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On ergodic properties of some flows on moduli spaces of flat and hyperbolic surfaces and their uses on counting problems

Etienne Bonnafoux

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INSTITUT
POLYTECHNIQUE
DE PARIS

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Thèse de doctorat



On ergodic properties of some flows on moduli spaces and their uses on counting problems

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'École polytechnique

École doctorale n°574 École doctorale de mathématiques Hadamard (EDMH)
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Thèse présentée et soutenue à Palaiseau, le 5 Juillet 2023, par

ETIENNE BONNAFOUX

Composition du Jury :

Anton Zorich Distinguished Professor of Mathematics, Université Paris 7	Président
Valérie Berthé Directrice de Recherche, Université Paris Cité (IRIF)	Rapporteur
Athreya Jayadev Professor, University of Washington	Rapporteur
Charles Favre Directeur de Recherche, Ecole polytechnique (CMLS)	Examineur
Elise Goujard Maitresse de conférence, Université de Bordeaux	Examineur
Carlos Matheus Silva Santos Directeur de Recherche, Ecole polytechnique (CMLS)	Directeur de thèse

So He said, "Go forth and stand on the mountain before the Lord."
 And behold, the Lord was passing by!
 And a great and strong wind was rending the mountains
 and breaking in pieces the rocks
 before the Lord:
 but the Lord was not in the wind.
 And after the wind an earthquake,
 but the Lord was not in the earthquake.
 After the earthquake a fire,
 but the Lord was not in the fire;
 and after the fire a sound of a gentle blowing.

Kings 1
19:11-12

L'Éternel dit: Sois, et tiens-toi sur la montagne devant l'Éternel!
 Et voici, l'Éternel passa.
 Et devant l'Éternel,
 il y eut un vent fort et violent qui déchirait les montagnes
 et brisait les rochers:
 l'Éternel n'était pas dans le vent.
 Et après le vent, ce fut un tremblement de terre:
 l'Éternel n'était pas dans le tremblement de terre.
 Et après le tremblement de terre, un feu:
 l'Éternel n'était pas dans le feu.
 Et après le feu, un murmure doux et léger.

Premier Livre des Rois
19:11-12

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Chapter 1

Introduction en français

Le principe de cette introduction est non seulement d'annoncer le contenu des différents chapitres, mais également de vulgariser les différents sujets présents dans cette thèse. Le lecteur plus curieux pourra suivre les liens pour découvrir les nombreuses ramifications de ces sujets.

1.1 Geometries plate et hyperbolique

Au 4^{ème} siècle avant notre ère, le mathématicien grec Euclide voulu poser les bases de la géométrie en partant de cinq axiomes. Cette vision-là conduit à la fameuse géométrie plate connue de tous. Dans ce cadre, le théorème de Pythagore ou encore l'existence d'une unique parallèle à une droite passant par un point donné sont des énoncés vrais.

Au XIX^{ème} siècle, les travaux de Carl Gauss et de Nikolai Lobatchevski, entre autre, élargissent cette vision en proposant de considérer un espace courbé. Une conséquence est par exemple que la somme des angles d'un triangle au quelle on soustrait π n'est plus forcément égale à 0. Dans le cas où la courbure est constante, cette différence est même proportionnelle à l'aire du triangle. C'est-à-dire pour un triangle ayant comme angles α, β et γ et aire Δ

$$\alpha + \beta + \gamma - \pi = C\Delta.$$

Le coefficient de proportionnalité C permet de calculer la courbure. Si cette différence est positive on dit être en courbure positive et de même si la différence est négative, nous serons en courbure négative.

Ce dernier cas va tout au long de ce manuscrit nous intéresser. Nous allons considérer soit le cas où la courbure est constante, soit toujours nulle, nous travaillons alors en *géométrie plate*, soit négative, égale partout à -1 . Nous dirons alors que nous étudions la *géométrie hyperbolique*.

1.2 Surfaces compactes et à volume fini

Pendant l'Antiquité et le Moyen Age, une grande partie de la géométrie était faite dans le plan, c'est-à-dire sur une feuille de papier ou sur un tableau. Cependant, vers le XIX^{ème} siècle, les mathématiciens ont commencé à vouloir considérer des surfaces et à les classifier. Les plus simples sont dites compactes c'est-à-dire qu'il ne manque aucun point à leurs surfaces et que nous pourrions les ranger dans une boîte en carton pourvue que celle-ci fusse assez grande. Dans cette famille, nous pouvons citer la sphère, comme la surface d'un ballon, le tore, représenté par un donut, le tore à deux trous, comme des menottes ou encore le tore à trois trous, tel un bretzel. Le lecteur généralisera facilement la construction. Le nombre de "trou" est appelé *genre* de la surface.

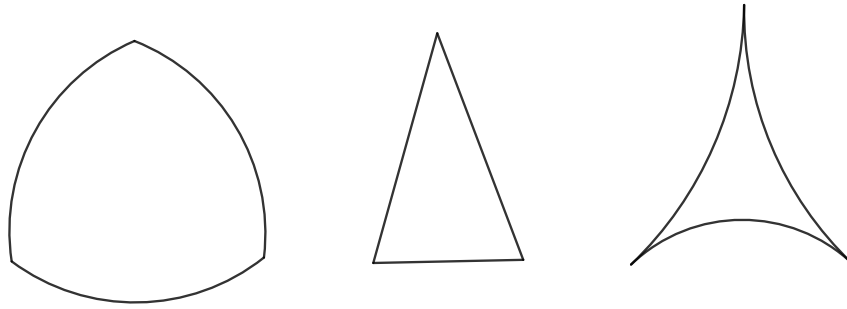


FIGURE 1.1 : Trois triangles avec différentes courbures, de gauche à droite positive, nulle et négative.

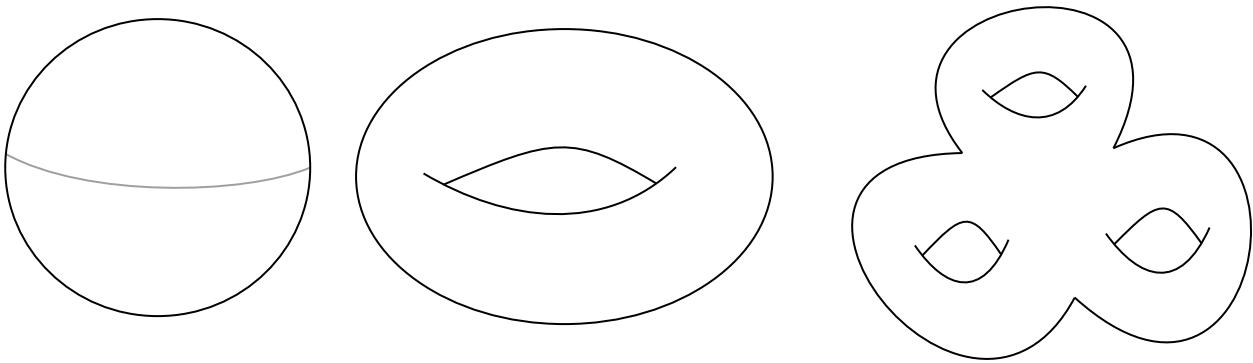


FIGURE 1.2 : Une sphere, un tore et une surface de genre 3

Toutes les surfaces ne sont pas compactes. Leurs géométries peuvent alors devenir plus compliquées. Celles qui vont nous intéresser par la suite sont dite à volume fini. Leurs constructions se font comme suit : en partant d'une surface compacte on choisit n points sur sa surface. On retire alors ces points. La surface devient alors non compacte. Nous pourrions si on voulait, et ça sera le cas en géométrie hyperbolique, tirer la surface autour de ces points manquants pour en faire une *pointe* allant vers l'infini. Dans la suite, S désignera une surface, compacte ou non.

Une fois que l'on a cette notion de surface nous pouvons la relier à la notion de courbure vu à la section précédente. Par un théorème nommé théorème d'uniformisation de Klein-Poincaré pour chaque surface, il n'existe qu'une seule géométrie à courbure constante. Pour la sphère, il faut mettre une courbure strictement positive, que nous n'étudierons pas ici. Pour le tore, la géométrie plate est naturelle. Pour les surfaces de plus grand genre ou pour le tore dont on aurait retiré plus de trois points, la géométrie hyperbolique est alors le bon objet. Une fois une surface S fixé, X désignera une métrique hyperbolique pour celle-ci. La présentation de telles surfaces occupera la Section 3.1.

Cependant, la géométrie hyperbolique étant moins intuitive, nous pourrions vouloir conti-

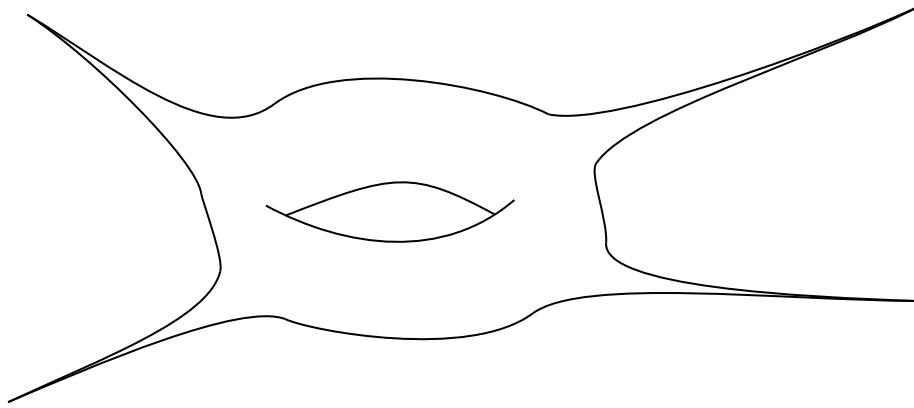


FIGURE 1.3 : Une surface non compacte, de genre 1 avec 4 pointes.

nuer à faire de la géométrie plate même sur des surfaces compliquées avec des genres plus grands que 2. Dans ce cas, il nous faut concentrer toute la courbure sur des points spéciaux appelés *singularités*. Si un observateur se tient debout sur ces points un bras tendus devant lui et qu'il commence à tourner sur lui même, il devra faire plus de 720° avant de se retrouver dans sa position première, contre 360° pour un point régulier.

Cette approche, étudiée à partir du XX^{ème} siècle, est appelé étude des *surface de translation*. Leur étude a notamment été initié pour comprendre les billards. De même pour une surface, nous noterons ω une métrique plate pour celle-ci. La Section 3.2 est dédié à ce sujet.

1.3 Espaces des Modules

Dans l'étude des géométries des surfaces, il est intéressant de les considérer en famille. Une fois une surface S fixée, de genre g et avec n pointes, nous pouvons considérer, dans un ensemble, toutes les métriques hyperboliques applicables à cette surface. Si $3g - 3 + n > 0$, cet ensemble est non vide et même infini. Nous l'appellerons alors *espace de Teichmüller* de la surface et sera dénoté par $\mathcal{T}(S)$. Cependant, de par l'existence de certaines applications généralisant les symétries appelées *difféotopies*, l'espace de Teichmüller contient des surfaces semblables à difféotopies près. Une fois les métriques différents par une difféotopie identifiée entre elles, nous obtenons un ensemble plus petit nommé *espace des modules* et noté $\mathcal{M}(S)$. Si l'on imagine que la surface S est faite de pâte à modeler, choisir X dans $\mathcal{M}(S)$ revient à modeler cette surface, avec pour contrainte d'avoir une courbure strictement négative et constante.

Comprendre cet espace a été et est un travail important de toute une branche des mathématiques. Il a été muni de différente métrique telle que la métrique de Teichmüller, de Weil-Peterson ou encore de Thurston. Les deux premières donne naissance à des flots géodésiques. On peut voir ces flots comme des déformations continues de la métrique hyperbolique d'une surface.

La même chose peut être faite pour les surfaces de translations. Une fois une surface choisie, nous pouvons spécifier le nombre de singularités et leurs types, c'est-à-dire, pour chaque singularité le nombre de tours de 360° supplémentaire à faire par rapport à la normale et considérer toutes les métriques plates sur cette surface ayant les singularités prescrites. Nous obtenons alors un autre espace des modules noté $\Omega_g(\bar{\kappa})$ où g est le genre de la surface et $\bar{\kappa}$ le vecteur précisant les singularités.

Un lien existe entre ces deux objets, il a été découvert par Mirzakhani et nous en reparlerons un peu plus bas et est détaillé dans la Section 3.3.

1.4 Géodésiques sur les surfaces et Liens selles

De même qu'un corps ayant une vitesse initiale non nulle et soumis à aucune force extérieure se déplace en ligne droite dans le plan, nous pouvons nous intéresser au mouvement qu'il aurait dans les mêmes conditions, mais sur une surface avec une géométrie hyperbolique. Les trajectoires effectuées sont appelées *géodésiques*. Lorsque l'objet revient au même endroit avec la même vitesse qu'au moment initiale après avoir effectué un trajet, on dit que la géodésique est fermée. L'étude de ces géodésiques fermés a été et reste un grand thème de recherche actuel. En effet, ces géodésiques interviennent notamment dans la *formule de traces de Selberg* qui relie l'analyse à la géométrie, les opérateurs différentielles aux géodésiques fermées [marklof2004selberg].

Parmi ces géodésiques fermés, il existe une sous-famille appelée simple. Elles sont caractérisées par le fait que la trajectoire ne se recoupe pas de façon transverse avant de revenir au point de départ.

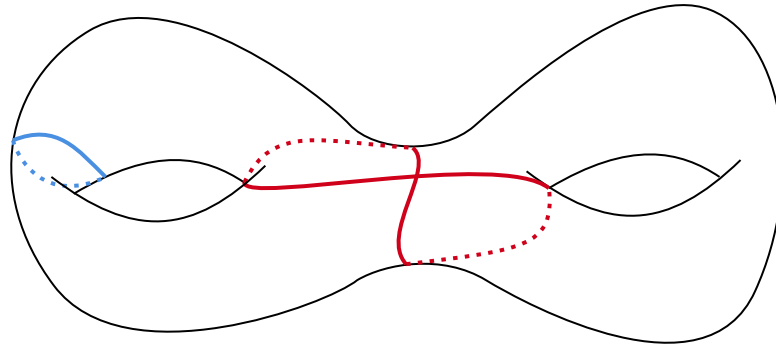


FIGURE 1.4 : Deux géodésiques fermées sur une surface de genre 2. Celle en bleu est simple contrairement à celle rouge.

Il nous est nécessaire d'introduire un objet plus compliqué qu'une géodésique simple fermée qui nous sera utile par la suite, cet objet est la lamination mesurée. Nous pouvons considérer sur la surface une famille de trajectoires géodésiques, pas nécessairement fermés, mais qui ne s'auto-intersecte pas ni qui s'intersecte deux-à-deux, cet objet est une *lamination*. Elle peut être difficile à appréhender, en effet la coupe d'une lamination par un arc peut-être un Cantor, un objet entre la ligne et le point. Puis nous pouvons considérer que cette lamination soit munie d'une mesure transverse, c'est-à-dire qu'à tout arc sur la surface nous associons un nombre représentant la masse déposée par la lamination sur cet arc. L'ensemble d'une lamination et de sa mesure est appelé *lamination mesurée*. Cet objet nous servira à définir le flot du tremblement de terre.

Revenons maintenant aux surfaces plates. Ici les géodésiques deviennent des lignes droites. Cependant, un objet plus fondamental à étudier ici que les géodésiques simples fermées sont les *liens selles*. Ce sont les géodésiques reliant deux singularités (qui peuvent être la même) et qui n'en rencontre pas une autre sur le trajet. À chaque lien selle, nous pouvons associer un vecteur dans le plan qui représente la direction et la longueur de ce lien selle. Il est appelé *vecteur d'holonomie*.

1.5 Action de $SL(2, \mathbb{R})$ sur les surfaces de translations

Pour étudier la géométrie des surfaces de translation, une bonne méthode est de regarder comment nous pouvons les déformer. Il existe trois familles principales de déformations pour ces surfaces qui préservent leurs aires.

La première appelée *flot géodésique* écrase la surface dans une direction et l'étire dans la direction opposée.

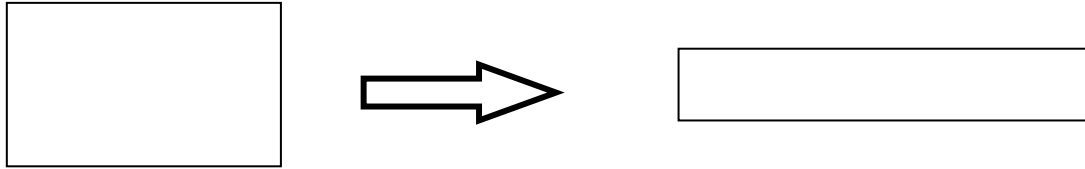


FIGURE 1.5 : Un rectangle soumis au flot géodésique.

La seconde appelée *flot horocyclique* décale le haut de la surface parallèlement au bas et laisse le bas invariant.

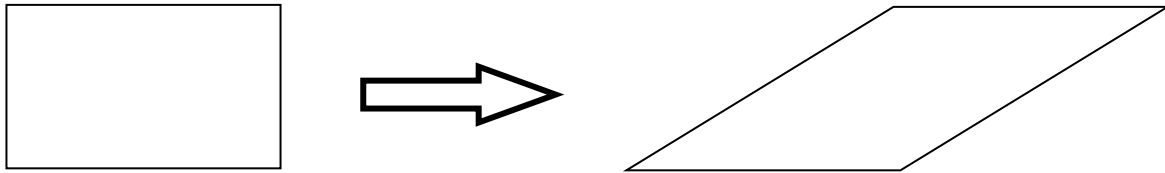


FIGURE 1.6 : Le même avec le flot horocyclique

La troisième appelée *rotation*, comme son nom l'indique, tourne la surface.

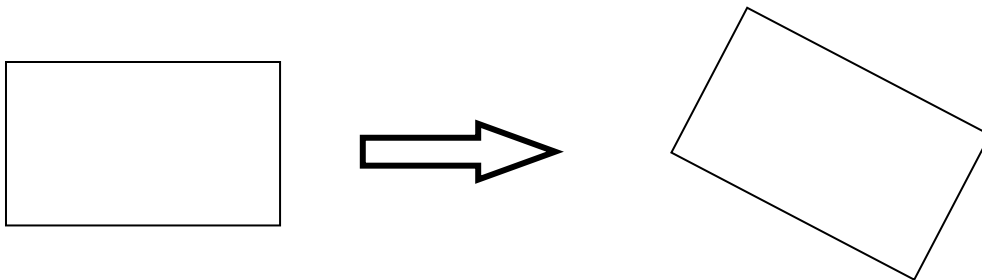


FIGURE 1.7 : Et la rotation.

Ces trois familles combinées forment ce que nous appelons l'*action de $SL(2, \mathbb{R})$* sur l'espace des modules et sera plus étudiée à la Section 4.1. L'action de $SL(2, \mathbb{R})$ est connue pour être ergodique, c'est-à-dire que tout ensemble que cette action laisserai invariant est soit de mesure nulle (donc négligeable) soit de mesure pleine (quasiment tout l'espace). Avila, Gouëzel et Yoccoz [AGY06] démontrèrent un résultat plus fort encore le flot géodésique est exponentiellement mélangeant et le flot horocyclique polynomialement mélangeant pour une classe de fonction qui se comporte bien pour la rotation.

1.6 Le flot du tremblement de terre

Nous abordons maintenant le flot du tremblement de terre, le personnage principal de notre histoire.

Le flot du tremblement de terre est un objet étudié depuis quelques décennies mais qui reste encore assez mystérieux. Il a été introduit par Thurston dans un cours à Princeton dans les années 1976-1977.

Il est construit en prenant une géométrie hyperbolique X et une lamination mesurée λ et en cisillant la surface le long de λ avec une force proportionnelle à la fois à la mesure de λ et à un temps t choisi. Ainsi l'on obtient une nouvelle géométrie notée $E^t(X, \lambda)$. La forme qu'aurait un arc traversant la lamination lui vaut sa dénomination. La construction exacte est donnée dans la Section 4.2.

L'intérêt que la communauté mathématique lui a porté quand Kerckhoff l'a utilisé pour résoudre la conjecture de la réalisation de Nielsen en 1983 [Ker83]. Cette conjecture peut s'énoncer sous la forme suivante :

Tout sous-groupe fini du groupe des difféotopies a un point fixe dans l'espace de Teichmüller.

Depuis le flot du tremblement de terre a été utilisé dans un autre domaine, celui des problèmes de comptage asymptotiques. En effet, il est intimement lié au flot horocyclique agissant sur les surfaces par une conjugaison mesurable trouvée par Mirzakhani [Mir08a]. De cette conjugaison on peut déduire que, par rapport à une mesure très naturelle ce flot est *ergodique*.

Cette ergodicité a été utilisée, par Mirzakhani, pour avoir l'asymptotique du nombre de courbes simple fermées dans l'orbite du groupe des difféotopies d'une autre courbe simple fermée donnée. Dans le même esprit, nous pouvons citer Liu [Liu19] et Arana-Herrera [Ara20] qui dans deux travaux indépendants, ont montré un résultat similaire pour l'asymptotique des multicourbes avec un suivi de la taille de chaque composante. Finalement notons aussi le résultat de Calderon et Arana-Herrera [AC22] sur la distribution de la forme de la surface complémentaire d'une multicourbe quand la taille de celle-ci devient grande. Ces trois résultats utilisent fortement l'ergodicité du tremblement de terre.

L'idée naturelle serait donc d'avoir plus d'information sur le tremblement de terre pour affiner ces résultats. Durant la Section 4.3, nous présenterons un résultat original permettant d'affirmer que la vitesse de mélange du tremblement de terre, pour plusieurs classes de fonctions Lipschitz.

Notre travail utilise une idée déjà présente chez Burns, Masur, Matheus et Wikilson [Bur+17] et consistant à définir un lieu puis une famille de lieux dans l'espace des modules, tel que le flot du tremblement de terre mette un temps contrôlé pour rejoindre le premier lieu. En considérant des fonctions tests ayant pour support ces lieux-ci, et en estimant leurs intégrales et norme de Lipschitz, nous pouvons avoir une borne supérieure pour la vitesse de mélange [Bon22].

Cependant, le lecteur pourrait penser que la conjugaison de Mirzakhani présentée plus haut permettrait d'avoir une vitesse de mélange pour le flot du tremblement de terre, car on connaît celui pour le flot horocyclique. Cependant, cette stratégie tombe sur un écueil. La conjugaison n'est pas continue. Or la vitesse de mélange pour le flot horocyclique est connue pour des classes de fonctions dont on contrôle la dérivée par rapport au flot rotationnel. Celui-ci n'est pas bien transporté par la conjugaison.

Un phénomène illustrant bien ce phénomène de discontinuité est le résultat de Arana-Herrera et Wright [AF22] affirmant qu'il n'existe pas d'automorphisme de la variété conique qui renormaliserait le tremblement de terre, sauf trivialement lui-même. De l'autre côté de la rivière, du côté plat, il est bien connu que le flot géodésique renormalise le flot horocyclique.

Une approche différente est d'essayer de décrire le flot du tremblement de terre dans différents systèmes de coordonnées. Cependant, au mieux des connaissances de l'auteur, les descriptions connues à ce jour se résument à des topologies simples, souvent le tore époinché, et à des tremblements de terre selon des courbes simple particulières ou des laminations particulières. Citons notamment le travail de Garden [Gar22] qui donne l'expression du tremblement de terre dans la

variété de caractère de $SL(2, \mathbb{C})$ du tore épointé selon une géodésique simple fermé. Également, Asaka [Asa20] a calculé l'action du tremblement de terre le long de certaine géodésique simple fermée pour le tore épointé dans un système de coordonnées appelé les *shear coordinates*.

Une autre façon d'avoir une vitesse de mélange pour ce flot serait de le considérer comme partie d'une action plus grande. En effet, le flot horocyclique peut se décrire comme l'action du groupe des matrices unipotentes de $SL(2, \mathbb{R})$. Cela permet, connaissant la vitesse de mélange de l'action des matrices diagonales, de déterminer par conjugaison celle du flot horocyclique.

Plusieurs travaux vont dans ce sens. L'action du flot du tremblement de terre peut être étendue à une action de groupe plus large. Ainsi si on considère l'action réduite aux laminations maximales le *stretch flow* décrit par Thurston [Thu98] se conjugue avec le flot du tremblement de terre. Une autre approche est d'étendre le flot du tremblement de terre à un domaine analytique de \mathbb{C} en ajoutant ce qui est appelé *grafting*, de nombreux travaux traitent ce sujet [mcmullen1998complex]. Finalement Bonsante, Mondello et Schlenker [bonsante2012cyclic ; bonsante2013cyclic] on décrit une action du cercle sur le produit de l'espace de Teichmüller par lui-même. Lors ce qu'un des arguments tend vers une lamination dans la compactification de Thurston, l'action tend vers le flot du tremblement de terre.

Finalement finissons ce paragraphe en citant différents travaux concernant le flot du tremblement de terre. Fu [Fu15] à utiliser le lemme de Borel Cantelli pour avoir une estimation de la vitesse à laquelle le tremblement de terre fait des excursions dans la pointe de l'espace des modules du tore épointé.

Papadopoulos [PAPADOPOULOS1991147] a montré que le flot du tremblement de terre normalisé s'étendait en un flot pour la compactification de Thurston et à donner une description géométrique de cette extension.

Jiang et Su [JS15] se sont intéressés à la convergence de tremblement de terre pour la compactification de Gardiner-Masur. Leur résultat est que si la lamination est uniquement ergodique ou est une géodésique simple fermée avec un poids alors le tremblement de terre va converger vers sa classe projective.

Bonsante et Schlenker [BS12] ont montré que si deux laminations sont remplissantes, c'est-à-dire que toute géodésique simple fermée a une intersection non nulle avec au moins l'une d'elle, alors la composition de leurs tremblements de terre a un point fixe dans l'espace de Teichmüller.

1.7 Comptage asymptotique

Une question importante est notamment de dénombrer le nombre de géodésiques fermés ayant une taille plus petite qu'une certaine quantité L donnée sur une surface S munie d'une métrique hyperbolique X . Nous noterons cette quantité $c(X, L)$. Cette tâche est en général trop compliqué, cependant nous pouvons avoir une idée de comment $c(X, L)$ grandit quand L devient très large. En effet, les travaux de Delsarte, Huber et Selberg ont montré que

$$c(X, L) \sim \frac{e^L}{L}.$$

Une référence standard sur le sujet est le livre de Buser [Bus10]

De même, nous pouvons nous intéresser à dénombrer ces géodésiques simples fermées. Nous noterons $s(X, L)$ le nombre de géodésique simple fermée sur une surface S munie d'une métrique hyperbolique X . Avoir une estimation de cette quantité prise bien plus de temps que pour $d(X, L)$, et il a fallu attendre pour que Mirzakhani [Mir08b] prouve que

$$s(X, L) \sim CB(X)L^{6g-6+2n}$$

où g est le genre de la surface, n le nombre de point enlevés, $B(\cdot)$ une fonction dépendant de la métrique et C une constante globale. Ce résultat est présenté à la Section 5.1.

Le comptage asymptotique de lien selle dont le vecteur d'Holonomie est de longueur inférieure à L , quantité notée $N(L, \omega)$ a également une longue histoire. Tout d'abord Masur [Mas90] montra que

$$c_1 L^2 \leq N(L, \omega) \leq c_2 L^2.$$

Veech [Vee98] continua en montrant qu'il existe une constante $c = c(\mathcal{H})$ tel que pour presque toute surface ω

$$\lim_{L \rightarrow \infty} \int_{\mathcal{H}} \left| \frac{N(L, \omega)}{L^2} - c \right| = 0.$$

Finalement Masur et Eskin [EM01] démontrèrent l'asymptotique quadratique pour presque toute surface. Ainsi pour presque tout ω

$$N(L, \omega) \sim cL^2.$$

Une seconde étape à été le comptage de pair de lien selle avec aire virtuelle bornée. C'est-à-dire que nous voulons considérer les liens selles de longueur plus petite que L et dont l'aire du parallélogramme formé par les vecteurs d'holonomie est plus petit que A . Nous noterons cette quantité $N_A(\omega, L)$. Athreya, Masur et Fairchild [AFM22] ont montré en 2022 que pour presque toute surface pour la mesure de Masur-Veech

$$N_A(\omega, L) \sim c(A)L^2.$$

Dans la Section 5.2 nous exposerons un résultat que nous avons démontré en 2023 [bonnafoux2022pairs]. Nous montrons qu'une asymptotique semblable pour toute mesure μ $SL(2, \mathbb{R})$ -invariante est démontrée modulo un lemme technique. De plus, nous pouvons trouver un terme d'erreur, c'est-à-dire qu'il existe une constante κ tel que pour μ -presque toute surface ω

$$\frac{N_A(\omega, L)}{L^2} = c_\mu(A) + O_\omega(L^{-\kappa})$$

Finalement, nous avons démontré la continuité de la constante $c_\mu(A)$ quand une famille de mesure μ_n converge faiblement vers une mesure μ_∞ .

Chapter 2

Introduction in English

The purpose of this introduction is not only to announce the content of the various chapters but also to popularize the different subjects present in this thesis. The more curious reader can follow the links to discover the many ramifications of these subjects.

2.1 Flat and Hyperbolic Geometries

In the 4th century BCE, the Greek mathematician Euclid wanted to establish the foundations of geometry starting from five axioms. This vision led to the famous flat geometry known to all. In this framework, the Pythagorean theorem or the existence of a unique parallel to a line passing through a given point are true statements.

In the 19th century, the work of Carl Gauss and Nikolai Lobachevsky, among others, expanded this vision by proposing to consider a curved space. One consequence is, for example, that the sum of the angles of a triangle from which we subtract π is no longer necessarily equal to 0. In the case where the curvature is constant, this difference is even proportional to the area of the triangle. That is, for a triangle having angles α, β and γ and area Δ

$$\alpha + \beta + \gamma - \pi = C\Delta.$$

The proportionality coefficient C allows us to calculate the curvature. If this difference is positive, we say that we are in positive curvature and likewise if the difference is negative, we will be in negative curvature.

Throughout this manuscript, we will be interested in the latter case. We will consider either the case where the curvature is constant and always zero, then we work in *flat geometry*, or negative, everywhere equal to -1 . We will then say that we are studying *hyperbolic geometry*.

2.2 Compact surfaces and finite volume surfaces

During Antiquity and the Middle Ages, a large part of geometry was done in the plane, i.e., on a sheet of paper or on a board. However, in the 19th century, mathematicians began to want to consider surfaces and classify them. The simplest ones are called compact, which means that there is no point missing from their surfaces and that we could put them in a cardboard box provided it was large enough. In this family, we can cite the sphere, such as the surface of a ball, the torus, represented by a donut, the double torus, like handcuffs, or the triple torus, like a pretzel. The reader can easily generalize the construction. The number of "holes" is called the *genus* of the surface.

Not all surfaces are compact. Their geometries can then become more complicated. The ones that will interest us later are said to be of finite volume. They are constructed as follows: starting from a compact surface, we choose n points on its surface. We then remove these points. The surface then becomes non-compact. We could, if we wanted to, and this will be the case in hyperbolic geometry, pull the surface around these missing points to make it a *cusp* going towards infinity. Hereafter, S will denote a surface, compact or not.

Once we have this notion of surfaces, we can relate it to the notion of curvature seen in the previous section. By a theorem called *Klein-Poincaré uniformization theorem*, for each surface, there is only one geometry with constant curvature. For the sphere, we need to put a strictly positive curvature, which we will not study here. For the torus, the flat geometry is natural. For surfaces of higher genus or for the torus from which more than three points have been removed, hyperbolic geometry is the appropriate object. Once a surface S is fixed, X will denote a hyperbolic metric for it. The presentation of such surfaces will occupy Section 3.1.

However, since hyperbolic geometry is less intuitive, we might want to continue making flat geometry even on complicated surfaces with higher genus than 2. In this case, we must concentrate all the curvature on special points called *singularities*. If an observer stands on these points with arms outstretched in front of him and starts turning himself, he will have to turn more than 720° before returning to his original position, compared to 360° for a regular point.

This approach, studied since the 20th century, is called the study of *translation surfaces*. Their study was initiated notably to understand billiard. Similarly, for a surface, we will denote ω a flat metric for it. Section 3.2 is dedicated to this subject.

2.3 Moduli spaces

In the study of surface geometries, it is interesting to consider them in families. Once a surface S is fixed, with genus g and n punctures, we can consider in a set all hyperbolic metrics applicable to this surface. If $3g - 3 + n > 0$, this set is not empty and even infinite. We will then call it the *Teichmüller space* of the surface and denote it by $\mathcal{T}(S)$. However, due to the existence of certain applications generalizing symmetries which group is called *mapping class group*, the Teichmüller space contains surfaces that are similar up these applications. Once the metrics differing by an element of the mapping class group are identified with each other, we obtain a smaller set called the *moduli space* and denoted by $\mathcal{M}(S)$. If we imagine that the surface S is made of modeling clay, choosing X in $\mathcal{M}(S)$ amounts to modeling this surface, with the constraint of having a strictly negative and constant curvature.

Understanding this space has been and continues to be an important task of an entire branch of mathematics. It has been equipped with different metrics such as the Teichmüller metric, the Weil-Petersson metric, and the Thurston metric. The first two give rise to geodesic flows. These flows can be seen as continuous deformations of the hyperbolic metric of a surface.

The same thing can be done for translation surfaces. Once a surface is chosen, we can specify the number of singularities and their types, i.e., for each singularity, the number of additional 360° turns to make relative to the normal, and consider all flat metrics on this surface having the prescribed singularities. We then obtain another moduli space denoted by $\Omega_g(\bar{\kappa})$, where g is the genus of the surface and $\bar{\kappa}$ is the vector specifying the singularities.

There is a link between these two objects, discovered by Mirzakhani, which we will discuss later and is detailed in Section 3.3.

2.4 Geodesic on hyperbolic surfaces an saddle connection

In the same way that a body with a non-zero initial velocity and under no external force moves in a straight line in the plane, we can be interested in the motion it would have under the same conditions but on a surface with hyperbolic geometry. The trajectories are called *geodesics*. When the object returns to the same place with the same velocity as at the initial moment after completing a journey, the geodesic is said to be closed. The study of these closed geodesics has been and remains a major current research topic. Indeed, these geodesics intervene notably in the *Selberg trace formula*, which relates analysis to geometry, differential operators to closed geodesics [marklof2004selbergra].

Among these closed geodesics, there is a sub-family called simple. They are characterized by the fact that the trajectory does not intersect transversely before returning to the starting point.

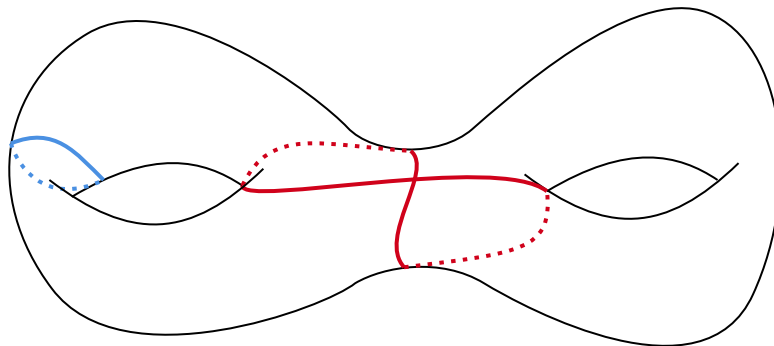


Figure 2.1: Topological representation of two closed curves on a surface of genus 2. The blue one is simple, unlike the red one.

