Universal solutions of the reflection equation, the q-Onsager algebra and applications.

Guillaume Lemarthe

UNIVERSITÉ DE TOURS

ÉCOLE DOCTORALE : *MIPTIS*

Institut Denis Poisson

THÈSE présentée par :

Guillaume LEMARTHE

soutenue le : **15 Décembre 2023**

pour obtenir le grade de : **Docteur de l'université de Tours**

Discipline/Spécialité : Physique

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Thèse dirigée par :

M. Pascal Baseilhac	Directeur de recherche CNRS, Université de Tours

Co-encadrée par :

M. Azat Gainutdinov	Chargé de recherche CNRS, Université de Tours

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Rapporteurs

M. Paul Terwilliger	Professeur, Université du Wisconsin-Madison – USA
M. Bart Vlaar	Associate Professor, BIMSA – Chine

Jury

M. Pascal Baseilhac	Directeur de recherche CNRS, Université de Tours
M. Azat Gainutdinov	Chargé de recherche CNRS, Université de Tours
M. Oleg Lisovyi (Président du jury)	Professeur, Université de Tours
Mme Véronique Terras	Directrice de recherche CNRS, Université de Paris-Saclay
M. Paul Terwilliger	Professeur, Université du Wisconsin-Madison – USA
M. Bart Vlaar	Associate Professor, BIMSA – Chine
M. Robert Weston	Professeur, Université de Heriot-Watt – Écosse

Remerciements

Je tiens tout d'abord à exprimer ma gratitude envers Azat et Pascal pour le temps qu'ils ont consacré à travers leurs nombreuses discussions et relectures, notamment ces derniers temps pour la finalisation de la thèse. J'ai rencontré Pascal en première année de Licence à l'occasion d'un projet scientifique. Quelques années plus tard, il a proposé un sujet de stage de Master 2 qui par la suite a mené à cette thèse. C'est à cette occasion qu'il m'a présenté Azat comme étant 'l'électron libre du labo'. En effet, Azat était très actif en recherche avec de nombreuses collaborations engendrant de fréquents déplacements. Cependant, il a toujours su s'organiser afin de trouver du temps pour discuter. Il attache par ailleurs une grande importance à expliquer les choses de la manière la plus pédagogique possible. Azat est aussi devenu papa en cours de route ;) La thèse s'est déroulée de manière relativement détendue, en grande partie grâce à la zénitude de Pascal. Une fois les rendez-vous avec lui terminés, ces derniers débouchaient fréquemment sur des discussions variées offrant ainsi une perspective différente sur le monde qui nous entoure.

Ensuite, je remercie Paul Terwilliger ainsi que Bart Vlaar pour avoir accepté d'être rapporteur de ma thèse, et en particulier pour leurs commentaires sur la thèse. Lire le manuscrit et rédiger un rapport en si peu de temps doit certainement demander beaucoup de travail. Je tiens également à remercier Oleg Lisoyvi, Véronique Terras et Robert Weston d'avoir accepté de faire partie de mon jury.

Je remercie les collègues et amis du bureau avec qui j'ai partagé plusieurs années au quotidien. On était bien ici, avec nos habitudes, à se demander qui allait se dévouer pour préparer la Bialetti. Puis, sans trop se concerter, on s'est motivé à soutenir à peu près en même temps. Thomas, merci pour les suggestions de séries et films. Je garderai longtemps le souvenir de ce kiwi bien trop mûr que tu m'as donné/lancé et qui a fini explosé entre mes mains et sur mon ordi alors que je t'avais dit à plusieurs reprises ne pas en vouloir. Merci à toi Romain de m'avoir initié aux champis (même si les premiers que tu m'as indiqués comme comestibles avaient 'le goût du placo' – agaric jaunissant). A mes débuts des fléchettes, j'étais vraiment au niveau zéro, j'ai bien appris avec toi et maintenant tu as du mal à gagner une partie :) Merci aussi à Yegor, le *tovarish* biélorusse, qui m'a enseigné quelques bases ainsi que certaines 'subtilités' de la langue russe. Tu as beaucoup aimé chercher des jeux de mots dans ton apprentissage du français. Maintenant, même moi je pense naturellement à eux, nickel !

En la qualité de représentant des doctorants, j'ai été amené à accueillir les nouveaux arrivants ou à les aider dans certaines démarches administratives. Chacun d'entre eux m'a apporté quelque chose et ce serait un peu long d'en faire la liste. Merci donc à ceux qui sont déjà bien installés depuis quelques années, à savoir: Antonin, Léa, Yohan, Théo, Julien, Maxime, Romane, Igor, Andrii, May, Duy. Merci aussi aux petits nouveaux pour les moments passés: Jad, Marion et Vagif, bon courage pour vos thèses qui débutent ! Enfin, une pensée pour ceux qui ont déjà quitté le laboratoire: Ayman et Eda, Hakim, Abraham, Andreas, Jean-David, Florestan, et d'autres que j'oublie.

Un petit mot pour la postérité (en espérant tout de même que cela perdure), et à titre informatif au cas où certains ne soient pas encore au courant: depuis quelques années, en plus de servir de lieu de travail, le bureau E1360 fait également office de Q.G. du Club Amande. C'est un espace convivial où l'on peut trouver des viennoiseries et des gâteaux faits maison, à condition de contribuer soi-même de temps en temps. Désolé à ceux qui ont pris un peu de poids pendant cette période, mais ça valait le coup :)

Je voudrais aussi remercier les membres du laboratoire pour leur accueil. Ayant fait mes études à l'université de Tours, j'ai eu un grand nombre de ces personnes en tant qu'enseignant. Au début, il était difficile de les tutoyer, mais à force de passer du temps avec eux, notamment au déjeuner, j'ai appris à les connaître. Concernant un sujet qui peut parfois être sensible, j'espère que vous allez parvenir à trouver une organisation durable pour la gestion du stock du café. Au sujet de l'entretien, le détartrage de la machine est un peu long à faire mais n'est pas si difficile :)

A présent, quelques mots pour la personne qui, par passion envers les potins, n'hésite pas à en imaginer et à partir à la chasse aux informations pour voir si ses suppositions sont correctes. J'ai nommé Laetitia. Merci pour les pauses cafés matinales lorsque les lieux étaient encore presque déserts.

Finalement, je remercie ma famille pour leur soutien tout au long de la thèse.

Résumé

L'objectif de cette thèse est de développer un cadre permettant d'obtenir des opérateurs K de spin-j avec paramètre spectral, solutions de l'équation de réflexion et appartenant à $B \otimes \mathrm{End}(\mathbb{C}^{2j+1})((u))$, où B est une algèbre comodule sur une algèbre de Hopf H. A partir de ces opérateurs K, nous obtenons des relations-TT universelles satisfaites par des *matrices de transfert universelles* dans la sous-algèbre commutante de B qu'ils génèrent. Ce nouveau cadre proposé est spécialisé à $B = \mathcal{A}_q$, l'extension centrale de l'algèbre q-Onsager, et $H = \mathcal{L}U_q sl_2$, l'algèbre quantique à boucle de sl_2, offrant une approche universelle à diverses chaînes de spins quantiques ouvertes (spin-j, spin-j_1,j_2 alternantes, ...) avec conditions aux bords génériques. Les trois problèmes suivants sont étudiés et résolus.

Premièrement, nous introduisons une définition axiomatique d'une matrice K-universelle $\mathfrak{K} \in B \otimes H$ satisfaisant une équation de réflexion universelle. Elle requiert la donnée d'une paire de twist composée d'un twist ψ (un certain automorphisme de H) et d'un twist de Drinfeld. Cette définition étend les travaux de Balagović-Kolb et Appel-Vlaar. Pour $H = \mathcal{L}U_q sl_2$ et une certaine paire de twist fixée, nous montrons que l'évaluation de la composante tensorielle H de $\mathfrak{K}$ à une représentation d'évaluation formelle (de dimension infinie) de spin-j mène à des opérateurs K avec paramètre spectral $\mathbf{K}^{(j)}(u)$.

Deuxièmement, en nous inspirant de l'évaluation d'un des axiomes de la matrice K-universelle, et en considérant les produits tensoriels des représentations d'évaluation formelle de $\mathcal{L}U_q sl_2$ dans les cas où apparaissent une sous-représentation de spin-j, nous introduisons un opérateur K de spin-j fusionné $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q \otimes \mathrm{End}(\mathbb{C}^{2j+1})((u))$. Ces derniers sont construits à l'aide d'opérateurs d'entrelacs de $\mathcal{L}U_q sl_2$. Ils sont une généralisation spin-j de l'opérateur K de spin-1/2 introduit par Baseilhac et Shigechi en 2009. Nous montrons indépendamment du cadre universel, que toutes ces opérateurs K fusionnés satisfont les équations de réflexion. Ensuite, nous conjecturons une relation de proportionnalité entre $\mathcal{K}^{(j)}(u)$ et l'évaluation de la matrice K-universelle $\mathbf{K}^{(j)}(u)$. Elle est appuyée en montrant que $\mathcal{K}^{(j)}(u)$ vérifie un ensemble de relations similaires à l'évaluation des axiomes de $\mathfrak{K}$.

Troisièmement, une fonction génératrice de spin-j (appelée matrice de transfert universelle) dans la sous-algèbre commutante de $\mathcal{A}_q$ est obtenue grâce aux operateurs K fusionnés $\mathcal{K}^{(j)}(u)$ que nous avons construits. Sous réserve que la conjecture reliant $\mathbf{K}^{(j)}(u)$ et $\mathcal{K}^{(j)}(u)$ soit vraie, nous prouvons que ces fonction génératrices satisfont des relations-TT universelles qui permettent de réduire récursivement le problème spectral de spin-j à celui de spin-1/2. En spécialisant $\mathcal{A}_q$ à certaines représentations de N produits tensoriels, nous retrouvons les matrices de transfert de diverses chaines de spin ainsi que les relations-TT qu'elles satisfont, et aussi des symétries non triviales pour les hamiltoniens correspondants.

Mots clés: Algèbres quantiques affines, algèbre q-Onsager, chaînes de spins quantiques, équation de réflexion, opérateurs K, matrices R et K universelles, relations-TT universelles.

Abstract

The goal of this thesis is to develop a framework for obtaining spin-j K-operators with spectral parameter and which are solutions to the reflection equation. They belong to $B \otimes \mathrm{End}(\mathbb{C}^{2j+1})((u))$, where B is a comodule algebra over a Hopf algebra H. From these K-operators, we derive universal TT-relations satisfied by *universal tranfer matrices* in the commutative subalgebra of B they generate. The proposed new framework is specialized to $B = \mathcal{A}_q$, the central extension of the q-Onsager algebra, and $H = \mathcal{L}U_q sl_2$, the quantum loop algebra of sl_2, providing a universal approach to various quantum spin chains (spin-j, alternating spin-j_1, j_2, etc.) with generic open boundary conditions. The following three problems are studied and solved.

Firstly, we introduce an axiomatic definition of a universal K-matrix $\mathfrak{K} \in B \otimes H$ satisfying a universal reflection equation. It requires the data of a pair of twists consisting of a twist ψ (a certain automorphism of H) and a Drinfeld twist. This definition extends the work of Balagović-Kolb and Appel-Vlaar. For $H = \mathcal{L}U_q sl_2$ and a certain fixed pair of twists, we show that the evaluation of the tensor component in H of $\mathfrak{K}$ at a formal (infinite-dimensional) evaluation representation of spin-j leads to K-operators with spectral parameters $\mathbf{K}^{(j)}(u)$.

Secondly, drawing inspiration from the evaluation of one of the axioms of the universal K-matrix and considering the tensor products of formal evaluation representations of $\mathcal{L}U_q sl_2$ in the cases where a spin-j sub-representation appears, we introduce fused spin-j K-operators $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q \otimes \mathrm{End}(\mathbb{C}^{2j+1})((u))$. These operators are constructed using $\mathcal{L}U_q sl_2$-intertwining operators. They are a spin-j generalization of the spin-1/2 K-operator introduced by Baseilhac and Shigechi in 2009. We also show independently of this universal framework that all of these fused K-operators satisfy the reflection equations. Then, we conjecture a proportionality relation between $\mathcal{K}^{(j)}(u)$ and the evaluated universal K-matrix $\mathbf{K}^{(j)}(u)$. This conjecture is supported by showing that $\mathcal{K}^{(j)}(u)$ satisfies a set of relations similar to the evaluated axioms of $\mathfrak{K}$.

Thirdly, a generating function for spin-j (called universal transfer matrix) in the commutative subalgebra of $\mathcal{A}_q$ is obtained using the fused K-operators $\mathcal{K}^{(j)}(u)$ that we constructed. Provided that the conjecture relating $\mathbf{K}^{(j)}(u)$ and $\mathcal{K}^{(j)}(u)$ holds, we prove that these generating functions satisfy universal-TT relations which allow to reduce recursively the spectral problem of spin-j to the one for spin-1/2. By specializing $\mathcal{A}_q$ to certain N-tensor product representations, we recover the transfer matrices of various spin chains along with the TT-relations they satisfy, and also find non-trivial symmetries for the corresponding hamiltonians.

Keywords: Quantum affine algebras, q-Onsager algebra, quantum spin-chains, reflection equation, K-operators, universal R- and K-matrices, universal TT-relations.

Contents

Introduction

The scope of this thesis lies within the research field of quantum integrable systems. This field has received increasing interest for nearly eighty years in both physics and mathematics. Quantum integrable systems are a class of models in physics known for their exact solvability, meaning that one can obtain the relevant physical quantities of the model without approximation. Among these quantities are, for example, energy levels and system states, correlation functions that indicate the probability of a system transitioning from one state to another, the S-matrix used to describe particle scattering processes. These systems are described by a *Hamiltonian*, which is an operator acting on a Hilbert space. Obtaining the spectrum and eigenstates of the Hamiltonian is essential for characterizing the energy states of the physical system. Note that this is achieved using mathematical tools, some of them will be detailed in this manuscript.

The introduction is organized as follows. Firstly, we present the model of quantum spin chains considered in this thesis. Secondly, we review some of the mathematical approaches employed in the context of quantum integrable systems, with a particular emphasis on certain tools that play a central role in the approach considered here. Finally, we present the results obtained in this thesis.

Spin chains model

Among the integrable models in physics, we will focus on models of *quantum spin chains* with arbitrary spin values. The simplest case for spin-1/2 is the XXX model of Heisenberg introduced in 1928 in [H28]. Spins are quantities that appear in quantum mechanics and possess intrinsic magnetic moment. They are often represented by magnets that can have different orientations. For example, an electron has a spin-1/2 that can be directed upwards or downwards. Among the reference models, one can find, for instance, the chain with periodic boundary conditions, where at each site, an electron interacts with those situated on its sides. First, these interactions can be expressed using Pauli matrices:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{0.0.1}$$

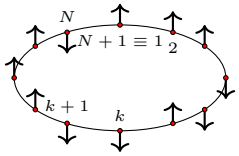
The most general chain of N spin-1/2 particles with nearest-neighbor interactions is given by the XYZ chain, which is described by the Hamiltonian

$$\mathcal{H}^{per}_{XYZ} = \sum_{k=1}^{N} J_x \sigma^x_k \sigma^x_{k+1} + J_y \sigma^y_k \sigma^y_{k+1} + J_z \sigma^z_k \sigma^z_{k+1} \ , \qquad \sigma^i_1 \equiv \sigma^i_{N+1} \ , \tag{0.0.2}$$

where we use the standard notation

$$\sigma^i_k = \underbrace{\mathbb{I}_2 \otimes \ldots \otimes \mathbb{I}_2}_{k-1} \otimes \sigma^i \otimes \underbrace{\mathbb{I}_2 \otimes \ldots \otimes \mathbb{I}_2}_{N-k} \ , \tag{0.0.3}$$

with $\mathbb{I}_2$ being the 2×2 identity matrix. In this context, J_x, J_y, and J_z are scalars referred to as anisotropy factors. They determine whether a direction in space is preferred during the interaction. For example, the XXX chain corresponds to the case where $J_x = J_y = J_z$, while the XXZ chain has $J_x = J_y \neq J_z$. Let us note that long-range interactions, i.e. interactions occurring between a spin at site k and those at sites $k+m$ with $m > 1$, are neglected here. Finally, the model (0.0.2) is subject to periodic boundary conditions, meaning that the spin at site 1 interacts with the one at site N. This is reflected in the second equality in (0.0.2), and it is referred to as a closed spin chain because it can be arranged in a circular fashion as follows:



Closed spin chain.

The Hamiltonian acts on the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$, which is formed as the tensor product of N spin-1/2 space representations, it is thus a matrix of size $(2^N \times 2^N)$. It is clear that, due to its size, obtaining the eigenvalues and eigenvectors becomes increasingly challenging as N increases. In particular, direct computation becomes infeasible when considering the thermodynamic limit with $N \to \infty$.

In order to solve spin chains, various methods have been developed, such as the Quantum Inverse Scattering Method (QISM) [STF79]. The latter allows for the establishment of an integrability criterion. This method relies on a R-matrix that acts on two sites. In the case of the XXZ chain, it encodes the interactions between the spins. More generally, it is a complex-valued operator in $\mathrm{End}(V^{(j_1)} \otimes V^{(j_2)})$, where $V^{(j_k)}$ are the spin spaces. It depends on parameters u_k, $u_\ell \in \mathbb{C}^*$, and is denoted as $R_{k\ell}(u_k, u_\ell)$. Finally, the R-matrix satisfies the Yang-Baxter equation [M64, BZ66, Ya67, Ba72], which belongs to the space $\mathrm{End}(V^{(j_1)} \otimes V^{(j_2)} \otimes V^{(j_3)})$

$$R_{12}(u_1, u_2) R_{13}(u_1, u_3) R_{23}(u_2, u_3) = R_{23}(u_2, u_3) R_{13}(u_1, u_3) R_{12}(u_1, u_2) \ . \tag{0.0.4}$$

Depending on the considered model, the R-matrices associated with the XYZ, XXZ, XXX spin chains are expressed in terms of elliptic, trigonometric, and rational functions, respectively.

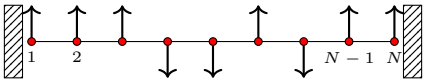
In the case of closed spin chains, starting from the R-matrix, a transfer matrix is constructed,

which generates a finite set of mutually commuting quantities. Among these, the Hamiltonian is included, implying that these quantities are conserved. This transfer matrix generates a sufficient number of conserved quantities, ensuring the integrability of the system. The diagonalization of the transfer matrix is notably performed by exploiting the underlying algebraic structure. In practice, this allows for the simultaneous diagonalization of all conserved quantities, not just the Hamiltonian. Details of this construction will be described later. Furthermore, these quantities are of prime importance in the formulation of the Generalized Gibbs Ensemble for the study of non-equilibrium systems [P13]. They are necessary for accurately describing the system at equilibrium after being perturbed by a 'quantum quench' [VR16].

In addition to closed spin chains, open chains are also considered. Instead of periodic boundary conditions, these open chains have diagonal, upper triangular, lower triangular, generic, and other types of boundary conditions. These conditions refer to the form of the Pauli matrices acting on sites 1 and N of the chain. For example, the Hamiltonian of the XYZ spin-1/2 chain with generic and integrable boundary conditions was derived in [IK94], and it is given by:

$$\begin{aligned}\mathcal{H}^{gen}_{XYZ} = &\sum_{k=1}^{N-1}\Big(J_x\sigma^x_k\sigma^x_{k+1} + J_y\sigma^y_k\sigma^y_{k+1} + J_z\sigma^z_k\sigma^z_{k+1}\Big) \\ &+ b_z\sigma^z_1 + b_+\sigma^+_1 + b_-\sigma^-_1 + \bar{b}_z\sigma^z_N + \bar{b}_+\sigma^+_N + \bar{b}_-\sigma^-_N \ ,\end{aligned} \tag{0.0.5}$$

where $b_\pm$, b_z, $\bar{b}_\pm$, $\bar{b}_z$ are scalars in $\mathbb{C}$. For instance, in relation to the various types of boundary conditions mentioned above, when $b_\pm = \bar{b}_\pm = 0$, we have diagonal boundary conditions, and when $b_- = \bar{b}_- = 0$, they are of the upper triangular type. The terms left and right boundaries are often used to refer to sites 1 and N, respectively. Finally, open spin chains can be represented as follows:



Open spin chain.

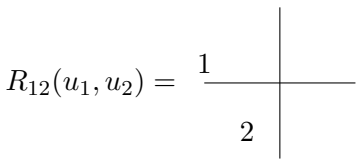
In this thesis, we will study the algebraic structures associated with XXZ spin chains of arbitrary spin j with generic boundary conditions. These are generalizations of the model (0.0.5) for $J_x = J_y \neq J_z$. Here, the boundary conditions are encoded through the K-matrix introduced by Sklyanin in [Sk88]. It is a complex-valued operator in $\mathrm{End}(V^{(j)})$. This K-matrix satisfies the reflection equation, which belongs to the space $\mathrm{End}(V^{(j_1)} \otimes V^{(j_2)})$

$$R_{12}(u_1,u_2)K_1(u_1)R_{21}(u_2,-u_1)K_2(u_2) = K_2(u_2)R_{12}(u_1,-u_2)K_1(u_1)R_{21}(-u_2,-u_1) \ . \tag{0.0.6}$$

The QISM mentioned earlier for closed spin chains has been extended to open spin chains in [Sk88]. Here as well, a transfer matrix generates the conserved quantities of the system. However, in this case, it is constructed using both the R and K-matrices, and the Hamiltonian is also derived. The specifics of this construction will be described later.

To conclude this section, we recall the physical interpretation of the Yang-Baxter and reflection equations, given respectively by (0.0.4) and (0.0.6), in the context of particle scattering. Consider

two particles labeled 1 and 2 that interact with each other. There are two states, referred to as incoming and outgoing, describing the system before and after the interaction. This interaction is encoded in a S-matrix, which is viewed as an R-matrix in our case, and thus satisfies the Yang-Baxter equation. In this context, the parameters u_1 and u_2 represent the rapidities[1] of the two particles. The R-matrix and the Yang-Baxter equation (0.0.4) are depicted graphically:



R-matrix representation.

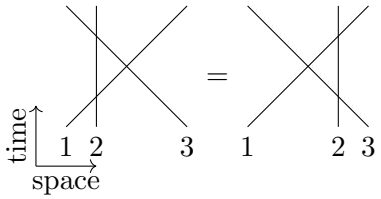
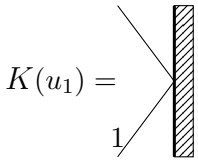


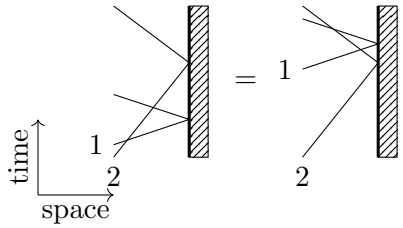
Illustration of the Yang-Baxter equation.

The left side of the second equality is interpreted as follows. First, the particle 1 interacts with the second, then the first particle interacts with the third, and finally, the second particle interacts with the third. The interpretation of the right side of the equation. Therefore, the Yang-Baxter equation implies that these two scattering processes are equivalent. According to this interpretation, an example of a solution to the Yang-Baxter equations was obtained in [ZZ79].

Then, for open spin chains, the K-matrix and the reflection equation (0.0.6) are represented as



K-matrix representation.



Reflection equation representation.

Consider the left-hand term of the reflection equation above. Two particles interact through an R-matrix, then the particle labeled as 1 bounces off the wall. It interacts again with the second particle, and then the second hits the wall and is reflected. Similarly, the right-hand term can be interpreted physically. Finally, the reflection equation indicates that these two processes are equivalent. According to this interpretation, an example of a solution to the reflection equation was obtained in [GZ93].

In the literature, two research directions in quantum integrable systems are commonly considered and involve the use of

- a representation-dependent approach;
- a universal approach.

The first approach consists in choosing a physically interesting model and using various methods to solve it. These methods include the algebraic Bethe Ansatz (ABA) or the Separation of Variables

[1] Each particle of mass m has a rapidity u that satisfies $m\,ch(u) = p_0$, $m\,sh(u) = p_1$, where p_i are components of the two-dimensional energy-momentum vector. They satisfy $p_0^2 - p_1^2 = m^2$, known as the *energy–momentum relation*.

(SoV), as described in [B31, Sk88] and [Ba72, Sk92, Sk95, KMNT16], respectively. Each model is associated with an R-matrix and a K-matrix, and this approach typically requires solving each case individually, considering different spin values and boundary conditions.

The second approach aims to establish a more general resolution method that can handle many spin chains simultaneously, such as XXZ chains with arbitrary spins on each site and various boundary conditions. The unification of these chains is achieved through the use of a non-abelian associative algebra of infinite dimension, namely the *alternating* central extension of the q-Onsager algebra denoted as $\mathcal{A}_q$, as described in [BS09, T21a]. A generating function that we call *universal transfer matrix* can be constructed in a commutative subalgebra of $\mathcal{A}_q$. By choosing certain finite-dimensional representations of tensor product type, this generating function specializes to the transfer matrices of these chains. The idea is to focus on this more fundamental structure to solve these different systems without the need to treat them one by one. This will be precisely studied in this thesis.

Representation-dependent approach

Now that we have reviewed what open and closed spin-1/2 chain models are, we can pose two questions:

(i) How to obtain open or closed spin chains with arbitrary value of spin-j ?

(ii) How to solve these systems ?

To address (i), as for the spin-1/2 case, the basic elements for constructing an arbitrary spin-j chain are the R and K-matrices as well. As will be explained in Section 1.2, from these matrices, a dressed K-matrix (also known as a double row monodromy matrix) is defined, which takes the form [Sk88]

$$R_{a1}(u)\cdots R_{aN}(u)K_a(u)R_{aN}(u)\cdots R_{a1}(u)\ , \tag{0.0.7}$$

where a indicates an auxiliary space used in the construction of the spin chain. The transfer matrix is constructed by taking the trace over the auxiliary space of the product of (0.0.7) with the dual K-matrix also acting on the auxiliary space. Recall that the Hamiltonian is extracted from this transfer matrix. Therefore, it is sufficient to find spin-j solutions to the Yang-Baxter and reflection equations. To do so, one way is to directly solve these equations, but this is not an easy task. This way, the spin-1 R and K-matrices for the XXZ chain were obtained in [ZF80] and [IOZ96], respectively. However, for higher spins, a direct resolution becomes significantly more challenging and requires an indirect approach.

Solutions to the Yang-Baxter and reflection equations can be also obtained through a fusion procedure known since the late 1970s. The spin-j R and K-matrices are constructed from spin-1/2 solutions of these equations. This procedure was developed for the R-matrix in [Ka79, KRS81, KR87], and it was later extended to the K-matrices in [MN92]. These fused matrices are obtained recursively by considering tensor products of fundamental representations and then projecting onto a sub-representation of $SU(2)$ corresponding to a higher spin. For example, the spin-1 XXZ chain with periodic boundary conditions was constructed in [ZF80] through a fusion procedure, and the one with generic boundary conditions was obtained in [IOZ96]. It was verified that the solutions obtained by direct resolution of the Yang-Baxter and reflection equations for spin-1 coincide with

those obtained through fusion. More recently, compact forms of the spin-j R and K-matrices in terms of spin-1/2 solutions were proposed in [FNR07].

To address (ii), we review some methods used in solving spin chains, namely, the Algebraic Bethe Ansatz (ABA) and the Separation of Variables (SoV) method, as discussed in [B31, Sk88] and [Ba72, Sk92, Sk95, KMNT16].

Firstly, the Bethe Ansatz was introduced by Bethe in [B31] to solve the spin-1/2 XXX chain. Eigenstates are constructed in terms of quasi-particles whose rapidities satisfy the *Bethe equations* (highly transcendental equations). The ABA was later developed for closed chains in [STF79] and open chains in [Sk88]. This method relies on the existence of a reference state from which all other eigenstates are built. However, it is worth noting that these methods do not apply to all integrable models, as we will see below.

The spectrum and eigenvectors of the transfer matrix obtained using the ABA are expressed in terms of solutions to the Bethe equations. These equations can be solved at the free fermionic point $q = i$ (corresponding to the XX chain), but in general, they are only solved numerically. The SoV method is also used for spin chains, but its application is not straightforward. The starting point for the SoV method is to consider a transfer matrix with inhomogeneities. In general, this matrix does not have an interpretation in terms of a local spin chain. However, its analysis is carried out using the T-Q equations, whose solution leads to the spectrum and eigenvectors of this transfer matrix. It is worth noting that, once again, these results are expressed in terms of solutions to the Bethe equations. Then, by taking the homogeneous limit, i.e. setting the inhomogeneities to the same value, the transfer matrix with inhomogeneities becomes the usual transfer matrix. Thus, in this limit, the results obtained apply to spin chains.

However, these methods have limitations, which we briefly outline. The ABA does not work systematically because its application requires the existence of a reference state (or pseudo-vacuum), and this state may not exist in the case of open chains. For example, the reference state is unique and corresponds to the highest weight state for the spin-1/2 XXZ chain with diagonal boundary conditions, and its resolution was accomplished in [Sk88]. However, for the chain with non-diagonal boundary conditions, the highest weight state (when all spins are oriented upwards) is no longer a reference state in the ABA sense. In other words, it is not annihilated by the upper triangular components of the monodromy matrix.

To circumvent this issue, a gauge transformation is used to construct a new state that satisfies properties analogous to the reference state with respect to the new elements of the double monodromy matrix [CLSW02]. For certain constraints on the previously identified boundary conditions in [N02], the diagonalization procedure of the transfer matrix using the ABA is then very similar to the case with diagonal boundary conditions. However, for generic boundary conditions, the ABA diagonalization procedure does not work directly. It requires a modified version of the ABA (MABA), which was introduced in [BC13, BP14]. It should be noted, though, that the MABA procedure comes with several challenges. For instance, it requires the introduction of so-called 'inhomogeneous' Bethe equations, which are still not fully understood, and their thermodynamic limit remains an open problem [BC13, BP14].

Furthermore, the fusion procedure mentioned in (i) has enabled the construction of the spin-j transfer matrix, forming a hierarchy of mutually commuting generating functions. It satisfies a recurrence relation between the transfer matrices of lower-spin chains, known as *TT-relations*,

which have been conjectured but verified only for a few spin-j values [MN92, Zh95, FNR07]. These relations are different for each spin chain (due to different boundary conditions and spin values) because each model is associated with R and K-matrices from which the transfer matrices are constructed. This motivates the study of a universal approach aiming to obtain an algebraic version of the TT-relations, which, after specialization to certain finite-dimensional representations, would reduce to the TT-relations of various spin chains.

Universal approach

Now, let us turn our attention to the second research direction, which involves the study of the underlying algebraic structures of quantum integrable systems. We start by recalling some important mathematical contributions that have applications in these systems, starting with the concept of quantum groups that emerged in the 1980s. Then, the universal R-matrix associated with a quantum group is also reviewed. Just as universal R-matrices are associated with quantum groups, it is expected that universal K-matrices are also associated with certain comodule algebras over quantum groups. Some recent advancements in universal K-matrices are also discussed.

Universal R-matrix

The characteristic feature of a quantum group is that it is a deformation of a classical object (a Lie algebra or an algebra of functions on a Lie group) with a certain parameter q, and its specialization to $q = 1$ leads to the same classical object. The first example that appeared in the literature is the algebra $U_q sl_2$, which is a deformation of the universal enveloping algebra of sl_2. It was obtained by Kulish and Reshetikhin in [KR83] by studying the quantum inverse scattering method. The algebra $U_q sl_2$ appears, for example, as a hidden symmetry of the Hamiltonian of the spin-1/2 XXZ chain with specific boundary conditions [PS90]. Jimbo and Drinfeld independently formalized the concept of quantum groups [J85, Dr86]. More generally, from Lie algebras $\mathfrak{g}$ and their affine extensions $\hat{\mathfrak{g}}$, quantum groups are constructed, namely $U_q(\mathfrak{g})$ and $U_q(\hat{\mathfrak{g}})$, which correspond to deformations of the enveloping algebra associated with $\mathfrak{g}$ and $\hat{\mathfrak{g}}$. Specifically, a quantum group is a Hopf algebra (a bialgebra with an antipode) that is neither commutative nor co-commutative. It is associated with a coproduct $\Delta\colon H \to H \otimes H$, from which an opposite coproduct $\Delta^{op} \equiv \mathfrak{p} \circ \Delta$ is defined, where $\mathfrak{p}$ is the permutation operator satisfying $\mathfrak{p}(a \otimes b) = b \otimes a$, for all $a, b \in H$. The non co-commutativity of H is reflected in the fact that Δ is not symmetric, i.e. $\Delta \neq \Delta^{op}$.

Historically, the Yang-Baxter equation with a spectral parameter (0.0.4) was introduced in [M64, BZ66, Ya67, Ba72]. In the absence of a spectral parameter in the Yang-Baxter equation, a presentation of $H = U_q(\mathfrak{g})$ was studied in [FRT87]. The corresponding R-matrices are derived from more general objects that we now recall. A particular case of a Hopf algebra, known as a quasi-triangular Hopf algebra, involves a universal R-matrix denoted as $\mathfrak{R}$, which is an element in (the completion of) $H \otimes H$ satisfying certain axioms [Dr86]:

$$\begin{aligned} \mathfrak{R}\Delta(x) &= \Delta^{op}(x)\mathfrak{R} \ , \qquad \text{for all } x \in H \ , \\ (\Delta \otimes \mathrm{id})(\mathfrak{R}) &= \mathfrak{R}_{13}\mathfrak{R}_{23} \ , \\ (\mathrm{id} \otimes \Delta)(\mathfrak{R}) &= \mathfrak{R}_{13}\mathfrak{R}_{12} \ . \end{aligned}$$

The universal R-matrix satisfies an algebraic (non-matrix) version of the Yang-Baxter equation,

which belongs to $H \otimes H \otimes H$:

$$\mathfrak{R}_{12}\mathfrak{R}_{13}\mathfrak{R}_{23} = \mathfrak{R}_{23}\mathfrak{R}_{13}\mathfrak{R}_{12} \ .$$

Since then, explicit examples of universal R-matrices have been found. In [Dr86] there was introduced a procedure allowing to construct a quasi-triangular Hopf algebra out of any Hopf algebra, which is now known as the quantum double construction. In particular, an expression for the universal R-matrix associated with $H = U_q(\mathfrak{g})$ has been given using this construction, see e.g. [CP94, Sec. 8.3]. For the quantum affine algebra $H = U_q(\hat{\mathfrak{g}})$, a factorized expression of the universal R-matrix has been found in [KT92a, Da98]. It is expressed as an infinite product of q-exponentials involving the root vectors of $H = U_q(\hat{\mathfrak{g}})$. In this case, $\mathfrak{R}$ is an element belonging to the completion of the tensor product of two copies of $U_q\widehat{sl}_2$.

For $H = U_q(\mathfrak{g})$, evaluating the second component of $\mathfrak{R}$ to a finite-dimensional representation leads to a L-operator without a spectral parameter, denoted as $\mathcal{L} \in H \otimes \mathrm{End}(V^{(j)})$. This L-operator satisfies an RLL equation belonging to $H \otimes \mathrm{End}(V^{(j_1)}) \otimes \mathrm{End}(V^{(j_2)})$ [FRT87]:

$$R\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2\mathcal{L}_1R \ . \tag{0.0.8}$$

Evaluating these L-operators lead to R-matrices whose entries are scalars. For example, the evaluation of $\mathfrak{R}$ for $H = U_q sl_2$ is carried out in [CP95, Sec. 6.4]. Furthermore, the evaluation of the second axiom of $\mathfrak{R}$ gives the action of the coproduct on this operator, which has the form[2]

$$(\Delta \otimes \mathrm{id})(\mathcal{L}) = \big(\mathcal{L}\big)_{[1]}\big(\mathcal{L}\big)_{[2]} \ , \tag{0.0.9}$$

that belongs to $H \otimes H \otimes \mathrm{End}(V^{(j)})$.

In this thesis, we consider two types of representations for $H = U_q\widehat{sl}_2$ with zero central charge[3]. The first is a finite-dimensional evaluation representation indexed by a nonzero complex number called the evaluation parameter, see Section 3.2.1. The second is an analogous version, but of infinite dimension, and in this case, u is a formal parameter. In the literature, this representation is known as *quantum loop modules* [CG03], see Section 3.2.2. In the following, we use this latter representation, which we call the formal evaluation representation. It should be noted that the operators we evaluate through this representation will either be Laurent polynomials or power series in u with coefficients belonging to B or H. For simplicity, in the introduction we omit the notation $H((u))$ and write just H instead, etc.

For $H = U_q\widehat{sl}_2$, the specialization of the second tensor component of the universal R-matrix on the formal evaluation representations leads to a L-operator with a spectral parameter that satisfies the following equation [RS90, FR92]:

$$R_{12}(u_1, u_2)\mathcal{L}_1(u_1)\mathcal{L}_2(u_2) = \mathcal{L}_2(u_2)\mathcal{L}_1(u_1)R_{12}(u_1, u_2) \ , \tag{0.0.10}$$

and the action of the coproduct of H on the L-operator is given by

$$(\Delta \otimes \mathrm{id})(\mathcal{L}(u)) = \big(\mathcal{L}(u)\big)_{[1]}\big(\mathcal{L}(u)\big)_{[2]} \ . \tag{0.0.11}$$

The evaluation of $\mathfrak{R}$ requires to treat u as a formal parameter to avoid convergence problems. This

[2] Here, the indices [1] and [2] are associated with the first and second tensor components, respectively.

[3] The central charge c is defined by $K_0K_1 = q^c$, where we set $c = 0$.

was suggested in [Dr86, Sec. 13] and later studied for $H = U_q(\hat{\mathfrak{g}})$ in [FR92, Sec. 4], see [He17, Sec. 1] for a review. The specialization of the remaining first tensor component of $\mathcal{L}(u)$ directly leads to R-matrix solutions of the Yang-Baxter equation (0.0.4). Thus, we have

$$\begin{array}{l}\text{L-operators}\\ \in H \otimes \mathrm{End}(V^{(j)})\end{array} \xrightarrow{\text{rep. of } H} \begin{array}{l}\text{R-matrices}\\ \in \mathrm{End}(V^{(j_1)}) \otimes \mathrm{End}(V^{(j)})\end{array}$$

Furthermore, the coproduct provides the action of H on the tensor product of its representations, and subsequently on the spin chain by applying the coproduct successively. If we specialize the copies of H in (0.0.11) to the evaluation representations, we obtain a product of R-matrices from which the transfer matrix is constructed. For $H = \mathcal{L}U_q sl_2$, using the factorized form of [KT92a], a detailed evaluation of $\mathfrak{R}$ for spin-1/2 is provided in [BGKNR12]. In this case, the simplest specialization, i.e. taking a two-dimensional representation for each component of the tensor product, leads to the R-matrix associated with the spin-1/2 XXZ chain (up to a scalar function that is meromorphic in the spectral parameter).

Universal K-matrix

In the following, we will need an algebraic structure associated with the Hopf algebras H, called a comodule algebra B. It is equipped with a coassociative coaction, which is an algebra homomorphism $\delta\colon B \to B \otimes H$ and is compatible with H in the sense that it satisfies certain relations involving δ as well as the coproduct and counit. If in addition B is a subalgebra of H, and δ agrees with the restriction to B of the coproduct of H denoted Δ_B, then it is referred to as a coideal subalgebra. Special cases, known as quantum symmetric pairs (H, B), were introduced in [Le99, Ko12]. For example, there exists an injective algebra homomorphism from the q-Onsager algebra O_q to $U_q\widehat{sl}_2$, allowing us to treat O_q as a coideal subalgebra of $H = U_q\widehat{sl}_2$ [BB09, BB12]. This coideal is generated by $\mathcal{W}_0$ and $\mathcal{W}_1$ that satisfy the q-Dolan/Grady relations [T99, B04]. In this thesis, a central example of a comodule algebra over $H = U_q\widehat{sl}_2$ is the alternating central extension of the q-Onsager algebra denoted by $\mathcal{A}_q$ [BS09, T21a]. Actually $\mathcal{A}_q$ is the central extension of O_q with an infinite number of algebraically independent central elements. To the best of our knowledge, the algebra $\mathcal{A}_q$ cannot be realized as a coideal subalgebra of $H = U_q\widehat{sl}_2$.

Historically, the reflection equation with a spectral parameter (0.0.6) was first introduced in [Sk88], and the version without a spectral parameter has been studied, for example, in [KSS92]. Analogously to the universal R-matrix leading to the L-operators and R-matrices, one may wonder if there exists a universal K-matrix associated with H-comodule algebras, in particular with coideal subalgebras in H, which would lead to K-operators and K-matrices. K-operators are analogues of L-operators, and we will explain below what they are precisely and why they are of interest.

The concept of a universal K-matrix is not new; it has been studied in [CG92, KSS92, tDHO98, DKM02]. More recently, significant progress has been made in the works of [BW13, BKo15, Ko17, AV20]. Each of the definitions of universal K-matrices introduced in these recent references has different axioms that were chosen to serve different purposes. We can highlight three directions corresponding to where a universal K-matrix belongs to:

(i) $H = U_q(\mathfrak{g})$, in [BKo15];

(ii) $B \otimes H$, where B is a coideal subalgebra of $H = U_q(\mathfrak{g})$, in [Ko17];

(iii) $H = U_q(\hat{\mathfrak{g}})$, with the data of a certain automorphism of H, called 'twist', in [AV20].

Let us clarify that (i) and (iii) imply a choice of B as a coideal subalgebra over H to form a quantum symmetric pair (H, B), and the universal K-matrix satisfies intertwining relations (twisted in (iii)) with respect to B (and not H as the R-matrix).

Let us start with direction (ii). Kolb introduced a universal K-matrix $\mathscr{K} \in B \otimes H$ that satisfies certain axioms [Ko17, Def. 2.7]:

$$\begin{aligned} \mathscr{K}\,\Delta_B(b) &= \Delta_B(b)\mathscr{K}\ , \qquad \text{for all } b \in B, \\ (\Delta_B \otimes \mathrm{id})(\mathscr{K}) &= \mathfrak{R}_{32}\mathscr{K}_{13}\mathfrak{R}_{23}\ , \\ (\mathrm{id} \otimes \Delta)(\mathscr{K}) &= \mathfrak{R}_{32}\mathscr{K}_{13}\mathfrak{R}_{23}\mathscr{K}_{12}\ . \end{aligned} \tag{0.0.12}$$

It satisfies an algebraic (non-matrix) version of the reflection equation that belongs to $B \otimes H \otimes H$

$$\mathfrak{R}_{32}\mathscr{K}_{13}\mathfrak{R}_{23}\mathscr{K}_{12} = \mathscr{K}_{12}\mathfrak{R}_{32}\mathscr{K}_{13}\mathfrak{R}_{23}\ .$$

The specialization of its second tensor component in H, to a finite-dimensional representation leads to K-operators that do not depend on a spectral parameter, denoted as $\mathcal{K} \in B \otimes \mathrm{End}(V^{(j)})$, satisfying a reflection equation of the form

$$R_{12}\mathcal{K}_1 R_{21}\mathcal{K}_2 = \mathcal{K}_2 R_{12}\mathcal{K}_1 R_{21}\ , \tag{0.0.13}$$

which belongs to $B \otimes \mathrm{End}(V^{(j_1)}) \otimes \mathrm{End}(V^{(j_2)})$. Moreover, the evaluation of the second relation in eq. (0.0.12) takes the form

$$(\Delta_B \otimes \mathrm{id})(\mathcal{K}) = \big(\mathcal{L}\big)_{[2]}\big(\mathcal{K}\big)_{[1]}\big(\mathcal{L}\big)_{[2]}\ . \tag{0.0.14}$$

Next, let us clarify that direction (i) is a special case of (ii). Indeed, applying the counit to the first component of $\mathscr{K}$, we obtain an object $k \in H$ that was introduced by Balagović and Kolb in [BKo15]. Its specialization to a finite-dimensional representation leads to K-matrices (with scalar entries).

However, axiomatics of directions (i) and (ii) is not suitable for the reflection equation with spectral parameter (0.0.6). To change this, Appel and Vlaar proposed in [AV20] a twisted version of the approach (i) in [BKo15], this is direction (iii). Their definition of universal K-matrix is associated with a coideal subalgebra of $H = U_q(\hat{\mathfrak{g}})$ and includes a pair of consistent twists. It was shown in [AV20] that such a universal K-matrix exists for a large class of coideals known as quantum symmetric pairs. However, no explicit expression of this universal K-matrix is known and the existence result requires a specific form of twists. Furthermore, it was shown in [AV22] that these universal K-matrices are well-defined on finite-dimensional evaluation representations of $H = U_q(\hat{\mathfrak{g}})$ and provide K-matrix solutions of (0.0.6).

Note that both works of [BKo15, AV20] are significantly inspired by quasi K-matrix construction [BW13] for certain examples of quantum symmetric pairs. Moreover, the axiomatics of [BKo15] was also inspired by [tDHO98]. Besides, universal automorphisms have been constructed in [KY19] that allow to construct universal solutions of generalized reflection equations as defined by [C92].

In this thesis, the algebra of primary interest for us is $B = \mathcal{A}_q$, the alternating central extension of the q-Onsager algebra, because of its applications to open spin-chains through K-operators with spectral parameters as it will be discussed below. It is important to note that we do not have any explicit form of a K-operator for O_q, while it is known for the comodule algebra $B = \mathcal{A}_q$ [BS09]. To

the best of our knowledge, $\mathcal{A}_q$ is not a coideal subalgebra of $H = U_q\widehat{sl}_2$ or any other quasi-triangular Hopf algebra, and so we cannot use axiomatics and results of [AV20]. These axiomatics need slight adjustment in order to treat comodule algebras and they will be given in the present text.

K-operators with spectral parameter

Now, let us get back to our initial goal of unifying various spin chains through a universal approach. The key objects to achieve this are the K-operators with spectral parameter. They belong to $B \otimes \mathrm{End}(V^{(j)})$, where B is an algebra, and they satisfy a reflection equation in $B \otimes \mathrm{End}(V^{(j_1)}) \otimes \mathrm{End}(V^{(j_2)})$

$$R_{12}(u_1,u_2)\mathcal{K}_1(u_1)R_{21}(u_2,-u_1)\mathcal{K}_2(u_2) = \mathcal{K}_2(u_2)R_{12}(u_1,-u_2)\mathcal{K}_1(u_1)R_{21}(-u_2,-u_1)\ , \qquad (0.0.15)$$

where $R(u_1,u_2)$ is a solution of the Yang-Baxter equation (0.0.4). In this case, we call $\mathcal{K}(u)$ a K-operator of spin-j. In other words, unlike K-matrices, which have scalar entries, K-operators are matrices with entries belonging to B. In the following, B will be typically a comodule algebra such as the alternating central extension of the q-Onsager algebra $\mathcal{A}_q$. These K-operators are generalizations of K-matrices because, when taking one-dimensional representations of B, they lead to K-matrices. Indeed, we classified all one-dimensional representations of $\mathcal{A}_q$ in Section 3.5.3, and found that the image of the spin-1/2 K-operator is proportional to the most general K-matrix of the open XXZ spin-1/2 chain [dVR94, GZ93]. Let us now explain why these K-operators are important for the universal approach.

Recall that the Hamiltonian of a quantum spin-chain is obtained through a transfer matrix constructed using the dressed K-matrices of the form (0.0.7). As we now recall, these dressed K-matrices can be obtained by applying the coaction to the entries of the K-operator and taking certain representations of B and H. We first request that the coaction of B satisfies[4]

$$(\delta \otimes \mathrm{id})(\mathcal{K}(u)) = \big(\mathcal{L}(u)\big)_{[2]}\big(\mathcal{K}(u)\big)_{[1]}\big(\mathcal{L}(u)\big)_{[2]}\ . \qquad (0.0.16)$$

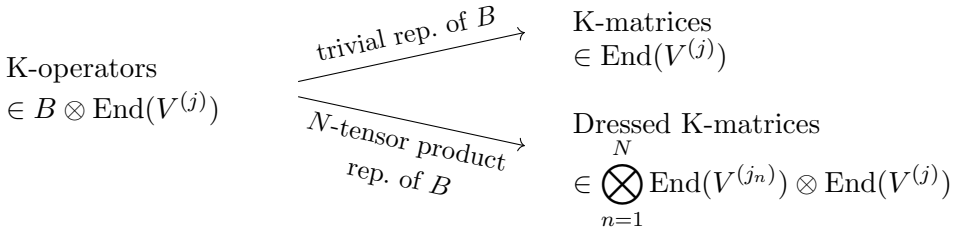
Due to its relation with open spin-chains and because we are mostly interested in such physical models, we call the above coaction the *physical coaction*. Then, by successively iterating δ on (0.0.16), we find that the resulting matrix belongs to

$$B \otimes H \otimes \ldots \otimes H \otimes \mathrm{End}(\mathbb{C}^{2j+1})\ .$$

To show the relation between K-operators and dressed K-matrices, we take a one-dimensional representation of B (corresponding to the boundary conditions at one end of the spin-chain) and a spin-j_k representation for each copy of H (corresponding to the bulk interaction in the spin-chain). The resulting image is indeed (0.0.7), as it will be shown in Section 4.4.1. In other words, the coaction (0.0.16) allows to define an action of B on the spin chain in such a way that it is also compatible with the form of dressed K-matrices and therefore with the structure of transfer matrices. The K-operators can thus be seen as a generalization of the dressed K-matrices given by (0.0.7). This was observed in [BK05a, BK05b], while studying the XXZ spin-1/2 chain with generic boundary conditions, that the dressed K-matrix is expressed as a spin-1/2 K-operator with

[4]Here, the indices [1] and [2] are respectively associated with the space of B and H.

its entries specialized to spin-chains representations. In summary, the K-operators specialize as follows:



It is also worth mentioning that a presentation of $H = U_q(\mathfrak{g})$ and $H = U_q(\hat{\mathfrak{g}})$ has been proposed in [FRT87, RS90] through the L-operators satisfying (0.0.8) and (0.0.10). Similarly, K-operators associated with a comodule algebra B and satisfying a spectral parameter version of (0.0.13) also provide an FRT type presentation of B. For instance, a spin-1/2 K-operator associated with $\mathcal{A}_q$ has been introduced in [BS09] and gives its FRT type presentation.

We now give some examples of applications of these K-operators.

First of all, K-operators play a central role in the q-Onsager approach used to study quantum spin chains. This approach draws inspiration from the method used by Onsager in [O44] (and later by Davies in [D90]) to solve the two-dimensional Ising model with zero magnetic field and periodic boundary conditions. This is a classical statistical mechanics model that corresponds to the one-dimensional quantum Ising spin chain. In brief, the q-Onsager approach relies on the use of a transfer matrix constructed from the K-operator of $\mathcal{A}_q$, combined with the representation theory of $\mathcal{A}_q$. The algebraic structure of the transfer matrix for spin-1/2 was studied in [BK07], and it was expressed in terms of the commutative subalgebra of $\mathcal{A}_q$. This highlights the commutative property of the transfer matrix. It should be noted that the specialization to tensor product representations of the generating function of this commutative subalgebra in $\mathcal{A}_q$ is proportional to the transfer matrix of the XXZ spin-1/2 chain with generic boundary conditions [BK07]. This has been particularly useful for studying the spectral problem of this spin chain within the framework of q-Onsager representation theory.

Furthermore, $\mathcal{A}_q$ is strongly connected to its degenerate versions such as the augmented q-Onsager algebra [IT09, BB12], triangular q-Onsager and $U_q sl_2$-invariant q-Onsager algebras [BB16, Ts19], corresponding to different limits of boundary conditions (diagonal, triangular, etc.). Moreover, $\mathcal{A}_q$ encompasses a broad class of integrable models through various quotients. In fact, all known examples of open XXZ spin chains of size N with integrable boundary conditions (special, diagonal, triangular, general) are associated with $\mathcal{A}_q$ or its degenerate versions.

Finally, it is worth noting that in the thermodynamic limit, the corresponding image of $\mathcal{A}_q$ (or its degenerate versions) becomes a non-abelian symmetry of the spin chain.

In this thesis, our goal is to generalize the results discussed above, namely: the study of spin chains with arbitrary spin-j (not just spin-1/2), using an algebraic framework based on $\mathcal{A}_q$ and its various quotients, as well as the representation theory of $U_q\widehat{sl}_2$. The key objects to achieve this are the spin-j K-operators associated with $\mathcal{A}_q$. We aim to construct them and provide an interpretation in terms of a universal K-matrix. Indeed, we expect to define a universal K-matrix using axioms in such a way that its specialization yields K-operators, and that the specialization of one of its axioms provides an appropriate form for the coaction as in (0.0.16). The goal is to construct a

general framework that allows the development of universal TT-relations belonging to $\mathcal{A}_q$*, which would then specialize to the conjectured TT-relations for transfer matrices in the literature such as* [FNR07, CYSW14].

The figure below schematically presents the different approaches and associated tools that have been discussed in the introduction.

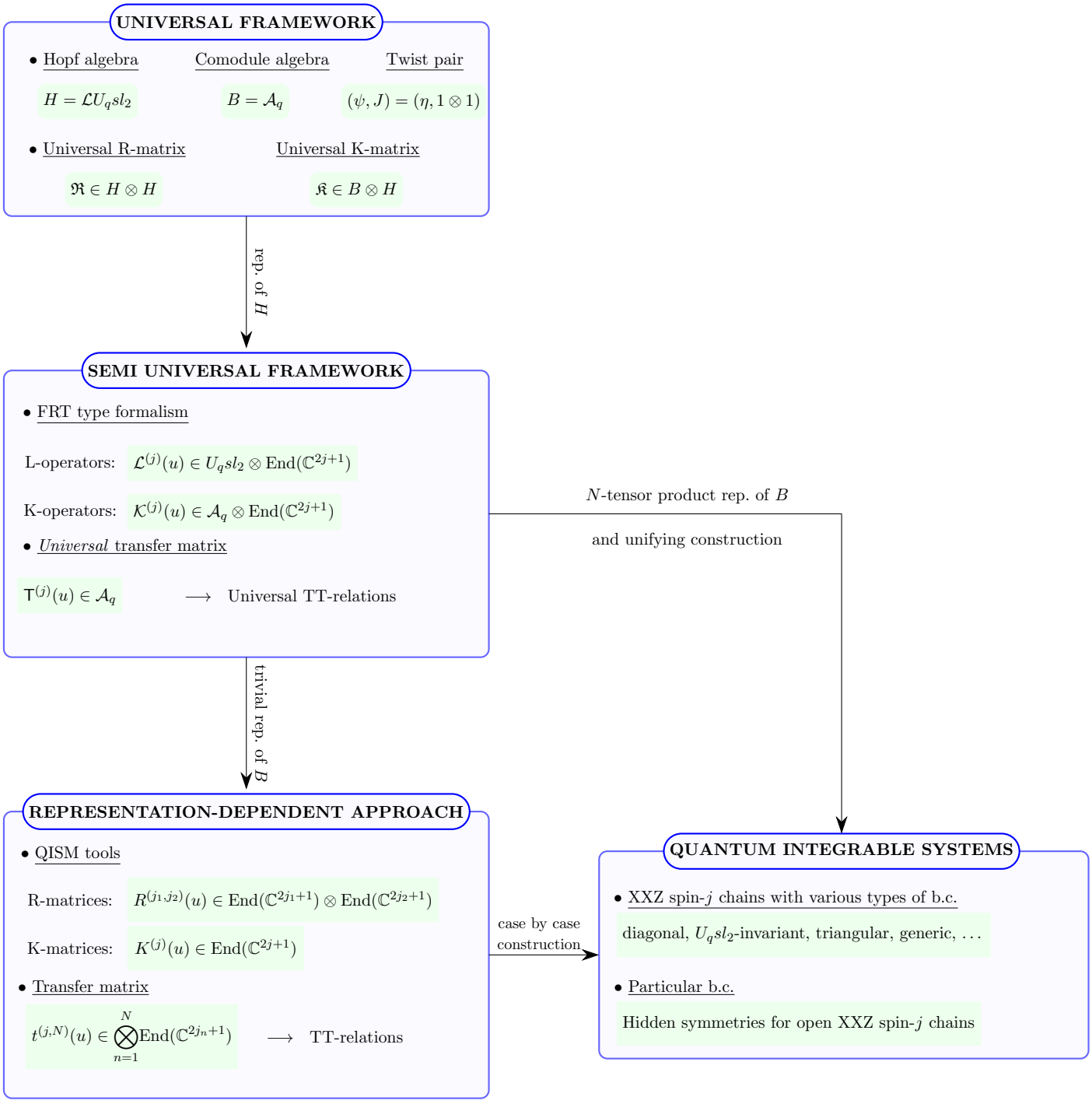


Figure 3: Schematic view of the two different approaches to study quantum spin-chains.

The following three problems are studied and solved in this thesis:

- Problem 1:

  The different universal K-matrices, as defined in [BKo15, Ko17, AV20, AV22], do not allow for the treatment of K-operators that are solutions of the reflection equation with a spectral parameter (0.0.15) and with a 'physical' coaction, that is of the form (0.0.16).

- Problem 2:

  The spin-1/2 K-operator for $\mathcal{A}_q$ was introduced in [BS09], but its spin-j version has never been constructed.

- Problem 3:

  One can construct a generating function for spin-1/2 in the commutative subalgebra of $\mathcal{A}_q$ from its spin-1/2 K-operator (2.1.5). This object has indeed been studied for the spin-1/2 XXZ chain in [BK05a, BK05b], but for a quotient of $\mathcal{A}_q$. Moreover, a generating function for spin-j in a commutative subalgebra of $\mathcal{A}_q$ has never been constructed, and its connection to spin chains remains to be investigated. And finally, a universal version of TT-relations was never studied.

Results

The following points summarize the main results of this thesis and address the three problems stated above.

- Result 1:

  In Definition 3.1.7, we introduce an axiomatic definition of a universal K-matrix $\mathfrak{K} \in B \otimes H$ where B is a comodule algebra over a Hopf algebra H, and we demonstrate that it satisfies a universal reflection equation belonging to $B \otimes H \otimes H$

  $$\mathfrak{K}_{12}(\mathfrak{R}^{\psi})_{32}\mathfrak{K}_{13}\mathfrak{R}_{23} = \mathfrak{R}^{\psi\psi}_{32}\mathfrak{K}_{13}\mathfrak{R}^{\psi}_{23}\mathfrak{K}_{12} \ .$$

  This universal K-matrix also requires the specification of a pair of twists composed of a twist ψ (a certain automorphism of H) and a Drinfeld twist $J \in H \otimes H$. Furthermore, these axioms correspond to a twisted version of (0.0.12), and we recover as special cases those given in directions (i), (ii), and (iii), see Section 3.1.3. For example, when we impose that B is a coideal subalgebra of H and apply the counit to the first tensor component of $\mathfrak{K}$, our axioms correspond to those of direction (iii) from [AV20]. This new definition of a universal K-matrix addresses Problem 1 for the following reasons.

  First, we consider $H = \mathcal{L}U_q sl_2$, a pair of twists $(\psi, J) = (\eta, 1 \otimes 1)$, where η is defined in (3.1.12), and an arbitrary comodule algebra B, assuming that $\mathfrak{K}$ exists for this pair of twists. Then, in Section 3.2.5, we define a spin-j K-operator belonging to $B((u^{-1})) \otimes \mathrm{End}(V^{(j)})$.

  $$\mathbf{K}^{(j)}(u) = (\mathrm{id} \otimes \pi^{j}_{u^{-1}})(\mathfrak{K}) \ ,$$

Here, $\pi_u^j\colon H \to \mathrm{End}(V^{(j)})$ is the formal evaluation representation of H. It is an analogue of the finite-dimensional evaluation representation of H, but it is not of highest weight type and has infinite dimension. These representations of quantum affine algebras are known as 'quantum loop modules' [CG03]. It is important that the twist satisfies a certain relation with the evaluation representation of H (which is the case for η), namely, the action of the twist has the effect of inverting the spectral parameter. In this case, $\mathbf{K}^{(j)}(u)$ satisfies the spectral parameter reflection equation (0.0.15). Moreover, evaluating one of our axioms gives the coaction of the K-operator, and it takes the desired form (0.0.16). Later, we will focus on the case of $B = \mathcal{A}_q$.

- Result 2:

  Drawing inspiration from the evaluation of one of the axioms of the universal K-matrix, we define a spin-j K-operator denoted as $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q \otimes \mathrm{End}(\mathbb{C}^{2j+1})$ through a fusion procedure without making any assumptions about the existence of $\mathfrak{K}$. The latter is constructed recursively based on the spin-1/2 K-operator discussed earlier.

$$\mathcal{K}^{(j)}(u) = \mathcal{F}^{(j)}_{\langle 12\rangle}\mathcal{K}_1^{(\frac{1}{2})}(uq^{-j+\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{-j+1})\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{\frac{1}{2}})\mathcal{E}^{(j)}_{\langle 12\rangle} \ ,$$

  and we show in Theorem 3.5.2 that it satisfies (0.0.15). This addresses Problem 2. More precisely, the formula above notably involves

$$\mathcal{E}^{(j)}\colon \mathbb{C}^{2j+1}_u \to \mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j}_{u_2} \ , \quad u_1 = uq^{-j+\frac{1}{2}}, \quad u_2 = uq^{\frac{1}{2}} \ ,$$

  that intertwines the action of $\mathcal{L}U_q sl_2$ on the evaluation representations and its pseudo-inverse.

$$\mathcal{F}^{(j)}\colon \mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j}_{u_2} \to \mathbb{C}^{2j+1}_u \ .$$

  The conditions imposed on u_1 and u_2 ensure that the tensor product of the evaluation representations of $\mathcal{L}U_q sl_2$ admits a spin-j sub-representation. This reducibility condition is expressed in terms of the ratio of the evaluation parameters of these tensor products [CP91, Sect. 4.9]. Using this criterion, we explicitly construct $\mathcal{E}^{(j)}$ and $\mathcal{F}^{(j)}$ by determining their matrix expressions, as described in Appendix C.1. In Section 3.3.2, we analyze the tensor products of the formal evaluation representations of infinite dimension and obtain the intertwining operator $\mathcal{E}^{(j)}$ and $\mathcal{F}^{(j)}$ in this formal framework with the parameter u being a formal variable. As an example, we compute $\mathcal{K}^{(1)}(u)$ and verify that it satisfies the reflection equation (0.0.15) using a Poincaré–Birkhoff–Witt (PBW) basis, as discussed in Section 2.1.2.

  Additionally, we have proposed a relation (under the form of a conjecture) that links the K-operators obtained through evaluation and the spin-j K-operator associated with $\mathcal{A}_q$ constructed through the fusion procedure outlined above, as stated in Conjecture 1. This conjecture suggests that these two operators are proportional to a central and invertible element in $\mathcal{A}_q((u^{-1}))$, which satisfies a functional relation and has a specific coaction. In particular, this conjecture is supported by showing independently that the fused K-operator satisfies a set of relations arising from the evaluation of the axioms of the universal K-matrix.

  Finally, we studied images of the fused K-operators $\mathcal{K}^{(j)}(u)$ under finite-dimensional representations of $B = \mathcal{A}_q$. For a one-dimensional representation of $\mathcal{A}_q$, the corresponding images

become fused K-matrices, see Definition 3.5.5. In particular, it reproduces known K-matrices such as the one from [dVR94, GZ93]. Furthermore, using spin-chain representations of $\mathcal{A}_q$, we also obtain dressed K-matrices of the form (0.0.7) out of $\mathcal{K}^{(j)}(u)$ for any value of spins, see Lemma 4.4.2.

- Result 3:

  To address Problem 3, we introduce a spin-j universal transfer matrix denoted as $\mathsf{T}^{(j)}(u) \in \mathcal{A}_q$, which is constructed using the spin-j K-operator $\mathcal{K}^{(j)}(u)$. Assuming that the conjecture linking $\mathbf{K}^{(j)}(u)$ and $\mathcal{K}^{(j)}(u)$ is true, we show that $\mathsf{T}^{(j)}(u)$ satisfies

$$\mathsf{T}^{(j)}(u) = \mathsf{T}^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\mathsf{T}^{(\frac{1}{2})}(uq^{j-\frac{1}{2}}) + f_0^{(j)}(u)\mathsf{T}^{(j-1)}(uq^{-1}) \ , \tag{0.0.17}$$

  where $f_0^{(j)}(u)$ is a central element in $\mathcal{A}_q$ that is explicitly calculated, as detailed in Theorem 4.2.5. We call them universal TT-relations. Independently of the universal K-matrix framework, we prove that (0.0.17) holds for $j = 1, 3/2$, using a PBW basis of $\mathcal{A}_q$.

  Moreover, these universal TT-relations can be applied to numerous spin chain models. As previously mentioned, the algebra $\mathcal{A}_q$ unifies many models through its various quotients [BB12, BB16]. We show in Proposition 4.4.4 that, by choosing a representation of $\mathcal{A}_q$, $\mathsf{T}^{(j)}(u)$ reduces to a standard transfer matrix that can have an arbitrary value of spin at each site. For instance, we recover the spin-1 Hamiltonian of the XXZ spin chain with generic boundary conditions [IOZ96], the conjectured TT-relations of the spin-j XXZ chain with generic boundary conditions [FNR07], or the alternating spin chain [CYSW14].

  Finally, based on this universal approach, we investigate the hidden symmetries of the XXZ spin chain with arbitrary spin-j for various boundary conditions. Typically, in the representation-dependent approach, the symmetries of the Hamiltonians are obtained on a case-by-case basis, as seen for the XXZ spin chain with special boundary conditions (such as $U_q sl_2$-invariant) in [PS90]. However, with the universal approach, we use the underlying structure of $\mathsf{T}^{(j)}(u)$ and the universal TT-relations (0.0.17) to determine hidden symmetries for many spin chains simultaneously. Specifically, we consider operators belonging to $\mathcal{A}_q$ that commute with $\mathsf{T}^{(j)}(u)$ (or more simply with $\mathsf{T}^{(\frac{1}{2})}(u)$ thanks to (0.0.17)). By construction, their representations commute with the corresponding Hamiltonian derived from the image of $\mathsf{T}^{(j)}(u)$. This is studied in Section 4.5.

The last chapter is dedicated to the perspectives. First, we discuss similar results for the alternating central extension of the positive part of $U_q\widehat{sl}_2$ [T19, B20]. We then address several questions that are analogous to the above problems discussed for $\mathcal{A}_q$, and give elements of response. Secondly, we gave an axiomatic definition of a universal K-matrix $\mathfrak{K} \in B \otimes H$, but we do not have any explicit expressions. Thus, we give some perspectives to construct such $\mathfrak{K}$ for a comodule algebra $B = \mathcal{A}_q$ and $B = O_q$ over $H = \mathcal{L}U_q sl_2$.

Notations.

Hopf algebra H	
$U_q sl_2$	quantum enveloping algebra for sl_2,
$U_q \widehat{sl}_2$	quantum affine algebra for sl_2,
$\mathcal{L}U_q sl_2$	quantum loop algebra for sl_2.

Comodule algebra B	
O_q	q-Onsager algebra,
$\mathcal{A}_q$	alternating central extension of the q-Onsager algebra,
$\mathcal{A}_q^{(N)}$	quotient algebra of $\mathcal{A}_q$,
$\varphi^{(N)}$	algebra map from $\mathcal{A}_q$ to $\mathcal{A}_q^{(N)}$.

Fusion/reduction procedure	
ev_u	(formal) evaluation map from $\mathcal{L}U_q sl_2$ to $U_q sl_2$,
π_u^j	(formal) evaluation representations from $\mathcal{L}U_q sl_2$ to $\mathrm{End}(\mathbb{C}^{2j+1})$,
$\mathcal{E}^{(j+\frac{1}{2})}$,	$\mathcal{L}U_q sl_2$-intertwiner for fusion $(j+1/2) \to (1/2, j)$,
$\bar{\mathcal{E}}^{(j-\frac{1}{2})}$	$\mathcal{L}U_q sl_2$-intertwiner for reduction $(j-1/2) \to (1/2, j)$,
$\mathcal{F}^{(j+\frac{1}{2})}$	pseudo-inverse of $\mathcal{E}^{(j+\frac{1}{2})}$,
$\bar{\mathcal{F}}^{(j-\frac{1}{2})}$	pseudo-inverse of $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$.

Spin-j solutions of YB and RE	
$\mathcal{L}^{(j)}(u)$	fused L-operator in $U_q sl_2[u, u^{-1}] \otimes \mathrm{End}(\mathbb{C}^{2j+1})$,
$R^{(j_1,j_2)}(u)$	fused R-matrix in $\mathrm{End}(\mathbb{C}^{2j_1+1}) \otimes \mathrm{End}(\mathbb{C}^{2j_2+1})$,
$\mathcal{K}^{(j)}(u)$	fused K-operator for $\mathcal{A}_q$,
$\mathcal{K}^{(j,N)}(u)$	fused K-operator for $\mathcal{A}_q^{(N)}$,
$K^{(j)}(u)$	fused K-matrix,
$K^{+(j)}(u)$	fused dual K-matrix.

Universal R- and K-matrices	
$\mathfrak{R}$	universal R-matrix in $H \otimes H$,
$\mathbf{L}^{\pm}(u)$	affine L-operators in $\mathcal{L}U_q sl_2((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^2)$,
$\mathbf{L}^{(j)}(u)$	spin-j L-operator in $U_q sl_2((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^{2j+1})$,
$\mathcal{R}^{(j_1,j_2)}(u)$	spin-j R-matrix in $\mathrm{End}(\mathbb{C}^{2j_1+1}) \otimes \mathrm{End}(\mathbb{C}^{2j_2+1})$,
$\mathfrak{K}$	universal K-matrix in $B \otimes H$,
(ψ, J)	twist pair: a Hopf algebra automorphism ψ, and a Drinfeld twist J,
$\mathbf{K}^{(j)}(u)$	spin-j K-operator for $\mathcal{A}_q$.

Transfer matrices	
$\mathsf{T}^{(j)}(u)$	spin-j generating function in the commutative subalgebra of $\mathcal{A}_q$,
$\mathsf{T}^{(j,N)}(u)$	spin-j generating function in the commutative subalgebra of $\mathcal{A}_q^{(N)}$,
$t^{(j,\{j_n\})}(u)$	spin-j transfer matrix in $\bigotimes_{n=1}^{N} \mathrm{End}(\mathbb{C}^{2j_n+1})$,

Conventions.

We denote the set of natural numbers by $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the positive integers by $\mathbb{N}_+ = \{1, 2, \ldots\}$.

All algebras are considered over the field of complex numbers $\mathbb{C}$, if not stated otherwise. Let $q \in \mathbb{C}^*$, and we assume that q is not a root of unity. The q-commutator is

$$\big[X, Y\big]_q = qXY - q^{-1}YX$$

and $\big[X, Y\big] = \big[X, Y\big]_1 = XY - YX$. We denote the q-numbers by $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. We also denote by $\mathbb{I}_{2j}$ the $2j \times 2j$ identity matrix.

All generating functions considered in this thesis like $\mathsf{T}^{(j)}(u)$ or $\Gamma(u)$ are formal Laurent series in u^{-1} with coefficients in the corresponding algebra like $\mathcal{A}_q$. In other words, they are of the form

$$\sum_{n \in \mathbb{Z}} f_n u^{-n} \ ,$$

where $f_n \in \mathcal{A}_q$ and all but finitely many f_n for $n < 0$ vanish. In this case, we use the notation $\mathcal{A}_q((u^{-1}))$, and the convention that every rational function of the form $1/p(u)$, where $p(u)$ is a Laurent polynomial, is expanded in u^{-1}.

Chapter 1

Construction of quantum integrable spin-chains

In this chapter, the basic material to construct and study integrable closed and open spin-chains is reviewed. As recalled in the introduction, the closed spin-chains are associated with the Yang-Baxter algebra [M64, BZ66, Ya67, Ba72]. For the open spin-chains, the associated algebra is the (dual) reflection algebra [Sk88]. Then, the Hamiltonian of the integrable closed spin-chains is derived from a transfer matrix whose building blocks are the R- and K-matrices. As an example, the Hamiltonian of the XXZ spin-$\frac{1}{2}$ chain with generic boundary conditions is constructed. Setting the boundary parameters to certain values, one obtains the special ($U_q sl_2$-invariant), diagonal, upper or lower triangular boundary conditions.

This formalism extends to higher spin-j for the construction of quantum integrable spin-j chains. We thus review a fusion procedure that allows to consider spin-j chains. This method aims to obtain higher spin R- and K-matrices that satisfy a fused Yang-Baxter equation and a fused reflection equation. These fused matrices are defined recursively and their building blocks are the spin-$\frac{1}{2}$ R- and K-matrices. The procedure relies on the assumption that the specialization of the R-matrix at some special point degenerates into a projector. For the fused R-matrix, this method originates from [KRS81]. A few years after the introduction of the reflection algebra, the fusion procedure of the R-matrices has been extended to the case of K-matrices in [MN92] using the same projectors. This allows to define a fused transfer matrix from which the Hamiltonian is derived. According to the algebra considered, these fused transfer matrices of spin-j satisfy a recurrence relations known under the name of the TT-relations. For the Yang-Baxter algebra, it was conjecture in [KS82]. For the reflection equation algebra, it was conjectured in [MN92, Zh95]. In this chapter, we review the representation-dependent approach for the construction of quantum integrable systems, as illustrated in the left part of Fig. 3.

This chapter is organized as follows.

In Section 1.1, the basic material to describe closed spin-chains is recalled, namely: the Yang-Baxter algebra whose defining relation is the Yang-Baxter equation, the monodromy matrix as well as the transfer matrix, see Definitions 1.1.1, 1.1.5, 1.1.6. Then, we consider the R-matrix associated to the XXZ spin-$\frac{1}{2}$ with periodic boundary conditions and recall the derivation of the Hamiltonian

from the transfer matrix.

In Section 1.2, the latter formalism is extended to describe closed spin-chains. This requires the introduction of a K-matrix (and its dual). In particular, we recall the reflection algebra and its dual algebra, whose defining relations are respectively the reflection equation and the dual reflection equation, see Definitions 1.2.1, 1.2.3. Such systems are described with the help of a double-row monodromy matrix and a transfer matrix. As an application, the XXZ spin-$\frac{1}{2}$ Hamiltonian with generic boundary conditions is derived from the latter transfer matrix.

In Section 1.3, we review the construction of the fused R- and K-matrices introduced in [KRS81] and in [MN92], respectively. The spin-j fused R- and K-matrices are then written in compact form and solely in terms of the fundamental R- and K-matrices. Consequently, the spin-$\frac{1}{2}$ transfer matrix associated to open spin-chains, given in Definition 1.2.7, is generalized to a spin-j auxiliary space using the latter fused matrices, see Definition 1.3.4. Then, we recall the fusion hierarchy of the transfer matrices, the so-called *TT-relations*, for the case of the XXZ spin-j chains with generic boundary conditions [YNZ05, FNR07]. Finally, in order to illustrate this procedure, we consider the fusion of the six-vertex R-matrix and the Ghoshal-Zamolodchikov scalar K-matrix given in (1.1.5) and (1.2.3), respectively. The resulting fused matrices are compared with the spin-1 solutions of the Yang-Baxter equation and the reflection equation obtained in the literature. Then, the Hamiltonian of the XXZ spin-1 chain with generic boundary conditions is derived from the spin-1 transfer matrix.

1.1 Closed spin-chains

Consider a spin-chain of size N whose spin interaction is between nearest neighbors. Here, we study systems with periodic boundary conditions, which means that the particle at site N interacts with the first one, i.e. $\sigma_i^{N+1} \equiv \sigma_i^1$, $i = x, y, z$.

In this section, we recall the algebraic elements that allow us to describe systems with periodic boundary conditions. More precisely, we recall the definitions of the Yang-Baxter algebra, the monodromy matrix and the transfer matrix. Then, we recall how the Hamiltonian is derived from the transfer matrix.

1.1.1 Yang-Baxter algebra

We first recall the definition of the R-matrix which originates from the independent work of C.N. Yang [Ya67] and R.J. Baxter [Ba72].

Definition 1.1.1. *The operator-valued function* $R\colon \mathbb{C} \otimes \mathbb{C} \to \mathrm{End}(V_1 \otimes V_2)$ *satisfies the so-called Yang-Baxter equation*

$$R_{12}(u_1, u_2) R_{13}(u_1, u_3) R_{23}(u_2, u_3) = R_{23}(u_2, u_3) R_{13}(u_1, u_3) R_{12}(u_1, u_2) \ , \tag{1.1.1}$$

where u_1, u_2, $u_3 \in \mathbb{C}^*$. *A solution of this equation is called an R-matrix.*

Example 1.1.2. *Set* $V_1 = V_2 = V_3 = \mathbb{C}^2$ *in* (1.1.1).

(i) An R-matrix satisfying the property $R(u_1, u_2) = R(u_1 - u_2)$ is called a *difference form* R-

matrix. In this case, the equation (1.1.1) reads[5]

$$R_{12}(\lambda_1-\lambda_2)R_{13}(\lambda_1-\lambda_3)R_{23}(\lambda_2-\lambda_3)=R_{23}(\lambda_2-\lambda_3)R_{13}(\lambda_1-\lambda_3)R_{12}(\lambda_1-\lambda_2)\ , \quad (1.1.2)$$

where $\lambda_i\in\mathbb{C}$. For instance, for λ and $\eta\in\mathbb{C}$,

$$R^{\mathrm{df}}(\lambda)=\begin{pmatrix}\sinh(\lambda+\eta) & 0 & 0 & 0\\ 0 & \sinh(\lambda) & \sinh(\eta) & 0\\ 0 & \sinh(\eta) & \sinh(\lambda) & 0\\ 0 & 0 & 0 & \sinh(\lambda+\eta)\end{pmatrix}\ , \quad (1.1.3)$$

is a solution of the Yang-Baxter equation.

(ii) An R-matrix satisfying the property $R(u_1,u_2)=R(u_1/u_2)$ is called a *rational form* R-matrix. In this case, the equation (1.1.1) becomes

$$R_{12}(u_1/u_2)R_{13}(u_1/u_3)R_{23}(u_2/u_3)=R_{23}(u_2/u_3)R_{13}(u_1/u_3)R_{12}(u_1/u_2)\ , \quad (1.1.4)$$

where $u_i\in\mathbb{C}^*$. For instance,

$$R(u)=\begin{pmatrix}uq-u^{-1}q^{-1} & 0 & 0 & 0\\ 0 & u-u^{-1} & q-q^{-1} & 0\\ 0 & q-q^{-1} & u-u^{-1} & 0\\ 0 & 0 & 0 & uq-u^{-1}q^{-1}\end{pmatrix}\ , \quad (1.1.5)$$

is a solution of the Yang-Baxter equation.

Note that $R^{df}(u)$ and $R(u)$ are related by the parametrization $q=\exp(\eta)$ and $u=\exp(\lambda)$.

In this thesis, we mainly use rational form R-matrices.

Definition 1.1.3. *Let $R(u)\in\mathrm{End}(V\otimes V)$ be a solution of the Yang-Baxter equation (1.1.1).* The Yang-Baxter algebra is generated by $\mathcal{T}_{ij}^{(n)}$ for $n\in\mathbb{Z}$ and $1\le i,j\le\dim(V)$. The defining relation is the so-called RTT relation

$$R_{12}(u/v)\mathcal{T}_1(u)\mathcal{T}_2(v)=\mathcal{T}_2(v)\mathcal{T}_1(u)R_{12}(u/v)\ , \quad (1.1.6)$$

where

$$\mathcal{T}(u)=\sum_{i,j=1}^{\dim(V)}\sum_{n=-\infty}^{\infty}u^n\mathcal{T}_{ij}^{(n)}\otimes E_{ij}\ , \quad (1.1.7)$$

and E_{ij} denotes the square matrix of size $(\dim(V)^2\times\dim(V)^2)$ with 1 in the entry (i,j) and 0 everywhere else.

Example 1.1.4. *Set $V=\mathbb{C}^2$ in (1.1.6).* Then, a solution of the RTT relation with $R(u)$ in (1.1.5)

[5] In the following, we use the parameters λ or λ_i for the difference form R-matrices.

and with the substitution $\mathcal{T}(u) \to \mathcal{L}(u)$ is given by

$$\mathcal{L}(u) = \begin{pmatrix} uq^{\frac{1}{2}}K^{\frac{1}{2}} - u^{-1}q^{-\frac{1}{2}}K^{-\frac{1}{2}} & (q-q^{-1})F \\ (q-q^{-1})E & uq^{\frac{1}{2}}K^{-\frac{1}{2}} - u^{-1}q^{-\frac{1}{2}}K^{\frac{1}{2}} \end{pmatrix} \in U_q sl_2[u,u^{-1}] \otimes \mathrm{End}(\mathbb{C}^2) \ , \quad (1.1.8)$$

where E, F and $K^{\pm\frac{1}{2}}$ are the generators of the quantum algebra $U_q sl_2$, see Appendix 3.3 for its definition. It satisfies the so-called RLL equation

$$R_{12}(u/v)\mathcal{L}_1(u)\mathcal{L}_2(v) = \mathcal{L}_2(v)\mathcal{L}_1(u)R_{12}(u/v) \ . \quad (1.1.9)$$

A solution of this equation is called an L-operator.

