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SORBONNE UNIVERSITÉ
École doctorale de Sciences Mathématiques de Paris Centre
Laboratoire de Probabilités, Statistique et Modélisation (UMR 8001)

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Discipline : Mathématiques

présentée par
Emilien BODIOT

Conditions aux bords pour les champs gaussiens markoviens discrets : une approche opéradique

sous la direction de **Damien SIMON** et **Thierry LÉVY**

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Conditions aux bords pour les champs gaussiens markoviens discrets : une approche opéradique

Résumé

Les chaînes de Markov ont le bon goût de se réduire à de l'algèbre linéaire. En effet, les lois de ces champs unidimensionnels sont localement données par leurs matrices de transition et leurs conditions aux bords par des vecteurs. La mesure invariante d'une chaîne de Markov correspond au vecteur propre de Perron-Frobenius de sa matrice de transition. De tels ponts entre probabilités et algèbre n'existent pas dans la littérature pour les champs markoviens en plus grandes dimensions. Les récents travaux de D. Simon comblent ce manque et proposent une description algébrique des bords invariants pour les champs markoviens discrets sur le réseau carré \mathbb{Z}^2 . Cette théorie s'accompagne d'objets algébriques nouveaux, encore peu compris et qui n'ont pas encore été mis en évidence dans le cadre d'une application non triviale. Dans ce manuscrit, l'objectif est d'exhiber et de comprendre ces structures dans le cas particulier des champs gaussiens markoviens sur le réseau carré \mathbb{Z}^2 . Ce faisant, nous donnons une description algébrique des bords invariants pour de tels champs, qui constituent la brique de base de nombreux modèles en théorie des champs, motivant ainsi le présent travail. Ce manuscrit est naturellement divisé en deux parties.

Dans un premier temps, nous nous intéressons au cas unidimensionnel des champs gaussiens markoviens sur \mathbb{Z} . Nous décomposons les conditions aux bords invariantes dans une base non triviale issue de l'étude précise des singularités d'une fonction méromorphe Ψ à valeurs matricielles. Cet apport s'accompagne d'un algorithme simple permettant le calcul explicite des conditions aux bords invariantes. Nous en profitons pour montrer que, sous ces conditions aux bords, nous retrouvons les quantités d'intérêt habituellement calculées par transformée de Fourier (énergie libre, fonction de corrélation et autres). Par ailleurs, nous appliquons nos résultats pour obtenir une version "invariante" du théorème limite de Szegő dans le cas simple des polynômes trigonométriques à valeurs matricielles.

Dans un second travail, mené conjointement avec Damien Simon, nous abordons le cas des champs gaussiens markoviens sur le réseau carré \mathbb{Z}^2 et exhibons de nouveaux objets et structures de bord, qui constituent le premier exemple non trivial de la théorie développée par D. Simon. Pour ce faire, nous bâtissons des outils originaux, dont cette récente théorie est dépourvue. Ces approches sont basées sur des techniques classiques de mécanique statistique et de calcul gaussien mais aussi sur des méthodes nouvelles. En particulier, nous revisitons la très classique matrice de transfert, objet unidimensionnel, que nous adaptons pour produire tous les objets purement bidimensionnels introduits par D. Simon. Les constructions opéradiques de D. Simon sont ici réalisées sur des espaces de formes quadratiques par l'intermédiaire de compléments de Schur. Nous montrons que tous ces objets vérifient des équations de type "vecteurs propres" à des morphismes d'opérade près, assurant que, une fois recombinaisonnés, ces objets constituent bien le bord invariant recherché pour nos champs gaussiens.

Mots-clés : théorie des champs, propriété de Markov, conditions aux bords, champs gaussiens, théorie des opérades

Boundary conditions for discrete Gaussian Markov fields: an operadic approach

Abstract

Markov chains reduce to linear algebra. Indeed, the laws of these one-dimensional fields are locally given by their transition matrices and their boundary conditions by vectors. In particular, one knows that the invariant measure associated to a Markov chain corresponds to the Perron-Frobenius eigenvector of its transition matrix. Such bridges between probability theory and algebra don't exist in the literature for higher dimensional Markov fields. The work of D. Simon fills this gap by proposing an algebraic description of invariant boundary conditions associated to discrete Markov fields defined on the square lattice \mathbb{Z}^2 . This theory comes with new algebraic objects which have not been constructed on any non trivial model yet. In the present manuscript, the main objective is to exhibit and understand these structures in the particular case of Gaussian Markov fields on the lattice \mathbb{Z}^2 . By doing so, we give an algebraic description of invariant boundary conditions for such fields. This thesis is naturally divided into two parts.

We start by focusing on one-dimensional Gaussian Markov fields on \mathbb{Z} . We decompose invariant boundary conditions in a non trivial basis obtained from the precise study of a meromorphic and matrix-valued function Ψ . This brings a simple algorithm for computing explicitly the invariant boundary conditions. Moreover, under these boundary conditions, we show that we recover the values of classical quantities of interest, such as free energy or correlation function, generally computed using Fourier transform. As a little application, we give an "invariant" version of the Szegő's limit theorem in the simple case of matrix-valued trigonometric polynomials.

Secondly, in a joint work with Damien Simon, we study the case of Gaussian Markov fields on the square lattice \mathbb{Z}^2 and exhibit new objects and boundary structures for Gaussian fields. It is the very first non trivial example for the theory introduced by D. Simon. To do so, we give new tools, missing in the theory. These approaches are based on classical tools from statistical mechanics and Gaussian calculus but also from original methods. For instance, we adapt the so-called transfer matrix, a one-dimensional object, to produce all the two-dimensional objects introduced by D. Simon. In this context, operadic constructions of D. Simon are given by Schur complements on quadratic forms spaces. We show that these objects verify "eigenvector-type" equations up to an operadic morphism. This ensures that, once being combined, they produce the desired invariant boundary associated to our Gaussian fields.

Keywords: field theory, Markov property, boundary conditions, gaussian fields, operad theory

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Chapter 1

Introduction

1.1 Motivations premières et principaux protagonistes

1.1.1 Théorie statistique des champs

Ce manuscrit s'inscrit dans la théorie statistique des champs. Cette branche des mathématiques, à l'interface avec la physique, a pour but de décrire des structures complexes à l'aide des probabilités. Étant données des contraintes microscopiques, on souhaite être capable de comprendre les phénomènes macroscopiques qui en sont issus. Se donner un champ statistique $\phi_M : M \rightarrow E$, sur une certaine variété ou un certain réseau/graphes M à valeurs dans un certain espace d'état E , c'est se donner la mesure suivante, appelée mesure de Gibbs,

$$d\mu_M^{\text{Gibbs}} = \frac{1}{Z_M} e^{-S_M(\phi_M)} d\phi_M. \quad (1.1.1)$$

Lorsque le contexte est clair, on écrira simplement $\phi := \phi_M$. On parle de champ discret lorsque M est un réseau ou un graphe et de champ continu quand M est une variété. La dimension du champ est donnée par la dimension de M . Arrêtons-nous quelques instants sur les différents protagonistes que cette équation fait intervenir. La fonction de partition, notée Z_M est une constante de normalisation qui assure que nous manipulons effectivement une mesure de probabilité. La mesure de référence $d\phi_M$ est bien souvent une hypothétique mesure de Lebesgue (ou de comptage). Enfin, la fonctionnelle S_M est appelée action du champ. La quantité $S_M(\phi_M)$ décrit l'énergie de la configuration ϕ_M . Moralement, les probabilités favorisent les configurations de basses énergies, c'est le principe de moindre action. De même que la mesure de référence $d\phi_M$, l'action est une quantité que l'on détermine en fonction du modèle que l'on cherche à étudier. Par exemple, supposer que M est un graphe fini, S_M une forme quadratique positive et $d\phi_M$ est la mesure produit de Lebesgue conduit à considérer un champ gaussien fini.

Le problème d'existence d'une telle mesure. Cependant, si M n'est pas un graphe fini mais infini ou une variété, une telle mesure n'a pas de chance d'exister rigoureusement sous cette forme. En cause, l'absence de mesure de Lebesgue en dimension infinie. Fort heureusement, pour certains champs, cette difficulté peut être contournée. C'est le cas des champs gaussiens. On pensera par exemple à la mesure de Wiener qui définit, au travers du mouvement brownien, une mesure gaussienne sur les fonctions continues sur $M = [0, T]$. En

plus grande dimension, sur un domaine $M \subset \mathbb{R}^d$, on pourra aussi penser au champ libre gaussien continu. Attention tout de même, dès que $d \geq 2$, ce dernier n'est plus une fonction mais presque sûrement une distribution. Il arrive cependant que l'on veuille s'affranchir du cas gaussien et ainsi définir d'autres champs sur des réseaux infinis ou des variétés. Alors, la question de l'existence de cette mesure de Gibbs devient inévitable. Une vaste partie de la littérature s'intéresse d'ailleurs à montrer que ces mesures existent et qu'on peut les construire. Pour ce faire, un certain nombre de méthodes peuvent être mises en place. On peut se servir du champ libre gaussien comme mesure de référence que l'on vient perturber par une densité apportant des interactions, c'est le cas en gravité quantique de Liouville, voir [DKRV16] par exemple. On peut aussi donner un sens à (1.1.1) comme mesure invariante d'un processus stochastique, on parle alors de quantification stochastique. Ce dernier explore le potentiel associé à S_M par une dynamique de Langevin et, en temps long, converge vers sa mesure invariante donnée par (1.1.1). Par exemple, de telles idées ont été utilisées pour définir la mesure de Gibbs associée au champ Φ_3^4 , dont l'action est proche de celle du champ libre en dimension 3 mais fait intervenir la puissance 4 du champ, voir [MW17]. Enfin, si le champ est markovien, on peut vouloir construire une telle mesure en volume infini par théorème d'extension de Kolmogorov pour peu que l'on se soit donné le bon jeu de marginales cohérentes. S'il est aisé de le faire pour les chaînes de Markov au moyen des mesures invariantes, il est compliqué d'imaginer le faire pour des champs en plus grande dimension. La motivation principale de ce manuscrit est d'emprunter ce dernier chemin et de donner des marginales cohérentes au travers d'une étude algébrique fine des bords.

En ce qui nous concerne, pour la majeure partie de ce manuscrit, nous considérerons des réseaux ou des graphes finis et l'existence de cette mesure ne sera pas l'objet de nos préoccupations. Notre intérêt est plutôt de tirer partie de la propriété de Markov et ainsi d'étudier la notion de bords invariants.

Propriété de Markov. En général l'action S_M possède une structure purement locale. Dans le cas des champs discrets sur réseaux, S_M est alors une somme ne faisant intervenir que les valeurs du champ entre nœuds voisins. On parle de modèle avec interactions entre "plus proches voisins". Par exemple, dans le cas du modèle d'Ising, qui décrit le ferromagnétisme, nous avons (dans une version simplifiée sans champ externe)

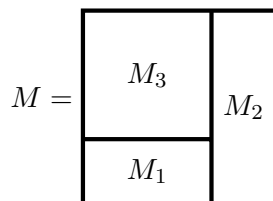
$$S_M(\phi) = - \sum_{i \sim j} J(i, j) \phi_i \phi_j$$

où M est un certain réseau, $\phi : M \rightarrow \{-1, 1\}$ et $J : M \times M \rightarrow \mathbb{R}$, $(i, j) \mapsto J(i, j)$ est l'interaction entre $i \sim j$ des nœuds voisins du réseau (reliés par une arête). D'une manière générale, le lecteur pourra se référer à [Bax16] ou [FV17] pour de plus amples détails sur les modèles sur réseaux. Pour les champs continus, l'action S_M s'exprime comme une intégrale sur M faisant intervenir des opérateurs locaux. Par exemple, dans le cas du champ libre gaussien sur un domaine $M \subset \mathbb{R}^d$ à valeurs réelles, l'action donnée par

$$S_M(\phi) = \frac{1}{2} \int_M \|\nabla \phi(x)\|^2 dx$$

fait intervenir le gradient du champ en le point x . C'est une donnée locale qui retranscrit l'interaction entre "voisins infinitésimalement proches". Néanmoins, des complications analytiques apparaissent très vite : existence de la mesure de Gibbs (1.1.1) ou encore régularité du champ ϕ rendant une telle écriture mal définie. Cependant, d'un point de vue probabiliste, que ce soit pour les champs discrets ou continus, la propriété de localité de S_M implique la propriété de Markov. Grossièrement, conditionnellement à la valeur du champ sur les bords de domaines d'intérieurs disjoints de M , les variables aléatoires à l'intérieur de

ces domaines sont indépendantes. Par exemple, si ϕ_M est un champ au dessus de M , un sous graphe rectangulaire du réseau carré \mathbb{Z}^2 que l'on partitionne en trois régions M_1 , M_2 et M_3 comme suit



alors la localité de S_M implique qu'il est possible de factoriser la mesure de Gibbs (1.1.1) au dessus des zones M_1 , M_2 et M_3 de la manière suivante

$$d\mu_M^{\text{Gibbs}} = e^{-S_{M_1}(\phi_{M_1})} e^{-S_{M_2}(\phi_{M_2})} e^{-S_{M_3}(\phi_{M_3})} d\phi_M.$$

Dès lors, les valeurs du champ évaluées en les zones M_1 , M_2 et M_3 sont indépendantes sachant les frontières représentées par des lignes noires. Inversement, il est possible de montrer que la loi de tout champ markovien sur un graphe quelconque M se factorise en des poids élémentaires au dessus des sous-graphes complets de M et s'écrit donc comme une mesure de Gibbs avec interactions entre plus proches voisins, c'est le théorème de Hammersley-Clifford.

Profitons-en pour attirer l'attention du lecteur sur le fait suivant. La construction d'un champ sur M à partir de champs sur M_1 , M_2 et M_3 , tels que dans la précédente figure, nécessite de faire coïncider les valeurs des champs le long des frontières qui séparent les trois régions. D'un point de vue probabiliste, cette nécessité est très contraignante. En dimension 1 déjà, recoller des processus stochastiques au dessus d'un point en y faisant coïncider leur valeur n'est pas une chose aisée. Ici, il s'agit de le faire au dessus de frontières qui sont étendues.

Cas unidimensionnel. Le cas des champs markoviens discrets de dimension 1 constitue, par sa nature algébrique simple, l'exemple le plus facile de la théorie des champs. Servons-nous en pour illustrer ce qui vient d'être dit et motiver nos futures préoccupations. Étant donné $P \in \mathbb{N}^*$, soit $M = \{0, \dots, P\}$ et $\phi : M \rightarrow E$, un champ unidimensionnel, discret et markovien à valeurs dans E un certain ensemble fini. Nous savons que la mesure de Gibbs (1.1.1) associée au champ peut se factoriser au travers d'une famille de poids élémentaires $A_i : M \times M \rightarrow \mathbb{R}_+$, pour $i \in \{0, \dots, P-1\}$, qui décrit les interactions entre proches voisins et sa densité contre la mesure produit de comptage s'écrit alors

$$\frac{1}{Z_P} A_0(\phi_0, \phi_1) \cdots A_{P-1}(\phi_{P-1}, \phi_P)$$

avec Z_P la fonction de partition. Supposons que $A_i = A$ pour tout $i \in \{0, \dots, P-1\}$. Nous sommes alors dans le cas unidimensionnel d'une chaîne de Markov homogène de matrice de transition A . On peut représenter graphiquement la situation par le dessin du graphe suivant

$$\begin{array}{ccccccc} \phi_0 & \phi_1 & \phi_2 & \cdots & \phi_P & & \\ \bullet & \bullet & \bullet & \cdots & \bullet & & \\ & A & A & A & A & & \end{array} \quad (1.1.2)$$

Le champ ϕ est défini au dessus des nœuds du graphe ci-dessus et sa loi est localement donnée par A qui décore les arêtes entre proches voisins. Conditionner sur la valeur du champ en un nœud interne au segment factorise la loi du champ en deux composantes indépendantes, c'est une conséquence de la propriété de Markov. En quelque sorte, la

propriété de Markov induit (géométriquement) des découpes (des partitions) du segment. Calculer la loi marginale sur un sous-ensemble de nœuds nécessite de sommer sur les autres et cela fait naturellement apparaître des puissances de la matrice A . Par exemple, si P est pair, la chaîne de Markov restreinte aux nœuds pairs a pour matrice de transition A^2 . Graphiquement, on a alors fusionné deux segments de tailles 1 en les recollant au dessus du nœud qu'ils partagent pour en former un de taille 2 :

$$\begin{array}{ccc} \bullet \text{---} \bullet & \bullet \text{---} \bullet & \longrightarrow & \bullet \text{---} \bullet \\ A & A & & A^2 \end{array} \quad (1.1.3)$$

Les propriétés probabilistes de Markov se traduisent donc géométriquement par des découpes ou des fusions de domaines et algébriquement par des matrices attachées à ces domaines que l'on peut recoller en faisant leur produit. L'un des atouts majeurs de l'algèbre linéaire est la simple description des bords. Pourvoir notre champ de conditions aux bords c'est le munir de conditions supplémentaires décrivant le comportement des valeurs en ϕ_0 et ϕ_P . Pour ce faire, il suffit de se donner un vecteur ligne à gauche B_L pour ϕ_0 et un vecteur colonne B_R pour ϕ_P à droite (le vecteur $B_R = (1, 1, \dots, 1)^t$ pour les chaînes orientées en temps) et alors considérer la mesure de Gibbs dont la densité est donnée par

$$\frac{1}{Z_P(B_L, B_R)} B_L(\phi_0) A(\phi_0, \phi_1) \cdots A(\phi_{P-1}, \phi_P) B_R(\phi_P).$$

Dit autrement, B_L et B_R donnent les densités de bords. En général, nous aimons particulièrement bien que ces conditions aux bords soient données par les mesures invariantes. Nous allons le voir, cela simplifie certains calculs. Pour faire simple, plaçons-nous dans le cas générique en supposant la chaîne irréductible et apériodique. Il est bien connu que la croissance exponentielle de A est contrôlée par la valeur propre de Perron-Frobenius Λ_{PF} (qui pourrait tout à fait être différente de 1 étant donné que la matrice A n'est pas forcément supposée stochastique) et que les mesures invariantes de la chaîne sont les vecteurs propres associés. On pose alors B_L et B_R respectivement les vecteurs propres de Perron-Frobenius à gauche et à droite de A . Alors, pour tout $M' = \llbracket k, l \rrbracket \subset M$,

$$\begin{aligned} & \sum_{\phi_i; i \in M \setminus M'} B_L(\phi_0) A(\phi_0, \phi_1) \cdots A(\phi_{P-1}, \phi_P) B_R(\phi_P) \\ &= \Lambda_{PF}^{k+P-l} B_L(\phi_k) A(\phi_k, \phi_{k+1}) \cdots A(\phi_{l-1}, \phi_l) B_R(\phi_l) \end{aligned}$$

par propriété des vecteurs propres. Autrement dit, la marginale sur M' du champ défini sur M de poids A et de conditions aux bords B_L et B_R est un champ de poids A et de conditions aux bords inchangées B_L et B_R . On parle de marginales cohérentes. En conséquence, sous de telles conditions aux bords, il est possible de construire le champ sur tout \mathbb{Z} grâce au théorème d'extension de Kolmogorov, c'est la mesure de Gibbs en volume infini du champ. Par ailleurs, ces conditions aux bords rendent simples les calculs de bon nombre de quantités d'intérêt. Par exemple, la fonction de partition s'écrit simplement $Z_P(B_L, B_R) = \Lambda^P$ (à constante multiplicative près) et l'énergie libre, habituellement calculée au travers de la limite thermodynamique

$$f = \lim_{P \rightarrow \infty} \frac{1}{P} \log Z_P(B_L, B_R)$$

n'est plus une limite ici, on a directement $f = \log \Lambda$. En résumé, les champs unidimensionnels ont le bon goût de se réduire à de l'algèbre linéaire. Les problèmes de bords invariants se ramènent à la recherche de la plus grande valeur propre de A et de ses vecteurs propres à gauche et à droite associés. Le chapitre 2 de ce manuscrit est d'ailleurs dédié à l'étude de ces bords invariants dans le cas où le champ ϕ est un champ gaussien.

Le coût de la montée en dimension. Si les champs unidimensionnels sont simples à traiter, les champs en plus grandes dimensions sont, quant à eux, plus difficiles à aborder. Bien sûr, l'approche classique par matrice de transfert, qui scinde la dimension d en $(d-1)+1$, permet de se ramener à un champ markovien unidimensionnel quitte à élargir grandement l'espace d'état. Une telle approche brise cependant la symétrie du modèle en privilégiant une dimension au profit des autres. L'information portée par ces dernières se retrouve mélangée sans qu'il ne soit possible de faire machine arrière. Par ailleurs, la montée en dimension s'accompagne évidemment de difficultés analytiques supplémentaires. Par exemple, comme précédemment mentionné, le recollement de champs en dimension supérieure nécessite de les faire coïncider sur les bords le long desquels on recolle et, d'un point de vue probabiliste, c'est une chose délicate (particulièrement pour le recollement le long de bords de tailles infinies). Enfin, pour les champs continus, la montée en dimension met en lumière des difficultés techniques supplémentaires. Par exemple, si le mouvement brownien est presque sûrement continu, les réalisations du champ libre gaussien en dimension supérieure à 2 sont presque sûrement des distributions.

1.1.2 Conditions aux bords invariantes.

Nous venons de l'aborder dans le cas unidimensionnel, il convient de spécifier des conditions au bord pour tout champ défini au dessus d'une variété ou d'un graphe possédant un bord ∂M . Cela permet de décrire le comportement des valeurs prises par le champ sur ∂M . La discussion des bords en dimension supérieure ou égale à 2 est bien souvent esquivée au profit de bords déterministes (champ nul au bord par exemple) ou de conditions aux bords périodiques. Cependant, après avoir constaté la force des bords invariants à 1D, nous aimerions exporter l'idée en dimension supérieure. Si l'on considère un champ markovien, nous pourrions donc être tentés de définir une notion de bords invariants sous restrictions à des sous-domaines. Essayons d'en donner une définition, ne serait-ce qu'heuristique dans le cas d'un champ markovien discret sur \mathbb{Z}^d . Soit \mathcal{M} une famille de domaines finis de \mathbb{Z}^d qui contient une sous famille $(M_n)_{n \in \mathbb{N}}$ croissante pour l'inclusion et telle que $\cup_n M_n = \mathbb{Z}^d$. Heuristiquement, on pourrait imaginer ajouter un terme d'énergie sur le bord à la mesure de Gibbs (1.1.1) tel que pour tout $M \in \mathcal{M}$, le champ markovien restreint sur M ait pour loi

$$d\mu_M^{\text{inv}} = \frac{1}{Z_M} e^{-S_M(\phi_M) - B_{\partial M}(\phi_{\partial M})} d\phi_M \quad (1.1.4)$$

avec $(S_M)_{M \in \mathcal{M}}$ une famille cohérente au sens où, pour tout $M' \subset M$, $S_{M'}$ est bien donnée par la restriction de S_M sur M' . Pour assurer la cohérence des mesures (1.1.4), il faut de plus que la famille $(B_{\partial M})_{M \in \mathcal{M}}$ soit cohérente sous marginalisation. C'est-à-dire que, pour tous $M' \subset M$,

$$\int_{\text{Champ}(M \setminus M')} e^{-S_M(\phi_M) - B_{\partial M}(\phi_{\partial M})} d\phi_{M \setminus M'} = \Lambda^{\text{Vol}(M \setminus M')} e^{-S_{M'}(\phi_{M'}) - B_{\partial M'}(\phi_{\partial M'})} \quad (1.1.5)$$

où $\text{Vol}(M \setminus M')$ est le volume de la partie que l'on a effacée en prenant la marginale. Autrement dit, la restriction sur M' du champ défini sur M et de condition au bord $B_{\partial M}$ est un champ dont les interactions sont données par la restriction de S_M sur M' et de condition au bord $B_{\partial M'}$. On dira alors que la condition au bord $B_{\partial M}$ est invariante. L'intérêt de cette notion de bords invariants est double. D'un point de vue théorique, les marginales cohérentes fournies par ces conditions aux bords rendent possible la définition de la mesure de Gibbs en volume infini. En effet, le théorème d'extension de Kolmogorov assure l'existence de la mesure de Gibbs sur \mathbb{Z}^d tout entier. D'un point de vue pratique, cela rend de nombreux calculs simples et exacts. En particulier, le calcul de la fonction de partition se déduit

directement

$$Z_M = \Lambda^{\text{Vol}(M)}$$

à constante multiplicative près. Par ailleurs, la valeur de l'énergie libre, formellement définie par la limite thermodynamique

$$\lim_{\text{Vol}(M) \rightarrow \infty} \frac{1}{\text{Vol}(M)} \log Z_M = \log \Lambda$$

n'est en fait plus une limite. C'est normal, par le choix des bords invariants, nous avons "compris" le comportement du champ à l'infini. Néanmoins, contrairement au cas unidimensionnel, il n'existe pas à notre connaissance, en dimension supérieure ou égale à deux, une description algébrique précise de telles conditions aux bords $(B_{\partial M})_{M \in \mathcal{M}}$ cohérentes au niveau du cadre général abstrait des processus markoviens.

Champs markoviens de dimension supérieure, produit de matrices et théorie des opérades.

Nous l'avons vu, en dimension 1, la recherche de bords invariants se ramène à de l'algèbre linéaire. En plus grande dimension, les choses ne sont malheureusement pas si simples. Pour des graphes généraux, le théorème de Hammersley-Clifford assure la factorisation de la loi de tout champ markovien au dessus des sous-graphes complets. Cela n'a cependant pas été exploité algébriquement et jusqu'à très récemment, aucune structure de bord, analogue à celle de la dimension 1, n'était connue pour les champs en dimension deux et plus. Néanmoins, dans le cas des champs sur le réseau carré \mathbb{Z}^2 , de récents travaux [Sim23] comblent ce manque et donnent une description de $B_{\partial M}$ au moyen d'un Ansatz de produit de matrices et de l'utilisation de la théorie des opérades. Afin de mieux suivre les découps de domaines imposées par la propriété de Markov, les champs markoviens considérés par l'auteur sont définis au dessus des arêtes et non des nœuds du réseau carré. Un changement de graphe permet de se ramener au cas classique. Dans la chapitre 3, nous adopterons la même convention. Sur le réseau carré, à l'instar de la matrice de transition attachée à un segment de taille 1 en (1.1.2), les interactions entre proches voisins sont données par un opérateur

$$\boxed{A} \tag{1.1.6}$$

attaché à un carré de taille 1-par-1. À chacune des arêtes qui composent le bord de ce carré est attachée une variable aléatoire. Ces quatre variables aléatoires voisines sont couplées entre elles par A . On parle de poids de face. Comme pour le cas unidimensionnel, la propriété de Markov induit des découps et des fusions de domaines sur le réseau carré. Par exemple, tout comme le recollement de segments (1.1.3) passe par les puissances de A , nous pouvons recoller les poids de face

$$\begin{array}{ccc} \boxed{A} & \boxed{A} & \rightarrow & \boxed{A} & \boxed{A} \\ & & & & \end{array} \qquad \begin{array}{ccc} \boxed{A} & & \rightarrow & \boxed{A} \\ \boxed{A} & & & \boxed{A} \end{array} \tag{1.1.7}$$

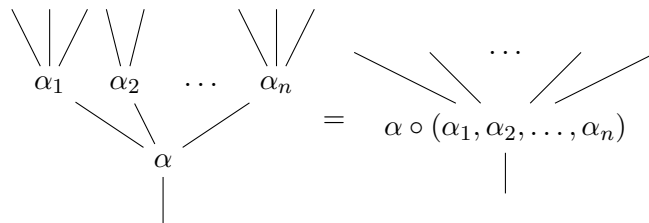
induisant des puissances "surfiques" des poids de face A . En témoignent ces dessins, nous avons à présent deux directions pour effectuer ces recollements. Chaque recollement conduit à une sommation ou une intégration sur un espace d'état et, en voyant les poids de face comme des tenseurs (dont les coefficients sont indexés par les 4 valeurs prises au bord de la face), cela donne des contractions d'indices de tenseurs (en dimension 1, c'est le simple produit de matrice). Le recours à la théorie des opérades permet de traiter rigoureusement ces nouvelles structures de découps et de fusions de domaines, en mettant

en avant, non pas la réalisation des opérations elles-mêmes mais leurs relations mutuelles (associativités). Cela permet la généralisation de nombreuses constructions d'algèbre linéaire. Leurs calculs concrets, en revanche, demanderont de revenir à la définition des produits. De plus amples détails peuvent être trouvés en section 1.1.3. Pour parler de bords invariants, l'idée développée par [Sim23] est de découper le plan \mathbb{Z}^2 en des domaines rectangulaires par des coupes horizontales et verticales successives appelées coupes guillotines. À chacun de ces domaines rectangulaires est alors associé un objet algébrique : les bandes, les demi-bandes, les coins, les rectangles... Tous ces objets algébriques, plus sophistiqués que de simples vecteurs propres, peuvent ensuite être recollés entre eux, comme dans (1.1.7), au moyen de produits dont la combinatoire est dictée par l'opérade guillotine, pour ainsi former le bord $B_{\partial M}$ voir (1.3.6). L'ensemble de ces objets munis de ces produits de recollements forment ce que l'on appelle une Guill_2 -algèbre. Ils satisfont par ailleurs des équations de type "vecteur propre" assurant que la condition au bord $B_{\partial M}$ est bien invariante sous restriction. Cependant, ces objets nouveaux sont encore peu compris. L'objectif de ce manuscrit est de donner et décrire ceux-ci dans le cas particulier des champs gaussiens markoviens sur le réseau carré. Cette étude sera menée dans le chapitre 3. À notre connaissance, la question des bords invariants n'est pas traitée dans la littérature ou, du moins, pas sous cette forme ni à ce niveau de description. Notons quand même l'existence d'une théorie conforme des bords en théorie conforme des champs, voir [Car04], ainsi que des algèbres à factorisation en théorie topologique des champs, voir [Gin15].

1.1.3 Opérades, opérades colorées et opérade guillotine

Le cœur de cette thèse réside en une description algébrique des bords des champs gaussiens markoviens au moyen d'objets nouveaux et de leurs relations introduits dans [Sim23]. Ces structures algébriques s'inscrivent dans le cadre de la théorie des opérades et nous profitons de cette section pour les aborder. Après avoir rapidement introduit le concept d'opérades, nous nous concentrerons sur l'opérade Guill_2 des partitions guillotines, particulièrement présente dans le chapitre 3. Nous n'allons cependant pas rentrer dans tous les détails. Aussi, nous invitons le lecteur à se référer à [Sim23] pour les définitions précises.

Opérades. La théorie des opérades généralise et étudie les lois de composition de manière abstraite et combinatoire. Moralement, une opérade est un ensemble d'opérations à plusieurs entrées que l'on peut composer. Le nombre d'entrées d'une opération est appelé l'arité. Bien souvent, ces opérations sont représentées par des arbres et leurs compositions par des greffages d'arbres comme suit



Dans ce dessin, l'opération α d'arité n est composée avec n opérations α_i pour former une opération $\alpha \circ (\alpha_1, \alpha_2, \dots, \alpha_n)$ d'arité $k_1 + k_2 + \dots + k_n$ où k_i est l'arité de α_i . Il n'y a, pour l'instant, pas d'ensemble que l'on combine par ces opérations. Ceux-ci iront se placer sur les feuilles des arbres et les résultats des opérations se liront aux racines. Une définition plus précise d'opérade est la suivante.

Définition 1.1.1 (Opérade symétrique). Une opérade symétrique est une suite d'ensembles $(P(n))_{n \in \mathbb{N}}$ (un élément de $P(n)$ est appelé opération d'arité n) telle que :

- (i) pour tout $n \in \mathbb{N}$ et toute suite finie $(k_i)_{1 \leq i \leq n}$ dans \mathbb{N} , il existe une fonction de composition

$$\begin{aligned} \circ : P(n) \times P(k_1) \times \dots \times P(k_n) &\rightarrow P(k_1 + \dots + k_n) \\ (\alpha, \alpha_1, \dots, \alpha_n) &\mapsto \alpha \circ (\alpha_1, \dots, \alpha_n) \end{aligned}$$

- (ii) il existe un élément $\text{id} \in P(1)$ appelé l'identité telle que, pour tout $\alpha \in P(n)$, $\alpha \circ (\text{id}, \dots, \text{id}) = \alpha$ et $\text{id} \circ \alpha = \alpha$;
- (iii) pour tout $n \in \mathbb{N}$, toutes suites $(k_i)_{1 \leq i \leq n}$ et $((l_{i,j})_{1 \leq j \leq k_i})_{1 \leq i \leq n}$, la fonction de composition satisfait les règles d'associativité suivantes, pour toutes opérations $\alpha \in P(n)$, $\alpha_i \in P(k_i)$ et $\alpha_{i,j} \in P(l_{i,j})$:

$$\begin{aligned} &(\alpha \circ (\alpha_1, \dots, \alpha_n)) \circ (\alpha_{1,1}, \dots, \alpha_{1,l_{1,k_1}}, \alpha_{2,1}, \dots, \alpha_{n,l_{n,k_n}}) \\ &= \alpha \circ (\alpha_1 \circ (\alpha_{1,1}, \dots, \alpha_{1,k_1}), \dots, (\alpha_n \circ (\alpha_{n,1}, \dots, \alpha_{n,k_n}))) \end{aligned} \quad (1.1.8)$$

- (iv) le groupe symétrique \mathfrak{S}_n agit sur $P(n)$ de sorte que

$$\alpha_\sigma \circ (\alpha_1, \dots, \alpha_n) = \alpha \circ (\alpha_{\sigma^{-1}(1)}, \dots, \alpha_{\sigma^{-1}(n)}). \quad (1.1.9)$$

Nous invitons le lecteur à se référer à [LV12] ou [MSS02] pour de plus amples détails sur la théorie des opérades. Lorsque l'opérade se réalise sur un ensemble, on parle d'algèbre sur une opérade. L'ensemble hérite alors des opérations et des relations de l'opérade.

Définition 1.1.2 (Algèbre sur une opérade). Soit P une opérade symétrique. Une P -algèbre est un ensemble E tel que :

- (i) pour tout $n \in \mathbb{N}$ et tout $\alpha \in P(n)$, il existe une application $E^n \rightarrow E$, $(u_1, \dots, u_n) \mapsto m_\alpha(u_1, \dots, u_n)$;
- (ii) pour tout $u \in E$, $m_{\text{id}}(u) = u$;
- (iii) pour tout $(\alpha, \alpha_{k_1}, \dots, \alpha_{k_n}) \in P(n) \times P(k_1) \times \dots \times P(k_n)$, et toute suite $(u_i)_{1 \leq i \leq k_1 + \dots + k_n}$,

$$m_{\alpha \circ (\alpha_1, \dots, \alpha_n)}(u_1, \dots, u_{k_1 + \dots + k_n}) = m_\alpha(v_1, \dots, v_n)$$

où les éléments $v_j \in E$ sont donnés par

$$v_j = m_{\alpha_j}(u_{k_1 + \dots + k_{j-1} + 1}, \dots, u_{k_1 + \dots + k_j})$$

- (iv) pour tout $\alpha \in P(n)$ et $(u_1, \dots, u_n) \in E^n$,

$$m_{\alpha_\sigma}(u_1, \dots, u_n) = m_\alpha(u_{\sigma^{-1}(1)}, \dots, u_{\sigma^{-1}(n)}).$$

Attention tout de même, aucune structure d'espace vectoriel ni de multilinéarité n'est supposée pour l'ensemble E et ses opérations m_α dans la précédente définition. Dès lors, celle-ci ne coïncide pas toujours avec la définition classique d'algèbre comme d'un espace vectoriel muni d'un produit. Ces opérations peuvent donc tout à fait être non-linéaires, comme ce sera le cas sur les formes quadratiques du chapitre 3 par exemple.

L'opérade associative $\text{Ass}(n)$. L'opérade associative $(\text{Ass}(n))_{n \in \mathbb{N}^*}$ est celle qui encode les monoïdes (ou les algèbres si une structure d'espace vectoriel supplémentaire est considérée). Cette opérade est simplement définie par $\text{Ass}(n) = \mathfrak{S}_n$ pour tout $n \geq 1$. La composition $\sigma \circ (\sigma_1, \dots, \sigma_n)$ est donnée par la permutation par bloc : chaque σ_i permute un bloc de taille k_i et σ permute les n blocs. On obtient alors une permutation de $k_1 + k_2 + \dots + k_n$ éléments. On note $m := \text{id} \in \mathfrak{S}_2$. Par récurrence, nous remarquons que cette opérade est générée par $\text{id} \in \text{Ass}(1)$ et $m \in \text{Ass}(2)$ (opération d'arité 2), sous l'action de \mathfrak{S}_n et la condition d'associativité

$$m \circ (m, \text{id}) = m \circ (\text{id}, m).$$

Réalisé sur un monoïde, $m : (a, b) \mapsto m(a, b) = ab$ est la loi de composition interne. Couplée avec l'action de \mathfrak{S}_n , elle génère tous les produits d'arité $n \in \mathbb{N}^*$, $m_\sigma : (a_1, \dots, a_n) \mapsto m_\sigma(a_1, \dots, a_n) = a_{\sigma(1)} \cdots a_{\sigma(n)}$. Comme aucune symétrie supplémentaire n'est supposée, l'ordre des a_i importe dans le produit. C'est la raison pour laquelle $\text{Ass}(n) = \mathfrak{S}_n$, il y a exactement $n!$ opérations d'arité n distinctes.

L'opérade commutative $\text{Com}(n)$. L'opérade commutative $(\text{Com}(n))_{n \in \mathbb{N}^*}$ est encore plus simple. En effet $\text{Com}(n) = \{m_n\}$ pour tout $n \in \mathbb{N}^*$. Elle correspond à un produit commutatif $m_n : (a_1, \dots, a_n) \mapsto m_n(a_1, \dots, a_n) = a_1 \cdots a_n$, i.e l'ordre des a_i n'importe pas et il n'y a donc qu'une seule opération d'arité n .

Opérades colorées. La structure opéradique qui nous intéresse, celle donnée par les partitions guillottes, est engendrée par le recollement de rectangles. Les rectangles peuvent être recollés entre eux s'ils partagent un côté de même longueur. C'est un critère de compatibilité qui nécessite de travailler avec la notion d'opérades colorées. Pour les composer, les opérations doivent alors être compatibles au regard de la couleur, ici la longueur des bords des rectangles. La définition d'opérade colorée est la suivante.

Définition 1.1.3 (Opérade colorée). Soit C un ensemble, appelé l'ensemble des couleurs. Une opérade colorée sur C est une suite d'ensembles $(P(c; c_1, \dots, c_n)_{(c, c_1, \dots, c_n) \in C^{n+1}})_{n \in \mathbb{N}}$ telle que :

- (i) pour tout $n \in \mathbb{N}$, toutes couleurs $(c, c_1, \dots, c_n) \in C^{n+1}$ et toute suite finie de couleurs $((c_i, c_1^{(i)}, \dots, c_{k_i}^{(i)}))_{1 \leq i \leq n}$, il existe une fonction de composition :

$$\circ : P(c; c_1, \dots, c_n) \times \left(\prod_{i=1}^n P(c_i; c_1^{(i)}, \dots, c_{k_i}^{(i)}) \right) \rightarrow P(c; c_1^{(1)}, \dots, c_{k_1}^{(1)}, \dots, c_{k_n}^{(n)})$$

$$(\alpha, \alpha_1, \dots, \alpha_n) \mapsto \alpha \circ (\alpha_1, \dots, \alpha_n)$$

- (ii) pour tout $c \in C$, il existe un élément $\text{id}_c \in P(c; c)$ tel que, pour tout $\alpha \in P(c; c_1, \dots, c_n)$, $\alpha \circ (\text{id}_{c_1}, \dots, \text{id}_{c_n}) = \alpha$ et $\text{id}_c \circ \alpha = \alpha$;
- (iii) (1.1.8) est vraie pour toutes opérations α , α_i et $\alpha_{i,j}$ compatibles au regard de la couleur, de sorte que les compositions soient définies;
- (iv) Le groupe symétrique \mathfrak{S}_n agit sur $P(c; c_1, \dots, c_n)$ de telle sorte que, pour tout $\alpha \in P(c; c_1, \dots, c_n)$, $\alpha_\sigma \in P(c; c_{\sigma(1)}, \dots, c_{\sigma(n)})$ et tel que (1.1.9) soit valide.

Le lecteur intéressé pourra se référer à [Yau16] pour une étude approfondie des opérades colorées. Lorsque l'opérade colorée se réalise sur une famille d'ensembles colorés, on parle d'algèbre sur une opérade colorée. La famille hérite alors des opérations et des relations de l'opérade et est soumise aux conditions de compatibilités au regard de la couleur.

Définition 1.1.4 (Algèbre sur une opérade colorée). Soit P une opérade colorée sur un ensemble de couleurs C . Une P -algèbre est une collection d'ensembles $(E_c)_{c \in C}$ telle que : pour tout $n \in \mathbb{N}$, tout $(c, c_1, \dots, c_n) \in C^{n+1}$, et pour tout $\alpha \in P(c; c_1, \dots, c_n)$, il existe une application

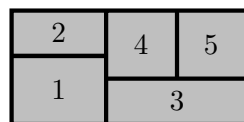
$$m_\alpha : E_{c_1} \times \dots \times E_{c_n} \rightarrow E_c$$

qui satisfait les mêmes conditions que dans la définition 1.1.2 aux conditions de compatibilités au regard de la couleur près.

Coupes et partitions guillottes. L'opérade Guill_2 , voir [Sim23], est l'opérade dont les opérations sont données par les partitions guillottes des rectangles. Abordons désormais le concept de coupes et de partitions guillottes. Étant donné un rectangle R de taille $(p, q) \in (\mathbb{N}^*)^2$ il est possible de le couper de part en part suivant une coupe, appelée coupe guillotte, verticale ou horizontale selon le dessin suivant

$$(1.1.10)$$

Le dessin de gauche est une partition du rectangle de taille (p, q) en deux sous rectangles respectivement de tailles (r, q) et $(p - r, q)$. Il en va de même pour le dessin de droite, qui représente une partition du même rectangle en deux rectangles de tailles respectives (p, s) et $(p, q - s)$. Ce sont deux *partitions guillottes* du rectangle (p, q) de taille 2. Chacun des sous rectangles, du dessin de droite par exemple, peut à son tour être découpé en partition guillotte par des coupes guillottes horizontales ou verticales successives. Par récurrence, on obtient une partition guillotte du rectangle R de taille (p, q) en $n \in \mathbb{N}^*$ rectangles R_i de taille (p_i, q_i) pour $i \in \{1, \dots, n\}$. On dit que cette partition guillotte a pour motif $((p, q); (p_1, q_1), \dots, (p_n, q_n))$ et on la note $\rho = (R_1, \dots, R_n)$. Voici un exemple de partition guillotte de taille 5 du rectangle précédent rectangle :



Composition de partitions guillottes. On peut composer les partitions guillottes de la manière suivante. Soit $\rho = (R_1, \dots, R_n)$ une partition guillotte du rectangle R et $\rho_i = (R_1^{(i)}, \dots, R_{k_i}^{(i)})$ une partition guillotte du i -ème rectangle de R_i pour tout $i \in \{1, \dots, n\}$. Alors

$$\rho \circ (\rho_1, \dots, \rho_n) = (R_1^{(1)}, \dots, R_{k_1}^{(1)}, \dots, R_1^{(n)}, \dots, R_{k_n}^{(n)}) \quad (1.1.11)$$

est encore une partition guillotte du rectangle R et est appelée la composition $\rho \circ (\rho_1, \dots, \rho_n)$. Voici un exemple graphique de composition de partition guillotte :

Il a fallu translater les rectangles pour les placer aux bons endroits mais nous ne rentrerons pas dans de tels détails que le lecteur pourra trouver dans [Sim23].

Opérade guillotine, générateurs et relations. Nous pouvons à présent définir l'opérade qui nous intéresse tout particulièrement.

Définition 1.1.5 (Opérade de partition guillotine). L'opérade de partition guillotine Guill_2 est l'opérade colorée sur l'ensemble des couleurs $(\mathbb{N}^*)^2$ dont les opérations d'arité n sont l'ensemble des partitions guillottes $\text{Guill}_2((p, q); (p_1, q_1), \dots, (p_n, q_n))$ ayant pour motif $((p, q); (p_1, q_1), \dots, (p_n, q_n))$ et dont les fonctions de composition sont données par (1.1.11). Le groupe symétrique agit en permutant les rectangles des partitions.

Le lecteur avisé aura sans doute pensé à l'opérades des petits cubes en voyant la précédente définition (voir [MSS02] par exemple). À la différence de cette dernière, la rigidité apportée par les couleurs de l'opérade Guill_2 empêche toute remise à l'échelle de la taille des rectangles au moment de la composition (comme c'est le cas pour l'opérade des petits cubes). La taille des rectangles, symbolisée dans l'opérade Guill_2 par la couleur, nous importe beaucoup. En effet, l'une des motivations premières de l'opérade Guill_2 est le calcul des fonctions de partition qui dépendent fortement de la taille (de l'aire et du périmètre) des rectangles sur lesquels on la calcule.

Lorsque nous considérerons une Guill_2 -algèbre $(E_c)_{c \in (\mathbb{N}^*)^2}$, il sera courant de représenter graphiquement les produits. Par exemple le produit m_ρ associé à une partition guillotine $\rho = (R_1, R_2, R_3)$ d'un rectangle R évalué sur $(u_1, u_2, u_3) \in E_{c_1} \times E_{c_2} \times E_{c_3}$ pourra s'écrire d'une des manières suivantes

$$m_\rho(u_1, u_2, u_3) = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & \\ \hline \end{array} (u_1, u_2, u_3) = \begin{array}{|c|c|} \hline u_2 & u_3 \\ \hline u_1 & \\ \hline \end{array} \quad (1.1.12)$$

où le dessin correspond à un rectangle R de taille $(3, 3)$ de partitions R_1 de taille $(2, 1)$, R_2 de taille $(2, 2)$ et R_3 de taille $(1, 3)$. On remarque que l'opérade Guill_2 se réalise simplement comme le recollement de rectangles le long de bords de même taille (compatibles pour la couleur). Ici, partant de trois petits rectangles R_1 , R_2 et R_3 nous en avons formé un rectangle R plus grand.

Par récurrence, il est clair que l'ensemble des partitions guillottes ayant pour motif $((p, q), (r, q), (p - r, q))$ et $((p, q), (p, s), (p, q - s))$ exposées en (1.1.10) génèrent l'opérade Guill_2 . Comme les couleurs seront toujours claires selon le contexte, nous notons simplement ces opérations guillottes élémentaires

$$m_{\text{WE}} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} : E_{(r,q)} \times E_{(p-r,q)} \rightarrow E_{(p,q)}$$

$$m_{\text{SN}} = \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} : E_{(p,s)} \times E_{(p,q-s)} \rightarrow E_{(p,q)}$$

où l'indice WE signifie "West-East" et SN signifie "South-North" afin de préciser la position de chacun des rectangles 1 et 2. Pour l'exemple, on remarquera que $m_\rho(u_1, u_2, u_3)$ tel que défini en (1.1.12), peut effectivement s'écrire à l'aide de m_{WE} et m_{SN} :

$$m_\rho(u_1, u_2, u_3) = \begin{array}{|c|c|} \hline u_2 & u_3 \\ \hline u_1 & \\ \hline \end{array} = m_{\text{WE}} \circ (m_{\text{SN}}(u_1, u_2), u_3).$$

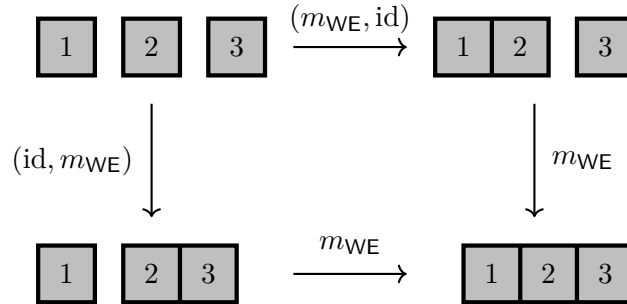
Toutes les relations satisfaites par les partitions guillottes élémentaires m_{WE} et m_{SN} sont générées par les relations suivantes

$$m_{\text{WE}} \circ (\text{id}, m_{\text{WE}}) = m_{\text{WE}} \circ (m_{\text{WE}}, \text{id}) \quad (1.1.13a)$$

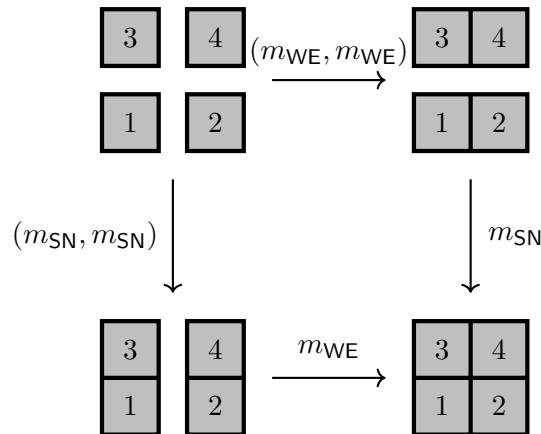
$$m_{\text{SN}} \circ (\text{id}, m_{\text{SN}}) = m_{\text{SN}} \circ (m_{\text{SN}}, \text{id}) \quad (1.1.13\text{b})$$

$$m_{\text{WE}} \circ (m_{\text{SN}}, m_{\text{SN}}) = m_{\text{SN}} \circ (m_{\text{WE}}, m_{\text{WE}}). \quad (1.1.13\text{c})$$

Les deux premières associativités ne concernent qu'un seul produit et donc seulement le recollement que dans une seule direction. Par exemple, nous pouvons représenter schématiquement la règle d'associativité de m_{WE} par le diagramme commutatif suivant



On peut commencer par recoller les rectangles 1 et 2 ensemble ou les rectangles 2 et 3 avant de recoller tout le monde ensemble, cela ne change pas le résultat. C'est une associativité au sens classique du terme. La troisième règle d'associativité de (1.1.13) quant à elle est moins commune. Elle mêle les produits dans les deux directions : on parle d'associativité carrée. C'est une associativité purement bi-dimensionnelle. À nouveau, nous pouvons la schématiser par le diagramme commutatif suivant



On peut commencer par recoller les rectangles 1, 2 et 3, 4 dans la direction Ouest-Est ou les rectangles 1, 3 et 2, 4 dans la direction Sud-Nord avant de recoller tout le monde ensemble, cela ne change pas le résultat.

Voici une brève conclusion de ce qu'il faut retenir. Une Guill_2 -algèbre est composée d'éléments attachés à des rectangles dont les tailles sont appelées les couleurs. Si les couleurs de deux éléments sont compatibles, i.e leurs rectangles partagent un bord de même taille, on peut multiplier ces deux éléments entre eux grâce aux opérations élémentaires m_{WE} et m_{SN} . Graphiquement, cela se traduit par le recollement de leurs rectangles associés. Ces opérations satisfont les trois règles d'associativité (1.1.13).

Opérateurs guillotine étendue. Comme nous allons le voir dans le chapitre 3, les Guill_2 -algèbres telles que précédemment exposées ne permettent qu'une réécriture du calcul de fonctions de partition sur des rectangles de tailles arbitraires. Pour être capable de parler des bords invariants des champs gaussiens, il faut disséquer le plan \mathbb{Z}^2 tout entier par des coupes guillottes infinies. En d'autres termes, il faut étendre les Guill_2 -algèbres à des formes

dégénérées : des rectangles de tailles infinies. En pratique, comme les couleurs représentent la longueur des bords des rectangles, il suffit d'élargir l'ensemble des couleurs $(\mathbb{N}^*)^2$ pour être capable de parler de longueurs infinies. On considérera alors l'ensemble des couleurs étendu

$$C_{LR} = (\mathbb{N}^* \cup \{\infty_W, \infty_E, \infty_{WE}\}) \times (\mathbb{N}^* \cup \{\infty_S, \infty_N, \infty_{SN}\}).$$

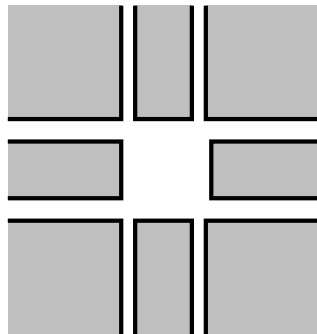
Nous invitons le lecteur à se référer au chapitre 3 ou à [Sim23] pour de plus amples détails sur cet ensemble de couleurs étendu. Par exemple, un rectangle ayant pour motif (∞_W, q) est appelé demi-bande à l'Ouest et est représentée par



C'est un rectangle de longueur horizontale infinie et de longueur verticale q . Le bord gauche de ce rectangle n'apparaît pas afin de représenter le fait qu'il ait été envoyé à l'infini à l'Ouest, d'où la notation ∞_W . Nous allons donc être amenés à recoller ces objets entre eux, toujours au moyen des opérations élémentaires m_{WE} et m_{SN} , étendus au nouveau jeu de couleurs. Par exemple le produit m_ρ associé à une partition guillotine $\rho' = (R_1, R_2, R_3)$ d'un rectangle R de taille $(\infty_W, 3)$ évalué sur $(u_1, u_2, u_3) \in E_{c_1} \times E_{c_2} \times E_{c_3}$ peut s'écrire d'une des manières suivantes

$$m_{\rho'}(u_1, u_2, u_3) = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & \\ \hline \end{array} (u_1, u_2, u_3) = \begin{array}{|c|c|} \hline u_2 & u_3 \\ \hline u_1 & \\ \hline \end{array} = m_{WE} \circ (m_{SN}(u_1, u_2), u_3).$$

Le dessin correspond au rectangle R de taille $(\infty_W, 3)$ de partitions R_1 de taille $(\infty_W, 1)$, R_2 de taille $(\infty_W, 2)$ et R_3 de taille $(1, 3)$. On remarque que, par convention, la couleur ∞_W l'emporte sur les couleurs finies (la longueur horizontale de R est ∞_W et non $\infty_W + 1$). Plus généralement, la structure Guill_2 étendue à ces nouvelles couleurs apporte aussi des demi-bandes à l'Est (∞_E, q) , au Sud (p, ∞_S) et au Nord (p, ∞_N) ainsi que des coins au Sud-Ouest (∞_W, ∞_S) , Sud-Est (∞_E, ∞_S) , Nord-Est (∞_E, ∞_N) et Nord-Ouest (∞_W, ∞_N) . On les représente comme dans le dessin suivant :



La zoologie ne s'arrête pas là, d'autres objets étendus pourront se dresser en travers de notre chemin. Par exemple la bande Ouest-Est doublement infinie (∞_{WE}, q) ou le demi-plan Sud (∞_{WE}, ∞_S) représentés ci-après

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \qquad \begin{array}{c} \text{---} \\ \text{---} \end{array} \qquad (1.1.14)$$

Notre objectif sera de construire ces objets dans le cas des champs gaussiens markoviens sur le réseau carré \mathbb{Z}^2 et de décrire leurs produits de recollement, prescrits par l'opérateur Guill_2 étendue. Ces objets permettront la description des bords invariants $\exp(-B_{\partial M}(\cdot))$

définis en (1.1.4). Plus précisément, à chacune des formes étendues sera associé un objet noté respectivement B^W , B^E , B^S , B^N pour les demi-bandes à l'Ouest, à l'Est, au Sud et au Nord et C^{SW} , C^{SE} , C^{NE} , et C^{NW} pour les coins Sud-Ouest, Sud-Est, Nord-Est et Nord-Ouest respectivement. Ces objets seront construits de sorte que pour tout rectangle M , la condition au bord se découpe de la manière suivante

$$\exp(-B_{\partial M}(\cdot)) = \begin{array}{|c|c|c|c|c|c|} \hline C^{NW} & B^N & B^N & \dots & B^N & C^{NE} \\ \hline B^W & & & & & B^E \\ \hline \vdots & & & & & \vdots \\ \hline B^W & & & & & B^E \\ \hline C^{SW} & B^S & B^S & \dots & B^S & C^{SE} \\ \hline \end{array}, \quad (1.1.15)$$

voir (1.3.6) pour plus de détails. Dans le dessin nous avons représenté le rectangle M autour duquel on vient enlacer les objets de demi-bandes et de coins. Les bandes et les coins sont recollés à leurs voisins grâce au bon produit prescrit par l'opérateur Guill_2 étendue et satisfaisant les règles d'associativité associées. Remarquons que, si le terme de gauche $\exp(-B_{\partial M}(\cdot))$ dépend évidemment de la forme de M , les objets de demi-bandes et de coins quant à eux n'en dépendent pas. Seul l'enchaînement des opérations algébriques que subissent les différents objets dépend de la forme précise de ∂M . Nous montrerons que ces objets satisfont des équations type vecteur propre, garantissant alors la cohérence des bords ainsi définis.