It is necessary for us to introduce an object more complicated than a simple closed geodesic that will be useful to us later, and that is the measured lamination. We can consider on the surface a family of geodesic trajectories, not necessarily closed, but which do not self-intersect or intersect pairwise. This object is a *lamination*. It can be difficult to apprehend, as the intersection of a lamination with an arc can be a Cantor set, an object between a line and a point. Then we can consider that this lamination is equipped with a transverse measure, meaning that we associate a number representing the mass deposited by the lamination on this arc to any arc on the surface. The set of a lamination and its measure is called a *measured lamination*. This object will be used to define the earthquake flow.

Let us now return to flat surfaces. Here, geodesics become straight lines. However, a more fundamental object to study here than simple closed geodesics are *saddle connections*. They are geodesics connecting two singularities (which can be the same) and which do not meet another one on the way. To each saddle connection, we can associate a vector in the plane that represents the direction and length of this saddle connection. It is called a *holonomy vector*.

2.5 Action of $SL(2, \mathbb{R})$ on translation surfaces

To study the geometry of translation surfaces, a good method is to look at how we can deform them. There are three main families of deformations for these surfaces that preserve their area.

The first, called the *geodesic flow*, crushes the surface in one direction and stretches it in the opposite direction.

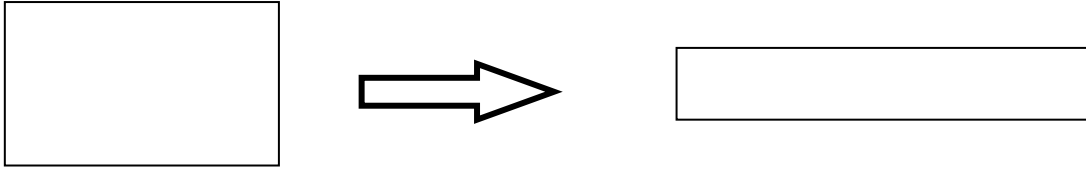


Figure 2.2: Action of geodesic flow on a rectangle.

The second, called the *horocyclic flow*, shifts the top of the surface parallel to the bottom and leaves the bottom invariant.

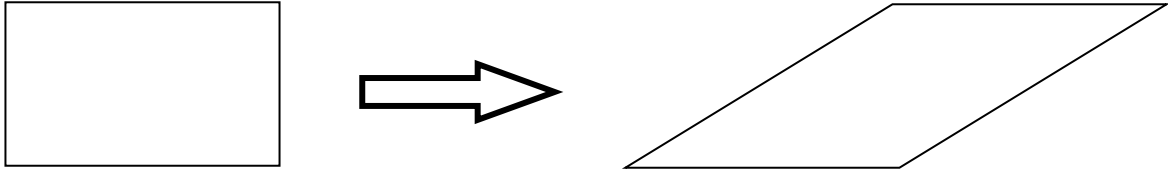


Figure 2.3: Action of horocyclic flow on a rectangle.

The third, called *rotation*, as its name suggests, rotates the surface.

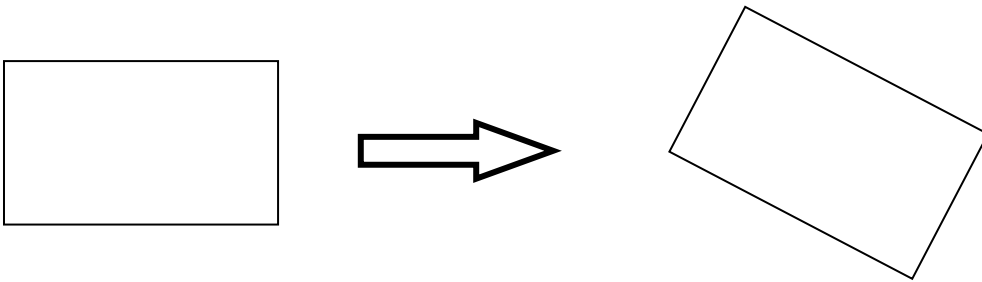


Figure 2.4: Action of rotational flow on a rectangle.

These three families combined form what we call the *action of $SL(2, \mathbb{R})$* on the moduli space and will be studied further in Section 2. The action of $SL(2, \mathbb{R})$ is known to be ergodic, meaning that any set that this action leaves invariant is either of measure zero (negligible) or of full measure (almost the entire space). Avila, Gouëzel and Yoccoz [AGY06] proved an even stronger result: the geodesic flow is exponentially mixing and the horocyclic flow is polynomially mixing for a class of functions that behave well under rotation.

2.6 The earthquake flow

We now turn to the earthquake flow, the main character of our story.

The earthquake flow is an object that has been studied for several decades but remains fairly mysterious. It was introduced by Thurston in a course at Princeton in the years 1976-1977.

It is constructed by taking a hyperbolic geometry X and a measured lamination λ and shearing the surface along λ with a force proportional to both the measure of λ and a chosen time t . Thus, a new geometry denoted $E^t(X, \lambda)$ is obtained. The shape of an arc crossing the lamination, after applying the earthquake flow, explains the name of the latter. The exact construction is given in Section 4.2.

The interest that the mathematical community has had in it began when Kerckhoff used it to solve the Nielsen realization conjecture in 1983 [Ker83]. This conjecture can be formulated as follows:

Every finite subgroup of the mapping class group has a fixed point in the Teichmüller space.

Since then, the earthquake flow has been used in nother area, such as asymptotic counting problems. Indeed, it is intimately related to the horocyclic flow acting on surfaces by a measurable conjugation found by Mirzakhani [Mir08a]. From this conjugation, we can deduce that, with respect to a very natural measure, this flow is *ergodic*.

This ergodicity was used, by Mirzakhani, to obtain the asymptotic number of simple closed curves in the orbit of the mapping class group of another given simple closed curve. In the same spirit, we can mention the works of Liu [Liu19] and Arana-Herrera [Ara20] who, in two independent works, showed a similar result for the asymptotic behavior of multicurves with tracking of the size of each component. Finally, we should note the result of Calderon and Arana-Herrera [AC22] on the distribution of the shape of the complementary surface of a multicurve when its size becomes large. These three results strongly use the ergodicity of the earthquake flow.

The natural idea would be to have more information on the earthquake flow to refine these results. In Section 2, we present an original result allowing us to state that the mixing rate of the earthquake flow for several classes of Lipschitz functions can be controlled.

Our work uses an idea already present in Burns, Masur, Matheus, and Wilson [Bur+17], consisting of defining a -a family of locus in the moduli space, such that the earthquake flow takes a controlled amount of time to reach a fixed given other locus. By considering test functions with these locus as their support, and estimating their integrals and Lipschitz norms, we can obtain an upper bound for the mixing rate [Bon22].

However, the reader might think that the Mirzakhani conjugation presented above would allow us to have a mixing rate for the earthquake flow since we know it for the horocyclic flow. However, this strategy runs into a problem. The conjugation is not continuous. And, the mixing rate for the horocyclic flow is known for classes of functions whose derivative with respect to the rotational flow is controlled. This is not well transported by the conjugation.

A phenomenon illustrating this discontinuity is the result of Arana-Herrera and Wright [AF22] stating that there is no automorphism preserving the orbifold structure that would renormalize the earthquake flow, except trivially itself. On the other side, the flat world, it is well known that the geodesic flow renormalizes the horocyclic flow.

A different approach is to try to describe the earthquake flow in different coordinate systems. However, to the best of the author's knowledge, the known descriptions to date are limited to simple topologies, often the punctured torus, and earthquake flows along particular simple curves or particular laminations. Notably, Garden [Gar22] gives the expression of the earthquake flow in the $SL(2, \mathbb{C})$ character variety of the punctured torus along a simple closed geodesic. Also, Asaka [Asa20] has calculated the action of the earthquake flow along certain simple closed geodesics for the punctured torus in shear coordinates.

Another way to have a mixing rate for this flow would be to consider it as part of a larger action. Indeed, as we have seen, the horocyclic flow can be described as the action of the group of unipotent matrices of $SL(2, \mathbb{R})$. This allows, knowing the mixing rate of the action of diagonal matrices, to determine by conjugation the one of the horocyclic flow.

Several works go in this direction. The action of the earthquake flow can be extended to a larger group action. Thus, if we consider the action reduced to maximal laminations, the *stretch flow* described by Thurston [Thu98] is conjugate to the earthquake flow. Another approach is to extend the earthquake flow to an analytic domain in \mathbb{C} by adding what is called *grafting*, and many works deal with this topic [mcmullen1998complex]. Finally, Bonsante, Mondello, and Schlenker [bonsante2012cyclic; bonsante2013cyclic] have described an action of the circle on the product of the Teichmüller space by itself. When one of the arguments tends toward a lamination in the Thurston compactification, the action tends toward the earthquake flow.

Finally, let us finish this paragraph by citing different works concerning the earthquake flow. Fu [Fu15] used the Borel-Cantelli lemma to obtain an estimate of the speed at which the earthquake flow makes excursions in the cusp of the moduli space of the punctured torus.

Papadopoulos [PAPADOPOULOS1991147] showed that the normalized earthquake flow extends to a flow for Thurston's compactification and gave a geometric description of this extension.

Jiang and Su [JS15] were interested in the convergence of the earthquake flow to the Gardiner-Masur compactification. Their result is that if the lamination is uniquely ergodic or is a simple closed geodesic with a weight, then the earthquake flow will converge to its projective class.

Bonsante and Schlenker [BS12] showed that if two laminations are filling, i.e., every simple closed geodesic intersects at least one of them, then the composition of their earthquake flows has a fixed point in Teichmüller space.

2.7 Asymptotic counting problems

An important question is to count the number of closed geodesics of length smaller than a certain quantity L on a surface S equipped with a hyperbolic metric X . We denote this quantity by $c(X, L)$. This task is generally too complicated, but we can have an idea of how $c(X, L)$ grows when L becomes very large. Indeed, the work of Delsarte, Huber, and Selberg has shown that

$$c(X, L) \sim \frac{e^L}{L}.$$

A standard reference on the topic is Buser's book [Bus10].

Similarly, we can be interested in counting these simple closed geodesics. We denote by $s(X, L)$ the number of simple closed geodesics on a surface S endowed with a hyperbolic metric X . Estimating this quantity took much longer than for $c(X, L)$, and it took until Mirzakhani [Mir08b] proved that

$$s(X, L) \sim CB(X)L^{6g-6+2n},$$

where g is the genus of the surface, n is the number of punctures, $B(\cdot)$ is a function depending on the metric, and C is a global constant. This result is presented in Section 5.1.

The asymptotic counting of saddle connections whose holonomy vector has length less than L , quantity denoted by $N(L, \omega)$, also has a long history. First, Masur [Mas90] showed that

$$c_1 L^2 \leq N(L, \omega) \leq c_2 L^2.$$

Veech [Vee98] continued by showing that there exists a constant $c = c(\mathcal{H})$ such that for almost every surface ω

$$\lim_{L \rightarrow \infty} \int_{\mathcal{H}} \left| \frac{N(L, \omega)}{L^2} - c \right| = 0.$$

Finally, Masur and Eskin [EM01] proved the quadratic asymptotics for almost every surface. Thus, for almost every ω ,

$$N(L, \omega) \sim cL^2.$$

A second step was the counting of saddle connection pairs with bounded virtual area. That is to say, we want to consider saddle connections of length smaller than L and whose

parallelogram area formed by the holonomy vectors is smaller than A . We denote this quantity $N_A(\omega, L)$. Athreya, Masur, and Fairchild [AFM22] showed in 2022 that for almost all surfaces with respect to the Masur-Veech measure,

$$N_A(\omega, L) \sim c(A)L^2.$$

In Section 5.2, we will present a result that we proved in 2023 [**bonnafoux2022pairs**]. We show that a similar asymptotic for any $SL(2, \mathbb{R})$ -invariant measure could be found if one demonstrates one technical lemma. Moreover, we can find an error term, that is, there exists a constant κ such that for μ -almost every surface ω ,

$$\frac{N_A(\omega, L)}{L^2} = c_\mu(A) + O_\omega(L^{-\kappa}).$$

Finally, we proved the continuity of the constant $c_\mu(A)$ when a family of measures μ_n weakly converges to a measure μ_∞ .

Chapter 3

Hyperbolic and Flat geometry

In this chapter, we set the scene. We will describe two conceptions of surface geometry, one with strictly negative constant curvature and one with constant vanishing curvature except for a finite set of points. In both worlds there are interesting metrics and measures that we will use in the two last chapters. We will begin in the first section by describing the hyperbolic world and in the second section we introduce flat geometry. In the last section of this chapter we will describe a bridge connecting these two visions of geometry.

3.1 Hyperbolic geometry and Teichmüller Theory

The first world that we will explore is the hyperbolic world. Many authors have written books on it. Let cite in a non-exhaustive way Kapovich [Kap09], Hubbard [Hub16] or Thurston [Book_Thurston].

3.1.1 The Hyperbolic half-plane

Our first model, the hyperbolic half-plane \mathbb{H}^2 is the set $\{x + iy \in \mathbb{C} : y > 0\}$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

This metric has a constant strictly negative curvature.

The geodesics are vertical lines or half circles which are perpendicular to the real axis.

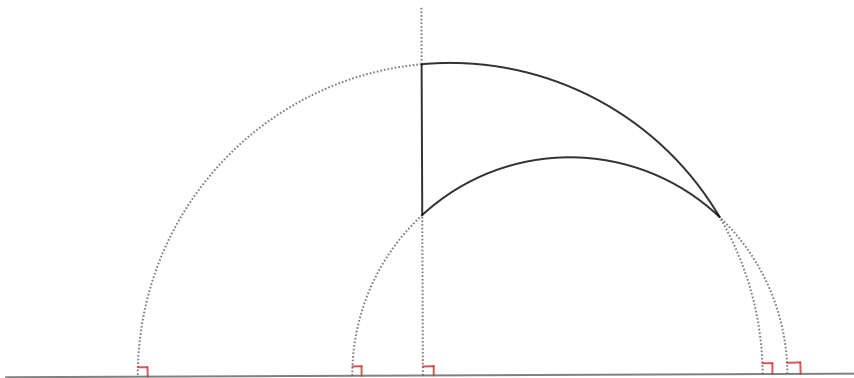


Figure 3.1: Three geodesics making a triangle in the hyperbolic plane. Note that the sum of the angles of the triangle is less than π .

A second useful model, that we will use later, is the hyperbolic disc $\mathbb{D}^2 = \{z = x + iy \in \mathbb{C}, x^2 + y^2 < 1\}$ with the metric

$$ds = \frac{2|dz|}{1 - |z|^2}.$$

One can switch from the hyperbolic half-plane model to the disk model by the Cayley transformation

$$z \mapsto i \frac{i - z}{i + z}.$$

This map preserves the angles between pair of geodesics and send the first metric to the second one.

Definition 3.1.1. A hyperbolic surface is a complete metric space such that every point has a neighbourhood isometric to an open set of the hyperbolic plane.

One common way to build such surfaces is by quotienting the half-plane \mathbb{H}^2 by orientation preserving isometries

One can easily describe these isometries. They are the *Möbius transformations* which for a matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

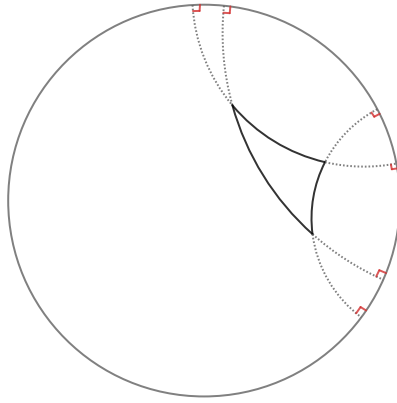


Figure 3.2: The same three geodesics but in the disk model.

associates the map such that for every $z \in \mathbb{H}$, $M \cdot z = \frac{az+b}{cz+d} \in \mathbb{H}$. The group generated by them is isomorphic to $PSL(2, \mathbb{R})$.

A *Fuchsian group* is a discrete subgroup of $PSL(2, \mathbb{R})$. When it is the case, the group of Möbius transformations it generates, acts properly and discontinuously on \mathbb{H}^2 .

In this kind of group, if Γ is a Fuchsian group, the quotient \mathbb{H}^2/Γ is a hyperbolic surface. A detailed survey on this group is given in the book [Kat92].

Example 3.1.1. If $\Gamma = SL(2, \mathbb{Z})$ we get the modular curve.

Example 3.1.2. If Γ is generated by two hyperbolic isometries (such that their traces is strictly greater than 2) with a parabolic commutator, the quotient surface will have the topology of a punctured torus.

Another way to build hyperbolic surfaces is by gluing pieces which already have a hyperbolic metric.

In the hyperbolic plane, it is possible to build hexagons with all right angles. A classical exercise of hyperbolic geometry indicates that, if one fix three positive real numbers $a, b, c \in \mathbb{R}_+^3$ there is only one such hexagon (modulo isometries) such that three non-consecutive sides have length a, b and c [Bus10].

Definition 3.1.2. A *pair of pants* is a hyperbolic surface, which is topologically a sphere minus three discs. It can be built by gluing two right-angle hexagons together. As a consequence of the previous remark the hyperbolic metric is determined by the lengths of the three geodesic components.

Remark. The length of one or more geodesic can go to zero. In this case the corresponding boundary will become a cusp, that is a locus isometric to $\{z \in \mathbb{H}, \text{Im}(z) > 1\}/z \mapsto z + 1$ with the hyperbolic metric.

These pairs of pants can be used as a basic element to construct every other hyperbolic surface. More precisely for all hyperbolic surfaces there is a way to see it into a disjoint union of pairs of pants as described in the next theorem.

Theorem 3.1.1 (see Chapter 3 of Buser's book [Bus10]). *Let S be a surface of genus g with n cusps. There is a set of $3g - 3 + n$ simple closed curves $(\gamma_1, \dots, \gamma_{3g-3+n})$ such that $S \setminus \cup_i \gamma_i$ is a disjoint collection of pairs of pants.*

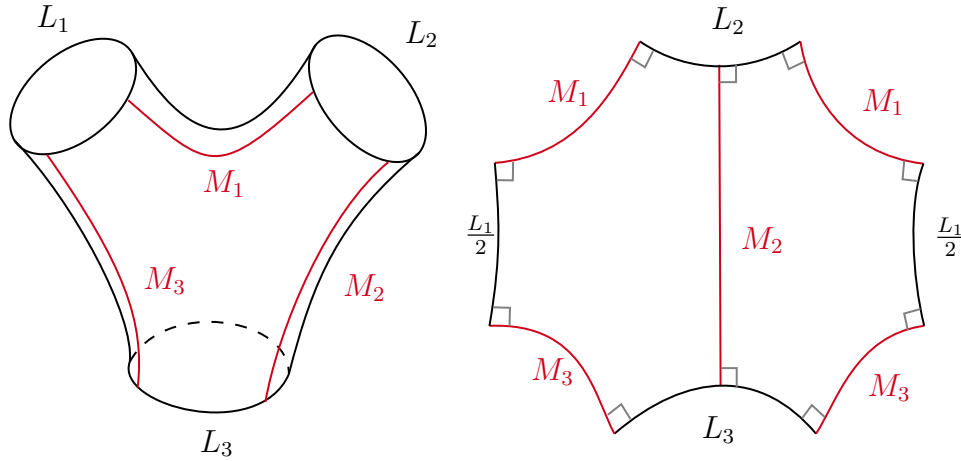


Figure 3.3: A pair of pants and the two right-angled hexagons which constitute it.

3.1.2 Teichmüller Space and Moduli space

Once we have defined what a hyperbolic surface is, we would like to consider them in families. More precisely, we fix the topological data, which are the genus and the number of cusps, and we consider all hyperbolic metrics that could be carried by this surface. This set, modulo some equivalence relation, can be equipped with different metrics and has been extensively studied for more than a century.

One of the first definitions in this direction is the following.

Definition 3.1.3. Let S be a surface of genus g , with n cusps. A *marking* of S is a couple (X, f) made of a closed Riemann surface X and a homeomorphism $f : S \rightarrow X$ which preserves the orientation. On the set of the markings S , we have an equivalence relation: $(X_1, f_1) \sim (X_2, f_2)$ if there exist $\alpha : X_1 \rightarrow X_2$ such that $f_2 \circ \alpha \circ f_1^{-1}$ is a homeomorphism of S preserving the orientation and isotopic to the identity map.

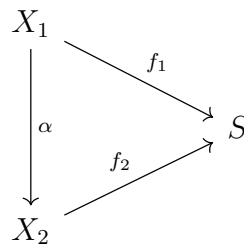


Figure 3.4: Two equivalent markings of a surface S .

The set of the equivalence classes of markings is the *Teichmüller space* $\mathcal{T}_{g,n}$. We will write $X \in \mathcal{T}_{g,n}$ (omitting the marking) in an abuse of notation.

There is a natural topology on $\mathcal{T}_{g,n}$ and several ways to define it: we postpone this to the next subsection when we will use more tools.

In order to do ergodic theory, it is more practical to work with a space supporting (interesting) probability measures. To do so we will consider a natural quotient of the Teichmüller space.

Definition 3.1.4. The mapping class group is the set of diffeomorphism preserving the orientation modulo the group of diffeomorphism isomorphic to the identity.

$$\text{MCG}(S) := \text{Diff}^+ / \text{Diff}^0.$$

To have a better understanding of this group, one can look at a special family in it, which are the so-called *Dehn twists*.

Definition 3.1.5 (Dehn twist). Let γ be a simple closed curve. There is a tubular neighborhood of γ called A homeomorphic to $[0; 1] \times S^1$. A *Dehn twist* around γ is the homeomorphism which is the identity out of A and is $(t, s) \mapsto (t, e^{2i\pi t}s)$ on A .

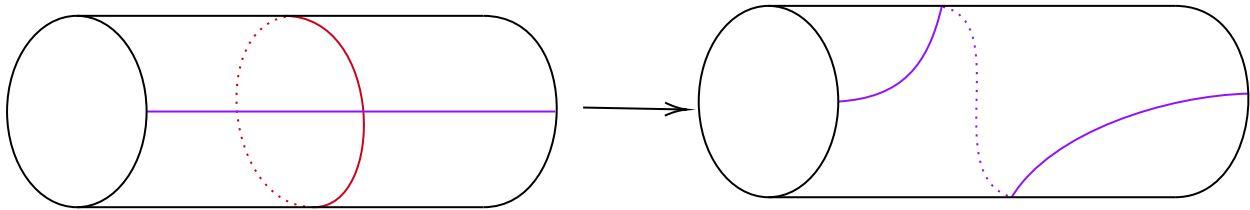


Figure 3.5: A Dehn twist around the red curve.

These elements are of great interest since they generate the (pure) mapping class group. More precisely

Theorem 3.1.2. *For a surface without cusp, the mapping class group is generated by Dehn twist and more precisely that one can choose only $2g + 1$ generators [Lic64].*

However, if the surface has cusps, Dehn twists generate the pure mapping class group that is the subgroup of elements which are isotopic to the identity near the cusps [farb2011].

We can give the final definition for this subsection.

Definition 3.1.6. For a surface S of genus g with n cusps, the moduli space is

$$\mathcal{M}_{g,n} := \mathcal{T}_{g,n} / \text{MCG}.$$

It is a non-compact orbifold (a variety locally modelled by finite group quotient of a Euclidean space) of dimension $6g - 6 + 2n$.

We want now to consider a bundle over this space via the notion of measured laminations.

3.1.3 Curves and laminations

Given S an orientable surface of genus g with n cusps, we call Γ the set of homotopy classes of (essential) simple closed curves. Then for each $X \in \mathcal{T}_{g,n}$ and for each $[\gamma] \in \Gamma$, there is only one representative which is a geodesic for the metric on X . Through this manuscript, when a $X \in \mathcal{T}_{g,n}$ is fixed, we will often take $\gamma \in \Gamma$, meaning that we take the geodesic representative for X .

Each $[\gamma] \in \Gamma$ induces a function on $\mathcal{T}_{g,n}$ by measuring its total arc length

$$l_\gamma(\cdot) : X \mapsto l_\gamma(X) \in \mathbb{R}^+.$$

Another quantity is the intersection number between two curves. Let $\gamma, \gamma' \in \Gamma^2$, if we fix two metrics $X, Y \in \mathcal{T}_{g,n}$, then the number of geometric intersections of the geodesic representatives for X and Y are the same. This allows us to consider the function counting the number of geometric intersection for two homotopic classes

$$i(\cdot, \cdot) : \gamma, \gamma' \mapsto i(\gamma, \gamma')$$

A very useful lemma in hyperbolic geometry links this intersection number to the length of the two curves: it is the so-called *collar lemma*.

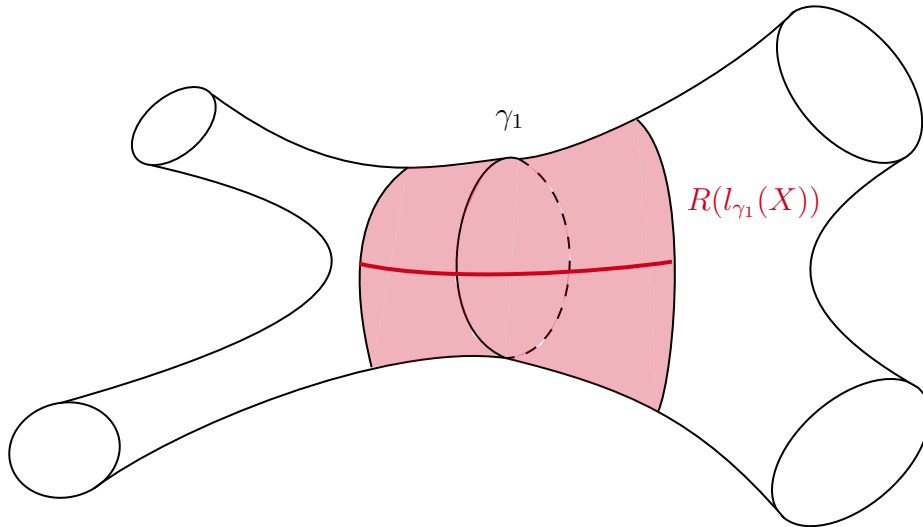


Figure 3.6: Around the geodesic γ_1 there is a annulus of length $R(l_{\gamma_1}(X))$.

Lemma 3.1.3. *If γ_1 and γ_2 are simple closed curves the collar lemma states (see Corollary 3.4 of [Mar16]) that:*

$$i(\gamma_1, \gamma_2) \leq \frac{l_{\gamma_1}(X)}{R(l_{\gamma_2}(X))}$$

where

$$R(x) := 2 \ln \coth(x/4) = 2 \ln \left(\frac{e^{x/2} + 1}{e^{x/2} - 1} \right).$$

It is a decreasing function with $\lim_{x \rightarrow 0} R(x) = +\infty$, $\lim_{x \rightarrow +\infty} R(x) = 0$ and $R \circ R(x) = x$

Proof. Take γ_1 a simple closed geodesic, and consider it in a family $\gamma_1, \dots, \gamma_{3g-3+n}$ of simple closed geodesics making a pair of pants decomposition.

Consider a pair of pants for which γ_1 is one of the boundary component, and without loss of generality assume that the two others boundary components are γ_2 and γ_3 . There is a neighborhood of γ_1 isomorphic to an annulus. The height of this annulus depends only on the length of γ_1 , γ_2 and γ_3 and that its minimal value $R(l_{\gamma_1})/2$ is attained when $l_{\gamma_2} = l_{\gamma_3} = 0$.

So if a simple closed geodesic cut γ_1 n times, it should pass n times through the total annulus of length $R(l_{\gamma_1})$ and so should have length at least $nR(l_{\gamma_1})$. \square

An easy extension of simple closed curves is given by *multicurves* formal weighted finite sum

$$\sum_{i=1}^k a_i \gamma_i \text{ where } (a_i) \in \mathbb{R}_+^k.$$

of disjoint simple closed curves. Length and intersection extend to them by linearity. That is

$$l_{\sum a_i \gamma_i}(X) = \sum a_i l_{\gamma_i}(X)$$

and

$$i\left(\sum a_i \gamma_i, \sum b_j \delta_j\right) = \sum_{i,j} a_i b_j i(\gamma_i, \delta_j).$$

A much more complex but interesting object is a measured lamination.

Definition 3.1.7. A *lamination* λ is a closed set made of a disjoint union of geodesic arcs.

Some of these laminations can be equipped with a *transverse measure*. For a lamination λ , a *transverse measure* is a weighting $\mu(\gamma)$ to each arc γ on the surface and such that

- Additivity: $\mu(\gamma_1 \cup \gamma_2) = \mu(\gamma_1) + \mu(\gamma_2)$
- Support: if $\gamma \cap \lambda = \emptyset$, then $\mu(\gamma) = 0$
- Transversality: if γ_1 and γ_2 are isotopic through arc with endpoints which are not on λ then $\mu(\gamma_1) = \mu(\gamma_2)$.

A pair (λ, μ) made of a lamination and a transverse measure is called a *measured lamination*. For $X \in \mathcal{T}_{g,n}$ we will write $\mathcal{ML}(X)$ the set of all measured laminations. We will often write λ for a measured lamination rather than (λ, μ) .

As for Γ , if λ is a lamination for a metric X , then for a new metric Y there is only one lamination which is isotopic to it. Identifying $\mathcal{ML}(X)$ with $\mathcal{ML}(Y)$ for two different metrics by these isotopies, we will, in an abuse of notation write \mathcal{ML} .

The length function extend to the measured laminations as well as the intersection number. In what follows, we will consider the bundle over the Teichmüller space

$$P\mathcal{T}_{g,n} = \mathcal{T}_{g,n} \times \mathcal{ML},$$

the unit length locus

$$P^1\mathcal{T}_{g,n} := \{(X, \lambda) \in P\mathcal{T}_{g,n}, l_\lambda(X) = 1\},$$

and the quotient

$$P^1\mathcal{M}_{g,n} := P^1\mathcal{T}_{g,n}/\text{MCG}.$$

3.1.4 Fenchel Nielsen coordinates and Weil-Petersson measure.

Later in this text, it will be comfortable to work in a certain system of coordinates.

We recall that, given a surface $S_{g,n}$, a *pair of pants decomposition* is a collection of $3g - 3 + n$ simple closed curves $\gamma_1, \gamma_2, \dots, \gamma_{3g-3+n}$ such that $S \setminus \cup \gamma_i$ is a disjoint union of pairs of pants.

Definition 3.1.8. Given a surface S and a pant decomposition $\gamma_1, \dots, \gamma_{3g-3+n}$, we have a map

$$\begin{aligned} S &\rightarrow \mathbb{R}_+^{3g-3+n} \times \mathbb{R}^{3g-3+n} \\ X &\mapsto (l_{\gamma_1}(X), \dots, l_{\gamma_{3g-3+n}}(X), \tau_{\gamma_1}(X), \dots, \tau_{\gamma_{3g-3+n}}(X)) \end{aligned}$$

This map is injective and is call the *Fenchel-Nielsen* coordinates [Hub16][Chapter 7].

We take the topology induced by this map on $\mathcal{T}_{g,n}$. As a consequence, this means that $\mathcal{T}_{g,n}$ is now a non-compact variety of dimension $6g - 6 + 2n$.

However, at the moduli space level, a folklore result, called Mumford's compactness criteria states that the sets

$$\{[X] \in \mathcal{M}_{g,n}, \forall \gamma \in \Gamma, l_\gamma(X) \geq \epsilon\}$$

are compact [**Mumford Compactness**] and are called *thick part* of the moduli space. Theirs complements, on the other hand are not compact and are called *thin part* of the moduli space and will be written as $\mathcal{M}_{g,n}^\epsilon$.

Given a pants decomposition $\gamma_1, \dots, \gamma_{3g-3}$, we have a natural symplectic 2-form on $\mathcal{T}_{g,n}$:

$$d\omega_{WP} = \sum_{i=1}^{3g-3} dl_{\gamma_i} \wedge d\tau_{\gamma_i},$$

where l denotes the length function and τ the twist. As it was discovered by Wolpert [Wol82; Wol83], this symplectic form does not depend on the choice of pants decomposition, and it induces a volume form μ_{WP} called the Weil-Petersson measure.

3.1.5 Metrics on Moduli space

We can consider different metrics on the Teichmüller spaces.

Thurston distance

Definition 3.1.9. The asymmetric Thurston distance (cf. [Thu98]) on $\mathcal{T}_{g,n}$ between Y and Y' is

$$d_{Th}^{asym}(Y, Y') := \log \left(\sup_{\lambda \in \mathcal{ML}} \frac{l_\lambda(Y)}{l_\lambda(Y')} \right)$$

By symmetrizing d_{Th}^{asym} , we get the Thurston distance:

$$d_{Th}(Y, Y') := \max(d_{Th}^{asym}(Y, Y'), d_{Th}^{asym}(Y', Y))$$

- This metric is not uniquely geodesic, there are pairs of points joined by more than one geodesic segment.
- It is a Finsler metric
- The metric is complete

For more on this metric see the survey of Papadopoulos and Th  ret [[papadopoulos:hal-00129729](#)].

Teichm  ller distance

The *Teichm  ller distance* d_{Teich} between $X, Y \in \mathcal{T}_{g,n}$ is defined to be $\frac{\log(K)}{2}$, where K is the minimum quasi-conformal dilatation of quasi-conformal homeomorphism between X and Y .

$$K := \inf_{h \equiv f_2 f_1^{-1}} \operatorname{ess\,sup} \frac{|\partial h / \partial z| + |\partial h / \partial \bar{z}|}{|\partial h / \partial z| - |\partial h / \partial \bar{z}|}$$

This metric can be computed in another way, which make it similar to the Thurston distance. For $\gamma \in \Gamma$, one define its *extremal length* as:

$$Ext_X(\gamma) = \sup_{\sigma} \frac{L_{\sigma}^2(\gamma)}{A(\gamma)}$$

where the supremum is over conformal metrics $\sigma(z)|dz|$,

$$L_{\sigma}(\gamma) = \inf_{\gamma' \equiv \gamma} \int_{\gamma'} \sigma(z) dz$$

and

$$A(\sigma) = \int_X \sigma^2(z) |ds|^2.$$

We have the following formula from Kerckhoff [[Kerckhoff1980TheAG](#)],

$$d_{Teich}(X, Y) = \sup_{\gamma \in \Gamma} \frac{1}{2} \log \frac{Ext_X(\gamma)}{Ext_Y(\gamma)}$$

Here are some facts about this metric.

- This metric is complete

- Between two points there is a unique Teichmüller geodesic which can be uniquely extended infinitely in both directions.
- The metric does not have negative curvature in the sense of Busemann [Masur1975OnAC] meaning that there are pairs of geodesic rays r_1, r_2 such that for some $t > 0$

$$d_{Teich}(r_1(t), r_2(t)) \geq \frac{1}{2}d_{Teich}(r_1(2t), r_2(2t)).$$

More than that there are pairs of geodesic rays which stay at a bounded distance apart.

A good survey on this metric was made by Masur [articleMasurSurvey].

Weil-Peterson distance

The cotangent space of $\mathcal{T}_{g,n}$ at a point X is the set of holomorphic quadratic differential. A Hermitian inner product is then defined by

$$\langle \phi, \psi \rangle = \int_S \phi \bar{\psi} ds^{-2}$$

where s is the hyperbolic metric.