Let $\pi^{\frac{1}{2}}$ be the spin-$\frac{1}{2}$ representation map of $U_q sl_2$

$$\pi^{\frac{1}{2}} : U_q sl_2 \to \mathrm{End}(\mathbb{C}^2) \ ,$$

defined by

$$\pi^{\frac{1}{2}}(E) = \sigma^+ \ , \qquad \pi^{\frac{1}{2}}(F) = \sigma^- \ , \qquad \pi^{\frac{1}{2}}(K^{\pm\frac{1}{2}}) = q^{\pm\frac{1}{2}\sigma^z} \ .$$

Note that the L-matrix (1.1.8) and the R-matrix (1.1.5) are related as follows:

$$R(u) = (\pi^{\frac{1}{2}} \otimes \mathrm{id})(\mathcal{L}(u)) \ . \quad (1.1.10)$$

1.1.2 Monodromy matrix

Now, in order to construct spin-chains, we recall the definition of the *monodromy matrix*. Introduce the inhomogeneous parameters $v_i \in \mathbb{C}^*$ with $i \in \mathbb{N}_+$.

Definition 1.1.5. *The monodromy matrix is defined for $N \in \mathbb{N}_+$ as*

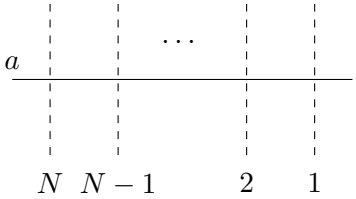
$$T_{a,N}(u) = R_{aN}(uv_N)R_{aN-1}(uv_{N-1})\cdots R_{a2}(uv_2)R_{a1}(uv_1) \ , \quad (1.1.11)$$

where a denotes the auxiliary space and integers $1, 2, \ldots, N$ label the quantum spaces. It satisfies the RTT relation

$$R_{12}(u/v)\ T_{1,N}(u)\ T_{2,N}(v) = T_{2,N}(v)\ T_{1,N}(u)\ R_{12}(u/v) \ . \quad (1.1.12)$$

Here, we consider only spin-$\frac{1}{2}$ auxiliary and quantum spaces. In the next section, we will define more general monodromy matrices with spin-j auxiliary space and spin-j_k for each quantum space.

Graphically, one represents the monodromy matrix as

$T_{a,N}(u) =$ 

. (1.1.13)

where a straight line stands for the auxiliary space a, while a dashed line represents the quantum space $V_{[n]}$.

1.1.3 Transfer matrix

Let us now define the transfer matrix from the monodromy matrix.

Definition 1.1.6. *The transfer matrix for periodic boundary systems is*

$$t^{(N)}(u) = \mathrm{tr}_a(T_{a,N}(u)) \ , \tag{1.1.14}$$

where $T_{a,N}(u)$ is given in (1.1.11).

It is a generating function for integrals of motion of a quantum system described by some representations of the Yang-Baxter algebra, see Definition 1.1.1. Indeed, provided the R-matrix is invertible, one has for $u, v \in \mathbb{C}$

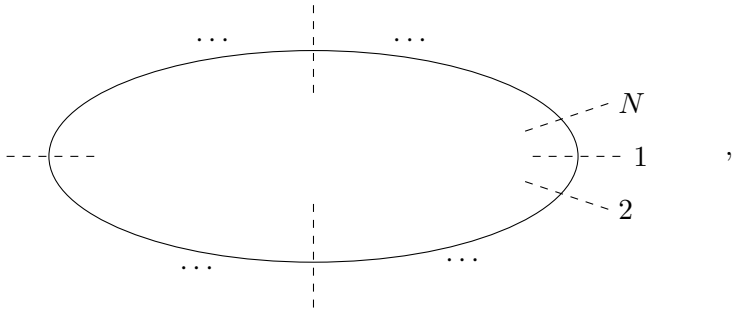
$$\left[t^{(N)}(u), t^{(N)}(v)\right] = 0 \ . \tag{1.1.15}$$

It is obtained as follows. Multiplying the RTT relation (1.1.12) from the right by $[R_{12}(u/v)]^{-1}$, one has

$$T_{1,N}(u)T_{2,N}(v) = [R_{12}(u/v)]^{-1}T_{2,N}(v)T_{1,N}(u)R_{12}(u/v) \ .$$

Then taking the trace over the space $V_1 \otimes V_2$, using the cyclicity of the trace and the fact that the trace of the tensor product is equal to the product of traces, one gets (1.1.15).

Graphically, due to (1.1.13), the transfer matrix is represented by



,

where the closed line stands for the trace over the auxiliary space.

Recall the Hamiltonian of the XXZ spin-$\frac{1}{2}$ chain with periodic boundary conditions is:

$$\mathcal{H}^{per}_{XXZ} = \sum_{k=1}^{N} \sigma_x^k \sigma_x^{k+1} + \sigma_y^k \sigma_y^{k+1} + \frac{q+q^{-1}}{2} \sigma_z^k \sigma_z^{k+1} \ , \qquad \text{with} \quad \sigma_\ell^1 \equiv \sigma_\ell^{N+1} \ . \tag{1.1.16}$$

We now recall how it is obtained from the transfer matrix. Due to the commutation (1.1.15), we have conserved quantities denoted $\mathcal{I}^{(n)}$. They are obtained by taking logarithmic derivatives of the transfer matrix [Lu76]

$$\mathcal{I}^{(n)} = \frac{d^n}{du^n} \left(\ln(t^{(N)}(u)) \right) \Big|_{u=1} \ . \tag{1.1.17}$$

Recall also the permutation matrix $\mathcal{P}_{12}^{(j_1,j_2)}$ that flips the space from $V^{(j_1)} \otimes V^{(j_2)}$ to $V^{(j_2)} \otimes V^{(j_1)}$. It acts as $\mathcal{P}^{(j_1,j_2)}(a \otimes b)\mathcal{P}^{(j_2,j_1)} = b \otimes a$. Consider $j_1 = j_2 = \frac{1}{2}$ and let $\mathcal{P}^{(\frac{1}{2},\frac{1}{2})} \equiv \mathcal{P}$, then one has for the R-matrix in (1.1.5)

$$\mathcal{P} = \frac{1}{q - q^{-1}} R(1) \ . \tag{1.1.18}$$

Proposition 1.1.7. *For the transfer matrix given in Definition* 1.1.6 and with the R-matrix from (1.1.5), we have

$$\mathcal{H}_{XXZ}^{per} = (q - q^{-1}) \frac{d}{du} \left(\ln(t^{(N)}(u)) \right) \Big|_{u=1,\{v_n\}=1} - N(\frac{q^2 - q^{-2}}{2}) \, \mathbb{I} \ , \tag{1.1.19}$$

where $\{v_n\} = \{v_1, v_2, \ldots, v_N\}$.

Proof. Let us set $v_n = 1$. The logarithmic derivative of the transfer matrix is obtained by compute the product $[t^{(N)}(1)]^{-1} t'^{(N)}(1)$. Recall that the R-matrix (1.1.5) at $u = 1$ specializes to the permutation matrix (1.1.18). First, for $u = 1$, computing the inverse of the transfer matrix starting with $t^{(N)}(1)$

$$\begin{aligned} t^{(N)}(1) &= (q - q^{-1})^N \mathrm{tr}_a(\mathcal{P}_{aN} \cdots \mathcal{P}_{a1}) \\ &= (q - q^{-1})^N \mathrm{tr}_a(\mathcal{P}_{aN} \cdots \mathcal{P}_{a1} \mathcal{P}_{aN}^2) \\ &= (q - q^{-1})^N \mathrm{tr}_a(\mathcal{P}_{NN-1} \cdots \mathcal{P}_{N1} \mathcal{P}_{aN}) \\ &= (q - q^{-1})^N \mathcal{P}_{NN-1} \cdots \mathcal{P}_{N1} \ , \end{aligned}$$

where we used that $\mathrm{tr}_a(\mathcal{P}_{ak}) = 1$. Then, one has

$$[t^{(N)}(1)]^{-1} = (q - q^{-1})^{-N} \mathcal{P}_{N1} \cdots \mathcal{P}_{NN-1} \ .$$

Secondly, one computes the derivative of the transfer matrix as follows

$$\begin{aligned} t'^{(N)}(1) &= (q - q^{-1})^{N-1} \sum_{k=1}^{N} \mathrm{tr}_a(\mathcal{P}_{aN} \cdots R'_{ak}(1) \cdots \mathcal{P}_{a1}) \\ &= (q - q^{-1})^{N-1} \sum_{k=1}^{N} \mathrm{tr}_a(\mathcal{P}_{aN} \cdots R'_{ak}(1) \cdots \mathcal{P}_{a1} \mathcal{P}_{aN}^2) \\ &= (q - q^{-1})^{N-1} \sum_{k=1}^{N} \mathrm{tr}_a(\mathcal{P}_{NN-1} \cdots R'_{Nk}(1) \cdots \mathcal{P}_{N1} \mathcal{P}_{aN}) \\ &= (q - q^{-1})^{N-1} \sum_{k=1}^{N} \mathcal{P}_{NN-1} \cdots R'_{Nk}(1) \cdots \mathcal{P}_{N1} \ . \end{aligned}$$

Finally, after straightforward calculations, one gets:

$$\begin{aligned}\frac{d}{du}\ln(t(u))\Big|_{u=1} &= [t^{(N)}(1)]^{-1}t'(1)\\ &= \frac{1}{q-q^{-1}}\sum_{k=1}^{N} H_{n\,n+1}\ ,\end{aligned}$$

where the Hamiltonian density is introduced

$$\begin{aligned}H_{n\,n+1} &= \mathcal{P}_{n\,n+1}R'_{n\,n+1}(1)\\ &= \sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \frac{q+q^{-1}}{2}(\sigma_n^z\sigma_{n+1}^z + \mathbb{I})\ ,\end{aligned} \tag{1.1.20}$$

and so (1.1.19) holds. □

1.2 Open spin-chains

Now, we construct open spin-chains, i.e. that has non-periodic boundary conditions. This requires the introduction of some algebras, namely the reflection algebra and its dual, as well as an analogue of the monodromy matrix (1.1.11) which is called the double-row monodromy matrix. Together, they allow to define a transfer matrix that takes into account the boundary conditions.

In this section, these notions are recalled and the XXZ spin-$\frac{1}{2}$ Hamiltonian with generic boundary conditions is derived.

1.2.1 Reflection equation algebra

We first recall the definition of the reflection algebra introduced by Sklyanin in [Sk88].

Definition 1.2.1 ([Sk88])**.** *Let $R(u) \in \mathrm{End}(V\otimes V)$ be a solution of the Yang-Baxter equation (1.1.1).* The reflection algebra, denoted $\mathcal{J}^-$, is generated by $\mathcal{K}_{ij}^{(n)}$ for $n \in \mathbb{Z}$ and $1 \le i,j \le \dim(V)$. The defining relation is the so-called reflection equation

$$R(u/v)\mathcal{K}_1(u)R(uv)\mathcal{K}_2(v) = \mathcal{K}_2(v)R(uv)\mathcal{K}_1(u)R(u/v)\ , \tag{1.2.1}$$

where

$$\mathcal{K}(u) = \sum_{i,j=1}^{\dim(V)}\sum_{n=-\infty}^{\infty} u^n\mathcal{K}_{ij}^{(n)}\otimes E_{ij}\ , \tag{1.2.2}$$

and we assume $\mathcal{K}_{ij}^{(n)}$ are all zero for $n > N$ for some fixed positive N. A solution of the reflection equation is called a K-matrix.

Example 1.2.2. *Set $V = \mathbb{C}^2$ in (1.2.1).* Then, a solution of the reflection equation for $R(u)$

in (1.1.5) and with the substitution $\mathcal{K}(u) \to K(u)$ is given by [GZ93, dVR94]

$$K(u) = \begin{pmatrix} u\varepsilon_+ + u^{-1}\varepsilon_- & \frac{k_+(u^2-u^{-2})}{q-q^{-1}} \\ \frac{k_-(u^2-u^{-2})}{q-q^{-1}} & u\varepsilon_- + u^{-1}\varepsilon_+ \end{pmatrix} , \tag{1.2.3}$$

where $k_\pm$ and $\varepsilon_\pm$ are scalars in $\mathbb{C}$.In other words, it defines a one-dimensional representation for $\mathcal{J}^-$.

Note that examples of K-operators (for which $\mathcal{K}_{ij}^{(n)}$ are not scalars) will be considered in Chapter 2. There are two presentations for an algebra that has an associated K-matrix. The first is the standard one which consists in giving the generators with their defining relations. The second is a Faddeev-Reshetikhin-Takhtajan (FRT) type presentation: given an R-matrix and a K-operator, the reflection equation is equivalent to the defining relations of the algebra.

In [Sk88], Sklyanin also introduced the dual reflection algebra as follows.

Definition 1.2.3 ([Sk88])**.** *Let $R(u) \in \mathrm{End}(V \otimes V)$ be a solution of the Yang-Baxter equation (1.1.1).* The dual reflection algebra, denoted $\mathcal{J}^+$, is generated by $\overline{\mathcal{K}}_{ij}^{(n)}$ for $n \in \mathbb{Z}$ and $1 \leq i, j \leq \dim(V)$. The defining relation is the so-called dual reflection equation

$$R(v/u)\overline{\mathcal{K}}_1^{t_1}(u)R(1/uvq^2)\overline{\mathcal{K}}_2^{t_2}(v) = \overline{\mathcal{K}}_2^{t_2}(v)R(1/uvq^2)\overline{\mathcal{K}}_1^{t_1}(u)R(v/u) , \tag{1.2.4}$$

where t_1 stands for the transposition on the space V_1, and

$$\overline{\mathcal{K}}(u) = \sum_{i,j=1}^{\dim(V)} \sum_{n=-\infty}^{\infty} u^n \overline{\mathcal{K}}_{ij}^{(n)} \otimes E_{ij} . \tag{1.2.5}$$

We also assume $\overline{\mathcal{K}}_{ij}^{(n)}$ are all zero for $n > N$ for some fixed positive N. A solution of the dual reflection equation is called a dual K-matrix.

Example 1.2.4. *Set $V = \mathbb{C}^2$ in (1.2.4).* Then, a solution of the dual reflection equation for $R(u)$ in (1.1.5) and with the substitution $\overline{\mathcal{K}}(u) \to K^+(u)$ is given by

$$K^+(u) = \begin{pmatrix} uq\overline{\varepsilon}_+ + u^{-1}q^{-1}\overline{\varepsilon}_- & \frac{\overline{k}_+(u^2q^2-u^{-2}q^{-2})}{q-q^{-1}} \\ \frac{\overline{k}_-(u^2q^2-u^{-2}q^{-2})}{q-q^{-1}} & uq\overline{\varepsilon}_- + u^{-1}q^{-1}\overline{\varepsilon}_+ \end{pmatrix} , \tag{1.2.6}$$

where $\overline{k}_\pm$ and $\overline{\varepsilon}_\pm$ are scalars in $\mathbb{C}$. It defines a one-dimensional representation for $\mathcal{J}^+$.

Remark 1.2.5. *Note that the R-matrix from (1.1.5)* satisfies the following properties

$$R^{t_{12}}(u) = R(u) , \tag{1.2.7}$$

$$R(u)R(u^{-1}) = -c(uq)c(uq^{-1})\, \mathbb{I}_4 , \tag{1.2.8}$$

$$R^{t_1}(u)R^{t_1}(u^{-1}q^{-2}) = -c(u)c(uq^2)\, \mathbb{I}_4 , \tag{1.2.9}$$

where t_{12} (resp. t_1) stands for the transposition applied to the space $V_1 \otimes V_2$ (resp. V_1). In this case, the reflection algebra and the dual reflection algebra are isomorphic [Sk88, Sec. 3]. An obvious

isomorphism $\varrho\colon \mathcal{J}^- \to \mathcal{J}^+$ is such that

$$\varrho(\mathcal{K}(u)) = \mathcal{K}(-u^{-1}q^{-1})^t \ . \tag{1.2.10}$$

In particular, $K^+(u)$ in (1.2.6) *is related to* $K(u)$ *in* (1.2.3) *by*

$$K^+(u) = \big(K(-u^{-1}q^{-1})\big)^t|_{\varepsilon_\pm \to -\overline{\varepsilon}_\mp, k_\pm \to -\overline{k}_\mp} \ .$$

In the physics literature, the K-matrix and its dual are respectively called the left and right boundary K-matrices because they encode boundary conditions at the corresponding ends of the spin-chains.

1.2.2 Double-row Monodromy Matrix

Now, in order to construct spin-chains with boundary conditions, we recall the definition of the *double-row monodromy matrix*.

Definition 1.2.6 ([Sk88])**.** *The double-row monodromy matrix is defined for* $N \in \mathbb{N}_+$ *as*

$$\mathbb{T}_{a,N}(u) = T_{a,N}(u)K_a(u)\widehat{T}_{a,N}(u) \ , \tag{1.2.11}$$

where $K(u)$ *is a solution of the reflection equation* (1.2.1), $T_{a,N}(u)$ is the monodromy matrix given in (1.1.11) and

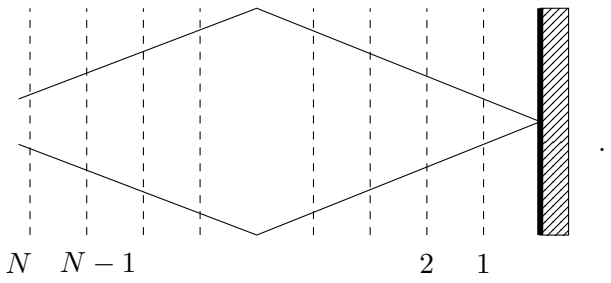
$$\widehat{T}_{a,N}(u) = R_{a1}(uv_1^{-1})R_{a2}(uv_2^{-1})\cdots R_{aN-1}(uv_{N-1}^{-1})R_{aN}(uv_N^{-1}) \ . \tag{1.2.12}$$

It satisfies the reflection equation

$$R_{12}(u/v)\ \mathbb{T}_{1,N}(u)\ R_{12}(uv)\ \mathbb{T}_{2,N}(v) = \mathbb{T}_{2,N}(v)\ R_{12}(uv)\ \mathbb{T}_{1,N}(u)\ R_{12}(u/v) \ . \tag{1.2.13}$$

In the literature, solutions of the reflection equation of the form (1.2.11) are called *dressed K-matrices*.

Graphically, one represents the double-row monodromy matrix as



. (1.2.14)

1.2.3 Transfer matrix

The study of the closed spin chains was done using the transfer matrix from Definition 1.1.6. Now, in order to take into account the boundary conditions, which are encoded into the K-matrices, we need an analogue of the latter transfer matrix. Recall the dual K-matrix $K^+(u)$ given in (1.2.6).

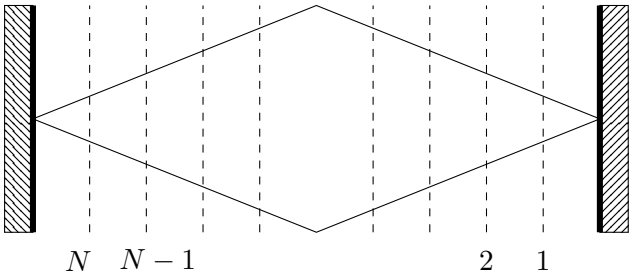
Definition 1.2.7. *The transfer matrix for open spin-chains is*

$$t^{(N)}(u) = \mathrm{tr}_a(K_a^+(u)\mathbb{T}_{a,N}(u)) \ , \tag{1.2.15}$$

where $\mathbb{T}_{a,N}(u)$ *is given in* (1.2.11).

Note that we keep the same notation for the transfer matrix $t^{(N)}(u)$ whether we are in the case of open or closed spin-chains. It will be clear which definition of the transfer matrix should be used, (1.1.14) or (1.2.15), according to the context.

Graphically, one represents the transfer matrix as



, (1.2.16)

where the left wall represents the matrix $K^+(u)$. The transfer matrix as defined in (1.2.15) forms a commutative family, i.e. the equation (1.1.15) also holds for open spin-chains. We refer the reader to [Sk88, Thm. 1] for the proof that this latter relation holds. Therefore, conserved quantities $\mathcal{I}^{(n)}$ are again extracted from the logarithmic derivative of the transfer matrix using the formula (1.1.17).

For instance, the Hamiltonian of the XXZ spin-$\frac{1}{2}$ with generic boundary conditions is given by [dVR93]:

$$\begin{aligned}\mathcal{H}_{XXZ}^{(\frac{1}{2})gen}(k_\pm,\varepsilon_\pm,\overline{k}_\pm,\overline{\varepsilon}_\pm) = \mathbf{b}^{(\frac{1}{2})} &+ \frac{2}{\varepsilon_+ + \varepsilon_-}\Big(\frac{q-q^{-1}}{4}(\varepsilon_+ - \varepsilon_-)\sigma_1^z + k_+\sigma_1^+ + k_-\sigma_1^-\Big) \\ &+ \frac{2}{\overline{\varepsilon}_+ + \overline{\varepsilon}_-}\Big(\frac{q-q^{-1}}{4}(\overline{\varepsilon}_+ - \overline{\varepsilon}_-)\sigma_N^z + \overline{k}_+\sigma_N^+ + \overline{k}_-\sigma_N^-\Big) \ ,\end{aligned} \tag{1.2.17}$$

where we introduced the bulk term

$$\mathbf{b}^{(\frac{1}{2})} = \sum_{k=1}^{N-1}\Big(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \frac{q+q^{-1}}{2}\sigma_k^z\sigma_{k+1}^z\Big) \tag{1.2.18}$$

and $k_\pm$, $\varepsilon_\pm$, $\overline{k}_\pm$, $\overline{\varepsilon}_\pm$ are scalars in $\mathbb{C}$. Then, as for the periodic spin-chains, the Hamiltonian is obtained from the transfer matrix with the K-matrices (1.2.3), (1.2.6) and the R-matrix (1.1.5)

Proposition 1.2.8. *For the transfer matrix given in Definition* 1.2.7 with the K-matrices (1.2.3), (1.2.6) and the R-matrix (1.1.5), we have

$$\mathcal{H}_{XXZ}^{(\frac{1}{2})gen}(k_\pm,\varepsilon_\pm,\overline{k}_\pm,\overline{\varepsilon}_\pm) - \mathcal{H}_0^{(\frac{1}{2})} = \frac{q-q^{-1}}{2}\frac{d}{du}\ln(t^{(N)}(u))\Big|_{u=1,\{v_n\}=1} \ , \tag{1.2.19}$$

where $\mathcal{H}_0^{(\frac{1}{2})}$ is some scalar $\in \mathbb{C}^*$.

Proof. We compute the r.h.s. of (1.2.19) and we set the inhomogeneities parameters v_n to 1. Firstly, using (1.1.18) one gets

$$[t^{(N)}(1)]^{-1} = \frac{(q-q^{-1})^{-2N}}{\mathrm{tr}(K^+(1))}\,[K_1(1)]^{-1}\ . \tag{1.2.20}$$

Secondly, we compute $t'(1)$ which denotes the derivative of $t(u)$ with respect to u evaluated at $u=1$. Note that there are $2N+2$ terms depending on u inside the trace of the transfer matrix, so one obtains:

$$\begin{aligned} t'(1) &= (q-q^{-1})^{2N}\Big(K_1(1)\mathrm{tr}(K^{+'}(1)) + K_1'(1)\mathrm{tr}(K^+(1))\Big) \\ &+ (q-q^{-1})^{2N-1}\mathrm{tr}(K^+(1))K_1(1)\Big(H_{12} + 2\sum_{n=2}^{N-1} H_{n\,n+1}\Big) \\ &+ (q-q^{-1})^{2N-1}\Big(\mathrm{tr}(K^+(1))H_{12}K_1(1) + 2K_1(1)\mathrm{tr}_a(K_a^+(1)H_{aN})\Big)\ . \end{aligned} \tag{1.2.21}$$

where $H_{n\ n+1}$ is given in (1.1.20). Then, using (1.2.20) and (1.2.21), one has:

$$\begin{aligned} \frac{d}{du}\ln(t(u))\Big|_{u=1} &= [t^{(N)}(1)]^{-1}t'(1) \\ &= \frac{\mathrm{tr}(K^{+'}(1))}{\mathrm{tr}(K^+(1))} + \frac{2}{q-q^{-1}}\sum_{n=2}^{N-1} H_{n\,n+1} + \frac{2}{q-q^{-1}}\frac{\mathrm{tr}_a(K_a^+(1)H_{aN})}{\mathrm{tr}(K^+(1))} \\ &+ [K_1(1)]^{-1}K_1'(1) + \frac{1}{q-q^{-1}}[K_1(1)]^{-1}H_{12}K_1(1) + \frac{1}{q-q^{-1}}H_{12}\ . \end{aligned} \tag{1.2.22}$$

Finally, after straightforward calculations, one finds that the claim follows. □

Now, several special cases are obtained by specializations of the Hamiltonian (1.2.17).

Example 1.2.9. *According to the boundary parameters $k_\pm$, $\varepsilon_\pm$, $\overline{k}_\pm$, $\overline{\varepsilon}_\pm$ chosen in (1.2.17),* the resulting Hamiltonian may exhibit some symmetries. These are often used in order to find an highest-weight vector which is required to apply the algebraic Bethe ansatz.

(i) The XXZ spin-$\frac{1}{2}$ chain with diagonal boundary conditions is defined as

$$\begin{aligned} \mathcal{H}_{XXZ}^{(\frac{1}{2})diag}(\varepsilon_\pm,\overline{\varepsilon}_\pm) &= \mathcal{H}_{XXZ}^{(\frac{1}{2})gen}(0,\varepsilon_\pm,0,\overline{\varepsilon}_\pm) \\ &= \mathbf{b}^{(\frac{1}{2})} + \frac{q-q^{-1}}{2}\left(\frac{\varepsilon_+-\varepsilon_-}{\varepsilon_++\varepsilon_-}\sigma_1^z + \frac{\overline{\varepsilon}_+-\overline{\varepsilon}_-}{\overline{\varepsilon}_++\overline{\varepsilon}_-}\sigma_N^z\right)\ . \end{aligned}$$

In this case, the K-matrix and its dual are diagonal. This Hamiltonian enjoys the $U(1)$ symmetry

$$\Big[\mathcal{H}_{XXZ}^{(\frac{1}{2})diag}(\varepsilon_\pm,\overline{\varepsilon}_\pm), S^z\Big] = 0\ , \qquad S^z = \sum_{k=1}^{N}\sigma_z^k\ . \tag{1.2.23}$$

(ii) The XXZ spin-$\frac{1}{2}$ chain with special ($U_q sl_2$-invariant) boundary condition is defined as [ABBBQ,

PS90]

$$\begin{aligned}\mathcal{H}_{XXZ}^{(\frac{1}{2})sp.} &= \mathcal{H}_{XXZ}^{(\frac{1}{2})gen}(0, \varepsilon_+ = 0, \varepsilon_-, 0, \overline{\varepsilon}_+, \overline{\varepsilon}_- = 0) \\ &= \mathbf{b}^{(\frac{1}{2})} + \frac{q - q^{-1}}{2}\left(-\sigma_1^z + \sigma_N^z\right) \ .\end{aligned}$$

Recall the quantum algebra $U_q sl_2$ from Definition 2.3.4. This Hamiltonian satisfies

$$[\mathcal{H}_{XXZ}^{(\frac{1}{2})sp.}, x] = 0 \ , \tag{1.2.24}$$

for any x in a N-tensor product[6] of spin-$\frac{1}{2}$ irreducible representation of $U_q sl_2$.

(iii) The XXZ spin-$\frac{1}{2}$ chains with either upper or lower triangular boundary conditions are defined as

$$\begin{aligned}\mathcal{H}_{XXZ}^{(\frac{1}{2})up.tr.}(k_+, \varepsilon_\pm, \overline{k}_+, \overline{\varepsilon}_\pm) &= \mathcal{H}_{XXZ}^{(\frac{1}{2})gen}(k_\pm, \varepsilon_\pm, \overline{k}_\pm, \overline{\varepsilon}_z)\big|_{k_- = \overline{k}_- = 0} \ , \\ \mathcal{H}_{XXZ}^{(\frac{1}{2})lo.tr.}(k_-, \varepsilon_\pm, \overline{k}_-, \overline{\varepsilon}_\pm) &= \mathcal{H}_{XXZ}^{(\frac{1}{2})gen}(k_\pm, \varepsilon_\pm, \overline{k}_\pm, \overline{\varepsilon}_z)\big|_{k_+ = \overline{k}_+ = 0} \ .\end{aligned}$$

Note that the name upper or lower triangular refers to the triangular form of the K-matrix and its dual. Such Hamiltonian does not enjoy $U(1)$ symmetry (1.2.23).

1.3 Towards integrable higher-spin chains

As we have seen in the previous sections, the FRT formalism is a general tool to construct integrable spin-chains. Now, one may ask how to construct Hamiltonians associated with either closed or open integrable spin-chains with higher spin-interactions. To do so, fusion techniques for the R- and K-matrices have been developed. In 1979 Karowski constructed scattering S-matrices in [Ka79] that describe the interaction of a spin-$\frac{1}{2}$ particle with a spin-j particle. Then, in 1981, Kulish, Reshetikhin and Sklyanin generalized this construction to obtain any spin-j R-matrices, see [KRS81]. Later, using the same formalism, Mezincescu and Nepomechie constructed fused K-matrices in [MN92]. In this section, we review the constructions of both fused R- and K-matrices which use (anti-)symmetric projectors $P^\pm$. Firstly, we recall the construction of the fused R-matrices following [KS82]. Secondly, using this formalism, fused K-matrices are constructed following [MN92]. Finally, this method is applied to the six-vertex R-matrix from (1.1.5) and the spin-$\frac{1}{2}$ scalar K-matrix given in (1.2.3). The Hamiltonian of the open XXZ spin-1 chain is also derived. In subsequent chapters, we will use a more recent fusion procedure [BLN15, LBG23], and the one of [KRS81, MN92] is reviewed in the present section for completeness.

1.3.1 Fusion of R- and K-matrices

Here, the goal is to obtain solutions of the Yang-Baxter equation (1.1.4) for any spin-j. They are obtained using a fusion procedure that we now describe. As we will see below, the fused R-matrices

[6] Recall the algebra homomorphism $\Delta\colon U_q sl_2 \to U_q sl_2 \otimes U_q sl_2$. Any word in $(U_q sl_2)^{\otimes N}$ is constructed using combinations of $\Delta^{N-1}(E)$, $\Delta^{N-1}(F)$, $\Delta^{N-1}(K^{\pm\frac{1}{2}})$ and any element $x \in \mathrm{End}(\mathbb{C}^2)^{\otimes N}$ is obtained from the latter words by mapping $E \to \sigma^+$, $F \to \sigma^-$, $K^{\pm\frac{1}{2}} \to q^{\pm\sigma^z}$.

are constructed from the *fundamental* R-matrix, i.e. the R-matrix that acts in $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, by considering a product of fundamental R-matrices. Let us now recall the construction of the fused spin-1 R-matrix following [KS82].

Notations.

Below we use the notation $\{12\}$ on the R- and K-matrices to indicate that the fusion is applied on the spaces labelled by 1 and 2. Introduce the notation

$$V_{\{2j\}} = V_1 \otimes \ldots \otimes V_{2j} = \underbrace{\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2}_{2j \text{ times}} \ . \tag{1.3.1}$$

Basic Ingredients.

Following [KS82], the fusion of the R-matrices relies on the following assumption. The specialization of $R(u)$ at $u = \beta^{-1}$, for some $\beta \in \mathbb{C}^*$, is proportional to a one-dimensional projector

$$P^-_{\{12\}} = \alpha R_{12}(\beta^{-1}) \ , \tag{1.3.2}$$

where $\alpha \in \mathbb{C}^*$ is fixed by the idempotent property of the projector, i.e. $(P^-_{\{12\}})^2 = P^-_{\{12\}}$. From this projector, two important relations are derived for the fusion. Firstly, we can define another projector

$$P^+_{\{12\}} = \mathbb{I} - P^-_{\{12\}} \ , \tag{1.3.3}$$

which projects onto a three-dimensional space. It is easy to verify that it satisfies the relation

$$P^+_{\{12\}} P^-_{\{12\}} = 0 \ . \tag{1.3.4}$$

Secondly, the specialization of the Yang-Baxter equation (1.1.4) and the latter relation imply that

$$P^-_{\{12\}} R_{13}(u) R_{23}(u\beta) P^+_{\{12\}} = 0 \ . \tag{1.3.5}$$

Indeed, setting $u_1 = u$, $u_2 = u\beta$, $u_3 = 1$ in (1.1.4), multiplying each side of this equation on the right by $P^+_{\{12\}}$, using (1.3.2) and (1.3.4), it follows (1.3.5).

Fusion of R-matrix.

Following [KS82], we introduce fused R-matrices.

Lemma 1.3.1. *The fused R-matrices*

$$R_{\{12\}3}(u) = P^+_{\{12\}} R_{13}(u) R_{23}(u\beta) P^+_{\{12\}} \ , \tag{1.3.6}$$

$$R_{1\{23\}}(u) = P^+_{\{23\}} R_{13}(u) R_{12}(u\beta) P^+_{\{23\}} \ , \tag{1.3.7}$$

satisfy the fused Yang-Baxter equations

$$R_{\{12\}3}(u/v) R_{\{12\}4}(u) R_{34}(v) = R_{34}(v) R_{\{12\}4}(u) R_{\{12\}3}(u/v) \ , \tag{1.3.8}$$

$$R_{12}(u/v)R_{1\{34\}}(u)R_{2\{34\}}(v) = R_{2\{34\}}(v)R_{1\{34\}}(u)R_{12}(u/v) \ . \tag{1.3.9}$$

Proof. We first show (1.3.8). Replace the fused R-matrices in the l.h.s. of (1.3.8) by (1.3.6). Remove the intermediate $P^+_{\{12\}}$ using (1.3.3) and (1.3.5). Then, reorder the R-matrices using the Yang-Baxter equation (1.1.4). Finally, using $\mathbb{I} = P^+_{\{12\}} + P^-_{\{12\}}$ and (1.3.5), one identifies the r.h.s. of (1.3.8). The equation (1.3.9) is shown similarly using now $P^-_{\{23\}}R_{13}(u)R_{12}(u\beta)P^+_{\{23\}} = 0$, instead of (1.3.5). □

Then, one defines fused R-matrices for any j_1, j_2

$$R_{\{12\}\{34\}}(u) = P^+_{\{34\}}R_{\{12\}4}(u)R_{\{12\}3}(u\beta)P^+_{\{34\}} \ , \tag{1.3.10}$$

that satisfy

$$R_{\{12\}\{34\}}(u/v)R_{\{12\}\{56\}}(u)R_{\{34\}\{56\}}(v) = R_{\{34\}\{56\}}(v)R_{\{12\}\{56\}}(u)R_{\{12\}\{34\}}(u/v) \ . \tag{1.3.11}$$

In a subsequent part, we will consider fusion of the K-matrices using the projector $P^+_{\{12\}}$. To this end, it will be necessary to use another fused R-matrix that we now introduce.

Lemma 1.3.2. *The fused R-matrices*

$$R_{3\{12\}}(u) = P^+_{\{21\}}R_{31}(u)R_{32}(u\beta)P^+_{\{21\}} \ , \tag{1.3.12}$$

where $P^+_{\{21\}} = P_{12}P^+_{\{12\}}P_{12}^{-1}$, *satisfy the fused Yang-Baxter equation*

$$R_{3\{12\}}(u/v)R_{4\{12\}}(u)R_{43}(v) = R_{43}(v)R_{4\{12\}}(u)R_{3\{12\}}(u/v) \ . \tag{1.3.13}$$

Proof. It is similar to the proof of Lemma 1.3.1. In particular, one needs to use the relations

$$\begin{aligned} R_{43}(v)R_{42}(u\beta)R_{32}(u\beta/v) &= R_{32}(u\beta/v)R_{42}(u\beta)R_{43}(v) \ , \\ R_{43}(v)R_{41}(u)R_{31}(u/v) &= R_{31}(u/v)R_{41}(u)R_{43}(v) \ , \end{aligned}$$

that come from the Yang-Baxter equation (1.1.4) by setting appropriately u, v and applying permutations. □

Fusion of K-matrix.

Consider a K-matrix that satisfies the following reflection equation [MN92]

$$R_{12}(u/v)K_1(u)R_{21}(uv)K_2(v) = K_2(v)R_{12}(uv)K_1(u)R_{21}(u/v) \ , \tag{1.3.14}$$

where

$$R_{21}(u) = \mathcal{P}_{12}R_{12}(u)\mathcal{P}_{12}^{-1} \ . \tag{1.3.15}$$

Now, we recall the construction of fused K-matrices that solve a reflection equation. It is clear that this reflection equation involves fused R-matrices as constructed in previous subsection, see Lemma 1.3.1 and Lemma 1.3.2. Here, we recall the fusion procedure for the K-matrices introduced by Mezincescu and Nepomechie in [MN92]. As discussed below, the fused K-matrices are constructed using the projector $P^+_{\{12\}}$.

The K-matrices are constructed from the *fundamental* R- and K-matrices as we now recall following [MN92]. This fusion is again based on the assumption that the R-matrix specializes to a one-dimensional projector as in (1.3.2).

Note that, relatively to the reflection algebra; see Definition 1.2.1, the formalism presented here can be applied not only to K-matrices (matrices with scalar entries) but also to K-operators (matrices with algebraic entries). Indeed, the formalism of Mezincescu and Nepomechie in [MN92] can be generalized to K-operators since it is enough that (1.3.2) holds. However, in this chapter only K-matrices are considered. Examples of K-operators, as well as the construction of fused K-operators, will be studied in Chapters 2 and 3. Following [MN92], we introduce fused K-matrices. We denote them by $K_{\{12\}}(u)$.

Lemma 1.3.3. *The fused K-matrix*

$$K_{\{12\}}(u) = P^{+}_{\{12\}} K_1(u) R_{21}(u^2\beta) K_2(u\beta) P^{+}_{\{21\}} \ , \tag{1.3.16}$$

satisfies the reflection equation

$$R_{\{12\}3}(u/v) K_{\{12\}}(u) R_{3\{12\}}(uv) K_3(v) = K_3(v) R_{\{12\}3}(u/v) K_{\{12\}}(u) R_{3\{12\}}(u/v) \ , \tag{1.3.17}$$

where $R_{\{12\}3}(u)$ and $R_{3\{12\}}(u)$ are respectively given by (1.3.6), (1.3.12).

Proof. Replace the fused R- and K-matrices in the l.h.s. of (1.3.17) by their expressions given in (1.3.6), (1.3.12) and (1.3.16), respectively. Remove the intermediate $P^{+}_{\{12\}}$ using

$$P^{-}_{\{21\}} R_{31}(uv) R_{32}(uv\beta) P^{+}_{\{21\}} = 0 \ , \tag{1.3.18}$$

$$P^{-}_{\{12\}} K_1(u) R_{21}(u^2\beta) K_2(u\beta) P^{+}_{\{21\}} = 0 \ , \tag{1.3.19}$$

which are obtained using (1.3.4) together with a specialization of the Yang-Baxter equation (1.1.4) and the reflection equation (1.3.14), respectively. Then, reorder the R- and K-matrices using (1.1.4) and (1.3.14). Finally, using $\mathbb{I} = P^{+}_{12} + P^{-}_{12}$ and the above relations, one identifies the r.h.s. of (1.3.17). □

Spin-j fused R- and K-matrices

The procedure detailed above can be repeatedly applied to construct spin-j R- and K-matrices. They are written solely in terms of the fundamental R- and K-matrices, i.e. of spin-$\frac{1}{2}$. For convenience, we now choose $R(u)$ and $K(u)$ given in (1.1.5), (1.2.3), respectively, as fundamental R- and K-matrices. Let $\{a\} = \{a_1, \dots, a_{2j_1}\}$ be a label of the space $V_{a_1} \otimes \dots \otimes V_{a_{2j_1}}$, with $V_{a_k} \equiv \mathbb{C}^2$. Following [KRS81, Zh95], introduce the symmetric projector $P^{+}_{\{a\}}$ as

$$P^{+}_{\{a\}} = \frac{1}{(2j_1)!} \prod_{k=1}^{2j_1} \left(\sum_{\ell=1}^{k} \mathcal{P}_{a_\ell, a_k} \right) \ , \tag{1.3.20}$$

where $\mathcal{P}$ is the permutation matrix given by

$$\mathcal{P} = \sum_{a,b=1}^{2} E_{ab} \otimes E_{ba} \ , \tag{1.3.21}$$

with $\mathcal{P}_{a_k,a_k} \equiv 1$. We now give the compact form of the fused R- and K-matrices. Recall the notation $V_{\{2j\}}$ from (1.3.1). Firstly, the fused R-matrices are expressed as [KR87, FNR07]

$$R^{(j_1,j_2)}_{\{a\}\{b\}}(u) = P^+_{\{a\}} P^+_{\{b\}} \prod_{k=1}^{2j_1} \prod_{\ell=1}^{2j_2} R_{a_k b_\ell}(uq^{k+\ell-j_1-j_2-1}) \ P^+_{\{a\}} P^+_{\{b\}} \in \mathrm{End}(V_{\{2j_1\}}) \otimes \mathrm{End}(V_{\{2j_2\}}) \ , \tag{1.3.22}$$

where $R^{(\frac{1}{2},\frac{1}{2})}_{a_k b_\ell}(u) \equiv R_{a_k b_\ell}(u)$ is the spin-$\frac{1}{2}$ R-matrix given in (1.1.5). The R-matrices in the product above are ordered in increasing value of k and ℓ. For instance, for $(j_1, j_2) = (\frac{1}{2}, \frac{3}{2})$ the fused R-matrix is

$$R^{(\frac{1}{2},\frac{3}{2})}_{\{a_1\}\{b_1,b_2,b_3\}}(u) = P^+_{\{b_1,b_2,b_3\}} R_{a_1 b_1}(uq^{-1}) R_{a_1 b_2}(u) R_{a_1 b_3}(uq) P^+_{\{b_1,b_2,b_3\}}$$

with

$$P^+_{\{b_1,b_2,b_3\}} = \frac{1}{6}(\mathcal{P}_{b_1 b_2} + \mathbb{I})(\mathcal{P}_{b_1 b_3} + \mathcal{P}_{b_2 b_3} + \mathbb{I}) \ .$$

Secondly, the fused K-matrices are expressed as [Zh95, FNR07]

$$K^{(j)}_{\{a\}}(u) = P^+_{\{a\}} \prod_{k=1}^{2j} \left\{ \left[\prod_{\ell=1}^{k-1} R_{a_\ell a_k}(u^2 q^{k+\ell-2j-1}) \right] K_{a_k}(uq^{k-j-\frac{1}{2}}) \right\} P^+_{\{a\}} \in \mathrm{End}(V_{\{2j\}}) \ , \tag{1.3.23}$$

where $K^{(\frac{1}{2})}_{a_k}(u) \equiv K_{a_k}(u)$ is given in (1.2.3). The products of braces are ordered in the order of increasing k. For instance, for $j = \frac{3}{2}$ one has

$$K^{(\frac{3}{2})}_{\{a_1,a_2,a_3\}}(u) = P^+_{\{a_1,a_2,a_3\}} K_{a_1}(uq^{-1} R_{a_1 a_2}(u^2 q^{-1}) K_{a_2}(u) R_{a_1 a_3}(u^2) R_{a_2 a_3}(u^2 q) K_{a_3}(uq) P^+_{\{a_1,a_2,a_3\}} \ .$$

Finally, let us point out that $R^{(j_1,j_2)}_{\{a\}\{b\}}(u) \in \mathrm{End}(V_{\{2j_1\}}) \otimes \mathrm{End}(V_{\{2j_2\}})$ and $K^{(j_1)}_{\{a\}}(u) \in \mathrm{End}(V_{\{2j_1\}})$. However, the fusion procedure has the effect of projecting the space $\mathbb{C}^{2^{\otimes N}}$ into $\mathbb{C}^{2j+1}$. In order to exhibit this structure, one can apply similarity transformations to transform the R-matrices in (1.3.22) into symmetric R-matrices, and similarly for the fused K-matrices, see [N02]. In Section 1.3.3, we apply the above formulas to construct spin-1 R- and K-matrices and apply such similarity transformations.

1.3.2 TT-relations for the open XXZ spin-chains

In the previous section, we recalled a procedure to obtain higher dimensional R- and K-matrices, using the formalism of [KS82, MN92] (i.e. with the projectors $P^\pm$). Recall that the essential object to construct spin-chains are transfer matrices. Here, we construct spin-j transfer matrices from the latter fused R- and K-matrices. TT-relations were proposed in order to derive transfer matrices of higher spin-chains from transfer matrices of lower spin-chains. Following [FNR07], we also recall

the TT-relations associated to the open XXZ spin-s chains.

Spin-j transfer matrices

Recall that the transfer matrix associated to the open XXZ spin-$\frac{1}{2}$ chain is given in Definition 1.2.7. In order to study higher spin-chains, it is thus necessary to consider a spin-j transfer matrix. Recall the fused R- and K-matrices from (1.3.22), (1.3.23). First, we need to define the spin-j analogues of the dual K-matrix $K^+(u)$ from (1.2.6). Due to Remark 1.2.5, the spin-j K-matrix solution of the dual reflection equation can be constructed as follows [Sk88]. Recall the notation $\{a\}$ above (1.3.20) and $c(u)$ from (1.3.33). Define

$$K^{+(j)}_{\{a\}}(u) = \frac{1}{f^{(j)}(u)} \left(K^{(j)}_{\{a\}}(-u^{-1}q^{-1})\right)^t \Big|_{\varepsilon_\pm \to -\bar{\varepsilon}_\mp, k_\pm \to -\bar{k}_\mp} , \tag{1.3.24}$$

with $K^{+(\frac{1}{2})}(u) \equiv K^+(u)$, where the normalization factor $f^{(\frac{1}{2})}(u) = 1$ and

$$f^{(j)}(u) = \prod_{k=1}^{2j-1} \prod_{\ell=1}^{k} -c(u^2 q^{k+\ell+2-2j}) c(u^2 q^{k+\ell-2j}) \tag{1.3.25}$$

is introduced for further convenience. We now introduce a generalization of the transfer matrix from Definition 1.1.6.

Definition 1.3.4. *Let j, $s \in \frac{1}{2}\mathbb{N}_+$. The transfer matrix for the XXZ open spin-chains with N sites is given by*

$$t^{(j,s)}(u) = \mathrm{tr}_a(K^{+(j)}_a(u) \mathbb{T}^{(j,s)}_{a,N}(u)) , \tag{1.3.26}$$

where the double-row monodromy matrix is

$$\mathbb{T}^{(j,s)}_{a,N}(u) = R^{(j,s)}_{aN}(uv_N^{-1}) \cdots R^{(j,s)}_{a1}(uv_1^{-1}) K^{(j)}_a(u) R^{(j,s)}_{a1}(uv_1) \cdots R^{(j,s)}_{aN}(uv_N) . \tag{1.3.27}$$

One can show for all u, $v \in \mathbb{C}^*$ that

$$\left[t^{(j_1,s)}(u), t^{(j_2,s)}(v)\right] = 0 . \tag{1.3.28}$$

The proof is done similarly to [Sk88, Thm. 1] and requires in particular that the fused R-matrices are invertible and satisfy the crossing unitarity property

$$[R^{(j,s)}(u)]^{t_1} [R^{(j,s)}(u^{-1}q^{-2})]^{t_1} \propto \mathbb{I}_{(2j+1)(2s+1)} ,$$

where t_1 denotes the transposition over the space V_1.

TT-relations

Recall the transfer matrix from Definition 1.3.4. It satisfies the so-called *TT-relations* for $j, s \in \frac{1}{2}\mathbb{N}_+$ [MN92, Zh95, YNZ05]

$$t^{(j,s)}(u) = t^{(j-\frac{1}{2},s)}(uq^{-\frac{1}{2}}) t^{(\frac{1}{2},s)}(uq^{j-\frac{1}{2}}) + \delta^{(s)}(uq^{j-\frac{3}{2}}) t^{(j-1,\{j_n\})}(uq^{-1}) , \tag{1.3.29}$$

where $\delta^{(s)}(u)$ is a product of various quantum determinants given by[7]:

$$\delta^{(s)}(u) = \left[\prod_{k=0}^{2s-1} c(uq^{s+\frac{3}{2}-k})c(uq^{-s-\frac{1}{2}+k})\right]^{2N} \frac{c(u^2)c(u^2q^4)}{c(u^2q)c(u^2q^3)} \tag{1.3.30}$$

$$\times \left(\varepsilon_+^+ + \varepsilon_-^2 + \varepsilon_+\varepsilon_-(u^2q^2 + u^{-2}q^{-2}) - k_+k_-\frac{c(u^2q^2)^2}{c(q)^2}\right) \tag{1.3.31}$$

$$\times \left(\overline{\varepsilon}_+^+ + \overline{\varepsilon}_-^2 + \overline{\varepsilon}_+\overline{\varepsilon}_-(u^2q^2 + u^{-2}q^{-2}) - k_+k_-\frac{c(u^2q^2)^2}{c(q)^2}\right) , \tag{1.3.32}$$

with the scalar function

$$c(u) = u - u^{-1} . \tag{1.3.33}$$

Let us make some important remarks concerning this relation. One can show analytically that it holds for small values of j. For instance, the case $j = 1$ was proven in [MN92]. For higher values of j, the proof of (1.3.29) relies on the following conjecture: the projectors $P^+_{\{a\}}$ defined in (1.3.20) and $P^-_{\{a\}} = \mathbb{I} - P^+_{\{a\}}$ satisfy

$$P^+_{\{12\ldots2j-1\}}P^-_{\{12\ldots2j\}}P^+_{\{12\ldots2j-1\}} = P^+_{\{12\ldots2j-2\}}P^-_{2j-1\,2j} + \sum_{k=1}^{2j-1} P^+_{k\,k+1}X^{(k)}P^-_{k\,k+1} , \tag{1.3.34}$$

for $j = \frac{3}{2}, 2, \ldots$, and for some set of matrices $\{X^{(1)}, \ldots, X^{(2j-1)}\}$. For more details, see [FNR07, App. A] where the relation (1.3.34) is verified up to $j = 3$. Let us mention that with a more recent fusion procedure described in Chapter 3, a different proof for the TT-relations will be given based on the existence of a universal K-matrix.

1.3.3 Application to the open XXZ spin-1 chain

Spin-1 R- and K-matrices

We now apply the fusion procedure of [KRS81, MN92] to construct spin-1 R- and K-matrices. Consider the spin-$\frac{1}{2}$ R- and K-matrices given in (1.1.5) and (1.2.3). The idempotent property of $P^-_{\{12\}}$ and the equation (1.3.2) hold for

$$\alpha = -\frac{1}{(2(q - q^{-1}))} , \qquad \beta = q , \tag{1.3.35}$$

and with

$$P^-_{\{12\}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{1.3.36}$$

[7] We fixed the inhomogeneities $v_n = 1$ for convenience.

From (1.3.3), one gets

$$P^{+}_{\{12\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{1.3.37}$$

Note that the rank of P^+ is 3 while the rank of P^- is 1.

Using the formulas (1.3.6), (1.3.10) and shifting $u \to uq^{-\frac{1}{2}}$ for later convenience, the fused R-matrices are given by

$$R^{(1,\frac{1}{2})}_{\{12\}3}(u) = P^{+}_{\{12\}} R_{13}(uq^{-\frac{1}{2}}) R_{23}(uq^{\frac{1}{2}}) P^{+}_{\{12\}} , \tag{1.3.38}$$

$$R^{(1,1)}_{\{12\}\{34\}}(u) = P^{+}_{\{34\}} R^{(1,\frac{1}{2})}_{\{12\}4}(uq^{-\frac{1}{2}}) R^{(1,\frac{1}{2})}_{\{12\}3}(uq^{\frac{1}{2}}) P^{+}_{\{34\}} . \tag{1.3.39}$$

Note that they are 8×8 and 16×16 matrices, respectively, with ranks 6 and 9. In order to exhibit their spin-1 forms, we need to rearrange them using some similarity transformations. Define the invertible matrices $A^{(1)}$ and $B^{(1)}$ by [N02]

$$A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} , \qquad B^{(1)} = diag\left(([2]_q)^{-\frac{1}{2}}, 1, ([2]_q)^{-\frac{1}{2}}, 1 \right) . \tag{1.3.40}$$

The actions of these matrices are respectively to change the basis and to make the R-matrices symmetric, i.e. invariant by transposition. Below, we apply similarity transformations on (1.3.38) and (1.3.39) to extract $R^{(1,\frac{1}{2})}(u) \in \mathrm{End}(\mathbb{C}^3 \otimes \mathbb{C}^2)$ and $R^{(1,1)}(u) \in \mathrm{End}(\mathbb{C}^3 \otimes \mathbb{C}^3)$. Here we removed the indices in brackets for the latter R-matrices because now they have a standard matrix size. Then, the conjugation actions of $A^{(1)}$ and $B^{(1)}$ on $R^{(1,\frac{1}{2})}_{\{12\}3}(u)$ read:

$$B^{(1)}_{\{12\}} A^{(1)}_{\{12\}} R^{(1,\frac{1}{2})}_{\{12\}3}(u) (A^{(1)}_{\{12\}})^{-1} (B^{(1)}_{\{12\}})^{-1} = \begin{pmatrix} R^{(1,\frac{1}{2})}(u) & 0 & 0 \\ 0 & \cdots & 0 \ 0 \\ 0 & \cdots & 0 \ 0 \end{pmatrix}$$

where

$$R^{(1,\frac{1}{2})}(u) = c(uq^{\frac{1}{2}}) \begin{pmatrix} c(uq^{\frac{3}{2}}) & 0 & 0 & 0 & 0 & 0 \\ 0 & c(uq^{-\frac{1}{2}}) & \sqrt{[2]_q}c(q) & 0 & 0 & 0 \\ 0 & \sqrt{[2]_q}c(q) & c(uq^{\frac{1}{2}}) & 0 & 0 & 0 \\ 0 & 0 & 0 & c(uq^{\frac{1}{2}}) & \sqrt{[2]_q}c(q) & 0 \\ 0 & 0 & 0 & \sqrt{[2]_q}c(q) & c(uq^{-\frac{1}{2}}) & 0 \\ 0 & 0 & 0 & 0 & 0 & c(uq^{\frac{3}{2}}) \end{pmatrix} , \tag{1.3.41}$$

with $c(u)$ from (1.3.33).

Similarly, we get

$$B^{(1)}_{\{34\}} A^{(1)}_{\{34\}} B^{(1)}_{\{12\}} A^{(1)}_{\{12\}} \ R^{(1,1)}_{\{12\}\{34\}}(u) \ (A^{(1)}_{\{12\}})^{-1} (B^{(1)}_{\{12\}})^{-1} (A^{(1)}_{\{34\}})^{-1} (B^{(1)}_{\{34\}})^{-1}$$

$$= \begin{pmatrix} R^{(1,1)}(u) & 0 \cdots & 0 \\ 0 & 0 \cdots & 0 \\ \vdots & \vdots \ddots & \vdots \\ 0 & 0 \cdots & 0 \end{pmatrix} , \tag{1.3.42}$$

where

$$R^{(1,1)}(u) = c(u)c(uq)^2 \begin{pmatrix} a_1(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2(u) & 0 & a_3(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_4(u) & 0 & a_5(u) & 0 & a_6(u) & 0 & 0 \\ 0 & a_3(u) & 0 & a_2(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_5(u) & 0 & a_7(u) & 0 & a_5(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_2(u) & 0 & a_3(u) & 0 \\ 0 & 0 & a_6(u) & 0 & a_5(u) & 0 & a_4(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_3(u) & 0 & a_2(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1(u) \end{pmatrix} \tag{1.3.43}$$

and with

$$a_1(u) = c(uq^2) , \qquad a_2(u) = c(u) , \qquad a_3(u) = c(q^2) , \qquad a_4(u) = \frac{c(u)c(uq^{-1})}{c(uq)} ,$$
$$a_5(u) = \frac{c(q^2)c(u)}{c(uq)} , \qquad a_6(u) = \frac{c(q)c(q^2)}{c(uq)} , \qquad a_7(u) = a_6(u) + a_2(u) .$$

Remark 1.3.5. *The spin-1 R-matrix given above corresponds to the R-matrix of the nineteen vertex model. It was originally obtained in* [ZF80] *by solving directly the Yang-Baxter equation.*

Now, we construct the spin-1 K-matrix using (1.3.16) (we shift the argument $u \to uq^{-\frac{1}{2}}$):

$$K^{(1)}_{\{12\}}(u) = P^{+}_{\{12\}} K_1(uq^{-\frac{1}{2}}) R_{12}(u^2) K_2(uq^{\frac{1}{2}}) P^{+}_{\{21\}} , \tag{1.3.44}$$

where we used that the R-matrix (1.1.5) enjoys the P-symmetry $R_{21}(u) = R_{12}(u)$. Note that it is a 4×4 matrix with rank 3. Here again, we apply the similarity transformations to get

$$B^{(1)}_{\{12\}} A^{(1)}_{\{12\}} K^{(1)}_{\{12\}}(u) (A^{(1)}_{\{12\}})^{-1} (B^{(1)}_{\{12\}})^{-1} = \frac{c(u^2q)}{c(q)} \begin{pmatrix} K^{(1)}(u) & 0 \\ 0 & 0 \end{pmatrix} , \tag{1.3.45}$$

with $K^{(1)}(u) \in \mathrm{End}(\mathbb{C}^3)$ given by

$$K^{(1)}(u) = \begin{pmatrix} x_1(u) & y_1(u) & z(u) \\ \tilde{y}_1(u) & x_2(u) & y_2(u) \\ \tilde{z}(u) & \tilde{y}_2(u) & x_3(u) \end{pmatrix} \tag{1.3.46}$$

$$z(u) = k_+^2 \frac{c(u^2)c(u^2q^{-1})}{c(q)}, \quad y_1(u) = k_+ c(u^2) \sqrt{q+q^{-1}} \left(uq^{-1/2} \varepsilon_+ + u^{-1} q^{1/2} \varepsilon_- \right) ,$$

$$x_1(u)=c(q)\big(\varepsilon_+^2u^2+\varepsilon_-^2u^{-2}+\varepsilon_+\varepsilon_-(q+q^{-1})\big)+k_+k_-c(u^2q^{-1}),\qquad x_3(u)=x_1(u)\big|_{\varepsilon_\pm\to\varepsilon_\mp},$$
$$x_2(u)=c(q)\big(\varepsilon_+^2+\varepsilon_-^2+\varepsilon_+\varepsilon_-c(u^{-2}q)\big)+k_+k_-\frac{(u^4+u^{-4}-q^2-q^{-2})}{q-q^{-1}},\qquad y_2(u)=y_1(u)\big|_{\varepsilon_\pm\to\varepsilon_\mp}\ ,$$

and $\tilde{y}_1(u)=y_1(u)\big|_{k_+\to k_-}$, $\tilde{y}_2(u)=y_2(u)\big|_{k_+\to k_-}$, $\tilde{z}(u)=z(u)\big|_{k_+\to k_-}$. Note that we used the notation $\langle 12\rangle$ for the K-matrix (1.3.46).

Remark 1.3.6. *The spin-1 K-matrix $K^{(1)}(u)$ coincides with the spin-1 K-matrix of class A from* [IOZ96] *provided one identifies* $\cos(\eta)\to\frac{q+q^{-1}}{2}$ *and*

$$\begin{aligned}&\zeta_-\to\varepsilon_+=-\varepsilon_-^{-1}\ ,\quad \mu_-\to\frac{k_+\sqrt{q+q^{-1}}}{q-q^{-1}}\ ,\quad \tilde{\mu}_-\to\frac{k_-\sqrt{q+q^{-1}}}{q-q^{-1}}\ ,\\ &\zeta_+\to\overline{\varepsilon}_+=-\overline{\varepsilon}_-^{-1}\ ,\quad \mu_+\to\frac{\overline{k}_+\sqrt{q+q^{-1}}}{q-q^{-1}}\ ,\quad \tilde{\mu}_+\to\frac{\overline{k}_-\sqrt{q+q^{-1}}}{q-q^{-1}}\ .\end{aligned}\tag{1.3.47}$$

It was obtained by solving directly the reflection equation.

Hamiltonian

Consider the transfer matrix (1.3.26) for $j=s=1$ with the spin-1 R- and K-matrices defined in (1.3.39), (1.3.44), (1.3.24). The transfer matrix forms a commutative family due to (1.3.28) and, similarly to Proposition 1.2.8, one extracts the Hamiltonian as follows:

$$\mathcal{H}_{XXZ}^{(1)gen}-\mathcal{H}_0^{(1)}\propto\frac{d}{du}\Big(\ln(t^{(1,1)}(u))\Big)\Big|_{u=1,\{v_n\}=1}\ .\tag{1.3.48}$$

Using the above relation, one finds that the Hamiltonian of the XXZ spin-1 chain with open boundary conditions is:

$$\begin{aligned}\mathcal{H}_{XXZ}^{(1)gen}(k_\pm,\varepsilon_\pm,\overline{k}_\pm,\overline{\varepsilon}_\pm)&=\mathbf{b}^{(1)}\\ &+\frac{1}{k_+k_--\varepsilon_+\varepsilon_-(q+q^{-1})-(\varepsilon_+^2+\varepsilon_-^2)}\\ &\times\Bigg(\Big(\varepsilon_+\varepsilon_-(q^{-1}-q)+k_+k_-\frac{q+q^{-1}}{q^{-1}-q}\Big)(s_1^z)^2+(\varepsilon_-^2-\varepsilon_+^2)s_1^z+\frac{k_+^2(s_1^+)^2+k_-^2(s_1^-)^2}{q-q^{-1}}\\ &+\frac{\sqrt{2(q+q^{-1})}}{q-q^{-1}}\Big(\varepsilon_+\Big(k_+[s_1^+,s_1^z]_{q^{\frac12}}+k_-[s_1^z,s_1^-]_{q^{\frac12}}\Big)+\varepsilon_-\Big(k_+[s_1^+,s_1^z]_{q^{-\frac12}}+k_-[s_1^z,s_1^-]_{q^{-\frac12}}\Big)\Big)\Bigg)\\ &+\frac{1}{\overline{k}_+\overline{k}_--\overline{\varepsilon}_+\overline{\varepsilon}_-(q+q^{-1})-(\overline{\varepsilon}_+^2+\overline{\varepsilon}_-^2)}\\ &\times\Bigg(\Big(\overline{\varepsilon}_+\overline{\varepsilon}_-(q^{-1}-q)+\overline{k}_+\overline{k}_-\frac{q+q^{-1}}{q^{-1}-q}\Big)(s_N^z)^2+(\overline{\varepsilon}_-^2-\overline{\varepsilon}_+^2)s_N^z+\frac{\overline{k}_+^2(s_N^+)^2+k_-^2(s_N^-)^2}{q-q^{-1}}\\ &+\frac{\sqrt{2(q+q^{-1})}}{q-q^{-1}}\Big(\overline{\varepsilon}_+\Big(\overline{k}_+[s_N^+,s_N^z]_{q^{\frac12}}+\overline{k}_-[s_N^z,s_N^-]_{q^{\frac12}}\Big)+\varepsilon_-\Big(\overline{k}_+[s_N^+,s_N^z]_{q^{-\frac12}}+\overline{k}_-[s_N^z,s_N^-]_{q^{-\frac12}}\Big)\Big)\Bigg)\end{aligned}\tag{1.3.49}$$

where the bulk term is

$$\mathbf{b}^{(1)} = \frac{2}{q^2-q^{-2}}\sum_{n=1}^{N-1}\Bigg(\vec{s}_n\cdot\vec{s}_{n+1} - (\vec{s}_n\cdot\vec{s}_{n+1})^2 - \frac{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2}{2}\left\{s_n^z s_{n+1}^z, s_n^+ s_{n+1}^- + s_n^- s_{n+1}^+\right\} + \frac{\left(q-q^{-1}\right)^2}{2}\left(s_n^z s_{n+1}^z - (s_n^z)^2(s_{n+1}^z)^2 + (s_n^z)^2 + (s_{n+1}^z)^2\right)\Bigg), \tag{1.3.50}$$

we also introduced the anti-commutator $\{A,B\} = AB+BA$, and with the spin-1 matrices:

$$s^x = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad s^y = \frac{i}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix}, \quad s^z = \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}, \tag{1.3.51}$$

$$s^+ = \sqrt{2}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}, \quad s^- = \sqrt{2}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}. \tag{1.3.52}$$

Note that the formula (1.2.22) is useful to compute the logarithmic derivative of the transfer matrix. However, it requires slight adjustments because here we use the spin-1 R- and K-matrices, and also the spin-1 permutation matrix $\mathcal{P}^{(1,1)} = \sum_{a,b=1}^{3} E_{ab}\otimes E_{ba}$.

Similarly to the spin-$\frac{1}{2}$ Hamiltonians given in Example 1.2.9, several special cases are obtained by specializations of the boundary parameters $k_\pm, \varepsilon_\pm, \overline{k}_\pm, \overline{\varepsilon}_\pm$ in the above spin-1 Hamiltonian.

Example 1.3.7. *The spin-1 analogues of the Hamiltonians from Example* 1.2.9 (i) and (ii) are given.

(i) Define the XXZ spin-1 chain with diagonal boundary conditions as

$$\begin{aligned}\mathcal{H}_{XXZ}^{(1)diag}(\varepsilon_\pm,\overline{\varepsilon}_\pm) &= \mathcal{H}_{XXZ}^{(1)gen}(0,\varepsilon_\pm,0,\overline{\varepsilon}_\pm)\\ &= \mathbf{b}^{(1)} + \frac{1}{\varepsilon_+\varepsilon_-(q+q^{-1})+\varepsilon_+^2+\varepsilon_-^2}\left(\varepsilon_+\varepsilon_-(q-q^{-1})(s_1^z)^2 + (\varepsilon_+^2-\varepsilon_-^2)s_1^z\right)\\ &\quad + \frac{1}{\overline{\varepsilon}_+\overline{\varepsilon}_-(q+q^{-1})+\overline{\varepsilon}_+^2+\overline{\varepsilon}_-^2}\left(\overline{\varepsilon}_+\overline{\varepsilon}_-(q-q^{-1})(s_N^z)^2 + (\overline{\varepsilon}_+^2-\overline{\varepsilon}_-^2)s_N^z\right).\end{aligned}$$

It satisfies

$$\left[\mathcal{H}_{XXZ}^{(1)diag}(\varepsilon_\pm,\overline{\varepsilon}_\pm), \sum_{k=1}^{N} s_k^z\right] = 0,$$

where s^z is given in (1.3.51).

(ii) Define the XXZ spin-1 chain with special ($U_q sl_2$-invariant) boundary conditions as

$$\begin{aligned}\mathcal{H}_{XXZ}^{(1)sp} &= \mathcal{H}_{XXZ}^{(1)gen}(0,\varepsilon_+=0,\varepsilon_-,0,\overline{\varepsilon}_+,\overline{\varepsilon}_-=0)\\ &= \mathbf{b}^{(1)} - s_1^z + s_N^z.\end{aligned}$$

For any x in a N-tensor product of spin-1 irreducible representation of $U_q sl_2$, this Hamiltonian

satisfies

$$\left[\mathcal{H}^{(1)sp}_{XXZ}, x\right] = 0 \ .$$

Now, it remains to find the eigenvalues and eigenvectors of the Hamiltonians constructed. Since, it goes beyond the scope of this thesis, we do not diagonalize it here. As discussed in the introduction, several tools are available. For instance, the ABA can be used if a reference state is known, see Appendix A for the diagonalization of the XXZ spin-$\frac{1}{2}$ chain with either periodic or diagonal boundary conditions following [STF79] and [Sk88]. For more general boundary conditions, when a reference state is not known, a generalization was proposed under the name of MABA, see [BC13, BP14].

In this chapter, we reviewed the representation-dependent approach for the construction of quantum integrable systems. Now, we want to consider a representation-independent approach to construct quantum integrable systems that correspond to the right part of Figure 3. As we will see in subsequent chapters, this corresponds to work with K-operators instead of K-matrices.

Chapter 2

The q-Onsager algebra and its alternating central extension

In this chapter, several presentations for the q-Onsager algebra O_q and its alternating central extension $\mathcal{A}_q$ are reviewed, as well as their comodule algebra structures. Among these presentations, the FRT type for $\mathcal{A}_q$ with its spin-1/2 K-operator will be of main importance in the following, see Definition 2.1.1. Briefly speaking, this chapter introduces the fundamental elements for studying the algebraic approach to quantum integrable systems as depicted in the right part of Figure 3. The algebra $\mathcal{A}_q$ is interesting from the point of view of quantum integrable systems because all known integrable boundary conditions for the XXZ spin-1/2 chains can be obtained from the specialization of the commutative subalgebra of $\mathcal{A}_q$. In order to consider any spin-j, fused K-operators for $\mathcal{A}_q$ of spin-j are required. The latter will be constructed in Chapter 3.

This chapter is organized as follows.

In Section 2.1, three presentations of the alternating central extension of the q-Onsager algebra, denoted $\mathcal{A}_q$, are reviewed. The PBW basis of $\mathcal{A}_q$ as well as its central elements are also given.

In Section 2.2, three presentations of the q-Onsager algebra, denoted O_q, are reviewed. This algebra is related with $\mathcal{A}_q$ by setting the central elements of $\mathcal{A}_q$ to some scalars. The PBW basis of O_q, written in terms of its root vectors, and its PBW basis are also given.

In Section 2.3, Hopf algebras are recalled as well as some examples such as the quantum algebra $U_q sl_2$, the quantum affine algebra $U_q \widehat{sl}_2$ and the quantum Loop algebra $\mathcal{L}U_q sl_2$.

Finally, in Section 2.4, the comodule algebra structures for O_q and $\mathcal{A}_q$ are given.

2.1 The alternating central extension of the q-Onsager algebra

In this section, we review three presentations for $\mathcal{A}_q$ from [BS09, BB17, T21a], as well as their Poincaré-Bikhoff-Witt bases. Then, using a K-operator coming from the FRT-type presentation for $\mathcal{A}_q$, we recall the center of this algebra which is extracted from the quantum-determinant of Sklyanin [Sk88]. Finally, we introduce a spin-1/2 generating function in the commutative subalgebra for $\mathcal{A}_q$.

2.1.1 Three presentations for $\mathcal{A}_q$

The alternating central extension of the q-Onsager algebra $\mathcal{A}_q$ is known to have three presentations that are now reviewed. The first one takes the form of a reflection algebra satisfied by a K-operator which entries are generating functions in the generators of $\mathcal{A}_q$ [BS09], recall Definition 1.2.1. It is called a FRT type presentation. This latter presentation is the one that plays a central role in this thesis. Note that part of the material in this subsection is taken from [BS09, BB17, T21a].

- *FRT type.*

First, recall the R-matrix associated with the tensor product of two-dimensional evaluation representations of $U_q\widehat{sl}_2$ is given in (1.1.5).

Definition 2.1.1 ([BS09]). *$\mathcal{A}_q$ is an associative algebra with alternating generators $\{\mathsf{W}_{-k}, \mathsf{W}_{k+1}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1} | k \in \mathbb{N}\}$. Introduce the generating functions:*

$$\mathcal{W}_+(u) = \sum_{k\in\mathbb{N}} \mathsf{W}_{-k} U^{-k-1} \ , \quad \mathcal{W}_-(u) = \sum_{k\in\mathbb{N}} \mathsf{W}_{k+1} U^{-k-1} \ , \tag{2.1.1}$$

$$\mathcal{G}_+(u) = \sum_{k\in\mathbb{N}} \mathsf{G}_{k+1} U^{-k-1} \ , \quad \mathcal{G}_-(u) = \sum_{k\in\mathbb{N}} \tilde{\mathsf{G}}_{k+1} U^{-k-1} \ , \tag{2.1.2}$$

where we use the shorthand notation $U = (qu^2 + q^{-1}u^{-2})/(q+q^{-1})$ and the expansion in u^{-1}:

$$U^{-1} = (q+q^{-1}) \sum_{\ell=0}^{\infty} (-1)^\ell (qu^2)^{-2\ell-1} \ . \tag{2.1.3}$$

The defining relations are given by

$$R^{(\frac{1}{2},\frac{1}{2})}(u/v)\ \mathcal{K}_1^{(\frac{1}{2})}(u)\ R^{(\frac{1}{2},\frac{1}{2})}(uv)\ \mathcal{K}_2^{(\frac{1}{2})}(v) \ = \ \mathcal{K}_2^{(\frac{1}{2})}(v)\ R^{(\frac{1}{2},\frac{1}{2})}(uv)\ \mathcal{K}_1^{(\frac{1}{2})}(u)\ R^{(\frac{1}{2},\frac{1}{2})}(u/v) \tag{2.1.4}$$

with the R-matrix (1.1.5) and the fundamental K-operator

$$\mathcal{K}^{(\frac{1}{2})}(u) = \begin{pmatrix} uq\mathcal{W}_+(u) - u^{-1}q^{-1}\mathcal{W}_-(u) & \frac{1}{k_-(q+q^{-1})}\mathcal{G}_+(u) + \frac{k_+(q+q^{-1})}{q-q^{-1}} \\ \frac{1}{k_+(q+q^{-1})}\mathcal{G}_-(u) + \frac{k_-(q+q^{-1})}{q-q^{-1}} & uq\mathcal{W}_-(u) - u^{-1}q^{-1}\mathcal{W}_+(u) \end{pmatrix} \ , \tag{2.1.5}$$

where $k_\pm \in \mathbb{C}^*$.

For convenience, introduce the parametrization:

$$\rho = k_+k_-(q+q^{-1})^2 \ \in \mathbb{C}^* \ . \tag{2.1.6}$$

Explicitly, one obtains the defining relations between the generating functions $\{\mathcal{W}_\pm(u), \mathcal{G}_\pm(u)\}$ as follows. First, insert the R-matrix (1.1.5) and the K-operator (2.1.5) in the reflection equation (2.1.4). Then, a comparison of the matrix entries of each sides of the latter equation yields [BS09, Def. 2.2]:

$$[\mathcal{W}_\pm(u), \mathcal{W}_\pm(v)] = 0 \ , \tag{2.1.7}$$

$$[\mathcal{W}_+(u), \mathcal{W}_-(v)] + [\mathcal{W}_-(u), \mathcal{W}_+(v)] = 0 \ , \tag{2.1.8}$$

$$[\mathcal{G}_\epsilon(u), \mathcal{W}_\pm(v)] + [\mathcal{W}_\pm(u), \mathcal{G}_\epsilon(v)] = 0, \quad \epsilon = \pm \ , \tag{2.1.9}$$

$$[\mathcal{G}_\pm(u), \mathcal{G}_\pm(v)] = 0 \ , \tag{2.1.10}$$

$$[\mathcal{G}_+(u), \mathcal{G}_-(v)] + [\mathcal{G}_-(u), \mathcal{G}_+(v)] = 0 \ , \tag{2.1.11}$$

$$\begin{aligned}(U - V)\,[\mathcal{W}_\pm(u), \mathcal{W}_\mp(v)] = {} & \frac{(q - q^{-1})}{\rho(q + q^{-1})}\,(\mathcal{G}_\pm(u)\mathcal{G}_\mp(v) - \mathcal{G}_\pm(v)\mathcal{G}_\mp(u)) \\ & + \frac{1}{(q + q^{-1})}\,(\mathcal{G}_\pm(u) - \mathcal{G}_\mp(u) + \mathcal{G}_\mp(v) - \mathcal{G}_\pm(v)) \ ,\end{aligned} \tag{2.1.12}$$

$$\begin{aligned}\mathcal{W}_\pm(u)\mathcal{W}_\pm(v) - \mathcal{W}_\mp(u)\mathcal{W}_\mp(v) + \frac{1}{\rho(q^2 - q^{-2})}\,[\mathcal{G}_\pm(u), \mathcal{G}_\mp(v)] \\ + \frac{1 - UV}{U - V}\,(\mathcal{W}_\pm(u)\mathcal{W}_\mp(v) - \mathcal{W}_\pm(v)\mathcal{W}_\mp(u)) = 0 \ ,\end{aligned} \tag{2.1.13}$$

$$\begin{aligned}U\,[\mathcal{G}_\mp(v), \mathcal{W}_\pm(u)]_q - V\,[\mathcal{G}_\mp(u), \mathcal{W}_\pm(v)]_q - (q - q^{-1})\,(\mathcal{W}_\mp(u)\mathcal{G}_\mp(v) - \mathcal{W}_\mp(v)\mathcal{G}_\mp(u)) \\ + \rho\,(U\mathcal{W}_\pm(u) - V\mathcal{W}_\pm(v) - \mathcal{W}_\mp(u) + \mathcal{W}_\mp(v)) = 0 \ ,\end{aligned} \tag{2.1.14}$$

$$\begin{aligned}U\,[\mathcal{W}_\mp(u), \mathcal{G}_\mp(v)]_q - V\,[\mathcal{W}_\mp(v), \mathcal{G}_\mp(u)]_q - (q - q^{-1})\,(\mathcal{W}_\pm(u)\mathcal{G}_\mp(v) - \mathcal{W}_\pm(v)\mathcal{G}_\mp(u)) \\ + \rho\,(U\mathcal{W}_\mp(u) - V\mathcal{W}_\mp(v) - \mathcal{W}_\pm(u) + \mathcal{W}_\pm(v)) = 0 \ .\end{aligned} \tag{2.1.15}$$

Lemma 2.1.2. *There exists an automorphism σ of $\mathcal{A}_q$ that sends*

$$\mathcal{W}_\pm(u) \mapsto \mathcal{W}_\mp(u) \ , \quad \mathcal{G}_\pm(u) \mapsto \mathcal{G}_\mp(u) \ . \tag{2.1.16}$$

Proof. It is obtained from the reflection equation (2.1.4). Indeed, note that the R-matrix is such that $R^{(\frac{1}{2},\frac{1}{2})}(u) = MR^{(\frac{1}{2},\frac{1}{2})}(u)M$, with $M = \sigma_x \otimes \sigma_x$. Consider the conjugation of the K-operator by σ_x. Its entries read: $(\sigma_x \mathcal{K}^{(\frac{1}{2})}(u)\sigma_x)_{i,j} = (\mathcal{K}^{(\frac{1}{2})}(u))_{3-i,3-j}$ for $1 \leq i, j \leq 2$. Then, multiplying (2.1.4) on both sides by $M \otimes M$, the automorphism σ follows. □

• *Alternating type.*

Now, from the equations (2.1.7)-(2.1.15) and using (2.1.1), (2.1.2), the relations between the alternating generators $\{\mathsf{W}_{-k}, \mathsf{W}_{k+1}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1} | k \in \mathbb{N}\}$ are extracted. The following is the second presentation of the algebra $\mathcal{A}_q$ [BS09, T21a]:

$$\big[\mathsf{W}_0, \mathsf{W}_{k+1}\big] = \big[\mathsf{W}_{-k}, \mathsf{W}_1\big] = \frac{1}{(q + q^{-1})}\big(\tilde{\mathsf{G}}_{k+1} - \mathsf{G}_{k+1}\big) \ , \tag{2.1.17}$$

$$\big[\mathsf{W}_0, \mathsf{G}_{k+1}\big]_q = \big[\tilde{\mathsf{G}}_{k+1}, \mathsf{W}_0\big]_q = \rho\mathsf{W}_{-k-1} - \rho\mathsf{W}_{k+1} \ , \tag{2.1.18}$$

$$\big[\mathsf{G}_{k+1}, \mathsf{W}_1\big]_q = \big[\mathsf{W}_1, \tilde{\mathsf{G}}_{k+1}\big]_q = \rho\mathsf{W}_{k+2} - \rho\mathsf{W}_{-k} \ , \tag{2.1.19}$$

$$\big[\mathsf{W}_{-k}, \mathsf{W}_{-\ell}\big] = 0 \ , \quad \big[\mathsf{W}_{k+1}, \mathsf{W}_{\ell+1}\big] = 0 \ , \tag{2.1.20}$$

$$\big[\mathsf{W}_{-k},\mathsf{W}_{\ell+1}\big]+\big[\mathsf{W}_{k+1},\mathsf{W}_{-\ell}\big]=0\ , \tag{2.1.21}$$
$$\big[\mathsf{W}_{-k},\mathsf{G}_{\ell+1}\big]+\big[\mathsf{G}_{k+1},\mathsf{W}_{-\ell}\big]=0\ , \tag{2.1.22}$$
$$\big[\mathsf{W}_{-k},\tilde{\mathsf{G}}_{\ell+1}\big]+\big[\tilde{\mathsf{G}}_{k+1},\mathsf{W}_{-\ell}\big]=0\ , \tag{2.1.23}$$
$$\big[\mathsf{W}_{k+1},\mathsf{G}_{\ell+1}\big]+\big[\mathsf{G}_{k+1},\mathsf{W}_{\ell+1}\big]=0\ , \tag{2.1.24}$$
$$\big[\mathsf{W}_{k+1},\tilde{\mathsf{G}}_{\ell+1}\big]+\big[\tilde{\mathsf{G}}_{k+1},\mathsf{W}_{\ell+1}\big]=0\ , \tag{2.1.25}$$
$$\big[\mathsf{G}_{k+1},\mathsf{G}_{\ell+1}\big]=0\ ,\quad \big[\tilde{\mathsf{G}}_{k+1},\tilde{\mathsf{G}}_{\ell+1}\big]=0\ , \tag{2.1.26}$$
$$\big[\tilde{\mathsf{G}}_{k+1},\mathsf{G}_{\ell+1}\big]+\big[\mathsf{G}_{k+1},\tilde{\mathsf{G}}_{\ell+1}\big]=0\ , \tag{2.1.27}$$

where $\ell\in\mathbb{N}$. Note that these two presentations were originally obtained in the context of quantum integrable systems [BK05a].

