macaulay2*). Moreover, it is straightforward that (ii) implies (iii).

Now, we assume that (iii) holds. The syzygies of degrees  $\mathbf{e}_1$  and  $\mathbf{e}_2$  can be respectively written as

$$s_0 a_1 - s_1 a_0 = \begin{vmatrix} s_0 & s_1 \\ a_0 & a_1 \end{vmatrix}, \quad t_0 b_1 - t_1 b_0 = \begin{vmatrix} t_0 & t_1 \\ b_0 & b_1 \end{vmatrix}, \quad (3.31)$$

for some linear forms  $a_i = a_i(\mathbf{y})$  and  $b_j = b_j(\mathbf{y})$ . Let  $P$  be a general point in  $\mathbb{P}_{\mathbb{C}}^3$ , and let  $(\alpha_0 : \alpha_1)$ ,  $(\beta_0 : \beta_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$  be the projective points such that

$$\begin{vmatrix} \alpha_0 & \alpha_1 \\ a_0(P) & a_1(P) \end{vmatrix} = 0, \quad \begin{vmatrix} \beta_0 & \beta_1 \\ b_0(P) & b_1(P) \end{vmatrix} = 0.$$

Thus, any point  $Q$  in the pullback  $\phi^{-1}(P)$  has the form

$$Q = (\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (u_0 : u_1)$$

for some  $(u_0 : u_1) \in \mathbb{P}_{\mathbb{C}}^1$ , since the point  $Q \times P$  in  $X \times \mathbb{P}_{\mathbb{C}}^3$  must vanish the polynomials in (3.31). Moreover, since  $\phi$  is dominant the restriction given by

$$\begin{array}{ccc}
\phi' : (\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times \mathbb{P}_{\mathbb{C}}^1 & \dashrightarrow & \mathbb{P}_{\mathbb{C}}^3 \\
(u_0 : u_1) & \mapsto & (f'_0 : f'_1 : f'_2 : f'_3),
\end{array} \quad (3.32)$$

where  $f'_n = f'_n(u_0, u_1) = f_n(\alpha_0, \alpha_1, \beta_0, \beta_1, u_0, u_1)$ , is an isomorphism of projective lines. In particular, there is a unique point in the pullback  $\phi^{-1}(P)$ , implying that  $\phi$  is birational. The syzygies of degrees  $\mathbf{e}_1$  and  $\mathbf{e}_2$  determine  $\phi^{-1}$  on the first two factors of  $X$ , and hence the degrees of the  $s$ - and  $t$ -surfaces. Moreover, there is no syzygy of degree  $\mathbf{e}_3$ . Therefore,  $\phi$  has type  $(1, 1, 2)$ .  $\square$

### 3.5.3. Birational maps of type $(1, 2, 2)$

Similarly to §3.5.2, we can restrict our analysis to the class of birational maps of type  $(1, 2, 2)$ .

**Proposition 3.5.5.** *Let  $\phi$  be dominant. The following are equivalent:*

(i)  $\phi$  is birational of type  $(1, 2, 2)$

(ii) The minimal graded free resolution of the base ideal  $B$  is

$$\begin{array}{ccccccc}
& & & R(-2, -1, -1) & & & \\
& & & \oplus & & & \\
& & & R(-2, -2, -1) & & & \\
0 \rightarrow & R(-2, -2, -2)^2 \rightarrow & & \oplus & \rightarrow & R(-1, -1, -1)^4 \xrightarrow{(f_0 \ f_1 \ f_2 \ f_3)} & B \rightarrow 0 \quad (3.33) \\
& & & R(-2, -1, -2) & & & \\
& & & \oplus & & & \\
& & & R(-1, -2, -2)^2 & & & 
\end{array}$$

(iii)  $\mathbf{f}$  has a syzygy of degree  $\mathbf{e}_1$ , but neither  $\mathbf{e}_2$  nor  $\mathbf{e}_3$ . Moreover,  $\mathbf{f}$  has a syzygy of degree either  $\mathbf{e}_1 + \mathbf{e}_2$  or  $\mathbf{e}_1 + \mathbf{e}_3$ , independent from the first one

*Proof.* The implication from (u) to (ii) follows from the computation of the minimal graded free resolutions of the base ideals of all the representatives listed in Theorem 3.4.8 (performed with the help of *Macaulay2*). Moreover, it is straightforward that (ii) implies (iii).

Now, we assume that (iii) holds. Without loss of generality, we suppose that  $\mathbf{f}$  has a syzygy of degree  $\mathbf{e}_1 + \mathbf{e}_2$ . The syzygies of degrees  $\mathbf{e}_1$  and  $\mathbf{e}_1 + \mathbf{e}_2$  can be respectively written as

$$s_0 a_1 - s_1 a_0 = \begin{vmatrix} s_0 & s_1 \\ a_0 & a_1 \end{vmatrix}, \quad (3.34)$$

$$t_0 (s_0 h_{01} + s_1 h_{11}) - t_1 (s_0 h_{00} + s_1 h_{10}) = \begin{vmatrix} t_0 & t_1 \\ s_0 h_{00} + s_1 h_{10} & s_0 h_{01} + s_1 h_{11} \end{vmatrix}, \quad (3.35)$$

for some linear forms  $a_i = a_i(\mathbf{y})$  and  $h_{ij} = h_{ij}(\mathbf{y})$ . Let  $P$  be a general point in  $\mathbb{P}_{\mathbb{C}}^3$ , and let  $(\alpha_0 : \alpha_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$  be the projective point such that

$$\begin{vmatrix} \alpha_0 & \alpha_1 \\ a_0(P) & a_1(P) \end{vmatrix} = 0.$$

Similarly, let  $(\beta_0 : \beta_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$  be the projective point such that

$$\begin{vmatrix} \beta_0 & \beta_1 \\ \alpha_0 h_{00}(P) + \alpha_1 h_{10}(P) & \alpha_0 h_{01}(P) + \alpha_1 h_{11}(P) \end{vmatrix} = 0.$$

Thus, any point  $Q$  in the pullback  $\phi^{-1}(P)$  has the form

$$Q = (\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (u_0 : u_1)$$

for some  $(u_0 : u_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$ , since the point  $Q \times P$  in  $X \times \mathbb{P}_{\mathbb{C}}^3$  must vanish the polynomials (3.34) and (3.35). Repeating the argument in the proof of Proposition 3.5.4 with the restriction map  $\phi'$ , it follows that there is a unique point in the pullback  $\phi^{-1}(P)$ , implying that  $\phi$  is birational. The syzygy of degree  $\mathbf{e}_1$  determines  $\phi^{-1}$  on the first factor of  $X$ , and hence the degree of the  $s$ -surfaces. Moreover, there are no syzygies of degrees  $\mathbf{e}_2$  nor  $\mathbf{e}_3$ . Therefore,  $\phi$  has type  $(1, 2, 2)$ .  $\square$

### 3.5.4. Birational maps of type $(2, 2, 2)$

Finally, we study the syzygies of birational maps of type  $(2, 2, 2)$ . In this class,  $\mathbf{f}$  does not have syzygies of degrees  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , or  $\mathbf{e}_3$ , and the analysis is more complicated. Moreover, the degree of the first syzygies alone is not enough to decide birationality, as Example 3.5.9 shows. However, from Proposition 3.5.8 it follows that birationality can still be decided from the first syzygies of  $\mathbf{f}$ , by checking a splitting condition on three related polynomials.

Before stating Proposition 3.5.8, we introduce some hypotheses on the first syzygies.

**Notation 3.5.6.** If  $\mathbf{f}$  has two independent syzygies of degree  $\mathbf{e}_1 + \mathbf{e}_2$ , these can be written as

$$s_0 t_0 g_{00} + s_1 t_0 g_{10} - s_0 t_1 g_{01} - s_1 t_1 g_{11}, \quad s_0 t_0 h_{00} + s_1 t_0 h_{10} - s_0 t_1 h_{01} - s_1 t_1 h_{11}, \quad (3.36)$$

for some linear forms  $g_{ij} = g_{ij}(\mathbf{y})$  and  $h_{ij} = h_{ij}(\mathbf{y})$ . We define  $z_1 = z_1(s_i, y_n)$  as

$$z_1 := \begin{vmatrix} s_0 g_{00} + s_1 g_{10} & s_0 g_{01} + s_1 g_{11} \\ s_0 h_{00} + s_1 h_{10} & s_0 h_{01} + s_1 h_{11} \end{vmatrix}.$$

The polynomial  $z_1$  is nonzero, since the syzygies (3.36) are independent by assumption. Moreover,

$$\begin{pmatrix} s_0 g_{00} + s_1 g_{10} & s_0 g_{01} + s_1 g_{11} \\ s_0 h_{00} + s_1 h_{10} & s_0 h_{01} + s_1 h_{11} \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \end{pmatrix} \xrightarrow{t_j \mapsto f_j} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

implying that  $z_1$  is a relation in the Rees ideal  $J$  of degree  $(2, 0, 0; 2)$ .

Similarly, if  $\mathbf{f}$  has syzygies of degrees  $\mathbf{e}_1 + \mathbf{e}_3$  and  $\mathbf{e}_2 + \mathbf{e}_3$ , we define the polynomials  $z_2 = z_2(t_j, y_n)$ ,  $z_3 = z_3(u_k, y_n)$  from the pairs of syzygies of degrees  $\mathbf{e}_2 + \mathbf{e}_3$  and  $\mathbf{e}_1 + \mathbf{e}_3$  respectively. These  $z_2, z_3$  lie in  $J$ , and have degrees  $(0, 2, 0; 2)$  and  $(0, 0, 2; 2)$ .

**Condition 3.5.7.** We assume that  $\mathbf{f}$  does not have syzygies of degrees  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ , or  $\mathbf{e}_3$ , but has a pair of independent syzygies in each of the degrees  $\mathbf{e}_1 + \mathbf{e}_2$ ,  $\mathbf{e}_1 + \mathbf{e}_3$ , and  $\mathbf{e}_2 + \mathbf{e}_3$ . Moreover, at least two of  $z_1, z_2$ , and  $z_3$  have a linear factor of degree  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ , respectively.

**Proposition 3.5.8.** *Let  $\phi$  be dominant. The following are equivalent:*

- (i)  $\phi$  is birational of type  $(2, 2, 2)$
- (ii) The minimal graded free resolution of the base ideal  $B$  is

$$\begin{array}{ccccccc} & & & & R(-2, -2, -1)^2 & & \\ & & & & \oplus & & \\ 0 & \rightarrow & R(-2, -2, -2)^3 & \rightarrow & R(-2, -1, -2)^2 & \rightarrow & R(-1, -1, -1)^4 \xrightarrow{(f_0 \ f_1 \ f_2 \ f_3)} B \rightarrow 0 \\ & & & & \oplus & & \\ & & & & R(-1, -2, -2)^2 & & \end{array} \quad (3.37)$$

and Condition 3.5.7 is satisfied.

*Proof.* The implication from (i) to (ii) follows from the computation of the minimal graded free resolutions of the base ideals of the two representatives listed in Theorem 3.4.10. Moreover, Condition 3.5.7 is always satisfied (these computations have been performed with the help of *Macaulay2*).

Now, we assume that Condition 3.5.7 holds. Without loss of generality, the polynomials  $z_1$  and  $z_2$  have linear factors  $l_1 = l_1(s_0, s_1)$  and  $l_2 = l_2(t_0, t_1)$ , i.e. we can write

$$z_1 = l_1 \begin{vmatrix} s_0 & s_1 \\ a_0 & a_1 \end{vmatrix}, \quad z_2 = l_2 \begin{vmatrix} t_0 & t_1 \\ b_0 & b_1 \end{vmatrix}, \quad (3.38)$$

for some quadratic forms  $a_i = a_i(\mathbf{y})$  and  $b_j = b_j(\mathbf{y})$ . Let  $P$  be a general point in  $\mathbb{P}_{\mathbb{C}}^3$ . We now prove that  $\phi^{-1}(P)$  has a unique point in the open set  $U \subset X$  where  $l_1$  and  $l_2$  are both nonzero. Given  $Q \in U \cap \phi^{-1}(P)$ , the point  $Q \times P$  in  $U \times \mathbb{P}_{\mathbb{C}}^3$  must vanish the two polynomials in (3.38). By assumption we have  $l_1(Q) \neq 0$  and  $l_2(Q) \neq 0$ , implying that  $Q \times P$  must vanish the second factor in each polynomial. In particular,  $Q$  must have the form

$$Q = (\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (u_0 : u_1)$$

for some  $(u_0 : u_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$ , where

$$\begin{vmatrix} \alpha_0 & \alpha_1 \\ a_0(P) & a_1(P) \end{vmatrix} = 0, \quad \begin{vmatrix} \beta_0 & \beta_1 \\ b_0(P) & b_1(P) \end{vmatrix} = 0.$$

Repeating the argument in the proof of Proposition 3.5.4 with the restriction map  $\phi'$ , it follows that the pullback  $\phi^{-1}(P)$  defines a unique point in  $U$ , implying that  $\phi$  is birational. Since  $\mathbf{f}$  has no syzygies of degrees  $\mathbf{e}_1, \mathbf{e}_2$  or  $\mathbf{e}_3$ ,  $\phi$  must have type  $(2, 2, 2)$ .  $\square$

**Example 3.5.9.** From Proposition 3.5.8, the minimal graded free resolution of the base ideal  $B$  of a trilinear birational map of type  $(2, 2, 2)$  is always (3.37). However, the rational map

$$\begin{aligned} \phi : \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^3 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto (s_0 t_0 u_1 : s_1 t_1 u_0 + s_1 t_0 u_1 + s_0 t_1 u_1 + s_1 t_1 u_1 : s_0 t_1 u_0 : s_1 t_0 u_0) \end{aligned}$$

has a resolution as (3.37) but is not birational, since Condition 3.5.7 is not satisfied.





# Chapter 4

## Construction and manipulation of birational trilinear volumes

In this chapter, we provide effective methods to construct and manipulate trilinear birational maps  $\phi : (\mathbb{P}_{\mathbb{R}}^1)^3 \dashrightarrow \mathbb{P}_{\mathbb{R}}^3$ . This is the first family of nonlinear birational transformations between 3D spaces that can be operated with enough flexibility for applications in geometric modeling (see [183, §7]). Interestingly, we find a connection between birationality and tensor rank that had remained unseen so far. This connection is the cornerstone of the results of this chapter.

In order to make our constructions geometrically intuitive, we adopt the standard setting in CAGD and define these rational maps by means of control points and weights. This point of view has also advantages from a purely algebraic perspective, since it facilitates the description of the parametric surfaces and makes the statements of our results lighter. Specifically, a trilinear rational map is defined by  $2 \times 2 \times 2 = 8$  control points  $\mathbf{P}_{ijk} = (1, x_{ijk}, y_{ijk}, z_{ijk})$  in  $\mathbb{R}^4$  and their associated (nonnegative) weights  $w_{ijk}$ , for each  $0 \leq i, j, k \leq 1$ . Thus, we study rational maps defined as

$$\begin{aligned} \phi : \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 &\dashrightarrow \mathbb{P}_{\mathbb{R}}^3 \\ (s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) &\mapsto \sum_{0 \leq i, j, k \leq 1} w_{ijk} \mathbf{P}_{ijk} b_i(s_0, s_1) b_j(t_0, t_1) b_k(u_0, u_1) \end{aligned} \quad (4.1)$$

where  $b_0(s_0, s_1) = s_0 - s_1$  and  $b_1(s_0, s_1) = s_1$  define the homogeneous Bernstein basis of linear polynomials.

**Construction of birational maps:** for general rational maps of type  $(1, 1, 1)$ ,  $(1, 1, 2)$ ,  $(1, 2, 2)$  and permutations, we describe the geometric constraints on the control points that are necessary for birationality, and present constructions for such constrained configurations (Constructions 1, 2, and 3). More importantly, for adequately constrained nets of control points, we prove that birationality is achieved if and only if a certain  $2 \times 2 \times 2$  tensor has rank one (Theorems 4.3.6, 4.4.7, 4.5.7). As a corollary, we derive formulas for computing exactly the weights that yield birationality, which are effective for the flexible construction of these transformations in applications. Additionally, we introduce a notion of “distance to birationality” for trilinear rational maps, and explain how to continuously deform birational maps. Finally, we provide explicit formulas for the inverse  $\phi^{-1}$  (Theorems 4.3.13, 4.4.19, and 4.5.15), and we give the defining equations of the irreducible components of the base loci.

### 4.1. Preliminaries

#### 4.1.1. Motivating by applications

To motivate our work and contributions, we list four questions of interest for applications that are formalized and answered throughout the chapter. This presentation is useful, since the main

sections of the chapter preserve the same structure, addressing these questions sequentially for the different types of trilinear maps. Specifically:

§4.3 deals with birational maps of type  $(1, 1, 1)$ , or *hexahedral* birational maps

§4.4 deals with birational maps of type  $(1, 1, 2)$  and permutations, or *pyramidal* birational maps

§4.5 deals with birational maps of type  $(1, 2, 2)$  and permutations, or *scaffold* birational maps

In this thesis, we do not discuss the construction of birational maps of type  $(2, 2, 2)$ . There are two reasons for this. The first one lies in Proposition 3.5.8, which establishes that the syzygies of the defining polynomials of a birational map of type  $(2, 2, 2)$  require additional splitting conditions, making their analysis more complicated when relying on syzygy-based birationality criteria. Secondly, the construction of birational maps of type  $(1, 2, 2)$  is already quite technical (see §4.5), and the geometric constraints on the control points are complicated due to the presence of quadric boundary surfaces. Because of this, we expect that  $(2, 2, 2)$  birational maps are not very suitable for CAGD, since they are fully quadratic. In particular, we expect that the class of hexahedral birational maps receives more attention, since it is the simplest to use in applications.

We align our approach with the following principle, which is common to all the existing algorithms for the construction of birational maps in geometric design (recall §1.1.1):

- (iii) Provide strategies for constructing (possibly constrained) nets of control points with sufficient flexibility, followed by the computation of weights that ensure birationality.

In this direction, the following is the first natural question.

**Question 4.1.1.** What constraints should be imposed on the control points to ensure the existence of weights that render  $\phi$  birational?

As we shall see, for a general choice of control points there are no weights that yield a birational map. Namely, achieving birationality necessitates specific control point arrangements, and Question 4.1.1 demands such constraints explicitly. This question leads us to discover various configurations of control points where birationality becomes possible: the classes of *hexahedral*, *pyramidal*, and *scaffold* rational maps (recall Definition 1.2.12). These configurations respectively correspond to the types  $(1, 1, 1)$ ,  $(1, 1, 2)$ ,  $(1, 2, 2)$  and their permutations.

**Question 4.1.2.** How “far” is  $\phi$  from being birational? how can we compute a birational approximation for  $\phi$ ?

The intuition behind this question is clear, but the question itself lacks precision. To formalize it, the first step is to introduce a notion of distance that provides a precise definition of the term “far”. In full generality, this presents a delicate challenge. Namely, the distance from  $\phi$  to each of the components of the locus of birational maps is in general different. Moreover, even when specifying a component, Question 4.1.2 remains highly challenging due to the fact that a point in the space of these rational maps is determined by 31 parameters (in the monomial basis:  $8 \times 4$  coefficients up to scalar; in the Bernstein basis: 3 coordinates  $\times$  8 control points + 8 weights up to scalar).

To tackle Questions 4.1.1 and 4.1.2, we establish a connection between birationality and tensor rank. Specifically, we prove that the simultaneous existence of the necessary syzygies for birationality (recall Theorem 3.5.1) occurs if and only if a certain tensor with a  $2 \times 2 \times 2$  format has rank one. This connection opens the door to the application of an extensive body of work related to tensor

rank and low-rank approximations to the study of birational maps. An important tool is the CP Decomposition (CPD) (e.g. [76, 126, 118, 136]), which enables us to efficiently compute rank-one approximations for tensors.

More specifically, for adequately constrained control points, we prove that  $\phi$  is birational if and only if a tensor of the form

$$W = \left( \frac{w_{ijk}}{\Delta_{ijk}} \right)_{0 \leq i, j, k \leq 1} \quad (4.2)$$

where  $\Delta_{ijk} = \Delta_{ijk}(\mathbf{P}_{ijk})$  are rational functions on the coordinates of the control points, has rank one. In particular, for each  $0 \leq i, j, k \leq 1$  it follows that

$$w_{ijk} = \alpha_i \beta_j \gamma_k \Delta_{ijk} \quad (4.3)$$

for some point  $(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1)$  in  $(\mathbb{P}_{\mathbb{R}}^1)^3$ . The formula (4.3) is useful for the computation of birational weights, and explicitly reveals the underlying geometric structure of the Segre variety associated with  $(\mathbb{P}_{\mathbb{R}}^1)^3$ . On the other hand, we rely on the Frobenius norm for tensors (recall Definition 2.2.20) to define our notion of “distance to birationality”. Namely, we measure it as the distance from  $W$  to the locus of rank-one tensors, i.e. the affine cone over the Segre variety  $\sigma(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1)$ . Remarkably, this distance depends solely on the weights of  $\phi$  (i.e. 8 parameters up to scalar). This is a design-oriented approach, since a designer will typically move the control points of a rational map but will rarely modify the weights. With this formulation, a “closest birational map” refers to a point in the affine cone of  $\sigma(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1)$  that minimizes the distance to  $W$ . Therefore, the problem of computing a closest birational map transfers to the computation of a rank-one CP decomposition for  $W$ . In the context of this thesis, we take advantage of the *Python* library *TensorFox* [32] to accomplish this task.

Although birational maps ensure injectivity on a Zariski open set, the most interesting application is the exact and fast computation of preimages avoiding numerical solving methods. This computation is performed through the inverse rational map, and therefore having explicit formulas for  $\phi^{-1}$  becomes imperative. The inverses of birational maps of the same type (i.e. lying on the same component) obey the same formulas. However, the behavior between distinct types has important differences, and distinct formulas are required.

**Question 4.1.3.** If  $\phi$  is birational, how can we compute  $\phi^{-1}$ ?

The last question concerns the modification of the parameters defining a birational map while preserving birationality. More specifically, we are interested in modifying these parameters continuously. The process of deformation is instrumental in practice, since applications typically require the manipulation of the control points until the designer achieves the desired shape. Specifically, we raise the following query.

**Question 4.1.4.** Given a birational map  $\phi_0$ , how can we “deform” it birationally to another birational map  $\phi_1$ ?

A deformation of  $\phi_0$  to  $\phi_1$  is simply a continuous map  $\delta : [0, 1] \rightarrow \mathcal{W} \times \mathcal{P}$  (in the usual topology), where  $\mathcal{W}$  and  $\mathcal{P}$  are respectively the sets of possible weights and control points, such that  $\delta(0) = \phi_0$  and  $\delta(1) = \phi_1$ . Question 4.1.4 asks for an explicit deformation  $\delta$  such that  $\delta(t) = \phi_t$  is birational for each  $0 \leq t \leq 1$ .

Actually, we answer a query more general than Question 4.1.4. Before we address it, we would like to exclude some degenerate configurations of control points, that are inherent to our formulation. To this purpose, we include the following remark, and a property that we always require.

**Remark 4.1.5** (Boundary surfaces might not be surfaces). In some degenerate cases, the image of the specialization of  $\phi$  to some parameters, say  $(s_0 : s_1) = (\lambda_0 : \lambda_1)$ , might not be a surface, but a curve or a point due to a contraction. For instance, this occurs if  $\mathbf{P}_{i00}$ ,  $\mathbf{P}_{i10}$ ,  $\mathbf{P}_{i01}$ , and  $\mathbf{P}_{i11}$  are colinear, for some  $i = 0, 1$ . Similarly, the images of the specializations to two parameters might not be lines.

Thus, according to Definition 1.2.11, the *boundary surfaces* and *lines* might not be, respectively, surfaces and lines. Even though these concepts are very useful to simplify our statements and formulas, as proven throughout the chapter, this problem is inherent to their definition. In this chapter, we avoid these degenerate cases for two reasons:

- (i) In general, they lack interest for applications
- (ii) Dealing with them would make our constructions and arguments cumbersome

Therefore, we will always require that  $\phi$  satisfies the following property.

**Property 1.** The following conditions are satisfied:

- (i)  $\Sigma_i$ ,  $T_j$ , and  $Y_k$  are smooth surfaces for every  $0 \leq i, j, k \leq 1$ , pairwise distinct
- (ii)  $s_{jk}$ ,  $t_{ik}$ , and  $u_{ij}$  are lines for every  $0 \leq i, j, k \leq 1$ , pairwise distinct
- (iii)  $\mathbf{P}_{ijk} = (1, x_{ijk}, y_{ijk}, z_{ijk})$  for every  $0 \leq i, j, k \leq 1$ , i.e. it lies in  $\mathbb{A}_{\mathbb{R}}^3$

Returning to Question 4.1.4, let  $\mathcal{P}_{(d_1, d_2, d_3)}$  be the set of admissible configurations that answers Question 4.1.1 for the type  $(d_1, d_2, d_3)$ . We provide a rational parametrization of the irreducible component labelled by  $(d_1, d_2, d_3)$  of the form

$$\begin{aligned} \Phi_{(d_1, d_2, d_3)} : (\mathbb{P}_{\mathbb{R}}^1)^3 \times \mathcal{P}_{(d_1, d_2, d_3)} &\dashrightarrow \mathcal{W} \times \mathcal{P}_{(d_1, d_2, d_3)} & (4.4) \\ (\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1) \times (\dots : \mathbf{P}_{ijk} : \dots) &\mapsto (\dots : w_{ijk} : \dots) \times (\dots : \mathbf{P}_{ijk} : \dots) = \\ &(\dots : \Delta_{ijk} \alpha_i \beta_j \gamma_k : \dots) \times (\dots : \mathbf{P}_{ijk} : \dots) \end{aligned}$$

In particular,  $\Phi_{(a, b, c)}$  yields all the weights that make  $\phi$  birational of type  $(d_1, d_2, d_3)$  for a valid net of control points. This parametrization explains how the continuous birational deformation can be performed. Specifically, a user can decide new control points and compute each new  $\Delta_{ijk}$  (which are continuous functions in the coordinates of the points) accordingly to update the weights.

Along this chapter, we illustrate all these applications with a variety of examples.

#### 4.1.2. Notation for this chapter

As usual, we write  $R = \mathbb{C}[\mathbf{s}] \otimes \mathbb{C}[\mathbf{t}] \otimes \mathbb{C}[\mathbf{u}] = \mathbb{C}[s_0, s_1] \otimes \mathbb{C}[t_0, t_1] \otimes \mathbb{C}[u_0, u_1]$  and  $\mathbb{C}[\mathbf{y}] = \mathbb{C}[y_0, y_1, y_2, y_3]$ , both rings standard graded. Additionally, when we refer to affine spaces  $\mathbb{A}_{\mathbb{C}}^3$  (or  $\mathbb{A}_{\mathbb{R}}^3$ ) in  $(\mathbb{P}_{\mathbb{C}}^1)^3$  and  $\mathbb{P}_{\mathbb{C}}^3$ , we respectively mean those defined by  $s_0 \neq 0$ ,  $t_0 \neq 0$ ,  $u_0 \neq 0$  and  $y_0 \neq 0$ . Furthermore, we denote by  $\mathfrak{N}$  the irrelevant ideal of  $R$ , namely  $\mathfrak{N} = (s_0, s_1) \cap (t_0, t_1) \cap (u_0, u_1)$ .

For the sake of simplicity, we use the standard monomial basis instead of the Bernstein basis. The purpose of this is to simplify the notation and statements. However, we retain the conventional formulation in CAGD, and define entries of a trilinear rational maps using control points and

weights. Specifically, for some  $\mathbf{P}_{ijk} = (1, x_{ijk}, y_{ijk}, z_{ijk})$  in  $\mathbb{R}^4$  and  $w_{ijk} \in \mathbb{C}$  (although we are primarily interested in real, nonnegative weights), we set

$$\begin{aligned} \phi : (\mathbb{P}_{\mathbb{C}}^1)^3 &\dashrightarrow \mathbb{P}_{\mathbb{C}}^3 \\ \mathbf{s} \times \mathbf{t} \times \mathbf{u} &\mapsto \mathbf{f} = (f_0 : f_1 : f_2 : f_3) = \sum_{0 \leq i, j, k \leq 1} w_{ijk} \mathbf{P}_{ijk} s_i t_j u_k . \end{aligned} \quad (4.5)$$

In the case that  $\phi$  is birational, we write

$$\begin{aligned} \phi^{-1} : \mathbb{P}_{\mathbb{C}}^3 &\dashrightarrow (\mathbb{P}_{\mathbb{C}}^1)^3 \\ \mathbf{y} &\mapsto \mathbf{g}_1 \times \mathbf{g}_2 \times \mathbf{g}_3 = (g_{01} : g_{11}) \times (g_{02} : g_{12}) \times (g_{03} : g_{13}) \end{aligned}$$

and denote by  $\mathbf{y} \mapsto \mathbf{f}$  the specialization  $y_n \mapsto f_n$  for each  $0 \leq n \leq 3$ , and by

$$\mathbf{s} \mapsto \mathbf{g}_1 \quad , \quad \mathbf{t} \mapsto \mathbf{g}_2 \quad , \quad \mathbf{u} \mapsto \mathbf{g}_3$$

the specializations respectively given by

$$s_i \mapsto g_{i1} \quad , \quad t_j \mapsto g_{j2} \quad , \quad u_k \mapsto g_{k3}$$

for each  $0 \leq i, j, k \leq 1$ . The following remark serves as a caveat for our choice of basis.

**Remark 4.1.6** (Standard basis instead of Bernstein basis). Due to our choice of basis, the intuition of a deformation of the unit cube in  $\mathbb{A}_{\mathbb{R}}^3$  (recall §1.2.3) is lost. Nevertheless, we recover the usual setting by means of the automorphism of  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  sending  $(s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1)$  to

$$(b_0(s_0, s_1) : b_1(s_0, s_1)) \times (b_0(t_0, t_1) : b_1(t_0, t_1)) \times (b_0(u_0, u_1) : b_1(u_0, u_1)) .$$

In all the examples of the section, we apply our results using the Bernstein basis, i.e. we apply the automorphism above. In particular, the *boundary surfaces* and *lines* (recall Definition 1.2.10) need to be adjusted to our notation. Specifically, in our setting the boundary surfaces are redefined as:

- (i)  $\Sigma_0$  and  $\Sigma_1$  are respectively defined by  $(s_0 : s_1) = (1 : 0)$  and  $(s_0 : s_1) = (0 : 1)$
- (ii)  $T_0$  and  $T_1$  are respectively defined by  $(t_0 : t_1) = (1 : 0)$  and  $(t_0 : t_1) = (0 : 1)$
- (iii)  $Y_0$  and  $Y_1$  are respectively defined by  $(u_0 : u_1) = (1 : 0)$  and  $(u_0 : u_1) = (0 : 1)$

And the boundary lines are redefined accordingly.

Additionally, we adopt Notation 1.2.18 (see §1.2.3) for the defining vectors and symmetric matrices of these boundary surfaces. This choice of notation is intentional: the boundary  $s$ -,  $t$ -, and  $u$ -surfaces are respectively defined by  $\boldsymbol{\sigma}_i$ ,  $\boldsymbol{\tau}_j$ , and  $\boldsymbol{u}_k$ . In particular, the defining polynomials of the boundary surfaces  $\Sigma_i$ ,  $T_j$ , and  $Y_k$  are respectively  $\boldsymbol{\sigma}_i(\mathbf{y})$ ,  $\boldsymbol{\tau}_j(\mathbf{y})$ , and  $\boldsymbol{u}_k(\mathbf{y})$  for each  $0 \leq i, j, k \leq 1$ . Additionally, given two vectors  $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$  and  $\mathbf{a} = (a_0, a_1, a_2, a_3)$  in  $\mathbb{C}^4$ , regarded as row vectors, we denote

$$\langle \boldsymbol{\alpha}, \mathbf{a} \rangle = \boldsymbol{\alpha} \cdot \mathbf{a}^T = \alpha_0 a_0 + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 .$$

On the other hand, for each  $0 \leq i, j, k \leq 1$  we define

$$\begin{aligned} \mathbf{A}_{jk}(s_0, s_1) &= w_{0jk} \mathbf{P}_{0jk} s_0 + w_{1jk} \mathbf{P}_{1jk} s_1 , \\ \mathbf{B}_{ik}(t_0, t_1) &= w_{i0k} \mathbf{P}_{i0k} t_0 + w_{i1k} \mathbf{P}_{i1k} t_1 , \\ \mathbf{C}_{ij}(u_0, u_1) &= w_{ij0} \mathbf{P}_{ij0} u_0 + w_{ij1} \mathbf{P}_{ij1} u_1 , \end{aligned}$$

which respectively define line isomorphisms to the boundary lines  $s_{jk}$ ,  $t_{ik}$ , and  $u_{ij}$ . Furthermore, we have the identities

$$\mathbf{f} = \sum_{0 \leq j, k \leq 1} \mathbf{A}_{jk}(s_0, s_1) t_j u_k = \sum_{0 \leq i, k \leq 1} \mathbf{B}_{ik}(t_0, t_1) s_i u_k = \sum_{0 \leq i, j \leq 1} \mathbf{C}_{ij}(u_0, u_1) s_i t_j .$$

## 4.2. Linear syzygies

In this section, we provide several characterizations for the existence of a linear syzygy for the defining polynomials of a trilinear rational map. More precisely, by ‘‘linear syzygies’’ we mean syzygies of degree either  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ . Additionally, the syzygies of  $B = (f_0, f_1, f_2, f_3)$  can be identified with linear polynomials in the Rees ideal of  $B$  (recall Remark 2.1.37), and we work with both perspectives indistinctly.

By Theorem 3.5.1, all birational maps admit such a syzygy except for the type  $(2, 2, 2)$ . Thus, all the birational maps considered in this chapter verify these characterizations for at least one parameter.

### 4.2.1. Rank characterization

The following lemma characterizes the existence of a linear syzygy of  $\mathbf{f}$  by means of a matrix rank condition. It is stated for syzygies of degree  $(1, 0, 0)$ , but it can be easily reformulated for syzygies of degrees  $(0, 1, 0)$  and  $(0, 0, 1)$ , as done in Remark 4.2.2. Notice that a syzygy of degree  $(1, 0, 0)$  implies that the parametric  $s$ -surfaces are planes (recall §3.1.1). In particular, for such a syzygy to exist, the boundary surfaces  $\Sigma_0$  and  $\Sigma_1$  are required to be planes.

**Lemma 4.2.1.** *Let  $\phi$  be dominant, and let  $\Sigma_0, \Sigma_1$  be planes. Then,  $\mathbf{f}$  has a syzygy of degree  $(1, 0, 0)$  if and only if*

$$\text{rank} \begin{pmatrix} w_{100} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{100} \rangle & w_{110} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{110} \rangle & w_{101} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{101} \rangle & w_{111} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{111} \rangle \\ w_{000} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{000} \rangle & w_{010} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{010} \rangle & w_{001} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{001} \rangle & w_{011} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{011} \rangle \end{pmatrix} = 1. \quad (4.6)$$

In particular, if (4.6) holds we find a point  $(\alpha_0 : \alpha_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$  (or  $\mathbb{P}_{\mathbb{R}}^1$  for real weights) such that

$$-\alpha_0 w_{1jk} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} \rangle = \alpha_1 w_{0jk} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle \quad (4.7)$$

for every  $0 \leq j, k \leq 1$ . Then, any syzygy of degree  $(1, 0, 0)$  of  $\mathbf{f}$  is proportional to

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(s_0, s_1) = \alpha_0 \boldsymbol{\sigma}_0 s_0 + \alpha_1 \boldsymbol{\sigma}_1 s_1 . \quad (4.8)$$

*Proof.* Suppose that  $\mathbf{f}$  has a syzygy of the form

$$\boldsymbol{\pi} = \boldsymbol{\pi}(s_0, s_1) = \boldsymbol{\pi}_0 s_0 + \boldsymbol{\pi}_1 s_1$$

for some  $\boldsymbol{\pi}_0, \boldsymbol{\pi}_1$  in  $\mathbb{R}^4$ . On the other hand, for each  $i = 0, 1$  we define

$$\mathbf{f}_i = \mathbf{f}_i(t_0, t_1, u_0, u_1) = \sum_{0 \leq j, k \leq 1} w_{ijk} \mathbf{P}_{ijk} t_j u_k ,$$

so that we can easily write  $\mathbf{f} = \mathbf{f}_0 s_0 + \mathbf{f}_1 s_1$ . In particular,  $\mathbf{f}_i$  defines a rational parametrization of  $\Sigma_i$ . With this notation, we find

$$\begin{aligned} \langle \boldsymbol{\pi}, \mathbf{f} \rangle &= s_0 \langle \boldsymbol{\pi}_0, \mathbf{f} \rangle + s_1 \langle \boldsymbol{\pi}_1, \mathbf{f} \rangle = s_0 (s_0 \langle \boldsymbol{\pi}_0, \mathbf{f}_0 \rangle + s_1 \langle \boldsymbol{\pi}_0, \mathbf{f}_1 \rangle) + s_1 (s_0 \langle \boldsymbol{\pi}_1, \mathbf{f}_0 \rangle + s_1 \langle \boldsymbol{\pi}_1, \mathbf{f}_1 \rangle) = \\ &= s_0^2 \langle \boldsymbol{\pi}_0, \mathbf{f}_0 \rangle + s_0 s_1 (\langle \boldsymbol{\pi}_0, \mathbf{f}_1 \rangle + \langle \boldsymbol{\pi}_1, \mathbf{f}_0 \rangle) + s_1^2 \langle \boldsymbol{\pi}_1, \mathbf{f}_1 \rangle = 0 . \end{aligned}$$

It follows immediately that  $\langle \boldsymbol{\pi}_i, \mathbf{f}_i \rangle = 0$  for both  $i = 0, 1$ . By hypothesis, the boundary surfaces  $\Sigma_0, \Sigma_1$  are both planes. Therefore,  $\mathbf{f}_i$  defines a bilinear rational parametrization to  $\Sigma_i$  and any vector  $\boldsymbol{\pi}$  satisfying  $\langle \boldsymbol{\pi}, \mathbf{f}_i \rangle = 0$  must be proportional to  $\boldsymbol{\sigma}_i$ , as  $\Sigma_i$  is defined by the ideal  $(\boldsymbol{\sigma}_i(\mathbf{y}))$ . In particular, we find  $\boldsymbol{\pi}_i = \alpha_i \boldsymbol{\sigma}_i$  for some non-zero  $\alpha_i$ . Hence, we obtain

$$\alpha_0 \langle \boldsymbol{\sigma}_0, \mathbf{f}_1 \rangle + \alpha_1 \langle \boldsymbol{\sigma}_1, \mathbf{f}_0 \rangle = \sum_{0 \leq j, k \leq 1} (\alpha_0 w_{1jk} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} \rangle + \alpha_1 w_{0jk} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle) t_j u_k = 0 ,$$

which is satisfied for some  $\alpha_0, \alpha_1$  if and only if (4.6) holds. Moreover, since  $\phi$  is dominant no row in (4.6) is identically zero. Thus,  $(\alpha_0 : \alpha_1) \in \mathbb{P}_{\mathbb{C}}^1$  is unique and  $\boldsymbol{\pi}$  is proportional to  $\boldsymbol{\sigma}$ .  $\square$

**Remark 4.2.2.** After the obvious modifications, we derive analogous results to Lemma 4.2.1 for syzygies of degree  $(0, 1, 0)$  and  $(0, 0, 1)$ . Specifically, let  $\phi$  be dominant.

(i) If  $T_0$  and  $T_1$  are planes, then  $\mathbf{f}$  has a syzygy of degree  $(0, 1, 0)$  if and only if

$$\text{rank} \begin{pmatrix} w_{010} \langle \boldsymbol{\tau}_0, \mathbf{P}_{010} \rangle & w_{110} \langle \boldsymbol{\tau}_0, \mathbf{P}_{110} \rangle & w_{011} \langle \boldsymbol{\tau}_0, \mathbf{P}_{011} \rangle & w_{111} \langle \boldsymbol{\tau}_0, \mathbf{P}_{111} \rangle \\ w_{000} \langle \boldsymbol{\tau}_1, \mathbf{P}_{000} \rangle & w_{100} \langle \boldsymbol{\tau}_1, \mathbf{P}_{100} \rangle & w_{001} \langle \boldsymbol{\tau}_1, \mathbf{P}_{001} \rangle & w_{101} \langle \boldsymbol{\tau}_1, \mathbf{P}_{101} \rangle \end{pmatrix} = 1 . \quad (4.9)$$

In particular, we find a unique  $(\beta_0 : \beta_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$  satisfying

$$-\beta_0 w_{i1k} \langle \boldsymbol{\tau}_0, \mathbf{P}_{i1k} \rangle = \beta_1 w_{i0k} \langle \boldsymbol{\tau}_1, \mathbf{P}_{i0k} \rangle ,$$

and therefore any syzygy of degree  $(0, 1, 0)$  of  $\mathbf{f}$  is proportional to

$$\boldsymbol{\tau} = \boldsymbol{\tau}(t_0, t_1) = \beta_0 \boldsymbol{\tau}_0 t_0 + \beta_1 \boldsymbol{\tau}_1 t_1 . \quad (4.10)$$

(ii) If  $Y_0$  and  $Y_1$  are planes, then  $\mathbf{f}$  has a syzygy of degree  $(0, 0, 1)$  if and only if

$$\text{rank} \begin{pmatrix} w_{001} \langle \boldsymbol{v}_0, \mathbf{P}_{001} \rangle & w_{101} \langle \boldsymbol{v}_0, \mathbf{P}_{101} \rangle & w_{011} \langle \boldsymbol{v}_0, \mathbf{P}_{011} \rangle & w_{111} \langle \boldsymbol{v}_0, \mathbf{P}_{111} \rangle \\ w_{000} \langle \boldsymbol{v}_1, \mathbf{P}_{000} \rangle & w_{100} \langle \boldsymbol{v}_1, \mathbf{P}_{100} \rangle & w_{010} \langle \boldsymbol{v}_1, \mathbf{P}_{010} \rangle & w_{110} \langle \boldsymbol{v}_1, \mathbf{P}_{110} \rangle \end{pmatrix} = 1 . \quad (4.11)$$

In particular, we find a unique  $(\gamma_0 : \gamma_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$  satisfying

$$-\gamma_0 w_{ij1} \langle \boldsymbol{v}_0, \mathbf{P}_{ij1} \rangle = \gamma_1 w_{ij0} \langle \boldsymbol{v}_1, \mathbf{P}_{ij0} \rangle$$

and therefore any syzygy of degree  $(0, 0, 1)$  of  $\mathbf{f}$  is proportional to

$$\boldsymbol{v} = \boldsymbol{v}(u_0, u_1) = \gamma_0 \boldsymbol{v}_0 u_0 + \gamma_1 \boldsymbol{v}_1 u_1 . \quad (4.12)$$

#### 4.2.2. Geometric characterizations

The previous rank conditions can be endowed with a more geometric meaning, related to the behavior of  $\phi$ . Specifically, we now establish connections between:

- (i) The linear syzygies of  $\mathbf{f}$
- (ii) The contractions of  $\phi$
- (iii) The geometry of  $\mathbb{C}\langle \mathbf{f} \rangle$  in the space of  $2 \times 2 \times 2$  tensors

To introduce formally the item (ii), we include the following definitions.



**Definition 4.2.3** (Pullback ideal). Let  $J = (h_1, \dots, h_r)$  be an homogeneous ideal in  $\mathbb{C}[\mathbf{y}]$ . We define the *pullback ideal* of  $J$  by  $\phi$  as the saturation of

$$J' = (h_1(\mathbf{f}), \dots, h_r(\mathbf{f})) \subset R$$

with respect to the irrelevant ideal  $\mathfrak{N} \subset R$ . Similarly, if  $\phi$  is birational and  $J = (h_1, \dots, h_r)$  is an homogeneous ideal in  $R$ , we define the *pullback ideal* of  $J$  by  $\phi^{-1}$  as the saturation of

$$J' = (h_1(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3), \dots, h_r(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)) \subset \mathbb{C}[\mathbf{y}]$$

with respect to the irrelevant ideal  $\mathfrak{m} = (y_0, y_1, y_2, y_3)$  of  $\mathbb{C}[\mathbf{y}]$ .