Here are some facts about this metric.

- Weil-Petersson metric is Kähler;
- it is an incomplete metric;
- its sectional curvatures are negative;
- The mapping class group acts by isometry with respect to this metric.

Proofs of these facts and other information could be found in [Masur1976TheEO], [Wolpert1986], [Wolpert87] and [tromba2012].

3.1.6 Dehn-Thurston coordinates

Among all pair of pants decomposition there is one with a leading role, the *standard basis*. It is described for a surface of genus g with n cusp as in the following pictures.

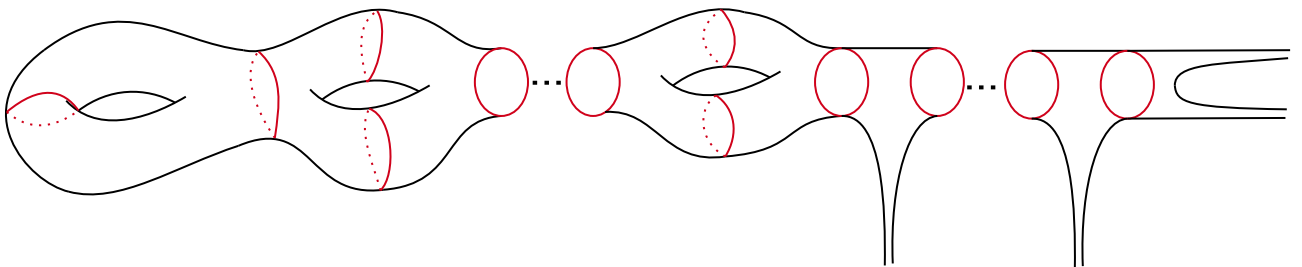


Figure 3.7: The standard basis in red.

Let's call $(\gamma_1, \dots, \gamma_{3g-3+n})$ the simple closed curves which composes it. If we denote by $i(\cdot, \cdot)$ the intersection number and by $t(\cdot)$ the twisting number on $\mathcal{ML} \times \mathcal{ML}$, then we have the following theorem [PH92]:

Theorem 3.1.4 (Dehn-Thurston coordinates). *The map*

$$\begin{aligned} \mathcal{ML} &\rightarrow (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3+n} \cup (\{0\} \times \mathbb{R}_{\geq 0})^{3g-3} \\ \lambda &\mapsto (i(\gamma_1, \lambda), t_{\gamma_1}(\lambda), \dots, i(\gamma_{3g-3}, \lambda), t_{\gamma_{3g-3}}(\lambda)) \end{aligned}$$

is a bijection.

Using this system of coordinates, we can pull back the L^∞ norm from \mathbb{R}^{6g-6} to \mathcal{ML} to get a distance d_{lam} .

In this context, we get distances on $\mathcal{T}_{g,n} \times \mathcal{ML}$ by setting

$$d_{\mathcal{T}_{g,n}} \times d_{lam}((X, \lambda), (X', \lambda')) = \max(d_{\mathcal{T}_{g,n}}(X, X'), d_{lam}(\lambda, \lambda'))$$

where $d_{\mathcal{T}_{g,n}}$ can be d_{Th} , d_{WP} or d_{Teich} .

Finally, we obtain a distance on $P^1\mathcal{M}_{g,n}$ with the formula

$$d_{P^1\mathcal{M}_{g,n}}([Y, \lambda], [Y', \lambda']) = \inf_{h \in Mod(S)} d_{Th} \times d_{lam}((Y, \lambda), (h.Y', h.\lambda')).$$

In the sequel, we will consider the space of bounded Lipschitz functions on $P^1\mathcal{M}_{g,n}$ equipped with the following norm:

$$\|f\|_{Lip} = \|f\|_{L^\infty} + \sup_{\substack{x, y \in P^1\mathcal{M}_{g,n}, \\ x \neq y}} \frac{|f(x) - f(y)|}{d_{P^1\mathcal{M}_{g,n}}(x, y)}.$$

3.1.7 Thurston measure

\mathcal{ML} possesses a family of measures depending on $X \in \mathcal{T}_{g,n}$:

$$\mu'_{Th}(X)(A) = \lim_{L \rightarrow \infty} \frac{\#\{\delta \in \mathcal{ML}(\mathbb{Z}) \cap A, l_\delta(X) \leq L\}}{L^{6g-6}}$$

where $\mathcal{ML}(\mathbb{Z})$ is the set of multicurves with integer coefficients. These measures are called Thurston measures and they can be projected to $P^1\mathcal{ML} = \mathcal{ML}/\mathbb{R}^{>0}$ by setting:

$$\mu_{Th}(X)(A) = \mu'_{Th}(X)(\{\lambda \in \mathcal{ML}, l_\lambda(X) \leq 1, [\lambda] \in A\}).$$

The product measure $\mu_{WP} \times \mu_{Th}$ on $P^1\mathcal{T}_{g,n}$ is $Mod(S)$ -invariant and, hence, it induces a measure on $P^1\mathcal{M}_{g,n}$ called ν .

Moreover, we will call $B(\cdot)$ the function indicating how the volume of the unit ball of the space of measured lamination changes in the moduli space.

Definition 3.1.10. For $X \in \mathcal{T}_{g,n}$ we consider $B(X) := \mu'_{Th}(\lambda \in \mathcal{ML}, l_\lambda(X) \leq 1)$. This function is $Mod(S)$ -invariant and therefore well-defined on $\mathcal{M}_{g,n}$. We keep the same notation $B(\cdot)$ for the function induced on $\mathcal{M}_{g,n}$.

Lemma 3.1.5. *The function B has a strictly positive lower bound on $\mathcal{M}_{g,n}$.*

Proof. First, the function B is continuous and never vanishes according to Proposition 3.2 of [Mir08b]. Secondly, on the complement of $\mathcal{M}_{g,n}^\epsilon$, according to Proposition 3.6 of [Mir08b] we have

$$1 \leq \prod_{\substack{\gamma \in \Gamma \\ l_\gamma(X) \leq \epsilon}} \frac{1}{l_\gamma(X) \log(l_\gamma(X))} \leq B(X)$$

for ϵ small enough. This completes the proof. \square

3.2 Flat Geometry

Let's now take a walk on the flat side. If we consider a surface, we could consider metrics with no curvature everywhere. According to Gauss-Bonnet such metrics exist only on the torus, however if we allow singularities we can equip surfaces of any genus with such metric. We will consider only orientable surfaces.

3.2.1 Translation surfaces

Definition 3.2.1. A translation surface (X, ω) is a pair made of a Riemann surface X and a non-zero holomorphic one form ω .

A more natural and intuitive approach is to say that it is a surface made by two-by-two identification of a finite collection of polygons in \mathbb{C} via translation map $z \mapsto z + a$ with $a \in \mathbb{C}$.

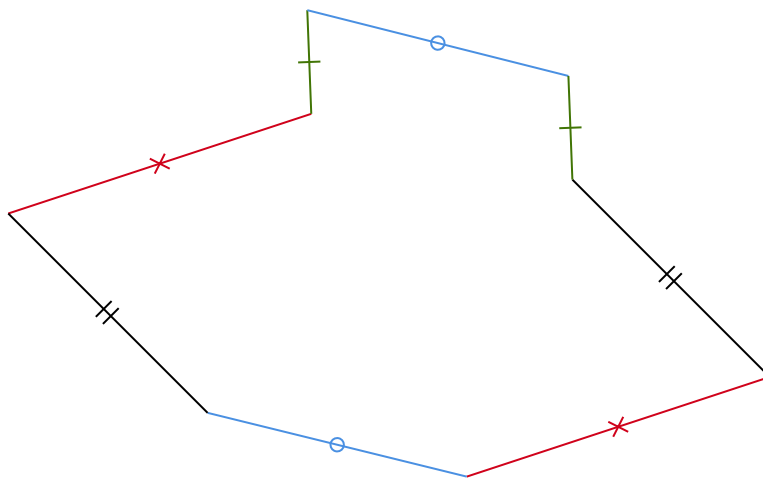


Figure 3.8: A model of translation surface with a polygon with sides identified two-by-two.

Remark. A more general notion is the half translation surfaces which are pair (X, q) with X a Riemann surface and q a quadratic differential. In this case the surface might not be orientable. The intuitive approach described above should be changed by considering identification with maps of the form $z \mapsto \pm z + a$.

Example 3.2.1. If the polygons involved in the construction of the translation surface are all squares with the same area, the surface is called a *square tiled surface*. They play an important role in flat geometry.

3.2.2 Moduli space and strata

We call Ω_g the moduli space of compact genus g area 1 translation surfaces. It follows from the Riemann-Roch theorem that any translation surface has $2g - 2$ zeros counted with multiplicity. So there is a natural stratification of Ω_g by the partition of $2g - 2$. We will write $\Omega_g(a_1, \dots, a_k)$ the locus where the partition is given by $a_1 + \dots + a_k$.

$$\Omega_g = \bigcup_{\sum_i a_i = 2g-2} \Omega_g(a_1, \dots, a_k)$$

We will also consider Ω'_g the moduli space of compact genus g translation surfaces without any area restriction.

3.2.3 Saddle connection

Definition 3.2.2. A *saddle connection* is a (flat-)geodesic between two zeros of a translation surface without any other zero in its inside.

Saddle connections can be used to give local coordinates inside a stratum of Ω_g . More concretely,

Definition 3.2.3. Let S be a surface of genus g and ω a holomorphic one form with zeros p_1, \dots, p_k . We can represent it by a finite collection of polygons with edges identified two-by-two and the zeros $\Sigma = \{p_i\}$ on the vertices.

We denote by $H_1(S, \Sigma, \mathbb{Z})$ the first relative homology group and by $H^1(S, \Sigma, \mathbb{C}) \equiv \text{Hom}(H_1(S, \Sigma, \mathbb{Z}), \mathbb{C})$ its cohomological dual. Then there is a neighborhood U (in the stratum \mathcal{H}) of (S, ω) such that for each $(S', \omega') \in U$, one can identify $H^1(S, \Sigma, \mathbb{C})$ with $H^1(S', \Sigma', \mathbb{C})$ using the Gauss-Manin connection. In this case in the neighborhood of (S, ω) , there are local coordinates given by

$$\Theta : \begin{array}{l} U \quad \rightarrow \quad H^1(S, \Sigma, \mathbb{C}) \\ (S', \omega') \mapsto (\gamma \rightarrow \int_\gamma \omega'). \end{array}$$

A short proof of the injectivity of this map is given in the survey of Matheus and Giovanni [forni2013introduction]. These coordinates are called *period coordinates*.

Remark. These coordinates equip Ω_g with a structure of local piecewise integral linear structures.

With this topology the strata $\Omega_g(a_1, \dots, a_k)$ are not always connected. The complete classification has been done by Kontsevitch and Zorich [KZ03]. In the half-translation surfaces case, look at Lanneau [lanneau2004hyperelliptic; lanneau2005connected] and also Chen and Möller [chen2012quadratic].

3.2.4 Masur-Veech volumes

With the piecewise integral linear structure of Ω'_g comes a Lebesgue measure μ'_{MV} . It is the Lebesgue measure with respect to the period coordinates and is called *Masur-Veech* measure.

Then by coning, a measure μ_{MV} , also named Masur-Veech measure, is defined on Ω_g , that is for every A measurable:

$$\mu_{MV}(A) = \mu'_{MV}(\{s\omega, 0 \leq s \leq 1, \omega \in A\}).$$

Of course, because of the scaling factor, the total volume of Ω'_g by μ'_{MV} is infinite, but independently Masur [MasurFoliation] and Veech [VeechIntervalExchange] showed that the measure of Ω_g by μ_{MV} is finite.

Remark. An equivalent definition of the measure is by counting the asymptotic number of square-tiled surfaces.

Theorem 3.2.1. Taking d to be the complex dimension of the stratum, if we define, for a Borel set A

$$\mu^{\text{sq.surf}}(A) = \lim_{N \rightarrow \infty} \frac{\#\{S \text{ square tiled surface} \in A, \text{ with } N \text{ squares}\}}{N^d}$$

then this quantity is proportional, with a factor independent of A , to $\mu_{MV}(A)$.

See Section 2 of [delecroix2021masur] for a short demonstration. It mostly relied on the invariance of the two volume forms by the Teichmüller flow (proved by Masur [masur_hamilton] for the first one and by Masur [MasurFoliation] and Veech [Vee82] for the second) and the ergodicity of the Teichmüller flow.

This property is one reason for which square tiled surfaces are extensively studied.

3.2.5 Multiscale Compactification

A crucial tool in chapter 3, will be the multi-scale compactification of strata introduced by Bainbridge, Chen, Grushevsky and Möller [Bai+18].

Let us recall quickly the compactification process. A multi-scale differential is the data of

- A nodal Riemann surface M ;
- A graph, where each vertex correspond to a component of the surface M , and each edge is a node between some components;
- Half-edge recording in which component the zeroes of the differential are;
- A level function l assigning a non-positive integer to each vertex. We take it to be surjective in $[-N, 0] \cap \mathbb{Z}$ for some N . The union of component with the same level, that is $l^{-1}(i)$ for a given i is called a *level subsurface* and denote by $X^{(i)}$;
- A positive integer b_e to each vertical edge which record the cone angle;
- A collection of meromorphic differentials on each component of the surface which are consistent with the previous data (we won't give here the meaning of this term, but we refer the reader to [Bai+18] for full definition);
- And a prong-matching record, giving the identification of the horizontal direction between two sides of a node.

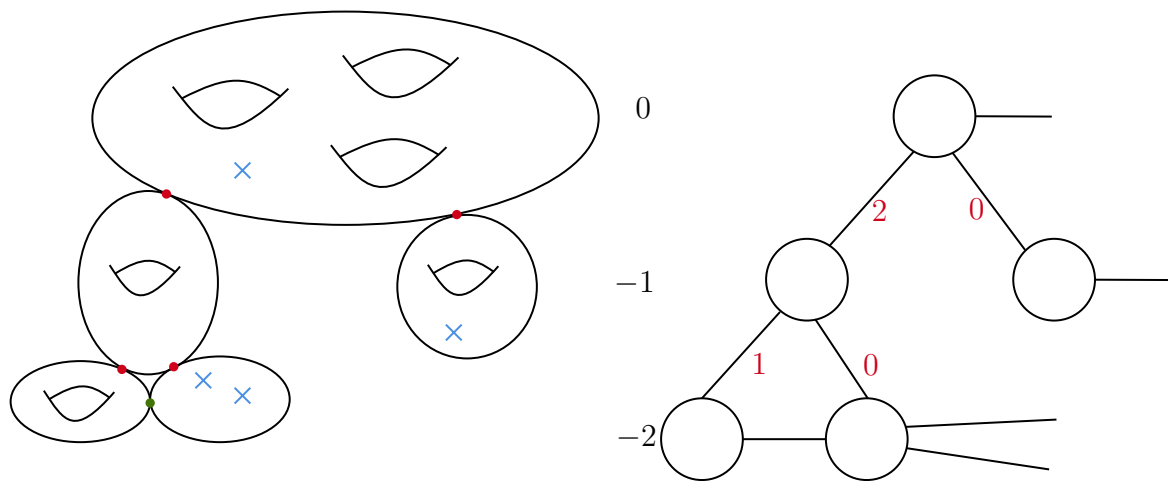


Figure 3.9: Data of a multiscale differential. A nodal surface with zeros as blue mark and an associated graph.

The compactified stratum is denoted $\bar{\mathcal{H}}$. On it, special neighborhoods are handy for computation. They are described by the following definition and theorem.

Definition 3.2.4. A connected, open subset $Q \subset \bar{\mathcal{H}}$ is said to be a *period coordinate chart* if it admits an injective map to \mathbb{C}^n that is locally linear with respect to the period coordinates.

Given \bar{X} in the boundary of $\bar{\mathcal{H}}$, one can find a good neighborhood supporting a complex-analytic system of coordinates such that

- in each component, there are *moduli parameters* $s_i \in \mathbb{C}$ that we take small, that is $0 < |s_i| < \epsilon$,

- for each node, there is a *smoothing parameter* $t \in \mathbb{C}$. We also take them with a restriction on the norm $0 < |t| < \epsilon$,
- if t is a horizontal node parameter, then we consider a restriction of the smoothing parameter

$$\arg(t) \in (\alpha, \alpha + \pi/4)$$

where we choose finitely many α such that the family of such intervals cover all the circle,

- if t_i is a scaling parameter for level i , with associate integer a_i , we find connected interval condition on $\arg(t_i)$ such that $\arg(t_i^{a_i})$ satisfies the same condition as in the previous point.

Dozier showed in Lemma 3.6 of [Doz20] that:

Theorem 3.2.2. *The previously defined neighborhoods are period coordinate charts, for ϵ small enough.*

Ordering of level subsurfaces

The ordering given by the level function might not be a total order. For this reason, we need to introduce some extra structure to organize the surface underlying a multi-scale differential.

Let \mathcal{S} be the set made of level subsurfaces and degenerating cylinders. Dozier defined a function $size_\omega(\cdot) : \mathcal{S} \rightarrow \mathbb{R}$ for every $\omega \in V$ with V a neighborhood as in Theorem 3.2.2.

Also, let \mathcal{O} be the set of orderings on \mathcal{S} that restricts to the ordering given by the level function and have the property that if C is a degenerating cylinder and X^i is the level subsurface at which the circumference curve of C lies, then $C \succ X^i$.

We say that $\succ \in \mathcal{O}$ is *consistent* with $\omega \in V$ if

$$size_\omega(Y_1) \geq size_\omega(Y_2) \implies Y_1 \succ Y_2.$$

Finally, we call a degenerating cylinder \succ -wide if it is \succ -greater than any level subsurface.

3.2.6 Delaunay triangulation

One last tool that we will use in Chapter 3 is Delaunay triangulation. Let's consider the *Voronoi decomposition* of a translation surface with each cell centered around singular point. Then the dual of this decomposition is the *Delaunay decomposition*. Any triangulation refining the Delaunay decomposition is called a *Delaunay triangulation*. Refer to [MS91] for complete definition.

One key point of these triangulations is the so-called efficiency property

Theorem 3.2.3. *Let S be a translation surface and T a Delaunay triangulation. Then every saddle connection β is homotopically equivalent to a path $P(\beta)$ in the Delaunay triangulation whose length satisfies*

$$|P(\beta)| \leq \sqrt{10}|\beta|.$$

Proof. See Lemma 2.3 of [ACM19]. □

Given $\omega \in \Omega'_g$ there is a closed neighborhood \mathcal{D}'_ω such that any flat surfaces in it share the same Delaunay decomposition. Some surfaces have multiple Delaunay decomposition, in this case we arbitrarily assigned one to them. We call *iso-Delaunay cell* the set obtained after the assignation step for ambiguous surfaces. These cells give disjoint cover $\Omega'_g = \cup \mathcal{D}$. One can show that this cover is actually finite, see Paragraph 7.1 of [nguyen2019volume].

3.3 The bridge between the two worlds

The two previous sections seem disconnected one from each other. The aim of this part is to link them with a powerful tool developed by Mirzakhani.

As mentioned previously we can consider flat surfaces as a pair made of a Riemann surfaces and a quadratic differential. This will be the point of view that we will use.

Theorem 3.3.1 ([Mir08a]). *There is a measurable conjugacy between two natural dynamical systems on $P^1\mathcal{M}_g$ and Ω_g (cf. Chapter 2 for the definitions).*

$$\begin{array}{ccc} P^1\mathcal{M}_g & \longrightarrow & P^1\mathcal{M}_g \\ F \downarrow & & \downarrow F \\ \Omega_g & \longrightarrow & \Omega_g \end{array}$$

Note that this conjugacy is not defined everywhere. It will be asked that the measured lamination is *maximal*, that is, all complementary regions are ideal triangles. In the image we find quadratic differentials in the principal stratum, which means with simple zeros.

We will describe the conjugacy F as a composition of three maps.

- First we will see a method to tight foliation to get a measured lamination and in the opposite map.
- Then we will describe quadratic differential by pair of measured foliations on the surface.
- Finally, we will use the so-called horocyclic foliation to associate a foliation to any hyperbolic metric.

3.3.1 Tightening map

A first correspondence, found by Thurston, exists between measured foliations and measured laminations. We will mostly follow the paper of Levitt [Lev83].

Definition 3.3.1. A *measured foliation* is a foliation of the surface with only prong type singularities and a transverse measure.

The set of all measured foliation is too big for our purpose, and (as in the definition of Teichmüller space) we shall consider a quotient of it. Let's describe a transformation leading to an adequate equivalence relation.

Definition 3.3.2. Given a measured foliation, a critical segment γ is an arc between two singularities along a leaf which is not a simple closed curve. There is a map f homotopic to the identity that collapse γ to a point x and is identity outside a neighborhood of γ which contain no other singularity. Doing so we reduce the number of singularities of the foliation and if the extremities of γ are singularities of order k_1 and k_2 , x is now a singularity of the new foliation of order $k_1 + k_2 - 2$. This action is called a *Whitehead move*.

Definition 3.3.3. We say that two foliations are equivalent if we can pass from one to the other by Whitehead moves or homeomorphism isotopic to the identity. We write \mathcal{MF} the set of equivalence classes of measured foliation.

We now give the main theorem of this subsection.

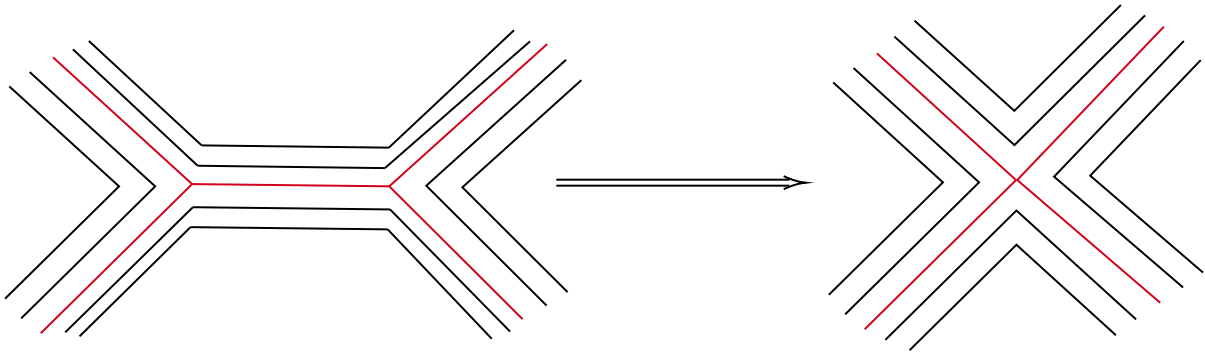


Figure 3.10: A Whitehead move collapsing the red arc.

Theorem 3.3.2. *Let X be a closed orientable hyperbolic surface and \mathcal{F} a foliation. There is a canonical geodesic lamination $\gamma(\mathcal{F})$ associated to \mathcal{F} . If \mathcal{F} and \mathcal{F}' are equivalent foliation then $\gamma(\mathcal{F}) = \gamma(\mathcal{F}')$. In the opposite direction given a geodesic lamination γ , one can find a foliation \mathcal{F} such that $\gamma(\mathcal{F}) = \gamma$ and it's unique up to equivalence.*

Definition 3.3.4. A *transverse curve* is a simple closed curve C which is never tangent to \mathcal{F} and contains no singularity of \mathcal{F} .

Remark. Since \mathcal{F} contains only saddle singularities, C cannot be contractible, therefore C is isotopic to a simple closed geodesic.

We will work on the universal cover of X , which is the Poincaré disc \mathbb{D} and we denote its “circle at infinity” by \mathbb{S}_∞ . We denote $p : \mathbb{D} \rightarrow X$ the universal cover and we define $\tilde{\mathcal{F}} = p^{-1}(\mathcal{F})$.

We will say that a foliation satisfies the condition (*) if the following statement is true:

If f_1 and f_2 are two compact homotopic leaves, then all leaves in the open annulus between them are also compact.

Lemma 3.3.3. *Let h be a leaf of $\tilde{\mathcal{F}}$. Each end of h converges to a point of \mathbb{S}_∞ and these two points at infinity can not be the same.*

Proof. First, we should notice that the behavior of leaf at infinity does not change if we take an equivalent foliation. Indeed, a homeomorphism ϕ on a compact fundamental domain isotopic to the identity can be extended to a homeomorphism $\tilde{\phi}$ on \mathbb{D} such that $dist(x, \tilde{\phi}(x)) \leq K$. This implies that $\tilde{\phi}$ extend as the identity on the boundary \mathbb{S}_∞ .

Then given a leaf h of $\tilde{\mathcal{F}}$, we take a half leaf h_0 . If $p(h_0)$ is compact or spirals toward a compact leaf of \mathcal{F} then the first part of the lemma is immediate.

Otherwise, $p(h)$ meet a transverse curve C infinitely often. With an isotopy we can take C to be a geodesic. Now h_0 can meet a connected component of $\tilde{C} = p^{-1}(C)$ only one time. Otherwise, there will be a disk bound by an arc of \tilde{C} and an arc of h_0 , which is impossible considering that C is transverse and that \mathcal{F} have no 1-type singularities.

Now every compact subset of \mathbb{D} meets a finite number of connected components of \tilde{C} so the limit set of h_0 must be on \mathbb{S}_∞ . This limit set is connected and non-empty. Moreover, it should not contain any end of a connected component of \tilde{C} . But the ends of connected components of \tilde{C} are dense in \mathbb{S}_∞ as \tilde{C} is the image of a geodesic by $\pi_1(X)$. This show the first point of the lemma.

The second assertion is clear if $p(h)$ is compact or if it meets a transverse curve C at least twice since then every connected component of \tilde{C} separates the end of h .

Otherwise, $p(h)$ spirals toward two compact leaf f_1 and f_2 . If $f_1 = f_2$ and the two endpoints of h are the same then there will be a singularity that would not be a saddle. On the other

hand, $f_1 \neq f_2$ is impossible since \mathcal{F} satisfies the condition (*). This contradiction proves the second part of the lemma. \square

We can now associate to every leaf h a geodesic $\gamma(h)$ by joining the endpoints. Then $\gamma(\tilde{\mathcal{F}}) = \cup_{h \in \mathcal{F}} \gamma(h)$ is a disjoint union of geodesic invariant by $\pi_1(X)$. We have to show that this set is closed to conclude that we have a lamination.

Lemma 3.3.4. $\gamma(\tilde{\mathcal{F}})$ is closed in $\bar{\mathbb{D}}$

Proof. Let $g_n = \gamma(h_n)$ be a sequence of geodesics in $\gamma(\tilde{\mathcal{F}})$ converging to a geodesic g . We want to show $g \in \gamma(\tilde{\mathcal{F}})$. We can suppose that all the g_n are distinct of g and are all on the same side.

Let L be the limit set in $\bar{\mathbb{D}}$. For each leaf m in $\gamma(\tilde{\mathcal{F}})$, we call \bar{m} the closure of m obtained by adding the two endpoints in \mathbb{S}_∞ . Then L meet at least one connected component of $\bar{\mathbb{D}} \setminus \bar{m}$. As the endpoints of all leaves of $\gamma(\tilde{\mathcal{F}})$ is a dense subset of \mathbb{S}_∞ , L contains a leaf h . Taking a half-leaf h_0 , we want to show that the end point is the same as one of g .

A first case is if there is a simple closed curve C transverse to \mathcal{F} which meets $p(h_0)$ infinitely often. If h_0 does not converge to the corresponding point at infinity then there would be a connected component of $p^{-1}(C)$ that contains the point of infinity of h_0 but does not contain the point of infinity of h_n which is impossible for large n .

A second case is if $p(h_0)$ spirals towards a compact leaf, then a closed leaf nearby to $p(h_0)$ also spirals towards the same compact leaf. Then h_0 converges to one of the points at infinity of g which is a point at infinity of h_n for n large.

Finally, if $p(h)$ is compact then $p(h_n)$ spirals toward it for large n , therefore $\gamma(h)$ and g have one point in common at infinity. If the second was different, by applying a transformation leaving $\gamma(h)$ invariant (but no g), we would separate h from the leaves h_n , a contradiction. \square

Now we want to exhibit an inverse construction which takes a lamination λ and gives a foliation μ . To do this we still consider $\tilde{\lambda}$ in the universal cover. We will suppose that every complementary region is an ideal polygon.

We can build a skeleton that it composed of edges between vertices and a chosen point in the center. After building the skeleton for every polygon we fill the complementary region. These regions are quadrangles delimited by edges previously built, two of its vertices are on the disk, the remaining ones lie in the center of two adjacent triangles.

This map is often called the "collapsing" map and its inverse the "tightening" map.

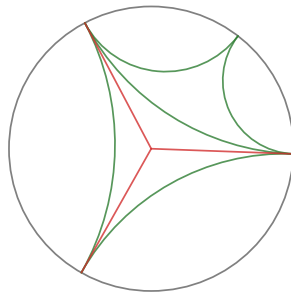


Figure 3.11: In red the skeleton of the ideal triangle in green.

The transverse measure to this foliation is uniform on every region between four edges of the skeleton.

This bridge between measured laminations and measured foliation allow us to compute intersection of a measured lamination with a measured foliation.

3.3.2 Pairs of foliations and quadratic differentials

For a quadratic differential q , one can define two measured foliations, the horizontal $h(q)$ and the vertical $v(q)$ foliations corresponding in local coordinates to $\text{Re}(z)$ and $\text{Im}(z)$ with the transverse measures

$$h_\nu = \int \text{Im}(\sqrt{q(z)} dz)$$

and

$$h_\mu = \int \text{Re}(\sqrt{q(z)} dz).$$

This gives a map to the space of pairs of foliations but it is not surjective. More precisely, define

$$\Delta = \{(\alpha, \beta) : i(\alpha, \gamma) = i(\beta, \gamma) = 0, \text{ for some } \gamma \in \mathcal{MF}\}.$$

Δ contains the diagonal (α, α) and is kind of "fat" diagonal.

Lemma 3.3.5. *For any $q \in \mathcal{QD}$, $(h(q), v(q)) \notin \Delta$*

Proof. Let's suppose that there is γ such as $i(h(q), \gamma) = i(v(q), \gamma) = 0$ for some $\gamma \in \mathcal{ML}$. Let's take a sequence of simple closed weighted curves γ_i converging to γ . By continuity of the intersection number we have that $i(h(q), \gamma_i) \rightarrow 0$. So there is a sequence of saddle connections in the same homotopy class as γ_i whose x -component is very small. The same argument applies in the vertical direction and this leads to a contradiction. \square

Theorem 3.3.6. *The map $q \mapsto (h(q), v(q))$ defines a homeomorphism $\mathcal{QD} \rightarrow \mathcal{MF} \times \mathcal{MF} \setminus \Delta$*

Proof. We can describe the inverse map. If we take two measured foliation h and v we can tighten them into laminations (which we also call h and v) as in the previous section. These two laminations do not share any leaf, otherwise we would have $(h, v) \in \Delta$ by considering the leaf as a geodesic. The complementary region of $h \cup v$ are compact polygons, i.e. they do not have a vertex on the boundary of the disk. Now we can fill the polygon to obtain the quadratic differential with a singularity in each complementary region of order the number of side of the polygon. \square

More details on the proof can be found in the book of Casson and Bleiber [CB88] Lemma 6.2 and or in the article of Hubbard and Masur [HM79]. The injectivity is discussed with more details in [Gar91] section 3.

We will note the inverse map $q(\cdot, \cdot)$.

3.3.3 The horocyclic foliation

Finally, the most important and crucial element is a map that, given a hyperbolic structure X and a lamination λ creates a measured foliation which is transverse to λ . We will note this map F_λ .

As indicated before, we impose for λ to be a maximal lamination i.e. the complementary region are ideal triangles. Indeed, we will construct the horocyclic measured foliation, first, for an ideal triangle:

Definition 3.3.5. Given an ideal triangle T on the hyperbolic disc \mathbb{D} , let $\{p_i\}_{i=1,2,3}$ be its three vertex. There is one and only one way to draw three circles $\{D_i\}_{i=1,2,3}$ passing through p_i , tangent to \mathbb{D} and tangent two by two.

For each D_i , we consider the family of disk obtained by homothety of center p_i and of factors $r \in [0, 1]$. The intersection of T with these three family is called the *horocyclic foliation* of T .

Moreover, this foliation come naturally with a transverse measure. For each arc γ crossing a set of leaves, its measure is the hyperbolic length of the segment made of the intersection of this set of leaves and the boundary of T . It is thus a measured foliation.

Remark. This foliation does not cover the totality of T , there is a triangular piece left at the end. We can find a full foliation by pinching the center which was not covered.

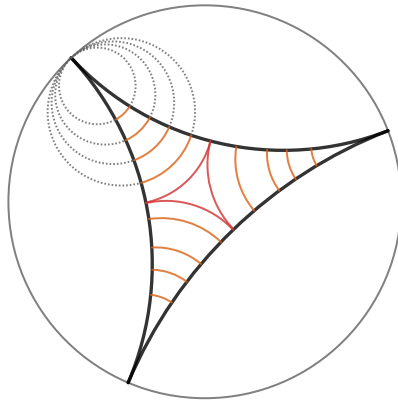


Figure 3.12: The horocyclic foliation: the three arcs of circle in red are tangents two-by-two.

To sum it up, if we have a hyperbolic metric X and a measured maximal lamination λ we construct the horocyclic foliation as follows:

- We take the problem into the universal cover,
- The surface is decomposed into an union of ideal triangles,
- We draw the horocyclic foliation for each triangle,
- Going back to the surface we have our measured horocyclic foliation for (X, λ) .

Remark. We can verify that $(\lambda, F_\lambda(X)) \notin \Delta$.

Remark. Another important remark is that this construction depend only on the topology of λ and not on its transverse measure.

We want to show that this construction is reversible. In the following μ will be a measured foliation and λ a measured lamination such that $(\mu, \lambda) \notin \Delta$.

We aim to construct $X \in \mathcal{T}_g$ whose horocyclic foliation is $F_\lambda(X) = \mu$.

The construction can be decomposed in the following steps.

- We want to recover X as a union of ideal triangles.
- Given two ideal triangles which made the surface we have to find how to position them one in relation to the other.
- To find it we will compute a Möbius transformation and an information called the *shear*.

- To do that we take two special points on edges of each triangle.
- We record how we pass from one to the other by following a path of leaf of μ and then a portion of one of the boundary of one ideal triangle.

This construction yields a system of coordinate called *shear coordinates* that has its own interest.

Let's note $\tilde{\mu}, \tilde{\lambda}$ the pre-images of μ and λ in \mathbb{D} .

We consider two triangles T_1 and T_2 that are complementary regions of $\tilde{\lambda}$. There are two edges which are border of the same connected component of $\mathbb{D}/T_1, T_2$. Let's call them *facing edges*.

For each triangle, considering the vertex which is not on the extremity of the facing edges, there is one arc coming from this vertex an orthogonal to the facing edge. Let's call p_1, p_2 the intersection of these arc and the facing edges.

Next, we take a segment A made of a leaf of $\tilde{\mu}$ that starts from p_1 and intersect the other facing edge in q_1 . The hyperbolic distance between p_2 and q_1 is called the shear.

We call v_1 and v_2 the two vectors (and their footprint) tangent to the facing edges at p_1 and q_1 .