• *Compact type.*

Let us now give the third presentation for $\mathcal{A}_q$, it is called the compact presentation, see[8] [T21b]. In this case, $\mathcal{A}_q$ is generated by W_0, W_1, $\{\mathsf{G}_{k+1}\}_{k\in\mathbb{N}}$ subject to the following defining relations:

$$[\mathsf{W}_0,[\mathsf{W}_0,[\mathsf{W}_0,\mathsf{W}_1]_q]_{q^{-1}}]=\rho[\mathsf{W}_0,\mathsf{W}_1]\ , \tag{2.1.28}$$
$$[\mathsf{W}_1,[\mathsf{W}_1,[\mathsf{W}_1,\mathsf{W}_0]_q]_{q^{-1}}]=\rho[\mathsf{W}_1,\mathsf{W}_0]\ , \tag{2.1.29}$$
$$[\mathsf{W}_1,\mathsf{G}_1]=[\mathsf{W}_1,[\mathsf{W}_1,\mathsf{W}_0]_q]\ , \tag{2.1.30}$$
$$[\mathsf{G}_1,\mathsf{W}_0]=[[\mathsf{W}_1,\mathsf{W}_0]_q,\mathsf{W}_0]\ , \tag{2.1.31}$$
$$[\mathsf{W}_1,\mathsf{G}_{k+1}]=\rho^{-1}[\mathsf{W}_1,[\mathsf{W}_1,[\mathsf{W}_0,\mathsf{G}_k]_q]_q]\ ,\qquad k\geq 1\ , \tag{2.1.32}$$
$$[\mathsf{G}_{k+1},\mathsf{W}_0]=\rho^{-1}[[[\mathsf{G}_k,\mathsf{W}_1]_q,\mathsf{W}_0]_q,\mathsf{W}_0]\ ,\qquad k\geq 1\ , \tag{2.1.33}$$
$$[\mathsf{G}_{k+1},\mathsf{G}_{\ell+1}]=0\ ,\qquad k,\ell\in\mathbb{N}\ , \tag{2.1.34}$$

where ρ is given by (2.1.6). The $\mathcal{A}_q$-generators in the above presentation are called *essential*.

Remark 2.1.3. *A variation of this presentation can be obtained* [T21b]. *The image of the alternating generators with respect to the automorphism σ from Lemma* 2.1.2 is given by:

$$\sigma(\mathsf{W}_{-k})=\mathsf{W}_{k+1}\ ,\qquad \sigma(\mathsf{W}_{k+1})=\mathsf{W}_{-k}\ ,\qquad \sigma(\mathsf{G}_{k+1})=\tilde{\mathsf{G}}_{k+1}\ ,\qquad \sigma(\tilde{\mathsf{G}}_{k+1})=\mathsf{G}_{k+1}\ ,$$

for $k\in\mathbb{N}$. Therefore, another compact presentation of $\mathcal{A}_q$ generated by $\{\mathsf{W}_0,\mathsf{W}_1,\tilde{\mathsf{G}}_{k+1}|k\in\mathbb{N}\}$ follows by applying σ to the relations (2.1.28)–(2.1.34).

2.1.2 PBW bases for $\mathcal{A}_q$

In the following, we will need Poincaré-Bikhoff-Witt bases that we now recall.

Definition 2.1.4. *Let $\mathcal{A}$ denote an algebra. A Poincaré-Bikhoff-Witt (PBW) basis for $\mathcal{A}$ consists of a subset $\Omega\subseteq\mathcal{A}$ and a linear order $<$ on Ω such that the following is a basis for the vector space $\mathcal{A}$:*

$$a_1a_2\cdots a_n\qquad n\in\mathbb{N}\ ,\qquad a_1,a_2,\ldots,a_n\in\Omega\ ,\qquad a_1\leq a_2\leq\cdots\leq a_n\ .$$

We now give the PBW basis for each presentation of $\mathcal{A}_q$.

[8] The defining relations of $\mathcal{A}_q$ given in (2.1.28)-(2.1.34) coincide with the compact presentation of $\mathcal{A}_q$ given in [T21b, Prop. 12.1] for the identification $\mathsf{W}_0\mapsto\mathcal{W}_0$, $\mathsf{W}_1\mapsto\mathcal{W}_1$, $\mathsf{G}_{k+1}\mapsto\mathcal{G}_{k+1}$, $\rho\mapsto-(q^2-q^{-2})^2$, for some $q\in\mathbb{C}^*$.

- $\{\mathcal{W}_\pm(u), \mathcal{G}_\pm(u)\}$

The first presentation of $\mathcal{A}_q$ in terms of the generating functions $\mathcal{W}_\pm(u)$ and $\mathcal{G}_\pm(u)$ will be of high importance for us, recall Definition 2.1.1. We will thus encounter various combinations of these generating functions that will need reordering. A PBW basis for $\mathcal{A}_q$ can be constructed in terms of these generating functions in the linear order $<$ that satisfies

$$\mathcal{W}_+(u) < \mathcal{G}_+(v) < \mathcal{G}_-(w) < \mathcal{W}_-(x)$$

where u, v, w, x are indeterminate. According to this chosen order, any words in $\{\mathcal{W}_\pm(u_i), \mathcal{G}_\pm(u_i)\}_{i\in\mathbb{N}}$ can be written in terms of ordered expressions that take the form:

$$\prod_{r=1}^{R} \mathcal{W}_+(u_r) \prod_{p=1}^{P} \mathcal{G}_+(u_p) \prod_{m=1}^{M} \mathcal{G}_-(u_m) \prod_{t=1}^{T} \mathcal{W}_-(u_t) \ , \qquad R, P, M, T \in \mathbb{N} \ . \tag{2.1.35}$$

The rewriting of any word of $\mathcal{A}_q$ in the PBW basis requires ordering relations that are given in the next lemma.

Lemma 2.1.5. *The following relations hold in $\mathcal{A}_q$:*

$$\begin{aligned}
\mathcal{G}_-(v)\mathcal{G}_+(u) &= \mathcal{G}_+(u)\mathcal{G}_-(v) + \rho(q^2-q^{-2})\Big(\mathcal{W}_+(u)\mathcal{W}_+(v) - \mathcal{W}_-(u)\mathcal{W}_-(v) \\
&\quad + \frac{1-UV}{U-V}\big(\mathcal{W}_+(u)\mathcal{W}_-(v) - \mathcal{W}_+(v)\mathcal{W}_-(u)\big)\Big) \ , \qquad (2.1.36)\\
\mathcal{W}_-(v)\mathcal{W}_+(u) &= \mathcal{W}_+(u)\mathcal{W}_-(v) + \frac{1}{V-U}\Big(\frac{(q-q^{-1})}{\rho(q+q^{-1})}\big(\mathcal{G}_+(u)\mathcal{G}_-(v) - \mathcal{G}_+(v)\mathcal{G}_-(u)\big) \\
&\quad + \frac{1}{q+q^{-1}}\big(\mathcal{G}_+(u) - \mathcal{G}_-(u) + \mathcal{G}_-(v) - \mathcal{G}_+(v)\big)\Big) \ , \qquad (2.1.37)\\
\mathcal{W}_-(v)\mathcal{G}_+(u) &= \frac{q}{U-V}\Big(\big(Uq^{-1}-Vq\big)\mathcal{G}_+(u)\mathcal{W}_-(v) - (q-q^{-1})\Big(\mathcal{W}_+(u)\mathcal{G}_+(v) \\
&\quad - \mathcal{W}_+(v)\mathcal{G}_+(u) - U\mathcal{G}_+(v)\mathcal{W}_-(u)\Big) + \rho\big(U\mathcal{W}_-(u) - V\mathcal{W}_-(v) - \mathcal{W}_+(u) + \mathcal{W}_+(v)\big)\Big) \ , \qquad (2.1.38)\\
\mathcal{W}_-(v)\mathcal{G}_-(u) &= \frac{q^{-1}}{V-U}\Big(\big(Vq^{-1}-Uq\big)\mathcal{G}_-(u)\mathcal{W}_-(v) - (q-q^{-1})\big(\mathcal{W}_+(u)\mathcal{G}_-(v) \\
&\quad - \mathcal{W}_+(v)\mathcal{G}_-(u) - U\mathcal{G}_-(v)\mathcal{W}_-(u)\big) + \rho\big(U\mathcal{W}_-(u) - V\mathcal{W}_-(v) - \mathcal{W}_+(u) + \mathcal{W}_+(v)\big)\Big) \ , \qquad (2.1.39)\\
\mathcal{G}_+(v)\mathcal{W}_+(u) &= \frac{q}{U-V}\Big(\big(Uq-Vq^{-1}\big)\mathcal{W}_+(u)\mathcal{G}_+(v) - (q-q^{-1})\big(\mathcal{G}_+(v)\mathcal{W}_-(u) \\
&\quad + \rho\big(U\mathcal{W}_+(u) - V\mathcal{W}_+(v) - \mathcal{W}_-(u) + \mathcal{W}_-(v)\big)\Big) \ , \qquad (2.1.40)\\
\mathcal{G}_-(v)\mathcal{W}_+(u) &= \frac{1}{q(V-U)}\Big(\big(Vq-Uq^{-1}\big)\mathcal{W}_+(u)\mathcal{G}_-(v) - (q-q^{1})\big(\mathcal{G}_-(v)\mathcal{W}_-(u) \\
&\quad - \mathcal{G}_-(u)\mathcal{W}_-(v) + V\mathcal{W}_+(v)\mathcal{G}_-(u)\big) + \rho\big(U\mathcal{W}_+(u) - V\mathcal{W}_+(v) - \mathcal{W}_-(u) + \mathcal{W}_-(v)\big)\Big) \ , \qquad (2.1.41)
\end{aligned}$$

where $V = (qv^2 + q^{-1}v^{-2})/(q+q^{-1})$.

Proof. The first two ordering relations (2.1.36), (2.1.37), are obtained directly from (2.1.9) and (2.1.10).

The third relation (2.1.38) follows from (2.1.11) and (2.1.13) by replacing the element which is not in the chosen order. The relations (2.1.39)–(2.1.41) are derived similarly. □

Below, PBW bases for the other two presentations of $\mathcal{A}_q$ are given but since we mostly work with the first one, i.e. with the generating functions $\mathcal{W}_\pm(u)$, $\mathcal{G}_\pm(u)$, we omit ordering relations for the alternating generators and the essential ones.

- $\{\mathsf{W}_{-k}, \mathsf{W}_{k+1}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1} | k \in \mathbb{N}\}$

Different types of PBW bases for $\mathcal{A}_q$ are known [T21a]. For instance, a PBW basis for $\mathcal{A}_q$ can be constructed in terms of the alternating generators

$$\{\mathsf{W}_{-k}\}_{k\in\mathbb{N}}\ ,\quad \{\mathsf{G}_{\ell+1}\}_{\ell\in\mathbb{N}}\ ,\quad \{\tilde{\mathsf{G}}_{m+1}\}_{m\in\mathbb{N}}\ ,\quad \{\mathsf{W}_{n+1}\}_{n\in\mathbb{N}}\ ,$$

in the linear order $<$ that satisfies[9]

$$\mathsf{W}_{-k} < \mathsf{G}_{\ell+1} < \tilde{\mathsf{G}}_{m+1} < \mathsf{W}_{n+1}\ , \qquad k,\ell,m,n \in \mathbb{N}\ . \tag{2.1.42}$$

2.1.3 The center of $\mathcal{A}_q$

The center of $\mathcal{A}_q$ has been studied in details in [T21a]. In the FRT presentation, the center is generated by the so-called quantum determinant associated with the K-operator (2.1.5). It is a generating function for central elements of $\mathcal{A}_q$, given by [Sk88]:

$$\Gamma(u) = \mathrm{tr}_{12}\big(P^-_{12}\mathcal{K}_1^{(\frac{1}{2})}(u)R_{12}^{(\frac{1}{2},\frac{1}{2})}(qu^2)\mathcal{K}_2^{(\frac{1}{2})}(uq)\big)\ , \tag{2.1.43}$$

where $P^- = (1-\mathcal{P})/2$ and with the R-matrix (1.1.5).

Proposition 2.1.6 ([BB17, T21a]). *The quantum determinant of the fundamental K-operator*

$$\Gamma(u) = \frac{(u^2q^2 - u^{-2}q^{-2})}{2(q-q^{-1})}\left(\Delta^{(\frac{1}{2})}(u) - \frac{2\rho}{q-q^{-1}}\right)\ , \tag{2.1.44}$$

with

$$\begin{aligned}\Delta^{(\frac{1}{2})}(u) = &-(q-q^{-1})(q^2+q^{-2})\Big(\mathcal{W}_+(u)\mathcal{W}_+(uq) + \mathcal{W}_-(u)\mathcal{W}_-(uq)\Big)\\ &+ (q-q^{-1})(u^2q^2+u^{-2}q^{-2})\Big(\mathcal{W}_+(u)\mathcal{W}_-(uq) + \mathcal{W}_-(u)\mathcal{W}_+(uq)\Big)\\ &- \frac{(q-q^{-1})}{\rho}\Big(\mathcal{G}_+(u)\mathcal{G}_-(uq) + \mathcal{G}_-(u)\mathcal{G}_+(uq)\Big) - \ \mathcal{G}_+(u) - \mathcal{G}_+(uq) - \mathcal{G}_-(u) - \mathcal{G}_-(uq)\ ,\end{aligned} \tag{2.1.45}$$

is such that $\big[\Gamma(u), \mathcal{K}_{mn}^{(\frac{1}{2})}(v)\big] = 0$. *The coefficients of* $\Gamma(u) \in \mathcal{A}_q((u^{-1}))$ *generate the center of* $\mathcal{A}_q$.

[9] For a different choice of ordering, the proof of the PBW basis is given in [T21a].

We can expand $\Delta^{(\frac{1}{2})}(u)$ as a formal power series in u^{-1}

$$\Delta^{(\frac{1}{2})}(u) = \sum_{k=0}^{\infty} u^{-2k-2} c_{k+1} \Delta_{k+1} \ , \tag{2.1.46}$$

with

$$c_k = -\frac{(q+q^{-1})^k (q^k + q^{-k})}{q^{2k}} \ . \tag{2.1.47}$$

Explicit expressions for the coefficients Δ_{k+1} in terms of the alternating generators of $\mathcal{A}_q$ can be found in [BB17, Lem. 2.1]. For instance, the first elements read:

$$\Delta_1 = \mathsf{G}_1 + \tilde{\mathsf{G}}_1 - (q-q^{-1})(\mathsf{W}_0\mathsf{W}_1 + \mathsf{W}_1\mathsf{W}_0) \ , \tag{2.1.48}$$

$$\Delta_2 = \mathsf{G}_2 + \tilde{\mathsf{G}}_2 - \frac{q^2-q^{-2}}{q^2+q^{-2}}(q^{-1}\mathsf{W}_0\mathsf{W}_2 + q\mathsf{W}_2\mathsf{W}_0 + q^{-1}\mathsf{W}_1\mathsf{W}_{-1} + q\mathsf{W}_{-1}\mathsf{W}_1) \tag{2.1.49}$$

$$+ \frac{q-q^{-1}}{q^2+q^{-2}}\big((q^2+q^{-2})(\mathsf{W}_0^2 + \mathsf{W}_1^2) + \frac{\tilde{\mathsf{G}}_1\mathsf{G}_1 + \mathsf{G}_1\tilde{\mathsf{G}}_1}{\rho}\big) \ . \tag{2.1.50}$$

We see that the constant term of $\Delta^{(\frac{1}{2})}(u) - 2\rho/(q-q^{-1})$ is $-2\rho/(q-q^{-1})$ and so it is invertible. Therefore, by [T21d, Lem. 4.1] it follows that $\Gamma(u)$ is invertible too.

Remark 2.1.7. *A central element of $\mathcal{A}_q((t))$ denoted $\mathcal{Z}(t)$ has been proposed in* [T21a, Def. 8.4]. *It is easily checked that adapting its expression to our conventions, we have the correspondence*

$$\frac{\Gamma(uq^{-\frac{1}{2}})}{c(u^2q)} \to \mathcal{Z}(t) \ , \tag{2.1.51}$$

together with

$$\rho \to -(q^2-q^{-2})^2 \ , \qquad q^{-1}u^{-2} \to t \ , \qquad \mathcal{W}_{\mp}(uq^{\frac{1}{2}}) \to S\mathcal{W}^{\pm}(S) \ , \qquad \mathcal{W}_{\mp}(uq^{-\frac{1}{2}}) \to T\mathcal{W}^{\pm}(T)$$
$$\mathcal{G}_{+}(uq^{\frac{1}{2}}) + \rho/(q-q^{-1}) \to \mathcal{G}(S) \ , \qquad \mathcal{G}_{-}(uq^{-\frac{1}{2}}) + \rho/(q-q^{-1}) \to \tilde{\mathcal{G}}(T) \ .$$

To conclude this subsection, let us point out the unitarity property of the fundamental K-operator $\mathcal{K}^{(\frac{1}{2})}(u)$ given in (2.1.5), that will be used in Section 3.6.

Lemma 2.1.8.

$$\mathcal{K}^{(\frac{1}{2})}(u^{-1})\mathcal{K}^{(\frac{1}{2})}(u) = \mathcal{K}^{(\frac{1}{2})}(u)\mathcal{K}^{(\frac{1}{2})}(u^{-1}) = \frac{\Gamma(uq^{-1})}{c(u^{-2})}\,\mathbb{I}_2 \ , \tag{2.1.52}$$

where $c(u)$ and $\Gamma(u)$ are respectively given in (1.3.33), (2.1.44).

Proof. Note that by definition of the generating functions in (2.1.1), (2.1.2), one has $\mathcal{W}_{\pm}(u^{-1}) = \mathcal{W}_{\pm}(uq^{-1})$ and $\mathcal{G}_{\pm}(u^{-1}) = \mathcal{G}_{\pm}(uq^{-1})$. Then, using the ordering relations of $\mathcal{A}_q$ given in Lemma 2.1.5, by straightforward calculation one gets (2.1.52). □

We call the above property (2.1.52) the unitarity property of the fundamental K-operator.

Remark 2.1.9. *Since $\Gamma(u)$ is invertible, it follows that $\mathcal{K}^{(\frac{1}{2})}(u)$ is invertible too and its inverse is given by:*

$$\left[\mathcal{K}^{(\frac{1}{2})}(u)\right]^{-1} = \frac{c(u^{-2})}{\Gamma(uq^{-1})}\,\mathcal{K}^{(\frac{1}{2})}(u^{-1}) \ . \tag{2.1.53}$$

2.1.4 Commutative subalgebra

There exists an abelian subalgebra of $\mathcal{A}_q$, denoted $\mathcal{I}$, and it is now recalled. The latter is obtained from the FRT type presentation of $\mathcal{A}_q$. First, define the generating function for $\mathcal{A}_q$ as

$$\mathsf{T}^{(\frac{1}{2})}(u) = \mathrm{tr}(K^+(u)\mathcal{K}(u)) \in \mathcal{A}_q \ , \tag{2.1.54}$$

where $K^+(u)$ and $\mathcal{K}(u) \equiv \mathcal{K}^{(\frac{1}{2})}(u)$ are respectively given in (1.2.6), (2.1.5). Note that it forms a commutative family [Sk88]

$$[\mathsf{T}^{(\frac{1}{2})}(u), \mathsf{T}^{(\frac{1}{2})}(v)] = 0 \ , \quad u, v \in \mathbb{C} \ .$$

By a direct computation, one finds that the latter generating function is expressed as a linear combination of the generators $\{\mathcal{I}_{2k+1} | k \in \mathbb{N}\}$ [BK05b, BB12]

$$\mathsf{T}^{(\frac{1}{2})}(u) = (u^2q^2 - u^{-2}q^{-2})\,(\mathcal{I}(u) + \mathcal{I}_0) \qquad \text{with} \quad \mathcal{I}(u) = \sum_{k\in\mathbb{N}} \mathcal{I}_{2k+1} U^{-k-1} \ , \tag{2.1.55}$$

where

$$\mathcal{I}_{2k+1} = \overline{\varepsilon}_+ \mathsf{W}_{-k} + \overline{\varepsilon}_- \mathsf{W}_{k+1} + \frac{1}{q^2 - q^{-2}} \left(\frac{\overline{k}_+}{k_+} \mathsf{G}_{k+1} + \frac{\overline{k}_-}{k_-} \tilde{\mathsf{G}}_{k+1} \right) \ , \tag{2.1.56}$$

with $\overline{\varepsilon}_\pm, \overline{k}_\pm \in \mathbb{C}$, $k_\pm \in \mathbb{C}^*$ and

$$\mathcal{I}_0 = \frac{\rho}{(q - q^{-1})(q^2 - q^{-2})} \left(\frac{\overline{k}_+}{k_+} + \frac{\overline{k}_-}{k_-} \right) . \tag{2.1.57}$$

Then, $\mathcal{I}$ is generated by the elements $\{\mathcal{I}_{2k+1} | k \in \mathbb{N}\}$, i.e.

$$[\mathcal{I}_{2k+1}, \mathcal{I}_{2\ell+1}] = 0 \ , \quad k, \ell \in \mathbb{N} \ .$$

2.2 The q-Onsager algebra O_q

Let us now review the q-Onsager algebra. As for $\mathcal{A}_q$, we give several presentations of O_q as well as its PBW basis. Finally, the definition of the Onsager algebra is given. It is known to be obtained from the limit $q = 1$ of the q-Onsager algebra.

2.2.1 Three presentations for O_q

The q-Onsager algebra O_q originally appeared in the context of P- and Q-polynomial scheme [T99] and later on in quantum integrable systems, in the study of the open XXZ spin-$\frac{1}{2}$ chain [B04]. It was understood later that $\mathcal{A}_q$ is isomorphic to the alternating central extension of the q-Onsager

algebra [BB17, T21a], i.e. we have an isomorphism of algebras

$$\mathcal{A}_q \cong O_q \otimes Z \ . \tag{2.2.1}$$

Here, Z denotes the center of $\mathcal{A}_q$ which is generated by the elements Δ_{k+1} from (2.1.46). Therefore, the algebra O_q is obtained from $\mathcal{A}_q$ by fixing the Δ_{k+1}'s to some scalars as we now recall.

• *FRT type.*

This presentation takes the form of a reflection equation with the same K-operator as for $\mathcal{A}_q$, see Definition 2.1.1, but in addition the quantum determinant in (2.1.44) is fixed as

$$\Gamma(u) \mapsto \frac{(u^2q^2 - u^{-2}q^{-2})}{2(q-q^{-1})}\left(\delta^{(\frac{1}{2})}(u) - \frac{2\rho}{q-q^{-1}}\right) \ , \tag{2.2.2}$$

with

$$\delta^{(\frac{1}{2})}(u) = 2\sum_{k=0}^{\infty} u^{-2k-2} c_{k+1}\delta_{k+1} \ , \tag{2.2.3}$$

for certain scalars $\delta_{k+1} \in \mathbb{C}$ and with c_k in (2.1.47), or equivalently we set $\Delta_{k+1} = 2\delta_{k+1}$. Note that Δ_k are algebraically independent and so all δ_k can be fixed independently. Due to the isomorphism (2.2.1), any choice of scalars δ_k gives isomorphic quotients O_q.

• *Alternating type.*

Another presentation of alternating type is given in terms of alternating generators $\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1}\}_{k\in\mathbb{N}}$ [BB17, T21a]. The defining relations are the same as for $\mathcal{A}_q$, see (2.1.17)-(2.1.27), and in addition the central elements Δ_{k+1} from (2.1.46) are now fixed to $2\delta_{k+1}$.

Define the image of the alternating generators of $\mathcal{A}_q$ into O_q [BB17, T21c]:

$$\mathsf{W}_{-k} \mapsto \mathcal{W}_{-k} \ , \quad \mathsf{W}_{k+1} \mapsto \mathcal{W}_{k+1} \ , \quad \mathsf{G}_{k+1} \mapsto \mathcal{G}_{k+1}, \quad \tilde{\mathsf{G}}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1} \ . \tag{2.2.4}$$

In O_q, given k fixed the above alternating generators can be recursively expressed as combinations of lower ones $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_{-1}, \mathcal{W}_2, \mathcal{G}_1, \tilde{\mathcal{G}}_1, \ldots$ [BB17, Prop. 3.1]. Let $\{\delta_k | k \in \mathbb{N}\}$ be fixed scalars. Let $i, j, k, \ell \in \mathbb{N}_+$ and denote $\overline{k} = k \mod 2$ and $[\frac{k}{2}] = \frac{k-\overline{k}}{2}$. One has:

$$\begin{aligned}\mathcal{G}_{k+1} = &-\sum_{\ell=0}^{[\frac{k}{2}]} \sum_{i+j=2\ell+1-\overline{k+1}} \frac{f_{ij}^{(k)}}{2}\mathbb{W}_{ij} - \sum_{\ell=0}^{[\frac{k}{2}]-\overline{k+1}} \sum_{i+j=2\ell+\overline{k+1}} \frac{e_{ij}^{(k)}}{2}\mathbb{F}_{ij} - \sum_{\ell=0}^{[\frac{k}{2}]-1} \frac{d_\ell^{(k)}}{2}\mathbb{G}_{2(l+1)-\overline{k+1}} \\ &+ \frac{(q+q^{-1})}{2}\big[\mathcal{W}_{k+1}, \mathcal{W}_0\big] + \delta_{k+1} \ ,\end{aligned} \tag{2.2.5}$$

where

$$\begin{aligned}\mathbb{G}_{i+1} &= \mathcal{G}_{i+1} + \tilde{\mathcal{G}}_{i+1} \ , \\ \mathbb{W}_{ij} &= (q-q^{-1})(\mathcal{W}_{-i}\mathcal{W}_{j+1} + \mathcal{W}_{i+1}\mathcal{W}_{-j}) \ , \\ \mathbb{F}_{ij} &= (q-q^{-1})\Big((q^2+q^{-2})(\mathcal{W}_{-i}\mathcal{W}_{-j} + \mathcal{W}_{i+1}\mathcal{W}_{j+1}) + \frac{1}{\rho}(\mathcal{G}_{j+1}\tilde{\mathcal{G}}_{i+1} + \tilde{\mathcal{G}}_{j+1}\mathcal{G}_{i+1})\Big) \ ,\end{aligned}$$

and

$$
\begin{aligned}
e_{ij}^{(k)} &= -c_{k+1}^{-1} \sum_{m=0}^{[\frac{k}{2}]-\frac{i+j+\overline{k+1}}{2}} w^i_{[\frac{k}{2}]-\frac{i+j+\overline{k+1}}{2}-m} w^j_m q^{-2j-4m-2} \ , \\
f_{ij}^{(k)} &= c_{k+1}^{-1} \sum_{m=0}^{[\frac{k}{2}]-\frac{i+j+\overline{k+1}-1}{2}} w^i_{[\frac{k}{2}]-\frac{i+j+\overline{k+1}-1}{2}-m} \Big(w^j_m + w^j_{m-1}\Big) q^{-2j-4m} \ , \\
d_\ell^{(k)} &= -c_{k+1}^{-1} w^{2\ell+1-\overline{k+1}}_{[\frac{k}{2}]-\ell} (1+q^{-2k-2}) \ ,
\end{aligned}
$$

with c_{k+1} from given in (2.1.47) and

$$
w^i_m = (-1)^m \frac{(m+i)!}{m!i!} (q+q^{-1})^{i+1} q^{-i-2m-1} \ .
$$

Also, one has:

$$
\mathcal{W}_{-k-1} = \frac{1}{\rho} \big[\mathcal{W}_0, \mathcal{G}_{k+1}\big]_q + \mathcal{W}_{k+1} \ . \tag{2.2.6}
$$

The expressions for the other alternating generators are given by $\mathcal{W}_{k+1} = \Omega(\mathcal{W}_{-k})$, $\tilde{\mathcal{G}}_{k+1} = \Omega(\mathcal{G}_{k+1})$, where the automorphism Ω of O_q is such that:

$$
\Omega(\mathcal{W}_{-k}) = \mathcal{W}_{k+1} \ , \quad \Omega(\mathcal{W}_{k+1}) = \mathcal{W}_{-k} \ , \quad \Omega(\mathcal{G}_{k+1}) = \tilde{\mathcal{G}}_{k+1} \ , \quad \Omega(\tilde{\mathcal{G}}_{k+1}) = \mathcal{G}_{k+1} \ .
$$

Iterating (2.2.5), (2.2.6), the alternating generators can be written as polynomials of $\mathcal{W}_0, \mathcal{W}_1$ and scalars δ_{k+1}. For instance,

$$
\begin{aligned}
\mathcal{G}_1 &= q\mathcal{W}_1\mathcal{W}_0 - q^{-1}\mathcal{W}_0\mathcal{W}_1 + \delta_1 \ , \qquad (2.2.7) \\
\mathcal{W}_{-1} &= \frac{1}{\rho}\left((q^2+q^{-2})\mathcal{W}_0\mathcal{W}_1\mathcal{W}_0 - \mathcal{W}_0^2\mathcal{W}_1 - \mathcal{W}_1\mathcal{W}_0^2\right) + \mathcal{W}_1 + \frac{\delta_1(q-q^{-1})}{\rho}\mathcal{W}_0 \ , \\
\mathcal{G}_2 &= \frac{1}{\rho(q^2+q^{-2})}\Big((q^{-3}+q^{-1})\mathcal{W}_0^2{\mathcal{W}_1}^2 - (q^3+q){\mathcal{W}_1}^2\mathcal{W}_0^2 + (q^{-3}-q^3)(\mathcal{W}_0{\mathcal{W}_1}^2\mathcal{W}_0 + \mathcal{W}_1\mathcal{W}_0^2\mathcal{W}_1) \\
&\quad - (q^{-5}+q^{-3}+2q^{-1})\mathcal{W}_0\mathcal{W}_1\mathcal{W}_0\mathcal{W}_1 + (q^5+q^3+2q)\mathcal{W}_1\mathcal{W}_0\mathcal{W}_1\mathcal{W}_0 + \rho(q-q^{-1})(\mathcal{W}_0^2+{\mathcal{W}_1}^2)\Big) \\
&\quad + \frac{\delta_1(q-q^{-1})}{\rho}\big(q\mathcal{W}_1\mathcal{W}_0 - q^{-1}\mathcal{W}_0\mathcal{W}_1\big) + \delta_2 - \frac{\delta_1^2(q-q^{-1})}{\rho(q^2+q^{-2})} \ , \\
\mathcal{W}_{-2} &= w_1\mathcal{W}_0^3{\mathcal{W}_1}^2 + w_2{\mathcal{W}_0}^2\mathcal{W}_1^2\mathcal{W}_0 + w_3\big({\mathcal{W}_1}\mathcal{W}_0^2\mathcal{W}_1\mathcal{W}_0 + \mathcal{W}_0^2\mathcal{W}_1\mathcal{W}_0\mathcal{W}_1\big) + w_4\mathcal{W}_0\mathcal{W}_1\mathcal{W}_0\mathcal{W}_1\mathcal{W}_0 \\
&\quad + w_5{\mathcal{W}_1}^2{\mathcal{W}_0}^3 + w_6\mathcal{W}_0{\mathcal{W}_1}^2{\mathcal{W}_0}^2 + w_7\mathcal{W}_1{\mathcal{W}_0}^3\mathcal{W}_1 \\
&\quad + w_8\mathcal{W}_0\mathcal{W}_1^2 + w_9\mathcal{W}_1^2\mathcal{W}_0 + w_{10}\big(\mathcal{W}_0^2\mathcal{W}_1 + \mathcal{W}_1\mathcal{W}_0^2\big) + \ w_{11}\mathcal{W}_1\mathcal{W}_0\mathcal{W}_1 + w_{12}\mathcal{W}_0\mathcal{W}_1\mathcal{W}_0 \\
&\quad + \ w_{13}\mathcal{W}_0^3 + w_{14}\mathcal{W}_0 + \ w_{15}\mathcal{W}_1 \ ,
\end{aligned}
$$

where

$$
w_1 = \frac{1}{\rho^2}, \quad w_2 = -\frac{[2]_q[8]_q}{\rho^2[4]_q^2}, \quad w_3 = -\frac{[4]_q}{\rho^2[2]_q}, \quad w_4 = \frac{1}{\rho^2}\left(\frac{[2]_q[3]_q[8]_q}{[4]_q^2} + 1\right),
$$

$$w_5 = \frac{1}{\rho^2[3]_q}, \quad w_6 = -\frac{[2]_q[8]_q}{\rho^2[3]_q[4]_q}, \quad w_7 = \frac{[2]_q^2}{\rho^2[3]_q[4]_q}, \quad w_8 = -\frac{1}{\rho}, \quad w_9 = -\frac{1}{\rho[3]_q},$$
$$w_{10} = -\frac{(q-q^{-1})}{\rho^2}\delta_1, \quad w_{11} = \frac{1}{\rho^2[3]_q}\left(\frac{[2]_q[3]_q[8]_q}{[4]_q^2}+1\right), \quad w_{12} = \frac{(q-q^{-1})[4]_q}{\rho^2[2]_q}\delta_1,$$
$$w_{13} = \frac{1}{\rho}\frac{(q-q^{-1})^2[2]_q}{[4]_q}, \quad w_{14} = 1 - \frac{(q-q^{-1})^2[2]_q}{\rho^2[4]_q}\delta_1^2 + \frac{(q-q^{-1})}{\rho}\delta_2, \quad w_{15} = \frac{(q-q^{-1})}{\rho}\delta_1.$$

- *Compact type.*

Recall that, in the compact presentation, the algebra $\mathcal{A}_q$ is generated by $\mathsf{W}_0, \mathsf{W}_1$, $\{\mathsf{G}_{k+1}\}_{k\in\mathbb{N}}$. It was found in [BB16] that the generators $\{\mathsf{G}_{k+1}\}_{k\in\mathbb{N}}$ are expressed solely in terms of W_0, W_1 and central elements $\Delta_{k+1} \in \mathcal{A}_q$. As a consequence, the compact presentation of O_q is generated by $\mathcal{W}_0$, $\mathcal{W}_1$ as we now recall.

Definition 2.2.1 ([T99, B04]). *The q-Onsager algebra O_q is generated by $\mathcal{W}_0$, $\mathcal{W}_1$, subject to the defining relations:*

$$[\mathcal{W}_0, [\mathcal{W}_0, [\mathcal{W}_0, \mathcal{W}_1]_q]_{q^{-1}}] = \rho[\mathcal{W}_0, \mathcal{W}_1], \tag{2.2.8}$$
$$[\mathcal{W}_1, [\mathcal{W}_1, [\mathcal{W}_1, \mathcal{W}_0]_q]_{q^{-1}}] = \rho^*[\mathcal{W}_1, \mathcal{W}_0], \tag{2.2.9}$$

where $\rho^* = \rho$ is given in (2.1.6). The equations (2.2.8), (2.2.9) are called the q-Dolan/Grady relations.

In addition to the FRT, alternating or compact type, a presentation of Lusztig's type for O_q in terms of real and imaginary root vectors $\{B_{n\delta+\alpha_0}\}_{n\in\mathbb{N}}, \{B_{n\delta+\alpha_1}\}_{n\in\mathbb{N}}, \{B_{n\delta}\}_{n\in\mathbb{N}_+}$ is known [BK17]. This will be recalled in the next subsection while introducing the PBW basis for O_q.

2.2.2 PBW basis for O_q

Two PBW bases are known for the q-Onsager algebra. From the perspective of the alternating type presentation, a PBW basis for O_q is conjectured to be obtained by the elements [BB17, Conj. 1] (see also [T21c, Conj. 16.2])

$$\{\mathcal{W}_{-k}\}_{k\in\mathbb{N}}, \qquad \{\mathcal{G}_{\ell+1}\}_{\ell\in\mathbb{N}}, \qquad \{\mathcal{W}_{m+1}\}_{m\in\mathbb{N}}$$

in the linear order $<$ that satisfies

$$\mathcal{W}_{-k} < \mathcal{G}_{\ell+1} < \mathcal{W}_{m+1} \quad k,\ell,m \in \mathbb{N}.$$

From the perspective of Drinfeld type presentation for O_q, there exists a PBW basis that involves the root vectors of O_q [BK17]

$$\{B_{n\delta+\alpha_0}\}_{n\in\mathbb{N}}, \qquad \{B_{n\delta+\alpha_1}\}_{n\in\mathbb{N}}, \qquad \{B_{n\delta}\}_{n\in\mathbb{N}_+}. \tag{2.2.10}$$

These elements are recursively defined as follows [BK17, T21a]. Writing $B_\delta = q^{-2}\mathcal{W}_1\mathcal{W}_0 - \mathcal{W}_0\mathcal{W}_1$ we have

$$B_{\alpha_0} = \mathcal{W}_0, \qquad B_{\delta+\alpha_0} = \mathcal{W}_1 + \frac{q[B_\delta, \mathcal{W}_0]}{(q-q^{-1})(q^2-q^{-2})}, \tag{2.2.11}$$

$$B_{n\delta+\alpha_0} = B_{(n-2)\delta+\alpha_0} + \frac{q[B_\delta, B_{(n-1)\delta+\alpha_0}]}{(q-q^{-1})(q^2-q^{-2})} \qquad n \geq 2 \tag{2.2.12}$$

and

$$B_{\alpha_1} = \mathcal{W}_1, \qquad B_{\delta+\alpha_1} = \mathcal{W}_0 - \frac{q[B_\delta, \mathcal{W}_1]}{(q-q^{-1})(q^2-q^{-2})}, \tag{2.2.13}$$

$$B_{n\delta+\alpha_1} = B_{(n-2)\delta+\alpha_1} - \frac{q[B_\delta, B_{(n-1)\delta+\alpha_1}]}{(q-q^{-1})(q^2-q^{-2})} \qquad n \geq 2. \tag{2.2.14}$$

Moreover for $n \geq 1$,

$$B_{n\delta} = q^{-2}B_{(n-1)\delta+\alpha_1}\mathcal{W}_0 - \mathcal{W}_0 B_{(n-1)\delta+\alpha_1} + (q^{-2}-1)\sum_{\ell=0}^{n-2} B_{\ell\delta+\alpha_1}B_{(n-\ell-2)\delta+\alpha_1}\ . \tag{2.2.15}$$

By [BK17, Prop. 5.12] the elements $\{B_{n\delta}\}_{n\in\mathbb{N}_+}$ mutually commute. Assuming q is transcendental over $\mathbb{K}$, then a PBW basis for O_q is obtained by the elements (2.2.10) in any linear order [BK17, Thm. 4.5].

Let us now briefly discuss the specialization $q = 1$ for the q-Onsager algebra and its alternating central extension $\mathcal{A}_q$. The former is the Onsager Lie algebra O and the latter is its alternating central extension $\mathcal{A}$.

Firstly, the FRT type presentation for O takes the form of a non-standard classical Yang-Baxter algebra (a classical version of the reflection equation) [BBC18, Thm. 1]. For $\mathcal{A}$, it is given in [BC18, Prop. 4.1]. It was proven in [BC18, Thm. 2] that these two FRT presentations are isomorphic, i.e. $\mathcal{A} \cong O$. Secondly, the alternating presentation for $\mathcal{A}$ was obtained in [BC18, Def. 4.1], see also [T21e, Rem. 5.10] for the limit $q = 1$ of $\mathcal{A}_q$. Finally, the compact presentation for O is generated by A_0, A_1, subject to the Dolan/Grady relations [Pe87]

$$[A_0,[A_0,[A_0,A_1]]] = 16[A_0,A_1]\ , \quad [A_1,[A_1,[A_1,A_0]]] = 16[A_1,A_0]\ .$$

These relations are the $q = 1$ limit of the q-Dolan/Grady relations (2.2.8), (2.2.9), see [T21e, Rem. 5.5].

2.3 Hopf algebras

Let us now recall the definition of a Hopf algebra. It can be found in any quantum groups book, see for instance [KS12, Chap. 1]. We also give some examples of Hopf algebras, namely: the quantum algebra $U_q sl_2$, the quantum affine algebra $U_q\widehat{sl}_2$ and the quantum loop algebra $\mathcal{L}U_q sl_2$.

2.3.1 Definition

Let us first briefly recall the notion of an algebra and a coalgebra. An algebra $\mathcal{A}$ over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space with two linear maps $\mu\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ and $\iota\colon \mathbb{K}\to\mathcal{A}$, called respectively the multiplication and the unit, that satisfy:

$$\mu\circ(\mathrm{id}\otimes\mu)=\mu\circ(\mu\otimes\mathrm{id})\ ,\qquad \mu\circ(\mathrm{id}\otimes\iota)=\mathrm{id}=\mu\circ(\iota\otimes\mathrm{id})\ .$$

A coalgebra $\mathcal{C}$ over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space with two linear maps: $\Delta\colon \mathcal{C}\to\mathcal{C}\otimes\mathcal{C}$ and $\epsilon\colon \mathcal{C}\to\mathbb{K}$, called respectively the coproduct and the counit, that satisfy:

$$(\mathrm{id}\otimes\Delta)\circ\Delta=(\Delta\otimes\mathrm{id})\circ\Delta\ ,\qquad (\mathrm{id}\otimes\epsilon)\circ\Delta=\mathrm{id}=(\epsilon\otimes\mathrm{id})\circ\Delta\ .$$

Definition 2.3.1. *A Hopf algebra is a $\mathbb{K}$-vector space H such that:*

(i) H is an algebra and a coalgebra over a field $\mathbb{K}$.

(ii) the coproduct and the counit are algebra homomorphisms.

(iii) H is equipped with a $\mathbb{K}$-linear map $S\colon H\to H$, called the antipode, that obeys

$$\mu\circ(S\otimes id)\circ\Delta=\mu\circ\epsilon=\mu\circ(id\otimes S)\circ\Delta\ .$$

Remark 2.3.2. *The above point (ii) is equivalent to the condition* [KS12, Prop. 3] *that the multiplication and the unit are coalgebra homomorphisms.*

Remark 2.3.3. *A bialgebra is the same as a Hopf algebra but without antipode.*

2.3.2 $U_q sl_2$, $U_q\widehat{sl}_2$ and $\mathcal{L}U_q sl_2$

We first recall the definition of $U_q sl_2$.

Definition 2.3.4. *The algebra $U_q sl_2$ is a Hopf algebra generated by the elements $E, F, K^{\pm\frac{1}{2}}$ satisfying:*

$$\begin{aligned} K^{\frac12}E&=qEK^{\frac12}\ , & K^{\frac12}F&=q^{-1}FK^{\frac12}\ ,\\ [E,F]&=\frac{K-K^{-1}}{q-q^{-1}}\ , & K^{\frac12}K^{-\frac12}&=K^{-\frac12}K^{\frac12}=1\ . \end{aligned}\tag{2.3.1}$$

Let $\Delta\colon U_q sl_2\to U_q sl_2\otimes U_q sl_2$, $\epsilon\colon U_q sl_2\to\mathbb{C}$ and $S\colon U_q sl_2\to U_q sl_2$ be respectively the coproduct, the counit and the antipode. They are given by:

$$\begin{gathered}\Delta(E)=E\otimes K^{-\frac12}+K^{\frac12}\otimes E\ ,\qquad \Delta(F)=F\otimes K^{-\frac12}+K^{\frac12}\otimes F\ ,\\ \Delta(K^{\pm\frac12})=K^{\pm\frac12}\otimes K^{\pm\frac12}\ ,\end{gathered}\tag{2.3.2}$$

and

$$\epsilon(E)=\epsilon(F)=0\ ,\qquad \epsilon(K^{\pm\frac12})=1\ ,\tag{2.3.3}$$

$$S(E)=-q^{-1}E\ ,\qquad S(F)=-qF\ ,\qquad S(K^{\pm\frac12})=K^{\mp\frac12}\ .\tag{2.3.4}$$

We also recall that the Casimir central element of $U_q sl_2$ is given by

$$\begin{aligned} C &= (q-q^{-1})^2 FE + qK + q^{-1}K^{-1} \\ &= (q-q^{-1})^2 EF + q^{-1}K + qK^{-1} \ . \end{aligned} \tag{2.3.5}$$

Note that the monomials $\{E^r K^{\pm\frac{s}{2}} F^t \mid r,s,t \in \mathbb{N}\}$ form a basis for $U_q sl_2$, see for instance [KS12, Chap. 3].

Now, the Drinfeld-Jimbo presentation of the quantum affine algebra $U_q\widehat{sl}_2$ is recalled [J85, Dr86].

Definition 2.3.5. *Define the 2×2 extended Cartan matrix a_{ij}, $i,j \in \{0,1\}$ with $a_{ii}=2$, $a_{ij}=-2$ for $i \neq j$. The quantum affine algebra $U_q\widehat{sl}_2$ is a Hopf algebra generated by the elements $E_i, F_i, K_i^{\pm\frac{1}{2}}$, $i \in \{0,1\}$ satisfying:*

$$K_i^{\frac{1}{2}} E_j = q^{\frac{a_{ij}}{2}} E_j K_i^{\frac{1}{2}} \ , \qquad K_i^{\frac{1}{2}} F_j = q^{-\frac{a_{ij}}{2}} F_j K_i^{\frac{1}{2}} \ , \qquad [E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q-q^{-1}} \ , \tag{2.3.6}$$

$$K_i^{\frac{1}{2}} K_i^{-\frac{1}{2}} = K_i^{-\frac{1}{2}} K_i^{\frac{1}{2}} = 1 \ , \qquad K_0^{\frac{1}{2}} K_1^{\frac{1}{2}} = K_1^{\frac{1}{2}} K_0^{\frac{1}{2}} \ , \tag{2.3.7}$$

with the q-Serre relations:

$$[E_i,[E_i,[E_i,E_j]_q]_{q^{-1}}] = 0 \ , \qquad [F_i,[F_i,[F_i,F_j]_q]_{q^{-1}}] = 0 \ . \tag{2.3.8}$$

Let $\Delta\colon U_q\widehat{sl}_2 \to U_q\widehat{sl}_2 \otimes U_q\widehat{sl}_2$, $\epsilon\colon U_q\widehat{sl}_2 \to \mathbb{C}$ and $S\colon U_q\widehat{sl}_2 \to U_q\widehat{sl}_2$ be respectively the coproduct, the counit and the antipode. They are given by:

$$\begin{gathered} \Delta(E_i) = E_i \otimes K_i^{\frac{1}{2}} + K_i^{-\frac{1}{2}} \otimes E_i \ , \qquad \Delta(F_i) = F_i \otimes K_i^{\frac{1}{2}} + K_i^{-\frac{1}{2}} \otimes F_i \ , \\ \Delta(K_i^{\pm\frac{1}{2}}) = K_i^{\pm\frac{1}{2}} \otimes K_i^{\pm\frac{1}{2}} \ , \end{gathered} \tag{2.3.9}$$

and

$$\epsilon(E_i) = \epsilon(F_i) = 0 \ , \qquad \epsilon(K_i^{\pm\frac{1}{2}}) = 1 \ , \tag{2.3.10}$$

$$S(E_i) = -qE_i \ , \qquad S(F_i) = -q^{-1}F_i \ , \qquad S(K_i^{\pm\frac{1}{2}}) = K_i^{\mp\frac{1}{2}} \ . \tag{2.3.11}$$

The element $K_0^{\frac{1}{2}} K_1^{\frac{1}{2}}$ is central. The algebra $U_q\widehat{sl}_2$ has an automorphism ν defined by

$$\nu(E_i) = E_i K_i^{\frac{1}{2}} \ , \qquad \nu(F_i) = K_i^{-\frac{1}{2}} F_i \ , \qquad \nu(K_i^{\pm\frac{1}{2}}) = K_i^{\pm\frac{1}{2}} \ . \tag{2.3.12}$$

Besides the above definition, other presentations for $U_q\widehat{sl}_2$ exist. The so-called Drinfeld second presentation (Drinfeld loop presentation) was found later in [Dr88]. It is given in terms of generators $\{x_m^{\pm}, h_r, K^{\pm 1}, c^{\pm\frac{1}{2}1} | m \in \mathbb{Z}, r \in \mathbb{Z}^*\}$ and relations. In [RS90], a FRT presentation for $U_q\widehat{sl}_2$ was obtained. It involves affine L-operators satisfying several relations similar to the RLL equation (1.1.9) that will be discussed more in Section 3.2.3. Moreover, a PBW basis was constructed using the root-vectors of $U_q\widehat{sl}_2$, see [Da93, Be94].

In this thesis we will mainly work with Definition 2.3.5 and the FRT presentation with the

affine L-operators. Their calculation, the PBW basis and the root vectors of $U_q\widehat{sl}_2$ are recalled in Appendix B.

Finally, the quantum Loop algebra is recalled.

Definition 2.3.6. *The quantum loop algebra $\mathcal{L}U_qsl_2$ is the unital associative $\mathbb{C}$-algebra generated by the elements $E_i, F_i, K_i^{\pm\frac{1}{2}}$; $i \in \{0,1\}$ which satisfy the defining relations of the $U_q\widehat{sl}_2$ algebra (2.3.6)-(2.3.8)* with the extra relation

$$K_0^{\frac{1}{2}} K_1^{\frac{1}{2}} = 1 = K_1^{\frac{1}{2}} K_0^{\frac{1}{2}} \ . \tag{2.3.13}$$

The quantum loop algebra $\mathcal{L}U_qsl_2$ is a Hopf algebra with the coproduct as in (2.3.9), counit in (2.3.10) and antipode (2.3.11) with the substitution $K_0^{\frac{1}{2}} = K_1^{-\frac{1}{2}}$.

2.3.3 Evaluation map

Let us now recall the evaluation map $\mathsf{ev}_u \colon \mathcal{L}U_qsl_2 \to U_qsl_2$ where $u \in \mathbb{C}^*$. First, consider the algebra map $\varphi \colon \mathcal{L}U_qsl_2 \to U_qsl_2$ defined by the equations:

$$\begin{aligned} &\varphi(E_0) = F \ , \qquad \varphi(F_0) = E \ , \qquad \varphi(K_0^{\frac{1}{2}}) = K^{-\frac{1}{2}} \ , \\ &\varphi(E_1) = E \ , \qquad \varphi(F_1) = F \ , \qquad \varphi(K_1^{\frac{1}{2}}) = K^{\frac{1}{2}} \ . \end{aligned} \tag{2.3.14}$$

Let ϕ_u be the principal gradation automorphism $\phi_u \colon \mathcal{L}U_qsl_2 \to \mathcal{L}U_qsl_2$, where $u \in \mathbb{C}^*$ [BDGZ93]. It is defined by

$$\begin{aligned} &\phi_u(E_0) = u^{-1}E_0 \ , \qquad \phi_u(F_0) = uF_0 \ , \qquad \phi_u(K_0^{\frac{1}{2}}) = K_0^{\frac{1}{2}} \ , \\ &\phi_u(E_1) = u^{-1}E_1 \ , \qquad \phi_u(F_1) = uF_1 \ , \qquad \phi_u(K_1^{\frac{1}{2}}) = K_1^{\frac{1}{2}} \ . \end{aligned} \tag{2.3.15}$$

Then the evaluation map[10] ev_u is defined by the composition

$$\mathsf{ev}_u = \varphi \circ \phi_u \ . \tag{2.3.16}$$

Its action on the generators of $\mathcal{L}U_qsl_2$ are obtained from (2.3.14) and (2.3.15).

2.4 Comodule algebra structure

Recall the definition of a right comodule algebra.

Definition 2.4.1. *B is a right comodule algebra over a Hopf algebra H if there exists an algebra map $\delta \colon B \to B \otimes H$, which we call right coaction, such that the coassociativity and counital conditions hold*

$$(id \otimes \Delta) \circ \delta = (\delta \otimes id) \circ \delta \ , \qquad (id \otimes \epsilon) \circ \delta = id \ . \tag{2.4.1}$$

In this thesis, we will only work with right comodule algebras, and we will often simply call them comodule algebras.

[10] For more general evaluation map, see for instance [BGKNR12, eq. (4.32)]. Here, we set $s_0 = s_1 = -1$.

2.4.1 Comodule algebra structure for $\mathcal{A}_q$

We conjecture that the algebra $\mathcal{A}_q$ is a right comodule algebra over $H = \mathcal{L}U_q sl_2$ with coaction

$$\delta\colon \mathcal{A}_q \to \mathcal{A}_q \otimes \mathcal{L}U_q sl_2$$

such that its values on first 2 generators are

$$\delta(\mathsf{W}_0) = 1 \otimes \left(k_+ q^{\frac{1}{2}} E_1 K_1^{\frac{1}{2}} + k_- q^{-\frac{1}{2}} F_1 K_1^{\frac{1}{2}}\right) + \mathsf{W}_0 \otimes K_1 \ , \tag{2.4.2}$$

$$\delta(\mathsf{W}_1) = 1 \otimes \left(k_+ q^{-\frac{1}{2}} F_0 K_0^{\frac{1}{2}} + k_- q^{\frac{1}{2}} E_0 K_0^{\frac{1}{2}}\right) + \mathsf{W}_1 \otimes K_0 \ . \tag{2.4.3}$$

Obtaining the coaction for all the generators of $\mathcal{A}_q$ requires a detailed study. At present, we do not have expressions of this coaction for all the generators of $\mathcal{A}_q$. Nevertheless, in this thesis, it will be sufficient to consider an evaluated coaction

$$\delta_w\colon \mathcal{A}_q \to \mathcal{A}_q \otimes U_q sl_2 \ ,$$

such that $\delta_w = (\mathrm{id} \otimes \mathsf{ev}_w) \circ \delta$, with $w \in \mathbb{C}^*$. Note that in the next chapter, we will use a formal evaluation map in order to consider w as a formal variable and not a nonzero complex scalar, see Section 3.2.2.

- $\{\mathcal{W}_\pm(u), \mathcal{G}_\pm(u)\}$

The proof of the following proposition is postponed to the end of Section 3.5.

Proposition 2.4.2. *The evaluated coaction of the generating functions (2.1.1), (2.1.2) is*

$$\begin{aligned}
\delta_w(\mathcal{W}_\pm(u)) = {} & \frac{U^{-1}}{q+q^{-1}} \mathcal{W}_\mp(u) \otimes \left((q-q^{-1})^2 S_\pm S_\mp - q(K^{\pm 1} - K^{\mp 1})\right) - \frac{U^{-1}}{q+q^{-1}} (w^2 + w^{-2}) \mathcal{W}_\pm(u) \otimes 1 \\
& + \frac{(q-q^{-1})U^{-1}}{k_+ k_- (q+q^{-1})^2} \left(k_+ q^{\pm\frac{1}{2}} \mathcal{G}_+(u) \otimes (w^{\mp 1} S_+ K^{\pm\frac{1}{2}}) + k_- q^{\mp\frac{1}{2}} \mathcal{G}_-(u) \otimes (w^{\pm 1} S_- K^{\pm\frac{1}{2}})\right) \\
& + U^{-1} \left(1 \otimes (k_+ q^{\pm\frac{1}{2}} w^{\mp 1} S_+ K^{\pm\frac{1}{2}}) + 1 \otimes (k_- q^{\mp\frac{1}{2}} w^{\pm 1} S_- K^{\pm\frac{1}{2}})\right) + \mathcal{W}_\pm(u) \otimes K^{\pm 1}, \qquad (2.4.4) \\
\delta_w(\mathcal{G}_\pm(u)) = {} & \frac{k_\mp}{k_\pm} \frac{(q-q^{-1})^2}{(q+q^{-1})} U^{-1} \mathcal{G}_\mp(u) \otimes S_\mp^2 - \frac{U^{-1}}{q+q^{-1}} \mathcal{G}_\pm(u) \otimes (w^{-2} K^{\pm 1} + w^2 K^{\mp 1}) + \mathcal{G}_\pm(u) \otimes 1 \\
& + (q^2 - q^{-2}) \Big(k_\mp q^{\mp\frac{1}{2}} (\mathcal{W}_+(u) - U^{-1} \mathcal{W}_-(u)) \otimes (w^{\mp 1} S_\mp K^{\frac{1}{2}}) \\
& \qquad + k_\mp q^{\pm\frac{1}{2}} (\mathcal{W}_-(u) - U^{-1} \mathcal{W}_+(u)) \otimes (w^{\pm 1} S_\mp K^{-\frac{1}{2}})\Big) \\
& + \frac{k_+ k_- (q+q^{-1}) U^{-1}}{(q-q^{-1})} 1 \otimes \left(\frac{k_\mp}{k_\pm} (q-q^{-1})^2 S_\mp^2 - (w^{-2} K^{\pm 1} + w^2 K^{\mp 1})\right) \ , \qquad (2.4.5)
\end{aligned}$$

where we used the shorthand notation $S_+ \equiv E$, $S_- \equiv F$. We note that these expressions were first obtained in [BS09, Prop. 2.2][11].

[11] Typos in [BS09] corrected (a prefactor was missing).

- $\{\mathsf{W}_{-k}, \mathsf{W}_{k+1}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1} | k \in \mathbb{N}\}$

Proposition 2.4.3. *The evaluated coaction for the alternating generators are given by*

$$\delta_w(\mathsf{W}_0) = 1 \otimes (k_+ q^{\frac{1}{2}} w^{-1} E K^{\frac{1}{2}} + k_- q^{-\frac{1}{2}} w F K^{\frac{1}{2}}) + \mathsf{W}_0 \otimes K \ , \tag{2.4.6}$$

$$\delta_w(\mathsf{W}_1) = 1 \otimes (k_+ q^{-\frac{1}{2}} w E K^{-\frac{1}{2}} + k_- q^{\frac{1}{2}} w^{-1} F K^{-\frac{1}{2}}) + \mathsf{W}_1 \otimes K^{-1} \ , \tag{2.4.7}$$

$$\begin{aligned}\delta_w(\mathsf{G}_1) = {} & \mathsf{G}_1 \otimes 1 + (q^2 - q^{-2}) k_- (q^{-\frac{1}{2}} \mathsf{W}_0 \otimes (w^{-1} F K^{\frac{1}{2}}) + q^{\frac{1}{2}} \mathsf{W}_1 \otimes (w F K^{-\frac{1}{2}})) \\ & + \frac{k_+ k_- (q + q^{-1})}{q - q^{-1}} 1 \otimes \big(\frac{k_-}{k_+} (q - q^{-1})^2 F^2 - (w^{-2} K + w^2 K^{-1}) \big) \ ,\end{aligned} \tag{2.4.8}$$

$$\begin{aligned}\delta_w(\tilde{\mathsf{G}}_1) = {} & \frac{k_+ k_- (q + q^{-1})}{q - q^{-1}} 1 \otimes \big(\frac{k_+}{k_-} (q - q^{-1})^2 E^2 - (w^{-2} K^{-1} + w^2 K) \big) \big) \\ & + \tilde{\mathsf{G}}_1 \otimes 1 + (q^2 - q^{-2}) k_+ (q^{\frac{1}{2}} \mathsf{W}_0 \otimes (w E K^{\frac{1}{2}}) + q^{-\frac{1}{2}} \mathsf{W}_1 \otimes (w^{-1} E K^{-\frac{1}{2}})) \ ,\end{aligned} \tag{2.4.9}$$

and for $k \in \mathbb{N}_+$

$$\begin{aligned}\delta_w(\mathsf{W}_{-k}) = {} & \frac{1}{q + q^{-1}} \mathsf{W}_k \otimes \big((q - q^{-1})^2 E F - q (K - K^{-1}) \big) + \mathsf{W}_{-k} \otimes K - \frac{w^2 + w^{-2}}{q + q^{-1}} \mathsf{W}_{-k+1} \\ & + \frac{q - q^{-1}}{k_+ k_- (q + q^{-1})^2} \big(k_+ q^{\frac{1}{2}} \mathsf{G}_k \otimes (w^{-1} E K^{\frac{1}{2}}) + k_- q^{-\frac{1}{2}} \tilde{\mathsf{G}}_k \otimes (w F K^{\frac{1}{2}}) \big) \ ,\end{aligned} \tag{2.4.10}$$

$$\begin{aligned}\delta_w(\mathsf{W}_{k+1}) = {} & \frac{1}{q + q^{-1}} \mathsf{W}_{-k+1} \otimes \big((q - q^{-1})^2 F E - q (K^{-1} - K) \big) + \mathsf{W}_{k+1} \otimes K^{-1} - \frac{w^2 + w^{-2}}{q + q^{-1}} \mathsf{W}_k \\ & + \frac{q - q^{-1}}{k_+ k_- (q + q^{-1})^2} \big(k_+ q^{-\frac{1}{2}} \mathsf{G}_k \otimes (w E K^{-\frac{1}{2}}) + k_- q^{\frac{1}{2}} \tilde{\mathsf{G}}_k \otimes (w^{-1} F K^{-\frac{1}{2}}) \big) \ ,\end{aligned} \tag{2.4.11}$$

$$\begin{aligned}\delta_w(\mathsf{G}_{k+1}) = {} & (q^2 - q^{-2}) k_- \Big(q^{-\frac{1}{2}} (\mathsf{W}_{-k} - \mathsf{W}_k) \otimes (w^{-1} F K^{\frac{1}{2}}) + q^{\frac{1}{2}} (\mathsf{W}_{k+1} - \mathsf{W}_{-k+1}) \otimes (w F K^{-\frac{1}{2}}) \Big) \\ & + \frac{k_- (q - q^{-1})^2}{k_+ (q + q^{-1})} \tilde{\mathsf{G}}_k \otimes F^2 - \frac{1}{q + q^{-1}} \mathsf{G}_k \otimes (w^{-2} K + w^2 K^{-1}) + \mathsf{G}_{k+1} \otimes 1 \ ,\end{aligned} \tag{2.4.12}$$

$$\begin{aligned}\delta_w(\tilde{\mathsf{G}}_{k+1}) = {} & (q^2 - q^{-2}) k_+ \Big(q^{\frac{1}{2}} (\mathsf{W}_{-k} - \mathsf{W}_k) \otimes (w E K^{\frac{1}{2}}) + q^{-\frac{1}{2}} (\mathsf{W}_{k+1} - \mathsf{W}_{-k+1}) \otimes (w^{-1} E K^{-\frac{1}{2}}) \Big) \\ & + \frac{k_+ (q - q^{-1})^2}{k_- (q + q^{-1})} \mathsf{G}_k \otimes E^2 - \frac{1}{q + q^{-1}} \tilde{\mathsf{G}}_k \otimes (w^{-2} K^{-1} + w^2 K) + \tilde{\mathsf{G}}_{k+1} \otimes 1 \ .\end{aligned} \tag{2.4.13}$$

Proof. Replace the generating functions $\mathcal{W}_\pm(u)$, $\mathcal{G}_\pm(u)$ in the equations (2.4.4), (2.4.5) by their expressions given in (2.1.1), (2.1.2). Then, the result follows by extracting the modes. □

- $\{\mathsf{W}_0, \mathsf{W}_1, \mathsf{G}_{k+1} | k \in \mathbb{N}\}$

The evaluated coaction for the generators of the compact form of $\mathcal{A}_q$ are extracted from Proposition 2.4.3.

Proposition 2.4.4. *The action of the evaluated coaction map on the essential generators of* $\mathcal{A}_q$ *are given by* (2.4.6), (2.4.7) and

$$\delta_w(\mathsf{G}_1) = k_-^2 (q^2 - q^{-2}) 1 \otimes F^2 + \mathsf{G}_1 \otimes 1 + \frac{k_+ k_- (q + q^{-1})}{q - q^{-1}} \otimes (w^2 K^{-1} + w^{-2} K) \tag{2.4.14}$$

$$
\begin{aligned}
&+\frac{q-q^{-1}}{k_+(q+q^{-1})}\big(q^{-\frac{1}{2}}[\mathsf{W}_0,\mathsf{G}_1]_q\otimes(w^{-1}FK^{\frac{1}{2}})+q^{\frac{1}{2}}[\mathsf{G}_1,\mathsf{W}_1]_q\otimes(wFK^{-\frac{1}{2}})\big)\ ,\\
\delta_w(\mathsf{G}_{k+1}) &= \frac{k_-}{k_+}\frac{(q-q^{-1})^2}{q+q^{-1}}(\mathsf{G}_k-\frac{(q+q^{-1})}{\rho}\big[\mathsf{W}_0,\big[\mathsf{W}_0,\mathsf{G}_k\big]_q\big])\otimes F^2 \qquad (2.4.15)\\
&+\frac{(q-q^{-1})}{k_+(q+q^{-1})}\big(q^{-\frac{1}{2}}w^{-1}\big[\mathsf{W}_0,\mathsf{G}_{k+1}\big]_q\otimes FK^{\frac{1}{2}}+q^{\frac{1}{2}}w\big[\mathsf{G}_{k+1},\mathsf{W}_1\big]_q\otimes FK^{-\frac{1}{2}}\big)\\
&-\frac{\mathsf{G}_k}{q+q^{-1}}\otimes(w^2K^{-1}+w^{-2}K)+\mathsf{G}_{k+1}\otimes 1\ ,
\end{aligned}
$$

for $k\in\mathbb{N}_+$.

Proof. Recall Proposition 2.4.3 with $\delta_w(\mathsf{G}_1)$ and $\delta_w(\mathsf{G}_{k+1})$ given in (2.4.8), (2.4.12). The goal is to express the r.h.s. of the latter relations solely in terms of $\{\mathsf{W}_0,\mathsf{W}_1,\mathsf{G}_{k+1}|k\in\mathbb{N}\}$. We first show the equation (2.4.15). It is proven using (2.1.17)–(2.1.19) and $[\mathsf{W}_0,\mathsf{W}_k]=-\rho^{-1}[\mathsf{W}_0,[\mathsf{W}_0,\mathsf{G}_k]_q]$, which is obtained using in the r.h.s. of the latter relation (2.1.18) for $k\to k-1$ and $]\mathsf{W}_0,\mathsf{W}_{-k}]=0$. The equation (2.4.14) is shown similarly. □

2.4.2 Comodule algebra structure for O_q

For O_q, the coaction map $\delta\colon O_q\to O_q\otimes\mathcal{L}U_qsl_2$ is given by (2.4.2), (2.4.3) but with the substitution $\mathsf{W}_0\to\mathcal{W}_0$ and $\mathsf{W}_1\to\mathcal{W}_1$ [BB12].

In this chapter the q-Onsager algebra as well as its alternating central extension $\mathcal{A}_q$ were both reviewed with several different presentations, and their comodule algebra structure were also given. In this thesis, the FRT presentation for $\mathcal{A}_q$ with its spin-$\frac{1}{2}$ K-operator given in Definition 2.1.1 will be of main importance. Indeed, in the next chapter we introduce an axiomatic definition of a 2-legs universal K-matrix associated with a comodule algebra B, and $\mathcal{A}_q$ will be our main example. On one hand, from this universal K-matrix, spin-j K-operators for $\mathcal{A}_q$ are obtained by specialization to formal evaluation representations of $\mathcal{L}U_qsl_2$. On the other hand, independently of this universal framework, fused K-operators for $\mathcal{A}_q$ are constructed using a fusion procedure different from the one presented in Section 1.3. This one is based on $\mathcal{L}U_qsl_2$-intertwiners between formal evaluation representations and their tensor products. Then, comparing spin-j K-operators and fused K-operators, some information on this universal K-matrix is extracted and reported in Conjecture 1.

Chapter 3

Universal K-matrix and fused K-operators for $\mathcal{A}_q$

In this chapter we solve the first two problems addressed in this thesis:

- Problem 1:

  The different universal K-matrices, as defined in [BKo15, Ko17, AV20, AV22] (recall the discussion of these constructions in Introduction), do not allow for the treatment of K-operators solutions to the reflection equation with spectral parameter (0.0.15) and with a *physical* coaction of the form (0.0.16).

- Problem 2:

  The spin-1/2 K-operator for the alternating central extension of the q-Onsager algebra $\mathcal{A}_q$ was introduced in [BS09], recall Definition 2.1.1, but its spin-j version has never been constructed.

In short, this chapter is divided into two parts: Sections 3.1 and 3.2 tackle the first problem, while Sections 3.3-3.6 address the second problem, offering an interpretation of the fused K-operators constructed by fusion in relation to the evaluation of a universal K-matrix.

This chapter is organized as follows.

In Section 3.1, the formalism of universal R-matrix is first reviewed following [Dr86]. In order to handle K-operators with a spectral parameter, we consider a mild modification of the universal K-matrix axioms from [Ko17, AV20], see Definition 3.1.7. In our framework, the universal K-matrix $\mathfrak{K}$ belongs to $B \otimes H$, where B is a comodule algebra over a Hopf algebra H. It satisfies a ψ-twisted reflection equation, where ψ is an automorphism of H, see Proposition 3.1.8.

In Section 3.2, we consider the choice where H is the quantum loop algebra $\mathcal{L}U_q sl_2$, and introduce formal evaluation representations, which are infinite-dimensional and not of highest-weight type.

Spin-j (affine) L-operators and K-operators are defined as evaluations of $\mathfrak{R}$ and $\mathfrak{K}$ in Section 3.2.3. It is important that these evaluations make only sense for formal parameter u. An importance is given for the choice of ψ, an automorphism of H that forms a twist pair together with a Drinfeld twist $J \in H \otimes H$, see Definition 3.1.5. Indeed, for $H = \mathcal{L}U_q sl_2$, $\psi = \eta$ defined in (3.1.12) and $J = 1 \otimes 1$, the specialization of the ψ-twisted reflection equation leads to the reflection equation with spectral parameter (3.2.37) that appears in the study of open quantum spin-chain models. Finally, the Hopf algebra structure of H, as well as the comodule algebra structure of B, are described in terms of the spin-$\frac{1}{2}$ affine L-operators and K-operator.

Section 3.3 is devoted to a detailed analysis of the tensor product of evaluation representations of $\mathcal{L}U_q sl_2$. They are of finite dimension and indexed by a non-zero complex number u called evaluation parameter. Sub-representations emerge from the tensor product representation of $\mathcal{L}U_q sl_2$ for special values of the evaluation parameters, and we find explicit expressions for the corresponding $\mathcal{L}U_q sl_2$-intertwining operators: the fusion/reduction operators $(j \pm \frac{1}{2}) \to (\frac{1}{2}, j)$, where we used the notation $(j) \equiv \mathbb{C}^{2j+1}$ and $(\frac{1}{2}, j) \equiv \mathbb{C}^2 \otimes \mathbb{C}^{2j+1}$. This analysis is then extended to the case where u is treated as a formal parameter, which is necessary for a careful treatment of the constructions of fused L- and K-operators.

In Section 3.4, we assume the existence of a universal K-matrix $\mathfrak{K}$ for a comodule algebra B and a certain twist pair, to explore consequences of universal K-matrix axioms on the properties of K-operators. Considering the sub-representation associated with the 'fusion' $(j + \frac{1}{2}) \to (\frac{1}{2}, j)$ of formal evaluation representations, it leads to a fusion relation satisfied by the spin-j L- and K-operators, see Propositions 3.4.1 and 3.4.8. This relation expresses a spin-j L-operator (resp. K-operator) in terms of L-operators (resp. K-operators) of lower spins recursively. The sub-representation corresponding to 'reduction' $(j - \frac{1}{2}) \to (\frac{1}{2}, j)$ leads similarly to Propositions 3.4.3 and 3.4.9 that express a spin-j L-operator (resp. K-operator) in terms of L-operators (resp. K-operators) of higher spins recursively.

In Section 3.5, we consider the choice of the comodule algebra $B = \mathcal{A}_q$, the alternating central extension of the q-Onsager algebra, over $H = \mathcal{L}U_q sl_2$. This time, we do not assume the existence of a universal K-matrix. Instead, we draw inspiration from the fusion relation satisfied by the spin-j K-operators established in the previous section to give a new proposal for fused K-operators generated by the fundamental one of spin-1/2 found by Baseilhac and Shigechi in 2009 [BS09]. More precisely, we introduce fused K-operators of spin-j in Definition 3.5.1 based on the fundamental K-operator (2.1.5) from the reflection algebra presentation of $\mathcal{A}_q$, see Theorem 2.1.1. We prove in Theorem 3.5.2 that they satisfy a reflection equation with spectral parameter. We also give compact expressions for the fused R-matrices and the fused K-operators in (3.5.9), (3.5.10), solely in terms of the fundamental R-matrix and K-operator. Then, we classify one-dimensional representations for $\mathcal{A}_q$ in Proposition 3.5.3, and by applying these results to the fused K-operators, we get fused K-matrices in Definition 3.5.5.

In Section 3.6, the precise relation between the spin-j K-operators from Section 3.4 and the fused K-operators from Section 3.5 leads to Conjecture 1, with a few supporting evidence discussed. Remarkably, the fused K-operators constructed by a fusion procedure turns out to satisfy analogs of the evaluated axioms of the universal K-matrix.

This chapter is based on our paper [LBG23]. Since all the proofs are already given in details there, we omit most of them here and we refer the interested reader to the latter reference.

3.1 Universal R- and K-matrices

Firstly, we recall the definition of a quasi-triangular Hopf algebra H with the associated universal R-matrix that satisfies the universal Yang-Baxter equation. Then, inspired by the works [Ko17] and [AV20], we define in Section 3.1.3 a universal K-matrix associated with H, a pair of its consistent twists (ψ, J), and its comodule algebra B – this is an element in $B \otimes H$ that satisfies a universal reflection equation (also called ψ-twisted reflection equation), see Proposition 3.1.8.

3.1.1 Universal R-matrix

Let H be a Hopf algebra with coproduct $\Delta\colon H \to H \otimes H$, the counit $\epsilon\colon H \to \mathbb{C}$ and the antipode $S\colon H \to H$, which are subject to consistency conditions[12]. We denote the opposite coproduct $\Delta^{op} = \mathfrak{p} \circ \Delta$, where $\mathfrak{p}$ is the permutation operator[13]. Here we use the notation $\mathfrak{R}_{12} = \mathfrak{R} \otimes 1$, $\mathfrak{R}_{23} = 1 \otimes \mathfrak{R}$, $\mathfrak{R}_{13} = \mathfrak{p}_{23}(\mathfrak{R}_{12})$.

Definition 3.1.1 ([Dr86]). *For a Hopf algebra H, an invertible element $\mathfrak{R} \in H \otimes H$ is called universal R-matrix if it satisfies*

$$\mathfrak{R}\Delta(x) = \Delta^{op}(x)\mathfrak{R} \ , \qquad \forall x \in H \ , \tag{R1}$$

$$(\Delta \otimes id)(\mathfrak{R}) = \mathfrak{R}_{13}\mathfrak{R}_{23} \ , \tag{R2}$$

$$(id \otimes \Delta)(\mathfrak{R}) = \mathfrak{R}_{13}\mathfrak{R}_{12} \ . \tag{R3}$$

If such $\mathfrak{R}$ exists, then the pair $(H, \mathfrak{R})$ is called a quasi-triangular Hopf algebra.

We note that the universal R-matrix necessarily satisfies

$$(S \otimes \mathrm{id})(\mathfrak{R}) = \mathfrak{R}^{-1} = (\mathrm{id} \otimes S)(\mathfrak{R}) \ , \tag{3.1.1}$$

$$(\epsilon \otimes \mathrm{id})(\mathfrak{R}) = 1 = (\mathrm{id} \otimes \epsilon)(\mathfrak{R}) \ . \tag{3.1.2}$$

Using the relations (R1)–(R3) one can show that the universal R-matrix satisfies the universal Yang-Baxter equation (without evaluation parameter):

$$\mathfrak{R}_{12}\mathfrak{R}_{13}\mathfrak{R}_{23} = \mathfrak{R}_{23}\mathfrak{R}_{13}\mathfrak{R}_{12} \ . \tag{3.1.3}$$

It is well-known that the universal R-matrix coming from a quasi-triangular Hopf algebra gives a way to generate R-matrices on tensor product of representations, via evaluations as we will see in Section 3.2 [Dr86].

3.1.2 Drinfeld twists and twist pairs

Appel and Vlaar introduced the notion of a cylindrical bialgebra [AV20, Def. 2.3] which is a quasi-triangular bialgebra with a universal solution of a twisted reflection equation in $H \otimes H$. This approach was further elaborated in [AV22] to study K-matrix solutions of the parameter-dependent reflection equation for the case of finite-dimensional representations of quantum affine algebras.

[12] We refer to [CP95, Chap. 4] for the axioms of a Hopf algebra.
[13] Here we define $\mathfrak{p}(x \otimes y) = y \otimes x$, for all x,y in H.

In this section, first we review known results of [AV20]. Then, inspired by [AV20, Prop. 2.7], we introduce a universal K-matrix $\mathfrak{K} \in B \otimes H$ that satisfies a new set of axioms (K1)-(K3).

The following is an extension of [AV20, Def. 2.2] from bialgebras[14] to Hopf algebras.

Definition 3.1.2. *Let $(H, \mathfrak{R})$ be a quasi-triangular Hopf algebra and $\psi\colon H \to H$ an algebra automorphism. The ψ-twisting of $(H, \mathfrak{R})$ is the quasi-triangular Hopf algebra $(H^{\psi}, \mathfrak{R}^{\psi\psi})$ obtained from $(H, \mathfrak{R})$ by pullback through ψ, i.e. H^{ψ} is the Hopf algebra with same multiplication, new coproduct, counit and antipode[15]:*

$$\Delta^{\psi} = (\psi \otimes \psi) \circ \Delta \circ \psi^{-1}, \qquad \epsilon^{\psi} = \epsilon \circ \psi^{-1}, \qquad S^{\psi} = \psi \circ S \circ \psi^{-1} \ , \tag{3.1.4}$$

and the universal R-matrix is given by

$$\mathfrak{R}^{\psi\psi} = (\psi \otimes \psi)(\mathfrak{R}) \ . \tag{3.1.5}$$

In what follows, we also use the opposite version of the ψ-twisting of $(H, \mathfrak{R})$. Let H^{cop} be the Hopf algebra with the coproduct Δ^{op}, the antipode S^{-1} and the other structure maps are the same as H. Also, the pair $(H^{cop,\psi}, \mathfrak{R}_{21}^{\psi\psi})$, where $H^{cop,\psi}$ is the ψ-twisting of H^{cop}, is a quasi-triangular Hopf algebra [LBG23, Lem. 2.3].

To introduce the concept of universal K-matrix, we also need another type of twists, called Drinfeld twists:

Definition 3.1.3 ([Dr86, Dr89a])**.** *A Drinfeld twist of a Hopf algebra H is an invertible element $J \in H \otimes H$ satisfying the property*

$$(\epsilon \otimes id)(J) = 1 = (id \otimes \epsilon)(J) \tag{3.1.6}$$

and the cocycle identity

$$(J \otimes 1)(\Delta \otimes id)(J) = (1 \otimes J)(id \otimes \Delta)(J) \ . \tag{3.1.7}$$

Example 3.1.4. *In the literature, there are two natural choices of Drinfeld twists* [AV20]*: $J = 1 \otimes 1$ and $J = \mathfrak{R}_{21}\mathfrak{R}$. It is straightforward to check using the universal R-matrix axioms* (R1)–(R3) that (3.1.6) and (3.1.7) indeed hold for both of them.

Given a Drinfeld twist J one obtains a new quasi-triangular Hopf algebra $(H_J, \mathfrak{R}_J)$ with the coproduct [Dr89b]

$$\Delta_J(x) = J\Delta(x)J^{-1} \ , \qquad \forall x \in H \ , \tag{3.1.8}$$

and the universal R-matrix

$$\mathfrak{R}_J = J_{21}\mathfrak{R}J^{-1} \ . \tag{3.1.9}$$

At this point, we introduced two ways of deforming a quasitriangular Hopf algebra $(H, \mathfrak{R})$ with coproduct Δ, either using the twist ψ, or a Drinfeld twist J. The following definition of a quasi-triangular pair imposes some compatibility conditions for these two deformations.

Definition 3.1.5 ([AV20])**.** *Let $(H, \mathfrak{R})$ be a quasi-triangular algebra. A twist pair (ψ, J) is the datum of an algebra automorphism $\psi\colon H \to H$ and a Drinfeld twist $J \in H \otimes H$ such that*

[14] Recall that a bialgebra is a Hopf algebra as defined in Definition 2.3.1 but it does not have an antipode S.