Geometrically, the pullback ideal by  $\phi$  defines the closure of the union of the base locus of  $\phi$  and the “strict pullback” of the scheme  $S$  defined by  $J$ , i.e. the closure in  $(\mathbb{P}_{\mathbb{C}}^1)^3$  of the pullback  $(\phi|_U)^{-1}(S)$  where  $U \subset (\mathbb{P}_{\mathbb{C}}^1)^3$  is an open set where  $\phi$  becomes a morphism. If  $\phi$  is birational, there are some subvarieties of  $\mathbb{P}_{\mathbb{C}}^3$  of codimension greater than one that pullback to surfaces in  $(\mathbb{P}_{\mathbb{C}}^1)^3$ . The following definition gives a name to this phenomenon.

**Definition 4.2.4** (Contractions and blow-ups). We say that  $\phi$  *contracts* a surface  $S \subset (\mathbb{P}_{\mathbb{C}}^1)^3$  if the image  $\phi(S)$  is either a curve or a point. If  $\phi$  is birational, we say that  $\phi^{-1}$  *blows up*  $\phi(S)$  to  $S$ .

**Remark 4.2.5.** Notice that  $\phi$  contracts a surface  $S$  to a curve (or a point)  $C$  if and only if the pullback by  $\phi$  of the defining ideal of  $C$  is principal. If  $\phi$  is birational,  $\phi$  blows up a curve (or a point)  $C$  to a surface  $S$  if and only if the pullback by  $\phi^{-1}$  of the defining ideal of  $C$  is principal.

The following remark explains how the existence of a linear syzygy of  $\mathbf{f}$  can be characterized as a specific contraction of  $\phi$ .

**Notation 4.2.6.** Given indices  $0 \leq i, j, k \leq 1$ , we denote their converse by  $i^*, j^*, k^*$ . More explicitly, we set  $0^* = 1$  and  $1^* = 0$ .

**Remark 4.2.7** (Contraction of bilinear surfaces). With the hypotheses of Lemma 4.2.1, we find

$$\langle \sigma_i, \mathbf{f} \rangle = s_{i^*} \sum_{0 \leq j, k \leq 1} w_{i^*jk} \langle \sigma_i, \mathbf{P}_{i^*jk} \rangle t_j u_k = s_{i^*} g_i(t_0, t_1, u_0, u_1).$$

In particular, the coefficients of  $g_i$  in the monomial basis coincide with the entries of the  $(i+1)$ -th row in (4.6). Therefore, if (4.6) is satisfied, we find a  $(\alpha_0 : \alpha_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$  such that  $g = \alpha_i g_i$ . In particular, the pullback by  $\phi$  of the ideal defining the line  $s = \Sigma_0 \cap \Sigma_1$  is principal, and generated by the polynomial  $g$  of degree  $(0, 1, 1)$ . Geometrically,  $\mathbf{f}$  admits a syzygy of degree  $(1, 0, 0)$  if and only if  $\phi$  contracts a surface of degree  $(0, 1, 1)$  in  $(\mathbb{P}_{\mathbb{C}}^1)^3$  to the line  $s$ .

Repeating the argument with the rank conditions (4.9) and (4.11), we conclude that  $\mathbf{f}$  admits syzygies of degrees  $(0, 1, 0)$  and  $(0, 0, 1)$  if and only if there are surface of degrees  $(1, 0, 1)$  and  $(1, 1, 0)$  that are contracted to the lines  $t = T_0 \cap T_1$  and  $u = Y_0 \cap Y_1$ , respectively.

**Remark 4.2.8** (Lines in  $\sigma(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^3)$ ). If  $\mathbf{f}$  admits a syzygy of degree  $(1, 0, 0)$  as (4.8), then by the previous Remark 4.2.7 there is a polynomial  $g$  of degree  $(0, 1, 1)$  such that

$$(\lambda_0 : \lambda_1) \mapsto \langle \lambda_0 \sigma_0 + \lambda_1 \sigma_1, \mathbf{f} \rangle = (\lambda_0 \alpha_1 s_0 + \lambda_1 \alpha_0 s_1) g(t_0, t_1, u_0, u_1)$$

is a line isomorphism  $\mathbb{P}_{\mathbb{R}}^1 \rightarrow \sigma(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^3)$ , where  $\sigma(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^3)$  is the Segre variety in the space of  $2 \times 2 \times 2$  tensors, or equivalently trilinear polynomials, representing the polynomials with one linear factor of degree  $(1, 0, 0)$  (see §2.2.3). In particular, its image is the Segre embedding of the line  $\mathbb{P}_{\mathbb{R}}^1 \times \{g\}$ .

Therefore,  $\mathbf{f}$  admits a syzygy of degree  $1 \times 0 \times 0$  if and only if the linear system  $\mathbb{C}\langle \mathbf{f} \rangle$  contains a line of the form  $\sigma(\mathbb{P}_{\mathbb{R}}^1 \times \{g\})$  for some polynomial  $g$  of degree  $(0, 1, 1)$ .

### 4.3. Hexahedral birational maps

In this section, we study the first and simplest family of trilinear birational maps: the class of hexahedral birational maps.

**Definition 4.3.1** (Hexahedral rational map). A trilinear rational map is hexahedral if it satisfies Property 1 and all the boundary surfaces are planes.

The most interesting case in applications, that gives the name to this family, is when the control points define a quadrilateral-faced hexahedron. Interestingly, hexahedral birational maps are the direct generalization to 3D of the 2D quadrilateral birational maps studied in [185, 28], since the minimal graded free resolution of the base ideal is Hilbert-Burch (recall Propositions 2.2.25 and 3.5.2). In Chapter 5, we generalize the construction of these two families of birational transformations to an arbitrary dimension. In particular, we will derive analogous results to those presented in this section.

#### 4.3.1. Construction of control points

Following the motivating questions listed in §4.1.1, we begin describing the geometric constraints on the control points necessary for birationality. Furthermore, we propose an effective construction for such constrained nets of control points.

Firstly, we make the following straightforward observation.

**Remark 4.3.2.** If  $\phi$  is a birational map of type  $(1, 1, 1)$  satisfying Property 1, then it is hexahedral. Namely, by definition the parametric surfaces (and in particular the boundary surfaces) are planes.

Remark 4.3.2 combined with Theorem 4.3.6, stated and proven in §4.3.2, provides an answer to Question 4.1.1 for birational maps of type  $(1, 1, 1)$ . More precisely, when  $\phi$  is hexahedral we can compute weights that render  $\phi$  birational of type  $(1, 1, 1)$ . Specifically, we find the following corollary.

**Corollary 4.3.3.** *The set*

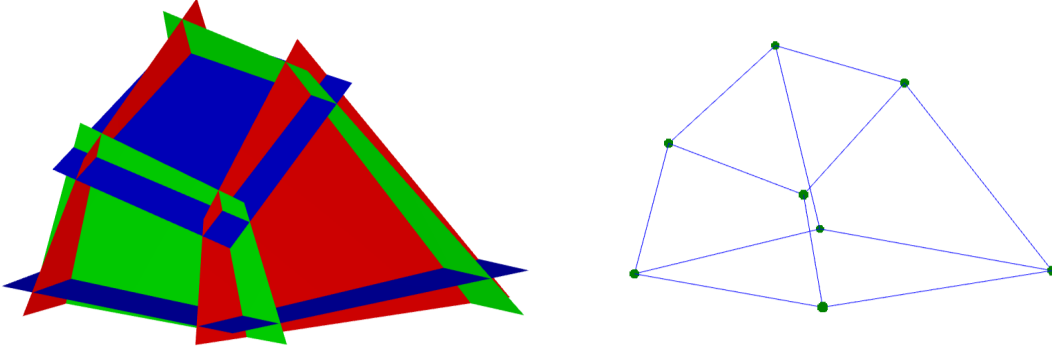
$$\mathcal{P}_{(1,1,1)} = \{ \{ \mathbf{P}_{ijk} \}_{0 \leq i,j,k \leq 1} : \phi \text{ is hexahedral} \}$$

*answers Question 4.1.1. More explicitly, for every net of control points in  $\mathcal{P}_{(1,1,1)}$  we can find weights that render  $\phi$  birational of type  $(1, 1, 1)$ .*

In practice, nets of control points in  $\mathcal{P}_{(1,1,1)}$  can be given indirectly if, instead of listing the constrained control points explicitly, a user decides boundary planes  $\Sigma_i, T_j, Y_k$  under no geometric constraint. The generation of a net of control points using Construction 1 is illustrated in Figure 4.1.

**Construction 1** (Control points for hexahedral rational maps). A net of control points in  $\mathcal{P}_{(1,1,1)}$  can be generated as follows:

1. For each  $0 \leq i, j, k \leq 1$ , choose planes  $\Sigma_i, T_j, Y_k$  such that  $\Sigma_i \cap T_j \cap Y_k$  is a point in  $\mathbb{A}_{\mathbb{R}}^3$
2. Define  $\mathbf{P}_{ijk} = \Sigma_i \cap T_j \cap Y_k$



**Figure 4.1.:** the control points of a hexahedral rational map, computed using Construction 1. The boundary planes  $\Sigma_0, \Sigma_1$  appear in red,  $T_0, T_1$  appear in green, and  $Y_0, Y_1$  appear in blue.

### 4.3.2. Effective computation of weights

In this section, we establish for the first time our birationality criterion based on a rank-one condition on a  $2 \times 2 \times 2$  tensor, for the class of hexahedral rational maps. This is the main result of the section.

**Notation 4.3.4.** Let  $\phi$  be hexahedral. We introduce the following notation:

- (i) For each  $0 \leq i, j, k \leq 1$ , the boundary planes  $\Sigma_i$ ,  $T_j$ , and  $Y_k$  are respectively defined by the vectors

$$\boldsymbol{\sigma}_i = (\sigma_{0i} \ \sigma_{1i} \ \sigma_{2i} \ \sigma_{3i}) \ , \ \boldsymbol{\tau}_j = (\tau_{0j} \ \tau_{1j} \ \tau_{2j} \ \tau_{3j}) \ , \ \boldsymbol{v}_k = (v_{0k} \ v_{1k} \ v_{2k} \ v_{3k}) \ .$$

- (ii) If  $\phi$  is hexahedral,  $\mathbf{P}_{ijk}$  can be expressed as the wedge of the vectors  $\boldsymbol{\sigma}_i$ ,  $\boldsymbol{\tau}_j$ , and  $\boldsymbol{v}_k$  defining the boundary planes. More specifically, we have

$$\mathbf{P}_{ijk} = \Delta_{ijk}^{-1} \boldsymbol{\sigma}_i \wedge \boldsymbol{\tau}_j \wedge \boldsymbol{v}_k \ , \quad (4.13)$$

where

$$\Delta_{ijk} = \begin{vmatrix} \sigma_{1i} & \sigma_{2i} & \sigma_{3i} \\ \tau_{1j} & \tau_{2j} & \tau_{3j} \\ v_{1k} & v_{2k} & v_{3k} \end{vmatrix} \quad (4.14)$$

for each  $0 \leq i, j, k \leq 1$ .

**Remark 4.3.5** (Nonzero and positive  $\Delta_{ijk}$ ). Notice that  $\Delta_{ijk}$  is zero if and only if  $\mathbf{P}_{ijk}$  lies at the plane  $y_0 = 0$  “at  $\infty$ ” in  $\mathbb{P}_{\mathbb{R}}^3$ , a possibility excluded by Property 1. Namely,  $\Delta_{ijk}$  is the first maximal minor of the  $3 \times 4$  matrix with  $\boldsymbol{\sigma}_i$ ,  $\boldsymbol{\tau}_j$ , and  $\boldsymbol{v}_k$  as rows, and the four  $3 \times 3$  minors, with alterned sign, determine  $\mathbf{P}_{ijk}$ .

On the other hand, the rows of the determinant  $\Delta_{ijk}$  can be regarded as the normal vectors of the affine boundary planes. In the most interesting case for geometric modeling, when these boundary planes define a quadrilateral-faced hexahedron, the nonnegativity of  $\Delta_{ijk}$  can be ensured by choosing all these normals “pointing to the same side” of the hexahedron. More precisely, we can always choose  $\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1$  such that  $\langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} - \mathbf{P}_{0jk} \rangle, \langle \boldsymbol{\sigma}_1, \mathbf{P}_{1jk} - \mathbf{P}_{0jk} \rangle \geq 0$  for some (and hence all)  $0 \leq j, k \leq 1$ .

The following theorem relates birationality, the existence of linear syzygies, and tensor rank. It is our key ingredient to answer the questions posed in §4.1.1 for the class of hexahedral birational maps.

**Theorem 4.3.6.** *Let  $\phi$  be hexahedral. Then,  $\phi$  is birational of type  $(1, 1, 1)$  if and only if the  $2 \times 2 \times 2$  tensor*

$$W = \left( \frac{w_{ijk}}{\Delta_{ijk}} \right)_{0 \leq i, j, k \leq 1} \quad (4.15)$$

has rank one.

*Proof.* By Theorem 3.5.1,  $\phi$  is birational of type  $(1, 1, 1)$  if and only if  $\mathbf{f}$  admits syzygies of degrees  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ . Equivalently, by Lemma 4.2.1 and Remark 4.2.2,  $\phi$  is birational if and only if the rank conditions (4.6), (4.9), and (4.11) are simultaneously satisfied.

We rewrite the rank conditions (4.6), (4.9), and (4.11) so that the matrices involved have the same entries. For any  $0 \leq i, j, k \leq 1$ , we can write

$$\Delta_{0jk} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle = \boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_0 \wedge \boldsymbol{\tau}_j \wedge \mathbf{v}_k = -\Delta_{1jk} \Delta_{1jk}^{-1} \boldsymbol{\sigma}_0 \wedge \boldsymbol{\sigma}_1 \wedge \boldsymbol{\tau}_j \wedge \mathbf{v}_k = -\Delta_{1jk} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} \rangle. \quad (4.16)$$

In particular, (4.6) can be equivalently written as

$$\text{rank} \begin{pmatrix} w_{100} \Delta_{100}^{-1} & w_{110} \Delta_{110}^{-1} & w_{101} \Delta_{101}^{-1} & w_{111} \Delta_{111}^{-1} \\ w_{000} \Delta_{000}^{-1} & w_{010} \Delta_{010}^{-1} & w_{001} \Delta_{001}^{-1} & w_{011} \Delta_{011}^{-1} \end{pmatrix} = 1. \quad (4.17)$$

With a similar argument, we derive

$$\Delta_{i0k} \langle \boldsymbol{\tau}_1, \mathbf{P}_{i0k} \rangle = -\Delta_{i1k} \langle \boldsymbol{\tau}_0, \mathbf{P}_{i1k} \rangle, \quad \Delta_{ij0} \langle \mathbf{v}_1, \mathbf{P}_{ij0} \rangle = -\Delta_{ij1} \langle \mathbf{v}_0, \mathbf{P}_{ij1} \rangle,$$

and (4.9) and (4.11) can be respectively rewritten as

$$\text{rank} \begin{pmatrix} w_{010} \Delta_{010}^{-1} & w_{110} \Delta_{110}^{-1} & w_{011} \Delta_{011}^{-1} & w_{111} \Delta_{111}^{-1} \\ w_{000} \Delta_{000}^{-1} & w_{100} \Delta_{100}^{-1} & w_{001} \Delta_{001}^{-1} & w_{101} \Delta_{101}^{-1} \end{pmatrix} = 1, \quad (4.18)$$

$$\text{rank} \begin{pmatrix} w_{001} \Delta_{001}^{-1} & w_{101} \Delta_{101}^{-1} & w_{011} \Delta_{011}^{-1} & w_{111} \Delta_{111}^{-1} \\ w_{000} \Delta_{000}^{-1} & w_{100} \Delta_{100}^{-1} & w_{010} \Delta_{010}^{-1} & w_{110} \Delta_{110}^{-1} \end{pmatrix} = 1. \quad (4.19)$$

In particular, the matrices in (4.17), (4.18), and (4.19) are the three flattenings (recall Definition 2.2.3 and Example 2.2.5) of the tensor  $W$  in the statement. By Lemma 2.2.16, they have all rank one if and only if  $W$  has rank one.  $\square$

**Corollary 4.3.7.** *Let  $\phi$  be hexahedral. Then,  $\phi$  is birational of type  $(1, 1, 1)$  if and only if*

$$w_{ijk} = \alpha_i \beta_j \gamma_k \Delta_{ijk} \quad (4.20)$$

for some  $(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1)$  in  $(\mathbb{P}_{\mathbb{C}}^1)^3$ .

*Proof.* The tensor  $W$  in (4.15) has rank one if and only if it lies in the Segre embedding of  $(\mathbb{P}_{\mathbb{R}}^1)^3$  (see §2.2).  $\square$

**Remark 4.3.8** (Nonzero, real, and positive weights). In particular, we must have  $\alpha_i \neq 0$ ,  $\beta_j \neq 0$ , and  $\gamma_k \neq 0$  for every  $0 \leq i, j, k \leq 1$ . Namely, if  $\alpha_i = 0$  then the weights  $w_{i00}$ ,  $w_{i10}$ ,  $w_{i01}$ , and  $w_{i11}$  are also zero, and the image of  $\phi$  lies in  $\Sigma_{i^*}$ . The obvious observations hold for the other indices.

In applications, a designer is typically interested in rational maps with real coefficients. Because of this, we will in general choose  $(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1)$  in  $(\mathbb{P}_{\mathbb{R}}^1)^3$ . Even more, nonnegative weights are preferred in order to preserve the property that  $\phi([0, 1]^3)$  lies in the convex hull of the control points. If the control points define a quadrilateral-faced hexahedron, positive weights can be ensured by choosing  $\alpha_i, \beta_j, \gamma_k > 0$  and by Remark 4.3.5.

### 4.3.3. Distance to birationality

We now address the question of measuring the distance of a trilinear rational map to the locus of birational maps. Interestingly, Theorem 4.3.6 provides a way to quantify this distance. Specifically, we introduce the following definition.

**Definition 4.3.9** (Distance to birationality: hexahedral). Let  $\phi$  be hexahedral with real weights, and let  $W$  in  $\mathbb{R}^{2 \times 2 \times 2}$  be as in Theorem 4.3.6. We define the *distance to birationality* of  $\phi$ , denoted by  $\text{dist}_{\text{bir}}(\phi)$ , as the relative distance from  $W$  to the affine cone over the Segre variety  $\sigma(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1)$ , with the Frobenius norm (Definition 2.2.20) in  $\mathbb{R}^{2 \times 2 \times 2}$ . More explicitly,

$$\text{dist}_{\text{bir}}(\phi) := \min_{P \in V} \frac{\|W - P\|}{\|W\|} \quad (4.21)$$

**Remark 4.3.10.** Although  $\phi$  is defined up to nonzero constants for the vector of weights  $\mathbf{w} = (w_{ijk})_{0 \leq i, j, k \leq 1}$  and the defining vectors of the boundary planes  $\boldsymbol{\sigma}_i, \boldsymbol{\tau}_j, \boldsymbol{\nu}_k$ , any choice of such constants yield proportional tensors  $W$  (the determinant is multilinear). In particular, for nonzero constants  $\lambda, \mu$ , we can write

$$\min_{P \in V} \frac{\|\lambda W - P\|}{\|\lambda W\|} = \min_{P \in V} \frac{\|\lambda W - \lambda \mu^{-1} P\|}{\|\lambda W\|} = \min_{P \in V} \frac{\|W - \mu^{-1} P\|}{\|W\|} = \min_{P \in V} \frac{\|\mu W - P\|}{\|\mu W\|},$$

and  $\text{dist}_{\text{bir}}(\phi)$  is thus well defined.

Therefore, in order to compute a birational map that approximates  $\phi$ , we can solve the minimization problem

$$\min_P \text{dist}_{\text{bir}}(M - P) \text{ where } P = \lambda \alpha \otimes \beta \otimes \gamma \text{ for some } \alpha, \beta, \gamma \text{ in } \mathbb{R}^2 \text{ and } \lambda \neq 0 \quad (4.22)$$

or simply

$$\min_P \|M - P\| \text{ where } P = \lambda \alpha \otimes \beta \otimes \gamma \text{ for some } \alpha, \beta, \gamma \text{ in } \mathbb{R}^2 \text{ and } \lambda \neq 0, \quad (4.23)$$

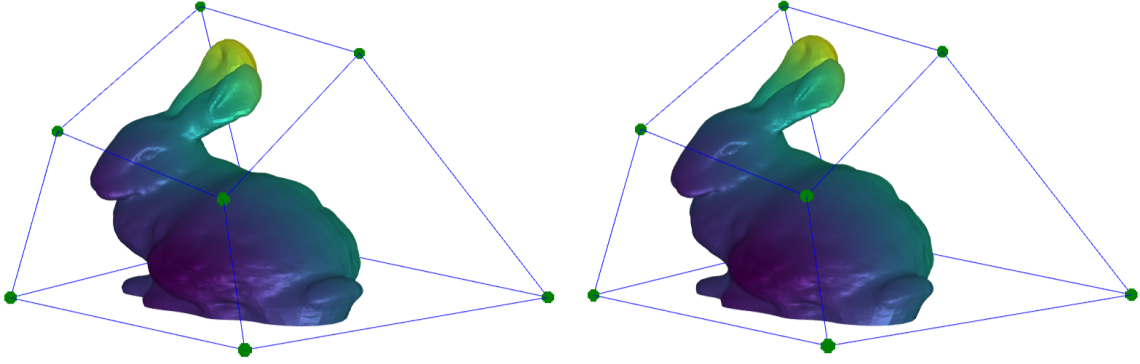
i.e. we can compute a rank-one CP decomposition of the tensor  $W$  (see §2.2.2.2).

**Remark 4.3.11.** Since the Frobenius norm is actually the Euclidean norm in  $\mathbb{R}^8$ , an optimal solution to (4.23) is among the orthogonal projections of  $W$  onto the affine cone over  $\sigma(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1)$ . By [88, Example 8.2], for a general  $W$  there are exactly 6 orthogonal projections onto this variety.

In the following example, we explicitly compute a birational map that approximates a hexahedral rational map that is not birational.

**Example 4.3.12** (Computation of a birational approximation: hexahedral). Consider the boundary planes defined by the vectors (with exact rational entries!)

$$\begin{aligned} \boldsymbol{\sigma}_0 &= (0.16 \quad -0.45 \quad -0.07 \quad -0.14) \quad , \quad \boldsymbol{\sigma}_1 = (1.25 \quad -0.63 \quad -0.32 \quad -0.63) \quad , \\ \boldsymbol{\tau}_0 &= (0 \quad 0 \quad 0 \quad 1) \quad , \quad \boldsymbol{\tau}_1 = (-1.18 \quad 0.18 \quad 0.51, 1) \quad , \\ \boldsymbol{\nu}_0 &= (0 \quad 0 \quad 1 \quad 0) \quad , \quad \boldsymbol{\nu}_1 = (-1.17 \quad 0.1 \quad 0.8 \quad 0.54) \quad . \end{aligned}$$



**Figure 4.2.:** deformations of the Stanford bunny [172] using the hexahedral birational maps of Example 4.3.12. To recover the intuition of a deformation of the unit cube, we have replaced the monomial basis by the Bernstein basis (see §4.1.2). In the left image, we use  $w_{ijk}^0 = 1$  for every  $0 \leq i, j, k \leq 1$ . The right image showcases the effect of utilizing the birational weights extracted from the rational rank-one approximation  $W^1$  of the tensor  $W^0$ . These weight adjustments subtly influence the deformation, while simultaneously ensuring the existence of an inverse transformation.

For each  $0 \leq i, j, k \leq 1$ , we can use the formula (4.13) and express  $\mathbf{P}_{ijk} = \Sigma_i \cap T_j \cap Y_k$  as the exterior product

$$\mathbf{P}_{ijk} = \Delta_{ijk}^{-1} \boldsymbol{\sigma}_i \wedge \boldsymbol{\tau}_j \wedge \boldsymbol{\nu}_k \quad (4.24)$$

where each  $\Delta_{ijk}$  can be computed using (4.14). Specifically, we find the exact rational numbers

$$\begin{aligned} \Delta_{000} &= 0.45, \quad \Delta_{100} = 0.63, \quad \Delta_{010} = 0.4248, \quad \Delta_{110} = 0.5166, \\ \Delta_{001} &= 0.353, \quad \Delta_{101} = 0.472, \quad \Delta_{011} = 0.222854, \quad \Delta_{111} = 0.271012. \end{aligned}$$

Suppose that we initialize the weights as  $w_{ijk}^0 = 1$  for every  $0 \leq i, j, k \leq 1$ . Then, the tensor

$$W^0 = \left( \frac{w_{ijk}^0}{\Delta_{ijk}} \right)_{0 \leq i, j, k \leq 1}$$

given by

$$W_{\{k=0\}}^0 = \begin{pmatrix} 2.2222 \dots & 2.3540 \dots \\ 1.5873 \dots & 1.9357 \dots \end{pmatrix}, \quad W_{\{k=1\}}^0 = \begin{pmatrix} 2.8328 \dots & 4.4872 \dots \\ 2.1186 \dots & 3.6898 \dots \end{pmatrix}$$

does not have rank one. Using the *Python* library *TensorFox* [32], we compute the rational rank-one approximation of  $W^0$

$$\begin{aligned} W^1 &= \left( \frac{w_{ijk}^1}{\Delta_{ijk}^1} \right)_{0 \leq i, j, k \leq 1} = \\ &(\alpha_0, \alpha_1) \otimes (\beta_0, \beta_1) \otimes (\gamma_0, \gamma_1) = (1.56, 1.24) \otimes (1.12, 1.65) \otimes (1.02, 1.71), \end{aligned}$$

which yields the tensor

$$W_{\{k=0\}}^1 = \begin{pmatrix} 1.7821 \dots & 2.6254 \dots \\ 1.4165 \dots & 2.0869 \dots \end{pmatrix}, \quad W_{\{k=1\}}^1 = \begin{pmatrix} 2.9877 \dots & 4.4015 \dots \\ 2.3748 \dots & 3.4986 \dots \end{pmatrix}.$$

In particular, using  $W^1$  we can upper bound the distance to birationality of  $\phi$  as

$$\text{dist}_{\text{bir}}(\phi) \leq \frac{\|W^0 - W^1\|}{\|W^0\|} = 0.0847 \dots$$

Furthermore, by Corollary 1.2.16 recomputing  $w_{ijk}^1 = \alpha_i \beta_j \gamma_k \Delta_{ijk}^1$  renders  $\phi$  birational. Specifically, we find the birational weights

$$\begin{aligned} w_{000}^1 &= 0.8019 \dots, & w_{100}^1 &= 0.8924 \dots, & w_{010}^1 &= 1.1153 \dots, & w_{110}^1 &= 1.0781 \dots, \\ w_{001}^1 &= 1.0546 \dots, & w_{101}^1 &= 1.1209 \dots, & w_{011}^1 &= 0.9809 \dots, & w_{111}^1 &= 0.9481 \dots \end{aligned} \quad (4.25)$$

Notice that, although we show approximations, the weights are exact since  $\alpha_i, \beta_j, \gamma_k$  and  $\Delta_{ijk}$  are rational numbers. In Figure 4.2, we present a comparison between the deformations resulting from the original rational map, with uniform weights  $w_{ijk} = 1$  for every  $0 \leq i, j, k \leq 1$ , and its birational approximation, utilizing the weights specified in (4.25).

#### 4.3.4. Inverse rational map and base locus

At this point, we address the most important computational advantage of birational maps: the efficient (and exact over  $\mathbb{Q}$ ) computation of preimages.

The following result presents explicit formulas for the inverse  $\phi^{-1}$  of a hexahedral birational map. Moreover, we provide a minimal set of generators for the base ideal, and explain the contractions and blow-ups.

**Theorem 4.3.13.** *Let  $\phi$  be hexahedral, with weights as in Corollary 4.3.7. Then,  $\phi^{-1}$  is given by*

$$\mathbf{y} \mapsto (-\alpha_1 \boldsymbol{\sigma}_1(\mathbf{y}) : \alpha_0 \boldsymbol{\sigma}_0(\mathbf{y})) \times (-\beta_1 \boldsymbol{\tau}_1(\mathbf{y}) : \beta_0 \boldsymbol{\tau}_0(\mathbf{y})) \times (-\gamma_1 \boldsymbol{\nu}_1(\mathbf{y}) : \gamma_0 \boldsymbol{\nu}_0(\mathbf{y})).$$

Moreover, the base locus of  $\phi$  is the curve  $C \subset (\mathbb{P}_{\mathbb{C}}^1)^3$  defined by the ideal

$$B = \left( \sum_{0 \leq j, k \leq 1} w_{0jk} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle t_j u_k, \sum_{0 \leq i, k \leq 1} w_{i0k} \langle \boldsymbol{\tau}_1, \mathbf{P}_{i0k} \rangle s_i u_k, \sum_{0 \leq i, j \leq 1} w_{ij0} \langle \boldsymbol{\nu}_1, \mathbf{P}_{ij0} \rangle s_i t_j \right),$$

and the base locus of  $\phi^{-1}$  is the union of the lines  $s = \Sigma_0 \cap \Sigma_1$ ,  $t = T_0 \cap T_1$ , and  $u = Y_0 \cap Y_1$ . Additionally:

- $\phi$  contracts the surfaces in  $(\mathbb{P}_{\mathbb{C}}^1)^3$  defined by the first, second, and third generators of  $B$  to the lines  $s$ ,  $t$ , and  $u$ , respectively
- If the lines  $s$ ,  $t$ , and  $u$  mutually do not intersect, then  $\phi$  blows up  $C$  to the unique smooth quadric  $Q \subset \mathbb{P}_{\mathbb{C}}^3$  through the lines  $s$ ,  $t$ , and  $u$

*Proof.* By Corollary 4.3.7, a hexahedral rational map is birational if and only if the weights satisfy (4.3.7) for some  $(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1)$  in  $(\mathbb{P}_{\mathbb{C}}^1)^3$ . In particular, we can write

$$\begin{aligned} \mathbf{f} &= \sum_{0 \leq i, j, k \leq 1} w_{ijk} \mathbf{P}_{ijk} s_i t_j u_k = \sum_{0 \leq i, j, k \leq 1} (\alpha_i \beta_j \gamma_k \Delta_{ijk}) \left( \Delta_{ijk}^{-1} \boldsymbol{\sigma}_i \wedge \boldsymbol{\tau}_j \wedge \boldsymbol{\nu}_k \right) s_i t_j u_k = \\ &= \sum_{0 \leq i, j, k \leq 1} (\alpha_i \boldsymbol{\sigma}_i s_i) \wedge (\beta_j \boldsymbol{\tau}_j t_j) \wedge (\gamma_k \boldsymbol{\nu}_k u_k) = \boldsymbol{\sigma}(s_0, s_1) \wedge \boldsymbol{\tau}(t_0, t_1) \wedge \boldsymbol{\nu}(u_0, u_1), \end{aligned}$$

where  $\sigma(s_0, s_1)$ ,  $\tau(t_0, t_1)$ , and  $\mathbf{v}(u_0, u_1)$  are as in Lemma 4.2.1 and Remark 4.2.2. Therefore,  $\mathbf{f}$  admits the linear syzygies

$$\begin{aligned}\sigma(s_0, s_1; \mathbf{y}) &= \alpha_0 \sigma_0(\mathbf{y}) s_0 + \alpha_1 \sigma_1(\mathbf{y}) s_1 = s_0 g_{11}(\mathbf{y}) - s_1 g_{01}(\mathbf{y}) , \\ \tau(t_0, t_1; \mathbf{y}) &= \beta_0 \tau_0(\mathbf{y}) t_0 + \beta_1 \tau_1(\mathbf{y}) t_1 = t_0 g_{12}(\mathbf{y}) - t_1 g_{02}(\mathbf{y}) , \\ \mathbf{v}(u_0, u_1; \mathbf{y}) &= \gamma_0 \mathbf{v}_0(\mathbf{y}) u_0 + \gamma_1 \mathbf{v}_1(\mathbf{y}) u_1 = u_0 g_{13}(\mathbf{y}) - u_1 g_{03}(\mathbf{y})\end{aligned}$$

and  $\phi^{-1}$  is thus given by

$$\mathbf{y} \mapsto (g_{01}(\mathbf{y}) : g_{11}(\mathbf{y})) \times (g_{02}(\mathbf{y}) : g_{12}(\mathbf{y})) \times (g_{03}(\mathbf{y}) : g_{13}(\mathbf{y})) .$$

Regarding the base locus of  $\phi$ , we find

$$\sigma_0(\mathbf{f}) = \alpha_1 s_1 a \quad , \quad \sigma_1(\mathbf{f}) = -\alpha_0 s_0 a$$

where  $a = a(t_0, t_1, u_0, u_1)$  is proportional to the first generator of  $B$  (recall Remark 4.2.7). Similarly, we find

$$\tau_j(\mathbf{f}) = (-1)^j \beta_{j^*} t_{j^*} b \quad , \quad \mathbf{v}_k(\mathbf{f}) = (-1)^k \gamma_{k^*} u_{k^*} c \quad ,$$

where  $b$  and  $c$  are the second and third generators of  $B$ . Hence, the graded component  $V = (B \cap \mathfrak{N})_{(1,1,1)}$  lies in  $\mathbb{C}\langle \mathbf{f} \rangle$ . Since  $\dim_{\mathbb{C}} V = 4$ , it follows that  $V = \mathbb{C}\langle \mathbf{f} \rangle$  and  $B$  defines the base locus of  $\phi$ . On the other hand, the base ideal of  $\phi^{-1}$  is the intersection

$$(g_{01}, g_{11}) \cap (g_{02}, g_{12}) \cap (g_{03}, g_{13}) \quad ,$$

which defines the union of the lines  $s$ ,  $t$ , and  $u$ . Furthermore, by Remark 4.2.7 the first, second, and third generators of  $B$  define the surfaces contracted by  $\phi$  to  $s$ ,  $t$ , and  $u$ , respectively.

Finally, the pullback of  $B$  by  $\phi^{-1}$  (recall Definition 4.2.3) must yield a principal ideal  $J \subset \mathbb{C}[\mathbf{y}]$ , since as  $\phi$  is birational (the composition yields the identity on an open set) we find

$$(f_0(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3), f_1(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3), f_2(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3), f_3(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)) = q(\mathbf{y}) (y_0, y_1, y_2, y_3) \quad ,$$

for some  $q(\mathbf{y})$  homogeneous and quadratic. If the lines  $s$ ,  $t$ , and  $u$  mutually do not intersect, then  $J$  is generated by a quadratic form that vanishes at the three of them. Therefore, by Lemma 2.2.23  $J$  defines the unique smooth quadric through the lines.  $\square$

**Remark 4.3.14** (Base locus in practical applications). One of the primary applications of birational transformations in CAGD is to ensure injectivity within a specific domain. While birational maps are injective in an open (Zariski) set, they may lose this property on the contracted loci, and are not even defined on the base locus. Therefore, having control over the base locus through its explicit defining equations can prove valuable in practice. In particular, these equations serve as a means to test whether the birational map is injective within a particular domain.

**Example 4.3.15.** Using the data of Example 4.3.12, the (exact!) inverse rational map of  $\phi$  is given by

$$\mathbf{y} \mapsto (1.24 \sigma_1(\mathbf{y}) : 1.56 \sigma_0(\mathbf{y})) \times (1.65 \tau_1(\mathbf{y}) : 1.12 \tau_0(\mathbf{y})) \times (1.71 \mathbf{v}_1(\mathbf{y}) : 1.02 \mathbf{v}_0(\mathbf{y}))$$

Moreover, since the lines  $s = \Sigma_0 \cap \Sigma_1$ ,  $t = T_0 \cap T_1$ , and  $u = Y_0 \cap Y_1$  do not intersect, the base locus of  $\phi$  is blown-up to the quadric  $Q$  through  $s$ ,  $t$ ,  $u$ , defined by the quadratic polynomial

$$\sum_{0 \leq i, j \leq 1} w_{ij1} \langle \mathbf{v}_0, \mathbf{P}_{ij1} \rangle (\alpha_{i^*} \sigma_{i^*}(\mathbf{y})) (\beta_{j^*} \tau_{j^*}(\mathbf{y})) = \alpha_0 \alpha_1 \beta_0 \beta_1 \gamma_1 \sum_{0 \leq i, j \leq 1} \Delta_{ij1} y_{ij1} \sigma_{i^*}(\mathbf{y}) \tau_{j^*}(\mathbf{y})$$



which, up to nonzero constant, is

$$q(\mathbf{y}) = 86855198 y_0^2 - 83702178 y_0 y_1 + 10747080 y_1^2 - 61871537 y_0 y_2 + 34161786 y_1 y_2 \\ + 10516557 y_2^2 - 154343278 y_0 y_3 + 93619656 y_1 y_3 + 50553377 y_2 y_3 + 58824388 y_3^2 .$$

Additionally, the pullback polynomial  $q(\mathbf{f})$  yields a trilinear polynomial with the factors

$$h_1(\mathbf{t}, \mathbf{u}) = 3907008000 t_0 u_0 - 1190646000 t_1 u_0 - 376698560 t_0 u_1 + 47450751 t_1 u_1 , \\ h_2(\mathbf{s}, \mathbf{u}) = 221972400 s_0 u_0 - 39841200 s_1 u_0 - 112388133 s_0 u_1 + 8111542 s_1 u_1 , \\ h_3(\mathbf{s}, \mathbf{t}) = 222986400 s_0 t_0 - 435507520 s_1 t_0 - 368641338 s_0 t_1 + 721224457 s_1 t_1 ,$$

which respectively define the surfaces contracted to the lines  $s$ ,  $t$ , and  $u$ .

**Remark 4.3.16** (Polynomial hexahedral birational maps). Polynomial maps that are nonrational, i.e. the entries are given by polynomials and not rational functions, are sometimes preferred for applications. In the conventional formulation in CAGD, using the Bernstein basis, this case corresponds to  $w_{ijk} = 1$  for every  $0 \leq i, j, k \leq 1$  since

$$f_0 = \sum_{0 \leq i, j, k \leq 1} w_{ijk} b_i(s_0, s_1) b_j(t_0, t_1) b_k(u_0, u_1) = \sum_{0 \leq i, j, k \leq 1} b_i(s_0, s_1) b_j(t_0, t_1) b_k(u_0, u_1) = s_0 t_0 u_0 ,$$

and in the affine chart  $\mathbb{A}_{\mathbb{R}}^3$  of  $(\mathbb{P}_{\mathbb{R}}^1)^3$  given by  $s_0 \neq 0$ ,  $t_0 \neq 0$ , and  $u_0 \neq 0$  we can set  $f_0 = 1$ . Therefore, in this chart  $\phi : \mathbb{A}_{\mathbb{R}}^3 \rightarrow \mathbb{A}_{\mathbb{R}}^3$  restricts to a polynomial morphism. From our former results, we derive the following corollary.

**Corollary 4.3.17.** *Let  $\phi$  be hexahedral, and  $w_{ijk} = 1$  for every  $0 \leq i, j, k \leq 1$ . If the control points define a parallelepiped, then  $\phi$  is birational of type  $(1, 1, 1)$ .*

*Proof.* For the same considerations in Theorem 4.3.13,  $\phi$  is birational of type  $(1, 1, 1)$  if and only if the rank conditions (4.6), (4.9), and (4.11) are simultaneously satisfied. Since  $w_{ijk} = 1$  for every  $0 \leq i, j, k \leq 1$ , this is equivalent to the matrices

$$\begin{pmatrix} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} \rangle \\ \langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle \end{pmatrix}_{0 \leq j, k \leq 1} , \quad \begin{pmatrix} \langle \boldsymbol{\tau}_0, \mathbf{P}_{i1k} \rangle \\ \langle \boldsymbol{\tau}_1, \mathbf{P}_{i0k} \rangle \end{pmatrix}_{0 \leq i, k \leq 1} , \quad \begin{pmatrix} \langle \boldsymbol{\nu}_0, \mathbf{P}_{ij1} \rangle \\ \langle \boldsymbol{\nu}_1, \mathbf{P}_{ij0} \rangle \end{pmatrix}_{0 \leq i, j \leq 1}$$

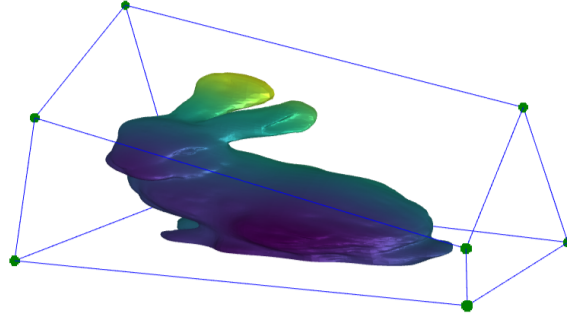
having rank one. On the other hand, we can write

$$\langle \boldsymbol{\sigma}_i, \mathbf{P}_{ijk} \rangle = \|\boldsymbol{\sigma}_i\| \langle \|\boldsymbol{\sigma}_i\|^{-1} \boldsymbol{\sigma}_i, \mathbf{P}_{ijk} \rangle = \|\boldsymbol{\sigma}_i\| \text{sdist}(\Sigma_i, \mathbf{P}_{ijk}) ,$$

where  $\text{sdist}(\cdot, \cdot)$  is the signed Euclidean distance. Hence,

$$\text{rank} \begin{pmatrix} \|\boldsymbol{\sigma}_0\| \text{sdist}(\Sigma_0, \mathbf{P}_{1jk}) \\ \|\boldsymbol{\sigma}_1\| \text{sdist}(\Sigma_1, \mathbf{P}_{0jk}) \end{pmatrix}_{0 \leq j, k \leq 1} = \text{rank} \begin{pmatrix} \text{sdist}(\Sigma_0, \mathbf{P}_{1jk}) \\ \text{sdist}(\Sigma_1, \mathbf{P}_{0jk}) \end{pmatrix}_{0 \leq j, k \leq 1} .$$

In particular, the rank is one if the boundary planes  $\Sigma_0, \Sigma_1$  are parallel. The analogous observations hold for the remaining two matrices.  $\square$



**Figure 4.3.:** a birational deformation of the hexahedral birational map of Example 4.3.12. The boundary plane  $\Sigma_1$ , previously defined by the vector  $\sigma_1 = (1.25 - 0.63 - 0.32 - 0.63)$ , is updated to the plane defined by  $\sigma_1 = (2.31 - 0.84 - 0.2 - 0.32)$ , yielding new control points. Additionally, the weights are also updated to preserve birationality.

#### 4.3.5. Deformation of birational maps

The construction of nets of control points for hexahedral rational maps of §4.3.1 and Corollary 4.3.7 yield a dominant rational parametrization of the irreducible component of trilinear birational maps of type  $(1, 1, 1)$ . Unlike the dominant rational parametrization that follows from the geometric classification of §3.4, based on the composition of a representative with the automorphisms in  $\text{Aut}((\mathbb{P}_{\mathbb{C}}^1)^3)$  and  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^3)$  (used in the proof of Corollary 3.4.12), we provide a parametrization where a designer grasps the geometry of the constructed map. Namely, birational maps are parametrized by means of the boundary planes and weights.