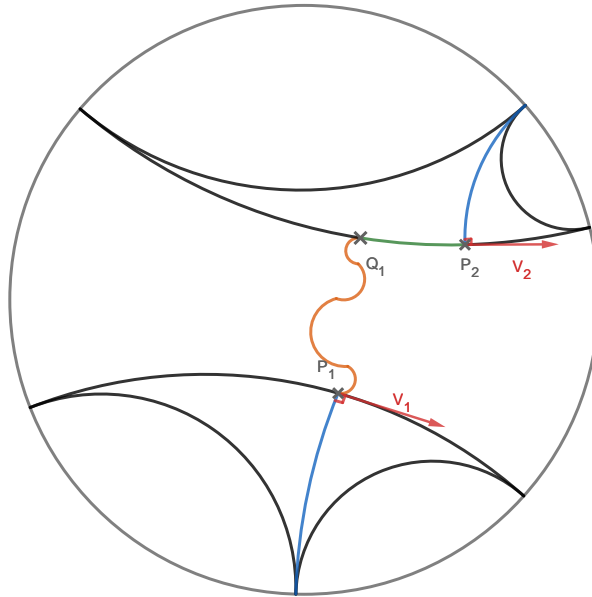


Figure 3.13: After following horocyclic leaves, we can compute the shear coordinate as the distance in green.

We now want to compute the Möbius transformation that sends v_1 to v_2 .

Let I be the set of all triangles in \mathbb{D} that A meets. For each $i \in I$ we can define v_i^+ and v_i^- the vectors tangent to the edge of the corresponding triangle at the intersection of the edges and A . Note that I is a countable totally ordered but non well ordered set. So if we take S_i the Möbius transformation which takes v_i^- to v_i^+ , we have to give a meaning to the expression

$$\prod_{i \in I} S_i$$

Definition 3.3.6. Given a countable totally ordered set of indices I and elements S_i in a Banach algebra, we say that $\prod_i S_i$ is *well defined* and equal to S if for any increasing chain

$$I_0 \subset I_1 \subset \dots \subset I$$

with $\cup_k I_k = I$ we have $\lim_{k \rightarrow \infty} \prod_{i \in I_k} S_i = S$.

Lemma 3.3.7. *For element s_i in a Banach algebra indexed by a countable totally ordered set, if $\sum \|s_i\| < \infty$, then $\prod(1 + s_i)$ is well-defined.*

Proof. For $1 \leq m \leq n$, we have

$$\left\| \prod_{i=1}^n (1 + s_i) - \prod_{i=1, i \neq m}^n (1 + s_i) \right\| \leq \|s_m\| \left\| \prod_{i=1, i \neq m}^n (1 + s_i) \right\| \leq \|s_m\| \prod_{i=1}^n (1 + \|s_i\|)$$

On the other hand, the assumption $\sum \|s_i\| < \infty$ we have that $\prod(1 + \|s_i\|) \leq C < \infty$, so removing or adding $1 + s_m$ produce a change bound by $\|s_m\|C$. \square

Now we want to apply this lemma to $S_i - Id$, with Id being the identity matrix.

Lemma 3.3.8. *For the previous S_i , if we note $s_i = S_i - Id$ we have $\sum \|s_i\| < \infty$.*

Proof. Each S_i is conjugate to a horocycle transformation of time one. The conjugacy is made by a geodesic flow along the edges of the triangle. We can compute

$$\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & e^{-t} \\ 0 & 0 \end{pmatrix}$$

So the norm of s_i is inversely correlated to the amount of geodesic flow used in the conjugation. Now we can partition the indices set I into finitely many subsets (I_k) according to which spike of the lamination the arc of A crosses. Then for a spike the sum of $\|s_i\|$ with $i \in I_k$ is finite, indeed the distance between two neighboring crossing is bounded below by a constant and so the amount of time we follow the geodesic flow increases at least linearly. Finally, the norm of the s_i should decrease geometrically. \square

So we can conclude that there is an unique Möbius transformation S equal to the meaningful expression $\prod_i S_i$.

Now we can conclude the proof. There exist without any hyperbolic structure X given, topological classes for $\tilde{\mu}$ and $\tilde{\lambda}$. We choose one arbitrary ideal triangle T_1 in the lamination. For every other triangle T_2 , the Möbius transformation and the shear are data that can be computed only using the transverse measure of $\tilde{\mu}$. So we can place T_2 , and the other triangle. The closure of this set gives the lamination $\tilde{\lambda}$. $\tilde{\lambda}$ will be preserved by a Fuchsian group Γ and we will have $X = \mathbb{D}/\Gamma$.

To sum it up, given $(X, \lambda) \in \mathcal{T}_g \times \mathcal{ML}$, then we get a pair of foliations by considering $(F_\lambda(X), \lambda)$, which itself can be seen as a quadratic differential.

3.3.4 Remark on the non-smoothness of the conjugacy

The fact that the horocyclic foliation construction depend only on the support of λ and not its measure, creates discontinuities. A visual example is given below.

Suppose that we have a maximal measured lamination λ , such that its pre-image in the universal cover contains a quadrilateral and its diagonal. If the weight, given by the measure, on the leaf is small compare to the total mass, then as a measured lamination it is close to another one with the second diagonal as a leaf with the same weight.

However, things change when we apply the conjugacy, at the moment where we compute the horocyclic foliation. As this map only sees the support of λ , the horocyclic foliations could be quite different as in the example given by figure 3.3.4.

As a consequence the conjugacy F is not a continuous map.

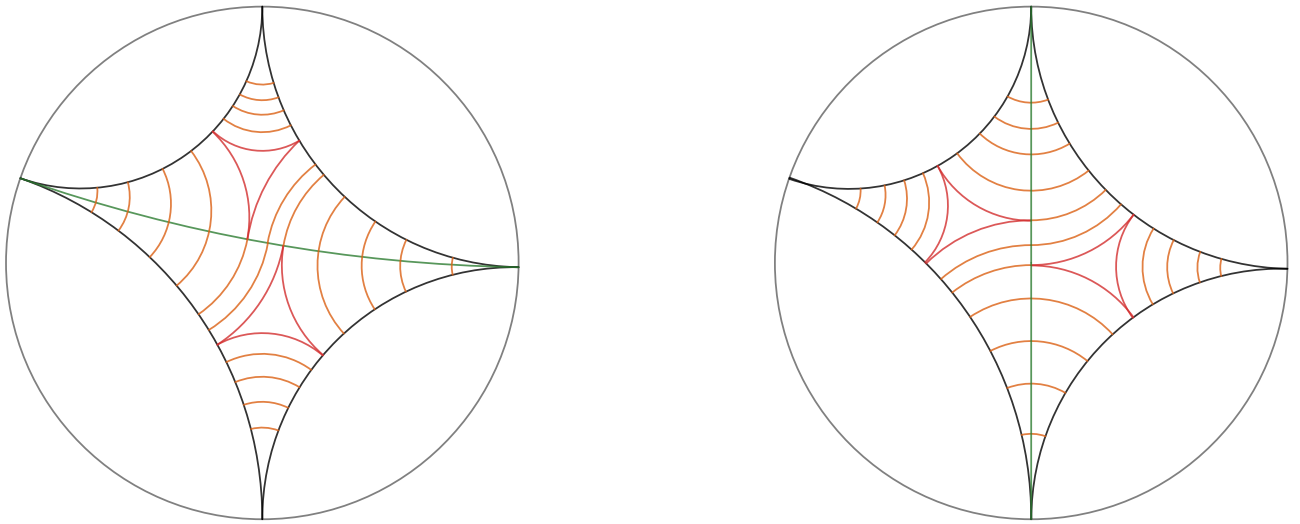


Figure 3.14: When the green leaf switches between the two diagonals of the quadrilateral, the horocyclic foliation changes.

Chapter 4

The horocyclic flow and Earthquake flow

In this chapter, once the scene has been set, we will introduce the actors. Indeed, what interests us are the dynamical systems on the moduli spaces described in the previous chapter. For the flat geometry, there is an action of $SL(2, \mathbb{R})$ which has been extensively studied in the last decades. However, in the hyperbolic part, there is a natural flow call Earthquake flow which is not yet fully understood. We will define it in the second section of this chapter. Then, in the third section, we will prove that its rate of mixing is less than polynomial with a degree given by the topology.

4.1 The action of $SL(2, \mathbb{R})$ on flat surfaces

4.1.1 Definition

We will now study the action of the group $SL(2, \mathbb{R})$ on the moduli space Ω'_g of flat surfaces. Given a flat surface in Ω'_g one can see it as a Abelian differential ω on a Riemann surface X . Taking charts on which $\omega = dz$, one can postcompose them by matrices of $SL(2, \mathbb{R})$ to get a new surface. This action preserves the stratification of Ω'_g by the orders of multiplicity of the zeros. It also preserves the area of the surface and so give an action on Ω_g . This action can be easily visualized by drawing the corresponding polygons on the plane and making $SL(2, \mathbb{R})$ act on it linearly.

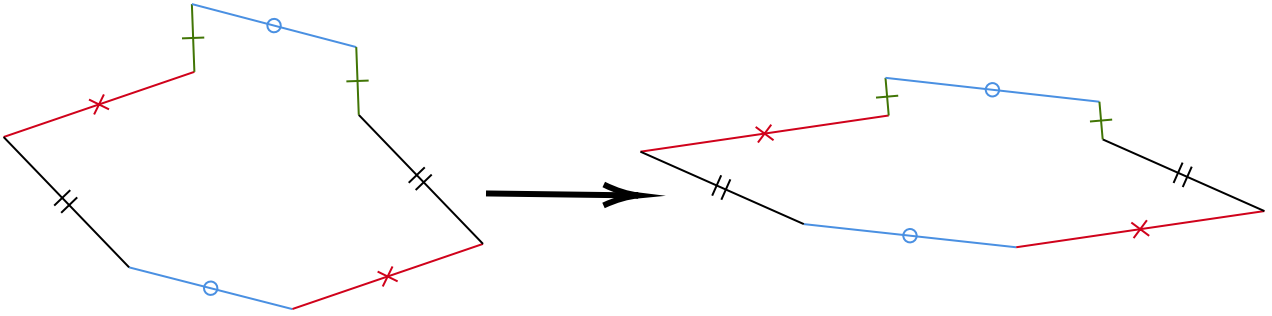


Figure 4.1: The action of the geodesic flow on a translation surface.

Remark. Any matrix $A \in SL(2, \mathbb{R})$ can be decomposed as follows

$$A = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

The flow generated by the family of matrices of the form

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

is called the geodesic flow and is denoted by g_t .

The flow generated by the family of matrices of the form

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

is called the horocyclic flow and is denoted by h_s .

The flow generated by the family of matrices of the form

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

is called the rotational flow and is denoted by r_θ .

In general, a connected component of a stratum $\mathcal{H} \subset \Omega_g$ carries many $SL(2, \mathbb{R})$ -invariant measures. As it was famously shown by Eskin and Mirzakhani [EM18], such measures are supported on affine orbifolds, here denoted by \mathcal{M} . Roughly speaking, this mean that in any point $p \in \mathcal{M}$, there is a neighborhood U such that $\mathcal{M} \cap U$ is mapped by local period coordinates on a subspace defined by real linear equations.

Moreover, the action of the geodesic and the horocyclic flow are mixing with respect to the Masur-Veech measure and the rate of mixing is known.

4.1.2 Rate of mixing

By coding the geodesic flow with an abstract system with prescribed properties, Avila, Gouëzel and Yoccoz showed the following theorem.

Theorem 4.1.1. *The geodesic flow restricted to any connected component of any stratum of the Ω_g is exponentially mixing for Hölder observables [AGY06] with respect to the Masur Veech measure.*

This result has been extended for any $SL(2, \mathbb{R})$ -invariant probability measure by Avila and Gouëzel [avila2013small].

The notion of Hölder observables is given for a norm defined in their article [AR12]. This theorem has, in particular, two consequences.

Remark. With the previous theorem, and a link given by Ratner's work [Rat87], one can deduce that the representation of $SL(2, \mathbb{R})$ on $L_0^2(\mathcal{H}, \mu)$ has a spectral gap. This is explained in Appendix B of [AGY06].

A second remark is that we can deduce the rate of mixing for the horocyclic flow.

Theorem 4.1.2. *The horocyclic flow restricted to any connected component of any stratum of Ω_g is polynomially mixing for K -smooth functions with respect to any probability $SL(2, \mathbb{R})$ -invariant measure.*

In the previous statement, the notion of K -smooth function has the following meaning:

Definition 4.1.1. A function $f \in L^2(\mathcal{H}, \mu)$ is called K -smooth of degree one if

$$\pi_{\mathcal{H}}(\theta)f := \lim_{\theta \rightarrow 0} \frac{1}{\theta} (r_{\theta}^* f - f)$$

exists with respect to the $L^2(\mathcal{H}, \mu)$ -norm, where r_{θ}^* is the action of rotation matrices.

Similarly, a function is K -smooth of degree d , for $d \in \mathbb{N}^*$ if

$$\pi_{\mathcal{H}}^d(\theta)f := \lim_{\theta \rightarrow 0} \frac{1}{\theta^d} (r_{\theta}^* f - f).$$

Without any surprise, a function is called K -smooth if it is K smooth of degree d for any $d \in \mathbb{N}^*$.

Remark. K -smooth functions are dense in $L^2(\mathcal{H}, \mu)$.

Ratner proved the following theorem about unitary representations of $SL(2, \mathbb{R})$.

Theorem 4.1.3. *Let T be a non-trivial irreducible unitary representation of $SL(2, \mathbb{R})$ in a Hilbert space H with spectral gap $\lambda > 0$. Let $v, w \in L^2(H)$ be K -smooth vectors of degree $p > 0$, $\langle w, 1 \rangle = 0$ and $C(t) = \langle v, w \circ h_t \rangle$. Then for all $|t| \geq 1$ and some $E > 0$*

$$|C(t)| \leq E b_{\lambda}(t)^{\alpha(p)}$$

where $b_{\lambda}(t)$ is

1. $\min\left(\frac{\ln(t)}{t}, \frac{1}{t(1+\sqrt{1+4\lambda})}\right)$ if $\lambda \geq 1/4$
2. $\min\left(\frac{\ln(t)}{t^{1-\sqrt{1+4\lambda}}}, \frac{1}{t^{1-\sqrt{1+4\lambda}}(1+\sqrt{1+4\lambda})}\right)$ if $\lambda \in]0, 1/4[$

and $\alpha(p)$ is

1. 1 if $p \geq 3$
2. $\frac{2p}{2p+1}$ if $2 \leq p < 3$
3. $\frac{2p}{2p+3}$ if $1 \leq p < 2$
4. $\frac{p}{p+3}$ if $0 < p < 1$

As previously said, in our case λ is strictly positive and we can deduce the rate of mixing for the horocyclic flow.

4.2 The earthquake Flow

4.2.1 Definition and existence

We will define one of the main objects of this thesis, the earthquake flow.

Definition 4.2.1. The earthquake flow $\hat{E}^t(\cdot, \cdot) : P^1\mathcal{T}_{g,n} \rightarrow P^1\mathcal{T}_{g,n}$ is a flow defined in the following way:

- For $(X, \gamma) \in \mathcal{T}_{g,n} \times \Gamma$, $\hat{E}^t(X, \gamma)$ is a twist around γ of length t .
- For $X \in \mathcal{T}_{g,n}$, $\sum_{i=1}^n \gamma_i \in \mathcal{ML}(\mathbb{Z})$ and $(a_i) \in \mathbb{R}^n$, then

$$\hat{E}^t\left(X, \sum a_i \gamma_i\right) = \hat{E}^{a_1 t}(\cdot, \gamma_1) \circ \hat{E}^{a_2 t}(\cdot, \gamma_2) \circ \cdots \circ \hat{E}^{a_n t}(X, \gamma_n)$$

(note that the order of the composition is irrelevant since the different terms are commuting).

- For $(X, \lambda) \in \mathcal{T}_{g,n} \times \mathcal{ML}$, the flow is obtained by continuous extension: since the space of weighted multicurves is dense in the space of measured laminations, the earthquake along $\lambda \in \mathcal{ML}$ is the limit of earthquake along weighted multicurves converging to λ .

Because $\hat{E}^t(\cdot, \cdot)$ commutes with the natural $Mod(S)$ -action, we have a flow $E^t([\cdot, \cdot])$ on $P^1\mathcal{M}_{g,n}$ also called earthquake flow. For more details on $E^t([\cdot, \cdot])$, see [Ker83].

Remark. If the lamination is just a simple closed curve γ then $\hat{E}^{t\gamma(X)}(X, \gamma)$ is just a Dehn twist around γ . Moreover, if we take a decomposition in a pair of pants that contains γ , it is just a translation in the coordinate of the twist of γ .

I would like to take a bit of time to prove the extension from the second to third bullet point, from multicurves to measure lamination. To show that it is well-defined, we need to show that if we take two sequence (α_n) and (α'_n) of multicurves which converge to the same lamination, then the sequences of earthquake map along this multicurves converge to the same map. Let's sketch this result below following [Ker83].

To begin with, let's give two definition of topologies, one for \mathcal{ML} and one for $\mathcal{T}_{g,n}$ which are equivalent to the previously given one but more practical.

Definition 4.2.2. Given a geodesic arc A and $\lambda \in \mathcal{ML}$ we will write $v_\lambda(A)$ for the vector

$$(i(A, \lambda), \theta(A, \lambda))$$

with $i(\cdot, \cdot)$ the total intersection mass and $\theta(\cdot, \cdot)$ the average angle of intersection.

We say that $\lambda, \lambda' \in \mathcal{ML}$ are ϵ -close along δ -subarc of A if A is a disjoint union of subarc A_i whose lengths are less than δ and such that

$$\sum_i |v_\lambda(A_i) - v_{\lambda'}(A_i)| < \epsilon.$$

Definition 4.2.3. Given $\{\phi_i\}$ a set of generators of the fundamental group $\pi^1(S)$ of the surface, and choosing continuously a representation of $\pi^1 S$ for each $X \in \mathcal{T}_{g,n}$, we say that $Y, Y' \in \mathcal{T}_{g,n}$ are ϵ -close if

$$|\phi_i(Y) - \phi_i(Y')| < \epsilon$$

for all i , where the $\phi_i(Y)$ are the elements of $PSL(2, \mathbb{R})$ representing the chosen generators of Y .

With these two definitions, Kerckhoff has shown that:

Theorem 4.2.1. *Let $X \in \mathcal{T}_{g,n}$, $\lambda \in \mathcal{ML}$, $T > 0$ and $\delta > 0$ there is a neighborhood U of λ in \mathcal{ML} such that $\forall \gamma, \gamma' \in (\Gamma \times \mathbb{R}^+) \cap U$ and for all $t \leq T$, $\hat{E}^t(X, \gamma)$ and $\hat{E}^t(X, \gamma')$ are in the same δ -neighborhood of $\mathcal{T}_{g,n}$.*

We give below a few hints on the proof of this theorem.

First we will work on universal covering of the surface, the half-plane \mathbb{H} . If $v \in T^1\mathbb{H}$ we will call $\gamma(v)$ the geodesic passing by the base-point of v and with v as tangent vector at this point. On $T^1\mathbb{H}$ we will use the product distance with the hyperbolic metric and the usual metric on S^1 .

We will use a flow that reproduces the earthquake flow on a hyperbolic surface on its universal cover.

If $v, w \in T^1\mathbb{H}$, we will call $\mathcal{E}_{\gamma(v)}^t(w)$ the element of $T^1\mathbb{H}$ that we get by translating w by a distance t parallel to $\gamma(v)$. This map will also be called earthquake map.

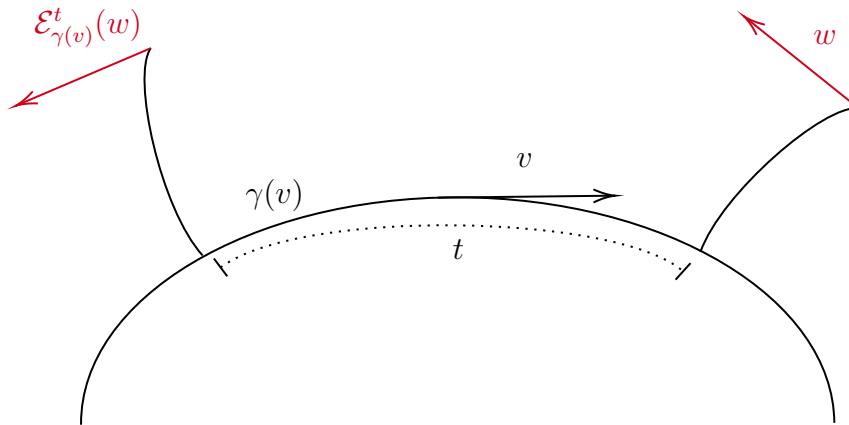


Figure 4.2: Definition of \mathcal{E}

We have two useful lemmas to control the distortion of the earthquake map. We will not prove them, but their proofs are in [Ker83].

Lemma 4.2.2. *Let l and l' be two disjoint geodesics and $x \in l$, $y \in l'$ two points at most ϵ apart. Then if v and v' are the two vectors tangents to l and l' respectively at points x and y , we have $d(v, v') < C\epsilon$ for a universal constant C .*

Lemma 4.2.3. *Let v and v' be two vectors such that $d(v, v') < \epsilon$, let denote $\gamma = \gamma(v)$ and $\gamma' = \gamma(v')$. Let $w \in T^1\mathbb{H}$, then*

1. $d(\mathcal{E}_{\gamma}^t(w), \mathcal{E}_{\gamma'}^t(w)) \leq Kt\epsilon$
2. $d(\mathcal{E}_{\gamma}^t(w), w) \leq Kt$

for all $t \leq T$ and for a constant K depending on T and on the distance between the base-point of v and w .

With these two lemma, we can describe what happens if we change a discrete lamination by a simple closed curve that average it.

Lemma 4.2.4. *Let $x, y \in \mathbb{H}$, $v \in T^1\mathbb{H}$ based at y and \bar{A} the geodesic from x to y . Suppose γ is a discrete lamination with equal measure on each leaf whose intersection with \bar{A} is included*

in a subarc A . Let l be a single geodesic intersecting A with angle equal to the average angle of the intersections of γ and A and with mass equal to $\mu = i(A, \gamma)$.

If A has length less than δ then for every $T \in \mathbb{R}^+$ the distance between $\mathcal{E}_\gamma^t v$ and $\mathcal{E}_l^t v$ is less than $Kt\mu\delta$, for all $t \leq T$ and K is a constant depending only on $T\mu$ and $d(x, y)$.

Proof. Let $\gamma_1, \gamma_2, \dots, \gamma_n$ be the leaves of γ . The point x is connected to $\mathcal{E}_\gamma^t y$ by a staircase path going along component of $\bar{A} \setminus \gamma$ and subarc of γ_i . Let denote the successive components A_0, A_1, \dots, A_n and δ_i the length of A_i . So the staircase path is moving by δ_0 along A_0 then by μ/n along l_1 , and so on.

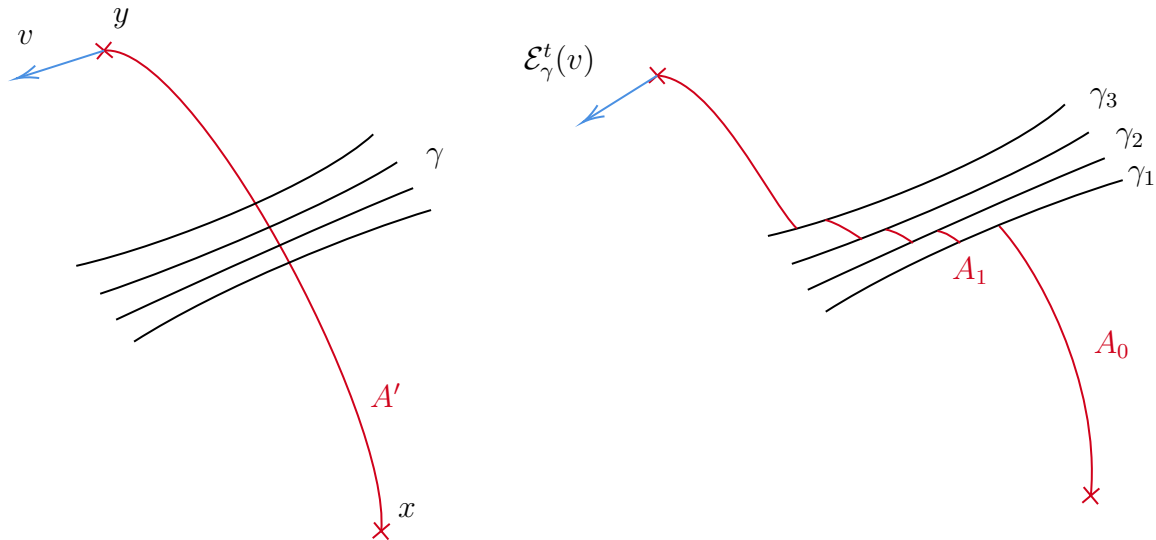


Figure 4.3: The situation in the proof of Lemma 4.2.4.

We now alter the path by replacing the shearing along γ_n distance μ/n by a shearing along γ_{n-1} distance $2\mu/n$. The change is less than $Kt\frac{\mu}{n}C\delta_{n-1}$ by lemma 4.2.2 and lemma 4.2.3. Then we change the shearing by a shearing along γ_{n-2} of distance $3\mu/n$ and we continue until we shear a distance μ along γ_1 . The total change is less than $KtC \sum_{i=1}^{n-1} \frac{i\mu}{n} \delta_{n-i}$ which is less than $KCt\mu\delta$. We now pass from γ_1 to l with lemma 4.2.2 and 4.2.3, and we obtain the new lemma. \square

We need a final lemma to conclude of the well-founded definition of the earthquake flow. In this case we control the difference after shearing by two simple geodesic going through the same point of the arc.

Lemma 4.2.5. *Let x, y, v , and \bar{A} be as above, if l and l' are geodesics of \mathbb{H} with measure μ and μ' such that $l \cup \bar{A} = l' \cup \bar{A} = p \notin \{x, y\}$ and the difference between the vectors tangent of l and l' at p of length μ and μ' is less than ϵ .*

Then for any T , $d(\mathcal{E}_l^t(v), \mathcal{E}_{l'}^t(v)) < Kt\epsilon$, for $t \leq T$ and K a constant which depends only on $d(x, y)$ and $T\mu$

Finally, this give the theorem which control the distance between two earthquake paths.

Proposition 4.2.6. *Let $\nu \in \mathcal{ML}$ be a lamination and let x, y be in \mathbb{H} , A be the geodesic from x to y and $v \in T^1\mathbb{H}$ be based at y , and x and y do not lie on the atomic part of ν . Then for any ϵ, T , there is a neighborhood U of ν in \mathcal{ML} such that for all $\gamma, \bar{\gamma}$ weighted multicurve in U , $d(E_{t\gamma}v, E_{t\bar{\gamma}}v) < Kt\epsilon$, for all $t \leq T$, K a constant depending only on $d(x, y)$ and $Ti(\nu, A)$*

Corollary. *The earthquake flow is well-defined along any lamination $\nu \in \mathcal{ML}$ and for all time t .*

Remark. The earthquake flow is an isometry outside the support of the lamination and is continuous outside the atomic part, i.e. the simple closed geodesics of the lamination.

Remark. Thurston showed that given two points in the Teichmüller space, there is a lamination λ such that the earthquake flow from one point with respect to λ reaches the other point (see also [Ker83]).

4.2.2 Control of lengths along earthquake orbits

We can control the length function along the earthquake in few ways. We will give three results in this direction.

A first lemma indicates the behavior of length functions along an earthquake orbit.

Lemma 4.2.7 ([Ker83]). *Let $\gamma \in \Gamma$, $\lambda \in \mathcal{ML}$ and $X \in \mathcal{T}_{g,n}$ then the function*

$$l_\gamma(\hat{E}^t(X, \lambda))$$

is convex. More precisely, it is constant if $i(\lambda, \gamma) = 0$ and strictly convex otherwise.

Then one can bound the variation of the length of a multicurve during the earthquake flow in terms of the intersection number between the multicurve and the lamination directing the earthquake.

Lemma 4.2.8 (cf. [Ker83] Corollary 3.4). *Let $(X, \lambda) \in P^1\mathcal{T}_{g,n}$ and $\gamma \in \Gamma$, the variation of the length of the curve is bounded by:*

$$\left| \frac{d}{dt} l_\gamma(\hat{E}^t(X, \lambda)) \right| < i(\lambda, \gamma).$$

Finally, a last lemma by Y. Minsky and B. Weiss [MW02], frames the length of a multicurve during the earthquake between two parallel lines. To do that, we need some notations.

Given ρ a positive real number, $\lambda \in \mathcal{ML}$, $X \in \mathcal{T}_{g,n}$ and $\gamma \in \Gamma$, let

$$J_\gamma^{\lambda, X}(\rho) := \{t \in \mathbb{R}, l_\gamma(\hat{E}_t(X, \lambda)) \leq \rho\},$$

$$\epsilon_\gamma^{\lambda, X} = \min_{t \in \mathbb{R}} l_\gamma(\hat{E}_t(X, \lambda))$$

and t_γ a time such that $l_\gamma(\hat{E}_{t_\gamma}(X, \lambda)) = \epsilon_\gamma^{\lambda, X}$. If $i(\lambda, \gamma) = 0$ then any time will do otherwise, if $i(\lambda, \gamma) > 0$ then by the convexity of length function along the earthquake orbit, this time is unique.

Lemma 4.2.9 (cf. [MW02], Lemma 5.2). *There are constants $\rho \leq K_g^{sys}$ and $C_{lem4.2.9}$, depending only on S , such that for any $(X, \lambda) \in P^1\mathcal{T}_{g,n}$, $\gamma \in \Gamma$ and all $t \in J_\gamma^{\lambda, X}(\rho)$,*

$$i(\lambda, \gamma)|t - t_\gamma| - C_{lem4.2.9}\epsilon_\gamma^{\lambda, X} \leq l_\gamma(\hat{E}^t(X, \lambda)) \leq i(\lambda, \gamma)|t - t_\gamma| + \epsilon_\gamma^{\lambda, X}$$

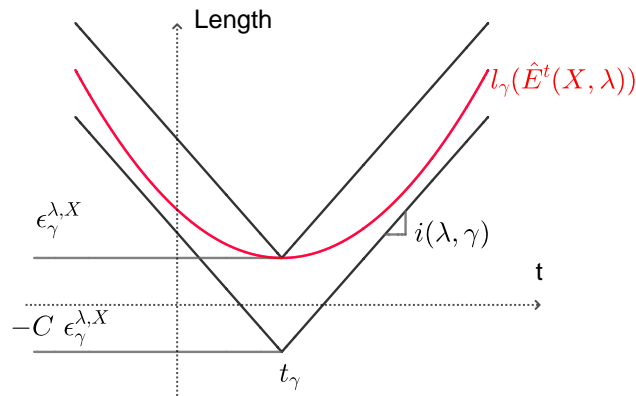


Figure 4.4: The length of a curve is controlled alongside the earthquake flow during a small time near its minimum.

4.2.3 Mirzakhani conjugacy

One key feature of the earthquake flow is its link to the horocyclic flow.

The aim of this part is to complete of Theorem 3.3.1 by adding that the conjugacy transports the horocyclic flow to the earthquake flow.

Theorem 4.2.10. *There is a measurable conjugacy F between the earthquake flow $(\lambda, X) \mapsto (\lambda, E_{t\lambda}(X))$ on $P^1\mathcal{M}_g$ and the horocyclic flow on Ω_g .*

$$\begin{array}{ccc}
 P^1\mathcal{M}_g & \xrightarrow{E_t} & P^1\mathcal{M}_g \\
 F \downarrow & & \downarrow F \\
 \Omega_g & \xrightarrow{u_t} & \Omega_g
 \end{array}$$

Lemma 4.2.11. *Denote by $Shear_X(T_1, T_2)$ the shear for two triangles joined by an arc A of the horocyclic foliation on the hyperbolic surface X . Then*

$$Shear_{E_{t\lambda}(X)}(T_1, T_2) = Shear_X(T_1, T_2) + t\lambda(A)$$

where $\lambda(A)$ denote the transverse measure of A and t is sufficiently small.

Proof. T_1 and T_2 are separated by infinitely many leaves of $\tilde{\lambda}$. We want to understand how T_2 moved relatively to T_1 by the action of the earthquake $E_{t\lambda}$. We can approximate the measured lamination between the two triangles by a discrete one. If we follow the earthquake path along a leaf γ of $\tilde{\lambda}$ between T_1 and T_2 by an amount t , then this changes the shear coordinate between T_1 and T_2 by t . Indeed, each arc of the horocyclic foliation which have endpoint on γ see its endpoints translated by t . Similarly, if the earthquake moves finitely many leaves of λ with measure a_i , the shear changes by precisely $t \sum a_i$. So taking a limit, we have the lemma for an arbitrary measured foliation. \square

We will now show that Mirzakhani's map conjugates the earthquake flow to the horocyclic flow. Let \mathcal{ML}_0 denote the set of maximal lamination and \mathcal{QD}_0 the locus of quadratic differentials with simple zeros and no horizontal saddle connection. We begin with an easy lemma.

Lemma 4.2.12. *An arc joining two singularities of a quadratic differential has in its isotopy class a path made of horizontal and vertical arcs between two singularities of the quadratic differential.*

Corollary. *Suppose q_t is a path in the space of quadratic differential such that for every t_0 and every path γ on q_{t_0} joining two singularities, the period $x_t + iy_t$ of γ satisfies*

$$\frac{d}{dt}\Big|_{t=t_0} x_t = y_{t_0}, \quad \frac{d}{dt}\Big|_{t=t_0} y_t = 0$$

Then q_t is an orbit of the horocyclic flow.

Proof. We have

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ty \\ y \end{pmatrix}$$

And we can integrate the two equations to find the linear action. □

We can now prove Theorem 4.2.10.

Proof. We want to show that

$$q(\lambda, F_\lambda(E_{t\lambda}(X)))$$

is a horocyclic flow path. Here λ is firstly seen as a foliation, and then as a lamination.

We pick an arbitrary time t_0 and look at the derivative of the path, called γ . The coordinate y_t is constant equal to $\lambda(\gamma)$ and the derivative of x_t is by lemma 4.2.11 equal to y_t . We conclude using the previous lemma. □

The corollary of this theorem is the following ergodicity result.

Corollary. *The earthquake flow is ergodic with respect to ν .*

Proof. The horocyclic flow is ergodic with respect to the Masur-Veech measure. As this measure is transported to the measure ν , we get our result. □

As the horocyclic flow is polynomially mixing with respect to the Masur-Veech measure, one can hope to find the same property for the earthquake flow. This could be used for getting more precise results on asymptotic counting problems which will be presented in chapter 3.

The first idea to have the rate of mixing for the earthquake flow is to use the known rate of mixing of the horocyclic flow and transport it with Mirzakhani's conjugacy described in Theorem 4.2.10.

Unfortunately a Lipschitz function on $P^1\mathcal{M}_{g,n}$ will, in the general case, not be K -smooth once transported by Mirzakhani conjugacy. This fact come from that two measure laminations can be close under the topology described in subsection 3.1.6 but have different topologies for theirs underlying laminations.

4.2.4 The earthquake flow is not renormalizable

Arana-Herrera and Wright gave a negative answer to the following question to the smooth conjugacy problem between the horocyclic and earthquake flows.