[15] It is enough to verify that $\epsilon^{\psi} \circ S^{\psi} = \epsilon^{\psi}$.

$(H^{cop,\psi}, \mathfrak{R}_{21}^{\psi\psi}) = (H_J, \mathfrak{R}_J)$, *i.e. such that* $\epsilon^\psi = \epsilon$,

$$\begin{aligned} \Delta^{op,\psi}(x) &= J\Delta(x)J^{-1} \ , \qquad \forall x \in H \ , \\ \mathfrak{R}_{21}^{\psi\psi} &= J_{21}\mathfrak{R}J^{-1} \ , \end{aligned} \tag{3.1.10}$$

where

$$\Delta^{op,\psi} = (\psi \otimes \psi) \circ \Delta^{op} \circ \psi^{-1}. \tag{3.1.11}$$

Example 3.1.6. *For* $H = \mathcal{L}U_q sl_2$, *the pair* $(\psi, J) = (\eta, 1 \otimes 1)$ *forms a twist pair with the automorphism* η *defined by*

$$\begin{aligned} &\eta(E_0) = F_1 \ , \qquad \eta(E_1) = F_0 \ , \qquad \eta(K_0^{\frac{1}{2}}) = K_1^{-\frac{1}{2}} \ , \\ &\eta(F_1) = E_0 \ , \qquad \eta(F_0) = E_1 \ , \qquad \eta(K_1^{\frac{1}{2}}) = K_0^{-\frac{1}{2}} \ . \end{aligned} \tag{3.1.12}$$

3.1.3 Universal K-matrix

Let (ψ, J) be a twist pair for a Hopf algebra H and B is a right comodule algebra over H. Inspired by [AV20] we define a universal K-matrix $\mathfrak{K} \in B \otimes H$. Recall the definition of a comodule algebra B with coaction $\delta \colon B \to B \otimes H$ from Definition 2.4.1. Here we use the notation $\mathfrak{K}_{12} = \mathfrak{K} \otimes 1$, $\mathfrak{K}_{13} = \mathfrak{p}_{23} \circ \mathfrak{K}_{12}$.

Definition 3.1.7. *We say that* $\mathfrak{K} \in B \otimes H$ *is universal K-matrix if the following relations hold for all* $b \in B$*:*

$$\mathfrak{K}\delta(b) = \delta^\psi(b)\mathfrak{K} \ , \qquad \textit{with } \delta^\psi = (id \otimes \psi) \circ \delta \ , \tag{K1}$$

$$(\delta \otimes id)(\mathfrak{K}) = (\mathfrak{R}^\psi)_{32}\mathfrak{K}_{13}\mathfrak{R}_{23} \ , \tag{K2}$$

$$(id \otimes \Delta)(\mathfrak{K}) = J_{23}^{-1}\mathfrak{K}_{13}\mathfrak{R}_{23}^{\psi}\mathfrak{K}_{12} \ , \tag{K3}$$

where

$$\mathfrak{R}^\psi = (\psi \otimes id)(\mathfrak{R}) \ . \tag{3.1.13}$$

A universal reflection equation is then derived using the axioms (K1)–(K3).

Proposition 3.1.8 ([LBG23, Prop. 2.11])**.** *Let* (ψ, J) *be a twist pair. The universal K-matrix satisfies the* ψ*-twisted reflection equation*

$$\mathfrak{K}_{12}(\mathfrak{R}^\psi)_{32}\mathfrak{K}_{13}\mathfrak{R}_{23} = \mathfrak{R}_{32}^{\psi\psi}\mathfrak{K}_{13}\mathfrak{R}_{23}^{\psi}\mathfrak{K}_{12} \ . \tag{3.1.14}$$

Let us now make several remarks concerning the above definition. First, we note that δ^ψ defines a coalgebra structure on B over H^ψ. Therefore, by (K1) $\mathfrak{K}$ intertwines two actions of B on $B \otimes H$, given by δ and δ^ψ respectively. By analogy with (R1), we call (K1) the twisted intertwining relation. Then, from above axioms and some properties of the comodule and Hopf algebras structure, we find:

Remark 3.1.9. *From the axioms* (K2)–(K3) of the universal K-matrix, we get some relations on the level of the algebra.

(i) We provide a consistency check of the axioms (K2) and (K3). From coassociativity property (2.4.1), we have:

$$(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\circ(\delta\otimes\mathrm{id})(\mathfrak{K}) = (\delta\otimes\mathrm{id}\otimes\mathrm{id})\circ(\delta\otimes\mathrm{id})(\mathfrak{K})\ . \tag{3.1.15}$$

This relation is checked using (R2) and (K2). Indeed, the l.h.s. equals $(\mathfrak{R}^{\psi})_{43}(\mathfrak{R}^{\psi})_{42}\mathfrak{K}_{14}\mathfrak{R}_{24}\mathfrak{R}_{34}$ where we used $(\Delta\otimes\mathrm{id})((\mathfrak{R}^{\psi})_{21}) = (\mathfrak{R}^{\psi})_{32}(\mathfrak{R}^{\psi})_{31}$, while the r.h.s. gives the same expression using twice (K2).

The counital property in (2.4.1) is checked using (K3) and (3.1.2) as follows:

$$(\mathrm{id}\otimes\epsilon\otimes\mathrm{id})\circ(\delta\otimes\mathrm{id})(\mathfrak{K}) = (\mathrm{id}\otimes\epsilon\otimes\mathrm{id})((\mathfrak{R}^{\psi})_{32}\mathfrak{K}_{13}\mathfrak{R}_{23}) = \mathfrak{K}\ . \tag{3.1.16}$$

(ii) Recall that the coproduct and the counit of a Hopf algebra satisfy

$$(\epsilon\otimes\mathrm{id})\circ\Delta = \mathrm{id} = (\mathrm{id}\otimes\epsilon)\circ\Delta\ . \tag{3.1.17}$$

Then, using (K3), (3.1.2), (3.1.6), and the fact that ϵ is an algebra homomorphism we obtain:

$$\mathfrak{K} = [(\mathrm{id}\otimes\mathrm{id}\otimes\epsilon)\circ(\mathrm{id}\otimes\Delta)](\mathfrak{K}) = (\mathrm{id}\otimes\mathrm{id}\otimes\epsilon)(J_{23}^{-1}\mathfrak{K}_{13}\mathfrak{R}^{\psi}_{23}\mathfrak{K}_{12}) = ([(\mathrm{id}\otimes\epsilon)(\mathfrak{K})]\otimes 1)\,\mathfrak{K}\ . \tag{3.1.18}$$

Applying again $(\mathrm{id}\otimes\epsilon)$ on (3.1.18), we find that $(\mathrm{id}\otimes\epsilon)(\mathfrak{K})$ is an idempotent. If in addition, $\mathfrak{K}$ is invertible, it follows:

$$(\mathrm{id}\otimes\epsilon)(\mathfrak{K}) = 1\ , \tag{3.1.19}$$

which is the analogue of (3.1.2) for the universal R-matrix.

Finally, we consider some special cases of Definition 3.1.7.

Remark 3.1.10. *The previous definitions of universal K-matrices from the works* [Ko17, AV20] *can be obtained as special cases of Definition* 3.1.7.

(i) It is easy to check that $(\mathrm{id}, \mathfrak{R}_{21}^{-1})$ is a twist pair. In particular, one finds that (K1)–(K3), for $\psi = \mathrm{id}$ and $J = \mathfrak{R}_{21}^{-1}$, correspond to the axioms for the universal K-matrix defined in [Ko17, Def. 2.7].

(ii) Assume that B is a right coideal subalgebra of H, that is $\delta = \Delta|_B$, then define:

$$\mathcal{K} = (\epsilon\otimes\mathrm{id})(\mathfrak{K})\ , \qquad \mathcal{K}\in H\ .$$

Then, applying the counit on the first tensor factor of the universal K-matrix in (K1)–(K3) yields:

$$\mathcal{K}\,b = \psi(b)\mathcal{K}\ , \qquad \forall b\in B\ , \tag{3.1.20}$$

$$\mathfrak{K} = (\mathfrak{R}^{\psi})_{21}\mathcal{K}_2\mathfrak{R}\ , \tag{3.1.21}$$

$$\Delta|_B(\mathcal{K}) = J^{-1}\mathcal{K}_2\mathfrak{R}^{\psi}\mathcal{K}_1\ . \tag{3.1.22}$$

We recall that in this setting of B a coideal subalgebra of H, Appel and Vlaar introduced in [AV20] one-component universal K-matrices $k\in H$. The formulas (3.1.20) and (3.1.22),

with the identification $\mathcal{K} = k$, correspond respectively to the defining relations of the so-called cylindrically invariant subalgebra B and the cylindrical pair $(H, \mathfrak{R})$, see [AV20, Def. 2.3]. We thus get the one-component universal K-matrix of [AV20] from our 2-component $\mathfrak{K}$. The opposite is actually also true: starting from a solution $\mathcal{K}$ of (3.1.20) and (3.1.22), and assuming that $\mathfrak{K}$ defined by (3.1.21) lies in $B \otimes H$, one can check that $\mathfrak{K}$ satisfies the axioms (K1)-(K3). Indeed, checking (K1) and (K2) is straightforward, while for (K3) we use the equalities

$$(\mathrm{id} \otimes \Delta)[(\mathfrak{R}^{\psi})_{21}] = J_{23}^{-1}(\mathfrak{R}^{\psi})_{31}(\mathfrak{R}^{\psi})_{21}J_{23} \ ,$$
$$(\mathfrak{R}^{\psi})_{21}\mathfrak{R}^{\psi}_{23}\mathfrak{R}_{13} = \mathfrak{R}_{13}\mathfrak{R}^{\psi}_{23}(\mathfrak{R}^{\psi})_{21} \ .$$

The first one is obtained using

$$\Delta(\psi(x)) = J^{-1}[(\psi \otimes \psi) \circ \Delta^{op}(x)]J$$

with

$$(\mathrm{id} \otimes \Delta^{op})(\mathfrak{R}_{21}) = \mathfrak{R}_{31}\mathfrak{R}_{21} \ ,$$

while we applied $(\mathrm{id} \otimes \psi \otimes \mathrm{id}) \circ \mathfrak{p}_{12}$ to the universal Yang-Baxter equation (3.1.3) to get the second equality. Then, we get

$$\begin{aligned}(\mathrm{id} \otimes \Delta)(\mathfrak{K}) &= J_{23}^{-1}\ (\mathfrak{R}^{\psi})_{31}\mathcal{K}_3\ (\mathfrak{R}^{\psi})_{21}\mathfrak{R}^{\psi}_{23}\mathfrak{R}_{13}\ \mathcal{K}_2\mathfrak{R}_{12} \\ &= J_{23}^{-1}\ (\mathfrak{R}^{\psi})_{31}\mathcal{K}_3\ \mathfrak{R}_{13}\ \mathfrak{R}^{\psi}_{23}\ (\mathfrak{R}^{\psi})_{21}\ \mathcal{K}_2\mathfrak{R}_{12} \\ &= J_{23}^{-1}\mathfrak{K}_{13}\mathfrak{R}^{\psi}_{23}\mathfrak{K}_{12}\end{aligned}$$

which is indeed (K3).

Let us comment on the point (ii) of the above remark. If $B = O_q$, the q-Onsager algebra, then an explicit formula for our universal K-matrix in $O_q \otimes \mathcal{L}U_q sl_2$ could be deduced from (3.1.21) provided $\mathcal{K}$ is known. Note that there are existence results for $\mathcal{K}$ for certain choice of twists [AV20]. However, no expressions for $\mathcal{K}$ in O_q, for example in terms of root vectors, are known. In the following, we fix the twist pair $(\psi, J) = (\eta, 1 \otimes 1)$ from Example 3.1.6 which actually differs from [AV20, Sec. 9.5]. We make this choice because η is easy to work with and it allows to obtain the standard reflection equation from the ψ-twisted reflection equation (3.1.14) in our choice of gradation. Let us mention that in the next sections, we define K-operators out of $\mathfrak{K} \in \mathcal{A}_q \otimes \mathcal{L}U_q sl_2$, and derive fusion formulas satisfied by the K-operators for the twist pair $(\eta, 1 \otimes 1)$. As it will be studied in Chapter 4, these K-operators are involved in the construction of physical transfer matrices. In view of these applications, it is not clear whether the choice of the twist pair of Appel-Vlaar from [AV20, Sec. 9.5] produces the physical transfer matrices discussed in Chapter 4. Indeed, following the approach in (ii) for $B = O_q$ and using their twist pair, the fusion formula is affected by non-trivial J and it provides a priori different spin-j K-operators.

Furthermore, if B is a comodule algebra over H and not a coideal subalgebra of H, as in our main case of interest $B = \mathcal{A}_q$, then the formalism from [AV20, AV22] is not suitable. That is why in this section we developed a slight generalization for the two-leg universal K-matrices, and we study its consequences in the next sections.

3.2 Evaluation of universal R- and K-matrices

We introduced a universal K-matrix $\mathfrak{K} \in B \otimes H$ in Definition 3.1.7, where B is a comodule algebra over H. Now, we need to define a K-operator as the 1-component evaluation of $\mathfrak{K}$ such that it satisfies the reflection equation with spectral parameter (0.0.15), and whose coaction is *physical*, i.e. of the form (0.0.16). Therefore, a study of the evaluation of $\mathfrak{R}$ (and its twisted versions), as well as $\mathfrak{K}$, is required. The spectral parameters naturally arise in quantum affine algebras from evaluation representations. For instance, the evaluation representation for $H = U_q\widehat{sl}_2$ was introduced in [J86, CP91], via an algebra map from $U_q\widehat{sl}_2 \to U_q sl_2$ with a spectral parameter u. Representations of $U_q\widehat{sl}_2$ can thus be obtained by pulling back representations of $U_q sl_2$ by the algebra map. It works similarly for $U_q(\widehat{sl}_n)$ [J86], but it is not the case for any $U_q(\hat{\mathfrak{g}})$, see the remark below [CP94, Prop. 4.1]. For clarity of presentation and because we will mainly work with this specific case, we consider $\hat{\mathfrak{g}} = \widehat{sl}_2$, and in particular its quotient: the quantum loop algebra $H = \mathcal{L}U_q sl_2$, recall Definition 2.3.6. As reviewed below, it has an evaluation representation map $\pi^j_u \colon H \to \mathrm{End}(V^{(j)})$. We want the twist automorphism ψ to satisfy

$$\pi^j_u \circ \psi = \pi^j_{u^{-1}} \ . \tag{3.2.1}$$

Such twist indeed exists, denoted as η and defined by (3.1.12). Together with the trivial Drinfeld twist, it forms a twist pair $(\psi, J) = (\eta, 1 \otimes 1)$ from Definition 3.1.5. We now briefly recall the evaluation representation of $\mathcal{L}U_q sl_2$ to define π^j_u.

In the following, in order to evaluate universal R- and K-matrices, we will need formal evaluation representations which are infinite-dimensional. For pedagogical reasons, we first recall the standard finite-dimensional case, i.e. when u is a non-zero complex number.

3.2.1 $\mathcal{L}U_q sl_2$ evaluation representation

Recall the evaluation map $\mathsf{ev}_u \colon \mathcal{L}U_q sl_2 \to U_q sl_2$ from Section 2.3.3. Recall also that finite-dimensional irreducible representations of $U_q sl_2$ are labelled by a non-negative integer or half-integer j, with the dimension of the representation being $2j+1$. Let $V^{(j)}$ denotes the $(2j+1)$-dimensional space spanned by $|j,m\rangle$ with $m \in \{-j, -j+1, \ldots, j-1, j\}$, then

$$E\,|j,m\rangle = A_{j,m}\,|j,m+1\rangle\,, \qquad F\,|j,m\rangle = B_{j,m}\,|j,m-1\rangle\ , \qquad K^{\pm\frac{1}{2}}\,|j,m\rangle = q^{\pm m}\,|j,m\rangle\ , \tag{3.2.2}$$

with

$$A_{j,m} = \sqrt{[j-m]_q\,[j+m+1]_q}\,, \qquad B_{j,m} = \sqrt{[j+m]_q\,[j-m+1]_q}\,. \tag{3.2.3}$$

Let π^j be the representation map of $U_q sl_2$, such that $\pi^j \colon U_q sl_2 \to \mathrm{End}(\mathbb{C}^{2j+1})$. Then, given $u \in \mathbb{C}^*$, we define the evaluation representations $\pi^j_u \colon \mathcal{L}U_q sl_2 \to \mathrm{End}(\mathbb{C}^{2j+1})$ by

$$\pi^j_u = \pi^j \circ \mathsf{ev}_u\ . \tag{3.2.4}$$

3.2.2 Formal evaluation representation of $\mathcal{L}U_q sl_2$

In the next sections, when we discuss L- and K-operators, and in calculations of 1-component evaluation of the universal R- and K-matrix, we actually work with formal parameter u, not a

complex number, because the evaluated expressions would not be well-defined if u was a number as in the evaluation representations considered above. Actually, in the evaluation of the universal R-matrix, such considerations were already accounted for in [Dr86, Sec. 13] and [FR92, Sec. 4], see [He17, Sec. 1] for a review. Similarly, for the evaluation of the universal K-matrix, similar precautions should be taken. Therefore, we need to introduce a formal-parameter analogue of the (finite-dimensional) evaluation representations from (3.2.4). In the literature, these representations of quantum affine algebras are also known [CG03] under the name *quantum loop modules*.

From now on, we assume that $u^{\pm 1}$ is formal, and keeping for brevity the same notations, we define the formal evaluation representations π^j_u as

$$\pi^j_u = \pi^j \circ \mathsf{ev}_u \colon \ \mathcal{L}U_q sl_2 \to \mathrm{End}(\mathbb{C}^{2j+1})[u^{\pm 1}] \ . \tag{3.2.5}$$

where the formal evaluation map

$$\mathsf{ev}_u \colon \mathcal{L}U_q sl_2 \to U_q sl_2[u^{\pm 1}] \ , \tag{3.2.6}$$

is defined by same formulas as in (2.3.16) for the principal gradation but with $u^{\pm 1}$ formal variables. We thus have explicitly on generators:

$$\begin{aligned} &\pi^j_u(E_0) = u^{-1}\pi^j(F) \ , \qquad \pi^j_u(F_0) = u\pi^j(E) \ , \qquad \pi^j_u(K_0^{\frac{1}{2}}) = \pi^j(K^{-\frac{1}{2}}) \ , \\ &\pi^j_u(E_1) = u^{-1}\pi^j(E) \ , \qquad \pi^j_u(F_1) = u\pi^j(F) \ , \qquad \pi^j_u(K_1^{\frac{1}{2}}) = \ \pi^j(K^{\frac{1}{2}}) \ . \end{aligned} \tag{3.2.7}$$

The map (3.2.5) and formulas (3.2.7) define the action of $\mathcal{L}U_q sl_2$ on the space $\mathbb{C}^{2j+1}[u^{\pm 1}]$ of Laurent polynomials in u with coefficients being vectors in $\mathbb{C}^{2j+1}$. For example, the action of E_0 is given by

$$\pi^j_u(E_0)(xu^n) = \pi^j(F)(x)u^{n-1} \ , \qquad x \in \mathbb{C}^{2j+1} \ , \ n \in \mathbb{Z} \ ,$$

and extended linearly to all elements in $\mathbb{C}^{2j+1}[u^{\pm 1}]$, and similarly for the other generators.

Here, we use the same notation for the corresponding representation $\pi^j_u \colon \mathcal{L}U_q sl_2 \to \mathrm{End}\big(\mathbb{C}^{2j+1}[u^{\pm 1}]\big)$, and call it *formal evaluation representation*. We denote the corresponding infinite-dimensional module by $\mathbb{C}^{2j+1}_u$. In what follows, we will often use the notation $\mathbb{C}^{2j+1}_{zu}$, for $z \in \mathbb{C}^*$, which means that the $\mathcal{L}U_q sl_2$ action is on the same underlying vector space $\mathbb{C}^{2j+1}[u^{\pm 1}]$ but one needs to replace u in the action (3.2.7) by zu. Let $\{x_k\}_{0 \le k \le 2j}$ be a basis in the spin-j module $\mathbb{C}^{2j+1}$ over $U_q sl_2$ such that x_0 is the highest weight vector, or $|j, j\rangle$ in the notations of (3.2.2), and $x_k := \pi^j(F)^k x_0$. Then the vectors

$$x_{k,n} := x_k u^n \ , \qquad 0 \le k \le 2j \ , \ n \in \mathbb{Z} \ , \tag{3.2.8}$$

form a basis in $\mathbb{C}^{2j+1}_u$ with the action of $\mathcal{L}U_q sl_2$ up to scalars described as follows: E_0 changes the index (k, n) by $(+1, -1)$, while F_0 changes it in the opposite way $(k, n) \to (k-1, n+1)$, and similarly for E_1, F_1, and finally K_0 and K_1 do not change the index (k, n), they act as K^{-1} and K, respectively, on x_k.

For the $\mathcal{L}U_q sl_2$ action in the basis (3.2.8), let us consider an example of the $j = 1$ case in more details. The action of $\mathcal{L}U_q sl_2$ is better to present diagrammatically as in Figure 3.1 where each node corresponds to some $x_{k,n}$ while edges are the actions of E_i and F_i described up to scalars above.

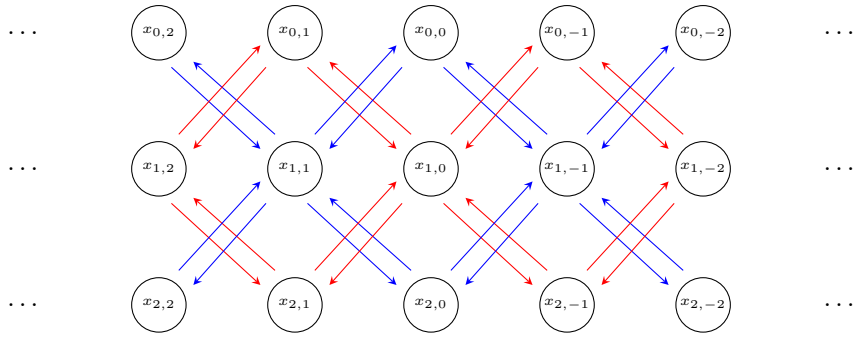


Figure 3.1: Quantum loop module $\mathbb{C}^3_u$ of spin-1 where the red and blue diagrams correspond to irreducible components of the $\mathcal{L}U_q sl_2$ action.

As we can see from this example, the formal-evaluation representation $\mathbb{C}^3_u$ is fully reducible due to the presence of two different sub-diagrams, one is blue and the other is red. Each of these sub-diagrams describes an irreducible action. It is not hard to see that this action arises from the other version of formal-evaluation representation in the homogeneous gradation. Recall [BDGZ93] that the homogeneous gradation action is as in (3.2.7) for E_0 and F_0, while E_1 and F_1 act just as E and F, respectively. Let us denote the corresponding $\mathcal{L}U_q sl_2$-module on the vector space $\mathbb{C}^{2j+1}[u^{\pm 1}]$ by $\widetilde{\mathbb{C}}^{2j+1}_u$. It is clear from the action in the basis of $\widetilde{\mathbb{C}}^{2j+1}_u$ analogous to $x_k u^n$ that this module is irreducible, while in our example $\mathbb{C}^3_u$ is decomposed as $\mathbb{C}^3_u \cong \widetilde{\mathbb{C}}^3_u \oplus \widetilde{\mathbb{C}}^3_u$.

We have a similar result for arbitrary spin j: indeed, the diagram for $j = \frac{1}{2}$ is obtained by simply deleting the last row of nodes labeled by $x_{2,n}$ and all arrows attached to them, so we get a disjoint union of two zig-zag's, while for $j > 1$ one just repeats the obvious pattern of arrows which gives again two independent sub-diagrams of different color. Let us reformulate this observation as the following lemma.

Lemma 3.2.1. *$\mathbb{C}^{2j+1}_u$ is a direct sum of two copies of $\widetilde{\mathbb{C}}^{2j+1}_u$.*

This effect of decomposition of the action in one gradation via actions in the other gradation is in contrast to the finite-dimensional evaluation representations where the choice of gradation is not really important.

3.2.3 Evaluation of the universal R-matrix.

The universal R-matrix $\mathfrak{R}$ is an element of a completion of $U_q\widehat{sl}_2 \otimes U_q\widehat{sl}_2$ having the form of a product of infinite series over root vectors [KT92a, Theorem 1], see the expression in our conventions in Appendix B. Let us now consider formal evaluation representations applied to the second component of the universal R-matrix which results in the so-called affine L-operators and R-matrices.

The spin-$\frac{1}{2}$ *affine* L-operators[16] are defined as follows [FR92]:

$$\mathbf{L}^+(u) = (\mathrm{id} \otimes \pi^{\frac{1}{2}}_{u^{-1}})(\mathfrak{R}) \ , \qquad \mathbf{L}^-(u) = [(\mathrm{id} \otimes \pi^{\frac{1}{2}}_{u^{-1}})(\mathfrak{R}_{21})]^{-1} \ . \tag{3.2.9}$$

[16] We call these operators affine in order to distinguish them from the usual L-operators defined in (3.2.12). The spin-j affine L-operators are defined using $\pi^j_{u^{-1}}$ instead of $\pi^{\frac{1}{2}}_{u^{-1}}$ in (3.2.9). However, we will only need the spin-1/2 ones.

The calculation of such specialization is tedious, so it is given in Appendix B in Section B.2.2, with their final expressions in (B.2.18) and (B.2.25). We thus see that they belong in $\mathcal{L}U_q sl_2((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^2)$. Considering the modes of the entries of $\mathbf{L}^{\pm}(u)$, they generate $H = \mathcal{L}U_q sl_2$ and satisfy the Yang-Baxter algebra relations:

$$\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(u/v)\mathbf{L}_1^{\pm}(u)\mathbf{L}_2^{\pm}(v) = \mathbf{L}_2^{\pm}(v)\mathbf{L}_1^{\pm}(u)\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(u/v) \ , \tag{3.2.10}$$

$$\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(u/v)\mathbf{L}_1^{\pm}(u)\mathbf{L}_2^{\mp}(v) = \mathbf{L}_2^{\mp}(v)\mathbf{L}_1^{\pm}(u)\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(u/v) \ . \tag{3.2.11}$$

These relations follow from the evaluation of the universal Yang-Baxter equation (3.1.3).

Secondly, applying the evaluation map as defined in (3.2.6), to the first tensor component of $\mathfrak{R}$ and the formal evaluation representation to the second, one obtains L-operators.

Definition 3.2.2. *For $j \in \frac{1}{2}\mathbb{N}$:*

$$\mathbf{L}^{(j)}(u_1/u_2) = (\mathsf{ev}_{u_1} \otimes \pi^j_{u_2})(\mathfrak{R}) \quad \in U_q sl_2((u_2/u_1)) \otimes \mathrm{End}(\mathbb{C}^{2j+1}) \ . \tag{3.2.12}$$

We call $\mathbf{L}^{(j)}(u)$ the spin-j L-operator.

Note that $\mathbf{L}^{(0)}(u) = 1$ by (3.1.2). Evaluating the first component of $\mathbf{L}^{(j)}(u)$ on a finite-dimensional representation of $U_q sl_2$ we get the R-matrix. For any spin j_1, j_2, we denote the R-matrix by

$$\begin{aligned}\mathcal{R}^{(j_1,j_2)}(u_1/u_2) &= (\pi^{j_1}_{u_1} \otimes \pi^{j_2}_{u_2})(\mathfrak{R}) \\ &= (\pi^{j_1} \otimes \mathrm{id})(\mathbf{L}^{(j_2)}(u_1/u_2)) \ .\end{aligned} \tag{3.2.13}$$

We recall that the L-operators satisfy the RLL relations. Indeed, applying $(\mathsf{ev}_{u_1} \otimes \pi^{j_2}_{u_2} \otimes \pi^{j_3}_{u_3})$ to (3.1.3), one finds[17]

$$\mathbf{L}_1^{(j_2)}(u_1/u_2)\mathbf{L}_2^{(j_3)}(u_1/u_3)\mathcal{R}_{12}^{(j_2,j_3)}(u_2/u_3) = \mathcal{R}_{12}^{(j_2,j_3)}(u_2/u_3)\mathbf{L}_2^{(j_3)}(u_1/u_3)\mathbf{L}_1^{(j_2)}(u_1/u_2) \ . \tag{3.2.14}$$

Recall also that the R-matrix satisfies the Yang-Baxter equation. It is found by applying $(\pi^{j_1} \otimes \mathrm{id} \otimes \mathrm{id})$ to the above equation and setting $u_3 = 1$:

$$\mathcal{R}_{12}^{(j_1,j_2)}(u_1/u_2)\mathcal{R}_{13}^{(j_1,j_3)}(u_1)\mathcal{R}_{23}^{(j_2,j_3)}(u_2) = \mathcal{R}_{23}^{(j_2,j_3)}(u_2)\mathcal{R}_{13}^{(j_1,j_3)}(u_1)\mathcal{R}_{12}^{(j_1,j_2)}(u_1/u_2) \ . \tag{3.2.15}$$

An explicit computation of the spin-$\frac{1}{2}$ L-operator $\mathbf{L}^{(\frac{1}{2})}(u)$ and the R-matrix for $j_1 = j_2 = \frac{1}{2}$ as the evaluation of the universal R-matrix can be found in [BGKNR12, eqs. (4.62), (4.53)]. Similar computations for affine L-operators $\mathbf{L}^{\pm}(u)$ are done in Appendix B.3. For $j = \frac{1}{2}$ in (3.2.12), the L-operator is given by:

$$\mathbf{L}^{(\frac{1}{2})}(u) = \mu(u)\mathcal{L}^{(\frac{1}{2})}(u) \ , \tag{3.2.16}$$

with $\mathcal{L}^{(\frac{1}{2})}(u)$ from (1.1.8), where the 'normalization' $\mu(u)$ is the following function of u^{-1}:

$$\mu(u) = u^{-1}q^{-\frac{1}{2}}e^{\Lambda(u^{-2}q^{-1})} \ , \tag{3.2.17}$$

[17] The RLL equation belongs to the triple product $U_q sl_2 \otimes \mathrm{End}(\mathbb{C}^{2j_1+1}) \otimes \mathrm{End}(\mathbb{C}^{2j_2+1})$, and thus the L-operator should be written as $\mathbf{L}^{(j)}_{0i}(u)$ but here we omit the label 0 corresponding to $U_q sl_2$.

and $\Lambda(u)$ can be written as a power series (B.3.1) in u^{-1} with coefficients $C_k \in U_q sl_2$ being certain polynomials in the Casimir element, see (B.2.5) and (B.2.6) for more details. Therefore, $\mathbf{L}^{(\frac{1}{2})}(u)$ is indeed in $U_q sl_2((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^2)$.

For $j_1 = j_2 = \frac{1}{2}$ in (3.2.13), the corresponding R-matrix is given by:

$$\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(u) = \pi^{\frac{1}{2}}(\mu(u)) R^{(\frac{1}{2},\frac{1}{2})}(u) \ , \tag{3.2.18}$$

where $R^{(\frac{1}{2},\frac{1}{2})}(u)$ is given in (1.1.5) and with

$$\pi^j(\mu(u)) = u^{-1} q^{-\frac{1}{2}} \exp\left(\sum_{k=1}^{\infty} \frac{q^{k(2j+1)} + q^{-k(2j+1)}}{1+q^{2k}} \frac{u^{-2k}}{k} \right) \ , \tag{3.2.19}$$

where we used the evaluation of the coefficients C_k of $\Lambda(u)$ given in (B.3.1), see [BGKNR12, eq. (4.59)]. Note that $R^{(\frac{1}{2},\frac{1}{2})}(u)$ coincides with the expression in (C.3.1) for $j = \frac{1}{2}$.

We now recall a special central element in $U_q sl_2$, called the quantum determinant $\gamma(u)$. It is given by [Sk88]:

$$\begin{aligned} \gamma(u) &= \mathrm{tr}_{12}\big(\mathcal{P}_{12}^{-} \mathcal{L}_1^{(\frac{1}{2})}(u) \mathcal{L}_2^{(\frac{1}{2})}(uq)\big) \\ &= u^2 q^2 + u^{-2} q^{-2} - C \ , \end{aligned} \tag{3.2.20}$$

where tr_{12} stands for the trace over $V_1 \otimes V_2$ and where C is the Casimir element of $U_q sl_2$ defined in (2.3.5). Here, as usual, the permutation matrix $\mathcal{P}_{12} \equiv \mathcal{P}$ with $\mathcal{P} = R^{(\frac{1}{2},\frac{1}{2})}(1)/(q-q^{-1})$ for the R-matrix (1.1.5) and $\mathcal{P}_{12}^{-} = (1-\mathcal{P})/2$.

By straightforward calculations, one finds that the L-operator $\mathbf{L}^{(\frac{1}{2})}(u)$ given in (3.2.16) satisfies a unitarity property:

$$\mathbf{L}^{(\frac{1}{2})}(u^{-1}) \mathbf{L}^{(\frac{1}{2})}(u) = \mathbf{L}^{(\frac{1}{2})}(u) \mathbf{L}^{(\frac{1}{2})}(u^{-1}) = \mathbf{c}(u) \mathbb{I}_2 \ , \qquad \text{with } \mathbf{c}(u) = -\gamma(uq^{-1}) \mu(u) \mu(u^{-1}) \ , \tag{3.2.21}$$

where the quantum determinant $\gamma(u)$ is given in (3.2.20). Note that $\mathbf{c}(u)$ is invariant by the inversion of its argument, i.e. $\mathbf{c}(u) = \mathbf{c}(u^{-1})$.

3.2.4 Hopf algebra structure

Given a Hopf algebra H, it is known that the coproduct, counit and antipode can be formulated solely in terms of the affine L-operators [RS90] defined in (3.2.9). Recall the counit and antipode for $\mathcal{L}U_q sl_2$ are respectively given by (2.3.10) and (2.3.11). Recall also the action of the counit and the antipode on the universal R-matrix are given by (3.1.1), (3.1.2). Then, regarding the coproduct's action on the affine L-operators, it is obtained from the evaluation of the second axiom of the

universal R-matrix (R2). The Hopf algebra structure of $\mathcal{L}U_q sl_2$ can be described as follows[18] [RS90]

$$(\Delta \otimes \mathrm{id})(\mathbf{L}^{\pm}(u)) = \left(\mathbf{L}^{\pm}(u)\right)_{[1]} \left(\mathbf{L}^{\pm}(u)\right)_{[2]} \ , \tag{3.2.23}$$

$$(S \otimes \mathrm{id})(\mathbf{L}^{\pm}(u)) = (\mathbf{L}^{\pm}(u))^{-1} \ , \tag{3.2.24}$$

$$(\epsilon \otimes \mathrm{id})(\mathbf{L}^{\pm}(u)) = 1 \ . \tag{3.2.25}$$

As a consequence, for the usual L-operator we have

$$(\Delta \otimes \mathrm{id})(\mathbf{L}^{(j)}(u)) = \left(\mathbf{L}^{(j)}(u)\right)_{[1]} \left(\mathbf{L}^{(j)}(u)\right)_{[2]} \ , \tag{3.2.26}$$

$$(S \otimes \mathrm{id})(\mathbf{L}^{(j)}(u)) = (\mathbf{L}^{(j)}(u))^{-1} \ , \tag{3.2.27}$$

$$(\epsilon \otimes \mathrm{id})(\mathbf{L}^{(j)}(u)) = 1 \ . \tag{3.2.28}$$

3.2.5 Evaluation of the universal K-matrix.

Recall that we fixed $H = \mathcal{L}U_q sl_2$ without specifying its comodule algebra B. Assume that a universal K-matrix $\mathfrak{K}$ exists for a choice of B and the twist pair $(\psi, J) = (\eta, 1 \otimes 1)$ where η is defined in (3.1.12). Here $\mathfrak{K}$ is in $B \otimes \mathcal{L}U_q sl_2$, or rather in an appropriate completion of the tensor product. For example, $\mathfrak{K}$ might be written in the form of an infinite product over root vectors, as it the case for the universal R-matrices. For the root vectors of $B = O_q$, see [BK17], and their relations with the alternating generators of $B = \mathcal{A}_q$ in [T21c], and (B.1.1)-(B.1.2) for $H = \mathcal{L}U_q sl_2$. Then, we can consider its evaluation with the formal evaluation representations on the second tensor component:

Definition 3.2.3. *For $j \in \frac{1}{2}\mathbb{N}$, introduce*

$$\mathbf{K}^{(j)}(u) = (id \otimes \pi^{j}_{u^{-1}})(\mathfrak{K}) \quad \in B((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^{2j+1}) \ . \tag{3.2.29}$$

We call $\mathbf{K}^{(j)}(u)$ the spin-j K-operator.

Similarly to the case of L-operators, we consider u as a formal variable and assume that $\mathbf{K}^{(\frac{1}{2})}(u)$ is in $B((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^2)$. By Proposition 3.1.8, the universal K-matrix $\mathfrak{K}$ satisfies the ψ-twisted reflection equation (3.1.14). We now show that the evaluation of this equation leads to a reflection equation with spectral parameter. To do so, we need to evaluate the ψ-twisted universal R-matrices (3.1.13). Firstly, note that $\psi = \eta$ from (3.1.12) is such that (recall the definition in (3.2.6))

$$\mathsf{ev}_u \circ \eta = \mathsf{ev}_{u^{-1}} \ , \tag{3.2.30}$$

which implies (3.2.1) for $\psi = \eta$. Then the evaluations of the ψ-twisted universal R-matrices read:

$$(\pi^{j_1}_{u_1} \otimes \pi^{j_2}_{u_2})(\mathfrak{R}^{\eta}) = \mathcal{R}^{(j_1,j_2)}(1/(u_1 u_2)) \ , \tag{3.2.31}$$

$$(\pi^{j_1}_{u_1} \otimes \pi^{j_2}_{u_2})((\mathfrak{R}^{\eta})_{21}) = \mathcal{R}^{(j_2,j_1)}_{21}(1/(u_1 u_2)) \ , \tag{3.2.32}$$

[18] The index $[j]$ characterizes the 'quantum space' $V_{[j]}$ on which the entries of $\mathbf{L}^{\pm}(u)$ act. With respect to the ordering $V_{[1]} \otimes V_{[2]}$, one has:

$$((T)_{[1]}(T')_{[2]})_{ij} = \sum_{k=1}^{2} (T)_{ik} \otimes (T')_{kj} \ . \tag{3.2.22}$$

$$(\pi^{j_1}_{u_1} \otimes \pi^{j_2}_{u_2})((\mathfrak{R}_{21})^{\eta}) = \mathcal{R}^{(j_2,j_1)}_{21}(u_1u_2) \ , \tag{3.2.33}$$

$$(\pi^{j_1}_{u_1} \otimes \pi^{j_2}_{u_2})(\mathfrak{R}^{\eta\eta}_{21}) = \mathcal{R}^{(j_2,j_1)}_{21}(u_1/u_2) \ , \tag{3.2.34}$$

$$(\pi^{j_1}_{u_1} \otimes \pi^{j_2}_{u_2})(\mathfrak{R}^{\eta\eta}) = \mathcal{R}^{(j_1,j_2)}(u_2/u_1) \ , \tag{3.2.35}$$

where

$$\mathcal{R}^{(j_2,j_1)}_{21}(u) = \mathcal{P}^{(j_2,j_1)}\mathcal{R}^{(j_2,j_1)}(u)\mathcal{P}^{(j_1,j_2)} \ . \tag{3.2.36}$$

Applying $\mathfrak{p}_{23}$ to (3.1.14) leads to $\mathfrak{K}_{13}(\mathfrak{R}^{\eta})_{23}\mathfrak{K}_{12}\mathfrak{R}_{32} = \mathfrak{R}^{\eta\eta}_{23}\mathfrak{K}_{12}(\mathfrak{R}^{\eta})_{32}\mathfrak{K}_{13}$. Finally, applying $(\mathrm{id} \otimes \pi^{j_1}_{u_1^{-1}} \otimes \pi^{j_2}_{u_2^{-1}})$ to the latter equation using (3.2.31)-(3.2.35), it follows that the K-operator (3.2.29) satisfies the reflection equation[19]

$$\begin{aligned}\mathcal{R}^{(j_1,j_2)}_{12}(u_1/u_2)\mathbf{K}^{(j_1)}_1(u_1)\mathcal{R}^{(j_2,j_1)}_{21}(u_1u_2)\mathbf{K}^{(j_2)}_2(u_2) \\ = \mathbf{K}^{(j_2)}_2(u_2)\mathcal{R}^{(j_1,j_2)}_{12}(u_1u_2)\mathbf{K}^{(j_1)}_1(u_1)\mathcal{R}^{(j_2,j_1)}_{21}(u_1/u_2) \ ,\end{aligned} \tag{3.2.37}$$

for any value of j_1, j_2.

We show in [LBG23, Sec. 4.1.4] that the evaluation of the universal R-matrix satisfies

$$(\mathsf{ev}_{u_1^{-1}} \otimes \pi^{j}_{u_2^{-1}})(\mathfrak{R}) = (\mathsf{ev}_{u_1} \otimes \pi^{j}_{u_2})(\mathfrak{R}_{21}) \ , \tag{3.2.38}$$

which implies the P-symmetry

$$\mathcal{R}^{(j_2,j_1)}_{21}(u) = \mathcal{R}^{(j_1,j_2)}(u) \ . \tag{3.2.39}$$

Note that a proof of the above relation that does not use (3.2.38) can be found in [RSV14, Lem. 2.1]. Then, due to the P-symmetry (3.2.39), the relations (3.2.32)-(3.2.34) become

$$(\pi^{j_1}_{u_1} \otimes \pi^{j_2}_{u_2})((\mathfrak{R}^{\eta})_{21}) = \mathcal{R}^{(j_1,j_2)}(1/(u_1u_2)) \ , \tag{3.2.40}$$

$$(\pi^{j_1}_{u_1} \otimes \pi^{j_2}_{u_2})((\mathfrak{R}_{21})^{\eta}) = \mathcal{R}^{(j_1,j_2)}(u_1u_2) \ , \tag{3.2.41}$$

$$(\pi^{j_1}_{u_1} \otimes \pi^{j_2}_{u_2})(\mathfrak{R}^{\eta\eta}_{21}) = \mathcal{R}^{(j_1,j_2)}(u_1/u_2) \ , \tag{3.2.42}$$

and the reflection equation (3.2.37) becomes the standard reflection equation

$$\begin{aligned}\mathcal{R}^{(j_1,j_2)}(u_1/u_2)\mathbf{K}^{(j_1)}_1(u_1)\mathcal{R}^{(j_1,j_2)}(u_1u_2)\mathbf{K}^{(j_2)}_2(u_2) \\ = \mathbf{K}^{(j_2)}_2(u_2)\mathcal{R}^{(j_1,j_2)}(u_1u_2)\mathbf{K}^{(j_1)}_1(u_1)\mathcal{R}^{(j_1,j_2)}(u_1/u_2) \ .\end{aligned} \tag{3.2.43}$$

Let us also mention that applying one-dimensional representations for B, the K-operators specialize to K-matrices. This will be precisely studied for $B = \mathcal{A}_q$ in Section 3.5.3.

3.2.6 Comodule algebra structure

Consider the subalgebra in B generated by the matrix entries of the K-operator $\mathbf{K}^{(\frac{1}{2})}(u)$. They satisfy the reflection equation (3.2.43) for $j_1 = j_2 = \frac{1}{2}$. Similarly to the coproduct of $\mathcal{L}U_q sl_2$

[19] As for the L-operator $\mathbf{L}^{(j)}(u)$, the K-operator should be written as $\mathbf{K}^{(j)}_{0i}(u)$ but here we use standard notation $\mathbf{K}^{(j)}_{i}(u)$ where we omit the label 0 corresponding to B.

discussed above, the coaction for this subalgebra can be expressed in terms of L- and K-operators. Applying $(\mathrm{id}\otimes\mathrm{id}\otimes\pi^j_{u^{-1}})$ on the second axiom of the universal K-matrix (K2), we get:

Proposition 3.2.4 ([LBG23, Prop. 4.18]). *The coaction map $\delta\colon B\to B\otimes\mathcal{L}U_qsl_2$ restricted to the subalgebra generated by the matrix entries of $\mathbf{K}^{(\frac{1}{2})}(u)$ is such that*

$$(\delta\otimes id)(\mathbf{K}^{(\frac{1}{2})}(u)) = \left([\mathbf{L}^-(u^{-1})]^{-1}\right)_{[2]}\left(\mathbf{K}^{(\frac{1}{2})}(u)\right)_{[1]}\left(\mathbf{L}^+(u)\right)_{[2]}. \tag{3.2.44}$$

In the case when B is generated by the matrix entries of the K-operator $\mathbf{K}^{(\frac{1}{2})}(u)$, eq. (3.2.44) expresses the coaction map for B solely in terms of L- and K-operators. This is the case when $B=\mathcal{A}_q$ and it will be discussed in Section 3.6.

We finally consider the evaluated coaction $\delta_w=(\mathrm{id}\otimes\mathsf{ev}_w)\circ\delta\colon B\to B\otimes U_qsl_2$, for a formal parameter w, applied to the matrix elements of the spin-j K-operator. The evaluated coaction is obtained by taking the image of (K2) under the evaluation map $(\mathrm{id}\otimes\mathsf{ev}_w\otimes\pi^j_{u^{-1}})$ using (3.2.12), (3.2.30), the evaluation

$$\mathbf{L}^{(j)}(u_2/u_1) = (\mathsf{ev}_{u_1}\otimes\pi^j_{u_2})(\mathfrak{R}_{21}) \tag{3.2.45}$$

and it is given by

$$(\delta_w\otimes\mathrm{id})(\mathbf{K}^{(j)}(u)) = \left(\mathbf{L}^{(j)}(u/w)\right)_{[2]}\left(\mathbf{K}^{(j)}(u)\right)_{[1]}\left(\mathbf{L}^{(j)}(uw)\right)_{[2]}. \tag{3.2.46}$$

Whereas the action of $(\mathrm{id}\otimes\pi^j_{v^{-1}})$ on (K1) gives

$$\mathbf{K}^{(j)}(v)(\mathrm{id}\otimes\pi^j)[\delta_{v^{-1}}(b)] = (\mathrm{id}\otimes\pi^j)[\delta_v(b)]\mathbf{K}^{(j)}(v)\ . \tag{3.2.47}$$

We call it the twisted intertwining relation for $\mathbf{K}^{(j)}(u)$.

In summary, the results obtained up to here are the following:

- Firstly, we defined axiomatically a universal K-matrix $\mathfrak{K}\in B\otimes H$, where B is a comodule algebra over H, see Definition 3.1.7.
- Secondly, we introduced formal evaluation representations of $\mathcal{L}U_qsl_2$ which are adapted to work with L- and K-operators. These infinite-dimensional representations were known as quantum loop modules [CG03], but the framework developed here for L- and K-operators is new to our knowledge.
- Thirdly, we defined spin-j K-operators, $\mathbf{K}^{(j)}(u)\in B((u^{-1}))\otimes\mathrm{End}(\mathbb{C}^{2j+1})$, as the formal evaluation of the second tensor component of the universal K-matrix, see Definition 3.2.29. Provided the Hopf algebra automorphism ψ satisfies (3.2.1), we have shown that $\mathbf{K}^{(j)}(u)$ satisfies the reflection equation with spectral parameter (3.2.37) (or (3.2.43) if the R-matrix enjoys the P-symmetry (3.2.39)).
- Finally, the comodule algebra structure of B using K-operators is obtained from the evaluation of the second axiom of the universal K-matrix (K2). This leads to the desired form, see (3.2.44) and (3.2.46).

Therefore, the elements presented here provide a solution to Problem 1, addressing and resolving the stated issue.

Now, we want to solve Problem 2 which consists in constructing a fused K-operator, $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q \otimes \mathrm{End}(\mathbb{C}^{2j+1})$ that is a spin-j generalization of the spin-1/2 case in (2.1.5). It would also be desirable to give a clear interpretation of these fused K-operators in terms of the universal K-matrix where the comodule algebra over the Hopf algebra is now fixed to $B = \mathcal{A}_q$ and $H = \mathcal{L}U_q sl_2$.

The strategy is the following. We begin by studying the tensor product representations of $\mathcal{L}U_q sl_2$ and use the reducibility criteria on the evaluation parameters [CP91] to identify sub-representations of spin-$(j+1/2)$ and spin-$(j-1/2)$. Then, we construct $\mathcal{L}U_q sl_2$-intertwiners explicitly for the corresponding sub-representations which is a new result to the best of our knowledge. This construction is then extended for tensor product of formal evaluation representations of $\mathcal{L}U_q sl_2$ and this is also a new result. These intertwiners are used to show that the spin-j K-operators $\mathbf{K}^{(j)}(u)$ satisfy fusion and reduction properties. These properties inspire us to recursively define fused K-operators $\mathcal{K}^{(j)}(u)$ from (2.1.5), and we show that they satisfy a reflection equation with a spectral parameter in Theorem 3.5.2. Finally, we propose an interpretation of $\mathcal{K}^{(j)}(u)$ in terms of a universal K-matrix, which is expressed as Conjecture 1, establishing a relationship between $\mathcal{K}^{(j)}(u)$ and $\mathbf{K}^{(j)}(u)$. Supporting evidence are also given.

3.3 Tensor product representations of $\mathcal{L}U_q sl_2$

In principle, as we have seen in the previous section, the evaluations of a universal R- or K-matrix lead to a L-operator or K-operator, respectively. Although the root vectors of O_q are known [BK17] as well as their relations with the alternating generators of $\mathcal{A}_q$ [T21c], the universal K-matrix $\mathfrak{K} \in \mathcal{A}_q \otimes \mathcal{L}U_q sl_2$ for the twist pair $(\psi, J) = (\eta, 1 \otimes 1)$, where η is defined in (3.1.12), is not known, even its existence is an open problem. In further sections we give evidence on the existence of such universal K-matrix by considering its relation with K-operators that are independently constructed using a fusion procedure. This construction is based on the analysis of the tensor product of evaluation representations of $\mathcal{L}U_q sl_2$. The reducibility criteria in terms of ratios of the evaluation parameters for these tensor products are known [CP91, Sec. 4.9]. In this section, we study the sub-quotient structure of these tensor products in more details, and construct explicitly the corresponding intertwining operators. We also extend the above study for tensor product of formal evaluation representations of $\mathcal{L}U_q sl_2$, i.e. when the evaluation parameters are formal. It is important to note that the latter are the principal elements in the construction of fused L- and K-operators for any spin-j, and the formal versions are necessary to give a careful treatment for this construction. In the next section, they are built from the fundamental (spin-1/2) L- and K-operators.

For pedagogical reasons, let us start with the case where u is a non-zero complex number. We study now tensor products of these representations

$$(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^{j}_{u_2}) \circ \Delta \colon \mathcal{L}U_q sl_2 \to \mathrm{End}(\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2}) \ , \tag{3.3.1}$$

where π^j_u is defined in (3.2.4), and look at special points in the evaluation parameters space so that a proper sub-representation emerges. The strategy is the following: we first construct basis vectors

$\{w_k\}, \{v_\ell\}$ for the decomposition with respect to the subalgebra generated by $\{E_1, F_1, K_1\}$. Then, we study the action of $\{E_0, F_0, K_0\}$ on these basis vectors. We find that there are only two ratios of evaluation parameters up to a sign when we get a proper sub-representation, and we also construct explicitly the corresponding intertwining maps.

3.3.1 Analysis of the tensor product representation of $\mathcal{L}U_q sl_2$

Consider first the subalgebra generated by $\{E_1,\ F_1,\ K_1\}$ and construct basis vectors $\{w_k\}, \{v_\ell\}$, where $k = 0,\ 1,\ \ldots,\ 2j+1$, $\ell = 0,\ 1,\ \ldots,\ 2j-1$ and $j \in \frac{1}{2}\mathbb{N}_+$, corresponding to the tensor product decomposition $\mathbb{C}^2 \otimes \mathbb{C}^{2j+1} = \mathbb{C}^{2j+2} \oplus \mathbb{C}^{2j}$. We denote by w_0 and v_0 the highest weight vectors of the corresponding spins-$(j+\frac{1}{2})$ and $(j-\frac{1}{2})$. These are defined by the relations

$$[(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^{j}_{u_2})\Delta(E_1)]w_0 = 0\ , \qquad [(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^{j}_{u_2})\Delta(K_1)]w_0 = q^{2j+1}w_0\ , \tag{3.3.2}$$

$$[(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^{j}_{u_2})\Delta(E_1)]v_0 = 0\ , \qquad [(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^{j}_{u_2})\Delta(K_1)]v_0 = q^{2j-1}v_0\ . \tag{3.3.3}$$

Solutions to these equations are uniquely determined, up to a scalar, by

$$w_0 = |\uparrow\rangle \otimes |j, j\rangle\ , \qquad v_0 = |\uparrow\rangle \otimes |j, j-1\rangle - \frac{u_1}{u_2} q^{-j-\frac{1}{2}} A_{j,j-1}\, |\downarrow\rangle \otimes |j, j\rangle\ , \tag{3.3.4}$$

with $|\uparrow\rangle = |\frac{1}{2}, \frac{1}{2}\rangle, |\downarrow\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$, and where $A_{j,m}$ is given in (3.2.3). The other basis vectors are constructed via the action of F_1:

$$w_k = \left[(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^{j}_{u_2})\Delta(F_1)\right]^k w_0\ , \qquad v_\ell = \left[(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^{j}_{u_2})\Delta(F_1)\right]^\ell v_0\ , \tag{3.3.5}$$

where $0 \le k \le 2j+1$ and $0 \le \ell \le 2j-1$, and we set these vectors to zero if their index is outside of the indicated range.

Now, we study the action of the generators E_0 and F_0 on these basis vectors, and we begin with the action on w_k's:

$$\begin{aligned} \left[(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^{j}_{u_2})\Delta(E_0)\right] w_k &= e_{1,k}(u_1,u_2)w_{k+1} + e_{2,k}(u_1,u_2)v_k\ , \\ \left[(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^{j}_{u_2})\Delta(F_0)\right] w_k &= f_{1,k}(u_1,u_2)w_{k-1} + f_{2,k}(u_1,u_2)v_{k-2}\ , \end{aligned} \tag{3.3.6}$$

where the coefficients are

$$e_{1,k}(u_1,u_2) = u_1^{-2} + [2j]_q u_2^{-2}\ , \qquad e_{2,k}(u_1,u_2) = \frac{q^{2j+\frac{1}{2}}(u_1^2 - u_2^2 q^{-2j-1})\sqrt{[2j]_q}}{u_1^2 u_2\,[2j+1]_q}\ , \tag{3.3.7}$$

$$f_{1,k}(u_1,u_2) = \left(u_1^2 + u_2^2[2j]_q\right)\frac{[2j-k+2]_q[k]_q}{[2j+1]_q}\ , \tag{3.3.8}$$

$$f_{2,k}(u_1,u_2) = \frac{(u_1^2 - u_2^2 q^{-2j-1})u_2 q^{2j+\frac{1}{2}}\sqrt{[2j]_q}[k]_q[k-1]_q}{[2j+1]_q}\ . \tag{3.3.9}$$

We next look at the action on v_ℓ's:

$$\begin{aligned}\left[(\pi^{\frac{1}{2}}_{u_1}\otimes\pi^{j}_{u_2})\Delta(E_0)\right]v_\ell &= \bar{e}_{1,\ell}(u_1,u_2)v_{\ell+1}+\bar{e}_{2,\ell}(u_1,u_2)w_{\ell+2}\ ,\\ \left[(\pi^{\frac{1}{2}}_{u_1}\otimes\pi^{j}_{u_2})\Delta(F_0)\right]v_\ell &= \bar{f}_{1,\ell}(u_1,u_2)v_{\ell-1}+\bar{f}_{2,\ell}(u_1,u_2)w_{\ell}\ .\end{aligned} \tag{3.3.10}$$

where the coefficients are

$$\bar{e}_{1,\ell}(u_1,u_2)=\frac{u_2^{-2}[2j+2]_q-u_1^{-2}}{[2j+1]_q}\ ,\qquad \bar{e}_{2,\ell}(u_1,u_2)=\frac{u_1^2-u_2^2q^{2j+1}}{q^{2j+\frac{1}{2}}u_1^2u_2^3\sqrt{[2j]_q}[2j+1]_q}\ ,$$
$$\bar{f}_{1,\ell}(u_1,u_2)=(u_1^2-u_2^2[2j+2]_q)\frac{[\ell-2j]_q[\ell]_q}{[2j+1]_q}\ ,\qquad \bar{f}_{2,\ell}(u_1,u_2)=\frac{(u_1^2-u_2^2q^{2j+1})[\ell-2j]_q[2j+1-\ell]_q}{u_2q^{2j+\frac{1}{2}}\sqrt{[2j]_q}[2j+1]_q}\ .$$

We first note that in (3.3.6) and (3.3.10) the coefficients with indices *2,k* and *2,ℓ*, respectively, correspond to contribution in the action that mixes the two spin components. In order to find proper sub-representations, we thus need to analyse the roots of these functions $e_{2,k}$, $f_{2,k}$, $\bar{e}_{2,\ell}$ and $\bar{f}_{2,\ell}$. First, for generic values of u_1/u_2 these functions do not vanish which corresponds to the well known fact that this tensor product is irreducible. For later convenience, we present this action diagrammatically below in Figure 3.2 where the red (resp. blue) arrows correspond to the action of F_0 (resp. E_0), and the branching points correspond to the linear combinations from (3.3.6) and (3.3.10).

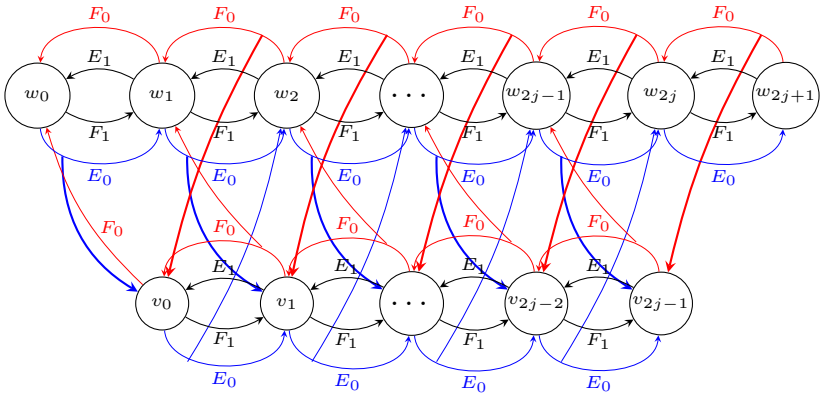
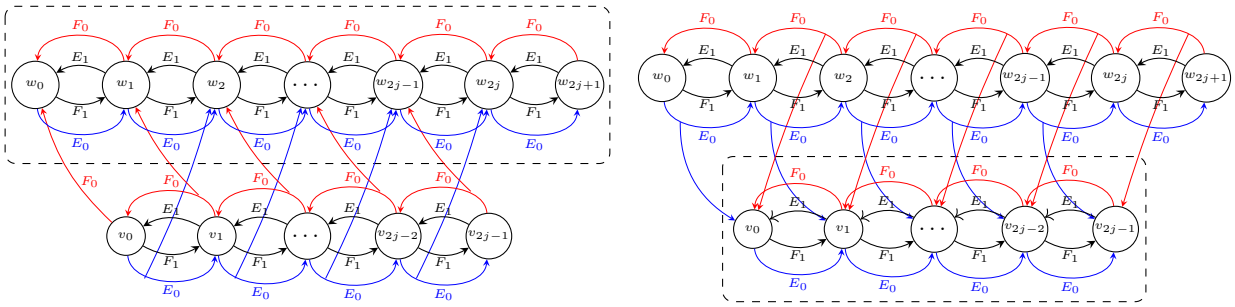


Figure 3.2: Action of E_i, F_i on w_k, v_ℓ for generic values of u_1 and u_2.

From the expression of the coefficients (3.3.7) and (3.3.9), one finds that $e_{2,k}(u_1,u_2)=f_{2,k}(u_1,u_2)=0$ iff $u_1/u_2=\pm q^{-j-\frac{1}{2}}$ and $\bar{e}_{2,\ell}(u_1,u_2)=\bar{f}_{2,\ell}(u_1,u_2)=0$ iff $u_1/u_2=\pm q^{j+\frac{1}{2}}$. Bold arrows in Figure 3.2 are used to emphasize that they disappear after fixing the ratio $u_1/u_2=\pm q^{-j-\frac{1}{2}}$. However we do not have simultaneously these four coefficients equal to zero so we do not have a direct sum decomposition. Instead, by fixing the ratio u_1/u_2 to the special values we have a sub-representation as depicted below by the dotted rectangles.



$u_1/u_2 = \pm q^{-j-\frac{1}{2}}$. $u_1/u_2 = \pm q^{j+\frac{1}{2}}$.

Figure 3.3: Action of E_i, F_i on w_k, v_ℓ for fixed values of u_1/u_2.

We thus find that fixing $u_1/u_2 = \pm q^{-j-\frac{1}{2}}$ (resp. $u_1/u_2 = \pm q^{j+\frac{1}{2}}$) gives a spin-$(j+\frac{1}{2})$ (resp. spin-$(j-\frac{1}{2})$) sub-representation.

Now, in order to prepare the fusion construction, we need to find which space the two sub-representations depicted above are isomorphic to. First, consider the spin-$(j+\frac{1}{2})$ sub-representation from the left part of Fig. 3.3. We find that $\mathbb{C}^2_{u_2 q^{-j-1/2}} \otimes \mathbb{C}^{2j}_{u_2}$ is isomorphic to $\mathbb{C}^{2j+1}_u$ when $u_2 = uq^{\frac{1}{2}}$. We thus introduce a $\mathcal{L}U_q sl_2$-intertwiner for $j \in \frac{1}{2}\mathbb{N}$

$$\mathcal{E}^{(j+\frac{1}{2})} \colon \mathbb{C}^{2j+1}_u \to \mathbb{C}^2_{uq^{-j}} \otimes \mathbb{C}^{2j}_{uq^{\frac{1}{2}}} \ ,$$

and compute explicitly its matrix expression in Lemma C.1.1.

Then, consider the right part of Fig. 3.2 with the spin-$(j-\frac{1}{2})$ sub-representation. We find that the space $\mathbb{C}^2_{u_2 q^{j+1/2}} \otimes \mathbb{C}^{2j}_{u_1}$ is isomorphic to $\mathbb{C}^{2j-1}_u$ when $u_2 = uq^{\frac{1}{2}}$. Similarly to $\mathcal{E}^{(j+\frac{1}{2})}$, we introduce a $\mathcal{L}U_q sl_2$-intertwiner for $j \in \frac{1}{2}\mathbb{N}_+$

$$\bar{\mathcal{E}}^{(j-\frac{1}{2})} \colon \mathbb{C}^{2j-1}_u \to \mathbb{C}^2_{uq^{j+1}} \otimes \mathbb{C}^{2j}_{uq^{\frac{1}{2}}} \ ,$$

and give its matrix expression in Lemma C.2.1.

Let us note that $\mathcal{E}^{(j+\frac{1}{2})}$ and $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ will be respectively used in the fusion and reduction of L- and K-operators. However, in order to give a careful treatment for this construction, one needs to consider u as a formal parameter and not a complex. Recall that we have already introduced the formal evaluation representation of $\mathcal{L}U_q sl_2$ in Section 3.2.2. The idea is now to extend the above analysis on tensor product representations of $\mathcal{L}U_q sl_2$, and construct $\mathcal{L}U_q sl_2$-intertwiners for a formal parameter u.

3.3.2 Tensor product of formal evaluation representations of $\mathcal{L}U_q sl_2$

Recall the formal evaluation representations of $\mathcal{L}U_q sl_2$ are defined by π^j_u from (3.2.5). We now want to study tensor products of these infinite-dimensional representations[20]

$$(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^j_{u_2}) \circ \Delta \colon \mathcal{L}U_q sl_2 \to \mathrm{End}(\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2}) \ , \tag{3.3.11}$$

to establish the $\mathcal{L}U_q sl_2$-intertwining properties analogous to Lemmas C.1.1 and C.2.1 on this formal level. First, note that as a vector space $\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2}$ is isomorphic to $\mathbb{C}^2 \otimes \mathbb{C}^{2j+1}[u_1^{\pm 1}, u_2^{\pm 1}]$ – the space of Laurent polynomials in two variables u_1 and u_2 with coefficients in the $2(2j+1)$-dimensional vector space – or equivalently, to the product of $\mathbb{C}^2 \otimes \mathbb{C}^{2j+1}$ and $\mathbb{C}[u_1^{\pm 1}, u_2^{\pm 1}]$. Below, we will use these identifications implicitly. Then, the action (3.3.11) can be written as

$$a\big(\boldsymbol{v} f(u_1,u_2)\big) = a(\boldsymbol{v}) f(u_1,u_2) = \Big[(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^j_{u_2}) \circ \Delta(a)\Big](\boldsymbol{v}) f(u_1,u_2) \tag{3.3.12}$$

for any $a \in \mathcal{L}U_q sl_2$, $\boldsymbol{v} \in \mathbb{C}^2 \otimes \mathbb{C}^{2j+1}$ and $f \in \mathbb{C}[u_1^{\pm 1}, u_2^{\pm 1}]$. In what follows, we will shortly write the action as on the left-hand side instead of lengthy but more precise expression on the right-hand side.

The difference from the finite-dimensional case where u_i's were generic complex numbers is that these infinite-dimensional tensor products are not irreducible anymore. Indeed, for every non-zero complex number z the Laurent polynomial $u_1 - z^{-1}u_2$ generates a proper ideal in the algebra of Laurent polynomials because its inverse lives in a bigger ring of rational functions in u_1 and u_2. Similarly, for the module $\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2}$ consider for a fixed $z \in \mathbb{C}^*$ the linear span of vectors of the following form:

$$I_z := \langle\ \boldsymbol{v}\,(u_1 - z^{-1}u_2) f(u_1,u_2) \mid \boldsymbol{v} \in \mathbb{C}^2 \otimes \mathbb{C}^{2j+1}\,,\ f \in \mathbb{C}[u_1^{\pm 1}, u_2^{\pm 1}]\ \rangle\ , \tag{3.3.13}$$

which can be thought of as 'ideal' generated by the Laurent polynomials $u_1 - z^{-1}u_2$.

Analysing the action of F_i and E_i on these vectors, recall (3.3.12), it is clear that I_z is a proper $\mathcal{L}U_q sl_2$-submodule of $\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2}$.

We can then define a so-called partial specialisation "identifying u_2 with zu_1" of the module $\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2}$ as the quotient by I_z from (3.3.13). We will use the following short-hand notations for this quotient:

$$\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{zu_1} := \Big(\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2}\Big)/I_z\ , \tag{3.3.14}$$

when the same formal parameter appears in both tensor factors. And let us denote the canonical projection

$$p^j_z :\ \mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2} \twoheadrightarrow \mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{zu_1}\ , \tag{3.3.15}$$

which is a $\mathcal{L}U_q sl_2$-intertwining operator. We also note that the quotient module $\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{zu_1}$ can be identified with $\mathbb{C}^2 \otimes \mathbb{C}^{2j+1}[u_1^{\pm 1}]$ as a vector space. Indeed, first note that the images $p^j_z(x \otimes y\, u_1^n u_2^m)$, for any $x \in \mathbb{C}^2$ and $y \in \mathbb{C}^{2j+1}$ are all proportional to $(x \otimes y)u_1^{n+m}$, therefore every element in the quotient space $\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{zu_1}$ is a (finite) linear combination of the monomials $(x \otimes y)u_1^n$, for $n \in \mathbb{Z}$,

[20] Note that the homomorphism $(\pi^{\frac{1}{2}}_{u_1} \otimes \pi^j_{u_2}) \circ \Delta$ sends $\mathcal{L}U_q sl_2$ to $\mathrm{End}(\mathbb{C}^2_{u_1}) \otimes \mathrm{End}(\mathbb{C}^{2j+1}_{u_2})$ which is a proper subalgebra in $\mathrm{End}(\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2})$, and this algebra embedding is implicitly used in (3.3.11).

which makes it identical to $\mathbb{C}^2 \otimes \mathbb{C}^{2j+1}[u_1^{\pm 1}]$.

In the quotient modules (3.3.14), we can now identify for appropriate values of z infinite-dimensional submodules isomorphic to certain quantum loop modules. First, denote by W^j the subspace in $\mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2}$ spanned by vectors $w_{k,n,m}$ obtained via the iterated action of F_1 on $w_0 u_1^n u_2^m$ where w_0 is the product (3.3.4) of highest-weight vectors in $U_q sl_2$-modules of spin-$\frac{1}{2}$ and j:

$$w_{k,n,m} := F_1^k(w_0)\, u_1^n u_2^m \ , \qquad 0 \leq k \leq 2j+1 \ , \ n,m \in \mathbb{Z} \ . \tag{3.3.16}$$

Note that as in the finite-dimensional situation if $k > 2j+1$ then the corresponding vectors $w_{k,n,m}$ are just zero. We will also denote $w_k := w_{k,0,0}$.

We claim that the image of W^j under the projection p_z^j at $z = q^{j+\frac{1}{2}}$ is a $\mathcal{L}U_q sl_2$-submodule isomorphic to the quantum loop module $\mathbb{C}^{2j+2}_{uq^j}$ of spin-$(j+\frac{1}{2})$. Indeed, first the action of E_1 preserves W^j because using the commutator relation between E_1 and F_1 and the fact that all vectors $w_{0,n,m}$ are annihilated by E_1 we get

$$E_1(w_{k,n,m}) \sim w_{k-1,n,m} \ ,$$

and so the image of W^j under p_z^j is equally preserved by the action of E_1 because p_z^j commutes with all E_i and F_i. Let us now analyse the action of E_0 and F_0 on the vectors $w_{k,n,m}$. Observing that $w_{k,n,m} = w_k u_1^n u_2^m$, we get for the action the expressions similar to (3.3.6):

$$\begin{aligned} E_0(w_{k,n,m}) &= E_0(w_k) u_1^n u_2^m = e_{1,k}(u_1,u_2) w_{k+1,n,m} + e_{2,k}(u_1,u_2) v_k u_1^n u_2^m \ , \\ F_0(w_{k,n,m}) &= F_0(w_k) u_1^n u_2^m = f_{1,k}(u_1,u_2) w_{k-1,n,m} + f_{2,k}(u_1,u_2) v_{k-2} u_1^n u_2^m \ , \end{aligned} \tag{3.3.17}$$

where v_k is defined as in (3.3.5) and the coefficients are as in (3.3.7)-(3.3.9). Note that the images $p_z^j(w_{k,n,m})$ are all proportional to $p_z^j(w_k) u_1^{n+m}$, so the vectors $w_{k,n} := p_z^j(w_k) u_1^n$, for $n \in \mathbb{Z}$ and $0 \leq k \leq 2j+1$, form a basis in the image of W^j under the projection p_z^j. Then, the action on these basis elements $w_{k,n}$ at $z = q^{j+\frac{1}{2}}$ simplifies to:

$$\begin{aligned} E_0(w_{k,n}) &= q^{-2j}[2j+1]_q \, w_{k+1,n-2} \ , \\ F_0(w_{k,n}) &= q^{2j}[2j+2-k]_q \, w_{k-1,n+2} \ . \end{aligned} \tag{3.3.18}$$

We thus see that $\mathcal{L}U_q sl_2$ acts on $p_z^j(W^j)$. We are just left to identify the quantum loop module $p_z^j(W^j)$ is isomorphic to. First note from (3.3.18) that the action of E_0 and F_0 changes the index n by ± 2 while the action of E_1 and F_1 keeps it unchanged. Therefore $p_z^j(W^j)$ is decomposed onto a direct sum of two $\mathcal{L}U_q sl_2$-modules: one generated by $w_{0,0}$ and the other by $w_{0,1}$, precisely as in Lemma 3.2.1. Following the same idea as in the proof of Lemma C.1.1 we identify this module with the quantum loop module $\mathbb{C}^{2j+2}_{u_1 q^j}$ which proves the above claim about the image of W^j under the projection p_z^j.

We can now construct the corresponding intertwining operator (for which again for brevity we use the same notation as in the finite-dimensional situation)

$$\mathcal{E}^{(j+\frac{1}{2})} \colon \mathbb{C}^{2j+2}_{uq^j} \to \mathbb{C}^2_u \otimes \mathbb{C}^{2j+1}_{uq^{j+1/2}}$$

by identifying the quotient space $\mathbb{C}^2_u \otimes \mathbb{C}^{2j+1}_{uq^{j+1/2}}$ with $\mathbb{C}^2 \otimes \mathbb{C}^{2j+1}[u^{\pm 1}]$ as discussed above, and simply acting on spin states by the same matrix as in (C.1.3) with the matrix entries as in Lemma C.1.1.

3.4 Spin-j L- and K-operators

In this section, we define spin-j L- and K-operators as evaluations of universal R- and K-matrices for $H = \mathcal{L}U_q sl_2$ and B a comodule algebra for a certain twist pair. Using the intertwining operators studied in the previous section, we show that the spin-j L- and K-operators satisfy certain properties named as 'fusion' and 'reduction'.

3.4.1 Spin-j L-operators

While the L-operators have been known for a long time, to our knowledge, the approach we introduce below has not been considered. It is worth mentioning the paper [RSV14], which also uses intertwiners to construct fused L-operators. For practical applications, the explicit expressions for intertwiners are needed. Here, the $\mathcal{L}U_q sl_2$-intertwiners are explicitly given in terms of the $B_{j,m}$ from (3.2.3), see Lemma C.1.1.

L-operators and fusion $(j+\frac{1}{2}) \to (\frac{1}{2}, j)$.

We study a so-called fusion relation for L-operators that relates $\mathbf{L}^{(j+\frac{1}{2})}(u)$ to $\mathbf{L}^{(j)}(u)$ and $\mathbf{L}^{(\frac{1}{2})}(u)$. For this, we evaluate the equation (R3) on the second and third tensor components for a special choice of evaluation parameters. Recall that in the previous section we obtained a spin-$(j+\frac{1}{2})$ sub-representation in the tensor product (3.3.11) of formal evaluation representations of $\mathcal{L}U_q sl_2$. The corresponding intertwining operator $\mathcal{E}^{(j+\frac{1}{2})}$ is fixed by Lemma C.1.1, and we introduce its pseudo-inverse

$$\mathcal{F}^{(j+\frac{1}{2})} \colon \mathbb{C}^2_{uq^{-j}} \otimes \mathbb{C}^{2j}_{uq^{\frac{1}{2}}} \to \mathbb{C}^{2j+1}_u \ ,$$

whose matrix entries are given in (C.1.7), (C.1.8). Then, inserting the product $\mathcal{F}^{(j+\frac{1}{2})}\mathcal{E}^{(j+\frac{1}{2})} = \mathbb{I}_{2j+2}$ and using the intertwining property (C.1.4), fusion relations satisfied by L-operators and R-matrices are exhibited in the next propositions (where we use the notation $\langle 12 \rangle$ to indicate which spaces are fused, that is to say, where the intertwiner acts). The following formula is an important result in the thesis. We provide its proof for the reader's convenience, aiming to facilitate the understanding of the application of the intertwining operators constructed above.

Proposition 3.4.1 ([LBG23, Prop. 4.4]). *The L-operators (3.2.12) satisfy for $j \in \frac{1}{2}\mathbb{N}$:*

$$\mathbf{L}^{(j+\frac{1}{2})}(u) = \mathcal{F}^{(j+\frac{1}{2})}_{\langle 12 \rangle} \mathbf{L}^{(j)}_2(uq^{-\frac{1}{2}}) \mathbf{L}^{(\frac{1}{2})}_1(uq^j) \mathcal{E}^{(j+\frac{1}{2})}_{\langle 12 \rangle} . \tag{3.4.1}$$

Proof. By definition of the L-operator we have $\mathbf{L}^{(j+\frac{1}{2})}(w/u) = (\mathsf{ev}_w \otimes \pi_u^{j+\frac{1}{2}})(\mathfrak{R})$. Using the pseudo-inverse property $\mathcal{F}^{(j+\frac{1}{2})}\mathcal{E}^{(j+\frac{1}{2})} = \mathbb{I}_{2j+2}$, we get

$$\mathbf{L}^{(j+\frac{1}{2})}(w/u) = (1 \otimes \mathcal{F}^{(j+\frac{1}{2})}\mathcal{E}^{(j+\frac{1}{2})}) \left[(\mathsf{ev}_w \otimes \pi_u^{j+\frac{1}{2}})(\mathfrak{R}) \right]$$

$$
\begin{aligned}
&= (1 \otimes \mathcal{F}^{(j+\frac{1}{2})}) \left[(\mathsf{ev}_w \otimes \pi^{\frac{1}{2}}_{uq^{-j}} \otimes \pi^{j}_{uq^{\frac{1}{2}}})(\mathrm{id} \otimes \Delta)(\mathfrak{R}) \right] (1 \otimes \mathcal{E}^{(j+\frac{1}{2})}) \\
&= (1 \otimes \mathcal{F}^{(j+\frac{1}{2})}) \left[(\mathsf{ev}_w \otimes \pi^{\frac{1}{2}}_{uq^{-j}} \otimes \pi^{j}_{uq^{\frac{1}{2}}})(\mathfrak{R}_{13}\mathfrak{R}_{12}) \right] (1 \otimes \mathcal{E}^{(j+\frac{1}{2})}) \\
&= (1 \otimes \mathcal{F}^{(j+\frac{1}{2})}) \mathbf{L}^{(j)}_2(q^{-\frac{1}{2}}w/u) \mathbf{L}^{(\frac{1}{2})}_1(q^j w/u) (1 \otimes \mathcal{E}^{(j+\frac{1}{2})}) \ .
\end{aligned}
$$

The second equality is obtained using the intertwining property (C.1.4):

$$
(1 \otimes \mathcal{E}^{(j+\frac{1}{2})}) \left[(\mathrm{id} \otimes \pi^{j+\frac{1}{2}}_u)(\mathfrak{R}) \right] = \left[(\mathrm{id} \otimes \pi^{\frac{1}{2}}_{uq^{-j}} \otimes \pi^{j}_{uq^{\frac{1}{2}}}) \circ (\mathrm{id} \otimes \Delta)(\mathfrak{R}) \right] (1 \otimes \mathcal{E}^{(j+\frac{1}{2})}) \ . \tag{3.4.2}
$$

Then, the third equality is due to (R3) and the last one follows by definition of the L-operator. □

From this proposition, we see that $\mathbf{L}^{(j)}(u)$ for $j \in \frac{1}{2}\mathbb{N}_+$ is in $U_q sl_2((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^{2j+1})$. For the next proposition, we give only a sketch of the proof, the reader is referred to [LBG23] for a more detailed proof.

Proposition 3.4.2 ([LBG23, Prop. 4.5])**.** *The R-matrices* (3.2.13) satisfy

$$
\mathcal{R}^{(j_1,j_2)}(u) = \mathcal{F}^{(j_1)}_{\langle 12 \rangle} \mathcal{R}^{(\frac{1}{2},j_2)}_{13}(uq^{-j_1+\frac{1}{2}}) \mathcal{R}^{(j_1-\frac{1}{2},j_2)}_{23}(uq^{\frac{1}{2}}) \mathcal{E}^{(j_1)}_{\langle 12 \rangle} \ , \tag{3.4.3}
$$

where

$$
\mathcal{R}^{(\frac{1}{2},j+\frac{1}{2})}(u) = \mathcal{F}^{(j+\frac{1}{2})}_{\langle 23 \rangle} \mathcal{R}^{(\frac{1}{2},j)}_{13}(uq^{-\frac{1}{2}}) \mathcal{R}^{(\frac{1}{2},\frac{1}{2})}_{12}(uq^{j}) \mathcal{E}^{(j+\frac{1}{2})}_{\langle 23 \rangle} \ . \tag{3.4.4}
$$

Sketch of Proof. The eq. (3.4.4) is obtained by applying $(\pi^{1/2} \otimes \mathrm{id})$ on (3.4.1), while (3.4.3) is proved by fusing the first component of $\mathcal{R}^{(\frac{1}{2},j_2+\frac{1}{2})}(u)$ similarly to the proof of Proposition 3.4.1 but using (R2). □

Note that the construction of fused L-operators and R-matrices, first uses the pseudo-inverse property $\mathcal{F}^{(j+\frac{1}{2})}\mathcal{E}^{(j+\frac{1}{2})} = \mathbb{I}_{2j+2}$ and then the intertwining property (C.1.4). Even if $\mathcal{F}^{(j+\frac{1}{2})}$ is not unique, taking different expressions for $\mathcal{F}^{(j+\frac{1}{2})}$ yields the same L-operators and R-matrices.

L-operators and reduction $(j-\frac{1}{2}) \to (\frac{1}{2}, j)$.