By (4.13), the control points of a hexahedral rational map are defined by 6 nonzero vectors  $\sigma_0, \sigma_1, \tau_0, \tau_1, \nu_0$ , and  $\nu_1$  defining the boundary planes, i.e. by a point in  $\mathcal{G}_{(1,1,1)} := ((\mathbb{P}_{\mathbb{R}}^3)^2)^3$ . In particular, we have the dominant rational parametrization

$$\begin{aligned} \Phi_{(1,1,1)} : (\mathbb{P}_{\mathbb{R}}^1)^3 \times \mathcal{G}_{(1,1,1)} &\dashrightarrow \mathcal{W} \times \mathcal{P}_{(1,1,1)} & (4.26) \\ (\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1) & & \\ \times &\mapsto (\dots : \Delta_{ijk} \alpha_i \beta_j \gamma_k : \dots) \times (\dots : \Delta_{ijk}^{-1} \sigma_i \wedge \tau_j \wedge \nu_k : \dots) & \\ (\sigma_0, \sigma_1) \times (\tau_0, \tau_1) \times (\nu_0, \nu_1) & & \end{aligned}$$

Therefore,  $\phi$  can be deformed by adjusting the control points deciding new boundary planes, updating the  $\Delta_{ijk}$  accordingly, and adjusting the weights as a point in  $(\mathbb{P}_{\mathbb{R}}^1)^3$ .

The following is an example of a deformation of the hexahedral birational map constructed in Example 4.3.12.

**Example 4.3.18** (Deformation of a hexahedral birational map). Starting where we left it at Example 4.3.12, suppose that we want to update the control points, as a user frequently does during the design process. However, we want to do so preserving birationality. In particular, the property of being hexahedral must(!) be preserved. If a single control point is moved, the new rational map is no longer hexahedral. Therefore, several control points must be moved simultaneously, making the update cumbersome.

An easier approach is to update the boundary planes instead, and compute the control points accordingly using the identity (4.24). In our example, we update

$$\sigma_1 = (1.25 \quad -0.63 \quad -0.32 \quad -0.63) \longrightarrow \sigma_1 = (2.31 \quad -0.84 \quad -0.2 \quad -0.32) .$$

In particular, for each  $0 \leq j, k \leq 1$  the values of  $\Delta_{1jk} \rightarrow \Delta_{1jk}^2$  must be recomputed accordingly using (4.14). Therefore, if we recompute the weights using

$$w_{1jk} = \alpha_1 \beta_j \gamma_k \Delta_{1jk}^2$$

for each  $0 \leq j, k \leq 1$  and the same  $(\alpha_0, \alpha_1) \otimes (\beta_0, \beta_1) \otimes (\gamma_0, \gamma_1)$  as in Example 4.3.12, birationality is preserved.

## 4.4. Pyramidal birational maps

In this section, we study the second family of birational maps: the class of pyramidal birational maps. This name is motivated by the geometry of the control points of birational maps of type  $(2, 1, 1)$ ,  $(1, 2, 1)$  and  $(1, 1, 2)$  (see Proposition 4.4.3).

**Definition 4.4.1** (Pyramidal rational map). A trilinear rational map is pyramidal if it satisfies Property 1 and for one of the three parameters:

- (i) The four boundary lines intersect at a point
- (ii) The two boundary surfaces are quadrics

**Remark 4.4.2.** In Definition 4.4.1, we require that a pair of boundary surfaces are smooth quadrics to exclude hexahedral rational maps.

### 4.4.1. Construction of control points

Once again, we start explaining the necessary constraints on the control points for birationality to be possible. Firstly, we prove that general birational maps of type  $(1, 1, 2)$ ,  $(1, 2, 1)$ , and  $(2, 1, 1)$  are pyramidal. Without loss of generality, we can restrict to the first type.

**Proposition 4.4.3.** *If  $\phi$  is a general birational map of type  $(1, 1, 2)$ , then it is pyramidal. In particular, the boundary  $u$ -lines meet at a point  $\mathbf{V}$  (in  $\mathbb{P}_{\mathbb{R}}^3$ ).*

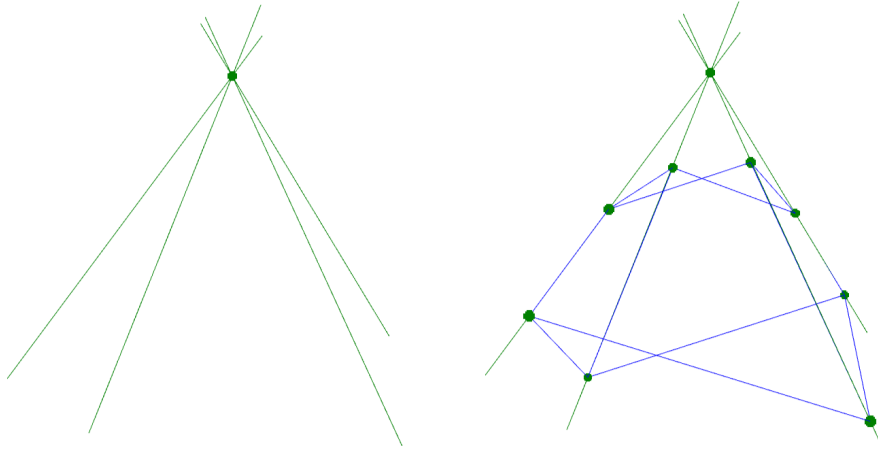
*Proof.* By Theorem 3.4.6,  $\phi$  has a base point defined by the ideal

$$(a(s_0, s_1), b(t_0, t_1), e(u_0, u_1)) \tag{4.27}$$

for some linear  $a(s_0, s_1)$ ,  $b(t_0, t_1)$ , and  $e(u_0, u_1)$  in  $R$ . In particular, the pullback ideal by  $\phi$

$$(a(g_{01}, g_{11}), b(g_{02}, g_{12}), e(g_{03}, g_{13})) \tag{4.28}$$

is principal in  $\mathbb{C}[\mathbf{y}]$ . More specifically, the blow-up  $\pi : X = \text{Bl}_Z(\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1) \rightarrow (\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1)$  of  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  along the base locus  $Z$  of  $\phi$  is isomorphic to the graph of  $\phi$  (see §2.1.4). In particular, the exceptional divisor of this blow-up has two irreducible components, which are Cartier divisors (i.e. they have codimension one in  $X$ ), since  $X$  can be obtained by blowing up  $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$  along the curve in  $Z$  and the base point in (4.27), subsequently. Hence, the image of each of these



**Figure 4.4.:** the generation of the control points of a pyramidal rational map, computed using Construction 2. The left image shows the choice of the boundary  $u$ -lines through a common point. The right image illustrates a choice of two control points per line.

irreducible components, by the morphism  $X \rightarrow \mathbb{P}_{\mathbb{C}}^3$  that extends  $\phi$ , is a surface in  $\mathbb{P}_{\mathbb{C}}^3$ . By definition, the surface associated to the irreducible point (4.27) in  $Z$  is defined by the pullback ideal in (4.28). Therefore, the linear polynomials  $a(g_{01}, g_{11})$  and  $b(g_{02}, g_{12})$  must be proportional. Geometrically, this means that the boundary planes  $\Sigma_0, \Sigma_1, T_0$ , and  $T_1$  are not independent, or equivalently, they intersect at a point  $\mathbf{V}$ . Since  $u_{ij} = \Sigma_i \cap T_j$ , we find

$$u_{00} \cap u_{10} \cap u_{01} \cap u_{11} = \Sigma_0 \cap \Sigma_1 \cap T_0 \cap T_1 = \mathbf{V} ,$$

and  $\phi$  is pyramidal. □

Proposition 4.4.3 combined with Theorem 4.4.7, presented and proven in §4.4.2, provides an answer to Question 4.1.1 for trilinear birational maps of type  $(1, 1, 2)$ . Specifically, we find the following corollary.

**Corollary 4.4.4.** *The set*

$$\mathcal{P}_{(1,1,2)} = \{ \{ \mathbf{P}_{ijk} \}_{0 \leq i,j,k \leq 1} : \phi \text{ is pyramidal} \}$$

*answer Question 4.1.1. More explicitly, for every net of control points in  $\mathcal{P}_{(1,1,2)}$  we can compute weights that render  $\phi$  birational of type  $(1, 1, 2)$ .*

In applications, nets of control points in  $\mathcal{P}_{(1,1,2)}$  can be generated using the following construction. The generation of a net of control points using Construction 2 is illustrated in Figure 4.4.

**Construction 2** (Control points for pyramidal rational maps). A net of control points in  $\mathcal{P}_{(1,1,2)}$  can be generated as follows:

1. Choose a point  $\mathbf{V}$  in  $\mathbb{P}_{\mathbb{R}}^3$  (green point in the left image of Figure 4.4). Points at  $\infty$  are allowed, meaning that the boundary  $u$ -lines are parallel
2. For each  $0 \leq i, j \leq 1$ , choose an affine line  $u_{ij}$  (green lines in Figure 4.4) through  $\mathbf{V}$
3. Choose two distinct affine points  $\mathbf{P}_{ij0}$  and  $\mathbf{P}_{ij1}$  on  $u_{ij}$ , different from  $\mathbf{V}$

#### 4.4.2. Effective computation of weights

In this section, we derive an analogous birationality criterion to Theorem 4.3.6 for pyramidal birational maps. To properly state this result, we introduce some necessary notation. Without loss of generality, we can assume that  $\phi$  has type  $(1, 1, 2)$ . Because of this choice, when we refer to a “pyramidal rational map”, we actually refer to a “pyramidal rational map of type  $(1, 1, 2)$ ”.

**Notation 4.4.5.** Let  $\phi$  be pyramidal. We introduce the following notation:

- (i) For each  $0 \leq i, j \leq 1$ , the boundary planes  $\Sigma_i$  and  $T_j$  are defined by the vectors  $\sigma_i$  and  $\tau_j$  in  $\mathbb{R}^4$ , respectively. Moreover, we denote  $s = \Sigma_0 \cap \Sigma_1$  and  $t = T_0 \cap T_1$ .
- (ii) For each  $k = 0, 1$ ,  $Y_k$  is defined by the symmetric matrix  $\mathbf{v}_k$  in  $\mathbb{R}^{4 \times 4}$ .
- (iii) On the other hand, there is a plane  $\Pi$  through the lines  $s$  and  $t$ , defined by a vector  $\pi$  in  $\mathbb{R}^4$ . Namely, we can write

$$\mathbf{V} = \Sigma_0 \cap \Sigma_1 \cap T_0 \cap T_1 = (\Sigma_0 \cap \Sigma_1) \cap (T_0 \cap T_1) = s \cap t ,$$

so the lines  $s, t$  intersect. In particular, we find  $(\lambda_0 : \lambda_1)$  and  $(\mu_0 : \mu_1)$  in  $\mathbb{P}_{\mathbb{R}}^1$  such that

$$\pi = \lambda_0 \sigma_0 + \lambda_1 \sigma_1 = \mu_0 \tau_0 + \mu_1 \tau_1 .$$

Additionally, for each  $0 \leq i, j, k \leq 1$  we define

$$\Delta_{ijk} = \frac{1}{\langle \pi, \mathbf{P}_{ijk} \rangle} ,$$

as well as the  $2 \times 2 \times 2$  tensor

$$W = \left( \frac{w_{ijk}}{\Delta_{ijk}} \right)_{0 \leq i, j, k \leq 1} . \quad (4.29)$$

**Remark 4.4.6** (Nonzero and positive  $\Delta_{ijk}$ ). Notice that  $\Delta_{ijk}$  is not defined if and only if  $\mathbf{P}_{ijk} = \mathbf{V}$ . Property 1 excludes this possibility, since all the boundary lines are distinct. Specifically, if  $\mathbf{P}_{ijk} = \mathbf{V}$  for some  $0 \leq i, j, k \leq 1$  we find

$$s_{jk} = \overline{\mathbf{P}_{0jk} \mathbf{P}_{1jk}} = \overline{\mathbf{P}_{i^*jk} \mathbf{V}} = \overline{\mathbf{P}_{i^*j0} \mathbf{P}_{i^*j1}} = u_{i^*k} ,$$

and similarly  $t_{ik} = u_{j^*k}$ . Furthermore, if all the control points are left at “one side of  $\Pi$ ”, i.e. if they lie in one of the two halfspaces defined by  $\Pi$  in  $\mathbb{A}_{\mathbb{R}}^3$ , then the vector  $\pi$  can be chosen so that  $\langle \pi, \mathbf{P}_{ijk} \rangle > 0$ , and hence  $\Delta_{ijk} > 0$ , for every  $0 \leq i, j, k \leq 1$ .

We now prove the first main result of the section. It is a characterization of birationality for pyramidal maps relying on tensor rank.

**Theorem 4.4.7.** *Let  $\phi$  be pyramidal. The following are equivalent:*

- (i)  $\phi$  is birational of type  $(1, 1, 2)$
- (ii) The tensor  $W$  has rank one

(iii) The trilinear polynomial  $\langle \boldsymbol{\pi}, \mathbf{f} \rangle$  has three linear factors

*Proof.* Clearly, (ii) and (iii) are equivalent, since

$$\langle \boldsymbol{\pi}, \mathbf{f} \rangle = \sum_{0 \leq i, j, k \leq 1} w_{ijk} \langle \boldsymbol{\pi}, \mathbf{P}_{ijk} \rangle s_i t_j u_k = \sum_{0 \leq i, j, k \leq 1} \frac{w_{ijk}}{\Delta_{ijk}} s_i t_j u_k .$$

Therefore, it is sufficient to prove that (i) and (ii) are equivalent. By Theorem 3.5.1,  $\phi$  is birational of type (1, 1, 2) if and only if  $\mathbf{f}$  admits syzygies of degrees (1, 0, 0), (0, 1, 0), and not (0, 0, 1). Since  $Y_0, Y_1$  are smooth quadrics,  $\mathbf{f}$  cannot have a syzygy of degree (0, 0, 1). Therefore, by Lemma 4.2.1 and Remark 4.2.2,  $\phi$  is birational if and only if the rank conditions (4.6) and (4.9) hold simultaneously.

Once more, we rewrite these rank conditions in terms of the flattenings of  $W$ . Namely, since

$$\langle \boldsymbol{\sigma}_i, \mathbf{P}_{ijk} \rangle = 0 \quad , \quad \langle \boldsymbol{\tau}_j, \mathbf{P}_{ijk} \rangle = 0$$

for every  $0 \leq i, j, k \leq 1$ , we can write

$$\begin{aligned} \lambda_{i^*} \langle \boldsymbol{\sigma}_{i^*}, \mathbf{P}_{ijk} \rangle &= \langle \lambda_0 \boldsymbol{\sigma}_0 + \lambda_1 \boldsymbol{\sigma}_1, \mathbf{P}_{ijk} \rangle = \langle \boldsymbol{\pi}, \mathbf{P}_{ijk} \rangle , \\ \mu_{j^*} \langle \boldsymbol{\tau}_{j^*}, \mathbf{P}_{ijk} \rangle &= \langle \mu_0 \boldsymbol{\tau}_0 + \mu_1 \boldsymbol{\tau}_1, \mathbf{P}_{ijk} \rangle = \langle \boldsymbol{\pi}, \mathbf{P}_{ijk} \rangle . \end{aligned}$$

Therefore, (4.6) and (4.9) are respectively equivalent to

$$\text{rank} \begin{pmatrix} w_{1jk} \Delta_{1jk}^{-1} \\ w_{0jk} \Delta_{0jk}^{-1} \end{pmatrix}_{0 \leq j, k \leq 1} = 1 \quad , \quad \text{rank} \begin{pmatrix} w_{i1k} \Delta_{i1k}^{-1} \\ w_{i0k} \Delta_{i0k}^{-1} \end{pmatrix}_{0 \leq i, k \leq 1} = 1 ,$$

which are the first two flattenings of  $W$ . By Lemma 2.2.16, the result follows.  $\square$

**Corollary 4.4.8.** *Let  $\phi$  be pyramidal. Then,  $\phi$  is birational of type (1, 1, 2) if and only if*

$$w_{ijk} = \alpha_i \beta_j \gamma_k \Delta_{ijk} \tag{4.30}$$

for some  $(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1)$  in  $(\mathbb{P}_{\mathbb{C}}^1)^3$ .

*Proof.* Analogous to the proof of Corollary 4.3.7.  $\square$

**Remark 4.4.9** (Nonzero, real, and positive weights). The same observations as in Remark 4.3.8, combined with Remark 4.5.6, apply.

**Remark 4.4.10** (Deformation of pyramidal birational maps). Construction 2, combined with Theorem 4.4.7, provides an effective way to deform pyramidal birational maps, as explained in §4.3.5. In particular, it offers users the flexibility to continuously deform the net of control points by choosing new positions for  $\mathbf{V}$ , adjusting the lines  $u_{ij}$  for each  $0 \leq i, j \leq 1$ , and making choices regarding the control points. Furthermore, Corollary 4.4.8 asserts that birational weights can be selected by adjusting specific parameters that depend rationally on the coordinates of the control points.

#### 4.4.2.1. The contractions of a pyramidal birational map

From Corollary 4.4.8, we can recover useful information about the contractions of  $\phi$ . In particular, we have the following result.

**Corollary 4.4.11.** *Let  $\phi$  be pyramidal with weights as in Corollary 4.4.8, and define the polynomials*

$$a(s_0, s_1) = \begin{vmatrix} s_0 & s_1 \\ -\alpha_1 & \alpha_0 \end{vmatrix}, \quad b(t_0, t_1) = \begin{vmatrix} t_0 & t_1 \\ -\beta_1 & \beta_0 \end{vmatrix}, \quad c(u_0, u_1) = \begin{vmatrix} u_0 & u_1 \\ -\gamma_1 & \gamma_0 \end{vmatrix}.$$

Then:

- (i)  $\phi$  contracts the surface  $a(s_0, s_1) = 0$  to the line  $t$
- (ii)  $\phi$  contracts the surface  $b(t_0, t_1) = 0$  to the line  $s$
- (iii)  $\phi$  contracts the surface  $c(u_0, u_1) = 0$  to the point  $\mathbf{V}$
- (iv) The base locus of  $u$ -surfaces, defined by  $Y_0 \cap Y_1$ , contains the lines  $s$  and  $t$

*Proof.* By (4.30), the pullback by  $\phi$  of the plane  $\Pi$  factorizes as

$$\langle \pi, \mathbf{f} \rangle = \sum_{0 \leq i, j, k \leq 1} (\alpha_i s_i) (\beta_j t_j) (\gamma_k u_k) = a(s_0, s_1) b(t_0, t_1) c(u_0, u_1).$$

Since the boundary lines  $t_{i0}, t_{i1}$  lie in  $\Sigma_i$  for each  $i = 0, 1$ , and by definition  $\Sigma_i \cap \Pi = \Sigma_0 \cap \Sigma_1 = s$ , all the boundary  $t$ -lines intersect  $\Pi$  at  $s$ . Thus, for each  $0 \leq i, k \leq 1$  the point  $\mathbf{B}_{ik}(-\beta_1, \beta_0)$  lies in  $s$ , and  $\phi$  contracts  $b(t_0, t_1) = 0$  to  $s$ . Similarly, it follows that  $\phi$  contracts  $a(s_0, s_1) = 0$  to  $t$ . Additionally, all the boundary  $u$ -lines meet  $\Pi$  at  $\mathbf{V}$ , and  $\phi$  contracts  $c(u_0, u_1) = 0$  to  $\mathbf{V}$ .

On the other hand, by Theorem 3.5.1, since  $\phi$  is birational of type  $(1, 1, 2)$  it admits syzygies

$$\sigma(s_0, s_1; \mathbf{y}) = \begin{vmatrix} s_0 & s_1 \\ \sigma'_0(\mathbf{y}) & \sigma'_1(\mathbf{y}) \end{vmatrix}, \quad \xi(s_0, s_1, u_0, u_1; \mathbf{y}) = \begin{vmatrix} s_0 & s_1 \\ \xi_0(u_0, u_1; \mathbf{y}) & \xi_1(u_0, u_1; \mathbf{y}) \end{vmatrix}$$

of degrees  $(1, 0, 0; 1)$  and  $(1, 0, 1; 1)$ , respectively. In particular, the inverse for the parameter  $(u_0 : u_1)$ , which defines the pencil of  $u$ -surfaces, is defined by

$$\begin{vmatrix} \sigma'_0(\mathbf{y}) & \sigma'_1(\mathbf{y}) \\ \xi_0(u_0, u_1; \mathbf{y}) & \xi_1(u_0, u_1; \mathbf{y}) \end{vmatrix}.$$

Since the ideal  $B_s = (\sigma'_0(\mathbf{y}), \sigma'_1(\mathbf{y})) = (\sigma_0(\mathbf{y}), \sigma_1(\mathbf{y}))$  defines  $s$ , it follows that all the  $u$ -surfaces contain  $s$ . Similarly, we deduce that all the  $u$ -surfaces contain  $t$ .  $\square$

**Remark 4.4.12** (Computation of the quadric boundary surfaces). It is convenient to keep the following parallelism in mind. In the same manner that

“three (not aligned) points determine a unique plane”,

by Lemma 2.2.23 we have that

“three (mutually nonintersecting) lines determine a unique (smooth) quadric”.

In the same way that the defining vector of a plane can be computed as the wedge of three points on it, we can compute the defining symmetric matrix of a smooth quadric as the wedge of the vectors in  $\mathbb{C}^{10}$  (or  $\mathbb{R}^{10}$ ) given as the image by the Veronese map (in degree two, with four homogeneous variables) of three distinct points per line, for three lines.

By Corollary 4.4.11, the lines  $s, t$  lie on the boundary  $u$ -surfaces  $Y_0, Y_1$ . Therefore,  $Y_k$  can be extracted from the lines  $s, s_{0k}, s_{1k}$  or  $t, t_{0k}, t_{1k}$ .

We can extract further geometric information about a pyramidal birational map. To this purpose, we introduce the following notation.

**Notation 4.4.13.** Let  $\phi$  be pyramidal. With Notation 4.4.5, for each  $k = 0, 1$  define

$$\epsilon_k = \sum_{0 \leq i, j \leq 1} (-1)^{i+j} \Delta_{ijk} ,$$

and for each  $0 \leq i, j \leq 1$ , define the point

$$\mathbf{Q}_{ij} = \epsilon_1 \Delta_{ij0} \mathbf{P}_{ij0} - \epsilon_0 \Delta_{ij1} \mathbf{P}_{ij1} .$$

The points  $\mathbf{Q}_{00}, \mathbf{Q}_{01}, \mathbf{Q}_{10},$  and  $\mathbf{Q}_{11}$  lie in a common plane  $\Theta$ , defined by a vector  $\boldsymbol{\theta}$  in  $\mathbb{R}^4$ , since by definition we have the relation

$$\mathbf{Q}_{00} - \mathbf{Q}_{10} - \mathbf{Q}_{01} + \mathbf{Q}_{11} = 0 ,$$

implying  $\det(\mathbf{Q}_{00}, \mathbf{Q}_{10}, \mathbf{Q}_{01}, \mathbf{Q}_{11}) = 0$ .

The following corollary asserts that the pullback by  $\phi$  of the plane  $\Theta$  admits a factor of degree  $(0, 0, 1)$ .

**Corollary 4.4.14.** *Let  $\phi$  be pyramidal, with weights as in Corollary 4.4.8. With Notation 4.4.13, define the polynomial*

$$e(u_0, u_1) = \begin{vmatrix} u_0 & u_1 \\ -\gamma_1 \epsilon_1 & \gamma_0 \epsilon_0 \end{vmatrix} .$$

*Then, we find a bilinear polynomial  $h = h(s_0, s_1, t_0, t_1)$  such that  $\langle \boldsymbol{\theta}, \mathbf{f} \rangle = e(u_0, u_1) h(s_0, s_1, t_0, t_1)$ . In particular:*

- (i) *The image by  $\phi$  of the surface  $e(u_0, u_1) = 0$  is dense in  $\Theta$*
- (ii)  *$\phi$  contracts the surface  $h(s_0, s_1, t_0, t_1) = 0$  to a plane conic  $C$  in  $\Theta$*
- (iii) *The rank-two quadric  $\Theta \cup \Pi$  lies in the pencil of  $u$ -surfaces*

*Proof.* For each  $0 \leq i, j \leq 1$ , we find

$$\begin{aligned} \mathbf{C}_{ij}(-\gamma_1 \epsilon_1, \gamma_0 \epsilon_0) &= \alpha_i \beta_j (\gamma_0 \Delta_{ij0} \mathbf{P}_{ij0} (-\gamma_1 \epsilon_1) + \gamma_1 \Delta_{ij1} \mathbf{P}_{ij1} (\gamma_0 \epsilon_0)) \\ &= \alpha_i \beta_j \gamma_0 \gamma_1 (-\Delta_{ij0} \mathbf{P}_{ij0} \epsilon_1 + \Delta_{ij1} \mathbf{P}_{ij1} \epsilon_0) = -\alpha_i \beta_j \gamma_0 \gamma_1 \mathbf{Q}_{ij} . \end{aligned}$$

Since  $\Theta$  contains  $\mathbf{Q}_{ij}$  for every  $0 \leq i, j \leq 1$ , it follows that the pullback polynomial  $\langle \boldsymbol{\theta}, \mathbf{f} \rangle$  admits  $e(u_0, u_1)$  as a factor. In particular, the specialization of  $\phi$  to  $(u_0 : u_1) = (-\gamma_1 \epsilon_1 : \gamma_0 \epsilon_0)$  yields a bilinear birational map to the plane  $\Theta$ . The image of this rational map is dense in  $\Theta$ , since otherwise the points  $\mathbf{Q}_{00}, \mathbf{Q}_{10}, \mathbf{Q}_{01},$  and  $\mathbf{Q}_{11}$  are colinear and  $\phi$  would not be dominant. Therefore, one of



the quadrics in the pencil of  $u$ -surfaces contains  $\Theta$ . Since by Corollary 4.4.11 all the  $u$ -surfaces contain  $\Pi$ , it follows that the rank-two quadric  $\Theta \cup \Pi$  is a  $u$ -surface. In particular, the base locus of the  $u$ -surfaces is supported in  $\Theta \cup \Pi$ . Thus, the intersection of any general  $u$ -surface with  $\Theta$  is a plane conic  $C$ , and the base locus  $Y_0 \cap Y_1$  of the  $u$ -surfaces is  $s \cup t \cup C$ . In particular, the pullback of the defining ideal of  $C$  is principal, and its generator must divide  $\langle \theta, \mathbf{f} \rangle$  since  $C \subset \Theta$ . Moreover, for each  $k = 0, 1$  we have

$$\mathbf{v}_k(\mathbf{f}) = u_k^* a(s_0, s_1) b(t_0, t_1) c(u_0, u_1) h'(s_0, s_1, t_0, t_1)$$

since  $a$ ,  $b$ , and  $c$  respectively define the pullbacks of  $s$ ,  $t$ , and  $\mathbf{V}$  (recall Corollary 4.4.11). Hence,  $h'$  has degree  $(1, 1, 0)$ , and we conclude that  $h$  and  $h'$  are proportional.  $\square$

**Example 4.4.15** (Geometry of a pyramidal birational map). Consider the control points

$$\begin{aligned} \mathbf{P}_{000} &= \left(1 \ -2 \ -2 \ -\frac{1}{2}\right), \quad \mathbf{P}_{100} = (1 \ 2 \ -2 \ 0), \quad \mathbf{P}_{010} = (1 \ -2 \ 2 \ 0), \quad \mathbf{P}_{110} = \left(1 \ 2 \ 2 \ -\frac{1}{2}\right), \\ \mathbf{P}_{001} &= \left(1 \ -\frac{2}{3} \ -\frac{2}{3} \ \frac{5}{2}\right), \quad \mathbf{P}_{101} = \left(1 \ \frac{1}{2} \ -\frac{1}{2} \ 3\right), \quad \mathbf{P}_{011} = \left(1 \ -\frac{1}{2} \ \frac{1}{2} \ 3\right), \quad \mathbf{P}_{111} = \left(1 \ \frac{2}{3} \ \frac{2}{3} \ \frac{5}{2}\right). \end{aligned}$$

All the boundary  $u$ -lines, appearing as green lines in Figure 4.5, meet at the point  $\mathbf{V} = (1 \ 0 \ 0 \ 4)$ , so the control points define a pyramidal map, depicted in Figure 4.5. In particular, the boundary  $s$ - and  $t$ -surfaces are defined by the vectors

$$\begin{aligned} \boldsymbol{\sigma}_0 &= (32 \ 17 \ 1 \ -8), \quad \boldsymbol{\sigma}_1 = (-32 \ 17 \ 1 \ 8), \\ \boldsymbol{\tau}_0 &= (32 \ 1 \ 17 \ -8), \quad \boldsymbol{\tau}_1 = (-32 \ 1 \ 17 \ 8) \end{aligned}$$

and the boundary  $u$ -surfaces by the symmetric matrices

$$\mathbf{v}_0 = \begin{pmatrix} 1024 & 0 & 0 & 2048 \\ 0 & 17 & 145 & 0 \\ 0 & 145 & 17 & 0 \\ 2048 & 0 & 0 & -1088 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 3584 & 0 & 0 & -1088 \\ 0 & -17 & -145 & 0 \\ 0 & -145 & -17 & 0 \\ -1088 & 0 & 0 & 320 \end{pmatrix}.$$

In particular, the plane  $\Pi$ , appearing in red in Figure 4.5, through the lines  $s = \Sigma_0 \cap \Sigma_1$  and  $t = T_0 \cap T_1$ , appearing as red lines in Figure 4.5, is defined by the vector

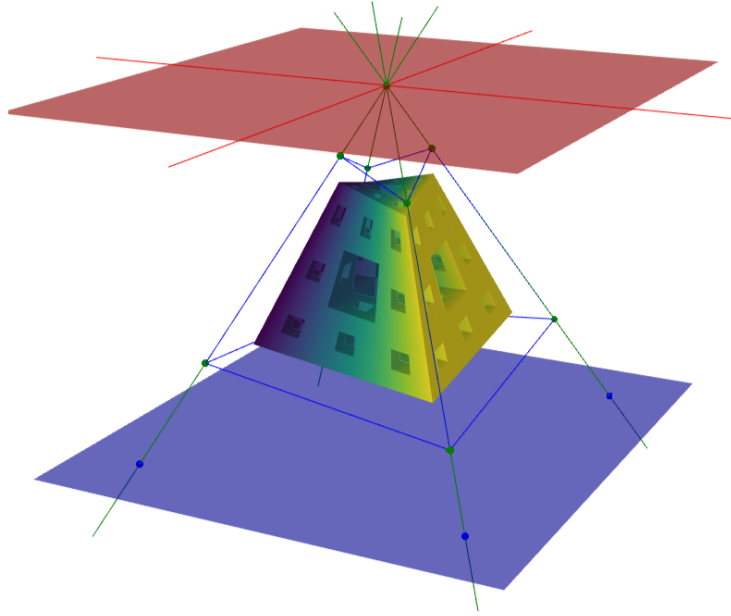
$$\boldsymbol{\pi} = (4 \ 0 \ 0 \ -1).$$

Similarly, using Notation 4.4.13 we compute  $(\epsilon_0 : \epsilon_1) = (1 : 12)$ , as well as the points

$$\begin{aligned} \mathbf{Q}_{00} &= \left(1 \ -\frac{22}{9} \ -\frac{22}{9} \ -\frac{3}{2}\right), \quad \mathbf{Q}_{10} = \left(1 \ \frac{11}{4} \ -\frac{11}{4} \ -\frac{3}{2}\right), \\ \mathbf{Q}_{01} &= \left(1 \ -\frac{11}{4} \ \frac{11}{4} \ -\frac{3}{2}\right), \quad \mathbf{Q}_{11} = \left(1 \ \frac{22}{9} \ \frac{22}{9} \ -\frac{3}{2}\right), \end{aligned}$$

which span the plane  $\Theta$ , appearing in blue in Figure 4.5, defined by the vector

$$\boldsymbol{\theta} = (3 \ 0 \ 0 \ 2).$$



**Figure 4.5.:** the geometric elements involved in the pyramidal map of Example 4.4.15. Specifically, we have replaced the monomial basis by the Bernstein basis, to recover the geometric intuition of a deformation of the unit cube. The lines  $s = \Sigma_0 \cap \Sigma_1$  and  $t = T_0 \cap T_1$  appear in red, and lie on the plane  $\Pi$ , also in red. The point  $\mathbf{Q}_{ij}$  appears in blue for each  $0 \leq i, j \leq 1$ , and lies on the plane  $\Theta$ , also in blue. If the pyramidal map is birational, the base locus of its inverse consists of the union of the lines  $s$ ,  $t$  and a plane conic  $C$  supported on  $\Theta$ .

#### 4.4.3. Distance to birationality

We now extend the notion of distance to birationality, introduced in §4.3.3, to pyramidal rational maps. The following definition is the immediate rephrasing of Definition 4.3.9.

**Definition 4.4.16** (Distance to birationality: pyramidal). Let  $\phi$  be pyramidal with real weights, let  $V \subset \mathbb{R}^{2 \times 2 \times 2}$  be the affine cone over  $\sigma(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1)$ , and let  $W$  be as in Theorem 4.4.7. We define the *distance to birationality* of  $\phi$ , denoted by  $\text{dist}_{\text{bir}}(\phi)$ , as

$$\text{dist}_{\text{bir}}(\phi) := \min_{P \in V} \frac{\|W - P\|}{\|W\|} \quad (4.31)$$

Example 1.2.8 in §1.2.3 shows an example of the computation of a birational approximation to a pyramidal map that is not birational, according to Definition 4.5.12.

#### 4.4.4. Inverse rational map and base locus

In this subsection, we derive formulas for the inverse of a pyramidal birational map, as well as the defining equations of the irreducible components of the base locus.

**Notation 4.4.17.** By Corollary 4.4.14, the rank-two quadric  $\Pi \cup \Theta$  is a  $u$ -surface. Therefore, for some  $(\nu_0 : \nu_1)$  in  $\mathbb{P}_{\mathbb{R}}^1$ , the quadric  $\Pi \cup \Theta$  is defined by the symmetric matrix

$$\nu_0 \mathbf{u}_0 + \nu_1 \mathbf{u}_1. \quad (4.32)$$

**Remark 4.4.18.** Notice that the points  $(\lambda_0 : \lambda_1)$ ,  $(\mu_0 : \mu_1)$ , and  $(\nu_0 : \nu_1)$  in  $\mathbb{P}_{\mathbb{R}}^1$  are easy to compute. Namely, to compute  $(\lambda_0 : \lambda_1)$  and  $(\mu_0 : \mu_1)$  we can just choose a general point  $P$  in  $\Pi$  and respectively solve the linear equations

$$\lambda_0 \boldsymbol{\sigma}_0(P) + \lambda_1 \boldsymbol{\sigma}_1(P) = 0 \quad , \quad \mu_0 \boldsymbol{\tau}_0(P) + \mu_1 \boldsymbol{\tau}_1(P) = 0 \quad .$$

Similarly, to compute  $(\nu_0 : \nu_1)$  we can take a general  $P$  in  $\Theta$ , and solve

$$\nu_0 \boldsymbol{u}_0(P) + \nu_1 \boldsymbol{u}_1(P) = 0 \quad .$$

**Theorem 4.4.19.** Let  $\phi$  be pyramidal, with weights as in Corollary 4.4.8. With the notation of Corollaries 4.4.11 and 4.4.14, the inverse rational map  $\phi^{-1}$  is given by

$$(\alpha_1 \lambda_1 \boldsymbol{\sigma}_1(\mathbf{y}) : \alpha_0 \lambda_0 \boldsymbol{\sigma}_0(\mathbf{y})) \times (\beta_1 \mu_1 \boldsymbol{\tau}_1(\mathbf{y}) : \beta_0 \mu_0 \boldsymbol{\tau}_0(\mathbf{y})) \times (\gamma_1 \nu_1 \boldsymbol{u}_1(\mathbf{y}) : \gamma_0 \nu_0 \boldsymbol{u}_0(\mathbf{y}))$$

Moreover, the base locus of  $\phi$  is defined by the ideal

$$B = (a, b, e) \cap (c, h) \quad , \quad (4.33)$$

and the base locus of  $\phi^{-1}$  is the union of the lines  $s, t$  and the plane conic  $C \subset \Theta$ . Additionally, we have the following:

- (i)  $\phi$  blows up the base point  $(a, b, e)$  to the plane  $\Pi$
- (ii) If  $h$  is irreducible,  $\phi$  blows up the base curve  $(c, h)$  to the cone through  $C$  of vertex  $\mathbf{V}$

*Proof.* First, we deduce the inverse for the parameters  $(s_0 : s_1)$  and  $(t_0 : t_1)$ . We have

$$\begin{pmatrix} w_{1jk} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} \rangle \\ w_{0jk} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle \end{pmatrix}_{0 \leq j, k \leq 1} = \begin{pmatrix} \lambda_0^{-1} w_{1jk} \langle \boldsymbol{\pi}, \mathbf{P}_{1jk} \rangle \\ -\lambda_1^{-1} w_{0jk} \langle \boldsymbol{\pi}, \mathbf{P}_{0jk} \rangle \end{pmatrix}_{0 \leq j, k \leq 1} = \begin{pmatrix} \lambda_0^{-1} \alpha_1 \beta_j \gamma_k \\ -\lambda_1^{-1} \alpha_0 \beta_j \gamma_k \end{pmatrix}_{0 \leq j, k \leq 1} \quad ,$$

and by Lemma 4.2.1, it follows that

$$\boldsymbol{\sigma}(s_0, s_1; \mathbf{y}) = \alpha_0 \lambda_0 \boldsymbol{\sigma}_0(\mathbf{y}) s_0 - \alpha_1 \lambda_1 \boldsymbol{\sigma}_1(\mathbf{y}) s_1 = \begin{vmatrix} s_0 & s_1 \\ \alpha_1 \lambda_1 \boldsymbol{\sigma}_1(\mathbf{y}) & \alpha_0 \lambda_0 \boldsymbol{\sigma}_0(\mathbf{y}) \end{vmatrix} \quad (4.34)$$

is syzygy of  $\mathbf{f}$  of degree  $(1, 0, 0)$ , which yields the inverse for  $(s_0 : s_1)$ . Similarly, it follows that

$$\boldsymbol{\tau}(t_0, t_1; \mathbf{y}) = \beta_0 \mu_0 \boldsymbol{\tau}_0(\mathbf{y}) t_0 - \beta_1 \mu_1 \boldsymbol{\tau}_1(\mathbf{y}) t_1 = \begin{vmatrix} t_0 & t_1 \\ \beta_1 \mu_1 \boldsymbol{\tau}_1(\mathbf{y}) & \beta_0 \mu_0 \boldsymbol{\tau}_0(\mathbf{y}) \end{vmatrix} \quad (4.35)$$

is also a syzygy of  $\mathbf{f}$  of degree  $(0, 1, 0)$ , which yields the inverse for  $(t_0 : t_1)$ . On the other hand, the relation of multidegree  $(0, 0, 1; 2)$

$$\boldsymbol{u}(u_0, u_1; \mathbf{y}) = \begin{vmatrix} u_0 & u_1 \\ \gamma_1 \nu_1 \boldsymbol{u}_1(\mathbf{y}) & \gamma_0 \nu_0 \boldsymbol{u}_0(\mathbf{y}) \end{vmatrix} \quad (4.36)$$

vanishes after the specialization  $\mathbf{y} \mapsto \mathbf{f}$ , and hence lies in the Rees ideal of  $\phi$ . Specifically, by definition  $\boldsymbol{u}(u_0, u_1; \mathbf{f})$  vanishes for

$$(u_0 : u_1) = (1 : 0) \quad , \quad (u_0 : u_1) = (0 : 1) \quad . \quad (4.37)$$

Additionally, if  $e(u_0, u_1) = 0$ , or equivalently if  $(u_0 : u_1) = (-\gamma_1 \epsilon_1 : \gamma_0 \epsilon_0)$ , we find

$$\begin{vmatrix} -\gamma_1 \epsilon_1 & \gamma_0 \epsilon_0 \\ \gamma_1 \epsilon_1 \nu_1 \mathbf{v}_1(\mathbf{y}) & \gamma_0 \epsilon_0 \nu_0 \mathbf{v}_0(\mathbf{y}) \end{vmatrix} = -\gamma_0 \gamma_1 \epsilon_0 \epsilon_1 (\nu_0 \mathbf{v}_0(\mathbf{y}) + \nu_1 \mathbf{v}_1(\mathbf{y})) ,$$

which defines the rank-two quadric  $\Pi \cup \Theta$  (recall Notation 4.4.17). Moreover, as the image of  $e(u_0, u_1) = 0$  lies in  $\Theta$ , the specialization  $\mathbf{v}(u_0, u_1; \mathbf{f})$  vanishes at  $(u_0 : u_1) = (-\gamma_1 \epsilon_1 : \gamma_0 \epsilon_0)$ . On the other hand, the pullback  $\mathbf{v}(u_0, u_1; \mathbf{f})$  is either zero or a polynomial of degree  $(2, 2, 3)$  in  $R$ . Moreover, it can be regarded as polynomial of degree  $(0, 0, 2)$ , since by the contractions of  $\phi$  described in Corollaries 4.4.11 and 4.4.14 the pullback polynomial  $\mathbf{v}_k(\mathbf{f})$  is proportional to the product

$$u_k^* a(s_0, s_1) b(t_0, t_1) c(u_0, u_1) h(s_0, s_1, t_0, t_1) ,$$

as  $Y_0 \cap Y_1 = s \cup t \cup C$ . Nevertheless, this polynomial of degree  $(0, 0, 2)$  admits the two roots in (4.37), as well as  $(u_0 : u_1) = (-\gamma_1 \epsilon_1 : \gamma_0 \epsilon_0)$ . Therefore,  $\mathbf{v}(u_0, u_1; \mathbf{f})$  must be identically zero.

Regarding the base locus of  $\phi$ , the vector space  $V = B_{(1,1,1)}$  contains the polynomials

$$s_0 b c , s_1 b c , a t_1 c , a t_0 c , e h ,$$

which are respectively proportional to the polynomials in  $\mathbb{C}\langle \mathbf{f} \rangle$  given by

$$\langle \sigma_1, \mathbf{f} \rangle , \langle \sigma_0, \mathbf{f} \rangle , \langle \tau_0, \mathbf{f} \rangle , \langle \tau_1, \mathbf{f} \rangle , \langle \theta, \mathbf{f} \rangle .$$

Additionally we have  $\dim_{\mathbb{C}} V = 4$ , implying that  $V = \mathbb{C}\langle \mathbf{f} \rangle$ . On the other hand, by Corollaries 4.4.11 and 4.4.14 it follows that the base locus of  $\phi^{-1}$ , defined by

$$(\sigma_0(\mathbf{y}), \sigma_1(\mathbf{y})) \cap (\tau_0(\mathbf{y}), \tau_1(\mathbf{y})) \cap (\nu_0(\mathbf{y}), \nu_1(\mathbf{y})) = (\nu_0(\mathbf{y}), \nu_1(\mathbf{y})) ,$$

is  $s \cup t \cup C$ . Finally, since  $(a, b, e)$  is prime its pullback by  $\phi^{-1}$  is a principal ideal  $\mathbb{C}\langle \mathbf{y} \rangle$ . In particular,  $a(\mathbf{f})$  (resp.  $b(\mathbf{f})$ ) define the plane  $\Pi$ . Hence,  $\phi$  blows up the base point  $(a, b, e)$  to  $\Pi$ . On the other hand, if  $h$  is irreducible, the ideal  $(c, h)$  is also prime. Hence, the pullback  $J$  by  $\phi^{-1}$  of  $(c, h)$  is principal. Additionally, since the point  $\mathbf{V}$  is a singular point in  $Y_0 \cap Y_1 = s \cup t \cup C$  and  $\phi$  contracts  $c = 0$  to  $\mathbf{V}$ , it follows that  $J$  defines a quadric with a singular point at  $\mathbf{V}$ . Therefore,  $J$  defines the cone through  $C$  of vertex  $\mathbf{V}$ .  $\square$

In the following example, we explicitly compute the inverse of a pyramidal birational map with the same control points as in Example 4.4.15. Moreover, we provide the defining equations of the irreducible components of the base locus, and the blown-up surfaces.