Theorem 4.2.13. *There does not exist an orbifold conjugacy $P^1\mathcal{M}_{g,n} \rightarrow \mathcal{Q}^1\mathcal{M}_g$ between the earthquake flow and the horocyclic flow [AF22].*

This result is got by studying the normalizers of the flow. The set of the normalizers of a flow ϕ^t on the orbifold $P^1\mathcal{M}_{g,n}$ is

$$N(\phi) = \{F \in \text{Aut}(P^1\mathcal{M}_{g,n}) : \exists \epsilon \in \{-1, 1\}, s \in \mathbb{R}, \phi^t \circ F = F \circ \phi^{\epsilon e^{2s}t}\}.$$

For the horocyclic flow, for each $t \in \mathbb{R}$, the geodesic flow at time t belong to the normalizer. On the contrary, for the earthquake flow, Arana-Herrera and Wright got the following result :

Theorem 4.2.14. *The normalizer of the earthquake flow inside the group of orbifold automorphism of $P^1\mathcal{M}_{g,n}$ is the flow itself.*

4.3 The rate of mixing of the earthquake flow

In this section we exclude very fast rates of mixing of the earthquake flow. In fact, we prove that the earthquake flow is at most polynomially mixing for some classes of Lipschitz function defined by different metrics.

Theorem 4.3.1. *Suppose that there are constants d and C so that*

$$\left| \int fg \circ E^t d\nu - \int f d\nu \int g d\nu \right| < C \frac{1}{t^d} \|f\|_{Lip} \|g\|_{Lip} \quad (4.1)$$

for all Lipschitz functions $f, g : P^1\mathcal{M}_g \rightarrow \mathbb{R}$ and all times $t \geq 1$ (where $\|\cdot\|_{Lip}$ will be defined in Subsection 4.3.1 below). Then $d \leq 6g - 5$.

To do that we will need to define and analyze some functions.

4.3.1 The systole function

Definition 4.3.1. On $\mathcal{T}_{g,n}$, the quantity $\inf_{\delta \in \Gamma} l_\delta(X) := l_{sys}(X)$ is always positive. It is called the systolic length of X and any curve γ realizing this minimum is called a systole of X . (In general, X could have multiple systoles.)

Observe that $l_{sys}(X) = l_{sys}(h.X)$ for every $h \in Mod(S)$. In particular, l_{sys} is also well-defined on $\mathcal{M}_{g,n}$ (and we will use the same notation for both functions on $\mathcal{T}_{g,n}$ and $\mathcal{M}_{g,n}$).

Lemma 4.3.2. *The function l_{sys} is bounded above on $\mathcal{T}_{g,n}$ by a constant K_g^{sys} .*

Proof. Any $X \in \mathcal{T}_{g,n}$ has hyperbolic area equal to $-2\pi\chi(S)$. So, if we pick any point $x \in X$, the set $\{y \in X, d(y, x) \leq R\}$ can't be homeomorphic to a disk for a large enough R (otherwise, its area will eventually become bigger than the area of the surface). This gives an upper bound for the systole function which depends only on the topology of the surface. \square

We need also to control the regularity of this function with respect to the different distances described above.

Lemma 4.3.3. *For each $\epsilon > 0$, the function l_{sys} is Lipschitz for the Thurston distance d_{Th} , on $\mathcal{M}_{g,n}^\epsilon = \{l_{sys}(X) \geq \epsilon, X \in \mathcal{M}_{g,n}\}$ (and we denote this Lipschitz constant $C_{sys,\epsilon}^{Lip}$).*

Proof. Take $[Y], [Y'] \in \mathcal{M}_{g,n}^\epsilon$, a systole $\gamma \in \Gamma$ of Y , a systole $\gamma' \in \Gamma$ of Y' . We have

$$e^{-d_{Th}(Y,Y')} l_\delta(Y) \leq l_\delta(Y') \leq e^{d_{Th}(Y,Y')} l_\delta(Y), \forall \delta \in \Gamma.$$

Hence,

$$\left| 1 - \frac{l_\gamma(Y)}{l_\gamma(Y')} \right| \leq \max(1 - e^{-d_{Th}(Y,Y')}, e^{d_{Th}(Y,Y')} - 1).$$

On the other hand,

$$\begin{aligned} |l_{sys}(Y) - l_{sys}(Y')| &= l_{sys}(Y') \left| 1 - \frac{l_{sys}(Y)}{l_{sys}(Y')} \right| \\ &\leq K_g^{sys} \left| 1 - \frac{l_\gamma(Y)}{l_\gamma(Y')} \right| \\ &= K_g^{sys} \left| 1 - \frac{l_\gamma(Y)}{l_\gamma(Y')} + \frac{l_\gamma(Y)}{l_\gamma(Y')} - \frac{l_\gamma(Y)}{l_\gamma(Y')} \right| \\ &\leq K_g^{sys} \left| 1 - \frac{l_\gamma(Y)}{l_\gamma(Y')} \right| + K_g^{sys} \left| \frac{l_\gamma(Y)}{l_\gamma(Y')} - \frac{l_\gamma(Y)}{l_\gamma(Y')} \right| \\ &\leq K_g^{sys} \left| 1 - \frac{l_\gamma(Y)}{l_\gamma(Y')} \right| + \frac{K_g^{sys} l_\gamma(Y)}{l_\gamma(Y')} \left| 1 - \frac{l_\gamma(Y')}{l_\gamma(Y)} \right| \\ &\leq K_g^{sys} \left| 1 - \frac{l_\gamma(Y)}{l_\gamma(Y')} \right| + \frac{(K_g^{sys})^2}{\epsilon} \left| 1 - \frac{l_\gamma(Y')}{l_\gamma(Y)} \right|. \end{aligned}$$

Since $l_{\gamma'}(Y') \leq l_{\gamma}(Y')$ (as γ' is a systole of Y') and

$$l_{\gamma}(Y') \leq e^{d_{Th}(Y,Y')} l_{\gamma}(Y) \leq e^{d_{Th}(Y,Y')} l_{\gamma'}(Y) \leq e^{2d_{Th}(Y,Y')} l_{\gamma'}(Y'),$$

we also have

$$\left| 1 - \frac{l_{\gamma}(Y')}{l_{\gamma'}(Y')} \right| \leq e^{2d_{Th}(Y,Y')} - 1.$$

By combining these estimates with the bound

$$|l_{sys}(Y) - l_{sys}(Y')| \leq 2K_g^{sys},$$

we get

$$\frac{|l_{sys}(Y) - l_{sys}(Y')|}{d_{Th}(Y, Y')} \leq \frac{\min(2K_g^{sys}, K_g^{sys} \max(1 - e^{-d_{Th}(Y, Y')}, e^{d_{Th}(Y, Y')} - 1)) + \frac{(K_g^{sys})^2}{\epsilon} (e^{2d_{Th}(Y, Y')} - 1)}{d_{Th}(Y, Y')}.$$

This completes the proof because the maximum over all $d_{Th}(Y, Y') \in]0, +\infty[$ of the right hand side is finite and it gives an upper bound of the Lipschitz constant of $l_{sys}(\cdot)|_{\mathcal{M}_{g,n}^{\epsilon}}$. \square

Lemma 4.3.4. *For each $\epsilon > 0$, the function l_{sys} is Lipschitz for the Teichmüller distance d_{Teich} , on $\mathcal{M}_{g,n}^{\epsilon} = \{l_{sys}(X) \geq \epsilon, X \in \mathcal{M}_{g,n}\}$ (and we denote this Lipschitz constant $C_{sys,\epsilon}^{Lip}$).*

Proof. We have by Wolpert's inequality ([Liu+10]) that for any $\gamma \in \Gamma$ and any $X, Y \in \mathcal{T}_{g,n}$,

$$\frac{l_{\gamma}(X)}{l_{\gamma}(Y)} \leq e^{2d_{Teich}(X,Y)}.$$

The same proof as the previous lemma can be repeated mutatis mutandis. \square

Lemma 4.3.5. *For each $\epsilon > 0$, the function l_{sys} is Lipschitz for the Weil-Petersson distance d_{WP} , on $\mathcal{M}_{g,n}^{\epsilon} = \{l_{sys}(X) \geq \epsilon, X \in \mathcal{M}_{g,n}\}$ (and we denote this Lipschitz constant $C_{sys,\epsilon}^{Lip}$).*

Proof. As indicated in Wolpert's survey [wolpert2007weil] on the thick part of moduli space, the Weil-Petersson metric and the Teichmüller metric are comparable. As we are interested only on the thick part, this lemma is a consequence of lemma 4.3.4. \square

Finally, we need to control the difference of two intersection numbers with a given curve.

Lemma 4.3.6. *For a simple closed curve γ and two measured laminations λ and λ' we have*

$$|i(\lambda, \gamma) - i(\lambda', \gamma)| \leq C_{int,\gamma}^{Lip} d_{lam}(\lambda, \lambda')$$

Proof. If γ is part of the standard basis, then there is nothing to prove.

Otherwise, one can obtain a basis which contain γ by applying Dehn twists, and transformations of the first and second kind as described in [PH92] and showed in Figures 4.5 and 4.6.

The actions of these transformations and the actions of Dehn twists are piecewise linear in the Dehn-Thurston coordinates and so the intersection remain linear. \square

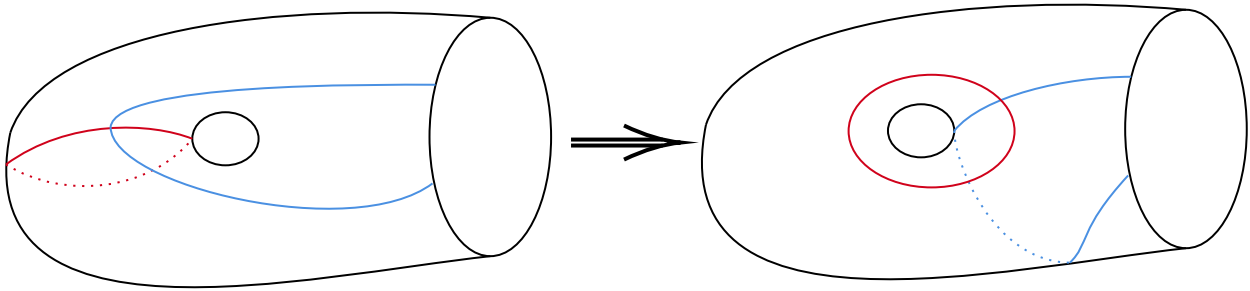


Figure 4.5: The first transformation

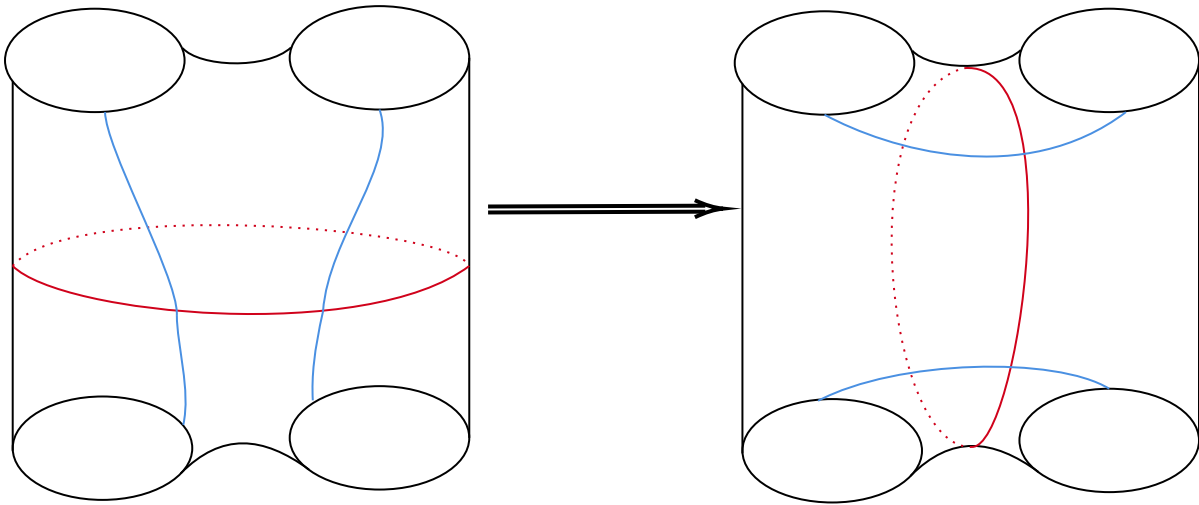


Figure 4.6: The second transformation

4.3.2 Control of the time of escaping the thin part

We will use the control given by the lemma 4.2.9

We now fix ϵ_0 such that $\epsilon_0(2 + C_{control}) < \frac{\rho}{2}$, and we consider a parameter $\mu > 0$. Since we will keep ϵ_0 fixed and will decrease μ in what follows, we will often omit the dependence on ϵ_0 of several constants in most estimates below.

Let

$$\Omega_{\epsilon_0, \mu} = \{[X, \lambda] \in P^1\mathcal{M}_{g,n}, l_{sys}(X) \in [\frac{\epsilon_0}{2}; \epsilon_0] \text{ and } i(\lambda, \gamma) \in [0, \mu] \text{ when } l_\gamma(X) \in [\frac{\epsilon_0}{2}; \epsilon_0]\} .$$

Moreover, let D be the set $\{[X, \lambda] \in \mathcal{PM}^1, l_{sys}(X) > \rho/2\}$. Obviously this set is non-empty (because $\rho < K_{sys}^g$) and has a positive ν -measure.

Given $[X, \lambda] \in \Omega_{\epsilon_0, \mu}$ and $\gamma \in \Gamma$ such that $\tilde{\epsilon} := l_\gamma(X) \in [\frac{\epsilon_0}{2}; \epsilon_0]$, if $\epsilon_\gamma^{\lambda, X} = \epsilon_\gamma$ is attained along the earthquake flow orbit (X, λ) at time t_γ , then the inequalities in Lemma 4.2.9 for $t = 0$ yield

$$\tilde{\epsilon} - \epsilon_\gamma \leq i(\lambda, \gamma)|t_\gamma| \leq \tilde{\epsilon} + C_{control}\epsilon_\gamma.$$

Because $0 < \epsilon_\gamma \leq \tilde{\epsilon}$ and $[X, \lambda] \in \Omega_{\epsilon_0, \mu}$, we have $i(\lambda, \gamma)|t_\gamma| \in [0; \tilde{\epsilon}(1 + C_{control})]$. By combining these estimates with Lemma 4.2.9, it follows that, for any $t \in J_\gamma^{\lambda, X}(\rho)$, one has

$$l_\gamma(\hat{E}_t(X, \lambda)) \leq \tilde{\epsilon}(2 + C_{control}) + i(\lambda, \gamma)|t| \leq \epsilon_0(2 + C_{control}) + \mu|t| \leq \frac{\rho}{2} + \mu|t|.$$

Hence, for all $|t| \leq \frac{\rho}{2\mu} := t_{lim}$, we have that $l_\gamma(\hat{E}_t(X, \lambda)) \leq \rho$. In particular, $E_t([X, \lambda]) \notin D$ for all $|t| \leq t_{lim}$.

Therefore, if $f_{\epsilon_0, \mu}$ is a Lipschitz positive function with support included in $\Omega_{\epsilon_0, \mu}$ and with $\int f_{\epsilon_0, \mu} d\nu > 0$, and g is a positive Lipschitz function with support in D , $K_1 = \int g$ and $K_2 = \|g\|_{Lip}$, then

$$\left| \int f_{\epsilon_0, \mu} \circ E_{t_{lim}} g d\nu - \int f_{\epsilon_0, \mu} d\nu \int g d\nu \right| = K_1 \int f_{\epsilon_0, \mu}. \quad (4.2)$$

Thus, if the earthquake flow were polynomially mixing with degree d in the sense of the inequality 4.1, then we would have:

$$K_1 \int f_{\epsilon_0, \mu} d\nu < K_2 \|f_{\epsilon_0, \mu}\|_{Lip} \frac{C}{t_{lim}^d}. \quad (4.3)$$

In the remainder of this section, we will contradict this estimate (for $d > 6g - 5$) by constructing $f_{\epsilon_0, \mu}$ with a small Lipschitz norm relatively to the integral $\int f_{\epsilon_0, \mu} d\nu$.

4.3.3 Estimate of the Lipschitz norm of $f_{\epsilon_0, \mu}$.

We will define $\hat{f}_{\epsilon_0, \mu}$ on $P^1\mathcal{T}_{g,n}$ as a product of piecewise linear function as follows: $\hat{f}_{\epsilon_0, \mu} = g_{\epsilon_0} j_{\mu}$ with

$$g_{\epsilon_0}(X) = \begin{cases} 0 & \text{if } l_{sys}(X) \notin [\epsilon_0/2; \epsilon_0] \\ 1 & \text{if } l_{sys}(X) \in [\frac{4\epsilon_0}{6}, \frac{5\epsilon_0}{6}] \\ \frac{6}{\epsilon_0}(l_{sys}(X) - \frac{\epsilon_0}{2}) & \text{if } l_{sys} \in [\frac{\epsilon_0}{2}, \frac{4\epsilon_0}{6}] \\ \frac{6}{\epsilon_0}(l_{sys}(X) - \epsilon_0) & \text{if } l_{sys} \in [\frac{5\epsilon_0}{6}, \epsilon_0] \end{cases}$$

and

$$j_{\mu}(X, \lambda) = \prod_{\gamma \in \Gamma, \frac{\epsilon_0}{2} < l_{\gamma}(X) \leq \epsilon_0} j_{\mu, \gamma}(\lambda)$$

where

$$j_{\mu, \gamma}(\lambda) = \begin{cases} 0 & \text{if } i(\lambda, \gamma) > \mu \\ 1 & \text{if } i(\lambda, \gamma) \leq \mu/2 \\ \frac{2}{\mu}(\mu - i(\lambda, \gamma)) & \text{if } i(\lambda, \gamma) \in [\mu/2; \mu]. \end{cases}$$

We will show that:

Lemma 4.3.7. $\hat{f}_{\epsilon_0, \mu}$ is Lipschitz and, for μ small enough,

$$\|\hat{f}_{\epsilon_0, \mu}\|_{Lip} \leq \frac{C_f^{Lip}}{\mu}.$$

Proof. As $\|\hat{f}_{\epsilon_0, \mu}\|_{L^\infty} \leq 1$, our task is reduced to bound

$$\sup_{\substack{(X, \lambda) \in P^1\mathcal{T}_{g,n}, \\ (X_n, \lambda_n) \in P^1\mathcal{T}_{g,n} \setminus \{(X, \lambda)\}, \\ (X_n, \lambda_n) \rightarrow (X, \lambda)}} \lim_{n \rightarrow \infty} \frac{|\hat{f}_{\epsilon_0, \mu}(X, \lambda) - \hat{f}_{\epsilon_0, \mu}(X_n, \lambda_n)|}{d_{Th} \times d_{lam}((X, \lambda), (X_n, \lambda_n))}.$$

In order to avoid difficulties which can occur when the curves in the product defining j_{μ} are not the same for X and X_n , we divide our analysis into three cases.

If $(X, \lambda) \notin \text{supp}(\hat{f}_{\epsilon_0, \mu})$ then eventually $(X_n, \lambda_n) \notin \text{supp}(\hat{f}_{\epsilon_0, \mu})$ and we are done.

If $(X, \lambda) \in \text{supp}(\hat{f}_{\epsilon_0, \mu})$ let $d_X = \min(l_{\gamma}(X) - \frac{\epsilon_0}{2}, \epsilon_0 - l_{\gamma}(X))$ where γ a curve is a systole.

If $d_X > 0$ eventually $l_{\gamma}(X_n) \in [\frac{\epsilon_0}{2}, \epsilon_0]$ for all n large enough. In this case we have

$$|\hat{f}_{\epsilon_0, \mu}(X, \lambda) - \hat{f}_{\epsilon_0, \mu}(X_n, \lambda_n)| \leq |g_{\epsilon_0}(X) - g_{\epsilon_0}(X_n)| \|j_{\mu}\|_{L^\infty} + |j_{\mu}(X, \lambda) - j_{\mu}(X_n, \lambda_n)| \|g_{\epsilon_0}\|_{L^\infty}.$$

Since $\|g_{\epsilon_0}\|_{L^\infty} = \|j_\mu\|_{L^\infty} = 1$, Lemma 4.3.3 ensures $\|g_{\epsilon_0}\|_{Lip} = \frac{6}{\epsilon_0} C_{sys, \epsilon_0}^{Lip} + 1$. On the other hand, Lemma 4.3.6 guarantees that

$$|j_\mu(X, \lambda) - j_\mu(X_n, \lambda_n)| \leq \frac{2}{\mu} |i(\lambda, \gamma) - i(\lambda_n, \gamma)| \leq \frac{2C_{int, \gamma}^{Lip}}{\mu} d_{lam}(\lambda, \lambda_n).$$

Because $X \in \mathcal{T}_{g, n}$ with $l_{sys}(X) \in [\epsilon_0/2, \epsilon_0]$ has a finite number of curves of length less than ϵ_0 , we can define

$$C_{int, X}^{Lip} = \max_{l_\gamma(X) \leq \epsilon_0} C_{int, \gamma}^{Lip}.$$

Therefore, if Δ is the intersection of a fundamental domain of the $Mod(S)$ -action on $\mathcal{T}_{g, n}$ intersect with $l_{sys}^{-1}([\epsilon_0/2, \epsilon_0])$, then we can use the compactness of Δ to define

$$C_{int}^{Lip} = \max_{X \in \Delta} C_{int, X}^{Lip}$$

and to conclude this second case.

Finally, in the third case when $d_X = 0$, one has $\hat{f}_{\epsilon_0, \mu}(X, \lambda) = g_{\epsilon_0}(X) = 0$. Since $\hat{f}_{\epsilon_0, \mu}(X_n, \lambda_n) \leq g_{\epsilon_0}(X_n) \leq \|g_{\epsilon_0}\|_{Lip} d_{Th}(X, X_n)$, we are also done in this case.

By taking these three cases into account, we see that there is a constant C_f^{Lip} such that

$$\|\hat{f}_{\epsilon_0, \mu}\|_{Lip} \leq \frac{C_f^{Lip}}{\mu}$$

for μ small enough. □

The function $\hat{f}_{\epsilon_0, \mu}$ is $Mod(S)$ -invariant and therefore it naturally descends to a function $f_{\epsilon_0, \mu}$ on $P^1\mathcal{M}_{g, n}$ with the same Lipschitz norm.

4.3.4 Estimate of the integral of $f_{\epsilon_0, \mu}$.

We will now bound the integral $\int f_{\epsilon_0, \mu} d\nu$.

If γ_1 and γ_2 are simple closed curves, we recall that the collar lemma 3.1.3 states that:

$$i(\gamma_1, \gamma_2) \leq \frac{l_{\gamma_1}(X)}{R(l_{\gamma_2}(X))}$$

where R is a decreasing function with $\lim_{x \rightarrow 0} R(x) = +\infty$ and $\lim_{x \rightarrow +\infty} R(x) = 0$. Then, we can partially extend this inequality for multicurves (by linearity) and for measured lamination (by continuity), that is:

$$i(\gamma, \lambda) \leq \frac{l_\lambda(X)}{R(l_\gamma(X))}.$$

Now in our case, if γ is such that $\frac{\epsilon_0}{2} < l_\gamma(X) \leq \epsilon_0$ we have for every curve δ :

$$i(\gamma, \delta) \leq \frac{l_\delta(X)}{R(\epsilon_0)}.$$

Assuming that μ is small enough so that $\mu R(\epsilon_0) \leq 1$, we get

$$\{\delta, l_\delta(X) \leq \mu R(\epsilon_0) L\} \subset \{\delta, i(\delta, \gamma) \leq \mu L, l_\delta(X) \leq L\}$$

for any $L > 0$. Indeed $i(\delta, \gamma) \leq \frac{1}{R(\epsilon_0)} l_\delta(X) \leq \mu L$. So, if $X \in \mathcal{T}_{g,n}$ and $\delta_1, \dots, \delta_k$ are the curves such that $\frac{\epsilon_0}{2} < l_{\delta_j}(X) \leq \epsilon_0$, we get that:

$$\begin{aligned} (\mu R(\epsilon_0))^{6g-6} \frac{\#\{\delta, l_\delta(X) \leq \mu R(\epsilon_0)L\}}{(\mu R(\epsilon_0)L)^{6g-6}} &= \frac{\#\{\delta, l_\delta(X) \leq \mu R(\epsilon_0)L\}}{L^{6g-6}} \\ &\leq \frac{\#\{\delta, i(\delta, \delta_j) \leq \mu L, j \in [1, k], l_\delta(X) \leq L\}}{L^{6g-6}} \end{aligned}$$

and, by taking the limit $L \rightarrow \infty$, we find that:

$$(\mu R(\epsilon_0))^{6g-6} B(X) \leq \mu_{Th}(X) \{\lambda \in \mathcal{ML}, i(\lambda, \delta_j) \leq \mu, j \in [1, k]\},$$

where $B(X)$ is the Thurston volume of the unit ball in the space of lamination (cf. Definition 3.1.10 above).

In this way, we derive that

$$\begin{aligned} \int f_{\epsilon_0, \mu} d\nu &\geq \int \mathbb{1}_{l_{sys}(X) \in [\frac{4\epsilon_0}{6}, \frac{5\epsilon_0}{6}]} \mathbb{1}_{j_\mu(X, \lambda)=1} d\nu \\ &= \int_{\Theta} \mathbb{1}_{l_{sys}(X) \in [\frac{4\epsilon_0}{6}, \frac{5\epsilon_0}{6}]} \int_{\mathcal{ML}} \mathbb{1}_{i(\lambda, \delta_j) \leq \mu/2, j \in [1, k]} d\mu_{Th}(X)(\lambda) d\mu_{WP}(X) \\ &\geq K_{vol} \mu^{6g-6} \end{aligned}$$

where Θ is a fundamental domain of the $Mod(S)$ -action on $\mathcal{T}_{g,n}$ and

$$K_{vol} = \left(\frac{R(\epsilon_0)}{2} \right)^{6g-6} \int_{\Theta} \mathbb{1}_{l_{sys}(X) \in [\frac{4\epsilon_0}{6}, \frac{5\epsilon_0}{6}]} B(X) d\mu_{WP}(X). \quad (4.4)$$

4.3.5 End of the proof of the main result

We now prove Theorem 4.3.1. Keeping the same notations and objects as before, we suppose that the earthquake flow is polynomially mixing in the sense of the inequality 4.1.

Recall that the inequality (4.3) yields

$$\int f_{\epsilon_0, \mu} d\nu < K_{fin} \|f_{\epsilon_0, \mu}\|_{Lip} \mu^d, \quad (4.5)$$

where $d \in \mathbb{R}^{+*}$ and K_{fin} a constant.

By injecting the estimates on $\|\hat{f}_{\epsilon_0, \mu}\|_{Lip}$ and $\int f_{\epsilon_0, \mu} d\nu$ from the two previous subsections, we have:

$$\mu^{6g-5} < \frac{K_{fin} C_f^{Lip}}{K_{vol}} \mu^d.$$

Since this inequality is false (for μ small enough) when $d > 6g - 5$ the proof of Theorem 4.3.1 is complete.

Chapter 5

Asymptotic Counting problems

In this final part, the two actors of the previous chapters, the action of $SL(2, \mathbb{R})$ on flat surfaces and the earthquake flow on hyperbolic surfaces, will be used to answer asymptotic counting problems. We begin, in the first section, by recalling a theorem of Mirzakhani on the growth of the set of simple closed geodesics with bounded length when this constraint is getting larger. This result uses the ergodicity of the earthquake flow. Then, in the last section, the ergodic properties of the action of $SL(2, \mathbb{R})$ are used to nearly obtain an asymptotic counting of pair of saddle connections with bounded virtual area.

5.1 Asymptotic counting in hyperbolic geometry

5.1.1 Mirzakhani's result

The ergodicity of the earthquake flow has been used to demonstrate many results on asymptotic countings. We expose here few of these results and gives a quick overview of the demonstration of one.

For $X \in \mathcal{T}_{g,n}$ and $\gamma \in \Gamma$, let $s(X, \gamma, L) = \#\{\delta \in \gamma \cdot MCG, l_\delta(X) \leq L\}$. This function count the number of simple closed curves in the mapping class group orbit of γ whose length is less than L . This function is mapping class group invariant on the first two variables and so give a function $s([X], [\gamma], \cdot)$ on $\mathcal{M}_{g,n} \times \Gamma/MCG$ that we will write with the same notation.

Remark. An equivalent way of saying that γ and δ are in the same mapping class group orbit is that each connected component of $S \setminus \gamma$ can be paired with a connected component of $S \setminus \delta$ with the same topology.

Mirzakhani showed the following theorem ([Mir08b] Theorems 1.1 and 1.2)

Theorem 5.1.1. *Let $\gamma \in \Gamma$ and $X \in \mathcal{T}_{g,n}$ then*

$$\frac{s(X, \gamma, L)}{L^{6g-6+2n}} \rightarrow \frac{r(\gamma)B(X)}{b_{g,n}}$$

where $r(\gamma)$ is a constant depending only on the mapping class group orbit of γ , B is the volume of the unit length ball of \mathcal{ML} for the measure μ_{T_h} as described in Definition 3.1.10 and $b_{g,n}$ is the integral of $B(\cdot)$ over the moduli space.

We sketch the proof to underline the role of the earthquake flow.

Proof. The proof use a methode called unfolding and averaging. Let $\gamma \in \Gamma$ and $X \in \mathcal{T}_{g,n}$. In the first step we "unfold" the function $s(\cdot, \gamma, L)$ to a function on an intermediate quotient between $\mathcal{T}_{g,n}$ and $\mathcal{M}_{g,n}$.

$$\begin{aligned} s(X, \gamma, L) &= \sum_{\alpha \in MCG \cdot \gamma} 1_{[0,1]} \left(\frac{1}{L} l_\alpha(X) \right) \\ &= \sum_{\phi \in MCG/Stab(\gamma)} 1_{[0,1]} \left(\frac{1}{L} l_{\phi \cdot \gamma}(X) \right) \\ &= \sum_{\phi \in MCG/Stab(\gamma)} 1_{[0,1]} \left(\frac{1}{L} l_\gamma(\phi \cdot X) \right) \end{aligned}$$

where $Stab(\gamma) = \{\phi \in MCG, \phi \cdot \gamma = \gamma\}$.

Definition 5.1.1. Let $\tilde{\mu}_{WP}([X])$ be the local pushforward of the Weil-Peterson measure on $\mathcal{T}_{g,n}/Stab(\gamma)$. Then we define

$$\tilde{\mu}_{\gamma,L} := 1_{[0,1]} \left(\frac{1}{L} l_\gamma(X) \right) \tilde{\mu}_{WP}([X]).$$

It is a measure on $\mathcal{T}_{g,n}/Stab(\gamma)$ with total mass

$$m_{\gamma,L} := \tilde{\mu}_{\gamma,L}(\mathcal{T}_{g,n}/Stab(\gamma))$$

Then, the second step is to "average" the function to a neighborhood of $[X]$. Let η_ϵ be a positive function with support in $B([X], \epsilon)$ (the ball can be taken with respect to any metrics for $\mathcal{T}_{g,n}$ described above) and of measure 1 for $\hat{\mu}_{WP}$.

$$\begin{aligned} s(X, \gamma, L) &\sim \int_{\mathcal{M}_{g,n}} \eta_\epsilon(Y) s(Y, \gamma, L) d\hat{\mu}_{WP} \\ &= \int_{\mathcal{T}_{g,n}/Stab(\gamma)} \eta_\epsilon(Y) 1_{[0,1]}\left(\frac{1}{L} l_\gamma(\phi \cdot Y)\right) \frac{d\tilde{\mu}_{WP}}{m_{\gamma,L}} m_{\gamma,L} \\ &= \int_{\mathcal{T}_{g,n}/Stab(\gamma)} \eta_\epsilon(Y) \frac{d\tilde{\mu}_{\gamma,L}}{m_{\gamma,L}} m_{\gamma,L} \end{aligned}$$

where \sim indicates a small error depending on ϵ .

Let's consider the family of measure $\tilde{\nu}_{\gamma,L} = \delta_{\gamma/l_\gamma(Y)} \times \frac{\tilde{\mu}_{\gamma,L}(Y)}{m_{\gamma,L}}$. They share good properties as described below.

Remark. All of this measured are invariant with respect to the earthquake flow and by design with total mass 1.

Moreover, Minsky and Weiss also prove that the earthquake flow is non-divergent in the following sense

Theorem 5.1.2 ([MW02] Theorem E2). *For every $\delta > 0$ there exist $\epsilon > 0$ such that for any $(X, \lambda) \in P^1\mathcal{M}_{g,n}$ exactly one of the following holds :*

1. $\liminf_{T \rightarrow \infty} \frac{Leb\{t \in [0, T] \mid l_{sys}(E^t(X, \lambda)) \geq \epsilon\}}{T} > 1 - \delta$
2. *There is a simple closed curve γ on the surface with $i(\lambda, \gamma) = 0$ and $l_\gamma(X) < \epsilon$.*

This theorem implies that if λ is a lamination such that $i(\lambda, \gamma) \neq 0$ for every $\gamma \in \Gamma$ (which is a generic condition) then one can control the proportion of time that the earthquake flow spend in the thin part of the moduli space.

This non-divergence property and the constant mass of these measure implies that a weak-star limit of this family exists. Then Mirzakhani also proved that

Theorem 5.1.3 (Theorem 5.9 of [Mir07a]). *Any weak limit of $\tilde{\nu}_{\gamma,L}$ are continuous with respect to ν .*

Then combining Theorem 5.1.2, theorem 5.1.3 and the remark 5.1.1, by ergodicity of the earthquake flow we have what is called the equidistribution of the counting measure which is,

$$\frac{\tilde{\nu}_{\gamma,L}}{m_{\gamma,L}} \rightarrow \frac{\tilde{\nu}}{b_{g,n}}.$$

Then integrating over the space of measured laminations \mathcal{ML} we get

$$\frac{\tilde{\mu}_{\gamma,L}}{m_{\gamma,L}} \rightarrow \frac{\tilde{\mu}B(Y)}{b_{g,n}}.$$

Finally, we have to give an estimate for $m_{\gamma,L}$. This can be done with the following theorem of Mirzakhani which gives a formula to integrate a class of function depending only on length of simple closed curves with respect to the Weil-Petersson measure.