1.1.4 Champs gaussiens

Comme nous l'avons mentionné, cette thèse se focalise tout particulièrement sur l'étude des bords des champs gaussiens markoviens. Formellement, un champ gaussien est un champ dont l'action est donnée par une forme quadratique symétrique et positive. Par exemple, dans le cas d'un champ gaussien markovien discret sur un graphe M à valeurs réelles,

$$S_M(\phi) = -\frac{1}{2}Q_M(\phi) = -\frac{1}{2}\sum_{i\sim j} J(i,j)\phi_i\phi_j - \frac{1}{2}\sum_i J(i,i)\phi_i^2$$

où Q_M est une forme quadratique symétrique et définie positive et $J : M \times M \rightarrow \mathbb{R}$, $(i,j) \mapsto J(i,j)$ traduit le couplage entre i et j . Nous rappelons d'ailleurs que $i \sim j$ signifie que i et j sont voisins dans le graphe M , i.e reliés par une arête. Cela assure que le champ est bien markovien. On l'observe d'ailleurs très bien sur la matrice de couplage $J = (J(i,j))_{i,j \in M}$ associée à Q_M qui contient beaucoup de zéros. On parle de matrice creuse. Sous ces hypothèses, la mesure de Gibbs associée, définie en (1.1.1), est gaussienne et markovienne. S'il ne fait aucun doute que cette mesure est bien définie quand M est fini, un peu plus de travail est requis dans le cas où M ne l'est pas. En cause, l'absence de mesure de Lebesgue en dimension infinie. Pour les champs gaussiens markoviens continus, l'action est donnée quant à elle par une intégrale faisant intervenir des dérivées. L'analyse est encore un peu plus compliquée (les réalisations du champ sont presque sûrement des distributions). Ces difficultés peuvent être contournées si l'on construit le champ, non pas par sa mesure de Gibbs (1.1.1), mais directement par une isométrie du bon espace de Hilbert vers l'espace L^2 d'un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$. Se donner le bon espace de Hilbert c'est, en quelque sorte, se donner la bonne matrice de covariance, c'est-à-dire l'inverse de la matrice de couplage J , voir [Jan97] par exemple. De fait, la question de l'existence est rarement un problème lorsque l'on considère des champs gaussiens. Néanmoins, outre cas

particuliers, la propriété de Markov est bien souvent plus difficile à observer sur la matrice de covariance que sur la matrice creuse J . Pour cette dernière raison, et avec le souhait de rester proche de la propriété de Markov, nous ne définirons pas les champs gaussiens markoviens par isométrie L^2 dans ce manuscrit. Nous maintiendrons donc la définition des champs par leur mesure de Gibbs (1.1.1) pour des sous-graphes finis du réseau carré \mathbb{Z}^2 . Par notre compréhension algébrique des bords nous serons ensuite capables de les construire sur \mathbb{Z}^2 tout entier par théorème d'extension de Kolmogorov.

Les champs gaussiens ont été largement étudiés, le lecteur intéressé pourra par exemple se référer à [Jan97]. Ils constituent un exemple de base à la théorie statistique des champs. Nous en présentons brièvement quelques-uns.

Champs gaussiens discrets et homogènes. Les champs gaussiens discrets et homogènes forment une grande famille des champs gaussiens, la plus largement étudiée. Le champ est dit homogène si les termes de couplages donnés par la matrice J ne dépendent en réalité que des distances entre les nœuds du graphe, i.e $J(i, j) = J(i - j)$. Pour peu que le réseau soit suffisamment régulier, cette invariance par translation permet le recours à la transformée de Fourier. À ce propos, nous recommandons chaleureusement la section 13.3 de [Geo11] pour les champs gaussiens homogènes. Les mesures de Gibbs en volume infini sur \mathbb{Z}^d tout entier y sont décrites. En particulier, l'existence des mesures de Gibbs en volume infini est garantie par le fait qu'une certaine fonction \hat{J} ne s'annule pas sur le tore. Cette fonction est la transformée de Fourier des interactions données par la matrice de couplage J . Dans le présent manuscrit, l'analogue de cette fonction est notée Ψ_Q dans les chapitre 2 et 3. Elles vérifient toutes deux cette hypothèse de non annulation sur le tore, assurant que les champs gaussiens que l'on manipule possèdent effectivement une mesure de Gibbs en dimension infinie.

Champs libres gaussiens discrets (massiques) ou cristaux harmoniques. Le champ libre gaussien discret (massique), parfois simplement appelé GFF discret ou éventuellement cristal harmonique, est un champ gaussien markovien sur $M \subset \mathbb{Z}^d$ fini à valeurs dans \mathbb{R} avec

$$Q_M(\phi) = \frac{1}{2d} \sum_{i \sim j} (\phi_i - \phi_j)^2 + m^2 \sum_i \phi_i^2.$$

où $m \geq 0$ est appelée la masse. Lorsque $m = 0$, on dit que le champ est sans masse. Si $d = 1$, il s'agit du processus d'Ornstein-Uhlenbeck discrétisé dans le cas massique et du mouvement Brownien discrétisé dans le cas non massique. Cette forme quadratique est la compétition de deux termes. L'interaction entre proches voisins d'une part, qui assure la propriété de Markov. Un terme qui pénalise les grandes valeurs en favorisant celles proches de 0 d'autre part. Ce terme est celui qui assure l'existence d'une mesure de Gibbs en volume infini. On le voit très bien lorsque $d = 1$: dans le cas Ornstein-Uhlenbeck $m \neq 0$, le champ admet une mesure de probabilité invariante quand, dans le cas Brownien $m = 0$ il n'en admet pas. D'ailleurs, comme notre objectif est de décrire les mesures de Gibbs en volume infini, c'est vers ce type de champs gaussiens massiques que nous nous tournerons. Ces termes d'interactions ne sont, en réalité, qu'apparents. En effet, avec conditions aux bords périodiques et alors par homogénéité du GFF, la matrice associée à cette forme quadratique est diagonalisable par transformée de Fourier discrète. En d'autres termes, quitte à travailler dans l'espace des phases, toutes les particules sont indépendantes. Cela justifie l'appellation "libre" du champ. Lorsque que $m = 0$, l'action du GFF avec conditions au bord nulles s'écrit

$$Q_M(\phi) = \langle \phi, -\Delta \phi \rangle$$

avec Δ le Laplacien discret. C'est l'énergie de Dirichlet de la configuration ϕ . L'inverse de Δ est donné par la fonction de Green discrète G , où $G(x, y)$ est le temps moyen passé en y par la marche aléatoire symétrique sur \mathbb{Z}^d partant de x avant de toucher le bord de M . Le GFF est d'ailleurs souvent appelé le champ gaussien dont la matrice de covariance est donnée par la fonction de Green. Nous invitons le lecteur désireux d'en apprendre plus au sujet du champ libre à s'en référer à [FV17].

Champ libre gaussien continu. À l'instar du GFF discret, le GFF continu sur un domaine $M \subset \mathbb{R}^d$ est un champ dont l'action est donnée par

$$S_M(\phi) = -\frac{1}{2} \int_M \phi(x)(-\Delta\phi(x))dx.$$

Cependant, en l'absence d'une mesure de Lebesgue $d\phi$ en dimension infinie, la mesure de Gibbs (1.1.1) est mal définie. C'est là que la définition par isométrie L^2 est utile. En effet, motivé par son homologue discret, le GFF continu est défini par isométrie L^2 comme le champ gaussien dont la matrice de covariance est donnée par la fonction de Green continue. Comme cette thèse ne se focalise que sur les champs gaussiens discrets, nous n'irons pas plus loin dans les détails analytiques du GFF continu, comme par exemple le fait que ses réalisations sont presque sûrement des distributions. Nous recommandons cependant au lecteur intéressé de s'en référer à [She07] ou [WP20].

Pourquoi les champs gaussiens comme modèle jouet ? Discutons à présent les motivations qui nous poussent à choisir les modèles gaussiens comme modèle jouet à la théorie développée dans [Sim23]. L'intérêt que nous portons envers les champs gaussien discrets est multiple.

Le premier est d'ordre pragmatique. Comme précédemment mentionné, les objets algébriques décrits dans [Sim23] sont sophistiqués et encore mal compris. Ils n'ont, pour l'heure, pas encore été calculés explicitement dans le cadre d'un modèle non trivial. Il est donc préférable de commencer par s'attaquer à un modèle simple. C'est là qu'est l'atout majeur du modèle gaussien. En plus de la structure de la dimension deux, ce modèle jouit de la structure additionnelle gaussienne. Cela ouvre les portes à tout un tas d'outils calculatoires gaussiens : espaces de Hilbert, produit scalaire, projections, décomposition en base... Cette structure favorise les calculs tractables et simples. Pour faire simple et comme nous allons le voir, avec un peu de calculs de transformée de Fourier, tout ce dont on rêve de calculer peut se calculer dans le monde gaussien. De plus, l'existence de la mesure de Gibbs (1.1.1) n'est jamais un problème pour les processus gaussien, il "suffit" de se donner le bon espace de Hilbert. Toutes ces raisons font du modèle gaussien un exemple de premier choix pour décrire les nouveaux objets de [Sim23].

Le second intérêt que nous portons aux modèles gaussiens discrets est d'ordre théorique. Ils forment un bon premier pas vers les champs continus tels que le champ gaussien libre massique (massive GFF dans la littérature). Ces modèles sont pauvres d'un point de vue physique à cause de leur manque d'interaction. Cependant, ils sont souvent la brique de base de modèles plus complexes comme en gravité quantique de Liouville voir par exemple [BP21] ou en théories Φ_d^4 des champs de dimension d dont l'action fait intervenir la puissance 4 du champ, voir [Tal22] ou [Pes18]. À notre connaissance, il n'existe pour l'heure aucune description algébrique des bords invariants pour les champs massiques. Souvent, cette discussion des bords invariants est évitée au profit de conditions aux bords nulles ou de variétés sans bords.

Enfin, le troisième intérêt que nous portons aux champs gaussiens discrets est d'ordre ludique. Comme nous allons le voir, le modèle gaussien regorge d'identités remarquables et

de propriétés intéressantes non triviales. Surprenamment, ce fait s'observe déjà à 1D. Cela justifie d'ailleurs la présence du premier chapitre de ce manuscrit, dédié à la dimension un uniquement.

1.1.5 Objectifs du présent manuscrit

Comme déjà mentionné à de multiples reprises, l'objectif de cette thèse est de décrire les conditions aux bords invariantes des modèles gaussiens markoviens bidimensionnels discrets. Cette description algébrique consiste en une découpe guillotine des bords en des objets satisfaisant des équations de type "vecteurs propres" et ce, dans le monde abstrait des Guill_2 -algèbres, introduites en section 1.1.3.

Processus gaussiens markoviens unidimensionnels. Afin de préparer au mieux la dimension 2, nous consacrerons le chapitre 2 aux champs gaussiens markoviens unidimensionnels, déjà porteurs d'un certain nombre de résultats intéressants. En témoigne l'étude précise d'objets "indépendants de la dimension" comme, par exemple, la fonction Ψ_Q voir (2.2.1) dans le chapitre 2 et (3.2.8) dans le chapitre 3, l'accent sera mis sur une réécriture plus adaptée au changement de dimension. Cela nous permettra ensuite de transporter plus facilement nos résultats sur les bords des champs unidimensionnels aux bords des champs bidimensionnels.

Champs gaussiens markoviens sur le réseau \mathbb{Z}^2 . Dans un deuxième temps, nous aborderons le cas des champs bidimensionnels et nous construirons les objets de bords de [Sim23] dans ce contexte. Il s'agit du premier exemple non trivial des structures développées par [Sim23]. Les objets construits satisfont des équations de type "vecteurs propres" dans un sens bidimensionnel prescrit par l'opérade Guill_2 . Cependant, ni la théorie des opérades ni l'article de référence [Sim23] ne fournissent les outils d'une théorie des valeurs propres en dimension plus grande que 1. Nous ne pourrions donc pas avoir recours aux méthodes de diagonalisation classiques : polynômes caractéristiques, espaces propres et autres. Une grande partie du travail sera donc de trouver des alternatives à ces techniques.

Nous aurons par exemple recours à la méthode par matrice de transfert. Quitte à élargir l'espace d'états, cette méthode permet de se ramener au cas unidimensionnel. Elle nous sera donc particulièrement utile, à l'aide du chapitre 2, pour deviner un certain nombre d'objets de bord purement unidimensionnels : bandes doublement-infinies et demi-plan représentées en (1.1.14). Cette approche qui favorise une dimension plutôt que l'autre ne permet cependant pas de considérer simultanément les deux dimensions. C'est précisément le rôle des demi-bandes et des coins d'apporter cette information manquante. À la différence de la matrice de transfert, ces objets sont purement bidimensionnels. Une difficulté surmontée par le présent manuscrit sera de s'affranchir de l'approche par matrice de transfert en "coupant" ou "repliant" les objets de bandes doublement-infinies et demi-plans pour former partiellement des objets de demi-bandes et coins comme dans (1.1.15) (des techniques supplémentaires permettent de les former complètement). Nous tâcherons, tout au long du chapitre 3, de préciser la nature des objets considérés : objets purement unidimensionnels ou bidimensionnels, termes de couplage entre dimensions et autres.

Nous le verrons, les nombreux outils gaussiens ne sont pas toujours adaptés à la propriété de Markov. Par exemple, l'absence de mesure de référence en dimension infinie nous empêche une définition directe des recollements entre objets étendus comme dans (1.1.15) au niveau des densités gaussiennes. Pour contourner ce problème, nous relèverons tous les calculs d'intégrales gaussiennes au niveau de leurs formes quadratiques, voir (1.3.3) pour une discussion à propos de ce sujet.

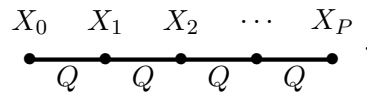
Nous sommes désormais en mesure d'aborder les principales contributions de ce manuscrit et d'annoncer plus précisément le contenu de chacun des chapitres.

1.2 Processus gaussiens markoviens unidimensionnels : décomposition harmonique des conditions aux bords invariants

Le chapitre 2 correspond à l'article [Bod23]. Cependant, pour une meilleure cohérence du présent manuscrit, des légers changements de notation ainsi que de rares ajouts pourront être observés dans le chapitre 2. Dans ce chapitre, nous étudions les conditions aux bords invariants pour les champs gaussiens markoviens unidimensionnels sur le réseau \mathbb{Z} . Ce modèle dissimule de nombreuses identités remarquables et constitue un amuse-bouche incontournable à la dimension deux. Nous apportons de nouveaux outils calculatoires pour le calcul explicite des bords invariants. Plus précisément, nous décomposons les conditions aux bords dans une base non triviale issue d'une étude précise des singularités d'une fonction inhérente au modèle Ψ_Q^{-1} . Cette décomposition apporte un algorithme simple pour le calcul explicite des mesures invariants. Nous en profiterons aussi pour retrouver les valeurs d'énergie libre et de fonctions de corrélation habituellement calculées par transformée de Fourier. Cette gymnastique entre bords invariants et calculs de Fourier sera fortement présente en dimension supérieure.

1.2.1 Le modèle

Le modèle est le suivant. Considérons une composante connexe du réseau \mathbb{Z} de longueur $P \in \mathbb{N}^*$ dont les nœuds sont décorés par des variables aléatoires $X_0, X_1, \dots, X_P \in \mathbb{C}^d$. Les arêtes quant à elles sont décorées par un poids $Q \in M_{2d}(\mathbb{C})$, une matrice hermitienne et définie positive. Le dessin suivant représente la situation



La collection de ces variables aléatoires (X_0, \dots, X_P) est un processus gaussien markovien homogène à valeurs dans \mathbb{C}^d si sa loi de probabilité est donnée par

$$\frac{1}{Z_P(Q)} e^{-\frac{1}{2} Q_P(x_0, \dots, x_P)} d(x_k)_{k \in \{0, \dots, P\}} \quad (1.2.1)$$

pour tout $(x_0, \dots, x_P) \in (\mathbb{C}^d)^{P+1}$, où $Z_P(Q)$ est la fonction de partition, $d(x_k)_{k \in \{0, \dots, P\}}$ la mesure de Lebesgue produit sur $(\mathbb{C}^d)^{P+1}$ et

$$Q_P(x_0, \dots, x_P) = \sum_{k=0}^{P-1} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^* Q \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}. \quad (1.2.2)$$

L'action du champ étant quadratique, ce champ est évidemment gaussien. De plus, on remarque que la variable x_k , associée au nœud $k \in \{1, \dots, P-1\}$, n'est couplée qu'à ses voisins x_{k-1} et x_{k+1} . Cela assure que le champ est markovien spatialement. La situation est légèrement différente pour les variables de bord x_0 et x_P qui ne possèdent qu'un seul voisin. On peut les pourvoir de conditions aux bords, c'est-à-dire d'hypothèses supplémentaires dictant leur comportement. Imaginons vouloir donner une condition aux bords gaussienne.

Le champ gaussien markovien avec condition aux bords B_L et $B_R \in M_d(\mathbb{C})$ (deux matrices hermitiennes définie positive) a pour loi

$$\frac{1}{Z_P(Q, B_L, B_R)} e^{-\frac{1}{2}Q_P(x_0, \dots, x_P) - \frac{1}{2}x_0^* B_L x_0 - \frac{1}{2}x_P^* B_R x_P} d(x_k)_{k \in \{0, \dots, P\}}. \quad (1.2.3)$$

Ici L et R signifient respectivement "Left" et "Right" pour signifier qu'il s'agit de la condition au bord gauche et au bord droit. De même, nous écrivons la décomposition de la matrice Q en des blocs de taille $d \times d$ de la manière suivante

$$Q = \begin{pmatrix} Q_{LL} & Q_{LR} \\ Q_{RL} & Q_{RR} \end{pmatrix}$$

Ces blocs traduisent les couplages entre proches voisins. Par exemple x_k est couplé avec son voisin de droite x_{k+1} au travers du bloc Q_{LR} et son voisin de gauche x_{k-1} au travers du bloc Q_{RL} .

1.2.2 Motivations classiques et conditions aux bords

Les motivations classiques et le problème des bords. Comme pour la plupart des modèles de mécanique statistique, il est d'usage de porter notre regard sur certaines quantités d'intérêt telles que la fonction de partition, l'énergie libre ou encore la fonction de corrélation. Prenons pour exemple l'énergie libre définie par

$$f := \lim_{P \rightarrow \infty} \frac{1}{P} \log Z_P(Q, B_L, B_R).$$

Si la fonction de partition Z_P dépend des conditions aux bords choisies, l'énergie libre quant à elle ne devrait pas en dépendre. En effet, heuristiquement, on s'attend à des décorrélations exponentielles qui empêcheraient le processus de se souvenir des conditions aux bords initialement choisies. Dès lors, si l'énergie libre ne dépend pas des conditions aux bords, il est judicieux de les choisir au mieux pour faciliter le calcul. Deux méthodes sont alors particulièrement efficaces : les conditions aux bords périodiques et invariantes (Perron-Frobenius). Parlons brièvement de leur spécificités.

Conditions aux bords périodiques. Identifier les nœuds de bords x_0 et x_P conduit à travailler avec conditions aux bords périodiques : c'est le processus gaussien markovien sur $\mathbb{Z}/P\mathbb{Z}$. Le processus est alors invariant par translation et la transformée de Fourier permet de calculer explicitement la plupart des quantités d'intérêt. En particulier, l'énergie libre se calcule explicitement :

$$f^{\text{per}} = \log(2\pi)^d - \frac{1}{2\pi} \int_0^{2\pi} \log \det \Psi_Q(e^{i\theta}) d\theta \quad (1.2.4)$$

où Ψ_Q est une fonction à valeurs matricielle construite à partir des blocs de Q , voir section 2.2. Bien qu'étant une méthode très efficace, la transformée de Fourier est un calcul global qui occulte complètement la propriété locale de Markov du système.

Conditions aux bords invariantes. À la différence de l'approche par Fourier, la recherche de bords invariants repose quant à elle sur la propriété de Markov et d'un calcul local. Un calcul d'intégrale gaussienne montre que, si une matrice $B_L \in M_d(\mathbb{C})$ hermitienne et définie positive satisfaisant l'équation de point fixe non linéaire (issu d'un complément de Schur noté S_L)

$$B_L = S_L(B_L, Q) := Q_{RR} - Q_{RL}(B_L + Q_{LL})^{-1}Q_{LR}, \quad (1.2.5)$$

alors pour tout $(x_1, \dots, x_P) \in \mathbb{C}^{dP}$,

$$\int_{\mathbb{C}^d} e^{-\frac{1}{2}Q_P(x_0, \dots, x_P) - \frac{1}{2}x_0^* B_L x_0} dx_0 = \Lambda e^{-\frac{1}{2}Q_{P-1}(x_1, \dots, x_P) - \frac{1}{2}x_1^* B_L x_1}$$

avec $\Lambda = (2\pi)^d \det(B_L + Q_{LL})^{-1}$. Dit avec des mots, après avoir fait "avancer" la chaîne à gauche en intégrant sur x_0 , le processus obtenu est toujours un processus gaussien markovien de poids Q avec condition au bord gauche B_L (de longueur $(P-1)$ dû à l'effacement de x_0). Il s'agit de la condition au bord gauche invariante. Une discussion analogue peut être faite pour la condition au bord droit invariante B_R . Alors, par récurrence, $Z_P(Q, B_L, B_R) = \Lambda^P$ à constance multiplicative près et l'énergie libre vérifie

$$f^{\text{inv}} = \log \Lambda. \quad (1.2.6)$$

Ce calcul d'énergie libre est exact au sens où ce n'est plus une limite. Effectivement, en nous munissant de la condition invariante à gauche nous avons compris comment le processus "vient de l'infini" et plus aucune asymptotique n'est à calculer. Néanmoins, la valeur explicite de l'énergie libre repose à présent sur notre capacité à résoudre l'équation (1.2.5). Si $d = 1$ il s'agit seulement de trouver les racines d'un polynôme de degré 2. Cependant, dès que $d > 1$ les choses se compliquent sérieusement à cause du caractère non linéaire de l'équation. De plus, comme précédemment mentionné, l'énergie libre ne devrait pas dépendre du choix des conditions aux bords et nous devrions avoir $f^{\text{per}} = f^{\text{inv}}$. Cependant, écrites telles quelles, ce fait n'est pas clair à première vue. La première expression fait intervenir une intégrale de contour et la fonction Ψ_Q quand la deuxième s'exprime simplement en fonction d'un déterminant dans lequel cette dernière fonction n'apparaît pas directement. Comprendre ce lien profond entre l'approche par conditions aux bords périodiques et invariantes et, ce faisant, résoudre explicitement l'équation (1.2.5) est l'objectif du chapitre 2 de ce manuscrit.

1.2.3 Principales contributions et idées clefs

La nouveauté apportée par le chapitre 2 est l'étude précise de la fonction

$$\begin{aligned} \Psi_Q : \mathbb{C}^* &\rightarrow M_d(\mathbb{C}) \\ z &\mapsto Q_{LL} + Q_{RR} + Q_{LR}z + Q_{RL}z^{-1} \end{aligned}$$

et, plus précisément, de s'intéresser aux points de \mathbb{C}^* où celle-ci n'est pas inversible

$$\mathcal{S}(\Psi_Q) = \{z \in \mathbb{C}^*; \det \Psi_Q(z) = 0\}.$$

On appelle ces points les zéros de Ψ_Q . Ils apparaissent lorsque l'on s'intéresse à calculer les fonctions harmoniques associées à l'énergie de Dirichlet (1.2.2). Comptés avec multiplicité, ces zéros sont au nombre de $2d$. Pour faire simple, la multiplicité de chacun des points est supposée égale à 1 et on a donc exactement $2d$ zéros de Ψ_Q . De plus, une vérification directe montre que si $w \in \mathcal{S}(\Psi_Q)$ alors $1/\bar{w} \in \mathcal{S}(\Psi_Q)$. En d'autres termes, ces zéros viennent par paires : à chaque zéro dans le disque unité est associé un zéro en dehors. Il y a donc d zéros dans le disque unité et d en dehors. On introduit alors les notations suivantes

$$\mathcal{S}_{<1}(\Psi_Q) := \{z \in \mathcal{S}(\Psi_Q); |z| < 1\} \quad \mathcal{S}_{>1}(\Psi_Q) := \{z \in \mathcal{S}(\Psi_Q); |z| > 1\}.$$

En chacun de ces zéros, Ψ_Q n'est pas inversible et, par conséquent, son noyau est non trivial. On fait l'hypothèse qu'en chacun de ces points, le noyau est de dimension 1 :

$$\ker \Psi_Q(w) = \text{Vect}_{\mathbb{C}}(u_w)$$

pour tout $w \in \mathcal{S}(\Psi_Q)$ avec $u_w \in \mathbb{C}^d \setminus \{0\}$. Cette hypothèse ainsi que celle de la multiplicité des zéros permettent surtout d'alléger certains calculs de résidus matriciels et preuves d'analyse complexe. Nous pensons d'ailleurs que des résultats similaires aux nôtres peuvent être établis dans le cas général. Cela nécessiterait cependant une étude plus fine des noyaux de la fonction Ψ_Q et de ses zéros au prix d'une analyse complexe plus technique avec des pôles matriciels d'ordres supérieurs. Par souci de simplicité, nous nous cantonnerons donc à cette hypothèse. À chacun des zéros de Ψ_Q est donc associé un vecteur. Nous avons alors d points dans le disque unité ainsi que d vecteurs qui leur sont associés. Il en va de même pour les d points en dehors du disque unité. L'un des résultats importants du chapitre 2 est alors le suivant.

Théorème 1.2.1. *Sous certaines hypothèses, les familles $(u_w)_{w \in \mathcal{S}_{<1}(\Psi_Q)}$ et $(u_w)_{w \in \mathcal{S}_{>1}(\Psi_Q)}$ forment chacune une base de \mathbb{C}^d .*

Ce résultat est la conséquence d'un calcul de résidu sur la fonction Ψ_Q^{-1} , voir le théorème 2.2.1 pour plus de détails. Ces bases, non triviales, permettent de décomposer simplement les conditions au bords invariants. Il s'agit du second résultat principal du chapitre 2 rappelé ci-dessous.

Théorème 1.2.2. *Soit $W_{>1} \in M_d(\mathbb{C})$ diagonale dans la base $(u_w)_{w \in \mathcal{S}_{>1}(\Psi_Q)}$ telle que $W_{>1}u_w = wu_w$ pour tout $w \in \mathcal{S}_{>1}(\Psi_Q)$. Alors, sous certaines hypothèses,*

$$B_L = Q_{RR} + Q_{RL}W_{>1}^{-1}$$

est une matrice hermitienne et définie positive qui résout l'équation 1.2.5.

Un résultat analogue décrit le bord invariant à droite, noté B_R , au besoin de la base $(u_w)_{w \in \mathcal{S}_{<1}(\Psi_Q)}$. Ces résultats sont démontrés en détails dans le chapitre 2, voir théorèmes 2.1.4 et 2.5.1. Cependant, donnons l'idée en quelques lignes. Par définition de $W_{>1}$,

$$Q_{LL} + Q_{RR} + Q_{LR}W_{>1} + Q_{RL}W_{>1}^{-1} = 0.$$

Pour le voir, il suffit d'évaluer la précédente expression sur chacun des éléments la base $(u_w)_{w \in \mathcal{S}_{>1}(\Psi_Q)}$. Alors, il suffit de calculer directement le membre de droite de (1.2.5) en utilisant

$$Q_{LL} + Q_{RR} + Q_{RL}W_{>1}^{-1} = -Q_{LR}W_{>1}.$$

Cette mécanique simple est très utile en pratique et sera d'ailleurs largement utilisée en dimension supérieure. Par exemple, cela rend explicite la formule de la valeur propre

$$\Lambda = (2\pi)^d \det(B_L + Q_{LL}) = (2\pi)^d \det(-Q_{LR}W_{>1}) = (-2\pi)^d \det(Q_{LR}) \prod_{w \in \mathcal{S}_{>1}(\Psi_Q)} w$$

et donc de l'énergie libre f^{inv} associée, s'exprimant ainsi en fonction de Ψ_Q comme f^{per} calculée avec conditions aux bords périodiques, voir (1.2.4). Il ne reste ensuite qu'un pas pour montrer que ces énergies libres sont effectivement égales, voir le lemme 2.2.5. Plus généralement, nous montrons dans le chapitre 2 que la plupart des quantités d'intérêt calculées par transformée de Fourier en limite thermodynamique (quand la taille du système tend vers l'infini) sont retrouvées avec les conditions aux bords invariants. Entre autres, la mesure de Gibbs en volume infinie peut être construite directement par le théorème d'extension de Kolmogorov, donnant ainsi l'existence du processus sur \mathbb{Z} tout entier.

En plus de donner une formule explicite pour l'expression du bord invariant, le théorème précédent nous indique un algorithme simple permettant de la calculer. En pratique, il suffit de trouver les racines d'un polynôme de degré $2d$, les zéros Ψ_Q , puis de calculer les noyaux associés (par pivot de Gauss par exemple) et enfin construire $W_{>1}$ puis B_L .

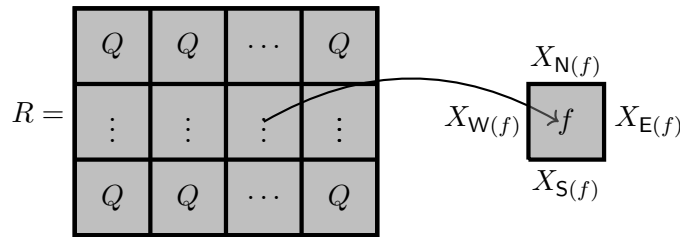
Le théorème précédent est important pour la description des bords des champs gaussiens mais apporte aussi un résultat intéressant en analyse des opérateurs de Toeplitz. En effet, par une application directe de ce théorème, dans un cas simple, le théorème 2.8.2 donne une version "invariante" du théorème de Szegő qui décrit l'asymptotique des déterminants de matrices de Toeplitz qui grandissent en taille, voir [Sze15].

1.3 Structure de bords des champs gaussiens markoviens bidimensionnels discrets

Le chapitre 3 est la version bientôt soumise d'un article en fin de préparation. Il s'agit d'un travail en collaboration avec Damien Simon. Dans ce dernier, nous étudions la structure algébrique des bords des champs gaussiens markoviens bidimensionnels discrets. Ces champs sont construits à l'aide de briques de base, les poids gaussiens. Ceux-ci ont le bon goût de former une Guill_2 -algèbre comme présentée en section 1.1.3. Nous étendons cette structure à des formes dégénérées de demi-bandes et de coins. Ces objets satisfont des équations de type "vecteur propre" à morphisme près. Une fois recollés entre eux, au moyen des produits prescrits par l'opérade, ils permettent de construire les conditions aux bords invariantes pour des champs vivant au dessus de rectangles de tailles arbitraires. Le chapitre 3 se veut être une description précise de ces nouvelles structures.

1.3.1 Le modèle

Le modèle bidimensionnel est analogue à celui du chapitre 2. Considérons un rectangle R de taille finie du réseau carré \mathbb{Z}^2 dont les arêtes sont décorées par des variables aléatoires $(X_e)_{e \in \text{Edges}(R)}$. Les variables associées à des arêtes horizontales (resp. verticales) sont à valeurs dans \mathbb{C}^{d_1} (resp. \mathbb{C}^{d_2}). Les faces quant à elles sont décorées par un poids $Q \in M_{2d_1+2d_2}(\mathbb{C})$, une matrice hermitienne et définie positive. Le dessin suivant représente la situation. Nous y avons représenté une face $f \in \text{Faces}(R)$ et chacune des quatre variables qui l'entourent : celle au Sud $X_{S(f)} \in \mathbb{C}^{d_1}$, au Nord $X_{N(f)} \in \mathbb{C}^{d_1}$, à l'Ouest $X_{W(f)} \in \mathbb{C}^{d_2}$ et à l'Est $X_{E(f)} \in \mathbb{C}^{d_2}$.



La collection de ces variables aléatoires $(X_e)_{e \in \text{Edges}(R)}$ est un champ gaussien markovien sur R si sa loi de probabilité est donnée par

$$\frac{1}{Z_R(Q)} e^{-\frac{1}{2} Q_R((x_e)_{e \in \text{Edges}(R)})} d(x_e)_{e \in \text{Edges}(R)} \quad (1.3.1)$$

pour tout $(x_e)_{e \in \text{Edges}(R)}$, où $Z_R(Q)$ est la fonction de partition, $d(x_e)_{e \in \text{Edges}(R)}$ la mesure de Lebesgue produit et

$$Q_R((x_e)_{e \in \text{Edges}(R)}) = \sum_{f \in \text{Faces}(R)} \begin{pmatrix} x_{S(f)} \\ x_{N(f)} \\ x_{W(f)} \\ x_{E(f)} \end{pmatrix}^* Q \begin{pmatrix} x_{S(f)} \\ x_{N(f)} \\ x_{W(f)} \\ x_{E(f)} \end{pmatrix}. \quad (1.3.2)$$

À nouveau, l'action étant quadratique, ce champ est évidemment gaussien. De plus, on remarque que la variable x_e , pour $e \in \text{Edges}(R)$, n'est couplée qu'à elle-même et ses voisines. Cela assure que le champ est markovien spatialement. La situation est légèrement différente pour les variables de bord qui ont moins de voisines que les autres. On peut les munir d'une condition au bord. Imaginons vouloir donner une condition au bord gaussienne. On note ∂R l'ensemble des arêtes formant le bord de R . Le champ gaussien markovien sur R avec condition au bord $B_{\partial R}$ (une forme quadratique définie positive) a pour loi

$$\frac{1}{Z_R(Q, B_{\partial R})} e^{-\frac{1}{2}Q_R((x_e)_{e \in \text{Edges}(R)}) - \frac{1}{2}B_{\partial R}((x_e)_{e \in \partial R})} d(x_e)_{e \in \text{Edges}(R)}. \quad (1.3.3)$$

C'est bel et bien un champ gaussien avec condition au bord gaussienne. On note

$$g_R((x_e)_{e \in \partial R}) = \exp\left(-\frac{1}{2}B_{\partial R}((x_e)_{e \in \partial R})\right)$$

la densité gaussienne non-normalisée de bord. Comme pour le cas unidimensionnel, nous introduisons la décomposition par blocs de Q

$$Q = \begin{pmatrix} Q_{SS} & Q_{SN} & Q_{SW} & Q_{SE} \\ Q_{NS} & Q_{NN} & Q_{NW} & Q_{NE} \\ Q_{WS} & Q_{WN} & Q_{WW} & Q_{WE} \\ Q_{ES} & Q_{EN} & Q_{EW} & Q_{EE} \end{pmatrix}. \quad (1.3.4)$$

Le bloc Q_{ab} traduit le couplage d'une arête en position a sur une face avec sa voisine en position b pour tout $a, b \in \{S, N, W, E\}$. Le modèle est complètement symétrique lorsque que Q est invariant sous l'action du groupe diédral, c'est-à-dire

$$Q = \begin{pmatrix} T & A & U & U \\ A & T & U & U \\ U & U & T & A \\ U & U & A & T \end{pmatrix}. \quad (1.3.5)$$

Motivation et décomposition des bords en partitions guillotines. L'objectif est de trouver une famille $(g_R)_R$ de densités gaussiennes de bord qui soit invariante, et donc cohérente sous restriction. En d'autres termes que, pour tous rectangles $R \subset R'$, la restriction $(X_e)_{e \in \text{Edges}(R)}$ du champ gaussien markovien $(X_e)_{e \in \text{Edges}(R')}$ de poids de face Q et densité de bord $g_{R'}$ soit un champ gaussien markovien de poids de face Q et densité de bord g_R . Cela permettrait par ailleurs de définir la mesure de Gibbs en volume infini directement par le théorème d'extension de Kolmogorov. Néanmoins, comme les périmètres des rectangles sont variables, les poids de bords agissent sur des espaces de tailles différentes et il n'est donc pas facile de trouver une telle famille $(g_R)_R$. On préférera donc un codage des conditions aux bords par des objets qui ne dépendent pas de la forme des rectangles. Pour ce faire, en suivant les idées développées dans [Sim23], il s'agit de décomposer les densités de bords g_R en des objets locaux aux formes rectangulaires dégénérées et possédant une structure de Guill_2 -algèbre étendue, comme définie en section 1.1.3. Plus précisément, nous cherchons des objets de demi-bandes à l'Ouest, à l'Est, au Sud, au Nord respectivement notées B^W, B^E, B^S, B^N ainsi que des objets de coins Sud-Ouest, Sud-Est, Nord-Est et Nord-Ouest respectivement

notés C^{SW} , C^{SE} , C^{NE} , et C^{NW} tels que pour tout rectangle de taille $(p, q) \in (\mathbb{N}^*)^2$ on ait

$$g_R \left(\begin{array}{c} y_1 \ y_2 \ \dots \ y_p \\ \boxed{} \\ z_q \\ \vdots \\ w_1 \\ \\ x_1 \ x_2 \ \dots \ x_p \\ \\ z_1 \\ \vdots \\ w_1 \\ \\ x_1 \ x_2 \ \dots \ x_p \end{array} \right) = \begin{array}{|c|c|c|c|c|c|} \hline C^{NW} & B^N(y_1) & B^N(y_2) & \dots & B^N(y_p) & C^{NE} \\ \hline B^W(w_q) & & & & & B^E(z_q) \\ \hline \vdots & & & & & \vdots \\ \hline B^W(w_1) & & & & & B^E(z_1) \\ \hline C^{SW} & B^S(x_1) & B^S(x_2) & \dots & B^S(x_p) & C^{SE} \\ \hline \end{array} \quad (1.3.6)$$

Dans le dessin de droite, les bandes et les coins sont recollés à leurs voisins grâce au bon produit prescrit par l'opérade Guill_2 étendue. Pour assurer la cohérence des densités de bords, ces objets locaux doivent satisfaire des équations de type "vecteur propre" à morphisme Φ près. Nous reviendrons sur ce point au moment opportun, après l'équation (1.3.28).

Une brève comparaison avec le cas 1D. Comme pour le modèle 1D du chapitre 2, la plupart des quantités d'intérêt telles que la fonction de partition, l'énergie libre ou la fonction de corrélation se calculent aisément à l'aide de la transformée de Fourier sous conditions aux bords périodiques. Par exemple, l'énergie libre est donnée par

$$f^{\text{per}} = \log(2\pi)^{d_1+d_2} - \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} \log \det \Psi_Q(e^{i\theta_1}, e^{i\theta_2}) d\theta_1 d\theta_2. \quad (1.3.7)$$

Là encore, ces quantités sont exprimées au travers d'une fonction $\Psi_Q : \mathbb{C}^* \times \mathbb{C}^* \rightarrow M_{d_1+d_2}(\mathbb{C})$. Néanmoins, l'ensemble des zéros associés

$$\mathcal{S}(\Psi_Q) = \{(z, w) \in \mathbb{C}^* \times \mathbb{C}^*; \det \Psi_Q(z, w) = 0\} \quad (1.3.8)$$

est maintenant une courbe algébrique et, sauf dans les cas triviaux, est de cardinal infini. Cette remarque de cardinalité va de pair avec la dimension des formes quadratiques de bord $B_{\partial R}$. Ce sont maintenant des objets étendus dont la dimension croît exponentiellement avec le périmètre du rectangle. À 1D nous avions à reconstruire des matrices de bord B_L et B_R de dimension $d \times d$ chacune et nous avions en notre possession $2d$ zéros. Maintenant, nous avons des formes quadratiques de bord qui grandissent en taille et une infinité de zéro. Nous aimerions cependant être toujours capables de construire les objets de bord de (1.3.6) associés à $B_{\partial R}$ en utilisant cette courbe algébrique $\mathcal{S}(\Psi_Q)$.

1.3.2 La Guill_2 -algèbre des poids gaussiens

Comme l'objectif du chapitre 3 est de construire une Guill_2 -algèbre étendue, il est sans doute préférable de connaître la Guill_2 -algèbre de base que l'on cherche à étendre, ici, celle des poids gaussiens. Pour tout couple $(p, q) \in (\mathbb{N}^*)^2$, on introduit l'ensemble

$$\mathcal{Q}_{p,q} = \{Q \in M_{2pd_1+2qd_2}(\mathbb{C}); Q^* = Q \text{ et } Q \text{ est définie positive}\}.$$

Les champs gaussiens markoviens bidimensionnels sont bâtis à l'aide de poids vivant sur les faces. Ces poids gaussiens bidimensionnels sont définis pour tout $(\alpha, Q) \in \mathbb{R} \times \mathcal{Q}_{p,q}$ par des densités

$$e_{\alpha, Q} : (\mathbb{C}^{d_1})^{2p} \times (\mathbb{C}^{d_2})^{2q} \rightarrow \mathbb{R} \quad (1.3.9)$$

$$(x_S, x_N, x_W, x_E) \mapsto \alpha \exp \left(-\frac{1}{2} \begin{pmatrix} x_S \\ x_N \\ x_W \\ x_E \end{pmatrix}^* Q \begin{pmatrix} x_S \\ x_N \\ x_W \\ x_E \end{pmatrix} \right) =: x_W \begin{array}{|c|} \hline x_N \\ \hline e_{\alpha, Q} \\ \hline x_S \\ \hline \end{array} x_E$$

On note $\mathcal{D}_{p,q} = \{e_{\alpha,Q}; (\alpha, Q) \in \mathbb{R} \times \mathcal{Q}_{p,q}\}$ l'ensemble des poids gaussiens de taille $(p, q) \in (\mathbb{N}^*)^2$. Les poids gaussiens viennent naturellement avec deux produits permettant de les recoller, formant alors un nouveau poids gaussien,

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} : \mathcal{D}_{p_1,q} \times \mathcal{D}_{p_1,q} \rightarrow \mathcal{D}_{p_1+p_2,q}$$

$$\begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} : \mathcal{D}_{p,q_1} \times \mathcal{D}_{p,q_2} \rightarrow \mathcal{D}_{p,q_1+q_2}$$

respectivement définis de la manière suivante (en prenant garde à respecter les bons jeux de couleurs pour la bonne définition des produits)

$$x_W \begin{array}{|c|c|} \hline x_N^{(1)} & x_N^{(2)} \\ \hline e_{\alpha_1,Q_1} & e_{\alpha_2,Q_2} \\ \hline x_S^{(1)} & x_S^{(2)} \\ \hline \end{array} x_E = \int_{(\mathbb{C}^{d_2})^q} e_{\alpha_1,Q_1}(x_S^{(1)}, x_N^{(1)}, x_W, v) e_{\alpha_2,Q_2}(x_S^{(2)}, x_N^{(2)}, v, x_E) dv$$

$$\begin{array}{|c|} \hline x_N \\ \hline x_W^{(2)} \begin{array}{|c|} \hline e_{\alpha_2,Q_2} \\ \hline \end{array} x_E^{(2)} \\ \hline x_W^{(1)} \begin{array}{|c|} \hline e_{\alpha_1,Q_1} \\ \hline \end{array} x_E^{(1)} \\ \hline x_S \\ \hline \end{array} = \int_{(\mathbb{C}^{d_1})^p} e_{\alpha_1,Q_1}(x_S, v, x_W^{(1)}, x_E^{(1)}) e_{\alpha_2,Q_2}(v, x_N, x_W^{(2)}, x_E^{(2)}) dv.$$

Ces deux produits élémentaires respectivement dans la direction Ouest-Est et Sud-Nord, vérifient les règles d'associativités (1.1.13). Dès lors, les poids gaussiens munis de ces produits élémentaires forment alors une Guill₂-algèbre. Nous l'avons vu en section 1.1.3, le recollement de rectangles en produit de plus grands. Dans les cas des poids gaussiens, cela se traduit par la montée en dimension des espaces vectoriels du bord. Par exemple, le côté Sud, après recollement Ouest-Est, est formé au travers de la concaténation $(x_S^{(1)}, x_S^{(2)}) \mapsto x_S^{(1)} \oplus x_S^{(2)}$.

Un premier fait intéressant du chapitre 3 est le suivant.

Proposition 1.3.1. *Pour tous poids gaussiens e_{α_1,Q_1} et e_{α_2,Q_2} , compatibles pour que les produits soient bien définis,*

$$\begin{array}{|c|c|} \hline e_{\alpha_1,Q_1} & e_{\alpha_2,Q_2} \\ \hline \end{array} = e_{\alpha_1\alpha_2\gamma_{WE}(Q_1,Q_2), S_{WE}(Q_1,Q_2)} \quad (1.3.10)$$

$$\begin{array}{|c|} \hline e_{\alpha_2,Q_2} \\ \hline e_{\alpha_1,Q_1} \\ \hline \end{array} = e_{\alpha_1\alpha_2\gamma_{SN}(Q_1,Q_2), S_{SN}(Q_1,Q_2)} \quad (1.3.11)$$

où les quantités S_{WE} , S_{SN} , γ_{WE} et γ_{SN} sont décrites en proposition 3.3.1.

Ce résultat est le fruit d'un calcul direct d'intégrale gaussienne. Comme les poids gaussiens sont caractérisés par leurs paramètres, ce résultat stipule que l'on peut relever la structure de Guill₂-algèbre des poids gaussiens sur l'espace de leurs paramètres $\mathbb{R} \times (\mathcal{Q}_{p,q})_{(p,q) \in (\mathbb{N}^*)^2}$, dont les produits sont données par γ_{ab} et S_{ab} pour $ab = WE$ ou $ab = SN$. Les produits γ_{ab} sont les déterminants qui sortent après calcul des intégrales gaussiennes. Ils sont responsables du poids total du champ gaussien, ce sont eux qui donnent la fonction de partition du champ Z_R . Les produits S_{WE} et S_{SN} , opérations au niveau des formes quadratiques, sont appelés produits de Schur car reliés à la notion de complément de Schur. On pourra les noter de la manière suivante

$$\begin{array}{|c|c|} \hline Q_1 & Q_2 \\ \hline \end{array} = S_{WE}(Q_1, Q_2) \quad \begin{array}{|c|} \hline Q_2 \\ \hline Q_1 \\ \hline \end{array} = S_{SN}(Q_1, Q_2). \quad (1.3.12)$$

Soulignons simplement que, à la différence des produits sur les poids gaussiens, les produits de Schur ne sont pas linéaires en les matrices. Cela n'a cependant aucune incidence pour les motivations qui sont les nôtres. La suite d'ensemble $(\mathcal{Q}_{p,q})_{(p,q) \in (\mathbb{N}^*)^2}$ munie de ces produits est une Guill₂-algèbre.

La Guill₂-algèbre engendrée par un élément. On note

$$Q^{[p,q]} = \begin{array}{|c|c|c|c|} \hline Q & Q & \cdots & Q \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline Q & Q & \cdots & Q \\ \hline \end{array} \quad (1.3.13)$$

la puissance surfacique d'une forme quadratique Q . C'est une forme quadratique sur $\mathbb{C}^{2pd_1} \oplus \mathbb{C}^{2qd_2}$, l'espace vectoriel associé au bord du rectangle R de taille (p, q) . Elle peut, par exemple, être obtenue après complément de Schur sur toutes les arêtes intérieures horizontales puis verticales. Au niveau des poids gaussiens, on la retrouve par le calcul de l'intégrale

$$\int_{\mathbb{C}^{p(q-1)d_1} \oplus \mathbb{C}^{q(p-1)d_2}} e^{-\frac{1}{2}Q_R((x_e)_{e \in \text{Edges}(R)})} d(x_e)_{e \in \text{Edges}(R) \setminus \partial R} = \gamma e^{-\frac{1}{2}Q^{[p,q]}((x_e)_{e \in \partial R})} \quad (1.3.14)$$

au dessus de chacune des arêtes de l'intérieur de R , avec $\gamma \in \mathbb{R}$. À Q fixé, $(Q^{[p,q]})_{(p,q) \in (\mathbb{N}^*)^2}$ hérite d'une structure de Guill₂-algèbre, celle engendrée par Q .

1.3.3 Objectif du chapitre 3 et premières difficultés

L'objectif principal chapitre 3 est d'étendre la structure guillotine des poids gaussiens $(\mathcal{D}_{p,q})_{(p,q) \in (\mathbb{N}^*)^2}$ à une structure guillotine étendue et de trouver alors des objets dégénérés de demi-bandes et de coins permettant de découper la condition au bord comme dans (1.3.6). Il faudra ensuite s'assurer que ces objets vérifient des équations type "vecteurs propres" pour que la condition au bord, alors construite, soit invariante. Par exemple, il faut que la demi-bande soit invariante sous l'action du poids gaussien $e_{\alpha,Q}$ induit par son recollement. Cependant, cette action conduit à un allongement de la demi-bande et il n'est donc pas possible de comparer directement la demi-bande allongée avec la demi-bande initiale car elle ne sont pas définies sur le même espace. L'équation type "vecteur propre" fait donc intervenir un application Φ qui permet de ramener la bande allongée dans le bon espace. Schématiquement, l'équation type "vecteur propre" pour la demi-bande Ouest B^W se traduit par le diagramme commutatif suivant

$$\begin{array}{ccc} \boxed{B^W} & \xrightarrow{\sim e_{\alpha,Q}} & \boxed{B^W \mid e_{\alpha,Q}} \\ & \searrow \times \Lambda & \downarrow \Phi \\ & & \Lambda \boxed{B^W} \end{array}$$

où $\Lambda \in \mathbb{R}$ est la valeur propre de B^W . Nous le verrons, l'application Φ est un morphisme, c'est-à-dire qu'elle respecte les recollements, voir le théorème 1.3.1. Nous invitons le lecteur à remarquer que la résolution d'une telle équation est plus délicate qu'en dimension 1. Pour les champs unidimensionnels markoviens de manière générale, comme précédemment

mentionné en section 1.1.1, les conditions aux bords sont données par les vecteurs propres de Perron-Frobenius à gauche et à droite de la matrice de transition (c'est ce que nous faisons d'ailleurs dans le monde gaussien dans le chapitre 2). Par exemple, quand on cherche la condition invariante à gauche, il n'y a alors que deux inconnues : le vecteur et sa valeur propre. En dimension 2, quand on cherche la condition invariante à morphisme près pour la demi-bande Ouest par exemple, en plus de l'objet invariant B^W et sa valeur propre Λ , l'espace sur lequel il est défini ainsi que le morphisme Φ associé sont, eux aussi, inconnus. Cela rend la tâche particulièrement ardue.

Quelques difficultés. Considérons donc un champ gaussien markovien dont le poids de face est donné par $Q \in \mathcal{Q}_{1,1}$. Naïvement, pour trouver la demi-bande à l'Ouest par exemple, nous pourrions essayer de recoller une infinité de poids gaussiens $e_{1,Q}$ et espérer trouver un poids gaussien limite

$$\boxed{\cdots \quad e_{\alpha,Q} \quad e_{\alpha,Q}} = e_{\alpha_\infty, B^W}.$$

Il va sans dire que la quantité α_∞ , issue d'un produit infini de déterminants, a de grande chance de diverger ou de tendre vers 0. Cela rendrait le poids gaussien limite mal défini ou simplement nul. En oubliant cette première difficulté, ne serait-ce que la forme quadratique limite B^W amène déjà son lot d'interrogations. Nous l'avons vu en section 1.3.2, recoller des poids gaussiens successivement élargit l'espace vectoriel qui vit sur les bords. Comme B^W est le fruit d'une infinité de recollements successifs, elle devrait agir sur un espace vectoriel de dimension infinie. Il faudrait donc munir les demi-lignes infinies de la bande d'un espace de Hilbert \mathcal{W}_W , lequel ? Par ailleurs, pour réaliser (1.3.6), il faudra être capable d'empiler des demi-bandes à l'Ouest et donc de faire leur produit. Formellement, en s'appuyant sur ceux précédemment introduits, ce produit devrait s'écrire

$$\boxed{\begin{array}{c} e_{\alpha_\infty, B^W} \\ e_{\alpha_\infty, B^W} \end{array}} = \int_{\mathcal{W}_W} e_{\alpha_\infty, B^W}(\cdot, v, \cdot) e_{\alpha_\infty, B^W}(v, \cdot, \cdot) dv.$$

Là encore, une nouvelle difficulté technique apparaît. L'absence de mesure de Lebesgue en dimension infinie rend cette intégrale mal définie. Une idée pourrait de définir les poids gaussiens contre une autre mesure de référence, plus permissive avec la grande dimension. Nous pourrions par exemple nous munir d'une mesure gaussienne de référence. La section 3.5.3 explique pourquoi une telle chose n'est pas possible. L'approche naïve telle que décrite ici semble donc compromise.

Une solution tout aussi naïve. Les poids gaussiens $(\mathcal{D}_{p,q})_{(p,q) \in (\mathbb{N}^*)^2}$ et leurs produits semblent s'étendre difficilement à la dimension infinie. Cependant, nous l'avons vu, leur structure guillotine peut se relever sur l'espace de leurs paramètres $\mathbb{R} \times (\mathcal{Q}_{p,q})_{(p,q) \in (\mathbb{N}^*)^2}$. Nous pouvons même aller plus loin et simplement oublier les déterminants donnés par les γ_{ab} pour ne plus considérer que les formes quadratiques colorées $(\mathcal{Q}_{p,q})_{(p,q) \in (\mathbb{N}^*)^2}$. Cette structure Guill_2 ne fait intervenir ni intégrales ni déterminants et ne concerne plus que des produits de Schur sur des matrices de dimensions finies. On peut, naïvement, se reposer la même question que précédemment : que devient ce produit pour des opérateurs en dimension infinie ? En fait, comme nous allons le voir dans le chapitre 3, ils s'exportent particulièrement bien en dimension infinie. Pour cette raison, nous abandonnons (pour un temps) l'espace des poids gaussiens ainsi que l'espace de leurs paramètres pour ne plus nous concentrer que sur les formes quadratiques colorées $(\mathcal{Q}_{p,q})_{(p,q) \in (\mathbb{N}^*)^2}$ et leur extension en grandes dimensions. Ce faisant, nous abandonnons toute mesure de référence et tout déterminant associé. Dès lors, il n'y aura pas non plus de calcul de fonction de partition et

la définition d'invariance à morphisme près ne fera pas intervenir de valeur propre, comme c'est le cas dans [Sim23]. Cependant, la restriction aux formes quadratiques est, quant à elle, suffisante à la décomposition des bords invariants et l'écriture (1.3.6) sera donc relevée au niveau de celles-ci. Nous pouvons d'ailleurs, dans un deuxième temps, redescendre sur l'espace des poids gaussiens, en prenant en compte les renormalisations apportées par γ_{WE} et γ_{SN} définies proposition 3.3.1.

1.3.4 Principales contributions et idées clefs

L'apport principal du chapitre 3 est la construction de la Guill₂-algèbre étendue des formes quadratiques gaussiennes $(\mathcal{Q}_{p,q})_{(p,q) \in (\mathbb{N}^*)^2}$. En réalité, comme nous nous intéressons au bord d'un champ gaussien markovien de poids de face Q fixé, nous allons nous restreindre à l'extension de la Guill₂-algèbre engendrée par Q . Pour deviner les objets de demi-bandes et de coins il faut donc être capable de calculer des puissances surfaciques $Q^{[p,q]}$ de grandes tailles, voir (1.3.13). Par exemple, on s'attend à ce que les demi-bandes horizontales se déduisent de $Q^{[\infty,q]}$. Calculer une telle puissance directement n'est pas envisageable car, à partir d'un moment, les calculs ne sont plus tractables. Il faut donc trouver une manière plus astucieuse de le faire. Le chapitre 3 en propose deux : l'approche par décompositions harmoniques et l'approche par matrice de transfert. Par ces deux approches, on "devine" les formes quadratiques associées aux demi-bandes et aux coins au moyen de calculs non rigoureux de limites de grandes tailles. Une fois devinées, nous prescrivons ces formes quadratiques et montrons qu'elle satisfont bien les identités voulues d'invariance à morphisme près.