**Example 4.4.20** (Inverse of a pyramidal birational map). Continuing with Example 4.4.15, using Remark 4.4.18 we compute

$$(\lambda_0 : \lambda_1) = (1 : -1) , \quad (\mu_0 : \mu_1) = (1 : -1) , \quad (\nu_0 : \nu_1) = (1 : 1) .$$

If we consider weights as in Corollary 4.4.7, defined by the “easy” choice of

$$(\alpha_0 : \alpha_1) \times (\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1) = (1 : 1) \times (1 : 1) \times (1 : 1) ,$$

the inverse rational map  $\phi^{-1}$  is defined by

$$\mathbf{y} \mapsto (-\sigma_1(\mathbf{y}) : \sigma_0(\mathbf{y})) \times (-\tau_1(\mathbf{y}) : \tau_0(\mathbf{y})) \times (12 \nu_1(\mathbf{y}) : \nu_0(\mathbf{y})) .$$

In particular, the base ideal  $B$  of  $\phi$  is

$$(a, b, e) \cap (c, h) = (s_0 + s_1, t_0 + t_1, u_0 + 12u_1) \cap (u_0 + u_1, 16s_0t_0 + 27s_1t_0 + 27s_0t_1 + 16s_1t_1)$$

and furthermore:

- (i)  $\phi$  contracts  $(s_0 : s_1) = (-1 : 1)$  to the line  $t$
- (ii)  $\phi$  contracts  $(t_0 : t_1) = (-1 : 1)$  to the line  $s$
- (iii)  $\phi$  contracts  $(u_0 : u_1) = (-1 : 1)$  to the point  $\mathbf{V}$
- (iv)  $\phi$  restricts to a (plane) bilinear birational map  $\phi : (\mathbb{P}_{\mathbb{R}}^1)^2 \dashrightarrow \Theta$  at  $(u_0 : u_1) = (-12 : 1)$
- (v)  $\phi$  contracts  $h = 0$  to a plane conic  $C \subset \Theta$

Additionally,  $\phi$  blows up the base point  $(a, b, e)$  to the plane  $\Pi$ , and since  $h(\mathbf{s}, \mathbf{t}) = 16s_0t_0 + 27s_1t_0 + 27s_0t_1 + 16s_1t_1$  is irreducible, it also blows up the base curve defined by  $(c, h)$  to the cone defined by the quadratic equation  $h(\mathbf{g}_1, \mathbf{g}_2)$ , which is proportional to

$$44032y_0^2 - 187y_1^2 - 3190y_1y_2 - 187y_2^2 - 22016y_0y_3 + 2752y_3^2.$$

**Remark 4.4.21** (Polynomial pyramidal birational maps). Similarly to the case of hexahedral birational maps, the following corollary is straightforward.

**Corollary 4.4.22.** *Let  $\phi$  be pyramidal, and  $w_{ijk} = 1$  for every  $0 \leq i, j, k \leq 1$ . If the pairs  $\Sigma_0, \Sigma_1$  and  $T_0, T_1$  define parallel planes, then  $\phi$  is birational of type  $(1, 1, 2)$ .*

*Proof.* By the proof of Corollary 4.3.17, each of the rank conditions (4.6), (4.9) are satisfied if the corresponding pair of boundary surfaces are parallel planes. By Theorem 3.5.1,  $\phi$  is birational of type either  $(1, 1, 2)$ ,  $(1, 2, 1)$ , or  $(2, 1, 1)$  if and only if exactly two of these rank conditions hold.  $\square$

## 4.5. Scaffold birational maps

We now study the third family of birational maps: the class of scaffold birational maps. Again, this name comes from the resemblance of the geometry of trilinear birational maps of type  $(1, 2, 2)$ ,  $(2, 1, 2)$  and  $(2, 2, 1)$ . A ‘‘scaffold’’ typically consists of four lines  $s_{00}, s_{10}, s_{01}, s_{11}$  that support a sequence of planar platforms, like the  $s$ -parametric planes.

**Definition 4.5.1** (Scaffold rational map). A trilinear rational map is scaffold if it satisfies Property 1 and all the boundary surfaces are quadrics except for one of the three parameters, for which:

- (i) The two boundary surfaces are planes, that intersect at a line  $\ell$
- (ii) The four boundary lines intersect two lines  $r_0, r_1$ , and these two intersect  $\ell$

**Remark 4.5.2.** In Definition 4.5.1, we require that two pairs of boundary surfaces are smooth quadrics to exclude the possible types  $(1, 1, 2)$ ,  $(1, 2, 1)$ ,  $(2, 1, 1)$ , and  $(1, 1, 1)$ .

### 4.5.1. Construction of control points

As for pyramidal maps, we begin discussing the necessary constraints on the control points for birationality. Namely, a general birational map of type either  $(1, 2, 2)$ ,  $(2, 1, 2)$ , or  $(2, 2, 1)$  must be scaffold.

**Proposition 4.5.3.** *If  $\phi$  is a general birational map of type  $(1, 2, 2)$ , then it is scaffold. Moreover,*

$$T_0 \cap T_1 = s \cup x_1 \cup x_2 \cup y, \quad Y_0 \cap Y_1 = s \cup x_1 \cup x_2 \cup z,$$

where:

- (i)  $x_1, x_2$  are the unique two lines that meet  $s_{jk}$  for every  $0 \leq j, k \leq 1$  (Lemma 2.2.24)
- (ii)  $y$  is the line through the points  $u_{00} \cap u_{01}$  and  $u_{10} \cap u_{11}$
- (iii)  $z$  is the line through the points  $t_{00} \cap t_{01}$  and  $t_{10} \cap t_{11}$

*Proof.* We start proving that  $\phi$  is scaffold. By Theorem 3.4.8,  $\phi$  has two base points defined by the ideals

$$(a_1(\mathbf{s}), b_1(\mathbf{t}), c_1(\mathbf{u})), \quad (a_2(\mathbf{s}), b_2(\mathbf{t}), c_2(\mathbf{u})) \quad (4.38)$$

for some linear  $a_r(\mathbf{s}), b_r(\mathbf{t}), c_r(\mathbf{u})$  in  $R$ , for each  $r = 1, 2$ . In particular, the pullback ideals by  $\phi^{-1}$

$$(a_1(\mathbf{g}_1), b_1(\mathbf{g}_2), c_1(\mathbf{g}_3)), \quad (a_2(\mathbf{g}_1), b_2(\mathbf{g}_2), c_2(\mathbf{g}_3)) \quad (4.39)$$

are principal in  $\mathbb{C}[\mathbf{y}]$ , for similar considerations as in the proof of Proposition 4.4.3. Therefore, we can write

$$b_r(\mathbf{g}_2) = a_r(\mathbf{g}_1) F_r(\mathbf{y}), \quad c_r(\mathbf{g}_3) = a_r(\mathbf{g}_1) G_r(\mathbf{y})$$

for some linear forms  $F_r(\mathbf{y}), G_r(\mathbf{y})$  in  $\mathbb{C}[\mathbf{y}]$ . Now, we look at the base loci of the pencils of  $t$ - and  $u$ -surfaces, respectively  $Z_t$  and  $Z_u$ . These are defined by the ideals in  $\mathbb{C}[\mathbf{y}]$

$$\begin{aligned} B_t = (\mathbf{g}_2) &= (g_{02}, g_{12}) = (b_1(\mathbf{g}_2), b_2(\mathbf{g}_2)) = (a_1(\mathbf{g}_1) F_1(\mathbf{y}), a_2(\mathbf{g}_1) F_2(\mathbf{y})) = \\ &= (a_1(\mathbf{g}_1), a_2(\mathbf{g}_1)) \cap (a_1(\mathbf{g}_1), F_2(\mathbf{y})) \cap (F_1(\mathbf{y}), a_2(\mathbf{g}_1)) \cap (F_1(\mathbf{y}), F_2(\mathbf{y})) \end{aligned}$$

and

$$\begin{aligned} B_u = (\mathbf{g}_3) &= (g_{03}, g_{13}) = (c_1(\mathbf{g}_3), c_2(\mathbf{g}_3)) = (a_1(\mathbf{g}_1) G_1(\mathbf{y}), a_2(\mathbf{g}_1) G_2(\mathbf{y})) = \\ &= (a_1(\mathbf{g}_1), a_2(\mathbf{g}_1)) \cap (a_1(\mathbf{g}_1), G_2(\mathbf{y})) \cap (G_1(\mathbf{y}), a_2(\mathbf{g}_1)) \cap (G_1(\mathbf{y}), G_2(\mathbf{y})). \end{aligned}$$

In particular, these base loci are the unions of four lines. Since general quadrics in each of these pencils are smooth, and therefore doubly ruled (Lemma 2.2.22), for each of the decompositions above we find two lines in each of the rulings. Specifically, if three lines belong to the same ruling, then they mutually do not intersect. By Lemma 2.2.23, all the  $t$ - and  $u$ -quadrics would be the unique quadric through these three lines. Moreover, the ideal

$$(a_1(\mathbf{g}_1), a_2(\mathbf{g}_1)) = (g_{01}, g_{11})$$

defines the line  $s$ . In particular, the line  $\ell$  defined by  $(F_1, F_2)$  lies in the same ruling of  $s$  for a general  $t$ -surface, since by definition  $s$  intersects the other two lines.

On the other hand, the line  $u_{ij}$  lies in  $T_j$  for each  $0 \leq i, j \leq 1$ . Moreover, it intersects  $s$  since  $u_{ij} \subset \Sigma_i$  and this is a plane. Hence,  $s$  and  $u_{ij}$  must belong to different rulings in  $T_j$ , implying that  $\ell$  intersects  $u_{ij}$  as well. Since  $u_{i0}, u_{i1}$  lie in  $\Sigma_i$  and  $\ell$  does not,  $\ell$  must meet the point  $u_{i0} \cap u_{i1}$  for each  $i = 0, 1$ . Therefore, it follows that  $\ell = y$ .

We conclude proving that  $x_1, x_2$  are the remaining two lines in  $Z_t$ . Since the line  $s_{jk}$  does not meet  $s$  for any  $0 \leq j, k \leq 1$ , it lies in the same ruling as  $s$  in  $T_j$ . Hence, the lines defined by the ideals  $(a_1(\mathbf{g}_1), F_2)$  and  $(F_1, a_2(\mathbf{g}_1))$  must intersect the four  $s_{jk}$ 's, and they are  $x_1$  and  $x_2$ .

With a similar argument, it follows that  $(G_1, G_2)$  defines the line  $z$ , and the ideals  $(a_1(\mathbf{g}_1), G_1)$  and  $(G_2, a_2(\mathbf{g}_1))$  define  $x_1$  and  $x_2$ .  $\square$

Proposition 4.5.3 combined with Theorem 4.5.7, presented and proven in §4.5.2, provides an answer to Question 4.1.1 for trilinear birational maps of type  $(1, 2, 2)$ . Specifically, we find the following corollary.

**Corollary 4.5.4.** *The set*

$$\mathcal{P}_{(1,2,2)} = \{ \{ \mathbf{P}_{ijk} \}_{0 \leq i,j,k \leq 1} : \phi \text{ is scaffold} \}$$

answer Question 4.1.1. More explicitly, for every net of control points in  $\mathcal{P}_{(1,2,2)}$  we can compute weights that render  $\phi$  birational of type  $(1, 2, 2)$ .

In practice, nets of control points in  $\mathcal{P}_{(1,2,2)}$  can be generated using the following construction. The generation of a net of control points using Construction 3 is illustrated in Figure 4.6.

**Construction 3** (Control points for scaffold rational maps). A net of control points in  $\mathcal{P}_{(1,2,2)}$  can be generated as follows:

1. Choose a line  $s$  in  $\mathbb{P}_{\mathbb{R}}^3$  (red line in Figure 4.6). Lines at  $x_0 = 0$  are acceptable, meaning that the boundary planes  $\Sigma_0, \Sigma_1$  are parallel
2. Choose two skew lines  $x_1, x_2$  in  $\mathbb{P}_{\mathbb{R}}^3$  intersecting  $s$  (black lines in Figure 4.6)
3. For each  $0 \leq j, k \leq 1$ , choose an affine line  $s_{jk}$  intersecting both  $x_1, x_2$  (green lines in Figure 4.6)
4. Choose two planes  $\Sigma_0, \Sigma_1$  containing  $s$
5. For each  $0 \leq i, j, k \leq 1$ , define  $\mathbf{P}_{ijk} = \Sigma_i \cap s_{jk}$  (green points in Figure 4.6)

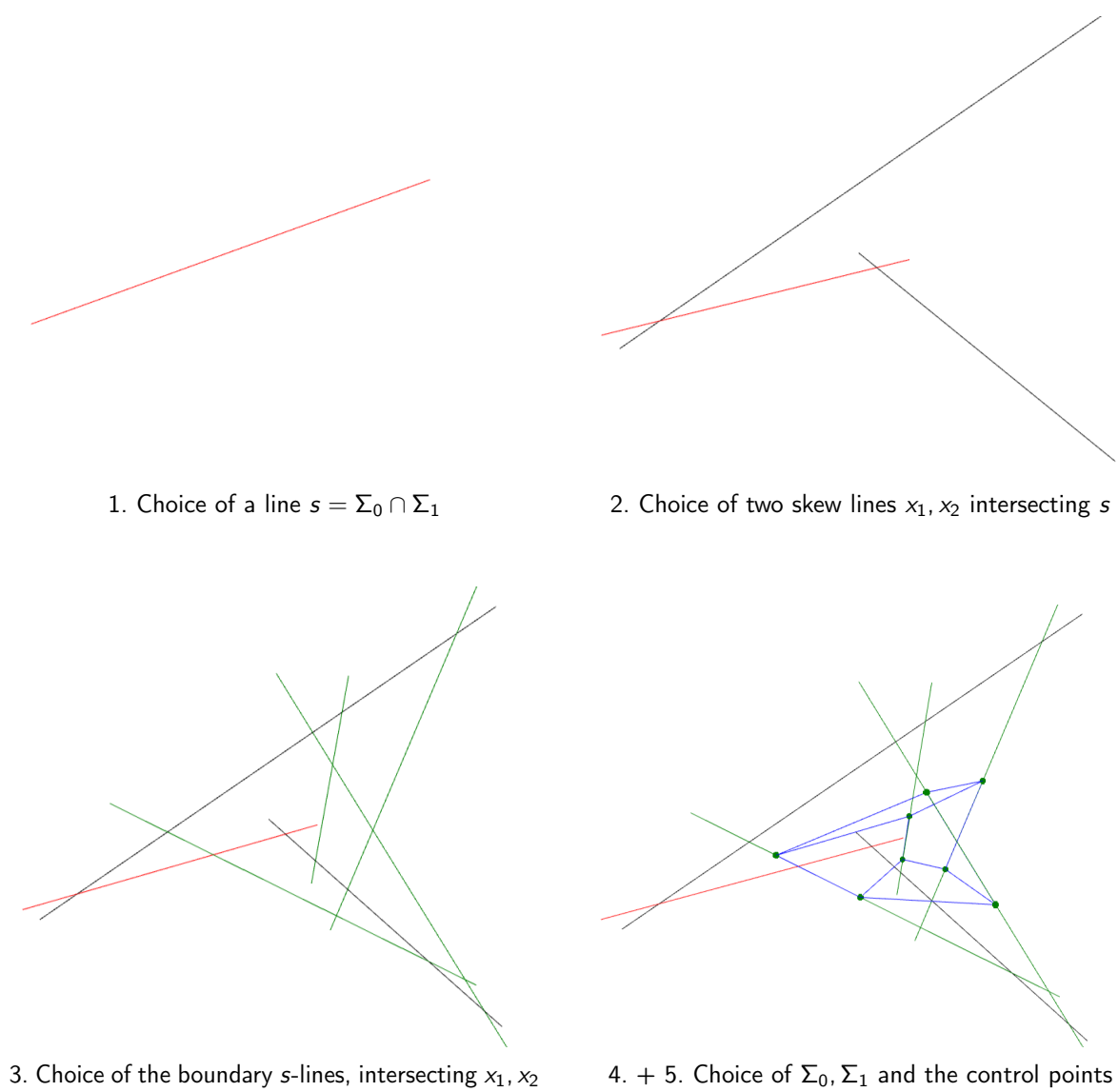
#### 4.5.2. Effective computation of weights

As in the previous sections, here we derive a rank-based birationality criterion for scaffold rational maps. Without loss of generality, we assume that  $\phi$  has type  $(1, 2, 2)$ , and refer to “scaffold rational maps” instead of “scaffold rational maps of type  $(1, 2, 2)$ ”.

In order to state our birationality criterion, we introduce some necessary notation.

**Notation 4.5.5.** Let  $\phi$  be scaffold.

- (i) For each  $i = 0, 1$ , the boundary plane  $\Sigma_i$  is defined by the vector  $\boldsymbol{\sigma}_i$  in  $\mathbb{R}^4$ , and  $s = \Sigma_0 \cap \Sigma_1$
- (ii) For each  $0 \leq j, k \leq 1$ , the boundary quadrics  $T_j, Y_k$  are defined by the symmetric matrices  $\boldsymbol{\tau}_j$  and  $\boldsymbol{\nu}_k$  in  $\mathbb{R}^{4 \times 4}$ .



**Figure 4.6.:** Generation of the control points of a scaffold map, using Construction 3.



- (iii) For each  $r = 1, 2$ ,  $\Sigma_{(r)}$  is the plane through  $s$  and  $x_r$ , defined by a vector  $\sigma_{(r)}$  in  $\mathbb{R}^4$ . In particular, we find  $(\lambda_{0(r)} : \lambda_{1(r)})$  in  $\mathbb{P}_{\mathbb{C}}^1$  such that

$$\sigma_{(r)} = \lambda_{0(r)} \sigma_0 + \lambda_{1(r)} \sigma_1 .$$

- (iv) As in Proposition 4.5.3,  $y$  is the line through the points  $u_{00} \cap u_{01}$  and  $u_{00} \cap u_{10}$
- (v) As in Proposition 4.5.3,  $z$  is the line through the points  $t_{00} \cap t_{01}$  and  $t_{00} \cap t_{10}$
- (vi) We denote by  $Q$  be the unique smooth quadric through the lines  $s, x, y$  (recall Lemma 2.2.23). Since by Propoposition 4.5.3 the base loci of the  $t$ - and  $u$ -surfaces are respectively the unions of lines  $s \cup x_1 \cup x_2 \cup y$  and  $s \cup x_1 \cup x_2 \cup z$ , it follows that  $Q$  lies in the pencils of both a  $t$ - and a  $u$ -surfaces. Therefore, we find points  $(\mu_0 : \mu_1)$  and  $(\nu_0 : \nu_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$  such that the symmetric matrices

$$\mu_0 \tau_0 + \mu_1 \tau_1 \quad , \quad \nu_0 \mathbf{u}_0 + \nu_1 \mathbf{u}_1$$

define  $Q$ .

- (vii) For each  $r = 1, 2$ ,  $B_r$  is the plane through  $y$  and  $x_r$ , defined by a vector  $\beta_r$  in  $\mathbb{R}^4$ . Additionally, for each  $0 \leq i, j, k \leq 1$  we define

$$\Delta_{ijk}(r, y) = \frac{1}{\langle \beta_r, \mathbf{P}_{ijk} \rangle} ,$$

as well as the tensor of format  $2 \times 2 \times 2$

$$W(r, y) = \left( \frac{w_{ijk}}{\Delta_{ijk}(r, y)} \right)_{0 \leq i, j, k \leq 1} . \quad (4.40)$$

- (viii) For each  $r = 1, 2$ ,  $\Gamma_r$  is the plane through  $z$  and  $x_r$ , defined by a vector  $\gamma_r$  in  $\mathbb{R}^4$ . Additionally, for each  $0 \leq i, j, k \leq 1$  we define

$$\Delta_{ijk}(r, z) = \frac{1}{\langle \gamma_r, \mathbf{P}_{ijk} \rangle} ,$$

as well as the tensor of format  $2 \times 2 \times 2$

$$W(r, z) = \left( \frac{w_{ijk}}{\Delta_{ijk}(r, z)} \right)_{0 \leq i, j, k \leq 1} . \quad (4.41)$$

**Remark 4.5.6** (Nonzero and positive  $\Delta_{ijk}$ ). Notice that  $\Delta_{ijk}(r, y)$  and  $\Delta_{ijk}(r, z)$  are not defined if and only if  $\mathbf{P}_{ijk}$  lies on  $x_r$ . Nevertheless, Property 1 excludes this possibility. Specifically, if  $\mathbf{P}_{ijk}$  lies on  $x_r$ , then  $\Sigma_i = \Sigma_{(r)}$ . Since  $\Sigma_i$  contains all the control points  $\mathbf{P}_{i00}$ ,  $\mathbf{P}_{i10}$ ,  $\mathbf{P}_{i01}$ , and  $\mathbf{P}_{i11}$ , it follows that all these lie on  $x_r$ . Therefore,  $\Sigma_i$  is not a surface but the line  $x_r$ .

Furthermore, if all the control points are left at "one side of  $B_r$  and  $\Gamma_r$ ", i.e. if they all respectively lie in one of the two halfspaces defined by each plane in  $\mathbb{A}_{\mathbb{R}}^3$ , then the vectors  $\beta_r$  and  $\gamma_r$  can be chosen so that  $\Delta_{ijk} > 0$  for every  $0 \leq i, j, k \leq 1$ .

As in the former sections, we now prove a characterization of birationality for scaffold rational maps, relying on tensor rank.

**Theorem 4.5.7.** *Let  $\phi$  be scaffold. The following are equivalent:*

- (i)  $\phi$  is birational of type  $(1, 2, 2)$
- (ii) One of the tensors  $W(1, y)$ ,  $W(2, y)$ ,  $W(1, z)$ , and  $W(2, z)$  has rank one
- (iii) All the tensors  $W(1, y)$ ,  $W(2, y)$ ,  $W(1, z)$ , and  $W(2, z)$  have rank one
- (iv) All the trilinear polynomials  $\langle \boldsymbol{\beta}_1, \mathbf{f} \rangle$ ,  $\langle \boldsymbol{\beta}_2, \mathbf{f} \rangle$ ,  $\langle \boldsymbol{\gamma}_1, \mathbf{f} \rangle$ ,  $\langle \boldsymbol{\gamma}_2, \mathbf{f} \rangle$  have three linear factors

*Proof.* It is straightforward that (iii) and (iv) are equivalent, since for each  $r = 1, 2$  we have

$$\begin{aligned} \langle \boldsymbol{\beta}_r, \mathbf{f} \rangle &= \sum_{0 \leq i, j, k \leq 1} w_{ijk} \langle \boldsymbol{\beta}_r, \mathbf{P}_{ijk} \rangle s_i t_j u_k = \sum_{0 \leq i, j, k \leq 1} \frac{w_{ijk}}{\Delta_{ijk}(r, y)} s_i t_j u_k, \\ \langle \boldsymbol{\gamma}_r, \mathbf{f} \rangle &= \sum_{0 \leq i, j, k \leq 1} w_{ijk} \langle \boldsymbol{\gamma}_r, \mathbf{P}_{ijk} \rangle s_i t_j u_k = \sum_{0 \leq i, j, k \leq 1} \frac{w_{ijk}}{\Delta_{ijk}(r, z)} s_i t_j u_k. \end{aligned}$$

Therefore, it is sufficient to prove the equivalence between (i), (ii), and (iii). First, we prove that (ii) implies (i). Without loss of generality, let us assume that

$$W(1, y) = (\alpha_{i(1)} \beta_{j(1)} \gamma_k)_{0 \leq i, j, k \leq 1}$$

for some  $(\alpha_{0(1)} : \alpha_{1(1)}) \times (\beta_{0(1)} : \beta_{1(1)}) \times (\gamma_0 : \gamma_1)$  in  $(\mathbb{P}_{\mathbb{C}}^1)^3$ , or equivalently

$$\langle \boldsymbol{\beta}_1, \mathbf{f} \rangle = a_1(s_0, s_1) \cdot b_1(t_0, t_1) \cdot c(u_0, u_1),$$

where

$$a_1(s_0, s_1) = \begin{vmatrix} s_0 & s_1 \\ -\alpha_{1(1)} & \alpha_{0(1)} \end{vmatrix}, \quad b_1(t_0, t_1) = \begin{vmatrix} t_0 & t_1 \\ -\beta_{1(1)} & \beta_{0(1)} \end{vmatrix}, \quad c(u_0, u_1) = \begin{vmatrix} u_0 & u_1 \\ -\gamma_1 & \gamma_0 \end{vmatrix}.$$

In particular, the images by  $\phi$  of the surfaces respectively defined by

$$(s_0 : s_1) = (-\alpha_{1(1)} : \alpha_{0(1)}) \quad , \quad (u_0 : u_1) = (-\gamma_1 : \gamma_0)$$

lie in the plane  $B_1$ . We make the following two observations:

- (i) Since  $\phi$  is scaffold, all the  $s$ -boundary lines meet the plane  $B_1$  at the line  $x_1$ . In particular, the specialization of  $\mathbf{A}_{jk}$  at  $(s_0 : s_1) = (-\alpha_{1(1)} : \alpha_{0(1)})$  yields a point in  $x_1$  for each  $0 \leq j, k \leq 1$ . Therefore,  $\phi$  contracts the surface  $(s_0 : s_1) = (-\alpha_{1(1)} : \alpha_{0(1)})$  to  $x_1$ . In particular, the pullback ideal of  $x_1$  is principal, and  $a_1$  divides its generator.
- (ii) Similarly, all the  $u$ -boundary lines meet the plane  $B_1$  at the line  $y$ . In particular, the specialization of  $\mathbf{C}_{ij}$  at  $(u_0 : u_1) = (-\gamma_1 : \gamma_0)$  yields a point in  $y$  for each  $0 \leq i, j \leq 1$ . Therefore,  $\phi$  contracts the surface  $(u_0 : u_1) = (-\gamma_1 : \gamma_0)$  to  $y$ . In particular, the pullback ideal of  $y$  is principal, and  $c$  divides its generator.

On the other hand, for each  $0 \leq j, k \leq 1$  the isomorphism of projective lines  $\mathbb{P}_{\mathbb{C}}^1 \rightarrow s_{jk}$  defined by  $\mathbf{A}_{jk}$  is completely determined by the image of the three points

$$(1 : 0) \mapsto \mathbf{P}_{0jk} \quad , \quad (0 : 1) \mapsto \mathbf{P}_{1jk} \quad , \quad (-\alpha_{1(1)} : \alpha_{0(1)}) \mapsto s_{jk} \cap x_1.$$

Additionally, the line isomorphism  $\sigma : \mathbb{P}_{\mathbb{C}}^1 \rightarrow (\mathbb{P}_{\mathbb{C}}^3)^{\vee}$  defined by

$$(1 : 0) \mapsto \sigma_0, \quad (0 : 1) \mapsto \sigma_1, \quad (-\alpha_{1(1)} : \alpha_{0(1)}) \mapsto \sigma_{(1)}$$

yields a syzygy of degree  $(1, 0, 0)$  of  $\mathbf{P}_{ij}$ , since  $\langle \sigma_0, \mathbf{P}_{0jk} \rangle = \langle \sigma_1, \mathbf{P}_{1jk} \rangle = 0$  and  $\Sigma_{(1)}$  contains  $x_1$ . Thus,  $\sigma$  also defines a syzygy of degree  $(1, 0, 0)$  of  $\mathbf{f}$ . In particular, as the line  $x_2$  intersects  $s$ , there is a point  $(\alpha_{0(2)} : \alpha_{1(2)})$  in  $\mathbb{P}_{\mathbb{C}}^1$  for which  $\sigma(-\alpha_{1(2)}, \alpha_{0(2)}) = \sigma_{(2)}$ . For the same observation as before,  $\phi$  contracts the surface  $(s_0 : s_1) = (-\alpha_{1(2)} : \alpha_{0(2)})$  to  $x_2$ . Hence, since  $B_2$  contains the two lines  $y$  and  $x_2$ , we can write

$$\langle \beta_2, \mathbf{f} \rangle = a_2(s_0, s_1) \cdot b_2(t_0, t_1) \cdot c(u_0, u_1)$$

for

$$a_2(s_0, s_1) = \begin{vmatrix} s_0 & s_1 \\ -\alpha_{1(2)} & \alpha_{0(2)} \end{vmatrix}, \quad b_2(t_0, t_1) = \begin{vmatrix} t_0 & t_1 \\ -\beta_{1(2)} & \beta_{0(2)} \end{vmatrix}$$

and some  $(\beta_{0(2)} : \beta_{1(2)})$  in  $\mathbb{P}_{\mathbb{C}}^1$ . Therefore, by definition

$$\beta = \beta(s_0, s_1, t_0, t_1) = a_2(s_0, s_1) b_2(t_0, t_1) \beta_1 - a_1(s_0, s_1) b_1(t_0, t_1) \beta_2$$

is a syzygy of  $\mathbf{f}$  of degree  $(1, 1, 0)$ . By hypothesis,  $\mathbf{f}$  does not admit syzygies of degrees  $(0, 1, 0)$  and  $(0, 0, 1)$  since  $T_j$  and  $Y_k$  are smooth quadrics, for each  $0 \leq j, k \leq 1$ . Therefore, by Theorem 3.5.1 it follows that  $\phi$  is birational of type  $(1, 2, 2)$ .

We conclude proving that (i) implies (iii), since the implication from (iii) to (ii) is straightforward. If  $\phi$  is birational of type  $(1, 2, 2)$ , it admits a syzygy

$$\sigma(s_0, s_1) = \sigma'_0 s_0 + \sigma'_1 s_1$$

for some  $\sigma'_i$  in  $\mathbb{C}^4$ , such that  $\mathbb{C}\langle \sigma'_0, \sigma'_1 \rangle = \mathbb{C}\langle \sigma_0, \sigma_1 \rangle$  by Lemma 4.2.1. In particular, for each  $r = 1, 2$  we find a point  $(\alpha_{0(r)} : \alpha_{1(r)})$  in  $\mathbb{P}_{\mathbb{C}}^1$  such that the plane defined by  $\sigma(-\alpha_{1(r)}, \alpha_{0(r)})$  contains  $x_r$ . As all the boundary  $s$ -lines intersect  $x_r$ , it follows that  $\phi$  contracts  $(s_0 : s_1) = (-\alpha_{1(r)}, \alpha_{0(r)})$  to  $x_r$ . In particular, the pullback ideal of  $x_r$  is principal, and its generator is divisible by

$$a_r(s_0, s_1) = \begin{vmatrix} s_0 & s_1 \\ -\alpha_{1(r)} & \alpha_{0(r)} \end{vmatrix}.$$

On the other hand, by Proposition 4.5.3 the intersection of the boundary  $t$ -surfaces is the union

$$s \cup x_1 \cup x_2 \cup y,$$

which is defined by the ideal

$$B_t = (\tau_0(\mathbf{y}), \tau_1(\mathbf{y})) \subset \mathbb{C}[\mathbf{y}].$$

Since  $\phi$  is birational with a quadratic inverse for  $(t_0 : t_1)$ , the pullback of  $B_t$  is principal and defined by a polynomial of degree  $(2, 1, 2)$ . Specifically, this polynomial is proportional to the product

$$g(t_0, t_1, u_0, u_1) \cdot a_1(s_0, s_1) \cdot a_2(s_0, s_1) \cdot c(u_0, u_1)$$

where:

- (i)  $g$  has degree  $(0, 1, 1)$  and defines the pullback of  $s$  (Remark 4.2.7)

- (ii)  $a_r$  has degree  $(1, 0, 0)$  and defines the pullback of  $x_r$ , for each  $r = 1, 2$
- (iii)  $c$  has degree  $(0, 0, 1)$  and defines the pullback of  $y$

Therefore, as the plane  $B_r$  contains the lines  $y$  and  $x_r$ , the pullback polynomial  $\langle \beta_r, \mathbf{f} \rangle$  admits the linear factors  $a_r$  and  $c$ , and must have rank one. With an analogous argument, we conclude that the pullback polynomials  $\langle \gamma_1, \mathbf{f} \rangle$  and  $\langle \gamma_2, \mathbf{f} \rangle$  have rank one as well.  $\square$

**Corollary 4.5.8.** *Let  $\phi$  be scaffold. Then,  $\phi$  is birational of type  $(1, 2, 2)$  if and only if one of the identities*

$$w_{ijk} = \alpha_{i(1)} \beta_{j(1)} \gamma_k \Delta_{ijk}(1, y) \delta(1, y) , \quad (4.42)$$

$$w_{ijk} = \alpha_{i(2)} \beta_{j(2)} \gamma_k \Delta_{ijk}(2, y) \delta(2, y) , \quad (4.43)$$

$$w_{ijk} = \alpha_{i(1)} \beta_j \gamma_{k(1)} \Delta_{ijk}(1, z) \delta(1, z) , \quad (4.44)$$

$$w_{ijk} = \alpha_{i(2)} \beta_j \gamma_{k(2)} \Delta_{ijk}(2, z) \delta(2, z) , \quad (4.45)$$

holds for every  $0 \leq i, j, k \leq 1$ , for some

$$(\beta_0 : \beta_1) \times (\gamma_0 : \gamma_1) \in (\mathbb{P}_{\mathbb{C}}^1)^2 , \quad (\alpha_{0(r)} : \alpha_{1(r)}) \times (\beta_{0(r)} : \beta_{1(r)}) \times (\gamma_{0(r)} : \gamma_{1(r)}) \in (\mathbb{P}_{\mathbb{C}}^1)^3$$

and nonzero constants  $\delta(1, y), \delta(2, y), \delta(1, z), \delta(2, z)$ . In particular:

- (i)  $\phi$  contracts  $(s_0 : s_1) = (-\alpha_{1(r)} : \alpha_{0(r)})$  to the line  $x_r$ , for each  $r = 1, 2$
- (ii)  $\phi$  contracts  $(t_0 : t_1) = (-\beta_1 : \beta_0)$  to the line  $z$
- (iii)  $\phi$  contracts  $(u_0 : u_1) = (-\gamma_1 : \gamma_0)$  to the line  $y$

Notice that, by (the proof of) Theorem 4.5.7, if one of the identities (4.42)-(4.45) holds, then the other three hold as well.

**Remark 4.5.9** (Nonzero, real, and positive weights). The same observations as in Remark 4.3.8, combined with Remark 4.5.6, apply.

**Remark 4.5.10.** Extending a bit further on the last sentence of Corollary 4.5.8, it does not follow directly from the statement of Theorem 4.5.7, but rather from its proof.

The reason is the dependence between the rank-one factorizations of the four tensors. For example, the point  $(\alpha_{0(r)} : \alpha_{1(r)})$  appears in the factorizations of  $W_{ijk}(r, y)$  and  $W_{ijk}(r, z)$ , since these tensors encode the pullback polynomials  $\langle \beta_r, \mathbf{f} \rangle$  and  $\langle \gamma_r, \mathbf{f} \rangle$ , and both vanish at the line  $x_r$ .

**Remark 4.5.11** (Deformation of scaffold birational maps). Once again, Construction 3 and Corollary 4.5.8 provide an effective approach to deform scaffold birational maps, in analogy to §4.3.5. Namely, the lines  $s, x_1, x_2$  can be chosen freely, as well as the lines  $s_{jk}$  for each  $0 \leq j, k \leq 1$ , and the boundary planes  $\Sigma_0, \Sigma_1$ . Additionally, Corollary 4.5.8 can be used to preserve birational weights, by recomputing some parameters depending rationally on the control points.

### 4.5.3. Distance to birationality

We now provide a notion of distance to birationality for scaffold rational maps. Notice that this time, by Theorem 4.5.7, the tensor that characterizes birationality is not unique. This is not a problem since Theorem 4.5.7 guarantees that the four possible tensors have rank one simultaneously. Nevertheless, we need to make a choice of one tensor to measure the distance to be birational.

**Definition 4.5.12** (Distance to birationality: scaffold). Let  $\phi$  be pyramidal with real weights, let  $V \subset \mathbb{R}^{2 \times 2 \times 2}$  be the affine cone over  $\sigma(\mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1 \times \mathbb{P}_{\mathbb{R}}^1)$ , and maintain the notation of Theorem 4.4.7. Let either  $r = 1$  or  $r = 2$ . We define the *distance to birationality of  $\phi$  with respect to  $(r, y)$* , denoted by  $\text{dist}_{\text{bir}}(\phi; r, y)$ , as

$$\text{dist}_{\text{bir}}(\phi; r, y) := \min_{P \in V} \frac{\|W(r, y) - P\|}{\|W(r, y)\|}. \quad (4.46)$$

Similarly, we define the *distance to birationality of  $\phi$  with respect to  $(r, z)$* , denoted by  $\text{dist}_{\text{bir}}(\phi; r, z)$ , as

$$\text{dist}_{\text{bir}}(\phi; r, z) := \min_{P \in V} \frac{\|W(r, z) - P\|}{\|W(r, z)\|}. \quad (4.47)$$

**Example 4.5.13** (Computation of a birational approximation: scaffold). We can use Construction 3 to generate a net of scaffold control points. Namely, consider the points

$$\mathbf{S}_1 = (1 \quad -3 \quad -5 \quad 0) \quad , \quad \mathbf{S}_2 = (1 \quad \frac{5}{2} \quad -3 \quad 0)$$

and define the lines  $s = \overline{\mathbf{S}_1 \mathbf{S}_2}$ . Similarly, define the points

$$\mathbf{X}_1 = (1 \quad -3 \quad 1 \quad \frac{36}{7}) \quad , \quad \mathbf{X}_2 = (1 \quad -3 \quad \frac{1}{2} \quad \frac{9}{2})$$

and define the lines  $x_1 = \overline{\mathbf{S}_1 \mathbf{X}_1}$  and  $x_2 = \overline{\mathbf{S}_2 \mathbf{X}_2}$ . Additionally, consider

$$\begin{aligned} \mathbf{S}_{00(1)} &= (1 \quad -3 \quad -4 \quad -\frac{6}{17}) & \mathbf{S}_{00(2)} &= (1 \quad -1 \quad -\frac{17}{22} \quad \frac{63}{22}) \\ \mathbf{S}_{10(1)} &= (1 \quad -3 \quad -\frac{3}{4} \quad -\frac{3}{2}) & \mathbf{S}_{10(2)} &= (1 \quad \frac{23}{18} \quad -\frac{20}{9} \quad 1) \\ \mathbf{S}_{01(1)} &= (1 \quad -3 \quad -3 \quad -\frac{12}{17}) & \mathbf{S}_{01(2)} &= (1 \quad -\frac{5}{2} \quad \frac{2}{11} \quad \frac{45}{11}) \\ \mathbf{S}_{11(1)} &= (1 \quad -3 \quad \frac{3}{5} \quad -\frac{168}{85}) & \mathbf{S}_{11(2)} &= (1 \quad -\frac{9}{14} \quad -1 \quad \frac{18}{7}) \end{aligned}$$

and define  $s_{jk} = \overline{\mathbf{S}_{jk(1)} \mathbf{S}_{jk(2)}}$  for each  $0 \leq j, k \leq 1$ . If the boundary planes  $\Sigma_0, \Sigma_1$  are defined by the vectors

$$\boldsymbol{\sigma}_0 = (0 \quad 0 \quad 0 \quad 1) \quad , \quad \boldsymbol{\sigma}_1 = (43 \quad -4 \quad 11 \quad -43/2)$$

we can define  $\mathbf{P}_{ijk} = \Sigma_i \cap s_{jk}$ . Preserving rational entries at this point yields very large fractions. Because of this, we start working over the reals, which is the most likely scenario for applications. The points

$$\begin{aligned} \mathbf{Y}_1 &= (1 \quad -3.43612 \quad 1.36111 \quad 0) \quad , \quad \mathbf{Y}_2 = (1 \quad -4.11591 \quad 1.44148 \quad 3.42001) \\ \mathbf{Z}_1 &= (1 \quad -3.6532 \quad -4.39411 \quad 0) \quad , \quad \mathbf{Z}_2 = (1 \quad -9.44588 \quad -3.84022 \quad 1.73094) \end{aligned}$$

span the lines  $y = \overline{\mathbf{Y}_1 \mathbf{Y}_2}$  and  $z = \overline{\mathbf{Z}_1 \mathbf{Z}_2}$ . Therefore, the defining vectors of the planes involved in the definition of the  $2 \times 2 \times 2$  determining birationality are

$$\boldsymbol{\beta}_1 = (74.5524 \quad 22.3023 \quad 1.5290 \quad 4.3323), \quad \boldsymbol{\beta}_2 = (24.2113 \quad 15.2903 \quad 20.8123 \quad 2.5007),$$

$$\boldsymbol{\gamma}_1 = (9.7868 \quad 1.1664 \quad 1.2575 \quad 3.5629), \quad \boldsymbol{\gamma}_2 = (42.2475 \quad -2.6839 \quad 11.8459 \quad -12.4938).$$

Suppose that we initialize the weights as  $w_{ijk}^0 = 1$  for every  $0 \leq i, j, k \leq 1$ . Then, the tensor

$$W^0(1, y) = \left( \frac{w_{ijk}^0}{\Delta_{ijk}^0(1, y)} \right)_{0 \leq i, j, k \leq 1}$$

given by

$$W_{\{k=0\}}^0(1, y) = \begin{pmatrix} 6.9649 & 62.3906 \\ 27.6214 & 94.2357 \end{pmatrix}, \quad W_{\{k=1\}}^0(1, y) = \begin{pmatrix} 5.4152 & 30.3459 \\ 19.0501 & 57.8150 \end{pmatrix}$$

does not have rank one. By Theorem 4.5.7, the tensors  $W^0(2, y)$ ,  $W^0(1, z)$ , and  $W^0(2, z)$  do not have rank one as well. Using *TensorFox*, we compute a rank-one approximation of  $W^0(1, y)$  as

$$W(1, y) = \left( \frac{w_{ijk}}{\Delta_{ijk}^1(1, y)} \right)_{0 \leq i, j, k \leq 1} = \delta(1, y) \cdot (\alpha_{0(1)}, \alpha_{1(1)}) \otimes (\beta_{0(1)}, \beta_{1(1)}) \otimes (\gamma_0, \gamma_1) =$$

$$134.5308 \cdot (0.5145, 0.8574) \otimes (0.2467, 0.9690) \otimes (0.8633, 0.5046),$$

which yields the tensor

$$W_{\{k=0\}} = \begin{pmatrix} 14.7430 & 57.9094 \\ 24.5710 & 96.5128 \end{pmatrix}, \quad W_{\{k=1\}} = \begin{pmatrix} 8.6180 & 33.8507 \\ 14.3629 & 56.4161 \end{pmatrix}.$$

In particular, we find the upper bound

$$\text{dist}_{\text{bir}}(\phi; 1, y) \leq \frac{\|W^0(1, y) - W(1, y)\|}{\|W^0(1, y)\|} = 0.0881.$$

Interestingly, when we compute optimal rank-one approximations for the tensors  $W(2, y)$ ,  $W(1, z)$ , and  $W(2, z)$ , we obtain the decompositions

$$W(2, y) \propto (0.9661, 0.2578) \otimes (0.9971, 0.0760) \otimes (0.8632, 0.5047)$$

$$W(1, z) \propto (0.5145, 0.8574) \otimes (0.5257, 0.8506) \otimes (0.6326, 0.7744)$$

$$W(2, z) \propto (0.9664, 0.2568) \otimes (0.5260, 0.8504) \otimes (0.3955, 0.9184),$$

which yield numerically close factors as expected by Corollary 4.5.8, namely coinciding up to the fourth decimal digit. Therefore, we can define

$$(\alpha_{0(2)}, \alpha_{1(2)}) = (0.9664, 0.2568), \quad (\beta_{0(2)}, \beta_{1(2)}) = (0.9971, 0.0760)$$

$$(\gamma_{0(1)}, \gamma_{1(1)}) = (0.6326, 0.7744), \quad (\gamma_{0(2)}, \gamma_{1(2)}) = (0.3955, 0.9184)$$

$$(\beta_0, \beta_1) = (0.5260, 0.8504).$$

In particular, we compute the weights

$$\begin{aligned} w_{000} &= 2.1167, \quad w_{100} = 0.8895, \quad w_{010} = 0.9281, \quad w_{110} = 1.0241, \\ w_{001} &= 1.5913, \quad w_{101} = 0.7539, \quad w_{011} = 1.1154, \quad w_{111} = 0.9758 \end{aligned} \quad (4.48)$$

that render  $\phi$  birational. **Figure 1.2**, shown in Chapter 1, illustrates the deformation of the unit cube by  $\phi$ , the weights specified in (4.48). To recover the geometric intuition for CAGD, we have replaced the monomial basis by the Bernstein basis (see §4.1.2). Additionally, **Figure 4.7** depicts the base locus of the inverse rational map (see Theorem 4.5.15), and some of the planes involved in the definition of the tensors of 4.5.7.