Theorem 5.1.4 ([Mir07b]). *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $g_\gamma(X) = \sum_{\alpha \in MCG, \gamma} g(l_\alpha(X))$ then*

$$\int_{\mathcal{M}_{g,n}} g_\gamma(X) d\hat{\mu}_{WP}(X) = \int_x g(|x|) V_{g,n}^\gamma(x) dx$$

With $V_{g,n}$ an explicit polynomial of degree $L^{6g-7+2n}$

We can conclude that $\frac{m_{\gamma,L}}{L^{6g-6+2n}} \rightarrow r(\gamma)$. Now we have all the element to finish the computation

$$\begin{aligned} c(X, \gamma, L) &\sim \int_{\mathcal{T}_{g,n}/Stab(\gamma)} \eta_\epsilon(Y) \frac{d\tilde{\mu}_{\gamma,L}}{m_{\gamma,L}} \frac{m_{\gamma,L}}{L^{6g-6+2n}} L^{6g-6+2n} \\ &\sim \int_{\mathcal{T}_{g,n}/Stab(\gamma)} \eta_\epsilon(Y) \frac{B(Y)}{b_{g,n}} d\tilde{\mu}_{WP} r(\gamma) L^{6g-6+2n} \\ &\sim \frac{B(X)}{b_{g,n}} r(\gamma) L^{6g-6+2n} \end{aligned}$$

□

5.1.2 Other results

With the same philosophy, Arana-Herrera [Ara20] and Liu [Liu19] demonstrated independently the following result about equidistribution of multicurves

Theorem 5.1.5. *If X is an hyperbolic surface of genus g with n punctures, $\gamma := (\gamma_1, \dots, \gamma_k)$ an ordered closed multicurve on X with $k \geq 1$ components and a system of weights $b := (b_1, \dots, b_k) \in \mathbb{R}_{>0}^k$, then if*

$$s(X, \gamma, b, L) := \#\{\alpha := (\alpha_1, \dots, \alpha_k) \in MCG(X) \cdot \gamma \mid l_{\alpha_i}(X) \leq b_i L, \forall i \in [1, k]\}$$

there is a constant $c(X, \gamma, b,)$ such that,

$$c(X, \gamma, b) := \lim_{L \rightarrow \infty} \frac{c(X, \gamma, b, L)}{L^{6g-6+2n}}.$$

5.2 Asymptotic counting of pair of saddle connection

Getting back to the flat world, we will study one other asymptotic counting problem, precisely the counting of pair of saddle connection.

To each saddle connection γ is associated its holonomy vector

$$z_\gamma = \int_\gamma \omega \in \mathbb{C}.$$

The length of γ is then defined as $|z_\gamma|$.

We will count pairs of saddle connections with constraint length and *virtual area* which is defined by the equation:

$$|z_\beta \wedge z_\gamma| = |\operatorname{Re}(z_\beta) \operatorname{Im}(z_\gamma) - \operatorname{Im}(z_\beta) \operatorname{Re}(z_\gamma)|$$

for two saddle connections γ and β .

For ω a translation surface, we call Λ_ω the set in \mathbb{C} of all holonomy vectors of the saddle connections in ω . This is a discrete locally finite set.

5.2.1 History and prior results

The saddle connections are keystones for studying the geometry of flat surfaces.

Concerning the counting of these objects, Masur [Mas90] managed to bound the function $N(\omega, R) = \#\Lambda_\omega \cap B(0, R)$. More precisely, he showed that for every $\omega \in \mathcal{H}$, there are $c_1(\omega)$ and $c_2(\omega)$ such that

$$c_1(\omega)R^2 \leq \#N(\omega, R) \leq c_2(\omega)R^2.$$

Later on, Veech [Vee98] found an L^1 -quadratic asymptotic formula: namely, there is a $c > 0$ such that for μ_{MV} -almost every ω ,

$$\lim_{R \rightarrow \infty} \int_{\mathcal{H}} \left| \frac{N(\omega, R)}{R^2} - c \right| d\mu(\omega) = 0$$

This result was subsequently improved by Eskin and Masur [EM01] who showed that, for every μ an ergodic $SL(2, \mathbb{R})$ -invariant measure, there is a $c > 0$ such that for μ -almost every $\omega \in \mathcal{H}$

$$\lim_{R \rightarrow \infty} \frac{N(\omega, R)}{R^2} = c.$$

More recently Athreya, Fairchild and Masur [AFM22] extended this result to a counting of pairs of saddle connections with a constraint on the virtual area. More concretely, consider the following counting function

$$N_A(\omega, R) = \#\{(z, w) \in (\Lambda_\omega \cap B(0, R))^2 : |z \wedge w| \leq A, |w| \leq |z|\}.$$

Theorem 5.2.1. *There is a constant $c(A)$ such that for μ_{MV} -almost every $\omega \in \mathcal{H}$*

$$\lim_{R \rightarrow \infty} \frac{N_A(\omega, R)}{R^2} = c(A).$$

In the following, we will generalize this result to other $SL(2, \mathbb{R})$ -invariant measures. These results are part of the article [bonnafoux2022pairs].

5.2.2 General $SL(2, \mathbb{R})$ -invariant measures

As previously said, the group $SL(2, \mathbb{R})$ acts on Ω_g and preserve μ_{MV} (which restrictions on each stratum component are ergodics with respect to its action). We recall that g_t denotes its diagonal action and r_θ the rotations.

In general, a connected component of a stratum \mathcal{H} carries many $SL(2, \mathbb{R})$ -invariant measures. As it was famously shown by Eskin and Mirzakhani [EM18], such measures are supported on affine orbifolds, here denoted by \mathcal{M} . Roughly speaking, this mean that in any point $p \in \mathcal{M}$, there is a neighborhood U such that $\mathcal{M} \cap U$ is map by local period coordinates on a subspace defined by real linear equations.

The goal would be to prove that theorem 5.2.1 also holds for this family of measures. Which is

Conjecture 1. *Given μ an ergodic $SL(2, \mathbb{R})$ -invariant probability measure, there is a $c_\mu(A)$ such that for μ -almost every $\omega \in \mathcal{H}$*

$$\lim_{R \rightarrow \infty} \frac{N_A(\omega, R)}{R^2} = c_\mu(A).$$

We will discuss elements which are needed to prove Conjecture 1 and prove most of them. Only Lemma 5.2.18 is still unproved.

5.2.3 Power saving error term

Nevertheless, we will demonstrate that a power error term can be added to the estimation of Theorem 5.2.1.

Theorem 5.2.2. *There is a $c(A)$ and a $\kappa > 0$ such that for μ_{MV} -almost every $\omega \in \mathcal{H}$*

$$\frac{N_A(\omega, R)}{R^2} = c(A) + O_\omega(R^{-\kappa}).$$

In the case where Conjecture 1 would be demonstrated to be true, the same result will hold for every ergodic $SL(2, \mathbb{R})$ -invariant measure.

Theorem 5.2.3. *Suppose Conjecture 1 is true. Given μ an ergodic $SL(2, \mathbb{R})$ -invariant probability measure, there is a $c_\mu(A)$ and a $\kappa > 0$ such that for μ -almost every $\omega \in \mathcal{H}$*

$$\frac{N_A(\omega, R)}{R^2} = c_\mu(A) + O_\omega(R^{-\kappa}).$$

5.2.4 Continuity of the constants

Finally, still supposing Conjecture 1 to be true, we will show that these constant are, to some extent, continuous with respect to the measures.

Theorem 5.2.4. *Suppose Conjecture 1 is true. If a sequence of ergodic $SL(2, \mathbb{R})$ -invariant probability measure μ_n weakly converge to a probability $SL(2, \mathbb{R})$ -invariant measure ergodic μ , then for any $A > 0$*

$$c_{\mu_n}(A) \rightarrow c_\mu(A)$$

The proof occupies the rest of the section.

More precisely we begin, in Section 5.2.7 and 5.2.9, to discuss lemma which should be demonstrated to prove that

$$\lim_{R \rightarrow \infty} \frac{N_A(\omega, R)}{R^2} = c_\mu(A).$$

We will demonstrate them all but one. Our argument will follow closely the proof of Athreya, Fairchild and Masur, with three significant changes, as indicated in Subsection 5.2.6 below. Then in Section 5.2.10 we will derive the power saving error term $O_\omega(R^{-\kappa})$, based on tools developed by Nevo, Rühr and Weiss [NRW20]. Finally, continuity properties of the constants $c_\mu(A)$ are discussed in Section 5.2.11.

For our purpose, it is important to recall that there is a notion of independence for the saddle connection attached to each affine orbifold \mathcal{M} .

Definition 5.2.1. Saddle connections s_1, \dots, s_k on a surface $\omega \in \mathcal{M}$ are said to be \mathcal{M} -independent if their relative homology classes define linearly independent functionals (over \mathbb{C}) on the linear subspace $T_\omega \mathcal{M} \subset T_\omega \mathcal{H} \cong H^1(\omega, \Sigma; \mathbb{C})$ where Σ stand for the set of 0 of ω .

5.2.5 Siegel-Veech transform

As remarked by Athreya, Fairchild and Masur in Section 1.4.3 of [AFM22] one key point to show theorem 1 is to extend a result on the integrability of the Siegel-Veech transform of bounded compact functions which we recall now.

Let $B_c(X)$ be the space of bounded measurable functions with compact support on a space X . The Siegel-Veech transform for a function $f \in B_c(\mathbb{R})$ is defined as

$$\hat{f}(\omega) = \sum_{z \in \Lambda_\omega} f(z).$$

Sometimes for readability, we will also use the notation f^{SV} .

In this setting Athreya, Cheung and Masur [ACM19] proved that:

Theorem 5.2.5. *There is a $\kappa > 0$ such that for every $f \in B_c(\mathbb{C})$, $\hat{f} \in L^{2+\kappa}(\mathcal{H}, \mu_{MV})$.*

As it turns out, our goal in the next subsection, will be to show the following extension of Theorem 5.2.5.

Theorem 5.2.6. *For every $SL(2, \mathbb{R})$ -invariant measure μ , there is a $\kappa > 0$ such that for every $f \in B_c(\mathbb{C})$, $\hat{f} \in L^{2+\kappa}(\mathcal{H}, \mu)$.*

As it is discussed in the beginning of Section 2.2 of [AFM22], this theorem extends to function of two variables. For a function $g \in B_c(\mathbb{C}^2)$, the Siegel-Veech transform is defined as

$$\hat{g}(\omega) = \sum_{z_1, z_2 \in \Lambda_\omega^2} g(z_1, z_2).$$

An easy corollary of the previous theorem is

Corollary. *For every μ $SL(2, \mathbb{R})$ -invariant measure, there is a $\kappa' > 0$ such that for every $h \in B_c(\mathbb{C}^2)$, $\hat{h} \in L^{1+\kappa'}(\mathcal{H}, \mu)$.*

5.2.6 Review of the proof of Theorem 5.2.1

As our argument towards Theorem 1 follow closely the proof of Theorem 5.2.1, we will give a quick landscape on it and comment on the points which need to be adapted.

To begin, Athreya, Fairchild and Masur focus on partial counting functions

$$N_A^*(\omega, R) = N_A(\omega, R) - N_A(\omega, R/2).$$

These functions count the number of pairs of saddle connections such that their period coordinates are in

$$D_A(R, R/2) := \{(z, w) \in \mathbb{C}^2 : R/2 < |z| < R, |w| < |z|, |w \wedge z| \leq A\}$$

Estimates on these function can be extended to $N_A(\omega, e^t)$ using geometric series argument with a suitable control on a upper bound on $N_A^*(\omega, R)$ given by Proposition 3.2 in their article.

Then they describe a set

$$R_A(\mathcal{T}) := \{(z, w) \in \mathbb{C}^2, 1/2 \leq \text{Im}(z) \leq 1, |\text{Re}(z)| \leq \text{Im}(z), |\text{Im}(w)| \leq \text{Im}(z), |w \wedge z| \leq A\}$$

called the fibered trapezoid, which have the property that its characteristic function h_A satisfies

$$|N_A^*(\omega, R) - \pi e^{2t} (A_t \hat{h}_A)(\omega)| = |m_t(\omega) + \sum_{i=1}^4 e_t^i(\omega)| \quad (5.1)$$

where A_t is the following averaging operator

$$A_t(h)(p) = \frac{1}{2\pi} \int_0^{2\pi} h(g_t r_{\theta} p) d\theta$$

and m_t is called *main term* and the e_t^i *error terms*.

These functions are defined as the difference of the two terms in the left hand side of equation (5.1) estimated on various loci. The definitions of these loci are the following

- *Main part*

$$M_t = \left\{ (z, w) \in D_A \left(\sqrt{\frac{\cosh(2t)}{2}}, e^t \right) : |w| < |z|(1 + e^{-4t})^{-1/2} \right\},$$

- *Bottom of the trapezoid*

$$E_t^1 = D_A \left(e^t/2, \sqrt{\frac{\cosh(2t)}{2}} \right),$$

- *The norm of w is greater than $|z|(1 + e^{-4t})^{-1/2}$*

$$E_t^2 = \left\{ (z, w) \in D_A \left(\sqrt{\frac{\cosh(2t)}{2}}, e^t \right) : |w| > |z|(1 + e^{-4t})^{-1/2} \right\},$$

- *Top of the trapezoid*

$$E_t^3 = \{(z, w) \in \mathbb{C}^2 : (A_t h_A)(z, w) > 0, |z| > e^t\},$$

- The averaging operator is positive but $(z, w) \notin D_A(e^t/2, e^t)$

$$E_t^4 = \{(z, w) \in \mathbb{C}^2 : (A_t h_A)(z, w) > 0, e^t/2 < |z| < e^t, |w| > |z|\}.$$

After that the error functions are defined as

$$e_t^i(\omega) = (\chi_{E_t^i} \cdot (\chi_{D_A(e^t/2, e^t)} - \pi e^{2t}(A_t h_A)))^{SV}(\omega)$$

and

$$m_t(\omega) = (\chi_{M_t} \cdot (\chi_{D_A(e^t/2, e^t)} - \pi e^{2t}(A_t h_A)))^{SV}(\omega).$$

where χ are characteristic functions (this notation will be used all along this article).

To control the limit of $(A_t \hat{h}_A)$ as $t \rightarrow \infty$, they use Nevo ergodic theorem [Nev17] stating that

Theorem 5.2.7. *Suppose μ is an ergodic $SL(2, \mathbb{R})$ -invariant probability measure on \mathcal{H} . Assume $f \in L^{1+\kappa}(\mathcal{H}, \mu)$ for some $\kappa > 0$, and that f is K -finite, that is, $f_\theta(\omega) := f(r_\theta \omega)$ the span of the functions $\{f_\theta : \theta \in [0, 2\pi[\}$ is finite-dimensional. Let $\eta \in C_c(\mathbb{R})$ be a continuous non negative bump function with compact support and of unit integral. Then for μ -almost every $\omega \in \mathcal{H}$,*

$$\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} \eta(t-s)(A_s f)(\omega) ds = \int_{\mathcal{H}} f d\mu$$

Using this theorem they get a result on the limit of the averaging operator.

Theorem 5.2.8 (Proposition 2.2 of [AFM22]). *For $\phi \in C_c(\mathbb{C}^2)$, for μ_{MV} -almost every $\omega \in \mathcal{H}$, the circle average of $\hat{\phi}$ converge*

$$\lim_{t \rightarrow \infty} A_t \hat{\phi}(\omega) = \int_{\mathcal{H}} \hat{\phi} d\mu_{MV}$$

To adapt this theorem to our case of an arbitrary $SL(2, \mathbb{R})$ -invariant ergodic measure μ , we need to show that there is a $\kappa > 0$ such that for any $f \in B_c(X)$, $f \in L^{2+2\kappa}(\mathcal{H}, \mu)$ which will imply that for any $h \in B_c(\mathbb{C}^2)$, $\hat{h} \in L^{1+\kappa}(\mathcal{H}, \mu)$. For the Masur-Veech measure, this statement is the main result of [ACM19]. In our setting, this is the main content of Section 5.2.7.

Once this is done, taking a bump function g_ϵ (whose existence is established in Lemma 3.4 of [AFM22]) such that

$$|A_t \hat{h}_A(\omega) - A_t \hat{g}_\epsilon(\omega)| \leq \epsilon$$

and

$$\left| \int_{\mathcal{H}} \hat{g}_\epsilon - \hat{h}_A d\mu \right| \leq \epsilon$$

we get that

$$|A_t \hat{h}_A(\omega) - \int_{\mathcal{H}} \hat{h}_A d\mu| \leq |A_t \hat{h}_A(\omega) - A_t \hat{g}_\epsilon(\omega)| + |A_t \hat{g}_\epsilon(\omega) - \int_{\mathcal{H}} \hat{g}_\epsilon d\mu| + \left| \int_{\mathcal{H}} \hat{g}_\epsilon - \hat{h}_A d\mu \right| \leq 3\epsilon$$

for t big enough, using the extension for μ of Theorem 5.2.8 on $A_t \hat{g}_\epsilon$ for the second term.

It remains to prove that for almost every ω ,

$$|m_t(\omega)| = o(e^{2t}) \tag{5.2}$$

and for $i = 1, 2, 3, 4$

$$|e_t^i(\omega)| = o(e^{2t}). \tag{5.3}$$

For this sake, in the case of the Masur-Veech measure, Athreya, Fairchild and Masur started by estimating the volume of the set with two non-homologous small saddle connections (cf. Lemma 5.2 of [AFM22]). In our current setting this lemma has an equivalent form thanks to Dozier [Doz20].

Next, Athreya, Fairchild and Masur have two lemmas (cf. Lemmas 5.3 and 5.4 of [AFM22]) bounding the function $N(\omega, L)$. As they show in their paper, these lemmas essentially follow paragraph 3.6.2 and 3.6.3 of [ACM19] and [EM01]. Since they are stated for every $\omega \in \mathcal{H}$, we don't need to modify them here.

Afterwards, Athreya, Fairchild and Masur bound (cf. Lemma 5.5 of [AFM22]) the Masur-Veech integral of a counting function in the thin part of the moduli space (where some saddle connections are small). This mostly uses the estimate of Lemma 5.2 (which as it was previously said is extended by Dozier's result [Doz20] and can be used to have similar result of Lemma 5.2) except a couple of technical terms should be adapted to be extended to any $SL(2, \mathbb{R})$ -invariant measure.

Furthermore, Lemmas 5.6 and 5.7 of [AFM22] are technical lemmas of a geometric nature, controlling the orbit of a point in \mathbb{C} under geodesic flow and rotation: this means that we can still use it.

Moreover, they need to control the Masur-Veech volume of different loci on the thick part of the stratum related to the error terms (cf. Lemma 5.9 of [AFM22]). In the case of an arbitrary $SL(2, \mathbb{R})$ -invariant ergodic measure, we shall need to add extra arguments. This will be done in Section 5.2.9.

In the end, using all these facts, we can follow in a straightforward way the same computation of Athreya, Fairchild and Masur in Section 5.9 of [AFM22] to get the bounds on the main term (5.2) and the error terms (5.3) and *a fortiori* combining the different limits the desired result.

In summary, our task for proving the quadratic asymptotics in Theorem 1 is reduced to show the integrability of the Siegel-Veech transform with respect to any $SL(2, \mathbb{R})$ -invariant ergodic measure (cf. Theorem 5.2.6) which will be done in Section 5.2.7 and to adapt three lemmas (Lemma 5.2, 5.5 and 5.9 of [AFM22]) which will be the content of Section 5.2.9.

5.2.7 Siegel-Veech transforms are in $L^{2+2\kappa}(\mathcal{H}, \mu)$

In this subsection, we now prove Theorem 5.2.6.

We use the same strategy as in [ACM19] by starting with the characteristic function of the disc of radius a small ϵ_0 centered at 0.

Theorem 5.2.9. *Let $\epsilon_0 > 0$ and let f be the characteristic function of the disc of radius ϵ_0 centered at 0. Then there is a $\kappa > 0$ such that \hat{f} is in $L^{2+2\kappa}(\mathcal{H}, \mu)$.*

Remark. ϵ_0 should be taken small enough later, in the last paragraph of the proof called **"The shortest non-parallel saddle connection is longer than a power of the shortest and the shortest is a cylinder curve and the height of the cylinder is at most ϵ_0 ".**

We discuss the integrability of this function on several loci of \mathcal{H} . For all of them, but the last one, the arguments are the same as in [ACM19]. We will briefly remind them to concentrate on the last one.

Thick part

On the locus with no saddle connection with length smaller than ϵ_0 , we have $\hat{f} = 0$.

No short loops

On the set where there is a saddle connection of length less than ϵ_0 , but no homotopic nontrivial closed curves of length less than ϵ_0 , following paragraph 3.4 of [ACM19], \hat{f} is bounded on this set. As the measure μ is of finite volume, this concludes this case.

Short loops

On the case where there are short loops of length smaller than ϵ_0 , we will subdivide into four cases $\Omega_0, \Omega_1, \Omega_2, \Omega_3$.

On each, we want to show that

$$\sum_{k \geq 1} \mu(\Omega_i(k)) < +\infty$$

where $\Omega_i(k) := \{\hat{f} \geq k^{1/q}\} \cap \Omega_i$, for a $q > 2$ that we will fix latter.

For the rest of this section we will call γ the smallest saddle connection and $|\epsilon|$ the length of the smallest saddle connection which is non parallel with γ . Note that in the case of two saddle connections non-parallelism implies \mathcal{M} -independence as explain in § 1.4 of the article of Dozier [Doz20].

Theorem 5.1 of [EM01] ensures that if N is the dimension of relative homology, for every $\delta < \frac{1}{N}$ there is a C such that

$$\hat{f}(\omega) \leq \frac{C}{|\gamma|^{1+\delta}}.$$

We can deduce that if $\hat{f} \geq k^{1/q}$ there is a c such that

$$|\gamma| \leq ck^{-\frac{1}{q(1+\delta)}}.$$

Choose for the rest of the section, any δ and p such that

$$0 < \delta < p < \frac{1}{N}.$$

We will fix them later.

The shortest non-parallel saddle connection is shorter that a power of the shortest one

Let

$$\Omega_0(k) = \{(X, \omega) \in \mathcal{H}, \hat{f}(\omega) \geq k^{1/q}, |\epsilon| \leq |\gamma|^p\}.$$

The regularity property of the measure μ proved by Dozier (Theorem 1.1 of [Doz20]), knowing non-parallelism and hence the \mathcal{M} -independence of (γ, ϵ) , ensures that

$$\mu(\Omega_0(k)) = O(|\gamma|^{2+2p}) = O(k^{-\frac{2(1+p)}{q(1+\delta)}}),$$

which is summable if $q < 2\frac{1+p}{1+\delta}$

The shortest non-parallel saddle connection is longer that a power of the shortest and the shortest is not a cylinder curve

Let

$$\Omega_1(k) = \{(X, \omega) \in \mathcal{H}, \hat{f}(\omega) \geq k^{1/q}, |\epsilon| \geq |\gamma|^p, \gamma \text{ is not on the boundary of a cylinder}\}$$

Let's recall a lemma from Athreya, Cheung and Masur [ACM19].

Lemma 5.2.10 (Lemma 3.2 of [ACM19]). *Suppose γ is the shortest saddle connection on (X, ω) . Let β be a saddle connection (with an orientation) such that the path $P(\beta)$ in the Delaunay triangulation given by Theorem 3.2.3 follows edges parallel to γ more than $2M + 1$ times, where M is the total number of triangles in the Delaunay triangulation. Then there is a cylinder C with γ on its boundary and β crosses C .*

As in this case γ is not on the boundary of a cylinder, every saddle connection β such that the path $P(\beta)$ follow $2M$ times a edge parallel with γ must follow an edge non parallel with γ after that, and hence of length at least $|\epsilon|$. Thus any saddle connection with length less than ϵ_0 can be decomposed as $O(|\epsilon|^{-1})$ edges of the triangulation and hence decomposed in a basis of $H_1(X, \omega, \Sigma)$ with coefficient that are $O(|\epsilon|^{-1})$. The number of saddle connection smaller than ϵ_0 is then $O(|\epsilon|^{-N})$ where N is the dimension of $H_1(X, \omega, \Sigma)$.

To summary

$$\hat{f}(X, \omega) = O(|\epsilon|^{-N}) = O(|\gamma|^{-Np}).$$

The result recalled at the beginning of subparagraph "short loops" gives that, if \hat{f} has to be bigger than $k^{1/q}$, it is needed that

$$|\gamma| = O(k^{\frac{-1}{qNp}}).$$

Thus, using the regularity of the measure μ ,

$$\mu(\Omega_1(k)) = O(|\gamma|^2) = O(k^{\frac{-2}{qNp}}),$$

which is summable if $q < \frac{2}{Np}$.

The shortest non-parallel saddle connection is longer that a power of the shortest and the shortest is a cylinder curve and the height of the cylinder is at least ϵ_0

Let

$$\Omega_2(k) = \{(X, \omega) \in \mathcal{H}, \hat{f}(\omega) \geq k^{1/q}, |\epsilon| \geq |\gamma|^p, \gamma \text{ is on the boundary of a cylinder of height } \geq \epsilon_0\}$$

Here the same lemma 5.2.10 as in the previous paragraph is applied, and we get that a curve that follow $2M$ times edges parallel to γ should after cross the cylinder whose height is ϵ_0 . As the number of parallel edges is bounded, after a controlled number of step in the path, the path should cross the cylinder of height ϵ_0 , and in this case has a length more than ϵ_0 , or take a non-parallel edge of length $|\epsilon|$. In the second case a discussion similar to the previous case implies that

$$\mu(\Omega_2(k)) = O(k^{\frac{-2}{qNp}}),$$

The shortest non-parallel saddle connection is longer that a power of the shortest and the shortest is a cylinder curve and the height of the cylinder is at most ϵ_0

Let

$$\Omega_3(k) = \{(X, \omega) \in \mathcal{H}, \hat{f}(\omega) \geq k^{1/q}, |\epsilon| \geq |\gamma|^p, \gamma \text{ is on the boundary of a cylinder of height } \leq \epsilon_0\}.$$

For every point $X \in \bar{\mathcal{H}}$ choose a neighborhood which is a period coordinate chart as in theorem 3.2.2. As the space is compact, we can extract a finite covering. This family separates into two kinds, the first one which are contained in the open locus of the compactification \mathcal{H} , and the others which have a point on the boundaries $\bar{\mathcal{H}} - \mathcal{H}$.

We fix ϵ_0 small enough, such that $\Omega_3(k)$ is included in a finite union of neighborhoods of the second type for every k .

Let V be one of these neighborhoods. In this neighborhood we pick an ordering \succ of the subsurface. There is only a finite number of choice of neighborhood and of ordering.

We are interested in the intersection of V with $\Omega_3(k)$ and the set of surfaces consistent with \succ . Let's call this set $B_{\Omega_3(k), V}^\succ$.

Let η be a saddle connection included in the cylinder C , joining two zeros on the boundaries of C and crossing γ only once. η and γ are \mathcal{M} -independent since they are not parallel.

We add $\alpha_1, \dots, \alpha_{k'}$, other saddle connections with representatives living at level at most $X^{(0)}$ such that $\gamma, \eta, \alpha_1, \dots, \alpha_{k'}$ generate $H_{X^0, \succ}$ the subspace of elements of $H_1(X, \Sigma; \mathbb{C})$ that lie at level $X^0 \succ$. They do not cross any \succ -wide cylinder.

Then for each \succ -wide cylinder, one by one, in the order given by \succ , if the circumference curve is independent of the previous basis we add a curve joining two zeros and crossing no other \succ -wide cylinder (this is always doable). These saddle connections are called $\alpha_{k'+1}, \dots, \alpha_{m'}$.

Then we can extract an \mathcal{M} -independent basis by removing, in the order of appearance in $\gamma, \eta, \alpha_1, \dots, \alpha_{k'}, \alpha_{k'+1}, \dots, \alpha_{m'}$, the \mathcal{M} -dependent saddle connections. We get a new basis $\gamma, \eta, \beta_1, \dots, \beta_k, \beta_{k+1}, \dots, \beta_m$.

Next we bound the period coordinates of this basis. To do this we recall two lemmas from Dozier [Doz20] (Lemma 5.3 and Lemma 5.4).

Lemma 5.2.11. *For α a relative homology class defined on the surfaces in V , there exists a constant $C > 0$ with the following property. Fix $X \in V$ of area 1 such that \succ is consistent with X . Let Y be the level of α with respect to \succ . Then*

$$|\alpha(X)| \leq C \text{size}_X(Y).$$

Lemma 5.2.12. *Let β be a relative homology class defined on the surface in V with the following properties:*

- β has a representative that crosses exactly one closed curve α that is the core curve of a degenerating cylinder. We let Y be the level subsurface containing α .
- β has a representative such that the level subsurfaces which the representative intersects all lie at or below the level of Y .

Then there exists a family of rectangle $\mathcal{R}(z) \subset \mathbb{C}$, of area bounded above by some R (depending on V and β) with the following property. For any $X \in V$ of area 1, we have,

$$\beta(X) \in \mathcal{R}(\alpha(X)).$$

Furthermore, the rectangles have the property that $s\mathcal{R}(z) \subset \mathcal{R}(sz)$ for any $z \in \mathbb{C}$ and $0 < s \leq 1$.

Using these two lemmas and prior information on γ and η , we can state that:

- The period coordinates of γ live in a disk of radius $ck^{-\frac{1}{q(1+\delta)}}$,
- The period coordinates of η live in a rectangle centered at the origin with one side of length $|\gamma|$ and one side ϵ_0 , we call $R(z_\gamma, \epsilon_0)$ this rectangle,
- All β_i with $1 \leq i \leq k$, with lemma 5.2.11 are bound by

$$|\beta_i(X')| \leq C \text{size}_{X^0}(X') \leq K$$

because the size of the top subsurface is bounded in V ,

- All β_i with $k + 1 \leq i \leq m$ are in $\mathcal{R}(\gamma_i(X'))$ with γ_i the circumference curves of the \succ -wide cylinders they cross, and $\mathcal{R}(\cdot)$ the rectangles given by lemma 5.2.12. There are linear functions f_i such that $\gamma_i = f_i(\gamma, \eta, \beta_1, \dots, \beta_k)$ for all $k < i \leq m$. Indeed γ_i belong to a sub-surface X^i below the level X^0 and $H_{X^0 \succ}$ is generated by $\gamma, \eta, \beta_1, \dots, \beta_k$. We call \mathcal{R}_i the rectangle $\mathcal{R}(f_i(\gamma, \eta, \beta_1, \dots, \beta_k))$.

We can now integrate using Fubini. To shorten the notation we call the period coordinate of β_i $z_i := z_{\beta_i}$.

$$\begin{aligned}
\mu(\{B_{\Omega_3(k), V}^\succ\}) &= \mu'(\{sX : 0 \leq s \leq 1, X \in B_{\Omega_3(k), V}^\succ\}) \\
&= \text{Leb}(\{s(z_\gamma, z_\eta, z_1, \dots, z_k) \in \mathbb{C}^{k+2}, 0 \leq s \leq 1, z_\gamma \in B(0, ck^{-\frac{1}{q(1+\delta)}}), \\
&\quad z_\eta \in R(z_\gamma, \epsilon_0), z_i \in B(0, K), 1 \leq i \leq m, z_j \in \mathcal{R}_j, k+1 \leq j \leq m\}) \\
&\leq \text{Leb}(\{s(z_\gamma, z_\eta, z_1, \dots, z_k) \in \mathbb{C}^{k+2}, 0 \leq s \leq 1, sz_\gamma \in B(0, sck^{-\frac{1}{q(1+\delta)}}), \\
&\quad sz_\eta \in R(sz_\gamma, s\epsilon_0), sz_i \in B(0, K), 1 \leq i \leq m, sz_j \in s\mathcal{R}_j, k+1 \leq j \leq m\}) \\
&\leq \text{Leb}(\{(z_\gamma, z_\eta, z_1, \dots, z_k) \in \mathbb{C}^{k+2}, z_\gamma \in B(0, ck^{-\frac{1}{q(1+\delta)}}), \\
&\quad sz_\eta \in R(z_\gamma, \epsilon_0), z_i \in B(0, K), 1 \leq i \leq m, z_j \in \mathcal{R}_j, k+1 \leq j \leq m\}) \\
&= \int_{B(0, ck^{-\frac{1}{q(1+\delta)}})} \int_{R(z_\gamma, \epsilon_0)} \int_{B(0, K)} \int_{\mathcal{R}_j} dz_m \wedge d\bar{z}_m \cdots dz_k \wedge d\bar{z}_k \cdots dz_\eta \wedge d\bar{z}_\eta dz_\gamma \wedge d\bar{z}_\gamma \\
&= O(k^{-\frac{3}{q(1+\delta)}})
\end{aligned}$$

Finally summing over all choices of ordering and of neighborhood, we get that

$$\mu(\Omega_3(k)) = O(k^{-\frac{3}{q(1+\delta)}})$$

which is summable if $q < \frac{3}{1+\delta}$.

5.2.8 End of the proof

In order to have the required integrability, we should now adjust the variables according to three conditions, namely:

- $q < \frac{2}{Np}$
- $q < 2\frac{1+p}{1+\delta}$
- $q < \frac{3}{1+\delta}$

So we fix p and δ as

$$\delta = \frac{1}{16N} < p = \frac{1}{8N} < \frac{1}{N}$$

This yields the condition $q < 2 + C(N)$ with $C(N)$ some positive constant. Overall choosing q between 2 and $2 + C(N)$ we have shown the $2 + \kappa$ -integrability of the Siegel-Veech transform for a $\kappa > 0$.

Corollary. *For every $R > 0$ the characteristic function of $B(0, R)$ is in $L^{2+\kappa}(\mathcal{H}, \mu)$.*

Proof. The proof is in [ACM19] at theorem 3.3. We reproduce it here for completeness.

Cover the disk of radius R with sectors of angle $\frac{\epsilon_0^2}{R^2}$. By linearity is enough to show that for each sector, the characteristic function f has its Siegel-Veech transform $\hat{f} \in L^{2+\kappa}(\mathcal{H}, \mu)$.

Let $(g_t)_{t \in \mathbb{R}}$ be the diagonal action and $(r_\theta)_{\theta \in \mathbb{R}}$ the rotational action of $SL(2, \mathbb{R})$ on the strata of the moduli space. Moreover, let $t_0 = \log(\frac{R}{\epsilon_0})$ and θ_0 be the center angle of the sector. For any translation surface, if we apply $g_{t_0} r_{-\theta_0}$, any saddle connection with period coordinate in the sector will have in the new geometry a length less than ϵ_0 . Calling h the characteristic function of the disk of radius ϵ_0 , since this two flow are measure preserving we have

$$\int \hat{f}^{2+\kappa} d\mu < \int \hat{h}^{2+\kappa} d\mu < +\infty.$$

□

The proof of Theorem 5.2.6 is now an easy consequence of the previous corollary.

Proof of Theorem 1.4. Indeed, take any $f \in B_c(\mathbb{R}^2)$. By assumption there is a R such that $|f| \leq \|f\|_\infty \chi_{B(0,R)}$, with $\chi_{B(0,R)}$ the characteristic function on the ball $B(0, R)$. Positivity of the Siegel-Veech transform yields the result. □

A corollary of Theorem 5.2.6 is that we can extend Theorem 5.2.8 to any ergodic $SL(2, \mathbb{R})$ -invariant measure μ .

Theorem 5.2.13. *For any ergodic $SL(2, \mathbb{R})$ -invariant measure μ and for any $\phi \in C_c(\mathbb{C}^2)$, for μ -almost every $\omega \in \mathcal{H}$, the circle average of $\hat{\phi}$ converge*

$$\lim_{t \rightarrow \infty} A_t \hat{\phi}(\omega) = \int_{\mathcal{H}} \hat{\phi} d\mu$$

Proof. The proof is identical to the proof of Proposition 2.2 of [AFM22], but using the that $\phi \in L^{1+\kappa}(\mathcal{H}, \mu)$ with a general μ . □

5.2.9 Bounds on the volume of some sets

To complete the discussion of the quadratic asymptotic for $N_A(\omega, R)$, we need to revisit the proof of three technical lemmas, namely Lemma 5.2, 5.5 and 5.9 of [AFM22].