Approche par décompositions harmoniques. La première approche pour calculer de grandes puissances surfaciques de Q est de s'appuyer sur un changement de variable "harmonique + indépendant". Considérons notre champ $(X_e)_{e \in \text{Edges}(R)}$ de poids de face Q , défini en (1.3.1), avec conditions au bord $(x_e)_{e \in \partial R}$ fixée. C'est un vecteur gaussien. Notons $X_{\partial R} = (X_e)_{e \in \partial R}$. Alors, par la propriété de projection orthogonale des vecteurs gaussiens, pour tout $e \in \text{Edges}(R)$, on peut écrire $X_e = \mathbb{E}[X_e | X_{\partial R} = x_{\partial R}] + X_e^{(0)}$ avec $X_e^{(0)}$ indépendante de l'espérance conditionnelle. On note $x_e^{(H)} = \mathbb{E}[X_e | X_{\partial R} = x_{\partial R}]$, c'est l'extension harmonique de la condition aux bords $(x_e)_{e \in \partial R}$ et c'est une quantité fixée. Alors, par le changement de variable affine $x_e = x_e^{(H)} + x_e^{(0)}$ dans l'intégrale suivante

$$\begin{aligned} & \int_{\mathbb{C}^{p(q-1)d_1} \oplus \mathbb{C}^{q(p-1)d_2}} e^{-\frac{1}{2}Q_R((x_e)_{e \in \text{Edges}(R)})} d(x_e)_{e \in \text{Edges}(R) \setminus \partial R} \\ &= e^{-\frac{1}{2}Q_R((x_e^{(H)})_{e \in \text{Edges}(R)})} \int_{\mathbb{C}^{p(q-1)d_1} \oplus \mathbb{C}^{q(p-1)d_2}} e^{-\frac{1}{2}Q_R((x_e^{(0)})_{e \in \text{Edges}(R)})} d(x_e^{(0)})_{e \in \text{Edges}(R) \setminus \partial R} \end{aligned}$$

grâce à la propriété d'indépendance, voir la proposition 3.3.2 pour plus de détails. Comme l'intégrale qui reste est simplement un nombre, on en déduit que, si R est de taille (p, q) ,

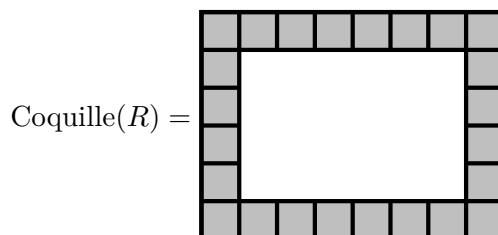
$$Q^{[p,q]}(x_{\partial R}) = Q_R((x_e^{(H)})_{e \in \text{Edges}(R)})$$

par identification en utilisant (1.3.14). Comme l'extension harmonique $(x_e^{(H)})_{e \in \text{Edges}(R)}$ est solution d'une équation de type Laplacien nul, voir (3.3.12), le théorème de Stokes apporte la précision suivante.

Proposition 1.3.2. *La puissance surfacique $Q^{[p,q]}$ est égale à*

$$\begin{aligned} Q^{[p,q]}(x_{\partial R}) &= \sum_{e \in \partial R_N} \langle x_e | D_{e,S}^{(1)}(x^{(H)}) \rangle + \sum_{e \in \partial R_S} \langle x_e | D_{e,N}^{(1)}(x^{(H)}) \rangle \\ &+ \sum_{e \in \partial R_W} \langle x_e | D_{e,E}^{(2)}(x^{(H)}) \rangle + \sum_{e \in \partial R_E} \langle x_e | D_{e,W}^{(2)}(x^{(H)}) \rangle \end{aligned}$$

Les ensembles ∂R_a sont les bords a du rectangle R où $a \in \{S, N, W, E\}$. Plutôt que d'introduire trop de notations, nous expliquons succinctement le rôle des quantités $D_{e,a}^{(i)}(x^{(H)})$ et invitons le lecteur à se référer à la proposition 3.3.3 pour de plus amples détails. Pour une arête $e \in \partial R$ fixée, les quantités $D_{e,a}^{(i)}(x^{(H)})$ ne font pas intervenir toute l'extension harmonique $(x_e^{(H)})_{e \in \text{Edges}(R)}$ mais simplement la sous-famille associée aux arêtes proches voisines de $e \in \partial R$. En d'autres termes, la partie de l'extension harmonique qui apparaît vraiment dans la précédente proposition est $(x_e^{(H)})_{e \in \text{Coquille}(R)}$ avec $\text{Coquille}(R)$ l'ensemble des arêtes représentées ci-après



C'est donc bien un théorème de Stokes pour le champ gaussien markovien de poids de face Q , localisant sur les bords de R un calcul sur son intérieur. Cette approche fournit donc une formule pour la puissance surfacique $Q^{[p,q]}$ de taille arbitraire. Elle est explicite pour peu que l'on soit capable de calculer l'extension harmonique $(x^{(H)})_e$. Ce calcul est donné par les propositions 3.3.4 et 3.3.5 dans le cas où $d_1 = d_2 = 1$ et Q est invariant par action du groupe diédral, c'est-à-dire

$$Q = \begin{pmatrix} t & a & u & u \\ a & t & u & u \\ u & u & t & a \\ u & u & a & t \end{pmatrix}$$

avec t, a et $u \in \mathbb{C}$. Comme $d_1 = d_2 = 1$, la décomposition par blocs de Q introduite en (1.3.5) est ici notée avec des lettres minuscules. Sous cette hypothèse, la solution harmonique est bâtie à partir de la courbe algébrique \mathcal{S}_Q définie en (1.3.8), ce que nous souhaitons. Dans la section 3.3.3, un raisonnement heuristique permet ensuite de deviner les formes quadratiques associées aux formes semi-infinies des demi-bandes et des coins en prenant des limites de grandes tailles.

Approche par matrice de transfert. Alternativement à la précédente approche, comme cela est proposé par la section 3.4, nous pouvons utiliser la très classique méthode de matrice de transfert afin de deviner les formes quadratiques associées aux bandes.

La méthode par matrice de transfert est une technique classique de la mécanique statistique qui consiste en une réduction du problème bidimensionnel à un problème unidimensionnel. On se ramène alors à un problème d'algèbre linéaire, quitte à payer le prix d'un espace d'état plus grand dans la direction transverse. Pour ce faire, on identifie deux bords opposés du rectangle R et on travaille sur le cylindre fini, tranche par tranche. Pour les modèles gaussiens, ce maintien de la périodicité dans une direction permet l'utilisation de la transformée de Fourier le long de celle-ci. Plus précisément, on dénote $B_Q^{WE,p} \in M_{2pd_1}(\mathbb{C})$ la matrice obtenue à partir de $Q^{[p,1]}$ après un complément de Schur sur les arêtes Ouest et Est identifiées, graphiquement on a

$$B_Q^{WE,p} = \bullet \begin{array}{|c|c|c|c|} \hline Q & Q & \dots & Q \\ \hline \end{array} \bullet \cdot$$

On peut empiler ces bandes par un produit de Schur dans la direction Sud-Nord comme suit

$$S_{\text{SN}}(B_Q^{WE,p}, B_Q^{WE,p}) = \bullet \begin{array}{|c|} \hline B_Q^{WE,p} \\ \hline B_Q^{WE,p} \\ \hline \end{array} \bullet .$$

On note $\omega_p = \exp(2i\pi/p)$. Grâce à l'invariance par translation offerte par l'identification des bords, il s'avère que $B_Q^{WE,p}$ est une matrice circulante et donc diagonalisable dans la base discrète de Fourier en une matrice dont les blocs diagonaux sont donnés par $(\widehat{B}_Q^{WE}(\omega_p^k))_{k=0}^{p-1}$ où $\widehat{B}_Q^{WE} : S^1 \rightarrow M_{2d_1}(\mathbb{C})$ est une fonction définie en (3.4.16). Le fait particulièrement intéressant est que le produit de Schur S_{SN} est indépendant du choix de base. Autrement dit, pour calculer $S_{\text{SN}}(B_Q^{WE,p}, B_Q^{WE,p})$, il suffit de le calculer sur son spectre, i.e calculer $S_{\text{SN}}(\widehat{B}_Q^{WE}(\omega_p^k), \widehat{B}_Q^{WE}(\omega_p^k))$ pour tout $k \in \{0, \dots, p-1\}$. On passe alors de calculer le produit de Schur de matrices de dimensions $2pd_1 \times 2pd_1$ à calculer p produits de Schur de matrices de dimensions $2d_1 \times 2d_1$. En dimension infinie, l'analogie de ce changement de base est donnée par l'identification de Fourier entre $L^2(S^1 \rightarrow \mathbb{C}^{d_1})$ et $l^2(\mathbb{Z} \rightarrow \mathbb{C}^{d_1})$. Alors, la matrice diagonale par blocs, dont les blocs sont $\widehat{B}_Q^{WE}(\omega_p^k)$, devient l'opérateur de multiplication

$$M(\widehat{B}_Q^{WE}) : L^2(S^1 \rightarrow \mathbb{C}^{d_1}) \rightarrow L^2(S^1 \rightarrow \mathbb{C}^{d_1}) \quad (1.3.15)$$

$$f \mapsto \widehat{B}_Q^{WE} f \quad (1.3.16)$$

donné par l'action ponctuelle de $\widehat{B}_Q^{WE}(e^{i\theta})$ sur $f(e^{i\theta})$. La matrice de bande B_Q^{WE} quant à elle devient donc l'opérateur de convolution $C(\widehat{B}_Q^{WE}) : l^2(\mathbb{Z} \rightarrow \mathbb{C}^{d_1}) \rightarrow l^2(\mathbb{Z} \rightarrow \mathbb{C}^{d_1})$ donné par $M(\widehat{B}_Q^{WE})$ écrit dans la base l^2 . Dès lors, on en déduit l'opérateur de la bande doublement infinie et son espace associé. On pose

$$\mathcal{W}_{\text{WE}} := l^2(\mathbb{Z} \rightarrow \mathbb{C}^{d_1}) \quad (1.3.17)$$

l'espace de la ligne doublement infinie horizontale et l'opérateur de bande doublement infinie est donc donné par

$$\boxed{C(\widehat{B}_Q^{WE})} \in \mathcal{Q}_{\infty\text{WE}} \quad (1.3.18)$$

où $\mathcal{Q}_{\infty\text{WE}}$ est l'espace des formes quadratiques étendues à la forme de la bande doublement infinie :

$$\mathcal{Q}_{\infty\text{WE}} := \{B \in \text{End}(\mathcal{W}_{\text{WE}} \oplus \mathcal{W}_{\text{WE}}); B^* = B \text{ et } B \text{ est défini positif}\}. \quad (1.3.19)$$

On peut empiler ces bandes et, de la même manière que pour le cas fini, le produit de Schur se ramène à un produit de Schur ponctuel sur la densité spectral \widehat{B}_Q^{WE}

$$S_{\text{SN}}(C(\widehat{B}_Q^{WE}), C(\widehat{B}_Q^{WE})) = \frac{\boxed{C(\widehat{B}_Q^{WE})}}{\boxed{C(\widehat{B}_Q^{WE})}} = \boxed{C(S_{\text{SN}}(\widehat{B}_Q^{WE}, \widehat{B}_Q^{WE}))}$$

voir proposition 3.4.1 pour de plus amples détails. L'avantage de travailler sur la densité spectrale est que, pour tout $\theta \in [0, 2\pi]$ fixé, les recollements Schur Sud-Nord successifs $\widehat{B}_Q^{WE}(e^{i\theta})$ décrivent une dynamique 1D bien comprise par le chapitre 2. En particulier, à l'instar de la matrice B_L invariante pour Q en équation (1.2.5), il est possible de trouver une application $\widehat{G}_S : S^1 \rightarrow M_{d_1}(\mathbb{C})$ telle que pour tout $\theta \in [0, 2\pi]$, $\widehat{G}_S(e^{i\theta})$ soit une condition de bord invariante pour $\widehat{B}_Q^{WE}(e^{i\theta})$. Cela fournit un nouvel objet étendu, le demi-plan Sud

donné par $C(\widehat{G}_S)$. Ce dernier est invariant sous l'action de la bande doublement infinie, graphiquement

$$S_{\text{SN}}(C(\widehat{G}_S), C(\widehat{B}_Q^{\text{WE}})) = \frac{C(\widehat{B}_Q^{\text{WE}})}{C(\widehat{G}_S)} = C(\widehat{G}_S) \in \mathcal{Q}_{\infty_{\text{WE}}, \infty_S} \quad (1.3.20)$$

où $\mathcal{Q}_{\infty_{\text{WE}}, \infty_S}$ est l'espace des formes quadratiques étendues à la forme de du demi-plan :

$$\mathcal{Q}_{\infty_{\text{WE}}, \infty_S} := \{B \in \text{End}(\mathcal{W}_{\text{WE}}); B^* = B \text{ et } B \text{ est défini positif}\}. \quad (1.3.21)$$

Ces calculs sont précisés dans la section 3.4.3. L'approche par matrice de transfert permet donc de deviner certaines formes dégénérées et leurs produits de Schur étendus. Nous ne l'avons pas fait, mais des calculs analogues mènent aux bandes verticales et aux demi-plans associés. Des calculs plus poussés avec l'ajout d'observables permettent d'ailleurs de couper la bande doublement infinie en deux et d'obtenir une formule pour les formes quadratiques associés aux demi-bandes. Nous n'irons pas plus loin et nous contenterons de donner son expression. Notons d'ailleurs que ces expressions rejoignent celles donnée par l'heuristique issue de l'approche harmonique 3.3.3.

Maintenant que nous avons des expressions pour les puissances surfaciques de grandes tailles pour Q et donc des objets dégénérés, il est temps de jouer avec. Dans la section 3.5.1.2 du chapitre 3, c'est que que nous faisons avec les demi-bandes.

Demi-bandes. Les demi-bandes sont étudiées dans la section 3.5.1.2 dans le cas où Q est invariant sous action du groupe diédral, voir (1.3.5). Cela permet d'alléger l'analyse de manière conséquente, notamment pour la bonne définition de l'empilement des demi-bandes grâce à un opérateur appelé opérateur "Fold". Nous pensons que, au prix d'une analyse plus coûteuse et d'une zoologie plus touffue, les résultats qui vont suivre devraient s'étendre au cas général. Cependant, notre objectif principal étant l'application de la théorie développée par [Sim23] et non l'étude exhaustive des bords des champs gaussiens en toute généralité, le cas (1.3.5) nous convient très bien. Concentrons-nous sur les demi-bandes horizontales. On note $\mathbb{Z}_{>0}$ et $\mathbb{Z}_{\leq 0}$ respectivement les ensembles des entiers strictement positifs et négatifs au sens large. Les espaces de demi-lignes infinies sont donnés par

$$\mathcal{W}_W = l^2(\mathbb{Z}_{>0} \rightarrow \mathbb{C}^{d_1}) \quad \mathcal{W}_E = l^2(\mathbb{Z}_{\leq 0} \rightarrow \mathbb{C}^{d_1}).$$

Les demi-bandes sont des opérateurs représentés par

$$\boxed{B_Q^{W, \infty}} \in \mathcal{Q}_{\infty_W, 1} \quad \boxed{B_Q^{E, \infty}} \in \mathcal{Q}_{\infty_E, 1} \quad (1.3.22)$$

où pour tout $q \in \mathbb{N}^*$, $\mathcal{Q}_{\infty_W, q}$ et $\mathcal{Q}_{\infty_E, q}$ sont les espaces des formes quadratiques étendues aux formes des demi-bandes Ouest et Est :

$$\mathcal{Q}_{\infty_W, q} := \{B \in \text{End}(\mathcal{W}_W \oplus \mathcal{W}_W \oplus \mathbb{C}^{qd_2}); B^* = B \text{ et } B \text{ est défini positif}\} \quad (1.3.23)$$

$$\mathcal{Q}_{\infty_E, q} := \{B \in \text{End}(\mathcal{W}_E \oplus \mathcal{W}_E \oplus \mathbb{C}^{qd_2}); B^* = B \text{ et } B \text{ est défini positif}\}. \quad (1.3.24)$$

L'espace $\mathcal{W}_W \oplus \mathcal{W}_W$ correspond aux demi-lignes infinies Sud et Nord de la demie bande et \mathbb{C}^{qd_2} à l'arête de taille finie q en bout de bande. Dans la section 3.5.1.2, nous étendons les produits guillotines, donnés par les compléments de Schur, à ces formes dégénérées. Nous en dressons ici la liste dans laquelle nous donnons les propriétés et résultats principaux associés. Remarquons que ces produits de Schur étant maintenant définis sur des opérateurs, il s'agit de s'assurer de leur bonne définition.

Demi-bandes : recollement longitudinal. Le recollement de demi-bandes opposées comme le suivant

$$S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty}) := \boxed{\begin{array}{c} B_Q^{W,\infty} \\ \hline B_Q^{E,\infty} \end{array}} \quad (1.3.25)$$

est défini à l'aide d'un produit de Schur $\mathcal{Q}_{\infty W, q} \times \mathcal{Q}_{\infty E, q} \rightarrow \mathcal{Q}_{\infty WE}$, voir section 3.5.1.3 pour une définition précise. Ce produit est défini le long d'une arête de taille finie q et ne requiert donc aucune technicité analytique particulière, le produit de Schur est bien défini. Par ailleurs, la proposition 3.5.2 donne

$$\boxed{\begin{array}{c} B_Q^{W,\infty} \\ \hline B_Q^{E,\infty} \end{array}} = \boxed{C(\widehat{B}_Q^{WE})}.$$

Autrement dit, le recollement des demi-bandes Ouest et Est redonne la bande doublement infinie (1.3.18) et permet ainsi de retrouver les objets obtenus classiquement par matrice de transfert. Néanmoins, il y a davantage d'informations dans les demi-bandes puisque la présence d'une arête verticale met en évidence des interactions entre les arêtes d'orientation différentes.

Demi-bandes : recollement transversal. Le recollement des demi-bandes voisines, ici à l'Ouest

$$S_{SN}(B_Q^{W,\infty}, B_Q^{W,\infty}) := \boxed{\begin{array}{c} B_Q^{W,\infty} \\ \hline B_Q^{W,\infty} \end{array}} \quad (1.3.26)$$

est défini grâce à un produit de Schur $\mathcal{Q}_{\infty W, q_1} \times \mathcal{Q}_{\infty W, q_2} \rightarrow \mathcal{Q}_{\infty W, q_1+q_2}$, voir la section 3.5.1.4 pour les définitions précises. Ce produit étant défini le long d'une demi-ligne infinie, quelques précautions analytiques sont à prendre pour la bonne définition du produit de Schur. C'est là que l'hypothèse d'invariance diédrale (1.3.5) nous est utile. Les symétries qu'elle impose nous permettent "d'inverser" l'opération (1.3.25) et, en quelque sorte, de replier la bande doublement infinie en la demi-bande :

$$\boxed{C(\widehat{B}_Q^{WE})} \xrightarrow[\text{F}(\cdot)]{\text{Repliement}} \boxed{B_Q^{W,\infty}}$$

grâce à un opérateur "Fold" noté F . Pour faire simple, en se permettant un léger abus, $F(C(\widehat{B}_Q^{WE})) = B_Q^{W,\infty}$. Cette égalité est en fait rigoureusement vraie pour le sous-bloc de $B_Q^{W,\infty}$ qui couple le Sud et le Nord, voir la définition 3.5.1. Évalué sur nos objets, l'opérateur fold est particulièrement agréable car il respecte les produits de Schur au sens où

$$F(S_{SN}(\cdot, \cdot)) = S_{SN}(F(\cdot), F(\cdot)). \quad (1.3.27)$$

Dès lors, le recollement transversal des demi-bandes est automatiquement bien défini et vérifie (toujours avec ce léger abus)

$$\boxed{\begin{array}{c} B_Q^{W,\infty} \\ \hline B_Q^{W,\infty} \end{array}} = \text{F} \left(\boxed{\begin{array}{c} C(\widehat{B}_Q^{WE}) \\ \hline C(\widehat{B}_Q^{WE}) \end{array}} \right).$$

Un sens précis est donné à cette équation dans la section 3.5.1.4.

Demi-bandes : action du poids de face Q . Enfin, le dernier recollement à étudier est celui de l'action d'un rectangle de taille fini au bout de la demi-bande, comme par exemple

$$S_{WE}(B_Q^{W,\infty}, Q) = \boxed{B_Q^{W,\infty} \mid Q} \quad (1.3.28)$$

Ce produit de Schur $\mathcal{Q}_{\infty,w,q} \times \mathcal{Q}_{p,q} \rightarrow \mathcal{Q}_{\infty,w,q}^{(1)}$ est défini en section 3.5.1.5. Ce recollement nous intéresse tout particulièrement car c'est sous cette action que nous aimerions voir $B_Q^{W,\infty}$ être invariant. À la manière de l'invariance de B_L pour le modèle 1D, voir (1.2.5), nous souhaiterions

$$S_{WE}(B_Q^{W,\infty}, Q) = B_Q^{W,\infty}$$

de sorte que nous ayons même $S_{WE}(B_Q^{W,\infty}, Q^{[p,1]}) = B_Q^{W,\infty}$ pour l'action récursive $p \in \mathbb{N}^*$ fois de Q . Il n'y a aucune chance de voir ce souhait se réaliser. En effet, à cause des recollements successifs, $S_{WE}(B_Q^{W,\infty}, Q^{[p,1]})$ agit sur un espace de demi-lignes allongées :

$$S_{WE}(B_Q^{W,\infty}, Q^{[p,1]}) \in \text{End}(\mathcal{W}_S \oplus \mathbb{C}^{pd_1} \oplus \mathcal{W}_W \oplus \mathbb{C}^{pd_1} \oplus \mathbb{C}^{d_2}) =: \mathcal{Q}_{\infty,w,1}^{(p)},$$

cet opérateur ne peut donc pas directement être comparé à $B_Q^{W,\infty}$ qui agit sur un espace vectoriel "plus petit". En réalité, comme proposé par [Sim23], la bonne notion est l'invariance à morphisme près, noté graphiquement,

$$\Phi_{p,1}^W \left(\boxed{B_Q^{W,\infty} \mid Q^{[p,1]}} \right) = \boxed{B_Q^{W,\infty}}. \quad (1.3.29)$$

Autrement, quitte à la ramener dans le bon espace, la demi-bande allongée sous l'action de Q est invariante. Les morphismes $(\Phi_{p,q})_{(p,q) \in (\mathbb{N}^*)^2}$ sont des applications qui respectent les recollements en le sens suivant

$$\Phi_{p,q}^W \left(\begin{array}{|c|} \hline \boxed{S_{WE}(B_Q^{W,\infty}, Q^{[p,1]})} \\ \hline \vdots \\ \hline \boxed{S_{WE}(B_Q^{W,\infty}, Q^{[p,1]})} \\ \hline \end{array} \right) = \begin{array}{|c|} \hline \boxed{\Phi_{p,1}^W(S_{WE}(B_Q^{W,\infty}, Q^{[p,1]})} \\ \hline \vdots \\ \hline \boxed{\Phi_{p,1}^W(S_{WE}(B_Q^{W,\infty}, Q^{[p,1]})} \\ \hline \end{array} \quad (1.3.30)$$

pour tout $(p, q) \in (\mathbb{N}^*)^2$ où le dessin représente q empilements de demi-bandes. Les théorèmes 3.5.1 et 3.5.2 donnent ce morphisme. Ils sont résumés en le théorème suivant.

Théorème 1.3.1. *La famille $\Phi^W := (\Phi_{p,q}^W)_{(p,q) \in (\mathbb{N}^*)^2}$ d'applications telle que, pour tout $(p, q) \in (\mathbb{N}^*)^2$,*

$$\Phi_{p,q}^W : \text{End}(\mathcal{W}_W \oplus \mathbb{C}^{pd_1} \oplus \mathcal{W}_W \oplus \mathbb{C}^{pd_1} \oplus \mathbb{C}^{qd_2}) \rightarrow \text{End}(\mathcal{W}_W \oplus \mathcal{W}_W \oplus \mathbb{C}^{qd_2}) \quad (1.3.31)$$

$$M \mapsto \left(D_p^W \oplus D_p^W \oplus I_{d_2}^{\oplus q} \right) M \left(D_p^W \oplus D_p^W \oplus I_{d_2}^{\oplus q} \right)^{-1}$$

vérifie la propriété de morphisme (1.3.30). De plus, $B_Q^{W,\infty}$ est invariante sous action de Q à Φ^W près, autrement dit (1.3.29) est vraie.

Les opérateurs D_p^W sont des opérateurs de "décalage" définis en (3.5.31). Ce théorème est le plus important de la section. Le produit de Schur $S_{WE}(B_Q^{W,\infty}, Q^{[p,1]})$ est a priori compliqué. Il s'accompagne d'un allongement des demi-lignes infinies et agit sur les blocs de la demi-bande et donc sur l'opérateur "Fold" F d'une manière non triviale. Il est remarquable que, en fin de compte, cette action Schur de Q sur la demi-bande se retrouve conjuguée

à l'identité pas un "changement de base" donné par $(D_p^W \oplus D_p^W \oplus I_{d_2}^{\oplus q})$. Il est tout aussi remarquable et agréable que le morphisme Φ^W ne dépende pas de Q . Il est possible de construire de tels morphismes pour les demi-bandes à l'Est, au Sud et au Nord. Ce sont ces morphismes qui rendent la décomposition de $B_{\partial R}$ 1.3.6 invariante (au niveau des formes quadratiques).

Coins. Les coins sont étudiés dans la section 3.5.2 dans le cas où Q est invariant sous action du groupe diédral, voir (1.3.5). Le coin Sud-Ouest, par exemple, est un opérateur représenté par

$$\boxed{C_Q^{SW}} \in \mathcal{Q}_{\infty_W, \infty_S}$$

où $\mathcal{Q}_{\infty_S, \infty_W}$ est l'espace des formes quadratiques étendues au coin Sud-Ouest :

$$\mathcal{Q}_{\infty_W, \infty_S} := \{C \in \text{End}(\mathcal{W}_W \oplus \mathcal{W}_S); C^* = C \text{ et } C \text{ est défini positif}\}$$

avec $\mathcal{W}_W = l^2(\mathbb{Z}_{>0} \rightarrow \mathbb{C}^{d_1})$ et $\mathcal{W}_S = l^2(\mathbb{Z}_{>0} \rightarrow \mathbb{C}^{d_2})$ les espaces correspondant respectivement aux demi-lignes infinies Nord et Est du bord du coin. Par sa nature géométrique, le coin est un objet difficile d'accès. Cet objet purement bi-dimensionnel est, à la différence des demi-bandes, infini dans deux directions et vérifie donc des propriétés légèrement différentes. Il est cependant toujours possible de le deviner au moyen des calculs harmoniques présentés en début de cette sous-section. Ces derniers nécessitent à présent la prise de deux limites, que l'on s'attend à voir commuter, et sont exposés en 3.3.3. Alternativement, les symétries imposées par l'hypothèse d'invariance diédrale permettent de deviner le coin directement grâce à l'opérateur de "Fold". Heuristiquement, on s'attend à ce que celui-ci s'obtienne comme le repliement du demi-plan Sud au moyen du même opérateur fold $F(\cdot)$:

$$\boxed{C(\hat{G}_S)} \xrightarrow[\text{F}(\cdot)]{\text{Repliement}} \boxed{C_Q^{SW}}$$

En réalité, cette opération ne permet de retrouver que le sous-bloc Nord du coin. Un repliement du demi-plan Ouest permettra de retrouver le sous-bloc Est. Des méthodes complémentaires permettent de trouver les blocs manquants établissant les interactions entre le bord Nord et Est du coin.

Dans la section 3.5.2, nous étendons les produits guillotines, donnés par les compléments de Schur, à ces formes dégénérées de coins. Nous en dressons ici la liste dans laquelle nous donnons les propriétés et résultats principaux associés. À nouveau, ces produits de Schur étant définis en dimension infinie, une partie du travail consiste à s'assurer de leur bonne définition. Là encore, ce sont les propriétés de l'opérateur "Fold" qui facilitent l'analyse.

Coins : recollement de coins opposés. Il est possible de recoller deux coins opposés. Notons C_Q^{SE} le coin Sud-Est. Par exemple,

$$S_{WE}(C_Q^{SW}, C_Q^{SE}) := \boxed{C_Q^{SW} \mid C_Q^{SE}}$$

est défini à l'aide d'un produit de Schur $\mathcal{Q}_{\infty_W, \infty_S} \times \mathcal{Q}_{\infty_E, \infty_S} \rightarrow \mathcal{Q}_{\infty_W, \infty_{SN}}$, voir section 3.5.2.2. Ce produit est défini le long d'une demi-ligne infinie et nécessite donc quelques précautions analytiques pour s'assurer de sa bonne définition. C'est le rôle de la proposition 3.5.4 qui montre par ailleurs que

$$\boxed{C_Q^{SW} \mid C_Q^{SE}} = \boxed{C(\hat{G}_S)} \quad (1.3.32)$$

Autrement dit, le recollement des coins Sud-Ouest et Sud-Est redonne le demi-plan Sud précédemment construit comme objet invariant sous action de la bande doublement-infinie Ouest-Est, voir (1.3.20). Notons tout de même que les coins détiennent d'avantage d'informations que le demi-plan car ils mettent en évidence les interactions en les deux dimensions.

Coin : action des demi-bandes. Il est possible d'agir sur le coin avec des demi-bandes voisines, au Nord comme à l'Est du coin, comme par exemple,

$$S_{SN}(C_Q^{SW}, B_Q^{W,\infty}) := \begin{array}{|c|} \hline B_Q^{W,\infty} \\ \hline C_Q^{SW} \\ \hline \end{array} \quad S_{SE}(C_Q^{SW}, B_Q^{S,\infty}) := \begin{array}{|c|c|} \hline C_Q^{SW} & B_Q^{S,\infty} \\ \hline \end{array} \quad (1.3.33)$$

Concentrons-nous sur le produit de gauche, des remarques similaires s'appliquent à celui de droite. Ce recollement est défini au moyen d'un produit de Schur $\mathcal{Q}_{\infty W, \infty S} \times \mathcal{Q}_{\infty W, 1} \rightarrow \mathcal{Q}_{\infty W, \infty S}^{(1)}$, où

$$\mathcal{Q}_{\infty W, \infty S}^{(1)} := \{C \in \text{End}(\mathcal{W}_W \oplus \mathcal{W}_S \oplus \mathbb{C}^{d_2}); C^* = C \text{ et } C \text{ est défini positif}\}.$$

L'espace de demi-ligne Sud est allongé par l'action successive des demi-bandes Ouest sur le coin : \mathcal{W}_S devient $\mathcal{W}_S \oplus \mathbb{C}^{d_2}$ (et même $\mathcal{W}_S \oplus \mathbb{C}^{qd_2}$ quand on agit récursivement q fois). Cependant, s'il n'est pas directement possible de comparer le coin allongé C_Q^{SW} lui-même, nous nous attendons tout de même à observer un phénomène d'invariance comme pour la demi-bande. En effet, se permettant de légers abus, comme l'opérateur "Fold" respecte les produits par (1.3.27), nous pouvons déplier le produit Sud-Nord, appliquer la propriété d'invariance (1.3.20) du demi-plan sous action de la bande doublement infinie et replier à nouveau pour obtenir le coin. Graphiquement,

$$\begin{array}{|c|} \hline B_Q^{W,\infty} \\ \hline C_Q^{SW} \\ \hline \end{array} \text{''} = \text{''} F \left(\begin{array}{|c|} \hline C(\widehat{B}_Q^{WE}) \\ \hline C(\widehat{G}_S) \\ \hline \end{array} \right) = F \left(\begin{array}{|c|} \hline C(\widehat{G}_S) \\ \hline \end{array} \right) \text{''} = \text{''} \begin{array}{|c|} \hline C_Q^{SW} \\ \hline \end{array} \text{''}$$

Bien sûr, comme précédemment mentionné, l'opérateur "Fold" ne permet de relier le coin et le demi-plan que partiellement et cet argument seul ne suffit pas. Cependant, il semblerait que la demi-bande laisse le coin Sud-Ouest invariant. Le théorème 3.5.3 clarifie ce point.

Théorème 1.3.2. *Pour tout $q \in \mathbb{N}^*$, on introduit l'application*

$$\Phi_{0,q}^{SW} : \text{End}(\mathcal{W}_W \oplus \mathcal{W}_S \oplus \mathbb{C}^{qd_2}) \rightarrow \text{End}(\mathcal{W}_W \oplus \mathcal{W}_S) \quad (1.3.34)$$

$$M \mapsto (I_{\mathcal{W}_W} \oplus D_q^S) M (I_{\mathcal{W}_W} \oplus D_q^S)^{-1}$$

où D_q^S est un opérateur de "décalage" dans la direction Sud défini en (3.5.31). Alors, pour tout $q \in \mathbb{N}^*$, nous avons

$$\Phi_{0,q}^{SW} \left(\begin{array}{|c|} \hline B_Q^{W,\infty} \\ \hline \vdots \\ \hline B_Q^{W,\infty} \\ \hline C_Q^{SW} \\ \hline \end{array} \right) = \begin{array}{|c|} \hline C_Q^{SW} \\ \hline \end{array} \quad (1.3.35)$$

où le dessin de gauche représente q demi-bandes empilées par des produits Schur sur le coin Sud-Ouest.

Un résultat analogue peut être énoncé pour l'action de la demi-bande Sud sur le coin Sud-Ouest au moyen des opérateurs de décalage Ouest $(D_p^W)_p$. Ces résultats remarquables traduisent la propriété d'invariance bi-dimensionnelle du coin à morphisme près : l'action des demi-bandes dans une direction ou l'autre laisse le coin invariant (à l'application Φ^{SW} près). C'est d'ailleurs ce qui différencie le coin de la demi-bande dont la propriété d'invariance ne concerne qu'une seule direction. Il est possible de vérifier que les applications Φ^{SW} possèdent des propriétés de morphismes relativement à l'opérateur Guill₂, respectant ainsi les produits et les jeux de couleurs, comme définies dans [Sim23].

Conclusion. Dans le chapitre 3 de ce manuscrit nous avons étendu la Guill₂-algèbre des formes quadratiques à de nouvelles couleurs nous permettant de parler de formes étendues comme les demi-bandes et les coins ainsi que de leurs recollements. Ces objets nous permettent de définir des conditions aux bords invariante pour notre champ sur n'importe quel rectangle R comme défini en (1.3.3). Rappelons que la condition au bord gaussienne s'écrit $g_R(-\frac{1}{2}B_{\partial R}(\cdot))$ avec $B_{\partial R}$ une forme quadratique définie positive sur le bord de R . À l'aide des opérateurs de demi-bandes à l'Ouest $B_Q^{W,\infty}$, à l'Est $B_Q^{E,\infty}$, au Sud $B_Q^{S,\infty}$, au Nord $B_Q^{N,\infty}$ ainsi que les opérateurs de coins au Sud-Ouest C_Q^{SW} , au Sud-Est C_Q^{SE} , au Nord-Est C_Q^{NE} et au Nord-Ouest C_Q^{NW} nous décomposons la matrice $B_{\partial R}$ associée à la forme quadratique $B_{\partial R}$ de bord de la manière suivante

$$B_{\partial R} = \begin{array}{|c|c|c|c|c|c|} \hline C_Q^{\text{NW}} & B_Q^{N,\infty} & B_Q^{N,\infty} & \dots & B_Q^{N,\infty} & C_Q^{\text{NE}} \\ \hline B_Q^{W,\infty} & & & & & B_Q^{E,\infty} \\ \hline \vdots & & & R & & \vdots \\ \hline B_Q^{W,\infty} & & & & & B_Q^{E,\infty} \\ \hline C_Q^{\text{SW}} & B_Q^{S,\infty} & B_Q^{S,\infty} & \dots & B_Q^{S,\infty} & C_Q^{\text{SE}} \\ \hline \end{array} \quad (1.3.36)$$

Dans ce dessin, les opérateurs que nous avons construits sont recollés entre eux, au moyen des produits de Schur, tout autour du rectangle évidé R . De plus, les propriétés d'associativité et d'invariance à morphisme près garantissent la cohérence sous restriction de ces bords. Graphiquement, pour tout $R' \subset R$:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|} \hline C_Q^{\text{NW}} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & C_Q^{\text{NE}} \\ \hline B_Q^{W,\infty} & Q & Q & Q & Q & Q & Q & Q & Q & Q & Q & B_Q^{E,\infty} \\ \hline B_Q^{W,\infty} & Q & Q & Q & Q & Q & Q & Q & Q & Q & Q & B_Q^{E,\infty} \\ \hline B_Q^{W,\infty} & Q & Q & & & & & & Q & Q & Q & B_Q^{E,\infty} \\ \hline B_Q^{W,\infty} & Q & Q & & & & & & Q & Q & Q & B_Q^{E,\infty} \\ \hline B_Q^{W,\infty} & Q & Q & & & & & & Q & Q & Q & B_Q^{E,\infty} \\ \hline B_Q^{W,\infty} & Q & Q & Q & Q & Q & Q & Q & Q & Q & Q & B_Q^{E,\infty} \\ \hline B_Q^{W,\infty} & Q & Q & Q & Q & Q & Q & Q & Q & Q & Q & B_Q^{E,\infty} \\ \hline B_Q^{W,\infty} & Q & Q & Q & Q & Q & Q & Q & Q & Q & Q & B_Q^{E,\infty} \\ \hline C_Q^{\text{SW}} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & C_Q^{\text{SE}} \\ \hline \end{array}$$

Ce dessin représente le produit des opérateurs de bords, disposés tout autour du bord de R , avec la forme quadratique Q attachée à chacune des faces du réseau $R \setminus R'$. Les propriétés d'invariance à morphisme près assurent que ce produit donne

$$\begin{array}{|c|c|c|c|c|c|c|}
 \hline
 C_Q^{NW} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & B_Q^{N,\infty} & C_Q^{NE} \\
 \hline
 B_Q^{W,\infty} & & & & & & B_Q^{E,\infty} \\
 \hline
 B_Q^{W,\infty} & & & R' & & & B_Q^{E,\infty} \\
 \hline
 B_Q^{W,\infty} & & & & & & B_Q^{E,\infty} \\
 \hline
 C_Q^{SW} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & B_Q^{S,\infty} & C_Q^{SE} \\
 \hline
 \end{array} = B_{\partial R'}$$

On retrouve alors bien la condition de bord $B_{\partial R'}$. De plus, comme la matrice $B_{\partial R}$ caractérise la loi sur le bord, nous avons bel et bien une famille de marginales cohérentes et le théorème d'extension de Kolmogorov assure l'existence du champ sur \mathbb{Z}^2 tout entier.

À propos de la fonction de partition. Comme nous travaillons uniquement au niveau des formes quadratiques et non plus au niveau des poids gaussiens, il n'est pas possible d'observer la fonction de partition ni la valeur propre, comme désiré dans (1.1.5). Dans le cas du modèle gaussien, ces quantités sont issues des déterminants des formes quadratiques qui apparaissent à chaque intégrale calculée, ce sont les γ_{ab} de la proposition 1.3.1. Si nous voulons faire apparaître la fonction de partition et donc la valeur propre associée aux demi-bandes et aux coins, nous devons tenir compte des déterminants issus de leurs recollements. Comme ce sont des opérateurs en dimension infinie, un contrôle analytique précis de déterminants de Fredholm est alors nécessaire. En section 3.5.6, nous exposons ce phénomène par des calculs de limites de déterminants non rigoureux et nous montrons (heuristiquement) comment il est possible de renormaliser les demi-bandes pour observer leur valeur propre associée.

1.3.5 Conclusion et extensions possibles

En conclusion, le présent manuscrit exhibe pour la première fois les objets de [Sim23] dans le cas des champ gaussiens markoviens sur la réseau \mathbb{Z}^2 . Plus précisément, cela est fait dans le cas où le poids de face est associé à une forme quadratique Q invariante sous action du groupe diédral, voir (1.3.5). Cette hypothèse permet d'alléger considérablement l'analyse. Cependant, nous pensons que, au prix de preuves plus techniques, et d'une zoologie plus touffue, nos résultats devraient s'étendre au cas général. Il serait intéressant de le montrer rigoureusement. Discutons à présent les possibles autres extensions du présent travail.

Vers des théorèmes de convergence et d'existence. Nous avons construit l'équivalent des mesures invariantes en dimension supérieure mais nous n'avons établi aucun théorème de convergence comme c'est le cas pour les chaînes de Markov avec le théorème ergodique par exemple. Il en va de même d'un point de vue algébrique. En dimension 1 et sous les hypothèses d'apériodicité et d'irréductibilité, le théorème de Perron-Frobenius établit la croissance exponentielle de la matrice de transition A , $A^n \sim_{n \rightarrow \infty} \Lambda_{PF}^n B_R (B_L)^t$ où Λ_{PF} , B_L et B_R sont respectivement la valeur, le vecteur à gauche et le vecteur à droite de Perron-Frobenius de A . Dans ce manuscrit, nous n'avons cependant montré aucune convergence de cet acabit pour les puissances surfaciques $Q^{[p,q]}$. Cela pourrait être une piste d'investigation intéressante. Par ailleurs, le théorème de Perron-Frobenius assure aussi l'existence des

conditions aux bords invariantes pour les processus unidimensionnels. Que devient ce fait en dimension supérieure ? Quelles hypothèses un poids de base (non obligatoirement gaussien) doit-il satisfaire pour assurer l'existence de demi-bandes et de coins décrivant le bord invariant du champ associé ? Cette question algébrique est très intéressante d'un point de vue probabiliste. Rappelons en effet que, en dimension 1, la mesure invariante est obtenue par les temps moyens d'occupation des différents sites lors d'une excursion. Les conditions de bords données dans cette thèse n'ont pour l'heure aucune interprétation probabiliste de ce genre. Par ailleurs, pour des modèles autres que le modèle gaussien, nous nous attendons à ce que la question des transitions de phase, absente en dimension 1, ait un impact sur les objets de bord. En les points de transition, ces derniers devraient avoir des propriétés remarquables.

Vers d'autres géométries. Dans cette thèse nous nous sommes arrêtés sur les champs gaussiens définis sur le réseau carré \mathbb{Z}^2 . Comme mentionné par [Sim23], grâce à l'opérateur Guill_d des partitions guillotines en dimension d , le présent travail devrait pouvoir s'exporter aux réseaux de dimensions supérieures \mathbb{Z}^d . Nous nous attendons quand même à ce que la variété des objets de bord devienne particulièrement touffue à mesure que d augmente, rendant leur classification fastidieuse. Par ailleurs, il est aussi envisageable de s'affranchir du réseau carré pour s'intéresser à d'autres réseaux : réseaux triangulaires, graphes isoradiaux ou autres. Le modèle du Laplacien massique sur graphes isoradiaux a d'ailleurs été étudié dans [BDTR17]. Pour ces nouvelles géométries, la zoologie des objets de bords sera évidemment bousculée, conduisant à d'autres formes que des demi-bandes et des coins. Par exemple, pour le réseau triangulaire, remarquons que nous devons faire face à deux poids de base, et non plus un, représentés de la manière suivante



Bien sûr, une fois recollés, ceux-ci redonnent le poids de base associé au carré de taille 1-par-1. Cependant, la combinatoire permise par ces nouvelles formes est nettement plus diversifiée que celle associée au réseau carré.

Vers une théorie perturbative. Le présent manuscrit fournit une brique de base intéressante pour l'exploration des bords des champs avec interactions. Nous pourrions par exemple imaginer ajouter un terme du champ à la puissance 4 dans la mesure de Gibbs (1.3.1) du champ gaussien markovien discret. Dès lors, nous nous trouverions dans le cas d'une théorie perturbative pour un champ " ϕ_2^4 discret" dont les conditions aux bords de la brique gaussienne de base sont bien comprises par ce manuscrit. Il serait intéressant d'observer la perturbation apportée par ce terme d'ordre 4 sur les objets de demi-bandes et de coins.

Vers des limites d'échelles. Une piste d'investigation alternative serait d'étudier les limites d'échelles du modèle gaussien discret. Nous savons que, correctement renormalisé, ce dernier devrait converger vers le champ libre massique. Que deviennent- alors les objets étudiés par ce manuscrit ? Dans le cas discret, le bord est donné par le produit d'empilement de matrices (1.3.6). Dans le continu, les rectangles de \mathbb{Z}^2 deviennent des rectangles de \mathbb{R}^2 et les produits d'empilements deviennent infinitésimaux. Dès lors, la décomposition aux bords (1.3.6) devraient devenir une équation différentielle. La basse régularité des champs indiquent que cette équation pourraient, en général, se rapprocher d'équations différentielles rugueuses, voir [FH20] pour de plus amples détails sur la théorie des chemins rugueux. L'auteur de cette thèse s'attend par ailleurs à voir apparaître les signatures en dimensions

supérieures de [GTZ10] (cas simplicial) ou [GLNO22] (cas "cubical"), étudiées par ce dernier en début de doctorat mais dont les travaux n'apparaissent pas dans le présent manuscrit.

Vers d'autres modèles. Enfin, il serait fortement intéressant de se pencher sur des modèles autres pour lesquels nous pourrions essayer d'appliquer les nouvelles méthodes apportées par ce manuscrit. Que devient par exemple le repliement de la matrice de transfert ? Outre cela, notons que le cas "bosonique" gaussien devrait pouvoir s'exporter, par le formalisme des espaces de Fock aux champs "fermioniques" comme les processus déterminantaux tels que le 6-vertex ou autres. Enfin, il serait tout aussi envisageable d'étudier les modèles intégrables, exploitant par exemple des équations de type Yang-Baxter. En effet, comme c'est le cas pour les modèles gaussiens, de nombreux calculs explicites sont possibles pour les modèles intégrables, voir [Bax16] pour de plus amples détails.

Note additionnelle : Nous avons choisi de ne pas présenter dans ce manuscrit le travail inachevé de la première année de thèse autour d'une approche rugueuse des champs stochastiques en dimension plus grande que 1. Il s'agissait d'étudier les constructions algébriques de G. Ginot et al. [GTZ10] d'un point de vue stochastique en introduisant des notions de régularité. Nous avons abouti (travail non publié) à la même obstruction de concaténation que celle présentée dans [GLNO22], avant de réorienter notre travail vers le présent sujet.

Chapter 2

One-dimensional discrete Gaussian Markov processes: Harmonic decomposition of invariant boundary conditions

Abstract

We study invariant boundary conditions for one dimensional discrete Gaussian Markov processes, basic toy models of spatial Markov processes in statistical mechanics. More precisely, we give a decomposition of boundary objects in a non trivial basis from the study of a meromorphic matrix-valued function Ψ (inherent to the model) and its singularities. This provides a simple algorithm for the explicit computation of invariant measures. As an application, we give an "eigen" version of Szegő limit theorem for matrix valued trigonometric polynomials.

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2.1 Introduction and results

This chapter corresponds to the submitted article [Bod23]. Nevertheless, for a better coherence of the present manuscript we slightly modified some notations and we added some content, not present in the online version (only in section 2.4).

Gaussian Markov fields have been widely studied and arise in numerous research areas such as quantum field theory, statistical physics, statistics... For instance, continuous Gaussian Free Field appears in Liouville quantum gravity, see [BP21], harmonic crystal in physics, and discrete Gaussian Markov fields in statistics, see [RH05]. Often, boundary conditions are taken to be periodic or deterministic (like zero boundary conditions). Rarely, invariant boundary conditions are studied.

In this article we will focus on the study of invariant boundary conditions for Gaussian Markov Processes on the lattice \mathbb{Z} . Of course, it is well-known that most of interesting physical quantities, such as free energy and correlation functions, are easily and explicitly computable using Fourier transform when working with periodic boundary conditions (due to invariance by translation). Nevertheless, these global computations totally hide the Markov property and the associated rich structure from which the model inherits. Another method is to take advantage of the Markov property and look for invariant measures. However, their explicit expression requires to solve tedious non-linear equations, which is hard in practice. Nevertheless, this local approach also leads to the computation of the same interesting physical quantities. It is remarkable since we do not know any link between invariant measures and Fourier transform.

The main objective of this article is to link these approaches together and, more importantly, to provide alternative computation tools and simple algorithms for the explicit calculation of invariant boundaries. To do so, we will investigate algebraic and recursive properties of the model and highlight its links with variety of other fields such as residue calculus, discrete harmonic analysis, Toeplitz analysis, etc... This will lead us to a decomposition of boundary objects in a non trivial basis from the study of a meromorphic matrix-valued function Ψ (inherent to the model) and its singularities. This article may also be viewed as a warm up for forthcoming work on the higher dimensional case.

After introducing and discussing the model, we expose the main results of this article.

2.1.1 Definitions and classical facts

This section is devoted to define the model and expose classical computations and results.

2.1.1.1 Definition of the model

The lattice \mathbb{Z} . In the whole article, \mathbb{Z} will be considered as a lattice composed of vertices V and unoriented edges E ,

$$V = \mathbb{Z}$$

$$E = \{\{x, y\} \subset V; |x - y| = 1\}.$$

For any $e \in E$, we denote $L(e) = \min e$ the left vertex of e and $R(e) = \max e$ the right vertex of e . We say that D is a (finite) domain of \mathbb{Z} if D is a (finite) subset of E . We denote its associated vertices $V(D)$, formally,

$$V(D) = \{v \in V; v = L(e) \text{ or } v = R(e) \text{ for } e \in D\}.$$

Moreover, we write ∂D the set of its boundary vertices defined as the vertices which belong exactly to one edge in D , that is

$$\partial D = \{v \in V(D); \exists! y \in V(D) \text{ s.t. } \{v, y\} \in E(D)\}.$$

A partition of D is a finite collection of non-empty domains $(D_i)_{i \in I}$ such that $D_i \cap D_j = \emptyset$ for $i \neq j$ and $\cup_{i \in I} D_i = D$.

We say that D is connected if D is only composed of consecutive edges, formally D is of form

$$D = \{\{L_D + k, L_D + k + 1\} \in E; k \in \{0, \dots, R_D - L_D - 1\}\}$$

for $L_D < R_D$ left and right vertices of D . Up to relabelling edges of D , we identify D with

$$D = \{\{k, k + 1\} \in E; k \in \{0, \dots, l(D) - 1\}\}$$

and $l(D) = R_D - L_D$ is the size of D .

Markov property on \mathbb{Z} . We will only consider non-oriented Markov processes as we are, due to physical motivations, more interested in spatial processes rather than processes indexed by time. We recall the definition of the spatial Markov property.

Definition 2.1.1 (Spatial Markov). Let (G, \mathcal{G}) be a measurable space, D a finite domain of \mathbb{Z} and $X^{(D)} = (X_v^{(D)})_{v \in V(D)}$ a set of (G, \mathcal{G}) -valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $X^{(D)}$ has the Markov property on D if and only if for all partition $(D_i)_{i \in I}$ of D and any bounded measurable function $h_i : G^{|D_i|} \rightarrow \mathbb{C}$, we have

$$\mathbb{E} \left[\prod_{i \in I} h_i(X^{(D_i)}) \middle| \sigma(X_v; v \in \cup_{i \in I} \partial D_i) \right] = \prod_{i \in I} \mathbb{E}[h_i(X^{(D_i)}) | \sigma(X_v; v \in \partial D_i)].$$

Remark 2.1.1. If D is not connected we can always write D as a disjoint union of its connected components which induces a rewriting of ∂D as a disjoint union of boundaries of its connected components. Then, $X^{(D)}$ can be decomposed in independent Markov processes defined on connected components of D . Therefore, without loss of generality, we restrict ourselves to the case of connected domains $D = \{\{k, k + 1\}; k \in \{0, \dots, P - 1\}\}$ for the study Markov processes on \mathbb{Z} .

Gaussian weights and Gaussian boundary weights. For all $d \in \mathbb{N}^*$, $\alpha \in \mathbb{R}_+^*$, $Q \in M_{2d}(\mathbb{C})$ and $x, y \in \mathbb{C}^d$, we denote

$$e_{\alpha, Q}(x, y) := \alpha \exp \left(-\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^* Q \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

and for all $\beta \in \mathbb{R}_+^*$ and $B \in M_d(\mathbb{C})$,

$$e'_{\beta, B}(x) = \beta \exp \left(-\frac{1}{2} x^* B x \right),$$

(cf. equations (2.4.1) and (2.4.3) for details). Constants α and β do not play an important role in the model but we will keep them for potential renormalizations (see theorem 2.1.2 for instance). Moreover, we will often refer to the block decomposition of $Q \in M_{2d}(\mathbb{C})$,

$$Q = \begin{pmatrix} Q_{LL} & Q_{LR} \\ Q_{RL} & Q_{RR} \end{pmatrix} \quad (2.1.1)$$

with Q_{IJ} ($d \times d$)-square matrices for all $I, J \in \{L, R\}$.

Gaussian Markov processes on \mathbb{Z} .

Definition 2.1.2 (1D Gaussian Markov Processes). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $d \in \mathbb{N}^*$ and D a connected domain of \mathbb{Z} of size $l(D) = P \in \mathbb{N}^*$. A \mathbb{C}^d -valued Gaussian Markov process on D of weights $(e_{\alpha^{(k)}, Q^{(k)}})_{k=0}^{P-1}$ with boundary conditions $(e'_{\beta_L, B_L}, e'_{\beta_R, B_R})$ is a collection of \mathbb{C}^d -valued random variables $X^{(D)} = (X_k^{(D)})_{k=0}^P \in \mathbb{C}^{d(P+1)}$, indexed by vertices of D , with density (with respect to product Lebesgue measure) given by

$$\begin{aligned} g_{X^{(D)}}(x^{(D)}) &= \frac{1}{Z_D(\alpha^{(\bullet)}, Q^{(\bullet)}, \beta_L, B_L, \beta_R, B_R)} e'_{\beta_L, B_L}(x_0) \left(\prod_{k=0}^{P-1} e_{\alpha^{(k)}, Q^{(k)}}(x_k, x_{k+1}) \right) e'_{\beta_R, B_R}(x_P) \\ &= \frac{\beta_L \beta_R \prod_{k=0}^{P-1} \alpha^{(k)}}{Z_D(\alpha^{(\bullet)}, Q^{(\bullet)}, \beta_L, B_L, \beta_R, B_R)} \exp \left(-\frac{1}{2} (x^{(P)})^* \mathbf{Q}_{(Q^{(\bullet)}, B_L, B_R)}^{(D)} x^{(P)} \right) \end{aligned}$$

for all $x^{(D)} = (x_k^{(D)})_{k=0}^P \in \mathbb{C}^{d(P+1)}$ with $\alpha^{(k)} \in \mathbb{R}_+^*$ for all k , $\beta_L, \beta_R \in \mathbb{R}_+^*$ and the matrices $(Q^{(k)})_{k=0}^{P-1}$, B_L and B_R are chosen so that

$$\mathbf{Q}_{(Q^{(\bullet)}, B_L, B_R)}^{(D)} = \begin{pmatrix} B_L + Q_{LL}^{(0)} & Q_{LR}^{(0)} & 0 & 0 & \cdots & 0 \\ Q_{RL}^{(0)} & Q_{RR}^{(0)} + Q_{LL}^{(1)} & Q_{LR}^{(1)} & 0 & \cdots & 0 \\ 0 & Q_{RL}^{(1)} & Q_{RR}^{(1)} + Q_{LL}^{(2)} & Q_{LR}^{(2)} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & Q_{RL}^{(P-2)} & Q_{RR}^{(P-2)} + Q_{LL}^{(P-1)} & Q_{LR}^{(P-1)} \\ 0 & 0 & \cdots & 0 & Q_{RL}^{(P-1)} & Q_{RR}^{(P-1)} + B_R \end{pmatrix} \quad (2.1.2)$$

is Hermitian and positive definite.