#### 4.5.4. Inverse rational map and base locus

**Notation 4.5.14.** Assuming weights as in Corollary 4.5.8, for each  $r = 1, 2$  we define the polynomials

$$a_r(\mathbf{s}) = \begin{vmatrix} s_0 & s_1 \\ -\alpha_{1(r)} & \alpha_{0(r)} \end{vmatrix}, \quad b_r(\mathbf{t}) = \begin{vmatrix} t_0 & t_1 \\ -\beta_{1(r)} & \beta_{0(r)} \end{vmatrix}, \quad c_r(\mathbf{u}) = \begin{vmatrix} u_0 & u_1 \\ -\gamma_{1(r)} & \gamma_{0(r)} \end{vmatrix},$$

as well as

$$b(\mathbf{t}) = \begin{vmatrix} t_0 & t_1 \\ -\beta_1 & \beta_0 \end{vmatrix}, \quad c(\mathbf{u}) = \begin{vmatrix} u_0 & u_1 \\ -\gamma_1 & \gamma_0 \end{vmatrix}.$$

The following result provides the formulas for the inverse of a scaffold birational map. As for the classes of hexahedral and pyramidal birational maps, we also provide the defining equations of the irreducible components of the base locus, and define the contractions and blow-ups.

**Theorem 4.5.15.** *Let  $\phi$  be scaffold. If  $\phi$  is birational of type  $(1, 2, 2)$  with weights as in Corollary 4.5.8, then the inverse rational map  $\phi^{-1}$  is given by*

$$(\alpha_{1(r)}\lambda_{1(r)}\boldsymbol{\sigma}_1(\mathbf{y}) : \alpha_{0(r)}\lambda_{0(r)}\boldsymbol{\sigma}_0(\mathbf{y})) \times (\beta_1\mu_1\boldsymbol{\tau}_1(\mathbf{y}) : \beta_0\mu_0\boldsymbol{\tau}_0(\mathbf{y})) \times (\gamma_1\nu_1\mathbf{u}_1(\mathbf{y}) : \gamma_0\nu_0\mathbf{u}_0(\mathbf{y}))$$

where any  $r = 1, 2$  is valid. Moreover, the base locus of  $\phi$  is defined by

$$B = (b, c) \cap (a_1, b_1, c_1) \cap (a_2, b_2, c_2),$$

and the base locus of  $\phi^{-1}$  is the union of the lines  $s, y, z, x_1$  and  $x_2$ . Additionally, we have the following:

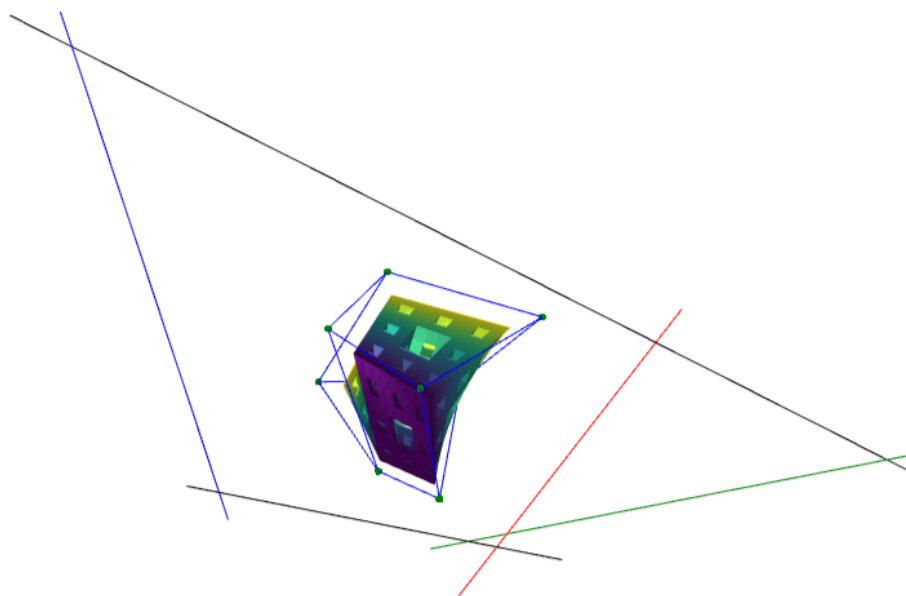
- (i) The base line  $(b, c)$  is blown-up to  $Q$
- (ii) The base point  $(a_r, b_r, c_r)$  is blown-up to  $\Sigma_{(r)}$ , for each  $r = 1, 2$

*Proof.* Let either  $r = 1$  or  $r = 2$ . By Lemma 4.2.1, the relation

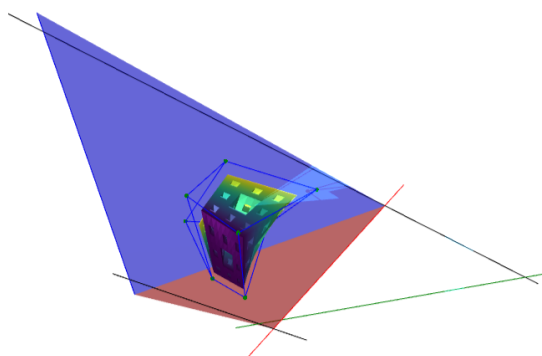
$$\boldsymbol{\sigma}(s_0, s_1; \mathbf{y}) = \begin{vmatrix} s_0 & s_1 \\ \alpha_{1(r)}\lambda_{1(r)}\boldsymbol{\sigma}_1(\mathbf{y}) & \alpha_{0(r)}\lambda_{0(r)}\boldsymbol{\sigma}_0(\mathbf{y}) \end{vmatrix}$$

is the syzygy of  $\mathbf{f}$  of degree  $(1, 0, 0)$ . Specifically, we have

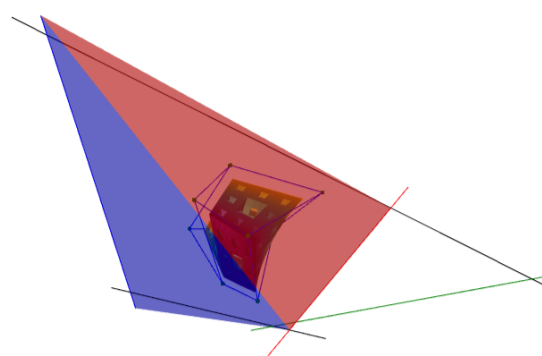
$$(\alpha_{0(r)}\lambda_{0(r)} \quad \alpha_{1(r)}\lambda_{1(r)}) \begin{pmatrix} w_{1jk} \langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} \rangle \\ w_{0jk} \langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle \end{pmatrix}_{0 \leq j, k \leq 1} = 0,$$



(a) the lines involved in the scaffold birational map of Example 4.5.13. Specifically, the line  $s$  appears in red, the lines  $x_1, x_2$  appear in black, the line  $y$  appears in blue, and the line  $z$  appears in green. The base locus of the  $t$ -surfaces, is the union  $s \cup x_1 \cup x_2 \cup y$ . Similarly, the base locus of the  $u$ -surfaces is  $s \cup x_1 \cup x_2 \cup z$ .



(b) the planes involved in the scaffold birational map of Example 4.5.13. The image shows the planes  $\Sigma_{(1)}$ , in red, and  $B_2$ , in blue. In particular, the union  $\Sigma_{(1)} \cup B_2$  is a rank-two quadric in the pencil of  $t$ -surfaces.



(c) The image shows the planes  $\Sigma_{(2)}$ , in red, and  $B_1$ , in blue. Again, the union  $\Sigma_{(2)} \cup B_1$  is a rank-two quadric in the pencil of  $t$ -surfaces. The planes  $\Gamma_1$  and  $\Gamma_2$ , not shown on the image, are defined by the line  $z$  and the two lines  $x_1, x_2$ .

Figure 4.7.: The geometry of a scaffold birational map.



since the plane  $\Sigma_{(r)}$  defined by  $\sigma_{(r)} = \lambda_{0(r)}\sigma_0 + \lambda_{1(r)}\sigma_1$  contains the point  $\mathbf{A}_{jk}(-\alpha_{1(r)}, \alpha_{0(r)})$  in  $x_r$ , for every  $0 \leq j, k \leq 1$ . On the other hand, the relation of multidegree  $(0, 1, 0; 2)$

$$\tau(t_0, t_1; \mathbf{y}) = \begin{vmatrix} t_0 & t_1 \\ \beta_1 \mu_1 \tau_1(\mathbf{y}) & \beta_0 \mu_0 \tau_0(\mathbf{y}) \end{vmatrix} \quad (4.49)$$

vanishes after the specialization  $\mathbf{y} \mapsto \mathbf{f}$ , and hence lies in the Rees ideal of  $\phi$ . Specifically, by definition  $\tau(t_0, t_1; \mathbf{f})$  for

$$(t_0 : t_1) = (1 : 0) \quad , \quad (t_0 : t_1) = (0 : 1) \quad . \quad (4.50)$$

Additionally, if  $(t_0 : t_1) = (-\beta_1 : \beta_0)$  then

$$\begin{vmatrix} -\beta_1 & \beta_0 \\ \beta_1 \mu_1 \tau_1(\mathbf{y}) & \beta_0 \mu_0 \tau_0(\mathbf{y}) \end{vmatrix} = -\beta_0 \beta_1 (\mu_0 \tau_0(\mathbf{y}) + \mu_1 \tau_1(\mathbf{y}))$$

defines the smooth quadric  $Q$ . In particular, since  $\mathbf{f}$  contracts  $(t_0 : t_1) = (-\beta_1 : \beta_0)$  to the line  $y$ , the specialization  $\tau(t_0, t_1; \mathbf{f})$  vanishes at  $(t_0 : t_1) = (-\beta_1 : \beta_0)$ . On the other hand, the pullback  $\tau(t_0, t_1; \mathbf{f})$  is either zero or a polynomial of degree  $(2, 3, 2)$  in  $R$ . Furthermore, if it is nonzero then it can be regarded as a polynomial of degree  $(0, 2, 0)$ , since by Proposition 4.5.3  $T_0 \cap T_1 = s \cup x_1 \cup x_2 \cup y$ , and thus it admits the factors

$$a_1(s_0, s_1) \quad , \quad a_2(s_0, s_1) \quad , \quad c(u_0, u_1) \quad , \quad g(t_0, t_1, u_0, u_1) \quad ,$$

of degrees  $(1, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 0, 1)$ , and  $(0, 1, 1)$  respectively. However, we have seen that it admits the two roots in (4.50) as well as  $(t_0 : t_1) = (-\beta_1 : \beta_0)$ . Therefore, it must be identically zero. With a similar argument, we conclude that the relation of multidegree  $(0, 0, 1; 2)$

$$\nu(u_0, u_1; \mathbf{y}) = \begin{vmatrix} u_0 & u_1 \\ \gamma_1 \nu_1 \nu_1(\mathbf{y}) & \gamma_0 \nu_0 \nu_0(\mathbf{y}) \end{vmatrix} \quad (4.51)$$

lies in the Rees ideal of  $\phi$ , and hence defines the inverse for  $(u_0 : u_1)$ .

The base locus of  $\phi^{-1}$  follows immediately from Proposition 4.5.3. Regarding the base locus of  $\phi$ , we find  $\dim_{\mathbb{C}} B_{(1,1,1)} = 4$ . Moreover,  $\mathbb{C}\langle \mathbf{f} \rangle$  contains the linearly independent polynomials

$$a_1 b_1 c \quad , \quad a_2 b_2 c \quad , \quad a_1 b c_1 \quad , \quad a_2 b c_2 \quad ,$$

since these are respectively proportional to

$$\langle \beta_1, \mathbf{f} \rangle \quad , \quad \langle \beta_2, \mathbf{f} \rangle \quad , \quad \langle \gamma_1, \mathbf{f} \rangle \quad , \quad \langle \gamma_2, \mathbf{f} \rangle \quad .$$

Therefore, we find  $B_{(1,1,1)} = \mathbb{C}\langle \mathbf{f} \rangle$ . On the other hand, the three components of  $B$  are prime. Hence, the pullback by  $\phi^{-1}$  of each of them is a principal ideal  $\mathbb{C}[\mathbf{y}]$ . Specifically, the pullback polynomials by  $\phi^{-1}$  of  $b, c$  are

$$b(\mathbf{g}_2) = \begin{vmatrix} -\beta_1 & \beta_0 \\ \beta_1 \mu_1 \tau_1(\mathbf{y}) & \beta_0 \mu_0 \tau_0(\mathbf{y}) \end{vmatrix} \quad , \quad c(\mathbf{g}_3) = \begin{vmatrix} -\gamma_1 & \gamma_0 \\ \gamma_1 \nu_1 \nu_1(\mathbf{y}) & \gamma_0 \nu_0 \nu_0(\mathbf{y}) \end{vmatrix} \quad ,$$

which define the smooth quadric  $Q$ . Similarly,

$$a_r(\mathbf{g}_1) = \begin{vmatrix} \alpha_1 & \alpha_0 \\ \alpha_1 \lambda_1 \sigma_1(\mathbf{y}) & \alpha_0 \lambda_0 \sigma_0(\mathbf{y}) \end{vmatrix} = -\alpha_0 \alpha_1 \langle \sigma_{(r)}, \mathbf{y} \rangle \quad ,$$

which must be the generator of the pullback of the base point indexed by  $(r)$ , since the other two specializations are quadratic polynomials divisible by  $a_r(\mathbf{g}_1)$ . In particular, the base point is blown-up to  $\Sigma_{(r)}$ .  $\square$

**Example 4.5.16** (Inverse of a pyramidal birational map). Continuing with Example 4.5.13, the symmetric matrices defining the boundar  $t$ - and  $u$ -surfaces are

$$\begin{aligned} \boldsymbol{\tau}_0 &= \begin{pmatrix} 1.0000 & 0.1069 & 0.1480 & -0.1148 \\ 0.1069 & -0.0285 & 0.0373 & -0.0430 \\ 0.1480 & 0.0373 & 0.0102 & 0.0117 \\ -0.114 & -0.0430 & 0.0117 & -0.0159 \end{pmatrix} & \boldsymbol{\tau}_1 &= \begin{pmatrix} 1.0000 & 0.1948 & 0.3699 & 0.1718 \\ 0.1948 & -0.0449 & 0.0392 & 0.1016 \\ 0.3699 & 0.0392 & 0.1238 & 0.1813 \\ 0.1718 & 0.1016 & 0.1813 & 0.0332 \end{pmatrix} \\ \boldsymbol{v}_0 &= \begin{pmatrix} 1.0000 & -0.3862 & 0.5241 & 0.8512 \\ -0.3862 & 0.0632 & -0.1237 & 0.0793 \\ 0.5241 & -0.1237 & 0.2027 & 0.1895 \\ 0.8512 & 0.0793 & 0.1895 & -0.5532 \end{pmatrix} & \boldsymbol{v}_1 &= \begin{pmatrix} 1.0000 & 0.2114 & 0.0272 & -0.3843 \\ 0.2114 & -0.0479 & 0.0753 & -0.0926 \\ 0.0272 & 0.0753 & -0.0515 & -0.0545 \\ -0.3843 & -0.0926 & -0.0545 & 0.1044 \end{pmatrix} \end{aligned}$$

In particular, using Notation 4.5.5 we compute

$$(\lambda_{0(2)} : \lambda_{1(2)}) = (0.9923 : 0.1232) , \quad (\mu_0 : \mu_1) = (0.8871 : -0.4615) , \quad (\nu_0 : \nu_1) = (0.9963 : 0.0855) .$$

Therefore, the inverse rational map is given by

$$\begin{aligned} & (\alpha_{1(r)} \lambda_{1(r)} \boldsymbol{\sigma}_1(\mathbf{y}) : \alpha_{0(r)} \lambda_{0(r)} \boldsymbol{\sigma}_0(\mathbf{y})) \times (\beta_1 \mu_1 \boldsymbol{\tau}_1(\mathbf{y}) : \beta_0 \mu_0 \boldsymbol{\tau}_0(\mathbf{y})) \times (\gamma_1 \nu_1 \boldsymbol{v}_1(\mathbf{y}) : \gamma_0 \nu_0 \boldsymbol{v}_0(\mathbf{y})) \\ & \quad = \\ & (0.0316 \boldsymbol{\sigma}_1(\mathbf{y}) : 0.9590 \boldsymbol{\sigma}_0(\mathbf{y})) \times (-0.3924 \boldsymbol{\tau}_1(\mathbf{y}) : 0.4666 \boldsymbol{\tau}_0(\mathbf{y})) \times (0.0431 \boldsymbol{v}_1(\mathbf{y}) : 0.8600 \boldsymbol{v}_0(\mathbf{y})) \end{aligned}$$

Furthermore, the base ideal  $B$  of  $\phi$  is

$$\begin{aligned} & (b, c) \cap (a_1, b_1, c_1) \cap (a_2, b_2, c_2) = \\ & (0.5260 t_0 + 0.8504 t_1, 0.8632 u_0 + 0.5047 u_1) \cap \\ & (0.5145 s_0 + 0.8574 s_1, 0.2467 t_0 + 0.9690 t_1, 0.6326 u_0 + 0.7744 u_1) \cap \\ & (0.9664 s_0 + 0.2569 s_1, 0.9971 t_0 + 0.0760 t_1, 0.3955 u_0 + 0.9184 u_1) \end{aligned}$$

and we have the subsequent contractions specified in Corollary 4.5.8.

Notice that, since in this example we are working with floating point arithmetic, the defining equations of the irreducible components (and therefore the contractions) of the base locus cannot be recovered from the entries of  $\phi$ . Hence, having the explicit defining equations of these components is convenient for applications.



# Chapter 5

## Construction of multilinear birational maps with multilinear inverse

In this short chapter, we extend our treatment of hexahedral birational maps to the analogous transformations in arbitrary dimension, namely multilinear rational maps with multilinear inverse, as described in Definition 1.2.22. These birational maps are the higher-dimensional counterparts of the hexahedral birational maps, studied in §4.3, and bilinear birational maps, discussed in §2.2.4. Specifically, the minimal graded free resolution of their base ideal is Hilbert-Burch.

We adhere to the conventional formulation in CAGD, and define these transformations by means of control points and weights. Specifically, for  $n \geq 4$  we consider the multilinear rational map

$$\begin{aligned} \phi : (\mathbb{P}_{\mathbb{C}}^1)^n &\dashrightarrow \mathbb{P}_{\mathbb{C}}^n & (5.1) \\ (x_{01} : x_{11}) \times \dots \times (x_{0n} : x_{1n}) &\mapsto \mathbf{f} = \sum_{0 \leq i_1, \dots, i_n \leq 1} w_{i_1 \dots i_n} \mathbf{P}_{i_1 \dots i_n} b_{i_1}(x_{01}, x_{11}) \dots b_{i_n}(x_{0n}, x_{1n}) \end{aligned}$$

where  $\mathbf{P}_{i_1 \dots i_n} = (1, y_{1i_1 \dots i_n}, \dots, y_{ni_1 \dots i_n})$  lies in  $\mathbb{C}^{n+1}$ , and the weights  $w_{i_1 \dots i_n}$  are nonzero complex numbers, for each  $0 \leq i_1, \dots, i_n \leq 1$ .

**Construction of birational maps:** Assuming the necessary constraints on the control points, we prove that (5.1) admits a multilinear inverse if and only if a certain tensor of format  $2 \times n$  has rank one (Theorem 5.7), and we provide formulas for the weights that yield birationality. Moreover, we propose a construction for nets of control points satisfying the necessary constraints required for birationality. Additionally, we provide explicit formulas for the inverse  $\phi^{-1}$  and the defining equations of the base loci, and describe the blow-ups and contractions (Theorem 5.2.1).

### 5.1. Rank-based characterization of birationality

In all the chapter, we let  $n \geq 4$  and only treat standard gradings. We consider the variables  $\mathbf{x}_k = (x_{0k}, x_{1k})$  in  $\mathbb{P}_{\mathbb{C}}^1$ , for each factor indexed by  $1 \leq k \leq n$  in  $(\mathbb{P}_{\mathbb{C}}^1)^n$ , and  $\mathbf{y} = (y_0, \dots, y_n)$  in  $\mathbb{P}_{\mathbb{C}}^n$ . Moreover, when we refer to affine charts  $\mathbb{A}_{\mathbb{C}}^n$  in either  $(\mathbb{P}_{\mathbb{C}}^1)^n$  or  $(\mathbb{P}_{\mathbb{C}})^n$  we always refer to those defined by  $x_{0k} \neq 0$  for every  $1 \leq k \leq n$ , and  $y_0 \neq 0$ , respectively.

By definition, all the parametric hypersurfaces of a multilinear birational map with multilinear inverse are hyperplanes. Since the boundary hypersurfaces (recall Definition 1.2.23) are instances of these parametric hypersurfaces, these transformations must be hypercubic rational maps, according to Definition 1.2.23. This observation describes the necessary geometric constraints on the control points that are necessary for a multilinear inverse to exist.

In analogy to the construction of hexahedral rational maps, we can generate nets of control points for hypercubic rational maps indirectly through the boundary hyperplanes.

**Construction 4** (Control points for hypercubic rational maps). A net of control points for a hypercubic rational map can be generated as follows:

1. For each  $1 \leq k \leq n$  and  $i_k = 0, 1$ , choose hyperplanes  $X_{i_k k}$  in  $\mathbb{A}_{\mathbb{C}}^n$  such that the intersections

$$X_{i_1 1} \cap \dots \cap X_{i_n n}$$

define affine points, for every  $0 \leq i_1, \dots, i_n \leq 1$ .

2. Define  $\mathbf{P}_{i_1 \dots i_n} = X_{i_1 1} \cap \dots \cap X_{i_n n}$

**Notation 5.1.1.** Let  $\phi$  be  $n$ -cubic. We introduce the following notation:

- (i) For each  $1 \leq k \leq n$  and  $i_k = 0, 1$ , the boundary hyperplane  $X_{i_k k}$  is defined by

$$\mathbf{x}_{i_k k}(\mathbf{y}) := \langle \mathbf{x}_{i_k k}, \mathbf{y} \rangle = \mathbf{x}_{i_k k} \cdot \mathbf{y}^T = 0$$

for some vector  $\mathbf{x}_{i_k k} = (x_{0i_k k}, \dots, x_{ni_k k})$  in  $\mathbb{C}^{n+1}$

- (ii) We denote  $x_k = X_{0k} \cap X_{1k}$ , which is a  $(n-2)$ -plane in  $\mathbb{P}_{\mathbb{C}}^n$

- (iii) For each  $0 \leq i_1, \dots, i_n \leq 1$  we define

$$\Delta_{i_1 \dots i_n} = \begin{vmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix} \quad (5.2)$$

In particular, we find the identity

$$\mathbf{P}_{i_1 \dots i_n} = \Delta_{i_1 \dots i_n}^{-1} \mathbf{x}_{i_1 1} \wedge \dots \wedge \mathbf{x}_{i_n n}$$

- (iv) For each  $1 \leq k \leq n$  and  $0 \leq i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n \leq 1$ , we define the product

$$B_{i_1 \dots i_{k-1} i_{k+1} \dots i_n}^k = B_{i_1 \dots i_{k-1} i_{k+1} \dots i_n}^k(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{x}_{k+1}, \dots, \mathbf{x}_n) = \prod_{\substack{1 \leq j \leq n \\ j \neq k}} b_j(x_{0j}, x_{1j})$$

**Remark 5.1.2** (Nonzero  $\Delta_{i_1 \dots i_n}$ ). As in Remark 4.3.5,  $\Delta_{i_1 \dots i_n}$  is zero if and only if  $\mathbf{P}_{i_1 \dots i_n}$  lies in the “hyperplane at  $\infty$ ” defined by  $y_0 = 0$ . However, by the hypothesis  $\mathbf{P}_{i_1 \dots i_n} = (1, y_{i_1 \dots i_n 1}, \dots, y_{i_1 \dots i_n n})$  we are excluding this possibility.

The following result is the direct generalization of Lemma 4.2.1 to dimension  $n$ . It characterizes the existence of linear syzygies of  $\mathbf{f}$ .

**Lemma 5.1.3.** *Let  $\phi$  be dominant, and let  $X_{0k}, X_{1k}$  be hyperplanes. Then,  $\mathbf{f}$  has a syzygy of degree  $\mathbf{e}_k$  if and only if*

$$\text{rank} \left( \begin{array}{c} w_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n} \langle \mathbf{x}_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n}, \mathbf{P}_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n} \rangle \\ w_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n} \langle \mathbf{x}_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n}, \mathbf{P}_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n} \rangle \end{array} \right)_{1 \leq i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n \leq 1} = 1, \quad (5.3)$$

where the matrix has size  $2 \times 2^{n-1}$ . If (5.3) holds, we find a point  $(\alpha_{0k} : \alpha_{1k})$  in  $\mathbb{P}_{\mathbb{R}}^1$  such that

$$\begin{aligned} -\alpha_{0k} w_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n} \langle \mathbf{x}_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n}, \mathbf{P}_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n} \rangle = \\ \alpha_{1k} w_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n} \langle \mathbf{x}_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n}, \mathbf{P}_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n} \rangle \end{aligned} \quad (5.4)$$

for every  $1 \leq i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n \leq 1$ . Then, any syzygy of degree  $e_k$  of  $\mathbf{f}$  is proportional to

$$\mathbf{x}_k(x_{0k}, x_{1k}) = \alpha_{0k} \mathbf{x}_0 b_0(x_{0k}, x_{1k}) + \alpha_{1k} \mathbf{x}_1 b_1(x_{0k}, x_{1k}). \quad (5.5)$$

*Proof.* The proof is the direct translation of Lemma 4.2.1 to dimension  $n$ .  $\square$

In analogy to trilinear rational maps, the rank condition (5.3) has a geometric interpretation in terms of the contractions of  $\phi$ . Specifically, we say that  $\phi$  contracts a hypersurface  $S$  in  $(\mathbb{P}_{\mathbb{C}}^1)^n$  if the dimension of  $\phi(S)$  is strictly less than  $n - 1$ .

**Remark 5.1.4** (Contraction of  $(n - 1)$ -linear hypersurfaces). Let  $i_k = 0, 1$ . By definition, if the boundary surface  $\Pi_{i_k k}$  is a plane, we have

$$\begin{aligned} \langle \mathbf{x}_{i_k k}, \mathbf{f} \rangle = \\ b_{i_k}^*(x_{0k}, x_{1k}) \sum_{0 \leq i_1 \dots i_{k-1} i_{k+1} \dots i_n \leq 1} w_{i_1 \dots i_{k-1} i_k^* i_{k+1} \dots i_n} \langle \mathbf{x}_{i_k k}, \mathbf{P}_{i_1 \dots i_{k-1} i_k^* i_{k+1} \dots i_n} \rangle B_{i_1 \dots i_{k-1} i_{k+1} \dots i_n}^k \end{aligned} \quad (5.6)$$

where  $0^* = 1$  and  $1^* = 0$ . In particular, the sum in the right-hand of (5.6) is a multilinear polynomial

$$g_{i_k} = g_{i_k}(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{x}_{k+1}, \dots, \mathbf{x}_n),$$

whose coefficients in the multilinear Bernstein basis determine the  $(i_k + 1)$ -row of the matrix in Lemma 5.1.3. Therefore, the rank of this matrix is one if and only if the polynomials  $g_0$  and  $g_1$  are proportional, or equivalently, if the image by  $\phi$  of the hypersurface in  $(\mathbb{P}_{\mathbb{C}}^1)^n$  defined by  $g_0 = g_1 = 0$  lies in the  $(n - 2)$ -plane  $x_k = X_{0k} \cap X_{1k}$ .

Geometrically,  $\mathbf{f}$  admits a syzygy of degree  $e_k$  if and only if  $\phi$  contracts a surface of degree  $1 - e_k$  to the  $(n - 2)$ -plane  $x_k$ .

The following is the first main result of the chapter, which asserts that birationality can be achieved by a rank-one condition on a tensor of format  $2 \times n$ .

**Theorem 5.1.5.** *Let  $\phi$  be  $n$ -cubic. Then,  $\phi$  is birational with multilinear inverse if and only if the tensor with  $2 \times n$  format*

$$W = \left( \frac{w_{i_1 \dots i_n}}{\Delta_{i_1 \dots i_n}} \right)_{0 \leq i_1, \dots, i_n \leq 1} \quad (5.7)$$

has rank one.

*Proof.* By definition,  $\phi$  is birational with multilinear inverse if and only if  $\mathbf{f}$  admits syzygies of degrees  $\mathbf{e}_1, \dots, \mathbf{e}_n$ . Equivalently, by Lemma 5.1.3  $\mathbf{f}$  admits these syzygies if and only if the rank condition (5.3) is satisfied for every  $1 \leq k \leq n$ .

We rewrite the rank condition (5.3) so that the matrices involved have the same entries for every  $1 \leq k \leq n$ . Namely, for any  $0 \leq i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n \leq 1$  we have

$$\begin{aligned} \Delta_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n} \langle \mathbf{x}_{1k}, \mathbf{P}_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n} \rangle &= \mathbf{x}_{1k} \wedge \mathbf{x}_{i_1 1} \dots \wedge \mathbf{x}_{i_{k-1} k-1} \wedge \mathbf{x}_{0k} \wedge \mathbf{x}_{i_{k+1} k+1} \wedge \dots \wedge \mathbf{x}_{i_n n} \\ &= -\mathbf{x}_{0k} \wedge \mathbf{x}_{i_1 1} \dots \wedge \mathbf{x}_{i_{k-1} k-1} \wedge \mathbf{x}_{1k} \wedge \mathbf{x}_{i_{k+1} k+1} \wedge \dots \wedge \mathbf{x}_{i_n n} \\ &= -\Delta_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n} \langle \mathbf{x}_{0k}, \mathbf{P}_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n} \rangle. \end{aligned}$$

Hence, we have

$$\left| \langle \mathbf{x}_{0k}, \mathbf{P}_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n} \rangle_{\Delta_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n}} - \langle \mathbf{x}_{1k}, \mathbf{P}_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n} \rangle_{\Delta_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n}} \right| = 0 ,$$

and (5.3) can be equivalently written as

$$\text{rank} \left( \begin{array}{c} w_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n} \Delta_{i_1 \dots i_{k-1} 1 i_{k+1} \dots i_n}^{-1} \\ w_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n} \Delta_{i_1 \dots i_{k-1} 0 i_{k+1} \dots i_n}^{-1} \end{array} \right)_{1 \leq i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n \leq 1} = 1 . \quad (5.8)$$

Therefore, the matrix in (5.8) is the  $k$ -flattening of the tensor  $W$  in the statement, and  $\phi$  admits a multilinear inverse if and only if all these flattenings have rank one. However, this is equivalent to  $W$  having rank one.  $\square$

**Corollary 5.1.6.** *Let  $\phi$  be  $n$ -cubic. Then,  $\phi$  is birational with multilinear inverse if and only if*

$$w_{i_1 \dots i_n} = \alpha_{i_1 1} \dots \alpha_{i_n n} \Delta_{i_1 \dots i_n} \quad (5.9)$$

for some  $(\alpha_{01} : \alpha_{11}) \times \dots \times (\alpha_{0n} : \alpha_{1n})$  in  $(\mathbb{P}_{\mathbb{C}}^1)^n$ .

**Remark 5.1.7** (Nonzero, real, and positive weights). Since  $\phi$  is dominant, we must have  $\alpha_{i_k k} \neq 0$  for every  $0 \leq i, j, k \leq 1$ . Namely, if  $\alpha_{i_k k} = 0$  then by Corollary 5.1.6 all the weights indexed by  $i_k$  at the  $k$ -th position are zero, and the image of  $\phi$  lies in  $X_{i_k^* k}$ .

Notice that, if the boundary hyperplanes are chosen over the reals, then  $(\alpha_{01} : \alpha_{11}) \times \dots \times (\alpha_{0n} : \alpha_{1n})$  can be chosen in  $(\mathbb{P}_{\mathbb{R}}^1)^n$  to achieve real weights. Furthermore, if  $\alpha_{i_k k} > 0$  for every  $1 \leq k \leq n$  and  $i_k = 0, 1$ , by Remark 5.1.2 we can take all the weights positive, preserving the property that  $\phi([0, 1]^n)$  lies in the convex hull of the control points.

## 5.2. Inverse rational map and base locus

The following result provides the explicit expression of the multilinear inverse of a birational  $n$ -cubic map. Furthermore, it describes the base loci of the birational map and its inverse.

**Theorem 5.2.1.** *Let  $\phi$  be  $n$ -cubic, with weights as in Corollary 5.1.6. Then, maintaining the notation of (1.14), the inverse  $\phi^{-1}$  is given by*

$$\mathbf{y} \mapsto (\alpha_{01} \mathbf{x}_{01}(\mathbf{y}) - \alpha_{11} \mathbf{x}_{11}(\mathbf{y}) : \alpha_{01} \mathbf{x}_{01}(\mathbf{y})) \times \dots \times (\alpha_{0n} \mathbf{x}_{0n}(\mathbf{y}) - \alpha_{1n} \mathbf{x}_{1n}(\mathbf{y}) : \alpha_{0n} \mathbf{x}_{0n}(\mathbf{y})) , \quad (5.10)$$

and the base ideal  $B = (f_0, \dots, f_n)$  in  $R$  has a Hilbert-Burch minimal  $\mathbb{Z}^n$ -graded free resolution

$$0 \rightarrow \bigoplus_{i=1}^n R(-\mathbf{1} - \mathbf{e}_i) \rightarrow R(-\mathbf{1})^{n+1} \xrightarrow{(f_0 \dots f_n)} B \rightarrow 0 , \quad (5.11)$$

where  $R = \mathbb{C}[\mathbf{x}_1] \otimes \dots \otimes \mathbb{C}[\mathbf{x}_n]$ . Moreover,  $B$  is generated by the  $(n-1)$ -linear polynomials

$$\sum_{0 \leq i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n \leq 1} \left( \prod_{\substack{1 \leq j \leq n \\ j \neq k}} \alpha_{i_j j} \right) \Delta_{i_1 \dots i_{k-1} i_k i_{k+1} \dots i_n} \langle \mathbf{x}_{i_k^* k}, \mathbf{P}_{i_1 \dots i_{k-1} i_k i_{k+1} \dots i_n} \rangle B_{i_1 \dots i_{k-1} i_{k+1} \dots i_n}^k \quad (5.12)$$

for each  $1 \leq k \leq n$ , where each possible index  $i_k = 0$  or  $i_k = 1$  is valid, and the base locus of  $\phi^{-1}$  is the union of all the  $(n-2)$ -planes

$$\bigcup_{1 \leq k \leq n} x_k.$$

Moreover,  $\phi$  contracts the surface  $S_k \subset (\mathbb{P}_{\mathbb{C}}^1)^n$  defined by (5.12) to  $x_k$ , for each  $1 \leq k \leq n$ .

*Proof.* By Corollary 5.1.6, a  $n$ -cubic rational map is birational with multilinear inverse if and only if the weights satisfy (4.3.7) for some  $(\alpha_{01} : \alpha_{11}) \times \dots \times (\alpha_{0n} : \alpha_{1n})$  in  $(\mathbb{P}_{\mathbb{C}}^1)^n$ . Therefore, we can write

$$\begin{aligned} \mathbf{f} &= \sum_{0 \leq i_1, \dots, i_n \leq 1} w_{i_1 \dots i_n} \mathbf{P}_{i_1 \dots i_n} b_{i_1}(x_{01}, x_{11}) \dots b_{i_n}(x_{n1}, x_{n1}) = \\ &= \sum_{0 \leq i_1, \dots, i_n \leq 1} (\alpha_{i_1 1} \dots \alpha_{i_n n} \Delta_{i_1 \dots i_n}) (\Delta_{i_1 \dots i_n} \mathbf{x}_{i_1 1} \wedge \dots \wedge \mathbf{x}_{i_n n}) b_{i_1}(x_{01}, x_{11}) \dots b_{i_n}(x_{n1}, x_{n1}) = \\ &= \sum_{0 \leq i_1, \dots, i_n \leq 1} (\alpha_{i_1 1} \mathbf{x}_{i_1 1} b_{i_1}(x_{01}, x_{11})) \wedge \dots \wedge (\alpha_{i_n n} \mathbf{x}_{i_n n} b_{i_n}(x_{0n}, x_{1n})) = \mathbf{x}_1(x_{01}, x_{11}) \wedge \dots \wedge \mathbf{x}_n(x_{0n}, x_{1n}) \end{aligned}$$

where  $\mathbf{x}_k(x_{0k}, x_{1k})$  is as in Lemma 5.2.1. Therefore,  $\mathbf{f}$  admits the linear syzygy

$$\begin{aligned} \mathbf{x}(x_{0k}, x_{1k}; \mathbf{y}) &= \alpha_{0k} \mathbf{x}_{0k}(\mathbf{y}) b_0(x_{0k}, x_{1k}) + \alpha_{1k} \mathbf{x}_{1k}(\mathbf{y}) b_1(x_{0k}, x_{1k}) \\ &= x_{0k} \alpha_{0k} \mathbf{x}_{0k}(\mathbf{y}) + x_{1k} (\alpha_{1k} \mathbf{x}_{1k}(\mathbf{y}) - \alpha_{0k} \mathbf{x}_{0k}(\mathbf{y})) \end{aligned}$$

for each  $1 \leq k \leq n$ , and  $\phi^{-1}$  is as (5.10). Since  $\mathbf{f}$  admits  $n$  independent linear syzygies, of degrees  $\mathbf{e}_1, \dots, \mathbf{e}_n$  respectively, by the Hilbert-Burch theorem (Theorem 2.1.24) it follows that  $\mathbf{f}$  is proportional to the tuple of signed minors of the matrix of first syzygies of  $\mathbf{f}$ . Hence, the minimal graded resolution (5.11) follows. Regarding the base locus of  $\phi$ , we find

$$\mathbf{x}_{0k}(\mathbf{f}) = \alpha_{1k} b_1(x_{0k}, x_{1k}) g_k, \quad \mathbf{x}_{1k}(\mathbf{f}) = -\alpha_{0k} b_0(x_{0k}, x_{1k}) g_k$$

where  $g = g(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{x}_{k+1}, \dots, \mathbf{x}_n)$  is proportional to  $(n-1)$ -linear polynomials encoded by the rows of the matrix in (5.3) (recall Remark 5.1.4), or equivalently, to the polynomial (5.12). Hence, the graded component  $V = (B \cap \mathfrak{N})_1$  lies in  $\mathbb{C}\langle \mathbf{f} \rangle$ . Since  $\dim_{\mathbb{C}} V = n+1$ , it follows that  $V = \mathbb{C}\langle \mathbf{f} \rangle$  and  $B$  defines the base locus of  $\phi$ . On the other hand, the base ideal of  $\phi^{-1}$  is the intersection ideal

$$(\mathbf{x}_{01}(\mathbf{y}), \mathbf{x}_{11}(\mathbf{y})) \cap \dots \cap (\mathbf{x}_{0n}(\mathbf{y}), \mathbf{x}_{1n}(\mathbf{y}))$$

defines the union of the  $(n-2)$ -planes  $x_k$ , for every  $1 \leq k \leq n$ . Furthermore, by Remark 5.1.4 the hypersurface in  $(\mathbb{P}_{\mathbb{C}}^1)^n$  defined by the polynomial (5.12) is contracted to  $x_k$ .  $\square$

**Remark 5.2.2** (Polynomial hypercubic birational maps with multilinear inverse). In analogy to Remark 4.3.16,  $\phi$  restricts to a polynomial map in  $\mathbb{A}_{\mathbb{R}}^n$  (choosing real weights) if  $w_{i_1 \dots i_n} = 1$  for every  $0 \leq i_1, \dots, i_n \leq 1$ . Using our results, we find the following straightforward class of birational maps with multilinear inverse. Specifically, *parallelepiped*, or *paralotope*, is a hypercube where the facets are pairwise parallel.

**Corollary 5.2.3.** *Let  $\phi$  be  $n$ -cubic and  $w_{i_1 \dots i_n} = 1$  for every  $0 \leq i_1, \dots, i_n \leq 1$ . If the control points define a parallelepiped, then  $\phi$  is birational with multilinear inverse.*



*Proof.* For each  $1 \leq k \leq n$  and  $i_k = 0, 1$ , we can write

$$\langle \mathbf{x}_{i_k k}, \mathbf{P}_{i_1 \dots i_{k-1} i_k^* i_{k+1} \dots i_n} \rangle = \|\mathbf{x}_{i_k k}\| \text{sdist} (X_{i_k k}, \mathbf{P}_{i_1 \dots i_{k-1} i_k^* i_{k+1} \dots i_n}) ,$$

where  $\text{sdist}(\cdot, \cdot)$  denotes the signed distance in  $\mathbb{A}_{\mathbb{R}}^n$ . In particular, if the control points define a parallelepiped, we find  $\text{sdist} (X_{i_k k}, \mathbf{P}_{i_1 \dots i_{k-1} i_k^* i_{k+1} \dots i_n}) = \lambda_k$  for some nonzero constant  $\lambda_k$ . Hence, the rank condition (5.3) is satisfied.  $\square$

To conclude the chapter, we construct a multilinear birational map in dimension five with a multilinear inverse. Furthermore, we compute explicitly the syzygies from the formulas of Theorem 5.2.1.