We now consider spin-$(j-\frac{1}{2})$ sub-representations in the tensor product of formal evaluation representations of $\mathcal{L}U_q sl_2$, with the corresponding intertwining operator $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ is fixed by Lemma C.2.1, and we introduce its pseudo-inverse

$$
\bar{\mathcal{F}}^{(j-\frac{1}{2})} \colon \mathbb{C}^{2j-1}_u \to \mathbb{C}^{2}_{uq^{j+1}} \otimes \mathbb{C}^{2j}_{uq^{\frac{1}{2}}} \ ,
$$

whose matrix entries are given in (C.2.7).

Proposition 3.4.3 ([LBG23, Prop. 4.7])**.** *The L-operators* (3.2.12) satisfy for $j \in \frac{1}{2}\mathbb{N}_+$:

$$
\mathbf{L}^{(j-\frac{1}{2})}(u) = \bar{\mathcal{F}}^{(j-\frac{1}{2})}_{\langle 12 \rangle} \mathbf{L}^{(j)}_2(uq^{-\frac{1}{2}}) \mathbf{L}^{(\frac{1}{2})}_1(uq^{-j-1}) \bar{\mathcal{E}}^{(j-\frac{1}{2})}_{\langle 12 \rangle} . \tag{3.4.5}
$$

Sketch of Proof. The proof is similar to Proposition 3.4.1 but using now the intertwining relation (C.2.4) satisfied by $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ instead of the one satisfied by $\mathcal{E}^{(j+\frac{1}{2})}$ given in (C.1.4). □

Fused L-operators and fused R-matrices

Recall that the spin-j L-operators $\mathbf{L}^{(j)}(u)$ and R-matrices $\mathcal{R}^{(j_1,j_2)}(u)$ are derived through evaluations of the universal R-matrix. Additionally, fusion properties have been established using $\mathcal{L}U_q sl_2$-intertwiners, as detailed in Propositions 3.4.1 and 3.4.2. Now, we aim to construct the spin-j generalizations of $\mathcal{L}^{(\frac{1}{2})}(u)$ and $R^{(\frac{1}{2},\frac{1}{2})}(u)$ from equations (1.1.8) and (1.1.5) through fusion. The goal is to link them with $\mathbf{L}^{(j)}(u)$ and $\mathcal{R}^{(j_1,j_2)}(u)$, thereby circumventing the difficulty of evaluating $\mathfrak{R}$ for increasing values of j.

A higher spin generalization of $\mathcal{L}^{(\frac{1}{2})}(u)$ from (1.1.8) is obtained as follows. Starting from the fundamental L-operator $\mathcal{L}^{(\frac{1}{2})}(u)$ given in (1.1.8), for any $j \in \frac{1}{2}\mathbb{N}_+$ define the fused L-operators $\mathcal{L}^{(j)}(u) \in U_q sl_2 \otimes \mathrm{End}(\mathbb{C}^{2j+1})$ as:

$$\mathcal{L}^{(j+\frac{1}{2})}(u) = \mathcal{F}^{(j+\frac{1}{2})}_{\langle 12\rangle} \mathcal{L}^{(j)}_2(uq^{-\frac{1}{2}}) \mathcal{L}^{(\frac{1}{2})}_1(uq^{j}) \mathcal{E}^{(j+\frac{1}{2})}_{\langle 12\rangle}. \tag{3.4.6}$$

Although not needed here, it can be proven directly by induction that $\mathcal{L}^{(j)}(u)$'s satisfy the Yang-Baxter equation (3.2.14), where $\mathbf{L}^{(j)}(u)$ are replaced by $\mathcal{L}^{(j)}(u)$. We now give the relations between the spin-j L-operators (3.2.12), obtained by evaluation of the universal R-matrix, and the fused L-operators (3.4.1).

Lemma 3.4.4 ([LBG23, Lem. 4.8])**.** *The spin-j L-operators and the fused L-operators are related as follows:*

$$\mathbf{L}^{(j)}(u) = \mu^{(j)}(u) \mathcal{L}^{(j)}(u) \ , \tag{3.4.7}$$

where

$$\mu^{(j)}(u) = \prod_{k=0}^{2j-1} \mu(uq^{j-\frac{1}{2}-k}) \tag{3.4.8}$$

is central in $U_q sl_2$.

Above, we have shown that the L-operators $\mathbf{L}^{(j)}(u)$'s satisfy both fusion and reduction relations. As the L-operator is the evaluation of the universal R-matrix (3.2.12), the expression (3.4.1) with j replaced by $j-1$ equals (3.4.5). By the consistency of the fusion and reduction relations (3.4.1) and (3.4.5), respectively, one gets a functional relation satisfied by the central element $\mu(u)$ from (3.2.17).

Lemma 3.4.5 ([LBG23, Lem. 4.9])**.** *The following relation holds:*

$$\mu(u)\mu(uq)\gamma(u) = 1 \ , \tag{3.4.9}$$

where $\gamma(u)$ is given in (3.2.20).

Corollary 3.4.6. *The quantum determinant of the L-operator $\mathbf{L}^{(\frac{1}{2})}(u)$ is such that*

$$\mathrm{tr}_{12}\big(\mathcal{P}^{-}_{12} \mathbf{L}^{(\frac{1}{2})}_1(u) \mathbf{L}^{(\frac{1}{2})}_2(uq)\big) = 1 \ . \tag{3.4.10}$$

We note that Lemma 3.4.5 provides an independent derivation of[21] [BGKNR12, eq. (4.60)]. We do not give a general solution to the functional equation (3.4.9). Solutions for spin-representations of $U_q sl_2$ can be easily constructed. For instance, for $\pi^{\frac{1}{2}}(\mu(u))$ the 'minimal' solution is given by (3.2.19).

We now introduce fused R-matrices (by analogy with (3.4.3) and (3.4.4))

$$R^{(j_1,j_2)}(u) = \mathcal{F}^{(j_1)}_{\langle 12\rangle} R^{(\frac{1}{2},j_2)}_{13}(uq^{-j_1+\frac{1}{2}}) R^{(j_1-\frac{1}{2},j_2)}_{23}(uq^{\frac{1}{2}}) \mathcal{E}^{(j_1)}_{\langle 12\rangle} \,, \tag{3.4.11}$$

for $j_1 \geq 1$ and where

$$R^{(\frac{1}{2},j+\frac{1}{2})}(u) = \mathcal{F}^{(j+\frac{1}{2})}_{\langle 23\rangle} R^{(\frac{1}{2},j)}_{13}(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},\frac{1}{2})}_{12}(uq^{j}) \mathcal{E}^{(j+\frac{1}{2})}_{\langle 23\rangle} \,, \tag{3.4.12}$$

with $R^{(\frac{1}{2},\frac{1}{2})}(u)$ given in (1.1.5), and show that (3.4.12) agrees with (C.3.1).

Lemma 3.4.7 ([LBG23, Lem. 4.11])**.** *The R-matrices* (3.2.13) and the fused R-matrices (3.4.11) are related by

$$\mathcal{R}^{(j_1,j_2)}(u) = f^{(j_1,j_2)}(u) R^{(j_1,j_2)}(u) \,, \tag{3.4.13}$$

where

$$f^{(j_1,j_2)}(u) = u^{-4j_1j_2} q^{-2j_1j_2} \exp\left(\sum_{k=1}^{\infty} \frac{q^{2k}+q^{-2k}}{1+q^{2k}} [2j_1]_{q^k} [2j_2]_{q^k} \frac{u^{-2k}}{k} \right) \,, \tag{3.4.14}$$

and (3.4.12) agrees with (C.3.1).

Unitarity properties of L-operators

Recall that the spin-1/2 L-operators $\mathbf{L}^{(\frac{1}{2})}(u)$ from (3.2.12) satisfies the unitarity property (3.2.21). More generally, by induction one gets the unitarity property for any spin-j:

$$\mathbf{L}^{(j)}(u^{-1})\mathbf{L}^{(j)}(u) = \mathbf{L}^{(j)}(u)\mathbf{L}^{(j)}(u^{-1}) = \left(\prod_{k=0}^{2j-1} \mathbf{c}(uq^{-j+\frac{1}{2}+k}) \right) \mathbb{I}_{2j+1} \,, \tag{3.4.15}$$

where $\mathbf{c}(u)$ is given in (3.2.21). It follows from (3.4.15) and (3.2.13) that both $\mathcal{R}^{(j_1,j_2)}(u)\mathcal{R}^{(j_1,j_2)}(u^{-1})$ and $\mathcal{R}^{(j_1,j_2)}(u^{-1})\mathcal{R}^{(j_1,j_2)}(u)$ are equal and proportional to the identity matrix for any j_1 and j_2. Because $\mathcal{R}^{(j_1,j_2)}(u)$ is proportional to $R^{(j_1,j_2)}(u)$ due to (3.4.13), we also have

$$R^{(j_1,j_2)}(u)R^{(j_1,j_2)}(u^{-1}) = R^{(j_1,j_2)}(u^{-1})R^{(j_1,j_2)}(u) \propto \mathbb{I}_{(2j_1+1)(2j_2+1)} \,. \tag{3.4.16}$$

3.4.2 Spin-j K-operators

Recall that the spin-j K-operators $\mathbf{K}^{(j)}(u)$ are defined as the 1-component evaluation of a universal K-matrix, see Definition 3.2.3. Now, using the intertwining operators constructed in Section 3.3 and the third axiom of the universal K-matrix (K3), we propose certain fusion and reduction relations satisfied by $\mathbf{K}^{(j)}(u)$.

[21] It is an exponential version of [BGKNR12] with the identification $\tau \to u$, $e^{\Lambda(q^{-1}\tau^s)} \to \mu(u)uq^{\frac{1}{2}}$, $s \to -2$, $s_0 \to -1$, $s_1 \to -1$.

K-operators and fusion $(j+\frac{1}{2}) \to (\frac{1}{2}, j)$.

We follow here the same approach used for L-operators based on sub-representations in the tensor product of formal evaluation representations of $\mathcal{L}U_q sl_2$), now considering (K3) instead of (R3). Recall the intertwining operator $\mathcal{E}^{(j+\frac{1}{2})}$ is fixed by Lemma C.1.1 and its pseudo-inverse $\mathcal{F}^{(j+\frac{1}{2})}$ is given in (C.1.6) with (C.1.7), (C.1.8). The following formula is one of the main result of the thesis, and so we give its detailed proof.

Proposition 3.4.8 ([LBG23, Prop. 4.14]). *The K-operators* (3.2.29) satisfy for $j \in \frac{1}{2}\mathbb{N}$:

$$\mathbf{K}^{(j+\frac{1}{2})}(u) = \mathcal{F}^{(j+\frac{1}{2})}_{\langle 12\rangle} \mathbf{K}_2^{(j)}(uq^{-\frac{1}{2}}) \mathcal{R}^{(\frac{1}{2},j)}(u^2 q^{j-\frac{1}{2}}) \mathbf{K}_1^{(\frac{1}{2})}(uq^j) \mathcal{E}^{(j+\frac{1}{2})}_{\langle 12\rangle}. \tag{3.4.17}$$

Proof. By definition, we have $\mathbf{K}^{(j+\frac{1}{2})}(u) = (\mathrm{id} \otimes \pi^{j+\frac{1}{2}}_{u^{-1}})(\mathfrak{K})$. Using the pseudo-inverse property $\mathcal{F}^{(j+\frac{1}{2})}\mathcal{E}^{(j+\frac{1}{2})} = \mathbb{I}_{2j+2}$, we get

$$\begin{aligned}
(\mathrm{id} \otimes \pi^{j+\frac{1}{2}}_{u^{-1}})(\mathfrak{K}) &= (1 \otimes \mathcal{F}^{(j+\frac{1}{2})}\mathcal{E}^{(j+\frac{1}{2})})(\mathrm{id} \otimes \pi^{j+\frac{1}{2}}_{u^{-1}})(\mathfrak{K}) \\
&= (1 \otimes \mathcal{F}^{(j+\frac{1}{2})})[(\mathrm{id} \otimes \pi^{\frac{1}{2}}_{u^{-1}q^{-j}} \otimes \pi^{j}_{u^{-1}q^{\frac{1}{2}}}) \circ (\mathrm{id} \otimes \Delta)(\mathfrak{K})](1 \otimes \mathcal{E}^{(j+\frac{1}{2})}) \\
&= (1 \otimes \mathcal{F}^{(j+\frac{1}{2})})[(\mathrm{id} \otimes \pi^{\frac{1}{2}}_{u^{-1}q^{-j}} \otimes \pi^{j}_{u^{-1}q^{\frac{1}{2}}})(\mathfrak{K}_{13}\mathfrak{K}^{\eta}_{23}\mathfrak{K}_{12})](1 \otimes \mathcal{E}^{(j+\frac{1}{2})}) \ .
\end{aligned}$$

The second equality is obtained using the intertwining property (C.1.4):

$$(1 \otimes \mathcal{E}^{(j+\frac{1}{2})})(\mathrm{id} \otimes \pi^{j+\frac{1}{2}}_{u^{-1}})(\mathfrak{K}) = [(\mathrm{id} \otimes \pi^{\frac{1}{2}}_{u^{-1}q^{-j}} \otimes \pi^{j}_{u^{-1}q^{\frac{1}{2}}}) \circ (\mathrm{id} \otimes \Delta)(\mathfrak{K})](1 \otimes \mathcal{E}^{(j+\frac{1}{2})}) \ . \tag{3.4.18}$$

Then, the third equality is due to (K3) and finally, from the definition of the K-operator (3.2.29) and the evaluation of the twisted universal R-matrix (3.2.31), the claim follows. □

Using the power series assumption on $\mathbf{K}^{(\frac{1}{2})}(u)$, we see that $\mathbf{K}^{(j)}(u)$ is also a formal power series in u^{-1}, i.e. it is in $B((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^{2j+1})$.

K-operators and reduction $(j-\frac{1}{2}) \to (\frac{1}{2}, j)$.

The following proposition is also an important result but since it is proved similarly to the reduction relation (3.4.5) for the L-operators, we skip its proof. Recall the intertwining operator $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ is fixed by Lemma C.2.1 and its pseudo-inverse $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$ is given in (C.2.6) with (C.2.7).

Proposition 3.4.9 ([LBG23, Prop. 4.16]). *The K-operators* (3.2.29) satisfy for $j \in \frac{1}{2}\mathbb{N}_+$:

$$\mathbf{K}^{(j-\frac{1}{2})}(u) = \bar{\mathcal{F}}^{(j-\frac{1}{2})}_{\langle 12\rangle} \mathbf{K}_2^{(j)}(uq^{-\frac{1}{2}}) \mathcal{R}^{(\frac{1}{2},j)}(u^2 q^{-j-\frac{3}{2}}) \mathbf{K}_1^{(\frac{1}{2})}(uq^{-j-1}) \bar{\mathcal{E}}^{(j-\frac{1}{2})}_{\langle 12\rangle} \ . \tag{3.4.19}$$

Recall that we assumed that the universal K-matrix exists for a given choice of B and the twist pair $(\eta, 1 \otimes 1)$. Therefore, the K-operator for a given spin is unique, that is, similarly to the case of the L-operator, we obtain the same operator $\mathbf{K}^{(j)}(u)$ either using the fusion for $(j) \to (\frac{1}{2}, j-\frac{1}{2})$ or using the reduction $(j) \to (\frac{1}{2}, j+\frac{1}{2})$.

Remark 3.4.10. *Consider the opposite coproduct $\Delta^{op} = \mathfrak{p} \circ \Delta$ with the definition (2.3.9).* It follows from (K3):

$$(\mathrm{id} \otimes \Delta^{op})(\mathfrak{K}) = \mathfrak{K}_{12}(\mathfrak{R}^{\psi})_{32}\mathfrak{K}_{13} \ . \tag{3.4.20}$$

Recall that we obtained for Δ^{op} an intertwining relation in (C.2.8) where $\mathcal{E}^{(j+\frac{1}{2})}$ is fixed as in (C.1.5), see Remark C.2.2. Thus, we also take $\mathcal{F}^{(j+\frac{1}{2})}$ as defined in (C.1.7), (C.1.8). Then, using (C.2.8) and (3.4.20), we obtain a new fusion relation for any $j \in \frac{1}{2}\mathbb{N}$:

$$\mathbf{K}^{(j+\frac{1}{2})}(u) = \mathcal{F}^{(j+\frac{1}{2})}_{\langle 12\rangle}\mathbf{K}^{(\frac{1}{2})}_1(uq^{-j})\mathcal{R}^{(\frac{1}{2},j)}(u^2q^{-j+\frac{1}{2}})\mathbf{K}^{(j)}_2(uq^{\frac{1}{2}})\mathcal{E}^{(j+\frac{1}{2})}_{\langle 12\rangle}. \tag{3.4.21}$$

Similarly, we also have the intertwining relation for Δ^{op} given in (C.2.9) with $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ fixed as in (C.2.5), see Remark C.2.2, and we take $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$ as defined in (C.2.6) with (C.2.7). Then, using (3.4.20) and (C.2.9), we obtain a new reduction relation for any $j \in \frac{1}{2}\mathbb{N}_+$:

$$\mathbf{K}^{(j-\frac{1}{2})}(u) = \bar{\mathcal{F}}^{(j-\frac{1}{2})}_{\langle 12\rangle}\mathbf{K}^{(\frac{1}{2})}_1(uq^{j+1})\mathcal{R}^{(\frac{1}{2},j)}(u^2q^{j+\frac{3}{2}})\mathbf{K}^{(j)}_2(uq^{\frac{1}{2}})\bar{\mathcal{E}}^{(j-\frac{1}{2})}_{\langle 12\rangle}. \tag{3.4.22}$$

Here, we assumed the existence of a universal K-matrix $\mathfrak{K} \in B \otimes \mathcal{L}U_q sl_2$. From our new axiomatic definition of $\mathfrak{K}$, fusion and reduction properties were exhibited. Now, we want to put aside this universal framework and construct fused K-operators $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q \otimes \mathrm{End}(\mathbb{C}^{2j+1})$ that are spin-j versions of the spin-1/2 K-operator for $\mathcal{A}_q$ from (2.1.5). To achieve this, recall first the fused L-operators from (3.4.6). This construction is guided by the evaluation of the third axiom of the universal R-matrix, as indicated in equation (R3), leading to Proposition 3.4.1. Similarly, the approach to constructing $\mathcal{K}^{(j)}(u)$ is to take inspiration from the evaluation of the action of the coproduct on the second tensor component of the universal K-matrix, as exhibited by the fusion property of $\mathbf{K}^{(j)}(u)$ presented in Proposition 3.4.8. In short, this amounts to consider the latter proposition and replace $\mathbf{K}^{(j)}(u) \to \mathcal{K}^{(j)}(u)$ and show that the corresponding fusion formula indeed satisfies reflection equations for all possible values of spins.

3.5 Fused K-operators for $\mathcal{A}_q$

In this section, we consider the comodule algebra $B = \mathcal{A}_q$ and related 'fused' K-operators. Contrary to the previous section, here we do not assume the existence of a universal K-matrix. In Section 3.5.1, fused K-operators $\mathcal{K}^{(j)}(u)$ built from the fundamental K-operator by analogy with (3.4.21) are shown to satisfy the reflection equation (3.2.43) for all $j \in \frac{1}{2}\mathbb{N}_+$ where $\mathbf{K}^{(j)}(u)$ is replaced by $\mathcal{K}^{(j)}(u)$. This is the main result in this section. We then introduce fused K-matrices in Definition 3.5.5 out of the fused K-operators $\mathcal{K}^{(j)}(u)$. It is done by classifying one-dimensional representations for $\mathcal{A}_q$ in Proposition 3.5.3, and applying these representations to $\mathcal{K}^{(j)}(u)$. We also establish the unitarity and invertibility properties of $\mathcal{K}^{(j)}(u)$ in Section 3.5.5. In preparation to the discussion in Section 3.5.6, we calculate the evaluated coaction for $\mathcal{A}_q$ and also establish the twisted intertwining relations for the fused K-operators which are similar to (3.2.47).

3.5.1 Fused K-operators for $\mathcal{A}_q$

Recall $R^{(\frac{1}{2},\frac{1}{2})}(u)$ and the fundamental K-operator $\mathcal{K}^{(\frac{1}{2})}(u)$, given respectively by (1.1.5) and by (2.1.5), satisfy the reflection equation (2.1.4). By analogy with (3.4.21), we now construct fused K-operators $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q \otimes \mathrm{End}(\mathbb{C}^{2j+1})$.

Definition 3.5.1. *For $j \in \frac{1}{2}\mathbb{N}_+$, the fused K-operators for $\mathcal{A}_q$ are*

$$\mathcal{K}^{(j+\frac{1}{2})}(u) = \mathcal{F}^{(j+\frac{1}{2})}_{\langle 12\rangle} \mathcal{K}^{(\frac{1}{2})}_1(uq^{-j}) R^{(\frac{1}{2},j)}(u^2 q^{-j+\frac{1}{2}}) \mathcal{K}^{(j)}_2(uq^{\frac{1}{2}}) \mathcal{E}^{(j+\frac{1}{2})}_{\langle 12\rangle}, \tag{3.5.1}$$

with $\mathcal{K}^{(\frac{1}{2})}(u)$ defined in (2.1.5).

Theorem 3.5.2. *The fused K-operators $\mathcal{K}^{(j)}(u)$ satisfy the reflection equation for any $j_1,\ j_2 \in \frac{1}{2}\mathbb{N}_+$:*

$$\begin{aligned} R^{(j_1,j_2)}(u_1/u_2)\mathcal{K}^{(j_1)}_1(u_1) R^{(j_1,j_2)}(u_1u_2)\mathcal{K}^{(j_2)}_2(u_2) = \\ \mathcal{K}^{(j_2)}_2(u_2) R^{(j_1,j_2)}(u_1u_2)\mathcal{K}^{(j_1)}_1(u_1) R^{(j_1,j_2)}(u_1/u_2)\ . \end{aligned} \tag{3.5.2}$$

The proof is done by induction on $j_1,\ j_2$. For $(j_1, j_2) = (\frac{1}{2}, \frac{1}{2})$, the reflection equation (3.5.2) holds for $\mathcal{K}^{(\frac{1}{2})}(u)$ due to [BS09]. The proof is divided into three parts and consists of three lemmas. We first show the case $(j_1, j_2) = (\frac{1}{2}, j + \frac{1}{2})$, assuming the equation (3.5.2) holds for $(j_1, j_2) = (\frac{1}{2}, j)$. Then, we prove the case $(j + \frac{1}{2}, \frac{1}{2})$, assuming (3.5.2) holds for $(j_1, j_2) = (j, \frac{1}{2})$. Finally, the generic case (j_1, j_2) follows. We refer the reader to [LBG23, Sec. 5.2] for a detailed proof.

3.5.2 Examples of fused K-operators

In this subsection we compute explicitly the spin-1 K-operator for $\mathcal{A}_q$, and give compact forms for $\mathcal{K}^{(j)}(u)$, as well as for the fused R-matrices defined by (3.5.1) and (3.4.12), respectively.

Spin-1 fused K-operator

The expressions of $\mathcal{E}^{(j+\frac{1}{2})}$, $\mathcal{F}^{(j+\frac{1}{2})}$ in (C.1.1), (C.1.2), for $j = \frac{1}{2}$ read:

$$\mathcal{E}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{[2]_q}} & 0 \\ 0 & \frac{1}{\sqrt{[2]_q}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathcal{F}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{[2]_q}}{2} & \frac{\sqrt{[2]_q}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.5.3}$$

From (3.4.12), the fused R-matrix reads

$$R^{(\frac{1}{2},1)}(u) = \mathcal{F}^{(1)}_{\langle 23\rangle} R^{(\frac{1}{2},\frac{1}{2})}_{13}(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},\frac{1}{2})}_{12}(uq^{\frac{1}{2}}) \mathcal{E}^{(1)}_{\langle 23\rangle} \tag{3.5.4}$$

and is given explicitly by

$$R^{(\frac{1}{2},1)}(u) = c(uq^{\frac{1}{2}}) \begin{pmatrix} c(uq^{\frac{3}{2}}) & 0 & 0 & 0 & 0 & 0 \\ 0 & c(uq^{\frac{1}{2}}) & 0 & c(q)\sqrt{[2]_q} & 0 & 0 \\ 0 & 0 & c(uq^{-\frac{1}{2}}) & 0 & c(q)\sqrt{[2]_q} & 0 \\ 0 & c(q)\sqrt{[2]_q} & 0 & c(uq^{-\frac{1}{2}}) & 0 & 0 \\ 0 & 0 & c(q)\sqrt{[2]_q} & 0 & c(uq^{\frac{1}{2}}) & 0 \\ 0 & 0 & 0 & 0 & 0 & c(uq^{\frac{3}{2}}) \end{pmatrix} , \tag{3.5.5}$$

where $c(u)$ is the scalar function given in (1.3.33). From (3.5.1), the fused K-operator is given by:

$$\mathcal{K}^{(1)}(u) = \mathcal{F}^{(1)}_{\langle 12\rangle} \mathcal{K}_1^{(\frac{1}{2})}(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},\frac{1}{2})}(u^2) \mathcal{K}_2^{(\frac{1}{2})}(uq^{\frac{1}{2}}) \mathcal{E}^{(1)}_{\langle 12\rangle} \ . \tag{3.5.6}$$

Using the above expressions, one finds that the entries $\mathcal{K}^{(1)}_{mn}(u)$ are explicitly given by:

$$\begin{aligned}
\mathcal{K}^{(1)}_{11}(u) &= \big(c(q)^{-1} + \rho^{-1}\mathcal{G}_+(uq^{-\frac{1}{2}})\big)\big(\rho + c(q)\mathcal{G}_-(uq^{\frac{1}{2}})\big) \\
&\quad + c(u^2q)\big(uq^{\frac{1}{2}}\mathcal{W}_+(uq^{-\frac{1}{2}}) - u^{-1}q^{-\frac{1}{2}}\mathcal{W}_-(uq^{-\frac{1}{2}})\big)\big(uq^{\frac{3}{2}}\mathcal{W}_+(uq^{\frac{1}{2}}) - u^{-1}q^{-\frac{3}{2}}\mathcal{W}_-(uq^{\frac{1}{2}})\big), \\
\mathcal{K}^{(1)}_{12}(u) &= \frac{(q+q^{-1})^{-\frac{3}{2}}}{k_-}\Big(c(u^2)\big(\rho c(q)^{-1} + \mathcal{G}_+(uq^{-\frac{1}{2}})\big)\big(uq^{\frac{3}{2}}\mathcal{W}_+(uq^{\frac{1}{2}}) - u^{-1}q^{-\frac{3}{2}}\mathcal{W}_-(uq^{\frac{1}{2}})\big) \\
&\quad + \big(\rho + c(q)\mathcal{G}_+(uq^{-\frac{1}{2}})\big)\big(uq^{\frac{3}{2}}\mathcal{W}_-(uq^{\frac{1}{2}}) - u^{-1}q^{-\frac{3}{2}}\mathcal{W}_-(uq^{\frac{1}{2}})\big) \\
&\quad + c(u^2q)\big(uq^{\frac{1}{2}}\mathcal{W}_+(uq^{-\frac{1}{2}}) - u^{-1}q^{-\frac{1}{2}}\mathcal{W}_-(uq^{-\frac{1}{2}})\big)\big(\rho c(q)^{-1} + \mathcal{G}_+(uq^{\frac{1}{2}})\big)\Big), \\
\mathcal{K}^{(1)}_{13}(u) &= \frac{c(u^2)}{k_-^2 c(q^2)^2}\big(\rho + c(q)\mathcal{G}_+(uq^{-\frac{1}{2}})\big)\big(\rho + c(q)\mathcal{G}_+(uq^{\frac{1}{2}})\big), \\
\mathcal{K}^{(1)}_{21}(u) &= \frac{c(q)^{-1}}{2k_+\sqrt{q+q^{-1}}}\Big(c(u^2q)\big(\rho + c(q)\mathcal{G}_-(uq^{-\frac{1}{2}})\big)\big(uq^{\frac{3}{2}}\mathcal{W}_+(uq^{\frac{1}{2}}) - u^{-1}q^{-\frac{3}{2}}\mathcal{W}_-(uq^{\frac{1}{2}})\big) \\
&\quad + \big(q^{-\frac{1}{2}}(u^{-3} + u(-2+q^2))\mathcal{W}_-(uq^{-\frac{1}{2}}) + q^{\frac{1}{2}}(u^3 + u^{-1}(-2+q^{-2}))\mathcal{W}_+(uq^{-\frac{1}{2}})\big)\Big) \\
&\quad \times \big(\rho + c(q)\mathcal{G}_-(uq^{\frac{1}{2}})\big), \\
\mathcal{K}^{(1)}_{22}(u) &= \frac{c(u^2q)}{2c(q)^2\rho}\Big(\big(\rho + c(q)\mathcal{G}_+(uq^{-\frac{1}{2}})\big)\big(\rho + c(q)\mathcal{G}_-(uq^{\frac{1}{2}})\big) + \big(\rho + c(q)\mathcal{G}_-(uq^{-\frac{1}{2}})\big)\big(\rho + c(q)\mathcal{G}_+(uq^{\frac{1}{2}})\big)\Big) \\
&\quad + \frac{1}{2}\big(q^{-\frac{1}{2}}(u^{-3} + u(-2+q^2))\mathcal{W}_+(uq^{-\frac{1}{2}}) + q^{\frac{1}{2}}(u^{-1}(-2+q^2) + u^3)\mathcal{W}_-(uq^{-\frac{1}{2}})\big) \\
&\quad \times \big(uq^{\frac{3}{2}}\mathcal{W}_+(uq^{\frac{1}{2}}) - u^{-1}q^{-\frac{3}{2}}\mathcal{W}_-(uq^{\frac{1}{2}})\big) \\
&\quad + \frac{1}{2}\big(q^{-\frac{1}{2}}(u^{-3} + u(-2+q^2))\mathcal{W}_-(uq^{-\frac{1}{2}}) + q^{\frac{1}{2}}(u^{-1}(-2+q^2) + u^3)\mathcal{W}_+(uq^{-\frac{1}{2}})\big) \\
&\quad \times \big(uq^{\frac{3}{2}}\mathcal{W}_-(uq^{\frac{1}{2}}) - u^{-1}q^{-\frac{3}{2}}\mathcal{W}_+(uq^{\frac{1}{2}})\big),
\end{aligned} \tag{3.5.7}$$

$$\begin{aligned}
\mathcal{K}^{(1)}_{23}(u) &= \sigma(\mathcal{K}^{(1)}_{21}(u))|_{k_\pm \to k_\mp} \ , \qquad \mathcal{K}^{(1)}_{31}(u) = \sigma(\mathcal{K}^{(1)}_{13}(u))|_{k_\pm \to k_\mp} \ , \\
\mathcal{K}^{(1)}_{32}(u) &= \sigma(\mathcal{K}^{(1)}_{12}(u))|_{k_\pm \to k_\mp} \ , \qquad \mathcal{K}^{(1)}_{33}(u) = \sigma(\mathcal{K}^{(1)}_{11}(u))|_{k_\pm \to k_\mp} \ ,
\end{aligned}$$

where σ is defined in (2.1.16) and $\{\mathcal{W}_\pm(u), \mathcal{G}_\pm(u)\}$ are given in (2.1.1)-(2.1.2).

The fused K-operator (3.6.1) for $j = 1$ satisfies the reflection equation:

$$R^{(\frac{1}{2},1)}(u/v)\mathcal{K}_1^{(\frac{1}{2})}(u)R^{(\frac{1}{2},1)}(uv)\mathcal{K}_2^{(1)}(v) = \mathcal{K}_2^{(1)}(v)R^{(\frac{1}{2},1)}(uv)\mathcal{K}_1^{(\frac{1}{2})}(u)R^{(\frac{1}{2},1)}(u/v)\ . \tag{3.5.8}$$

Note that the latter equation can be independently checked using the ordering relations given in Lemma 2.1.5.

Spin-j fused K-operator

From the fusion formulas (3.4.12) and (3.5.1) it is clear that the fused R-matrices and K-operators can be expressed only in terms of the fundamental K-operator and R-matrix, and the maps $\mathcal{E}^{(j)}$ and $\mathcal{F}^{(j)}$. They are given by:

$$R^{(\frac{1}{2},j)}(u) = \left(\prod_{m=0}^{2j-2} \mathbb{I}_{2m+2} \otimes \mathcal{F}^{(j-\frac{m}{2})}\right)\left(\prod_{k=0}^{2j-1} R^{(\frac{1}{2},\frac{1}{2})}_{1\,2j+1-k}(uq^{-j+\frac{1}{2}+k})\right)\left(\prod_{m=0}^{2j-2} \mathbb{I}_{2(2j-1-m)} \otimes \mathcal{E}^{(1+\frac{m}{2})}\right)\ , \tag{3.5.9}$$

and

$$\begin{aligned}\mathcal{K}^{(j)}(u) = &\left(\prod_{m=0}^{2j-2} \mathbb{I}_{2m} \otimes \mathcal{F}^{(j-\frac{m}{2})}\right)\prod_{k=1}^{2j}\left\{\mathcal{K}_k^{(\frac{1}{2})}(uq^{k-j-\frac{1}{2}})\left[\prod_{\ell=0}^{2j-k-1} R^{(\frac{1}{2},\frac{1}{2})}_{k\,2j-\ell}(u^2q^{-2j+2k+\ell})\right]\right\}\\ &\times\left(\prod_{m=0}^{2j-2} \mathbb{I}_{2(2j-2-m)} \otimes \mathcal{E}^{(1+\frac{m}{2})}\right)\ ,\end{aligned} \tag{3.5.10}$$

where the product stands for the usual matrix product and the products are ordered from left to right in an increasing way in the indices.

3.5.3 K-matrices from one-dimensional representations of $\mathcal{A}_q$

Recall in Section 1.3, that we reviewed another fusion procedure that was developed for the R-matrix in [Ka79, KRS81], and for the K-matrix in [MN92]. Now, we want to compare this approach with the fusion procedure we developed here. Note, however, that we introduced fused K-operators while it was not considered in [MN92]. Therefore, we first need to construct fused K-matrices out of fused K-operators $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q \otimes \mathrm{End}(\mathbb{C}^{2j+1})$ defined in (3.5.1). It turns out that it is sufficient to apply one-dimensional representations on the first component to get fused K-matrices. All one-dimensional representations of $\mathcal{A}_q$ are classified in the following proposition.

Proposition 3.5.3. *We have the algebra map $\epsilon\colon \mathcal{A}_q \to \mathbb{C}$*

$$\epsilon(\mathsf{G}_{k+1}) = \epsilon(\tilde{\mathsf{G}}_{k+1}) = g_{k+1}\ , \tag{3.5.11}$$

$$\epsilon(\mathsf{W}_{k+1}) = g_0^{-1}\left(\sum_{m=0}^{k}(\overline{m}\varepsilon_+ + \overline{m+1}\varepsilon_-)g_{k-m}\right)\ , \qquad k \in \mathbb{N}\ , \tag{3.5.12}$$

$$\epsilon(\mathsf{W}_{-k}) = \epsilon(\mathsf{W}_{k+1})|_{\varepsilon_\pm \to \varepsilon_\mp}\ , \tag{3.5.13}$$

with $g_0 = \rho/(q-q^{-1})$, and for any scalars g_{k+1}, $\varepsilon_\pm \in \mathbb{C}$ for any $k \in \mathbb{N}$, and where we used the

notation $\overline{m} = m \mod 2$.

Proof. Recall the presentation of $\mathcal{A}_q$ with defining relations (2.1.17)-(2.1.27). In every one-dimensional representation of $\mathcal{A}_q$, the generators are evaluated to scalars, and so they commute. Therefore, the only non-trivial defining relations in such representations are (2.1.17)-(2.1.19). From (2.1.17), we conclude that $\epsilon(\mathsf{G}_{k+1}) = \epsilon(\tilde{\mathsf{G}}_{k+1})$. Moreover, from (2.1.18) and (2.1.19), the images $\epsilon(\mathsf{W}_\ell)$ for $\ell \in \mathbb{Z}\backslash\{0,1\}$ can be expressed in terms of $\epsilon(\mathsf{W}_0)$, $\epsilon(\mathsf{W}_1)$ and $\epsilon(\mathsf{G}_{k+1})$. Indeed, setting $\epsilon(\mathsf{W}_0) = \varepsilon_+$ and $\epsilon(\mathsf{W}_1) = \varepsilon_-$, the recursion relations (2.1.18), (2.1.19) have a unique solution given by (3.5.12)-(3.5.13). Finally, the scalars $\varepsilon_\pm$ and $\epsilon(\mathsf{G}_{k+1}) = g_{k+1}$ have no more constraints, therefore they parameterize one-dimensional representations. □

We now compute the image of the fundamental K-operator $\mathcal{K}^{(\frac{1}{2})}(u)$ given in (2.1.5) under the map $(\epsilon \otimes \mathrm{id})$, recall the notation $U = (u^2q + u^{-2}q^{-1})/(q + q^{-1})$ and $c(u)$ from (1.3.33). We find that the corresponding image is proportional to the spin-1/2 K-matrix of [dVR94, GZ93].

Proposition 3.5.4.

$$(\epsilon \otimes id)(\mathcal{K}^{(\frac{1}{2})}(u)) = \varsigma^{(\frac{1}{2})}(u) K^{(\frac{1}{2})}(u) \ , \tag{3.5.14}$$

where $K^{(\frac{1}{2})}(u) \equiv K(u)$ *is from* (1.2.3) and

$$\varsigma^{(\frac{1}{2})}(u) = \frac{c(q^2)}{\rho c(u^2)} \left(\sum_{k=0}^{\infty} U^{-k} g_k \right) \ . \tag{3.5.15}$$

Proof. We compute the l.h.s. of the first relation in (3.5.14) using Proposition 3.5.3 and the expressions (2.1.1)-(2.1.2) of the generating functions $\mathcal{G}_\pm(u)$, $\mathcal{W}_\pm(u)$. Firstly, from (2.1.2) and (3.5.11) we have

$$\epsilon(\mathcal{G}_\pm(u)) = \sum_{k=0}^{\infty} U^{-k} g_k - g_0 \ . \tag{3.5.16}$$

Secondly, using (2.1.1) and (3.5.12) we find that the image of $\mathcal{W}_-(u)$ under the map ϵ is

$$\begin{aligned}
\epsilon(\mathcal{W}_-(u)) &= \rho^{-1}(q - q^{-1}) \sum_{k=0}^{\infty} \sum_{m=0}^{k} U^{-k-1} g_{k-m} (\overline{m}\varepsilon_+ + \overline{m+1}\varepsilon_-) \\
&= \rho^{-1}(q - q^{-1}) \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} U^{-k-1} g_{k-m} (\overline{m}\varepsilon_+ + \overline{m+1}\varepsilon_-) \\
&= \rho^{-1}(q - q^{-1}) \sum_{m=0}^{\infty} U^{-m-1} (\overline{m}\varepsilon_+ + \overline{m+1}\varepsilon_-) \sum_{k=0}^{\infty} U^{-k} g_k \ .
\end{aligned}$$

Then, we get

$$\epsilon(\mathcal{W}_-(u)) = \frac{q - q^{-1}}{\rho} \left(\frac{\varepsilon_- U^{-1} + \varepsilon_+ U^{-2}}{1 - U^{-2}} \right) \left(\sum_{k=0}^{\infty} U^{-k} g_k \right) \ , \tag{3.5.17}$$

$$\epsilon(\mathcal{W}_+(u)) = \epsilon(\mathcal{W}_-(u))|_{\varepsilon_\pm \to \varepsilon_\mp} \ , \tag{3.5.18}$$

where the second relation follows from (3.5.13). Finally, applying $(\epsilon \otimes \mathrm{id})$ to (2.1.5) and using (3.5.16)-(3.5.18), one gets the r.h.s. of (3.5.14). □

More generally, we define fused K-matrices $K^{(j)}(u) \in \mathrm{End}(\mathbb{C}^{2j+1})$ as follows:

Definition 3.5.5. *The fused K-matrices are given for any $j \in \frac{1}{2}\mathbb{N}_+$ by*

$$(\epsilon \otimes id)(\mathcal{K}^{(j)}(u)) = \varsigma^{(j)}(u) K^{(j)}(u) \ , \tag{3.5.19}$$

where

$$\varsigma^{(j)}(u) = \prod_{k=0}^{2j-1} \varsigma^{(\frac{1}{2})}(uq^{j-\frac{1}{2}-k}) \ , \tag{3.5.20}$$

and with $\varsigma^{(\frac{1}{2})}(u)$ from (3.5.15).

By construction, they satisfy the reflection equation

$$\begin{aligned} R^{(j_1,j_2)}(u_1/u_2) K_1^{(j_1)}(u_1) R^{(j_1,j_2)}(u_1 u_2) K_2^{(j_2)}(u_2) = \\ K_2^{(j_2)}(u_2) R^{(j_1,j_2)}(u_1 u_2) K_1^{(j_1)}(u_1) R^{(j_1,j_2)}(u_1/u_2) \ . \end{aligned} \tag{3.5.21}$$

Note that the fused K-matrices $K^{(j)}(u)$ satisfy a direct analogue of the fusion formula (3.5.1) for $\mathcal{K}^{(j)}(u)$.

Proposition 3.5.6. *The fused K-matrices of spin-j satisfy*

$$K^{(j)}(u) = \mathcal{F}^{(j)}_{\langle 12\rangle} K_1^{(\frac{1}{2})}(uq^{-j+\frac{1}{2}}) R^{(\frac{1}{2},j-\frac{1}{2})}(u^2 q^{-j+1}) K_2^{(j-\frac{1}{2})}(uq^{\frac{1}{2}}) \mathcal{E}^{(j)}_{\langle 12\rangle} \ , \tag{3.5.22}$$

where $K^{(\frac{1}{2})}(u) \equiv K(u)$ is defined in (1.2.3).

Proof. Start from the definition of the fused K-matrices (3.5.19), then replace $\mathcal{K}^{(j)}(u)$ by the fusion formula, and finally simplify using the expression of $\varsigma^{(j)}(u)$ in (3.5.20). □

The above proposition will be important for the construction of the spin-j generating function in a commutative subalgebra of $\mathcal{A}_q$, as it will be discussed in Section 4.1, and it also explains the form of $\varsigma^{(j)}(u)$. Moreover, the fusion formula (3.5.22) for K-matrices will be more convenient to use since it is defined recursively from the fundamental K-matrix (1.2.3), and $R^{(\frac{1}{2},\frac{1}{2})}(u)$ from (1.1.5).

3.5.4 Comparison to other fusion approaches

We are now able to compare the fusion procedure for R- and K-matrices introduced in [Ka79, KRS81] and [MN92], respectively, with our own approach. In their formalism, the analogue of our formulas (3.5.9) and (3.5.10), are given in (1.3.22) and (1.3.23). Recall also that the resulting R- and K-matrices are not of the *good* size, as discussed in Section 1.3.3. For instance, the R-matrix $R^{(1,1)}_{\{12\}\{34\}}(u)$ is a (16×16) matrix, but after applying some similarity transformations, its spin-1 structure is exhibited in (1.3.42), in a block of correct size 9×9. We have also checked that this block expression is equal to our formula (3.4.11) for $(j_1, j_2) = (1, 1)$. The spin-1 K-matrix is computed using (3.5.19) for $j = 1$, with $\mathcal{K}^{(1)}(u)$ from (3.5.6)-(3.5.7). Then, after simplification, we find that

the expression of $K^{(1)}(u)$ coincides with the one given in (1.3.46). The latter was obtained via the fusion procedure from [MN92] after taking similarity transformations. We have also computed both R- and K-matrices up to $j = 2$ using the two fusion approaches, and they turned out to match (after applying similarity transformations and extracting the block of correct size). We thus expect that these approaches lead to the same R- and K-matrices for a general j.

More precisely, comparing the fusion procedure introduced in [Ka79, KRS81], [MN92] with our fusion construction, the advantages of the latter are the following:

- By construction, our fused R-matrices and K-operators (including K-matrices) are directly of the correct size, i.e. $R^{(j_1,j_2)}(u) \in \mathrm{End}(\mathbb{C}^{2j_1+1}) \otimes \mathrm{End}(\mathbb{C}^{2j_2+1})$.
- The fusion technique developed here has a clear interpretation in representation theory since it uses $\mathcal{L}U_q sl_2$-intertwining operators $\mathcal{E}^{(j+\frac{1}{2})}$.
- The fusion formulas (3.4.6) and (3.5.1) draw inspiration from the fusion properties of the spin-j L- and K-operators satisfied by the evaluations of the universal R- and K-matrices, see (3.4.1) and (3.4.21). Thus, it has a clear interpretation in the context of universal R- and K-matrices.

3.5.5 Unitarity and invertibility properties

We now discuss the unitarity and invertibility properties of the fused K-operators $\mathcal{K}^{(j)}(u)$ given in (3.5.1). Recall that $\mathcal{K}^{(\frac{1}{2})}(u)$ satisfies the unitarity property and is invertible, see Lemma 2.1.8 and Remark 2.1.9, respectively. We generalize these properties for any spin-j.

Proposition 3.5.7 ([LBG23, Prop. 5.12])**.** *Let*

$$\widehat{\mathcal{K}}^{(j+\frac{1}{2})}(u) = \mathcal{F}^{(j+\frac{1}{2})}_{\langle 12\rangle} \widehat{\mathcal{K}}^{(j)}_2(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},j)}(u^2 q^{j-\frac{1}{2}}) \widehat{\mathcal{K}}^{(\frac{1}{2})}_1(uq^j) \mathcal{E}^{(j+\frac{1}{2})}_{\langle 12\rangle} \ , \tag{3.5.23}$$

for $j \in \frac{1}{2}\mathbb{N}_+$ and with $\widehat{\mathcal{K}}^{(\frac{1}{2})}(u) \equiv \mathcal{K}^{(\frac{1}{2})}(u)$. Then

$$\begin{aligned} \mathcal{K}^{(j)}(u)\widehat{\mathcal{K}}^{(j)}(u^{-1}) &= \left(\prod_{k=0}^{2j-1} \frac{\Gamma(uq^{-j-\frac{1}{2}+k})}{c(u^{-2}q^{2j-1-2k})}\right) \left(\prod_{k=0}^{2j-2} \prod_{\ell=0}^{2j-k-2} c(u^2 q^{2j-1-2k-\ell}) c(u^{-2}q^{1-k+\ell})\right) \mathbb{I}_{2j+1} \\ &= \widehat{\mathcal{K}}^{(j)}(u^{-1})\mathcal{K}^{(j)}(u) \ , \end{aligned} \tag{3.5.24}$$

where $\mathcal{K}^{(j)}(u)$, $\Gamma(u)$ and $c(u)$ are respectively given in (3.5.1), (2.1.44), (1.3.33).

Remark 3.5.8. *The spin-j fused K-operator $\mathcal{K}^{(j)}(u)$ is invertible and its inverse is given by:*

$$\left[\mathcal{K}^{(j)}(u)\right]^{-1} = \left[\prod_{k=0}^{2j-1} \frac{\Gamma(uq^{-j-\frac{1}{2}+k})}{c(u^{-2}q^{2j-1-2k})}\right]^{-1} \left[\prod_{k=0}^{2j-2} \prod_{\ell=0}^{2j-k-2} c(u^2 q^{2j-1-2k-\ell}) c(u^{-2}q^{1-k+\ell})\right]^{-1} \widehat{\mathcal{K}}^{(j)}(u^{-1}) \ . \tag{3.5.25}$$

Remark 3.5.9. *By direct calculations we have checked for $j = 1, \frac{3}{2}, 2$ that $\widehat{\mathcal{K}}^{(j)}(u)$ is equal to $\mathcal{K}^{(j)}(u)$ defined in (3.5.1)* and we expect this equality holds for any j. Note that $\mathcal{K}^{(j)}(u)$ and $\widehat{\mathcal{K}}^{(j)}(u)$, are direct analogs of the spin-j K-operators $\mathbf{K}^{(j)}(u)$ defined in (3.4.21) and (3.4.17), respectively.

3.5.6 Twisted intertwining relations and evaluated coaction of fused K-operators

The fused K-operators $\mathcal{K}^{(j)}(u)$ are expected to have a simple relation with the spin-j K-operators $\mathbf{K}^{(j)}(u)$ as will be discussed in Section 3.6, similarly to the relations between $\mathbf{L}^{(j)}(u)$ and $\mathcal{L}^{(j)}(u)$. Therefore, the evaluated coaction for $\mathcal{K}^{(\frac{1}{2})}(u)$ is expected to be of the form (3.2.46) up to appropriate normalization.

Lemma 3.5.10 ([LBG23, Lem. 5.15]). *The evaluated coaction $\delta_w\colon \mathcal{A}_q \to \mathcal{A}_q \otimes U_q sl_2$ is such that*[22]

$$(\delta_w \otimes id)(\mathcal{K}^{(\frac{1}{2})}(u)) = \frac{U^{-1}}{q+q^{-1}} \left(\mathcal{L}^{(\frac{1}{2})}(u/w)\right)_{[2]} \left(\mathcal{K}^{(\frac{1}{2})}(u)\right)_{[1]} \left(\mathcal{L}^{(\frac{1}{2})}(uw)\right)_{[2]} . \tag{3.5.26}$$

Note that using above lemma, one obtains the evaluated coactions of the generating functions $\mathcal{W}_\pm(u), \mathcal{G}_\pm(u)$ of $\mathcal{A}_q$, see Proposition 2.4.2. Now, using (2.4.4), (2.4.5) we can compute the evaluated coaction of the quantum determinant $\Gamma(u)$ from (2.1.44):

$$\delta_w(\Gamma(u)) = \frac{1}{(u^2q + u^{-2}q^{-1})(u^2q^3 + u^{-2}q^{-3})} \Gamma(u) \otimes \gamma(u/w)\gamma(uw) , \tag{3.5.27}$$

where $\gamma(u)$ is given in (3.2.20). Here we used the ordering relations of $\mathcal{A}_q$ in Lemma 2.1.5 and the PBW basis of $U_q sl_2$, see Section 2.3.2.

The following result is a natural generalization of Lemma 3.5.10.

Proposition 3.5.11 ([LBG23, Prop. 5.16]). *The evaluated coaction of $\mathcal{K}^{(j)}(u)$ for $j \in \frac{1}{2}\mathbb{N}_+$ is given by*

$$(\delta_w \otimes id)(\mathcal{K}^{(j)}(u)) = \left(\prod_{p=1}^{2j} \frac{U^{-1}}{q+q^{-1}}\Big|_{u=uq^{j+\frac{1}{2}-p}}\right) \times \left(\mathcal{L}^{(j)}(u/w)\right)_{[2]} \left(\mathcal{K}^{(j)}(u)\right)_{[1]} \left(\mathcal{L}^{(j)}(uw)\right)_{[2]} . \tag{3.5.28}$$

In the next section, we will also need the twisted intertwining relations satisfied by the fused K-operators. For the fundamental K-operator (2.1.5), the twisted intertwining relations have been given in [BS09, Prop. 4.2]. This result is now extended to higher values of j.

Proposition 3.5.12 ([LBG23, Prop. 5.17]). *The following relation holds for any $j \in \frac{1}{2}\mathbb{N}_+$ and all $b \in \mathcal{A}_q$:*

$$\mathcal{K}^{(j)}(v)(id \otimes \pi^j)[\delta_{v^{-1}}(b)] = (id \otimes \pi^j)[\delta_v(b)]\mathcal{K}^{(j)}(v) . \tag{3.5.29}$$

Having introduced fused K-operators for $\mathcal{A}_q$ in Definition 3.5.1 and shown that they satisfy the reflection equation with spectral parameter (3.5.2) in Theorem 3.5.2, the Problem 2 is then solved. However, we want to go further in order to find an interpretation of $\mathcal{K}^{(j)}(u)$ in terms of the universal K-matrix $\mathfrak{K}$. Recall the relationship between the fused L-operators $\mathcal{L}^{(j)}(u)$ and those obtained by evaluating the universal R-matrix, as established in Lemma 3.4.4. We now look for an analogous relation between $\mathcal{K}^{(j)}(u)$ and $\mathbf{K}^{(j)}(u)$. As we will see, due to the absence of an explicit expression for a universal K-matrix $\mathfrak{K} \in \mathcal{A}_q \otimes \mathcal{L}U_q sl_2$, we can only propose such a relation and provide supporting evidence.

[22] Here, the index [1] is associated with the space for $\mathcal{A}_q$, and [2] for $U_q sl_2$, and we use the convention $((T)_{[2]}(T')_{[1]}(T'')_{[2]})_{ij} = \sum_{k,\ell=1}^{2} (T')_{k\ell} \otimes (T)_{ik}(T'')_{\ell j}$.

3.6 Fused K-operators and the universal K-matrix for $\mathcal{A}_q$

In this section, we assume there exists a universal K-matrix for $\mathcal{A}_q$. We are interested in the precise relationship between the fused K-operators $\mathcal{K}^{(j)}(u)$ constructed in the previous section using (3.5.1) and the spin-j K-operators defined in (3.2.29) through the evaluation of the universal K-matrix. By analogy with Lemma 3.4.4 relating spin-j L-operators (3.2.12) and fused L-operators (3.4.1), we propose:

Conjecture 1. *For $j \in \frac{1}{2}\mathbb{N}$, we have*

$$\mathbf{K}^{(j)}(u) = \nu^{(j)}(u)\mathcal{K}^{(j)}(u) \ , \tag{3.6.1}$$

where $\mathcal{K}^{(j)}(u)$ is defined in (3.5.1) with

$$\nu^{(j)}(u) = \left(\prod_{m=0}^{2j-1} \nu(uq^{j-\frac{1}{2}-m})\right)\left(\prod_{k=0}^{2j-2}\prod_{\ell=0}^{2j-k-2} \pi^{\frac{1}{2}}(\mu(u^2q^{2j-2-2k-\ell}))\right) \ . \tag{3.6.2}$$

Here $\pi^{\frac{1}{2}}(\mu(u))$ is given by (3.2.19) and $\nu(u) \equiv \nu^{(\frac{1}{2})}(u)$ is an invertible central element in $\mathcal{A}_q((u^{-1}))$, defined by the functional relation

$$\pi^{\frac{1}{2}}(\mu(u^2q))\nu(u)\nu(uq)\Gamma(u) = 1 \ , \tag{3.6.3}$$

where $\Gamma(u)$ is given in (2.1.44), and has the evaluated coaction

$$\delta_w(\nu(u)) = (u^2q + u^{-2}q^{-1})\nu(u) \otimes \mu(u/w)\mu(uw) \ . \tag{3.6.4}$$

Supporting evidence for Conjecture 1 are now presented. Afterwards, we derive from Conjecture 1 certain properties of the fused K-operators for $j \geq 0$.

3.6.1 Supporting evidence

For the clarity of the presentation, let us define:

$$\tilde{\mathbf{K}}^{(j)}(u) = \nu^{(j)}(u)\mathcal{K}^{(j)}(u) \quad \text{for} \ \ j \in \tfrac{1}{2}\mathbb{N} \ , \tag{3.6.5}$$

where we assume $\nu^{(\frac{1}{2})}(u)$ is an invertible central element in $\mathcal{A}_q((u^{-1}))$. Importantly, it is not assumed that $\tilde{\mathbf{K}}^{(j)}(u)$ is obtained from the evaluation of a universal K-matrix.

We provide supporting evidence for Conjecture 1. We show that $\tilde{\mathbf{K}}^{(j)}(u)$ for all j satisfy the following systems of equations:

$$\tilde{\mathbf{K}}^{(j)}(v)(\mathrm{id} \otimes \pi^j)[\delta_{v^{-1}}(b)] = (\mathrm{id} \otimes \pi^j)[\delta_v(b)]\tilde{\mathbf{K}}^{(j)}(v) \ , \tag{K1'}$$

$$(\delta_w \otimes \mathrm{id})(\tilde{\mathbf{K}}^{(j)}(u)) = \left(\mathbf{L}^{(j)}(u/w)\right)_{[2]}\left(\tilde{\mathbf{K}}^{(j)}(u)\right)_{[1]}\left(\mathbf{L}^{(j)}(uw)\right)_{[2]} \ , \tag{K2'}$$

$$\tilde{\mathbf{K}}^{(j)}(u) = \mathcal{F}^{(j)}_{\langle 12\rangle}\tilde{\mathbf{K}}_1^{(\frac{1}{2})}(uq^{-j+\frac{1}{2}})\mathcal{R}^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{-j+1})\tilde{\mathbf{K}}_2^{(j-\frac{1}{2})}(uq^{\frac{1}{2}})\mathcal{E}^{(j)}_{\langle 12\rangle} \ , \tag{K3'}$$

where $\mathcal{R}^{(\frac{1}{2},j)}(u)$ is given in (3.4.4), if and only if (3.6.2) and (3.6.4) hold. Here, (K1'), (K2'), (K3') are direct analogs of (3.2.47), (3.2.46), (3.4.21), respectively. We will also show that $\tilde{\mathbf{K}}^{(j)}(u)$ satisfies the reflection equation (3.2.43) where $\mathbf{K}^{(j)}(u)$ is replaced by $\tilde{\mathbf{K}}^{(j)}(u)$.

We assume that $\nu(u)$ is an invertible central element in $\mathcal{A}_q((u^{-1}))$.

Lemma 3.6.1 ([LBG23, Lem. 6.1]). *The K-operators $\tilde{\mathbf{K}}^{(j)}(u)$ for all $j \in \frac{1}{2}\mathbb{N}$ satisfy the fusion relation (K3') if and only if $\nu^{(j)}(u)$ takes the form (3.6.2), and so they are central.*

In what follows, we will assume that (K3') holds, so in particular all $\nu^{(j)}(u)$ are central. Then, using Proposition 3.5.12, the twisted intertwining relation for $\tilde{\mathbf{K}}^{(j)}(u)$ is immediate:

Lemma 3.6.2 ([LBG23, Lem. 6.2]). *The K-operators $\tilde{\mathbf{K}}^{(j)}(u)$ for all $j \in \frac{1}{2}\mathbb{N}$ satisfy (K1').*

We also show that the evaluated coaction (K2') holds for $\tilde{\mathbf{K}}^{(j)}(u)$.

Lemma 3.6.3 ([LBG23, Lem. 6.3]). *The K-operators $\tilde{\mathbf{K}}^{(j)}(u)$ for all $j \in \frac{1}{2}\mathbb{N}$ satisfy (K2') if and only if $\delta_w(\nu(u))$ is given by (3.6.4).*

Finally, assuming (K3') holds so that $\nu^{(j)}(u)$ are central, see Lemma 3.6.1, it is easy to see that the fused K-operators $\tilde{\mathbf{K}}^{(j)}(u)$ satisfy the reflection equation due to Theorem 3.5.2.

Lemma 3.6.4 ([LBG23, Lem. 6.4]). *The K-operators $\tilde{\mathbf{K}}^{(j)}(u)$ satisfy the reflection equation (3.2.43) where $\mathbf{K}^{(j)}(u)$ is replaced by $\tilde{\mathbf{K}}^{(j)}(u)$ for any $j_1, j_2 \in \frac{1}{2}\mathbb{N}_+$.*

In summary, the evidence supporting this conjecture can be summarized in Figure 3.4.

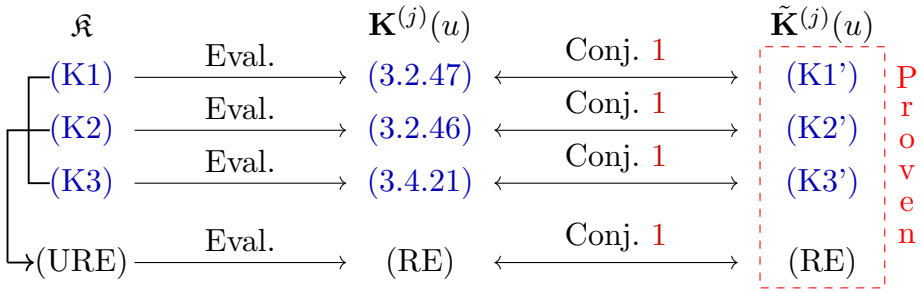


Figure 3.4: Supporting evidence for Conjecture 1, where the double-sided arrows connecting two equations signifies the equality of the latter under the assumption that Conjecture 1 is true.

Functional relation on $\nu(u)$

We have seen in Lemma 3.6.1 that the relation (K3') fixes the normalization factor $\nu^{(j)}(u)$ as (3.6.2). Here we show that the analog of the reduction relation (3.4.22) for $\tilde{\mathbf{K}}^{(j)}(u)$ leads to the functional relation (3.6.3). Recall the functional relation on $\mu(u)$ in (3.4.9) was obtained by comparing the fusion relation with the reduction relation satisfied by the spin-j L-operators, see Lemma 3.4.5. We proceed similarly for $\tilde{\mathbf{K}}^{(j)}(u)$.

Proposition 3.6.5 ([LBG23, Prop. 6.5]). *The K-operators $\tilde{\mathbf{K}}^{(j)}(u)$ satisfy (K3') and*

$$\tilde{\mathbf{K}}^{(j-\frac{1}{2})}(u) = \bar{\mathcal{F}}^{(j-\frac{1}{2})}_{\langle 12\rangle}\tilde{\mathbf{K}}^{(\frac{1}{2})}_1(uq^{j+1})\mathcal{R}^{(\frac{1}{2},j)}(u^2q^{j+\frac{3}{2}})\tilde{\mathbf{K}}^{(j)}_2(uq^{\frac{1}{2}})\bar{\mathcal{E}}^{(j-\frac{1}{2})}_{\langle 12\rangle}\ , \tag{K3"}$$

for $j = 1$ if and only if $\nu(u)$ satisfies the functional relation (3.6.3).

Remark 3.6.6. *As a consistency check, we observe that the evaluated coaction in* (3.6.4) respects the functional relation (3.6.3) on $\nu(u)$. Indeed, using (3.5.27) and the functional relations (3.4.9) and (3.6.3), we obtain

$$\delta_w(\nu(u))\delta_w(\nu(uq))\delta_w(\Gamma(u))\pi^{\frac{1}{2}}(\mu(u^2q)) = 1 \otimes 1 \ . \tag{3.6.6}$$

Coaction

We propose a right coaction for the components of $\tilde{\mathbf{K}}^{(\frac{1}{2})}(u)$:

$$(\delta \otimes \mathrm{id})(\tilde{\mathbf{K}}^{(\frac{1}{2})}(u)) = \left([\mathbf{L}^-(u^{-1})]^{-1}\right)_{[2]} \left(\tilde{\mathbf{K}}^{(\frac{1}{2})}(u)\right)_{[1]} \left(\mathbf{L}^+(u)\right)_{[2]} \ , \tag{3.6.7}$$

where $\mathbf{L}^{\pm}(u)$ are defined in (3.2.9), which is the direct analog of (3.2.44). First of all, we show that the coaction as defined by (3.6.7) respects the relations satisfied by the components of $\tilde{\mathbf{K}}^{(\frac{1}{2})}(u)$. Recall that, due to Lemma 3.6.4, these relations are

$$\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(u/v)\tilde{\mathbf{K}}_1^{(\frac{1}{2})}(u)\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(uv)\tilde{\mathbf{K}}_2^{(\frac{1}{2})}(v) = \tilde{\mathbf{K}}_2^{(\frac{1}{2})}(v)\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(uv)\tilde{\mathbf{K}}_1^{(\frac{1}{2})}(u)\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(u/v) \ . \tag{3.6.8}$$

We finally show that δ defined in (3.6.7) is coassociative and counital, see (2.4.1). Firstly, we check the coassociativity:

$$\begin{aligned}
(\delta \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id})(\tilde{\mathbf{K}}^{(\frac{1}{2})}(u)) &= (\delta \otimes \mathrm{id} \otimes \mathrm{id}) \left(\left([\mathbf{L}^-(u^{-1})]^{-1}\right)_{[2]} \left(\tilde{\mathbf{K}}^{(\frac{1}{2})}(u)\right)_{[1]} \left(\mathbf{L}^+(u)\right)_{[2]} \right) \\
&= \left([\mathbf{L}^-(u^{-1})]^{-1}\right)_{[3]} \left([\mathbf{L}^-(u^{-1})]^{-1}\right)_{[2]} \left(\tilde{\mathbf{K}}^{(\frac{1}{2})}(u)\right)_{[1]} \left(\mathbf{L}^+(u)\right)_{[2]} \left(\mathbf{L}^+(u)\right)_{[3]} \\
&= (\mathrm{id} \otimes \Delta \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id})(\tilde{\mathbf{K}}^{(\frac{1}{2})}(u)) \ ,
\end{aligned}$$

where the coproduct is given in (3.2.23) and we used $(\Delta \otimes \mathrm{id})([\mathbf{L}^{\mp}(u)]^{-1}) = (\mathbf{L}^{\mp}(u))_{[2]}^{-1} (\mathbf{L}^{\mp}(u))_{[1]}^{-1}$. Secondly, the condition with the counit is checked:

$$\begin{aligned}
(\mathrm{id} \otimes \epsilon \otimes \mathrm{id}) \circ (\delta \otimes \mathrm{id})(\tilde{\mathbf{K}}^{(\frac{1}{2})}(u)) &= (\mathrm{id} \otimes \epsilon \otimes \mathrm{id}) \circ \left(\left([\mathbf{L}^{\mp}(u^{-1})]^{-1}\right)_{[2]} \left(\tilde{\mathbf{K}}^{(\frac{1}{2})}(u)\right)_{[1]} \left(\mathbf{L}^{\pm}(u)\right)_{[2]} \right) \\
&= \tilde{\mathbf{K}}^{(\frac{1}{2})}(u) \ ,
\end{aligned}$$

where we used (3.2.25).

To conclude this section, let us comment on the coaction of $\mathcal{A}_q$, that is expected to be closely related with the coaction of O_q. By assumption, $\nu(u) = \sum_{k=0}^{\infty} \nu_k u^{-k}$ where $\nu_k \in Z(\mathcal{A}_q)$. From (3.6.3) it is easily checked that ν_0 is a scalar satisfying

$$-\frac{\nu_0^2 \rho q^{\frac{1}{2}}}{(q-q^{-1})^2} = 1 \ . \tag{3.6.9}$$

Provided Conjecture 1 holds, the coaction of $\mathcal{A}_q$ is given by (3.6.7) with (3.6.5), (2.1.5) and (B.2.18), (B.2.25), recall the discussion in Section 3.2.6 and Proposition 3.2.4. A comparison of the leading

term u^{-1} of the matrix entries $(1,1)$ and $(2,2)$ of both sides of (3.6.7) gives:

$$\delta(\mathsf{W}_0) = 1 \otimes \left(k_+ q^{\frac{1}{2}} E_1 K_1^{\frac{1}{2}} + k_- q^{-\frac{1}{2}} F_1 K_1^{\frac{1}{2}} \right) + \mathsf{W}_0 \otimes K_1 \ , \tag{3.6.10}$$

$$\delta(\mathsf{W}_1) = 1 \otimes \left(k_+ q^{-\frac{1}{2}} F_0 K_0^{\frac{1}{2}} + k_- q^{\frac{1}{2}} E_0 K_0^{\frac{1}{2}} \right) + \mathsf{W}_1 \otimes K_0 \ . \tag{3.6.11}$$

Note that these equations indeed agree with the coaction of O_q given in (2.4.2), (2.4.3), where we used $\mathcal{A}_q \cong O_q \otimes Z$. To construct the coaction of $\mathsf{W}_{-k}, \mathsf{W}_{k+1}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1}$ for general values of k, the properties of the generating function $\nu(u)$ need to be investigated further starting from the functional relation (3.6.3).

3.6.2 Comments

Based on the supporting evidence given in the previous subsection, we believe Conjecture 1 is correct. Some straightforward consequences are now pointed out. Firstly, some relations among the fused K-operators (3.5.1) are derived, and will be used in the next chapter.

Proposition 3.6.7 ([LBG23, Prop. 6.8]). *Assume Conjecture 1.* Then, the following relations hold for any $j \in \frac{1}{2}\mathbb{N}_+$:

$$\begin{aligned} &\bar{\mathcal{F}}^{(j-\frac{1}{2})}_{\langle 12\rangle} \mathcal{K}_1^{(\frac{1}{2})}(uq^{j+1}) R^{(\frac{1}{2},j)}(u^2 q^{j+\frac{3}{2}}) \mathcal{K}_2^{(j)}(uq^{\frac{1}{2}}) \bar{\mathcal{E}}^{(j-\frac{1}{2})}_{\langle 12\rangle} = \\ &\qquad \left(\prod_{k=0}^{2j-2} c(u^2 q^{2j-1-k}) c(u^2 q^{2j+1-k}) \right) \Gamma(uq^j) \mathcal{K}^{(j-\frac{1}{2})}(u) \ , \end{aligned} \tag{3.6.12}$$

$$\begin{aligned} &\bar{\mathcal{F}}^{(j-\frac{1}{2})}_{\langle 12\rangle} \mathcal{K}_2^{(j)}(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},j)}(u^2 q^{-j-\frac{3}{2}}) \mathcal{K}_1^{(\frac{1}{2})}(uq^{-j-1}) \bar{\mathcal{E}}^{(j-\frac{1}{2})}_{\langle 12\rangle} = \\ &\qquad \left(\prod_{k=0}^{2j-2} c(u^2 q^{-2j+2+k}) c(u^2 q^{-2j+k}) \right) \Gamma(uq^{-j-1}) \mathcal{K}^{(j-\frac{1}{2})}(u) \ , \end{aligned} \tag{3.6.13}$$

where $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ is fixed by Lemma C.2.1 and $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$ is given in (C.2.6) with (C.2.7).

Secondly, we analyze the spin-0 K-operator $\mathbf{K}^{(0)}(u)$ and the analog of the quantum determinant (2.1.43) for the spin-$\frac{1}{2}$ K-operator $\mathbf{K}^{(\frac{1}{2})}(u)$.

Proposition 3.6.8 ([LBG23, Prop. 6.9]). *Assume Conjecture 1,* then $\mathbf{K}^{(0)}(u) = 1$. Furthermore, the quantum determinant of the K-operator $\mathbf{K}^{(\frac{1}{2})}(u)$ is equally 1:

$$\mathrm{tr}_{12}\big(\mathcal{P}_{12}^{-} \mathbf{K}_1^{(\frac{1}{2})}(u) \mathcal{R}_{12}^{(\frac{1}{2},\frac{1}{2})}(qu^2) \mathbf{K}_2^{(\frac{1}{2})}(uq)\big) = 1 \ . \tag{3.6.14}$$

Proposition 3.6.9 ([LBG23, Prop. 6.10]). *Assume Conjecture 1,* then $\widehat{\mathcal{K}}^{(j)}(u)$ from (3.5.23) is equal to the fused K-operator $\mathcal{K}^{(j)}(u)$ defined in (3.5.1).

3.6.3 Solutions to the twisted intertwining relations

We conclude this section by mentioning another method to obtain solutions of the Yang-Baxter and reflection equations. Recall that solutions of these equations have been obtained using the second

and third axioms of the universal R- and K-matrices. Indeed, the evaluations of (R2)-(R3) lead to the fused formulas for the L-operators and R-matrices, as well as the Hopf algebra structure of H; while (K2)-(K3) lead to the fused K-operators and the comodule algebra structure of B, see Propositions 3.4.1, 3.4.2, Sec. 3.2.4, and Propositions 3.4.8, 3.2.4, respectively. Now, let us discuss the interest of the evaluation of the first axiom of the universal R- and K-matrices, (R1) and (K1).

Firstly, for the R-matrices, the idea is to demand that they satisfy a set of intertwining relations with respect to the quantum affine algebra action on $V^{(j_1)} \otimes V^{(j_2)}$, given by its coproduct Δ. For $H = \mathcal{L}U_q sl_2$ and for any $x \in \mathcal{L}U_q sl_2$, these relations read

$$R^{(j_1,j_2)}(u/v)[(\pi_u^{j_1} \otimes \pi_v^{j_2})\Delta(x)] = [(\pi_u^{j_1} \otimes \pi_v^{j_2})\Delta^{op}(x)]R^{(j_1,j_2)}(u/v) \ .$$

It is then enough to solve them to obtain solutions of the Yang-Baxter equation. Note that this technique has been initiated in [KR83] without this universal framework. The solutions were found to be unique (up to an overall scalar function), and thus they should come from the evaluation of the universal R-matrix. In the universal framework, the solution of (R1)–(R3) of the form [KT92b, eq. (42)] is unique [KT92b, Theorem 7.1], and is given by [KT92b, eq. (58)]. For the expression of the universal R-matrix in a factorized form, see Section B.1.2.

Secondly, the K-operators satisfy a set of twisted intertwining relations with respect to the comodule algebra action on $B \otimes V^{(j_1)}$ given by its coaction δ. For $B = \mathcal{A}_q$ and $H = \mathcal{L}U_q sl_2$, the twisted intertwining relations are given in Proposition 3.5.12 which are the direct analogs of the evaluated axiom (K1) from (3.2.47). Importantly, it is sufficient to consider the relations for $b = \mathsf{W}_0, \mathsf{W}_1$. We consider a rather general ansatz for $\mathcal{K}^{(j)}(u)$, it is a $(2j+1) \times (2j+1)$ matrix with entries in $\mathcal{A}_q$ that are linear combinations of monomials of the form (recall the ordering (2.1.42))

$$f(u) \prod_{r=1}^{R} \mathcal{W}_+(uq^{a_r}) \prod_{p=1}^{P} \mathcal{G}_+(uq^{b_p}) \prod_{m=1}^{M} \mathcal{G}_-(uq^{c_m}) \prod_{t=1}^{T} \mathcal{W}_-(uq^{d_t}) \ , \qquad R+P+M+T \leq 2j \quad (3.6.15)$$

and for some choice of $a_r, b_p, c_m, d_t \in \frac{1}{2}\mathbb{Z}$, and $f(u)$ is a Laurent polynomial in u of maximal degree $2(R+P+M+T)$. Then, solving the twisted intertwining relations for the spin-1/2 and 1 K-operators, we indeed find that their expressions match with the fused ones derived using the fusion formula (3.5.1). The details of these calculations can be found in [LBG23, Sec. 7]. Based on these evidences, we conjecture that spin-j K-operator solutions of (3.5.29) are unique (up to an overall scalar factor), and coincide with the expression provided in (3.5.1). However, due to the non-commutative nature of $\mathcal{A}_q$, it appears challenging to express the entries of $\mathcal{K}^{(j)}(u)$ for a general j using the twisted intertwining relations.

Note that the K-matrices should also satisfy a set of twisted intertwining relations but for the action of a certain coideal subalgebra in the quantum affine algebra. In this case, this method has been initiated in [MN97, DM01] where the coideal subalgebra is now known as the q-Onsager algebra O_q [T99, B04]. Let us also notice that recursion formulas for the entries of the spin-j K-matrices have been obtained this way in [DM01], and they are already quite complicated.

Finally, let us compare the two methods for obtaining K-operator solutions of the reflection equations with spectral parameter (3.5.2), namely: the fusion procedure and the solving of the twisted intertwining relations. The latter method is more complicated than the use of the fusion procedure. Indeed, it quickly becomes tedious to solve the twisted intertwining relations due to the

repeated use of the ordering relations [LBG23, eq. (7.2)] that serve to express each term in the PBW basis of $\mathcal{A}_q$. In contrast, the fusion formula (3.5.1) for $\mathcal{K}^{(j)}(u)$ can be more easily applied since the intertwining operators $\mathcal{E}^{(j+\frac{1}{2})}$ and their pseudo-inverses $\mathcal{F}^{(j+\frac{1}{2})}$ are explicitly known.

Summary of main results

Recall that Problem 1 was addressed in Sections 3.1 and 3.2, see the summary at the end of Section 3.2. The results obtained in Sections 3.3-3.6 are the following:

- Firstly, we studied tensor product of evaluation representations of $\mathcal{L}U_q sl_2$, to highlight the reducibility criteria that gives rise either to a spin-$(j+1/2)$ or a spin-$(j-1/2)$ sub-representation as depicted in Fig. 3.3. Then, we constructed explicitly the corresponding $\mathcal{L}U_q sl_2$-intertwining operators $\mathcal{E}^{(j+\frac{1}{2})}$ and $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$, as well as their pseudo-inverses $\mathcal{F}^{(j+\frac{1}{2})}$ and $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$ in Sections C.1 and C.2. Importantly, we have also extended this analysis to the formal evaluation representations of $\mathcal{L}U_q sl_2$, when u is a formal parameter.

- Secondly, using the $\mathcal{L}U_q sl_2$-intertwiners, we find that the spin-j K-operators $\mathbf{K}^{(j)}(u)$ from Definition 3.2.29, satisfy fusion properties such as (3.4.21). This inspires us to define fused K-operators $\mathcal{K}^{(j)}(u)$ in Definition 3.5.1, independently of the framework of the universal K-matrix. Then, we show by induction in Theorem 3.5.2 that they satisfy a reflection equation with spectral parameter. Applying one-dimensional representations for $\mathcal{A}_q$ to these fused K-operators, we obtain spin-j K-matrices in Definition 3.5.5. We also find that they satisfy a fusion formula in Proposition 3.5.6.

- Finally, we give an interpretation of these fused K-operators $\mathcal{K}^{(j)}(u)$ in terms of the spin-j K-operators $\mathbf{K}^{(j)}(u)$ coming from the evaluation of the universal K-matrix $\mathfrak{K} \in B \otimes H$. We propose that they are proportional (up to a central element $\nu(u) \in \mathcal{A}_q((u^{-1})))$, see Conjecture 1, and we give supporting evidence as depicted in Figure 3.4.

Therefore, the elements outlined above offer a solution to Problem 2.

Chapter 4

Universal TT-relations and applications

Recall that every solution of the reflection equation enables the construction of a generating function for mutually commuting elements [Sk88]. This was reviewed for the transfer matrix for open spin-chains in Section 1.2.3. In the previous chapter, we constructed fused K-operators for $\mathcal{A}_q$, and thus one may ask how to explicitly construct such generating functions in a commutative subalgebra of $\mathcal{A}_q$. For the case $j = 1/2$, the generating function $\mathsf{T}^{(\frac{1}{2})}(u)$ is computed in [BB12], see its expression in (2.1.55), and the corresponding commutative subalgebra is known. It is denoted as $\mathcal{I}$ and is generated by the elements $\{\mathcal{I}_{2k+1} | k \in \mathbb{N}\}$, as discussed in Section 2.1.4. We call this generating function a universal transfer-matrix because of its link to quantum spin-chains. Indeed, using spin-chain representations for $\mathcal{A}_q$, it was found that $\mathsf{T}^{(\frac{1}{2})}(u)$ specializes to the transfer matrix of the XXZ spin-1/2 chain with generic boundary conditions [BK05a, BK05b]. However, for a general j such generating functions and the corresponding commutative subalgebras are not known, as well as the connection with the spin-j transfer matrices, generalizing [BK05a, BK05b] for $j = 1/2$. In this chapter, we solve the third problem of the thesis.

- Problem 3:

  One can construct a generating function for spin-1/2 in the commutative subalgebra of $\mathcal{A}_q$ from its spin-1/2 K-operator (2.1.5). This object has indeed been studied for the spin-1/2 XXZ chain in [BK05a, BK05b], but for a quotient of $\mathcal{A}_q$. Moreover, a spin-j version of this generating function has never been constructed, and its connection to spin chains remains to be investigated.

The main result of this chapter is the construction of universal-transfer-matrix generating functions for every spin value j, that we denote $\mathsf{T}^{(j)}(u) \in \mathcal{A}_q((u^{-1}))$, and the derivation of universal TT-relations for $\mathcal{A}_q$, see eq. (4.1.4) and Theorem 4.2.5. They are recurrence relations satisfied by the spin-j generating functions. The universal TT-relations are proven for any spin-j provided the evaluations of the universal K-matrix $\mathbf{K}^{(j)}(u)$ are proportional to $\mathcal{K}^{(j)}(u)$ by a factor $\nu^{(j)}(u)$, which is an invertible central element in $\mathcal{A}_q((u^{-1}))$ defined by (3.6.2), see Conjecture 1. For brevity, when we refer to Conjecture 1 in the following, we actually mean that there exists a universal K-matrix

$\mathfrak{K} \in \mathcal{A}_q \otimes \mathcal{L}U_q sl_2$ such that the proportionality relations hold. A consequence of these universal TT-relations is that all $\mathsf{T}^{(j)}(u)$ are polynomials in $\mathsf{T}^{(\frac{1}{2})}(u)$ and so they all belong to the same commutative subalgebra $\mathcal{I}$ as for $j = 1/2$. We also show independently for small values $j = 1, 3/2$, that the universal TT-relations indeed hold using a PBW basis for $\mathcal{A}_q$ from Section 2.1.2.

Then, by considering spin-chain realizations which are certain tensor product representations of $\mathcal{A}_q$, we show that $\mathsf{T}^{(j)}(u)$ gives rise to a general open spin-j transfer matrix with any value of spin at each site. In this case, the universal TT-relations become the TT-relations for actual physical transfer matrices. These TT-relations allow to reduce recursively the spectral problem of spin-j to the one for spin-1/2. Finally, from the algebraic structure of $\mathsf{T}^{(j)}(u)$ and using the PBW basis of $\mathcal{A}_q$, we obtain non-trivial symmetries for the corresponding Hamiltonians, i.e. operators that commute with the Hamiltonians.