Moreover, the normalizing constant $Z_D(\alpha^{(\bullet)}, Q^{(\bullet)}, \beta_L, B_L, \beta_R, B_R)$, the so-called partition function in statistical mechanics, is given by

$$\begin{aligned} Z_D(\alpha^{(\bullet)}, Q^{(\bullet)}, \beta_L, B_L, \beta_R, B_R) &= \int_{\mathbb{C}^{d(P+1)}} e'_{\beta_L, B_L}(x_0) \left(\prod_{k=0}^{P-1} e_{\alpha^{(k)}, Q^{(k)}}(x_k, x_{k+1}) \right) e'_{\beta_R, B_R}(x_P) dx^{(D)} \\ &= (2\pi)^{d(P+1)} \det \left(\mathbf{Q}_{(Q^{(\bullet)}, B_L, B_R)}^{(D)} \right)^{-1} \beta_L \beta_R \prod_{k=0}^{P-1} \alpha^{(k)}. \end{aligned}$$

We say that this process is homogeneous of weight (α, Q) if $\alpha^{(k)} = \alpha$ and $Q^{(k)} = Q$ for all $k \in \{0, \dots, P-1\}$.

It is clear, from the factorized form, that any Gaussian Markov Process has indeed the Markov property. This justifies, as long as we stick to finite dimension, defining Gaussian processes by their density rather than by their covariance matrices on which the Markov property is less easy to read.

Remark 2.1.2. Our processes are \mathbb{C}^d -valued circularly symmetric ($e^{i\phi}X \sim X$ for any $\phi \in [-\pi, \pi]$) only to simplify discrete Fourier transform computations. We could have chosen \mathbb{R}^d -valued processes and most of results remain the same (up to square roots on determinants).

Remark 2.1.3. In the definition of Gaussian Markov Processes 2.1.2, the only assumption on the matrices involved is $Q_{(Q^{(\bullet)}, B_L, B_R)}^{(D)}$ being Hermitian and positive definite. This ensures that the random vector is indeed a multivariate Gaussian on $\mathbb{C}^{d(P+1)}$. Of course, this global assumption implies local properties such as

$$\begin{pmatrix} Q_{RR}^{(k-1)} + Q_{LL}^{(k)} & Q_{LR}^{(k)} \\ Q_{RL}^{(k)} & Q_{RR}^{(k)} + Q_{LL}^{(k+1)} \end{pmatrix}$$

being Hermitian and positive definite for $k \in \{1, \dots, P-2\}$. Nevertheless, this is not a property on $Q^{(k)}$ itself but on $Q^{(k)}$ surrounded by its neighbors $Q^{(k-1)}$ and $Q^{(k+1)}$. In a view of giving a consistent algebraic structure to Gaussian weights, we are interested in considering $Q^{(k)}$ (thus $e_{\alpha^{(k)}, Q^{(k)}}$) on its own. What property the matrices $(Q^{(k)})_k$ should satisfy? Same question holds for B_L and B_R . The assumption that $(Q^{(k)})_k$, B_L and B_R are Hermitian and positive definite basically allows to do whatever we want in the "Gaussian world": marginalization, computing observables etc... Moreover, the (homogeneous) model admits a unique infinite volume Gibbs measure if the inverse of a certain function Ψ , defined in section 2.2, is not singular on the unit circle, see section 13.3 of [Geo11]. This is automatic when $(Q^{(k)})_k$, B_L and B_R are assumed to verify Hermitian and positive definite properties. We will therefore work with the following set.

The set $H_d^+(\mathbb{C})$. For any $d \in \mathbb{N}^*$, we introduce the following set

$$H_d^+(\mathbb{C}) = \{M \in M_d(\mathbb{C}); M \text{ is Hermitian positive definite}\}. \quad (2.1.3)$$

In this article, due to the latter discussion, we will only be considering matrices $Q^{(k)} \in H_{2d}^+(\mathbb{C})$, and $B_L, B_R \in H_d^+(\mathbb{C})$.

Particular cases. To highlight the previous assumption, here are some specific cases covered or not by the model.

- Discretised Ornstein-Uhlenbeck is a special case of the model we are considering. Let $d = 1$ and take $m > 0$. The Gaussian Markov process with weight $e_{\alpha, Q}$ for some $\alpha \in \mathbb{R}_+^*$ and

$$Q = \begin{pmatrix} 1 + \frac{m^2}{2} & -1 \\ -1 & 1 + \frac{m^2}{2} \end{pmatrix} \in H_2^+(\mathbb{C}).$$

is a discretised Ornstein-Uhlenbeck. It corresponds to the Hamiltonian

$$H(x) = \frac{1}{2} \sum_k (x_{k+1} - x_k)^2 + \frac{m^2}{2} \sum_k x_k^2$$

of a one-dimensional massive Gaussian free field of mass m , see [FV17] for instance.

- Discretised Brownian motion (or one-dimensional harmonic crystal) does not verify the positive definite assumption and is therefore not covered by the model. Indeed, discretised Brownian motion requires to work with following kernel

$$\mathbf{e}(x_k, x_{k+1}) \sim e^{-\frac{1}{2}|x_{k+1}-x_k|^2} = \exp\left(-\frac{1}{2}\begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^* \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}\right)$$

and $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \notin H_2^+(\mathbb{C})$ since it is not full rank. However, not covering the Brownian case is not a problem as we are interested, in this article, in invariant probability measures (which discretised Brownian motion does not have).

Homogeneous case. Except if the contrary is mentioned, we will stick to the homogeneous case. Let $l(D) = P \in \mathbb{N}^*$ be the size of D . In that case, $X^{(D)}$, the partition function, and the matrix $\mathbf{Q}_{(Q, B_L, B_R)}^{(D)}$ are simply denoted, respectively, $X^{(P)}$, $Z_P(\alpha, Q, \beta_L, B_L, \beta_R, B_R)$ and $\mathbf{Q}_{(Q, B_L, B_R)}^{(P)}$. We say that $X^{(P)}$ is a (homogeneous) Gaussian Markov process of size P .

The full rank assumption. Remark that if Q_{LR} is not full rank, so is Q_{RL} and x and $y \in \mathbb{C}^d$ are only coupled by $\mathbf{e}_{\alpha, Q}(x, y)$ through $x_{\ker(Q_{RL})^\perp}$ and $y_{\ker(Q_{LR})^\perp}$, their orthogonal projection onto, respectively, $\ker(Q_{RL})^\perp$ and $\ker(Q_{LR})^\perp$. Therefore, in most of the results we adopt the following assumption.

Assumption I. The matrix Q satisfies

$$\text{rank}(Q_{LR}) = d. \quad (2.1.4)$$

Section 2.7 is devoted to the extension of these results in the case where Q_{LR} is not full rank. This only makes computations much heavier and does not change the nature of the results.

2.1.1.2 Classical motivations and the problem with boundaries

Given a Gaussian Markov process of weight $\mathbf{e}_{\alpha, Q}$ with boundary conditions $(\mathbf{e}'_{\beta_L, B_L}, \mathbf{e}'_{\beta_R, B_R})$, we could be interested in computing the following limit

$$f(\alpha, Q, \beta_L, B_L, \beta_R, B_R) = \lim_{P \rightarrow \infty} \frac{1}{P} \log Z_P(\alpha, Q, \beta_L, B_L, \beta_R, B_R), \quad (2.1.5)$$

the so-called free energy. Another quantity of interest is the correlation function in the limit $P \rightarrow \infty$. That is, for $n \leq P$ and for any $v_1, \dots, v_n \in V(D)$, the computation of

$$\lim_{P \rightarrow \infty} \mathbb{E}_{B_L, B_R}[X_{v_1}^{(P)} \dots X_{v_n}^{(P)}], \quad (2.1.6)$$

which, for Gaussian processes reduces to the limit of the covariance matrix by Wick theorem. Being able to define the model on the whole lattice \mathbb{Z} is also one of the main motivation when dealing with statistical physics problems. This point is particularly easy here since for an infinite dimensional Gaussian measure to exist, one only need to give its covariance operator. Therefore, the computation of the correlation function will lead to the existence of the associated infinite volume Gibbs measure.

The choice of boundary conditions. Remark that, no matter which quantity we wish to compute, we face the problem of specifying the boundary conditions B_L and B_R . As explained below, some choices are particularly efficient. Nevertheless, because we expect exponential decorrelation (see remark 2.2.4), these quantities should not be depending on the initial boundary settings we choose. The present paper discusses the similarities between two common boundary conditions: periodic and invariant boundary conditions.

2.1.1.3 Various tools of study

Let us briefly present various ways to compute those quantities.

Via Discrete Fourier Transform. One common approach is to work with Gaussian Markov Processes on the discrete torus $\mathbb{Z}/P\mathbb{Z}$. Then, with the help of discrete Fourier transform, computations become easier.

Notation. Let $P \in \mathbb{N}^*$, and $k \in \mathbb{N}$. We write

$$\omega_P^k := e^{\frac{2i\pi k}{P}}. \quad (2.1.7)$$

Under periodic boundary conditions, we have the following theorem which answers most of classical questions.

Theorem 2.1.1. *Let $P \in \mathbb{N}^*$, $\alpha \in \mathbb{R}_+^*$, and $Q \in H_{2d}^+(\mathbb{C})$. Let $X^{(P),\text{per}} = (X_k^{(P),\text{per}})_{k \in \mathbb{Z}/P\mathbb{Z}}$ be a \mathbb{C}^d -valued Gaussian Markov process on $\mathbb{Z}/P\mathbb{Z}$ with weight $e_{\alpha,Q}$. Denote for all $k \in \mathbb{Z}/P\mathbb{Z}$, the Fourier transform*

$$\widehat{X}_k^{(P)} = \frac{1}{\sqrt{P}} \sum_{i \in \mathbb{Z}/P\mathbb{Z}} X_i^{(P),\text{per}} \omega_P^{ik}.$$

Then, the r.v. $(\widehat{X}_k^{(P)})_{k \in \mathbb{Z}/P\mathbb{Z}}$ are independent and each $\widehat{X}_k^{(P)}$ has density against Lebesgue measure

$$\frac{1}{2\pi \det \Psi_Q(\omega_P^k)^{-1}} \exp\left(-\frac{1}{2} x^* \Psi_Q(\omega_P^k) x\right)$$

where $\Psi_Q : \mathbb{C}^ \rightarrow M_d(\mathbb{C})$ is a matrix-valued function built from blocks of Q , non singular on the unit circle S^1 , see (2.2.1). We then have*

$$Z_P^{\text{per}}(\alpha, Q) = \alpha^P (2\pi)^{dP} \prod_{k=0}^{P-1} \det \Psi_Q(\omega_P^k)^{-1}$$

so that the free energy $f^{\text{per}}(\alpha, Q)$ is equal to

$$f^{\text{per}}(\alpha, Q) = \lim_{P \rightarrow \infty} \frac{1}{P} \log Z_P^{\text{per}}(\alpha, Q) = \log(\alpha(2\pi)^d) - \frac{1}{2\pi} \int_0^{2\pi} \log \det \Psi_Q(e^{i\theta}) d\theta. \quad (2.1.8)$$

Moreover, for any $k, l \in \mathbb{Z}$ fixed,

$$\lim_{P \rightarrow \infty} \mathbb{E}[X_k^{(P),\text{per}} X_l^{(P),\text{per}}] = C_{l-k}(\Psi_Q^{-1})$$

where $C_n(\Psi_Q^{-1})$ is the n -th Fourier coefficient of Ψ_Q^{-1} for any $n \in \mathbb{Z}$, see (2.2.4).

Proof. This follows from a discrete Fourier change of variable and (2.1.8) is a Riemann sum. See section 2.3 or chapter 13 of [Geo11] for more details. \square

The function Ψ_Q . As we will see, the function Ψ_Q , that appears in the previous theorem, plays an important role in the model and the whole section 2.2 is dedicated to its complete study.

Via Invariant Measure Approach. Another classical approach is to take advantage of the Markov property and look for left (or right) invariant measure. For instance, that is finding B_L such that

$$\int_{\mathbb{C}^d} e'_{\beta_L, B_L}(x) e_{\alpha, Q}(x, y) dx = \Lambda e'_{\beta_L, B_L}(y)$$

for some eigenvalue Λ . We start with the following definition.

Definition 2.1.3 (Left and right Schur-invariant matrices). Let $Q \in H_{2d}^+(\mathbb{C})$. A matrix $B_L \in H_d^+(\mathbb{C})$ is left Schur-invariant for Q if it satisfies the following non linear equation

$$B_L = Q_{RR} - Q_{RL}(B_L + Q_{LL})^{-1}Q_{LR} \quad (2.1.9)$$

where we recognize a Schur complement. Similarly, a matrix $B_R \in H_d^+(\mathbb{C})$ is right Schur-invariant for Q if it satisfies

$$B_R = Q_{LL} - Q_{LR}(Q_{RR} + B_R)^{-1}Q_{RL}. \quad (2.1.10)$$

Standard Gaussian computations give the following theorem.

Theorem 2.1.2 (Left and right eigen-boundary conditions). Let $\alpha \in \mathbb{R}_+^*$, $Q \in H_{2d}^+(\mathbb{C})$ and B_L (resp. B_R) left Schur-invariant (resp. right Schur-invariant) for Q . Then

$$\int_{\mathbb{C}^d} e'_{\beta_L, B_L}(x) \cdot e_{\alpha, Q}(x, y) dx = \alpha(2\pi)^d \det(B_L + Q_{LL})^{-1} e'_{\beta_L, B_L}(y) \quad (2.1.11)$$

for all $y \in \mathbb{C}^d$. In that case, we say that e'_{β_L, B_L} is a left eigen-boundary density for $e_{\alpha, Q}$. Similarly,

$$\int_{\mathbb{C}^d} e_{\alpha, Q}(x, y) \cdot e'_{\beta_R, B_R}(y) dy = \alpha(2\pi)^d \det(Q_{RR} + B_R)^{-1} e'_{\beta_R, B_R}(x). \quad (2.1.12)$$

for all $x \in \mathbb{C}^d$. In that case, we say that e'_{β_R, B_R} is a right eigen-boundary density for $e_{\alpha, Q}$. Moreover, the partition function of any homogeneous Gaussian Markov process of weight $e_{\alpha, Q}$ with boundary conditions $(e'_{\beta_L, B_L}, e'_{\beta_R, B_R})$ on any finite component D of size $P \in \mathbb{N}^*$ is

$$Z_P(\alpha, Q, \beta_L, B_L, \beta_R, B_R) = \Lambda^P \kappa \quad (2.1.13)$$

with the eigenvalue $\Lambda = \alpha(2\pi)^d \det(B_L + Q_{LL})^{-1} = \alpha(2\pi)^d \det(Q_{RR} + B_R)^{-1}$ and $\kappa = \langle e'_{\beta_L, B_L}, e'_{\beta_R, B_R} \rangle_{L^2(\mathbb{C}^d)} = 1$ (with the good choices of β_L and β_R). Then,

$$f^{\text{eigen}}(\alpha, Q) := f(\alpha, Q, \beta_L, B_L, \beta_R, B_R) = \log \Lambda. \quad (2.1.14)$$

Proof. This is a direct consequence of properties 2.4.3, 2.4.4 and 2.4.5, see section 2.4 for algebraic details. \square

This theorem first reduces the research of invariant measures to the understanding of the non-linear fix point relations (2.1.9) and (2.1.10). Solutions are, in general, difficult to compute explicitly due to the non-linearity of these equations. Our main occupation will be to simplify these computations using a well-chosen decomposition. This theorem also shows that the computation of the partition function and the free energy is particularly simple under eigen-boundary conditions. Remark that, with this approach, the free energy is not a limit anymore.

Remark 2.1.4. Take $(D_n)_{n \in \mathbb{N}}$ an increasing sequence of finite connected components such that

$$\bigcup_{n \in \mathbb{N}} D_n = \mathbb{Z}.$$

The family of homogeneous Gaussian Markov processes $(X^{(D_n)})_{n \in \mathbb{N}}$ of weight $e_{\alpha, Q}$ with left and right eigen-boundary conditions defines a consistent family of probability measures. Kolmogorov extension theorem ensures the existence of a stochastic process X on a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose marginal laws are the $X^{(D_n)}$. The law of X is the infinite volume Gibbs measure associated to this problem.

What is the link? Fourier transform and eigen-boundary conditions provide two methods for computing the same quantities. Nevertheless one doesn't easily see the link between these two approaches. Fourier transform is a global computation which totally hides local Markov property whereas invariant measure is a local computation based on the Markov property of the model. Moreover, Fourier transform provides explicit formulas (see theorem 2.1.1) when invariant measure approach is based on equations (2.1.9) and (2.1.10), which are difficult to solve explicitly. The main goal of this article is to reconcile these methods and to specify the underlying structure which describes the links that they share. By doing so, we will provide a simple algorithm for solving explicitly equations (2.1.9) and (2.1.10).

A step towards higher dimension. Establishing bridges between Fourier and eigen-boundary conditions for one dimensional Markov processes is interesting in itself but it becomes even more relevant in higher dimension. In higher dimension, Fourier transform approach keeps working and is well understood. Moreover, recent work from [Sim23] uses operad theory to give a coherent notion of eigen-boundary ("invariant-boundary") conditions for higher dimensional Markov Processes. Due to geometrical properties, eigen-boundary objects are much more sophisticated than in dimension one and are not well understood yet. For higher dimensional Gaussian Markov processes, our understanding of Fourier method should be converted into the understanding of those new boundary objects via bridges between the two approaches. The present article can be viewed as a warm-up for upcoming work on higher dimensional Gaussian processes using this novel approach.

2.1.2 Main theorems and structure of the article

We now expose the structure of the present paper.

2.1.2.1 Study of the meromorphic function Ψ_Q

As it can be seen in theorem 2.1.1, the function Ψ_Q plays an important role: one can compute many quantities of interest from this single function. It is therefore fully studied in section 2.2, which contains every relevant technical results. In particular, we will focus on the points of \mathbb{C}^* where Ψ_Q is not invertible, simply called the zeros of Ψ_Q , see lemma 2.2.4. We will also construct two non trivial bases of \mathbb{C}^d , indexed by zeros of Ψ_Q , see definition 2.2.2, along which many objects related to the model can be decomposed. This paper establishes a dictionary between zeros of Ψ_Q , together with their associated bases, and various probability laws of the model.

2.1.2.2 From periodic boundary conditions to translation invariant Gaussian Markov on \mathbb{Z}

Section 2.3 gives more details for the periodic case and the structures behind theorem 2.1.1 obtained with Fourier transform. This is common material, for instance, see [Geo11], chapter 13 for homogeneous Gaussian fields. We will compute marginal laws under thermodynamic limit, see (2.3.5), and show that the Markov property is preserved, see theorem 2.3.1. This gives a natural candidate for the Hilbert space which "carries" the infinite volume Gibbs measure.

2.1.2.3 Markov and Schur complement: algebraic formulas

Motivated by theorem 2.1.2, geometric and algebraic structures of $(Z_P(\alpha, Q))_{\alpha, Q, P}$ or, equivalently, the ones given by weights $(e_{\alpha, Q})_{\alpha, Q}$, are studied in section 2.4. There is a natural associative product m which allows to glue them together. Moreover, there is a non linear product S at the level of Hermitian matrices and a matrix K such that we have the lift

$$m(e_{\alpha, Q}, e_{\tilde{\alpha}, \tilde{Q}}) = e_{(2\pi)^d \alpha \tilde{\alpha} \det K, S(Q, \tilde{Q})},$$

see property 2.4.1. We will also have a look at algebraic properties of Gaussian boundary densities $e'_{\beta, B}$ on which Gaussian weights $e_{\alpha, Q}$ naturally act. All those objects form a set $\mathcal{Q}_{\text{Gauss}}^{\text{ext}}(\mathbb{C}^d)$ obeying to the following result.

Theorem 2.1.3. *The set $\mathcal{Q}_{\text{Gauss}}^{\text{ext}}(\mathbb{C}^d)$ is an algebra over the extended Guill_1 operad.*

2.1.2.4 Eigen-boundary conditions

Section 2.5 focuses on the study of eigen-boundary conditions. In view of a higher dimension generalization, we wish to give a "dimension-free" recipe for the computation of B_L and B_R satisfying the non linear fix point relations (2.1.9) and (2.1.10). This section aims at a better understanding of these equations and especially, at their translation in terms of Ψ_Q (to establish links with Fourier). We briefly recall that we denote $\mathcal{S}(\Psi_Q) = \mathcal{S}_{<1}(\Psi_Q) \sqcup \mathcal{S}_{>1}(\Psi_Q)$ the set of zeros of Ψ_Q inside and outside the unit disk, see lemma 2.2.4 and equation (2.2.11). To each of these sets is associated a non-trivial basis of \mathbb{C}^d , denoted $(u_w)_{w \in \mathcal{S}_{<1}(\Psi_Q)}$ and $(u_w)_{w \in \mathcal{S}_{>1}(\Psi_Q)}$, see definition 2.2.2. We now expose the main theorem of the paper, proven in section 2.5.

Theorem 2.1.4. *Let $Q \in H_{2d}^+(\mathbb{C})$ and $W_{>1} \in M_d(\mathbb{C})$ any diagonalizable matrix. Under assumptions I and II, the following assertions are equivalent*

- (i) *The matrix $B_L = Q_{RR} + Q_{RL}W_{>1}^{-1}$ is left Schur-invariant for Q .*
- (ii) *The matrix $W_{>1}$ is diagonal in the basis $(u_w)_{w \in \mathcal{S}_{>1}(\Psi_Q)}$, with $W_{>1}u_w = wu_w$ for all $w \in \mathcal{S}_{>1}(\Psi_Q)$.*
- (iii) *Let $\mathbf{x} \in \mathbb{C}^d$. The sequence $(x_k)_{k=0}^{-\infty}$ with $x_k = W_{>1}^{-k}x_0$ is solution to discrete Dirichlet-type problem*

$$\begin{cases} A_{RL}x_{-k-1} + (A_{LL} + A_{RR})x_{-k} + A_{LR}x_{-k+1} = 0 \text{ for all } k > 0 \\ x_0 = \mathbf{x} \\ \lim_{k \rightarrow -\infty} x_k = 0 \end{cases} \quad (2.1.15)$$

Moreover, in those cases, the eigenvalue associated to e'_{β_L, B_L} is

$$\Lambda = \alpha(2\pi)^d (-1)^d \frac{\prod_{w \in \mathcal{S}_{<1}} w}{\det(Q_{RL})} \quad (2.1.16)$$

An analog theorem for right eigen-boundary condition B_R involving zeros inside the unit disk $\mathcal{S}_{<1}(\Psi_Q)$ is given in the same section, see theorem 2.5.1. Here are a few comments concerning these two theorems. First of all, we remark from (2.1.16) that the eigenvalue is entirely determined by zeros of Ψ_Q . By (2.1.14), taking the logarithm gives an expression for the free energy in terms of these zeros and which, as we will see in (2.2.12), is equal to the free energy computed with periodic boundary conditions. Those theorems also give an explicit construction of left and right Schur-invariant elements B_L and B_R and reduce the eigen-boundary problem to a linear algebra problem. The construction of B_L is performed from zeros of Ψ_Q outside the unit disk together with their associated basis whereas the construction of B_R is performed from the zeros inside the unit disk. Finally we remark the emergence of new matrices $W_{<1}$ and $W_{>1}$ which solve the discrete harmonic problem (iii) associated to the tridiagonal operator generated by Q . In some sense, the function Ψ_Q , and especially its zeros with their associated bases, is what links Fourier with invariant boundary conditions.

In the same section we will show that, under eigen-boundary conditions provided by these theorems, we recover the marginal laws computed through discrete Fourier transform, see lemma 2.5.6.

Interlude: solving invariant equations. Apart from any probabilistic considerations, theorems 2.1.4 and 2.5.1 provide an efficient algorithm to solve Schur-invariant equations (2.1.9) and (2.1.10). Of course, these equations are easily solvable in the case $d = 1$. It is just a matter of finding the roots of a second order polynomial with complex coefficients. Nevertheless, when $d > 1$, this becomes quite difficult. We have to solve a non-linear system of $d(d+1)/2$ equations (by hermitian property) involving multivariate polynomials. Nevertheless, theorems 2.1.4 and 2.5.1 give a very simple step by step algorithm to build a solution:

- Find $\mathcal{S}(\Psi_Q)$. This amounts to solve $\det(\Psi_Q) = 0$, that is finding the $2d$ roots a polynomial of order $2d$.
- For any $w \in \mathcal{S}(\Psi_Q)$, find a basis of $\ker \Psi_Q(w)$. This can be performed by Gaussian elimination.
- Build $W_{>1}$ and $W_{<1}$ in the basis $(u_w)_{w \in \mathcal{S}_{>1}(\Psi_Q)}$ and $(u_w)_{w \in \mathcal{S}_{<1}(\Psi_Q)}$ respectively.

2.1.2.5 Identities in law under Gibbs measure

Under the infinite volume Gibbs measure provided by eigen-boundary conditions given by theorems 2.1.4 and 2.5.1, we give, in section 2.6, some identities in law under equilibrium. Motivated by the point (iii) of theorem 2.1.4 and 2.5.1, we link the matrices $W_{<1}$ and $W_{>1}$ with conditional expectations (known to be harmonic). In particular, we will see that conditional expectations are given by two semi-groups generated by zeros of Ψ_Q , see theorem 2.6.1. In this theorem, we will also compute conditional variances.

2.1.2.6 The case Q_{LR} non-invertible

The main results of this paper are established under the assumption that Q_{LR} is full rank, see assumption I. In section 2.7, we show how to extend these results to the case where Q_{LR} is non-invertible. This does not change the nature of the results but makes computations much heavier. In this new context, the bases indexed by zeros of Ψ_Q now need to be completed with the bases of $\ker Q_{LR}$ and $\ker Q_{RL}$ to form basis of \mathbb{C}^d . We will build left and right Schur-invariant boundary conditions in terms of matrices similar to $W_{<1}$ and $W_{>1}$

defined in theorem 2.1.4 and 2.5.1. These matrices are now diagonal in the bases indexed by zeros of Ψ_Q completed with the bases $\ker Q_{LR}$ and $\ker Q_{RL}$. Especially, they are no longer invertible (their kernels are respectively $\ker(Q_{RL})$ and $\ker(Q_{LR})$).

2.1.2.7 Application: probabilistic representation of Szegő limit theorem

Szegő limit theorem is a famous result concerning the asymptotic behavior of determinant of Toeplitz matrices. In section 2.8, we propose an "eigen" version of this theorem in a simple case as an application.

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2.2 Study of the meromorphic function Ψ_Q

Let $d \in \mathbb{N}^*$ and $Q \in H_{2d}^+(\mathbb{C})$ with block decomposition previously given in (2.1.1). Let Ψ_Q be the matrix-valued function defined by

$$\begin{aligned} \Psi_Q(z) : \mathbb{C}^* &\rightarrow M_d(\mathbb{C}) \\ z &\mapsto \Psi_Q(z) = Q_{LL} + Q_{RR} + Q_{LR}z + Q_{RL}z^{-1}. \end{aligned} \quad (2.2.1)$$

If Q is clear from the context, we skip the index and write simply Ψ . Note that, for all $z \in \mathbb{C}^*$, $\Psi(z)$ is obtained from Q through

$$\Psi(z) = \begin{pmatrix} I_d \\ \bar{z}^{-1}I_d \end{pmatrix}^* Q \begin{pmatrix} I_d \\ zI_d \end{pmatrix} \quad (2.2.2)$$

with I_d the d -dimensional identity matrix. This section is devoted to the study of this function Ψ . First, we remark that we have the following lemma.

Lemma 2.2.1. *For all $z \in \mathbb{C}^*$,*

$$\Psi(1/\bar{z}) = \Psi(z)^*. \quad (2.2.3)$$

Proof. It is a direct computation with the use of Hermitian property of Q . \square

This last lemma shows that the case $|z| = 1$ is particular. We will therefore split the study of the function Ψ into three subsections. Firstly, we will study Ψ on the unit circle and deduce, from the positive definite and Hermitian properties, the existence of an inverse Ψ^{-1} . Moreover, we will show that the Fourier coefficients of this inverse follow a second order recursive relation. Secondly, we will study Ψ outside the unit circle and focus on its zeros, the points where it is not invertible. In particular, we will see that the free energies computed via Fourier and via eigen-boundary conditions are both equal to a sum over these zeros. Finally, we will introduce two non trivial bases of \mathbb{C}^d , indexed by zeros of Ψ . With the help of residue calculus, we will decompose Ψ^{-1} , everywhere it is invertible, and its Fourier coefficients on these bases. The bases indexed by zeros of Ψ play an important role in the description of eigen-boundary conditions, see theorems 2.1.4 and 2.5.1 and in probability laws under infinite volume Gibbs measure, see theorem 2.6.1.

2.2.1 Behavior on the unit circle

For this whole subsection, $d \in \mathbb{N}^*$ and $Q \in H_{2d}^+(\mathbb{C})$ are fixed.

Invertibility on the unit circle. We start by showing that Ψ is invertible on the unit circle.

Lemma 2.2.2 (Hermitian property and positive definiteness). *For all $\theta \in [0, 2\pi]$, $\Psi(e^{i\theta})$ is Hermitian and positive definite of dimension d . Therefore, for all $\theta \in [0, 2\pi]$, $\Psi(e^{i\theta})$ has a Hermitian and positive definite inverse $\Psi(e^{i\theta})^{-1}$. Finally, $\theta \mapsto \Psi(e^{i\theta})^{-1}$ is continuous.*

Proof. Hermitian property is direct from (2.2.3) and positive definite property follows from (2.2.2) together with positive definite property of Q . Continuity is direct from continuity of Ψ and inverse formula using adjugate matrix. \square

Fourier coefficients of Ψ^{-1} . Motivated by the covariance matrix computed in theorem 2.1.1, we are interested in the Fourier coefficients of Ψ^{-1} . We first recall the definition of matrix-valued Fourier coefficients.

Definition 2.2.1 (Fourier coefficients of a matrix). Let $d \in \mathbb{N}^*$ and, for all $\theta \in [0, 2\pi]$, let $\Phi(\theta)$ be a matrix of dimension $d \times d$. Suppose $\Phi_{i,j} \in L^1([0, 2\pi])$ for all $i, j \in \{1, \dots, d\}$. For all $k \in \mathbb{Z}$ we define the k -th Fourier coefficient of Φ as the matrix $C_k \in M_d(\mathbb{C})$ such that

$$(C_k(\Phi))_{i,j} = \frac{1}{2\pi} \int_0^{2\pi} (\Phi(\theta))_{i,j} e^{-ik\theta} d\theta$$

for all $i, j \in \{1, \dots, d\}$. We will often write

$$C_k(\Phi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(\theta) e^{-ik\theta} d\theta. \quad (2.2.4)$$

Notation. As a continuous function, $\Psi^{-1} \in L^1([0, 2\pi])$. In the following we will denote

$$C_k := C_k(\Psi^{-1}) \quad (2.2.5)$$

for all $k \in \mathbb{Z}$ the Fourier coefficients of Ψ^{-1} . They obey to the following linear relation.

Lemma 2.2.3. *For all $k \in \mathbb{Z}$, we have*

$$C_k^* = C_{-k} \quad (2.2.6)$$

and

$$C_k(Q_{LL} + Q_{RR}) + C_{k+1}Q_{RL} + C_{k-1}Q_{LR} = I_d \delta_{k,0}. \quad (2.2.7)$$

Proof. The first identity is due to Hermitian property of Ψ^{-1} . By definition of the inverse, for all $\theta \in [0, 2\pi]$,

$$\Psi(e^{i\theta})^{-1} \Psi(e^{i\theta}) = I_d.$$

Using Fourier decomposition $\Psi(e^{i\theta})^{-1} = \sum_{k \in \mathbb{Z}} C_k e^{ik\theta}$ and the definition of $\Psi(e^{i\theta})$ we get

$$\sum_{k \in \mathbb{Z}} (C_k(Q_{LL} + Q_{RR}) + C_{k+1}Q_{RL} + C_{k-1}Q_{LR}) e^{ik\theta} = I_d$$

and uniqueness of Fourier decomposition gives the result. \square

Remark 2.2.1. Fourier coefficients $(C_k)_{k \in \mathbb{Z}}$ follow a second order recursive relation. This can be seen as a consequence of the Markov property of the model or, in other words, of the "nearest neighbor" structure of the model. This can be seen in the fact that $\mathbf{Q}_{(Q, B_\bullet)}^{(D)}$ is tridiagonal, see (2.1.2): it is a "nearest neighbor" operator, generated by Q with boundary conditions B_L and B_R .

Remark 2.2.2. Taking Hermitian transpose of (2.2.7) together with (2.2.6) we also get,

$$(Q_{LL} + Q_{RR})C_k + Q_{RL}C_{k+1} + Q_{LR}C_{k-1} = I_d \delta_{k,0}$$

for all $k \in \mathbb{Z}$.

2.2.2 Behavior on \mathbb{C}^*

For this whole subsection, $d \in \mathbb{N}^*$ and $Q \in H_{2d}^+(\mathbb{C})$ are fixed. Except if the contrary is mentioned, we also work under assumption I. The matrix Ψ is not invertible everywhere on \mathbb{C}^* . The following lemma describes the singularities of Ψ^{-1} .

Lemma 2.2.4. *Suppose assumption I fulfilled. The equation $\det \Psi(z) = 0$ for $z \in \mathbb{C}^*$ admits $2d$ solutions counted with multiplicity. We let*

$$\mathcal{S}(\Psi) = \{z \in \mathbb{C}^*; \det \Psi(z) = 0\}, \quad (2.2.8)$$

abusively called the set of zeros of Ψ . For all $w \in \mathcal{S}(\Psi)$, $1/\bar{w} \in \mathcal{S}(\Psi)$. Moreover, for all $z \in \mathbb{C}^$,*

$$\det \Psi(z) = \det(Q_{LR})z^{-d} \prod_{w \in \mathcal{S}(\Psi)} (z - w). \quad (2.2.9)$$

Proof. Remark that $z\Psi(z)$ is a matrix polynomial of degree 2. Coming back to the definition of determinant we get that, for all $z \in \mathbb{C}^*$,

$$z^d \det \Psi(z) = \det z\Psi(z)$$

so that

$$z^d \det \Psi(z) = \sum_{\sigma \in \mathfrak{S}_d} \epsilon(\sigma) \prod_{i=1}^d \left(Q_{LR}z^2 + (Q_{LL} + Q_{RR})z + Q_{RL} \right)_{i, \sigma(i)}$$

is a polynomial of degree $2d$ and 0 is not a root since Q_{RL} is invertible. Therefore, counted with multiplicity, the equation $\det \Psi(z) = 0$ admits $2d$ solutions on \mathbb{C}^* . It is clear that the coefficient in front of z^{2d} is $\det(Q_{LR})$ so we get (2.2.9). Remark that the constant term is $\det(Q_{RL})$. Finally, take $w \in \mathcal{S}(\Psi)$, the identity (2.2.3) gives $1/\bar{w} \in \mathcal{S}(\Psi)$. \square

A direct consequence of the latter proof is

$$\frac{\prod_{w \in \mathcal{S}_{<1}} w}{\det(Q_{RL})} = \frac{\prod_{w \in \mathcal{S}_{<1}} \bar{w}}{\det(Q_{LR})}. \quad (2.2.10)$$

Notation. Since Q is fixed, so is Ψ , and we will often simply write \mathcal{S} instead of $\mathcal{S}(\Psi)$. We can decompose $\mathcal{S} = \mathcal{S}_{<1} \sqcup \mathcal{S}_{>1}$ with

$$\mathcal{S}_{<1} = \{w \in \mathcal{S}; |w| < 1\} \text{ and } \mathcal{S}_{>1} = \{w \in \mathcal{S}; |w| > 1\} \quad (2.2.11)$$

zeros inside and outside the unit disk. We recall that there is no zero on the unit circle thanks to lemma 2.2.2.

Remark 2.2.3. Previous lemma tells that Ψ is not invertible on d points inside the unit disk and on d points outside counted with multiplicity. In particular, Ψ admits the following Birkhoff decomposition

$$\Psi = (\Psi^+)^* \Psi^+$$

with Ψ^+ continuous and invertible on the unit disk, see [CG13]. We will however use a more linear version, see theorem 2.2.1, involving bases indexed by zeros of Ψ inside and outside the unit disk.

Link with free energy. As an application of the previous lemma, and to highlight the role played by \mathcal{S} , we propose to show that the free energy computed with periodic boundary conditions $f^{\text{per}}(\alpha, Q)$ (see (2.1.8)) and the free energy computed with eigenboundary conditions $f^{\text{eigen}}(\alpha, Q)$ (see (2.1.14)) are equal and can be expressed in terms of zeros of Ψ . This is a first link between the two approaches. Nevertheless, to link the Fourier approach with the invariant matrices B_L and B_R we will need more than just the zeros of Ψ : we will need the bases indexed by zeros of Ψ , see definition 2.2.2 and theorems 2.1.4 and 2.5.1.

Lemma 2.2.5. *Let $\alpha > 0$. Under assumptions I and II, we have*

$$f^{\text{per}}(\alpha, Q) = f^{\text{eigen}}(\alpha, Q). \quad (2.2.12)$$

Proof. Using theorem 2.1.4 (proven in section 2.5) and equations (2.1.8) and (2.1.14), we only have to show that

$$-\frac{1}{2\pi} \int_0^{2\pi} \log \det \Psi(e^{i\theta}) d\theta = \log \left((-1)^d \frac{\prod_{w \in \mathcal{S}_{<1}} w}{\det(Q_{RL})} \right). \quad (2.2.13)$$

For all $\theta \in [0, 2\pi]$, using (2.2.9) we have,

$$\begin{aligned} \det \Psi(e^{i\theta}) &= \det(Q_{LR}) e^{-id\theta} \prod_{w \in \mathcal{S}_{<1}} (e^{i\theta} - w) \prod_{w \in \mathcal{S}_{<1}} (e^{i\theta} - \frac{1}{\bar{w}}) \\ &= (-1)^d \frac{\det(Q_{LR})}{\prod_{w \in \mathcal{S}_{<1}} \bar{w}} \prod_{w \in \mathcal{S}_{<1}} (1 - we^{-i\theta}) \prod_{w \in \mathcal{S}_{<1}} (1 - \bar{w}e^{i\theta}). \end{aligned}$$

In particular, since $\det \Psi(e^{i\theta}) > 0$ for all $\theta \in [0, 2\pi]$ by lemma 2.2.2, we have

$$(-1)^d \frac{\det(Q_{LR})}{\prod_{w \in \mathcal{S}_{<1}} \bar{w}} > 0.$$

Denote S^1 the unit circle, for all $u \in \mathbb{C}^*$ such that $|u| < 1$,

$$\int_0^{2\pi} \log(1 - ue^{i\theta}) d\theta = \int_{\mathcal{C}_1} \frac{1}{iz} \log(1 - uz) dz = 0.$$

Indeed, \log is the principal value, continuous onto $\mathbb{D}(1, |u|)$ the disk centered at 1 with radius $|u|$. Therefore, the function $\frac{1}{iz} \log(1 - uz)$ as a unique pole on $\mathbb{D}(0, 1)$, the disk centered at 0 with radius 1, and the associated residue is equal to 0. A change of variable also gives

$$\int_0^{2\pi} \log(1 - ue^{-i\theta}) d\theta = 0.$$

Take $u = w$ or $u = \bar{w}$ and the result follows from the fact that logarithm of product is the sum of logarithms up to a multiple of $2i\pi$ and from (2.2.10). \square

Symmetry property. We end this section by giving little results concerning symmetry properties of Ψ . This will be convenient for deducing theorem 2.1.4 from theorem 2.5.1 with a symmetric argument, see section 2.5. Letting I_d the d -dimensional identity matrix, $S = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}$ acts by conjugation on Q in the following way

$$SQS^{-1} = \begin{pmatrix} Q_{RR} & Q_{RL} \\ Q_{LR} & Q_{LL} \end{pmatrix}$$

which reverses direction of edges. For all $z \in \mathbb{C}^*$, we have

$$\Psi_{SQS^{-1}}(z) = \Psi_Q\left(\frac{1}{z}\right).$$

We deduce the following result.

Lemma 2.2.6. *Let $Q \in H_{2d}^+(\mathbb{C})$. For all $z \in \mathcal{S}(\Psi_Q)$, $\ker \Psi_{SQS^{-1}}(\bar{z}) = \ker \Psi_Q(1/\bar{z})$. Moreover, we have*

$$\mathcal{S}(\Psi_{SQS^{-1}}) = \overline{\mathcal{S}(\Psi_Q)}$$

where $\overline{\mathcal{S}(\Psi_Q)} = \{\bar{z}, z \in \mathcal{S}(\Psi_Q)\}$.

Proof. The first identity is obvious thanks to the previous remark. If $z \in \mathcal{S}(\Psi_Q)$ then $1/\bar{z} \in \mathcal{S}(\Psi_Q)$ and, thanks to the previous remark, $\bar{z} \in \mathcal{S}(\Psi_{SQS^{-1}})$. \square

2.2.3 The bases indexed by zeros of Ψ .

Once again, for this whole subsection $d \in \mathbb{N}^*$ and $Q \in H_{2d}^+(\mathbb{C})$ are fixed. We will now expose an important notion of this paper, the bases indexed by zeros of Ψ . These bases are involved in most of the interesting quantities of the model, especially when looking for left and right Schur-invariant boundaries, see theorems 2.1.4 and 2.5.1. Besides the fact that we work under assumption I, we also make the following one.

Assumption II. For any root w of the polynomial $z^d \det \Psi(z)$, studied in lemma 2.2.4, let $\text{mult}(w)$ be its multiplicity. To make things easier, we make the following assumption

$$\text{mult}(w) = \dim \ker \Psi(w) = 1 \tag{2.2.14}$$

for all $w \in \mathcal{S}$. In that case, $|\mathcal{S}| = 2d$ and has exactly d non zero elements inside the unit disk and d outside. For all $w \in \mathcal{S}$, we fix $u_w \neq 0$ such that

$$\ker \Psi(w) = \text{Vect}_{\mathbb{C}}(u_w). \tag{2.2.15}$$

Additional work might be needed for the general case. We only give an example where $\dim \ker \Psi(w) > 1$.

Example 1. Suppose $Q_{LL} = Q_{RR} = I_d$ and Q_{LR} Hermitian, thus diagonalized by a unitary matrix. Write $(a_i)_{i=1}^d$ its real eigenvalues. In that case, up to a change of basis, for all $z \in \mathbb{C}^*$, $\Psi(z)$ is diagonal with diagonal entries given by $\phi_i(z) = 1 + a_i z + a_i z^{-1}$ for $i \in \{1, \dots, d\}$. If any eigenvalue is of multiplicity greater than 1, the associated kernel is of dimension greater than 1.

Definition 2.2.2 (Bases indexed by zeros of Ψ). Under assumptions I and II, we prove in theorem 2.2.1 that the families $(u_w)_{w \in \mathcal{S}_{<1}}$ and $(u_w)_{w \in \mathcal{S}_{>1}}$ are bases of \mathbb{C}^d . They are called the bases indexed by zeros of Ψ , respectively, inside the unite disk and outside the unit disk.

We introduce the following notation.

Notation. Every $x \in \mathbb{C}^d$ can be decomposed in the basis $(u_w)_{w \in \mathcal{S}_{<1}}$, resp. $(u_w)_{w \in \mathcal{S}_{>1}}$, as

$$\sum_{w \in \mathcal{S}_{<1}} x(w) u_w \text{ resp. } \sum_{w \in \mathcal{S}_{>1}} x(w) u_w. \tag{2.2.16}$$

Before giving the main theorem of this section, we start by giving the following lemma, which is direct from (2.2.3).

Lemma 2.2.7. *Let $w \in \mathcal{S}$. Then, $u_w \in \ker \Psi(w)$ if and only if $u_w \in \operatorname{Im} \Psi(1/\bar{w})^\perp$.*

The next theorem decomposes Ψ^{-1} along the vectors indexed by zeros of Ψ which are proven to form bases of \mathbb{C}^d .

Theorem 2.2.1. *Assume I and II. For all $w \in \mathcal{S}$, $\langle u_{1/\bar{w}}, \Psi'(w)u_w \rangle \neq 0$. Then let $\alpha_w \in \mathbb{C}^*$ and $P_w \in M_d(\mathbb{C})$ defined as*

$$\alpha_w = \frac{1}{\langle u_{1/\bar{w}}, \Psi'(w)u_w \rangle} \text{ and } P_w x = \langle u_{1/\bar{w}}, x \rangle u_w$$

$w \in \mathcal{S}$ and for all $x \in \mathbb{C}^d$. For all $z \in \mathbb{C}^* \setminus \mathcal{S}$, we have

$$\Psi(z)^{-1} = \sum_{w \in \mathcal{S}} \frac{\alpha_w}{z - w} P_w \tag{2.2.17}$$

$$= \sum_{w \in \mathcal{S}_{<1}} \left(\frac{\alpha_w}{z - w} P_w + \frac{\alpha_{1/\bar{w}}}{z - 1/\bar{w}} P_{1/\bar{w}} \right) \tag{2.2.18}$$

Moreover, $(u_w)_{w \in \mathcal{S}_{<1}}$ and $(u_w)_{w \in \mathcal{S}_{>1}}$ are bases of \mathbb{C}^d .

Proof. The proof is based on residue calculus. First, recall

$$\Psi(z)^{-1} = \frac{1}{\det \Psi(z)} \operatorname{adj}(\Psi(z)) \tag{2.2.19}$$

for all $z \in \mathbb{C}^* \setminus \mathcal{S}$, where $\operatorname{adj}(\Psi(z))$ is the adjugate matrix of $\Psi(z)$. We know that $\det \Psi(z)$ is a linear combination of z^i for $i \in \{-d, \dots, d\}$ and the coefficients of order d and $(-d)$ are non zero under assumption I, due to lemma 2.2.4. Since $\operatorname{adj}(\Psi(z))$ is a linear combination of z^i with $i \in \{-d+1, \dots, d-1\}$, it comes

$$\lim_{|z| \rightarrow 0} \Psi(z)^{-1} = 0 \text{ and } \lim_{|z| \rightarrow \infty} \Psi(z)^{-1} = 0.$$

We let \mathcal{C}_R be the circle centered at origin of radius $R > 0$. Take R sufficiently large so that $\mathcal{S} \subset \mathbb{D}(0, R)$, the disk centered at origin of radius R . For any $z \in \mathbb{C}^* \setminus \mathcal{S}$, we have

$$\begin{aligned} \int_{\mathcal{C}_R} \Psi(u)^{-1} \frac{1}{u - z} du &= 2\pi i \left(\Psi(z)^{-1} + \sum_{w \in \mathcal{S} \cup \{0\}} \operatorname{Res}(\Psi^{-1}, w) \frac{1}{w - z} \right) \\ &= 2\pi i \left(\Psi(z)^{-1} + \sum_{w \in \mathcal{S}} \operatorname{Res}(\Psi^{-1}, w) \frac{1}{w - z} \right) \end{aligned}$$

where we used the limit at 0 previously computed. Moreover, using the limit of $\Psi(u)^{-1}$ when $|u| \rightarrow \infty$, we get

$$\int_{\mathcal{C}_R} \Psi(u)^{-1} \frac{1}{u - z} du = \operatorname{Res} \left(\Psi^{-1} \frac{1}{\bullet - z}, \infty \right) = 0.$$

We deduce,

$$\Psi(z)^{-1} = \sum_{w \in \mathcal{S}} \operatorname{Res}(\Psi^{-1}, w) \frac{1}{z - w}.$$

To get (2.2.17), we only have to show that, for all $w \in \mathcal{S}$,

$$\operatorname{Res}(\Psi^{-1}, w) = \lim_{u \rightarrow w} (u - w) \Psi(u)^{-1} = \alpha_w P_w.$$

Fix any $w \in \mathcal{S}$. We are therefore interested in a Taylor expansion of (2.2.19). We first have, for any $u \rightarrow w$,

$$\det \Psi(u) = (\det \Psi(w))'(u - w) + o(u - w) \quad (2.2.20)$$

and we check that $(\det(\Psi(w)))' \neq 0$ as w is assumed to be of multiplicity 1 by assumption II. We also have that, when $u \rightarrow w$,

$$\text{adj}(\Psi(u)) = \text{adj}(\Psi(w)) + (\text{adj}(\Psi(w)))'(u - w) + o(u - w). \quad (2.2.21)$$

We need to check that none of these terms is zero. We have, for any $z \in \mathbb{C}^*$,

$$\Psi(z)\text{adj}(\Psi(z)) = \det(\Psi(z))I_d, \text{ and } \text{adj}(\Psi(z))\Psi(z) = (\det \Psi(z))I_d. \quad (2.2.22)$$

It first gives

$$\Psi(w)\text{adj}(\Psi(w)) = \text{adj}(\Psi(w))\Psi(w) = 0.$$

So that, by assumption II, and lemma 2.2.7, $\text{adj}(\Psi(w))$ is of form $\text{adj}(\Psi(w)) = \beta_w P_w$ with $\beta_w \in \mathbb{C}$. Now, taking the derivative of (2.2.22) in w gives

$$\beta_w \Psi'(w)P_w + \Psi(w)(\text{adj}(\Psi(w)))' = (\det \Psi(w))'I_d.$$

This is the sum of two linear maps: the first one is of rank at most 1 and the second one of rank at most $d - 1$. It forces $\beta_w \neq 0$ and $(\text{adj}(\Psi(w)))' \neq 0$. Now, multiplying this identity on the left by $u_{1/\bar{w}}^*$ and using lemma 2.2.7, gives for all $x \in \mathbb{C}^d$,

$$\beta_w \langle u_{1/\bar{w}}, \Psi'(w)u_w \rangle \langle u_{1/\bar{w}}, x \rangle = (\det \Psi(w))' \langle u_{1/\bar{w}}, x \rangle$$

so that $\langle u_{1/\bar{w}}, \Psi'(w)u_w \rangle \neq 0$ and $\beta_w = \alpha_w (\det \Psi(w))'$. Combining (2.2.20) and (2.2.21) with (2.2.19), we end up with

$$\Psi(u)^{-1} = \frac{1}{u - w} \alpha_w P_w + \frac{1}{(\det \Psi(w))'} (\text{adj}(\Psi(w)))' + o(1)$$

as $u \rightarrow w$ and (2.2.17) is proved. The second expression (2.2.18) follows from a change of variable in the sum (2.2.17). We finally show that $(u_w)_{w \in \mathcal{S}_{<1}}$ is a basis. If $(u_w)_{w \in \mathcal{S}_{<1}}$ is not a basis, there exists $x \in (\text{Vect}((u_w)_{w \in \mathcal{S}_{<1}}))^\perp$ non-zero so that for all $\theta \in [0, 2\pi]$,

$$\langle x, \Psi(e^{i\theta})^{-1}x \rangle = \sum_{w \in \mathcal{S}_{<1}} \left(\frac{\alpha_w}{e^{i\theta} - w} \langle x, P_w x \rangle + \frac{\alpha_{1/\bar{w}}}{e^{i\theta} - 1/\bar{w}} \langle x, P_{1/\bar{w}} x \rangle \right) = 0$$

which is in contradiction with positive definiteness of $\Psi(e^{i\theta})^{-1}$. Then, $(u_w)_{w \in \mathcal{S}_{<1}}$ is a basis of \mathbb{C}^d . With similar arguments, so is $(u_w)_{w \in \mathcal{S}_{>1}}$. \square

We recall that the Fourier coefficients of Ψ^{-1} are denoted $(C_k)_{k \in \mathbb{Z}}$, see (2.2.5). An application of the latter lemma is the decomposition of these Fourier coefficients along the bases indexed by zeros of Ψ .

Corollary 2.2.1. *Under assumptions and notations of theorem 2.2.1, for all $k \geq 0$, we have*

$$C_k = \sum_{w \in \mathcal{S}_{>1}} -\frac{\alpha_w}{w^{k+1}} P_w \text{ and } C_{-k} = \sum_{w \in \mathcal{S}_{<1}} w^{k-1} \alpha_w P_w. \quad (2.2.23)$$

Proof. Let $k \geq 0$, and denote S^1 the complex unit circle. We have, from theorem 2.2.1,

$$C_{-k} = \frac{1}{2i\pi} \sum_{w \in \mathcal{S}} \alpha_w P_w \int_{C_1} \frac{z^{k-1}}{z - w} dz.$$

A little care for the case $k = 0$ and residues theorem give the result. The identity for C_k is deduced from using $C_k = C_{-k}^*$, see (2.2.6), together with the identity

$$\Psi'(w)w^2 = -\Psi'(1/\bar{w})^*$$

derived from (2.2.1) and using hermitian property of Q . \square

Remark 2.2.4. As mentioned in theorem 2.1.1, the Fourier coefficients $(C_k)_{k \in \mathbb{Z}}$ give the covariance matrix in thermodynamic limit. The latter corollary ensures that these covariances decrease exponentially.

Short conclusion. We introduced $(u_w)_{w \in \mathcal{S}_{<1}}$ and $(u_w)_{w \in \mathcal{S}_{>1}}$ two families that parametrize Ψ^{-1} and its Fourier coefficients $(C_k)_{k \in \mathbb{Z}}$. Other objects related to the model will be decomposed along these families in sections 2.5 and 2.6.

2.3 From periodic boundary conditions to translation invariant Gaussian Markov on \mathbb{Z}

Motivated by further comparisons with the eigen-boundary case, this section deals with basics results concerning the periodic case and structures behind the theorem 2.1.1. We will see that the matrices involved when considering periodic Gaussian Markov processes are block-circulant and therefore diagonal in the basis of Fourier. This allows an easy computation of the correlation function in the thermodynamic limit. Moreover, this gives a natural candidate for the Hilbert space which carries the infinite dimensional Gaussian process on \mathbb{Z} which, as it will be shown, verifies the Markov property.

2.3.1 Circulant and multiplication operators.

We start by recalling the definitions of block-circulant and multiplication operators. Let $d, P \in \mathbb{N}^*$. In the following, we will be considering the Hilbert spaces $(\mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{C}^d, \langle \cdot, \cdot \rangle_{\mathbb{C}^{dP}})$ and $(l^2(\mathbb{Z} \rightarrow \mathbb{C}^d), \langle \cdot, \cdot \rangle_{l^2})$ with $\langle \cdot, \cdot \rangle_{\mathbb{C}^{dP}}$ and $\langle \cdot, \cdot \rangle_{l^2}$ usual inner products. In these contexts, a dP -dimensional block-matrix B (resp. an infinite dimensional block operator B) with d -dimensional blocks is

$$B = (B_{k,l})_{k,l \in \{0, \dots, P-1\}} \text{ (resp. } B = (B_{k,l})_{k,l \in \mathbb{Z}})$$

with $B_{k,l} \in M_d(\mathbb{C})$ for all k and l .

Circulant and multiplication operators on $\mathbb{Z}/P\mathbb{Z}$. Let $\Phi : S^1 \rightarrow M_d(\mathbb{C})$ continuous. Its discrete Fourier coefficients are defined in the following way

$$C_k^{(P)}(\Phi) = \frac{1}{\sqrt{P}} \sum_{i=0}^{P-1} \Phi(\omega_P^i) \omega_P^{-ki} \in M_d(\mathbb{C}) \quad (2.3.1)$$

for all $k \in \{0, \dots, P-1\}$. This is, of course, the discrete version of the Fourier coefficients introduced in (2.2.4). We denote $C^{(P)}(\Phi) = (C_{l-k}^{(P)}(\Phi))_{k,l \in \{0, \dots, P-1\}} \in M_{dP}(\mathbb{C})$ the block-circulant matrix generated by discrete Fourier coefficients of Φ . Such a matrix is block-diagonalizable via discrete Fourier transform into the multiplication operator on $\mathbb{Z}/P\mathbb{Z}$ generated by Φ :

$$\begin{aligned} M^{(P)}(\Phi) : (\mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{C}^d) &\rightarrow (\mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{C}^d) \\ (x_k)_{k=0}^{P-1} &\mapsto \left(\Phi(\omega_P^k) x_k \right)_{k=0}^{P-1}. \end{aligned}$$

Circulant and multiplication operators on \mathbb{Z} . Let $\Phi : S^1 \rightarrow M_d(\mathbb{C})$ continuous and for all $k \in \mathbb{Z}$, $C_k(\Phi) \in M_d(\mathbb{C})$ its k -th Fourier coefficient already defined in (2.2.4). We denote $C(\Phi) = (C_{l-k}(\Phi))_{k,l \in \mathbb{Z}}$ the infinite dimensional block-circulant operator acting on $l^2(\mathbb{Z} \rightarrow \mathbb{C}^d)$. This operator, via Fourier correspondence between $l^2(\mathbb{Z} \rightarrow \mathbb{C}^d)$ and $L^2(S^1 \rightarrow \mathbb{C}^d)$, is linked to

$$M(\Phi) : L^2(S^1 \rightarrow \mathbb{C}^d) \rightarrow L^2(S^1 \rightarrow \mathbb{C}^d) \\ f \mapsto \Phi f$$

the multiplication operator on $L^2(S^1 \rightarrow \mathbb{C}^d)$ generated by Φ .