**Example 5.2.4** (Birational deformation to a Klee-Minty 5-cube). We illustrate our results by explicitly constructing a birational deformation from the unit 5-cube to the Klee-Minty 5-cube [124, 82], which is defined by the inequalities in  $\mathbb{A}_{\mathbb{R}}^5 = \text{Spec}(\mathbb{R}[x_1, \dots, x_5])$

$$x_1 \leq 5 \quad , \quad 2^2 x_1 + x_2 \leq 5^2 \quad , \quad \dots \quad , \quad 2^5 x_1 + 2^4 x_2 + 2^3 x_3 + 2^2 x_4 + x_5 \leq 5^5$$

and  $x_k \geq 0$  for each  $1 \leq k \leq 5$ . Thus, the boundary hyperplanes are defined by the vectors in  $\mathbb{R}^5$

$$\begin{aligned} \mathbf{x}_{01} &= (5 \quad -1 \quad 0 \quad 0 \quad 0) \quad , \quad \mathbf{x}_{11} = (0 \quad 1 \quad 0 \quad 0 \quad 0) \quad , \\ \mathbf{x}_{02} &= (5^2 \quad -4 \quad -1 \quad 0 \quad 0) \quad , \quad \mathbf{x}_{12} = (0 \quad 0 \quad 1 \quad 0 \quad 0) \quad , \\ \mathbf{x}_{03} &= (5^3 \quad -8 \quad -4 \quad -1 \quad 0) \quad , \quad \mathbf{x}_{13} = (0 \quad 0 \quad 0 \quad 1 \quad 0) \quad , \\ \mathbf{x}_{04} &= (5^4 \quad -16 \quad -8 \quad -4 \quad -1) \quad , \quad \mathbf{x}_{14} = (0 \quad 0 \quad 0 \quad 0 \quad 1) \quad , \\ \mathbf{x}_{05} &= (5^5 \quad -32 \quad -16 \quad -8 \quad -4) \quad , \quad \mathbf{x}_{15} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1) \quad . \end{aligned}$$

In particular, the control points can be computed as

$$\mathbf{P}_{i_1 i_2 i_3 i_4 i_5} = \Delta_{i_1 i_2 i_3 i_4 i_5}^{-1} \mathbf{x}_{i_1 1} \wedge \mathbf{x}_{i_2 2} \wedge \mathbf{x}_{i_3 3} \wedge \mathbf{x}_{i_4 4} \wedge \mathbf{x}_{i_5 5} \quad (5.13)$$

where  $\Delta_{i_1 i_2 i_3 i_4 i_5}$  is as in (5.2), for each  $0 \leq i_1, i_2, i_3, i_4, i_5 \leq 1$ . Therefore, by Corollary (5.1.6), the computation of the weights

$$w_{i_1 i_2 i_3 i_4 i_5} = \alpha_{i_1 1} \alpha_{i_2 2} \alpha_{i_3 3} \alpha_{i_4 4} \alpha_{i_5 5} \Delta_{i_1 i_2 i_3 i_4 i_5} \quad ,$$

for any  $(\alpha_{01} : \alpha_{11}) \times (\alpha_{02} : \alpha_{12}) \times (\alpha_{03} : \alpha_{13}) \times (\alpha_{04} : \alpha_{14}) \times (\alpha_{05} : \alpha_{15})$  in  $(\mathbb{P}_{\mathbb{C}}^1)^5$ , where all the  $(\alpha_{0k} : \alpha_{1k})$  are distinct from  $(1 : 0)$  and  $(0 : 1)$ , renders  $\phi$  birational. In particular, the choice

$$\begin{aligned} (\alpha_{01} : \alpha_{11}) &= (1 : 1) \quad , \quad (\alpha_{02} : \alpha_{12}) = (0.5 : 3) \quad , \quad (\alpha_{03} : \alpha_{13}) = (2 : 0.1) \quad , \\ (\alpha_{04} : \alpha_{14}) &= (0.12 : 7) \quad , \quad (\alpha_{05} : \alpha_{15}) = (3 : 1.25) \quad , \end{aligned}$$

renders all the weights nonnegative, and moreover  $\mathbf{f}$  admits the linear syzygies

$$\begin{aligned} \mathbf{x}_1(\mathbf{x}_1) &= \mathbf{x}_{01} b_0(\mathbf{x}_1) + \mathbf{x}_{11} b_1(\mathbf{x}_1) \quad , & \mathbf{x}_2(\mathbf{x}_2) &= 0.5 \mathbf{x}_{02} b_0(\mathbf{x}_2) + 3 \mathbf{x}_{12} b_1(\mathbf{x}_2) \\ \mathbf{x}_3(\mathbf{x}_3) &= 2 \mathbf{x}_{03} b_0(\mathbf{x}_3) + 0.1 \mathbf{x}_{13} b_1(\mathbf{x}_3) \quad , & \mathbf{x}_4(\mathbf{x}_4) &= 0.12 \mathbf{x}_{04} b_0(\mathbf{x}_4) + 7 \mathbf{x}_{14} b_1(\mathbf{x}_4) \\ \mathbf{x}_5(\mathbf{x}_5) &= 3 \mathbf{x}_{05} b_0(\mathbf{x}_5) + 1.25 \mathbf{x}_{15} b_1(\mathbf{x}_5) \quad . \end{aligned}$$

By the Hilbert-Burch theorem (Theorem 2.1.24), the vector of polynomials  $\mathbf{f}$  defining  $\phi$  can be recovered, up to scalar, as the wedge

$$\chi_1(\mathbf{x}_1) \wedge \chi_2(\mathbf{x}_2) \wedge \chi_3(\mathbf{x}_3) \wedge \chi_4(\mathbf{x}_4) \wedge \chi_5(\mathbf{x}_5) .$$

Moreover, by the convex hull property of the Bernstein basis (recall §1.1.1), the image of the unit cube  $[0, 1]^5 \subset \mathbb{A}_{\mathbb{R}}^5$  is precisely the Klee-Minty 5-cube.



## **Part II**

# **POLYNOMIAL DATA ANALYSIS**



# Chapter 6

## Weak $(1 - \epsilon)$ -nets for polynomial superlevel sets

In this chapter, we prove an extension of the classical centerpoint theorem that encompasses sets defined by a single polynomial inequality of arbitrary degree. Additionally, we address the estimations of the size of  $(1 - \epsilon)$ -nets for the range systems defined by polynomial superlevel sets of bounded degree, and the Carathéodory number of real Veronese varieties.

**Metric geometry:** for any probability measure on  $\mathbb{R}^n$ , we prove that there exist  $n + 1$  points such that the nonnegativity region of every quadratic polynomial that is nonnegative on the points has probability at least  $2/((n + 1)(n + 2))$  (Theorem 6.2.1). Moreover, the number  $n + 1$  is the least possible. More generally, for a positive integer  $d$  and an absolutely continuous probability, we can always find  $\binom{n+2d}{n} - n - 1$  points such that the nonnegativity region of every polynomial of degree  $d$ , which is nonnegative at the points, has probability greater than zero (Theorem 6.2.2).

**Convex algebraic geometry:** We prove new bounds on the Carathéodory number of the real affine Veronese variety and the real Veronese cone (Theorem 6.2.3).

### 6.1. Preliminaries

The classical centerpoint theorem of Rado and Birch [169, 19], stated as Theorem 1.2.28, is an important result in computational geometry and data sciences, since it provides an extension of the one-dimensional median to higher-dimensional data. In the language of convexity, the centerpoint theorem states that the  $(\frac{1}{n+1})$ -floating body of any probability measure is not empty (see e.g. [10, 122, 156, 196] and the references therein for different perspectives and a large body of work around the centerpoint theorem and Tuckey depth).

Our goal in this chapter is to generalize this result by replacing half-spaces with superlevel sets of polynomials, i.e. the loci of points in  $\mathbb{A}_{\mathbb{R}}^n$  satisfying a polynomial inequality.

#### 6.1.1. Motivating example: disks and annuli

We begin with an example in  $\mathbb{A}_{\mathbb{R}}^2$ , that provides some intuition on our strategy and results. Specifically, we replace half-planes by disks and annuli. Given the family of polynomials of the form

$$f(\mathbf{x}) = f(x_1, x_2) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 (x_1^2 + x_2^2) \quad (6.1)$$

for some  $(\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3)$  in  $\mathbb{P}_{\mathbb{R}}^3$ , disks and annuli are the superlevel sets

$$(f \geq 0) = \{\mathbf{x} = (x_1, x_2) \in \mathbb{A}_{\mathbb{R}}^2 : f(\mathbf{x}) \geq 0\}.$$

Let  $X$  be a set of  $N$  points in  $\mathbb{A}_{\mathbb{R}}^2$ .

**Question 6.1.1.** Is there a centerpoint  $c = (a, b)$  in  $\mathbb{A}_{\mathbb{R}}^2$  such that, given a  $f$  as (6.1), if  $f(c) \geq 0$  then  $(f \geq 0)$  contains a fixed fraction of the points in  $X$ ?

The answer is negative, since we can take the disk centered at  $c$  of radius  $\epsilon > 0$  sufficiently small, defined by

$$-(x_1^2 + x_2^2) + 2a x_1 + 2b x_2 - (\epsilon^2 + a^2 + b^2) \geq 0$$

which does not contain any point different from  $c$ . Hence, this disk encloses at most an arbitrarily small fraction of the points in  $X$ .

We can reformulate the previous question, allowing more than one point.

**Question 6.1.2.** Are there two points  $c_1, c_2$  in  $\mathbb{A}_{\mathbb{R}}^2$  such that, given  $f$  as (6.1), if  $f(c_1) \geq 0$  and  $f(c_2) \geq 0$  then  $(f \geq 0)$  contains a fixed fraction of the points in  $X$ ?

In this case, the answer is affirmative. Namely, let

$$\begin{aligned} \psi : \mathbb{A}_{\mathbb{R}}^2 &\rightarrow \mathbb{A}_{\mathbb{R}}^3 \\ (x_1, x_2) &\mapsto (y_1, y_2, y_3) = (x_1, x_2, x_1^2 + x_2^2). \end{aligned}$$

The image of  $\psi$  is the paraboloid of revolution  $P$  defined by  $y_1^2 + y_2^2 - y_3 = 0$ . In particular, the pushforward of  $X$  yields a sample  $Y$  of  $N$  points in  $P$ . By the centerpoint theorem, there exists a point  $c$  in  $\mathbb{A}_{\mathbb{R}}^3$  such that every half-space  $H \subset \mathbb{A}_{\mathbb{R}}^3$  defined by

$$\alpha_0 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 \geq 0$$

containing  $c$  also contains (at least)  $N/4$  of the points in  $Y$ .

Unfortunately, this centerpoint  $c$  might not lie on  $P$ . Nevertheless,  $c$  does lie in the convex hull of  $Y$  (this follows either from the classical Helly's theorem, see for instance [160], or by the hyperplane separation theorem [34, Exercise 2.22]), and therefore in the convex hull of  $P$ . On the other hand, a general line  $\ell$  through  $c$  meets  $P$  at two points  $c_1, c_2$ . In particular, any half-space  $H$  containing the points  $c_1$  and  $c_2$  must also contain  $c$ . Therefore,  $H$  contains at least  $N/4$  of the points in  $Y$ . Finally, the pullback  $\psi^{-1}(H \cap P)$  is precisely the superlevel set of

$$f(x_1, x_2) := \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 (x_1^2 + x_2^2),$$

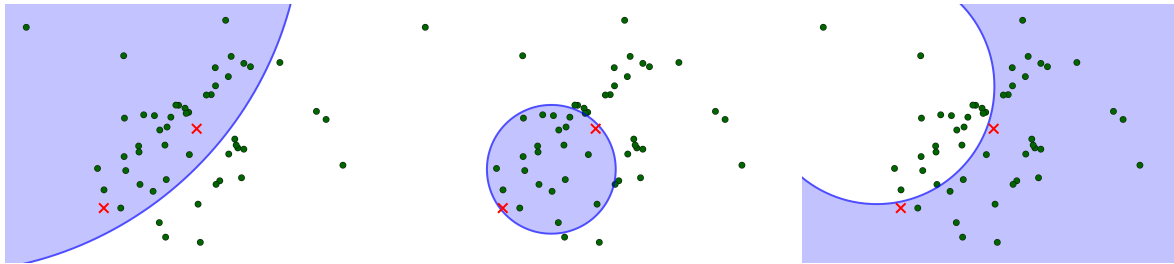
and Question 6.1.2 is answered affirmatively. **Figure 6.1** shows a sample  $X$  of points in  $\mathbb{A}_{\mathbb{R}}^2$  for which two centerpoints  $c_1, c_2$  have been found. In particular, any superlevel set  $(f \geq 0)$  with  $f$  as (6.1) encloses at least  $N/4$  points.

**Remark 6.1.3.** The same idea of the example above works in any dimension, for superlevel sets of polynomials

$$f(\mathbf{x}) = f(x_1, \dots, x_n) = \alpha_0 + \alpha_1 x_1 + \dots + \alpha_n x_n + \alpha_{n+1} \left( \sum_{1 \leq i \leq n} x_i^2 \right),$$

i.e. for  $n$ -dimensional balls and their complements. Namely, given a finite set  $X$  in  $\mathbb{A}_{\mathbb{R}}^n$  we can always find two centerpoints  $c_1, c_2$  such that every  $(f \geq 0)$  containing  $c_1, c_2$  also contains a fraction  $(n+2)^{-1}$  of the points in  $X$ .

In this chapter, we extend this strategy by considering the superlevel sets of all real polynomials of degree  $d$  in  $n$  variables. This task naturally relies on the study of Veronese varieties.



**Figure 6.1.:** A sample of  $N = 50$  (green) points in  $\mathbb{A}_{\mathbb{R}}^2$ . Every disk and annulus containing the two centerpoints (red crosses) contains at least  $N/4$  of the points.

### 6.1.2. Weak $(1 - \epsilon)$ nets for polynomial superlevel sets

We briefly recall the basic definitions that are necessary in our work. As usual in the field of measure theory, we replace the notation  $\mathbb{A}_{\mathbb{R}}^n$ , that we employed throughout Part I, by just  $\mathbb{R}^n$ . Additionally, in contrast to Part I, we start working over the usual topology in  $\mathbb{R}^n$ . Furthermore, we always consider the Borel  $\sigma$ -algebra on  $\mathbb{R}^n$ .

**Definition 6.1.4** (Borel  $\sigma$ -algebra). The *Borel  $\sigma$ -algebra* is the  $\sigma$ -algebra generated by the collection of all open subsets in  $\mathbb{R}^n$ , with the usual topology.

In particular, we only consider probability measures defined over the Borel  $\sigma$ -algebra. Similarly, we are interested in probability measures that are absolutely continuous with respect to the Lebesgue measure (see [63, §1.3] to recall the Lebesgue measure in  $\mathbb{R}^n$ ). More precisely, we have the following definition.

**Definition 6.1.5.** (Absolutely continuous measure, [63, §4.2]) A measure  $\mu$  on  $\mathbb{R}^n$  is *absolutely continuous with respect to the Lebesgue measure  $\nu$* , or simply *absolutely continuous*, if for every Borel set  $A$  such that  $\nu(A) = 0$  we have  $\mu(A) = 0$ .

The notion of a weak  $(1 - \epsilon)$ -net is central for our results, since it plays the role of the centerpoint for nonlinear polynomial inequalities. To formalize it, we require the following definition.

**Definition 6.1.6** (Range space). A *range space* in  $\mathbb{R}^n$  is a pair  $(\mu, F)$  where  $\mu$  is a probability measure on  $\mathbb{R}^n$  and  $F$  is a family of subsets of  $\mathbb{R}^n$ .

**Definition 6.1.7** (Weak  $\epsilon$ -net). Let  $0 \leq \epsilon \leq 1$ , and let  $(\mu, F)$  be a range space in  $\mathbb{R}^n$ . A *weak  $\epsilon$ -net* for  $(\mu, F)$  is a set  $X \subset \mathbb{R}^n$  such that every  $R$  in  $F$  satisfying  $\mu(R) \geq \epsilon$  has a nontrivial intersection with  $X$ , i.e.  $X \cap R \neq \emptyset$ .

The motivating examples for this definition are the families of closed half-spaces in  $\mathbb{R}^n$  and closed convex sets (e.g. [155, 160]). Weak  $\epsilon$ -nets were introduced in [11], to tackle the celebrated halving sets problem. Additionally,  $\epsilon$ -nets for half-spaces were introduced in the learning theory literature [197], where they remain an important concept closely connected to the Vapnik-Chervonenkis dimension. Furthermore, they are also relevant in computational geometry and approximation algorithms.

**Remark 6.1.8** (Centerpoint theorem and  $(1 - \epsilon)$ -nets). Importantly, the centerpoint theorem can be reformulated in the language of  $(1 - \epsilon)$ -nets. Specifically, given a probability measure  $\mu$  on  $\mathbb{R}^n$ , a centerpoint is a  $(1 - \epsilon)$ -net for  $\epsilon = (n + 1)^{-1}$ . Namely, every half-space  $H$  satisfying  $\mu(H) \geq 1 - \epsilon$  must contain the centerpoint, since the complement  $\mathbb{R}^n \setminus H$  has  $\mu(H) < \epsilon$ .



Since we are interested in nonlinear polynomial inequalities, the main object of study in this chapter are weak  $(1 - \epsilon)$ -nets for range space determined by the family of superlevel sets of polynomials.

**Definition 6.1.9** (Polynomial superlevel set). Let  $f$  in  $\mathbb{R}[x_1, \dots, x_n]$ . The *superlevel set* of  $f$ , denoted by  $(f \geq 0)$ , is

$$(f \geq 0) = \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : f(\mathbf{x}) \geq 0\}. \quad (6.2)$$

**Remark 6.1.10.** Half-spaces are the superlevel sets of linear polynomials.

**Notation 6.1.11.** Given  $\mathbf{x} = (x_1, \dots, x_n)$ , we denote by  $\mathbb{R}[\mathbf{x}]_{\leq d}$  the  $\mathbb{R}$ -vector space of polynomials of degree at most  $d$  in the variables of  $\mathbf{x}$ . Additionally, we define the collection of subsets of  $\mathbb{R}^n$

$$F_d := \{(f \geq 0) : f \in \mathbb{R}[\mathbf{x}]_{\leq d}\}.$$

In words,  $F_d$  is the family of superlevel sets of polynomials of degree at most  $d$ .

### 6.1.3. The Carathéodory number of Veronese varieties

We start recalling some elementary concepts in convex geometry.

**Definition 6.1.12** (Canonical simplex, convex combination, and convex hull).

(i) The  $(k - 1)$ -dimensional *canonical simplex* is the set

$$\Delta_{k-1} = \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k : \sum_{i=1}^k \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for every } 1 \leq i \leq k\}$$

(ii) A *convex combination* of the points  $x_1, \dots, x_k$  in  $\mathbb{R}^n$  is a sum

$$\lambda_1 x_1 + \dots + \lambda_k x_k$$

where  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$  lies in the  $(k - 1)$ -dimensional canonical simplex

(iii) The *convex hull* of a set  $X \subset \mathbb{R}^n$ , denoted by  $\text{conv}(X)$ , is the set of all the convex combinations of points in  $X$

The Carathéodory number of a set in  $\mathbb{R}^n$  is central in convex geometry.

**Definition 6.1.13** (Carathéodory number). Let  $X$  be a subset in  $\mathbb{R}^n$ . The Carathéodory number of  $X$ , denoted by  $\kappa(X)$ , is the smallest integer  $r$  such that every point in the convex hull of  $X$  can be written as a convex combination of  $r$  points in  $X$ .

In this chapter, we are interested in the Carathéodory number of the real affine Veronese variety  $V(n, d)$  and the real Veronese cone  $\hat{V}(n, d)$  (recall Definition 1.2.31). Remarkably, the Carathéodory number of  $\hat{V}(n, d)$  is simply

$$\max_{\mathbf{x} \in \text{conv}(\hat{V}(n, d))} \min\{r : \mathbf{x} = \sum_{i=1}^r \lambda_i \mathbf{x}_i^{\otimes d} \text{ where } \mathbf{x}_i \in \mathbb{R}^{n+1} \text{ and } \lambda_i \geq 0 \text{ for every } 1 \leq i \leq r\},$$

i.e. we can forget about the constraint  $\lambda_1 + \dots + \lambda_r = 1$  since  $\hat{V}(n, d)$  is a cone, and the coefficients can thus be “rescaled”. Recalling Definition 2.2.19, the Carathéodory number of the Veronese cone is the maximum nonnegative symmetric rank of a real symmetric tensor in the convex hull of  $\hat{V}(n, d)$ , since by definition  $\hat{V}(n, d)$  is the locus of rank-one symmetric tensors of format  $(n + 1)^{\otimes d}$  (recall §2.2.2 and Definition 2.2.14).

## 6.2. Contributions and related work

In this section, we present the main results of this chapter. Their proofs, among other technical results, are postponed to the following sections.

### 6.2.1. Statement of our results

Our first result is an extension of the centerpoint theorem for the superlevel sets of quadratic polynomials in  $\mathbb{R}^n$ , for any Borel probability measure. Remarkably, the number  $n + 1$  is sharp, meaning that the result is not true for any set with less than  $n + 1$  points (see Lemma 6.3.2).

**Theorem 6.2.1.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^n$ . There exists a set  $X$  of  $n + 1$  points in  $\mathbb{R}^n$  such that any quadratic  $f$  in  $\mathbb{R}[\mathbf{x}]$  that is nonnegative on every point of  $X$  satisfies*

$$\mu(f \geq 0) \geq \frac{2}{(n+2)(n+1)}.$$

Our second main result is an extension of the centerpoint theorem to arbitrary degree. However, we require that the probability measure is absolutely continuous.

**Theorem 6.2.2.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^n$  which is absolutely continuous with respect to the Lebesgue measure. For any  $\delta > 0$ , there exists a set  $X$  of at most  $\binom{n+2d}{n} - n - 1$  points in  $\mathbb{R}^n$  such that any  $f$  in  $\mathbb{R}[\mathbf{x}]_{\leq d}$  that is nonnegative on every point of  $X$  satisfies*

$$\mu(f \geq 0) \geq \left( \binom{n+2d}{2d} + 1 \right)^{-1} - \delta.$$

The previous results rely on new estimates for the Carathéodory numbers of the Veronese varieties. Thus, our next task is to provide bounds for these numbers. Specifically, from Lemma 6.3.2 and Proposition 6.3.3 we obtain a lower bound  $\kappa(V(n, 2d)) \geq \binom{n+d}{n}$ . On the other hand, the Carathéodory theorem [13, Theorem 2.3] provides the bound  $\kappa(V(n, d)) \leq \binom{n+d}{n}$ . We remark that the problem of estimating these Carathéodory numbers has been raised by several authors (e.g. [13, 12, 84], and [103, pp.10]). From a technical point of view, the bounds provided in the next theorem are our most interesting result.

**Theorem 6.2.3.** *Let  $n, d$  be positive integers. The Carathéodory numbers of the Veronese varieties satisfy the following:*

(i)  $\kappa(V(n, 2)) = n + 1$

(ii) (di Dio and Kummer, [84])  $\kappa(V(n, 2d)) \geq \binom{n+2d}{n} - n \binom{n+2d}{n} + \binom{n}{2}$

(iii)  $\kappa(\hat{V}(n+1, 2d)) \leq \binom{n+2d}{n} - n - 1$

(iv)  $\kappa(V(n, 2d)) \leq \kappa(V(n, 2d+1)) \leq \kappa(V(n, 2d+2))$

Item (i) follows from the spectral theorem for positive semidefinite matrices. Item (ii) is based on Noga Alon's Combinatorial Nullstellensatz [4]. While this result has been recently discovered in [84], we offer a direct geometric proof that does not necessitate familiarity with commutative algebra and Hilbert polynomials, as required in [84]. Item (iii) relies on a necessary conditions for a hyperplane to be a supporting hyperplane of the convex hull of  $\hat{V}(n+1, 2d)$ , which might have its own significance. Finally, (iv) is a straightforward observation:  $V(n, d)$  is a linear projection of  $V(n, d+1)$ .

### 6.2.2. Related work and discussion

Item (ii) of Theorem 6.2.3 is somehow disappointing. As explained in §6.1.3 before, the Carathéodory number of  $\hat{V}(n+1, d)$  is the maximum nonnegative symmetric rank of a symmetric tensor in the convex hull of the locus of rank-one tensors. Without the nonnegativity condition, [26, Theorem 3] shows that the real symmetric rank is not greater than twice the complex symmetric rank. The complex symmetric rank, in turn, is understood by the celebrated Alexander-Hirschowitz theorem [3]. Specifically, except for a finite number of cases the maximum complex symmetric rank is  $\lceil \frac{1}{n+1} \binom{n+d}{n} \rceil$ ; in the exceptions, it is  $\lceil \frac{1}{n+1} \binom{n+d}{n} \rceil + 1$ . We will explain these algebraic results in more detail, and explain the inequality  $\kappa(\hat{V}(n+1, d)) \leq \frac{2}{n+1} \binom{n+d}{n}$  for odd  $d$  in §6.4.2. However, this behavior is irrelevant to obtain upper bounds on  $\kappa(V(n, d))$  for both  $d$  even and odd. Moreover, the lower bound of item (ii) in Theorem 6.2.3 shows that, as  $d \rightarrow \infty$ , the multiplicative improvement over the bound given by the Carathéodory theorem is asymptotically like  $(1 - \frac{n}{2n})$ . If we fix  $d$  and let  $n$  go to infinity the situation is even more dire, since there is no asymptotical improvement over Carathéodory's bound.

As we mentioned before, item (ii) of our Theorem 6.2.3 is not new. The work [84] proves it with a less elementary argument using the Hilbert polynomials. Also, notice that [84] has a similar bound for the odd degree case, but considering the Veronese image of the cube  $[0, 1]^n$  instead of the whole  $\mathbb{R}^n$ .

### 6.3. $(1 - \epsilon)$ -nets for superlevel sets of polynomials

**Notation 6.3.1.** Given positive integers  $n, d$ , we set

$$m(n, d) := \binom{n+d}{n},$$

which is the dimension of the vector space of homogeneous polynomials of degree  $d$  in  $n+1$  variables.

In this section, we extend the strategy explained in §6.1.1 for disks and annuli to general polynomial superlevel sets of arbitrary degree. In few words, our strategy can be summarized as follows:

- (i) We pushforward a measure in  $\mathbb{R}^n$  using a Veronese embedding, so that we can identify the superlevel sets of polynomials in  $\mathbb{R}^n$  with halfspaces in  $\mathbb{R}^m$
- (ii) We apply the classical Carathéodory theorem in  $\mathbb{R}^m$  to find a centerpoint
- (iii) In order to pullback this centerpoint, we express it as a convex combination of points in the Veronese variety

The following result is a lower bound on the size of  $(1 - \epsilon)$ -nets for polynomial superlevel sets.

**Lemma 6.3.2.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^n$  which is absolutely continuous with respect to the Lebesgue measure, and let  $d$  be a positive integer. For every  $\epsilon > 0$ , a  $(1 - \epsilon)$ -net of the range space  $(\mu, F_{2d})$  has at least  $m(n, d)$  points.*

*Proof.* We proceed by contradiction. Let  $\epsilon > 0$ , and let  $X = \{c_1, \dots, c_k\}$  be a  $(1 - \epsilon)$ -net of  $(\mu, F_{2d})$  of size  $k < m(n, d)$ . Since the  $\mathbb{R}$ -vector space  $\mathbb{R}[\mathbf{x}]_{\leq d}$  has dimension  $m(n, d)$ , the evaluation linear

map

$$\begin{aligned} \mathbb{R}[\mathbf{x}]_{\leq d} &\rightarrow \mathbb{R}^k \\ f &\mapsto (f(c_1), \dots, f(c_k)) \end{aligned}$$

has a nontrivial kernel. In particular, we find a  $g$  in  $\mathbb{R}[\mathbf{x}]_{\leq d}$  that vanishes on every point of  $X$ . Hence, for any  $\delta > 0$  the polynomial  $f = -g^2 + \delta$  in  $\mathbb{R}[\mathbf{x}]_{\leq 2d}$  is positive on every point of  $X$ . Moreover, since  $\mu$  is absolutely continuous with respect to the Lebesgue measure and the measure of  $(f \geq 0)$  is arbitrarily small, we can find  $\delta = \delta(\epsilon)$  such that  $\mu((f \geq 0)) < \epsilon$ . Therefore,  $X$  is not a  $(1 - \epsilon)$ -net of  $(\mu, F_{2d})$ .  $\square$

The following results provides bounds for the size of a  $(1 - \epsilon)$ -net for  $F_d$  in terms of the Carathéodory number of the Veronese varieties.

**Proposition 6.3.3.** *Let  $\mu$  be a probability measure on  $\mathbb{R}^n$  and let  $d$  be a positive integer.*

- (i) *There exists a weak  $(1 - m(n, d)^{-1})$ -net of  $(\mu, F_d)$  of size at most  $\kappa(V(n, d))$*
- (ii) *If  $\mu$  is absolutely continuous with respect to the Lebesgue measure, then for every  $\delta > 0$  there exists a weak  $(1 - (m(n, 2d) + 1)^{-1} + \delta)$ -net of  $(\mu, F_{2d})$  of size at most  $\kappa(\hat{V}(n + 1, 2d))$*

*Proof.* We first prove (i). Let  $\nu$  be the probability measure on  $V(n, d)$  obtained by pushing forward  $\mu$  by the Veronese embedding  $v_d : \mathbb{R}^n \rightarrow \mathbb{R}^{m(n, d)-1}$ . By the centerpoint theorem, we find a point  $c$  in  $\mathbb{R}^{m(n, d)-1}$  such that every half-plane  $H$  in  $\mathbb{R}^{m(n, d)-1}$  containing  $c$  satisfies  $\nu(H) \geq m(n, d)^{-1}$ . By the hyperplane separation theorem [34, Exercise 2.22],  $c$  lies in the convex hull of  $V(n, d)$ . If we set  $k = \kappa(V(n, d))$ , we thus can write

$$c = \sum_{i=1}^k \lambda_i c_i,$$

where  $c_i$  lies on  $V(n, d)$  for every  $1 \leq i \leq k$ , and  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$  lies on the  $(k - 1)$ -dimensional canonical simplex. We conclude proving that

$$X = \{v_d^{-1}(c_i) : 1 \leq i \leq k\} \subset \mathbb{R}^n$$

is a  $(1 - m(n, d)^{-1})$ -net for  $(\mu, F_d)$ . The affine Veronese embedding  $v_d$  induces an isomorphism between the  $\mathbb{R}$ -vector spaces

$$\begin{aligned} \Phi : \mathbb{R}[\mathbf{x}]_{\leq d} &\rightarrow \mathbb{R}[\mathbf{y}]_{\leq 1} & (6.3) \\ f = \sum_{\substack{\mathbf{e}=(e_1, \dots, e_n), \\ |\mathbf{e}| \leq d}} \alpha_{\mathbf{e}} \mathbf{x}^{\mathbf{e}} &\mapsto L_f = \sum_{\substack{\mathbf{e}=(e_1, \dots, e_n), \\ |\mathbf{e}| \leq d}} \alpha_{\mathbf{e}} y_{\mathbf{e}} = \sum_{i=0}^{m(n, d)-1} \alpha_i y_i, \end{aligned}$$

where for simplicity we index the monomials of degree at most  $d$  in the variables  $\mathbf{x} = (x_1, \dots, x_n)$  with  $0 \leq i \leq m(n, d) - 1$ , and  $\mathbf{y} = (y_0, \dots, y_{m(n, d)-1})$ . In particular, we find that  $(f \geq 0)$  is the pullback  $v_d^{-1}(H_f \cap V(n, d))$ , where  $H_f = (L_f \geq 0)$ . Therefore,  $(f \geq 0)$  contains all the points in  $X$  if and only if  $H_f$  contains  $c_i$  for every  $1 \leq i \leq k$ . Hence, we find  $\mu(f \geq 0) \geq m(n, d)^{-1}$ .

Secondly, we prove (ii). As before, we let  $\nu$  be the probability measure on  $V(n, 2d)$  obtained by pushing forward  $\mu$  by the Veronese embedding. Moreover, if we identify  $\mathbb{R}^n$  with the hyperplane

$x_0 = 1$  in  $\mathbb{R}^{n+1}$ , we find  $V(n, 2d) \subset \hat{V}(n+1, 2d) \subset \mathbb{R}^{m(n,d)}$ . Thus, we can repeat the previous strategy applying the centerpoint theorem in  $\mathbb{R}^{m(n,d)}$ .

Let  $k = \kappa(\hat{V}(n+1, 2d))$ . Then, we find a set

$$Y = \{c_j = (c_{0j}, c_{1j}, \dots, c_{(m(n,d)-1)j}) \in \hat{V}(n+1, 2d) : 1 \leq j \leq k\}$$

such that

$$c = \sum_{j=1}^k \lambda_j c_j ,$$

where  $\lambda = (\lambda_1, \dots, \lambda_k)$  lies on the  $(k-1)$ -dimensional canonical simplex. Additionally, the centerpoint  $c$  must lie on  $y_0 = 1$ , implying the relation

$$\sum_{j=1}^k \lambda_j c_{0j} = 1 . \quad (6.4)$$

Since  $x_0^{2d}$  is nonnegative for any real number, it follows that  $\hat{V}(n+1, 2d)$  lies in the half-space  $y_0 \geq 0$ . Therefore, if  $c_{0j}$  is nonzero for every  $1 \leq j \leq k$  we can rewrite

$$c = \sum_{j=1}^k \lambda_j c_{0j} (c_{0j}^{-1} c_j) , \quad (6.5)$$

and the result follows by (6.4) and the fact that  $\hat{V}(n+1, 2d)$  is a cone. However, some  $c_j$  might lie on  $y_0 = 0$ . We now deal with this situation. Let

$$(Y_i = \{c_1^i, \dots, c_k^i\})_{i=1}^{\infty}$$

be a sequence of sets in  $\hat{V}(n+1, 2d) \cap \{y_0 > 0\}$  satisfying that

$$c_j^i \xrightarrow{i \rightarrow \infty} c_j$$

for each  $1 \leq j \leq k$ . Additionally, define

$$c^i = \sum_{j=1}^k \lambda_j c_j^i ,$$

so that  $c^i \xrightarrow{i \rightarrow \infty} c$ . Moreover, for each  $1 \leq i < \infty$  we can apply the same normalization as (6.5) to write  $c^i$  as a convex combination of  $k$  points in  $y_0 = 1$ .

Let  $\delta > 0$  and  $\epsilon = (m(n, 2d) + 1)^{-1} - \delta$ . We show that there exists some  $i = i(\delta, \mu)$  such that  $Y^i$  is a weak  $(1 - \epsilon)$ -net for  $(\mu, F_{2d})$ . We proceed by contradiction. For each  $0 \leq i < \infty$ , suppose that there exists a polynomial

$$f_i = \sum_{\substack{\mathbf{e}=(e_1, \dots, e_n) \\ |\mathbf{e}| \leq 2d}} \alpha_{\mathbf{e}i} \mathbf{x}^{\mathbf{e}}$$

which is nonnegative on every point of  $Y_i$  and satisfies

$$\mu((f_i \geq 0)) < \epsilon . \quad (6.6)$$

In particular  $f_i$  is nonzero, and we can assume

$$\|f_i\|^2 := \sum_{\substack{\mathbf{e}=(e_1, \dots, e_n) \\ |\mathbf{e}| \leq 2d}} (\alpha_{\mathbf{e}i})^2 = 1$$

so that  $f_i$  lies in the (compact) unit sphere. Therefore, the sequence  $(f_i)_{i=1}^\infty$  converges to some  $f$  uniformly on compacta. Moreover, since  $f_i(c_j^i) \geq 0$  it follows that  $f(c_j) \geq 0$  for every  $1 \leq j \leq k$ . In order to find a contradiction, we prove that

$$\mu(f \geq 0) \leq \epsilon < \frac{1}{m(n, 2d) + 1},$$

which is not possible by the isomorphism (6.3) since it implies the existence of a half-space  $H_f = (L_f \geq 0)$  containing the centerpoint  $c$  of measure strictly less than  $(m(n, 2d) + 1)^{-1}$ . Assume the contrary, i.e.  $\mu(f \geq 0) > \epsilon$ . Since  $\mu$  is absolutely continuous with respect to the Lebesgue measure and algebraic sets have measure zero, we find  $\mu((f \geq 0)) = \mu((f > 0))$ . On the other hand, as  $f_i \xrightarrow{i \rightarrow \infty} f$  pointwise it follows that

$$\chi_{(f \geq 0)} \leq \liminf_{i \rightarrow \infty} \chi_{(f_i \geq 0)},$$

where  $\chi_S$  is the indicator function of a subset  $S$  of  $\mathbb{R}^n$ . Additionally, by (6.6) and the Fatou lemma [47, Lemma 18.13] we derive

$$\int_{\mathbb{R}^n} \chi_{(f \geq 0)} d\mu = \mu((f \geq 0)) \leq \frac{1}{m(n, d) + 1} - \delta,$$

which is a contradiction. □

## 6.4. The Carathéodory number of Veronese varieties

The goal of this section is to derive the bounds for the Carathéodory numbers of the Veronese varieties presented in Theorem 6.2.3. Remarkably, the parity of the degree of the Veronese embedding matters. Because of this, we address the even and odd degrees separately.

### 6.4.1. Even degree

We begin by proving the following lemma, which is a direct consequence of the spectral theorem for real symmetric matrices.

**Lemma 6.4.1.**  $\kappa(\hat{V}(n+1, 2)) = n+1$ .

*Proof.* The Veronese cone  $\hat{V}(n+1, d)$  is the locus of symmetric rank-one tensors of format  $(n+1)^{\otimes d}$  (recall §2.2.2). In particular, if  $d = 2$  it is just the locus of real symmetric matrices.

Let  $M$  be a nonzero  $(n+1) \times (n+1)$  real symmetric matrix in the convex hull of  $\hat{V}(n+1, d)$ . We can write

$$M = \sum_{i=1}^k \lambda_i M_i$$

where  $M_i = \mathbf{x}_i \otimes \mathbf{x}_i \equiv \mathbf{x}_i \mathbf{x}_i^T$  for some  $\mathbf{x}_i$  in  $\mathbb{R}^{n+1}$  and  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$  is a point in the  $(k-1)$ -dimensional canonical simplex. As  $M_i$  is positive semidefinite for every  $1 \leq i \leq k$ ,  $M$  is a positive

combination of positive semidefinite matrices. Thus,  $M$  is also positive semidefinite. By the spectral theorem, we can write

$$M = \sum_{i=0}^n \mu_i \mathbf{y}_i \mathbf{y}_i^T = \sum_{i=0}^n \left( \frac{\mu_i}{\mu} \right) (\sqrt{\mu_i} \mathbf{y}_i)(\sqrt{\mu_i} \mathbf{y}_i)^T \equiv \sum_{i=0}^n \left( \frac{\mu_i}{\mu} \right) (\sqrt{\mu_i} \mathbf{y}_i) \otimes (\sqrt{\mu_i} \mathbf{y}_i)$$

where  $\mu_0, \mu_1, \dots, \mu_n$  are the eigenvalues of  $M$ ,  $\mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_n$  define an orthonormal basis of  $\mathbb{R}^{n+1}$ , and

$$\mu = \sum_{i=0}^n \mu_i. \quad (6.7)$$

By definition, the point

$$\boldsymbol{\mu} = \left( \frac{\mu_0}{\mu}, \frac{\mu_1}{\mu}, \dots, \frac{\mu_n}{\mu} \right) \quad (6.8)$$

lies on the  $n$ -dimensional canonical simplex. Hence,  $M$  can be written as a convex combination of  $n + 1$  points on  $\hat{V}(n + 1, 2)$ . Since we find positive semidefinite matrices of rank  $n + 1$ , the result follows.  $\square$

Now, we sequentially prove the bounds for the Carathéodory number of the Veronese varieties stated in Theorem 6.2.3.

*Proof of (i) in Theorem 6.2.3.* We embed  $\mathbb{R}^n$  in  $\mathbb{R}^{n+1}$  with the additional coordinate  $x_0 = 1$ . Then, the affine Veronese variety  $V = V(n, 2)$  consists of rank-one  $(n + 1) \times (n + 1)$  symmetric matrices  $\mathbf{x} \otimes \mathbf{x}$  for some nonzero  $\mathbf{x} = (1, x_1, \dots, x_n)$ , regarded as a column vector.

Similarly to the proof of Lemma 6.4.1, for any  $M$  in the convex hull of  $V$  there exist a sum of  $k \leq n + 1$  rank one positive matrices

$$M = E_1 + \dots + E_k.$$

If  $M$  is full rank and  $k = n + 1$ , then there are a lot of such decompositions, transformed by rotations of  $\mathbb{R}^{n+1}$  one into another. Because of this, one may also assume that the corner element  $(E_i)_{0,0}$  is positive for every  $1 \leq i \leq k$ . In order to achieve this, we just need to rotate so that the hyperplanes in  $\mathbb{R}^{n+1}$

$$\{\mathbf{x}^T E_i \mathbf{x} = 0\} = \{\mathbf{x}^T (\mathbf{e}_i \mathbf{e}_i^T) \mathbf{x} = 0\} = \{(\mathbf{x}^T \mathbf{e}_i)^2 = 0\} = \{\mathbf{x}^T \mathbf{e}_i = 0\}$$

do not contain the vector  $\mathbf{z} = (1, 0, \dots, 0)$ . If  $A$  is not full rank, then the decompositions into  $k = \text{rank}(M)$  matrices of rank one are transformed one into another by rotations of the  $k$ -dimensional quotient  $\mathbb{R}^{n+1} / \ker M$ . It is also possible to rotate so that  $\mathbf{z}$  is not inside any of the hyperplanes  $\{\mathbf{x}^T E_i \mathbf{x} = 0\}$ , since  $\mathbf{z}$  does not lie in  $\ker M$  (equivalently  $M_{0,0} \neq 0$ ) and the intersection of those hyperplanes is  $\ker A$ .

After this, it is possible to take out the positive corner elements of the  $E_i$  and write

$$M = \lambda_1 \mathbf{e}_1 \mathbf{e}_1^T + \dots + \lambda_k \mathbf{e}_k \mathbf{e}_k^T$$

with positive  $\lambda_i$  and vectors  $\mathbf{e}_i$  with first entry equal to one. Since  $M$  lies in the hyperplane  $H = \{M_{0,0} = 1\}$ , as well as  $\mathbf{e}_i \mathbf{e}_i^T$  for every  $1 \leq i \leq k$ , it follows that  $\lambda_1 + \dots + \lambda_k = 1$ . Hence,  $M$  is a convex combination of at most  $n + 1$  elements of  $V$ .  $\square$

*Proof of (ii) in Theorem 6.2.3.* Consider the univariate polynomial in the variable  $x$

$$F(x) = (x - 1) \dots (x - d) ,$$

and define the multivariate polynomial of degree  $D = 2d$  in the variables  $x_1, \dots, x_n$

$$P(\mathbf{x}) = P(x_1, \dots, x_n) = F(x_1)^2 + \dots + F(x_n)^2 .$$

By definition  $P$  is nonnegative, and its real zero set  $M \subset \mathbb{R}^n$  is the finite product  $\{1, \dots, d\}^n$ , of size  $d^n$ . The Veronese image  $v(M)$  is an intersection of  $V(n, D)$  with the support hyperplane corresponding to the polynomial  $P$ . Hence, we have  $\kappa(V(n, D)) \geq \kappa(v(M))$ . Since  $v(M)$  is a finite set, its Carathéodory number equals the dimension of its affine hull plus one, that is  $\dim v(M) + 1$ . This in turn equals the dimension of the vector space of restrictions of polynomials of degree at most  $D$  to  $M$ .

In order to understand the dimension of the space of such restrictions, let us analyze the kernel of the restriction map, that is the vector subspace  $\mathcal{Z}(M)$  of  $\mathbb{R}[x_1, \dots, x_n]_{\leq D}$  of polynomials vanishing on  $M$ . By Noga Alon's Combinatorial Nullstellensatz [4, Theorem 1.1], any polynomial  $Q$  in  $\mathcal{Z}(M)$  can be written as

$$Q(x_1, \dots, x_n) = S_1(x_1, \dots, x_n) F(x_1) + \dots + S_n(x_1, \dots, x_n) F(x_n) ,$$

where  $\deg S_i \leq \deg Q - d$  for each  $1 \leq i \leq n$ . When  $\deg Q \leq D$ , we have  $\deg S_i \leq d$  for every  $1 \leq i \leq n$ . The dimension of all possible combinations of the  $S_i$  is at most  $n \cdot m(n, d) = n \binom{n+d}{n}$ . Furthermore, the kernel of the linear map

$$(S_1, \dots, S_n) \mapsto S_1 F(x_1) + \dots + S_n F(x_n)$$

contains the  $\binom{n}{2}$  linearly independent Koszul-like vectors of polynomials

$$(0, \dots, \underbrace{F(x_j)}_{i\text{-th entry}}, \dots, -\underbrace{F(x_i)}_{j\text{-th entry}}, \dots, 0) .$$

Therefore, the kernel of the restriction map for polynomials of degree  $D = 2d$  has dimension at most  $n \binom{n+k}{k} - \binom{n}{2}$ . Thus, for the degree of the image of the restriction map we obtain

$$\kappa(V(n, 2d)) \geq m(n, 2d) - n \cdot m(n, d) + \binom{n}{2} = \binom{2d+n}{n} - n \binom{d+n}{n} + \binom{n}{2} .$$

Additionally, when  $D \rightarrow \infty$ , this is asymptotically

$$\kappa(V(n, 2d)) \geq \frac{(2d)^n}{n!} - n \frac{d^n}{2^n n!} = \left(1 - \frac{n}{2^n}\right) \frac{(2d)^n}{n!} = \left(1 - \frac{n}{2^n}\right) m(n, 2d) .$$

□

*Proof of (ii) in Theorem 6.2.3.* We proceed similarly to [13], where the Carathéodory number of the moment curve is computed. Let  $D = 2d$ , and consider  $\hat{V}(n+1, D)$ . In particular,  $\hat{V}(n+1, D)$  is a cone over its section with the hyperplane in  $\mathbb{R}^{m(n, D)}$  that is identified (by the Veronese embedding) with the homogeneous polynomial

$$(x_0^2 + x_1^2 + \dots + x_n^2)^d = 1 .$$



Namely, this hyperplane section corresponds to the restriction of homogeneous map

$$\begin{aligned} \hat{v} : \mathbb{R}^{n+1} &\rightarrow \mathbb{R}^{m(n,D)} \\ \mathbf{x} = (x_0, \dots, x_n) &\mapsto \mathbf{x}^{\otimes D}, \end{aligned} \quad (6.9)$$

to the unit sphere  $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ . Therefore, we may focus on estimating the Carathéodory number of  $\hat{v}(\mathbb{S}^n)$ , since it is the same as the Carathéodory number of  $\hat{V}(n+1, D)$ . This point of view has the advantage that  $v(\mathbb{S}^n)$ , as well as its convex hull, is compact.