This chapter is organized as follows.

In Section 4.1, we introduce spin-j dual K-matrices $K^{+(j)}(u)$ in (4.1.2), and show they satisfy fusion formula (4.1.3). This allows us to define in (4.1.4) the generating function $\mathsf{T}^{(j)}(u)$ as the trace over the spin-j auxiliary space of the product of the dual K-matrix with the spin-j K-operator $\mathcal{K}^{(j)}(u)$ from (3.5.1).

Section 4.2 is dedicated to the universal TT-relations. Firstly, they are proved under the assumption that Conjecture 1 holds. What sets this conjecture apart from previous works on TT-relations is the presence of a universal K-matrix that specializes to K-operators. This is sufficient to show TT-relations at an algebraic level (what we call *universal TT-relations*), without any additional assumption. On the contrary, in the literature, there are conjectured TT-relations that rely on assumptions concerning the specific representations considered, which are not universal. For instance, the proof of the TT-relations in [FNR07] relies on relations satisfied by the projectors (1.3.34). Secondly, we give a proof (with no assumptions made) of the universal TT-relations for the spin values $j = 1, \frac{3}{2}$. This uses commutation relations of PBW basis elements of $\mathcal{A}_q$ given in Section 2.1.2.

In Section 4.3, we recall spin-chain representations for $\mathcal{A}_q$, denoted $\vartheta^{(N)}$, where N is the number of tensor factors or length of the chain, that was introduced in [BK05b], and find that the images of the alternating generators of $\mathcal{A}_q$ satisfy linear relations (4.3.9)-(4.3.10). The latter imply that, on representations, the generating functions $\{\mathcal{W}_\pm(u), \mathcal{G}_\pm(u)\}$ for $\mathcal{A}_q$ truncate. Inspired by these linear relations, we introduce in Definition 4.3.1 the quotient algebra $\mathcal{A}_q^{(N)}$. We also describe the corresponding quotient map $\varphi^{(N)} \colon \mathcal{A}_q \to \mathcal{A}_q^{(N)}$, to provide a FRT type presentation of $\mathcal{A}_q^{(N)}$. Finally, in (4.3.28), we compute explicitly the spin-1/2 generating function $\mathsf{T}^{(\frac{1}{2})}(u)$ in $\mathcal{I}^{(N)}$, which is a commutative subalgebra of $\mathcal{A}_q^{(N)}$ generated by $\{\varphi^{(N)}(\mathcal{I}_{2k+1}) | k \in \mathbb{N}\}$.

In Section 4.4, we focus on the applications to the quantum integrable systems beyond the case spin-1/2. Firstly, we construct truncated K-operators $(\varphi^{(N)} \otimes \mathrm{id})(\mathcal{K}^{(j)}(u)) \sim \mathcal{K}^{(j,N)}(u) \in \mathcal{A}_q^{(N)}[u, u^{-1}] \otimes \mathrm{End}(\mathbb{C}^{2j+1})$, as well as the spin-$j$ generating function in $\mathcal{I}^{(N)}$, denoted $\mathsf{T}^{(j,N)}(u) \in \mathcal{A}_q^{(N)}((u^{-1}))$. Using the quotient map $\varphi^{(N)}$, $\mathsf{T}^{(j,N)}(u)$ are found to satisfy universal TT-relations in $\mathcal{A}_q^{(N)}((u^{-1}))$, see (4.4.14). Secondly, we consider the spin-chain representations of $\mathcal{A}_q$. The image of $\mathcal{K}^{(j,N)}(u)$ is found to form a double-row monodromy matrix (dressed K-matrix, recall the form in (1.2.11)) with arbitrary spin-j_n at each site and a spin-j auxiliary space. Moreover, we show in Proposition 4.4.5 that the universal TT-relations specialize to TT-relations satisfied by transfer

matrices $t^{(j,\{j_n\}_N)}(u)$ defined in (4.4.16). Importantly, the auxiliary space j is arbitrary as well as the spin of the quantum spaces j_n, and thus generalizes the case $j = 1/2$ from [BK05a, BK05b]. As an application, we set the spins j_n to some values, and we recover the conjectural TT-relations studied in the literature such as the XXZ spin-j chains with generic boundary conditions [FNR07], or the alternating spin chains [CYSW14]. The spins j_n have the same value j in the former case, and alternating values j_1, j_2 in the latter case. Finally, as a corollary of the TT-relations, we find that the transfer matrix $t^{(j,\{j_n\}_N)}(u)$ is a polynomial of degree $4Nj$ in the spin-chain representation of the generators of the q-Onsager algebra $\mathcal{W}_0$ and $\mathcal{W}_1$.

In the literature, symmetries of open XXZ spin-1/2 chains have been studied, see e.g. [ABBBQ, PS90]. For instance, these symmetries include the $U(1)$ symmetry with diagonal boundary conditions, or the $U_q sl_2$ symmetry with special boundary conditions, as recalled in Example 1.2.9. To conclude this chapter, we study *hidden* symmetries for open spin-j chains but using now the underlying algebraic structure of the generating functions $\mathsf{T}^{(j)}(u)$, and not directly the Hamiltonian. To do so, in Section 4.5 we study (in the algebra) commutation relations between elements of $\mathcal{A}_q$ and $\mathsf{T}^{(\frac{1}{2})}(u)$.

The results presented in this chapter form the foundation of an article currently in preparation, titled *TT-relations and the q-Onsager algebra*.

4.1 Generating function $\mathsf{T}^{(j)}(u)$ for $\mathcal{A}_q$

Here we introduce the material to construct the spin-j generating function $\mathsf{T}^{(j)}(u)$ in the commutative subalgebra of $\mathcal{A}_q$. In the next section, we show that they are solely expressed in terms of the fundamental one (of spin-1/2) $\mathsf{T}^{(\frac{1}{2})}(u)$, recall its expression in (2.1.55), and central elements of $\mathcal{A}_q$.

4.1.1 Spin-j fused K-matrices

The construction of $\mathsf{T}^{(j)}(u)$ requires fused K-operators $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^{2j+1})$ as well as fused dual K-matrices $K^{+(j)}(u) \in \mathrm{End}(\mathbb{C}^{2j+1})$ that satisfy the dual reflection equation from Definition 1.2.3

$$R^{(j_1,j_2)}(v/u)K_1^{+(j_1)}(u)R^{(j_1,j_2)}(1/uvq^2)K_2^{+(j_2)}(v) = \\ K_2^{+(j_2)}(v)R^{(j_1,j_2)}(1/uvq^2)K_1^{+(j_1)}(u)R^{(j_1,j_2)}(v/u)\ . \tag{4.1.1}$$

As shown in the previous chapter, the fused K-operators are given in Definition 3.5.1 and they satisfy the reflection equation, see Theorem 3.5.2. The dual K-matrices are obtained as follows.

Recall that we introduced spin-j K-matrices out of fused K-operators $\mathcal{K}^{(j)}(u)$ in Definition 3.5.5. Solutions of the dual reflection equation (4.1.1) can be constructed similarly as in Remark 1.2.5

$$K^{+(j)}(u) = \frac{1}{f^{(j)}(u)}\left(K^{(j)}(-u^{-1}q^{-1})\right)^t\Big|_{\varepsilon_\pm\to-\bar{\varepsilon}_\mp,k_\pm\to-\bar{k}_\mp}\ , \tag{4.1.2}$$

where the normalization factor $f^{(j)}(u)$ from (1.3.25) is used for further convenience. From (4.1.2) and the fusion formula (3.5.22) for the K-matrices, we find that the spin-j dual K-matrices satisfy

the fusion formula:

$$K^{+(j)}(u) = \frac{f^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})}{f^{(j)}(u)} \big[\mathcal{E}^{(j)}_{\langle 12\rangle}\big]^t K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1}) K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}}) \big[\mathcal{F}^{(j)}_{\langle 12\rangle}\big]^t. \quad (4.1.3)$$

4.1.2 $\mathsf{T}^{(j)}(u)$

Following [Sk88, Thm. 1], the spin-j generating function is built from the fused K-operator $\mathcal{K}^{(j)}(u)$ for $\mathcal{A}_q$ given by (3.5.1) and the fused dual K-matrix with scalar entries $K^{+(j)}(u)$ given by (4.1.2):

$$\mathsf{T}^{(j)}(u) = \mathrm{tr}_{V^{(j)}}(K^{+(j)}(u)\mathcal{K}^{(j)}(u)) \ , \quad (4.1.4)$$

where the trace is taken over the auxiliary space $V^{(j)} \equiv \mathbb{C}^{2j+1}$. Let us note that these generating functions can be seen as universal versions of the physical transfer matrices, e.g. the one for the open quantum spin-chains as reviewed in 1.2.3. Indeed, we will show in Section 4.4 that these power series specialize to transfer matrices acting on spin-chains.

In the next section, we show that $\mathsf{T}^{(j)}(u)$ satisfy universal TT-relations that involve various quantum determinants. Recall $\Gamma(u) \in \mathcal{A}_q((u^{-1}))$ is given in (2.1.44). We will also need the value of the quantum determinant of the dual spin-$\frac{1}{2}$ K-matrix. For $j = \frac{1}{2}$ in (4.1.2), one gets:

$$\begin{aligned}\Gamma_+(u) &= \mathrm{tr}_{12}(\mathcal{P}^-_{12} K_1^{+(\frac{1}{2})}(u) R_{12}^{(\frac{1}{2},\frac{1}{2})}(u^{-2}q^{-3}) K_2^{+(\frac{1}{2})}(uq)) \quad (4.1.5)\\ &= -c(u^2q^4)\left(\overline{\varepsilon}_+^2 + \overline{\varepsilon}_-^2 + \overline{\varepsilon}_+\overline{\varepsilon}_-(u^2q^2 + u^{-2}q^{-2}) - \overline{k}_+\overline{k}_- \frac{c(u^2q^2)^2}{(q-q^{-1})^2}\right) \ ,\end{aligned}$$

where $c(u)$ is given in (1.3.33). The quantum determinant associated with $K(u)$ from (1.2.3) is given by

$$\Gamma_-(u) = \Gamma_+(-u^{-1}q^{-2})\Big|_{\overline{\varepsilon}_\pm \to -\varepsilon_\mp, \overline{k}_\pm \to -k_\mp} \ . \quad (4.1.6)$$

Note that both $\Gamma(u)$ and $\Gamma_-(u)$ are related as follows:

$$\epsilon(\Gamma(u)) = \varsigma^{(\frac{1}{2})}(u)\varsigma^{(\frac{1}{2})}(uq)\,\Gamma_-(u) \ , \quad (4.1.7)$$

where ϵ is the one-dimensional representation of $\mathcal{A}_q$ defined in Proposition 3.5.3, and $\varsigma^{(\frac{1}{2})}(u)$ is given in (3.5.15).

Finally, we will also need some properties satisfied by the fused R-matrix $R^{(\frac{1}{2},j)}(u)$ given in (C.3.1). By straightforward calculations, one finds that it is symmetric and enjoys unitarity and crossing symmetry properties:

$$\Big[R^{(\frac{1}{2},j)}(u)\Big]^{t_{12}} = R^{(\frac{1}{2},j)}(u) \ , \quad (4.1.8)$$

$$R^{(\frac{1}{2},j)}(u)R^{(\frac{1}{2},j)}(u^{-1}) = \beta^{(j)}(u)\ \mathbb{I}_{4j+2} \ , \quad (4.1.9)$$

$$\Big[R^{(\frac{1}{2},j)}(u)\Big]^{t_1}\Big[R^{(\frac{1}{2},j)}(u^{-1}q^{-2})\Big]^{t_1} = \xi^{(j)}(u)\ \mathbb{I}_{4j+2} \ , \quad (4.1.10)$$

where t_{12} (resp. t_1) stands for the transposition applied to the space $V^{(j_1)} \otimes V^{(j_2)}$ (resp. $V^{(j_1)}$), and

$$\beta^{(j)}(u) = \prod_{k=0}^{2j-1} -c(uq^{j+\frac{1}{2}-k})c(uq^{-j-\frac{1}{2}+k}), \quad \xi^{(j)}(u) = \prod_{k=0}^{2j-1} -c(uq^{j-k-\frac{1}{2}})c(uq^{-j+k+\frac{5}{2}}). \tag{4.1.11}$$

4.2 Universal TT-relations for $\mathcal{A}_q$

In this section, we first show in Theorem 4.2.5 that $\mathsf{T}^{(j)}(u)$, defined in (4.1.4), satisfy universal TT-relations for any $j \in \frac{1}{2}\mathbb{N}_+$ provided Conjecture 1 is true. Then, an independent proof of the universal TT-relations is given for $j = 1, 3/2$ using a PBW basis for $\mathcal{A}_q$.

4.2.1 Proof of the universal TT-relations

For the proof of the universal TT-relations, a few properties satisfied by the intertwining operator $\mathcal{E}^{(j)}$ and its pseudo-inverse $\mathcal{F}^{(j)}$ are needed, recall their expressions are respectively given in Lemma C.1.4 and in (C.1.7)-(C.1.8). These properties are gathered in the next two lemmas. Recall also that $\mathcal{H}^{(j+\frac{1}{2})}(u)$ is a diagonal matrix that is useful to decompose $R^{(\frac{1}{2},j)}(u)$ at a special point $u = q^{j+\frac{1}{2}}$ in terms of $\mathcal{E}^{(j+\frac{1}{2})}$, $\mathcal{F}^{(j+\frac{1}{2})}$ and $\mathcal{H}^{(j+\frac{1}{2})}$ as in (C.3.5).

Lemma 4.2.1. *The following relations hold:*

$$\mathcal{H}_1^{(j)}\ \mathcal{E}^{(j)} = \left[\mathcal{F}^{(j)}\right]^t \mathcal{H}^{(j)}\ , \qquad \bar{\mathcal{E}}^{(j-\frac{1}{2})}\bar{\mathcal{F}}^{(j-\frac{1}{2})} + \mathcal{E}^{(j+\frac{1}{2})}\mathcal{F}^{(j+\frac{1}{2})} = \mathbb{I}_{4j+2}\ , \tag{4.2.1}$$

$$\left[\mathcal{F}^{(j)}\right]^t \mathcal{F}^{(j)} R^{(\frac{1}{2},j-\frac{1}{2})}(q^j) = \mathcal{H}_1^{(j)}\ \mathcal{E}^{(j)}\mathcal{F}^{(j)}\ , \qquad \mathcal{H}_1^{(j)}\ \mathcal{E}^{(j)}\left[\mathcal{E}^{(j)}\right]^t = R^{(\frac{1}{2},j-\frac{1}{2})}(q^j)\ , \tag{4.2.2}$$

where $\mathcal{H}_1^{(j)} = \prod_{k=0}^{2j}(q^{2j-k} - q^{-2j+k})$.

Proof. The relations in (4.2.1) are straightforwardly checked using the expressions of $\mathcal{E}^{(j+\frac{1}{2})}$, $\mathcal{F}^{(j+\frac{1}{2})}$, $\mathcal{H}^{(j+\frac{1}{2})}$ and $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$, $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$,$\bar{\mathcal{H}}^{(j-\frac{1}{2})}$ given in Sections C.1–C.3. Recall the decomposition of the R-matrix (C.3.1) at special points (C.3.5), (C.3.10). Since $R^{(\frac{1}{2},j)}(u)$ is symmetric (4.1.8), we have:

$$R^{(\frac{1}{2},j)}(q^{j+\frac{1}{2}}) = \left[\mathcal{F}^{(j+\frac{1}{2})}\right]^t \mathcal{H}^{(j+\frac{1}{2})}\left[\mathcal{E}^{(j+\frac{1}{2})}\right]^t\ , \quad R^{(\frac{1}{2},j)}(q^{-j-\frac{1}{2}}) = \left[\bar{\mathcal{F}}^{(j-\frac{1}{2})}\right]^t \bar{\mathcal{H}}^{(j-\frac{1}{2})}\left[\bar{\mathcal{E}}^{(j-\frac{1}{2})}\right]^t\ . \tag{4.2.3}$$

For the relations in (4.2.2), the first one is obtained as follows:

$$\left[\mathcal{F}^{(j)}\right]^t \mathcal{F}^{(j)} R^{(\frac{1}{2},j-\frac{1}{2})}(q^j) = \left[\mathcal{F}^{(j)}\right]^t \mathcal{F}^{(j)}\mathcal{E}^{(j)}\mathcal{H}^{(j)}\mathcal{F}^{(j)} = \left[\mathcal{F}^{(j)}\right]^t \mathcal{H}^{(j)}\mathcal{F}^{(j)} = \mathcal{H}_1^{(j)}\ \mathcal{E}^{(j)}\left[\mathcal{E}^{(j)}\right]^t\ , \tag{4.2.4}$$

where we used $\mathcal{F}^{(j+\frac{1}{2})}\mathcal{E}^{(j+\frac{1}{2})} = \mathbb{I}_{2j+2}$ and (4.2.1). The second relation in (4.2.2) is obtained from (4.2.3), using (4.2.1). □

Similarly for $\bar{\mathcal{E}}^{(j)}$, $\bar{\mathcal{F}}^{(j)}$ and $\bar{\mathcal{H}}^{(j)}$ given in Section C.2, one has:

Lemma 4.2.2. *The following relations hold:*

$$\bar{\mathcal{F}}_{12}^{(j)}\bar{\mathcal{H}}_1^{(j)}\bar{\mathcal{E}}^{(j)} = [\bar{\mathcal{F}}^{(j)}]^t\bar{\mathcal{H}}^{(j)}\ , \qquad \bar{\mathcal{F}}_{12}^{(j-1)}\bar{\mathcal{H}}_1^{(j-1)}\bar{\mathcal{E}}^{(j-1)}[\bar{\mathcal{E}}^{(j-1)}]^t = R^{(\frac{1}{2},j-\frac{1}{2})}(q^{-j})\ , \tag{4.2.5}$$

$$[\bar{\mathcal{F}}^{(j-1)}]^t \bar{\mathcal{F}}^{(j-1)} R^{(\frac{1}{2},j-\frac{1}{2})}(q^{-j}) = \bar{\mathcal{F}}_{12}^{(j-1)} \bar{\mathcal{H}}_1^{(j-1)} \ \bar{\mathcal{E}}^{(j-1)} \bar{\mathcal{F}}^{(j-1)} \ . \tag{4.2.6}$$

Proof. The first equation in (4.2.5) is proven directly. The other relations are obtained as in Lemma 4.2.1. □

The following Lemmas will be used for the proof of the universal TT-relations.

Lemma 4.2.3. *Assume the following from Conjecture 1* holds: the equation (3.2.29) and $\nu^{(j)}(u)$ given by (3.6.2) is invertible, then we have

$$\begin{aligned}
&\bar{\mathcal{F}}_{\langle 12\rangle}^{(j-\frac{1}{2})} K_2^{+(j)}(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},j)}(u^{-2}q^{-j-\frac{3}{2}}) K_1^{+(\frac{1}{2})}(uq^j) \bar{\mathcal{E}}_{\langle 12\rangle}^{(j-\frac{1}{2})} \\
&= \frac{f^{(j-\frac{1}{2})}(uq^{-1})}{f^{(j)}(uq^{-\frac{1}{2}})} \left(\prod_{k=0}^{2j-2} c(u^2 q^{2j-k}) c(u^2 q^{2j-2-k}) \right) \Gamma_+(uq^{j-1}) K^{+(j-\frac{1}{2})}(uq^{-1}) \ ,
\end{aligned} \tag{4.2.7}$$

where $f^{(j)}(u)$ is given in (1.3.25), $\Gamma(u)$ and $\Gamma_+(u)$ are defined in (2.1.43) and (4.1.5), respectively.

Proof. Recall that the relation (3.6.13) from Proposition 3.6.7 holds provided Conjecture 1 is true. Now, applying $(\epsilon \otimes \mathrm{id})$ to the latter equation using (3.5.19) and (4.1.7), one gets

$$\begin{aligned}
&\bar{\mathcal{F}}_{\langle 12\rangle}^{(j-\frac{1}{2})} K_2^{(j)}(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},j)}(u^2 q^{-j-\frac{3}{2}}) K_1^{(\frac{1}{2})}(uq^{-j-1}) \bar{\mathcal{E}}_{\langle 12\rangle}^{(j-\frac{1}{2})} \\
&= \left(\prod_{k=0}^{2j-2} c(u^2 q^{-2j+2+k}) c(u^2 q^{-2j+k}) \right) \Gamma_-(uq^{-j-1}) K^{(j-\frac{1}{2})}(u) \ ,
\end{aligned} \tag{4.2.8}$$

where we used that $\varsigma^{(j)}(uq^{-\frac{1}{2}}) = \varsigma^{(j-\frac{1}{2})}(u)\varsigma^{(\frac{1}{2})}(uq^{-j})$ due to (3.5.20). Then, taking the transpose of (4.2.8) and replacing $u \to -u^{-1}$, $\varepsilon_\pm \to -\bar{\varepsilon}_\mp$, $k_\pm \to -\bar{k}_\mp$, from (4.1.2) and (4.1.6) one has

$$\begin{aligned}
&[\bar{\mathcal{E}}_{\langle 12\rangle}^{(j-\frac{1}{2})}]^t K_1^{+(\frac{1}{2})}(uq^j) R^{(\frac{1}{2},j)}(u^{-2}q^{-j-\frac{3}{2}}) K_2^{+(j)}(uq^{-\frac{1}{2}}) [\bar{\mathcal{F}}_{\langle 12\rangle}^{(j-\frac{1}{2})}]^t \\
&= \frac{f^{(j-\frac{1}{2})}(uq^{-1})}{f^{(j)}(uq^{-\frac{1}{2}})} \left(\prod_{k=0}^{2j-2} c(u^2 q^{2j-k}) c(u^2 q^{2j-2-k}) \right) \Gamma_+(uq^{j-1}) K^{+(j-\frac{1}{2})}(uq^{-1}) \ .
\end{aligned} \tag{4.2.9}$$

Finally, using (4.2.5), Corollary C.3.4 and the dual reflection equation, the l.h.s. of (4.2.9) becomes:

$$\begin{aligned}
&\underline{\bar{\mathcal{H}}^{(j-\frac{1}{2})} \bar{\mathcal{F}}^{(j-\frac{1}{2})}} K_1^{+(\frac{1}{2})}(uq^j) R^{(\frac{1}{2},j)}(u^{-2}q^{-j-\frac{3}{2}}) K_2^{+(j)}(uq^{-\frac{1}{2}}) \bar{\mathcal{E}}^{(j-\frac{1}{2})} [\bar{\mathcal{H}}^{(j-\frac{1}{2})}]^{-1} \\
&= \bar{\mathcal{F}}^{(j-\frac{1}{2})} \underline{R^{(\frac{1}{2},j)}(q^{-j-\frac{1}{2}}) K_1^{+(\frac{1}{2})}(uq^j) R^{(\frac{1}{2},j)}(u^{-2}q^{-j-\frac{3}{2}}) K_2^{+(j)}(uq^{-\frac{1}{2}}) \bar{\mathcal{E}}^{(j-\frac{1}{2})}} [\bar{\mathcal{H}}^{(j-\frac{1}{2})}]^{-1} \\
&= \bar{\mathcal{F}}^{(j-\frac{1}{2})} K_2^{+(j)}(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},j)}(u^{-2}q^{-j-\frac{3}{2}}) K_1^{+(\frac{1}{2})}(uq^j) \underline{R^{(\frac{1}{2},j)}(q^{-j-\frac{1}{2}}) \bar{\mathcal{E}}^{(j-\frac{1}{2})}} [\bar{\mathcal{H}}^{(j-\frac{1}{2})}]^{-1} \ ,
\end{aligned}$$

and using again Corollary C.3.4, the equation (4.2.7) follows. □

Lemma 4.2.4. *The following relation holds:*

$$\begin{aligned}
&K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1}) K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}}) \bar{\mathcal{E}}_{\langle 12\rangle}^{(j-1)} \\
&= \bar{\mathcal{E}}_{\langle 12\rangle}^{(j-1)} \bar{\mathcal{F}}_{\langle 12\rangle}^{(j-1)} K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}}) R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1}) K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}}) \bar{\mathcal{E}}_{\langle 12\rangle}^{(j-1)} \ .
\end{aligned} \tag{4.2.10}$$

Proof. Inserting $\bar{\mathcal{H}}^{(j-1)}[\bar{\mathcal{H}}^{(j-1)}]^{-1} = \mathbb{I}_{2j-1}$ in the l.h.s. of (4.2.10) and using Corollary C.3.4, one gets:

$$\begin{aligned} &K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})\bar{\mathcal{E}}^{(j-1)}_{\langle 12\rangle}\bar{\mathcal{H}}^{(j-1)}[\bar{\mathcal{H}}^{(j-1)}]^{-1} \\ &= K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(q^{-j})\bar{\mathcal{E}}^{(j-1)}_{\langle 12\rangle}[\bar{\mathcal{H}}^{(j-1)}]^{-1}\ . \end{aligned} \tag{4.2.11}$$

Then, using the dual reflection equation (4.1.1) and (C.3.12), one finds that (4.2.11) equals

$$\bar{\mathcal{E}}^{(j-1)}_{\langle 12\rangle}\bar{\mathcal{F}}^{(j-1)}_{\langle 12\rangle}R^{(\frac{1}{2},j-\frac{1}{2})}(q^{-j})K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\bar{\mathcal{E}}^{(j-1)}_{\langle 12\rangle}[\bar{\mathcal{H}}^{(j-1)}]^{-1}.$$

Finally, using again (4.1.1) and Corollary C.3.4, the equation (4.2.10) follows. □

We now show the universal TT-relations for any spin-j. These relations are new, and we call them 'universal' because they belong in the algebra $\mathcal{A}_q$ and not in some representations of an algebra. As it will be shown in Section 4.4, they generalize the conjectural TT-relations already studied in the literature. Indeed, the latter are recovered by taking finite-dimensional representations for $\mathcal{A}_q$. Recall the expressions for the quantum determinants $\Gamma(u)$ and $\Gamma_+(u)$ given in (2.1.43), (4.1.5).

Theorem 4.2.5. *Assume Conjecture 1.* The following relation holds for all $j \in \frac{1}{2}\mathbb{N}_{>1}$:

$$\mathsf{T}^{(j)}(u) = \mathsf{T}^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\mathsf{T}^{(\frac{1}{2})}(uq^{j-\frac{1}{2}}) + \frac{\Gamma(uq^{j-\frac{3}{2}})\Gamma_+(uq^{j-\frac{3}{2}})}{c(u^2q^{2j})c(u^2q^{2j-2})}\mathsf{T}^{(j-1)}(uq^{-1}) \tag{4.2.12}$$

with $\mathsf{T}^{(0)}(u) = 1$.

Proof. Assuming Conjecture 1, another expression for the fused K-operators that agrees with Definition 3.5.1 is given by Proposition 3.6.9

$$\mathcal{K}^{(j)}(u) = \mathcal{F}^{(j)}_{\langle 12\rangle}\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})\mathcal{E}^{(j)}_{\langle 12\rangle}\ . \tag{4.2.13}$$

We underline the step of calculations that use (4.2.1), (4.2.2), the reflection equation or the dual reflection equation. Inserting (4.1.3) and (4.2.13) in (4.1.4), $\mathsf{T}^{(j)}(u)$ reads, up to an overall scalar factor, as follows:

$$\begin{aligned} &\mathrm{tr}\Big(\underline{[\mathcal{E}^{(j)}_{\langle 12\rangle}]^t}K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})[\mathcal{F}^{(j)}_{\langle 12\rangle}]^t \\ &\times \mathcal{F}^{(j)}_{\langle 12\rangle}\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})\underline{\mathcal{E}^{(j)}_{\langle 12\rangle}}\Big) \\ &\overset{(4.2.2)}{=} \frac{1}{\mathcal{H}_1^{(j)}}\mathrm{tr}\Big(K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})[\mathcal{F}^{(j)}_{\langle 12\rangle}]^t \\ &\times \mathcal{F}^{(j)}_{\langle 12\rangle}\underline{\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(q^j)}\Big) \\ &\overset{(3.5.2)}{=} \frac{1}{\mathcal{H}_1^{(j)}}\mathrm{tr}\Big(K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})\underline{[\mathcal{F}^{(j)}_{\langle 12\rangle}]^t} \\ &\times \underline{\mathcal{F}^{(j)}_{\langle 12\rangle}R^{(\frac{1}{2},j-\frac{1}{2})}(q^j)}\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\Big) \end{aligned}$$

$$\overset{(4.2.2)}{=} \mathrm{tr}\Big(K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})$$
$$\times \underline{\mathcal{E}^{(j)}_{\langle 12\rangle}\mathcal{F}^{(j)}_{\langle 12\rangle}}\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\Big)\ .$$

Then, from the second relation of (4.2.1) one has $\mathcal{E}^{(j)}_{\langle 12\rangle}\mathcal{F}^{(j)}_{\langle 12\rangle} = \mathbb{I}_{4j} - \bar{\mathcal{E}}^{(j-1)}_{\langle 12\rangle}\bar{\mathcal{F}}^{(j-1)}_{\langle 12\rangle}$, and taking into account the overall factor, one gets:

$$\mathsf{T}^{(j)}(u) = (a) + (b)\ , \tag{4.2.14}$$

where

$$(a) = \frac{f^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})}{f^{(j)}(u)}\mathrm{tr}\Big(K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})$$
$$\times K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\Big)\ ,$$
$$(b) = -\frac{f^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})}{f^{(j)}(u)}\mathrm{tr}\Big(K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})$$
$$\times \bar{\mathcal{E}}^{(j-1)}_{\langle 12\rangle}\bar{\mathcal{F}}^{(j-1)}_{\langle 12\rangle}\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\Big)\ .$$

Now, we rewrite the first term as follows:

$$(a) = \frac{f^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})}{f^{(j)}(u)}\mathrm{tr}\Big(\Big[K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\big[R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})\big]^{t_1}\Big]^{t_1}$$
$$\times\Big[\big[\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\big]^{t_1}\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\Big]^{t_1}\Big)$$
$$= \frac{f^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})}{f^{(j)}(u)}\mathrm{tr}\Big(\Big[K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\big[K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})\big]^{t_1}\big[R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})\big]^{t_1}\Big]^{t_1}$$
$$\times\Big[\big[R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\big]^{t_1}\big[\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})\big]^{t_1}\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\Big]^{t_1}\Big)\ .$$

Then, apply t_1 inside the trace and use the cyclicity of the trace to get:

$$(a) = \frac{f^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})}{f^{(j)}(u)}\mathrm{tr}\Big(\big[R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})\big]^{t_1}\big[R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\big]^{t_1}\big[\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})\big]^{t_1}$$
$$\times\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\big[K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})\big]^{t_1}\Big)\ ,$$

and using the crossing symmetry (4.1.10) we obtain

$$(a) = \frac{f^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})}{f^{(j)}(u)}\xi^{(j-\frac{1}{2})}(u^{-2}q^{-j-1})\mathsf{T}^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\mathsf{T}^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})$$
$$= \mathsf{T}^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\mathsf{T}^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})\ . \tag{4.2.15}$$

The latter equality is obtained using the expressions of $\xi^{(j)}(u)$, $f^{(j)}(u)$ and $c(u)$ given in (4.1.11), (1.3.25), (1.3.33), respectively. We now compute the second term in (4.2.14). Using (4.2.10) and the cyclicity of the trace, it reads

$$(b) = -\frac{f^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})}{f^{(j)}(u)}\mathrm{tr}\Big(\bar{\mathcal{F}}^{(j-1)}_{\langle 12\rangle}K_2^{+(j-\frac{1}{2})}(uq^{-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^{-2}q^{-j-1})K_1^{+(\frac{1}{2})}(uq^{j-\frac{1}{2}})$$
$$\times\,\bar{\mathcal{E}}^{(j-1)}_{\langle 12\rangle}\bar{\mathcal{F}}^{(j-1)}_{\langle 12\rangle}\mathcal{K}_1^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{j-1})\mathcal{K}_2^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\bar{\mathcal{E}}^{(j-1)}_{\langle 12\rangle}\Big)\ .$$

Finally, using the relations (3.6.12) and (4.2.7) we get

$$(b) = -\frac{f^{(j-1)}(uq^{-1})}{f^{(j)}(u)}\left(\prod_{k=0}^{2j-3}\prod_{\ell=1}^{4}c(u^2q^{2j-k-\ell})\right)\Gamma(uq^{j-\frac{3}{2}})\Gamma_+(uq^{j-\frac{3}{2}})\mathsf{T}^{(j-1)}(uq^{-1})$$
$$= \frac{\Gamma(uq^{j-\frac{3}{2}})\Gamma_+(uq^{j-\frac{3}{2}})}{c(u^2q^{2j})c(u^2q^{2j-2})}\mathsf{T}^{(j-1)}(uq^{-1})\ . \tag{4.2.16}$$

Therefore, inserting (4.2.15) and (4.2.16) in (4.2.14), the claim follows. □

Let us now write the generating function $\mathsf{T}^{(j)}(u)$ for some values of j. Recall that $\mathsf{T}^{(\frac{1}{2})}(u)$ is expressed in terms of elements of $\mathcal{A}_q$ denoted $\mathcal{I}_{2k+1}$, that form the commutative subalgebra $\mathcal{I}$, see eq. (2.1.56). The generating functions $\mathsf{T}^{(j)}(u)$ for $j = 1, \frac{3}{2}$, follow from the TT-relation (4.2.12). In terms of $\mathcal{I}(u)$ from (2.1.55), they read:

$$\mathsf{T}^{(1)}(u) = c(u^2q)c(u^2q^3)\left(\mathcal{I}(uq^{\frac{1}{2}})+\mathcal{I}_0\right)\left(\mathcal{I}(uq^{-\frac{1}{2}})+\mathcal{I}_0\right)+f_0^{(1)}(u)\ ,$$
$$\mathsf{T}^{(\frac{3}{2})}(u) = c(u^2q^4)c(u^2q^2)c(u^2)\Big(\mathcal{I}(uq)+\mathcal{I}_0\Big)\Big(\mathcal{I}(u)+\mathcal{I}_0\Big)\Big(\mathcal{I}(uq^{-1})+\mathcal{I}_0\Big)$$
$$+\,c(u^2q^4)f_0^{(1)}(uq^{-\frac{1}{2}})\Big(\mathcal{I}(uq)+\mathcal{I}_0\Big)+c(u^2)f_0^{(\frac{3}{2})}(u)\Big(\mathcal{I}(uq^{-1})+\mathcal{I}_0\Big)\ ,$$

where $c(u)$ is given in (1.3.33) and

$$f_0^{(1)}(u) = \frac{\Gamma(uq^{-\frac{1}{2}})\Gamma_+(uq^{-\frac{1}{2}})}{c(u^2q^2)c(u^2)}\ ,\qquad f_0^{(\frac{3}{2})}(u) = \frac{\Gamma(u)\Gamma_+(u)}{c(u^2q^3)c(u^2q)}\ . \tag{4.2.17}$$

More generally, the universal TT-relations (4.2.12) imply that $\mathsf{T}^{(j)}(u)$ is a polynomial of order $2j$ in the generating functions $\mathcal{I}(u)$ with shifted arguments, and coefficients that are central in $\mathcal{A}_q$. Furthermore, from the universal TT-relations, it follows that all $\mathsf{T}^{(j)}(u)$ (or rather their modes in the expansion in u) belong to the same commutative subalgebra $\mathcal{I}$ generated by the modes $\mathcal{I}_{2k+1}$ of $\mathsf{T}^{(\frac{1}{2})}(u)$.

Recall that the spin-$\frac{1}{2}$ generating function for $\mathcal{A}_q$ forms a commutative family [Sk88]

$$\big[\mathsf{T}^{(\frac{1}{2})}(u),\mathsf{T}^{(\frac{1}{2})}(v)\big] = 0\ ,\quad \text{for all } u,\,v\in\mathbb{C}^*$$

More generally, from Theorem 4.2.5 we have:

Corollary 4.2.6. *For all $u, v \in \mathbb{C}^*$ and $j, j' \in \frac{1}{2}\mathbb{N}_+$, one has:*

$$\left[\mathsf{T}^{(j)}(u), \mathsf{T}^{(j')}(v)\right] = 0 \ .$$

Remark 4.2.7. *The universal TT-relations* (4.2.12) is invariant under a change of normalization of $\mathcal{K}^{(j)}(u)$. Indeed, let $g^{(j)}(u)$ be a scalar function and $\mathcal{K}^{(j)}(u) \to g^{(j)}(u)\mathcal{K}^{(j)}(u)$. Recall $\mathcal{K}^{(j)}(u)$ is defined via the relation (3.5.1) and because $R^{(\frac{1}{2},j)}(u)$ and $\mathcal{K}^{(j)}(u)$ are invertible, see Remark 3.5.8, it follows that

$$g^{(j)}(u) = g^{(j-\frac{1}{2})}(uq^{\frac{1}{2}})g^{(\frac{1}{2})}(uq^{-j+\frac{1}{2}}) = \prod_{k=0}^{2j-1} g^{(\frac{1}{2})}(uq^{j-\frac{1}{2}-k}) \ . \tag{4.2.18}$$

Now, consider the universal TT-relations (4.2.12) under a change of normalization of $\mathcal{K}^{(j)}(u)$, it is given by

$$\begin{aligned}\mathsf{T}^{(j)}(u) = {} & \frac{g^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})g^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})}{g^{(j)}(u)}\mathsf{T}^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\mathsf{T}^{(\frac{1}{2})}(uq^{j-\frac{1}{2}}) \\ & + \frac{g^{(\frac{1}{2})}(uq^{j-\frac{3}{2}})g^{(\frac{1}{2})}(uq^{j-\frac{1}{2}})g^{(j-1)}(uq^{-1})}{g^{(j)}(u)}\frac{\Gamma(uq^{j-\frac{3}{2}})\Gamma_+(uq^{j-\frac{3}{2}})}{c(u^2q^{2j})c(u^2q^{2j-2})}\mathsf{T}^{(j-1)}(uq^{-1}) \ ,\end{aligned} \tag{4.2.19}$$

and using (4.2.18), the latter equation simplifies to (4.2.12).

4.2.2 Proof of the universal TT-relations using a PBW basis for $\mathcal{A}_q$

One may wonder whether the universal-TT relations can be proven for the fused K-operators constructed by fusion (3.5.1), without any additional assumptions. Using a PBW basis for $\mathcal{A}_q$ from Section 2.1.2, and the explicit expression of the fused K-operator, we show that (4.2.12) indeed holds for $j = 1, \frac{3}{2}$.

To check the universal TT-relations for $j = 1$, recall that $\mathsf{T}^{(\frac{1}{2})}(u)$ is expressed in (2.1.55) in terms of the generating functions $\mathcal{W}_\pm(u), \mathcal{G}_\pm(u)$ defined in (2.1.1), (2.1.2). Similarly, $\mathsf{T}^{(1)}(u)$ is computed using the spin-1 K-operator for $\mathcal{A}_q$ given in (3.5.7) and the fused dual K-matrix (4.1.3) for $j = 1$. Then, one finds that the difference

$$\mathsf{T}^{(1)}(u) - \mathsf{T}^{(\frac{1}{2})}(uq^{-\frac{1}{2}})\mathsf{T}^{(\frac{1}{2})}(uq^{\frac{1}{2}})$$

reads as a quadratic combination of the generating functions $\mathcal{W}_\pm(u), \mathcal{G}_\pm(u)$. Using the ordering relations of Lemma 2.1.5, this difference simplifies to

$$\Gamma(uq^{-\frac{1}{2}})\frac{\Gamma_+(uq^{-\frac{1}{2}})}{c(u^2q^2)c(u^2)} \ .$$

The universal TT-relation for $j = \frac{3}{2}$ is checked along the same line. The expressions being lengthy, a symbolic Mathematica program is used to simplify all expressions.

In summary, the results obtained up to here are the following:

- Firstly, we introduced in (4.1.2) spin-j dual K-matrices $K^{+(j)}(u)$ out of fused K-matrices, and found that they satisfy fusion formula (4.1.3). Then, from the latter and the fused K-operators $\mathcal{K}^{(j)}(u)$, we defined in (4.1.4) spin-j generating functions $\mathsf{T}^{(j)}(u)$.

- Secondly, in the framework of the universal K-matrix introduced in the previous chapter, we showed in Theorem 4.2.5 that $\mathsf{T}^{(j)}(u)$ satisfy universal TT-relations provided Conjecture 1 holds. More precisely, we only need part of this conjecture, namely: that $\mathcal{K}^{(j)}(u)$ and the 1-component evaluated universal K-matrix $\mathbf{K}^{(j)}(u)$ are proportional to an invertible central element $\nu^{(j)}(u) \in \mathcal{A}_q((u^{-1}))$, see (3.6.1) and (3.6.2). From the universal TT-relations, we found that all $\mathsf{T}^{(j)}(u)$ belong to $\mathcal{I}$, the commutative subalgebra of $\mathcal{A}_q$ generated by $\{\mathcal{I}_{2k+1} | k \in \mathbb{N}\}$, recall Section 2.1.4.

- Thirdly, independently of this universal framework and without any additional assumption, we also showed that $\mathsf{T}^{(j)}(u)$ satisfy universal TT-relations for $j = 1, \frac{3}{2}$. It was done using a PBW basis for $\mathcal{A}_q$ as well as the explicit expressions of $\mathcal{K}^{(j)}(u)$.

4.3 Quotients of $\mathcal{A}_q$

The rest of this chapter is dedicated to applications of the results described earlier for $\mathcal{A}_q$, within the context of quantum integrable spin-chains. Recall that the goal of this universal approach is to unify open spin-chains, and it goes as follows. In this section, we begin by recalling spin-chain representations of $\mathcal{A}_q$, which in turn inspire us to introduce in Definition 4.3.1 certain quotients of $\mathcal{A}_q$, denoted as $\mathcal{A}_q^{(N)}$. We also describe the quotient map $\varphi^{(N)} \colon \mathcal{A}_q \to \mathcal{A}_q^{(N)}$ and give a FRT type presentation for $\mathcal{A}_q^{(N)}$ in (4.3.20)–(4.3.27) in terms of the truncated $\mathcal{K}^{(\frac{1}{2},N)}(u) \sim (\varphi^{(N)} \otimes \mathrm{id})(\mathcal{K}^{(\frac{1}{2})}(u))$. Compared to the K-operator $\mathcal{K}^{(\frac{1}{2})}(u) \in \mathcal{A}_q((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^2)$, this truncated K-operator $\mathcal{K}^{(\frac{1}{2},N)}(u)$ has a simpler structure because $\mathcal{A}_q^{(N)}$ has finitely-many generators, and it belongs to $\mathcal{A}_q^{(N)}[u, u^{-1}] \otimes \mathrm{End}(\mathbb{C}^2)$. Note that it takes the form of a Laurent polynomial in u because the generating functions $\{\mathcal{W}_\pm(u), \mathcal{G}_\pm(u)\}$ truncate under the quotient map, and we also use an appropriate normalization factor in (4.3.20). In the next section, we use $\varphi^{(N)}$ to derive universal TT-relations satisfied by spin-j versions of $\mathsf{T}^{(\frac{1}{2},N)}(u) \in \mathcal{A}_q^{(N)}[u, u^{-1}]$. Then, from these relations and using the spin-chain representations, we recover in Section 4.4 the transfer matrices of various spin-chains along with the TT-relations they satisfy.

4.3.1 Spin-chain representations of $\mathcal{A}_q$

We now recall spin-chain representations of $\mathcal{A}_q$, that originally appeared in [BK05b] for the XXZ open spin-1/2 chain with generic boundary conditions. Let us introduce the following shorthand notation that corresponds to the spin-j irreducible representation of $U_q sl_2$, recall its definition in (3.2.2)

$$\pi^j(E) = S_+ \ , \quad \pi^j(F) = S_- \ , \quad \pi^j(K^{\pm\frac{1}{2}}) = q^{\pm\frac{S_3}{2}} \ , \tag{4.3.1}$$

with

$$(S_+)_{mn} = B_{j,j+1-m}\delta_{m,n-1} \ , \quad (S_-)_{mn} = B_{j,j+1-n}\delta_{m-1,n} \ , \quad (S_3)_{mn} = 2(j+1-n)\delta_{m,n} \ ,$$

for $m,n=1,2,...,2j+1$ and where $B_{j,m}$ is given in (3.2.3).

We now review the spin-chain representations of $\mathcal{A}_q$ introduced in [BK05b], with the corresponding representation map

$$\vartheta^{(N)}\colon \mathcal{A}_q \to \mathrm{End}(\mathbb{C}^{2j_N+1}\otimes\ldots\otimes\mathbb{C}^{2j_1+1})\ , \tag{4.3.2}$$

where $N\in\mathbb{N}_+$, and j_n denotes the $2j_n+1$ dimensional irreducible representation of $U_q sl_2$, with $n=1,2,\ldots,N$. Each space is indexed by a non-zero complex parameter v_n corresponding to the inhomogeneities of the quantum spin-chains and can be also identified with the evaluation parameters of $\mathcal{L}U_q sl_2$.

Recall the FRT-type presentation of $\mathcal{A}_q$ is given by the K-operator $\mathcal{K}^{(\frac{1}{2})}(u)$ in (2.1.5) satisfying the defining relation (2.1.4). Representations for $\mathcal{A}_q$ are thus obtained by taking images of $\mathcal{K}^{(\frac{1}{2})}(u)$ so that they satisfy (2.1.4), and they were actually found in [BK05b] as follows. First, introduce

$$L^{(\frac{1}{2})}_{[k]}(u)=(\pi^{j_k}\otimes\mathrm{id})(\mathcal{L}^{(\frac{1}{2})}(u))\ \in\mathrm{End}(\mathbb{C}^{2j_k+1})\otimes\mathrm{End}(\mathbb{C}^2)\ , \tag{4.3.3}$$

where $\mathcal{L}^{(\frac{1}{2})}(u)$ is given in (1.1.8). Then, the image of $\mathcal{K}^{(\frac{1}{2})}(u)$ is identified with the dressed K-matrix

$$(\vartheta^{(N)}\otimes\mathrm{id})(\mathcal{K}^{(\frac{1}{2})}(u))=f(u)\ L^{(\frac{1}{2})}_{[N]}(uv_N)\cdots L^{(\frac{1}{2})}_{[1]}(uv_1)K^{(\frac{1}{2})}(u)L^{(\frac{1}{2})}_{[1]}(uv_1^{-1})\cdots L^{(\frac{1}{2})}_{[N]}(uv_N^{-1}) \tag{4.3.4}$$

where $f(u)$ is a Laurent polynomial in u that will be determined later, with $v_n\in\mathbb{C}^*$ and $K^{(\frac{1}{2})}(u)$ is the K-matrix from (1.2.3). By construction it satisfies the reflection equation (2.1.4) with the substitution $\mathcal{K}^{(\frac{1}{2})}(u)\to(\vartheta^{(N)}\otimes\mathrm{id})(\mathcal{K}^{(\frac{1}{2})}(u))$, recall Section 1.2.2. We thus have a representation for $\mathcal{A}_q$ due to its FRT-type presentation.

Note that (4.3.4) can be seen as a 2×2 matrix with entries in $\mathrm{End}(\mathbb{C}^{2j_N+1}\otimes\ldots\otimes\mathbb{C}^{2j_1+1})$. Explicitly, for $k=0,1,\ldots,N-1$, the images of the alternating generators of $\mathcal{A}_q$ on N sites are given recursively by [BK05b]

$$\begin{aligned}
\vartheta^{(N)}(\mathsf{W}_{-k}) &= \frac{(w_0^{(j_N)}\mathbb{I}_{2j_N+1}-(q+q^{-1})q^{S_3})}{(q+q^{-1})}\otimes\vartheta^{(N-1)}(\mathsf{W}_k)-\frac{v_N^2+v_N^{-2}}{(q+q^{-1})}\mathbb{I}_{2j_N+1}\otimes\vartheta^{(N-1)}(\mathsf{W}_{-k+1})\\
&+\frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\vartheta^{(N)}(\mathsf{W}_{-k+1})+\ q^{S_3}\otimes\vartheta^{(N-1)}(\mathsf{W}_{-k})\\
&+\frac{(q-q^{-1})}{k_+k_-(q+q^{-1})^2}\Big(k_+v_Nq^{1/2}S_+q^{S_3/2}\otimes\vartheta^{(N-1)}(\mathsf{G}_k)+k_-v_N^{-1}q^{-1/2}S_-q^{S_3/2}\otimes\vartheta^{(N-1)}(\tilde{\mathsf{G}}_k)\Big)\ ,
\end{aligned} \tag{4.3.5}$$

$$\begin{aligned}
\vartheta^{(N)}(\mathsf{W}_{k+1}) &= \frac{(w_0^{(j_N)}\mathbb{I}_{2j_N+1}-(q+q^{-1})q^{-S_3})}{(q+q^{-1})}\otimes\vartheta^{(N-1)}(\mathsf{W}_{-k+1})-\frac{v_N^2+v_N^{-2}}{(q+q^{-1})}\mathbb{I}_{2j_N+1}\otimes\vartheta^{(N-1)}(\mathsf{W}_k)\\
&+\ q^{-S_3}\otimes\vartheta^{(N-1)}(\mathsf{W}_{k+1})+\frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\vartheta^{(N)}(\mathsf{W}_k)\\
&+\frac{(q-q^{-1})}{k_+k_-(q+q^{-1})^2}\Big(k_+v_N^{-1}q^{-1/2}S_+q^{-S_3/2}\otimes\vartheta^{(N-1)}(\mathsf{G}_k)+k_-v_Nq^{1/2}S_-q^{-S_3/2}\otimes\vartheta^{(N-1)}(\tilde{\mathsf{G}}_k)\Big)\ ,
\end{aligned} \tag{4.3.6}$$

$$\begin{aligned}\vartheta^{(N)}(\mathsf{G}_{k+1}) &= \frac{k_-(q-q^{-1})^2}{k_+(q+q^{-1})}S_-^2\otimes\vartheta^{(N-1)}(\tilde{\mathsf{G}}_k) - \frac{v_N^2q^{S_3}+v_N^{-2}q^{-S_3}}{(q+q^{-1})}\otimes\vartheta^{(N-1)}(\mathsf{G}_k) + \mathbb{I}_{2j_N+1}\otimes\vartheta^{(N-1)}(\mathsf{G}_{k+1})\\ &+(q^2-q^{-2})\Big(k_-v_Nq^{-1/2}S_-q^{S_3/2}\otimes\big(\vartheta^{(N-1)}(\mathsf{W}_{-k})-\vartheta^{(N-1)}(\mathsf{W}_k)\big)\\ &\qquad +k_-v_N^{-1}q^{1/2}S_-q^{-S_3/2}\otimes\big(\vartheta^{(N-1)}(\mathsf{W}_{k+1})-\vartheta^{(N-1)}(\mathsf{W}_{-k+1})\big)\Big)\\ &+\frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\vartheta^{(N)}(\mathsf{G}_k)\ ,\end{aligned}\tag{4.3.7}$$

$$\begin{aligned}\vartheta^{(N)}(\tilde{\mathsf{G}}_{k+1}) &= \frac{k_+(q-q^{-1})^2}{k_-(q+q^{-1})}S_+^2\otimes\vartheta^{(N-1)}(\mathsf{G}_k) - \frac{v_N^2q^{-S_3}+v_N^{-2}q^{S_3}}{(q+q^{-1})}\otimes\vartheta^{(N-1)}(\tilde{\mathsf{G}}_k) + \mathbb{I}_{2j_N+1}\otimes\vartheta^{(N-1)}(\tilde{\mathsf{G}}_{k+1})\\ &+(q^2-q^{-2})\Big(k_+v_N^{-1}q^{1/2}S_+q^{S_3/2}\otimes\big(\vartheta^{(N-1)}(\mathsf{W}_{-k})-\vartheta^{(N-1)}(\mathsf{W}_k)\big)\\ &\qquad +k_+v_Nq^{-1/2}S_+q^{-S_3/2}\otimes\big(\vartheta^{(N-1)}(\mathsf{W}_{k+1})-\vartheta^{(N-1)}(\mathsf{W}_{-k+1})\big)\Big)\\ &+\frac{(v_N^2+v_N^{-2})w_0^{(j_N)}}{(q+q^{-1})^2}\vartheta^{(N)}(\tilde{\mathsf{G}}_k)\ ,\end{aligned}\tag{4.3.8}$$

where

$$w_0^{(j_n)} = q^{2j_n+1}+q^{-2j_n-1}\ .$$

For the special case $k=0$, one has[23]

$$\vartheta^{(N)}(\mathsf{W}_k|_{k=0}) = 0\ ,\qquad \vartheta^{(N)}(\mathsf{W}_{-k+1}|_{k=0}) = 0\ ,$$
$$\vartheta^{(N)}(\mathsf{G}_k|_{k=0}) = \vartheta^{(N)}(\tilde{\mathsf{G}}_k|_{k=0}) = \frac{k_+k_-(q+q^{-1})^2}{q-q^{-1}}\,\mathbb{I}_{(2j_N+1)\times\ldots\times(2j_1+1)}\ ,$$

and also the initial conditions

$$\vartheta^{(0)}(\mathsf{W}_0)\equiv\varepsilon_+^{(0)},\qquad \vartheta^{(0)}(\mathsf{W}_1)\equiv\varepsilon_-^{(0)},\qquad \vartheta^{(0)}(\mathsf{G}_1)=\vartheta^{(0)}(\tilde{\mathsf{G}}_1)\equiv\varepsilon_+^{(0)}\varepsilon_-^{(0)}(q-q^{-1})\ .$$

Here $\varepsilon_\pm^{(0)}$ are identified with the boundary parameters $\varepsilon_\pm$ from the K-matrix (1.2.3). Note that we do not need the image of all generators of $\mathcal{A}_q$ because the image of the generating functions $\mathcal{W}_\pm(u),\mathcal{G}_\pm(u)$ truncate, as we now show.

The images of $\mathcal{A}_q$ on spin-chain representations satisfy the following linear relations [BK05b]

$$\sum_{k=0}^{N}d_k^{(N)}\vartheta^{(N)}(\mathsf{W}_{-k-\ell})+\overline{\ell+1}\,\varepsilon_+^{(N)}+\overline{\ell}\,\varepsilon_-^{(N)}=0\ ,\qquad \sum_{k=0}^{N}d_k^{(N)}\vartheta^{(N)}(\mathsf{G}_{k+1+\ell})=0\ ,\tag{4.3.9}$$

$$\sum_{k=0}^{N}d_k^{(N)}\vartheta^{(N)}(\mathsf{W}_{k+1+\ell})+\overline{\ell+1}\,\varepsilon_-^{(N)}+\overline{\ell}\,\varepsilon_+^{(N)}=0\ ,\qquad \sum_{k=0}^{N}d_k^{(N)}\vartheta^{(N)}(\tilde{\mathsf{G}}_{k+1+\ell})=0\ ,\tag{4.3.10}$$

[23] We use the formal convention $\vartheta^{(N)}(\mathsf{W}_k|_{k=0})\neq\vartheta^{(N)}(\mathsf{W}_{-k}|_{k=0})$ and $\vartheta^{(N)}(\mathsf{W}_{-k+1}|_{k=0})\neq\vartheta^{(N)}(\mathsf{W}_{k+1}|_{k=0})$ for any N.

for any $\ell \in \mathbb{N}$, with $\overline{\ell} = \ell \mod 2$ and

$$\begin{aligned} d_n^{(N)} &= (-1)^{N+1-n}(q+q^{-1})^n \sum_{k_1<\ldots<k_{N-n}=1}^{N} \alpha_{k_1}\cdots\alpha_{k_{N-n}}, \quad n=0,1,...,N\ , \\ \varepsilon_\pm^{(N)} &= w_0^{(j_N)}\varepsilon_\mp^{(N-1)} - (v_N^2+v_N^{-2})\varepsilon_\pm^{(N-1)}\ , \end{aligned} \tag{4.3.11}$$

and where

$$\alpha_1 = \frac{(v_1^2+v_1^{-2})w_0^{(j_1)}}{(q+q^{-1})} + \frac{\varepsilon_+^{(0)}\varepsilon_-^{(0)}(q-q^{-1})^2}{k_+k_-(q+q^{-1})}\ , \qquad \alpha_n = \frac{(v_n^2+v_n^{-2})w_0^{(j_n)}}{(q+q^{-1})} \quad \text{for} \quad n=2,3,\ldots,N\ .$$

The case $\ell = 0$ is proved in [BK05b]. The proof for $\ell \geq 1$ is done similarly by induction. Note that the relations (4.3.9), (4.3.10), can be interpreted as q-deformed analogs of Davies' relations [D90, D91] for the alternating central extension of the q-Onsager algebra. For $q = 1$, see [BC18, eq. (4.20)].

4.3.2 The quotient algebra $\mathcal{A}_q^{(N)}$

Recall the algebra $\mathcal{A}_q$ has infinitely-many generators $\{\mathsf{W}_{-k}, \mathsf{W}_{k+1}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1} | k \in \mathbb{N}\}$, and the spin-1/2 generating function $\mathsf{T}^{(\frac{1}{2})}(u)$ from (2.1.55) belongs to the commutative subalgebra $\mathcal{I}$. It is generated by $\{\mathcal{I}_{2k+1} | k \in \mathbb{N}\}$ from (2.1.56), and is thus infinite-dimensional. As we will see in the following, these elements are more fundamental than $\mathsf{T}^{(\frac{1}{2})}(u)$ itself. The spin-chain representations of $\mathcal{A}_q$ given by (4.3.5)-(4.3.8) simplify the algebraic structure of $\mathcal{A}_q$ and $\mathcal{I}$. Indeed, in [BK05b] the transfer matrix of the XXZ spin-1/2 chain is expressed in terms of images of $\mathcal{I}_{2k+1}$. In this case, they form a finite set of quantities in involution with $k = 0, 1 \ldots, N-1$, which corresponds to conserved quantities of the system. This motivates us to introduce some quotient algebras of $\mathcal{A}_q$, inspired by the spin-chain representations. These quotients will be used in further sections in order to describe integrable spin-j chains and their symmetries.

Let us now introduce two presentations for the quotients of $\mathcal{A}_q$. The first one amounts to define $\mathcal{A}_q^{(N)}$, with $N \in \mathbb{N}_+$, as the quotient of $\mathcal{A}_q$ by certain relations analogous to (4.3.9)-(4.3.10), while the second presentation takes the form of a reflection equation with a truncated K-operator $\mathcal{K}^{(\frac{1}{2},N)}(u) \in \mathcal{A}_q^{(N)}[u,u^{-1}] \otimes \mathrm{End}(\mathbb{C}^2)$.

$\{\mathcal{A}_q^{(N)}\}_{N\in\mathbb{N}_+}$ as quotients of $\mathcal{A}_q$

Let us introduce the quotient relations that define $\mathcal{A}_q^{(N)}$.

Definition 4.3.1. *Let N be a positive integer. $\mathcal{A}_q^{(N)}$ is the quotient of $\mathcal{A}_q$ by the relations:*

$$\sum_{k=0}^{N} d_k^{(N)} \mathsf{W}_{-k-\ell} + \overline{\ell+1}\,\varepsilon_+^{(N)} + \overline{\ell}\,\varepsilon_-^{(N)} = 0\ , \qquad \sum_{k=0}^{N} d_k^{(N)} \mathsf{G}_{k+1+\ell} = 0\ , \tag{4.3.12}$$

$$\sum_{k=0}^{N} d_k^{(N)} \mathsf{W}_{k+1+\ell} + \overline{\ell+1}\,\varepsilon_-^{(N)} + \overline{\ell}\,\varepsilon_+^{(N)} = 0\ , \qquad \sum_{k=0}^{N} d_k^{(N)} \tilde{\mathsf{G}}_{k+1+\ell} = 0\ , \tag{4.3.13}$$

for any $\ell \in \mathbb{N}$, where $d_k^{(N)}$, $\varepsilon_\pm^{(N)}$ are scalars, and we used the notation $\bar{\ell} = \ell \mod 2$.
Let $\{\mathcal{W}_{-k}^{(N)}, \mathcal{W}_{k+1}^{(N)}, \mathcal{G}_{k+1}^{(N)}, \tilde{\mathcal{G}}_{k+1}^{(N)} | k = 0, 1, ..., N-1\}$ denote the alternating generators of $\mathcal{A}_q^{(N)}$. The surjective homomorphism

$$\varphi^{(N)} \colon \mathcal{A}_q \to \mathcal{A}_q^{(N)} \tag{4.3.14}$$

is such that

$$\mathsf{W}_{-k} \mapsto \mathcal{W}_{-k}^{(N)}, \quad \mathsf{W}_{k+1} \mapsto \mathcal{W}_{k+1}^{(N)}, \quad \mathsf{G}_{k+1} \mapsto \mathcal{G}_{k+1}^{(N)}, \quad \tilde{\mathsf{G}}_{k+1} \mapsto \tilde{\mathcal{G}}_{k+1}^{(N)} \qquad k = 0, 1, ..., N-1. \tag{4.3.15}$$

For $k \geq N$, the image of the other generators of $\mathcal{A}_q$ follows from (4.3.12), (4.3.13).

The relations (4.3.12)-(4.3.13) are inspired by (4.3.9)-(4.3.10) for the spin-chain representations of $\mathcal{A}_q$. In particular, the former relations are automatically satisfied by taking spin-chain representations $\vartheta^{(N)}$ of $\mathcal{A}_q^{(N)}$, with the values for $d_k^{(N)}$, $\varepsilon_\pm^{(N)}$ from (4.3.12)-(4.3.13) fixed by (4.3.11). Therefore $\vartheta^{(N)}$ factors through $\mathcal{A}_q^{(N)}$

$$\begin{array}{ccc} \mathcal{A}_q & \xrightarrow{\varphi^{(N)}} & \mathcal{A}_q^{(N)} \\ & \searrow^{\vartheta^{(N)}} & \downarrow \vartheta \\ & & \mathrm{End}(\mathbb{C}^{2j_N+1} \otimes \ldots \otimes \mathbb{C}^{2j_1+1}) \end{array} ,$$

where the quotient map $\varphi^{(N)}$ is defined by Definition 4.3.1. For simplicity we keep the notation $\vartheta^{(N)}$ when we actually use ϑ.

FRT type presentation

Due to the relations (4.3.12), (4.3.13), in the quotient $\mathcal{A}_q^{(N)}$ the generating functions (2.1.1), (2.1.2) truncate to finite sums. Introduce:

$$\begin{aligned} \mathcal{W}_+^{(N)}(u) &= \sum_{k=0}^{N-1} P_{-k}^{(N)}(u) \mathcal{W}_{-k}^{(N)} , \qquad \mathcal{W}_-^{(N)}(u) = \sum_{k=0}^{N-1} P_{-k}^{(N)}(u) \mathcal{W}_{k+1}^{(N)} , \\ \mathcal{G}_+^{(N)}(u) &= \sum_{k=0}^{N-1} P_{-k}^{(N)}(u) \mathcal{G}_{k+1}^{(N)} , \qquad \mathcal{G}_-^{(N)}(u) = \sum_{k=0}^{N-1} P_{-k}^{(N)}(u) \tilde{\mathcal{G}}_{k+1}^{(N)} , \end{aligned} \tag{4.3.16}$$

where

$$P_{-k}^{(N)}(u) = -\frac{1}{q+q^{-1}} \sum_{n=k}^{N-1} d_{n+1}^{(N)} U^{n-k} . \tag{4.3.17}$$

Recall the map $\varphi^{(N)}$ with (4.3.15).

Lemma 4.3.2. *The action of $\varphi^{(N)}$ on the generators of $\mathcal{A}_q$ is given by*

$$\begin{aligned} h_0^{(N)}(u) \varphi^{(N)}(\mathcal{W}_+(u)) &= \mathcal{W}_+^{(N)}(u) + \frac{u^2 q + u^{-2} q^{-1}}{(u^2 - u^{-2})(u^2 q^2 - u^{-2} q^{-2})} \varepsilon_+^{(N)} + \frac{q + q^{-1}}{(u^2 - u^{-2})(u^2 q^2 - u^{-2} q^{-2})} \varepsilon_-^{(N)} , \\ h_0^{(N)}(u) \varphi^{(N)}(\mathcal{W}_-(u)) &= \mathcal{W}_-^{(N)}(u) + \frac{q + q^{-1}}{(u^2 - u^{-2})(u^2 q^2 - u^{-2} q^{-2})} \varepsilon_+^{(N)} + \frac{u^2 q + u^{-2} q^{-1}}{(u^2 - u^{-2})(u^2 q^2 - u^{-2} q^{-2})} \varepsilon_-^{(N)} , \end{aligned}$$

$$h_0^{(N)}(u)\varphi^{(N)}(\mathcal{G}_\pm(u)) = \mathcal{G}_\pm^{(N)}(u)\ ,$$

where

$$h_0^{(N)}(u) = -\frac{1}{q+q^{-1}}\left(\sum_{k=0}^{N} d_k^{(N)} U^k\right). \tag{4.3.18}$$

Proof. Firstly, consider the image of the generating functions $\mathcal{W}_\pm(u)$, $\mathcal{G}_\pm(u)$ given in (2.1.1), (2.1.2). For instance, from (2.1.1) one gets:

$$\mathcal{W}_+(u) \mapsto \sum_{k\in\mathbb{N}} \mathcal{W}_{-k}^{(N)} U^{-k-1} = \sum_{k=0}^{N-1} \mathcal{W}_{-k}^{(N)} U^{-k-1} + \sum_{k=N}^{\infty} \mathcal{W}_{-k}^{(N)} U^{-k-1}\ , \tag{4.3.19}$$

where $U = (qu^2 + q^{-1}u^{-2})/(q+q^{-1})$. Now, using the linear relation (4.3.12), it follows:

$$\begin{aligned}
\sum_{k=N}^{\infty} \mathcal{W}_{-k}^{(N)} U^{-k-1} &= \sum_{p=0}^{\infty} \mathcal{W}_{-N-p}^{(N)} U^{-N-p-1} \\
&= -\sum_{p=0}^{\infty}\sum_{k=0}^{N-1} \frac{d_k^{(N)}}{d_N^{(N)}} \mathcal{W}_{-k-p}^{(N)} U^{-N-p-1} - \sum_{p=0}^{\infty} \overline{p+1}\, U^{-p-1-N} \frac{\varepsilon_+^{(N)}}{d_N^{(N)}} - \sum_{p=0}^{\infty} \overline{p}\, U^{-p-1-N} \frac{\varepsilon_-^{(N)}}{d_N^{(N)}} \\
&= -\left(\sum_{k=0}^{N-1} \frac{d_k^{(N)}}{d_N^{(N)}} U^{k-N}\right) \mathcal{W}_+(u) + \sum_{k=1}^{N-1}\sum_{p=0}^{k-1} \frac{d_k^{(N)}}{d_N^{(N)}} \mathcal{W}_{-p}^{(N)} U^{-N-p-1+k} - \sum_{n=0}^{\infty} U^{-2n-1-N}\left(\frac{\varepsilon_+^{(N)}}{d_N^{(N)}} + U^{-1}\frac{\varepsilon_-^{(N)}}{d_N^{(N)}}\right) \\
&= -\left(\sum_{k=0}^{N-1} \frac{d_k^{(N)}}{d_N^{(N)}} U^{k-N}\right) \mathcal{W}_+(u) + \sum_{k=0}^{N-1} \mathcal{W}_{-k}^{(N)} \left(\sum_{p=k+1}^{N-1} \frac{d_p^{(N)}}{d_N^{(N)}} U^{p-N-k-1}\right) - \sum_{n=0}^{\infty} U^{-2n-1-N}\left(\frac{\varepsilon_+^{(N)}}{d_N^{(N)}} + U^{-1}\frac{\varepsilon_-^{(N)}}{d_N^{(N)}}\right).
\end{aligned}$$

Inserting the above expression into (4.3.19) and multiplying the result by $d_N^{(N)}$, one gets:

$$\begin{aligned}
\left(\sum_{k=0}^{N} d_k^{(N)} U^k\right) \mathcal{W}_+(u) &= \sum_{k=0}^{N-1} \mathcal{W}_{-k}^{(N)} \left(\sum_{n=k}^{N-1} d_{n+1}^{(N)} U^{n-k}\right) - \sum_{n=0}^{\infty} U^{-2n-1}\left(\varepsilon_+^{(N)} + U^{-1}\varepsilon_-^{(N)}\right) \\
&= \sum_{k=0}^{N-1} \mathcal{W}_{-k}^{(N)} \left(\sum_{n=k}^{N-1} d_{n+1}^{(N)} U^{n-k}\right) \\
&\quad - \frac{(q+q^{-1})(u^2q + u^{-2}q^{-1})}{(u^2-u^{-2})(u^2q^2 - u^{-2}q^{-2})}\varepsilon_+^{(N)} - \frac{(q+q^{-1})^2}{(u^2-u^{-2})(u^2q^2-u^{-2}q^{-2})}\varepsilon_-^{(N)}\ .
\end{aligned}$$

Dividing both sides by $-(q+q^{-1})$, the expression of $\varphi^{(N)}(\mathcal{W}_+(u))$ follows. The action of $\varphi^{(N)}$ on the other generating functions is obtained similarly. □

Let us now introduce the spin-$\frac{1}{2}$ K-operator for $\mathcal{A}_q^{(N)}$, denoted $\mathcal{K}^{(\frac{1}{2},N)}(u)$. Recall the spin-$\frac{1}{2}$ K-operator for $\mathcal{A}_q$ in (2.1.5), then taking its image by $(\varphi^{(N)} \otimes \mathrm{id})$ and using Lemma 4.3.2, we define

$$\mathcal{K}^{(\frac{1}{2},N)}(u) = (u^2-u^{-2}) h_0^{(N)}(u)(\varphi^{(N)} \otimes \mathrm{id})(\mathcal{K}^{(\frac{1}{2})}(u))\ . \tag{4.3.20}$$

Explicitly, it reads:

$$\mathcal{K}^{(\frac{1}{2},N)}(u) = \begin{pmatrix} \mathcal{A}^{(N)}(u) & \mathcal{B}^{(N)}(u) \\ \mathcal{C}^{(N)}(u) & \mathcal{D}^{(N)}(u) \end{pmatrix}, \tag{4.3.21}$$

with

$$\mathcal{A}^{(N)}(u) = u\varepsilon_+^{(N)} + u^{-1}\varepsilon_-^{(N)} + (u^2-u^{-2})\big(uq\mathcal{W}_+^{(N)}(u) - u^{-1}q^{-1}\mathcal{W}_-^{(N)}(u)\big)\ , \tag{4.3.22}$$

$$\mathcal{D}^{(N)}(u) = u\varepsilon_-^{(N)} + u^{-1}\varepsilon_+^{(N)} + (u^2-u^{-2})\big(uq\mathcal{W}_-^{(N)}(u) - u^{-1}q^{-1}\mathcal{W}_+^{(N)}(u)\big)\ , \tag{4.3.23}$$

$$\mathcal{B}^{(N)}(u) = \frac{(u^2-u^{-2})}{k_-}\Big(\frac{k_+k_-(u^2q+u^{-2}q^{-1})}{q-q^{-1}}P_0^{(N)}(u) + \frac{1}{q+q^{-1}}\mathcal{G}_+^{(N)}(u) + \omega_0^{(N)}\Big)\ , \tag{4.3.24}$$

$$\mathcal{C}^{(N)}(u) = \frac{(u^2-u^{-2})}{k_+}\Big(\frac{k_+k_-(u^2q+u^{-2}q^{-1})}{q-q^{-1}}P_0^{(N)}(u) + \frac{1}{q+q^{-1}}\mathcal{G}_-^{(N)}(u) + \omega_0^{(N)}\Big)\ , \tag{4.3.25}$$

where $h_0^{(N)}(u)$ is given in (4.3.18) and

$$\omega_0^{(N)} = -\frac{k_+k_-}{q-q^{-1}}d_0^{(N)}\ . \tag{4.3.26}$$

Note that $\mathcal{K}^{(\frac{1}{2},N)}(u)$ belongs to $\mathcal{A}_q^{(N)}[u,u^{-1}]\otimes \mathrm{End}(\mathbb{C}^2)$ because the truncated generating functions $\mathcal{W}_\pm^{(N)}(u)$, $\mathcal{G}_\pm^{(N)}(u)$ from (4.3.16) have only positive powers in U up to the power N. Using the fact that $\varphi^{(N)}$ is an algebra homomorphism and the definition of the truncated K-operator from (4.3.20), it is clear that $\mathcal{K}^{(\frac{1}{2},N)}(u)$ satisfies the reflection equation

$$R^{(\frac{1}{2},\frac{1}{2})}(u/v)\mathcal{K}_1^{(\frac{1}{2},N)}(u)R^{(\frac{1}{2},\frac{1}{2})}(uv)\mathcal{K}_2^{(\frac{1}{2},N)}(v) = \mathcal{K}_2^{(\frac{1}{2},N)}(v)R^{(\frac{1}{2},\frac{1}{2})}(uv)\mathcal{K}_1^{(\frac{1}{2},N)}(u)R^{(\frac{1}{2},\frac{1}{2})}(u/v). \tag{4.3.27}$$

Finally, we define the spin-1/2 generating function $\mathsf{T}^{(\frac{1}{2},N)}(u)$ for $\mathcal{A}_q^{(N)}$ similarly to $\mathsf{T}^{(\frac{1}{2})}(u)$ in (2.1.54) but with the substitution $\mathcal{K}(u)\to\mathcal{K}^{(\frac{1}{2},N)}(u)$ for the K-operator in (4.3.21). It is computed explicitly:

$$\begin{aligned}\mathsf{T}^{(\frac{1}{2},N)}(u) &= c(u^2)c(u^2q^2)\Big(\mathcal{I}^{(N)}(u) + h_0^{(N)}(u)\mathcal{I}_0\Big) \\ &\quad + \bar{\varepsilon}_+\Big((u^2q+u^{-2}q^{-1})\varepsilon_+^{(N)} + (q+q^{-1})\varepsilon_-^{(N)}\Big) + \bar{\varepsilon}_-\Big((u^2q+u^{-2}q^{-1})\varepsilon_-^{(N)} + (q+q^{-1})\varepsilon_+^{(N)}\Big)\ ,\end{aligned} \tag{4.3.28}$$

where $c(u)$ is given in (1.3.33) and

$$\mathcal{I}^{(N)}(u) = \sum_{k=0}^{N-1} P_{-k}^{(N)}(u)\mathcal{I}_{2k+1}^{(N)} \qquad \text{with} \qquad \mathcal{I}_{2k+1}^{(N)} = \varphi^{(N)}(\mathcal{I}_{2k+1})\ , \tag{4.3.29}$$

with $\mathcal{I}_{2k+1}$ from (2.1.56) and $P_{-k}^{(N)}(u)$ from (4.3.17). We thus notice that the commutative subalgebra in $\mathcal{A}_q^{(N)}$, extracted from $\mathsf{T}^{(\frac{1}{2},N)}(u)$, is generated by finitely-many elements $\{\mathcal{I}_{2k+1}^{(N)} | k = 0,1,\dots,N-1\}$.

4.4 From the generating functions for $\mathcal{A}_q$ to the transfer matrices

Now, recall the transfer matrix for open spin-chains has been reviewed in Section 1.2.3. In this section, we derive spin-j transfer matrices with arbitrary spin at each site, starting from the spin-j generating functions in the commutative subalgebra of $\mathcal{A}_q$ as follows

$$\underset{\in\mathcal{A}_q((u^{-1}))}{\mathsf{T}^{(j)}(u)} \xrightarrow{\varphi^{(N)}} \underset{\in\mathcal{A}_q^{(N)}((u^{-1}))}{\mathsf{T}^{(j,N)}(u)} \xrightarrow{\vartheta^{(N)}} \underset{\in\mathrm{End}(\mathbb{C}^{2j_N+1}\otimes\ldots\otimes\mathbb{C}^{2j_1+1})}{t^{(j,\{j_n\}_N)}(u)} \tag{4.4.1}$$

with $n = 1, 2, \ldots, N$. This is done by constructing first the fused K-operators for $\mathcal{A}_q^{(N)}$. Then the images of these latter under $\vartheta^{(N)}$ are expressed as dressed K-matrices, which in turn, makes the connection between the generating functions for $\mathcal{A}_q^{(N)}$ and the transfer matrices. Finally, the TT-relations satisfied by $t^{(j,\{j_n\}_N)}(u) \in \mathrm{End}(\mathbb{C}^{2j_N+1}\otimes\ldots\otimes\mathbb{C}^{2j_1+1})$ are then deduced from the universal TT-relations for $\mathcal{A}_q^{(N)}$.

4.4.1 From fused K-operators to dressed K-matrices

As we have already discussed, the algebra $\mathcal{A}_q$ finds application in quantum integrable systems through its quotients. Historically, the first application was the XXZ spin-1/2 chain with generic boundary conditions [BK05a, BK05b]. In spin-chain representations, a K-operator of the form $\mathcal{K}^{(\frac{1}{2},N)}(u)$ given in (4.3.21) was constructed out of dressed K-matrices, recall the form in (1.2.11). Now, in order to extend this to higher spin-j, the truncated fused K-operators for $\mathcal{A}_q^{(N)}$ are required. We denote them $\mathcal{K}^{(j,N)}(u)$, and they are constructed similarly to $\mathcal{K}^{(j)}(u)$ from Definition 3.5.1.

Definition 4.4.1. *The truncated fused K-operators for $\mathcal{A}_q^{(N)}$ are given by:*

$$\mathcal{K}^{(j,N)}(u) = \mathcal{F}^{(j)}_{\langle 12\rangle}\mathcal{K}_1^{(\frac{1}{2},N)}(uq^{-j+\frac{1}{2}})R^{(\frac{1}{2},j-\frac{1}{2})}(u^2q^{-j+1})\mathcal{K}_2^{(j-\frac{1}{2},N)}(uq^{\frac{1}{2}})\mathcal{E}^{(j)}_{\langle 12\rangle} , \tag{4.4.2}$$

with $\mathcal{K}^{(\frac{1}{2},N)}(u)$ in (4.3.20).

By construction they satisfy the reflection equation

$$\begin{aligned} R^{(j_1,j_2)}(u/v)\mathcal{K}_1^{(j_1,N)}(u)R^{(j_1,j_2)}(uv)\mathcal{K}_2^{(j_2,N)}(v) = \\ \mathcal{K}_2^{(j_2,N)}(v)R^{(j_1,j_2)}(uv)\mathcal{K}_1^{(j_1,N)}(u)R^{(j_1,j_2)}(u/v) . \end{aligned} \tag{4.4.3}$$

Then, using the expression of $\mathcal{K}^{(j)}(u)$ from (3.5.1), one shows by induction the spin-j analog of (4.3.20)

$$\mathcal{K}^{(j,N)}(u) = \left[\prod_{k=1}^{2j-1} c(u^2q^{j-\frac{1}{2}-k})h_0^{(N)}(uq^{j-\frac{1}{2}-k})\right](\varphi^{(N)}\otimes\mathrm{id})(\mathcal{K}^{(j)}(u)) . \tag{4.4.4}$$

The latter truncated fused K-operators are now related to the basic building ingredient for quantum integrable open spin chains, namely the so-called dressed K-matrices according to the standard terminology used in [Sk88] and related works, see (1.2.11).

Recall the fused L-operators $\mathcal{L}^{(j)}(u) \in U_qsl_2 \otimes \mathrm{End}(\mathbb{C}^{2j+1})$ are given by (3.4.6), with the funda-

mental L-operator $\mathcal{L}^{(\frac{1}{2})}(u)$ from (1.1.8). They satisfy the Yang-Baxter algebra

$$R^{(\frac{1}{2},j)}(u/v)\mathcal{L}_1^{(\frac{1}{2})}(u)\mathcal{L}_2^{(j)}(v) = \mathcal{L}_2^{(j)}(v)\mathcal{L}_1^{(\frac{1}{2})}(u)R^{(\frac{1}{2},j)}(u/v) \tag{4.4.5}$$

with the R-matrix (3.4.4). Define the evaluation of the L-operators for any $k \in \mathbb{N}_+$

$$L_{[k]}^{(j)}(u) = (\pi^{j_k} \otimes \mathrm{id})(\mathcal{L}^{(j)}(u)) \ \in \mathrm{End}(\mathbb{C}^{2j_k+1}) \otimes \mathrm{End}(\mathbb{C}^{2j+1}) \ , \tag{4.4.6}$$

where $\pi^j \colon U_q sl_2 \to \mathrm{End}(\mathbb{C}^{2j+1})$ is given in (4.3.1).

We now relate the fused K-operators for $\mathcal{A}_q$ to dressed K-matrices. Recall the fused K-matrix (3.5.22) with the fundamental K-matrix (1.2.3), and the spin-chain representations $\vartheta^{(N)}$ from (4.3.5)-(4.3.8). Let $v_1, ..., v_N$ be nonzero scalars.