2.3.2 Gaussian Markov processes on $\mathbb{Z}/P\mathbb{Z}$

When working with periodic boundary conditions, the matrix appearing in the density of the Gaussian Markov process is (tridiagonal) block-circulant and is therefore block-diagonalizable using discrete Fourier transform. We first briefly summarize this fact.

Let $Q \in H_{2d}^+(\mathbb{C})$ fixed and $\Psi := \Psi_Q$, see section 2.2. A Gaussian Markov process, on the periodic lattice $\mathbb{Z}/P\mathbb{Z}$, $X^{(P),\text{per}} = (X_k^{(P),\text{per}})_{k=0}^{P-1}$ has density proportional to

$$g_{X^{(P),\text{per}}}(x^{(P)}) \propto e^{-\frac{1}{2}(x^{(P)})^* Q_{Q,\text{per}}^{(P)} x^{(P)}} \quad (2.3.2)$$

for all $x^{(P)} = (x_k^{(P)})_{k=0}^{P-1} \in \mathbb{C}^{dP}$ where $Q_{Q,\text{per}}^{(P)} = C^{(P)}(\Psi)$ the dP -dimensional block-circulant matrix generated by Ψ :

$$Q_{Q,\text{per}}^{(P)} = \begin{pmatrix} T & Q_{LR} & 0 & 0 & \dots & 0 & Q_{RL} \\ Q_{RL} & T & Q_{LR} & 0 & \dots & 0 & 0 \\ 0 & Q_{RL} & T & Q_{LR} & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \ddots & T & Q_{LR} & 0 \\ Q_{LR} & 0 & \dots & 0 & 0 & Q_{RL} & T \end{pmatrix}, \quad (2.3.3)$$

with $T = Q_{LL} + Q_{RR}$. Recalling

$$\widehat{X}_k^{(P)} = \frac{1}{\sqrt{P}} \sum_{i \in \mathbb{Z}/P\mathbb{Z}} X_i^{(P),\text{per}} \omega_P^{ik},$$

we recover the result of the theorem 2.1.1, which states that $\widehat{X}^{(P)} = (\widehat{X}_k^{(P)})_{k=0}^{P-1}$ has density proportional to

$$g_{\widehat{X}^{(P)}}(x^{(P)}) \propto e^{-\frac{1}{2}(x^{(P)})^* M^{(P)}(\Psi) x^{(P)}}.$$

for all $x^{(P)} = (x_k^{(P)})_{k=0}^{P-1} \in \mathbb{C}^{dP}$. We end up with the following lemma.

Lemma 2.3.1. *The following assertions are equivalent*

- i) $X^{(P),\text{per}}$ is a centered Gaussian Markov process on $\mathbb{Z}/P\mathbb{Z}$ of weight $e_{\alpha,Q}$.
- ii) $X^{(P),\text{per}}$ is a centered Gaussian process on $H^{(P)} = (\mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{C}^d, \langle \cdot, C^{(P)}(\Psi^{-1}) \cdot \rangle_{\mathbb{C}^{dP}})$. In other words, for any $E \subset \mathbb{Z}/P\mathbb{Z}$ and $(\alpha_k)_{k \in E} \subset \mathbb{C}^d$ -valued sequence, we have

$$\mathbb{E} \left[\exp \left(i \sum_{k \in E} \text{Re} \langle \alpha_k, X_k^{(P),\text{per}} \rangle_{\mathbb{C}^d} \right) \right] = \exp \left(-\frac{1}{2} \sum_{(k,l) \in E^2} \langle \alpha_k, C_{l-k}^{(P)}(\Psi^{-1}) \alpha_l \rangle_{\mathbb{C}^d} \right).$$

iii) $\widehat{X}^{(P)}$ is a centered Gaussian process on $\widehat{H}^{(P)} = (\mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{C}^d, \langle \cdot, M^{(P)}(\Psi^{-1}) \cdot \rangle_{\mathbb{C}^{dP}})$.

Covariance matrices of $X^{(P),\text{per}}$ and $\widehat{X}^{(P)}$ and their inverses are summarized in the following commutative diagram

$$\begin{array}{ccc} \mathbf{Q}_{Q,\text{per}}^{(P)} & \xleftarrow{-1} & C^{(P)}(\Psi^{-1}) \\ \uparrow \text{DFT} & & \uparrow \text{DFT} \\ M^{(P)}(\Psi) & \xleftarrow{-1} & M^{(P)}(\Psi^{-1}) \end{array} \quad (2.3.4)$$

where "DFT" stands for Discrete Fourier Transform. Upper line (resp. lower) corresponds to $X^{(P),\text{per}}$ (resp. $\widehat{X}^{(P)}$) and left column (resp. right) corresponds to inverse of covariance matrices (resp. covariance matrices).

Remark 2.3.1. We insist that, for any P there exists such a diagram and Hilbert spaces, $H^{(P)}$ and $\widehat{H}^{(P)}$. However, none of them are connected either by any projection nor any restriction. See (2.5.12) for a similar discussion under eigen-boundary conditions instead of periodic boundary conditions.

2.3.3 Thermodynamic limit

Now that the case $\mathbb{Z}/P\mathbb{Z}$ is understood, we compute the marginal laws as $P \rightarrow \infty$ and discuss the Hilbert space obtained at the limit.

Convergence in law for finite dimensional marginal laws. We introduce

$$\begin{aligned} & (\mathbb{Z} \rightarrow \mathbb{C}^d) \rightarrow (\mathbb{Z}/P\mathbb{Z} \rightarrow \mathbb{C}^d) \\ & (\alpha_k)_{k \in \mathbb{Z}} \mapsto (\alpha_k^{(P)})_{k=0}^{P-1} = \begin{cases} \alpha_k^{(P)} = \alpha_k & \text{if } k \in [0, P/3[\\ \alpha_{P-k}^{(P)} = \alpha_{-k} & \text{if } k \in [1, P/3[\\ \alpha_k^{(P)} = 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

and, with abuse of notation, we let $X_\alpha^{(P),\text{per}} = X_{\alpha^{(P)}}^{(P),\text{per}} := \sum_{k=0}^{P-1} \langle \alpha_k^{(P)}, X_k^{(P),\text{per}} \rangle$ for all $(\alpha_k)_{k \in \mathbb{Z}} \mathbb{C}^d$ -valued sequence. Let $(\alpha_k)_{k \in \mathbb{Z}}$ a \mathbb{C}^d -valued sequence with bounded support associated to a sequence $(\alpha_k^{(P)})_{k=0}^{P-1}$. Lemma 2.3.1 gives

$$\lim_{P \rightarrow \infty} \mathbb{E} \left[\exp \left(\mathbf{i} \text{Re} X_\alpha^{(P),\text{per}} \right) \right] = \exp \left(-\frac{1}{2} \sum_{(k,l) \in \mathbb{Z}^2} \langle \alpha_k, C_{l-k}(\Psi^{-1}) \alpha_l \rangle \right). \quad (2.3.5)$$

Law of n consecutive vertices. From (2.3.5), we check that any $n \in \mathbb{N}^*$ consecutive vertices converge in law to a centered Gaussian random variable with block-Toeplitz covariance matrix $\Sigma^{(n-1)} = (C_{l-k}(\Psi^{-1}))_{k,l \in \{1, \dots, n\}}$. The Toeplitz property ensures that the limit marginal laws are translation invariant.

Remark 2.3.2. This convergence in law highlights the importance of lemma 2.2.2. Fourier coefficients of Ψ^{-1} are well defined because Ψ is positive definite on the unit circle. This guarantees that we can compute many quantities of interest using periodic boundary conditions.

Infinite volume Gibbs measure. We now give the infinite volume Gibbs measure associated to the Gaussian Markov process we constructed through thermodynamic limit. We let

$$H = L^2(S^1 \rightarrow \mathbb{C}^d)$$

the Hilbert space endowed with the following norm, for any $f \in H$,

$$\|f\|^2 = \langle f, M(\Psi^{-1})f \rangle_{L^2(S^1 \rightarrow \mathbb{C}^d)} \quad (2.3.6)$$

We introduce a Gaussian process X as an isometry from H to a probability space:

$$X : H \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Let $(e_i)_{i=1}^d$ be the canonical basis of \mathbb{C}^d . This defines a translation invariant Gaussian process on \mathbb{Z} through

$$X_k = (X_{k,i})_{i=1}^d = (X(e^{ik\bullet} e_i))_{i=1}^d \quad (2.3.7)$$

for all $k \in \mathbb{Z}$, where $e^{ik\bullet} : \theta \mapsto e^{ik\theta}$ for all $\theta \in [0, 2\pi]$. A direct computation shows that the process $(X_k)_{k \in \mathbb{Z}}$ has the same marginal laws as the ones of the limit Gaussian Markov process computed in (2.3.5). This gives the infinite volume Gibbs measure associated to the model. Remark that, due to infinite dimension, we are now forced to abandon densities and work with the (infinite dimensional) covariance matrix.

As for P finite, the following operators are involved

$$\begin{array}{ccc} \mathbf{Q} & \xleftarrow{-1} & C(\Psi^{-1}) \\ \uparrow \text{FT} & & \downarrow \text{FT} \\ M(\Psi) & \xleftarrow{-1} & M(\Psi^{-1}) \end{array}$$

where "FT" stands for Fourier Transform and $\mathbf{Q} = C(\Psi)$ the block-circulant operator generated by Ψ , see subsection 2.3.1. Of course, as probabilists, only right part is interesting since it gives covariance matrices. Nevertheless, tridiagonality of \mathbf{Q} is reminiscent of our construction of a "nearest neighbor" model and, as we saw in section 2.2, it leads to a second order recursive relation on Fourier coefficients of Ψ^{-1} . Therefore, we expect the Gaussian process X to satisfy the Markov property. Indeed, this can be proved either by showing that the thermodynamic limit preserves the Markov property or by working directly on the covariance matrix of X . This is the purpose of the following theorem.

Theorem 2.3.1 (Markov property). *The Gaussian process $(X_k)_{k \in \mathbb{Z}}$ defined in (2.3.7) is Markov.*

Proof. Since, for Gaussian random vectors, independence is equivalent to covariance being equal to 0, we only have to show that, for all $k < m < l$ we have

$$\text{Cov}_{\text{Gibbs}}(X_k, X_l | X_m) = 0. \quad (2.3.8)$$

In other words, X_k and X_l are independent given X_m . From invariant by translation, we can restrict ourselves to the case $m = 0$. From lemma 2.6.1, we get

$$\text{Cov}_{\text{Gibbs}}(X_k, X_l | X_0) = C_{l-k} - C_{-k} C_0^{-1} C_l$$

which is 0 from (2.5.8). □

Remark 2.3.3. As a Markov process, X must admit invariant boundary weights. They are determined by theorem 2.1.4 and 2.5.1, see lemma 2.5.6.

Remark 2.3.4. From a Gaussian weight $e_{\alpha,Q}$, and thus from the first-degree trigonometric polynomial Ψ_Q , we constructed a translation invariant Gaussian Markov process on \mathbb{Z} . Conversely, take a translation invariant Gaussian process with covariance matrix given by a multiplication operator $M(\tilde{\Psi}^{-1})$ with some function $\tilde{\Psi} : S^1 \rightarrow H_d^+(\mathbb{C})$ continuous. If we assume this process to be Markov, we can show that $\tilde{\Psi}$ is a trigonometric polynomial of order 1. Indeed, Markov property (2.3.8) implies that Fourier coefficients of $\tilde{\Psi}^{-1}$ follow a second order recursive relation similar to (2.2.7) whose coefficients are Fourier coefficients of $\tilde{\Psi}$.

2.4 Markov and Schur complement: algebraic structure

In this section we give standard algebraic properties for Gaussian weights and Gaussian boundary densities.

2.4.1 Gaussian weights

Definition 2.4.1 (Gaussian weights). Let $d \in \mathbb{N}^*$. We say that $e_{\alpha,Q} : \mathbb{C}^d \times \mathbb{C}^d \rightarrow \mathbb{R}_+$ is a Gaussian weight on \mathbb{C}^d if for all $x, y \in \mathbb{C}^d$,

$$e_{\alpha,Q}(x, y) = \alpha \exp\left(-\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^* Q \begin{pmatrix} x \\ y \end{pmatrix}\right) \quad (2.4.1)$$

for $Q \in H_{2d}^+(\mathbb{C})$ and $\alpha \in \mathbb{R}_+^*$. We denote $\text{Gauss}(\mathbb{C}^d)$ the set of all Gaussian weights on \mathbb{C}^d and $\mathcal{Q}_{\text{Gauss}}(\mathbb{C}^d)$ the \mathbb{C} -vector space spanned by elements of $\text{Gauss}(\mathbb{C}^d)$.

We use the terminology *weight* rather than *transition kernel* because we consider *unoriented* Markov Chains.

Remark 2.4.1. Gaussian weights correspond to densities of centered non-degenerated circularly-symmetric complex normal random variables.

Notation. Since the Gaussian weight $e_{\alpha,Q}$ is entirely determined by α and Q , we might simply write (α, Q) . The identification map is given by the following invertible map

$$\begin{aligned} \iota : \text{Gauss}(\mathbb{C}^d) &\rightarrow \mathbb{R}_+ \times H_{2d}^+(\mathbb{C}^d) \\ e_{\alpha,Q} &\mapsto (\alpha, Q). \end{aligned}$$

Recall that, for any $2d$ -dimensional square matrix Q we introduce its writing with d -dimensional block matrix, that is,

$$Q = \begin{pmatrix} Q_{LL} & Q_{LR} \\ Q_{RL} & Q_{RR} \end{pmatrix}$$

with Q_{IJ} ($d \times d$)-square matrices for all $I, J \in \{L, R\}$. Gaussian weights and related partition functions come with the following product.

Property 2.4.1. We define $m : \text{Gauss}(\mathbb{C}^d) \times \text{Gauss}(\mathbb{C}^d) \rightarrow \text{Gauss}(\mathbb{C}^d)$ the product of Gaussian weights as

$$m(e_{\alpha,Q}, e_{\tilde{\alpha},\tilde{Q}})(x, z) = \int_{\mathbb{C}^d} e_{\alpha,Q}(x, y) e_{\tilde{\alpha},\tilde{Q}}(y, z) dy$$

for $e_{\alpha,Q}, e_{\tilde{\alpha},\tilde{Q}} \in \text{Gauss}(\mathbb{C}^d)$ and for all $x, z \in \mathbb{C}^d$. We have,

$$m(e_{\alpha,Q}, e_{\beta,\tilde{Q}}) = e_{(2\pi)^d \alpha \tilde{\alpha} \det K, S(Q, \tilde{Q})}$$

where

$$S(Q, \tilde{Q}) = \begin{pmatrix} Q_{LL} - Q_{LR}KQ_{RL} & -Q_{LR}K\tilde{Q}_{LR} \\ -\tilde{Q}_{RL}KQ_{RL} & \tilde{Q}_{RR} - \tilde{Q}_{RL}K\tilde{Q}_{LR} \end{pmatrix}$$

and $K = (Q_{RR} + \tilde{Q}_{LL})^{-1}$. Moreover m is associative and, extended to $\mathcal{Q}_{\text{Gauss}}(\mathbb{C}^d)$ by linearity of integral, $(\mathcal{Q}_{\text{Gauss}}(\mathbb{C}^d), m)$ is an associative algebra.

Remark 2.4.2. We have $S(Q, \tilde{Q}) = M(Q, \tilde{Q})/K$ the Schur complement of the block K of the matrix

$$M(Q, \tilde{Q}) = \begin{pmatrix} Q_{LL} & 0 & Q_{LR} \\ 0 & \tilde{Q}_{RR} & \tilde{Q}_{RL} \\ Q_{RL} & \tilde{Q}_{LR} & K \end{pmatrix}.$$

Proof. Gaussian calculus. Associative property is given by Fubini's theorem. We only need to show that $S(Q, \tilde{Q}) \in H_{2d}^+(\mathbb{C})$. Hermitian property is immediate. Let's show positive definiteness. Let $p, q \in \mathbb{N}^*$ and $M \in H_{p+q}^+(\mathbb{C})$ with block decomposition

$$M = \begin{pmatrix} C & D \\ D^* & E \end{pmatrix}.$$

Then, for all non zero $x \in \mathbb{C}^d$,

$$x^*(M/E)x = x^*(C - DE^{-1}D^*)x = \begin{pmatrix} x \\ -E^{-1}D^*x \end{pmatrix}^* M \begin{pmatrix} x \\ -E^{-1}D^*x \end{pmatrix} > 0.$$

Take $M = M(Q, \tilde{Q})$. It is clear that $M \in H_{3d}^+(\mathbb{C})$ since for all $x, y, z \in \mathbb{C}^d$

$$\begin{pmatrix} x \\ z \\ y \end{pmatrix}^* M \begin{pmatrix} x \\ z \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}^* Q \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y \\ z \end{pmatrix}^* \tilde{Q} \begin{pmatrix} y \\ z \end{pmatrix} > 0.$$

Then, it comes $S(Q, \tilde{Q}) = M(Q, \tilde{Q})/K$ is positive definite. \square

Remark 2.4.3. There is no unit element for m in $\text{Gauss}(\mathbb{C}^d)$. If we wanted one, we would need to add degenerate Gaussians to our definition.

Property 2.4.2. Product $S : H_{2d}^+(\mathbb{C}) \rightarrow H_{2d}^+(\mathbb{C})$ makes $(H_{2d}^+(\mathbb{C}), S)$ a semi-group. Moreover, letting $m_S((\alpha, Q), (\tilde{\alpha}, \tilde{Q})) = (\alpha\tilde{\alpha} \det(Q_{RR} + \tilde{Q}_{LL})^{-1}, S(Q, \tilde{Q}))$, for all $(\alpha, Q), (\tilde{\alpha}, \tilde{Q}) \in \mathbb{R}_+ \times H_{2d}^+(\mathbb{C})$, the following diagram

$$\begin{array}{ccc} \text{Gauss}(\mathbb{C}^d) \times \text{Gauss}(\mathbb{C}^d) & \xrightarrow{m} & \text{Gauss}(\mathbb{C}^d) \\ \downarrow \iota \times \iota & & \downarrow \iota \\ (\mathbb{R}_+ \times H_{2d}^+(\mathbb{C})) \times (\mathbb{R}_+ \times H_{2d}^+(\mathbb{C})) & \xrightarrow{m_S} & \mathbb{R}_+ \times H_{2d}^+(\mathbb{C}) \end{array}$$

commutes.

Proof. Straightforward. \square

Remark 2.4.4. The product m is bilinear at the level of Gaussian weights but the product $S(Q, \tilde{Q})$ is not bilinear in Q and \tilde{Q} .

Schur power of a matrix. Let $Q \in H_{2d}^+(\mathbb{C})$, for any $n \in \mathbb{N}^*$ we define

$$Q^{[n]} = S(Q, S(Q, \dots, S(Q, Q))) \in H_{2d}^+(\mathbb{C}) \quad (2.4.2)$$

where the product S appears $n-1$ times. By associativity of S , this definition is unambiguous.

Operator point of view and symmetry. We remark that $e_{\alpha,Q}$ defines the kernel of the operator $e_{\alpha,Q}$ which naturally acts on $L^2(\mathbb{C}^d, dx)$ through

$$e_{\alpha,Q} : L^2(\mathbb{C}^d, dx) \rightarrow L^2(\mathbb{C}^d, dx)$$

$$f \mapsto \int_{\mathbb{C}^d} e_{\alpha,Q}(\bullet, y) f(y) dy.$$

Denote $e_{\alpha,Q}^*$ the adjoint operator of $e_{\alpha,Q}$ defined through the identity

$$\langle g, e_{\alpha,Q} f \rangle_{L^2(\mathbb{C}^d, dx)} = \langle e_{\alpha,Q}^* g, f \rangle_{L^2(\mathbb{C}^d, dx)}.$$

Since $\overline{e_Q(y, x)} = e_{\alpha,Q}(y, x) = e_{\alpha, SQS^{-1}}(x, y)$ with $S = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}$, we have

$$e_{\alpha,Q}^* = e_{\alpha, SQS^{-1}}.$$

2.4.2 Gaussian boundary weights

Definition 2.4.2 (Gaussian boundary weight/density). Let $d \in \mathbb{N}^*$. We say that $e'_{\beta,B} : \mathbb{C}^d \rightarrow \mathbb{R}_+$ is a Gaussian boundary weight/density on \mathbb{C}^d if for all $x \in \mathbb{C}^d$,

$$e'_{\beta,B}(x) = \beta e^{-\frac{1}{2}x^* B x} \quad (2.4.3)$$

for $B \in H_d^+(\mathbb{C})$ and $\beta \in \mathbb{R}_+^*$. We denote $\text{Gauss}^{\text{bc}}(\mathbb{C}^d)$ the set of all Gaussian densities on \mathbb{C}^d and $\mathcal{Q}_{\text{Gauss}^{\text{bc}}}(\mathbb{C}^d)$ the \mathbb{C} -vector space spanned by elements of $\text{Gauss}^{\text{bc}}(\mathbb{C}^d)$.

Remark 2.4.5. As for Gaussian weights, a Gaussian boundary weight $e'_{\beta,B}$ is uniquely determined by β and B so it might simply be written (β, B) .

We also remark that Gaussian weights naturally act on Gaussian boundary densities. This is the purpose of the following property.

Property 2.4.3. We define $m_L : \text{Gauss}^{\text{bc}}(\mathbb{C}^d) \times \text{Gauss}(\mathbb{C}^d) \rightarrow \text{Gauss}^{\text{bc}}(\mathbb{C}^d)$ as

$$m_L(e'_{\beta,B}, e_{\alpha,Q})(y) = \int_{\mathbb{C}^d} e'_{\beta,B}(x) e_{\alpha,Q}(x, y) dx$$

for all $e'_{\beta,B} \in \text{Gauss}^{\text{bc}}(\mathbb{C}^d)$, $e_{\alpha,Q} \in \text{Gauss}(\mathbb{C}^d)$ and all $y \in \mathbb{C}^d$. We have,

$$m_L(e'_{\beta,B}, e_{\alpha,Q}) = e_{(2\pi)^d \alpha \beta \det(B+Q_{LL})^{-1}, S_L(B, Q)}$$

where

$$S_L(B, Q) = Q_{RR} - Q_{RL}(B + Q_{LL})^{-1} Q_{LR}.$$

Moreover, m_L and m verify the following associativity property

$$m_L(m_L(\cdot, \cdot), id) = m_L(id, m(\cdot, \cdot))$$

Similarly we define $m_R : \text{Gauss}(\mathbb{C}^d) \times \text{Gauss}^{\text{bc}}(\mathbb{C}^d) \rightarrow \text{Gauss}^{\text{bc}}(\mathbb{C}^d)$ as

$$m_R(e_{\alpha,Q}, e'_{\beta,B})(x) = \int_{\mathbb{C}^d} e_{\alpha,Q}(x, y) e'_{\beta,B}(y) dy$$

for all $e_{\alpha,Q} \in \text{Gauss}(\mathbb{C}^d)$, $e'_{\beta,B} \in \text{Gauss}^{\text{bc}}(\mathbb{C}^d)$ and all $x \in \mathbb{C}^d$. We have,

$$m_R(e'_{\beta,B}, e_{\alpha,Q}) = e_{(2\pi)^d \alpha \beta \det(Q_{RR}+B)^{-1}, S_R(Q, B)}$$

where

$$S_R(Q, B) = Q_{LL} - Q_{LR}(Q_{RR} + B)^{-1} Q_{RL}.$$

Moreover, m_R and m verify the following associativity property

$$m_R(id, m_R(\cdot, \cdot)) = m_R(m(\cdot, \cdot), id).$$

Property 2.4.4. We define $m_{LR} : \text{Gauss}^{bc}(\mathbb{C}^d) \times \text{Gauss}^{bc}(\mathbb{C}^d) \rightarrow \mathbb{R}_+$ as

$$m_{LR}(e'_{\beta,B}, e'_{\tilde{\beta},\tilde{B}}) = \int_{\mathbb{C}^d} e'_{\beta,B}(x) e'_{\tilde{\beta},\tilde{B}}(x) dx = (2\pi)^d \beta \tilde{\beta} \det(B + \tilde{B})^{-1}$$

for all $e'_{\beta,B}$ and $e'_{\tilde{\beta},\tilde{B}} \in \text{Gauss}^{bc}(\mathbb{C}^d)$. Moreover, m_{LR} , m_L and m_R satisfy the following associativity rule

$$m_{LR}(id, m_R(\cdot, \cdot)) = m_{LR}(m_L(\cdot, \cdot), id).$$

Remark 2.4.6. To be even more precise, $\mathcal{Q}_{\text{Gauss}}^{\text{ext}}(\mathbb{C}^d) = (\mathcal{Q}_p)_{p \in \{\mathbb{N}^*, \infty_L, \infty_R, \infty_{LR}\}}$ with

$$\mathcal{Q}_p = \mathcal{Q}_{\text{Gauss}}(\mathbb{C}^d) \quad \forall p \in \mathbb{N}^*, \quad \mathcal{Q}_{\infty_L} = \mathcal{Q}_{\infty_R} = \mathcal{Q}_{\text{Gauss}^{bc}}(\mathbb{C}^d) \quad \text{and} \quad \mathcal{Q}_{\infty_{LR}} = \mathbb{R}_+.$$

together with m , m_L , m_R and m_{LR} is an algebra over the extended Guill_1 operad, see [Sim23]. It is not important as we stick to one dimensional Gaussian Markov processes but it will be relevant for chapter 3 dealing with higher dimensional Gaussian Markov processes.

Notation. Except if there is a possible confusion, we will write m , m_L and m_R with the symbol ".". For instance,

$$m_L(e'_{\beta,B}, e_{\alpha,Q}) = e'_{\beta,B} \cdot e_{\alpha,Q}.$$

2.4.3 Partition function

We now link the Gaussian weights and their associated partition function. Let D be a connected component of size $l(D) = P \in \mathbb{N}^*$ and $X^{(D)}$ a Gaussian Markov process on D of weights $(\alpha_k, Q_k)_{k=0}^{P-1}$, with boundary conditions (β_L, B_L) and (β_R, B_R) . We define

$$Z_D^{\text{int}}(\alpha_{\bullet}, Q_{\bullet}, x_0, x_P) = \int_{\mathbb{C}^{d(P-1)}} \prod_{k=0}^{P-1} e_{\alpha_k, Q_k}(x_k, x_{k+1}) dx_1 \dots dx_{P-1}.$$

Property 2.4.5. The function $Z_D^{\text{int}}(\alpha_{\bullet}, Q_{\bullet}) : (x, y) \mapsto Z_D^{\text{int}}(\alpha_{\bullet}, Q_{\bullet}, x, y)$ is an element of $\text{Gauss}(\mathbb{C}^d)$ and we have

$$Z_D^{\text{int}}(\alpha_{\bullet}, Q_{\bullet}) = e_{\alpha_0, Q_0} \cdot e_{\alpha_2, Q_2} \cdots e_{\alpha_{P-2}, Q_{P-2}} \cdot e_{\alpha_{P-1}, Q_{P-1}}$$

and

$$Z_D^{\text{int}}(\alpha_{\bullet}, Q_{\bullet}) = m(Z_{D_1}^{\text{int}}((\alpha_i)_{i=0}^{P_1-1}, (Q_i)_{i=0}^{P_1-1}), Z_{D_2}^{\text{int}}((\alpha_i)_{i=P_1}^{P_1+P_2-1}, (Q_i)_{i=P_1-1}^{P_1+P_2-1}))$$

for all D_1, D_2 connected components of sizes $l(D_1) = P_1$ and $l(D_2) = P_2$ which form a partition of D . Finally,

$$Z_D(\alpha_{\bullet}, Q_{\bullet}, \beta_{\bullet}, B_{\bullet}) = e'_{\beta_L, B_L} e_{\alpha_0, Q_0} \cdots e_{\alpha_{P-1}, Q_{P-1}} e'_{\beta_R, B_R}.$$

Proof. Under the latter notation, direct from definition of m , m_L and m_R and associative property (implied by Fubini's theorem). \square

Remark 2.4.7. Remark that, for homogeneous Gaussian Markov processes of weight (α, Q) ,

$$Z_D^{\text{int}}(\alpha, Q) = e_{\alpha^n, Q^{[n]}}. \quad (2.4.4)$$

2.5 Eigen-boundary conditions

In theorem 2.1.2 we gave a fixed point condition for matrices B_L and B_R to define eigen-boundary conditions. In this section, we will first prove theorems 2.1.4 and 2.5.1 which give an explicit recipe to recover such matrices in terms of zeros of Ψ_Q and their associated bases. Then, we will show that we recover the marginal laws computed in section 2.3 via periodic boundary conditions. Finally, as an application of theorems 2.1.4 and 2.5.1, we will link zeros of Ψ_Q with zeros of $\Psi_{Q^{[n]}}$ the n -th Schur power of Q , see (2.4.2).

We recall that we denote \mathcal{S} the set of zeros of Ψ_Q . The set \mathcal{S} is partitioned in $\mathcal{S}_{<1}$ and $\mathcal{S}_{>1}$ the zeros inside and outside the unit disk, see lemma 2.2.4. Moreover, to any of these sets is associated a basis, denoted respectively $(u_w)_{w \in \mathcal{S}_{<1}}$ and $(u_w)_{w \in \mathcal{S}_{>1}}$, see definition 2.2.2. This is, of course, under assumptions I and II.

2.5.1 Proof of Theorem 2.1.4

This subsection is devoted to the proofs of theorem 2.1.4 and of the following analogous theorem for right eigen-boundary condition.

Theorem 2.5.1. *Let $Q \in H_{2d}^+(\mathbb{C})$ and $W_{<1} \in M_d(\mathbb{C})$ any diagonalizable matrix. Under assumptions and I and II, the following assertions are equivalent*

- (i) *The matrix $B_R = Q_{LL} + Q_{LR}W_{<1}$ is right Schur-invariant for Q .*
- (ii) *The matrix $W_{<1}$ is diagonal in the basis $(u_w)_{w \in \mathcal{S}_{<1}}$ indexed by zeros of Ψ_Q , with $W_{<1}u_w = wu_w$ for all $w \in \mathcal{S}_{<1}$.*
- (iii) *Let $\mathbf{x} \in \mathbb{C}^d$. The sequence $(x_k)_{k=0}^{+\infty}$, with $x_k = W_{<1}^k x_0 \in \mathbb{C}^d$ for all $k \in \mathbb{N}^*$, is solution to the discrete Dirichlet-type problem*

$$\begin{cases} A_{RL}x_{k-1} + (A_{LL} + A_{RR})x_k + A_{LR}x_{k+1} = 0 \text{ for all } k > 0 \\ x_0 = \mathbf{x} \\ \lim_{k \rightarrow +\infty} x_k = 0 \end{cases} \quad (2.5.1)$$

Moreover, in these cases, the eigenvalue associated to e'_{β_R, B_R} is

$$\Lambda = \alpha(2\pi)^d (-1)^d \frac{\prod_{w \in \mathcal{S}_{<1}} w}{\det(Q_{RL})}. \quad (2.5.2)$$

We will first show the latter theorem and, with an argument using symmetry of the problem, the proof of theorem 2.1.4 will be a direct consequence. We start by giving three lemmas.

Lemma 2.5.1. *Let $Q \in H_{2d}^+(\mathbb{C})$ and take w_1 and w_2 in \mathcal{S} , we have*

$$\frac{1}{1 - \bar{w}_1 w_2} \begin{pmatrix} I_d \\ w_1 I_d \end{pmatrix}^* Q \begin{pmatrix} I_d \\ w_2 I_d \end{pmatrix} = Q_{LL} + Q_{LR}w_2 \quad (2.5.3)$$

Proof. Let w_1 and w_2 be in \mathcal{S} , we have

$$\begin{aligned} \frac{1}{1 - \bar{w}_1 w_2} \begin{pmatrix} I_d \\ w_1 I_d \end{pmatrix}^* Q \begin{pmatrix} I_d \\ w_2 I_d \end{pmatrix} &= \frac{1}{1 - \bar{w}_1 w_2} ((Q_{LL} + Q_{LR}w_2 + Q_{RL}\bar{w}_1 + Q_{RR}\bar{w}_1 w_2)) \\ &= \frac{1}{1 - \bar{w}_1 w_2} ((Q_{LL} + Q_{LR}w_2)(1 - \bar{w}_1 w_2) + \Psi(w_2)\bar{w}_1 w_2) \\ &= Q_{LL} + Q_{LR}w_2. \end{aligned}$$

□

We recall that, in the bases indexed by zeros of Ψ_Q , the coordinate of any $x \in \mathbb{C}^d$ associated to the vector u_w with $w \in \mathcal{S}_{<1}$ (resp. $w \in \mathcal{S}_{>1}$) is denoted $x(w)$, see (2.2.16).

Lemma 2.5.2. *Let $Q \in H_{2d}^+(\mathbb{C})$ and assumptions I and II fulfilled. Let $W \in M_d(\mathbb{C})$ such that $Wu_w = wu_w$ for all $w \in \mathcal{S}_{<1}$. Then, for all $x \in \mathbb{C}^d$, under notation (2.2.16),*

$$x^*(Q_{LL} + Q_{LR}W)x = \sum_{w_1, w_2 \in \mathcal{S}_{<1}} \overline{x(w_1)} u_{w_1}^* (Q_{LL} + Q_{LR}w_2) u_{w_2} x(w_2) \quad (2.5.4)$$

$$= \sum_{w_1, w_2 \in \mathcal{S}_{<1}} \overline{x(w_1)} \frac{1}{1 - \overline{w_1}w_2} \begin{pmatrix} u_{w_1} \\ w_1 u_{w_1} \end{pmatrix}^* Q \begin{pmatrix} u_{w_2} \\ w_2 u_{w_2} \end{pmatrix} x(w_2) \quad (2.5.5)$$

and $(Q_{LL} + Q_{LR}W)$ is Hermitian and positive definite.

Proof. The first equality is a direct consequence of the definition of W and the second one follows from (2.5.3). The Hermitian property is clear from (2.5.5), it remains to prove that B_R is positive definite. Since $|w_1| < 1$ and $|w_2| < 1$,

$$\frac{1}{1 - \overline{w_1}w_2} = \sum_{k=0}^{+\infty} \overline{w_1^k} w_2^k.$$

Therefore, (2.5.5) leads to

$$x^*(Q_{LL} + Q_{LR}W)x = \sum_{k=0}^{+\infty} \begin{pmatrix} W^k x \\ W^{k+1} x \end{pmatrix}^* Q \begin{pmatrix} W^k x \\ W^{k+1} x \end{pmatrix} > 0$$

since Q is Hermitian, so every term is strictly positive for every $x \neq 0$. \square

Lemma 2.5.3. *Let $Q \in H_{2d}^+(\mathbb{C})$ and $(Q_{LL} + Q_{LR}W) \in H_d^+(\mathbb{C})$ with W diagonalizable. For all $i \in \{1, \dots, d\}$ let u_{w_i} be the eigenvector of W with eigenvalue $w_i \in \mathbb{C}^*$. We suppose, for all i , $w_i \in \mathcal{S}$ and $u_{w_i} \in \ker \Psi(w_i)$. Then, for all $i \in \{1, \dots, d\}$, $w_i \in \mathcal{S}_{<1}$.*

Proof. We only have to show that every w_i is of module less than 1. The equation (2.5.3) and positive definiteness of $(Q_{LL} + Q_{LR}W)$ gives, for all $x \in \mathbb{C}^d$,

$$x^*(Q_{LL} + Q_{LR}W)x = \sum_{(w_i, w_j)} \overline{x(w_i)} \frac{1}{1 - \overline{w_i}w_j} \begin{pmatrix} u_{w_i} \\ w_i u_{w_i} \end{pmatrix}^* Q \begin{pmatrix} u_{w_j} \\ w_j u_{w_j} \end{pmatrix} x(w_j) > 0.$$

In particular, given $i \in \{1, \dots, d\}$, take $x(w_i) \neq 0$ and $x(w_j) = 0$ for all $j \neq i$, it comes

$$|x(w_i)|^2 \frac{1}{1 - |w_i|^2} \begin{pmatrix} u_{w_i} \\ w_i u_{w_i} \end{pmatrix}^* Q \begin{pmatrix} u_{w_i} \\ w_i u_{w_i} \end{pmatrix} > 0.$$

Since Q is positive definite, $|w_i| < 1$. \square

We are now ready to prove theorems 2.5.1 and 2.1.4.

Proof of Theorem 2.5.1. We start by showing that (i) \Leftrightarrow (ii). Suppose (i) and for all $i \in \{1, \dots, d\}$ let u_{w_i} be the eigenvector of $W_{<1}$ with eigenvalue $w_i \in \mathbb{C}^*$. Then, right Schur-invariance property (2.1.10) together with the fact that Q_{LR} is invertible lead to

$$Q_{LL} + Q_{RR} + Q_{LR}W_{<1} + Q_{RL}W_{<1}^{-1} = 0 \quad (2.5.6)$$

which, applied to any eigenvector u_{w_i} of $W_{<1}$ becomes

$$\Psi_Q(w_i)u_{w_i} = 0.$$

Therefore, $w_i \in \mathcal{S}$ and $u_{w_i} \in \ker \Psi_Q(w_i)$ and lemma 2.5.3 gives $w_i \in \mathcal{S}_{<1}$ so we recover (ii). Suppose (ii) and recall that, by theorem 2.2.1, $(u_w)_{w \in \mathcal{S}_{<1}}$ is a basis of \mathbb{C}^d . By construction of $W_{<1}$, we recover (2.5.6) which implies, since Q_{LR} is invertible, we get right invariance. Hermitian and positive definite properties of B_R are given by lemma 2.5.2. We then have (i). It is clear that (ii) \Leftrightarrow (iii). Note that $\lim_{k \rightarrow +\infty} x_k = 0$ if and only if every eigenvalue w is in $\mathcal{S}_{<1}$ (module strictly less than 1). Finally, from theorem 2.1.2,

$$\Lambda = \alpha(2\pi)^d \det(Q_{LL} + Q_{RR} + Q_{LR}W_{<1})$$

which, using (2.5.6), gives

$$\Lambda = \alpha(2\pi)^d \det(-Q_{RL}W_{<1}^{-1})^{-1} = \alpha(2\pi)^d (-1)^d \frac{\prod_{w \in \mathcal{S}_{<1}} w}{\det(Q_{RL})}.$$

□

Proof of Theorem 2.1.4. Theorem 2.1.4 can be deduced by remarking that if B_L is a left Schur-invariant for Q , then it is right Schur-invariant for SQS^{-1} . Applying theorem 2.5.1 for the matrix SQS^{-1} and using identities of lemma 2.2.6 we get the desired result. □

2.5.2 Comparison with Fourier approach

Now that we gave explicit formulas for left and right eigen-boundary conditions, we are ready to fully compare this approach with the periodic boundary condition approach discussed in section 2.3. Recall that the link between free energies has already been given in (2.2.12). In this section, we will first establish a correspondence between eigen-boundary conditions and Fourier coefficients of Ψ_Q^{-1} , denoted $(C_k)_{k \in \mathbb{Z}}$ (see (2.2.5)). Then, we will show that, under invariant boundary conditions, we do recover the marginal laws computed with periodic boundary conditions. Our construction of invariant boundaries B_L and B_R has an interesting application in terms of projective diagram discussed in the end of this section.

For this whole section, fix $\alpha \in \mathbb{R}_+^*$ and $Q \in H_{2d}^+(\mathbb{C})$ satisfying assumptions I and II. Let B_L and B_R be its associated eigen-boundary conditions, respectively given by theorems 2.1.4 and 2.5.1.

2.5.2.1 Invariant boundaries and Fourier coefficients of Ψ_Q^{-1}

The following lemma gives an explicit link between matrices $W_{>1}$ and $W_{<1}$ and $(C_k)_{k \in \mathbb{Z}}$, the Fourier coefficients of Ψ_Q^{-1} .

Lemma 2.5.4. *Let $W_{>1}$ and $W_{<1}$ respectively given by points (ii) of Theorems 2.1.4 and 2.5.1. For all $k \geq 0$, we have*

$$C_k C_0^{-1} = W_{>1}^{-k} \text{ and } C_{-k} C_0^{-1} = W_{<1}^k. \quad (2.5.7)$$

Proof. The case $k = 0$ is trivial. Recall that, for all $w \in \mathcal{S}$ and all $x \in \mathbb{C}^d$, $P_w x = \langle u_{1/\bar{w}}, x \rangle u_w$, see theorem 2.2.1. Let $k > 0$, corollary 2.2.1 and definition of $W_{<1}$ give

$$\begin{aligned} W_{<1}^k C_0 &= W_{<1}^k \sum_{w \in \mathcal{S}_{<1}} w^{-1} \alpha_w P_w \\ &= \sum_{w \in \mathcal{S}_{<1}} w^{k-1} \alpha_w P_w = C_{-k}. \end{aligned}$$

Proof for the other identity is similar. □

Remark 2.5.1. Take for instance $k, l \in \mathbb{N}$. From the previous lemma we get

$$C_{-(k+l)} = W_{<1}^{k+l} C_0 = W_{<1}^k W_{<1}^l C_0 = C_{-k} C_0^{-1} C_{-l}$$

for all $k, l \in \mathbb{N}$. More generally for all $k, l \in \mathbb{Z}$ such that $kl \geq 0$, we have

$$C_{k+l} = C_k C_0^{-1} C_l. \quad (2.5.8)$$

This formula is the Markov property for the infinite volume Gaussian process $(X_k)_{k \in \mathbb{Z}}$ defined in (2.3.7), see theorem 2.3.1.

We deduce the following lemma which gives alternative formulas for eigen-boundary conditions B_L and B_R .

Lemma 2.5.5. *The left and right Schur-invariant matrices are of the form*

$$B_L = Q_{RR} + Q_{RL} C_1 C_0^{-1} \text{ and } B_R = Q_{LL} + Q_{LR} C_{-1} C_0^{-1} \quad (2.5.9)$$

Moreover,

$$B_L + B_R = C_0^{-1}. \quad (2.5.10)$$

Proof. Direct consequence of lemma 2.5.4. Furthermore, we have

$$\begin{aligned} B_L + B_R &= Q_{LL} + Q_{RR} + C_0^{-1} C_{-1} Q_{LR} + C_0^{-1} C_1 Q_{RL} \\ &= C_0^{-1} (C_0 (Q_{LL} + Q_{RR}) + C_{-1} Q_{LR} + C_1 Q_{RL}) \\ &= C_0^{-1} \end{aligned}$$

using the recursive formula (2.2.7). \square

2.5.2.2 Recovering the marginal laws computed via Fourier transform.

In section 2.3, we computed marginal laws under thermodynamic limit for homogeneous Gaussian Markov processes with periodic boundary conditions, see (2.3.5). In particular, the limit in law of any $P + 1$ consecutive vertices is a centered Gaussian with covariance matrix given by

$$\Sigma^{(P)} = (C_{l-k})_{k,l \in \{1, \dots, P+1\}}, \quad (2.5.11)$$

the block-Toeplitz matrix generated by Fourier coefficients of Ψ_Q^{-1} . In this subsection, we show that we recover this result when taking eigen-boundary conditions B_L and B_R . More precisely, take $X^{(P)}$ the homogeneous Gaussian Markov process of size P with weights $e_{\alpha, Q}$ and eigen-boundary conditions B_L and B_R . Denote $\mathbf{Q}_{Q, \text{eigen}}^{(P)} := \mathbf{Q}_{(Q, B_L, B_R)}^{(P)}$ the matrix which appears in the distribution of $X^{(P)}$, see (2.1.2). The next lemma shows that $X^{(P)}$ is also a centered Gaussian with covariance matrix given by the block-Toeplitz matrix generated by Fourier coefficients of Ψ_Q^{-1} .

Lemma 2.5.6. *Under the eigen boundary conditions B_L and B_R , $X^{(P)}$ is a centered Gaussian vector with covariance matrix given by the block-Toeplitz matrix*

$$\Sigma^{(P)} = (C_{l-k})_{k,l \in \{1, \dots, P+1\}}.$$

Proof. Since $X^{(P)}$ is obviously a centered Gaussian, we only have to prove that the matrix $\mathbf{Q}_{Q, \text{eigen}}^{(P)}$ is the inverse of $\Sigma^{(P)}$. The recursive formula (2.2.7) gives

$$(\Sigma^{(P)} \mathbf{Q}_{Q, \text{eigen}}^{(P)})_{k,l} = C_{k-l-1} Q_{LR} + C_{k-l} (Q_{LL} + Q_{RR}) + C_{k-l+1} Q_{RL} = Id \delta_{k,l}$$

for all $1 < l < P + 1$. Moreover, using lemma 2.5.5,

$$\begin{aligned} (\Sigma^{(P)} \mathbf{Q}_{Q,\text{eigen}}^{(P)})_{k,1} &= C_{1-k}(B_L + Q_{LL}) + C_{2-k}Q_{RL} \\ &= C_{1-k}(C_0^{-1}C_{-1}Q_{LR} + Q_{RR} + Q_{LL}) + C_{2-k}Q_{RL} \\ &= C_{-k}Q_{LR} + C_{1-k}(Q_{LL} + Q_{RR}) + C_{2-k}Q_{RL} = Id\delta_{k,1} \end{aligned}$$

for all $1 \leq k \leq P + 1$ using equation (2.5.8). Similarly, we find,

$$(\Sigma^{(P)} \mathbf{Q}_{Q,\text{eigen}}^{(P)}(Q))_{k,P-1} = Id\delta_{k,P-1}.$$

□

Projective diagram. Given the covariance matrix $\Sigma^{(P)}$, we get the covariance matrix of the sub-connected component $\Sigma^{(P-1)}$ simply with a sub-block extraction, which we write $\Sigma^{(P-1)} = \text{Proj}(\Sigma^{(P)})$. Given $\mathbf{Q}_{Q,\text{eigen}}^{(P)}$ we could have also integrated the associated Gaussian distribution with respect to the left (or right) boundary variable to get the marginal distribution of the sub-connected component. We end up with a matrix $S(\mathbf{Q}_{Q,\text{eigen}}^{(P)})$ obtained from a Schur complement of $\mathbf{Q}_{Q,\text{eigen}}^{(P)}$. Thanks to eigen-boundary conditions this matrix is simply $\mathbf{Q}_{Q,\text{eigen}}^{(P-1)}$, see (2.1.9) and (2.1.10). We end up with the following projective diagram.

$$\begin{array}{ccccccc} \dots & \xrightarrow{S} & \mathbf{Q}_{Q,\text{eigen}}^{(P+1)} & \xrightarrow{S} & \mathbf{Q}_{Q,\text{eigen}}^{(P)} & \xrightarrow{S} & \mathbf{Q}_{Q,\text{eigen}}^{(P-1)} & \xrightarrow{S} & \dots \\ & & \uparrow^{-1} & & \uparrow^{-1} & & \uparrow^{-1} & & \\ \dots & \xrightarrow{\text{Proj}} & \Sigma^{(P+1)} & \xrightarrow{\text{Proj}} & \Sigma^{(P)} & \xrightarrow{\text{Proj}} & \Sigma^{(P-1)} & \xrightarrow{\text{Proj}} & \dots \end{array} \quad (2.5.12)$$

At the level of probability theory, this diagram does not have anything new. It is a simple consequence of marginalization under eigen-boundary conditions. Its translation at the level of matrices is more interesting. The family $(\mathbf{Q}_{Q,\text{eigen}}^{(P)})_{P \in \mathbb{N}^*}$ is a consistent family of matrices with respect to Schur "projections". The situation is different from the periodic case where we found a family of coherent but disconnected diagrams, see (2.3.4). Nevertheless, as $P \rightarrow \infty$ both approaches lead to the same operator $\mathbf{Q}_Q = C(\Psi_Q)$, the block-circulant operator generated by Ψ_Q .

2.5.3 Gluing property for Ψ_Q

We now give an application of theorems 2.1.4 and 2.5.1. Recall that for $n \in \mathbb{N}^*$, $Q^{[n]}$ is the n -th Schur power of $Q \in H_{2d}^+(\mathbb{C})$, see (2.4.2). A consequence of theorem 2.5.1 is an easy proof of the following result which links Ψ_Q with $\Psi_{Q^{[n]}}$.

Corollary 2.5.1. *Let $Q \in H_{2d}^+(\mathbb{C})$. Under assumptions I and II and assuming $|w| \neq |w'|$ for all $w \neq w' \in \mathcal{S}(\Psi_Q)$, we have, for all $n \in \mathbb{N}^*$,*

$$\begin{aligned} \mathcal{S}(\Psi_Q) &\rightarrow \mathcal{S}(\Psi_{Q^{[n]}}) \\ w &\mapsto w^n \end{aligned}$$

is a bijection. Moreover, $\Psi_{Q^{[n]}}(w^n)u_w = 0$ for all $w \in \mathcal{S}(\Psi_Q)$.

Proof. Let $Q \in H_{2d}^+(\mathbb{C})$ and $n \in \mathbb{N}^*$. Since Q_{LR} is invertible, we deduce by induction that $Q_{LR}^{[n]}$ is also invertible. Therefore, by lemma 2.2.4, $\mathcal{S}(\Psi_{Q^{[n]}})$ is composed of at most $2d$ elements. Take $W_{<1}$ as defined in theorem 2.5.1. Then, $(x_k)_{k \in \mathbb{N}} = (W_{<1}^k x_0)_{k \in \mathbb{N}}$ is solution

to the discrete Dirichlet type problem (2.5.1). Thanks to Gaussian elimination (Schur complement), we get in particular, for all $x_0 \in \mathbb{C}^d$

$$Q_{RL}^{[n]}x_0^{[n]} + (Q_{LL}^{[n]} + Q_{RR}^{[n]})x_1^{[n]} + Q_{LR}^{[n]}x_2^{[n]} = 0$$

with $x_k^{[n]} = (W_{<1}^n)^k x_0$ for any $k \in \mathbb{N}$. It implies that for any $w \in \mathcal{S}_{<1}(\Psi_Q)$

$$\Psi_{Q^{[n]}}(w^n)u_w = 0$$

and $w^n \in \mathcal{S}_{<1}(\Psi_{Q^{[n]}})$. A similar argument also leads to $\Psi_{Q^{[n]}}(w^n)u_w = 0$ for all $w \in \mathcal{S}_{>1}(\Psi_Q)$. We conclude by cardinality of $\mathcal{S}(\Psi_{Q^{[n]}})$. \square

2.6 Identities in law under infinite volume Gibbs measure

For this whole section, fix $\alpha \in \mathbb{R}_+^*$ and $Q \in H_{2d}^+(\mathbb{C})$ satisfying assumptions **I** and **II**. As previously discussed in remark 2.1.4, eigen-boundary conditions B_L and B_R , defined in lemma 2.5.5 or, equivalently, in theorems 2.1.4 and 2.5.1, provide an infinite volume Gibbs measure under which the homogeneous Gaussian Markov process $X = (X_k)_{k \in \mathbb{Z}}$ of weight (α, Q) is well defined. This section is devoted to the computation of conditional expectations and variances of this process in terms of matrices $W_{<1}$ and $W_{>1}$ given in theorems 2.1.4 and 2.5.1.

We start by recalling the following result concerning Gaussian random vectors.

Lemma 2.6.1. *Let $d, d' \in \mathbb{N}^*$, and (Y, Z) be a centered circularly symmetric Gaussian vector with $Y \in \mathbb{C}^d$ and $Z \in \mathbb{C}^{d'}$ and positive definite covariance matrix*

$$\Sigma = \begin{pmatrix} \Sigma_{YY} & \Sigma_{YZ} \\ \Sigma_{ZY} & \Sigma_{ZZ} \end{pmatrix}.$$

Then,

$$\mathbb{E}[Y|Z] = \Sigma_{YZ}\Sigma_{ZZ}^{-1}Z \text{ and } \text{Var}[Y|Z] = \Sigma_{YY} - \Sigma_{YZ}\Sigma_{ZZ}^{-1}\Sigma_{ZY}. \quad (2.6.1)$$

We recall that, in the bases indexed by zeros of Ψ_Q , the coordinate of any $x \in \mathbb{C}^d$ associated to the vector u_w with $w \in \mathcal{S}$ is denoted $x(w)$, see (2.2.16). We have the following theorem.

Theorem 2.6.1. *For all $l \in \mathbb{Z}$, $k \in \mathbb{N}$, we have*

$$\mathbb{E}_{\text{Gibbs}}[X_{l-k}|X_l] = C_k C_0^{-1} X_l = W_{>1}^{-k} X_l = \sum_{w \in \mathcal{S}_{>1}} w^{-k} X_l(w) u_w \quad (2.6.2)$$

and

$$\mathbb{E}_{\text{Gibbs}}[X_{l+k}|X_l] = C_{-k} C_0^{-1} X_l = W_{<1}^k X_l = \sum_{w \in \mathcal{S}_{<1}} w^k X_l(w) u_w. \quad (2.6.3)$$

Moreover, for all $k_1, k_2 \in \mathbb{N}$ such that $k_1 \leq k_2$, we have

$$\text{Cov}_{\text{Gibbs}}(X_{l+k_1}, X_{l+k_2}|X_l) C_0^{-1} = W_{>1}^{k_1-k_2} - W_{<1}^{k_1} W_{>1}^{-k_2}. \quad (2.6.4)$$

Proof. We start with conditional expectations and we only prove the second line (the proof for the first one is similar). Under eigen boundary conditions, lemma 2.5.6 gives $\text{Var}(X_l) = C_0$ and $\text{Cov}_{\text{Gibbs}}(X_{l+k}, X_l) = C_{-k}$ for all $l \in \mathbb{Z}$, $k \in \mathbb{N}$. Lemma 2.6.1 gives $\mathbb{E}_{\text{Gibbs}}[X_{l-k}|X_l] = C_{-k} C_0^{-1} X_l$. Lemma 2.5.4 together with definition of $W_{<1}$ gives the two other identities. Just as for conditional expectation, (2.6.4) is a direct consequence of lemmas 2.6.1, 2.5.6 and 2.5.4. \square

Remark 2.6.1. Given X_l , the process is no longer centered and invariant by translation. We remark that the conditional expectations are given by two semi-groups generated by the matrices $W_{<1}$ and $W_{>1}$. In the bases indexed by zeros of Ψ_Q , they are simply given by the semi-groups generated by multiplying each coordinate, respectively, by $w \in \mathcal{S}_{<1}$ and w^{-1} with $w \in \mathcal{S}_{>1}$. These conditional expectations decrease to zero with exponential rate. Now, consider $\text{Cov}_{\text{Gibbs}}(X_{l+k_1}, X_{l+k_2} | X_l)$ for $k_1, k_2 \in \mathbb{N}$. It is a sum of two terms. The first term, $W_{>1}^{k_1-k_2} C_0 = \text{Cov}_{\text{Gibbs}}(X_{l+k_1}, X_{l+k_2})$ (thanks to lemmas 2.5.4 and 2.5.6), only depends on the distance between X_{l+k_1} and X_{l+k_2} . This term is penalized with $(-W_{<1}^{k_1} W_{>1}^{-k_2} C_0)$ which depends on the distances of X_{l+k_1} and X_{l+k_2} from the conditioning random variable X_l . This penalization decreases with exponential rate as k_1 and k_2 get larger. In other words, when X_{l+k_1} and X_{l+k_2} are far away from X_l , their conditional law is close to their law without conditioning.