The first one is not hard to extend thanks to the work of Dozier [Doz20]. In our setting, it states that if μ is a $SL(2, \mathbb{R})$ -invariant measure then:

Lemma 5.2.14. *For all $\epsilon, \kappa > 0$, the μ -volume of the set $V_1(\epsilon, \kappa) \subset H$ of ω which have a saddle connection of length at most ϵ , and a non-homologous saddle connection with length at most κ is $O(\epsilon^2 \kappa^2)$*

Proof. The analog of this lemma for all $SL(2, \mathbb{R})$ -invariant measure is precisely the main result of Dozier, that is Theorem 1.1 of [Doz20]. □

The second one bounds the integral of the Siegel-Veech transform of a function with compact support on the thin part of the moduli space.

Lemma 5.2.15. *Let N be the maximal number of edge in a Delaunay triangulation and choose δ such that $\delta < 1/2N$. Let \mathcal{H}_ϵ be the locus of moduli space where the shortest saddle connection has length less than ϵ_1 . Let ψ be the Siegel-Veech transform of the characteristic function of the ball $B(0, L_0) \subset \mathbb{C}^2$ then*

$$\int_{\mathcal{H}_\epsilon} \psi d\mu = O(\epsilon^{1/N-2\delta}).$$

To prove this lemma we recall a lemma of Athreya, Fairchild and Masur [AFM22] bounding

$$N(\omega, R) := \Lambda_\omega \cap B(0, R)$$

with respect to the length of the smallest saddle connection.

Lemma 5.2.16 (Lemma 5.4 of [AFM22]). *For any $L_0 > 0$ and $\delta > 0$, there exist $C(\delta, L_0)$ such that for any $L < L_0$ and any surface (X, ω) in the stratum we have*

$$N(\omega, L) \leq C \left(\frac{L}{l(\omega)} \right)^{1+\delta}.$$

We should also control $N(\omega, R)$ with respect to the length of the shortest saddle connection non-parallel to γ .

Lemma 5.2.17. *For $\omega \in \mathcal{H}$, if γ is its shortest saddle connection and γ' is the shortest saddle connection non-parallel to γ , if γ does not bound a cylinder or if γ bound a cylinder of height at least ϵ_0 , then*

$$N(\omega, \epsilon_0)^2 = O(l(\gamma')^{-2N})$$

Proof. The proof has been done in Section 5.2.7 in the paragraphs

"The shortest non-parallel saddle connection is longer that a power of the shortest and the shortest is not a cylinder curve"

and

"The shortest non-parallel saddle connection is longer that a power of the shortest and the shortest is a cylinder curve and the height of the cylinder is at least ϵ_0 ". \square

We can now adapt the proof of Lemma 5.5 of [AFM22] to obtain our equivalent Lemma 5.2.15.

Proof of Lemma 5.2.15. Following the proof of Athreya, Fairchild and Masur, we choose a real $\sigma \in]0, 1[$, and we define three families of set exhausting \mathcal{H}_ϵ .

We will use the notation of the proof of theorem 5.2.9 that is γ is the shortest saddle connection of a translation surface ω and ϵ is the shortest saddle connection non-parallel with γ .

The first family is

$$F(j) =: \{\omega \in \mathcal{H}, \sigma^{j+1} \leq |\gamma| \leq \sigma^j, |\epsilon| \leq \sigma^{j/2N}\}.$$

In this case Lemma 5.2.14 say that

$$\mu(F(j)) = O(\sigma^{2j+2j/2N})$$

and Lemma 5.2.16 that for $\omega \in F(j)$

$$\psi(\omega) = O(\sigma^{-j(2+2\delta)}).$$

This gives

$$\int_{F(j)} \psi d\mu = O(\sigma^{j(1/N-2\delta)}).$$

The second family is

$$G(j) := \{\omega \in \mathcal{H}, \sigma^{j+1} \leq |\gamma| \leq \sigma^j, |\epsilon| \leq \sigma^{j/2N}, \gamma \text{ is not on the boundary of a cylinder or is on the boundary of a cylinder of height } \geq \epsilon_0\}.$$

In this case Lemma 5.2.14 gives that

$$\mu(G(j)) = O(\sigma^{2j})$$

and Lemma 5.2.17 indicates that for $\omega \in G(j)$

$$\psi(\omega) = O((\sigma^{j/2N})^{-2N}) = O(\sigma^{-j}).$$

And so

$$\int_{G(j)} \psi d\mu = O(\sigma^j).$$

Finally the third family is

$$H(j) := \{\omega \in \mathcal{H}, \sigma^{j+1} \leq |\gamma| \leq \sigma^j, \gamma \text{ is on the boundary of a cylinder of height } \leq \epsilon_0\}.$$

In this case Lemma 5.2.16 gives that for $\omega \in H(j)$

$$\psi(\omega) = O(\sigma^{-j(2+2\delta)}).$$

Moreover the measure of this set can be computed as the measure of the set $\Omega_3(k)$ is the last case of the proof of Theorem 5.2.9. Notice that in the computation of the volume of $\Omega_3(k)$ the length of the shortest non-parallel saddle connection does not interfere. So

$$\mu(H(j)) = O(\sigma^{3j})$$

and

$$\int_{H(j)} \psi d\mu = O(\sigma^{j(1-2\delta)}).$$

Finally choose j_0 so that $\sigma^{j_0+1} \leq \epsilon_1 \leq \sigma^{j_0}$ we have

$$\int_{\mathcal{H}_{\epsilon_1}} \psi d\mu = O\left(\sum_{j \geq j_0} \sigma^j + \sigma^{j(1-2\delta)} + \sigma^{j(1/N-2\delta)}\right) = O(\sigma^{j_0(1/N-2\delta)}) = O(\epsilon_1^{1/N-2\delta}).$$

□

The third one (Lemma 5.9 of [AFM22]) bounds the measure of a set described by four inequalities.

More precisely, given $L, \hat{\epsilon}, \epsilon'$ and $L' \in \{1/2, 1\}$ define $\Omega(\hat{\epsilon}, \epsilon', L, L')$ to be the set of surfaces ω such that ω is $\hat{\epsilon}$ -thick (that is has no saddle connection of length less than $\hat{\epsilon}$), and there are $(z, w) \in (\Lambda_\omega \cap B(0, L))^2$, where at least one of the following holds:

- $1 - \epsilon' \leq \frac{|\operatorname{Im}(w)|}{|\operatorname{Im}(z)|} \leq 1 + \epsilon'$
- $(1 - \epsilon')A \leq |z \wedge w| \leq (1 + \epsilon')A$

- $|\operatorname{Im}(z) - L'| < \epsilon'$
- $(1 - \epsilon') \operatorname{Im}(z) \leq |\operatorname{Re}(z)| \leq (1 + \epsilon') \operatorname{Im}(z)$

Lemma 5.2.18. *There exists C and $D > 0$ so that for all $\hat{\epsilon}, \epsilon'$*

$$\mu(\Omega(\hat{\epsilon}, \epsilon', L, L')) \leq C \frac{\epsilon'}{\hat{\epsilon}^D}$$

The idea to prove it, would be to use two kinds of cover of the compactification of the stratum $\bar{\mathcal{H}}$ and make them play together.

Ordered Period coordinate charts

We will first work with period coordinate charts, which have been defined in Definition 3.2.4. For each of them there is a finite number of ordering \succ . We will call ordered period coordinate chart, the choice of a period coordinate chart and an ordering. Each $X \in \bar{\mathcal{H}}$ is included in one of these open sets, and as the compactification is compact, we can extract a finite covering $\bar{\mathcal{H}} = \cup_a V_a$.

Now each V_a is a semianalytic set. Hence, its intersection with the affine submanifold \mathcal{M} is made of a finite number of connected components, see Lemma 4.5 of Dozier's work [Doz20]. We will name them $\cup_b V_a^{\mathcal{M},b} := \mathcal{M} \cap V_a$.

Fixing one of them, we can as in paragraph

The shortest non-parallel saddle connection is longer than a power of the shortest and the shortest is a cylinder curve and the height of the cylinder is at most ϵ_0
 take a basis $\alpha_1^{a,b}, \dots, \alpha_\ell^{a,b}$ generating $H_{X^0 \succ}$. By Lemma 5.2.11, their lengths are no more than a constant for every $X \in V_a^{\mathcal{M},b}$. Then, we can complete the basis $\alpha_{\ell+1}^{a,b}, \dots, \alpha_n^{a,b}$ with relative homology classes which cross one and only one \succ -wide cylinder. As we are on the $\hat{\epsilon}$ -thick part of Ω_g , the length of these cylinders can not be longer than $1/\hat{\epsilon}$, since we work with area 1 surfaces. We recall that this construction appear in Lemma 5.6 of Dozier's article [Doz20].

We take a \mathcal{M} -independent basis out of $\alpha_1^{a,b}, \dots, \alpha_n^{a,b}$ by taking it in their order of apparition. We call this basis $\beta_1^{a,b}, \dots, \beta_k^{a,b}$, and with the discussion of the previous paragraph, their lengths are between $\hat{\epsilon}$ and $1/\hat{\epsilon}$. We decompose the period coordinates as $\beta_j^{a,b} = x_j^{a,b} + iy_j^{a,b}$. The μ measure is the Lebesgue measure under their period coordinates up to a factor which is $O(\frac{1}{\hat{\epsilon}^k})$ because of the framing of the lengths of the $\beta_j^{a,b}$.

iso-Delaunay cells related basis

As previously said one can cover Ω_g by a finite number of iso-Delaunay cells $\cup_c \mathcal{D}_c$.

We consider the sets

$$\mathcal{M}_c^{\hat{\epsilon}} := \mathcal{D}_c \cap \Omega(\hat{\epsilon}, \epsilon', L, L') \cap \mathcal{M}.$$

Each of them can be made of an infinite number of connected components.

For each of \mathcal{D}_c , we choose a Delaunay triangulation refining the Delaunay decomposition. Then, for each connected component of each $\mathcal{M}_c^{\hat{\epsilon}}$, we take a maximal family of \mathcal{M} -independent saddle connections which are edges of the Delaunay triangulation. There are a finite number of choices, we can write $\mathcal{M}_c^{\hat{\epsilon}}$ as a finite union of sets of connected components for which we made the same choice. That is $\mathcal{M}_c^{\hat{\epsilon}} = \cup_{\bar{e}} \mathcal{M}_c^{\hat{\epsilon}}(\bar{e})$ where \bar{e} is a subset of edges of the Delaunay triangulation.

For one $\mathcal{M}_c^{\hat{\epsilon}}(\bar{e})$, we call $t_1^{c,\bar{e}}, t_2^{c,\bar{e}}, \dots, t_k^{c,\bar{e}}$ the chosen saddle connections.

The length of any $t_i^{c,\bar{e}}$ is bounded below by $\hat{\epsilon}$ and also bound above by $1/\hat{\epsilon}$. Indeed Theorem 4.4 of [MS91] indicates that the length of an edge of the Delaunay triangulation is less than the diameter of the surface which itself should be less than the inverse of the injectivity radius $\hat{\epsilon}$ for a surface of area 1.

In $\mathcal{M}_c^{\hat{\epsilon}}(\bar{e})$, we complete \bar{e} with $t_{k+1}^{c,\bar{e}}, \dots, t_n^{c,\bar{e}}$, other edges of the Delaunay triangulation, to have a basis of $H_1(X, \Sigma, \mathbb{C})$.

Change of basis

At this point, an argument is missing. We would like to use the following result :

Conjecture 2. *We can make change of basis with coefficient which are $O(1/\hat{\epsilon}^m)$ for some $m \in \mathbb{N}$. That is, for every $V_a^{\mathcal{M},b}$ and for every $\mathcal{M}_c^{\hat{\epsilon}}(\bar{e})$, there are coefficients $\sigma_{i,j}^{a,b,c,\bar{e}}$ such that*

$$t_i^{c,\bar{e}} = \sum_{j=1}^n \alpha_j^{a,b} \sigma_{i,j}^{a,b,c,\bar{e}}$$

with $|\sigma_{i,j}^{a,b,c,\bar{e}}| = O(1/\hat{\epsilon}^m)$.

As previously said, the numbers of $\mathcal{M}_c^{\hat{\epsilon}}(\bar{e})$ and $V_a^{\mathcal{M},b}$ are finite. However, as it is not known if the iso-Delaunay are real-analytics, these two families could intersect with an infinite number of connected component.

If Conjecture 2 would be proven to be true, our task of estimating the μ -measures of each $\Omega(\hat{\epsilon}, \epsilon', L, L') \cap V_a^{\mathcal{M},b}$ would be reduced to bound, for each a and b , the μ -measures of the finite collection $\Omega(\hat{\epsilon}, \epsilon', L, L') \cap V_a^{\mathcal{M},b} \cap \mathcal{M}_c^{\hat{\epsilon}}(\bar{e})$ of subsets of $\Omega(\hat{\epsilon}, \epsilon', L, L') \cap V_a^{\mathcal{M},b}$. Note that, even though c and \bar{e} range through a finite set, the subsets $\Omega(\hat{\epsilon}, \epsilon', L, L') \cap V_a^{\mathcal{M},b} \cap \mathcal{M}_c^{\hat{\epsilon}}(\bar{e})$ might have infinitely many connected components. Nonetheless, using that the conditions defining $\Omega(\hat{\epsilon}, \epsilon', L, L')$ impose nice constraints on iso-Delaunay triangulations (thanks to their efficiency) and the transitions between the iso-Delaunay and period charts bases are controlled, we shall see below that each of them is contained in a region given in the basis $\beta_1^{a,b}, \dots, \beta_k^{a,b}$ by the intersection of a $O(\epsilon')$ neighborhood of a certain number (which is polynomial in $1/\hat{\epsilon}$) of $(k-1)$ -dimensional hyperplanes with a ball of radius $O(1/\hat{\epsilon})$. In particular, the desired result would follow since we can control the μ -measure of such regions.

End of the proof assuming Conjecture 2

Proof. So let's fix $\Omega(\hat{\epsilon}, \epsilon', L, L') \cap V_a^{\mathcal{M},b} \cap \mathcal{M}_c^{\hat{\epsilon}}(\bar{e})$ and consider a surface $X \in \Omega(\hat{\epsilon}, \epsilon', L, L') \cap V_a^{\mathcal{M},b} \cap \mathcal{M}_c^{\hat{\epsilon}}(\bar{e})$.

Then if two saddle connections z, w have length less than L and their holonomy vectors satisfy the first inequality, we can decompose

$$z = \sum_{i=1}^n z'_i t_i^{c,\bar{e}} \text{ and } w = \sum_{i=1}^n w'_i t_i^{c,\bar{e}}$$

in the homology with the iso-Delaunay related basis, with integer coefficient.

The coefficients are $O(\frac{1}{\hat{\epsilon}})$ since the Delaunay triangulation is efficient.

We can now express them in the basis related to the period coordinate charts, fixed by $V_j^{\mathcal{M},a}$,

$$z = \sum_{i=1}^n \tilde{z}_i \alpha_i^{a,b} \text{ and } w = \sum_{i=1}^n \tilde{w}_i \alpha_i^{a,b}$$

with $|\tilde{z}_i| = O(\frac{1}{\hat{\epsilon}^{1+m}})$.

Since every $\alpha_i^{a,b}$, for $1 \leq i \leq n$ are linear combinations of the $\beta_j^{a,b}$ for $1 \leq j \leq k$, we can write

$$z = \sum_{i=1}^k z_i \beta_i^{a,b} \quad \text{and} \quad w = \sum_{i=1}^k w_i \beta_i^{a,b}$$

with coefficients which are still $O(\frac{1}{\hat{\epsilon}^{1+m}})$. We fix a pair of tuple of coefficients (z_i, w_i) with $z_i \neq w_i$, and which satisfy the maximum of the $O(\frac{1}{\hat{\epsilon}})$ conditions over the finite family of the $\mathcal{M}_i^{\hat{\epsilon}}(\bar{e})$. So we have a finite number (polynomial in $1/\hat{\epsilon}$) of admissible coefficients.

Now if we want w and z to satisfy the first condition in the definition of $\Omega(\hat{\epsilon}, \epsilon', L, L')$, we are interested in computing the volume of the set of surfaces, in $V_j^{\mathcal{M},a}$, whose basis β_j has holonomy vectors with imaginary parts (y_j) which can be combined into two holonomy vectors z and w of lengths less than L such that

$$\left| \frac{\sum_{i=1}^k z_i y_i}{\sum_{i=1}^k w_i y_i} - 1 \right| \leq \epsilon'$$

Since $\left| \sum_{i=1}^k w_i y_i \right| \leq L$, the previous condition is stronger than the condition

$$\left| \sum_{i=1}^k (z_i - w_i) y_i \right| \leq L\epsilon' \tag{5.4}$$

The vector (y_i) satisfying equation 5.4 are in the $L\epsilon'$ neighborhood of a $(k-1)$ -hyperplane in a k dimensional coordinate vector space. Moreover, each y_i should be less than $1/\hat{\epsilon}$, due to the length of the $\beta_j^{a,b}$. This volume can be computed as we did before, using Fubini and the \mathcal{M} -independence of the $\beta_j^{a,b}$. We obtain the desired result of $O(\frac{\epsilon'}{\hat{\epsilon}^{2k}})$. The number of choices of coefficients $z_i - w_i$ is $O(\frac{1}{\hat{\epsilon}^{1+m}})$. By summing the estimates for each choice of coefficient, then for each $V_a^{\mathcal{M},b}$, we get the bound of the first point.

The other points lead to same the kind of inequalities and are demonstrated similarly. \square

At this point we have shown that, assuming Conjecture 2, given μ an ergodic $SL(2, \mathbb{R})$ -invariant measure, there is a $c_\mu(A)$ such that for μ -almost every $\omega \in \mathcal{H}$

$$\lim_{R \rightarrow \infty} \frac{N_A(\omega, R)}{R^2} = c_\mu(A).$$

5.2.10 Effectivization of the counting

In this section we assume Conjecture 1 to be true in order to get the power saving error term of Theorem 5.2.3. Taking $\mu = \mu_{MV}$ yields Theorem 5.2.2.

We mainly need to control two quantities. The first one is the difference between the circle average and the integral of the Siegel-Veech transform of h_A . To do this we will use an effective version of Nevo ergodic theorem 5.2.7. The second quantity is the sum of the four error terms e_t^i .

Effective Nevo ergodic theorem

We will work with K -smooth functions as defined in the second chapter in Definition 4.1.1. For this class of function, there is natural norm.

Definition 5.2.2. We define the *Sobolev norm* by

$$S_K(f)^2 = \|f\|_2^2 + \|\pi_{\mathcal{H}}(\theta)f\|_2^2$$

The *K-Sobolev space* is then define as

$$\Sigma_K(\mathcal{H}) = \{f \in L^2(\mathcal{H}, \mu), S_K(f) < +\infty\}.$$

For a function g from \mathbb{C}^2 to \mathbb{R} we define

$$\partial_{\theta}g := \lim_{\theta \rightarrow 0} \frac{r_{\theta}^*g - g}{\theta}.$$

Note that $(\partial_{\theta}g)^{SV} = \pi_{\mathcal{H}}(\theta)\hat{g}$.

We can bound the Siegel-Veech transform of function with compact support and its S_K -norm with data from the original function.

Lemma 5.2.19. *Given $\alpha_1 > 2$, let $g : \mathbb{C}^2 \rightarrow \mathbb{R}$ be function with compact support and let $\epsilon > 0$. We have that*

$$|\hat{g}(\omega)| \leq C_{\text{supp}(g)} l(\omega)^{-\alpha_1} \|g\|_{\infty}$$

and if g is differentiable,

$$S_K \left(\hat{g}\chi_{l(\cdot) \geq \epsilon} - \int_{\mathcal{H}} \hat{g}\chi_{l(\cdot) \geq \epsilon} d\mu \right)^2 = O \left(\frac{\|g\|_{\infty}^2 + \|\partial_{\theta}g\|_{\infty}^2}{\epsilon^{2\alpha_1}} \right)$$

where the implied constant depend on the support.

Proof. For the first point, taking any surface $\omega \in \mathcal{H}$, if a pair of saddle connection are in $\text{supp}(g)$, then they both belong to a compact set $B \subset \mathbb{R}^2$. Hence using Lemma 5.2.16, we got

$$|\hat{g}(\omega)| \leq \|g\|_{\infty} (\#\Lambda(\omega) \cap B)^2 \leq C_{\text{supp}(g)} l(\omega)^{-\alpha_1} \|g\|_{\infty}$$

For the second point

$$\begin{aligned} \|\hat{g}\chi_{l(\cdot) \geq \epsilon} - \int_{\mathcal{H}} \hat{g}\chi_{l(\cdot) \geq \epsilon} d\mu\|_2^2 &\leq \int_{l(\cdot) \geq \epsilon} \hat{g}^2 d\mu + \left(\int_{\mathcal{H}} \hat{g}\chi_{l(\cdot) \geq \epsilon} d\mu \right)^2 \\ &\leq O \left(\int_{l(\cdot) \geq \epsilon} l(\omega)^{-2\alpha_1} \|g\|_{\infty}^2 d\mu \right) + O \left(\left(\int_{l(\cdot) \geq \epsilon} l(\omega)^{-\alpha_1} \|g\|_{\infty} d\mu \right)^2 \right) \\ &= O(\|g\|_{\infty}^2 \epsilon^{-2\alpha_1}) \end{aligned}$$

Where the second inequality comes from the first point of this lemma.

Then note that $\pi_{\mathcal{H}}(\theta)\chi_{l(\cdot) \geq \epsilon} = 0$ and $\pi_{\mathcal{H}}(\theta)\hat{g} = (\partial_{\theta}g)^{SV}$ and so

$$\pi_{\mathcal{H}}(\theta)\hat{g}\chi_{l(\cdot) \geq \epsilon} = (\partial_{\theta}g)^{SV} \chi_{l(\cdot) \geq \epsilon}.$$

So, in the same flavor as the previous estimate

$$\begin{aligned} \|\pi_{\mathcal{H}}(\theta)\hat{g}\chi_{l(\cdot) \geq \epsilon}\|_2^2 &= \int_{l(\cdot) \geq \epsilon} (\partial_{\theta}g)^{SV}^2 d\mu \\ &\leq \int_{l(\cdot) \geq \epsilon} l(\omega)^{-2\alpha_1} \|\partial_{\theta}g\|_{\infty}^2 d\mu \\ &= O(\epsilon^{-2\alpha_1} \|\partial_{\theta}g\|_{\infty}^2) \end{aligned}$$

This end the demonstration of this lemma. □

The following statement is an effective version of 5.2.7 which is included in Theorem 3.5 of [NRW20].

Theorem 5.2.20. *There is a $\lambda' > 0$ and a $C > 0$ such that for all $t > 1$ and any $f \in \Sigma_K(\mathcal{H})$, we have*

$$\|A_t f - \int_{\mathcal{H}} f d\mu\|_2^2 \leq C e^{-2\lambda' t} S_K(f)^2.$$

Furthermore, if $(t_n) \subset \mathbb{R}_+$ and η_1 satisfy

$$\sum_{n \in \mathbb{N}} e^{-\lambda' \eta_1 t_n} < \infty \tag{5.5}$$

then for almost all $\omega \in \mathcal{H}$ there is $n_0 = n_0(\omega)$ such that for all $n \geq n_0$

$$|A_{t_n} f(\omega) - \int_{\mathcal{H}} f d\mu| \leq C e^{-2(\eta - \frac{\eta_1}{2})\lambda' t_n} S_K(f)^2$$

with $\eta = \frac{1}{\lambda' + 1}$

From now on, we fix a sequence (t_n) and an η_1 satisfying equation 5.5. We will call $\lambda = \frac{\eta - \eta_1/2}{\lambda' + 1}$.

Circle average convergence

The aim of this subsection is to estimate

$$\left| A_t \hat{h}_A(\omega) - \int_{\mathcal{H}} \hat{h}_A d\mu \right|$$

To begin, we will need a bounding lemma (Lemma 5.2.22 below) which is based on the following theorem.

Theorem 5.2.21 (See [EM01] Thm 5.2 and Lem 5.5 and [Vee98] Cor 2.8). *For any $\omega \in \mathcal{H}$, and for any $1 \leq \alpha < 2$,*

$$\sup_{t > 0} A_t(l^{-\alpha})(\omega) < \infty.$$

The bound can be taken uniform as ω ranges over compact sets in \mathcal{H} . Moreover,

$$l(\cdot)^{-\alpha} \in L^1(\mathcal{H}, \mu).$$

Using this theorem we can bound the circle average of a Siegel-Veech transform of a function with compact support on the thin part.

Lemma 5.2.22. *There is an $\alpha_2 > 0$ such that for any $g : \mathbb{C}^2 \rightarrow \mathbb{R}$ be a positive function with compact support, any $\epsilon > 0$, any $\omega \in \mathcal{H}$ there is a constant $K(\omega, g)$ with*

$$|A_{t_n}(\hat{g}\chi_{l(\cdot) \leq \epsilon})| \leq \epsilon^{\alpha_2} K(\omega, g).$$

Moreover if $h \leq g$ then $K(\omega, h) \leq K(\omega, g)$.

Proof. Denote by $\chi_\epsilon := \chi_{l(\cdot) \leq \epsilon}$ and observe that, by taking α_2 such that $\alpha_2 \kappa < 2(1 + \kappa)$, one has

$$\begin{aligned} |A_t(\hat{g}\chi_\epsilon)(\omega)| &= \frac{1}{2\pi} \int_0^{2\pi} \hat{g}(g_t r_\theta \omega) \chi_\epsilon(g_t r_\theta \omega) d\theta \\ &\leq \frac{\epsilon^{\alpha_2}}{2\pi} \int_0^{2\pi} \hat{g}(g_t r_\theta \omega) l^{-\alpha_2}(g_t r_\theta \omega) \chi_\epsilon(g_t r_\theta \omega) d\theta \\ &\leq \frac{\epsilon^{\alpha_2}}{2\pi} \left(\int_0^{2\pi} l^{\frac{\alpha_2 \kappa}{1+\kappa}}(g_t r_\theta \omega) d\theta \right)^{\frac{\kappa+1}{\kappa}} \left(\int_0^{2\pi} \hat{g}(g_t r_\theta \omega)^{1+\kappa} \chi_\epsilon(g_t r_\theta \omega)^{1+\kappa} d\theta \right)^{\frac{1}{1+\kappa}} \end{aligned}$$

where the second inequality is the Hölder inequality. Then by Theorem 5.2.21 and the choice of α_2 the first integral of the last line is finite. The second integral, with Theorem 5.2.13, converges to

$$\int_{\mathcal{H}} \hat{g}^{1+\kappa} \chi_\epsilon^{1+\kappa} d\mu.$$

So the second integral is bound for all t by a quantity called $K'(\omega, g)$. This completes the argument.

The last assertion is obvious looking at what defined the constants. \square

As \hat{h}_A does not satisfy the hypothesis of Theorem 5.2.20, we will approximate \hat{h}_A by a family of function depending on two parameters. The first one will smooth h_A and the second will restrict its Siegel-Veech transform to the thick part of the moduli space.

Definition 5.2.3. Let $\eta : \mathbb{C}^2 \rightarrow [0, 1]$ be a smooth function with support in $B(0, 2)$ and equal to 1 on $B(0, 1)$ and such that

$$\int_{\mathbb{C}^2} \eta dLeb = 1.$$

Let

$$\eta_s(y) = \frac{1}{s^4} \eta\left(\frac{y}{s}\right)$$

and

$$R_{A,s}(\mathcal{T}) := \{x \in \mathbb{C}^2, \text{dist}(x, R_A(\mathcal{T})) \leq s\}$$

We consider

$$g_{A,s}(x) = (\chi_{R_{A,s}(\mathcal{T})} * \eta_s)(x) = \int_{\mathbb{C}^2} \eta_s(x - y) \chi_{R_{A,s}(\mathcal{T})}(y) dLeb(y)$$

$g_{A,s}$ is a function from \mathbb{C}^2 to $[0, 1]$ which is differentiable, with a bounded differential and compact support. An easy consequence is that the K -derivative, $\partial_\theta g_{A,s}$ is also bound and is $O(\frac{1}{s})$.

The supports of the function g_A^t are increasing meaning that if $s < t < 1$ then

$$\text{supp}(g_{s,A}) \subset \text{supp}(g_{t,A})$$

In particular $\text{supp}(g_{t,A}) \subset \text{supp}(g_{1,A})$ for every $t \leq 1$. We bound every function $g_{t,A}$ by the characteristic function of $\text{supp}(g_{1,A})$.

So using the first part of Lemma 5.2.19 we have that

$$|\hat{g}_{A,s}(\omega)| \leq Cl(\omega)^{-\alpha_1}$$

with no dependency on s for the constant.

This family of function also satisfy the following estimates.

Lemma 5.2.23. *There are $\kappa_1, \kappa_2, \kappa_3 > 0$ such that, for almost every $\omega \in \mathcal{H}$ there is a n_0 such that for $n \geq n_0(\omega)$ we have:*

- $\left| A_{t_n} \hat{h}_A(\omega) - A_{t_n} \hat{g}_{A,s}(\omega) \right| \leq O_\omega(s^{\kappa_1}) + O(s^{-\kappa_2})e^{-2\lambda t_n}$
- $\left| \int_{\mathcal{H}} \hat{g}_{A,s} d\mu - \int_{\mathcal{H}} \hat{h}_A d\mu \right| \leq O(s^{\kappa_3})$

Proof. Let's take

$$F_{A,s}(\mathcal{T}) := \{x \in \mathbb{C}^2, \text{dist}(x, \partial R_A(\mathcal{T})) \leq s\}$$

We consider

$$\psi_s(x) = (\chi_{F_{A,s}(\mathcal{T})} * \eta_s)(x)$$

We have that $0 \leq g_{A,s} - h_A \leq \psi_s$, $\psi_s \leq 1$, the support of ψ_s is compact, ψ_s is differentiable with bounded differential. So, calling χ_ϵ the characteristic function of the ϵ -thin part, we can say using Theorem 5.2.20 that for almost every $\omega \in \mathcal{H}$ and $n \geq n_0(\omega)$

$$\left| A_{t_n} \hat{h}_A(\omega) - A_{t_n} \hat{g}_{A,s}(\omega) \right| \leq \left| A_{t_n} \hat{\psi}_s(\omega) \chi_\epsilon \right| + \left| A_{t_n} \hat{\psi}_s(\omega) (1 - \chi_\epsilon) \right|$$

As $\psi_s \leq \psi_1$, by Lemma 5.2.22 we already have that

$$\left| A_{t_n} \hat{\psi}_s(\omega) \chi_\epsilon \right| = O(\epsilon^{\alpha_2}) K(\omega, \psi_s) \leq O(\epsilon^{\alpha_2}) K(\omega, \psi_1)$$

And using Theorem 5.2.20,

$$\left| A_{t_n} \hat{\psi}_s(\omega) (1 - \chi_\epsilon) \right| \leq \left| \int_{l(\cdot) \geq \epsilon} \hat{\psi}_s d\mu \right| + CS_K \left(\hat{\psi}_s(1 - \chi_\epsilon) - \int_{\mathcal{H}} \hat{\psi}_s(1 - \chi_\epsilon) d\mu \right)^2 e^{-2\lambda t_n}.$$

Using the second part of Lemma 5.2.19, we have easily that

$$S_K \left(\hat{\psi}_s(1 - \chi_\epsilon) - \int_{\mathcal{H}} \hat{\psi}_s(1 - \chi_\epsilon) d\mu \right)^2 = O\left(\frac{1}{\epsilon^{2\alpha_1} s^2}\right).$$

Moreover

$$\left| \int_{\mathcal{H}} \hat{g}_{A,s} d\mu - \int_{\mathcal{H}} \hat{h}_A d\mu \right| \leq \int_{\mathcal{H}} \hat{\psi}_s d\mu = \int_{l \leq \epsilon} \hat{\psi}_s d\mu + \int_{l \geq \epsilon} \hat{\psi}_s d\mu.$$

For the first integral by Lemma 5.2.15, if N is the maximal number of edge in a Delaunay triangulation and δ satisfies $\delta < \frac{1}{2N}$ then

$$\int_{l \leq \epsilon} \hat{\psi}_s d\mu = O(\epsilon^{1/N - 2\delta}).$$

Then by Lemma 5.2.19, for each ω in the ϵ thick part, $|\hat{\psi}_s(\omega)| = O_{\text{supp}(\psi_s)}(\frac{1}{\epsilon^{\alpha_1}}) = O_{\text{supp}(\psi_1)}(\frac{1}{\epsilon^{\alpha_1}})$. As the support of ψ_s is included in $\Omega(\epsilon_1, s, L, L')$ for a L big enough, Lemma 5.2.18 say that the measure of the support of $\hat{\psi}_s$ is $O(\frac{s}{\epsilon^D})$. Putting everything together we have that

$$\int_{\mathcal{H}} \hat{\psi}_s d\mu \leq O(\epsilon^{1/N - 2\delta}) + O\left(\frac{s}{\epsilon^{D+\alpha_1}}\right)$$

Taking $\epsilon = s^{\frac{1}{D+\alpha_1+1/N-2\delta}}$ we got that

$$\int_{\mathcal{H}} \hat{\psi}_s d\mu \leq O\left(s^{\frac{1/N-2\delta}{D+\alpha_1+1/N-2\delta}}\right).$$

This concludes the second point.

For the first point we have

$$\left| A_{t_n} \hat{h}_A(\omega) - A_{t_n} \hat{g}_{A,s}(\omega) \right| \leq O(\epsilon^{\alpha_2}) K(\omega) + O\left(\frac{s}{\epsilon^{D+\alpha_1}}\right) + O\left(\frac{1}{\epsilon^{2\alpha_1} s^2}\right) e^{-2\lambda t_n}$$

Taking here $\epsilon = s^{\frac{1}{D+2\alpha_1+\alpha_2}}$ we have

$$\left| A_{t_n} \hat{h}_A(\omega) - A_{t_n} \hat{g}_{A,s}(\omega) \right| \leq O_\omega\left(s^{\frac{\alpha_2}{D+2\alpha_1+\alpha_2}}\right) + O\left(\frac{1}{s^{\frac{\alpha_1}{D+2\alpha_1+\alpha_2}+2}}\right) \cdot e^{-2\lambda t_n}$$

□

We now fix an $\epsilon > 0$ and call χ_ϵ the characteristic function of $l^{-1}([0, \epsilon])$ on \mathcal{H} . We have that,

$$\begin{aligned} \left| A_t \hat{h}_A(\omega) - \int_{\mathcal{H}} \hat{h}_A d\mu \right| &\leq \left| A_t \hat{h}_A(\omega) - A_t \hat{g}_{A,s}(\omega) \right| + |A_t \hat{g}_{A,s} \chi_\epsilon(\omega)| \\ &\quad + \left| A_t \hat{g}_{A,s}(1 - \chi_\epsilon)(\omega) - \int_{\mathcal{H}} \hat{g}_{A,s}(1 - \chi_\epsilon) d\mu \right| \\ &\quad + \left| \int_{\mathcal{H}} \hat{g}_{A,s} \chi_\epsilon d\mu \right| + \left| \int_{\mathcal{H}} \hat{g}_{A,s} d\mu - \int_{\mathcal{H}} \hat{h}_A d\mu \right| \end{aligned} \quad (5.6)$$

In the previous inequality we approximated h_A by $g_{A,s}$ and then treated separately the thin and thick part of the moduli space. We can bound these terms with the following lemma.