Lemma 4.4.2. *The action of $(\vartheta^{(N)} \otimes id)$ on the fused K-operators $\mathcal{K}^{(j,N)}(u)$ from (4.4.2)* are expressed in terms of $L_{[k]}^{(j)}(u)$ as follows

$$\begin{aligned}(\vartheta^{(N)} \otimes \mathrm{id})(\mathcal{K}^{(j,N)}(u)) &= L_{[N]}^{(j)}(uv_N)\cdots L_{[1]}^{(j)}(uv_1)K^{(j)}(u)L_{[1]}^{(j)}(uv_1^{-1})\cdots L_{[N]}^{(j)}(uv_N^{-1}) \\ &\in \mathrm{End}(\mathbb{C}^{2j_N+1} \otimes \ldots \otimes \mathbb{C}^{2j_1+1} \otimes \mathbb{C}^{2j+1})\end{aligned} \tag{4.4.7}$$

with the identification $\varepsilon_\pm^{(0)} = \varepsilon_\pm$.

Proof. The proof is done by induction. The case $j = \frac{1}{2}$ holds by [BK05b]. Assume (4.4.7) holds for a fixed j. We now show the case $j + \frac{1}{2}$. For convenience, consider $N = 2$. From the formula of $\mathcal{K}^{(j,N)}(u)$ in (4.4.2), we have:

$$\begin{aligned}&(\vartheta^{(N)} \otimes \mathrm{id})(\mathcal{K}^{(j+\frac{1}{2},2)}(u)) \\ &= (\vartheta^{(N)} \otimes \mathrm{id})\left(\mathcal{F}_{\langle 12\rangle}^{(j+\frac{1}{2})}\mathcal{K}_1^{(\frac{1}{2},2)}(uq^{-j})R_{12}^{(\frac{1}{2},j)}(u^2q^{-j+\frac{1}{2}})\mathcal{K}_2^{(j,2)}(uq^{\frac{1}{2}})\mathcal{E}_{\langle 12\rangle}^{(j+\frac{1}{2})}\mathcal{H}_{\langle 12\rangle}^{(j+\frac{1}{2})}[\mathcal{H}_{\langle 12\rangle}^{(j+\frac{1}{2})}]^{-1}\right) \\ &= (\vartheta^{(N)} \otimes \mathrm{id})\left(\mathcal{F}_{\langle 12\rangle}^{(j+\frac{1}{2})}\mathcal{K}_1^{(\frac{1}{2},2)}(uq^{-j})R_{12}^{(\frac{1}{2},j)}(u^2q^{-j+\frac{1}{2}})\mathcal{K}_2^{(j,2)}(uq^{\frac{1}{2}})R_{12}^{(\frac{1}{2},j)}(q^{j+\frac{1}{2}})\mathcal{E}_{\langle 12\rangle}^{(j+\frac{1}{2})}[\mathcal{H}_{\langle 12\rangle}^{(j+\frac{1}{2})}]^{-1}\right) \\ &= (\vartheta^{(N)} \otimes \mathrm{id})\left(\mathcal{F}_{\langle 12\rangle}^{(j+\frac{1}{2})}R_{12}^{(\frac{1}{2},j)}(q^{j+\frac{1}{2}})\mathcal{K}_2^{(j,2)}(uq^{\frac{1}{2}})R_{12}^{(\frac{1}{2},j)}(u^2q^{-j+\frac{1}{2}})\mathcal{K}_1^{(\frac{1}{2},2)}(uq^{-j})\mathcal{E}_{\langle 12\rangle}^{(j+\frac{1}{2})}[\mathcal{H}_{\langle 12\rangle}^{(j+\frac{1}{2})}]^{-1}\right)\end{aligned} \tag{4.4.8}$$

where we used Corollary C.3.2 and the reflection equation (3.5.21). Below, we underline the steps of calculation and we use the shorthand notation:

$$\begin{aligned}&R_{k\ell} = R_{k\ell}^{(j_k,j_\ell)}(u_k/u_\ell) \ , \qquad \bar{R}_{k\ell} = R_{k\ell}^{(j_k,j_\ell)}(u_\ell/u_k) \ , \qquad \hat{R}_{k\ell} = R_{k\ell}^{(j_k,j_\ell)}(u_k u_\ell) \ , \\ &K_\ell = K_\ell^{(j_\ell)}(u_\ell) \ , \qquad L_{k[\ell]} = \left(L_k^{(j_k)}(u_k/v_\ell)\right)_{[\ell]} , \quad \hat{L}_{k[\ell]} = \left(L_k^{(j_k)}(u_k v_\ell)\right)_{[\ell]} \ .\end{aligned}$$

Recall that two operators that act on different auxiliary and quantum spaces commute, i.e.

$$\left[L_{k[\ell]}, L_{m[n]}\right] = 0 \ , \qquad \left[L_{k[\ell]}, K_m\right] = 0 \ , \qquad \text{for } k \neq m \text{ and } \ell \neq n \ . \tag{4.4.9}$$

Recall also that $L^{(j)}(u)$ and $R^{(\frac{1}{2},j)}(u)$ are invertible, their inverses are proportional to $L^{(j)}(u^{-1})$ and $R^{(\frac{1}{2},j)}(u^{-1})$, see (3.4.15) and (3.4.16). Then, the reflection equation and the Yang-Baxter algebra yield:

$$K_1\hat{R}_{12}K_2\bar{R}_{12} = \bar{R}_{12}K_2\hat{R}_{12}K_1 \ , \tag{4.4.10}$$

$$L_{2[2]}\hat{R}_{12}\hat{L}_{1[\ell]} = \hat{L}_{1[\ell]}\hat{R}_{12}L_{2[\ell]} \ , \tag{4.4.11}$$

$$\bar{R}_{12}\hat{L}_{2[\ell]}\hat{L}_{1[\ell]} = \hat{L}_{1[\ell]}\hat{L}_{2[\ell]}\bar{R}_{12} \ . \tag{4.4.12}$$

Fix $j_1 = \frac{1}{2}$, $j_2 = j$ and $u_1 = uq^{-j}$, $u_2 = uq^{\frac{1}{2}}$. Then from (4.4.8), assuming (4.4.7) holds for a fixed j, one has:

$$\begin{aligned}
(\vartheta^{(N)}\otimes \mathrm{id})(\mathcal{K}^{(j+\frac{1}{2},2)}(u)) &= \mathcal{F}_{\langle 12\rangle}\bar{R}_{12}\hat{L}_{2[2]}\hat{L}_{2[1]}K_2L_{2[1]}\underline{L_{2[2]}\hat{R}_{12}\hat{L}_{1[2]}}\hat{L}_{1[1]}K_1L_{1[1]}L_{1[2]}\mathcal{E}_{\langle 12\rangle}\mathcal{H}^{-1}_{\langle 12\rangle}\\
&\overset{(4.4.11)}{=} \mathcal{F}_{\langle 12\rangle}\bar{R}_{12}\hat{L}_{2[2]}\underline{\hat{L}_{2[1]}K_2L_{2[1]}\hat{L}_{1[2]}}\hat{R}_{12}\underline{L_{2[2]}\hat{L}_{1[1]}K_1L_{1[1]}}L_{1[2]}\mathcal{E}_{\langle 12\rangle}\mathcal{H}^{-1}_{\langle 12\rangle}\\
&\overset{(4.4.9)}{=} \mathcal{F}_{\langle 12\rangle}\bar{R}_{12}\hat{L}_{2[2]}\hat{L}_{1[2]}\hat{L}_{2[1]}K_2\underline{L_{2[1]}\hat{R}_{12}\hat{L}_{1[1]}}K_1L_{1[1]}L_{2[2]}L_{1[2]}\mathcal{E}_{\langle 12\rangle}\mathcal{H}^{-1}_{\langle 12\rangle}\\
&\overset{(4.4.11)}{=} \mathcal{F}_{\langle 12\rangle}\bar{R}_{12}\hat{L}_{2[2]}\hat{L}_{1[2]}\hat{L}_{2[1]}\underline{K_2\hat{L}_{1[1]}}\hat{R}_{12}\underline{L_{2[1]}K_1}L_{1[1]}L_{2[2]}L_{1[2]}\mathcal{E}_{\langle 12\rangle}\mathcal{H}^{-1}_{\langle 12\rangle}\\
&\overset{(4.4.12)}{=} \mathcal{F}_{\langle 12\rangle}\underline{\bar{R}_{12}\hat{L}_{2[2]}\hat{L}_{1[2]}\hat{L}_{2[1]}\hat{L}_{1[1]}}K_2\hat{R}_{12}K_1L_{2[1]}L_{1[1]}L_{2[2]}L_{1[2]}\mathcal{E}_{\langle 12\rangle}\mathcal{H}^{-1}_{\langle 12\rangle} \ .
\end{aligned}$$

Now, using the Yang-Baxter algebra (4.4.5), the reflection equation and Corollary C.3.2 we get:

$$\begin{aligned}
&(\vartheta^{(N)}\otimes \mathrm{id})(\mathcal{K}^{(j+\frac{1}{2},2)}(u))\\
&\quad= \mathcal{F}_{\langle 12\rangle}\hat{L}_{1[2]}\hat{L}_{2[2]}\mathcal{E}_{\langle 12\rangle}\mathcal{F}_{\langle 12\rangle}\hat{L}_{1[1]}\hat{L}_{2[1]}\mathcal{E}_{\langle 12\rangle}\mathcal{F}_{\langle 12\rangle}\underline{\bar{R}_{12}K_2\hat{R}_{12}K_1}L_{2[1]}L_{1[1]}L_{2[2]}L_{1[2]}\mathcal{E}_{\langle 12\rangle}\mathcal{H}^{-1}_{\langle 12\rangle}\\
&\overset{(4.4.10)}{=} L^{(j+\frac{1}{2})}_{[2]}(uv_2)L^{(j+\frac{1}{2})}_{[1]}(uv_1)\mathcal{F}_{\langle 12\rangle}K_1\hat{R}_{12}K_2\mathcal{E}_{\langle 12\rangle}\mathcal{F}_{\langle 12\rangle}\underline{\bar{R}_{12}L_{2[1]}L_{1[1]}L_{2[2]}L_{1[2]}}\mathcal{E}_{\langle 12\rangle}\mathcal{H}^{-1}_{\langle 12\rangle}\\
&\overset{(4.4.12)}{=} L^{(j+\frac{1}{2})}_{[2]}(uv_2)L^{(j+\frac{1}{2})}_{[1]}(uv_1)K^{(j+\frac{1}{2})}(u)\mathcal{F}_{\langle 12\rangle}L_{1[1]}L_{2[1]}\mathcal{E}_{\langle 12\rangle}\mathcal{F}_{\langle 12\rangle}L_{1[2]}L_{2[2]}\underline{\bar{R}_{12}\mathcal{E}_{\langle 12\rangle}\mathcal{H}^{-1}_{\langle 12\rangle}}\\
&\overset{(C.3.7)}{=} L^{(j+\frac{1}{2})}_{[2]}(uv_2)L^{(j+\frac{1}{2})}_{[1]}(uv_1)K^{(j+\frac{1}{2})}(u)L^{(j+\frac{1}{2})}_{[1]}(uv_1^{-1})L^{(j+\frac{1}{2})}_{[2]}(uv_2^{-1}) \ .
\end{aligned}$$

The proof extends to general N by induction on N. □

Recall that the image on spin-chain representations of the spin-1/2 K-operator was given in (4.3.4). Using above lemma and (4.3.20), one finds that $f(u) = (u^2-u^{-2})h_0^{(N)}(u)$ in (4.3.4). In the following, we study the relations between the transfer matrices and the spin-j version of $\mathsf{T}^{(\frac{1}{2},N)}(u) \in \mathcal{A}_q^{(N)}[u,u^{-1}]$ given in (4.3.28). The latter is defined by analogy with $\mathsf{T}^{(j)}(u)$ from (4.1.4)

$$\mathsf{T}^{(j,N)}(u) = \mathrm{tr}_{\mathbb{C}^{2j+1}}(K^{+(j)}(u)\mathcal{K}^{(j,N)}(u)) \ \in \mathcal{A}_q^{(N)}((u^{-1})) \ , \tag{4.4.13}$$

where the truncated fused K-operator $\mathcal{K}^{(j,N)}(u)$ and $K^{+(j)}(u)$ are given in (4.4.2) and (4.1.2), respectively. Note that the truncated fused K-operators $\mathcal{K}^{(j,N)}(u) \in \mathcal{A}_q^{(N)}[u,u^{-1}]\otimes \mathrm{End}(\mathbb{C}^{2j+1})$ while $\mathsf{T}^{(j,N)}(u) \in \mathcal{A}_q^{(N)}((u^{-1}))$. The latter has finitely many terms in $\mathcal{A}_q^{(N)}$ but the presence of factors in $1/c(u^2)$ in $K^{+(j)}(u)$ explains the fact that $\mathsf{T}^{(j,N)}(u)$ is a Laurent series in u^{-1} with coefficients in $\mathcal{A}_q^{(N)}$ (recall our conventions).

Using the invariance of the TT-relations of Theorem 4.2.5 under a change of normalization of $\mathcal{K}^{(j)}(u)$, see Remark 4.2.7, and applying the map $\varphi^{(N)}$ with (4.3.15) to (4.2.12), one finds:

$$\mathsf{T}^{(j,N)}(u) = \mathsf{T}^{(j-\frac{1}{2},N)}(uq^{-\frac{1}{2}})\mathsf{T}^{(\frac{1}{2},N)}(uq^{j-\frac{1}{2}}) + \frac{\Gamma^{(N)}(uq^{j-\frac{3}{2}})\Gamma_+(uq^{j-\frac{3}{2}})}{c(u^2q^{2j})c(u^2q^{2j-2})}\mathsf{T}^{(j-1,N)}(uq^{-1}) \ , \tag{4.4.14}$$

where $c(u)$ is given in (1.3.33) and

$$\begin{aligned}\Gamma^{(N)}(u) &= \mathrm{tr}_{12}\big(\mathcal{P}_{12}^{-}\mathcal{K}_1^{(\frac{1}{2},N)}(u)R^{(\frac{1}{2},\frac{1}{2})}(qu^2)\mathcal{K}_2^{(\frac{1}{2},N)}(uq)\big) \\ &= c(u^2)c(u^2q^2)h_0^{(N)}(u)h_0^{(N)}(uq)\varphi^{(N)}(\Gamma(u)) \ .\end{aligned} \tag{4.4.15}$$

4.4.2 Relations to transfer matrices

Recall the transfer matrices for open spin-chains was reviewed in Section 1.2.3. The explicit construction of local conserved quantities associated with various examples of quantum integrable open spin chains of length N with generic integrable boundary conditions (higher XXZ spin chains [FNR07], alternating spin chains [CYSW14], ...) is now revisited in light of previous results. For these models, it is well-known that all *local* conserved quantities are derived from a transfer matrix [Sk88]:

$$t^{(j,\{j_n\}_N)}(u) = \mathrm{tr}_{\mathbb{C}^{2j+1}}(K_a^{+(j)}(u)T_{a,N}^{(j,\{j_n\}_N)}(u)K_a^{(j)}(u)\hat{T}_{a,N}^{(j,\{j_n\}_N)}(u)) \ , \tag{4.4.16}$$

where the generalizations of the monodromy matrices from (1.1.11), (1.2.12) are

$$T_{a,N}^{(j,\{j_n\}_N)}(u) = R_{Na}^{(j_N,j)}(uv_N)\cdots R_{1a}^{(j_1,j)}(uv_1) \ , \tag{4.4.17}$$

$$\hat{T}_{a,N}^{(j,\{j_n\}_N)}(u) = R_{1a}^{(j_1,j)}(uv_1^{-1})\cdots R_{Na}^{(j_N,j)}(uv_N^{-1}) \ , \tag{4.4.18}$$

and with $R^{(j_1,j_2)}(u)$ given in (3.4.3). Note that v_n are the so-called inhomogeneities and they are exactly the non-zero complex parameters v_n introduced in the spin-chain representation of $\mathcal{A}_q$ given by (4.3.5)-(4.3.8). Here, we use the notation $\{j_n\}_N = (j_1, j_2, \ldots, j_N)$ that denotes the spin of the representation located at site n of the chain, whereas j denotes the spin of the auxiliary space 'a'.

Among the conserved quantities of interest, local ones such as the Hamiltonian are derived from the transfer matrix [Sk88] for a suitable choice of parameters $(j, \{j_n\}_N)$ and $\{v_n\}$. Typically we are interested in homogeneous case where all $j_n = j$ and $v_n = 1$, then the Hamiltonian is local. Up to an overall factor, adjusting the scalar term H_0 one has:

$$\mathcal{H}_{XXZ} - H_0 \sim \frac{d}{du}\ln(t^{(j,\{j_n\}_N)}(u))|_{u=1,v_n=1} \ . \tag{4.4.19}$$

Example 4.4.3. *Fixing j, j_n, and v_n to some values, we recover from (4.4.19)* the XXZ Hamiltonian with generic boundary conditions:

(i) For $j = j_n = \frac{1}{2}$, $v_n = 1\ \forall n$, we get $\mathcal{H}_{XXZ}^{(\frac{1}{2})gen}(k_\pm, \varepsilon_\pm, \overline{k}_\pm, \overline{\varepsilon}_\pm)$ given in (1.2.17).

(ii) For $j = j_n = 1$, $v_n = 1\ \forall n$, we get $\mathcal{H}_{XXZ}^{(1)gen}(k_\pm, \varepsilon_\pm, \overline{k}_\pm, \overline{\varepsilon}_\pm)$ given in (1.3.49).

Now, we relate the generating function (4.4.13) to the transfer matrix (4.4.16). To do so, first recall the images of the truncated fused K-operators are expressed in Lemma 4.4.2 in terms of evaluations of L-operators (4.4.6). Actually, they are proportional to the R-matrix from (3.4.3). Indeed, one finds

$$L^{(j)}_{[n]}(u) = \left[\prod_{k=0}^{2j_n-2}\prod_{\ell=0}^{2j-1} c(uq^{j+j_n-k-\ell-1})\right]^{-1} R^{(j_n,j)}(u)\ , \tag{4.4.20}$$

with

$$L^{(\frac{1}{2})}_{[n]}(u) = \left[\prod_{k=0}^{2j_n-2} c(uq^{j_n-k-\frac{1}{2}})\right]^{-1} R^{(j_n,\frac{1}{2})}(u)\ . \tag{4.4.21}$$

Recall $\vartheta^{(N)}$ is given by (4.3.5)-(4.3.8).

Proposition 4.4.4. *The spin-j generating function for $\mathcal{A}_q^{(N)}$ specializes to the transfer matrix (4.4.16)*

$$\vartheta^{(N)}(\mathsf{T}^{(j,N)}(u)) = \left[\prod_{n=1}^{N}\prod_{k=0}^{2j_n-2}\prod_{\ell=0}^{2j-1} c(uq^{j+j_n-k-\ell-1}v_n)c(uq^{j+j_n-k-\ell-1}v_n^{-1})\right]^{-1} t^{(j,\{j_n\}_N)}(u)\ . \tag{4.4.22}$$

Proof. It is straightforward starting from the definition of $\mathsf{T}^{(j,N)}(u)$ in (4.4.13), and using (4.4.7) with (4.4.20). □

Now, recall the quantum determinants $\Gamma_\pm(u)$, and $\beta^{(j)}(u)$ given respectively in (4.1.5), (4.1.6), (4.1.11).

Proposition 4.4.5. *The transfer matrix $t^{(j,\{j_n\}_N)}(u)$ from (4.4.16)* satisfies the TT-relation:

$$\begin{aligned} t^{(j,\{j_n\}_N)}(u) &= t^{(j-\frac{1}{2},\{j_n\}_N)}(uq^{-\frac{1}{2}})t^{(\frac{1}{2},\{j_n\}_N)}(uq^{j-\frac{1}{2}}) \\ &+ \left[\prod_{n=1}^{N}\beta^{(j_n)}(uq^{j-\frac{1}{2}}v_n)\beta^{(j_n)}(uq^{j-\frac{1}{2}}v_n^{-1})\right]\frac{\Gamma_-(uq^{j-\frac{3}{2}})\Gamma_+(u^{j-\frac{3}{2}})}{c(u^2q^{2j})c(u^2q^{2j-2})}t^{(j-1,\{j_n\}_N)}(uq^{-1})\ , \end{aligned} \tag{4.4.23}$$

where $\beta^{(j)}(u)$ is given in (4.1.11).

Proof. Recall the universal TT-relations satisfied by $\mathsf{T}^{(j,N)}(u)$ in (4.4.14), and that spin-chain representations for $\mathsf{T}^{(j,N)}(u)$ lead to transfer matrices in Proposition 4.4.4. The TT-relations (4.4.23) are obtained as the images of $\vartheta^{(N)}$ on (4.4.14). We thus only need to compute the image of the quantum determinant $\Gamma^{(N)}(u)$ from the first line of (4.4.15). Recall $T^{(j,\{j_n\}_N)}_{a,N}(u)$ is given in (4.4.17). One has [Sk88, Prop. 6]:

$$\begin{aligned} &\mathrm{tr}_{12}\big(\mathcal{P}^-_{12}T^{(\frac{1}{2},\{j_n\}_N)}_{1,N}(u)K_1^{-(\frac{1}{2})}(u)\hat{T}^{(\frac{1}{2},\{j_n\}_N)}_{1,N}(u)\ R^{(\frac{1}{2},\frac{1}{2})}_{12}(qu^2)T^{(\frac{1}{2},\{j_n\}_N)}_{2,N}(uq)K_2^{-(\frac{1}{2})}(uq)\hat{T}^{(\frac{1}{2},\{j_n\}_N)}_{2,N}(uq)\big) \\ &= \gamma(T^{(\frac{1}{2},\{j_n\}_N)}_N(u))\gamma(\hat{T}^{(\frac{1}{2},\{j_n\}_N)}_N(u))\Gamma_-(u) = \left[\prod_{n=1}^{N}\beta^{(j_n)}(uqv_n)\beta^{(j_n)}(uqv_n^{-1})\right]\Gamma_-(u) \end{aligned}$$

where $\beta^{(j)}(u)$ is given in (4.1.11) and $\gamma(T(u)) = \mathrm{tr}_{12}\big(\mathcal{P}_{12}^{-}T_1(u)T_2(uq)\big)$. From (4.4.15), it follows:

$$\vartheta^{(N)}(\Gamma^{(N)}(u)) = \frac{\left[\prod_{n=1}^{N}\beta^{(j_n)}(uqv_n)\beta^{(j_n)}(uqv_n^{-1})\right]\Gamma_-(u)}{\left[\prod_{n=1}^{N}\prod_{k=0}^{2j_n-2}c(uq^{j_n-k-\frac{1}{2}}v_n)c(uq^{j_n-k-\frac{1}{2}}v_n^{-1})c(uq^{j_n-k+\frac{1}{2}}v_n)c(uq^{j_n-k+\frac{1}{2}}v_n^{-1})\right]} \ . \quad (4.4.24)$$

After simplifications, using (4.4.22), (4.4.24), one finds that the image of the TT-relations (4.4.14) under $\vartheta^{(N)}$ maps to (4.4.23). □

By specialization of (4.4.23), one recovers some examples of TT-relations conjectured in the literature. For instance, the choice $j_n = j \in \frac{1}{2}\mathbb{N}_+$, $v_n = 1\,\forall n$, corresponds to the TT-relations satisfied by the transfer matrix of the spin-j XXZ open spin chain [FNR07, eq. (2.16)]. Whereas the choice $j_{2p} = s$, $j_{2p+1} = s'\ \forall p$ and $s, s' \in \frac{1}{2}\mathbb{N}_+$, corresponds to the TT-relations conjectured for the alternating spin chain of length N even [CYSW14, eq. (3.1)].

4.4.3 The case of the q-Onsager algebra

As we now describe, the transfer matrix can be written in terms of the two fundamental alternating generators. From the spin-chain representations (4.3.5)-(4.3.6) one has in particular:

$$\begin{aligned}\vartheta^{(N)}(\mathcal{W}_0^{(N)}) &= \left(k_+v_Nq^{1/2}S_+q^{S_3/2} + k_-v_N^{-1}q^{-1/2}S_-q^{S_3/2}\right)\otimes\mathbb{I}_{(2j_1+1)\times\ldots\times(2j_{N-1}+1)} \\ &\quad + q^{S_3}\otimes\vartheta^{(N)}(\mathcal{W}_0^{(N-1)}) \ , \qquad (4.4.25)\\ \vartheta^{(N)}(\mathcal{W}_1^{(N)}) &= \left(k_+v_N^{-1}q^{-1/2}S_+q^{-S_3/2} + k_-v_Nq^{1/2}S_-q^{-S_3/2}\right)\otimes\mathbb{I}_{(2j_1+1)\times\ldots\times(2j_{N-1}+1)} \\ &\quad + q^{-S_3}\otimes\vartheta^{(N)}(\mathcal{W}_1^{(N-1)}) \ . \qquad (4.4.26)\end{aligned}$$

These are *non-local* operators associated with the spin chain.

Recall that the defining relations of the q-Onsager algebra are the same as for $\mathcal{A}_q$, but in addition the central elements $\Delta_{k+1} \in \mathcal{A}_q$ are fixed to some scalars $2\delta_{k+1} \in \mathbb{C}$, see Section 2.2. In spin-chain representations, $\mathcal{A}_q$ factors through O_q because the images of the central elements Δ_{k+1} under $\vartheta^{(N)}$ are indeed scalars.

We now study the commutative subalgebra $\mathcal{I}$ in more details for the special case of the q-Onsager algebra O_q.

Remark 4.4.6. *Let* $\deg\colon O_q \to \mathbb{N}$ *denote the total degree of a polynomial in the alternating generators* $\mathcal{W}_0, \mathcal{W}_1$ *such that*

$$\deg(\mathcal{W}_k) = \deg(\mathcal{W}_{k+1}) = 2k+1 \ , \qquad \deg(\mathcal{G}_{k+1}) = \deg(\tilde{\mathcal{G}}_{k+1}) = 2k+2 \ .$$

Inserting the expressions (2.2.6) and (2.2.5) into (2.1.56), it follows that the coefficients $\mathcal{I}_{2k+1}$ of the generating function $\mathsf{T}^{(j)}(u)$ are polynomials in $\mathcal{W}_0, \mathcal{W}_1$, such that

$$\deg(\mathcal{I}_{2k+1}) = 2k+2 \ . \quad (4.4.27)$$

For instance for $k=0$ in (2.1.56), one has

$$\mathcal{I}_1 = \bar{\varepsilon}_+\mathcal{W}_0 + \bar{\varepsilon}_-\mathcal{W}_1 + \frac{1}{q^2-q^{-2}}\left(\frac{\overline{k}_-}{k_-}\big([\mathcal{W}_1,\mathcal{W}_0]_q+\delta_1\big) + \frac{\overline{k}_+}{k_+}\big([\mathcal{W}_0,\mathcal{W}_1]_q+\delta_1\big)\right) . \tag{4.4.28}$$

Finally, from (4.3.28) and using the explicit form of the $\mathcal{I}_{2k+1}$'s in terms of the alternating generators given by (2.1.56) and (4.4.27), it follows from Proposition 4.4.5:

Corollary 4.4.7. *The transfer matrix* $t^{(j,\{j_n\}_N)}(u)$ *from* (1.3.26) is a polynomial of total degree $4Nj$ in the images of the fundamental alternating generators $\vartheta^{(N)}(\mathcal{W}_0^{(N)})$, $\vartheta^{(N)}(\mathcal{W}_1^{(N)})$.

In the q-Onsager algebra, $\mathcal{I}_{2k+1}$ from (2.1.56) is expressed in terms of $\mathcal{W}_0$, $\mathcal{W}_1$ and δ_{k+1}. On the spin-chain representations, or under the image of $\vartheta^{(N)}$, the expressions of the latter scalars can be extracted as follows. Recall the quantum determinant $\Gamma(u)$ defined in (2.1.44) is fixed in O_q as (2.2.2). Therefore, its image on the spin-chains is

$$\vartheta^{(N)}(\Gamma(u)) = \frac{(u^2q^2-u^{-2}q^{-2})}{2(q-q^{-1})}\left(\delta^{(\frac{1}{2})}(u) - \frac{2\rho}{q-q^{-1}}\right) , \tag{4.4.29}$$

with $\delta^{(\frac{1}{2})}(u)$ from (2.2.3). Using (4.4.15) with (4.4.24), we get the following equation

$$\begin{aligned}
&\vartheta^{(N)}(\Gamma(u))\ c(u^2)c(u^2q^2)h_0^{(N)}(u)h_0^{(N)}(uq)\\
&\times\left[\prod_{n=1}^{N}\prod_{k=0}^{2j_n-2} c(uq^{j_n-k-\frac{1}{2}}v_n)c(uq^{j_n-k-\frac{1}{2}}v_n^{-1})c(uq^{j_n-k+\frac{1}{2}}v_n)c(uq^{j_n-k+\frac{1}{2}}v_n^{-1})\right]\\
&=\left[\prod_{n=1}^{N}\beta^{(j_n)}(uqv_n)\beta^{(j_n)}(uqv_n^{-1})\right]\ \Gamma_-(u) ,
\end{aligned}$$

where $\Gamma_-(u)$ is given in (4.1.6), and thus the scalars δ_{k+1} can be extracted by comparing the powers in u on both sides of this equation.

4.5 Hidden symmetries

In Chapter 1, the representation-dependent approach was considered with the spin-$\frac{1}{2}$ transfer matrix from Definition 1.2.7. As an example, the XXZ spin-$\frac{1}{2}$ Hamiltonian with generic boundary conditions was constructed in Proposition 1.2.8. An inconvenience of using this approach is that the algebraic structure is not transparent. As a consequence, since the Hamiltonian is generated from a transfer matrix, finding its symmetries is not necessarily an easy task. Through certain choices of the boundary parameters, some symmetries were exhibited in Example 1.2.9 for the spin-$\frac{1}{2}$, and in Example 1.3.7 for the spin-1. For instance $\mathcal{H}_{XXZ}^{(\frac{1}{2})diag}(\varepsilon_\pm,\bar{\varepsilon}_\pm)$ and $\mathcal{H}_{XXZ}^{(\frac{1}{2})sp.}$ are respectively invariant under the total magnetization S^z and the action of U_qsl_2 on N-fold tensor product of its fundamental representations. However, one can ask if there exists non-trivial symmetries for more general boundary conditions.

The goal of this section is to use a universal approach in order to find symmetries for a large class of models, such as the open XXZ spin-j chains or the alternating spin chains, in a more systematical

way. This is done by using the underlying structure of the generating function $\mathsf{T}^{(\frac{1}{2})}(u) \in \mathcal{A}_q((u^{-1}))$ together with the universal TT-relations satisfied by $\mathsf{T}^{(j)}(u)$. The idea is to find the centralizer of the commutative subalgebra $\mathcal{I}$, and the first step is to study commutation relations between $\mathcal{I}_{2k+1}$ and all generators W_{-k}, W_{k+1}, G_{k+1}, $\tilde{\mathsf{G}}_{k+1}$, but we explore now only a few of them.

4.5.1 Study of the exchange relations in $\mathcal{A}_q$

Now, recall that the abelian subalgebra $\mathcal{I}$ of $\mathcal{A}_q$ is generated by the elements $\{\mathcal{I}_{2k+1} | k \in \mathbb{N}\}$ given in (2.1.56). The spin-$\frac{1}{2}$ universal transfer matrix $\mathsf{T}^{(\frac{1}{2})}(u)$ is expressed in terms of these $\mathcal{I}_{2k+1}$, see (2.1.55). We now study commutation relations between them and the first generators of $\mathcal{A}_q$, namely: W_0 and W_1.

For convenience introduce

$$\mathcal{I}'_{2k+1} = \overline{\varepsilon}'_+ \mathsf{W}_{-k} + \overline{\varepsilon}'_- \mathsf{W}_{k+1} + \frac{1}{q^2 - q^{-2}} \left(\frac{\overline{k}'_+}{k'_+} \mathsf{G}_{k+1} + \frac{\overline{k}'_-}{k'_-} \tilde{\mathsf{G}}_{k+1} \right)$$

where $\overline{k}'_\pm, k'_\pm \in \mathbb{C}$ and $\overline{\varepsilon}'_\pm \in \mathbb{C}^*$. The only difference with $\mathcal{I}_{2k+1}$ is their parameters.

Proposition 4.5.1.

$$\mathsf{W}_0 \mathcal{I}_{2k+1} = \mathcal{I}'_{2k+1} \mathsf{W}_0 \ , \tag{4.5.1}$$

under the condition

$$\overline{\varepsilon}'_- = \overline{\varepsilon}_- = 0 \ , \quad \overline{\varepsilon}'_+ = \overline{\varepsilon}_+ \ , \quad \frac{\overline{k}_+}{k_+} = \frac{\overline{k}_-}{k_-} q^2 \ , \quad \frac{\overline{k}'_\pm}{k'_\pm} = \frac{\overline{k}_\pm}{k_\pm} q^{\mp 2} \ . \tag{4.5.2}$$

Proof. Using the defining relations of $\mathcal{A}_q$ given in (2.1.17)-(2.1.27) the l.h.s. of (4.5.1) becomes

$$\begin{aligned} \mathsf{W}_0 \mathcal{I}_{2k+1} =& \overline{\varepsilon}_+ \mathsf{W}_0 \mathsf{W}_{-k} + \overline{\varepsilon}_- \mathsf{W}_0 \mathsf{W}_{k+1} + \frac{1}{q^2 - q^{-2}} \Big(\frac{\overline{k}_+}{k_+} \mathsf{W}_0 \mathsf{G}_{k+1} + \frac{\overline{k}_-}{k_-} \mathsf{W}_0 \tilde{\mathsf{G}}_{k+1} \Big) \\ =& \left(\overline{\varepsilon}_+ \mathsf{W}_{-k} + \overline{\varepsilon}_- \mathsf{W}_{k+1} + \frac{1}{q^2 - q^{-2}} \Big(\frac{\overline{k}_+}{k_+} q^{-2} \mathsf{G}_{k+1} + \frac{\overline{k}_-}{k_-} q^2 \tilde{\mathsf{G}}_{k+1} \Big) \right) \mathsf{W}_0 \\ &+ \overline{\varepsilon}_- \frac{\tilde{\mathsf{G}}_{k+1} - \mathsf{G}_{k+1}}{q + q^{-1}} + \frac{1}{q^2 - q^{-2}} \rho \Big(\frac{\overline{k}_+}{k_+} q^{-1} - \frac{\overline{k}_-}{k_-} q \Big) \big(\mathsf{W}_{-k-1} - \mathsf{W}_{k+1} \big) \ . \end{aligned} \tag{4.5.3}$$

On the other hand,

$$\mathcal{I}'_{2k+1} \mathsf{W}_0 = \left(\overline{\varepsilon}'_+ \mathsf{W}_{-k} + \overline{\varepsilon}'_- \mathsf{W}_{k+1} + \frac{1}{q^2 - q^{-2}} \Big(\frac{\overline{k}'_+}{k'_+} \mathsf{G}_{k+1} + \frac{\overline{k}'_-}{k'_-} \tilde{\mathsf{G}}_{k+1} \Big) \right) \mathsf{W}_0 \ . \tag{4.5.4}$$

$\mathsf{W}_{k+1}, \mathsf{W}_{-k-1}$ are linearly independent from $\mathsf{W}_0, \mathsf{W}_{-k}, \mathsf{G}_{k+1}, \tilde{\mathsf{G}}_{k+1}$ then comparing the linear terms from both sides of (4.5.1) we get

$$\overline{\varepsilon}_- = 0 \ , \qquad \frac{\overline{k}_+}{k_+} = \frac{\overline{k}_-}{k_-} q^2 \ , \tag{4.5.5}$$

and the second line of (4.5.3) vanishes. Now the l.h.s. reads :

$$\mathsf{W}_0\mathcal{I}_{2k+1} = \left(\overline{\varepsilon}_+\mathsf{W}_{-k} + \frac{1}{q^2-q^{-2}}\Big(\frac{\overline{k}_+}{k_+}q^{-2}\mathsf{G}_{k+1} + \frac{\overline{k}_-}{k_-}q^2\tilde{\mathsf{G}}_{k+1}\Big)\right)\mathsf{W}_0 \ . \tag{4.5.6}$$

The conditions (4.5.2) follow from (4.5.5), using the PBW basis from (2.1.42) and by equating (4.5.4) with (4.5.6). □

Similarly an exchange relation for W_1 is obtained.

Proposition 4.5.2.

$$\mathsf{W}_1\mathcal{I}_{2k+1} = \mathcal{I}'_{2k+1}\mathsf{W}_1, \tag{4.5.7}$$

under the condition

$$\overline{\varepsilon}'_+ = \overline{\varepsilon}_+ = 0 \ , \quad \overline{\varepsilon}'_- = \overline{\varepsilon}_- \ , \quad \frac{\overline{k}_+}{k_+} = \frac{\overline{k}_-}{k_-}q^{-2} \ , \quad \frac{\overline{k}'_\pm}{k'_\pm} = \frac{\overline{k}_\pm}{k_\pm}q^{\pm 2} \ . \tag{4.5.8}$$

From the above propositions, we have:

Corollary 4.5.3. *The generators* W_0, W_1 *commute with* $\mathcal{I}_{2k+1}$ *from* (2.1.56) as follows.

(i) $$[\mathcal{I}_{2k+1}, \mathsf{W}_0] = 0 \ ,$$
iff $\overline{\varepsilon}_- = \overline{k}_\pm = 0$.

(ii) $$[\mathcal{I}_{2k+1}, \mathsf{W}_1] = 0 \ ,$$
iff $\overline{\varepsilon}_+ = \overline{k}_\pm = 0$.

Remark 4.5.4. *The above corollary can be understood differently. Recall* $\mathcal{I}_{2k+1}$ *from* (2.1.56). *In the first case, we have* $\mathcal{I}_{2k+1} = \mathsf{W}_{-k}$ *and the commutation follows from the first relation in* (2.1.20) *for* $\ell = 0$*; while in the second case* $\mathcal{I}_{2k+1} = \mathsf{W}_{k+1}$ *and the commutation relation is obtained from the second relation in* (2.1.20) *for* $\ell = 0$.

4.5.2 Application to quantum integrable spin-chains

The goal is now to find non-trivial symmetries for various Hamiltonians. To this end, it is helpful to keep in mind the picture with the maps given in (4.4.1).

Firstly, recall that the spin-$\frac{1}{2}$ generating function for $\mathcal{A}_q$ is expressed in terms of $\mathcal{I}_{2k+1}$ (2.1.55). Then, due to the universal TT-relations (4.2.12), it follows that $\mathsf{T}^{(j)}(u)$ is a polynomial of order $2j$ in $\mathsf{T}^{(\frac{1}{2})}(u)$. Therefore, the commutation relations from Corollary 4.5.3 imply that W_0 and W_1, respectively commute with $\mathsf{T}^{(j)}(u)$ for this specific choice of the boundary parameters. Note that $\vartheta^{(N)}(\mathsf{T}^{(j)}(u))$ is proportional to the transfer matrix $t^{(j,\{j_n\}_N)}(u)$ due to (4.4.22). *Consequently, the spin-chains representations for* W_0 *and* W_1 *commute respectively with the corresponding transfer matrix for any spin values* j, j_n *and inhomogeneities* v_n *for* $n = 1, 2, \dots N$. Below we study in more details the homogeneous case where all $j_n = j$ and $v_n = 1$.

1-boundary cases

Recall the Hamiltonians of the XXZ spin-1/2 and spin-1 with generic boundary conditions are constructed in Proposition 1.2.8 and equation (1.3.48). More generally, by taking the logarithmic derivative of the spin-j transfer matrix $t^{(j,\{j_n\}_N)}(u)$ given in (4.4.16), and setting all the spins $j_n = j$ as well as the inhomogeneities $v_n = 1$, one can extract the XXZ spin-j Hamiltonian with generic boundary conditions. We denote them by $\mathcal{H}^{(j)gen}_{XXZ}(k_\pm, \varepsilon_\pm, \overline{k}_\pm, \overline{\varepsilon}_\pm)$. Then, fixing the boundary parameters as:

$$\mathcal{H}^{(j)+}_{XXZ}(k_\pm, \varepsilon_\pm) = \mathcal{H}^{(j)gen}_{XXZ}(k_\pm, \varepsilon_\pm, \overline{k}_\pm = 0, \overline{\varepsilon}_+ = 0) \ ,$$
$$\mathcal{H}^{(j)-}_{XXZ}(k_\pm, \varepsilon_\pm) = \mathcal{H}^{(j)gen}_{XXZ}(k_\pm, \varepsilon_\pm, \overline{k}_\pm = 0, \overline{\varepsilon}_- = 0) \ ,$$

it follows from Corollary 4.5.3 that they enjoy the following symmetries

Proposition 4.5.5. *The following commutation relations hold for any $j \in \frac{1}{2}\mathbb{N}_+$*

$$\big[\mathcal{H}^{(j)-}_{XXZ}(k_\pm, \varepsilon_\pm), \vartheta^{(N)}(\mathsf{W}_0)|_{\{j_n\}_N = j}\big] = 0 \ , \qquad \big[\mathcal{H}^{(j)+}_{XXZ}(k_\pm, \varepsilon_\pm), \vartheta^{(N)}(\mathsf{W}_1)|_{\{j_n\}_N = j}\big] = 0 \ , \tag{4.5.9}$$

where $\vartheta^{(N)}(\mathsf{W}_0)$, $\vartheta^{(N)}(\mathsf{W}_1)$ are given in (4.4.25), (4.4.26).

Explicitly, for $j = \frac{1}{2}$, 1, the corresponding Hamiltonians read:

$$\mathcal{H}^{(\frac{1}{2})\pm}_{XXZ}(k_\pm, \varepsilon_\pm) = \mathbf{b}^{(\frac{1}{2})} + \frac{2}{\varepsilon_+ + \varepsilon_-}\Big(\frac{q - q^{-1}}{4}(\varepsilon_+ - \varepsilon_-)\sigma^z_1 + k_+\sigma^+_1 + k_-\sigma^-_1\Big) \tag{4.5.10}$$
$$\mp \frac{q - q^{-1}}{2}\sigma^z_N \ ,$$

$$\mathcal{H}^{(1)\pm}_{XXZ}(k_\pm, \varepsilon_\pm) = \mathbf{b}^{(1)} \tag{4.5.11}$$
$$+ \frac{1}{k_+k_- - \varepsilon_+\varepsilon_-(q + q^{-1}) - (\varepsilon^2_+ + \varepsilon^2_-)}$$
$$\times \Bigg(\Big(\varepsilon_+\varepsilon_-(q^{-1} - q) + k_+k_-\frac{q + q^{-1}}{q^{-1} - q}\Big)(s^z_1)^2 + (\varepsilon^2_- - \varepsilon^2_+)s^z_1 + \frac{k^2_+(s^+_1)^2 + k^2_-(s^-_1)^2}{q - q^{-1}}$$
$$+ \frac{\sqrt{2(q + q^{-1})}}{q - q^{-1}}\Big(\varepsilon_+\Big(k_+[s^+_1, s^z_1]_{q^{\frac{1}{2}}} + k_-[s^z_1, s^-_1]_{q^{\frac{1}{2}}}\Big) + \varepsilon_-\Big(k_+[s^+_1, s^z_1]_{q^{-\frac{1}{2}}} + k_-[s^z_1, s^-_1]_{q^{-\frac{1}{2}}}\Big)\Big)\Bigg)$$
$$\mp s^z_N \ ,$$

where the bulk terms $\mathbf{b}^{(\frac{1}{2})}$ and $\mathbf{b}^{(1)}$ are respectively given in (1.2.18) and (1.3.50), and with the spin-1 matrices from (1.3.51). Note that the left boundary parameters (site 1) are free.

To conclude this chapter, we discuss how the image of the q-Onsager generators can prove to be useful in diagonalizing Hamiltonians with the symmetries (4.5.9). Recall the Hamiltonian (1.2.23), which commutes with the total magnetization S^z, has been diagonalized using the algebraic Bethe Ansatz (ABA) in [Sk88], see the review in Section A.2. Let us consider the case where all the spin-values are set to $j_n = 1/2$. The images of W_0 and W_1 are similar in many ways to S^z, as we now see.

Firstly, the spin-1/2 total magnetization operator S^z has $N+1$ distinct eigenvalues degenerated C_N^n times, with $n = 0, 1, \ldots, N$, where C_N^n are binomial coefficients. The associated eigenvectors are the usual spin basis vectors of the space $(\mathbb{C}^2)^{\otimes N}$. In the ABA setting, the reference state for $\mathcal{H}_{XXZ}^{(\frac{1}{2})diag}$ from Example 1.2.9, is simply given by the highest eigenvector of S^z, i.e. when all spins are up (A.1.7).

Secondly, the images of the generators of the q-Onsager algebra $\vartheta^{(N)}(\mathsf{W}_0)$ and $\vartheta^{(N)}(\mathsf{W}_1)$ have been diagonalized in [B06, Prop. 3.3]. Each of them has $N+1$ distinct eigenvalues indexed by $n = 0, 1, \ldots, N$, degenerated C_N^n times, so the same degeneracy as for S^z. The associated eigenvectors are linear combinations of spins either up or down at each sites, i.e. all the eigenvectors have nonzero entries in canonical basis with respect to the standard spin basis. Let $q = e^{\phi}$, $\phi \in \mathbb{C}$ and $\alpha, \alpha^*, \eta, \eta' \in \mathbb{C}$ be certain parametrization of the boundary parameters $k_\pm, \varepsilon_\pm$ given in [BVZ17, eq. (3.3)] by

$$k_+ = -\frac{q-q^{-1}}{2} q^{\eta} \ , \quad k_- = \frac{q-q^{-1}}{2} q^{\eta'} \ , \quad \varepsilon_+ = q^{\frac{\eta+\eta'}{2}} \cosh\alpha \ , \quad \varepsilon_- = q^{\frac{\eta+\eta'}{2}} \cosh\alpha^* \ .$$

The non-degenerate eigenvector of $\vartheta^{(N)}(\mathsf{W}_0)$ with eigenvalue $\lambda_0^{(N)} = \cosh(\alpha + \phi N) e^{\phi(\frac{\eta+\eta'}{2})}$ is obtained recursively:

$$\psi_{0[1]}^{(N+1)} = \psi_{0[1]}^{(N)} \otimes \left(e^{\alpha + \phi(\frac{\eta-\eta'}{2} + N)} \left|\uparrow\right\rangle + \left|\downarrow\right\rangle \right) \ , \tag{4.5.12}$$

where $\left|\uparrow\right\rangle = (1,0)$, $\left|\downarrow\right\rangle = (0,1)$, with the initial value:

$$\psi_{0[1]}^{(1)} = e^{\alpha + \phi(\frac{\eta-\eta'}{2})} \left|\uparrow\right\rangle + \left|\downarrow\right\rangle \ . \tag{4.5.13}$$

It is the analog of the highest eigenvector of S^z. This vector can be used as a reference state in the setting of ABA. Indeed, the reference state, as a highest weight vector for a gauge transformed monodromy matrix, was constructed in [CLSW02, YZ07], and then observed in [BK07, eq. (63)] that it agrees with (4.5.12).

More generally, for any spin values at each site, $\vartheta^{(N)}(\mathsf{W}_0)$ and $\vartheta^{(N)}(\mathsf{W}_1)$ have also been studied in [BVZ17]. The images of the generators of the q-Onsager algebra have been first expressed in terms of q-difference operators acting in the linear space of multivariable polynomials of total degree $2(j_1 + j_2 + \ldots j_N)$, in the variables $z_1, z_2, \ldots, z_N$. Then, the corresponding spectral problem for W_0 and W_1 has been solved, with explicit expressions for the multivariable polynomial eigenvectors and eigenvalues that are given in terms of q-Pochhammer functions. The situation with degeneracy of eigenvalues of $\vartheta^{(N)}(\mathsf{W}_0)$ is similar to the spin-1/2 case. In particular, there is a non-degenerate eigenstate analogous to (4.5.12). We expect that it plays the role of a reference state for the corresponding ABA.

Chapter 5

Conclusion and perspectives

To conclude, we briefly recall what has been done in this thesis and then give some perspectives.

5.1 Conclusion

The first chapter is a review on how to construct quantum spin-chains using the formalism of the transfer matrix. The second chapter gathers known information about the q-Onsager algebra and its alternating central extension $\mathcal{A}_q$, such as their different presentations, comodule algebra structures, and PBW bases. The quantum algebras $U_q sl_2$, $U_q \widehat{sl}_2$, $\mathcal{L}U_q sl_2$ are also recalled. In short, Chapters 1 and 2 are basically review chapters that introduce the necessary material to address three problems answered in Chapters 3 and 4:

- Problem 1:

  The different universal K-matrices, as defined in [BKo15, Ko17, AV20, AV22] (recall the discussion of these constructions in Introduction), do not allow for the treatment of K-operators solutions to the reflection equation with spectral parameter (0.0.15) and with a *physical* coaction of the form (0.0.16).

- Problem 2:

  The spin-1/2 K-operator for the alternating central extension of the q-Onsager algebra $\mathcal{A}_q$ was introduced in [BS09], recall Definition 2.1.1, but its spin-j version has never been constructed.

- Problem 3:

  One can construct a spin-1/2 generating function in $\mathcal{I}$, the commutative subalgebra of $\mathcal{A}_q$ generated by $\{\mathcal{I}_{2k+1} | k \in \mathbb{N}\}$ given in (2.1.56), from its spin-1/2 K-operator (2.1.5). This object has indeed been studied for the spin-1/2 XXZ chain in [BK05a, BK05b], but for a quotient of $\mathcal{A}_q$. Moreover, a generating function for spin-j in the commutative subalgebra of

$\mathcal{A}_q$ has never been constructed, and its connection to spin chains remains to be investigated. And finally, a universal version of TT-relations was never studied.

Let us briefly summarize the answers of the above problems. Chapter 3 tackle the first two problems. Firstly, we introduced in Definition 3.1.7 a universal K-matrix $\mathfrak{K} \in B \otimes H$, where B is a comodule algebra over a Hopf algebra H, and show it satisfies a ψ-twisted reflection equation (3.1.14). For a pair of consistent twists and using a formal evaluation representation for $H = \mathcal{L}U_q sl_2$, we show that the evaluation of $\mathfrak{K}$ and one of its axioms lead respectively to a spectral-dependent K-operator (3.2.29) that satisfy the reflection equation (3.2.43) and has the *physical* coaction (3.2.46).

Secondly, studying the tensor product of formal evaluation representations of $\mathcal{L}U_q sl_2$, we constructed $\mathcal{L}U_q sl_2$-intertwining operators. The latter are used to obtain fusion and reduction properties satisfied by L- and K-operators. In particular, based on the results in Proposition 3.4.8 and Remark 3.4.10, and also independently of the above universal framework, we provided a new set of K-operator solutions to the spectral parameter dependent reflection equation, see Theorem 3.5.2. These solutions denoted $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q \otimes \mathrm{End}(\mathbb{C}^{2j+1})$ are expressed in terms of generating functions of the centrally extended q-Onsager algebra $\mathcal{A}_q$. We also gave explicit formulas for the fused R-matrices and the fused K-operators in (3.5.9), (3.5.10), whose expressions contain only the fundamental R-matrix and K-operator. Fused K-matrices were also obtained by applying one-dimensional representations of $\mathcal{A}_q$ to $\mathcal{K}^{(j)}(u)$, see Definition 3.5.5. Additionally, we proposed a relation between the K-operators coming from the evaluation of $\mathfrak{K}$ and the fused K-operators, see Conjecture 1. Supporting evidence was also provided.

Thirdly, Chapter 4 answers the last problem as follows. First, using the fused K-operators $\mathcal{K}^{(j)}(u)$, we have introduced spin-j generating functions in the commutative subalgebra of $\mathcal{A}_q$, denoted $\mathsf{T}^{(j)}(u)$. Then, we assumed the 1-component evaluation of $\mathfrak{K}$ is proportional to $\mathcal{K}^{(j)}(u)$, by an invertible central element $\nu^{(j)}(u) \in \mathcal{A}_q((u^{-1}))$, see the equation (3.6.1) with (3.6.2). Note that these assumptions are parts of Conjecture 1. Using them we have shown in Theorem 4.2.5 that $\mathsf{T}^{(j)}(u)$ satisfy recurrence relations (universal TT-relations) that involve central elements in $\mathcal{A}_q$. Finally, we considered applications of these relations in the context of quantum integrable systems and found that on quantum spin-chains $\mathsf{T}^{(j)}(u)$ are represented by the so-called *transfer matrices* that describe the physical evolution of the system, i.e. they generate the Hamiltonians. More precisely, we begin with recalling the spin-chains representations of $\mathcal{A}_q$, and noticed that the image of the generating functions for $\mathcal{A}_q$ truncates. This motivated us to refine the above analysis done for $\mathcal{A}_q$, by introducing $\mathcal{A}_q^{(N)}$ which is a quotient algebra of $\mathcal{A}_q$ by the relations (4.3.12)-(4.3.13). We found a quotient map $\varphi^{(N)} \colon \mathcal{A}_q \to \mathcal{A}_q^{(N)}$ in (4.3.15), that allowed us to get a FRT type presentation for $\mathcal{A}_q^{(N)}$, as well as truncated fused K-operators. Using the latter, we defined in (4.4.13) the analog for $\mathcal{A}_q^{(N)}$ of the spin-j generating functions in $\mathcal{I}$, denoted $\mathsf{T}^{(j,N)}(u)$, and found they satisfy universal TT-relations (4.4.14). Then, using the spin-chain representations (that automatically verifies the quotient relations of $\mathcal{A}_q^{(N)}$), we recover the transfer matrices of various spin-chains along with the TT-relations they satisfy. This universal approach is also useful to explore symmetries in spin-chains, as it was discussed in Section 4.5.

5.2 Perspectives

The work presented in this thesis opens the way for the following perspectives.

5.2.1 The alternating central extension of the positive part of $U_q\widehat{sl}_2$

In this thesis, we worked with K-operators that satisfy reflection equations. There exists, however, more general equations that we are now interested in. Indeed, the reflection algebra [C84, Sk88] can be viewed as a limiting case of the quadratic algebras introduced by Freidel and Maillet in [FM91]. For instance, as pointed out in [KS92], a Freidel-Maillet type equation can involve two different R-matrices (that might be associated to different Hopf algebras) and a K-operator. For convenience, we use the terminology K-operator for solutions of such equations. We hope this will not bring any confusion with the K-operator solutions of the reflection equation studied in this thesis. For instance, using this more general setting, a Freidel-Maillet type presentation for the so-called alternating central extension of the positive part of $U_q\widehat{sl}_2$ has been introduced in [B20].

Let us denote by U_q^+ the algebra generated by A, B that satisfy the q-Serre relations

$$[A,[A,[A,B]_q]_{q^{-1}}]=0\ ,\qquad [B,[B,[B,A]_q]_{q^{-1}}]=0\ .$$

Two embeddings of U_q^+ into $U_q\widehat{sl}_2$ are known, see the discussion in [B20, Sec. 2]. The image of the first one is the positive part of the Drinfeld-Jimbo presentation of $U_q\widehat{sl}_2$ with

$$A\mapsto E_0\ ,\qquad B\mapsto E_1\ ,$$

while the second corresponds to the positive part of the equitable presentation of $U_q\widehat{sl}_2$ generated by y_0^+, y_1^+, as introduced in [IT03], with

$$A\mapsto y_0^+\ ,\qquad B\mapsto y_1^+\ .$$

We now discuss the alternating central extension of U_q^+, referred to as $\mathcal{U}_q^+$, which bears a certain resemblance to $\mathcal{A}_q$. The precise relation between U_q^+ and $\mathcal{U}_q^+$ is studied in details in [T19]. The algebra $\mathcal{U}_q^+$ has a presentation in terms of generators $\{\mathsf{W}_{-k},\mathsf{W}_{k+1},\mathsf{G}_{k+1},\tilde{\mathsf{G}}_{k+1}|k\in\mathbb{N}\}$ and relations, see [T19, Def. 3.1]. In [B20], a so-called Freidel-Maillet type presentation has been introduced that we now review. Recall the fundamental R-matrix from (1.1.5) and let[24]

$$\bar{R}^{(\frac12,\frac12)}=\mathrm{diag}(1,q^{-1},q^{-1},1)\ , \tag{5.2.1}$$

that satisfies the Yang-Baxter equation (1.1.4) but with the substitution $R^{(\frac12,\frac12)}(u)\to\bar{R}^{(\frac12,\frac12)}$, and also

$$R_{12}^{(\frac12,\frac12)}(u/v)\bar{R}_{13}^{(\frac12,\frac12)}\bar{R}_{23}^{(\frac12,\frac12)}=\bar{R}_{23}^{(\frac12,\frac12)}\bar{R}_{13}^{(\frac12,\frac12)}R_{12}^{(\frac12,\frac12)}(u/v)\ . \tag{5.2.2}$$

[24]Notice that $\bar{R}^{(\frac12,\frac12)}$ is (up to a scalar factor) the spin-1/2 evaluation representation of the Cartan part of the universal R-matrix (B.1.4), given by $q^{\frac12 h_1\otimes h_1}$, as one can easily check from (B.2.17).

Introduce the generating functions:

$$\mathcal{W}_+(u) = \sum_{k\in\mathbb{N}} \mathsf{W}_{-k}U^{-k-1} \ , \quad \mathcal{W}_-(u) = \sum_{k\in\mathbb{N}} \mathsf{W}_{k+1}U^{-k-1} \ ,$$
$$\mathcal{G}_+(u) = \sum_{k\in\mathbb{N}} \mathsf{G}_{k+1}U^{-k-1} \ , \quad \mathcal{G}_-(u) = \sum_{k\in\mathbb{N}} \tilde{\mathsf{G}}_{k+1}U^{-k-1} \ ,$$

where the shorthand notation $U = qu^2/(q+q^{-1})$ is used. The defining relations of $\mathcal{U}_q^+$ are given by the so-called Freidel-Maillet equation[25] [FM91, eq. (14)]

$$R^{(\frac{1}{2},\frac{1}{2})}(u/v)\bar{\mathcal{K}}_1^{(\frac{1}{2})}(u)\bar{R}^{(\frac{1}{2},\frac{1}{2})}\bar{\mathcal{K}}_2^{(\frac{1}{2})}(v) = \bar{\mathcal{K}}_2^{(\frac{1}{2})}(v)\bar{R}^{(\frac{1}{2},\frac{1}{2})}\bar{\mathcal{K}}_1^{(\frac{1}{2})}(u)R^{(\frac{1}{2},\frac{1}{2})}(u/v) \ , \tag{5.2.3}$$

with the R-matrices (1.1.5), (5.2.1) and the spin-1/2 K-operator [B20]

$$\bar{\mathcal{K}}^{(\frac{1}{2})}(u) = \begin{pmatrix} uq\mathcal{W}_+(u) & \frac{1}{k_-(q+q^{-1})}\mathcal{G}_+(u) + \frac{k_+(q+q^{-1})}{(q-q^{-1})} \\ \frac{1}{k_+(q+q^{-1})}\mathcal{G}_-(u) + \frac{k_-(q+q^{-1})}{(q-q^{-1})} & uq\mathcal{W}_-(u) \end{pmatrix} \tag{5.2.4}$$

where $k_\pm \in \mathbb{C}^*$.

The algebra $\mathcal{U}_q^+$ is also a comodule algebra[26] over $H = \mathcal{L}U_qsl_2$, see [B20, Lem. 5.25] for the coaction. The center of $\mathcal{U}_q^+$ is generated by the quantum determinant $\bar{\Gamma}(u)$ [B20, Prop. 3.3]:

$$\bar{\Gamma}(u) = tr_{12}\big(\mathcal{P}_{12}^-\bar{\mathcal{K}}_1^{(\frac{1}{2})}(u)\bar{R}_{12}^{(\frac{1}{2},\frac{1}{2})}\bar{\mathcal{K}}_2^{(\frac{1}{2})}(uq)\big) \ . \tag{5.2.5}$$

Recall the FRT type presentation for the q-Onsager algebra O_q in Section 2.2.1. It is given by the reflection equation (2.1.4) satisfied by (2.1.5), and the quantum determinant $\Gamma(u)$ in (2.1.44) is set to a scalar as in (2.2.2). The situation is similar for U_q^+. Indeed, the Freidel-Maillet type presentation for U_q^+ is given by the Freidel-Maillet equation satisfied by (5.2.4), and setting the quantum determinant (5.2.5) to a scalar as in [B21, Lem. 4.5].

It is natural to pose similar questions to those for $\mathcal{A}_q$.

Questions

(i) What would be the universal K-matrix framework that leads after evaluation to K-operator solutions of the Freidel-Maillet equation (5.2.3) ?

(ii) How to construct fused K-operators for $\mathcal{U}_q^+$ that solve higher spin versions of Freidel-Maillet equations (5.2.3), see eq. (5.2.8) below ?

(iii) What are the universal TT-relations satisfied by the analogs of $\mathsf{T}^{(j)}(u)$ for $\mathcal{U}_q^+$?

(iv) What types of physical systems with mutually conserved quantities can we derive from these K-operators ?

[25] The equation (5.2.3) is a special case of [FM91, eq. (14)] with $A_{12} = D_{12} = R^{(\frac{1}{2},\frac{1}{2})}(u/v)$, $B_{12} = C_{12} = \bar{R}^{(\frac{1}{2},\frac{1}{2})}$, $T_1 = \bar{\mathcal{K}}_1^{(\frac{1}{2})}(u)$, $T_2 = \bar{\mathcal{K}}_2^{(\frac{1}{2})}(v)$.

[26] Actually it is a left comodule algebra in [B20], while in this thesis we always use right comodule algebras. A right coaction is similarly obtained by using in the latter reference the opposite coproduct.

We have already found solutions to some of these questions. The results presented here, actually constitute an upcoming paper *Fused K-operators and universal TT-relations for the positive part of* $U_q\widehat{sl}_2$. We now answer or give research directions for these questions.

Elements of response

(i) Recall that our motivation in establishing the axioms of $\mathfrak{K} \in B \otimes H$ in Definition 3.1.7 was firstly to construct K-operator solutions of the reflection equation with spectral parameter. Secondly, to ensure that the specialization of (K2) results in a physical coaction of the form (0.0.16). For $B = \mathcal{A}_q$, $H = \mathcal{L}U_q sl_2$ and the twist pair $(\psi, J) = (\eta, 1 \otimes 1)$, we indeed showed these two aspects.

Now, if we consider the comodule algebra $B = \mathcal{U}_q^+$ over $H = \mathcal{L}U_q sl_2$, the situation is different. Indeed, in this case, the K-operator satisfy now the Freidel-Maillet equation (5.2.3). Moreover, the coaction takes the form [B20]

$$(\delta \otimes \mathrm{id})(\bar{\mathcal{K}}^{(\frac{1}{2})}(u)) = \big(\bar{\mathcal{L}}^{(\frac{1}{2})}\big)_{[2]} \big(\bar{\mathcal{K}}^{(\frac{1}{2})}(u)\big)_{[1]} \big(\mathcal{L}^{(\frac{1}{2})}(u)\big)_{[2]} \,, \tag{5.2.6}$$

where $\bar{\mathcal{L}}^{(\frac{1}{2})}$ satisfies analog of the RLL equation (1.1.9) with $\bar{R}^{(\frac{1}{2},\frac{1}{2})}$, which is different from the physical coaction (0.0.16). Importantly, the algebra $\mathcal{U}_q^+$ involves two different R-matrices associated respectively to $H = \mathcal{L}U_q sl_2$ and its Cartan subalgebra. It is not possible to obtain the above coaction (5.2.6) by evaluating (K2) because there are no automorphism ψ of $\mathcal{L}U_q sl_2$ that can give a diagonal matrix out of the evaluation of $(\mathfrak{R}^{\psi})_{21}$. As a consequence, a modification of the axiom (K2) is needed to treat this more general framework.

(ii) We want to construct fused R-matrices and fused K-operators that satisfy

$$R_{12}^{(j_1,j_2)}(u/v)\bar{R}_{13}^{(j_1,j_3)}\bar{R}_{23}^{(j_2,j_3)} = \bar{R}_{23}^{(j_2,j_3)}\bar{R}_{13}^{(j_1,j_3)}R_{12}^{(j_1,j_2)}(u/v) \,, \tag{5.2.7}$$

$$R^{(j_1,j_2)}(u/v)\ \bar{\mathcal{K}}_1^{(j_1)}(u)\ \bar{R}^{(j_1,j_2)}\ \bar{\mathcal{K}}_2^{(j_2)}(v) = \bar{\mathcal{K}}_2^{(j_2)}(v)\ \bar{R}^{(j_1,j_2)}\ \bar{\mathcal{K}}_1^{(j_1)}(u)\ R^{(j_1,j_2)}(u/v) \,, \tag{5.2.8}$$

where $R^{(j_1,j_2)}(u)$ are given in (3.4.11). Recall the intertwining operator $\mathcal{E}^{(j+\frac{1}{2})}$ is fixed by Lemma C.1.1 and its pseudo-inverse $\mathcal{F}^{(j+\frac{1}{2})}$ is given in (C.1.6) with (C.1.7), (C.1.8). First, recall $\bar{R}^{(\frac{1}{2},\frac{1}{2})}$ is associated to the Cartan subalgebra of $\mathcal{L}U_q sl_2$, and so we can use same the intertwiner to construct fused R-matrices. Introduce

$$\bar{R}^{(j_1,j_2)} = \mathcal{F}_{\langle 12\rangle}^{(j_1)}\bar{R}_{13}^{(\frac{1}{2},j_2)}\bar{R}_{23}^{(j_1-\frac{1}{2},j_2)}\mathcal{E}_{\langle 12\rangle}^{(j_1)} \,, \tag{5.2.9}$$

with

$$\bar{R}^{(\frac{1}{2},j)} = \mathcal{F}_{\langle 23\rangle}^{(j)}\bar{R}_{13}^{(\frac{1}{2},j-\frac{1}{2})}\bar{R}_{12}^{(\frac{1}{2},\frac{1}{2})}\mathcal{E}_{\langle 23\rangle}^{(j)} \,, \tag{5.2.10}$$

and where $\bar{R}^{(\frac{1}{2},\frac{1}{2})}$ is given in (5.2.1). We can show by induction that they indeed satisfy (5.2.7). Then, we introduce fused K-operators $\bar{\mathcal{K}}^{(j)}(u) \in \mathcal{U}_q^+((u)) \otimes \mathrm{End}(\mathbb{C}^{2j+1})$ as

$$\bar{\mathcal{K}}^{(j)}(u) = \mathcal{F}_{\langle 12\rangle}^{(j)}\bar{\mathcal{K}}_1^{(\frac{1}{2})}(uq^{-j+\frac{1}{2}})\bar{R}^{(\frac{1}{2},j-\frac{1}{2})}\bar{\mathcal{K}}_2^{(j-\frac{1}{2})}(uq^{\frac{1}{2}})\mathcal{E}_{\langle 12\rangle}^{(j)} \,, \tag{5.2.11}$$

where $\bar{\mathcal{K}}^{(\frac{1}{2})}(u)$ is defined in (5.2.4). Similarly to the proof that fused K-operators $\mathcal{K}^{(j)}(u)$ satisfy the reflection equations (3.5.2), we can also prove by induction that $\bar{\mathcal{K}}^{(j)}(u)$ satisfy (5.2.8). Additionally, studying one-dimensional representations for $\mathcal{U}_q^+$, we also obtain K-matrix solutions of the Freidel-Maillet equation out of these K-operators, and find they satisfy a fusion formula. The fused K-matrices are thus given by (5.2.11) with the substitution $\bar{\mathcal{K}}^{(j)}(u) \to \bar{K}^{(j)}(u)$, and with the fundamental K-matrix

$$\bar{K}^{(\frac{1}{2})}(u) = \begin{pmatrix} u^{-1}\varepsilon_+ & \frac{k_+}{q-q^{-1}} \\ \frac{k_-}{q-q^{-1}} & u^{-1}\varepsilon_- \end{pmatrix} ,$$

where $k_\pm, \varepsilon_\pm \in \mathbb{C}$. Note that it is related with the K-matrix (1.2.3) from [GZ93, dVR94]

$$\bar{K}^{(\frac{1}{2})}(u) = u^{-2}K^{(\frac{1}{2})}(u) + O(u^{-3}) .$$

They satisfy the Freidel-Maillet equation (5.2.8) with $\bar{\mathcal{K}}^{(j)}(u) \to \bar{K}^{(j)}(u)$.

(iii) In order to construct the analog of $\mathsf{T}^{(j)}(u)$ for $\mathcal{U}_q^+$, we first introduce fused dual K-matrices as

$$\bar{K}^{+(j)}(u) = [\bar{K}^{(j)}(u)]^{-1}|_{k_\pm \to \bar{k}_\pm, \varepsilon_\pm \to \bar{\varepsilon}_\pm}$$

that satisfy the dual Freidel-Maillet equations

$$R^{(j_1,j_2)}(v/u)\bar{K}_1^{+(j_1)}(u)[\bar{R}^{(j_1,j_2)}]^{-1}\bar{K}_2^{+(j_2)}(u) = \bar{K}_2^{+(j_2)}(u)[\bar{R}^{(j_1,j_2)}]^{-1}\bar{K}_1^{+(j_1)}(u)R^{(j_1,j_2)}(v/u) .$$

Then, we define spin-j generating functions in $\bar{\mathcal{I}}$, a commutative subalgebra of $\mathcal{U}_q^+$, as

$$\bar{\mathsf{T}}^{(j)}(u) = \mathrm{tr}_{V^{(j)}}(\bar{K}^{+(j)}(u)\bar{\mathcal{K}}^{(j)}(u)) .$$

Note that $\bar{\mathcal{I}}$ is straightforwardly extracted from $\bar{\mathsf{T}}^{(\frac{1}{2})}(u)$. Recall $\bar{\Gamma}(u)$ is given in (5.2.5), and introduce the quantum determinant of $\bar{K}^+(u)$ as

$$\bar{\Gamma}^+(u) = \frac{\bar{k}_+\bar{k}_-}{(q-q^{-1})^2} - u^{-2}\bar{\varepsilon}_+\bar{\varepsilon}_- .$$

Using the PBW basis for $\mathcal{U}_q^+$ from [T19, Thm. 10.2], we show for $j = 1, 3/2$ that $\bar{\mathsf{T}}^{(j)}(u)$ satisfy the universal TT-relations

$$\bar{\mathsf{T}}^{(j)}(u) = \bar{\mathsf{T}}^{(j-\frac{1}{2})}(uq^{-\frac{1}{2}})\bar{\mathsf{T}}^{(\frac{1}{2})}(uq^{j-\frac{1}{2}}) + \bar{\Gamma}(uq^{j-\frac{3}{2}})\bar{\Gamma}^+(uq^{j-\frac{3}{2}})\bar{\mathsf{T}}^{(j-1)}(uq^{-1}) , \tag{5.2.12}$$

with $\bar{\mathsf{T}}^{(0)}(u) = 1$. As for $\mathcal{A}_q$, we also expect to find a proportionality relation between the fused K-operators $\bar{\mathcal{K}}^{(j)}(u)$ and a 1-component evaluation of a universal K-matrix in $\mathcal{U}_q^+ \otimes \mathcal{L}U_q sl_2$ that would allow to show (5.2.12) for any spin-j.

(iv) The last question is about specializations of the K-operators to quantum integrable systems. First recall that $\vartheta^{(N)}$, defined in (4.3.2), provides spin-representations for $\mathcal{A}_q$. Its analog for $\mathcal{U}_q^+$ is actually known and can be found in [B20, eqs. (4.16)-(4.19)]. Then, we can show that

the image of the fused K-operator $\bar{\mathcal{K}}^{(j)}(u)$ takes the form

$$\bar{L}^{(j)}_{[N]}\cdots\bar{L}^{(j)}_{[1]}\bar{K}^{(j)}(u)L^{(j)}_{[1]}(u)\cdots L^{(j)}_{[N]}(u) \quad \in \mathrm{End}(\mathbb{C}^{2j_N+1}\otimes\ldots\otimes\mathbb{C}^{2j_1+1}\otimes\mathbb{C}^{2j+1}) \ ,$$

where $\bar{L}^{(j)}_{[1]} \propto \bar{R}^{(j_1,j)}$ is the evaluation in spin-j representation of $\mathcal{L}^{(j)}$ that appears in (5.2.6). Note that it is a spin-j generalization of [B20, Prop. 4.1], and it is the analog of the image of $\mathcal{K}^{(j)}(u)$ on spin-chain representations given in (4.4.7). The first case to consider would be $j = 1/2$ and all the quantum space fixed to $j_n = 1/2$, with $n = 1, 2, \ldots N$. It is an interesting question to ask what are the interaction terms in the corresponding Hamiltonians, whether they are local or not.

5.2.2 Construction of a universal K-matrix

Recall Tolstoy and Khoroshkin first obtained a factorized form of the universal R-matrix associated with the untwisted affine Lie algebras in [KT92a] and they gave it explicitly for $H = U_q\widehat{sl}_2$ in [KT92b, eq. (58)]. Then, Damiani in [Da98] gave a detailed construction of $\mathfrak{R}$ for $H = U_q(\hat{\mathfrak{g}})$. The universal R-matrix is expressed in terms of the root vectors of H, see its explicit form for $H = \mathcal{L}U_q sl_2$ in our conventions in Appendix B. In the present thesis, we have introduced an axiomatic definition of a universal K-matrix $\mathfrak{K} \in B \otimes H$ in Definition 3.1.7. For our purpose, we mostly considered the case where $B = \mathcal{A}_q$ and $H = \mathcal{L}U_q sl_2$. Some question naturally arises.

(i) What is the explicit expression for a universal K-matrix $\mathfrak{K}$ for a comodule algebra $B = \mathcal{A}_q$ or $B = O_q$ over $H = \mathcal{L}U_q sl_2$?

Finding an explicit expression for a universal K-matrix that satisfies equations (K1)-(K3) is highly desirable. For instance, this is essential in order to establish Conjecture 1, which plays a crucial role in proving the universal TT-relations in Theorem 4.2.5. Let us now explore potential research directions for addressing the question mentioned above.

Elements of response

- Let us stress that we do not expect the universal K-matrix to be in $B \otimes \mathcal{L}U_q sl_2$, but rather in an appropriate completion of the tensor product. For the finite Dynkin diagrams case, one-leg universal K-matrices are known to be written in the form of finite product of infinite sums [DKo18], see [CG92, KS92] for $H = U_q sl_2$. In the affine case, $\mathfrak{K}$ is even expected to be of the form of an infinite product over root vectors of B, as it is the case for the universal R-matrices. In this perspective, recall the root vectors of $B = O_q$ are known [BK17], as well as their relations with the alternating generators of $B = \mathcal{A}_q$ [T21c].
  Let us consider the choice for the twist pair $(\psi, J) = (\eta, 1 \otimes 1)$, where η is defined by (3.1.12). By definition, the universal K-matrix should satisfy the axioms (K1)-(K3). The key equation that needs to be solved to derive an expression for $\mathfrak{K}$ is the twisted intertwining relation (K1), which is given by:
  $$\mathfrak{K}\ \delta(b) = \delta^{\eta}(b)\ \mathfrak{K}\ , \qquad \text{for all } b \in B\ . \tag{5.2.13}$$
  It is sufficient to consider the elements of the set $\{\mathsf{W}_0, \mathsf{W}_1, \Delta_{k+1}, k \in \mathbb{N}\}$ for b, where Δ_{k+1} are central elements. This follows from the fact that $\mathcal{A}_q$ is the central extension of O_q – it is

generated by W_0, W_1 and its center [T21a]. The coaction $\delta(\mathsf{W}_0)$ and $\delta(\mathsf{W}_1)$ are given for $\mathcal{A}_q$ by (2.4.2)-(2.4.3). However, the $\delta(\Delta_{k+1})$'s are not known. They should be extracted from the coaction of the quantum determinant $\Gamma(u)$, but this has not been studied yet. A first step on that direction would be to find the expression for $\delta(\Gamma(u))$. This could be reached as follows. Firstly, we choose an ansatz for $\delta(\Gamma(u))$ of the form

$$\Gamma(u) \otimes \sum \text{PBW elem. of } \mathcal{L}U_q sl_2$$

such that it gives $\delta_w(\Gamma(u))$, given in (3.5.27), after evaluation. Secondly, imposing the coassociativity and counital conditions given in Definition 2.4.1, we expect this sets the expression for $\delta(\Gamma(u))$.

In [KT92b], it is shown that a solution of the intertwining relation (R1) is unique and automatically satisfies (R2)-(R3). For the universal K-matrix, we expect that a solution of the above equation (5.2.13) similarly satisfies (K2)-(K3).

- Now, consider $B = O_q$, where the coaction is explicitly known. Let us recall that the universal R-matrix for $H = U_q\widehat{sl}_2$ was constructed using the following key ingredients [Da98]:
  - PBW bases of $U_q^{\pm}$;
  - the Killing form;
  - the braid group action;
  - the coproduct formulas for root vectors.

  The final expression of the universal R-matrix for $H = U_q\widehat{sl}_2$ is then given in terms of dual PBW bases of U_q^+ and U_q^- (dual with respect to the Killing form). The construction of a universal K-matrix could draw inspiration on this approach. Some of the above-mentioned key ingredients are known for O_q such as the root vectors and the braid group action [BK17]. What is missing is an appropriate concept of the Killing form and the explicit coaction expression for the root vectors of O_q.

- Recall that $B = O_q$ is a right coideal subalgebra defined by the embedding $O_q \to U_q\widehat{sl}_2$ [BB12]

$$\mathcal{W}_0 \to k_+E_1 + k_-q^{-1}F_1K_1 + \varepsilon_+K_1 \ , \tag{5.2.14}$$
$$\mathcal{W}_1 \to k'_-E_0 + k'_+q^{-1}F_0K_0 + \varepsilon_-K_0 \ , \tag{5.2.15}$$

  where $k_\pm, k'_\pm, \varepsilon_\pm \in \mathbb{C}$, and such that (2.2.8)-(2.2.9) are satisfied for

$$\rho = k_+k_-(q+q^{-1})^2 \ , \qquad \rho^* = k'_+k'_-(q+q^{-1})^2 \ .$$

  In this case, finding an expression of a universal K-matrix might be simpler. Indeed, recall that such case was discussed in the second item of Remark 3.1.10. Instead of solving (5.2.13), it is sufficient to solve for all $b \in B$:

$$\mathcal{K}b = \eta(b)\mathcal{K} \ , \tag{5.2.16}$$

  where $\mathcal{K} \in H$, or in an appropriate completion of H. If such a solution $\mathcal{K}$ exists and satis-

fies (3.1.22), then the two-leg universal K-matrix of the form

$$\mathfrak{K} = (\mathfrak{R}^{\eta})_{21}\mathcal{K}_2\mathfrak{R} \tag{5.2.17}$$

satisfies (K1)-(K3) as was shown in (ii) of Remark 3.1.10. Then, one should take an ansatz for $\mathcal{K}$ in terms of the root vectors of $U_q\widehat{sl}_2$, and insert this expression in (5.2.16) for b as in (5.2.14)-(5.2.15). In practice, this makes use of ordering relations [Da93] between the root vectors and E_i, F_i, K_i, $i = 0, 1$.

In [AV20], existence results were given for a universal K-matrix $\mathcal{K} \in B$, where B is a certain coideal subalgebra of H (a quantum symmetric pair), and with a specific choice of twist pair (different from ours). Notably, the q-Onsager case is discussed in [AV20, Sec. 9.5]. In [AV22], they show that $\mathcal{K}$ evaluated on finite-dimensional representations satisfies the spectral parameter dependent reflection equation (3.5.21). Recall that we do not know if a universal K-matrix $\mathfrak{K} \in B \otimes \mathcal{L}U_q sl_2$ exists for $B = O_q$ and our choice of twist pair $(\psi, J) = (\eta, 1 \otimes 1)$. It is thus desirable to show existence of $\mathfrak{K}$ with our twist pair, for instance, in the q-Onsager case. One idea is to gauge transform $\mathcal{K}$ from [AV20] to a new $\mathcal{K}$ that satisfies (5.2.16), and then construct $\mathfrak{K}$ with (5.2.17). In more details, in [AV20], the cylindrical structure has been introduced as the data of a quasi-triangular bialgebra $(H, \mathfrak{R}, \Delta)$, with a coideal subalgebra $B \subseteq H$ together with a pair of consistent twists. In order to apply existence results of [AV20], one should use a gauge transformation of the cylindrical structure, see [AV20, Rem. 8.11] and [AV22, Cor. 3.6.1], such that it results in the twist pair $(\eta, 1 \otimes 1)$. It is however not clear whether it is possible, and if so, how different will be Δ and J after the gauge transformation. In particular, the fusion formula for the K-operators might be affected by non-trivial J, as well as the transfer matrix from Section 1.2.3. It remains a challenge to find such gauge transformation.