Remark 2.6.2. As we already saw, the quantities $(W_{>1}^{-k} x_0)_{k=0}^{-\infty}$ and $(W_{<1}^k x_0)_{k=0}^{+\infty}$ solve the discrete harmonic problems (2.1.15) and (2.5.1) associated to the nearest neighbor operator generated by Q . Indeed, it is known that conditional expectations and harmonic extensions are similar objects for Gaussian Processes. Here, we recover that $(\mathbb{E}_{\text{Gibbs}}[X_{l+k} | X_l])_{k \in \mathbb{N}}$ is the harmonic extension of X_l .

2.7 The case Q_{LR} non-invertible

The previous sections dealt with the case Q_{LR} invertible, see assumption I. At the level of the function Ψ , studied in section 2.2, this assumption is only made to ensure that Ψ has $2d$ zeros counted with multiplicity, see lemma 2.2.4 and that, with theorem 2.2.1 we have two families of d vectors which form bases of \mathbb{C}^d , see definition 2.2.2. At the level of invariant boundaries this assumption was made to make simpler the resolution of, for instance, right Schur-invariance equation (2.1.10). When Q_{LR} is invertible, this equation is equivalent to (2.5.6), see proof of theorem 2.5.1, and this establish the link between invariance property and zeros of Ψ .

In this section, we show how to extend the results to the case where Q_{LR} is non-invertible. To do so, we will first study the function Ψ in this new context. As we will see, it has now $2q$ zeros with $q < d$. In particular, the bases indexed by zeros of Ψ now need to be completed with the bases of $\ker Q_{LR}$ and $\ker Q_{RL}$ to form bases of \mathbb{C}^d . Similarly to theorem 2.2.1, we get a decomposition of Ψ^{-1} and its Fourier coefficients along this new set of vectors. Secondly, we will build left and right Schur-invariant boundary conditions in terms of matrices similar to $W_{<1}$ and $W_{>1}$ defined in theorem 2.1.4 and 2.5.1. These matrices are now diagonal in the bases indexed by zeros of Ψ completed with the bases $\ker Q_{LR}$ and $\ker Q_{RL}$. Especially, they are no longer invertible (their kernels are respectively $\ker(Q_{RL})$ and $\ker(Q_{LR})$).

Assumption I'. For the whole section, let $d \in \mathbb{N}^*$ and fix $Q \in H_{2d}^+(\mathbb{C})$. Take $k \in \{1, \dots, d\}$. We will now work under the following assumption

$$\text{rank}(Q_{LR}) = d - k. \quad (2.7.1)$$

Fix $(u_i^{Q_{LR}})_{i=1}^k$ and $(u_i^{Q_{RL}})_{i=1}^k$ respectively basis of $\ker Q_{LR}$ and $\ker Q_{RL}$.

2.7.1 A short study of Ψ

The function Ψ has been fully studied in section 2.2 in the case Q_{LR} invertible. Recall that any point of \mathbb{C}^* where Ψ is not invertible is called a zero of Ψ , see (2.2.8). When Q_{LR} is not invertible, Ψ has less zeros. Moreover, the bases indexed by its zeros, defined in 2.2.2,

are no longer bases of \mathbb{C}^d . They need to be completed by the bases of $\ker Q_{LR}$ and $\ker Q_{RL}$. We start by giving a technical lemma which will be helpful for the whole section.

Lemma 2.7.1. *Let $\Phi : \mathbb{C} \rightarrow M_d(\mathbb{C})$ polynomial. Assume there exists $u \in \mathbb{C}$ such that $\text{rank}(\Phi(u)) = d - k$ for $k \geq 2$. Then,*

$$(\text{adj}\Phi(u))^{(l)} = 0 \quad (2.7.2)$$

for all $0 \leq l \leq k - 2$.

Proof. For any $n \in \mathbb{N}^*$ such that $n \leq d$, we let

$$E_n = \{1, \dots, d\} \times \{1, \dots, d - 1\} \times \dots \times \{1, \dots, d - n + 1\}$$

and $E_0 = \emptyset$. For any $n \geq 1$, let $I_n = (i_1, \dots, i_n) \in E_n$ and $J_n = (j_1, \dots, j_n) \in E_n$. If $n = 1$, for all $z \in \mathbb{C}$, $\Phi_{i_1, j_1}(z) \in M_{d-1}(\mathbb{C})$ is the matrix obtained from $\Phi(z)$ by erasing its i_1 -th row and j_1 -th column. If $n \geq 2$, we define, recursively, for all $z \in \mathbb{C}$, $\Phi_{I_n, J_n}(z) \in M_{d-n}(\mathbb{C})$ as the matrix obtained from $\Phi_{I_{n-1}, J_{n-1}}(z) \in M_{d-n+1}(\mathbb{C})$ by erasing its i_n -th row and j_n -th column, with $I_{n-1} = (i_1, \dots, i_{n-1})$ and $J_{n-1} = (j_1, \dots, j_{n-1})$. If $n = 0$ we keep $\Phi(z)$ by convention. If we show that, for all $0 \leq l < k - 1$, all $1 \leq n < k - l$, and any $I_{n-1} J_{n-1} \in E_{n-1}$,

$$\text{adj}(\Phi_{I_{n-1}, J_{n-1}}(u))^{(l)} = 0 \quad (2.7.3)$$

then (2.7.2) will follow from the specific case of $n = 1$, by convention. We prove (2.7.3) recursively on l . Take $l = 0$. Then,

$$\text{adj}\Phi_{I_{n-1}, J_{n-1}}(u) = 0$$

for $z = u$ and all $n < k$ and any $I_{n-1}, J_{n-1} \in E_{n-1}$. Indeed, $\text{adj}\Phi_{I_{n-1}, J_{n-1}}(u)$ is a matrix whose coefficients are proportional to minors of order $(d - n)$ of $\Phi(u)$ which are 0 for all $n < k$ (since $\text{rank}(\Phi(u)) = d - k$). Suppose (2.7.3) true for all $m \leq l$. Then, given $1 \leq n < k - l - 1$ (such n exists as long as $l < k - 2$) and any $I_{n-1} = (i_1, \dots, i_{n-1}) \in E_{n-1}$ and $J_{n-1} = (j_1, \dots, j_{n-1}) \in E_{n-1}$,

$$\begin{aligned} \left((\text{adj}\Phi_{I_{n-1}, J_{n-1}}(z))^{(l+1)} \right)_{i_n, j_n} &= (-1)^{i_n + j_n} (\det \Phi_{I_n, J_n}(z))^{(l+1)} \\ &= (-1)^{i_n + j_n} \text{Tr} \left(\sum_{m=0}^l \binom{l}{m} (\text{adj}\Phi_{I_n, J_n}(z))^{(m)} \Phi_{I_n, J_n}^{(l-m+1)}(z) \right). \end{aligned}$$

where $I_n = (i_1, \dots, i_{n-1}, i_n)$ and $J_n = (j_1, \dots, j_{n-1}, j_n)$, using Jacobi's formula and Leibniz's rule. For $z = u$ we apply (2.7.3) to $(\text{adj}\Phi_{I_n, J_n}(u))^{(m)}$ for all $m \leq l$ and this conclude. \square

Zeros of Ψ . Recall that we denote \mathcal{S} the set of zeros of Ψ , see (2.2.8). When Q_{RL} is invertible we saw that, counted with multiplicity, Ψ has $2d$ zeros. When $\text{rank}(Q_{RL}) = d - k$ is not invertible, there are at most $2(d - k)$ zeros counted with multiplicity. This is the purpose of the following lemma.

Lemma 2.7.2. *Let $\Phi(z) = z\Psi(z) = Q_{LR}z^2 + (Q_{LL} + Q_{RR})z + Q_{RL}$ for all $z \in \mathbb{C}$. Under assumption I', the polynomial*

$$P(z) = \det \Phi(z) \quad (2.7.4)$$

for all $z \in \mathbb{C}$ is of degree at most $2d - k$. Moreover 0 is of multiplicity at least k . Therefore, Ψ admits at most $2(d - k)$ zeros counted with multiplicity. Finally, for all $w \in \mathcal{S}$, $1/\bar{w} \in \mathcal{S}$.

Proof. We write $P(z) = \sum_{l=0}^{2d} p_l z^l$ for all $z \in \mathbb{C}$. Moreover, we have $\Phi(z) = z^2 \Phi(1/\bar{z})^*$ for all $z \in \mathbb{C}^*$ so that,

$$\det \Phi(z) = z^{2d} \det \Phi(1/\bar{z})^* = \sum_{l=0}^{2d} \overline{p_{2d-l}} z^l.$$

Thus, $p_l = \overline{p_{2d-l}}$ for all $l \in \{0, \dots, 2d\}$. Moreover, $P(0) = \det(Q_{RL}) = 0$ by hypothesis and for all $z \in \mathbb{C}$ and $l \geq 1$,

$$P^{(l)}(z) = \text{Tr} \left(\sum_{m=0}^{l-1} \binom{l-1}{m} (\text{adj} \Phi(z))^{(m)} \Phi^{(m-l+2)}(z) \right),$$

which is zero for $z = 0$ and any $l \leq k-1$ by (2.7.2). Then, $p_{2d-l} = \overline{p_l} = 0$ for all $l \leq k-1$ and P is of degree at most $2d-k$. Since for all $z \in \mathbb{C}^*$, $P(z) = 0$ if and only if $\det \Psi(z) = 0$, we deduce that Ψ admits at most $2(d-k)$ zeros. For the last assertion, see lemma 2.2.4. \square

Assumption II'. For any root x of P defined previously in (2.7.4), let $\text{mult}(x)$ be its multiplicity. To make things easier, we make the following assumptions

$$\text{mult}(w) = \dim \ker \Psi(w) = 1 \text{ for all } w \in \mathcal{S}, \text{ and } \text{mult}(0) = k. \quad (2.7.5)$$

In that case, P is exactly of degree $2d-k$ and Ψ has exactly $d-k$ non zero roots inside the unit disk and $d-k$ outside. For all $w \in \mathcal{S}$, we fix $u_w \neq 0$ such that

$$\ker \Psi(w) = \text{Vect}_{\mathbb{C}}(u_w). \quad (2.7.6)$$

The bases indexed by zeros of Ψ . Compared to definition 2.2.2, the families $(u_w)_{w \in \mathcal{S}_{<1}}$ and $(u_w)_{w \in \mathcal{S}_{>1}}$ are no longer bases of \mathbb{C}^d but, as we will see in theorem 2.7.1, the sequences $((u_w)_{w \in \mathcal{S}_{<1}}, (u_i^{Q_{RL}})_{i=1}^k)$ and $((u_w)_{w \in \mathcal{S}_{>1}}, (u_i^{Q_{LR}})_{i=1}^k)$ are, where $(u_i^{Q_{LR}})_{i=1}^k$ and $(u_i^{Q_{RL}})_{i=1}^k$ are respectively basis of $\ker Q_{LR}$ and $\ker Q_{RL}$. We first need two lemmas.

Lemma 2.7.3. *Let $\Phi(z) = z\Psi(z) = Q_{LR}z^2 + (Q_{LL} + Q_{RR})z + Q_{RL}$ for all $z \in \mathbb{C}$. Under assumptions I' and II',*

$$\text{adj} \Phi(z) = \sum_{l=k-1}^{2d-k-1} \Phi_l z^l \quad (2.7.7)$$

for all $z \in \mathbb{C}$ with. Moreover for all $l \in \{k-1, \dots, 2d-k-1\}$, $\Phi_l \in M_d(\mathbb{C})$ and

$$\Phi_l^* = \Phi_{2(d-1)-l}. \quad (2.7.8)$$

Finally, the matrices Φ_{k-1} and Φ_k are non zero and for all $x \in \mathbb{C}^d$,

$$\Phi_{k-1}x = \sum_{i,j=1}^k \alpha_{i,j} \langle u_{Q_{LR}}^{(i)}, x \rangle u_{Q_{RL}}^{(j)} \quad (2.7.9)$$

with $\alpha_{i,j} \in \mathbb{C}$ for all $i, j \in \{1, \dots, k\}$.

Proof. First of all, we remark that there exist matrices $\Phi_l \in M_d(\mathbb{C})$ for $l \in \{0, \dots, 2d-2\}$ such that for all $z \in \mathbb{C}$,

$$\text{adj} \Phi(z) = \sum_{l=0}^{2d-2} \Phi_l z^l.$$

Moreover, we have $\Phi(z) = z^2\Phi(1/\bar{z})^*$ for all $z \in \mathbb{C}^*$ so that,

$$\text{adj}\Phi(z) = z^{2(d-1)}\text{adj}\Phi(1/\bar{z})^* = \sum_{l=0}^{2d-2} \Phi_{2(d-1)-l}^* z^l.$$

Then, $\Phi_l^* = \Phi_{2(d-1)-l}$ for all $l \in \{k-1, \dots, 2d-k-1\}$. Finally, $\text{rank}(\Phi(0)) = \text{rank}(Q_{RL}) = d-k$ by assumption so we can use equation (2.7.2) to verify that $\Phi_l = 0$ for all $l \in \{0, \dots, k-2\}$ and therefore also for all $l \in \{2d-k, \dots, 2d-2\}$ thanks to what we just showed. Finally, recall that, for all $z \in \mathbb{C}$,

$$\Phi(z)\text{adj}\Phi(z) = \text{adj}\Phi(z)\Phi(z) = \det(\Phi(z))I_d$$

for I_d the d -dimensional identity matrix. The $(k-1)$ -th derivative, using equation (2.7.2) if $k \geq 2$, together with the assumption **II'**, gives for $z = 0$,

$$Q_{RL}\Phi_{k-1} = \Phi_{k-1}Q_{RL} = 0. \quad (2.7.10)$$

We deduce (2.7.9).

Taking the next derivative evaluated at $z = 0$ also gives

$$Q_{RL}\Phi_k + (Q_{LL} + Q_{RR})\Phi_{k-1} = (\det \Phi(0))^{(k)} I_d. \quad (2.7.11)$$

We have $\text{rank}(Q_{RL}\Phi_k) \leq \text{rank}(Q_{RL}) = d-k$ and, since $(Q_{LL} + Q_{RR})$ is invertible, $\text{rank}((Q_{LL} + Q_{RR})\Phi_{k-1}) = \text{rank}(\Phi_{k-1}) \leq k$ thanks to (2.7.9). Thanks to (2.7.11), we deduce that Φ_{k-1} and Φ_k are non zero. Moreover, we get that $\text{Im}(\Phi_{k-1}) = \ker Q_{RL}$. \square

Remark 2.7.1. Denote $T : \mathbb{C}^d \rightarrow \mathbb{C}^d$, $v \mapsto (Q_{LL} + Q_{RR})v$. We deduce from (2.7.11), that

$$\mathbb{C}^d = T(\ker Q_{RL}) \oplus \text{Im}Q_{RL}. \quad (2.7.12)$$

The following lemma gives asymptotic formulas for Ψ^{-1} which will be relevant for the residue calculus in the proof of theorem 2.7.1.

Lemma 2.7.4. *Under assumptions **I'** and **II'** and notations introduced in lemma 2.7.3, we have,*

$$\lim_{|z| \rightarrow 0} \Psi(z)^{-1} = \frac{\Phi_{k-1}}{p_{k-1}} \quad \text{and} \quad \lim_{|z| \rightarrow \infty} \Psi(z)^{-1} = \frac{\Phi_{2d-k-1}}{p_{2d-k}}. \quad (2.7.13)$$

Proof. We have, for all $z \in \mathbb{C}^*$,

$$\Psi(z)^{-1} = \frac{z\text{adj}(\Phi(z))}{\det(\Phi(z))}.$$

By lemma 2.7.2, we can write $\det \Phi(z) = z^k \tilde{P}(z)$ with \tilde{P} polynomial such that $\tilde{P}(0) = p_k \neq 0$. Using equation (2.7.7), we get the limit in 0. Again thanks to lemma 2.7.2, we know that $\det \Phi(z) \sim z^{2d-k} p_{2d-k}$ as $|z| \rightarrow \infty$. We get the limit at infinity with (2.7.7). \square

We are now ready to prove the following theorem, which is the extension of theorem 2.2.1 to the case where Q_{LR} is no longer invertible.

Theorem 2.7.1. *For all $w \in \mathcal{S}$, $\langle u_{1/\bar{w}}, \Psi'(w)u_w \rangle \neq 0$. Let for all $w \in \mathcal{S}$, $\alpha_w \in \mathbb{C}^*$ and $P_w \in M_d(\mathbb{C})$ defined as*

$$\alpha_w = \frac{1}{\langle u_{1/\bar{w}}, \Psi'(w)u_w \rangle} \quad \text{and} \quad P_w x = \langle u_{1/\bar{w}}, x \rangle u_w$$

for all $x \in \mathbb{C}^d$. Under assumptions **I'** and **II'**, for all $z \in \mathbb{C}^* \setminus \mathcal{S}$, we have

$$\Psi(z)^{-1} = \sum_{w \in \mathcal{S}} \frac{\alpha_w}{z-w} P_w + \frac{1}{p_{2d-k}} \Phi_{2d-k+1} \quad (2.7.14)$$

Moreover, $((u_w)_{w \in \mathcal{S}_{<1}}, (u_i^{Q_{RL}})_{i=1}^k)$ and $((u_w)_{w \in \mathcal{S}_{>1}}, (u_i^{Q_{LR}})_{i=1}^k)$ are bases of \mathbb{C}^d .

Proof. The proof is similar to the one of theorem 2.2.1. We let \mathcal{C}_R be the circle centered at the origin of radius R sufficiently large so that $\mathcal{S} \subset \mathbb{D}(0, R)$. For any $z \in \mathbb{C}^* \setminus \mathcal{S}$, we have

$$\int_{\mathcal{C}_R} \Psi(u)^{-1} \frac{1}{u-z} du = \pi i \left(\Psi(z)^{-1} + \sum_{w \in \mathcal{S}} \text{Res}(\Psi^{-1}, w) \frac{1}{w-z} \right)$$

where we used limit in 0 previously computed. Moreover, using the limit of $\Psi(u)^{-1}$ when $|u| \rightarrow \infty$ we get,

$$\int_{\mathcal{C}_R} \Psi(u)^{-1} \frac{1}{u-z} du = \text{Res} \left(\Psi^{-1} \frac{1}{\bullet - z}, \infty \right) = \frac{1}{p_{2d-k}} \Phi_{2d-k-1}.$$

We deduce,

$$\Psi(z)^{-1} = \sum_{w \in \mathcal{S}} \text{Res}(\Psi^{-1}, w) \frac{1}{z-w} + \frac{1}{p_{2d-k}} \Phi_{2d-k-1}.$$

To get (2.2.17), we only have to show that, for all $w \in \mathcal{S}$,

$$\text{Res}(\Psi^{-1}, w) = \lim_{u \rightarrow w} (u-w) \Psi(u)^{-1} = \alpha_w P_w.$$

This has already been done in proof of theorem 2.2.1. Finally, we recall from lemma 2.7.3 that for all $x \in \mathbb{C}^d$,

$$\Phi_{2d-k-1} x = \Phi_{k-1}^* x = \Phi_{k-1} x = \sum_{i,j=1}^k \overline{\alpha_{i,j}} \langle u_{Q_{RL}}^{(i)}, x \rangle u_{Q_{LR}}^{(j)}.$$

Now, just as for theorem 2.2.1, we prove that the families $((u_w)_{w \in \mathcal{S}_{<1}}, (u_i^{Q_{RL}})_{i=1}^k)$ and $((u_w)_{w \in \mathcal{S}_{>1}}, (u_i^{Q_{LR}})_{i=1}^k)$ are bases of \mathbb{C}^d . \square

Remark 2.7.2. As in corollary 2.2.1, a direct computation shows that

$$C_0 = \sum_{w \in \mathcal{S}_{<1}} w^{-1} \alpha_w P_w + \frac{1}{p_{2d-k}} \Phi_{2d-k+1} \quad (2.7.15)$$

and for all $k > 0$,

$$C_k = \sum_{w \in \mathcal{S}_{>1}} -\frac{\alpha_w}{w^{k+1}} P_w \text{ and } C_{-k} = \sum_{w \in \mathcal{S}_{<1}} w^{k-1} \alpha_w P_w. \quad (2.7.16)$$

which are now sub-blocks matrices.

2.7.2 Invariant boundaries

We are now ready to give the following theorem which give left and right Schur-invariant boundaries.

Theorem 2.7.2. *Under assumptions I' and II', let $W_{<1} \in M_d(\mathbb{C})$ such that $W_{<1} u_w = w u_w$ for all $w \in \mathcal{S}_{<1}$ and $W_{<1} u_i^{Q_{RL}} = 0$ for all $i \in \{1, \dots, k\}$. Then,*

$$B_R = Q_{LL} + Q_{LR} W_{<1} \in H_d^+(\mathbb{C}). \quad (2.7.17)$$

is right Schur-invariant for Q . Similarly, with an abuse of notation, let $W_{>1}^{-1} \in M_d(\mathbb{C})$ be the (non invertible) matrix such that $W_{>1}^{-1} u_w = (1/w) u_w$ for all $w \in \mathcal{S}_{>1}$ and $W_{>1}^{-1} u_i^{Q_{LR}} = 0$ for all $i \in \{1, \dots, k\}$. Then,

$$B_L = Q_{RR} + Q_{RL} W_{>1}^{-1} \quad (2.7.18)$$

is left Schur-invariant for Q .

Proof. Similarly to lemma 2.5.2, we prove that $B_R \in H_d^+(\mathbb{C})$ and $(Q_{RR} + B_R)$ is invertible. We now have to prove that B_R is right Schur-invariant, that is

$$B_R = Q_{LL} - Q_{LR}(Q_{RR} + B_R)^{-1}Q_{RL},$$

see (2.1.10). This equality is clear for any $v \in \ker(Q_{RL})$. For any $w \in \mathcal{S}_{<1}$, we have

$$(Q_{RR} + Q_{LL} + Q_{LR}W_{<1})u_w = -w^{-1}Q_{RL}u_w \quad (2.7.19)$$

by definition of u_w . Denote $f : \mathbb{C}^d \rightarrow \mathbb{C}^d$, $v \mapsto (Q_{RR} + B_R)v$. Then, using decomposition (2.7.12) together with (2.7.19), we get

$$\begin{aligned} f : \text{Span}((u_w)_{w \in \mathcal{S}_{<1}}) \oplus \ker Q_{RL} &\rightarrow \text{Im}(Q_{RL}) \oplus T(\ker Q_{RL}) \\ \sum_{w \in \mathcal{S}_{<1}} x(w)u_w + \sum_{i=1}^k x(i)u_i^{Q_{RL}} &\mapsto \sum_{w \in \mathcal{S}_{<1}} -w^{-1}x(w)Q_{RL}u_w + \sum_{i=1}^k x(i)Tu_i^{Q_{RL}}. \end{aligned}$$

Now, the restriction of f on $\text{Span}((u_w)_{w \in \mathcal{S}_{<1}})$, is clearly invertible with inverse given by $Q_{RL}u_w \mapsto -wu_w$ for all $w \in \mathcal{S}_{<1}$. We conclude that for any $w \in \mathcal{S}_{<1}$,

$$(Q_{LL} - Q_{LR}(Q_{RR} + B_R)^{-1}Q_{RL})u_w = (Q_{LL} + Q_{LR}W_{<1})u_w = B_R$$

and B_R is right Schur-invariant. The proof is similar for B_L . \square

Remark 2.7.3. We remark that this result is similar to the one obtained under assumption I, see 2.1.4 and 2.5.1. Nevertheless, we remark that the matrices $W_{<1}$ and $W_{>1}^{-1}$ are now 0 on $\ker Q_{RL}$ and $\ker Q_{LR}$ respectively. We can prove that any other result obtained in sections 2.5 and 2.6 is still true.

2.8 Application: probabilistic representation of Szegő limit theorem

In this section, after recalling the Szegő's limit theorem, we apply theorems 2.1.4 and 2.5.1 to get an "eigen" version of this theorem in a simple case. In particular, we will see that there is no limit to take anymore.

2.8.1 Szegő limit theorem for block Toeplitz matrices

We briefly recall Szegő Limit theorem. Let $\Phi : S^1 \rightarrow M_d(\mathbb{C})$ be a continuous function and denote $(C_k(\Phi))_{k \in \mathbb{Z}}$ its Fourier coefficients, see (2.2.4). We define the block-Toeplitz operator $T(\Phi)$ generated by Φ as the semi-infinite matrix with blocks

$$(T(\Phi))_{k,l} = C_{k-l}(\Phi)$$

for $1 \leq k, l < \infty$. For all $P \geq 1$, write $T_P(\Phi)$ the truncation of $T(\Phi)$ at order P , that is,

$$(T_P(\Phi))_{k,l} = C_{k-l}(\Phi)$$

for $1 \leq k, l \leq P + 1$. Under some assumptions for Φ , Szegő's limit theorem states

$$\det T_P(\Phi) \sim_{P \rightarrow \infty} \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log \det \Phi(e^{i\theta}) d\theta \right)^{P+1} \kappa(\Phi). \quad (2.8.1)$$

with $\kappa(\Phi) = \det T(\Phi)T(\Phi^{-1})$ for $T(\Phi)$ (resp. $T(\Phi^{-1})$) the Toeplitz operator associated to Φ (resp. Φ^{-1}).

This theorem, originally for scalar Toeplitz matrices, is due to Gabor Szegő in [Sze15]. Elegant proofs for block Toeplitz using operator theory can be found in [Wid74] and, more generally for Toeplitz operator theory, we advise the reader to refer to [BS13].

Remark 2.8.1. Let Φ be a trigonometric polynomial of order $N \in \mathbb{N}$. Take $P \geq N + 1$. Up to a little modification of $T_P(\Phi)$ making it block-circulant we recover a similar asymptotic behavior of its determinant using discrete Fourier transform like for Gaussian Markov Processes.

Remark 2.8.2. In the specific case where Φ is a trigonometric polynomial of order 1 with values in $H_d^+(\mathbb{C})$, a direct computation of $\kappa(\Phi)$ gives,

$$\det T_P(\Phi) \sim_{P \rightarrow \infty} \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \det \Phi(e^{i\theta}) d\theta\right)^P \det(C_0(\Phi^{-1}))^{-1}. \quad (2.8.2)$$

This expression is reminiscent of the free energy computed for periodic Gaussian Markov processes, see (2.1.8). We are therefore interested in an invariant version of this theorem. This is the purpose of the following subsection.

2.8.2 "Eigen" version of Szegő limit theorem for matrix valued trigonometric polynomials

We now give an "eigen" version of Szegő limit. Inspired by Gaussian Markov processes, we introduce the following Toeplitz matrices modified at the boundary.

Definition 2.8.1 (Toeplitz matrix with boundary conditions). Let G_L and $G_R \in M_d(\mathbb{C})$. The Toeplitz matrix with boundary conditions G_L and G_R generated by Φ is the following quasi-Toeplitz matrix

$$(T_P^{G_L, G_R}(\Phi))_{k,l} = \begin{cases} G_L + C_0(\Phi) & \text{for } k = l = 1 \\ G_R + C_0(\Phi) & \text{for } k = l = P + 1 \\ (T_P(\Phi))_{k,l} & \text{for } (k, l) \notin \{(1, 1), (P + 1, P + 1)\} \end{cases}.$$

Trigonometric polynomials of order 1. Let $\Phi(e^{i\theta}) = \Phi_0 + \Phi_1 e^{i\theta} + \Phi_{-1} e^{-i\theta} \in H_d^+(\mathbb{C})$ for all $\theta \in [0, 2\pi]$, any trigonometric polynomial of order 1. The function Φ can be treated as the function Ψ , in section 2.2. Especially, if Φ satisfies II, we can define, as in theorems 2.1.4 and 2.5.1, two matrices $W_{<1}(\Phi)$ and $W_{>1}(\Phi) \in M_d(\mathbb{C})$ which are diagonal in the bases indexed by zeros of Φ . Similarly to (2.1.9) and (2.1.10), these matrices provide invariant boundaries for Φ . For instance,

$$\Phi_1 W_{<1}(\Phi) = -\Phi_1 (\Phi_0 + \Phi_1 W_{<1}(\Phi))^{-1} \Phi_{-1}. \quad (2.8.3)$$

With this good choice of boundary conditions, we get the following result.

Theorem 2.8.1. Let $\Phi(e^{i\theta}) = \Phi_0 + \Phi_1 e^{i\theta} + \Phi_{-1} e^{-i\theta} \in H_d^+(\mathbb{C})$ for all $\theta \in [0, 2\pi]$ such that Φ_1 is invertible. Suppose that Φ satisfies assumptions II. Then, taking $G_L = \Phi_{-1} (W_{>1}(\Phi))^{-1}$ and $G_R = \Phi_1 W_{<1}(\Phi)$ for $W_{>1}(\Phi)$ and $W_{<1}(\Phi)$ as discussed above, for all $P \in \mathbb{N}^*$,

$$\det T_P^{G_L, G_R} = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \det \Phi(e^{i\theta}) d\theta\right)^P \kappa(G_L, G_R) \quad (2.8.4)$$

with $\kappa(G_L, G_R) = \det(G_L + G_R + \Phi_0)^{-1}$.

Proof. The result follows from successive Schur complements formulas and invariant properties of G_L and G_R , see for instance (2.8.3). We conclude using (2.2.12). \square

Remark 2.8.3. Thanks to (2.5.10), $\kappa(G_L, G_R) = \det(C_0(\Phi^{-1}))^{-1}$ and we obviously recover (2.8.2) computed with Szegő's limit theorem. Compared to Szegő's limit theorem (or to the block-circulant approach), there is no limit to take anymore. Nevertheless, this result only works when $\Phi \in H_d^+(\mathbb{C})$, which is due to the Gaussian framework in which we work, see (2.1.3).

Geometric interpretation of Szegő's limit theorem. From a statistical physics point of view, in Szegő's limit theorem, the term

$$\log \left(\exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log \det \Phi(e^{i\theta}) d\theta \right) / \kappa(\Phi) \right)$$

is the 0-dimensional free energy attached to a vertex under Gibbs measure of a Gaussian Markov process on \mathbb{Z} .

Trigonometric polynomials of order $N \geq 2$. When Φ is a trigonometric polynomial of order $N \geq 2$, it is still possible to modify the matrix at the boundary to obtain an "eigen" version of Szegő's limit theorem. This is the purpose of the following result.

Theorem 2.8.2. *Let $N \geq 2$ and $\Phi(e^{i\theta}) = \sum_{k=-N}^N \Phi_k e^{ik\theta} \in H_d^+(\mathbb{C})$ for all $\theta \in [0, 2\pi]$. We introduce the function $\tilde{\Phi}(e^{i\theta}) = \tilde{\Phi}_0 + \tilde{\Phi}_1 e^{i\theta} + \tilde{\Phi}_{-1} e^{-i\theta}$ assumed to be in $H_{Nd}^+(\mathbb{C})$ with $\tilde{\Phi}_0 = T_{N-1}(\Phi)$, $\tilde{\Phi}_1 = (C_{k-l-N}(\Phi))_{k,l \in \{0, \dots, N-1\}}$ and $\tilde{\Phi}_{-1} = \tilde{\Phi}_1^*$. Suppose that $\tilde{\Phi}_1$ is invertible and that $\tilde{\Phi}$ satisfies assumption II. Then, taking $\tilde{G}_L = \Phi_{-1}(W_{>1}(\Phi))^{-1}$ and $\tilde{G}_R = \Phi_1 W_{<1}(\Phi)$ for $W_{>1}(\tilde{\Phi})$ and $W_{<1}(\tilde{\Phi})$ as discussed above, for all $P \in \mathbb{N}^*$,*

$$\det T_{(N-1)P}^{\tilde{G}_L, \tilde{G}_R} = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log \det \tilde{\Phi}(e^{i\theta}) d\theta \right)^P \tilde{\kappa}(\tilde{G}_L, \tilde{G}_R) \quad (2.8.5)$$

with $\tilde{\kappa}(\tilde{G}_L, \tilde{G}_R) = \det(\tilde{G}_L + \tilde{G}_R + \tilde{\Phi}_0)^{-1}$.

Proof. The proof is similar to the one of theorem 2.8.1. □

Remark 2.8.4. Since our theorems are only applicable to trigonometric polynomials of order 2, we had to consider the function $\tilde{\Phi}$, which takes values in a higher dimensional vector space, rather than Φ itself. Therefore, we don't directly recover (2.8.1). Nevertheless, we expect

$$\det T_{(N-1)P}(\Phi) \sim_{P \rightarrow \infty} \det T_{(N-1)P}^{\tilde{G}_L, \tilde{G}_R}(\Phi)$$

so that

$$\exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log \det \tilde{\Phi}(e^{i\theta}) d\theta \right) = c^{1/P} \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log \det \Phi(e^{i\theta}) d\theta \right)^{N-1}$$

where

$$c = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log \det \Phi(e^{i\theta}) d\theta \right) \frac{\kappa(\Phi)}{\kappa(\tilde{G}_L, \tilde{G}_R)}.$$

We believe that this could be shown thanks to a particular study of Φ but is not the purpose of the paper.

Chapter 3

Boundary structure of two-dimensional Markov Gaussian random fields.

Abstract

Recent work from [Sim23] gives an algebraic description of invariant boundary conditions associated to discrete Markov fields defined on the square lattice \mathbb{Z}^2 . This theory comes with new algebraic objects which have not been constructed on any non trivial model yet. In this article, the main objective is to exhibit and understand these structures in the particular case of Gaussian Markov fields on the lattice \mathbb{Z}^2 . We give new tools, missing in the theory, for the explicit computation of such algebraic objects in this context. These approaches are based on classical methods from statistical mechanics and Gaussian calculus but also from original methods. For instance, we revisit the so-called transfer matrix, a one-dimensional object, by "folding" it with a "fold" operator to partially obtain a purely two-dimensional object: the half-strip. In a similar fashion, the eigenvector of the transfer matrix is folded into another two-dimensional object: the corner. In our context, half-strips and corners objects, introduced in [Sim23], are quadratic forms on Hilbert spaces which can be glued together. We give a precise meaning to these gluing products, called Schur product, and we study their combinatorics provided by the Guill_2 -operad introduced in [Sim23]. Moreover, as defined in [Sim23], we show that these objects verify "eigenvector-type" equations up to a morphism Φ (which respects the gluings). This ensures that, once being glued together, half-strips and corners give the desired invariant boundary condition. We show that, in Gaussian settings, this morphism Φ is simply given by a conjugation, explicitly given in this paper.

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3.1 Introduction

This chapter corresponds to an article soon to be submitted. It is a joint work with Damien Simon.

3.1.1 A brief description of the model and the results

3.1.1.1 Motivations

We consider the lattice \mathbb{Z}^2 seen as a planar graph with sets of vertices, unoriented edges and faces (V, E, F) given by

$$\begin{aligned} V &= \mathbb{Z}^2 \\ E &= E_h \cup E_v = \{(k, l), (k + 1, l)\}; k, l \in \mathbb{Z}\} \cup \{(k, l), (k, l + 1)\}; k, l \in \mathbb{Z}\} \\ F &= \{(k, l), (k + 1, l), (k, l + 1), (k + 1, l + 1)\}; k, l \in \mathbb{Z}\} \end{aligned}$$

The edges in E_h are the horizontal edges and the ones in the E_v the vertical edges. A *domain* (resp. finite domain) of \mathbb{Z}^2 is a sub-planar graph of \mathbb{Z}^2 obtained by considering a subset (resp. a finite subset) of the set F of faces and all the edges and vertices contained in these faces (and not more). The boundary ∂D of a domain D is defined as the set of edges that belongs to exactly one face. The set of horizontal (resp. vertical) faces of a domain D is written $E_h(D)$ (resp. $E_v(D)$).

In order to simplify indexing of mathematical objects, we introduce for each face $f \in F$ the following names. For a face $f = \{(k, l), (k + 1, l), (k, l + 1), (k + 1, l + 1)\}$,

- the edge $\{(k, l), (k + 1, l)\}$ lies on the *South* (abbreviated S),
- the edge $\{(k, l + 1), (k + 1, l + 1)\}$ lies on the *North* (abbreviated N),
- the edge $\{(k, l), (k, l + 1)\}$ lies on the *West* (abbreviated W),
- the edge $\{(k + 1, l), (k + 1, l + 1)\}$ lies on the *East* (abbreviated E).

The South edge (resp. North, West, East) is the North (resp. South, East, West) edge of a neighbouring face

We introduce two real or complex finite-dimensional Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 and attach to each horizontal edge in E_h a vector in \mathcal{H}_1 and to each vertical edge in E_v a vector

in \mathcal{H}_2 through maps $X^{(1)} : E_h \rightarrow \mathcal{H}_1$ and $X^{(2)} : E_v \rightarrow \mathcal{H}_2$. These vectors are chosen at random by using a centered normal distribution with a density w.r.t the Lebesgue measure on the spaces \mathcal{H}_i .

We are interested in random fields X on domains D with the Markov property, i.e. such that, given any partition of D into two domains D_1 and D_2 , the two restrictions of X to each sub-domains are independent *conditionally* to the values of the field on the boundary of the sub-domains. For a Gaussian process on the edges of D , this corresponds to a law density given by

$$\frac{1}{Z_D^{\text{bw}}(Q_\bullet; g_D)} g_D(x_{\partial D}) \exp\left(-\frac{1}{2} \sum_{f \in F(D)} x_{\partial f}^* Q_f x_{\partial f}\right) \quad (3.1.1)$$

where, for any domain E , $x_{\partial E} = (x_e)_{e \in \partial E}$ are the variables on the boundary of E . The matrices Q_f are Hermitian matrices on the face Hilbert space

$$\mathcal{H}_F(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_2$$

We may impose translation invariance in law on the processes by assuming that all the face Hermitian matrices Q_f are equal to a fixed matrix Q . In order to have a well-defined normal distribution on any domain D , we may assume that all the Q_f are positive definite.

The non-negative real-function g_D is called a *boundary weight* and the real number $Z_D^{\text{bw}}(Q_\bullet; g_D)$ the partition function of the domain whose value can be computed by imposing that the integral of the density (3.1.1) is equal to 1.

A major goal of statistical mechanics is the description of the thermodynamic limit, scaling limits or infinite-volume Gibbs measures. When exact computations are available, this is often performed by a careful study of the partition function Z_D and expectation values of well-chosen observables. In the present case, infinite-volume Gibbs measure corresponds to the definition of the Gaussian model on the whole planar graph \mathbb{Z}^2 . For such translation-invariant Gaussian Markov processes, the infinite-volume Gibbs measure can be defined directly through a white noise on the Fourier space (see section 3.4) and then on \mathbb{Z}^2 by inverse Fourier transform. However, this trick is inherently related to the Gaussian nature of the process and cannot be extended to other processes on the edges of \mathbb{Z}^2 . Other approaches to the construction of infinite-volume Gibbs measures are presented in [Geo11; FV17] and often rely on analytical constructions with a final use of Riesz's representation theorem. The most direct purely probabilistic approach consists in Kolmogorov's extension theorem. Such an approach is easy in dimension one since we can use boundary conditions given by the left and right Perron-Frobenius eigenvectors of the transition matrix, which are often easy to compute (see [Bod23] for Gaussian and Markov random fields on \mathbb{Z}). In dimension two, this approach requires to describe all the boundary weights (g_D) for a sufficiently large collection of domains D with arbitrary large sizes in a *consistent* way under restrictions.

A breakthrough in this direction is made in [Sim23] where consistent boundary weights g_R for rectangular domains R are built out of local objects, that satisfy eigenvalue-like equations in some abstract operadic sense, in such a way that Kolmogorov's extension theorem can be applied. Moreover the eigenvalue Λ computed for these objects corresponds to a partition function $Z_R^{\text{bw}}(Q; g_R)$ given by $\Lambda^{\text{Area}(R)}$. The purpose of the present paper is precisely to illustrate how all the concepts introduced in [Sim23] can be incarnated in the present Gaussian model and to how this framework is related to other approaches.

3.1.1.2 The geometry of domains and boundary weights

The idea developed in [Sim23] consists in cutting recursively the plane \mathbb{Z}^2 into domains by vertical and horizontal lines, called guillotine cuts, thereby producing half-planes, strips, half-strips, corners (quarters of plane) and rectangles (see figure 3.1).

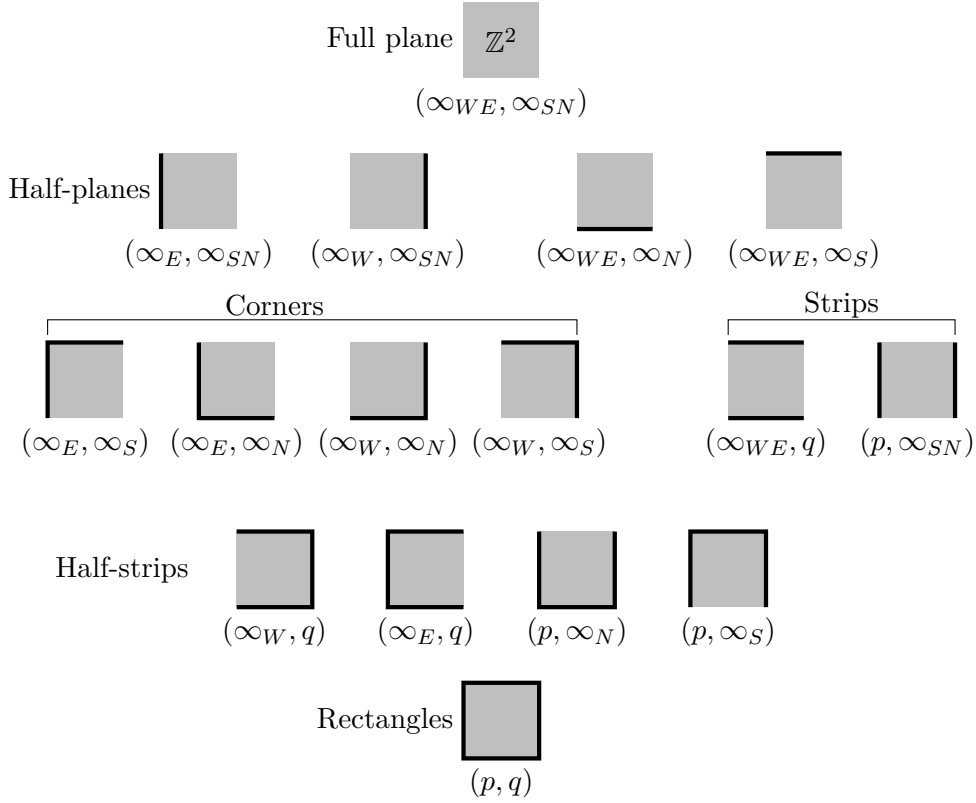


Figure 3.1: (extracted from [Sim23]) Five types of admissible patterns, with their pattern shape below. Any pattern in a line may appear in the guillotine partitions of some of the patterns above this line. All finite sizes p and q are elements of \mathbb{N}_1 .

We may then attach to each shape an algebraic object. Gluing two shapes into a larger shape then encodes a product of the corresponding algebraic objects. On a rectangle R with sizes $(1, 1)$, we associate the face Gaussian weight on four variables (the ones on the boundary edges of R):

$$\exp\left(-\frac{1}{2}x_{\partial R}^* Q x_{\partial R}\right)$$

Gluing two rectangles $R_1 = \{k, k+1\} \times \{l, l+1\}$ and $R_2 = \{k+1, k+2\} \times \{l, l+1\}$ produces a rectangle $R = \{k, k+2\} \times \{l, l+1\}$ with six boundary edges. Then the Gaussian weight associated to R is given by:

$$\int_{\mathcal{H}_2} \exp\left(-\frac{1}{2}x_{\partial R_1}^* Q x_{\partial R_1}\right) \exp\left(-\frac{1}{2}x_{\partial R_2}^* Q x_{\partial R_2}\right) dx_{\partial R_1 \cap \partial R_2} =: \alpha_{2,1} \exp\left(-\frac{1}{2}x_{\partial R}^* Q^{[2,1]} x_{\partial R}\right)$$

where the integration is performed on the variable associated to the edge between R_1 and R_2 . After integration, we recover a Gaussian density multiplied by a scalar.

Similar gluings can be performed vertically or use larger rectangles and one checks easily that, for any rectangle size (p, q) , one obtain an element

$$\alpha_{p,q} \exp\left(-\frac{1}{2}x_{\partial R}^* Q^{[p,q]} x_{\partial R}\right)$$

where $x_{\partial R}$ contains $2p$ variables in \mathcal{H}_1 and \mathcal{H}_2 . The algebraic rules behind these products can be encoded through an operadic structure with generalized associativities as described

in [Sim23]. Moreover, this is an elementary exercise based on Fubini's theorem to check that

$$Z_R^{\text{bw}}(Q; g_R) = \int_{\mathcal{H}_1^{2p} \oplus \mathcal{H}_2^{2q}} \alpha_{p,q} \exp\left(-\frac{1}{2} x_{\partial R}^* Q^{[p,q]} x_{\partial R}\right) g_R(x_{\partial R}) dx_{\partial R}$$

In order to achieve $Z_R^{\text{bw}}(Q; g_R) = \Lambda^{pq}$ for any rectangle with size (p, q) , [Sim23] also introduces objects associated to the following shapes:

- to each half-strip $] - \infty, k] \times \{l, l + 1\}$, an object $A_W : \mathcal{H}_2 \rightarrow \mathcal{B}_W$
- to each half-strip $]k, +\infty[\times \{l, l + 1\}$, an object $A_E : \mathcal{H}_2 \rightarrow \mathcal{B}_E$
- to each half-strip $\{k, k + 1\} \times] - \infty, l]$, an object $A_S : \mathcal{H}_2 \rightarrow \mathcal{B}_S$
- to each half-strip $\{k, k + 1\} \times]l, +\infty[$, an object $A_N : \mathcal{H}_2 \rightarrow \mathcal{B}_N$
- to each quarter of space, an object $U_{ab} \in \mathcal{B}_{ab}$ with $a \in \{S, N\}$ and $b \in \{W, E\}$.

such that, for any rectangle (p, q) ,

$$g_R \left(\begin{array}{c} y_1 \ y_2 \ \dots \ y_p \\ w_q \ \square \ z_q \\ \vdots \\ w_1 \ \square \ z_1 \\ x_1 \ x_2 \ \dots \ x_p \end{array} \right) = \begin{array}{|c|c|c|c|c|c|} \hline U_{\text{NW}} & A_{\text{N}}(y_1) & A_{\text{N}}(y_2) & \dots & A_{\text{N}}(y_p) & U_{\text{NE}} \\ \hline A_{\text{W}}(w_q) & \square & \square & \square & \square & A_{\text{E}}(z_q) \\ \hline \vdots & \square & \square & \square & \square & \vdots \\ \hline A_{\text{W}}(w_1) & \square & \square & \square & \square & A_{\text{E}}(z_1) \\ \hline U_{\text{SW}} & A_{\text{S}}(x_1) & A_{\text{S}}(x_2) & \dots & A_{\text{S}}(x_p) & U_{\text{SE}} \\ \hline \end{array} \quad (3.1.2)$$

where each gluing of neighbouring shapes along a half-line corresponds to a suitable product between the spaces \mathcal{B}_a and/or \mathcal{B}_{ab} . The local objects (in the sense that they depend only on at most one variable on one edge) have to be found such that the gluings of the half-strips with rectangles behave "nicely". For example, gluing $\{k, k + 1\} \times] - \infty, l]$ with $\{k, k + 1\} \times \{l, l + 1\}$ produce the half-strip $\{k, k + 1\} \times] - \infty, l + 1]$, which is the same as the initial half-strip shifted in the vertical direction. From an algebraic point of view, this corresponds to a requirement such as

$$\int_{\mathcal{H}_1} A_S(x) \exp\left(-\frac{1}{2} x_{\partial R}^* Q x_{\partial R}\right) dx_S \mapsto \Lambda A_S(x) \quad (3.1.3a)$$

on the boundary object A_S and more generally

$$\int_{\mathcal{H}_1^p} A_S(x_1) A_S(x_2) \dots A_S(x_p) \exp\left(-\frac{1}{2} x_{\partial R}^* Q^{[p,q]} x_{\partial R}\right) dx_S \mapsto \Lambda^{pq} A_S(x_1) A_S(x_2) \dots A_S(x_p) \quad (3.1.3b)$$

where a precise meaning has to be put on the arrows \mapsto . In particular, one observe that the l.h.s. is a function $\mathcal{H}_1^p \oplus \mathcal{H}_2^{2q} \rightarrow \mathcal{B}_S$ since the Gaussian weight is a function of the variables on the four sides of the rectangle and the integral only eliminates the variables on the bottom side. On the contrary, the r.h.s. is a function $\mathcal{H}_1^p \rightarrow \mathcal{B}_S$. The maps in the previous equations thus relate functions on different spaces such that the product structure on the $A_S(x_i)$ is preserved up to an overall scalar constant.

Precise constructions are made in [Sim23] and the purpose of the present paper is the rigorous construction of the spaces \mathcal{B}_a and \mathcal{B}_{ab} as well as the element $A_a(x)$ and U_{ab} together with the simplification maps up to an eigenvalue presented in (3.1.3): this is the purpose of section 3.5. The operadic structure in [Sim23] introduce also many other spaces associated

to other shapes, which have various relation with other tools that we also describe. In fact, due to analytical obstacles explained in section 3.5.3, the boundary objects A_a and U_{ab} will be lifted to quadratic forms denoted $B_Q^{a,\infty}$ and C_Q^{ab} respectively. These quadratic forms will be glued together as in representation (3.1.2) but in operator spaces denoted \mathcal{Q}_a and \mathcal{Q}_{ab} using non-linear operations, see section 3.5. We will thus drop the notations of [Sim23] used for boundary objects and employ these latter spaces.

3.1.2 Map of the paper

After this introduction, the present paper is divided into 4 sections.

In section 3.2 we give the basic definitions, useful in this article. After a precise definition of the two-dimensional model considered, we recall the operadic constructions of [Sim23]. We also give a quick reminder of the work of [Bod23] in one-dimensional settings, very useful for many computations.

In section 3.3 we describe the Guill_2 -algebra generated by elementary gluing product of Gaussian weights. We show that this structure can be lifted at the level of their quadratic forms through Schur products, see corollary 3.3.1. This is an important result as, due to analytical obstacles described in section 3.5.3, we will mainly work with quadratic forms rather than Gaussian weights. Elementary gluings applied to a single quadratic form Q gives rise to the definition of the surface power $Q^{[p,q]}$, related to the partition function on a rectangle of size (p, q) . In practice $Q^{[p,q]}$ is hard to compute. Nevertheless we show how to this quantity can be easily computed thanks to a harmonic decomposition, leading to a Stokes' formula, see Proposition 3.3.3. Roughly speaking, this result states that the harmonic expansion near the boundary of the rectangle of size (p, q) gives an explicit formula for $Q^{[p,q]}$. It is particularly interesting for the computation of large surface power, related to extended objects of [Sim23] like half-strips or corners. For instance, the limit $p \rightarrow \infty$ should be related to West/East strips operators. We thus give an explicit solution to the harmonic problem in a simple case, see Propositions 3.3.5 and 3.3.4. This gives heuristics to the limit of large surface power and help guessing half-strips and corners under non-rigorous computations, see section 3.3.3.

The section 3.4 is devoted to give classical statistical mechanics computation performed under periodic assumptions. We by start recalling the construction of the infinite-volume Gibbs measure associated with such translation-invariant Gaussian Markov processes. It can be defined directly through a white noise on the Fourier space and then on \mathbb{Z}^2 by inverse Fourier transform. Secondly, we apply the so-called transfer matrix approach to our model. This method, imposing periodicity in only one direction, allows to guess a first object of [Sim23]: the doubly-infinite strips, given by a positive definite convolution operator see (3.4.24) for instance. We define an infinite dimensional Schur product allowing to stack strips together and we show that they can be reduced to finite-dimensional Schur products performed on their spectral densities. The one-dimensional dynamic generated by doubly-infinite strips admits invariant boundary conditions, explicitly computed using the results of [Bod23], which give a second object of [Sim23]: the half-planes. In theorem 3.4.1 we show how these positive definite operators are related to a portion of the spectral curve (3.2.10). Doubly-infinite strips and half-planes are purely one-dimensional objects that will be very helpful for the next section.

The last section 3.5 exhibits, for the first time, purely two-dimensional boundary objects of [Sim23] in non-trivial settings. Following previous sections, it is performed at the level of quadratic forms. For analytical simplicity, we work under dihedral-invariance property, see (3.2.6). Based on the heuristics provided by section 3.3.3 and 3.4, we give a definition of half-strips operators, see Definitions 3.5.1 for instance. Thanks to symmetry assumptions, there is a correspondence between doubly-finite strips and some sub-blocks of half-strips

given by a "fold" operator (defined in (3.5.10)): the doubly-infinite strip can be folded to partially produce half-strips. The geometrical properties of half-strips operators allows many gluings (Schur product):

- gluing of opposite half-strips gives back doubly-infinite strips, see Proposition 3.5.2. As we just said, this operation is partially inverted by the fold operator;
- gluing of neighboring half-strips. We make sense of the associated infinite dimensional Schur product in Lemma 3.5.5 thanks to the properties of the fold operator (the product can be partially read on the "unfold" doubly-infinite strip, making the analysis easier);
- action of a face-weight Q at the end of the half-strip. This action lengthens the half-strip making the comparison with the original half-strip impossible: these are two operators acting on different spaces. Nevertheless, motivated by the definition of invariant boundary up to morphism in [Sim23], we show that half-strips operators associated to Q are invariant under the action of Q up to a morphism Φ , see theorems 3.5.1 and 3.5.2. Roughly speaking, the morphism Φ is a map which respects gluing products and sends back the lengthens half-strip to the original space, making the comparison with the original half-strip possible. We show that Φ is simply given by a conjugation involving the well-chosen operators.

These gluing product witnesses the two-dimensional nature of half-strips operators, as opposed to doubly-infinite strips (transfer matrix) which are purely one-dimensional objects.

In section 3.5.2 we give the definition of corner operators, see Definition 3.5.4 for instance. Diagonal block of this operator are obtained by folding half-planes operators. Additional tricks are required for guessing off-diagonal blocks, explained in the very end of section 3.5.1.2. As for half-strips, the geometrical properties of corners operators allows many gluings (Schur product):

- gluing of opposite corners gives back half-planes, see Proposition 3.5.4. As for half-strips, this operation is partially inverted by the fold operator;
- gluing of half-strips at one side of the corner. This lengthens the corner and can be performed in two directions. The morphism Φ can be extended to corners and Theorem 3.5.3 shows that the corner is invariant under the action of its associated half-strips up to Φ .

Half-strip and corner operators allow to define invariant boundary conditions for the Gaussian Markov fields we consider. Therefore, we obtain a family of consistent marginal and Kolmogorov's extension ensures the existence of the infinite volume Gibbs measure on the whole lattice \mathbb{Z}^2 . Nevertheless as we dropped Gaussian weights for focusing on quadratic forms, we lost any notion of partition function and eigenvalue. In section 3.5.6 we give a heuristic argument for recovering the eigenvalue associated to half-strips in term of non-rigorous analytical controls of Fredholm determinants.

3.2 Definitions and basic tools

3.2.1 The probabilistic model

3.2.1.1 Definitions

We start with the rigorous definition of the Gaussian Markov processes used in the present paper.

Definition 3.2.1. Let \mathcal{H}_1 and \mathcal{H}_2 be two finite-dimensional Hilbert spaces. Let D be a finite domain of \mathbb{Z}^2 . A $(\mathcal{H}_1, \mathcal{H}_2)$ -valued homogeneous Gaussian Markov on D is a process $(X_e)_{e \in E(D)}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that:

- (i) for any $e \in E_h(D)$, X_e is a \mathcal{H}_1 -valued random variable;
- (ii) for any $e \in E_v(D)$, X_e is a \mathcal{H}_2 -valued random variable
- (iii) there exists a Hermitian operator Q on $\mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_2$ such that the conditional law of $(X_e)_{e \in E(D)}$ knowing $(X_e)_{e \in \partial D}$ admits the following density $f_{D,Q}$ with respect to the product Lebesgue measure on the internal edges $e \in E(D) \setminus \partial D$:

$$f_{D,A,x_{\partial D}}(x_{E(D) \setminus \partial D}) = \frac{1}{Z_D(Q; x_{\partial D})} \prod_{f \in F(D)} \exp\left(-\frac{1}{2} x_{\partial f}^* Q x_{\partial f}\right) \quad (3.2.1)$$

In order to have notations that can be easily interpreted, any variable associated to a horizontal (resp. vertical) edge will be noted $X_e^{(1)}$ and $X_e^{(2)}$. Whenever a coordinate system is needed, the \mathcal{H}_1 -valued (resp. \mathcal{H}_2 -valued) variable $X_{\{(k,l),(k+1,l)\}}$ (resp. $X_{\{(k,l),(k,l+1)\}}$) will be noted $X_{k,l}^{(1)}$ (resp. $X_{k,l}^{(2)}$), where (k, l) corresponds to the left (resp. bottom) vertex of the edge.