Let  $\xi$  be a point in  $\text{conv}(\hat{v}(\mathbb{S}^n))$ . Additionally, let  $x$  be a point in  $\mathbb{S}^n$  and consider the maximum  $t$  such that

$$\xi - t\hat{v}(x) \in \text{conv} \hat{v}(\mathbb{S}^n),$$

which is attained since the convex hull is compact. We define the point

$$\eta = \xi - t\hat{v}(x),$$

which lies in the boundary of  $\text{conv} \hat{v}(\mathbb{S}^n)$ . By the Hahn-Banach theorem (see e.g. [157]), there exists a linear functional  $L$  that defines a supporting plane in  $\mathbb{R}^{m(n,D)}$  of  $\text{conv}(\hat{v}(\mathbb{S}^n))$  and contains  $\eta$ . Additionally, the Veronese map identifies  $L$  with a homogeneous polynomial  $P$  in of degree  $D$  in  $n+1$  variables. In particular, if

$$M = \{x \in \mathbb{S}^n \mid P(x) = 0\},$$

it follows that  $\eta$  lies in  $\text{conv} \hat{v}(M)$ . Notice that, if  $\dim \text{conv} \hat{v}(M) \leq N$ , by Carathéodory's theorem any point in  $\text{conv} \hat{v}(M)$  can be expressed as a convex combination of  $N+1$  points. Additionally, since  $\hat{v}$  yields an identification between homogeneous polynomials of degree  $D$  in  $\mathbb{R}^n$  and linear forms in  $\mathbb{R}^{m(n,D)}$ , we find

$$\dim \text{conv} \hat{v}(M) = \binom{n+2d}{n} - \dim \mathcal{Z}(M) - 1,$$

where  $\mathcal{Z}(M)$  is the vector space of homogeneous polynomials of degree  $D$  in  $n+1$  that vanish on  $M$ . Therefore, we derive

$$\kappa(\hat{v}(M)) \leq \binom{n+2d}{n} - \dim \mathcal{Z}(M)$$

and

$$\kappa(\hat{V}(n+1, 2d)) \leq \binom{n+2d}{n} - \dim \mathcal{Z}(M) + 1.$$

We conclude providing a lower bound for the dimension of  $\mathcal{Z}(M)$ . Notice that  $P$  lies in  $\mathcal{Z}(M)$  by definition. On the other hand, since  $M$  is the locus in  $\mathbb{R}^{n+1}$  where the minimum of  $P$  is attained, all the  $(n+1)^2$  products

$$x_j \frac{\partial P}{\partial x_j}$$

vanish on  $M$ . Nevertheless, there might be linear relations between these polynomials. If we have a nontrivial linear relation of the form

$$\alpha_0 x_0 \frac{\partial P}{\partial x_j} + \dots + \alpha_n x_n \frac{\partial P}{\partial x_j} = (\alpha_0 x_0 + \dots + \alpha_n x_n) \frac{\partial P}{\partial x_j} = 0$$

for some  $0 \leq j \leq n$ , it follows that  $\frac{\partial P}{\partial x_j} = 0$ . However, since  $P$  is homogeneous of degree  $2d$  there must be some  $0 \leq j_0 \leq n$  for which  $\frac{\partial P}{\partial x_{j_0}}$  is nonzero. In particular, all the products

$$x_0 \frac{\partial P}{\partial x_{j_0}}, \dots, x_n \frac{\partial P}{\partial x_{j_0}}$$

are linearly independent, and lie in  $\mathcal{Z}(M)$ . Finally, if  $P$  is linearly independent from these  $n + 1$  partial derivatives, we find

$$\dim \mathcal{Z}(M) \geq n + 2$$

implying the bound (iii) in Theorem 6.2.3

$$\kappa(\hat{V}(n + 1, 2d)) \leq \binom{n + 2d}{n} - n - 1 .$$

If this is not the case, assume without loss of generality that  $j_0 = 0$ . Then, we find the nontrivial linear relation

$$P = (\alpha_0 x_0 + \dots + \alpha_n x_n) \frac{\partial P}{\partial x_0} . \quad (6.10)$$

If  $\alpha_0 = 0$ , the restriction of the identity above to any line in  $\mathbb{R}^{n+1}$  of the form

$$x_1 = \beta_1, \dots, x_n = \beta_n ,$$

for some  $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$  in  $\mathbb{R}^n$ , becomes an ODE

$$\frac{\partial P}{\partial x_0} \propto P ,$$

with  $P$  as unknown, whose solutions are exponentials. Since  $P$  is a polynomial, this cannot be the case. Therefore, we have  $\alpha_0 \neq 0$ , and we can apply the inverse of the change of coordinates defined by

$$y_0 \mapsto x_0 + \frac{\alpha_1}{\alpha_0} x_1 + \dots + \frac{\alpha_n}{\alpha_0} x_n ,$$

and sending  $y_i \mapsto x_i$  for every  $1 \leq i \leq n$ . Hence, (6.10) becomes

$$\alpha_0 y_0 \frac{\partial P}{\partial y_0} = P ,$$

and any solution of this has the form

$$P(y_0, \dots, y_n) = y_0^{1/a} Q(y_1, \dots, y_n) .$$

Moreover, since  $P$  is a nonnegative polynomial, we have:

- (i)  $\frac{1}{a}$  must be an even positive integer  $2d'$
- (ii)  $Q$  must be a nonnegative of degree  $D - 2d' = 2(d - d')$

Additionally, we find some  $1 \leq k_0 \leq n$  such that the partial derivative

$$R_{k_0} = \frac{\partial Q}{\partial x_{k_0}}$$

is a nonzero polynomial that vanishes at  $M$ . In particular, all the products  $x_i x_j R_{k_0}$  yield linearly independent homogeneous polynomials of degree at most  $2d$  that vanish at  $M$ . Since there are  $\binom{n+2}{2}$  of such products, multiplying by a power of  $x_0$  if necessary to make them of degree  $2d$ , we improve the bound

$$\dim \mathcal{Z}(M) \geq \binom{n+2}{2} \geq n+2.$$

□

### 6.4.2. Odd degree

Regarding odd degrees, the bounds (iv) in Theorem 6.2.3 for the Carathéodory number of the affine Veronese variety follow from the fact that linear projections  $\pi : V(n, 2d) \rightarrow V(n, 2d-1)$  preserve convex combinations. For the Veronese cone in odd degree (not stated in Theorem 6.2.3, since we do not use to bound the size of  $(1 - \epsilon)$ -nets), a bound follows from the results of [26] and [3].

For the sake of completeness, we briefly include the statements of the results that yield this bound. Specifically, the following is a version of the celebrated Alexander–Hirschowitz, stated in the language of symmetric tensors.

**Theorem 6.4.2.** [3] *The symmetric rank of a general symmetric complex tensor of format  $(n+1)^{\otimes d}$  is  $\lceil \frac{1}{n+1} m(n, d) \rceil$ , except for finitely many pairs  $(n, d)$  (see [26, §3.1]).*

In general, over  $\mathbb{R}$  there are (Zariski dense) open sets, within the spaces of tensors of a given format, consisting of tensors that have a fixed rank. Each of these ranks is called a *typical rank*. This phenomenon does not occur over  $\mathbb{C}$ , where there is only one (Zariski dense) open set consisting of tensors of a fixed rank. In particular, this rank is called the *general rank*. Here, we do not discuss real typical ranks in detail (see e.g. [26, 126, 17], and interested readers can look at Table 3.3 in [126] for a list of the typical ranks for a number of formats), but we include the following result.

**Theorem 6.4.3.** ([26, Theorems 2 and 3]) *Given a specific tensor format, the maximum rank of a real tensor is at most twice the smallest typical rank. Moreover, the smallest typical (real) rank is the general rank of complex tensors of the same format.*

**Corollary 6.4.4.** (Follows from [26, §3.1]) *If  $d$  is odd, then*

$$\kappa(\hat{V}(n, d)) \leq \frac{2}{n+1} \binom{n+d}{d}$$

*except for finitely many pairs  $(n, d)$ .*

# Chapter 7

## Cylinders through four cocyclic points

In this short chapter, we provide algebraic certificates for the number of real circular cylinders through a particular configuration of five points in  $\mathbb{A}_{\mathbb{R}}^3$ : four of the five points are cocyclic, i.e. they lie in a common circumference (Theorem 7.3.5 and Corollary 7.3.7). Additionally, we give closed formulas for the defining equations of such cylinders (Corollary 7.3.6).

### 7.1. Projective definition of circular cylinders

Intuitively, a cylinder is a surface in  $\mathbb{A}_{\mathbb{R}}^3$  that is swept by parallel lines along a plane conic. If the conic is a circumference, we say that the cylinder is circular. In particular, cylinders are quadric surfaces. When we look at these quadrics in  $\mathbb{P}_{\mathbb{R}}^3$ , all the parallel lines meet at the “plane at  $\infty$ ”. Therefore, cylinders are just real rank-three quadrics, that is real cones, where the singular point lies at “ $\infty$ ”. To formalize ideas, we establish the following notation for the rest of the chapter.

**Notation 7.1.1.** We denote by  $H_i$  the plane in  $\mathbb{P}_{\mathbb{C}}^3 = \text{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3])$  defined by  $x_i = 0$ , for each  $0 \leq i \leq 3$ . As usual,  $\mathbb{A}_{\mathbb{C}}^3$  is defined by  $y_0 \neq 0$ .

In this chapter, we are interested only in circular cylinders. To define them formally, we utilize a projective characterization by means of the absolute conic, defined as follows.

**Definition 7.1.2** (Absolute conic). The *absolute conic* in  $H_0$ , denoted by  $K$ , is the curve defined by

$$x_1^2 + x_2^2 + x_3^2 = 0 .$$

The following is a useful property of the absolute conic.

**Remark 7.1.3.** (Rigid invariance of the absolute conic [49, Theorem 6.9.21], see also [49, §6.9]) The absolute conic is invariant under “rigid” transformations of  $\mathbb{A}_{\mathbb{C}}^3$ , i.e. those defined by

$$(y_1, y_2, y_3) \mapsto (y_1, y_2, y_3) \cdot M + v \tag{7.1}$$

where  $y_i = x_i/x_0$  for each  $1 \leq i \leq 3$ ,  $M$  is a  $\mathbb{R}^{3 \times 3}$  orthogonal matrix, and  $v$  is a vector in  $\mathbb{R}^3$ .

We begin explaining our definition of circular cylinder. After a rigid transformation of  $\mathbb{A}_{\mathbb{R}}^3$ , any cylinder can be transformed into the one defined by the equation

$$x_1^2 + x_2^2 = r^2 ,$$

for some  $r > 0$ . If we homogenize the previous equation using the variable  $x_0$ , we find

$$x_1^2 + x_2^2 = r^2 x_0 ,$$

which at the plane  $H_0$  restricts to

$$x_1^2 + x_2^2 = (x_1 - i x_2)(x_1 + i x_2) = 0 ,$$

which defines the two complex conjugate lines parametrized by

$$(0 : \alpha : \pm i \alpha : \beta) , \quad (7.2)$$

for  $(\alpha : \beta)$  in  $\mathbb{P}_{\mathbb{C}}^1$ . The following observations are in order:

- (i) The unique real point of the lines (7.2) is  $(0 : 0 : 0 : 1)$ , corresponding to the apex of the cylinder
- (ii) Each of these lines is tangent to the absolute conic  $K$  at the *circular point*  $(0 : 1 : \pm i : 0)$

Hence, this circular cylinder defines two tangent lines to the absolute conic at  $H_0$ . Furthermore, since  $K$  remains invariant under rigid transformations, every circular cylinder exhibits this property. In fact, circular cylinders are characterized by this property. To be more precise, circular cylinders precisely correspond to real rank-three quadrics that define tangent lines to the absolute conic at  $H_0$  (see [49, §6.9 and §10]). We adopt this characterization as our definition.

**Definition 7.1.4** (Circular cylinder). A cylinder  $C$  is *circular* if  $C \cap H_0$  consists of two lines tangent to  $K$ .

**Remark 7.1.5** (The axis of a cylinder). Equivalently, cylinders can be defined affinely by specifying a direction for their axis, a point on the axis, and a radius. In the majority of works focused on extracting circular cylinders from point sets, this perspective is adopted (e.g. [81, 42, 143]). Within our projective framework, we identify the direction  $v$  for the cylinder's axis with a point in  $H_0$ . To be more precise, we interpret an axis represented as  $v = (\alpha, \beta, \gamma)$  as the point  $(0 : \alpha : \beta : \gamma)$ .

## 7.2. The conic of admissible vertices

As cylinders are real cones with their singular point at  $H_0$ , they are determined by the apex and a section with a plane in  $\mathbb{P}_{\mathbb{C}}^3$ , say  $H_3$ . In particular, given a circular cylinder  $C \subset \mathbb{P}_{\mathbb{C}}^3$ , the section  $C \cap H_3$  defines a plane conic in  $H_3$ . When we focus on the cylinders through five points in  $\mathbb{A}_{\mathbb{R}}^3$ , and four of these points lie in the plane  $H_3$ , it follows that the section  $C \subset \mathbb{P}_{\mathbb{C}}^3$  belongs to the pencil of conics that the four coplanar points determine in  $H_3$ .

In this chapter, we focus specifically on configurations where four of the points are coplanar. Furthermore, in §7.3 we impose the additional constraint that the four coplanar points lie on a common circumference. However, for the present section, our primary objective is to derive the equation of a plane conic in  $H_0$ , relative to four coplanar points in  $H_3$ , that contains the apex of any circular cylinder passing through these four points

The following lemma describes explicitly the singular lines in  $H_0$  that are tangent to the absolute conic and have a singular real point  $V$ , as it arises in  $C \cap H_0$  when  $C$  is a circular cylinder.

**Lemma 7.2.1.** Let  $V = (\alpha : \beta : \gamma)$  in  $\mathbb{P}_{\mathbb{C}}^2$ . The conic singular at  $V$  that defines tangent lines to the absolute conic  $K$  is defined by

$$(x_1 \ x_2 \ x_3) \begin{pmatrix} \beta^2 + \gamma^2 & -\alpha \beta & -\alpha \gamma \\ -\alpha \beta & \alpha^2 + \gamma^2 & -\beta \gamma \\ -\alpha \gamma & -\beta \gamma & \alpha^2 + \beta^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 . \quad (7.3)$$

*Proof.* For each  $i = 1, 2$ , let  $\ell_i$  be the line through  $V$  that is tangent to  $K$ . Also, let  $d$  be the double line spanned by  $Q_1 = \ell_1 \cap K$  and  $Q_2 = \ell_2 \cap K$ , i.e. the double line of the polar of  $V$  with respect to  $K$ . The three conics  $K$ ,  $\ell_1 \cup \ell_2$ , and  $d$  have fixed tangencies to  $\ell_1, \ell_2$  respectively at  $Q_1, Q_2$ . Therefore, each conic lies in the pencil spanned by the other two.

The polar of  $V$  with respect to  $K$  is

$$F = F(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \alpha x + \beta y + \gamma z = 0. \quad (7.4)$$

Hence,  $d$  is defined by  $F^2 = 0$ . Without loss of generality, we can assume either  $\alpha \neq 0$ ,  $\beta \neq 0$ , or  $\gamma \neq 0$ . For simplicity, we choose  $\alpha \neq 0$ . A conic in the pencil spanned by  $K$  and  $d$  is defined by a matrix

$$M(\lambda_0, \lambda_1) = \lambda_0 \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} + \lambda_1 \begin{pmatrix} \alpha^2 & \alpha\beta & \alpha\gamma \\ \alpha\beta & \beta^2 & \beta\gamma \\ \alpha\gamma & \beta\gamma & \gamma^2 \end{pmatrix}$$

for some  $(\lambda_0 : \lambda_1)$  in  $\mathbb{P}_{\mathbb{C}}^1$ . Moreover,  $\ell_1 \cup \ell_2$  is a singular conic in this pencil. In particular, the determinant of its defining matrix vanishes. Since we have

$$\det M(\lambda_0, \lambda_1) = \alpha^4 \lambda_0^2 (\lambda_0 \alpha^2 + \lambda_1 (\alpha^2 + \beta^2 + \gamma^2)) \quad (7.5)$$

and the factor  $\lambda_0^2$  corresponds to the double line  $d$ , it follows that  $\ell_1 \cup \ell_2$  is defined by

$$(\lambda_0 : \lambda_1) = (\alpha^2 + \beta^2 + \gamma^2 : -\alpha^2).$$

□

At this point, we explicitly specify our selection of four coplanar points in  $\mathbb{A}_{\mathbb{R}}^3$ . More precisely, we establish the following notation.

**Notation 7.2.2.** Let  $Q_1, Q_2, Q_3, Q_4$  be four affine points in  $H_3$ , that do not define a rectangle. Applying a rigid transformation, we can assume that

$$Q_1 = (1 : 0 : 0 : 0), \quad Q_2 = (1 : 1 : 0 : 0), \quad Q_3 = (1 : \alpha_3 : \beta_3 : 0), \quad Q_4 = (1 : \alpha_4 : \beta_4 : 0)$$

for some real  $\alpha_3, \alpha_4, \beta_3, \beta_4$ .

The following result provides an explicit parametrization of the pencil of conics, in the plane  $H_3$ , defined by the points  $Q_1, Q_2, Q_3$ , and  $Q_4$ .

**Lemma 7.2.3.** *The pencil of conics in  $H_3$  through  $Q_1, Q_2, Q_3$ , and  $Q_4$  is parametrized by*

$$\phi : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^5$$

$$(\lambda_0 : \lambda_1) \mapsto (x_0 \ x_1 \ x_2) \begin{pmatrix} 0 & \lambda_1 & \mu_0 \lambda_0 + \mu_1 \lambda_1 \\ \lambda_1 & -2\lambda_1 & \lambda_0 \\ \mu_0 \lambda_0 + \mu_1 \lambda_1 & \lambda_0 & \rho_0 \lambda_0 + \rho_1 \lambda_1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

where

$$\rho_0 = \frac{-2(\alpha_3 - \alpha_4)}{(\beta_3 - \beta_4)}, \quad \rho_1 = \frac{-2((\alpha_3 \beta_4 (1 - \alpha_3) - \alpha_4 \beta_3 (1 - \alpha_4))}{\beta_3 \beta_4 (\beta_3 - \beta_4)} \quad (7.6)$$

and

$$\mu_0 = \frac{(\alpha_3 \beta_4 - \alpha_4 \beta_3)}{(\beta_3 - \beta_4)}, \quad \mu_1 = \frac{\alpha_3 \beta_4^2 (1 - \alpha_3) - \alpha_4 \beta_3^2 (1 - \alpha_4)}{\beta_3 \beta_4 (\beta_3 - \beta_4)}.$$

**Remark 7.2.4.** The hypothesis on the points  $Q_0, Q_1, Q_2,$  and  $Q_3$  of not being rectangular is necessary for the good definition of the constants  $\rho_0, \rho_1, \mu_0, \mu_1,$  i.e. to ensure that

$$\beta_3, \beta_4, \beta_3 - \beta_4$$

are all nonzero.

*Proof.* Let

$$F = F(x_0, x_1, x_2) = \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

define a conic through  $Q_1, Q_2, Q_3,$  and  $Q_4$ . Since  $F(q_1) = F(q_2) = 0$ , we find

$$A = 0, \quad B = -2D.$$

Moreover, since  $F(q_3) = F(q_4) = 0$  we also find

$$\begin{pmatrix} 2\alpha_3^2 D - 2\alpha_3 D - 2\alpha_3 \beta_3 F \\ 2\alpha_4^2 D - 2\alpha_4 D - 2\alpha_4 \beta_4 F \end{pmatrix} = \begin{pmatrix} \beta_3^2 & 2\beta_3 \\ \beta_4^2 & 2\beta_4 \end{pmatrix} \begin{pmatrix} C \\ E \end{pmatrix}.$$

Applying Cramer's rule, we derive

$$C = \frac{\begin{vmatrix} 2\alpha_3^2 D - 2\alpha_3 D - 2\alpha_3 \beta_3 F & 2\beta_3 \\ 2\alpha_4^2 D - 2\alpha_4 D - 2\alpha_4 \beta_4 F & 2\beta_4 \end{vmatrix}}{\begin{vmatrix} \beta_3^2 & 2\beta_3 \\ \beta_4^2 & 2\beta_4 \end{vmatrix}} = \rho_1 F + \rho_2 D.$$

Similarly, we derive

$$E = \frac{\begin{vmatrix} \beta_3^2 & 2\alpha_3^2 D - 2\alpha_3 D - 2\alpha_3 \beta_3 F \\ \beta_4^2 & 2\alpha_4^2 D - 2\alpha_4 D - 2\alpha_4 \beta_4 F \end{vmatrix}}{\begin{vmatrix} \beta_3^2 & 2\beta_3 \\ \beta_4^2 & 2\beta_4 \end{vmatrix}} = \mu_1 F + \mu_2 D.$$

□

The following is the definition of the conic of vertices, that vanishes at the "admissible" apexes, or axis directions, of circular cylinders through  $Q_1, Q_2, Q_3,$  and  $Q_4$ .

**Definition 7.2.5** (Conic of vertices). The *conic of vertices* relative to the points  $Q_1, Q_2, Q_3, Q_4$  is the curve in  $H_0$  defined by

$$G(x_1, x_2, x_3) = 2x_1^2 + \rho_1 x_2^2 + (2 + \rho_1)x_3^2 + 2\rho_0 x_1 x_2 = 0. \quad (7.7)$$

In the following lemma, we prove the aforementioned property of the conic of vertices.

**Lemma 7.2.6.** Let  $C$  be a circular cylinder through  $Q_1, Q_2, Q_3, Q_4$  of vertex  $V = (0 : \alpha : \beta : \gamma)$ . We have the following:

(i)  $V$  lies in the conic of vertices

(ii) The intersection  $C \cap H_3$  is the conic defined by the matrix

$$M(\alpha, \beta, \gamma) = \begin{pmatrix} 0 & \beta^2 + \gamma^2 & 2\mu_0 \alpha \beta + \mu_1(\beta^2 + \gamma^2) \\ \beta^2 + \gamma^2 & -2(\beta^2 + \gamma^2) & 2\alpha \beta \\ 2\mu_0 \alpha \beta + \mu_1(\beta^2 + \gamma^2) & 2\alpha \beta & 2\rho_0 \alpha \beta + \rho_1(\beta^2 + \gamma^2) \end{pmatrix} \quad (7.8)$$

*Proof.* By Definition 7.1.4, the intersection  $C \cap H_0$  is singular at  $V$  and tangent to  $K$ . Moreover, since  $C$  contains  $Q_1, Q_2, Q_3, Q_4$  the intersection  $C \cap H_3$  is a conic in the pencil defined by these four points in  $H_3$ . By Lemmas 7.2.3 and 7.2.3, the restriction of  $C$  to  $H_0 \cap H_3$  yields the matrices

$$\begin{pmatrix} -2\lambda_1 & \lambda_0 \\ \lambda_0 & \rho_0 \lambda_0 + \rho_1 \lambda_1 \end{pmatrix}, \begin{pmatrix} \beta^2 + \gamma^2 & -\alpha \beta \\ -\alpha \beta & \alpha^2 + \gamma^2 \end{pmatrix} \quad (7.9)$$

proportional, i.e.

$$\text{rank} \begin{pmatrix} -2\lambda_1 & \lambda_0 & \rho_0 \lambda_0 + \rho_1 \lambda_1 \\ \beta^2 + \gamma^2 & -\alpha \beta & \alpha^2 + \gamma^2 \end{pmatrix} = 1.$$

In particular, we derive

$$\begin{vmatrix} -2\alpha \beta & \beta^2 + \gamma^2 \\ \rho_1 \alpha \beta & \alpha^2 + \gamma^2 + \rho_2 \alpha \beta \end{vmatrix} = \alpha \beta G(\alpha, \beta, \gamma) = 0.$$

If either  $\alpha = 0$  or  $\beta = 0$ , the points  $Q_0, Q_1, Q_2, Q_3$  lie on parallel lines. Thus, the statement in (i) follows. On the other hand, we also find

$$\begin{vmatrix} \lambda_0 & \lambda_1 \\ 2\alpha \beta & \beta^2 + \gamma^2 \end{vmatrix} = 0,$$

and specializing in (7.2.2), the statement in (ii) follows.  $\square$

### 7.3. Classification of real circular cylinders

In this section, we perform our classification of the real cylinders passing through five points, where four of the points are cocyclic. Remarkably, we can characterize cocyclic configurations of four points with a simple algebraic condition.

**Lemma 7.3.1.** *Assume that  $\rho_0^2 \neq 2\rho_1$ . The following are equivalent:*

- (i) *The points  $Q_1, Q_2, Q_3, Q_4$  are cocyclic (recall Definition 1.2.33)*
- (ii)  $\rho_1 = -2$
- (iii) *The conic of vertices is singular*

*Proof.* The equivalence between the statements (ii) and (iii) follows from

$$\begin{vmatrix} 2 & \rho_0 & 0 \\ \rho_0 & \rho_1 & 0 \\ 0 & 0 & 2 + \rho_1 \end{vmatrix} = (2 + \rho_1)(2\rho_1 - \rho_0^2).$$



On the other hand, (i) is equivalent to

$$\begin{vmatrix} 1 & \alpha_1 & \beta_1 & \alpha_1^2 + \beta_1^2 \\ 1 & \alpha_2 & \beta_2 & \alpha_2^2 + \beta_2^2 \\ 1 & \alpha_3 & \beta_3 & \alpha_3^2 + \beta_3^2 \\ 1 & \alpha_4 & \beta_4 & \alpha_4^2 + \beta_4^2 \end{vmatrix} = \beta_3 \beta_4 (\beta_3 - \beta_4) - (\alpha_3 \beta_4 (1 - \alpha_3) - \alpha_4 \beta_3 (1 - \alpha_4)) = 0 \quad (7.10)$$

where  $Q_i = (1 : \alpha_i : \beta_i)$ , since this condition means the existence of an equation

$$F = F(x_0, x_1, x_2) = (x_1 - \alpha x_0)^2 + (x_2 - \beta x_0)^2 - r^2 x_0^2$$

such that  $F(Q_i) = 0$  for every  $1 \leq i \leq 4$ . On the other hand, by definition we have

$$\beta_3 \beta_4 (\beta_3 - \beta_4) \rho_1 + 2(\alpha_3 \beta_4 (1 - \alpha_3) - \alpha_4 \beta_3 (1 - \alpha_4)) = 0 ,$$

and (7.10) can be equivalently written as

$$\beta_3 \beta_4 (\beta_3 - \beta_4) (\rho_1 + 2) = 0 ,$$

and by the hypothesis on  $Q_1, Q_2, Q_3, Q_4$  we find that the statements (i) and (ii) are equivalent.  $\square$

**Notation 7.3.2.** If  $\rho_1 = -2$  we can factor

$$2G(x_1, x_2, x_3) = (2x_1 + (\rho_0 - \sigma)x_2)(2x_1 + (\rho_0 + \sigma)x_2) ,$$

where  $\sigma = \sqrt{\rho_0^2 + 4} \neq 0$ . Hence, the conic of vertices is the union  $L_0 \cup L_1$ , where  $L_0$  and  $L_1$  are the lines in  $H_0$  defined by the two factors of  $G$  respectively. In particular, for each  $i = 0, 1$ , we have the parametrizations

$$\begin{aligned} \psi_i : \mathbb{P}_{\mathbb{C}}^1 &\rightarrow L_i & (7.11) \\ (\lambda_0 : \lambda_1) &\mapsto ((\rho_0 + (-1)^i \sigma) \lambda_0 : -2 \lambda_0 : \lambda_1) . \end{aligned}$$

The common point  $L_0 \cap L_1 = (0 : 0 : 1) = \psi_i(0, 1)$  corresponds to the vertex of the ‘‘vertical’’ cylinder.

Up to this point, our discussion has focused solely on configurations involving four coplanar points. However, it is for five-point configurations that the count of circular cylinders becomes finite [81, 42, 143] over the complex numbers. As a result, we now introduce a fifth unconstrained point within  $\mathbb{A}_{\mathbb{R}}^3$ .

**Notation 7.3.3.** Let

$$Q_5 = (1 : \alpha_5 : \beta_5 : \gamma_5)$$

for some real  $\alpha_5, \beta_5, \gamma_5$  with  $\gamma_5$  nonzero. The central projection through  $Q_5$  induces an isomorphism

$$\begin{aligned} \Pi : H_0 &\rightarrow H_3 & (7.12) \\ (0 : x_1 : x_2 : x_3) &\mapsto (x_3 : \alpha_5 x_3 - \gamma_5 x_1 : \beta_5 x_3 - \gamma_5 x_2 : 0) \end{aligned}$$

By introducing the fifth point  $Q_5$ , we can now proceed to count the number of real circular cylinders, and derive their defining polynomials explicitly. To achieve this goal, we first introduce some polynomials that are closely related to the geometry of these five points.

**Notation 7.3.4.** For each  $i = 0, 1$ , we define the polynomials

$$\begin{aligned} h_{0i}(z) &= 4z \left( (-1)^i \sigma(2\mu_0 + 1) + 2\rho_0\mu_0 + \rho_0 - 2\mu_1 \right) , \\ h_{1i}(x, y, z) &= 2\sigma(-1)^i (\rho_0 y^2 + \rho_0 z^2 + 2\mu_0 y + 2xy) + \rho_0^2 y^2 + \rho_0^2 z^2 \\ &\quad + 2\rho_0\mu_0 y + 2\rho_0 xy + 2x^2 - 2\mu_1 y + 2y^2 + 4z^2 - 2x , \\ h_{2i}(x, y, z) &= z \left( (-1)^i \sigma(-2x + 1) - 2\rho_0 x + \rho_0 - 2\mu_1 + 4y \right) , \\ h_3(x, y) &= x^2 + y^2 - \mu_1 y - x . \end{aligned}$$

Additionally, define

$$H_i(\lambda_0, \lambda_1; x, y, z) = \lambda_0^3 h_{0i}(z) + \lambda_0^2 \lambda_1 h_{1i}(x, y, z) + \lambda_0 \lambda_1^2 h_{2i}(x, y, z) + \lambda_1^3 h_3(x, y) . \quad (7.13)$$

The following represents the main result of this chapter, establishing that the count of real circular cylinders through a five-point configuration of affine points, where four of them are cocyclic, coincides with the real roots in  $\mathbb{P}_{\mathbb{R}}^1$  of two bivariate homogeneous cubic equations.

**Theorem 7.3.5.** *Let  $Q_1, Q_2, Q_3, Q_4$  be cocyclic and not rectangular. Then,  $V$  is the vertex of a circular cylinder through the previous points and  $Q_5$  if and only if  $V = \psi_i(\lambda_0, \lambda_1)$ , where*

$$H_i(\lambda_0, \lambda_1) = 0 , \quad (7.14)$$

for some  $i = 0, 1$ . In particular, the real circular cylinders through the five points are in one-to-one correspondence with the real roots of  $H_0(\lambda_0, \lambda_1) = 0$  and  $H_1(\lambda_0, \lambda_1) = 0$ .

*Proof.* By Lemma 7.3.1, since  $Q_1, Q_2, Q_3, Q_4$  are cocyclic and they do not lie on parallel lines the conic of admissible vertices is the union of two lines parametrized by (7.11). Thus, if  $V$  is the vertex of a cylinder  $C$  through  $Q_1, \dots, Q_5$  we have

$$V = \psi_i(\lambda_0, \lambda_1) = (\lambda_0(\rho_1 + (-1)^i \sigma) : -2\lambda_0 : \lambda_1)$$

for some  $(\lambda_0 : \lambda_1)$  in  $\mathbb{P}_{\mathbb{R}}^1$  and  $i = 0, 1$ . By Lemma 7.2.6, the intersection  $C \cap H_3$  is defined by

$$(x_0 \ x_1 \ x_2) \cdot M(V) \cdot (x_0 \ x_1 \ x_2)^T = 0$$

where  $M = M(\alpha, \beta, \gamma)$  is as (7.8). Therefore,  $V$  is the vertex of a circular cylinder through the points if and only if the projection  $\Pi(Q_5)$  lies in  $C \cap H_3$ , i.e.

$$\Pi(Q_5) \cdot M(V) \cdot \Pi(Q_5)^T = -2\lambda_1 H_i(\lambda_0, \lambda_1) = 0 .$$

If  $\lambda_1 = 0$ , then  $V$  lies in  $H_0 \cap H_3$  and  $C \cap H_3$  consists of two (affinely) parallel lines, which is not possible by our hypothesis on  $Q_1, Q_2, Q_3, Q_4$ . Hence, the statement follows.  $\square$

The following are two corollaries of Theorem 7.3.5. Specifically, Corollary 7.3.6 permits the explicit computation of the defining polynomials of each of the real circular cylinders through the five points, given a real root  $(\lambda_0 : \lambda_1)$  of either  $H_0(\alpha_5, \beta_5, \gamma_5)$  or  $H_1(\alpha_5, \beta_5, \gamma_5)$ . On the other hand, Corollary 7.3.7 classifies the count of the real cylinders through the five points using the discriminant of bivariate, homogeneous, cubic polynomials.

**Corollary 7.3.6.** *With the hypotheses of Theorem 7.3.5, given a real root  $(\lambda_0 : \lambda_1)$  of (7.14) the equation of the circular cylinder in  $\mathbb{A}_{\mathbb{R}}^3$  through  $Q_1, Q_2, Q_3, Q_4$ , and  $Q_5$  of vertex  $V = \psi_i(\lambda_0, \lambda_1)$  is*

$$H(\lambda_0, \lambda_1; x, y, z) = 0. \quad (7.15)$$

*Proof.* Let  $C$  be the circular cylinder through the five points of vertex  $V$ . By Lemma 7.2.6, the intersection  $C \cap H_3$  is determined by  $Q_1, Q_2, Q_3, Q_4$ , and  $V$ . By definition, (7.15) defines the locus of points  $Q = (x, y, z)$  for which the projection  $\Pi_Q : H_0 \rightarrow H_3$  induced by  $Q$  sends  $V$  to a point in  $C \cap H_3$ . Therefore, (7.15) is the implicit equation of  $C$ .  $\square$

**Corollary 7.3.7.** *With the hypotheses of Theorem 7.3.5, let  $D_i = D_i(x, y, z)$  be the discriminant of  $H_i(\lambda_0, \lambda_1; x, y, z)$  with respect to  $\lambda_0, \lambda_1$ . Then, there are exactly*

- (i) *six circular cylinders through the points if and only if  $D_0(Q_5), D_1(Q_5) > 0$*
- (ii) *four circular cylinders through the points if and only if  $D_0(Q_5) \cdot D_1(Q_5) < 0$*
- (iii) *two circular cylinders through the points if and only if  $D_0(Q_5), D_1(Q_5) < 0$*

We close the chapter with a computational example. More precisely, we count the number of real circular cylinders through a five-point configuration of points in  $\mathbb{A}_{\mathbb{R}}^3$ , where four points are cocyclic, and compute their defining polynomials explicitly.

**Example 7.3.8.** Consider the points in  $\mathbb{A}_{\mathbb{R}}^3$

$$Q_1 = (0, 0, 0) \quad , \quad Q_2 = (1, 0, 0) \quad , \quad Q_3 = (-2.2336, 2.3888, 0) \quad , \quad Q_4 = (3.16, 3.41, 0) \quad ,$$

which are cocyclic since  $\rho_1 = -2$  and  $\rho_0^2 \neq 2\rho_1$  (Lemma 7.3.1). In particular, we compute the constants

$$\rho_0 = -10.5633 \quad , \quad \mu_0 = 14.8504 \quad , \quad \mu_1 = 5.4139 \quad .$$

Now, let  $Q_5 = (1, 1, 1)$ . For each  $i = 1, 2$ , let  $D_i$  be the discriminant of the polynomial in (7.13) with respect to  $\lambda_0, \lambda_1$ . Evaluating  $Q_5$  on each discriminant, we find

$$D_0(Q_5) < 0 \quad , \quad D_1(Q_5) < 0 \quad .$$

Therefore, by Corollary 7.3.7 there are exactly two circular cylinders through the five points. Moreover, we can compute these cylinders using Corollary 7.3.6. Specifically, the unique root of  $H_0(\lambda_0, \lambda_1; Q_5)$  in  $\mathbb{P}_{\mathbb{R}}^1$  is

$$R_1 = (0.4381 : -0.8989)$$

and the unique root of  $H_1(\lambda_0, \lambda_1; Q_5)$  in  $\mathbb{P}_{\mathbb{R}}^1$  is

$$R_2 = (0.2229 : -0.9749) \quad .$$

Therefore, the defining polynomials of the two cylinders through the five points are

$$\begin{aligned} C_1(x, y, z) &= -1.4165x^2 - 0.129509xy - 0.732485y^2 - 0.132873xz \\ &\quad + 1.41604yz - 0.696169z^2 + 1.4165x + 5.74553y - 5.47052z \quad , \\ C_2(x, y, z) &= -1.12009x^2 + 4.12761xy - 22.9206y^2 + 9.02801xz \\ &\quad + 0.847136yz - 22.1878z^2 + 1.12009x + 67.3605y - 36.2548z \end{aligned}$$

The circular cylinders through  $Q_1, Q_2, Q_3, Q_4$  and  $Q_5$  are illustrated in **Figure 7.1**.

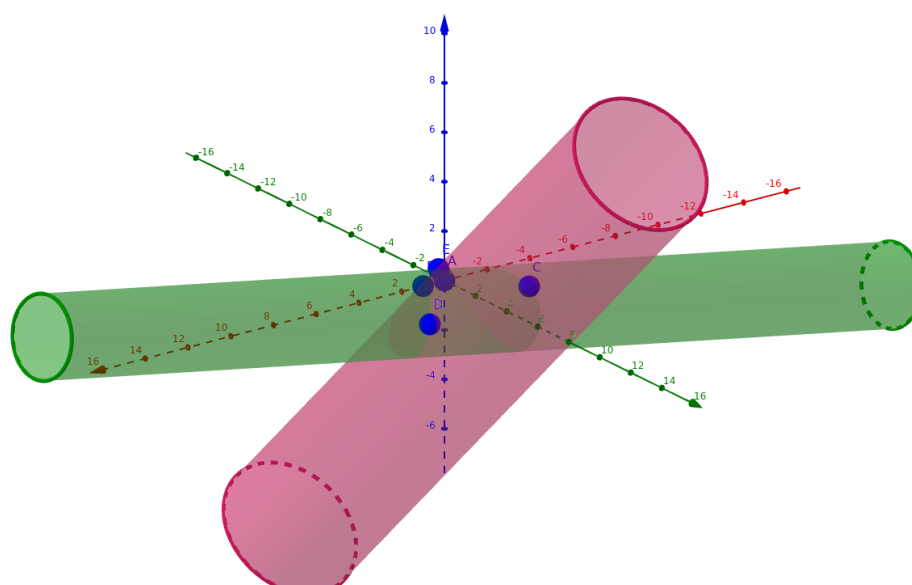


Figure 7.1.: The two circular cylinders through the five (blue) points in Example 7.3.8.



## Bibliography

- [1] Nir Ailon and Bernard Chazelle. “The fast Johnson-Lindenstrauss transform and approximate nearest neighbors”. In: *SIAM J. Comput.* 39.1 (2009), pp. 302–322. URL: <https://doi.org/10.1137/060673096>.
- [2] Maria Alberich-Carramiñana. *Geometry of the plane Cremona maps*. Vol. 1769. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2002, pp. xvi+257. ISBN: 3-540-42816-X. URL: <https://doi.org/10.1007/b82933>.
- [3] James Alexander and André Hirschowitz. “Polynomial interpolation in several variables”. In: *Journal of Algebraic Geometry* 4.2 (1995), pp. 201–222.
- [4] Noga Alon. “Combinatorial Nullstellensatz”. In: *Combin. Probab. Comput.* 8 (1999), pp. 7–29.
- [5] Noga Alon and Joel H Spencer. *The probabilistic method*. John Wiley & Sons, 2016.
- [6] Boris Aronov, Esther Ezra, and Micha Shair. “Small-size  $\epsilon$ -nets for axis-parallel rectangles and boxes”. In: *Proceedings of the forty-first annual ACM symposium on Theory of computing*. 2009, pp. 639–648.
- [7] Boris Aronov and Sariel Har-Peled. “On approximating the depth and related problems”. In: *SIAM Journal on Computing* 38.3 (2008), pp. 899–921.
- [8] Boris Aronov, Sariel Har-Peled, and Micha Sharir. “On approximate halfspace range counting and relative epsilon-approximations”. In: *Proceedings of the twenty-third annual symposium on Computational geometry*. 2007, pp. 327–336.
- [9] Edoardo Ballico, Kiryong Chung, and Sukmoon Huh. “Curves on Segre threefolds”. In: *Forum Math.* 32.1 (2020), pp. 63–78. URL: <https://doi.org/10.1515/forum-2019-0001>.
- [10] Imre Bárány, Matthieu Fradelizi, Xavier Goaoc, Alfredo Hubbard, and Günter Rote. “Random polytopes and the wet part for arbitrary probability distributions”. In: *Annales Henri Lebesgue* 3 (2020), pp. 701–715.
- [11] Imre Bárány, Zoltán Füredi, and László Lovász. “On the number of halving planes”. In: *Comb.* 10.2 (1990), pp. 175–183. URL: <https://doi.org/10.1007/BF02123008>.
- [12] Imre Bárány and Roman Karasev. “Notes about the Carathéodory number”. In: *Discrete & Computational Geometry* 48 (2012), pp. 783–792.
- [13] Alexander Barvinok. *A course in convexity*. Vol. 54. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2002, pp. x+366. ISBN: 0-8218-2968-8. URL: <https://doi.org/10.1090/gsm/054>.
- [14] Arnaud Beauville and Jérémy Blanc. “On Cremona transformations of prime order”. In: *Comptes Rendus Mathématique* 339.4 (2004), pp. 257–259.
- [15] Filippo Bergamasco, Mara Pistellato, Andrea Albarelli, and Andrea Torsello. “Cylinders extraction in non-oriented point clouds as a clustering problem”. In: *Pattern Recognition* 107 (2020), p. 107443.