Finally, let us mention a special case. Setting, $\rho = \rho^* = 0$, the q-Dolan-Grady relations reduce to the q-Serre relations (2.3.8) satisfied by $\mathcal{W}_0, \mathcal{W}_1$. Thus, O_q degenerates to U_q^+. If a solution of (5.2.16) for this positive part existed, it should necessarily take the form (5.2.17). However, from the expression of the universal R-matrix given in (B.1.4), it is clear that the first component of (5.2.17) also contains F_0, F_1 which do not belong to U_q^+. We thus claim, in this case, that the universal K-matrix defined with our axioms (K1)-(K3) does not exist for U_q^+. This leads us to a more general framework behind Freidel-Maillet equations (5.2.8), as discussed in (i) for the alternating central extension of U_q^+.

Appendix A

Some applications of the algebraic Bethe ansatz

In this appendix, we review the algebraic Bethe ansatz to obtain the eigenvalues and the corresponding eigenvectors of the transfer matrix. First, we apply it for the XXZ spin-1/2 chain with periodic boundary conditions [STF79], and then for the XXZ spin-1/2 chain with diagonal boundary conditions [Sk88].

A.1 Closed XXZ spin-$\frac{1}{2}$ chain with periodic boundary conditions

Recall that periodic spin-chains are studied using the Yang-Baxter algebra, the RTT equation and the transfer matrix, see Definitions 1.1.1, 1.1.5 and 1.1.6, respectively. In particular the Hamiltonian of this spin-chain is derived from the transfer matrix, see Prop. 1.1.7. Now, we diagonalize the transfer matrix using the algebraic Bethe ansatz.

Let us fix all the algebraic elements that we have introduced until now. Consider the R-matrix in (1.1.5) and let

$$a^{6V}(u) = uq - u^{-1}q^{-1} \ , \qquad b^{6V}(u) = u - u^{-1} \ , \qquad c^{6V} = q - q^{-1} \ . \tag{A.1.1}$$

The Hamiltonian of the system is given by (1.1.16) and it enjoys the $U(1)$ symmetry

$$\left[H^{per}_{XXZ}, S^z\right] = 0 \ , \qquad S^z = \sum_{k=1}^{N} \sigma_z^k \ . \tag{A.1.2}$$

The monodromy matrix defined in (1.1.11) with the R-matrix given in (1.1.5), can be seen as a two-by-two matrix as follows:

$$T_{a,N}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} \ , \tag{A.1.3}$$

where $\mathcal{A}(u), \mathcal{B}(u), \mathcal{C}(u)$ and $\mathcal{D}(u) \in \bigotimes_{i=1}^{N} \mathrm{End}(V_{[i]})$, with the quantum spaces $V_{[i]} = \mathbb{C}^2$ here. Then,

the transfer matrix in (1.1.14) reads

$$t^{(N)}(u) = \mathcal{A}(u) + \mathcal{D}(u) \ . \tag{A.1.4}$$

Now, in order to apply the algebraic Bethe ansatz, we need exchange relations between $\mathcal{A}(u)$, $\mathcal{B}(u)$, $\mathcal{C}(u)$ and $\mathcal{D}(u)$. They are obtained by inserting $T_{a,N}(u)$ of the form (A.1.3) in the RTT relation (1.1.6), and by comparing the resulting matrix entries. In particular, the following will be useful

$$\begin{aligned} &[\mathcal{B}(u), \mathcal{B}(v)] = 0 \ , \\ &\mathcal{A}(v)\mathcal{B}(u) = \frac{a^{6V}(u/v)}{b^{6V}(u/v)}\mathcal{B}(u)\mathcal{A}(v) - \frac{c^{6V}}{b^{6V}(u/v)}\mathcal{B}(v)\mathcal{A}(u) \ , \\ &\mathcal{D}(u)\mathcal{B}(v) = \frac{a^{6V}(u/v)}{b^{6V}(u/v)}\mathcal{B}(v)\mathcal{D}(u) - \frac{c^{6V}}{b^{6V}(u/v)}\mathcal{B}(u)\mathcal{D}(v) \ . \end{aligned} \tag{A.1.5}$$

Besides, we will also need the action of $\mathcal{A}(u)$ and $\mathcal{D}(u)$ on $\prod_{k=1}^{M}\mathcal{B}(u_k)$ with $u_k \in \mathbb{C}^*$.

Proposition A.1.1. *The following relations hold for $M = 1, 2, \dots, N$:*

$$\mathcal{A}(u)\prod_{k=1}^{M}\mathcal{B}(u_k) = \prod_{k=1}^{M}\frac{a^{6V}(u_k/u)}{b^{6V}(u_k/u)}\mathcal{B}(u_k)\,\mathcal{A}(u) + \sum_{n=1}^{M}\left(\frac{c^{6V}}{b^{6V}(u/u_n)}\mathcal{B}(u)\prod_{\substack{k=1\\k\neq n}}^{M}\frac{a^{6V}(u_n/u_k)}{b^{6V}(u_n/u_k)}\mathcal{B}(u_k)\,\mathcal{A}(u_n)\right) \ ,$$

$$\mathcal{D}(u)\prod_{k=1}^{M}\mathcal{B}(u_k) = \prod_{k=1}^{M}\frac{a^{6V}(u/u_k)}{b^{6V}(u/u_k)}\mathcal{B}(u_k)\,\mathcal{D}(u) + \sum_{n=1}^{M}\left(\frac{c^{6V}}{b^{6V}(u_n/u)}\mathcal{B}(u)\prod_{\substack{k=1\\k\neq n}}^{M}\frac{a^{6V}(u_k/u_n)}{b^{6V}(u_k/u_n)}\mathcal{B}(u_k)\,\mathcal{D}(u_n)\right) \ .$$

Proof. The proof is done by induction on M, using the exchange relations (A.1.5) and the property $b^{6V}(v/u) = -b^{6V}(u/v)$. □

The algebraic Bethe ansatz relies on the exists on a reference state called the *pseudo-vacuum state*. It is a highest weight state denoted $|\Uparrow\rangle$, that satisfies:

$$\begin{aligned} &\mathcal{A}(u)\,|\Uparrow\rangle = a(u)\,|\Uparrow\rangle \ , \\ &\mathcal{D}(u)\,|\Uparrow\rangle = d(u)\,|\Uparrow\rangle \ , \\ &\mathcal{C}(u)\,|\Uparrow\rangle = 0 \ , \end{aligned} \tag{A.1.6}$$

where $a(u)$ and $d(u)$ are some scalar functions. If such state exists[27], then one constructs eigenstates of the transfer matrix in the form $\prod_{k=1}^{M}\mathcal{B}(u_k)\,|\Uparrow\rangle$, $M = 1, 2, \dots, N$, called the Bethe states, provided the $\{u_k\}$ satisfy some relations called the Bethe equations.

Let us apply the ABA to the N sites periodic spin-$\frac{1}{2}$ XXZ chain. Firstly, due to the $U(1)$ symmetry of the Hamiltonian (A.1.2), an easy reference state exists and a solution of (A.1.6) is

[27] The relations in (A.1.6) correspond to a highest-weight state. Another possibility is to consider the reference state as a lowest weight state defined by $\mathcal{A}(u)\,|\Downarrow\rangle = \bar{a}(u)\,|\Downarrow\rangle$, $\mathcal{D}(u)\,|\Downarrow\rangle = \bar{d}(u)\,|\Downarrow\rangle$ and $\mathcal{B}(u)\,|\Downarrow\rangle = 0$.

given by

$$|\Uparrow\rangle = \bigotimes_{k=1}^{N} |\uparrow\rangle \ , \qquad \text{with} \quad |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ . \tag{A.1.7}$$

It corresponds to the completely ferromagnetic state with all spins up. Then, the action of $\mathcal{A}(u)$ and $\mathcal{D}(u)$ on this reference state gives

$$a(u) = (a^{6V}(u))^N \ , \qquad d(u) = (b^{6V}(u))^N \ . \tag{A.1.8}$$

Secondly, define the Bethe states for $M = 1, 2, \ldots, N$, by taking the eigenvectors of $t^{(N)}(u)$ of the form

$$|\psi^{(M)}; \{u_k\}\rangle = \prod_{k=1}^{M} \mathcal{B}(u_k) \, |\Uparrow\rangle \ . \tag{A.1.9}$$

Since $|\Uparrow\rangle$ is the completely ferromagnetic state, then M is interpreted as the number of excitations, so $|\psi^{(M)}; \{u_k\}\rangle$ is a state with M spins down. Finally, we solve the eigenvalue problem of the transfer matrix [STF79, TF79].

Theorem A.1.2. *The Bethe states $|\psi^{(M)}; \{u_k\}\rangle$ are the eigenvectors of the transfer matrix $t^{(N)}(u)$ in (A.1.4) with the associated eigenvalues*

$$\Lambda^{(M)}(u; \{u_k\}) = a(u) \prod_{k=1}^{M} \frac{a^{6V}(u_k/u)}{b^{6V}(u_k/u)} + d(u) \prod_{k=1}^{M} \frac{a^{6V}(u/u_k)}{b^{6V}(u/u_k)} \ , \tag{A.1.10}$$

where $a(u)$ and $d(u)$ are given in (A.1.8), and provided that the scalars u_n, $n = 1, 2, \ldots, M$ satisfy the Bethe equations

$$\frac{a(u_n)}{d(u_n)} = \prod_{\substack{k=1 \\ k \neq n}}^{M} \frac{a^{6V}(u_k/u_n)}{a^{6V}(u_n/u_k)} \ . \tag{A.1.11}$$

Proof. Compute $t^{(N)}(u) \, |\psi^{(M)}; \{u_k\}\rangle = (\mathcal{A}(u) + \mathcal{D}(u)) \, |\psi^{(M)}; \{u_k\}\rangle$ using (A.1.9) and Proposition A.1.1. Then, the eigenvalues $\Lambda^{(M)}(u; \{u_k\})$ are obtained by imposing the vanishing of the *unwanted terms* of the form

$$\mathcal{B}(u) \prod_{\substack{k=1 \\ k \neq n}}^{M} \mathcal{B}(u_k) \, |\Uparrow\rangle \ , \tag{A.1.12}$$

which amounts to impose the Bethe equations (A.1.11). □

Here we have treated the case of the XXZ spin-chain with periodic boundary conditions and we have solved the spectral problem via Bethe ansatz. In addition, there exists the so-called anti-periodic (or twisted) boundary conditions, defined[28] by $\sigma^i_{N+1} \equiv \sigma^x_1 \sigma^i_1 \sigma^x_1$, whose spectral problem cannot be solved by Bethe ansatz. Indeed, the $U(1)$ symmetry is broken and the transfer matrix does not have an obvious reference state. However, the transfer matrix can be diagonalized using the *off-diagonal Bethe ansatz* and the correlation functions, form factors are also obtained. As it goes beyond this thesis, we refer the interested reader to [ODBA] for more information on this topic.

[28] More generally, it is defined by $\sigma^i_{N+1} \equiv K_1 \sigma^i_1 K_i^{-1}$, where K is a two-by-two unitary matrix.

A.2 Open XXZ spin-$\frac{1}{2}$ chain with diagonal boundary conditions

We now review the application of the algebraic Bethe ansatz for the XXZ closed spin-1/2 chain with diagonal boundary conditions [Sk88]. The diagonalization of the transfer matrix defined in (1.2.15) can be done similarly to the case of the N sites periodic spin-$\frac{1}{2}$ XXZ chain studied in Appendix A.1. However, care must be taken in choosing the resolution method. Indeed, recall that the algebraic Bethe ansatz relies on the existence of a reference state.

For instance, consider the XXZ spin-$\frac{1}{2}$ chain with diagonal boundary conditions. As it is mostly used in the literature for this case, we adopt the trigonometric R-matrix (1.1.3). A solution of the reflection equation is given by [C84]:

$$K^d(\lambda,\xi) = \begin{pmatrix} \mathrm{sh}(\lambda+\xi) & 0 \\ 0 & \mathrm{sh}(-\lambda+\xi) \end{pmatrix} , \tag{A.2.1}$$

where $\xi \in \mathbb{C}$. Now, following [Sk88], consider the left and right boundary K-matrices defined respectively by

$$K^{d,-}(\lambda) = K^d(\lambda,\xi_-) , \qquad K^{d,+}(\lambda) = K^d(\lambda+\eta,\xi_+) , \tag{A.2.2}$$

where $\xi_\pm \in \mathbb{C}$. Note that these matrices are related with the ones in (1.2.3) and (1.2.6) by setting $k_\pm = \overline{k}_\pm = 0$, $q = \exp(\eta)$, $u = \exp(\lambda)$, $\varepsilon_\pm = \pm\exp(\pm\xi_-)$ and $\overline{\varepsilon}_\pm = \pm\exp(\pm\xi_+)$.

Then, the corresponding Hamiltonian of the system is

$$\mathcal{H}^{diag}_{XXZ} = \sum_{k=1}^{N-1} \left(\sigma^x_k\sigma^x_{k+1} + \sigma^y_k\sigma^y_{k+1} + \cosh(\eta)\sigma^z_k\sigma^z_{k+1} \right) + \coth(\xi_-)\sigma^z_1 + \coth(\xi_+)\sigma^z_N . \tag{A.2.3}$$

Note that it has $U(1)$ symmetry

$$\left[\mathcal{H}^{diag}_{XXZ}, S^z\right] = 0 , \tag{A.2.4}$$

where S^z is given in (A.1.2). In this case, a reference state exists so that the algebraic Bethe ansatz can be applied [Sk88]. The method is similar to the case of the closed spin-chains described in Section 1.1.

The analog of the exchange relations (A.1.5) are more complicated because the underlying algebra is now the reflection algebra. They are obtained as follows. Writing the double-row monodromy matrix as

$$\mathbb{T}_{a,N}(\lambda) = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix} , \tag{A.2.5}$$

the exchange relations between these operators are extracted by inserting (A.2.5) and the trigonometric R-matrix (1.1.3) in the reflection equation (1.2.13) with the substitution $R(uv) \to R^{df}(\lambda_1+\lambda_2)$, $R(u/v) \to R^{df}(\lambda_1-\lambda_2)$ and $\mathbb{T}_{a,N}(u) \to \mathbb{T}_{a,N}(\lambda_1)$. For instance, one has for λ_1, $\lambda_2 \in \mathbb{C}$:

$$[\mathcal{B}(\lambda_1),\mathcal{B}(\lambda_2)] = 0 ,$$
$$\mathcal{A}(\lambda_1)\mathcal{B}(\lambda_2) = \frac{\mathrm{sh}(\lambda_1+\lambda_2)\mathrm{sh}(\lambda_1-\lambda_2-\eta)}{\mathrm{sh}(\lambda_1+\lambda_2+\eta)\mathrm{sh}(\lambda_1-\lambda_2)}\mathcal{B}(\lambda_2)\mathcal{A}(\lambda_1)$$

$$+\frac{\mathrm{sh}(\eta)\mathrm{sh}(\lambda_1+\lambda_2)}{\mathrm{sh}(\lambda_1-\lambda_2)\mathrm{sh}(\lambda_1+\lambda_2+\eta)}\mathcal{B}(\lambda_1)\mathcal{A}(\lambda_2)-\frac{\mathrm{sh}(\eta)}{\mathrm{sh}(\lambda_1+\lambda_2+\eta)}\mathcal{B}(\lambda_1)\mathcal{D}(\lambda_2)\ ,$$

$$\mathcal{D}(\lambda_1)\mathcal{B}(\lambda_2)=\frac{\mathrm{sh}(\lambda_1-\lambda_2+\eta)\mathrm{sh}(\lambda_1+\lambda_2+2\eta)}{\mathrm{sh}(\lambda_1-\lambda_2)\mathrm{sh}(\lambda_1+\lambda_2+\eta)}\mathcal{B}(\lambda_2)\mathcal{D}(\lambda_1)-\frac{2\mathrm{sh}^2(\eta)\mathrm{ch}(\eta)}{\mathrm{sh}(\lambda_1-\lambda_2)\mathrm{sh}(\lambda_1+\lambda_2+\eta)}\mathcal{B}(\lambda_2)\mathcal{A}(\lambda_1)$$
$$+\frac{\mathrm{sh}(\eta)\mathrm{sh}(\lambda_1+\lambda_2+2\eta)}{\mathrm{sh}(\lambda_1-\lambda_2)\mathrm{sh}(\lambda_1+\lambda_2+\eta)}\mathcal{B}(\lambda_1)\mathcal{D}(\lambda_2)+\frac{\mathrm{sh}(\eta)\mathrm{sh}(\lambda_1-\lambda_2+2\eta)}{\mathrm{sh}(\lambda_1-\lambda_2)\mathrm{sh}(\lambda_1+\lambda_2+\eta)}\mathcal{B}(\lambda_1)\mathcal{A}(\lambda_2)\ .$$

Then, following [Sk88], introduce $\tilde{D}(\lambda)=\mathrm{sh}(2\lambda+\eta)\mathcal{D}(\lambda)-\mathrm{sh}(\eta)\mathcal{A}(\lambda)$ so that the last two relations slightly simplify as

$$\mathcal{A}(\lambda_1)\mathcal{B}(\lambda_2)=\frac{\mathrm{sh}(\lambda_1+\lambda_2)\mathrm{sh}(\lambda_1-\lambda_2-\eta)}{\mathrm{sh}(\lambda_1+\lambda_2+\eta)\mathrm{sh}(\lambda_1-\lambda_2)}\mathcal{B}(\lambda_2)\mathcal{A}(\lambda_1)$$
$$-\frac{\mathrm{sh}(\eta)}{\mathrm{sh}(2\lambda_2+\eta)\mathrm{sh}(\lambda_1+\lambda_2+\eta)}\mathcal{B}(\lambda_1)\tilde{\mathcal{D}}(\lambda_2)+\frac{\mathrm{sh}(2\lambda_2)\mathrm{sh}(\eta)}{\mathrm{sh}(2\lambda_2+\eta)\mathrm{sh}(\lambda_1-\lambda_2)}\mathcal{B}(\lambda_1)\mathcal{A}(\lambda_2)\ ,$$

$$\tilde{\mathcal{D}}(\lambda_1)\mathcal{B}(\lambda_2)=\frac{\mathrm{sh}(\lambda_1+\lambda_2+2\eta)\mathrm{sh}(\lambda_1-\lambda_2+\eta)}{\mathrm{sh}(\lambda_1-\lambda_2)\mathrm{sh}(\lambda_1+\lambda_2+\eta)}\mathcal{B}(\lambda_2)\tilde{\mathcal{D}}(\lambda_1)$$
$$+\frac{\mathrm{sh}(\eta)\mathrm{sh}(2\lambda_1+2\eta)\mathrm{sh}(2\lambda_2)}{\mathrm{sh}(2\lambda_2+\eta)\mathrm{sh}(\lambda_1+\lambda_2+\eta)}\mathcal{B}(\lambda_1)\mathcal{A}(\lambda_2)-\frac{\mathrm{sh}(\eta)\mathrm{sh}(2\lambda_1+2\eta)}{\mathrm{sh}(\lambda_1-\lambda_2)\mathrm{sh}(2\lambda_2+\eta)}\mathcal{B}(\lambda_1)\mathcal{D}(\tilde{\lambda}_2)\ .$$

Here, the transfer matrix is given by

$$t^{(N)}(\lambda)=\frac{\mathrm{sh}(2\lambda+2\eta)\mathrm{sh}(\lambda+\xi_+)}{\mathrm{sh}(2\lambda+\eta)}\mathcal{A}(\lambda)-\frac{\mathrm{sh}(\lambda+\eta-\xi_+)}{\mathrm{sh}(2\lambda+\eta)}\tilde{\mathcal{D}}(\lambda)\ . \tag{A.2.6}$$

Due to the $U(1)$ symmetry, a highest-weight state $|\Uparrow\rangle$ exists and it satisfies

$$\begin{aligned}\mathcal{A}(\lambda)\,|\Uparrow\rangle&=a(\lambda)\,|\Uparrow\rangle\ ,\\ \tilde{\mathcal{D}}(\lambda)\,|\Uparrow\rangle&=\tilde{d}(\lambda)\,|\Uparrow\rangle\ ,\\ \mathcal{C}(\lambda)\,|\Uparrow\rangle&=0\ ,\end{aligned} \tag{A.2.7}$$

so that one constructs Bethe states $|\psi^{(M)};\{\lambda_k\}\rangle$. As for the closed spin-chains, they are given by (A.1.7) and (A.1.9) with $u_k\to\lambda_k$, respectively. Therefore, the algebraic Bethe ansatz can be applied and we refer the reader to [Sk88] for details of the following.

Theorem A.2.1. *The Bethe states $|\psi^{(M)};\{\lambda_k\}\rangle$ are the eigenvectors of the transfer matrix $t^{(N)}(\lambda)$ in (A.2.6) with the associated eigenvalues*

$$\Lambda^{(M)}(\lambda;\{\lambda_k\})=a(\lambda)\frac{\mathrm{sh}(2\lambda+2\eta)}{\mathrm{sh}(2\lambda+\eta)}\mathrm{sh}(\lambda+\xi_+)\times\prod_{k=1}^{M}\frac{\mathrm{sh}(\lambda-\lambda_k-\eta)\mathrm{sh}(\lambda+\lambda_k)}{\mathrm{sh}(\lambda-\lambda_k)\mathrm{sh}(\lambda+\lambda_k+\eta)}$$
$$-\tilde{d}(\lambda)\frac{\mathrm{sh}(\lambda+\eta-\xi_+)}{\mathrm{sh}(2\lambda+\eta)}\times\prod_{j=1}^{M}\frac{\mathrm{sh}(\lambda-\lambda_k+\eta)\mathrm{sh}(\lambda+\lambda_k+2\eta)}{\mathrm{sh}(\lambda-\lambda_k)\mathrm{sh}(\lambda+\lambda_k+\eta)}\ ,$$

provided that the scalars λ_n, $n = 1, 2, \dots, M$ satisfy the Bethe equations

$$\frac{a(\lambda_n)}{\tilde{d}(\lambda_n)} \frac{\mathrm{sh}(2\lambda_n)\mathrm{sh}(\lambda_n + \xi_+)}{\mathrm{sh}(\xi_+ - \eta - \lambda_m)} = \prod_{\substack{k=1 \\ k \neq n}}^{M} \frac{\mathrm{sh}(\lambda_n - \lambda_k + \eta)\mathrm{sh}(\lambda_n + \lambda_k + 2\eta)}{\mathrm{sh}(\lambda_n - \lambda_k - \eta)\mathrm{sh}(\lambda_n + \lambda_k)} \ . \tag{A.2.8}$$

The algebraic Bethe ansatz is a useful technique for some systems but it has its limits. Here, we have considered diagonal boundary conditions so that the Hamiltonian satisfies the $U(1)$ symmetry (A.1.2) and the Bethe ansatz is applied. However, for more general boundary conditions such as (1.2.17), the $U(1)$ symmetry is broken, the reference state is no longer obvious and this method cannot be applied.

As discussed in the introduction, this problem can be solved as follows. For instance, using a set of gauge transformations, the non-diagonal[29] XXZ spin-$\frac{1}{2}$ chain was solved in [CLSW02]. Some techniques have been developed to solve models without $U(1)$ symmetry such as the modified algebraic Bethe ansatz, see instance [Be14, BP14, ABGP15]. Alternative approaches also exist, they use functional relations [G08] or the q-Onsager algebra [BK05b, BB12].

[29] The boundary conditions are not the most general because some constraints are imposed on the left and right K-matrices.

Appendix B

The universal R-matrix

In this appendix, we compute formal evaluations of the universal R-matrix. Firstly, we give the factorized expression of the universal R-matrix for $H = \mathcal{L}U_q sl_2$ in terms of root vectors [KT92a] using our conventions. Secondly, evaluations of the universal R-matrix are considered. In particular, we give expressions of the affine L-operators $\mathbf{L}^+(u)$ and $[\mathbf{L}^-(u^{-1})]^{-1}$, as defined in (3.2.9). Then, the spin-$\frac{1}{2}$ L-operator $\mathbf{L}^{(\frac{1}{2})}(u)$ is computed by evaluating $\mathbf{L}^+(u)$, and finally we compute the spin-$\frac{1}{2}$ R-matrix.

B.1 Construction of the universal R-matrix

B.1.1 Root vectors

Let us first recall the definition of the root vectors of $\mathcal{L}U_q sl_2$. We adapt the construction in [BGKNR12] to our choice of coproduct. Let us set

$$e_\alpha = E_1 K_1^{-\frac{1}{2}} , \qquad e_{\delta-\alpha} = E_0 K_0^{-\frac{1}{2}} , \qquad f_\alpha = K_1^{\frac{1}{2}} F_1 , \qquad f_{\delta-\alpha} = K_0^{\frac{1}{2}} F_0 . \tag{B.1.1}$$

The other root vectors are defined by the recursion relations:

$$\begin{aligned}
e'_{k\delta} &= q^{-1}[e_{\alpha+(k-1)\delta}, e_{\delta-\alpha}]_q , \\
e_{\alpha+k\delta} &= [2]_q^{-1}[e_{\alpha+(k-1)\delta}, e'_\delta] , \\
e_{\delta-\alpha+k\delta} &= [2]_q^{-1}[e'_\delta, e_{\delta-\alpha+(k-1)\delta}] , \\
f'_{k\delta} &= q[f_{\delta-\alpha}, f_{\alpha+(k-1)\delta}]_{q^{-1}} , \\
f_{\alpha+k\delta} &= [2]_q^{-1}[f'_\delta, f_{\alpha+(k-1)\delta}] , \\
f_{\delta-\alpha+k\delta} &= [2]_q^{-1}[f_{\delta-\alpha+(k-1)\delta}, f'_\delta] , \qquad k \in \mathbb{N}_+ .
\end{aligned} \tag{B.1.2}$$

The root vectors $e_{k\delta}$, $f_{k\delta}$ are defined via the generating functions

$$(q-q^{-1})\sum_{k=1}^{\infty} e_{k\delta} z^{-k} = \log\left(1+(q-q^{-1})\sum_{k=1}^{\infty} e'_{k\delta} z^{-k}\right) ,$$
$$-(q-q^{-1})\sum_{k=1}^{\infty} f_{k\delta} z^{-k} = \log\left(1-(q-q^{-1})\sum_{k=1}^{\infty} f'_{k\delta} z^{-k}\right) .$$

B.1.2 Factorized form of the universal R-matrix

Let $\{\alpha+k\delta\}_{k=0}^{\infty} \cup \{k\delta\}_{k=0}^{\infty} \cup \{\delta-\alpha+k\delta\}_{k=0}^{\infty}$ be the positive root system of $\widehat{sl}_2$. We choose the root ordering as

$$\alpha,\ \alpha+\delta,\ \ldots,\ \alpha+k\delta,\ \ldots,\delta,\ 2\delta,\ \ldots,\ \ell\delta,\ \ldots,\ \ldots,\ (\delta-\alpha)+m\delta,\ldots,\ (\delta-\alpha)+\delta,\ \delta-\alpha\ , \quad \text{(B.1.3)}$$

for any k, ℓ, $m \in \mathbb{N}$. Then, the universal R-matrix obtained by Khoroshkin and Tolstoy takes the following factorized form

$$\mathcal{R} = \mathcal{R}^{+}\mathcal{R}^{0}\mathcal{R}^{-} q^{\frac{1}{2}h_1\otimes h_1} , \quad \text{(B.1.4)}$$

where[30]

$$\mathcal{R}^{+} = \prod_{k=0}^{\overrightarrow{\infty}} \exp_{q^{-2}}\left((q-q^{-1}) e_{\alpha+k\delta}\otimes f_{\alpha+k\delta}\right) , \quad \text{(B.1.5)}$$

$$\mathcal{R}^{0} = \exp\left((q-q^{-1})\sum_{k=1}^{\infty}\frac{k}{[2k]_q} e_{k\delta}\otimes f_{k\delta}\right) , \quad \text{(B.1.6)}$$

$$\mathcal{R}^{-} = \prod_{k=0}^{\overleftarrow{\infty}} \exp_{q^{-2}}\left((q-q^{-1}) e_{\delta-\alpha+k\delta}\otimes f_{\delta-\alpha+k\delta}\right) , \quad \text{(B.1.7)}$$

with $q^{h_1} \equiv K_1$ and the q-exponential is

$$\exp_q(x) = 1+\sum_{k=1}^{\infty}\frac{x^k}{(k)_q!} , \quad (k)_q! = (1)_q(2)_q\cdots(k)_q,\ (k)_q = \frac{q^k-1}{q-1} . \quad \text{(B.1.8)}$$

We also notice that $[e_{k\delta}, e_{\ell\delta}] = 0 = [f_{k\delta}, f_{\ell\delta}]$ for any k, ℓ, and so the exponent in (B.1.6) can be also written in the form of semi-infinite product of exponents involving only one term $\sim e_{k\delta}\otimes f_{k\delta}$.

B.2 Evaluations of the universal R-matrix.

In the previous subsection, the explicit form of the universal R-matrix was recalled. It is expressed as a product of q-exponentials with root vectors in the arguments. Now, we evaluate the second tensor product component of the universal R-matrix by taking its image under the formal evaluation

[30] We use the notation $\prod_{k=0}^{\overleftarrow{n}} a(k) = a(n)a(n-1)\ldots a(0)$ and $\prod_{k=0}^{\overrightarrow{n}} a(k) = a(0)a(1)\ldots a(n)$, for any function $a(n)$.

representation. As it is well known, evaluations of the universal R-matrix lead to L-operators and R-matrices.

B.2.1 Evaluated root vectors

The action of formal evaluation representation defined in (3.2.5) on the first root vectors gives

$$\mathsf{ev}_u(e_{\delta-\alpha}) = u^{-1}FK^{\frac{1}{2}} \ , \qquad \mathsf{ev}_u(e_\alpha) = u^{-1}EK^{-\frac{1}{2}} \ , \qquad \mathsf{ev}_u(K_0^{\frac{1}{2}}) = K^{-\frac{1}{2}} \ , \tag{B.2.1}$$

$$\mathsf{ev}_u(f_{\delta-\alpha}) = uq^{-1}EK^{-\frac{1}{2}} \ , \qquad \mathsf{ev}_u(f_\alpha) = uq^{-1}FK^{\frac{1}{2}} \ , \qquad \mathsf{ev}_u(K_1^{\frac{1}{2}}) = K^{\frac{1}{2}} \ . \tag{B.2.2}$$

The image of the other root vectors of $\mathcal{L}U_qsl_2$ in (B.1.2) under the evaluation map are obtained by induction similarly to [BGKNR12, Sect. 4.4]. They are given for $k \in \mathbb{N}$ by:

$$\begin{aligned}
\mathsf{ev}_u(e_{\alpha+k\delta}) &= (-1)^k u^{-2k-1} q^{-k} EK^{-k-\frac{1}{2}} \ , \\
\mathsf{ev}_u(e_{\delta-\alpha+k\delta}) &= (-1)^k u^{-2k-1} q^{k} FK^{-k+\frac{1}{2}} \ , \\
\mathsf{ev}_u(f_{\alpha+k\delta}) &= (-1)^k u^{2k+1} q^{-k-1} FK^{k+\frac{1}{2}} \ , \\
\mathsf{ev}_u(f_{\delta-\alpha+k\delta}) &= (-1)^k u^{2k+1} q^{k-1} EK^{k-\frac{1}{2}} \ ,
\end{aligned} \tag{B.2.3}$$

and for $k \in \mathbb{N}_+$

$$\begin{aligned}
\mathsf{ev}_u(e'_{k\delta}) &= (-1)^{k-1} u^{-2k} [E,F]_{q^k} K^{-k+1} \ , \\
\mathsf{ev}_u(e_{k\delta}) &= \frac{(-1)^{k-1} u^{-2k}}{(q-q^{-1})k} (C_k - (q^k + q^{-k})K^{-k}) \ , \\
\mathsf{ev}_u(f'_{k\delta}) &= (-1)^{k-1} u^{2k} [E,F]_{q^{-k}} K^{k-1} \ , \\
\mathsf{ev}_u(f_{k\delta}) &= -\frac{(-1)^{k-1} u^{2k}}{(q-q^{-1})k} (C_k - (q^k + q^{-k})K^{k}) \ ,
\end{aligned} \tag{B.2.4}$$

where the elements C_k are defined by the generating function

$$\sum_{k=1}^{\infty} (-1)^{k-1} C_k \frac{z^{-k}}{k} = \log(1 + Cz^{-1} + z^{-2}) \ , \qquad z \in \mathbb{C} \ , \tag{B.2.5}$$

and where C is the central element of U_qsl_2 given in (2.3.5). For instance, by expanding (B.2.5) the first elements of C_k are obtained

$$C_1 = C \ , \qquad C_2 = C^2 - 2 \ , \qquad C_3 = C^3 - 3C \ , \qquad C_4 = C^4 - 4C^2 + 2 \ . \tag{B.2.6}$$

Recall E_{ab} is the matrix with zero everywhere except 1 in the entry (a,b). The matrix multiplication obeys

$$E_{ab}E_{cd} = \delta_{b,c} E_{ad} \ . \tag{B.2.7}$$

In this notation, the spin-$\frac{1}{2}$ finite-dimensional representation of U_qsl_2 reads

$$\pi^{\frac{1}{2}}(K^m) = q^m E_{11} + q^{-m} E_{22} \ , \qquad \pi^{\frac{1}{2}}(E) = E_{12} \ , \qquad \pi^{\frac{1}{2}}(F) = E_{21} \ , \tag{B.2.8}$$

and the central elements become

$$\pi^{\frac{1}{2}}(C) = (q^2 + q^{-2})\mathbb{I}_2 \ , \qquad \pi^{\frac{1}{2}}(C_k) = (q^{2k} + q^{-2k})\mathbb{I}_2 \ . \tag{B.2.9}$$

In order to obtain affine L-operators and R-matrices from the universal R-matrix, one also needs the image of the root vectors under the formal evaluation representation $\pi^{\frac{1}{2}}_u \colon \mathcal{L}U_q sl_2 \to \mathrm{End}(\mathbb{C}^2)[u^{\pm 1}]$ from (3.2.7), which is given by:

$$\begin{aligned}
\pi^{\frac{1}{2}}_u(e_{\alpha+k\delta}) &= (-1)^k u^{-2k-1} q^{\frac{1}{2}} E_{12} \ , \\
\pi^{\frac{1}{2}}_u(e_{\delta-\alpha+k\delta}) &= (-1)^k u^{-2k-1} q^{\frac{1}{2}} E_{21} \ , \\
\pi^{\frac{1}{2}}_u(f_{\alpha+k\delta}) &= (-1)^k u^{2k+1} q^{-\frac{1}{2}} E_{21} \ , \\
\pi^{\frac{1}{2}}_u(f_{\delta-\alpha+k\delta}) &= (-1)^k u^{2k+1} q^{-\frac{1}{2}} E_{12} \ , \\
\pi^{\frac{1}{2}}_u(e'_{k\delta}) &= (-1)^{k-1} u^{-2k}(qE_{11} - q^{-1}E_{22}) \ , \\
\pi^{\frac{1}{2}}_u(e_{k\delta}) &= (-1)^{k-1} u^{-2k} \frac{[k]_q}{k}(q^k E_{11} - q^{-k}E_{22}) \ , \\
\pi^{\frac{1}{2}}_u(f'_{k\delta}) &= (-1)^{k-1} u^{2k}(q^{-1}E_{11} - qE_{22}) \ , \\
\pi^{\frac{1}{2}}_u(f_{k\delta}) &= (-1)^{k-1} u^{2k} \frac{[k]_q}{k}(q^{-k}E_{11} - q^k E_{22}) \ .
\end{aligned} \tag{B.2.10}$$

Recall that $\mathbf{L}^{\pm}(u)$ are defined in (3.2.9). We now compute explicitly $\mathbf{L}^{+}(u)$ and $[\mathbf{L}^{-}(u^{-1})]^{-1}$.

B.2.2 The affine L-operators

$\mathbf{L}^{+}(u)$

Recall the factorized form of the universal R-matrix (B.1.4). We now compute the image of $\mathfrak{R}^{\pm}$, $\mathfrak{R}^0$, $q^{\frac{1}{2}h_1 \otimes h_1}$ under the action of $(\mathrm{id} \otimes \pi^{\frac{1}{2}}_{u^{-1}})$. First, from (B.1.5) and with (B.2.10) we get:

$$\begin{aligned}
(\mathrm{id} \otimes \pi^{\frac{1}{2}}_{u^{-1}})(\mathfrak{R}^{+}) &= \prod_{k=0}^{\overrightarrow{\infty}} \exp_{q^{-2}}\left((-u^{-2})^k u^{-1}(q - q^{-1}) q^{-\frac{1}{2}} e_{\alpha+k\delta} \otimes E_{21}\right) \\
&= 1 \otimes (E_{11} + E_{22}) + e^{+}(u) \otimes E_{21} \ ,
\end{aligned} \tag{B.2.11}$$

where

$$e^{+}(u) = (q - q^{-1}) q^{-\frac{1}{2}} u^{-1} \left(\sum_{k=0}^{\infty} (-u^{-2})^k e_{\alpha+k\delta} \right) \ . \tag{B.2.12}$$

Similarly, from (B.1.7) and with (B.2.10) we have:

$$\begin{aligned}
(\mathrm{id} \otimes \pi^{\frac{1}{2}}_{u^{-1}})(\mathfrak{R}^{-}) &= \prod_{k=0}^{\overleftarrow{\infty}} \exp_{q^{-2}}\left((-u^{-2})^k u^{-1}(q - q^{-1}) q^{-\frac{1}{2}} e_{\delta-\alpha+k\delta} \otimes E_{12}\right) \\
&= 1 \otimes (E_{11} + E_{22}) + f^{+}(u) \otimes E_{12} \ ,
\end{aligned} \tag{B.2.13}$$

where

$$f^+(u) = (q-q^{-1})q^{-\frac{1}{2}}u^{-1}\left(\sum_{k=0}^{\infty}(-u^{-2})^k e_{\delta-\alpha+k\delta}\right) . \tag{B.2.14}$$

A straightforward calculation from (B.1.6) and using (B.2.10) yields

$$(\mathrm{id}\otimes\pi^{\frac{1}{2}}_{u^{-1}})(\mathfrak{R}^0) = k^+(u)\otimes E_{11} + \tilde{k}^+(u)\otimes E_{22} , \tag{B.2.15}$$

where

$$\begin{aligned} k^+(u) &= \exp\left(-(q-q^{-1})\sum_{k=1}^{\infty}\frac{(-u^{-2}q^{-1})^k}{q^k+q^{-k}}e_{k\delta}\right) , \\ \tilde{k}^+(u) &= \exp\left((q-q^{-1})\sum_{k=1}^{\infty}\frac{(-u^{-2}q)^k}{q^k+q^{-k}}e_{k\delta}\right) . \end{aligned} \tag{B.2.16}$$

Then, using $\pi^{\frac{1}{2}}_u(h_1) = E_{11} - E_{22}$, we get

$$(\mathrm{id}\otimes\pi^{\frac{1}{2}}_{u^{-1}})(q^{\frac{1}{2}h_1\otimes h_1}) = K_1^{\frac{1}{2}}\otimes E_{11} + K_1^{-\frac{1}{2}}\otimes E_{22} . \tag{B.2.17}$$

Finally, combining (B.2.11)-(B.2.17), we get

$$\mathbf{L}^+(u) = \begin{pmatrix} k^+(u)K_1^{\frac{1}{2}} & k^+(u)f^+(u)K_1^{-\frac{1}{2}} \\ e^+(u)k^+(u)K_1^{\frac{1}{2}} & \tilde{k}^+(u)K_1^{-\frac{1}{2}} + e^+(u)k^+(u)f^+(u)K_1^{-\frac{1}{2}} \end{pmatrix} . \tag{B.2.18}$$

Note that from the definition of $e^+(u)$, $f^+(u)$, $k^+(u)$, $\tilde{k}^+(u)$ in (B.2.12), (B.2.14), (B.2.16) it is easy to see that $\mathbf{L}^+(u)$ is a formal power series in u^{-1}, i.e. $\mathbf{L}^+(u)$ is in $\mathcal{L}U_q sl_2((u^{-1}))\otimes \mathrm{End}(\mathbb{C}^2)$. We note that up to conventions, the expression (B.2.18) matches with the one given in [DF93, eq. (4.8)].

$[\mathbf{L}^-(u^{-1})]^{-1}$

Consider $\mathfrak{p}\circ\mathfrak{R}^{\pm} = \mathfrak{R}^{\pm}_{21}$, $\mathfrak{p}\circ\mathfrak{R}^0 = \mathfrak{R}^0_{21}$. We now compute their image under the action of $(\mathrm{id}\otimes\pi^{\frac{1}{2}}_{u^{-1}})$ to obtain the expression of $[\mathbf{L}^-(u^{-1})]^{-1}$ defined in (3.2.9). First, it follows from (B.1.5) and (B.2.10)

$$\begin{aligned} (\mathrm{id}\otimes\pi^{\frac{1}{2}}_u)(\mathfrak{R}^+_{21}) &= \prod_{k=0}^{\overrightarrow{\infty}}\exp_{q^{-2}}\left((-u^{-2})^k u^{-1}(q-q^{-1})q^{\frac{1}{2}}f_{\alpha+k\delta}\otimes E_{12}\right) \\ &= 1\otimes(E_{11}+E_{22}) + f^-(u)\otimes E_{12} , \end{aligned} \tag{B.2.19}$$

where

$$f^-(u) = (q-q^{-1})q^{\frac{1}{2}}u^{-1}\left(\sum_{k=0}^{\infty}(-u^{-2})^k f_{\alpha+k\delta}\right) . \tag{B.2.20}$$

Similarly, from (B.1.7) and using (B.2.10)

$$(\mathrm{id}\otimes\pi^{\frac{1}{2}}_u)(\mathfrak{R}^-_{21}) = \prod_{k=0}^{\overleftarrow{\infty}}\exp_{q^{-2}}\left((-u^{-2})^k u^{-1}(q-q^{-1})q^{\frac{1}{2}}f_{\delta-\alpha+k\delta}\otimes E_{21}\right)$$

$$= 1 \otimes (E_{11} + E_{22}) + e^-(u) \otimes E_{21} \ , \tag{B.2.21}$$

where

$$e^-(u) = (q - q^{-1}) q^{\frac{1}{2}} u^{-1} \left(\sum_{k=0}^{\infty} (-u^{-2})^k f_{\delta - \alpha + k\delta} \right) \ . \tag{B.2.22}$$

Then from (B.1.7) and using (B.2.10) we get

$$(\mathrm{id} \otimes \pi_u^{\frac{1}{2}})(\mathfrak{R}_{21}^0) = k^-(u) \otimes E_{11} + \tilde{k}^-(u) \otimes E_{22} \ , \tag{B.2.23}$$

where

$$\begin{aligned} k^-(u) &= \exp\left(-(q - q^{-1}) \sum_{k=1}^{\infty} \frac{(-u^{-2}q)^k}{q^k + q^{-k}} f_{k\delta} \right) \ , \\ \tilde{k}^-(u) &= \exp\left((q - q^{-1}) \sum_{k=1}^{\infty} \frac{(-u^{-2}q^{-1})^k}{q^k + q^{-k}} f_{k\delta} \right) \ . \end{aligned} \tag{B.2.24}$$

Finally, combining (B.2.19)-(B.2.23) and (B.2.17), we get

$$[\mathbf{L}^-(u^{-1})]^{-1} = \begin{pmatrix} k^-(u) K_1^{\frac{1}{2}} + f^-(u) \tilde{k}^-(u) e^-(u) K_1^{\frac{1}{2}} & f^-(u) \tilde{k}^-(u) K_1^{-\frac{1}{2}} \\ \tilde{k}^-(u) e^-(u) K_1^{\frac{1}{2}} & \tilde{k}^-(u) K_1^{-\frac{1}{2}} \end{pmatrix} . \tag{B.2.25}$$

Note that from the definition of $e^-(u)$, $f^-(u)$, $k^-(u)$, $\tilde{k}^-(u)$ in (B.2.22), (B.2.20), (B.2.24) it is easy to see that $[\mathbf{L}^-(u^{-1})]^{-1}$ is a formal power series in u^{-1}, i.e. $[\mathbf{L}^-(u^{-1})]^{-1}$ is in $\mathcal{L}U_q sl_2((u^{-1})) \otimes \mathrm{End}(\mathbb{C}^2)$. We note that up to conventions, the expression (B.2.25) matches with the one given in [DF93, eq. (4.9)].

B.3 The spin-$\frac{1}{2}$ L-operator $\mathbf{L}^{(\frac{1}{2})}(u)$

We now compute the spin-$\frac{1}{2}$ L-operator $\mathbf{L}^{(\frac{1}{2})}(u)$ defined in (3.2.12). It is obtained by taking the image of $\mathbf{L}^+(u)$ under the formal evaluation with $(\mathsf{ev}_v \otimes \mathrm{id})$.

Recall the expression of $\mathbf{L}^+(u)$ in (B.2.18). The spin-$\frac{1}{2}$ L-operator is then obtained by evaluating $e^+(u)$, $f^+(u)$, $k^+(u)$, $\tilde{k}^+(u)$ defined in (B.2.12), (B.2.14), (B.2.16). Let us first introduce the function [BGKNR12]

$$\Lambda(u) = \sum_{k=1}^{\infty} \frac{C_k}{(q^k + q^{-k})} \frac{u^k}{k} \ , \tag{B.3.1}$$

where the central elements C_k are defined by (B.2.5). Note that it satisfies

$$\Lambda(uq) + \Lambda(uq^{-1}) = -\log(1 - Cu + u^2) \ . \tag{B.3.2}$$

From the evaluated root vectors (B.2.1) and (B.2.4), we get:

$$\begin{aligned}
\mathsf{ev}_v(e^+(u)) &= (q-q^{-1})q^{-\frac{1}{2}}u^{-1}v^{-1}EK^{-\frac{1}{2}}\left(1-u^{-2}v^{-2}q^{-1}K^{-1}\right)^{-1} \ , \\
\mathsf{ev}_v(f^+(u)) &= (q-q^{-1})q^{-\frac{1}{2}}u^{-1}v^{-1}FK^{\frac{1}{2}}\left(1-u^{-2}v^{-2}qK^{-1}\right)^{-1} \ , \\
\mathsf{ev}_v(k^+(u)) &= e^{\Lambda(u^{-2}v^{-2}q^{-1})}\left(1-u^{-2}v^{-2}q^{-1}K^{-1}\right) \ , \\
\mathsf{ev}_v(\tilde{k}^+(u)) &= e^{-\Lambda(u^{-2}v^{-2}q)}\left(1-u^{-2}v^{-2}qK^{-1}\right)^{-1} \ .
\end{aligned} \tag{B.3.3}$$

For instance, let us now compute the evaluation of the matrix entry $(2,2)$ of $\mathbf{L}^+(u)$ in (B.2.18), it reads:

$$\begin{aligned}
\mathsf{ev}_v((\mathbf{L}^+(u))_{22}) &= \mathsf{ev}_v\Big(\tilde{k}^+(u)K_1^{-\frac{1}{2}} + e^+(u)k^+(u)f^+(u)K_1^{-\frac{1}{2}}\Big) \\
&= \Big(e^{-\Lambda(u^{-2}v^{-2}q)} + e^{\Lambda(u^{-2}v^{-2}q^{-1})}u^{-2}v^{-2}(q-q^{-1})^2EF\Big)\left(1-u^{-2}v^{-2}qK^{-1}\right)^{-1}K^{-\frac{1}{2}} \\
&= e^{\Lambda(u^{-2}v^{-2}q^{-1})}\Big(1+u^{-4}v^{-4}-u^{-2}v^{-2}(C-(q-q^{-1})^2EF)\Big)\left(1-u^{-2}v^{-2}qK^{-1}\right)^{-1}K^{-\frac{1}{2}} \ ,
\end{aligned}$$

where we used (B.3.2) on the third line, and where C is defined in (2.3.5). Then, we obtain

$$\mathsf{ev}_v((\mathbf{L}^+(u))_{22}) = e^{\Lambda(u^{-2}v^{-2}q^{-1})}\left(K^{-\frac{1}{2}} - u^{-2}v^{-2}q^{-1}K^{\frac{1}{2}}\right) \ .$$

Computing the other entries, we have

$$\mathbf{L}^{(\frac{1}{2})}(uv) = (\mathsf{ev}_v \otimes \mathrm{id})(\mathbf{L}^+(u)) = \mu(uv)\mathcal{L}^{(\frac{1}{2})}(uv) \ ,$$

where $\mu(u)$ and $\mathcal{L}^{(\frac{1}{2})}(u)$ are given respectively in (3.2.17) and (1.1.8).

Similarly, evaluating the affine L-operator in (B.2.25), we have

$$(\mathsf{ev}_v \otimes \mathrm{id})([\mathbf{L}^-(u^{-1})]^{-1}) = \mu(u/v)\mathcal{L}^{(\frac{1}{2})}(u/v) = \mathbf{L}^{(\frac{1}{2})}(u/v) \ .$$

B.3.1 The spin-$\frac{1}{2}$ R-matrix.

As we already mentioned, the universal R-matrix provides a way of obtaining R-matrices. So, to conclude this appendix, we compute the spin-$\frac{1}{2}$ R-matrix coming from the evaluation of $\mathfrak{R}$ with the following relation

$$\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(u/v) = (\pi_u^{\frac{1}{2}} \otimes \pi_v^{\frac{1}{2}})(\mathfrak{R}) \ .$$

Recall the spin-$\frac{1}{2}$ L-operator from (3.2.12). Then, due to the decomposition $\pi_u^{\frac{1}{2}} = \pi^{\frac{1}{2}} \circ \mathsf{ev}_u$, one has

$$\begin{aligned}
\mathcal{R}^{(\frac{1}{2},\frac{1}{2})}(u) &= (\pi^{\frac{1}{2}} \otimes \mathrm{id})(\mathbf{L}^{(\frac{1}{2})}(u)) \\
&= \pi^{\frac{1}{2}}(\mu(u))R^{(\frac{1}{2},\frac{1}{2})}(u) \ .
\end{aligned}$$

where we used the spin-$\frac{1}{2}$ finite-dimensional of $U_q sl_2$ in (4.3.1) and with $\pi^{\frac{1}{2}}(\mu(u))$, $R^{(\frac{1}{2},\frac{1}{2})}(u)$, given respectively in (3.2.19), (1.1.5).

Appendix C

Intertwining operators

C.1 The intertwining maps $\mathcal{E}^{(j+\frac{1}{2})}$ and $\mathcal{F}^{(j+\frac{1}{2})}$

We now study the spin-$(j+\frac{1}{2})$ sub-representation when $u_1/u_2 = q^{-j-\frac{1}{2}}$, and give the corresponding intertwining operator explicitly. Introduce the two linear operators $\mathcal{E}^{(j+\frac{1}{2})}$ and $\mathcal{F}^{(j+\frac{1}{2})}$ for any $j \in \frac{1}{2}\mathbb{N}$:

$$\mathcal{E}^{(j+\frac{1}{2})} \colon \mathbb{C}^{2j+2}_u \to \mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2} \ , \tag{C.1.1}$$

$$\mathcal{F}^{(j+\frac{1}{2})} \colon \mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2} \to \mathbb{C}^{2j+2}_u \ , \tag{C.1.2}$$

given by:

$$\mathcal{E}^{(j+\frac{1}{2})} = \sum_{a=1}^{4j+2} \sum_{b=1}^{2j+2} \mathcal{E}^{(j+\frac{1}{2})}_{a,b} E^{(2j,j)}_{a,b} \ , \qquad \mathcal{F}^{(j+\frac{1}{2})}\mathcal{E}^{(j+\frac{1}{2})} = \mathbb{I}_{2j+2} \ , \tag{C.1.3}$$

where $\mathcal{E}^{(j+\frac{1}{2})}_{a,b}$ are certain scalars, $E^{(j_1,j_2)}_{a,b}$ denotes the matrix of dimension $(2j_1+2)\times(2j_2+2)$ with 1 at position (a,b) and 0 otherwise. Here, we choose the bases of the source $\{|j+\frac{1}{2}, m\rangle\}$ with $m = j+\frac{1}{2}, j-\frac{1}{2}, \ldots, -j+\frac{1}{2}, -j-\frac{1}{2}$ and the target $\{|\uparrow\rangle \otimes |j,j\rangle, \ldots, |\uparrow\rangle \otimes |j,-j\rangle, |\downarrow\rangle \otimes |j,j\rangle, \ldots, |\downarrow\rangle \otimes |j,-j\rangle\}$.

We now calculate the coefficients $\mathcal{E}^{(j+\frac{1}{2})}_{a,b}$ from (C.1.3) provided $\mathcal{E}^{(j+\frac{1}{2})}$ is a $\mathcal{L}U_q sl_2$-intertwiner for the conditions $u_1/u_2 = q^{-j-\frac{1}{2}}$ and $u_2 = uq^{\frac{1}{2}}$, that was found in the previous subsection for $j \in \frac{1}{2}\mathbb{N}_+$. First of all for $j=0$, we have for any u that $\pi^0_u = \epsilon$, where the counit is defined in (2.3.10), i.e. the trivial representation of $\mathcal{L}U_q sl_2$. Then identifying $\mathbb{C}^2 \otimes \mathbb{C}$ with $\mathbb{C}^2$, it follows from (C.1.1), (C.1.2) and (C.1.3) that $\mathcal{E}^{(\frac{1}{2})} = \mathcal{F}^{(\frac{1}{2})} = \mathbb{I}_2$.

Lemma C.1.1 ([LBG23, Lem. 3.1]). *Let $u_1/u_2 = q^{-j-\frac{1}{2}}$ and $u_2 = uq^{\frac{1}{2}}$, then the map $\mathcal{E}^{(j+\frac{1}{2})}$*

in (C.1.3) is a $\mathcal{L}U_q sl_2$-intertwiner

$$\mathcal{E}^{(j+\frac{1}{2})}(\pi_u^{j+\frac{1}{2}})(x) = (\pi^{\frac{1}{2}}_{uq^{-j}} \otimes \pi^{j}_{uq^{\frac{1}{2}}})(\Delta(x))\mathcal{E}^{(j+\frac{1}{2})} \ , \qquad \forall x \in \mathcal{L}U_q sl_2 \ , \tag{C.1.4}$$

if and only if its entries are given for any $j \in \frac{1}{2}\mathbb{N}_+$ by

$$\mathcal{E}^{(j+\frac{1}{2})}_{1,1} = 1 \ , \qquad \mathcal{E}^{(j+\frac{1}{2})}_{1+n,1+n} = \prod_{p=0}^{n-1} \frac{B_{j,j-p}}{B_{j+\frac{1}{2},j+\frac{1}{2}-p}} \ , \qquad \mathcal{E}^{(j+\frac{1}{2})}_{2j+1+m,1+m} = [m]_q \frac{\mathcal{E}^{(j+\frac{1}{2})}_{m,m}}{B_{j+\frac{1}{2},j+\frac{3}{2}-m}} \ , \tag{C.1.5}$$

where $n = 1, 2, \ldots, 2j$, $m = 1, 2, \ldots, 2j+1$ and $B_{j,m}$ is given in (3.2.3), and all the other entries are zero.

Note that this intertwining property is of prior importance for the construction of fused L- and K-operators.

We now give an expression of $\mathcal{F}^{(j+\frac{1}{2})}$ which is a pseudo-inverse of $\mathcal{E}^{(j+\frac{1}{2})}$. It takes the form:

$$\mathcal{F}^{(j+\frac{1}{2})} = \sum_{a=1}^{2j+2} \sum_{b=1}^{4j+2} \mathcal{F}^{(j+\frac{1}{2})}_{a,b} E^{(j,2j)}_{a,b} \ , \tag{C.1.6}$$

where $\mathcal{F}^{(j+\frac{1}{2})}_{a,b}$ are scalars. The solution of $\mathcal{F}^{(j+\frac{1}{2})}\mathcal{E}^{(j+\frac{1}{2})} = \mathbb{I}_{2j+2}$ is not unique. For instance, we fix the entries of $\mathcal{F}^{(j+\frac{1}{2})}$ for $n = 2, 3, \ldots, 2j+1$ as follows:

$$\mathcal{F}^{(j+\frac{1}{2})}_{1,1} = 1 \ , \qquad \mathcal{F}^{(j+\frac{1}{2})}_{n,n+2j} = \frac{\mathcal{E}^{(j+\frac{1}{2})}_{n+2j,n}}{(\mathcal{E}^{(j+\frac{1}{2})}_{n,n})^2 + (\mathcal{E}^{(j+\frac{1}{2})}_{n+2j,n})^2} \ , \tag{C.1.7}$$

$$\mathcal{F}^{(j+\frac{1}{2})}_{2j+2,4j+2} = (\mathcal{E}^{(j+\frac{1}{2})}_{4j+2,2j+2})^{-1} \ , \qquad \mathcal{F}^{(j+\frac{1}{2})}_{n,n} = \frac{1 - \mathcal{F}^{(j+\frac{1}{2})}_{n,n+2j}\mathcal{E}^{(j+\frac{1}{2})}_{n+2j,n}}{\mathcal{E}^{(j+\frac{1}{2})}_{n,n}} \ , \tag{C.1.8}$$

and all other entries are zero. This choice is important because it allows the factorization of the R-matrix as in Lemma C.3.1 below. We finally note that any pseudo-inverse of $\mathcal{E}^{(j+\frac{1}{2})}$, in particular the one given above, is not a $\mathcal{L}U_q sl_2$-intertwiner because the sub-representation involved is not a direct summand, recall the structure in Fig. 3.3.

C.2 The maps $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ and $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$

We now study the spin-$(j-\frac{1}{2})$ sub-representation when $u_1/u_2 = q^{j+\frac{1}{2}}$. Introduce the two maps $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ and $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$ for any $j \in \frac{1}{2}\mathbb{N}_+$:

$$\bar{\mathcal{E}}^{(j-\frac{1}{2})} \colon \mathbb{C}^{2j}_u \to \mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2} \ , \tag{C.2.1}$$

$$\bar{\mathcal{F}}^{(j-\frac{1}{2})} \colon \mathbb{C}^2_{u_1} \otimes \mathbb{C}^{2j+1}_{u_2} \to \mathbb{C}^{2j}_u \ , \tag{C.2.2}$$

given by:

$$\bar{\mathcal{E}}^{(j-\frac{1}{2})} = \sum_{a=1}^{4j+2}\sum_{b=1}^{2j} \bar{\mathcal{E}}^{(j-\frac{1}{2})}_{a,b} E^{(2j,j-1)}_{a,b}, \qquad \bar{\mathcal{F}}^{(j-\frac{1}{2})}\bar{\mathcal{E}}^{(j-\frac{1}{2})} = \mathbb{I}_{2j} \ , \tag{C.2.3}$$

where $\bar{\mathcal{E}}^{(j-\frac{1}{2})}_{a,b}$ are certain scalars. The bases of the source and the target of $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ are respectively $\{|j-\frac{1}{2}, m\rangle\}$ with $m = j-\frac{1}{2}$, $j-\frac{3}{2}$, …, $-j+\frac{3}{2}$, $-j+\frac{1}{2}$ and $\{|\uparrow\rangle \otimes |j,j\rangle, \ldots, |\uparrow\rangle \otimes |j,-j\rangle, |\downarrow\rangle \otimes |j,j\rangle, \ldots, |\downarrow\rangle \otimes |j,-j\rangle\}$.

Lemma C.2.1 ([LBG23, Lem. 3.2]). *Let $u_1/u_2 = q^{j+\frac{1}{2}}$ and $u_2 = uq^{\frac{1}{2}}$, then the map $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ in (C.2.3)* is a $\mathcal{L}U_q sl_2$-intertwiner

$$\bar{\mathcal{E}}^{(j-\frac{1}{2})}(\pi_u^{j-\frac{1}{2}})(x) = (\pi^{\frac{1}{2}}_{uq^{j+1}} \otimes \pi^{j}_{\ uq^{\frac{1}{2}}})(\Delta(x))\bar{\mathcal{E}}^{(j-\frac{1}{2})} \ , \qquad \forall x \in \mathcal{L}U_q sl_2 \ , \tag{C.2.4}$$

if and only if its entries are given for any $j \in \frac{1}{2}\mathbb{N}_+$ by

$$\bar{\mathcal{E}}^{(j-\frac{1}{2})}_{2,1} = 1 \ , \qquad \bar{\mathcal{E}}^{(j-\frac{1}{2})}_{2+n,1+n} = \prod_{p=0}^{n-1} \frac{B_{j,j-p-1}}{B_{j-\frac{1}{2},j-\frac{1}{2}-p}} \ , \qquad \bar{\mathcal{E}}^{(j-\frac{1}{2})}_{2j+2+m,1+m} = \frac{[m-2j]_q}{B_{j,j-m}} \bar{\mathcal{E}}^{(j-\frac{1}{2})}_{2+m,1+m} \ , \tag{C.2.5}$$

where $n = 1, 2, \ldots, 2j-1$, $m = 0, 1, \ldots, 2j-1$ and $B_{j,m}$ is given in (3.2.3), and all the other entries are zero.

We now give expression of $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$ which is a pseudo-inverse of $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$. It takes the form:

$$\bar{\mathcal{F}}^{(j-\frac{1}{2})} = \sum_{a=1}^{2j}\sum_{b=1}^{4j+2} \bar{\mathcal{F}}^{(j-\frac{1}{2})}_{a,b} E^{(j-1,2j)}_{a,b} \ , \tag{C.2.6}$$

where $\bar{\mathcal{F}}^{(j-\frac{1}{2})}_{a,b}$ are scalars. The solution of $\bar{\mathcal{F}}^{(j-\frac{1}{2})}\bar{\mathcal{E}}^{(j-\frac{1}{2})} = \mathbb{I}_{2j}$ is not unique. Similarly to the fusion case above, we fix the entries of $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$ for $n = 1, 2, \ldots, 2j$ this way:

$$\bar{\mathcal{F}}^{(j-\frac{1}{2})}_{n,n+2j+1} = \frac{\bar{\mathcal{E}}^{(j-\frac{1}{2})}_{n+2j+1,n}}{(\bar{\mathcal{E}}^{(j-\frac{1}{2})}_{n+1,n})^2 + (\bar{\mathcal{E}}^{(j-\frac{1}{2})}_{n+2j+1,n})^2} \ , \qquad \bar{\mathcal{F}}^{(j-\frac{1}{2})}_{n,n+1} = \frac{1 - \bar{\mathcal{F}}^{(j-\frac{1}{2})}_{n,n+2j+1}\bar{\mathcal{E}}^{(j-\frac{1}{2})}_{n+2j+1,n}}{\bar{\mathcal{E}}^{(j-\frac{1}{2})}_{n+1,n}} \ , \tag{C.2.7}$$

and all other entries are zero. We note that, similarly to the previous case, $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$ is not an intertwiner.

In summary, imposing some conditions on the ratio of evaluation parameters as in Lemmas C.1.1 and C.2.1, the tensor product representation of $\mathcal{L}U_q sl_2$ admits a non-trivial sub-representation either of spin-$(j+\frac{1}{2})$ or of spin-$(j-\frac{1}{2})$. And we have constructed intertwining operators $\mathcal{E}^{(j+\frac{1}{2})}\colon \mathbb{C}^2 \otimes \mathbb{C}^{2j+1} \to \mathbb{C}^{2j+2}$ and $\bar{\mathcal{E}}^{(j-\frac{1}{2})}\colon \mathbb{C}^2 \otimes \mathbb{C}^{2j+1} \to \mathbb{C}^{2j}$, and their pseudo-inverses $\mathcal{F}^{(j+\frac{1}{2})}$ and $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$, respectively.

In what follows, we will need action on tensor product using opposite coproduct. For this new action[31], the corresponding intertwining operators appear at different evaluation parameters.

[31] Recall that for a bialgebra H, we can define another bialgebra H^{cop} with the coproduct Δ^{op}. Therefore, Δ^{op} also

Remark C.2.2. *Consider the opposite coproduct* $\Delta^{op} = \mathfrak{p} \circ \Delta$ *with the definition for* Δ *in* (2.3.9). Using now the action of $\mathcal{L}U_q sl_2$ on the tensor product given by Δ^{op}, the above sub-representation analysis is done similarly. Then, we find that the conditions on the evaluations parameter are different: for the spin-$(j+\frac{1}{2})$ sub-representation $u_1/u_2 = \pm q^{j+\frac{1}{2}}$ and $u = u_2 q^{\frac{1}{2}}$; and for the spin-$(j-\frac{1}{2})$ sub-representation $u_1/u_2 = \pm q^{-j-\frac{1}{2}}$ and $u = u_2 q^{\frac{1}{2}}$. However, it leads to the intertwining operator $\mathcal{E}^{(j+\frac{1}{2})}$ (resp. $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$) with the same matrix elements as in (C.1.5) (resp. in (C.2.5)), that satisfy

$$\mathcal{E}^{(j+\frac{1}{2})}(\pi_u^{j+\frac{1}{2}})(x) = (\pi^{\frac{1}{2}}_{uq^j} \otimes \pi^{j}_{uq^{-\frac{1}{2}}})(\Delta^{op}(x))\mathcal{E}^{(j+\frac{1}{2})} \ , \tag{C.2.8}$$

$$\bar{\mathcal{E}}^{(j-\frac{1}{2})}(\pi_u^{j-\frac{1}{2}})(x) = (\pi^{\frac{1}{2}}_{uq^{-j-1}} \otimes \pi^{j}_{uq^{-\frac{1}{2}}})(\Delta^{op}(x))\bar{\mathcal{E}}^{(j-\frac{1}{2})} \ . \tag{C.2.9}$$

C.3 Additional properties

We conclude this section with a few observations on relations between the intertwining operator $\mathcal{E}^{(j+\frac{1}{2})}$, its pseudo-inverse $\mathcal{F}^{(j+\frac{1}{2})}$ and the R-matrix. In the literature the expression of the R-matrix $R^{(\frac{1}{2},j)}(u) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^{2j+1})$ is known [KR83, DN02]. It reads:

$$\begin{aligned} R^{(\frac{1}{2},j)}(u) = \prod_{k=0}^{2j-2} c(uq^{j-\frac{1}{2}-k})\times \\ \Big((q-q^{-1})\left(\sigma_+ \otimes \pi^j(F) + \sigma_- \otimes \pi^j(E)\right) + uq^{\frac{1}{2}(\mathbb{I}_{4j+2}+\sigma_z\otimes\pi^j(H))} - u^{-1}q^{-\frac{1}{2}(\mathbb{I}_{4j+2}+\sigma_z\otimes\pi^j(H))}\Big) , \end{aligned} \tag{C.3.1}$$

where $\pi^j(E)$, $\pi^j(F)$ are given in (3.2.2), $\left(\pi^j(H)\right)_{mn} = 2(j+1-n)\delta_{m,n}$, with m, $n = 1, 2, \ldots, 2j+1$, and where we use the scalar function $c(u)$ in (1.3.33). Note that this R-matrix satisfies the unitarity property

$$R^{(\frac{1}{2},j)}(u)R^{(\frac{1}{2},j)}(u^{-1}) = \left(\prod_{k=0}^{2j-1} c(uq^{j+\frac{1}{2}-k})c(u^{-1}q^{j+\frac{1}{2}-k})\right)\mathbb{I}_{4j+2} \ . \tag{C.3.2}$$

Let $\mathcal{H}^{(j+\frac{1}{2})}$ and $\bar{\mathcal{H}}^{(j-\frac{1}{2})}$ be invertible diagonal matrices given by:

$$\mathcal{H}^{(j+\frac{1}{2})} = \mathrm{Diag}(\mathcal{H}_1^{(j+\frac{1}{2})}, \mathcal{H}_2^{(j+\frac{1}{2})}, \ldots, \mathcal{H}_{2j+2}^{(j+\frac{1}{2})}) \ , \tag{C.3.3}$$

$$\bar{\mathcal{H}}^{(j-\frac{1}{2})} = \mathrm{Diag}(\bar{\mathcal{H}}_1^{(j-\frac{1}{2})}, \bar{\mathcal{H}}_2^{(j-\frac{1}{2})}, \ldots, \bar{\mathcal{H}}_{2j}^{(j-\frac{1}{2})}) \ , \tag{C.3.4}$$

where $\mathcal{H}_m^{(j+\frac{1}{2})}$ and $\bar{\mathcal{H}}_n^{(j-\frac{1}{2})}$ are scalars.

Inspired by [BLN15], the R-matrix (C.3.1) admits two special points for which its rank drops below its maximum. Then at these points, the R-matrix decomposes in terms of the intertwining operator $\mathcal{E}^{(j+\frac{1}{2})}$ and the operator $\mathcal{F}^{(j+\frac{1}{2})}$, defined above, as follows:

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defines an action of the algebra H on the tensor product of H-modules.

Lemma C.3.1 ([LBG23, Lem. 3.5])**.** *The R-matrix* (C.3.1) at the point $u = q^{j+\frac{1}{2}}$ decomposes as:

$$R^{(\frac{1}{2},j)}(q^{j+\frac{1}{2}}) = \mathcal{E}^{(j+\frac{1}{2})}\mathcal{H}^{(j+\frac{1}{2})}\mathcal{F}^{(j+\frac{1}{2})} \ , \tag{C.3.5}$$

where $\mathcal{E}^{(j+\frac{1}{2})}$ is fixed by Lemma C.1.1, $\mathcal{F}^{(j+\frac{1}{2})}$ is given in (C.1.6) with (C.1.7), (C.1.8) and the entries of $\mathcal{H}^{(j+\frac{1}{2})}$ are

$$\begin{aligned} \mathcal{H}_1^{(j+\frac{1}{2})} &= \mathcal{H}_{2j+2}^{(j+\frac{1}{2})} = \left(\prod_{k=0}^{2j-2} c(q^{2j-k})\right)(q^{2j+1} - q^{-2j-1}) \ , \\ \mathcal{H}_n^{(j+\frac{1}{2})} &= \left(\prod_{k=0}^{2j-2} c(q^{2j-k})\right)\frac{(q-q^{-1})B_{j,-j-1+n}}{\mathcal{E}_{n,n}^{(j+\frac{1}{2})}\mathcal{F}_{n,n+2j}^{(j+\frac{1}{2})}} \ , \end{aligned} \tag{C.3.6}$$

for $n = 2, 3, \ldots, 2j+1$.

Then, with the decomposition (C.3.5) and using the pseudo-inverse property $\mathcal{F}^{(j+\frac{1}{2})}\mathcal{E}^{(j+\frac{1}{2})} = \mathbb{I}_{2j+2}$, we have:

Corollary C.3.2. *The following relations hold:*

$$\mathcal{E}^{(j+\frac{1}{2})}\mathcal{H}^{(j+\frac{1}{2})} = R^{(\frac{1}{2},j)}(q^{j+\frac{1}{2}})\mathcal{E}^{(j+\frac{1}{2})} \ , \tag{C.3.7}$$

$$\mathcal{H}^{(j+\frac{1}{2})}\mathcal{F}^{(j+\frac{1}{2})} = \mathcal{F}^{(j+\frac{1}{2})}R^{(\frac{1}{2},j)}(q^{j+\frac{1}{2}}) \ , \tag{C.3.8}$$

$$R^{(\frac{1}{2},j)}(q^{j+\frac{1}{2}}) = \mathcal{E}^{(j+\frac{1}{2})}\mathcal{F}^{(j+\frac{1}{2})}R^{(\frac{1}{2},j)}(q^{j+\frac{1}{2}}) \ . \tag{C.3.9}$$

We note that Lemma C.3.1 and Corollary C.3.2 are used many times to prove important results such as the reflection equation in Theorem 3.5.2.

Similarly to Lemma C.3.1, for the second special point we have:

Lemma C.3.3 ([LBG23, Lem. 3.7])**.** *The R-matrix* (C.3.1) at the point $u = q^{-j-\frac{1}{2}}$ is decomposed as:

$$R^{(\frac{1}{2},j)}(q^{-j-\frac{1}{2}}) = \bar{\mathcal{E}}^{(j-\frac{1}{2})}\bar{\mathcal{H}}^{(j-\frac{1}{2})}\bar{\mathcal{F}}^{(j-\frac{1}{2})} \ , \tag{C.3.10}$$

where $\bar{\mathcal{E}}^{(j-\frac{1}{2})}$ is fixed by Lemma C.2.1, $\bar{\mathcal{F}}^{(j-\frac{1}{2})}$ is given in (C.2.6) with (C.2.7) and the entries of $\bar{\mathcal{H}}^{(j-\frac{1}{2})}$ are

$$\bar{\mathcal{H}}_n^{(j-\frac{1}{2})} = \left(\prod_{k=0}^{2j-2} c(q^{-k-1})\right)\frac{(q-q^{-1})B_{j,-j+n}}{\bar{\mathcal{E}}_{n+2j+1,n}^{(j-\frac{1}{2})}\bar{\mathcal{F}}_{n,n+1}^{(j-\frac{1}{2})}} \ , \tag{C.3.11}$$

for $n = 1, 2, \ldots, 2j$.

Besides, from the above decomposition and $\bar{\mathcal{F}}^{(j-\frac{1}{2})}\bar{\mathcal{E}}^{(j-\frac{1}{2})} = \mathbb{I}_{2j}$, it follows:

Corollary C.3.4.

$$\begin{aligned} \bar{\mathcal{E}}^{(j-\frac{1}{2})}\bar{\mathcal{H}}^{(j-\frac{1}{2})} &= R^{(\frac{1}{2},j)}(q^{-j-\frac{1}{2}})\bar{\mathcal{E}}^{(j-\frac{1}{2})} \ , \\ \bar{\mathcal{H}}^{(j-\frac{1}{2})}\bar{\mathcal{F}}^{(j-\frac{1}{2})} &= \bar{\mathcal{F}}^{(j-\frac{1}{2})}R^{(\frac{1}{2},j)}(q^{-j-\frac{1}{2}}) \ , \\ R^{(\frac{1}{2},j)}(q^{-j-\frac{1}{2}}) &= \bar{\mathcal{E}}^{(j-\frac{1}{2})}\bar{\mathcal{F}}^{(j-\frac{1}{2})}R^{(\frac{1}{2},j)}(q^{-j-\frac{1}{2}}) \ . \end{aligned} \tag{C.3.12}$$

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Guillaume LEMARTHE

Solutions universelles de l'équation de réflexion, l'algèbre q-Onsager et applications.

Résumé

L'objectif de cette thèse est de développer un cadre permettant d'obtenir des opérateurs K de spin-j avec paramètre spectral, solutions de l'équation de réflexion et appartenant à $B \otimes \mathrm{End}(\mathbb{C}^{2j+1})((u))$, où B est une algèbre comodule sur une algèbre de Hopf H. A partir de ces opérateurs K, nous obtenons des relations-TT universelles satisfaites par des *matrices de transfert universelles* dans la sous-algèbre commutante de B qu'ils génèrent. Ce nouveau cadre proposé est spécialisé à $B = \mathcal{A}_q$, l'extension centrale de l'algèbre q-Onsager, et $H = \mathcal{L}U_q sl_2$, l'algèbre quantique à boucle de sl_2, offrant une approche universelle à diverses chaînes de spins quantiques ouvertes (spin-j, spin-j_1,j_2 alternantes, ...) avec conditions aux bords génériques. Les trois problèmes suivants sont étudiés et résolus.

Premièrement, nous introduisons une définition axiomatique d'une matrice K-universelle $\mathfrak{K} \in B \otimes H$ satisfaisant une équation de réflexion universelle. Elle requiert la donnée d'une paire de twist composée d'un twist ψ (un certain automorphisme de H) et d'un twist de Drinfeld. Cette définition étend les travaux de Balagović-Kolb et Appel-Vlaar. Pour $H = \mathcal{L}U_q sl_2$ et une certaine paire de twist fixée, nous montrons que l'évaluation de la composante tensorielle H de $\mathfrak{K}$ à une représentation d'évaluation formelle (de dimension infinie) de spin-j mène à des opérateurs K avec paramètre spectral $\mathbf{K}^{(j)}(u)$.

Deuxièmement, en nous inspirant de l'évaluation d'un des axiomes de la matrice K-universelle, et en considérant les produits tensoriels des représentations d'évaluation formelle de $\mathcal{L}U_q sl_2$ dans les cas où apparaissent une sous-représentation de spin-j, nous introduisons un opérateur K de spin-j fusionné $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q \otimes \mathrm{End}(\mathbb{C}^{2j+1})((u))$. Ces derniers sont construits à l'aide d'opérateurs d'entrelacs de $\mathcal{L}U_q sl_2$. Ils sont une généralisation spin-j de l'opérateur K de spin-1/2 introduit par Baseilhac et Shigechi en 2009. Nous montrons indépendamment du cadre universel, que toutes ces opérateurs K fusionnés satisfont les équations de réflexion. Ensuite, nous conjecturons une relation de proportionnalité entre $\mathcal{K}^{(j)}(u)$ et l'évaluation de la matrice K-universelle $\mathbf{K}^{(j)}(u)$. Elle est appuyée en montrant que $\mathcal{K}^{(j)}(u)$ vérifie un ensemble de relations similaires à l'évaluation des axiomes de $\mathfrak{K}$.

Troisièmement, une fonction génératrice de spin-j (appelée matrice de transfert universelle) dans la sous-algèbre commutante de $\mathcal{A}_q$ est obtenue grâce aux operateurs K fusionnés $\mathcal{K}^{(j)}(u)$ que nous avons construits. Sous réserve que la conjecture reliant $\mathbf{K}^{(j)}(u)$ et $\mathcal{K}^{(j)}(u)$ soit vraie, nous prouvons que ces fonction génératrices satisfont des relations-TT universelles qui permettent de réduire récursivement le problème spectral de spin-j à celui de spin-1/2. En spécialisant $\mathcal{A}_q$ à certaines représentations de N produits tensoriels, nous retrouvons les matrices de transfert de diverses chaines de spin ainsi que les relations-TT qu'elles satisfont, et aussi des symétries non triviales pour les hamiltoniens correspondants.

Mots clés: Algèbres quantiques affines, algèbre q-Onsager, chaînes de spins quantiques, équation de réflexion, opérateurs K, matrices R et K universelles, relations-TT universelles.

Abstract

The goal of this thesis is to develop a framework for obtaining spin-j K-operators with spectral parameter and which are solutions to the reflection equation. They belong to $B \otimes \mathrm{End}(\mathbb{C}^{2j+1})((u))$, where B is a comodule algebra over a Hopf algebra H. From these K-operators, we derive universal TT-relations satisfied by *universal tranfer matrices* in the commutative subalgebra of B they generate. The proposed new framework is specialized to $B = \mathcal{A}_q$, the central extension of the q-Onsager algebra, and $H = \mathcal{L}U_q sl_2$, the quantum loop algebra of sl_2, providing a universal approach to various quantum spin chains (spin-j, alternating spin-j_1, j_2, etc.) with generic open boundary conditions. The following three problems are studied and solved.

Firstly, we introduce an axiomatic definition of a universal K-matrix $\mathfrak{K} \in B \otimes H$ satisfying a universal reflection equation. It requires the data of a pair of twists consisting of a twist ψ (a certain automorphism of H) and a Drinfeld twist. This definition extends the work of Balagović-Kolb and Appel-Vlaar. For $H = \mathcal{L}U_q sl_2$ and a certain fixed pair of twists, we show that the evaluation of the tensor component in H of $\mathfrak{K}$ at a formal (infinite-dimensional) evaluation representation of spin-j leads to K-operators with spectral parameters $\mathbf{K}^{(j)}(u)$.

Secondly, drawing inspiration from the evaluation of one of the axioms of the universal K-matrix and considering the tensor products of formal evaluation representations of $\mathcal{L}U_q sl_2$ in the cases where a spin-j sub-representation appears, we introduce fused spin-j K-operators $\mathcal{K}^{(j)}(u) \in \mathcal{A}_q \otimes \mathrm{End}(\mathbb{C}^{2j+1})((u))$. These operators are constructed using $\mathcal{L}U_q sl_2$-intertwining operators. They are a spin-j generalization of the spin-1/2 K-operator introduced by Baseilhac and Shigechi in 2009. We also show independently of this universal framework that all of these fused K-operators satisfy the reflection equations. Then, we conjecture a proportionality relation between $\mathcal{K}^{(j)}(u)$ and the evaluated universal K-matrix $\mathbf{K}^{(j)}(u)$. This conjecture is supported by showing that $\mathcal{K}^{(j)}(u)$ satisfies a set of relations similar to the evaluated axioms of $\mathfrak{K}$.

Thirdly, a generating function for spin-j (called universal transfer matrix) in the commutative subalgebra of $\mathcal{A}_q$ is obtained using the fused K-operators $\mathcal{K}^{(j)}(u)$ that we constructed. Provided that the conjecture relating $\mathbf{K}^{(j)}(u)$ and $\mathcal{K}^{(j)}(u)$ holds, we prove that these generating functions satisfy universal-TT relations which allow to reduce recursively the spectral problem of spin-j to the one for spin-1/2. By specializing $\mathcal{A}_q$ to certain N-tensor product representations, we recover the transfer matrices of various spin chains along with the TT-relations they satisfy, and also find non-trivial symmetries for the corresponding hamiltonians.

Keywords: Quantum affine algebras, q-Onsager algebra, quantum spin-chains, reflection equation, K-operators, universal R- and K-matrices, universal TT-relations.