Obtaining the complete law of the process requires to take the expectation w.r.t. the boundary variables $x_{\partial D}$ whose law is written $\nu_{\partial D}$. For deterministic boundary conditions, $\nu_{\partial D}$ is a product of Dirac measure on each edge and is not absolutely continuous w.r.t. the Lebesgue measure on the spaces \mathcal{H}_i . If the boundary law $\nu_{\partial D}$ admits a density w.r.t. the product Lebesgue measure, then the complete law of the process admits a representation:

$$\frac{1}{Z_D^{\text{bw}}(Q; g_D)} g_D(x_{\partial D}) \prod_{f \in F(D)} \exp\left(-\frac{1}{2} x_{\partial f}^* Q x_{\partial f}\right)$$

whose partition function is given by

$$Z_D^{\text{bw}}(Q; g_D) = \int g_D(x_{\partial D}) Z_D(Q; x_{\partial D}) dx_{\partial D}$$

The representation in (3.2.1) as a product of face weights ensures automatically the Markov property.

Property 3.2.1 (Markov property). *For any partition of the domain D into subdomains (D_1, \dots, D_n) , the n processes $(X_e)_{e \in E(D_i)}$, $1 \leq i \leq n$, are independent conditionnally to the r.v. $(X_e)_{e \in \cup_i \partial D_i}$.*

The product representation (3.2.1) can be rewritten as

$$f_{D,A,x_{\partial D}}(x_{E(D) \setminus \partial D}) = \frac{1}{Z_D(A; x_{\partial D})} \exp\left(-\frac{1}{2} x_D^* \tilde{Q}_D x_D\right) \quad (3.2.2)$$

where \tilde{Q}_D defines a quadratic form on the space

$$\bigoplus_{e \in E_h(D)} \mathcal{H}_1 \oplus \bigoplus_{e \in E_v(D)} \mathcal{H}_2$$

through the following summation:

$$\tilde{Q}_D(x_D, x_D) = \sum_{f \in F(D)} x_{\partial f}^* Q x_{\partial f}. \quad (3.2.3)$$

3.2.1.2 Translation and dihedral invariance

The face operator Q has a block decomposition

$$Q = \begin{pmatrix} Q_{SS} & Q_{SN} & Q_{SW} & Q_{SE} \\ Q_{NS} & Q_{NN} & Q_{NW} & Q_{NE} \\ Q_{WS} & Q_{WN} & Q_{WW} & Q_{WE} \\ Q_{ES} & Q_{EN} & Q_{EW} & Q_{EE} \end{pmatrix} \quad (3.2.4)$$

on the face space $\mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_2$ where the first (resp. second, third and fourth) space corresponds to the variable on the South (resp. North, West and East) edge of the face. The Hermitian property then requires $Q_{ab} = Q_{ba}^*$. We might as well refer to the following decompositions

$$Q_{[ab],[ab]} = \begin{pmatrix} Q_{aa} & Q_{ab} \\ Q_{ba} & Q_{bb} \end{pmatrix} \quad Q_{[ab],[c]} = \begin{pmatrix} Q_{ac} \\ Q_{bc} \end{pmatrix} \quad Q_{[c],[ab]} = \begin{pmatrix} Q_{ca} & Q_{cb} \end{pmatrix} \quad (3.2.5)$$

for $a, b, c \in \{\text{S, N, W, E}\}$. The lattice \mathbb{Z}^2 is invariant under the translations and the dihedral group. This group is generated by the rotation r of angle $\pi/2$ and the orthogonal symmetry s_Δ w.r.t. the first diagonal; this group also acts on the face operator Q through

$$r \triangleright Q = \begin{pmatrix} Q_{WW} & Q_{WE} & Q_{WN} & Q_{WS} \\ Q_{EW} & Q_{EE} & Q_{EN} & Q_{ES} \\ Q_{NW} & Q_{NE} & Q_{NN} & Q_{NS} \\ Q_{SW} & Q_{SE} & Q_{SN} & Q_{SS} \end{pmatrix}$$

$$s_\Delta \triangleright Q = \begin{pmatrix} Q_{WW} & Q_{WE} & Q_{WS} & Q_{WN} \\ Q_{EW} & Q_{EE} & Q_{ES} & Q_{EN} \\ Q_{SW} & Q_{SE} & Q_{SS} & Q_{SN} \\ Q_{NW} & Q_{NE} & Q_{NS} & Q_{NN} \end{pmatrix}$$

The two generators permute the two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 .

In order to present lighter computations, we will often consider in this paper models that are invariant under the action of dihedral group.

Lemma 3.2.1. *A face operator Q is invariant under the dihedral group if and only if both Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 are equal to some Hilbert space \mathcal{H} and there exist three self-adjoint operators T , A and U acting on \mathcal{H} such that*

$$Q = \begin{pmatrix} T & A & U & U \\ A & T & U & U \\ U & U & T & A \\ U & U & A & T \end{pmatrix} \quad (3.2.6)$$

Since we consider here only finite domains D , translation invariance in law may be only formulated between different domains D and D' and we have the following lemma:

Lemma 3.2.2. *Let D and D' be two domains. If there exist a translation θ of \mathbb{Z}^2 such that $D' = \theta(D)$ and such that, for all boundary configuration $x_{\partial D}$,*

$$g_{D'}((x_{\theta(e)})_{e \in \partial D}) = g_D((x_e)_{e \in \partial D}) \quad (3.2.7)$$

then the process $(X_e)_{e \in D'}$ with face operator Q and boundary weight $g_{D'}$ and the translated process $(X_{\theta(e)})_{e \in D}$ with face operator Q and boundary weight g_D have the same law.

3.2.1.3 From finite domains to infinite-volume Gibbs measure

Defining the random field X on \mathbb{Z}^2 or other infinite domains requires a bit more of analysis. Since the process is Gaussian this can be done by introducing a suitable infinite-dimensional Hilbert space: this is the purpose of section 3.4. If one emphasizes on the Markov property and the representation (3.2.1) as a product of face weights with some additional boundary weights, the easiest way (in appearance) to obtain an infinite volume Gibbs measure on \mathbb{Z}^2 relies on Kolmogorov's extension theorem that we now formulate for our model.

Property 3.2.2 (Gibbs measure from extension theorem). *Let Q be a face operator. Let \mathcal{D} be a collection of finite domains of \mathbb{Z}^2 such that:*

- (i) *there exists an increasing (for the partial order of inclusion) sequence of domains $(D_n)_{n \in \mathbb{N}}$ with $\cup_n D_n = \mathbb{Z}^2$,*
- (ii) *there exists a collection of boundary weights $(g_D)_{D \in \mathcal{D}}$ such that, for any $D \subset D'$, the restriction $(X_e)_{e \in D}$ of the process $(X_e)_{e \in D'}$ with face operator Q and boundary weight g_D is a Gaussian Markov process on D with face operator Q and boundary weight g_D .*

Then there exists a process $(\tilde{X}_e)_{e \in E}$ on the whole lattice \mathbb{Z}^2 such that, for all $D \in \mathcal{D}$, the restriction of \tilde{X} to the edges of D is Gaussian Markov process with face operator Q and boundary weight g_D .

If the set \mathcal{D} is invariant by translation, i.e. for any translation θ of \mathbb{Z}^2 and any $D \in \mathcal{D}$, $\theta(D) \in \mathcal{D}$ and the face weights g_D and $g_{\theta(D)}$ satisfy (3.2.7), then the process \tilde{X} on \mathbb{Z}^2 is invariant in law under translations.

If moreover the set \mathcal{D} is invariant under the action of the dihedral group and also the face weights g_D and the face operator Q , the process \tilde{X} on \mathbb{Z}^2 is also invariant under the dihedral group.

The major difficulty in this construction, circumvented in [Sim23] for rectangles, relies in the construction of consistent weights $(g_D)_{D \in \mathcal{D}}$, which is made explicitly in the present paper.

3.2.1.4 Algebraic curves related to face operators Q .

The functions Ψ_Q , Ψ_Q^h and Ψ_Q^v . Since the extended process \tilde{X} on \mathbb{Z}^2 is a Gaussian process, translation invariance can be nicely described by Fourier transform. In this case, in the suitable Hilbert spaces, we may introduce the horizontal, vertical and bidirection Fourier transforms, for $\theta, \varphi \in [0, 2\pi)$,

$$\begin{aligned} \hat{X}_l^{(1),h}(\theta_1) &= \sum_{k \in \mathbb{Z}} X_{\{(k,l),(k+1,l)\}} e^{i\theta_1 k} & \hat{X}_l^{(2),h}(\theta_1) &= \sum_{k \in \mathbb{Z}} X_{\{(k,l),(k,l+1)\}} e^{i\theta_1 k} \\ \hat{X}_k^{v,(1)}(\theta_2) &= \sum_{l \in \mathbb{Z}} X_{\{(k,l),(k+1,l)\}} e^{i\theta_2 l} & \hat{X}_k^{v,(2)}(\theta_2) &= \sum_{l \in \mathbb{Z}} X_{\{(k,l),(k,l+1)\}} e^{i\theta_2 l} \\ \hat{X}^{(1)}(\theta_1, \theta_2) &= \sum_{(k,l) \in \mathbb{Z}^2} X_{\{(k,l),(k+1,l)\}} e^{i\theta_1 k + i\theta_2 l} & \hat{X}^{(2)}(\theta_1, \theta_2) &= \sum_{(k,l) \in \mathbb{Z}^2} X_{\{(k,l),(k,l+1)\}} e^{i\theta_1 k + i\theta_2 l} \end{aligned}$$

where the exponent (i) indicates that it belongs to \mathcal{H}_i and thus corresponds to horizontal and vertical edges.

Introducing (at least formally, see below for rigorous constructions) such expressions in (3.2.3) for $D = \mathbb{Z}^2$ indicates that \tilde{Q} acts block-diagonally on the Fourier modes through

$\Psi_Q^h(e^{i\theta_1})$, $\Psi_Q^v(e^{i\theta_2})$ and $\Psi_Q(e^{i\theta_1}, e^{i\theta_2})$ respectively. These blocks are defined through the following three functions:

$$\begin{aligned} \Psi_Q^h : \mathbb{C}^* &\rightarrow \text{End}(\mathcal{H}_1 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2) \\ z &\mapsto \begin{pmatrix} Q_{SS} & Q_{SN} & Q_{SW} + zQ_{SE} \\ Q_{NS} & Q_{NN} & Q_{NW} + zQ_{NE} \\ Q_{WS} + z^{-1}Q_{ES} & Q_{WN} + z^{-1}Q_{EN} & \phi_Q^{\text{WE}}(z) \end{pmatrix} \end{aligned} \quad (3.2.8a)$$

for the horizontal Fourier transform,

$$\begin{aligned} \Psi_Q^v : \mathbb{C}^* &\rightarrow \text{End}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_2) \\ w &\mapsto \begin{pmatrix} \phi_Q^{\text{SN}}(w) & Q_{SW} + w^{-1}Q_{NW} & Q_{SE} + w^{-1}Q_{NE} \\ Q_{WS} + wQ_{WN} & Q_{WW} & Q_{WE} \\ Q_{ES} + wQ_{EN} & Q_{EW} & Q_{EE} \end{pmatrix} \end{aligned} \quad (3.2.8b)$$

for the vertical Fourier transform and

$$\begin{aligned} \Psi_Q : \mathbb{C}^* \times \mathbb{C}^* &\rightarrow \text{End}(\mathcal{H}_1 \oplus \mathcal{H}_2) \\ (z, w) &\mapsto \begin{pmatrix} \phi_Q^{\text{SN}}(w) & Q_{SW} + zQ_{SE} + w^{-1}Q_{NW} + w^{-1}zQ_{NE} \\ Q_{WS} + z^{-1}Q_{ES} + wQ_{WN} + wz^{-1}Q_{EN} & \phi_Q^{\text{WE}}(z) \end{pmatrix} \end{aligned} \quad (3.2.8c)$$

for the bidirectional Fourier transform, where the reduced diagonal blocks are given by:

$$\begin{aligned} \phi_Q^{\text{WE}}(z) &= Q_{WW} + Q_{EE} + zQ_{WE} + z^{-1}Q_{EW} \in \text{End}(\mathcal{H}_2) \\ \phi_Q^{\text{SN}}(w) &= Q_{SS} + Q_{NN} + wQ_{SN} + w^{-1}Q_{NS} \in \text{End}(\mathcal{H}_1). \end{aligned}$$

These diagonal blocks are related to the meromorphic function Φ_A of [Bod23] and related to the one-dimensional processes in \mathcal{H}_1 and \mathcal{H}_2 living on columns and rows of the two-dimensional lattice \mathbb{Z}^2 . We might refer to the blocks of Ψ_Q through the following notation

$$\Psi_Q(z, w) = \begin{pmatrix} (\Psi_Q(z, w))_{1,1} & (\Psi_Q(z, w))_{1,2} \\ (\Psi_Q(z, w))_{2,1} & (\Psi_Q(z, w))_{2,2} \end{pmatrix}. \quad (3.2.9)$$

for all z and $w \in \mathbb{C}^*$.

The argument of the functions Ψ_Q , Ψ_Q^h and Ψ_Q^v are extended to the whole set \mathbb{C}^* since we will see below that values outside the unit circle are related to harmonic functions which will be fundamental in the next sections.

Property 3.2.3. *For any $(z, w) \in \mathbb{C}^* \times \mathbb{C}^*$, it holds:*

$$\begin{aligned} \Psi_Q^h(z)^* &= \Psi_Q^h(\bar{z}^{-1}) & \Psi_Q^v(w)^* &= \Psi_Q^v(\bar{w}^{-1}) \\ \Psi_Q(z, w)^* &= \Psi_Q(\bar{z}^{-1}, \bar{w}^{-1}) \end{aligned}$$

The spectral curves. We now introduce for each function Ψ_Q^h , Ψ_Q^v and Ψ_Q the following sets:

$$\mathcal{S}_Q^h = \{z \in \mathbb{C}^*; \det \Psi_Q^h(z) = 0\} \quad (3.2.10a)$$

$$\mathcal{S}_Q^v = \{w \in \mathbb{C}^*; \det \Psi_Q^v(w) = 0\} \quad (3.2.10b)$$

$$\mathcal{S}_Q = \{(z, w) \in \mathbb{C}^* \times \mathbb{C}^*; \det \Psi_Q(z, w) = 0\} \quad (3.2.10c)$$

which correspond to the parameters z and w for which the corresponding function is not invertible. The first two sets are finite sets whose cardinal is given by the degree of the Laurent polynomials $\Psi_Q^{h|v}$. The third set \mathcal{S}_Q is an algebraic curve.

Lemma 3.2.3. *If Q is positive definite, then for all θ_1 and $\theta_2 \in [0, 2\pi]$, the operators $\phi_Q^{\text{WE}}(e^{i\theta_1})$, $\phi_Q^{\text{SN}}(e^{i\theta_1})$, $\Psi_Q^h(e^{i\theta_1})$, $\Psi_Q^v(e^{i\theta_1})$ and $\Psi_Q(e^{i\theta_1}, e^{i\theta_2})$ are positive definite. Hence,*

$$\mathcal{S}_Q^h \cap S^1 = \mathcal{S}_Q^v \cap S^1 = \mathcal{S}_Q \cap (S^1 \times S^1) = \emptyset.$$

From the previous lemma we deduce that Ψ_Q is invertible on $S^1 \times S^1$ with continuous inverse $\Psi_Q^{-1} : S^1 \times S^1 \rightarrow \text{End}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and, similarly to (3.2.9), we let its block decomposition

$$\Psi_Q^{-1} = \begin{pmatrix} ((\Psi_Q^{-1})_{1,1}) & ((\Psi_Q^{-1})_{1,2}) \\ ((\Psi_Q^{-1})_{2,1}) & ((\Psi_Q^{-1})_{2,2}) \end{pmatrix}. \quad (3.2.11)$$

Fourier coefficients. Let \mathcal{H} and \mathcal{H}' be finite-dimensional Hilbert spaces and $\hat{a} : S^1 \times S^1 \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{H}')$ continuous. The bi-dimensional Fourier coefficient of order $(k, l) \in \mathbb{Z}^2$ of \hat{a} is denoted

$$C_{k,l}(\hat{a}) := \frac{1}{(2\pi)^2} \int_{[0, 2\pi]^2} \hat{a}(e^{i\theta_1}, e^{i\theta_2}) e^{-ik\theta_1} e^{-il\theta_2} d\theta_1 d\theta_2 \in \mathcal{L}(\mathcal{H}, \mathcal{H}'). \quad (3.2.12)$$

Moreover, we introduce its partial Fourier coefficients of order $k \in \mathbb{Z}$ as

$$C_{k,\bullet}(\hat{a}) := \frac{1}{2\pi} \int_0^{2\pi} \hat{a}(e^{i\theta}, \cdot) e^{-ik\theta} d\theta \quad C_{\bullet,k}(\hat{a}) := \frac{1}{2\pi} \int_0^{2\pi} \hat{a}(\cdot, e^{i\theta}) e^{-ik\theta} d\theta. \quad (3.2.13)$$

Recurrent in the paper, we make simpler the notation of the Fourier coefficients of Ψ_Q^{-1} , denoted

$$C_{k,l} := C_{k,l}(\Psi_Q^{-1}) \in \text{End}(\mathcal{H}_1 \oplus \mathcal{H}_2) \quad C_{k,l}^{i,j} := C_{k,l}((\Psi_Q^{-1})_{i,j}) \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i) \quad (3.2.14)$$

$$C_{k,\bullet} := C_{k,\bullet}(\Psi_Q^{-1}) \quad C_{\bullet,k} := C_{\bullet,k}(\Psi_Q^{-1}) \quad C_{k,\bullet}^{i,j} := C_{k,\bullet}((\Psi_Q^{-1})_{i,j}) \quad C_{\bullet,k}^{i,j} := C_{\bullet,k}((\Psi_Q^{-1})_{i,j}). \quad (3.2.15)$$

for all $i, j \in \{1, 2\}$ and $k, l \in \mathbb{Z}$.

The particular case of a face weight invariant under the dihedral group. If the face operator Q is invariant under the dihedral group and is parametrized as in (3.2.6), the expression of the Fourier transform can be simplified and we have

$$\Psi_Q(z, w) = \begin{pmatrix} 2T + (w + w^{-1})A & (1+z)(1+w^{-1})U \\ (1+z^{-1})(1+w)U & 2T + (z + z^{-1})A \end{pmatrix} \quad (3.2.16)$$

In the one-dimensional case $\mathcal{H} = \mathbb{C}$ with $T = t$, $A = a$ and $U = u$, computations are made easy and the spectral curve is given by

$$\mathcal{S}_Q = \pi^{-1}(\mathcal{S}'_Q)$$

with the following definitions:

$$\pi(z, w) = \frac{1}{2}(z + z^{-1}, w + w^{-1}) \\ \mathcal{S}'_Q = \left\{ (z, w) \in \mathbb{C}^*; (t + aw)(t + az) = u^2(1+z)(1+w) \right\}$$

3.2.1.5 One-dimensional Gaussian Markov processes

The boundary structure of 1D Gaussian Markov processes have been studied in [Bod23]. We briefly recall some results, important for future 2D computations. Let \mathcal{H} be a finite-dimensional complex Hilbert space. Similarly to the definition of two-dimensional Gaussian Markov fields 3.2.1, we define a \mathcal{H} -valued one-dimensional Gaussian Markov process through its density in terms of a Hermitian and positive definite operator Q^{1D} on $\mathcal{H} \oplus \mathcal{H}$ with block decomposition

$$Q^{1D} = \begin{pmatrix} Q_{LL}^{1D} & Q_{LR}^{1D} \\ Q_{RL}^{1D} & Q_{RR}^{1D} \end{pmatrix}.$$

The first (resp. second) space corresponds to the variable on the Left (resp. Right). Being given the left and the right invariant boundary conditions, and therefore consistent marginal laws, the unique infinite-dimensional Gibbs measure is given by Kolmogorov's extension theorem. For instance, the left invariant boundary is given by a self-adjoint and positive definite matrix B_L^{1D} , acting on \mathcal{H} , which satisfies the fix point relation

$$B_L^{1D} = Q_{RR}^{1D} - Q_{RL}^{1D} (B_L^{1D} + Q_{LL}^{1D})^{-1} Q_{LR}^{1D}. \quad (3.2.17)$$

Such a matrix is built after a precise study of the meromorphic function

$$\phi_{Q^{1D}}(z) = Q_{LL}^{1D} + Q_{RR}^{1D} + zQ_{LR}^{1D} + z^{-1}Q_{RL}^{1D} \in \text{End}(\mathcal{H})$$

for all $z \in \mathbb{C}^*$, together with its zeros

$$\mathcal{S}_{Q^{1D}}^{1D} := \{z \in \mathbb{C}^*; \det \phi_{Q^{1D}}(z) = 0\}$$

and their associated bases provided by the ones of $\ker \phi_{Q^{1D}}(z)$ for all $z \in \mathcal{S}_{Q^{1D}}^{1D}$. We briefly recall this fact. Let $\text{mult}_{Q^{1D}}(z)$ be the multiplicity of the root $z \in \mathcal{S}_{Q^{1D}}^{1D}$ with respect to the polynomial $\det(X\phi_{Q^{1D}}(X))$. Most of the results in [Bod23] are stated under the following assumptions.

Assumption (1D). We assume Q_{LR}^{1D} to be invertible and $\text{mult}_{Q^{1D}}(z) = \dim \ker \phi_{Q^{1D}}(z) = 1$ for all $z \in \mathcal{S}_{Q^{1D}}^{1D}$. Then, we fix $u_z \neq 0$ such that, for all $z \in \mathcal{S}_{Q^{1D}}^{1D}$,

$$\ker \phi_{Q^{1D}}(z) = \text{Vect}_{\mathbb{C}}(u_z).$$

In [Bod23], these assumptions are only made to simplify the proofs. See [Bod23] for related discussions and extensions. In this article, whenever 1D results are needed, we will work under these assumptions. The main theorem of [Bod23] states the following.

Theorem 3.2.1. *Under assumptions (1D), letting*

$$\mathcal{S}_{Q^{1D}}^{>1} = \{z \in \mathcal{S}_{Q^{1D}}^{1D}; |z| > 1\},$$

the family $(u_z)_{z \in \mathcal{S}_{Q^{1D}}^{>1}}$ is a basis of \mathcal{H} . We let $W_{Q^{1D}} \in \text{End}(\mathcal{H})$ be diagonal in this basis with $(W_{Q^{1D}})u_z = (1/z)u_z$ for all $z \in \mathcal{S}_{Q^{1D}}^{>1}$. Then, the self-adjoint and positive definite matrix $B_L^{1D} = Q_{RR}^{1D} + Q_{RL}^{1D}W_{Q^{1D}}$ satisfies (3.2.17).

Compared to [Bod23], there is a slight rewriting $W_{Q^{1D}} = W_{>1}^{-1}$ where $W_{>1}^{-1}$ is the operator defined in Theorem 1.3 of [Bod23]. The proof of this theorem is based on the following equation, very useful in practice

$$Q_{LL}^{1D} + Q_{RR}^{1D} + Q_{RL}^{1D}W_{Q^{1D}} + Q_{LR}^{1D}W_{Q^{1D}}^{-1} = 0. \quad (3.2.18)$$

There is of course an analogous theorem concerning right invariant boundary condition B_R^{1D} . In particular, if $Q_{LR} = Q_{RL}$, we have $B_R^{1D} = B_L^{1D}$. Otherwise, left and right invariant boundary conditions always verify

$$B_L^{1D} + B_R^{1D} = (C_0(\phi_{Q^{1D}}^{-1}))^{-1}, \quad (3.2.19)$$

where $C_0(\phi_{Q^{1D}}^{-1})$ is the 0-th Fourier coefficient of $\phi_{Q^{1D}}^{-1}$, see Lemma 5.5 of [Bod23].

Some additional properties. Here are some properties, under assumptions (1D), concerning the Fourier coefficients of $\phi_{Q^{1D}}^{-1}$, useful for future 2D computations. For all $k \in \mathbb{Z}$, we denote by $C_k(\phi_{Q^{1D}}^{-1})$ the k -th Fourier coefficient of $\phi_{Q^{1D}}^{-1}$. The lemma 5.4 of [Bod23] gives the following explicit formula

$$W_{Q^{1D}} = C_1(\phi_{Q^{1D}}^{-1})(C_0(\phi_{Q^{1D}}^{-1}))^{-1}. \quad (3.2.20)$$

For all $k \in \mathbb{Z}$, the self-adjoint property of Q^{1D} implies that

$$C_k(\phi_{Q^{1D}}^{-1})^* = C_{-k}(\phi_{Q^{1D}}^{-1}). \quad (3.2.21)$$

Moreover, these Fourier coefficients follow a second order recursive relation,

$$C_k(\phi_{Q^{1D}}^{-1})(Q_{LL}^{1D} + Q_{RR}^{1D}) + C_{k+1}(\phi_{Q^{1D}}^{-1})Q_{RL}^{1D} + C_{k-1}(\phi_{Q^{1D}}^{-1})Q_{LR}^{1D} = I_{\mathcal{H}}\delta_{k,0} \quad (3.2.22)$$

for all $k \in \mathbb{Z}$, see Lemma 2.3 of [Bod23]. Finally, for all $kl \geq 0$, the Remark 5.1 of [Bod23], gives

$$C_k(\phi_{Q^{1D}}^{-1})(C_0(\phi_{Q^{1D}}^{-1}))^{-1}C_l(\phi_{Q^{1D}}^{-1}) = C_{k+l}(\phi_{Q^{1D}}^{-1}). \quad (3.2.23)$$

This last equation is related to conditional covariances and it gives the Markov property for the process in thermodynamic limit, see Theorem 3.1 of [Bod23].

3.2.2 Operadic constructions

We recall here the basic construction of [Sim23] and precise them in the present Gaussian context. The reader may refer to [MSS02] for a general view on operads and [Yau16] for coloured operads.

3.2.2.1 Generalities on operads

An operad generalizes the notion of internal laws on sets by focusing on the intrinsic properties of these laws regardless of the underlying sets of objects. This can be formalized in the following way.

Definition 3.2.2 (coloured operad). Let C be a set, called the set of colours. A coloured operad over C is a collection of sets

$$P = (P(c; c_1, \dots, c_n))_{n \in \mathbb{N}, (c, c_1, \dots, c_n) \in C^{n+1}}$$

such that:

- (i) for any $n \in \mathbb{N}$, any colours $(c, c_1, \dots, c_n) \in C^{n+1}$ and any sequences $(c_1^{(i)}, \dots, c_{n_i}^{(i)}) \in C^{m_i}$ with $1 \leq i \leq n$, there is a composition function:

$$\begin{aligned} \circ : P(c; c_1, \dots, c_n) \times \prod_{i=1}^n P(c_i; c_1^{(i)}, \dots, c_{n_i}^{(i)}) &\rightarrow P(c; c^{(1)}c^{(2)} \dots c^{(n)}) \\ (\alpha, \alpha_1, \dots, \alpha_n) &\mapsto \alpha \circ (\alpha_1, \dots, \alpha_n) \end{aligned}$$

where $c^{(1)}c^{(2)} \dots c^{(n)}$ is the sequence of length $\sum_{i=1}^n n_i$ obtained by concatenation of the n sequences;

- (ii) for any $\alpha \in P(c; \mathbf{c})$ with $\mathbf{c} = (c_i)_{1 \leq i \leq n}$, any $\alpha_i \in P(c_i; \mathbf{c}^{(i)})$ with $\mathbf{c}^{(i)} = (c_j^{(i)})_{1 \leq j \leq m_i}$ with $1 \leq i \leq n$ and any $\alpha_{i,j} \in P(c_j^{(i)}; \mathbf{c}^{(i,j)})$, $1 \leq i \leq n$ and $1 \leq j \leq m_i$, it holds:

$$\begin{aligned} & (\alpha \circ (\alpha_1, \dots, \alpha_n)) \circ (\alpha_{1,1}, \dots, \alpha_{1,m_1}, \dots, \alpha_{n,1}, \dots, \alpha_{n,m_n}) \\ &= \alpha \circ (\alpha_1 \circ (\alpha_{1,1}, \dots, \alpha_{1,m_1}), \dots, \alpha_n \circ (\alpha_{n,1}, \dots, \alpha_{n,m_n})) \end{aligned}$$

- (iii) for any $c \in C$, there is an element $\text{id}_c \in P(c; c)$ such that, for any $\alpha \in P(c; c_1, \dots, c_n)$, $\alpha \circ (\text{id}_{c_1}, \dots, \text{id}_{c_n}) = \alpha$ and $\text{id}_c \circ \alpha = \alpha$.

- (iv) for any set $\alpha \in P(c; c_1, \dots, c_n)$ and any permutation $\sigma \in \mathfrak{S}_n$, there is an element $\alpha_\sigma \in P(c; c_{\sigma(1)}, \dots, c_{\sigma(n)})$ such that, for any sequence $(\beta_i)_{1 \leq i \leq n}$ with $\beta_i \in P(c_i; c_1^{(i)}, \dots, c_{n_i}^{(i)})$

$$\alpha_\sigma \circ (\beta_{\sigma(1)}, \dots, \beta_{\sigma(n)}) = \alpha \circ (\beta_1, \dots, \beta_n)$$

Definition 3.2.3 (structure over an operad). Let P be a coloured operad over a set of colours C . A P -algebra is a collection of sets $(E_c)_{c \in C}$ such that:

- (i) for any $n \in \mathbb{N}$ and for any sequence $(c, c_1, \dots, c_n) \in C^{n+1}$, for any $\alpha \in P(c; c_1, \dots, c_n)$, there is a map

$$m_\alpha : E_{c_1} \times \dots \times E_{c_n} \rightarrow E_c$$

- (ii) for any $n \in \mathbb{N}$, for any $c \in C$, for any sequence $(c_i)_{1 \leq i \leq n}$, for any finite sequences $(c_j^{(i)})_{1 \leq j \leq m_i}$, for any $\alpha \in P(c; c_1, \dots, c_n)$ and any $\alpha_i \in P(c_i; c_1^{(i)}, \dots, c_{m_i}^{(i)})$ and for any sequences of elements $(u_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m_i}$,

$$\begin{aligned} & m_{\alpha \circ (\alpha_1, \dots, \alpha_n)}(u_{1,1}, \dots, u_{1,m_1}, \dots, u_{n,1}, \dots, u_{n,m_n}) \\ &= m_\alpha(m_{\alpha_1}(u_{1,1}, \dots, u_{1,m_1}), \dots, m_{\alpha_n}(u_{n,1}, \dots, u_{n,m_n})) \end{aligned}$$

- (iii) for any $c \in C$ and $u \in E_c$, $m_{\text{id}_c}(u) = u$

- (iv) for any $c \in C$ and any finite sequence $(c_i)_{1 \leq i \leq n}$, for any $\alpha \in P(c; c_1, \dots, c_n)$ and any finite sequence $(u_i)_{1 \leq i \leq n}$ with $u_i \in E_{c_i}$,

$$m_{\alpha_\sigma}(u_{\sigma(1)}, \dots, u_{\sigma(n)}) = m_\alpha(u_1, \dots, u_n)$$

3.2.2.2 The guillotine operad in dimension 2

Geometry of guillotine cuts. In the present case, we will consider a particular operad, called the guillotine operad introduced in [Sim23], which encodes how the partition functions on large rectangles are computed out of partition functions on small rectangles (rewriting then the standard rules of integration) and, *at the same time*, provides a local-to-global construction of boundary weights.

We recall here the definition of a guillotine partition of a rectangle and a pointed guillotine partition and their equivalence classes.

Definition 3.2.4. Let R be a rectangular shape in \mathbb{Z}^2 , i.e. a cartesian product $\{x_W, x_W + 1, \dots, x_E\} \times \{y_S, y_S + 1, \dots, y_N\}$ with $x_W, x_E, y_S, y_N \in \mathbb{Z} \cup \{-\infty, +\infty\}$ and $x_W < x_E$ and $y_S < y_N$.

A vertical (resp. horizontal) guillotine cut on R is a integer $x_W < m < x_E$ (resp. $y_S < m < y_N$) such that it defines two rectangular shapes R_1 and R_2 with $R_1 \cup R_2 = R$ and $R_1 \cap R_2 = \{m\} \times \{y_S, \dots, y_N\}$ (resp. $\{x_W, \dots, x_E\} \times \{m\}$). The couple (R_1, R_2) is called an *elementary guillotine partition* of R .

A guillotine partition of R is either R itself or an elementary guillotine partition or a collection of rectangular shapes (R_1, \dots, R_k) with $k \geq 3$ such that there exists a permutation $\sigma \in \mathcal{S}_k$, an integer $j \in \{1, \dots, k-1\}$ and a guillotine cut that cuts R into two rectangles R' and R'' such that $(R_{\sigma(1)}, \dots, R_{\sigma(j)})$ is a guillotine partition of R' and $(R_{\sigma(j+1)}, \dots, R_{\sigma(k)})$ is a guillotine partition of R'' .

Definition 3.2.5. Two guillotine partitions (R_1, \dots, R_k) and (R'_1, \dots, R'_l) of a rectangular shape R that does not contain neither a horizontal nor a vertical line are *equivalent* if and only if $k = l$ and there exists a translation θ on \mathbb{Z}^2 such that, for all $1 \leq i \leq k$, $\theta(R_i) = R'_i$.

Definition 3.2.6. If R contains an full infinite horizontal line (resp. a vertical line, resp. both), the rectangle shape R is called *doubly-infinite* and a *pointed* version of R is a couple (R, x) (resp. (R, y) , resp. a 3-uplet (R, x, y)). A *pointed* guillotine partition of R is a couple (ρ, b) such that ρ is a guillotine partition of R and R a base point $(x, y$ or $(x, y))$.

Two pointed guillotine partitions (ρ, b) and (ρ', b') are equivalent if and only if there exists a translation θ on \mathbb{Z}^2 such that $\theta(\rho) = \rho'$ and $\theta(b) = b'$.

The size of a rectangular shape $\{x_W, x_W + 1, \dots, x_E\} \times \{y_S, y_S + 1, \dots, y_N\}$ is defined as

$$(p, q) = (x_E - x_W, y_N - y_S) \in C_{LR} = (\mathbb{N}_1 \cup \{\infty_W, \infty_E, \infty_{WE}\}) \times (\mathbb{N}_1 \cup \{\infty_S, \infty_N, \infty_{SN}\}) \quad (3.2.24)$$

with the conventions for finite u :

$$\begin{aligned} u - (-\infty) &= \infty_W & u - (-\infty) &= \infty_S \\ (+\infty) - x &= \infty_E & (+\infty) - x &= \infty_N \\ (+\infty) - (-\infty) &= \infty_{WE} & (+\infty) - (-\infty) &= \infty_{SN} \end{aligned}$$

A rectangular shape with a size equal to ∞_{WE} or ∞_{SN} in one direction contains a line in this direction and thus requires to be pointed. It is an easy exercise to check that two equivalent guillotine partitions are made of rectangular shapes with the same sizes. The following theorem is proved in [Sim23].

Theorem 3.2.2. For all integer $n \in \mathbb{N}$ and any sequence of colours $(c, c_1, \dots, c_n) \in C_{LR}^{n+1}$, let $\text{Guill}_2(c; c_1, \dots, c_n)$ be the set of equivalence classes of guillotine partitions (R_1, \dots, R_n) (resp. pointed guillotine partitions (R_1, \dots, R_n, b) if c contains at least a size equal to ∞_{WE} or ∞_{SN}) such that, for all $1 \leq i \leq n$, the size of R_i is equal to c_i . These sets define a coloured operad Guill_2^{LR} over the set of colours C_{LR} with compositions explicited in [Sim23].

Property 3.2.4. If the rectangular shapes are restricted to finite rectangles only, the colours are restricted to the set $C = \mathbb{N}_1^2$ and the sets $\text{Guill}_2(c; c_1, \dots, c_n)$ define a coloured operad $\text{Guill}_2^{(r)}$ over C , which can be seen as a sub-operad of the previous one Guill_2^{LR} .

Graphical notations. We adopt the graphical notations of [Sim23].

For a given Guill_2 -algebra $(E_c)_{c \in C}$, the product m_ρ associated to a guillotine partition $\rho = (R_1, R_2, R_3)$ of a rectangle R evaluated on $(u_1, u_2, u_3) \in E_{c_1} \times E_{c_2} \times E_{c_3}$ is written in one of the three following equivalent ways:

$$m_\rho(u_1, u_2, u_3) = \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & \\ \hline \end{array} (u_1, u_2, u_3) = \begin{array}{|c|c|} \hline u_2 & u_3 \\ \hline u_1 & \\ \hline \end{array}$$

where the drawing correspond a rectangle R of size 3 with partitions R_1 of size (2, 1), R_2 of size (2, 2) and R_3 of size (1, 3). The elements of the sets E_c are written either as

arguments with an order associated to the labelling of the inner rectangles or directly inside the rectangles to shorten the notation.

If a rectangular shape is infinite in one direction, the external line is not drawn. For example, the product associated to the elementary guillotine partition

$$\rho = ((-\infty, x-1] \times \{y, y+1, y+2\}, (-\infty, x] \times \{y+2, y+3\}, \{x-1, x\} \times \{y, y+1, y+2\})$$

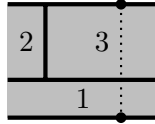
applied to $(a, b, c) \in E_{\infty_W, 2} \times E_{\infty_W, 1} \times E_{1, 2}$ is represented as

$$m_\rho(a, b, c) = \begin{array}{|c|c|} \hline 2 \\ \hline 1 & 3 \\ \hline \end{array} (a, b, c) = \begin{array}{|c|c|} \hline b \\ \hline a & c \\ \hline \end{array}$$

For rectangular shapes with a size equal to ∞_{WE} or ∞_{SN} that require a pointing in this direction, we add a thick point and a dotted line in the drawing. For example, the guillotine partition

$$\rho = (\mathbb{Z} \times \{y, y+1\}, (-\infty, x] \times \{y+1, y+2, y+3\}, [x, +\infty) \times \{y+1, y+2, y+3\})$$

of $\mathbb{Z} \times \{y, \dots, y+3\}$ with a pointing at $x+2$ is represented by



Guill₂^(r)-algebra of quasi-densities. Let $(M_1, \mathcal{M}_1, \mu_1)$ and $(M_2, \mathcal{M}_2, \mu_2)$ be two σ -finite measure spaces. We introduce, for any $(p, q) \in C$, the set $L_{p,q}^\infty$ defined as:

$$L_{p,q}^\infty = L^\infty(M_1^{2p} \times M_2^{2q})$$

Elements in these sets are called quasi-densities on the boundary of a rectangle (p, q) in the present paper. For the two type of elementary guillotine partitions with a vertical and a horizontal cut

$$\rho_{p_1, p_2 | q} = ([0, p_1] \times [0, q], [p_1, p_1 + p_2] \times [0, q])$$

$$\rho_{p | q_1, q_2} = ([0, p] \times [0, q_1], [0, p] \times [q_1, q_1 + q_2])$$

of the rectangle $[0, p_1 + p_2] \times [0, q]$, we define

$$m_{\rho_{p_1, p_2 | q}} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} : L_{p_1, q}^\infty \times L_{p_2, q}^\infty \rightarrow L_{p_1 + p_2, q}^\infty \quad (3.2.25a)$$

$$m_{\rho_{p | q_1, q_2}} = \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} : L_{p, q_1}^\infty \times L_{p, q_2}^\infty \rightarrow L_{p, q_1 + q_2}^\infty \quad (3.2.25b)$$

through the following integrals:

$$m_{\rho_{p_1, p_2 | q}}(f_1, f_2)(u_S^{(1)}, u_S^{(2)}, u_N^{(1)}, u_N^{(2)}, u_W, u_E) = \int_{M_2^q} f_1(u_S^{(1)}, u_N^{(1)}, u_W, v) f_2(u_S^{(2)}, u_N^{(2)}, v, u_E) dv \quad (3.2.25c)$$

$$m_{\rho_{p | q_1, q_2}}(f_1, f_2)(u_S, u_N, u_W^{(1)}, u_W^{(2)}, u_E^{(1)}, u_E^{(2)}) = \int_{M_1^p} f_1(u_S, v, u_W^{(1)}, u_E^{(1)}) f_2(v, u_N, u_W^{(2)}, u_E^{(2)}) dv \quad (3.2.25d)$$

with $u_a^{(i)} \in H_1^{p_i}$ and $u_a \in H_1^p$ for $a \in \{S, N\}$ and $u_b^{(i)} \in H_2^{q_i}$ and $u_b \in H_2^q$ for $b \in \{W, E\}$.

Theorem 3.2.3. *The elementary guillotine products (3.2.25) define a $\text{Guill}_2^{(r)}$ -algebra on the sets $(L_{p,q}^\infty)_{(p,q) \in C}$.*

This structure theorem is proved in [Sim23] and we encourage the reader to refer to it to understand how this reinterprets various probabilistic computations of marginals and partition functions of Markov processes that will use massively in the following sections.

Additional remarks. The present paper describes the construction of various sets $(E_c)_{c \in C_{LR}}$ related to Gaussian Markov processes from definition 3.2.1. In particular, we will have the following correspondence:

- the sets E_c with $c \in C$ (finite rectangles) are particular stable subspaces of the L_c^∞ and are related to Gaussian partition functions on rectangles of size c and will be lifted to the space of the parameters of the Gaussian Markov field, i.e. the face operators Q ; in particular, the products associated to guillotine partitions are just formal rewriting of standard probabilistic integration properties. This is mainly the content of section 3.3, in which we introduce additional harmonic tools to compute efficiently the Guill_2 -products.
- the sets E_c with $c \in C_{LR}$ such that some sizes are equal to ∞_{WE} or ∞_{SN} in some of the dimensions are related to traditional approaches by Fourier transform and transfer matrices. This is the main content of section 3.4.
- the sets E_c with $c \in C_{LR}$ without any size equal to ∞_{WE} or ∞_{SN} , which correspond to quarter of planes and half-strips, are related to the boundary weights g_R of the restrictions of the infinite volume Gauss-Markov process \tilde{X} to rectangles. This is the main content of section 3.5, which is completely new up to our knowledge.

3.3 Gluing rectangles: harmonic analysis

3.3.1 Gaussian weights and their gluings on rectangles

3.3.1.1 Heuristics

We first introduce several related coloured sets on C , define their products under elementary guillotine partitions and check their generalized associativities.

One of the theorems of [Sim23] states that partition functions on rectangles can be obtained from partition functions on the rectangles of a guillotine partition of the initial rectangle and, by considering the finest partition of the initial rectangle, as a surface product of the elementary Gaussian face weights.

Gluings of rectangles along a guillotine cut correspond to integration of Gaussian weights w.r.t. the variables associated to the edges of the guillotine cut. In our case, partial integration of Gaussian weights provides again a Gaussian weight on the remaining variables with a modified quadratic form using Schur complements. We may thus introduce the set of Gaussian weights, which will define an $\text{Guill}_2^{(r)}$ -algebra, as well as the set of parameters of these weights (the quadratic form in the exponential), which will define a second $\text{Guill}_2^{(r)}$ -operad. During the process of Gaussian integration, scalar "volumes" emerge during the computation and thus must be incorporated in these structures.

3.3.1.2 The coloured sets for Gaussian weights

Let \mathcal{H}_1 and \mathcal{H}_2 be two complex Hilbert spaces with dimensions d_1 and d_2 . For any $(p, q) \in C = \mathbb{N}_1 \times \mathbb{N}_1$, we first introduce the set of Hermitian positive definite operators on

the boundary direct sum of Hilbert spaces:

$$\mathcal{Q}_{p,q} = \left\{ Q \in \text{End} \left(H_1^{2p} \oplus H_2^{2q} \right); Q^* = Q \text{ and } Q \text{ positive definite} \right\} \quad (3.3.1)$$

In H_1^{2p} (resp. H_2^{2q}), the first p (resp. q) vectors correspond to the p (resp. q) vectors on a South (resp. West) boundary of a rectangle ordered from left to right (resp. bottom to top) and the last p (resp. q) vectors to the p (resp. q) vectors on the North (resp. East) boundary with the same ordering. We emphasize on the fact that these sets are not algebras nor vector spaces.

For each element $(\alpha, Q) \in \mathbb{R} \times \mathcal{Q}_{p,q}$, we define the Gaussian weight:

$$\begin{aligned} e_{\alpha,Q} : H_1^{2p} \oplus H_2^{2q} &\rightarrow \mathbb{R} \\ u &\mapsto \frac{\alpha}{(2\pi)^{2d_1+2d_2}} \exp \left(-\frac{1}{2} u^* Q u \right) \end{aligned} \quad (3.3.2)$$

and the set of Gaussian quasi-densities (i.e. they are not normalized so that their integral is one)

$$\mathcal{D}_{p,q} = \{ e_{\alpha,Q}; \alpha \in \mathbb{R}, Q \in \mathcal{Q}_{p,q} \} \quad (3.3.3)$$

It is trivial to check that, for all $(p, q) \in C$, the map $e : \mathbb{R} \times \mathcal{Q}_{p,q} \rightarrow \mathcal{D}_{p,q}$ is a bijection.

We recall the following fundamental Gaussian integration lemma.

Lemma 3.3.1 (Schur complement). *Let \mathcal{W} be a finite dimensional Hilbert space and let \mathcal{W}_1 and \mathcal{W}_2 be two subspaces of \mathcal{W} with dimensions d_1 and d_2 such that $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$. Let Q be a Hermitian positive definite operator on \mathcal{W} with a decomposition*

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

where Q_{ij} is an operator $\mathcal{W}_i \rightarrow \mathcal{W}_j$. Then,

$$\frac{1}{(2\pi)^{d_2}} \int_{\mathcal{W}_2} \exp \left(-\frac{1}{2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^* Q \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) du_2 = \det(Q_{22}^{-1}) \exp \left(-\frac{1}{2} u_1^* \bar{Q}_{11} u_1 \right) \quad (3.3.4)$$

where the matrix called the Schur complement

$$\bar{Q}_{11} = Q_{11} - Q_{12} Q_{22}^{-1} Q_{21}$$

is again Hermitian positive definite on \mathcal{W}_1 .

We may now use this lemma to prove the following property which is fundamental for the construction of the $\text{Guill}_2^{(r)}$ -algebra.

Property 3.3.1. *Let (p_1, q) and (p_2, q) be two rectangle sizes in C . For any $Q_1 \in \mathcal{Q}_{p_1,q}$ and any $Q_2 \in \mathcal{Q}_{p_2,q}$ there is a unique $\gamma_{\text{WE}}(Q_1, Q_2) \in \mathbb{R}_+^*$ and a unique element $S_{\text{WE}}(Q_1, Q_2) \in \mathcal{Q}_{p_1+p_2,q}$ such that, for all $\alpha_1, \alpha_2 \in \mathbb{R}$,*

$$\boxed{e_{\alpha_1, Q_1} \mid e_{\alpha_2, Q_2}} = e_{\alpha_1 \alpha_2 \gamma_{\text{WE}}(Q_1, Q_2), S_{\text{WE}}(Q_1, Q_2)} \quad (3.3.5)$$

Let (p, q_1) and (p, q_2) be two rectangle sizes in C . For any $Q_1 \in \mathcal{Q}_{p,q_1}$ and any $Q_2 \in \mathcal{Q}_{p,q_2}$ there is a unique $\gamma_{\text{SN}}(Q_1, Q_2) \in \mathbb{R}_+^$ and a unique element $S_{\text{SN}}(Q_1, Q_2) \in \mathcal{Q}_{p,q_1+q_2}$ such that, for all $\alpha_1, \alpha_2 \in \mathbb{R}$,*

$$\boxed{e_{\alpha_2, Q_2} \mid e_{\alpha_1, Q_1}} = e_{\alpha_1 \alpha_2 \gamma_{\text{SN}}(Q_1, Q_2), S_{\text{SN}}(Q_1, Q_2)} \quad (3.3.6)$$

Proof. We prove only the first equation (3.3.5) since (3.3.6) follows the same steps by permuting the horizontal and the vertical directions.

Since the Gaussian weights e_{α_i, Q_i} are elements of the spaces $L_{p_i, q}^\infty$, the elementary guillotine product in (3.3.5) corresponds to the following integral:

$$\boxed{e_{\alpha_1, Q_1} \mid e_{\alpha_2, Q_2}} (u_S, u_N, u_W, u_E) = \int_{H_2^q} e_{\alpha_1, Q_1}(u_S^{(1)}, u_N^{(1)}, u_W, v) e_{\alpha_2, Q_2}(u_S^{(2)}, u_N^{(2)}, v, u_E) dv$$

for all $u_S^{(1)}, u_N^{(1)} \in H_1^{p_1}$, $u_S^{(2)}, u_N^{(2)} \in H_1^{p_2}$, $u_W, u_E \in H_2^q$ with the convention with $u_a = (u_a^{(1)}, u_a^{(2)}) \in H_1^{p_1+p_2}$ for $a \in \{S, N\}$. Evaluating the integral is then an immediate consequence of lemma 3.3.1 by considering the Hilbert space:

$$\mathcal{W} = \mathcal{H}_1^{2(p_1+p_2)} \oplus \mathcal{H}_2^{3q}$$

with the following embedding convention of the variables in (3.3.5):

$$(u_S^{(1)}, u_S^{(2)}, u_N^{(1)}, u_N^{(2)}, u_W, v, u_E)$$

We then observe that the integrand of the l.h.s. of (3.3.5) is equal to $e_{\alpha\beta, \iota(Q_1, Q_2)}$ where $\iota(Q_1, Q_2)$ is a quadratic form on \mathcal{W} . We may now apply lemma 3.3.1 where \mathcal{W}_2 corresponds to the subspace of the vectors v and \mathcal{W}_1 corresponds to the subspace of the other vectors. We thus obtain the following formula in terms of sub-blocks of the operators Q_1 and Q_2 :

$$\gamma_{WE}(Q_1, Q_2) = \det(Q_{1,EE} + Q_{2,WW})^{-1}$$

and $S_{WE}(Q_1, Q_2)$ is the corresponding Schur complement, which one may writes in terms of the blocks of Q_1 and Q_2 . \square

Corollary 3.3.1 (guillotine structure of Gaussian weights and their parameters). *The subsets $\mathcal{D}_{p,q}$ of $L_{p,q}^\infty$ are stable under the guillotine products and thus define a sub-Guill $_2^{(r)}$ -algebra of L_\bullet^∞ . Moreover, the sets $(\mathcal{Q}_{p,q})_{(p,q) \in C}$ are also endowed with a Guill $_2^{(r)}$ -algebra given by the Schur complements.*

Proof. The first is a direct consequence of the previous proposition. The second part uses the fact that e is a bijection between $\mathcal{D}_{p,q}$ and $\mathbb{R} \times \mathcal{Q}_{p,q}$ and thus the Guill $_2^{(r)}$ -algebra of \mathcal{D}_\bullet can be lifted to $\mathbb{R} \times \mathcal{Q}_\bullet$ where the products are given γ_{ab} and S_{ab} . Finally, the projection $\mathbb{R} \times \mathcal{Q}_{p,q} \rightarrow \mathcal{Q}_{p,q}$ is compatible with the elementary guillotine products and we obtain the Guill $_2^{(r)}$ -algebra of \mathcal{Q}_\bullet with explicit expressions given by the Schur complements:

$$\boxed{Q_1 \mid Q_2}_{\mathcal{Q}} = S_{WE}(Q_1, Q_2) \quad \boxed{Q_2}_{\mathcal{Q}} = S_{SN}(Q_1, Q_2) \quad (3.3.7)$$

In all the paper, we will drop the index \mathcal{Q} in the guillotine products since in general the Guill $_2^{(r)}$ -algebra is clear from contexte. \square

A more precise look at (3.3.5) and (3.3.6) shows that the parameter space $\mathbb{R} \times \mathcal{Q}_\bullet$ should be better written $\mathbb{R} \triangleleft_\gamma \mathcal{Q}$ as a semi-direct product between the Guill $_2^{(r)}$ -algebra \mathcal{Q}_\bullet and the trivial Guill $_2^{(r)}$ -algebra \mathbb{R} where all the guillotine partition products are given by the usual product on \mathbb{R} . The maps γ_{SN} and γ_{SE} play the role of cocycles generalized at the operadic level, i.e. they are compatible with the higher associativities satisfied by the elementary guillotine products defined by the Schur complements S_{SN} and S_{SE} .

Here the higher associativities of the elementary guillotine products are obtained as a consequence of Fubini's theorem at the level of the sets L_\bullet^∞ . We leave to the reader the non-trivial exercise of checking these associativities at the linear algebra level of the Schur complement directly.

3.3.2 Surface powers, harmonic functions and Stoke's theorem

3.3.2.1 Surface powers and Schur complements

The partition function $Z_R(Q; x_{\partial R})$ of (3.2.1) on a rectangle of size (p, q) is a function of $L_{p,q}^\infty$ given by

$$Z_R(Q; \cdot) = \begin{array}{|c|c|c|c|} \hline e_{1,Q} & e_{1,Q} & \cdots & e_{1,Q} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline e_{1,Q} & e_{1,Q} & \cdots & e_{1,Q} \\ \hline \end{array} = e_{1,Q}^{[p,q]}$$

with p horizontal edges and q vertical edges in the partitions, where $a^{[p,q]}$ is the surface power notation of [Sim23] and corresponds to a guillotine partition along a grid with cells of the same sizes. This result is proved in [Sim23] but can be verified quickly by hand by integrating the Gaussian densities along all internal edges of R and successive uses of Fubini's theorem.

As a consequence of Gaussian integrations used in the previous section, we obtain

$$Z_R(Q; \cdot) = e_{\alpha_{p,q}, Q^{[p,q]}} \tag{3.3.8}$$

$$Q^{[p,q]} = \begin{array}{|c|c|c|c|} \hline Q & Q & \cdots & Q \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline Q & Q & \cdots & Q \\ \hline \end{array}_Q$$

where $Q^{[p,q]}$ can be computed either by a Schur complement on *all* the vertical edges or by successive Schur complement on successive guillotine cuts. In both cases, we obtain the coefficients γ_{ab} during the computations and thus the coefficient $\alpha_{p,q}$. This computation can be done on a computer but does not give much insight on the large size properties of $Q^{[p,q]}$ and $\alpha_{p,q}$. This is not a surprise since the guillotine structure only encodes the Markov property in a generic way on rectangles (but allows for a lot more on the boundaries, see section 3.5 below).

The coefficient $\alpha_{p,q}$ is equal to the partition function of Gaussian field with zero boundary condition on a rectangle of size (p, q) . It can be obtained directly by Fourier transform with sine modes in each direction if the model is invariant under reflections along the axes. Asymptotically, we expect it to grow exponentially with the area pq of the rectangle and it can be seen by convergence of Riemann sums to integrals in the Fourier space. For any "reasonable" boundary conditions, the additional quadratic term $Q^{[p,q]}$ on the boundary adds only a sub-leading correction — due to boundary conditions — to the partition function given by $\alpha_{p,q}$: it may thus appear at first sight as less interesting than $\alpha_{p,q}$ itself and the associated Fourier transform. However, in the previous operadic constructions, we see that the surface powers $Q^{[p,q]}$ with their Schur complements (3.3.7) are the basic objects to understand first since the components γ_{ab} appear only as cocycles with respect to \mathcal{Q}_\bullet . This induces an important change of point of view with respect to classical Fourier transform approaches to models such as the harmonic crystal.