- [16] Christine Berkesch, Daniel Erman, and Gregory G. Smith. “Virtual resolutions for a product of projective spaces”. In: *Algebr. Geom.* 7.4 (2020), pp. 460–481. URL: <https://doi.org/10.14231/ag-2020-013>.
- [17] Alessandra Bernardi, Grigoriy Blekherman, and Giorgio Ottaviani. “On real typical ranks”. In: *Boll. Unione Mat. Ital.* 11.3 (2018), pp. 293–307. URL: <https://doi.org/10.1007/s40574-017-0134-0>.
- [18] Pierre Bézier. “Procédé de définition numérique des courbes et surfaces non mathématiques”. In: *Automatisme* 13.5 (1968), pp. 189–196.
- [19] B. J. Birch. “On  $3N$  points in a plane”. In: *Proc. Cambridge Philos. Soc.* 55 (1959), pp. 289–293. URL: <https://doi.org/10.1017/s0305004100034071>.
- [20] Cinzia Bisi, Alberto Calabri, and Massimiliano Mella. “On plane Cremona transformations of fixed degree”. In: *J. Geom. Anal.* 25.2 (2015), pp. 1108–1131. URL: <https://doi.org/10.1007/s12220-013-9459-9>.
- [21] Jérémy Blanc. “Elements and cyclic subgroups of finite order of the Cremona group”. In: *Commentarii Mathematici Helvetici* 86.2 (2011), pp. 469–497.
- [22] Jérémy Blanc. “Finite abelian subgroups of the Cremona group of the plane”. In: *Comptes Rendus Mathématique* 344.1 (2007), pp. 21–26.
- [23] Jérémy Blanc. “Groupes de Cremona, connexité et simplicité”. In: *Ann. Sci. Éc. Norm. Supér. (4)* 43.2 (2010), pp. 357–364. URL: <https://doi.org/10.24033/asens.2123>.
- [24] Jérémy Blanc and Jean-Philippe Furter. “Topologies and structures of the Cremona groups”. In: *Ann. of Math. (2)* 178.3 (2013), pp. 1173–1198. URL: <https://doi.org/10.4007/annals.2013.178.3.8>.
- [25] Jérémy Blanc, Stéphane Lamy, and Susanna Zimmermann. “Quotients of higher-dimensional Cremona groups”. In: *Acta Math.* 226.2 (2021), pp. 211–318. URL: <https://doi.org/10.4310/acta.2021.v226.n2.a1>.
- [26] Grigoriy Blekherman and Zach Teitler. “On maximum, typical and generic ranks”. In: *Math. Ann.* 362.3-4 (2015), pp. 1021–1031. URL: <https://doi.org/10.1007/s00208-014-1150-3>.
- [27] Jacob A. Boswell and Vivek Mukundan. “Rees algebras and almost linearly presented ideals”. In: *J. Algebra* 460 (2016), pp. 102–127. URL: <https://doi.org/10.1016/j.jalgebra.2016.03.035>.
- [28] Nicolás Botbol, Laurent Busé, Marc Chardin, Seyed Hamid Hassanzadeh, Aron Simis, and Quang Hoa Tran. “Effective criteria for bigraded birational maps”. In: *J. Symbolic Comput.* 81 (2017), pp. 69–87. URL: <https://doi.org/10.1016/j.jsc.2016.12.001>.
- [29] Nicolás Botbol and Marc Chardin. “Castelnuovo Mumford regularity with respect to multi-graded ideals”. In: *J. Algebra* 474 (2017), pp. 361–392. URL: <https://doi.org/10.1016/j.jalgebra.2016.11.017>.
- [30] Nicolás Botbol and Alicia Dickenstein. “Implicitization of rational hypersurfaces via linear syzygies: a practical overview”. In: *J. Symbolic Comput.* 74 (2016), pp. 493–512. URL: <https://doi.org/10.1016/j.jsc.2015.09.001>.

- [31] Nicolás Botbol, Alicia Dickenstein, and Hal Schenck. “The simplest minimal free resolutions in  $\mathbb{P}^1 \times \mathbb{P}^1$ ”. In: *Commutative algebra*. Springer, Cham, [2021] ©2021, pp. 113–145. ISBN: 978-3-030-89693-5; 978-3-030-89694-2. URL: [https://doi.org/10.1007/978-3-030-89694-2\\_3](https://doi.org/10.1007/978-3-030-89694-2_3).
- [32] Felipe Bottega Diniz. “Tensor decomposition and algorithms, with applications to tensor learning”. PhD thesis. Universidade Federal do Rio de Janeiro, 2019.
- [33] Christos Boutsidis, Anastasios Zouzias, and Petros Drineas. “Random projections for  $k$ -means clustering”. In: *Advances in neural information processing systems* 23 (2010).
- [34] Stephen Boyd and Lieven Vandenbergh. *Convex optimization*. Cambridge University Press, Cambridge, 2004, pp. xiv+716. ISBN: 0-521-83378-7. URL: <https://doi.org/10.1017/CB09780511804441>.
- [35] Boris Bukh, Jiří Matoušek, and Gabriel Nivasch. “Lower bounds for weak epsilon-nets and stair-convexity”. In: *Israel J. Math.* 182 (2011), pp. 199–208. URL: <https://doi.org/10.1007/s11856-011-0029-1>.
- [36] Brent Burley and Walt Disney Animation Studios. “Physically-based shading at disney”. In: *Acm Siggraph*. Vol. 2012. vol. 2012. 2012, pp. 1–7.
- [37] Norbert Bus, Shashwat Garg, Nabil H Mustafa, and Saurabh Ray. “Tighter estimates for  $\epsilon$ -nets for disks”. In: *Computational Geometry* 53 (2016), pp. 27–35.
- [38] Laurent Busé. “On the equations of the moving curve ideal of a rational algebraic plane curve”. In: *J. Algebra* 321.8 (2009), pp. 2317–2344. URL: <https://doi.org/10.1016/j.jalgebra.2009.01.030>.
- [39] Laurent Busé, Fabrizio Catanese, and Elisa Postinghel. *Algebraic Curves and Surfaces*. Vol. 4. Springer International Publishing, 2023.
- [40] Laurent Busé and Falai Chen. “Determinantal tensor product surfaces and the method of moving quadrics”. In: *Trans. Amer. Math. Soc.* 374.7 (2021), pp. 4931–4952. URL: <https://doi.org/10.1090/tran/8358>.
- [41] Laurent Busé, Yairon Cid-Ruiz, and Carlos D’Andrea. “Degree and birationality of multi-graded rational maps”. In: *Proc. Lond. Math. Soc. (3)* 121.4 (2020), pp. 743–787. URL: <https://doi.org/10.1112/plms.12336>.
- [42] Laurent Busé, André Galligo, and Jiajun Zhang. “Extraction of cylinders and cones from minimal point sets”. In: *Graph. Models* 86 (2016), pp. 1–12. URL: <https://doi.org/10.1016/j.gmod.2016.05.003>.
- [43] Laurent Busé, Pablo González-Mazón, and Josef Schicho. “Tri-linear birational maps in dimension three”. In: *Math. Comp.* 92.342 (2023), pp. 1837–1866. URL: <https://doi.org/10.1090/mcom/3804>.
- [44] Laurent Busé and Thang Luu Ba. “The surface/surface intersection problem by means of matrix based representations”. In: *Comput. Aided Geom. Design* 29.8 (2012), pp. 579–598. URL: <https://doi.org/10.1016/j.cagd.2012.04.002>.
- [45] Serge Cantat. “The Cremona group in two variables”. In: *European Congress of Mathematics*. Eur. Math. Soc., Zürich, 2013, pp. 211–225.
- [46] Serge Cantat and Stéphane Lamy. “Normal subgroups in the Cremona group”. In: *Acta Math.* 210.1 (2013), pp. 31–94. URL: <https://doi.org/10.1007/s11511-013-0090-1>.



- [47] N. L. Carothers. *Real analysis*. Cambridge University Press, Cambridge, 2000, pp. xiv+401. ISBN: 0-521-49756-6. URL: <https://doi.org/10.1017/CB09780511814228>.
- [48] J Douglas Carroll and Jih-Jie Chang. "Analysis of individual differences in multidimensional scaling via an N-way generalization of "Eckart-Young" decomposition". In: *Psychometrika* 35.3 (1970), pp. 283–319.
- [49] Eduardo Casas-Alvero. *Analytic projective geometry*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2014, pp. xvi+620. ISBN: 978-3-03719-138-5. URL: <https://doi.org/10.4171/138>.
- [50] Guido Castelnuovo. *Le trasformazioni generatrici del gruppo cremoniano nel piano*. Turin R. Accad. d. Sci., 1901.
- [51] Federico Castillo, Yairon Cid-Ruiz, Binglin Li, Jonathan Montaña, and Naizhen Zhang. "When are multidegrees positive?" In: *Advances in Mathematics* 374 (2020), p. 107382.
- [52] Arthur Cayley. *On the theory of determinants*. Pitt Press, 1844.
- [53] Dominique Cerveau and Julie Déserti. *Transformations birationnelles de petit degré*. Vol. 19. Cours Spécialisés [Specialized Courses]. Société Mathématique de France, Paris, 2013, pp. viii+223. ISBN: 978-2-85629-770-4.
- [54] Thomas Chaperon and Francois Goulette. "A note on the construction of right circular cylinders through five 3D points". PhD thesis. MINES ParisTech, 2003.
- [55] Thomas Chaperon and François Goulette. "Extracting cylinders in full 3D data using a random sampling method and the Gaussian image". In: *Vision Modeling and Visualization Conference 2001 (VMV-01)*. 2001.
- [56] B. Chazelle, H. Edelsbrunner, M. Grigni, L. Guibas, M. Sharir, and E. Welzl. "Improved bounds on weak  $\epsilon$ -nets for convex sets". In: *Discrete Comput. Geom.* 13.1 (1995), pp. 1–15. URL: <https://doi.org/10.1007/BF02574025>.
- [57] Yairon Cid Ruiz. "Blow-up algebras in Algebra, Geometry and Combinatorics". PhD thesis. Universitat de Barcelona, 2019.
- [58] Yairon Cid-Ruiz. "Mixed multiplicities and projective degrees of rational maps". In: *J. Algebra* 566 (2021), pp. 136–162. URL: <https://doi.org/10.1016/j.jalgebra.2020.08.037>.
- [59] Yairon Cid-Ruiz, Oliver Clarke, and Fatemeh Mohammadi. "A study of nonlinear multiview varieties". In: *arXiv preprint arXiv:2112.06216* (2021).
- [60] Ciro Ciliberto, Francesco Russo, and Aron Simis. "Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian". In: *Adv. Math.* 218.6 (2008), pp. 1759–1805. URL: <https://doi.org/10.1016/j.aim.2008.03.025>.
- [61] Kenneth L Clarkson. "Applications of random sampling in computational geometry, II". In: *Proceedings of the fourth annual symposium on Computational geometry*. 1988, pp. 1–11.
- [62] Kenneth L Clarkson and Kasturi Varadarajan. "Improved approximation algorithms for geometric set cover". In: *Proceedings of the twenty-first annual symposium on Computational geometry*. 2005, pp. 135–141.
- [63] Donald L. Cohn. *Measure theory*. Second. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, New York, 2013, pp. xxi+457. ISBN: 978-1-4614-6955-1; 978-1-4614-6956-8. URL: <https://doi.org/10.1007/978-1-4614-6956-8>.

- [64] Sabine Coquillart. "Extended free-form deformation: A sculpturing tool for 3D geometric modeling". In: *Proceedings of the 17th annual conference on Computer graphics and interactive techniques*. 1990, pp. 187–196.
- [65] Teresa Cortadellas Benítez and Carlos D'Andrea. "Minimal generators of the defining ideal of the Rees algebra associated with a rational plane parametrization with  $\mu = 2$ ". In: *Canad. J. Math.* 66.6 (2014), pp. 1225–1249. URL: <https://doi.org/10.4153/CJM-2013-035-1>.
- [66] Teresa Cortadellas Benítez and Carlos D'Andrea. "Rational plane curves parameterizable by conics". In: *J. Algebra* 373 (2013), pp. 453–480. URL: <https://doi.org/10.1016/j.jalgebra.2012.09.034>.
- [67] Teresa Cortadellas Benítez and Carlos D'Andrea. "The Rees algebra of a monomial plane parametrization". In: *J. Symbolic Comput.* 70 (2015), pp. 71–105. URL: <https://doi.org/10.1016/j.jsc.2014.09.026>.
- [68] David Cox, J. William Hoffman, and Haohao Wang. "Syzygies and the Rees algebra". In: *J. Pure Appl. Algebra* 212.7 (2008), pp. 1787–1796. URL: <https://doi.org/10.1016/j.jpaa.2007.11.006>.
- [69] David A. Cox. *Applications of polynomial systems*. Vol. 134. CBMS Regional Conference Series in Mathematics. American Mathematical Society, Providence, RI, [2020] ©2020, pp. ix+250. ISBN: 978-1-4704-5137-0.
- [70] David A. Cox. "The moving curve ideal and the Rees algebra". In: *Theoret. Comput. Sci.* 392.1-3 (2008), pp. 23–36. URL: <https://doi.org/10.1016/j.tcs.2007.10.012>.
- [71] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxiv+841. ISBN: 978-0-8218-4819-7. URL: <https://doi.org/10.1090/gsm/124>.
- [72] Luigi Cremona. *Sulle trasformazione geometriche delle figure piane*. Tipi Gamberini e Parmeggiani, 1863.
- [73] Luigi Cremona. "Sulle trasformazioni razionali nello spazio". In: *Annali di Matematica Pura ed Applicata (1867-1897)* 5.1 (1871), pp. 131–162.
- [74] Paul De Casteljaou. "Courbes à pôles". In: *National Industrial Property Institute (France)* (1959).
- [75] Paul De Casteljaou. "Courbes et surfaces à pôles". In: *André Citroën, Automobiles SA, Paris* 66 (1963).
- [76] Lieven De Lathauwer, Bart De Moor, and Joos Vandewalle. "On the best rank-1 and rank- $(R_1, R_2, \dots, R_N)$  approximation of higher-order tensors". In: *SIAM J. Matrix Anal. Appl.* 21.4 (2000), pp. 1324–1342. URL: <https://doi.org/10.1137/S0895479898346995>.
- [77] Michel Demazure. "Sous-groupes algébriques de rang maximum du groupe de Cremona". In: *Ann. Sci. École Norm. Sup. (4)* 3 (1970), pp. 507–588. URL: [http://www.numdam.org/item?id=ASENS\\_1970\\_4\\_3\\_4\\_507\\_0](http://www.numdam.org/item?id=ASENS_1970_4_3_4_507_0).
- [78] Julie Déserti. *Some properties of the Cremona group*. Vol. 21. Ensaïos Matemáticos [Mathematical Surveys]. Sociedade Brasileira de Matemática, Rio de Janeiro, 2012, pp. ii+188. ISBN: 978-85-85818-58-6.
- [79] Julie Déserti. "Sur les automorphismes du groupe de Cremona". In: *Compos. Math.* 142.6 (2006), pp. 1459–1478. URL: <https://doi.org/10.1112/S0010437X06002478>.

- [80] Julie Déserti and Frédéric Han. “On cubic birational maps of  $\mathbb{P}_{\mathbb{C}}^3$ ”. In: *Bull. Soc. Math. France* 144.2 (2016), pp. 217–249. URL: <https://doi.org/10.24033/bsmf.2712>.
- [81] Olivier Devillers, Bernard Mourrain, Franco P. Preparata, and Philippe Trebuchet. “Circular cylinders through four or five points in space”. In: *Discrete Comput. Geom.* 29.1 (2003), pp. 83–104.
- [82] Antoine Deza, Eissa Nematollahi, and Tamás Terlaky. “How good are interior point methods? Klee-Minty cubes tighten iteration-complexity bounds”. In: *Math. Program.* 113.1 (2008), pp. 1–14. URL: <https://doi.org/10.1007/s10107-006-0044-x>.
- [83] Philipp J. di Dio and Mario Kummer. “The multidimensional truncated moment problem: Carathéodory numbers from Hilbert functions”. In: *Math. Ann.* 380.1-2 (2021), pp. 267–291. URL: <https://doi.org/10.1007/s00208-021-02166-x>.
- [84] Philipp J. di Dio and Konrad Schmüdgen. “The multidimensional truncated moment problem: Carathéodory numbers”. In: *J. Math. Anal. Appl.* 461.2 (2018), pp. 1606–1638. URL: <https://doi.org/10.1016/j.jmaa.2017.12.021>.
- [85] Igor Dolgachev. “*The Cremona group and its subgroups* [book review of 4256046]”. In: *Bull. Amer. Math. Soc. (N.S.)* 59.4 (2022), pp. 617–622.
- [86] Igor V. Dolgachev. *Classical algebraic geometry*. Cambridge University Press, Cambridge, 2012, pp. xii+639. ISBN: 978-1-107-01765-8. URL: <https://doi.org/10.1017/CB09781139084437>.
- [87] A. V. Doria, S. H. Hassanzadeh, and A. Simis. “A characteristic-free criterion of birationality”. In: *Adv. Math.* 230.1 (2012), pp. 390–413. URL: <https://doi.org/10.1016/j.aim.2011.12.005>.
- [88] Jan Draisma, Emil Horobet, Giorgio Ottaviani, Bernd Sturmfels, and Rekha R. Thomas. “The Euclidean distance degree of an algebraic variety”. In: *Found. Comput. Math.* 16.1 (2016), pp. 99–149. URL: <https://doi.org/10.1007/s10208-014-9240-x>.
- [89] David Eisenbud. *Commutative Algebra with a view towards Algebraic Geometry*. Graduate Texts in Mathematics, 150. Springer-Verlag, 1995.
- [90] David Eisenbud. *The geometry of syzygies*. Vol. 229. Graduate Texts in Mathematics. Springer-Verlag, New York, 2005, pp. xvi+243. ISBN: 0-387-22215-4.
- [91] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016, pp. xiv+616. ISBN: 978-1-107-60272-4; 978-1-107-01708-5. URL: <https://doi.org/10.1017/CB09781139062046>.
- [92] David Eisenbud and Joe Harris. *The geometry of schemes*. Vol. 197. Graduate Texts in Mathematics. Springer-Verlag, New York, 2000, pp. x+294. ISBN: 0-387-98638-3; 0-387-98637-5.
- [93] Gerald Farin. *Curves and surfaces for computer-aided geometric design*. Fourth. Computer Science and Scientific Computing. Academic Press, Inc., San Diego, CA, 1997, pp. xviii+429. ISBN: 0-12-249054-1.
- [94] Michael S. Floater. “The inverse of a rational bilinear mapping”. In: *Comput. Aided Geom. Design* 33 (2015), pp. 46–50. URL: <https://doi.org/10.1016/j.cagd.2015.01.002>.
- [95] Xiao-Ming Fu, Yang Liu, and Baining Guo. “Computing locally injective mappings by advanced MIPS”. In: *ACM Transactions on Graphics (TOG)* 34.4 (2015), pp. 1–12.

- [96] William Fulton. *Intersection theory*. Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998, pp. xiv+470.
- [97] James E. Gain and Neil A. Dodgson. "Preventing self-intersection under free-form deformation". In: *IEEE Transactions on Visualization and Computer Graphics* 7.4 (2001), pp. 289–298.
- [98] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1994, pp. x+523. ISBN: 0-8176-3660-9. URL: <https://doi.org/10.1007/978-0-8176-4771-1>.
- [99] Pablo González-Mazón and Laurent Busé. "Construction of birational trilinear volumes via tensor rank criteria". 2024. URL: <https://inria.hal.science/hal-03939273>.
- [100] Pablo González-Mazón, Alfredo Hubard, and Roman Karasev. *Weak  $(1 - \epsilon)$ -nets for polynomial superlevel sets*. 2023. arXiv: [2308.14060](https://arxiv.org/abs/2308.14060) [math.MG].
- [101] Daniel R. Grayson and Michael E. Stillman. *Macaulay2, a software system for research in algebraic geometry*. Available at <https://math.uiuc.edu/Macaulay2/>.
- [102] Gert-Martin Greuel and Gerhard Pfister. *A Singular introduction to commutative algebra*. extended. Springer, Berlin, 2008, pp. xx+689. ISBN: 978-3-540-73541-0.
- [103] Misha Gromov. "Geometric, algebraic, and analytic descendants of Nash isometric embedding theorems". In: *Bull. Amer. Math. Soc. (N.S.)* 54.2 (2017), pp. 173–245. URL: <https://doi.org/10.1090/bull/1551>.
- [104] Huy Tài Hà and Adam Van Tuyl. "The regularity of points in multi-projective spaces". In: *J. Pure Appl. Algebra* 187.1-3 (2004), pp. 153–167. URL: <https://doi.org/10.1016/j.jpaa.2003.07.006>.
- [105] Sarel Har-Peled. *Geometric approximation algorithms*. Vol. 173. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2011, pp. xii+362. ISBN: 978-0-8218-4911-8. URL: <https://doi.org/10.1090/surv/173>.
- [106] Sarel Har-Peled, Piotr Indyk, and Rajeev Motwani. "Approximate nearest neighbor: towards removing the curse of dimensionality". In: *Theory Comput.* 8 (2012), pp. 321–350. URL: <https://doi.org/10.4086/toc.2012.v008a014>.
- [107] Megumi Harada, Maryam Nowroozi, and Adam Van Tuyl. "Virtual resolutions of points in  $\mathbb{P}^1 \times \mathbb{P}^1$ ". In: *J. Pure Appl. Algebra* 226.12 (2022), Paper No. 107140, 18. URL: <https://doi.org/10.1016/j.jpaa.2022.107140>.
- [108] Robert M. Hardt. "Semi-algebraic local-triviality in semi-algebraic mappings". In: *Amer. J. Math.* 102.2 (1980), pp. 291–302. URL: <https://doi.org/10.2307/2374240>.
- [109] Joe Harris. *Algebraic geometry*. Vol. 133. Graduate Texts in Mathematics. Springer-Verlag, New York, 1992, pp. xx+328. ISBN: 0-387-97716-3. URL: <https://doi.org/10.1007/978-1-4757-2189-8>.
- [110] Richard A Harshman et al. "Foundations of the PARAFAC procedure: Models and conditions for an " explanatory" multimodal factor analysis". In: (1970).

- [111] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.
- [112] Seyed Hamid Hassanzadeh and Aron Simis. "Plane Cremona maps: saturation and regularity of the base ideal". In: *J. Algebra* 371 (2012), pp. 620–652. URL: <https://doi.org/10.1016/j.jalgebra.2012.08.022>.
- [113] David Haussler. "Decision theoretic generalizations of the PAC model for neural net and other learning applications". In: *The Mathematics of Generalization*. CRC Press, 2018, pp. 37–116.
- [114] David Haussler and Emo Welzl. "Epsilon-nets and simplex range queries". In: *Proceedings of the second annual symposium on Computational geometry*. 1986, pp. 61–71.
- [115] Manfred Herrmann, Eero Hyry, Jürgen Ribbe, and Zhongming Tang. "Reduction numbers and multiplicities of multigraded structures". In: *J. Algebra* 197.2 (1997), pp. 311–341. URL: <https://doi.org/10.1006/jabr.1997.7128>.
- [116] David Hilbert. *Hilbert's invariant theory papers*. Vol. VIII. Lie Groups: History, Frontiers and Applications. Math Sci Press, Brookline, MA, 1978, pp. ix+336. ISBN: 0-915692-26-0.
- [117] Frank L Hitchcock. "Multiple invariants and generalized rank of a p-way matrix or tensor". In: *Journal of Mathematics and Physics* 7.1-4 (1928), pp. 39–79.
- [118] David Hong, Tamara G Kolda, and Jed A Duersch. "Generalized canonical polyadic tensor decomposition". In: *SIAM Review* 62.1 (2020), pp. 133–163.
- [119] Hilda P Hudson. *Cremona transformations in plane and space*. Vol. 1927. Cambridge, 1927.
- [120] Eero Hyry. "The diagonal subring and the Cohen-Macaulay property of a multigraded ring". In: *Trans. Amer. Math. Soc.* 351.6 (1999), pp. 2213–2232. URL: <https://doi.org/10.1090/S0002-9947-99-02143-1>.
- [121] Roman Karasev and Benjamin Matschke. "Projective center point and Tverberg theorems". In: *Discrete Comput. Geom.* 52.1 (2014), pp. 88–101. URL: <https://doi.org/10.1007/s00454-014-9602-9>.
- [122] Roman N Karasev. "A topological central point theorem". In: *arXiv preprint arXiv:1011.1802* (2010).
- [123] Amit Khetan and Carlos D'Andrea. "Implicitization of rational surfaces using toric varieties". In: *J. Algebra* 303.2 (2006), pp. 543–565. URL: <https://doi.org/10.1016/j.jalgebra.2005.05.028>.
- [124] Victor Klee and George J. Minty. "How good is the simplex algorithm?" In: *Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin)*. Academic Press, New York-London, 1972, pp. 159–175.
- [125] Tamara G Kolda and Brett W Bader. *MATLAB tensor toolbox*. Tech. rep. Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA . . . , 2006.
- [126] Tamara G. Kolda and Brett W. Bader. "Tensor decompositions and applications". In: *SIAM Rev.* 51.3 (2009), pp. 455–500. URL: <https://doi.org/10.1137/07070111X>.
- [127] Tamara Gibson Kolda. *Multilinear operators for higher-order decompositions*. Tech. rep. Sandia National Laboratories (SNL), Albuquerque, NM, and Livermore, CA . . . , 2006.

- [128] János Komlós, János Pach, and Gerhard Woeginger. “Almost tight bounds for  $\epsilon$ -nets”. In: *Discrete & Computational Geometry* 7 (1992), pp. 163–173.
- [129] Rimvydas Krasauskas. “Toric surface patches”. In: *Advances in Computational Mathematics* 17 (2002), pp. 89–113.
- [130] Joseph B Kruskal. “Three-way arrays: rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics”. In: *Linear algebra and its applications* 18.2 (1977), pp. 95–138.
- [131] Andrey Kupavskii, Nabil Mustafa, and János Pach. “New Lower Bounds for  $\epsilon$ -nets”. In: *32nd Annual International Symposium on Computational Geometry (SoCG 2016)*. 2016, p. 54.
- [132] Andrew Kustin, Claudia Polini, and Bernd Ulrich. “The bi-graded structure of symmetric algebras with applications to Rees rings”. In: *J. Algebra* 469 (2017), pp. 188–250. URL: <https://doi.org/10.1016/j.jalgebra.2016.08.014>.
- [133] Andrew R. Kustin, Claudia Polini, and Bernd Ulrich. “Blowups and fibers of morphisms”. In: *Nagoya Math. J.* 224.1 (2016), pp. 168–201. URL: <https://doi.org/10.1017/nmj.2016.34>.
- [134] Andrew R. Kustin, Claudia Polini, and Bernd Ulrich. “Degree bounds for local cohomology”. In: *Proc. Lond. Math. Soc. (3)* 121.5 (2020), pp. 1251–1267. URL: <https://doi.org/10.1112/plms.12364>.
- [135] Andrew R. Kustin, Claudia Polini, and Bernd Ulrich. “Rational normal scrolls and the defining equations of Rees algebras”. In: *J. Reine Angew. Math.* 650 (2011), pp. 23–65. URL: <https://doi.org/10.1515/CRELLE.2011.002>.
- [136] Joseph M Landsberg. “Tensors: geometry and applications”. In: *Representation theory* 381.402 (2012), p. 3.
- [137] Daniel Lazard and Fabrice Rouillier. “Solving parametric polynomial systems”. In: *Journal of Symbolic Computation* 42.6 (2007), pp. 636–667.
- [138] Binh Huy Le and Zhigang Deng. “Smooth skinning decomposition with rigid bones”. In: *ACM Transactions on Graphics (TOG)* 31.6 (2012), pp. 1–10.
- [139] Huu Phuoc Le. “On solving parametric polynomial systems and quantifier elimination over the reals: algorithms, complexity and implementations”. PhD thesis. Sorbonne Université, 2021.
- [140] Huu Phuoc Le and Mohab Safey El Din. “Solving parametric systems of polynomial equations over the reals through Hermite matrices”. In: *J. Symbolic Comput.* 112 (2022), pp. 25–61. URL: <https://doi.org/10.1016/j.jsc.2021.12.002>.
- [141] Songxin Liang and David J. Jeffrey. “Automatic computation of the complete root classification for a parametric polynomial”. In: *J. Symbolic Comput.* 44.10 (2009), pp. 1487–1501. URL: <https://doi.org/10.1016/j.jsc.2009.05.003>.
- [142] Songxin Liang, David J. Jeffrey, and Marc Moreno Maza. “The complete root classification of a parametric polynomial on an interval”. In: *ISSAC 2008*. ACM, New York, 2008, pp. 189–195. ISBN: 978-1-59593-904-3. URL: <https://doi.org/10.1145/1390768.1390796>.
- [143] Daniel Lichtblau. “Cylinders through five points: computational algebra and geometry”. In: *J. Math. Res.* 4.6 (2012), pp. 65–82. URL: <https://doi.org/10.5539/jmr.v4n6p65>.

- [144] Mingxia Liu, Jun Zhang, Pew-Thian Yap, and Dinggang Shen. "View-aligned hypergraph learning for Alzheimer's disease diagnosis with incomplete multi-modality data". In: *Medical image analysis* 36 (2017), pp. 123–134.
- [145] Michael C. Loper. "What makes a complex a virtual resolution?" In: *Trans. Amer. Math. Soc. Ser. B* 8 (2021), pp. 885–898. URL: <https://doi.org/10.1090/btran/91>.
- [146] Diane Maclagan and Gregory G. Smith. "Multigraded Castelnuovo-Mumford regularity". In: *J. Reine Angew. Math.* 571 (2004), pp. 179–212. URL: <https://doi.org/10.1515/crll.2004.040>.
- [147] Jiri Matousek. "Reporting points in halfspaces". In: *Computational Geometry* 2.3 (1992), pp. 169–186.
- [148] Jiří Matoušek, Raimund Seidel, and Emo Welzl. "How to net a lot with little: Small  $\epsilon$ -nets for disks and halfspaces". In: *Proceedings of the sixth annual symposium on Computational geometry*. 1990, pp. 16–22.
- [149] Hideyuki Matsumura. *Commutative ring theory*. Second. Vol. 8. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1989, pp. xiv+320. ISBN: 0-521-36764-6.
- [150] Tom McReynolds, David Blythe, Brad Grantham, and Scott Nelson. "Advanced graphics programming techniques using OpenGL". In: *Computer Graphics* (1998), pp. 95–145.
- [151] Artibano Micali. "Sur les algèbres universelles". In: *Ann. Inst. Fourier (Grenoble)* 14 (1964), pp. 33–87. URL: [http://www.numdam.org/item?id=AIF\\_1964\\_\\_14\\_2\\_33\\_0](http://www.numdam.org/item?id=AIF_1964__14_2_33_0).
- [152] Mateusz Michałek and Bernd Sturmfels. *Invitation to nonlinear algebra*. Vol. 211. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, [2021] ©2021, pp. xiii+226. ISBN: 978-1-4704-5367-1. URL: <https://doi.org/10.1090/gsm/211>.
- [153] Nabil H Mustafa and Saurabh Ray. "An optimal extension of the centerpoint theorem". In: *Computational Geometry* 42.6-7 (2009), pp. 505–510.
- [154] Nabil H Mustafa and Saurabh Ray. "Improved results on geometric hitting set problems". In: *Discrete & Computational Geometry* 44 (2010), pp. 883–895.
- [155] Nabil H. Mustafa and Kasturi R. Varadarajan. *Epsilon-approximations and epsilon-nets*. 2017. arXiv: [1702.03676](https://arxiv.org/abs/1702.03676) [cs.CG].
- [156] Stanislav Nagy, Carsten Schütt, and Elisabeth M Werner. "Halfspace depth and floating body". In: *Statistics Surveys* 13 (2019), pp. 52–118.
- [157] Lawrence Narici and Edward Beckenstein. *Topological vector spaces*. Second. Vol. 296. Pure and Applied Mathematics (Boca Raton). CRC Press, Boca Raton, FL, 2011, pp. xviii+610. ISBN: 978-1-58488-866-6.
- [158] Andrew Nealen, Matthias Müller, Richard Keiser, Eddy Boxerman, and Mark Carlson. "Physically based deformable models in computer graphics". In: *Computer graphics forum*. Vol. 25. 4. Wiley Online Library. 2006, pp. 809–836.
- [159] Giorgio Ottaviani and Philipp Reichenbach. "Tensor rank and complexity". In: *arXiv preprint arXiv:2004.01492* (2020).
- [160] János Pach and Pankaj K Agarwal. *Combinatorial geometry*. John Wiley & Sons, 2011.

- [161] I. Pan, F. Ronga, and T. Vust. “Transformations birationnelles quadratiques de l’espace projectif complexe à trois dimensions”. In: *Ann. Inst. Fourier (Grenoble)* 51.5 (2001), pp. 1153–1187. URL: [http://aif.cedram.org/item?id=AIF\\_2001\\_\\_51\\_5\\_1153\\_0](http://aif.cedram.org/item?id=AIF_2001__51_5_1153_0).
- [162] Ivan Pan. “Sur le multidegré des transformations de Cremona”. In: *C. R. Acad. Sci. Paris Sér. I Math.* 330.4 (2000), pp. 297–300. URL: [https://doi.org/10.1016/S0764-4442\(00\)00142-7](https://doi.org/10.1016/S0764-4442(00)00142-7).
- [163] Ivan Pan. “Une remarque sur la génération du groupe de Cremona”. In: *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society* 30.1 (1999), pp. 95–98.
- [164] Ivan Pan and Francesco Russo. “Cremona transformations and special double structures”. In: *Manuscripta Math.* 117.4 (2005), pp. 491–510. URL: <https://doi.org/10.1007/s00229-005-0573-2>.
- [165] Michel Petitjean. “About the algebraic solutions of smallest enclosing cylinders problems”. In: *Appl. Algebra Engrg. Comm. Comput.* 23.3-4 (2012), pp. 151–164. URL: <https://doi.org/10.1007/s00200-012-0171-y>.
- [166] Les Piegl and Wayne Tiller. *The NURBS book*. Springer Science & Business Media, 1996.
- [167] Alexander Pilz and Patrick Schnider. “Extending the centerpoint theorem to multiple points”. In: *arXiv preprint arXiv:1810.10231* (2018).
- [168] Michael Rabinovich, Roi Poranne, Daniele Panozzo, and Olga Sorkine-Hornung. “Scalable locally injective mappings”. In: *ACM Transactions on Graphics (TOG)* 36.4 (2017), p. 1.
- [169] R. Rado. “A Theorem on General Measure”. In: *Journal of the London Mathematical Society* s1-21.4 (1946), pp. 291–300. eprint: <https://academic.oup.com/jlms/article-pdf/s1-21/4/291/2726046/s1-21-4-291.pdf>. URL: <https://doi.org/10.1112/jlms/s1-21.4.291>.
- [170] David Rees. “On a problem of Zariski”. In: *Illinois Journal of Mathematics* 2.1 (1958), pp. 145–149.
- [171] Bruce Reznick. “On the length of binary forms”. In: *Quadratic and higher degree forms*. Vol. 31. Dev. Math. Springer, New York, 2013, pp. 207–232. ISBN: 978-1-4614-7487-6; 978-1-4614-7488-3. URL: [https://doi.org/10.1007/978-1-4614-7488-3\\_8](https://doi.org/10.1007/978-1-4614-7488-3_8).
- [172] Robert Riener, Matthias Harders, Robert Riener, and Matthias Harders. *Introduction to virtual reality in medicine*. Springer, 2012.
- [173] David F Rogers and James Alan Adams. *Mathematical elements for computer graphics*. McGraw-Hill, Inc., 1989.
- [174] Ofer Ron, Leo Joskowicz, Charles Milgrom, and Ariel Simkin. “Computer-based periaxial rotation measurement for aligning fractured femur fragments from CT: a feasibility study”. In: *Computer Aided Surgery* 7.6 (2002), pp. 332–341.
- [175] Gerhard Roth and Martin D Levine. “Extracting geometric primitives”. In: *CVGIP: image understanding* 58.1 (1993), pp. 1–22.
- [176] Natan Rubin. “Stronger bounds for weak epsilon-nets in higher dimensions”. In: *STOC ’21—Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*. ACM, New York, [2021] ©2021, pp. 989–1002. URL: <https://doi.org/10.1145/3406325.3451062>.



- [177] Daniel Rueckert, Luke I Sonoda, Carmel Hayes, Derek LG Hill, Martin O Leach, and David J Hawkes. "Nonrigid registration using free-form deformations: application to breast MR images". In: *IEEE transactions on medical imaging* 18.8 (1999), pp. 712–721.
- [178] Francesco Russo and Aron Simis. "On birational maps and Jacobian matrices". In: *Compositio Math.* 126.3 (2001), pp. 335–358. URL: <https://doi.org/10.1023/A:1017572213947>.
- [179] Hal Schenck. *Computational algebraic geometry*. Vol. 58. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2003, pp. xiv+193. ISBN: 0-521-82964-X; 0-521-53650-2. URL: <https://doi.org/10.1017/CB09780511756320>.
- [180] Elmar Schömer, Jürgen Sellen, Marek Teichmann, and Chee Yap. "Smallest enclosing cylinders". In: *Proceedings of the twelfth annual symposium on computational geometry*. 1996, pp. 413–414.
- [181] F.-O. Schreyer, K. Hulek, and S. Katz. "Cremona transformations and syzygies." In: *Mathematische Zeitschrift* 209.3 (1992), pp. 419–444. URL: <http://eudml.org/doc/174372>.
- [182] Thomas W Sederberg and Falai Chen. "Implicitization using moving curves and surfaces". In: *Proceedings of the 22nd annual conference on Computer graphics and interactive techniques*. 1995, pp. 301–308.
- [183] Thomas W. Sederberg, Ronald N. Goldman, and Xuhui Wang. "Birational 2D free-form deformation of degree  $1 \times n$ ". In: *Comput. Aided Geom. Design* 44 (2016), pp. 1–9. URL: <https://doi.org/10.1016/j.cagd.2016.02.020>.
- [184] Thomas W Sederberg and Scott R Parry. "Free-form deformation of solid geometric models". In: *Proceedings of the 13th annual conference on Computer graphics and interactive techniques*. 1986, pp. 151–160.
- [185] Thomas W. Sederberg and Jianmin Zheng. "Birational quadrilateral maps". In: *Comput. Aided Geom. Design* 32 (2015), pp. 1–4. URL: <https://doi.org/10.1016/j.cagd.2014.11.001>.
- [186] Jean-Pierre Serre. "Le groupe de Cremona et ses sous-groupes finis". In: 332. 2010, Exp. No. 1000, vii, 75–100. ISBN: 978-2-85629-291-4.
- [187] Micha Sharir and Hayim Shaul. "Semialgebraic range reporting and emptiness searching with applications". In: *SIAM J. Comput.* 40.4 (2011), pp. 1045–1074. URL: <https://doi.org/10.1137/090765092>.
- [188] Aron Simis. "Cremona transformations and some related algebras". In: *J. Algebra* 280.1 (2004), pp. 162–179. URL: <https://doi.org/10.1016/j.jalgebra.2004.03.025>.
- [189] Aron Simis and Rafael H. Villarreal. "Combinatorics of Cremona monomial maps". In: *Math. Comp.* 81.279 (2012), pp. 1857–1867. URL: <https://doi.org/10.1090/S0025-5718-2011-02556-1>.
- [190] Aron Simis and Rafael H. Villarreal. "Constraints for the normality of monomial subrings and birationality". In: *Proc. Amer. Math. Soc.* 131.7 (2003), pp. 2043–2048. URL: <https://doi.org/10.1090/S0002-9939-02-06790-4>.
- [191] Aron Simis and Rafael H. Villarreal. "Linear syzygies and birational combinatorics". In: *Results Math.* 48.3-4 (2005), pp. 326–343. URL: <https://doi.org/10.1007/BF03323372>.

- [192] Laurent Sorber, Marc Van Barel, and Lieven De Lathauwer. "Optimization-based algorithms for tensor decompositions: Canonical polyadic decomposition, decomposition in rank-( $L_r, L_r, 1$ ) terms, and a new generalization". In: *SIAM Journal on Optimization* 23.2 (2013), pp. 695–720.
- [193] Olga Sorkine, Daniel Cohen-Or, Yaron Lipman, Marc Alexa, Christian Rössl, and H-P Seidel. "Laplacian surface editing". In: *Proceedings of the 2004 Eurographics/ACM SIGGRAPH symposium on Geometry processing*. 2004, pp. 175–184.
- [194] Aristeidis Sotiras, Christos Davatzikos, and Nikos Paragios. "Deformable medical image registration: A survey". In: *IEEE transactions on medical imaging* 32.7 (2013), pp. 1153–1190.
- [195] The Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>. 2018.
- [196] John W Tukey. "Mathematics and the picturing of data". In: *Proceedings of the International Congress of Mathematicians, Vancouver, 1975*. Vol. 2. 1975, pp. 523–531.
- [197] V. N. Vapnik and A. Ya. Chervonenkis. "On the uniform convergence of relative frequencies of events to their probabilities". In: *Measures of complexity*. Springer, Cham, 2015, pp. 11–30. ISBN: 978-3-319-21851-9; 978-3-319-21852-6.
- [198] Wolmer V Vasconcelos. *Arithmetic of blowup algebras*. Vol. 195. Cambridge University Press, 1994.
- [199] Wolmer V Vasconcelos. *Integral closure: Rees algebras, multiplicities, algorithms*. Springer, 2005.
- [200] Bin Wang, Longhua Wu, KangKang Yin, Uri M Ascher, Libin Liu, and Hui Huang. "Deformation capture and modeling of soft objects." In: *ACM Trans. Graph.* 34.4 (2015), pp. 94–1.
- [201] Kexin Wang and Anna Seigal. "Lower bounds on the rank and symmetric rank of real tensors". In: *J. Symbolic Comput.* 118 (2023), pp. 69–92. URL: <https://doi.org/10.1016/j.jsc.2023.01.004>.
- [202] Xuhui Wang, Yuhao Han, Qian Ni, Rui Li, and Ron Goldman. "Birational Quadratic Planar Maps with Generalized Complex Rational Representations". In: *Mathematics* 11.16 (2023), p. 3609.
- [203] Xuhui Wang, Meng Wu, Yuan Liu, and Qian Ni. "Constructing quadratic birational maps via their complex rational representation". In: *Comput. Aided Geom. Design* 85 (2021), Paper No. 101969, 11. URL: <https://doi.org/10.1016/j.cagd.2021.101969>.
- [204] G Alistair Watson. "Fitting enclosing cylinders to data in  $\mathbb{R}^n$ ". In: *Numerical Algorithms* 43 (2006), pp. 189–196.
- [205] Simon Winkelbach, Ralf Westphal, and Thomas Goesling. "Pose estimation of cylindrical fragments for semi-automatic bone fracture reduction". In: *Pattern Recognition: 25th DAGM Symposium, Magdeburg, Germany, September 10-12, 2003. Proceedings* 25. Springer. 2003, pp. 566–573.
- [206] Gang Xu, Bernard Mourrain, Régis Duvigneau, and André Galligo. "Analysis-suitable volume parameterization of multi-block computational domain in isogeometric applications". In: *Comput.-Aided Des.* 45.2 (2013), pp. 395–404. URL: <https://doi.org/10.1016/j.cad.2012.10.022>.

- 
- [207] Gang Xu, Bernard Mourrain, Régis Duvigneau, and André Galligo. “Optimal analysis-aware parameterization of computational domain in 3D isogeometric analysis”. In: *Comput.-Aided Des.* 45.4 (2013), pp. 812–821. URL: <https://doi.org/10.1016/j.cad.2011.05.007>.
- [208] Lu Yang, Xiaorong Hou, and Bican Xia. “A complete algorithm for automated discovering of a class of inequality-type theorems”. In: *Sci. China Ser. F* 44.1 (2001), pp. 33–49. URL: <https://doi.org/10.1007/BF02713938>.
- [209] Lu Yang and Bican Xia. “Real Solution Classification for Parametric Semi-Algebraic Systems.” In: *Algorithmic Algebra and Logic*. 2005, pp. 281–289.