Lemma 5.2.24. *There are $\alpha_3, C > 0$ such that*

- $S_K(\hat{g}_{A,s}(1 - \chi_\epsilon) - \int_{\mathcal{H}} \hat{g}_{A,s}(1 - \chi_\epsilon) d\mu)^2 \leq \epsilon^{-\alpha_1} O(\frac{1}{s^2})$
- $\int_{\mathcal{H}} \hat{g}_{A,s} \chi_\epsilon d\mu \leq C_3 \epsilon^{\alpha_3}$
- $|A_t(\hat{g}_{A,s} \chi_\epsilon)(\omega)| \leq \epsilon^{\alpha_2} K(\omega)$

Proof. The first point is just Lemma 5.2.19.

For the second point, choosing α_3 such that $\alpha_3 + \alpha_1 < 2$, with Theorem 5.2.21 we have

$$\begin{aligned} \int_{\mathcal{H}} \hat{g}_{A,s} \chi_\epsilon d\mu &\leq \int_{l(\omega \leq \epsilon)} \hat{g}_{A,s} d\mu \\ &\leq \int_{l(\omega \leq \epsilon)} C l(\omega)^{-\alpha_1} d\mu \\ &\leq \int_{l(\omega \leq \epsilon)} C \frac{\epsilon^{\alpha_3}}{l(\omega)^{\alpha_3}} l(\omega)^{-\alpha_1} d\mu \\ &\leq \epsilon^{\alpha_3} C \|l^{-\alpha_3 - \alpha_1}\|_1 \end{aligned}$$

The third point is an application of Lemma 5.2.22, in which we bound $g_{A,s}$ by $g_{A,1}$ to get rid of the dependency on s . □

Going back to inequality 5.6, we have that for almost every $\omega \in \mathcal{H}$ and $n \geq n_0(\omega)$

$$\begin{aligned}
\left| A_{t_n} \hat{h}_A(\omega) - \int_{\mathcal{H}} \hat{h}_A d\mu \right| &\leq \left| A_{t_n} \hat{h}_A(\omega) - A_{t_n} \hat{g}_{A,s}(\omega) \right| + |A_{t_n} \hat{g}_{A,s} \chi_\epsilon(\omega)| \\
&\quad + \left| A_{t_n} \hat{g}_{A,s}(1 - \chi_\epsilon)(\omega) - \int_{\mathcal{H}} \hat{g}_{A,s}(1 - \chi_\epsilon) d\mu \right| \\
&\quad + \left| \int_{\mathcal{H}} \hat{g}_{A,s} \chi_\epsilon d\mu \right| + \left| \int_{\mathcal{H}} \hat{g}_{A,s} d\mu - \int_{\mathcal{H}} \hat{h}_A d\mu \right| \\
&\leq O_\omega(s^{\kappa_1}) + O(s^{-\kappa_2}) e^{-2\lambda t_n} + \epsilon^{\alpha_2} K(\omega) \\
&\quad + C e^{-2\lambda t_n} S_K(\hat{g}_{A,s}(1 - \chi_\epsilon) - \int_{\mathcal{H}} \hat{g}_{A,s}(1 - \chi_\epsilon) d\mu)^2 \\
&\quad + C_3 \epsilon^{\alpha_3} + O(s^{\kappa_3}) \\
&\leq e^{-2\lambda t_n} O(s^{-\kappa_2} + \frac{1}{s^2 \epsilon_1}) + O_\omega(s^{\kappa_1} + s^{\kappa_3} + \epsilon^{\alpha_2} + \epsilon^{\alpha_1})
\end{aligned}$$

Putting $\epsilon = e^{-f_1 t_n}$ and $s = e^{-f_2 t_n}$ we have

$$\left| A_{t_n} \hat{h}_A(\omega) - \int_{\mathcal{H}} \hat{h}_A \right| \leq O_\omega(e^{-t_n(2\lambda - \kappa_2 f_2)} + e^{-t_n(2\lambda - 2f_2 - \alpha_1 f_1)} + e^{-f_2 \kappa_1 t_n} + e^{-f_2 \kappa_3 t_n} + e^{-f_1 \alpha_2 t_n} + e^{-f_1 \alpha_3 t_n})$$

Choosing wisely f_1 and f_2 we have for a $f > 0$

$$\left| A_{t_n} \hat{h}_A(\omega) - \int_{\mathcal{H}} \hat{h}_A \right| \leq O_\omega(e^{-t_n f}).$$

Error terms

Equation (4.10) of [AFM22] indicates that

$$\left| N_A^*(\omega, e^t) - \pi e^{2t} (A_t \hat{h}_A)(\omega) \right| \leq |m_t(\omega)| + \sum_{i=1}^4 |e_t^i(\omega)|$$

where the different terms are explained above.

Equation (5.4) of [AFM22] gives that $|m_t(\omega)| = O(e^{-2t})$. We will concentrate on the four other terms.

For the other error terms we choose $\hat{\epsilon} > 0$ to work separately on the thin and thick parts of the moduli space.

If ω is on the $\hat{\epsilon}$ thin part of the moduli space, according to Corollary 5.8 of [AFM22], we have that

$$|E_t^k \cap \Lambda_\omega^2| = O(\hat{\epsilon}^{1/N - 2\delta} e^{2t}).$$

We should now pick ϵ' satisfying every inequality in the proof of theorem 5.1 in [AFM22] which are on the beginning of each paragraph "Error term E_t^i ".

Taking larger bounds than these inequalities, we can find a K such that ϵ' satisfies all the conditions if

$$K e^{-2t} \leq \epsilon'.$$

To have estimates on the error terms on the thick part, it is needed to bound $|A_t h|$ where h is the characteristic function of $\Omega(\frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, L')$. As h is not K -smooth we will also approach it by a family of function defined below.

To do that let's call $\Theta(\hat{\epsilon}, \epsilon', L, L')$ the set of pair (w, z) in \mathbb{C}^2 such that

$$\hat{\epsilon} \leq \min(|w|, |z|) \leq \max(|w|, |z|) \leq L$$

and which verified at least one of these equations

- $1 - \epsilon' \leq \frac{|\operatorname{Im}(w)|}{|\operatorname{Im}(z)|} \leq 1 + \epsilon'$
- $(1 - \epsilon')A \leq |z \wedge w| \leq (1 + \epsilon')A$
- $|\operatorname{Im}(z) - L'| < \epsilon'$
- $(1 - \epsilon')\operatorname{Im}(z) \leq |\operatorname{Re}(z)| \leq (1 + \epsilon')\operatorname{Im}(z)$

We consider η_s the family of bump functions defined in the previous part and call $\xi_s = \eta_s * \chi_{\{(z,w), \operatorname{dist}((z,w), \Theta(\frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, L')) \leq s\}}$.
 ξ_s have the following properties

1. $\xi_s(z, w) \in [0, 1]$
2. $\xi_s \geq \chi_{\Theta(\frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon', L, L')}$
3. $\operatorname{supp}(\xi_s) \subset \Theta(\frac{\hat{\epsilon}}{\sqrt{8\pi}} - s, \epsilon' + \frac{s}{\hat{\epsilon}}, L + s, L') \subset \Theta(\frac{\hat{\epsilon}}{2\sqrt{8\pi}}, \epsilon' + \frac{s}{\hat{\epsilon}}, 2L, L')$ for s small enough
4. ξ_s is differentiable with its norm bound by $O(1/s)$.

Then h is bounded by the Siegel-Veech transform of these functions time the characteristic function of the $\frac{\hat{\epsilon}}{\sqrt{8\pi}}$ -thick part of the moduli space

$$\chi_{l(\geq \frac{\hat{\epsilon}}{\sqrt{8\pi}})} \hat{\xi}_s := \Xi_{s, \hat{\epsilon}}.$$

$\Xi_{s, \hat{\epsilon}}$ has its support on $\Omega(\frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon' + \frac{\sqrt{8\pi}s}{\hat{\epsilon}}, 2L, L')$ and is K -smooth.
Using the first part of Lemma 5.2.19 we have that for all $\omega \in \mathcal{H}$

$$|\Xi_{s, \hat{\epsilon}}(\omega)| \leq O(\hat{\epsilon}^{-\alpha_1}).$$

Using the second part of Lemma 5.2.19 we have that $S_K(\Xi_{s, \hat{\epsilon}} - \int_{\mathcal{H}} \Xi_{s, \hat{\epsilon}} d\mu)^2 = O(\frac{1}{\hat{\epsilon}^{2\alpha_1 s^2}})$
Moreover using Lemma 5.2.18 we got

$$\int_{\mathcal{H}} \Xi_{s, \hat{\epsilon}} d\mu \leq \|\Xi_{s, \hat{\epsilon}}\|_{\infty} \mu(\Omega(\frac{\hat{\epsilon}}{\sqrt{8\pi}}, \epsilon' + \frac{s}{\hat{\epsilon}}, 2L, L')) = O(\hat{\epsilon}^{-\alpha_1}) O(\frac{\epsilon' + \frac{s}{\hat{\epsilon}}}{\hat{\epsilon}^D}).$$

We have that for μ -almost every $\omega \in \mathcal{H}$ that for $n \geq n_0(\omega)$

$$|A_{t_n} \hat{h}| \leq |A_{t_n} \Xi_{s, \hat{\epsilon}}| \leq \int_{\mathcal{H}} \Xi_{s, \hat{\epsilon}} d\mu + CS_K(\Xi_{s, \hat{\epsilon}} - \int_{\mathcal{H}} \Xi_{s, \hat{\epsilon}} d\mu)^2 e^{-\lambda t_n} = O(\frac{\epsilon' + \frac{s}{\hat{\epsilon}}}{\hat{\epsilon}^{D+\alpha_1}} + \frac{e^{-\lambda t_n}}{\hat{\epsilon}^{2\alpha_1 s^2}}).$$

Taking $s = \epsilon' \hat{\epsilon}$ we have

$$|A_{t_n} \hat{h}| = O(\frac{\epsilon'}{\hat{\epsilon}^{D+\alpha_1}} + \frac{e^{-\lambda t_n}}{\epsilon'^2 \hat{\epsilon}^{2\alpha_1+2}})$$

Following the rest of the proof of Proposition 5.1 of Athreya, Fairchild and Masur in [AFM22], on the thick part of the moduli space, $|E_{t_n}^k \cap \Lambda_{\omega}^2|$ is bound by the product of two quantities.

The first one is a counting of acceptable sectors $\#I(\theta_i)$ which is shown to be $O(e^{2t} |A_t \hat{h}|)$.
The second one is the maximum number of pair of saddle connections in each sector which they showed to be $O(\frac{1}{\hat{\epsilon}^{1+\delta}})$.

With the previous computation we have that

$$\#I(\theta_i) = e^{2t_n} |A_{t_n} \hat{h}| = e^{2t_n} O\left(\frac{\epsilon'}{\hat{\epsilon}^{D+\alpha_1}} + \frac{e^{-\lambda t_n}}{\epsilon'^2 \hat{\epsilon}^{2\alpha_1+2}}\right)$$

and then in the thick part

$$|E_{t_n}^k \cap \Lambda_\omega^2| = e^{2t_n} O\left(\left(\frac{1}{\hat{\epsilon}}\right)^{1+\delta} \left(\frac{\epsilon'}{\hat{\epsilon}^{D+\alpha_1}} + \frac{e^{-\lambda t_n}}{\epsilon'^2 \hat{\epsilon}^{2\alpha_1+2}}\right)\right).$$

Grouping the thin and thick part we have

$$|E_{t_n}^k \cap \Lambda_\omega^2| = e^{2t_n} O\left(\hat{\epsilon}^{1/N-2\delta} + \left(\frac{1}{\hat{\epsilon}}\right)^{1+\delta} \left(\frac{\epsilon'}{\hat{\epsilon}^{D+\alpha_1}} + \frac{e^{-\lambda t_n}}{\epsilon'^2 \hat{\epsilon}^{2\alpha_1+2}}\right)\right).$$

Then by Lemma 4.1 of [AFM22], there is a T_0 such that for $t \geq T_0$,

$$A_t(h_A)(z, w)\pi e^{2t} \leq e^{2t} \arctan(e^{-2t}) \leq 2.$$

Then

$$|\chi_{D_A(e^t/2, e^t)} - A_t(h_A)(z, w)\pi e^{2t}| \leq 3.$$

As

$$e_t^i(\omega) = (\chi_{E_t^i} \cdot (\chi_{D_A(e_t^n/2, e^t)} - \pi e^{2t}(A_t h_A)))^{SV}(\omega)$$

and hence

$$|e_{t_n}^i(\omega)| = e^{2t_n} O\left(\hat{\epsilon}^{1/N-2\delta} + \left(\frac{1}{\hat{\epsilon}}\right)^{1+\delta} \left(\frac{\epsilon'}{\hat{\epsilon}^{D+\alpha_1}} + \frac{e^{-\lambda t_n}}{\epsilon'^2 \hat{\epsilon}^{2\alpha_1+2}}\right)\right).$$

So taking $\epsilon' = K e^{-\nu t}$ with $\nu = \min(\lambda/3, 2)$ and as $\hat{\epsilon}^{2\alpha_1+2}$ and $\hat{\epsilon}^{D+\alpha_1} \geq \hat{\epsilon}^{D+2\alpha_1+2}$, we have for some λ'

$$|e_{t_n}^i(\omega)| = e^{2t_n} O\left(\hat{\epsilon}^{1/N-2\delta} + \left(\frac{e^{-\lambda' t_n}}{\hat{\epsilon}^{D+2\alpha_1+3+\delta}}\right)\right)$$

Finally taking $\hat{\epsilon} = e^{-f_3 t}$ with $f_3 = \frac{\lambda'}{D+2\alpha_1+3+\delta+1/N-2\delta}$ we have shown that for each sequence (t_n) satisfying equation 5.5, for almost every $\omega \in \mathcal{H}$ there is a $n_0(\omega)$ such that for $n \geq n_0$, if $t_n > T_0$

$$|e_{t_n}^i(\omega)| = e^{2t_n} O(e^{-f_4 t})$$

with $f_4 > 0$. For technical reason, which will appear after, we choose $f_4 < \min(f, 2)$.

End of the estimation

Putting everything together we have for every sequence (t_n) respecting condition of equation 5.5 for almost every $\omega \in \mathcal{H}$, and for every $n \geq n_0(\omega)$,

$$\begin{aligned} \left| N_A^*(\omega, e^{t_n}) - \pi e^{2t_n} \int_{\mathcal{H}} h_A d\mu \right| &\leq \left| N_A^*(\omega, e^{t_n}) - \pi e^{2t_n} (A_{t_n} \hat{h}_A)(\omega) \right| + \left| \pi e^{2t_n} \int_{\mathcal{H}} h_A d\mu - \pi e^{2t_n} (A_t \hat{h}_A)(\omega) \right| \\ &\leq \left| N_A^*(\omega, e^{t_n}) - \pi e^{2t_n} (A_{t_n} \hat{h}_A)(\omega) \right| + |m_{t_n}(\omega)| + \sum_{i=1}^4 |e_{t_n}^i(\omega)| \\ &= e^{2t_n} O(e^{-f_4 t_n}) \end{aligned} \tag{5.7}$$

We consider the family of sequence $(\frac{t_n}{2^j})$. For each sequence there is a set of full measure \mathcal{H}_i on which the estimate 5.7 is true. Considering the set $\mathcal{H}_\infty = \bigcap_{j \in \mathbb{N}} \mathcal{H}_i$, which is also of full measure, we have the estimate 5.7 for all times $\frac{t_n}{2^j}$.

Fixing a natural number n , let's call $K = \lfloor \frac{t_n - \ln(T_0)}{\ln(2)} \rfloor$ such that $\frac{e^{t_n}}{2^K} > T_0$ we have

$$\begin{aligned} \frac{N_A(\omega, e^{t_n})}{e^{2t_n}} &= \sum_{j=0}^K \frac{N_A^*(\omega, e^{t_n}/2^j)}{e^{2t_n}} + \frac{N_A^*(\omega, e^{t_n}/2^K)}{e^{2t_n}} \\ &= \sum_{j=0}^K 2^{-2j} \frac{N_A^*(\omega, e^{t_n}/2^{2j})}{e^{2t_n}/2^{2j}} + 2^{-2K} \frac{N_A^*(\omega, e^{t_n}/2^{2K})}{e^{2t_n}/2^{2K}} \end{aligned}$$

As we have

$$\frac{N_A^*(\omega, e^{t_n}/2^{2j})}{e^{2t_n}/2^{2j}} = \pi \int_{\mathcal{H}} h_A d\mu + O\left(\frac{2^{f_4 j}}{e^{f_4 t_n}}\right)$$

so

$$\begin{aligned} \frac{N_A(\omega, e^{t_n})}{e^{2t_n}} &= \sum_{j=0}^K 2^{-2j} \left(\pi \int_{\mathcal{H}} h_A d\mu + O\left(\frac{2^{f_4 j}}{e^{f_4 t_n}}\right) \right) + 2^{-2K} \frac{N_A^*(\omega, e^{t_n}/2^{2K})}{e^{2t_n}/2^K} \\ &= 4 \left(\frac{1 - (1/4)^K}{3} \right) \pi \int_{\mathcal{H}} h_A d\mu + O\left(\frac{1}{e^{f_4 t_n}}\right) \frac{1 - 2^{(f_4 - 2)K}}{1 - 2^{f_4 - 2}} + 2^{-2K} \frac{N_A^*(\omega, e^{t_n}/2^{2K})}{e^{2t_n}/2^K} \end{aligned}$$

By proposition 3.2 and equation 1.1 of [AFM22] $\frac{N_A^*(\omega, e^{t_n}/2^{2K})}{e^{2t_n}/2^K}$ is bounded by a constant M . Moreover as $f_4 < 2$ we have also $\frac{1 - 2^{(f_4 - 2)K}}{1 - 2^{f_4 - 2}} \leq M'$

$$\left| \frac{N_A(\omega, e^{t_n})}{e^{2t_n}} - c_\mu(A) \right| \leq c_\mu(A)(1/4)^K + \frac{f(t_n)}{e^{f_4 t_n}} M' + 2^{-2K} M$$

as $K \geq \frac{t_n - \ln(T_0)}{\ln(2)} - 1$ we have

$$\begin{aligned} \left| \frac{N_A(\omega, e^{t_n})}{e^{2t_n}} - c_\mu(A) \right| &\leq c_\mu(A)(1/4)^{\frac{t_n - \ln(T_0)}{\ln(2)} - 1} + \frac{f(t_n)}{e^{f_4 t_n}} M' + 2^{-2(\frac{t_n - \ln(T_0)}{\ln(2)} - 1)} M \\ &= O(e^{-f_5 t_n}) \end{aligned}$$

with $f_5 = \min(2 \ln(2), f_4)$.

Estimate for all times

Let's take $t_n = \theta \log(n)$, which satisfy equation 5.5 for θ big enough, by monotony we have for a general time t such that $n \leq t \leq n + 1$ and $n \geq T_0$,

$$\begin{aligned} e^{2\theta \log(n)} (c_\mu(A) - O(e^{-f_5 \theta \log(n)})) &\leq N_A(\omega, e^{\theta \log(n)}) \leq N_A(\omega, e^{\theta \log(t)}) \\ &\leq N_A(\omega, e^{\theta \log(n+1)}) \leq e^{2\theta \log(n+1)} (c_\mu(A) + O(e^{-f_5 \theta \log(n+1)})) \end{aligned}$$

So that

$$n^{2\theta} \left(c_\mu(A) - O\left(\frac{1}{n^{f_5 \theta}}\right) \right) \leq N_A(\omega, t^\theta) \leq (n+1)^{2\theta} \left(c_\mu(A) + O\left(\frac{1}{(n+1)^{f_5 \theta}}\right) \right)$$

Then

$$\left(\frac{n}{t}\right)^{2\theta} \left(c_\mu(A) - O\left(\left(\frac{t}{n}\right)^{f_5 \theta} \frac{1}{t^{f_5 \theta}}\right) \right) \leq \frac{N_A(\omega, t^\theta)}{t^{2\theta}} \leq \left(\frac{n+1}{t}\right)^{2\theta} \left(c_\mu(A) + O\left(\left(\frac{t}{n+1}\right)^{f_5 \theta} \frac{1}{t^{f_5 \theta}}\right) \right)$$

But then as $1 - \frac{1}{t} \leq \frac{n}{t} \leq 1$ and $1 \leq \frac{n+1}{t} \leq 1 + \frac{1}{t}$ we have

$$\left| \frac{N_A(\omega, t^\theta)}{t^{2\theta}} - c_\mu(A) \right| = O\left(\frac{1}{t^{f_5\theta}} + \frac{1}{t^{2\theta}}\right)$$

as $f_5 \leq 2 \ln(2) \leq 2$, taking $s = t^\theta$, we can conclude that

$$\left| \frac{N_A(\omega, s)}{s^2} - c_\mu(A) \right| = O\left(\frac{1}{s^{f_5}}\right).$$

This concludes the proof of the existence of the power saving error term in Theorem 5.2.3.

5.2.11 Property of the constants $c_\mu(A)$

In this section, we also assume Conjecture 1 to be true. We aim to prove Theorem 5.2.4.

Fixing a $A > 0$ and a ergodic $SL(2, \mathbb{R})$ -invariant measure μ , the constant $c_\mu(A)$ remains mysterious. By the previous computation, it is equal to

$$c_\mu(A) := \frac{4}{3}\pi \int_{\mathcal{H}} \hat{h}_A d\mu = \frac{4}{3}\pi \int_{\mathbb{C}^2} h_A dm$$

where m is called *Siegel-Veech measure* and is invariant with respect to the $SL(2, \mathbb{R})$ action on \mathbb{C}^2 . We can decompose further as

$$\int_{\mathbb{C}^2} h_A dm = \int_{\mathbb{R} \setminus \{0\}} \left(\int_{SL(2, \mathbb{R})} h_A(tz, w) d\lambda(z, w) \right) d\nu(t) + \int_{\mathbb{P}^1(\mathbb{R})} \left(\int_{\mathbb{C}} h_A(z, sz) dz \right) d\rho(s) \quad (5.8)$$

where λ is the Haar measure on $SL(2, \mathbb{R})$ and $\nu = \nu(m)$ and $\rho = \rho(m)$ are decomposition of the measure m , on which little is known.

One could push the computation further. Let's recall that

$$h_A(z, w) = \chi_{\{(x,y), 1/2 \leq \text{Im}(x) \leq 1, |\text{Re}(x)| \leq \text{Im}(x), |\text{Im}(y)| \leq \text{Im}(x), |x \wedge y| \leq A\}}(z, w)$$

So $h(z, sz) = 0$ since the last condition is neither realized, and so the second integral in 5.8 is vanishing. Then we can compute the first integral using the Isawa decomposition of $SL(2, \mathbb{R})$. In this decomposition a general matrix is written as

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta)y^{1/2} + \sin(\theta)xy^{-1/2} & -\sin(\theta)y^{1/2} + xy^{-1/2}\cos(\theta) \\ \sin(\theta)y^{-1/2} & \cos(\theta)y^{-1/2} \end{pmatrix}$$

Its associated Haar measure is $dx dy d\theta$. To have short notations will give name to some subsets of $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{R}$.

$$\begin{aligned} S_1(t, y, \theta) &= \{1/2 \leq t \sin(\theta)y^{-1/2} \leq 1\} = \{t^2 \sin(\theta)^2 \leq y \leq 4t^2 \sin(\theta)^2\} \\ S_2(t, x, y, \theta) &= \{|t \cos(\theta)y^{1/2} + t \sin(\theta)xy^{-1/2}| \leq t \sin(\theta)y^{-1/2}\} \\ &= \{|\cos(\theta)/\sin(\theta)y + x| \leq 1\} \\ S_3(\theta, y, t) &= \{|\cos(\theta)y^{-1/2}| \leq t \sin(\theta)y^{-1/2}\} = \{|\cos(\theta)| \leq t \sin(\theta)\} \\ S_4(t, A) &= \{|t| \leq A\} \end{aligned}$$

Notice that the set S^2 does not really depend on t and that is measure along the x variable is always 2 and that the set S^3 does not depend on y and that is measure along the y variable is $3t^2 \sin(\theta)^2$.

Then we have

$$\begin{aligned}
& \int_{\mathbb{C}^2} h_A dm = \\
& \int_{\mathbb{R}^*} \int_0^{2\pi} \int_{\mathbb{R}^+} \int_{\mathbb{R}} 1_{S_1(t,y,\theta) \cap S_2(t,x,y,\theta) \cap S_3(\theta,y,t) \cap S_4(t,A)} dx dy d\theta d\nu(t) \\
& = \int_{[-A,A] \setminus \{0\}} \int_0^{2\pi} \int_{\mathbb{R}^+} \int_{\mathbb{R}} 1_{S_1(t,y,\theta) \cap S_2(t,x,y,\theta) \cap S_3(\theta,y,t)} dx dy d\theta d\nu(t) \\
& = 2 \int_{[-A,A] \setminus \{0\}} \int_0^{2\pi} \int_{\mathbb{R}^+} 1_{S_1(t,y,\theta) \cap S_3(\theta,y,t)} dy d\theta d\nu(t) \\
& = 6 \int_{[-A,A] \setminus \{0\}} t^2 \int_0^{2\pi} \sin(\theta)^2 1_{S_3(\theta,y,t)} d\theta d\nu(t) \\
& = 6 \int_{(0,A]} t^2 \int_{\arctan(1/t)}^{\pi - \arctan(1/t)} \sin(\theta)^2 d\theta d\nu(t) + 6 \int_{[-A,0)} t^2 \int_{\arctan(1/t) + \pi}^{2\pi - \arctan(1/t)} \sin(\theta)^2 d\theta d\nu(t) \\
& = 6 \int_{(0,A]} t^2 \left(\frac{\pi - 2 \arctan(1/t)}{2} + \frac{\sin(-2 \arctan(1/t)) - \sin(2 \arctan(1/t))}{4} \right) d\nu(t) \\
& \quad + 6 \int_{[-A,0)} t^2 \left(\frac{\pi - 2 \arctan(1/t)}{2} + \frac{\sin(-2 \arctan(1/t)) - \sin(2 \arctan(1/t))}{4} \right) d\nu(t) \\
& = 6 \int_{[-A,A] \setminus \{0\}} t^2 \left(\frac{\pi}{2} - \arctan(1/t) - \frac{t}{2(t^2 + 1)} \right) d\nu(t).
\end{aligned}$$

So understanding the variation of $c_\mu(A)$ when A is changing is equivalent of understanding the measures ν .

We can't compute them but we can still get a result of convergence for a *-weak limit of a sequence of ergodic $SL(2, \mathbb{R})$ -invariant probability measures.

Theorem 5.2.25. *Fixing an A , let (μ_n) be a sequence of $SL(2, \mathbb{R})$ -invariant probability ergodic measures on \mathcal{H} , such that $\mu_n \rightarrow \nu$ in the weak-* topology, where ν is another ergodic $SL(2, \mathbb{R})$ -invariant probability measure. Then the constants satisfy*

$$c_{\mu_n}(A) \rightarrow c_\nu(A).$$

The proof is similar to the proof for Siegel-Veech constants made by Dozier [Doz19]. We briefly recall it here.

Proof. Let's call $l(X)$ the length of the shortest saddle connection on the flat surface X and let $C_K = \{X \in \mathcal{H}, \frac{1}{l(X)} \leq K\}$. We take χ_K be a continuous function with image in $[0, 1]$ whose value is 1 on C_K and 0 on $\mathcal{H} - C_{K+1}$.

Then

$$c_{\mu_n}(A) := \frac{4}{3} \pi \int_{\mathcal{H}} \hat{h}_A d\mu_n = \frac{4}{3} \pi \left(\int_{\mathcal{H}} \hat{h}_A \chi_K d\mu_n + \int_{\mathcal{H}} \hat{h}_A (1 - \chi_K) d\mu_n \right)$$

We want to use *-weak convergence of the measure for the integral on the thick part of the stratum and to control the integral in the thin independently of the measure μ_n .

As h_A is bounded and compactly supported, \hat{h}_A is bounded by \hat{f}^2 where $f = \chi_B \|h\|_\infty$ with B a ball big enough to contain the union of the projections of the support of h via the maps $\mathbb{R}^4 \rightarrow \mathbb{R}^2$

$$(x, y) \mapsto x \text{ and } (x, y) \mapsto y$$

Eskin and Masur show that [EM01, Theorem 5.1] $\hat{f} < \frac{C}{l^{1+\delta}}$ for some C and $0 < \delta < 1/2$. Choosing a δ' in order that $\delta + \delta' < 1/2$ we obtain

$$\int_{\mathcal{H}} \hat{h}_A(1 - \chi_K) d\mu_n \leq \int_{\mathcal{H}-C_K} \frac{C}{l^{1+\delta}} d\mu_n \leq \frac{C}{K^{\delta'}} \int_{\mathcal{H}-C_K} \frac{1}{l^{1+\delta+\delta'}} d\mu_n$$

The last integral is finite by a lemma of the two previously cited authors [EM01, Lemma 5.5], and we can bound it from above by a constant independent of μ_n . First we recall a bound on circle average made by Dozier.

Lemma 5.2.26 (Proposition 1.1 of [Doz19]). *For any stratum \mathcal{H} and $0 < \delta < 1/2$, there exists a function $\alpha : \mathcal{H} \rightarrow \mathbb{R}_+$ and constants c_0, b such that for any $X \in \mathcal{H}$,*

$$\int_0^{2\pi} \frac{1}{l(g_t r_\theta X)^{1+\delta}} d\theta \leq c_0 e^{-(1-2\delta)t} \alpha(X) + b$$

Then, choosing any smoothing function ϕ non-negative, smooth and compactly supported with integral over \mathbb{R} equal to 1, we get by Nevo ergodic Theorem 5.2.7, for a generic $X \in \mathcal{H}$

$$\begin{aligned} \int_{\mathcal{H}-C_K} \frac{1}{l^{1+\delta+\delta'}} d\mu_n &\leq \int_{\mathcal{H}} \frac{1}{l^{1+\delta+\delta'}} d\mu_n \\ &= \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} \phi(t-s) \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{l(g_s r_\theta X)^{1+\delta}} d\theta \right) ds \\ &\leq b \end{aligned}$$

So picking an ϵ we can choose K big enough such that for all n

$$\int_{\mathcal{H}} \hat{h}_A(1 - \chi_K) d\mu_n \leq \frac{C}{K^{\delta'}} \int_{\mathcal{H}-C_K} \frac{1}{l^{1+\delta+\delta'}} d\mu_n \leq \frac{Cb}{K^{\delta'}} \leq \epsilon/2.$$

Then by $*$ -weak convergence of the measure there exist a N such that for $n \geq N$

$$\int_{\mathcal{H}} \hat{h}_A \chi_K d\mu_n \leq \epsilon/2.$$

Putting the two together gives the desired result. \square

This theorem has a direct consequence on the possible values of the constants $c_\mu(A)$ when μ varies on the space of ergodic $SL(2, \mathbb{R})$ -invariant probability measure, namely they all fall in a closed interval of $\mathbb{R}_{>0}$. Using the following proposition of Eskin, Mirzakhani, and Mohammadi

Theorem 5.2.27 (Theorem 2.3 of [EMM15]). *The space of ergodic $SL(2, \mathbb{R})$ -invariant probability measure is compact in the $*$ -weak topology.*

We get the following corollary.

Corollary. *For every A , there are two constants $0 < m(A) < M(A) < \infty$ such that for every ergodic $SL(2, \mathbb{R})$ -invariant measure μ*

$$m(A) \leq c_\mu(A) \leq M(A)$$

Proof. By contradiction taking a sequence of measure μ_n such that $c_{\mu_n}(A) \rightarrow \infty$ we can extract a subsequence $*$ -weakly converging to a measure μ . As the constant for this subsequence goes to ∞ and to the constant of the limit measure, we have a contradiction.

The lower bound is obtained by a similar proof. \square

5.2.12 Conclusion

Given an ergodic $SL(2, \mathbb{R})$ -invariant measure μ on \mathcal{H} , we showed in Section 5.2.9 the quadratic growth $N_A(\omega, R)$ for μ -almost every ω in \mathcal{H} assuming Conjecture 2. Then in Section 5.2.10 we proved that this quadratic growth admits a power saving error term. Finally, in this Section, we demonstrated the continuity of the constants $c_\mu(A)$ with respect to μ .

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Titre : Sur les propriétés ergodiques de flots sur les espaces de modules de surfaces et leurs utilisations pour des problèmes de comptage

Mots clés : Tremblement de terre, Lien selle, géométrie hyperbolique, surface de translation, théorie ergodique

Résumé : Cette thèse présente deux résultats alliant des outils de la théorie ergodique et de la géométrie, hyperbolique pour le premier et plate pour le second. Motivé par le théorème de Mirzakhani donnant une asymptotique polynomiale pour le nombre de géodésiques simple et fermée sur une surface hyperbolique, nous voulions mieux comprendre le comportement du flot du tremblement de terre. En effet, c'est son ergodicité que Mirzakhani utilise pour son théorème. L'idée est qu'une connaissance d'une vitesse de mélange pour le flot du tremblement de terre, pour une bonne classe d'observable, permettrait d'affiner le comptage de Mirzakhani par exemple avec un terme d'erreur. Notre premier résultat est un pas dans cette direction. Il borne la possible vitesse de mélange du tremblement de terre à une vitesse polynomiale dont le degré dépend de la topologie de la surface.

Pour connaître précisément sa vitesse de mélange, une idée serait de regarder l'action du groupe spécial linéaire d'ordre 2, sur les surfaces plates. En effet, il existe un lien entre le flot du tremblement de terre et le flot du sous-groupe unipotent appelé flot horocyclique. Dans cette optique nous nous sommes intéressés à affiner un résultat d'Athreya, Fairchild et Masur, donnant le comptage quadratique de paire de vecteurs d'holonomie de lien selle dont l'aire virtuelle est bornée par une constante donnée pour presque toute surface plate par rapport à une mesure naturelle appelée mesure de Masur-Veech. Une des clefs de voûte de leur résultat est un théorème ergodique dû à Névo. Nous avons approfondi ce résultat en donnant un terme d'erreur. Par ailleurs, nous discuterons d'un lemme nécessaire pour l'étendre à une famille plus large de mesure.

Title : On ergodic properties of some flows on moduli spaces and their uses on counting problems

Keywords : Earthquake flow, saddle connection, hyperbolic geometry, translation surfaces, ergodic theory

Abstract : This thesis presents two results combining tools from ergodic theory and geometry, hyperbolic for the first result and flat for the second. Motivated by Mirzakhani's theorem giving a polynomial asymptotics for the number of simple and closed geodesics on a hyperbolic surface, we wanted to better understand the behavior of the earthquake flow. Indeed, it is its ergodicity that Mirzakhani uses for her theorem. The idea is that a knowledge of a mixing rate for the earthquake flow, for a good class of observables, would allow to refine Mirzakhani's counting, with an error term for example. Our first result is a step in this direction. It bounds the possible mixing rate of the earthquake to a polynomial one whose degree depends on the surface topology.

To know precisely its mixing speed, an idea would be to look at the action of the special linear group of order 2, on flat surfaces. Indeed, there is a link between the earthquake flow and the flow of the unipotent subgroup called horocyclic flow. In this perspective we interested ourselves in refining a result of Athreya, Fairchild and Masur, giving the quadratic count of holonomy vectors of pair of saddle connection whose virtual area is bounded by a given constant for almost any flat surface with respect to a natural measure called Masur-Veech measure. One of the key points of their result is an ergodic theorem due to Nevo. We have deepened this result by giving an error term. Finally we will discuss a lemma which allow to extend the result to a larger family of measures.