We will now focus on the computation of surface powers $Q^{[p,q]}$ in \mathcal{Q}_\bullet through harmonic constructions and show how to understand the limit $p, q \rightarrow \infty$ in order to prepare section 3.5 and establish correspondences with the Fourier approach. In particular, we will relate the elementary guillotine products to Stoke's theorem in such a way that it can be extended to other rectangular shapes.

3.3.2.2 Harmonic description of the surface powers $Q^{[p,q]}$

We now consider a Gaussian Markov process as in definition 3.2.1 on a rectangular domain R of size (p, q) . The partition function $Z_R(Q; x_{\partial D})$ is then given by the integral formula:

$$Z_R(Q; x_{\partial D}) = \int_{\mathcal{H}_1^{p(q-1)} \oplus \mathcal{H}_2^{q(p-1)}} \exp\left(-\frac{1}{2} \sum_{f \in F(R)} x_{\partial f}^* Q x_{\partial f}\right) dx_{E(R) \setminus \partial R} \quad (3.3.9)$$

The quadratic form in the exponential is written \tilde{Q}_R as introduced in (3.2.3).

Property 3.3.2. *For any fixed $(x_e)_{e \in \partial R}$, let $(x_e^{(H)})_{e \in E(R)}$ be the unique function on the edges of R such that $\tilde{Q}_R(x^{(H)}, x^{(H)})$ is minimal and $x_e^{(H)} = x_e$ for $e \in \partial R$. Then,*

$$Z_R(Q; x_{\partial D}) = \exp\left(-\frac{1}{2} \tilde{Q}_R(x^{(H)}, x^{(H)})\right) Z_R(Q; 0)$$

Proof. The unicity of $x^{(H)}$ is deduced from the fact that \tilde{Q}_R is positive definite. We may then introduce the decomposition

$$x_e = x_e^{(H)} + x_e^{(0)}$$

where $x_e^{(0)} = 0$ for $e \in \partial R$. The optimality of $x^{(H)}$ implies that $x^{(0)}$ is orthogonal to $x^{(H)}$ for \tilde{Q}_R . From a probabilistic point of view, $x^{(H)}$ corresponds to the probability conditional density

$$x_e^{(H)} = \mathbb{E}_R[X_e | X_{\partial R} = x_{\partial R}]$$

and the previous decomposition corresponds to $X_e = \mathbb{E}_R[X_e | X_{\partial R} = x_{\partial R}] + X_e^{(0)}$ with $X_e^{(0)}$ independent of the conditional expectation.

The change of variable $x_{E(R) \setminus \partial R} = x_e^{(H)} + x_e^{(0)}$ where $x_e^{(H)}$ is a fixed quantity computed from the boundary conditions $x_{\partial D}$ specified in the l.h.s. of (3.3.9) provides directly

$$\begin{aligned} Z_R(Q; x_{\partial D}) &= \int_{\mathcal{H}_1^{p(q-1)} \oplus \mathcal{H}_2^{q(p-1)}} \exp\left(-\frac{1}{2} \tilde{Q}_R(x^{(H)} + x^{(0)}, x^{(H)} + x^{(0)})\right) dx_{E(R) \setminus \partial R}^{(0)} \\ &= \int_{\mathcal{H}_1^{p(q-1)} \oplus \mathcal{H}_2^{q(p-1)}} \exp\left(-\frac{1}{2} \tilde{Q}_R(x^{(H)}, x^{(H)}) - \frac{1}{2} \tilde{Q}_R(x^{(0)}, x^{(0)})\right) dx_{E(R) \setminus \partial R}^{(0)} \\ &= \exp\left(-\frac{\tilde{Q}_R(x^{(H)}, x^{(H)})}{2}\right) \int_{\mathcal{H}_1^{p(q-1)} \oplus \mathcal{H}_2^{q(p-1)}} \exp\left(-\frac{\tilde{Q}_R(x^{(0)}, x^{(0)})}{2}\right) dx_{E(R) \setminus \partial R}^{(0)} \end{aligned}$$

by orthogonality between $x^{(H)}$ and $x^{(0)}$. We then recognize $Z_R(Q; 0)$ inside the brackets. \square

By construction, there exists a linear operator, given by the Green function G_R , such that $x^{(H)} = G_R(x_{\partial D})$. We then obtain the following corollary by a trivial identification with (3.3.8).

Corollary 3.3.2. *The surface powers of Q and the coefficient $\alpha_{p,q}$ are given by*

$$Q^{[p,q]}(x, x) = \tilde{Q}_R(G_R(x), G_R(x)) \quad (3.3.10)$$

$$\alpha_{p,q} = \int_{\mathcal{H}_1^{p(q-1)} \oplus \mathcal{H}_2^{q(p-1)}} \exp\left(-\frac{1}{2} \tilde{Q}_R(x^{(0)}, x^{(0)})\right) dx_{E(R) \setminus \partial R}^{(0)} \quad (3.3.11)$$

The interesting part of (3.3.10) is that it can be further simplified by describing more precisely $x^{(H)}$ and the Green function. In order to simplify notations, we recall the following indexation of edges

$$x_{\{(k,l),(k+1,l)\}}^{(H)} = x_{k,l}^{(1),(H)} \in \mathcal{H}_1 \quad x_{\{(k,l),(k,l+1)\}}^{(H)} = x_{k,l}^{(2),(H)} \in \mathcal{H}_2$$

Lemma 3.3.2. *The unique optimizer $x^{(H)}$ of \tilde{Q}_R with boundary condition $x_{\partial R}$ is characterized by the following equations:*

$$0 = (Q_{SS} + Q_{NN})x_{k,l}^{(1),(H)} + Q_{SW}x_{k,l}^{(2),(H)} + Q_{SE}x_{k+1,l}^{(2),(H)} + Q_{SN}x_{k,l+1}^{(1),(H)} \quad (3.3.12a)$$

$$+ Q_{NW}x_{k,l-1}^{(2),(H)} + Q_{NE}x_{k+1,l-1}^{(2),(H)} + Q_{NS}x_{k,l-1}^{(1),(H)}$$

$$0 = (Q_{WW} + Q_{EE})x_{k,l}^{(2),(H)} + Q_{WS}x_{k,l}^{(1),(H)} + Q_{WN}x_{k,l+1}^{(1),(H)} + Q_{WE}x_{k+1,l}^{(2),(H)} \quad (3.3.12b)$$

$$+ Q_{ES}x_{k-1,l}^{(1),(H)} + Q_{EN}x_{k-1,l+1}^{(1),(H)} + Q_{EW}x_{k-1,l}^{(2),(H)}$$

for any edge $\{(k, l), (k+1, l)\}$ and $\{(k, l), (k, l+1)\}$ in $E(R) \setminus \partial R$.

These equations correspond to a zero Laplacian, with a quadratic energy $\tilde{Q}_R(x^{(H)}, x^{(H)})$. Using Stokes' theorem, we obtain

Property 3.3.3. *The surface power $Q^{[p,q]}$ is equal to*

$$Q^{[p,q]}(x_{\partial R}, x_{\partial R}) = \sum_{e \in \partial R_N} \langle x_e | D_{e,S}^{(1)}(x^{(H)}) \rangle + \sum_{e \in \partial R_S} \langle x_e | D_{e,N}^{(1)}(x^{(H)}) \rangle \quad (3.3.13)$$

$$+ \sum_{e \in \partial R_W} \langle x_e | D_{e,E}^{(2)}(x^{(H)}) \rangle + \sum_{e \in \partial R_E} \langle x_e | D_{e,W}^{(2)}(x^{(H)}) \rangle$$

with the following definition of $D_{e,a}^{(i)}(x)$ of a $(\mathcal{H}_1, \mathcal{H}_2)$ -valued function x on $E(\mathbb{Z}^2)$:

$$D_{(k,l),E}^{(2)}(x) = Q_{WS}x_{k,l}^{(1)} + Q_{WN}x_{k,l+1}^{(1)} + Q_{WW}x_{k,l}^{(2)} + Q_{WE}x_{k+1,l}^{(2)}$$

$$D_{(k,l),W}^{(2)}(x) = Q_{ES}x_{k-1,l}^{(1)} + Q_{EN}x_{k-1,l+1}^{(1)} + Q_{EW}x_{k-1,l}^{(2)} + Q_{EE}x_{k,l}^{(2)}$$

$$D_{(k,l),S}^{(1)}(x) = Q_{NS}x_{k,l-1}^{(1)} + Q_{NN}x_{k,l}^{(1)} + Q_{NW}x_{k,l-1}^{(2)} + Q_{NE}x_{k+1,l-1}^{(2)}$$

$$D_{(k,l),N}^{(1)}(x) = Q_{SS}x_{k,l}^{(1)} + Q_{SN}x_{k,l+1}^{(1)} + Q_{SW}x_{k,l}^{(2)} + Q_{SE}x_{k+1,l}^{(2)}$$

Proof. The proof is similar to the discrete Stokes' theorem. One has

$$Q^{[p,q]}(x, x) = \tilde{Q}_R(x^{(H)}, x^{(H)}) = \sum_{f \in F(R)} x^{(H)*}_{\partial f} Q x_{\partial f}^{(H)}$$

$$= \sum_{f \in F(R)} \sum_{e \in \partial f} x_e^{(H)*} D_{e,n_f(e)}(x^{(H)})$$

obtained by expanding $x^{(H)*}_{\partial f}$ along the four edges and by recognizing the quantities $D_{e,n_f(e)}(x^{(H)})$ on the components of $Q x_{\partial f}^{(H)}$. The index $n_f(e)$ indicates the direction in which R lies with respect to the edge e . All internal edges $e \in \{E(R) \setminus \partial R\}$ appear twice (once on the North and once on the South or once on the West and once on the East of a face) and edges on the boundaries appear only once.

$$Q^{[p,q]}(x, x) = \sum_{e \in \partial R} \langle x_e^{(H)} | D_{e,n_R(e)}^{(\bullet)}(x^{(H)}) \rangle$$

$$+ \sum_{e \in E_h(R) \setminus \partial R} \langle x_e^{(H)} | D_{e,S}^{(1)}(x^{(H)}) + D_{e,N}(x^{(H)}) \rangle$$

$$+ \sum_{e \in E_v(R) \setminus \partial R} \langle x_e^{(H)} | D_{e,W}^{(2)}(x^{(H)}) + D_{e,E}(x^{(H)}) \rangle$$

The characterization (3.3.12) of the harmonic function $x^{(H)}$ provides the cancellation of the terms in the sums over internal edges. \square

Corollary 3.3.3. *For any vertical guillotine cut that cuts a rectangle of size (p, q) into two rectangles of size (p_1, q) and (p_2, q) , it holds*

$$Q^{[p,q]}(x, x) = Q^{[p_1,q]}(\pi_1(x), \pi_1(x)) + Q^{[p_2,q]}(\pi_2(x), \pi_2(x)) \quad (3.3.14)$$

where $\pi_1(x)_W = x_W$, $\pi_2(x)_E = x_E$, $\pi_1(x)_N$ (resp $\pi_2(x)_N$) is the restriction of x_N to the left part of size p_1 (right part of size p_2 and similarly on the South and $\pi_1(x)_E = \pi_2(x)_W$ is the restriction to the guillotine cut of the harmonic solution $x^{(H)}$ with x_R as boundary condition.

Proof. This is a straightforward consequence of the Stokes-like formula (3.3.13). In the r.h.s. of (3.3.14), writing the sums of each element with (3.3.13) provides on all the sides but the guillotine cut the correct terms of the l.h.s. On the guillotine cut itself, for each edge, the sum over the opposite $D_{e, n_R(e)}^{(\bullet)}(x^{(H)})$ precisely provides the equation (3.3.12) and thus vanishes. \square

Equation (3.3.13) is particularly suited for various computations due to its local aspect as we will see below.

3.3.2.3 Harmonic functions for processes with the dihedral symmetry

Decomposition of the harmonic problem. Understanding the surface power of quadratic form $Q^{[p,q]}$ is then obtained by solving the harmonic equations (3.3.12) with boundary condition $x_{\partial R}$. The linearity of the harmonic equations show that we can compute $x^{(H)}$ on rectangles using the decomposition

$$x^{(H)} = x^{(H),WE} + x^{(H),SN}$$

where $x^{(H),WE}$ is the harmonic solution with boundary conditions

$$x_{\partial R_S}^{(H),WE} = 0 \quad x_{\partial R_N}^{(H),WE} = 0 \quad (3.3.15a)$$

$$x_{\partial R_W}^{(H),WE} = x_{\partial R_W} \quad x_{\partial R_E}^{(H),WE} = x_{\partial R_E} \quad (3.3.15b)$$

and $x^{(H),SN}$ is the harmonic solution with boundary conditions

$$x_{\partial R_S}^{(H),SN} = x_{\partial R_S} \quad x_{\partial R_N}^{(H),SN} = x_{\partial R_N} \quad (3.3.16a)$$

$$x_{\partial R_W}^{(H),SN} = 0 \quad x_{\partial R_E}^{(H),SN} = 0 \quad (3.3.16b)$$

A basis of harmonic functions and the spectral curve.

Lemma 3.3.3. *Let $(z, w) \in \mathcal{S}_Q$ defined in (3.2.10) and let $u = (u^{(1)}, u^{(2)}) \in \ker \Psi_Q(z, w)$. Then the function $H_{z,w}$ on $E(\mathbb{Z}^2)$ defined by:*

$$H_{z,w}(e) = \begin{cases} z^k w^l u^{(1)} \in \mathcal{H}_1 & \text{for } e \in E_h(\mathbb{Z}^2) \\ z^k w^l u^{(2)} \in \mathcal{H}_2 & \text{for } e \in E_v(\mathbb{Z}^2) \end{cases} \quad (3.3.17)$$

is solution of (3.3.12) on \mathbb{Z}^2 .

It is thus sufficient to decompose any arbitrary boundary conditions on the sets of harmonic functions $H_{z,w}$ to obtain the harmonic solution on the whole rectangle. Equations may however be quite involved and there may be redundant solutions: we have indeed a boundary space of dimension $2pd_1 + 2pd_2$ whereas the spectral curve \mathcal{S}_Q is uncountable. This situation contrasts strongly with the one-dimensional case described in [Bod23], in which boundary conditions on segments are decomposed in a unique way in a basis related to the finite spectral set that plays the role of \mathcal{S}_Q .

The situation becomes much simpler for operators Q with the dihedral symmetry and we now focus on this case in order to highlight best the operadic approach.

3.3.2.4 A representation of the harmonic solution with dihedral symmetry in the scalar case $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}$.

We consider here again (see (3.2.6)) the case of a 4-by-4 matrix Q given by

$$Q = \begin{pmatrix} t & a & u & u \\ a & t & u & u \\ u & u & t & a \\ u & u & a & t \end{pmatrix}$$

To make notations easier, we define $\omega_q = \exp(2i\pi/q)$ and, for any $r \in \{1, \dots, q\}$, the real number ζ_λ which is the unique real number with $|\zeta_r| > 1$ such that

$$(\zeta_r, \omega_{2q}^r) \in \mathcal{S}_Q$$

By dihedral symmetry, we have also $(\zeta_r^{\epsilon_1}, \omega_{2q}^{\epsilon_2 r}) \in \mathcal{S}_Q$ for any $\epsilon_1, \epsilon_2 \in \{-1, 1\}$, as well as $(\omega_{2q}^{\epsilon_2 r}, \zeta_r^{\epsilon_1}) \in \mathcal{S}_Q$.

For a given $r \in \{1, \dots, q\}$, the kernel of $\Psi_Q(\zeta_r^{\epsilon_1}, \omega_{2q}^{\epsilon_2 r})$ is one dimensional and given by vectors $(v_1(r)\zeta_r^{\epsilon_1/2}, v_2(r)\omega_{4q}^{\epsilon_2})$ where $v_1(r)$ and $v_2(r)$ do not depend on the ϵ_i and moreover $v_1(q) = 0$. We thus introduce the function $\kappa(r) = v_2(r)/v_1(r)$ if $r \neq p$ and $\kappa(q) = 1$.

Property 3.3.4. *Let R be the finite rectangle $[0, p] \times [0, q]$ of size (p, q) . Any solution of (3.3.12) on R with boundary conditions (3.3.15) has a unique representation:*

$$x_{k,l}^{(1),(H),WE} = \sum_{r=1}^{q-1} \sum_{\epsilon \in \{-1, 1\}} \alpha_{p,q}(r, \epsilon) \zeta_r^{\epsilon(k+1/2)} \sin\left(\frac{\pi r l}{q}\right) \quad (3.3.18a)$$

$$x_{k,l}^{(2),(H),WE} = \sum_{r=1}^q \sum_{\epsilon \in \{-1, 1\}} \alpha_{p,q}(r, \epsilon) \kappa(r) \zeta_r^{\epsilon k} \sin\left(\frac{\pi r(l+1/2)}{q}\right) \quad (3.3.18b)$$

where the coefficient $\alpha_{p,q}(r, \epsilon)$ are given in terms of the Fourier transform of the West and East boundary conditions defined by:

$$\hat{x}^W(r) = \sum_{l=0}^{q-1} x_{0,l}^{(2)} \sin\left(\frac{\pi r(l+1/2)}{q}\right) \quad \hat{x}^E(r) = \sum_{l=0}^{q-1} x_{p,l}^{(2)} \sin\left(\frac{\pi r(l+1/2)}{q}\right)$$

through

$$\alpha_{p,q}(r, 1) = \frac{2}{q} \kappa(r)^{-1} \frac{\hat{x}^E(r) - \hat{x}^W(r) \zeta_r^{-p}}{\zeta_r^p - \zeta_r^{-p}} \quad \alpha_{p,q}(r, -1) = \frac{2}{q} \kappa(r)^{-1} \frac{\hat{x}^W(r) \zeta_r^p - \hat{x}^E(r)}{\zeta_r^p - \zeta_r^{-p}} \quad (3.3.19)$$

These concrete formulae have far-reaching consequences for the guessing of the boundary spaces and boundary operators introduced in section 3.5 and thus we spend a few more lines in the following proof for the interpretation of the various variables.

The half-integers $l + 1/2$ for vertical edges and $k + 1/2$ for horizontal edges only reflects that another natural parametrization of the edges may consist in using their midpoints.

Proof. Solution (3.3.18) can be written as a linear combination of harmonic solutions $H_{z,w}$ described in lemma 3.3.3 and is thus harmonic inside the rectangle R . The shift by $\epsilon/2$ and $1/2$ in the sine modes together with the definition of $\kappa(r)$ ensures that the corresponding vectors belong to the kernel of $\Psi_Q(\zeta_r^\epsilon, \omega_{2q}^{\epsilon_2 r})$ where ϵ_2 corresponds to the two opposite complex exponentials of the sine function.

Checking the Dirichlet boundary condition on the North and the South is trivial by plugging $(k + 1/2, 0)$ and $(k + 1/2, q)$ in the first line of (3.3.18).

On the West and East side, we may use the formula

$$\sum_{l=0}^{q-1} \sin\left(\frac{\pi r(l+1/2)}{q}\right) \sin\left(\frac{\pi r'(l+1/2)}{q}\right) = \frac{q}{2} \delta_{r,r'}$$

for $1 \leq r, r' \leq q$. By considering the second line of (3.3.18) for $(0, l)$ (West side) and (p, l) (East side) and by using the Fourier transform, we obtain

$$\frac{2}{q} \widehat{x}^W(r) = \alpha_{p,q}(r, 1) \kappa(r) + \alpha_{p,q}(r, 1) \kappa(r) \quad \frac{2}{q} \widehat{x}^E(r) = \alpha_{p,q}(r, 1) \kappa(r) \zeta_r^p + \alpha_{p,q}(r, 1) \kappa(r) \zeta_r^{-p}$$

and hence the final formulae. \square

We may consider first the case of a rectangle of height $q = 1$. In this case, all the horizontal edges are boundary edges and their variables are set to zero (this corresponds to the vanishing of the first line of (3.3.18)). The second line contains then only the term $r = 1$ and thus the formula becomes simply

$$x_{k,0}^{(2),(H),WE} = \sum_{\epsilon \in \{-1,1\}} \alpha_{p,1}(1, \epsilon) \kappa(1) \zeta_1^{\epsilon k} \sin\left(\frac{\pi}{2}\right) = \sum_{\epsilon \in \{-1,1\}} \alpha_{p,1}(1, \epsilon) \zeta_1^{\epsilon k}$$

where ζ_1 is such that $(\zeta_1, -1) \in \mathcal{S}_Q$. We have

$$\Psi_Q(z, -1) = \begin{pmatrix} 2(t-a) & 0 \\ 0 & \phi_Q^{WE}(z) \end{pmatrix}$$

and the singularity of $\Psi_Q(z, -1)$ are created only the singularity of the one-dimensional term ϕ_Q^{WE} studied in detail in [Bod23]. Indeed, the horizontal North and South null boundary conditions imply that the variables $x_{k,0}^{(2)}$ in the rectangle is a one-dimensional Gaussian Markov process with elementary quadratic form given by the West-East sub-block of Q . In particular, one observes here that no Fourier transform is required in the vertical direction on the West and East boundaries.

The previous proposition has an equivalent in the vertical direction, that we formulate without any further proof since it is equivalent to the previous one.

Property 3.3.5. *Let R be the finite rectangle $[0, p] \times [0, q]$ of size (p, q) . Any solution of (3.3.12) on R with boundary conditions (3.3.16) has a unique representation:*

$$x_{k,l}^{(1),(H),SN} = \sum_{r=1}^p \sum_{\epsilon \in \{-1,1\}} \beta_{p,q}(r, \epsilon) \kappa(r)^{-1} \sin\left(\frac{\pi r(k+1/2)}{p}\right) \zeta_r^{\epsilon l} \quad (3.3.20a)$$

$$x_{k,l}^{(2),(H),SN} = \sum_{r=1}^{p-1} \sum_{\epsilon \in \{-1,1\}} \beta_{p,q}(r, \epsilon) \sin\left(\frac{\pi r k}{p}\right) \zeta_r^{\epsilon(l+1/2)} \quad (3.3.20b)$$

where the coefficient $\beta_{p,q}(r, \epsilon)$ are given in terms of the Fourier transform of the South and North boundary conditions defined by:

$$\widehat{x}^S(r) = \sum_{k=0}^{p-1} x_{k,0}^{(1)} \sin\left(\frac{\pi r(k+1/2)}{p}\right) \quad \widehat{x}^N(r) = \sum_{k=0}^{p-1} x_{k,q}^{(1)} \sin\left(\frac{\pi r(k+1/2)}{p}\right)$$

through

$$\beta_{p,q}(r, 1) = \frac{2}{p} \kappa(r) \frac{\widehat{x}^N(r) - \widehat{x}^S(r) \zeta_r^{-q}}{\zeta_r^q - \zeta_r^{-q}} \quad \beta_{p,q}(r, -1) = \frac{2}{p} \kappa(r) \frac{\widehat{x}^S(r) \zeta_r^q - \widehat{x}^N(r)}{\zeta_r^q - \zeta_r^{-q}} \quad (3.3.21)$$

Using the decomposition of $x^{(H)}$ into a West-East and a North-South component, we may now rewrite the surface power as a sum over terms localized on each boundary side.

Property 3.3.6. *The surface power $Q^{[p,q]}$ in (3.3.13) can be written:*

$$\begin{aligned} Q^{[p,q]}(x_{\partial R}, x_{\partial R}) &= \sum_{e \in \partial R_{WE}} \langle x_e | D_{e, n_R(e)}^{(2)}(x^{(H), WE}) \rangle + \sum_{e \in \partial R_{SN}} \langle x_e | D_{e, n_R(e)}^{(1)}(x^{(H), SN}) \rangle \\ &+ \sum_{e \in \partial R_{WE}} \langle x_e | D_{e, n_R(e)}^{(2)}(x^{(H), SN}) \rangle + \sum_{e \in \partial R_{SN}} \langle x_e | D_{e, n_R(e)}^{(1)}(x^{(H), WE}) \rangle \end{aligned} \quad (3.3.22)$$

where the first two terms do not contain any interaction between orthogonal boundaries but only between opposite boundaries.

We will see below that each term admits a nice limit when sizes of the rectangle R go to infinity in some suitable way. The practical exact formula of propositions 3.3.4 and 3.3.5 provides explicit guesses for the new spaces introduced in section 3.5.

The situation is similar to the one-dimensional case of obtaining Perron-Frobenius eigenvectors of a matrix A out of the exact computation of A^n by considering $n \rightarrow +\infty$; the main two differences are that we consider surface powers $Q^{[p,q]}$ for the "Schur" product and that the dimension two requires the presence of spaces along the transverse directions.

3.3.3 Heuristics for half-strips and corners

Following the operadic approach introduced in [Sim23], we wish to construct spaces $\mathcal{Q}_{\infty E, q}$ of quadratic forms and $\mathcal{D}_{\infty E, q}$ of Gaussian densities on the East half-strips of type $[0, +\infty) \times [0, q]$ with suitable eigenvector-like properties.

The idea is to study each term of (3.3.22) in the limit $p \rightarrow \infty$ using the representations (3.3.18) and (3.3.20) and their equations (3.3.19) and (3.3.21).

The present section is only heuristic in the sense that we will *guess* the boundary spaces $\mathcal{E}_{\infty E, q}$ from non-rigorous computations on finite rectangles by taking unjustified limits or double limits. However, this lack of rigor is not important since, once the solution spaces are guessed, [Sim23] provides a way to prove rigorously that these guessed solution spaces are indeed correct.

3.3.3.1 Hilbert spaces and their quadratic form for East half-strips

We expect the invariant boundary weights of a Gaussian Markov process to be also Gaussian. Moreover, from geometrical considerations and the possible construction of Gibbs measures from Gibbs measures on large torii, we expect it to be readable in the large size $p \rightarrow \infty$ limit of $Q^{[p,q]}$.

Following this credo, we expect the existence of a Hilbert space \mathcal{W}^E associated to the horizontal half-lines of East half-strips with a suitable quadratic forms $Q^{(\infty E, q)}$ on $\mathcal{W}^E \oplus \mathcal{W}^E \oplus \mathcal{H}_2^q$, associated to a space $\mathcal{Q}_{\infty E, q}$ with suitable operadic properties. We also expect that this quadratic form defines a Gaussian weight $\bar{\mathbf{e}}_{\beta, Q^{(\infty E, q)}}$ associated to another space $\mathcal{D}_{\infty E, q}$.

Here, there is no additional copy of \mathcal{H}_2^q since the East boundary disappears from the half-strip shape in the limit $p \rightarrow \infty$ (since there is no long range order). We thus have a block representation of $Q^{(\infty E, q)}$ given by:

$$Q^{(\infty E, q)} = \begin{pmatrix} Q_{SS}^{(\infty E, q)} & Q_{SN}^{(\infty E, q)} & Q_{SW}^{(\infty E, q)} \\ Q_{NS}^{(\infty E, q)} & Q_{NN}^{(\infty E, q)} & Q_{NW}^{(\infty E, q)} \\ Q_{WS}^{(\infty E, q)} & Q_{WN}^{(\infty E, q)} & Q_{WW}^{(\infty E, q)} \end{pmatrix} \quad (3.3.23)$$

West sub-block. The term $Q_{\text{WW}}^{(\infty\text{E},q)}$ can be conjectured easily, either by applying the result of [Bod23] to the Markov process on the vertical edges, or from the first term of (3.3.22) with $p \rightarrow \infty$.

Indeed, $D_{e,\text{E}}^{(2)}(x^{(H),\text{WE}})$ for $e \in \partial R_{\text{W}}$ requires only the knowledge of (3.3.18) for $k \in \{0, 1\}$:

$$\begin{aligned} & \sum_{e \in \partial R_{\text{W}}} \langle x_e \mid D_{e,\text{E}}^{(2)}(x^{(H),\text{WE}}) \rangle \\ &= \sum_{l=0}^{q-1} \langle x_{0,l}^{(2)} \mid Q_{\text{WS}} x_{0,l}^{(1),(H),\text{WE}} + Q_{\text{WN}} x_{0,l+1}^{(1),(H),\text{WE}} + Q_{(2),\text{WW}} x_{0,l} + Q_{\text{WE}} x_{1,l}^{(2),(H),\text{WE}} \rangle \end{aligned} \quad (3.3.24)$$

Since all the numbers ζ_r , lie outside the unit disk, we have for fixed boundary equations in the limit $p \rightarrow +\infty$:

$$\alpha_{p,q}(r, 1) \rightarrow 0 \qquad \alpha_{p,q}(r, -1) \rightarrow \frac{2}{q} \kappa(r)^{-1} \hat{x}^{\text{W}}(r)$$

and thus, for arbitrary finite k as $p \rightarrow \infty$, we obtain:

$$x_{k,l}^{(1),(H),\text{WE}} \rightarrow \frac{2}{q} \sum_{r=1}^{q-1} \kappa(r)^{-1} \hat{x}^{\text{W}}(r) \zeta_r^{-(k+1/2)} \sin\left(\frac{\pi r l}{q}\right) \quad (3.3.25a)$$

$$x_{k,l}^{(2),(H),\text{WE}} \rightarrow \frac{2}{q} \sum_{r=1}^{q-1} \hat{x}^{\text{W}}(r) \zeta_r^{-k} \sin\left(\frac{\pi r(l+1/2)}{q}\right) \quad (3.3.25b)$$

and thus

$$\begin{aligned} & \sum_{e \in \partial R_{\text{W}}} \langle x_e \mid D_{e,\text{E}}^{(2)}(x^{(H),\text{WE}}) \rangle \\ & \rightarrow \sum_{l=0}^{q-1} Q_{\text{WW}} |x_{0,l}^{(2)}|^2 + \sum_{l=0}^{q-1} \sum_{r=1}^{q-1} \frac{2}{q} \zeta_r^{-1} \overline{x_{0,l}^{(2)}} Q_{\text{WE}} \sin\left(\frac{\pi r(l+1/2)}{q}\right) \hat{x}^{\text{W}}(r) \\ & \quad + \sum_{l=0}^{q-1} \sum_{r=1}^{q-1} \frac{2}{q} \kappa(r)^{-1} \zeta_r^{-1/2} \overline{x_{0,l}^{(2)}} \left[Q_{\text{WS}} \sin\left(\frac{\pi r l}{q}\right) + Q_{\text{WN}} \sin\left(\frac{\pi r(l+1)}{q}\right) \right] \hat{x}^{\text{W}}(r) \end{aligned}$$

in which either the values $x_{0,l}^{(2)}$ can be replaced by their discrete Fourier transform $\hat{x}^{\text{W}}(r)$ or the Fourier transform $\hat{x}^{\text{W}}(r)$ can be replaced by the values $x_{0,l}^{(2)}$, in order to obtain

$$\langle x_{\partial R_{\text{W}}} \mid Q_{\text{WW}}^{(\infty\text{E},q)} x_{\partial R_{\text{W}}} \rangle$$

The South and North half-line spaces. In the limit $p \rightarrow +\infty$, the spaces \mathcal{H}_1^p grow and thus a scaling limit is required. At first sight, from the Fourier transform \hat{x}^{S} and \hat{x}^{N} used in the solution of harmonic functions, it may not be clear at first sight which modes are relevant in the limit $p \rightarrow \infty$ from the point of view of boundary weights and their ROPE representations in [Sim23]. In particular, this is not the continuous space limit, i.e. the limit of zero mesh $\epsilon\mathbb{Z}$, which is relevant: this would correspond to Fourier modes with finite r as $p \rightarrow \infty$, due to the k/p factor in the sine expressions in proposition 3.3.5. On the contrary, this is rather the values $x_{k,0}^{(1)}$ and $x_{k,q}^{(1)}$ with arbitrary large but finite distance k from the left boundary as $p \rightarrow \infty$; in particular, this corresponds to Fourier modes $r = \lfloor \alpha p \rfloor$ with fixed $\alpha \in (0, 1)$ as $p \rightarrow \infty$.

This scaling regime can be deduced from the finite correlation length in the model and the exponential decrease of the harmonic functions (3.3.25) far from the boundaries. Indeed,

writing down the fourth term of (3.3.22) shows the coupling between any term $x_{0,l}^{(2)}$ on the West and $x_{0,k}^{(1)}$ decreases with k : it may then be plausible that terms with k growing with p decorrelate asymptotically from the West boundary. This has to be confirmed by the study of the third term of (3.3.22) which describe the other part of the coupling between the West side and the North and South sides. From proposition 3.3.5, one observes that this term involves only sums of the following type

$$\sum_{l=0}^{q-1} \sum_{r=1}^p \frac{1}{p} f(l, r/p)$$

which are Riemann sums. They converge provided that the elements $x_{k,0}^{(1)}$ and $x_{k,0}^{(1)}$ behaves nicely enough at large k . More generally, in the Fourier basis, the only required scaling to ensure the convergences of the Riemann sums is

$$\beta_{p,q}(r, \epsilon) \simeq \frac{1}{p} \bar{\beta}_q \left(\frac{r}{p}, \epsilon \right) \quad (3.3.26)$$

In practice, the variables on the horizontal East half-line are not required to have a precise form. The ROPE representations of boundary weights introduced in [Sim23] integrate out these variables in the computation of the values of the boundary weights and thus any change of basis is irrelevant. We may then wish to replace the fields $x_{k,l}^{(1)}$ on the half-line by the continuous random fields $\bar{\beta}_q(\theta, \epsilon)$.

We then obtain the following convergence

$$\begin{aligned} & \sum_{e \in \partial R_W} \langle x_e | D_{e,E}^{(2)}(x^{(H),SN}) \rangle \\ & \rightarrow \sum_{l=0}^{q-1} \overline{x_{0,l}^{(2)}} \sum_{\epsilon \in \{-1,1\}} \int_0^1 \bar{\beta}_q(\alpha, \epsilon) \left[\frac{\sin(\frac{\pi\alpha}{2})}{\bar{\kappa}(\alpha)} \left(Q_{WS} \bar{\zeta}_\alpha^{\epsilon l} + Q_{WN} \bar{\zeta}_\alpha^{\epsilon(l+1)} \right) + Q_{WE} \sin(\pi\alpha) \bar{\zeta}_\alpha^{\epsilon(l+1/2)} \right] d\alpha \end{aligned}$$

as well as

$$\begin{aligned} & \sum_{e \in \partial R_S} \langle x_e | D_{e,N}^{(1)}(x^{(H),WE}) \rangle \\ & \rightarrow \sum_{k=0}^{\infty} \overline{x_{k,0}^{(1)}} \sum_{r=1}^{q-1} \hat{x}^W(r) \left[\sin\left(\frac{\pi r}{2q}\right) \left(Q_{SW} \zeta_r^{-k} + Q_{SE} \zeta_r^{-(k+1)} \right) + Q_{SN} \kappa(r)^{-1} \zeta_r^{-(k+1/2)} \sin\left(\frac{\pi r}{q}\right) \right] \\ & = 2 \sum_{r=1}^{q-1} \int_0^1 \overline{\hat{x}^S(\alpha)} \hat{x}^W(r) \left[\sin\left(\frac{\pi r}{2q}\right) \left(Q_{SW} + Q_{SE} \frac{1}{\zeta_r} \right) + Q_{SN} \frac{\sin\left(\frac{\pi r}{q}\right)}{\kappa(r) \zeta_r^{1/2}} \right] \Im \left(\frac{e^{i\pi\alpha/2}}{1 - \zeta_r e^{i\pi\alpha}} \right) d\alpha \end{aligned}$$

where, in this case, we have implicitly accepted the convergence of the series whenever the South boundary conditions is nice enough in a sense that has to be precised. We will show in the following section 3.4 how this Fourier space representations can be recovered rigorously from classical computations of Fourier transform of transfer matrices; the rigorous and general definitions of the half-strips and corner spaces are presented in section 3.5 with their additional properties.

The South-North block. The South-North block $Q_{\text{SN,SN}}^{(\infty\text{E},q)}$ is related to the second term of (3.3.22), which is given for finite p by:

$$\begin{aligned} & \sum_{k=0}^{p-1} \overline{x_{k,0}^{(1)}} \left(Q_{\text{SW}} x_{k,l}^{(2),(H)} + Q_{\text{SE}} x_{k+1,l}^{(2),(H)} + Q_{\text{SS}} x_{k,l}^{(1)} + Q_{\text{SN}} x_{k,l+1}^{(1),(H)} \right) \\ & + \sum_{k=0}^{p-1} \overline{x_{k,p}^{(1)}} \left(Q_{\text{NW}} x_{k,p-1}^{(2),(H)} + Q_{\text{NE}} x_{k+1,p-1}^{(2),(H)} + Q_{\text{NS}} x_{k,p-1}^{(1),(H)} + Q_{\text{NN}} x_{k,p}^{(1)} \right) \end{aligned}$$

Using the Fourier transform on this expression, it is easy to see it can be structured as a unique sum over $r \in \{1, \dots\}$ and two sums over ϵ and ϵ' with quadratic terms on the four terms $\beta_{p,q}(r, \bullet)$, r by r . One also observes that the sum over r becomes a Riemann sum in the scaling limit (3.3.26) and thus the expression can be written as

$$\int_0^\pi \begin{pmatrix} \overline{\beta}_q(\theta, +1) \\ \overline{\beta}_q(\theta, -1) \end{pmatrix}^* \tilde{Q}(\theta) \begin{pmatrix} \overline{\beta}_q(\theta, +1) \\ \overline{\beta}_q(\theta, -1) \end{pmatrix} d\theta$$

hence providing the South-North block $Q_{\text{SN,SN}}^{(\infty\text{E},q)}$ as a direct integral of 2-by-2 blocks on the South and North Fourier transforms. The exact expression of $\tilde{Q}(\theta)$ is provided in the next sections.

3.3.3.2 Heuristics for corners

On corners, the situation corresponds to a rectangle of size (p, q) where both p and q go to infinity simultaneously. Tackling this double limit rigorously from the harmonic solutions (3.3.18) and (3.3.20) is not trivial; however, the double Riemann sums together with the guessed scalings of the Fourier modes provides a nice candidate.

The same heuristic computation for the South-East corner can be performed in various ways: either by considering only the relevant terms in (3.3.22) (double limit) or by considering a East half-strip of width q with q growing to ∞ , or a South half-strip of width p growing to infinity, where each limit is taken by using the correct Riemann sum scaling.

We then consider a space \mathcal{W}_{NE} in the North East corner with a quadratic form

$$Q^{(\infty\text{E},\infty\text{N})} = \begin{pmatrix} Q_{\text{SS}}^{(\infty\text{E},\infty\text{N})} & Q_{\text{SW}}^{(\infty\text{E},\infty\text{N})} \\ Q_{\text{WS}}^{(\infty\text{E},\infty\text{N})} & Q_{\text{WW}}^{(\infty\text{E},\infty\text{N})} \end{pmatrix}$$

We consider now (3.3.22) while keeping only the West and South sides and sending $p, q \rightarrow \infty$ with the following scalings

$$\begin{aligned} \beta_{p,q}(r, 1) &= 0 & \beta_{p,q}(r, -1) &= \frac{1}{p} \tilde{\beta}_p(r/p, -1) \\ \alpha_{p,q}(r', 1) &= 0 & \alpha_{p,q}(r', -1) &= \frac{1}{q} \tilde{\beta}_p(r'/q, -1) \end{aligned}$$

and we obtain:

$$\begin{aligned} & \sum_{k \in \mathbb{N}} \langle x_{k,0}^{(1)} | D_{k,0,\text{N}}^{(1)}(x^{(H),\text{WE,E}}) \rangle + \sum_{k \in \mathbb{N}} \langle x_{k,0}^{(1)} | D_{k,0,\text{N}}^{(1)}(x^{(H),\text{SN,N}}) \rangle \\ & \sum_{l \in \mathbb{N}} \langle x_{0,l}^{(2)} | D_{0,l,\text{E}}^{(1)}(x^{(H),\text{WE,E}}) \rangle + \sum_{l \in \mathbb{N}} \langle x_{0,l}^{(2)} | D_{0,l,\text{N}}^{(1)}(x^{(H),\text{SN,N}}) \rangle \end{aligned}$$

It is then useful to replace the sums over k and l in the direct space by their Fourier modes $\hat{x}^{(1)}(\alpha_1)$ and $\hat{x}^{(2)}(\alpha_2)$ and one obtains for $Q^{(\infty\mathbb{E}, \infty\mathbb{N})}$ an expression of the type

$$\begin{aligned} & \int \langle x^{(1)}(\alpha_1) | q_S(\alpha_1) x^{(1)}(\alpha_1) \rangle d\alpha_1 + \int \langle x^{(2)}(\alpha_2) | q_W(\alpha_2) x^{(1)}(\alpha_2) \rangle d\alpha_2 \\ & + \iint \Re \langle x^{(1)}(\alpha_1) | q_{SW}(\alpha_1, \alpha_2) x^{(2)}(\alpha_2) \rangle d\alpha_1 d\alpha_2 \end{aligned}$$

where the three matrices $q_a(\alpha)$ and $q_{SW}(\alpha_1, \alpha_2)$ can be expressed from integrals on circles on the spectral curve \mathcal{S}_Q (see section 3.5 for rigorous explicit definitions).

3.4 Doubly-infinite or periodic shapes: Fourier analysis

3.4.1 Classical facts

Classically, most of relevant quantities (e.g partition function or correlation function), concerning the Gaussian Markov fields defined in 3.2.1 are computed with the help of Fourier transform under periodic boundary conditions. We briefly expose these facts. We recall, from lemma 3.2.3, that the matrix-valued function Ψ_Q defined in (3.2.8) is positive definite on $S^1 \times S^1$ and therefore invertible with a positive definite inverse Ψ_Q^{-1} . Let $p, q \in \mathbb{N}^*$ and denote $X^{\text{per}, p, q}$ the Gaussian Markov process on the discrete torus $\mathbb{T}_{p, q}^2 := \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$ with face-weight $Q \in \mathcal{Q}_{1,1}$. Any variable associated to a horizontal (resp. vertical) edge is denoted $X_e^{\text{per}, p, q, (1)} \in \mathcal{H}_1$ (resp. $X_e^{\text{per}, p, q, (2)} \in \mathcal{H}_2$). We let $d_i = \dim \mathcal{H}_i$ be the dimension of the Hilbert spaces associated to vertical and horizontal edges. We denote $\omega_p = \exp(2i\pi/p)$ and, for all $(k, l) \in \mathbb{T}_{p, q}^2$ and $i \in \{1, 2\}$,

$$\hat{X}_{k, l}^{\text{per}, p, q, (i)} = \frac{1}{\sqrt{pq}} \sum_{(n, m) \in \mathbb{T}_{p, q}^2} X_{n, m}^{\text{per}, p, q, (i)} \omega_p^n \omega_q^m.$$

A change of variable shows that, for all $(k, l) \in \mathbb{T}_{p, q}^2$, the random variable

$$\hat{X}_{k, l}^{\text{per}, p, q} := \left(\hat{X}_{k, l}^{\text{per}, p, q, (1)}, \hat{X}_{k, l}^{\text{per}, p, q, (2)} \right) \in \mathcal{H}_1 \oplus \mathcal{H}_2$$

is a centered Gaussian with covariance matrix given by $\Psi_Q^{-1}(\omega_p^k, \omega_q^l)$ and the random variables $(\hat{X}_{k, l}^{\text{per}, p, q})_{(k, l) \in \mathbb{T}_{p, q}^2}$ are independent. We deduce the following expression for the partition function

$$Z_{p, q}^{\text{per}}(Q) = (2\pi)^{pq(d_1 + d_2)} \prod_{k=0}^{p-1} \prod_{l=0}^{q-1} \det \Psi_Q^{-1}(\omega_p^k, \omega_q^l). \quad (3.4.1)$$

Therefore, recognizing Riemann sums, the free energy is given by

$$f^{\text{per}}(Q) := \lim_{p, q \rightarrow \infty} \frac{1}{pq} \log Z_{p, q}^{\text{per}}(Q) = \log((2\pi)^{d_1 + d_2}) - \frac{1}{(2\pi)^2} \int_{[0, 2\pi]^2} \log \det \Psi_Q(e^{i\theta_1}, e^{i\theta_2}) d\theta_1 d\theta_2. \quad (3.4.2)$$

Moreover, in thermodynamic limit, the marginal laws are centered Gaussian and for all $k, k', l, l' \in \mathbb{Z}$ and $i, j \in \{1, 2\}$, their covariance matrix is given by

$$\lim_{p, q \rightarrow \infty} \mathbb{E} \left[\overline{X_{k, l}^{\text{per}, p, q, (i)}} \otimes X_{k', l'}^{\text{per}, p, q, (j)} \right] = C_{k' - k, l' - l}^{i, j} \quad (3.4.3)$$

where $C_{k' - k, l' - l}^{i, j}$ is the Fourier coefficient of $(\Psi_Q^{-1})_{i, j}$ of order $(k' - k, l' - l)$, see (3.2.14). By Wick's theorem, this gives the correlation function in the thermodynamic limit.

Infinite-volume Gibbs measure. We now construct the infinite-dimensional Gibbs measure associated to the translation-invariant Gaussian Markov process with face-weight Q . This is a classical construction, see [Geo11] for instance. We consider the Hilbert space

$$\mathcal{G} = L^2(S^1 \times S^1 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2)$$

equipped with the following norm

$$\|f\|^2 = \frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} \langle f(e^{i\theta_1}, e^{i\theta_2}), \Psi_Q^{-1}(e^{i\theta_1}, e^{i\theta_2})f(e^{i\theta_1}, e^{i\theta_2}) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} d\theta_1 d\theta_2 = \langle f, \mathbf{M}(\Psi_Q^{-1})f \rangle_{L^2}$$

where $\mathbf{M}(\Psi_Q^{-1})$ is the multiplication operator associated with the symbol Ψ_Q^{-1} , see (3.4.19) below. Define the Gaussian field X^F as an isometry to a probability space:

$$X^F : \mathcal{G} \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

From this Gaussian field in Fourier space, we may define by Fourier transform a Gaussian Field on the edges of \mathbb{Z}^2 such that the variables on horizontal (resp. vertical) edges are \mathcal{H}_1 -valued (resp. \mathcal{H}_2 -valued) by considering the following variables:

- for any horizontal edge $\{(k, l), (k + 1, l)\}$, we define

$$X_{k,l}^{(1)} = X^F \left(\begin{pmatrix} \exp(ik \bullet_1 + il \bullet_2) \\ 0 \end{pmatrix} \right)$$

- for any vertical edge $\{(k, l), (k, l + 1)\}$, we define

$$X_{k,l}^{(2)} = X^F \left(\begin{pmatrix} 0 \\ \exp(ik \bullet_1 + il \bullet_2) \end{pmatrix} \right)$$

where $\exp(ik \bullet_1 + il \bullet_2) : S^1 \times S^1, (\theta_1, \theta_2) \mapsto \exp(ik\theta_1 + il\theta_2)$. The law of this Gaussian field is precisely the infinite-dimensional Gibbs measure associated to our model defined in 3.2.1. We check that for all $k, k', l, l' \in \mathbb{Z}$ and $i, j \in \{1, 2\}$,

$$\mathbb{E}^{\text{Gibbs}} \left[\overline{X_{k,l}^{(i)}} \otimes X_{k',l'}^{(j)} \right] = C_{k'-k, l'-l}^{i,j} \quad (3.4.4)$$

just as in (3.4.3), obtained under thermodynamic limit. In particular, the Gaussian field restricted to horizontal edges has a covariance matrix built with the family $(C_{k,l}^{1,1})_{k,l \in \mathbb{Z}}$ of Fourier coefficients of the sub-block $(\Psi_Q^{-1})_{1,1}$ of Ψ_Q^{-1} .

Conclusion and further motivations. The translation invariance property of the model together with Fourier transform computations give most of the answers to classical questions. It leads to an easy computation of the partition function, the free energy and the correlation function. Moreover, as we just saw, it even makes the construction of the infinite-volume Gibbs measure straightforward. Nevertheless, the Fourier transform is a global computation which totally hides the (local) Markov property of the model. The main motivation of this article is precisely to stay closed to the Markov property by constructing the local objects of [Sim23] in the Gaussian settings, see half-strips in section 3.5.1.2 or corners in section 3.5.2. These objects satisfy eigenvalue-like equations in such a way that one can build consistent boundary weights g_R for rectangular domains R , and therefore, construct the infinite-volume Gibbs measure from Kolmogorov's extension theorem. Moreover the

eigenvalue Λ_Q computed for these objects corresponds to a partition function $Z_R^{\text{bw}}(Q; g_R)$ given by $\Lambda_Q^{\text{Area}(R)}$. Then, the free energy should be equal to

$$f^{\text{eigen}}(Q) = \log \Lambda_Q,$$

without taking any limit anymore. Due to exponential de-correlation, we expect that the value of the free energy should not be depending on the boundary conditions. Therefore, we should have $f^{\text{per}}(Q) = f^{\text{eigen}}(Q)$. Then, one objective will be to show that the eigenvalue associated with these objects does verify

$$\Lambda_Q = (2\pi)^{d_1+d_2} \exp \left(-\frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} \log \det \Psi_Q(e^{i\theta_1}, e^{i\theta_2}) d\theta_1 d\theta_2 \right). \quad (3.4.5)$$

This will be related to a renormalization procedure at the level of an extended parameter space $\mathbb{R} \times \mathcal{Q}_\bullet$, with a precise control on the cocycles γ_{WE} and γ_{SN} , see section 3.5.6 for non-rigorous computations concerning this phenomenon. Nevertheless, as the extended quadratic forms space alone is not trivial to construct, we will firstly drop the parameter space $\mathbb{R} \times \mathcal{Q}_\bullet$ to focus on \mathcal{Q}_\bullet on its own. In that case, no eigenvalue will appear, see Theorem 3.5.1 for instance.

3.4.2 Doubly-infinite strips: transfer matrix approach

In section 3.3.3 we saw that solving the harmonic problem for rectangles of any size gives strong intuition for guessing extended objects of [Sim23]. In this section, we propose an alternative approach based on the so-called transfer matrix method in view of giving doubly-infinite strips. Transfer matrix method is a classical trick in statistical mechanics which consists in the reduction to a one-dimensional problem by working on finite cylinder of increasing size, rather than the torus, and thus maintaining periodicity in one direction. We are therefore able to use discrete Fourier transform along this direction. One big advantage is that Schur products in the direct space, which become high-dimensional with the size of the cylinder, are converted in low-dimensional Schur products in the space of Fourier. At the limit, this will lead to the definition of doubly-infinite strips together with their product defined via finite dimensional Schur products on the spectral density. Under symmetry assumptions, we will be able to fold doubly-infinite strips into half-strips using an operator of fold which respects Schur products on the spectral density.

3.4.2.1 Finite periodic strips

We denote $B_Q^{WE,p}$ the West-East cylindrical surface power of Q obtained from $Q^{[p,1]}$ after a Schur complement on the identified West and East edges, which corresponds, after a lift to the quadratic forms, to an integration w.r.t the random variable on this edge. Graphically, we denote it

$$B_Q^{WE,p} = \bullet \left[\begin{array}{|c|c|c|c|} \hline Q & Q & \dots & Q \\ \hline \end{array} \right] \bullet \cdot$$

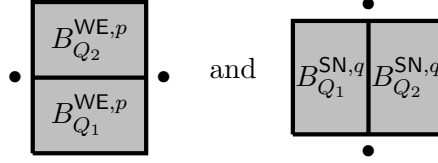
and it corresponds to the pattern shape cyl_{WE}^* of [Sim23] (section 3.5.6.2) with colours $(p, 1, h)$: we will assume all through this section that the base point is put on the vertical cut used to glue the left and the right sides of the previous drawing.

Similarly, we let $B_Q^{SN,q}$ be the South-North cylindrical surface power of Q obtained from $Q^{[1,q]}$ after the Schur complement which corresponds to the integration over the identified South and North edges. The quadratic form $B_Q^{WE,p}$ (resp. $B_Q^{SN,q}$) defines a one-dimensional

Gaussian Markov process with values in \mathcal{H}_1^p (resp. \mathcal{H}_2^q). With a slight abuse of notation, we will denote the associated one-dimensional product in the following way

$$S_{\text{SN}}(B_{Q_1}^{\text{WE},p}, B_{Q_2}^{\text{WE},p}) \quad \text{and} \quad S_{\text{WE}}(B_{Q_1}^{\text{SN},q}, B_{Q_2}^{\text{SN},q}).$$

or graphically as



These one-dimensional products are respectively compatible with South-North and West-East Schur products on the quadratic forms Q_1 and Q_2 in the sense of the following lemma.

Lemma 3.4.1. *We have*

$$S_{\text{SN}}(B_{Q_1}^{\text{WE},p}, B_{Q_2}^{\text{WE},p}) = B_{S_{\text{SN}}(Q_1, Q_2)}^{\text{WE},p} \quad \text{and} \quad S_{\text{WE}}(B_{Q_1}^{\text{SN},q}, B_{Q_2}^{\text{SN},q}) = B_{S_{\text{WE}}(Q_1, Q_2)}^{\text{SN},q}. \quad (3.4.6)$$

Proof. This is a consequence of associativity. \square

In other words, the previous lemma says that $B^{\text{WE},p} : ((Q_{1,q})_{q \in \mathbb{N}^*}, S_{\text{SN}}) \rightarrow (Q_p^{1D}, S_{\text{SN}})$, $Q \mapsto B_Q^{\text{WE},p}$ is a morphism of algebras (from LengthAss to Ass to be precise, see [Sim23]), where

$$Q_p^{1D} = \{Q \in \text{End}(\mathcal{H}_1^{2p}); Q^* = Q \text{ and } Q \text{ positive definite}\}.$$

For all $n \in \mathbb{N}^*$ and \mathcal{H} finite-dimensional Hilbert space, we let $\text{Diag}(M_k)_{k=1}^n \in \text{End}(\mathcal{H}^n)$ be the block-diagonal matrix with diagonal entries given by $M_k \in \text{End}(\mathcal{H})$ for all $k \in \{1, \dots, n\}$. A direct computation with a discrete Fourier change of variable gives the following lemma.

Lemma 3.4.2. *The quadratic forms $B_Q^{\text{WE},p}$ and $B_Q^{\text{SN},q}$ are block-circulant and therefore block diagonalizable in the basis of discrete Fourier (writing U_p and U_q for the associated change of bases):*

$$B_Q^{\text{WE},p} = U_p \widehat{B}_Q^{\text{WE},p} U_p^* \quad \text{and} \quad B_Q^{\text{SN},q} = U_q \widehat{B}_Q^{\text{SN},q} U_q^* \quad (3.4.7)$$

with $\widehat{B}_Q^{\text{WE},p}$ and $\widehat{B}_Q^{\text{SN},q}$ block diagonal with block entries respectively given by

$$\begin{aligned} (\widehat{B}_Q^{\text{WE},p})_{k,k} &= Q_{[\text{SN}],[\text{SN}]} - (Q_{[\text{SN}],[\text{W}]} + Q_{[\text{SN}],[\text{E}]} \omega_p^k) \phi_Q^{\text{WE}}(\omega_p^k)^{-1} (Q_{[\text{W}],[\text{SN}]} + Q_{[\text{E}],[\text{SN}]} \omega_p^{-k}) \\ &:= \widehat{B}_Q^{\text{WE}}(\omega_p^k) \end{aligned} \quad (3.4.8)$$

for all $k \in \{1, \dots, p\}$, and

$$\begin{aligned} (\widehat{B}_Q^{\text{SN},q})_{k,k} &= Q_{[\text{WE}],[\text{WE}]} - (Q_{[\text{WE}],[\text{S}]} + Q_{[\text{WE}],[\text{N}]} \omega_q^k) \phi_Q^{\text{SN}}(\omega_q^k)^{-1} (Q_{[\text{S}],[\text{WE}]} + Q_{[\text{N}],[\text{WE}]} \omega_q^{-k}) \\ &:= \widehat{B}_Q^{\text{SN}}(\omega_q^k) \end{aligned} \quad (3.4.9)$$

for all $k \in \{1, \dots, q\}$. Moreover Schur products are compatible with changes of bases in the sense that

$$S_{\text{SN}}(B_{Q_1}^{\text{WE},p}, B_{Q_2}^{\text{WE},p}) = U_p \left(\text{Diag} \left(S_{\text{SN}}(\widehat{B}_{Q_1}^{\text{WE}}(\omega_p^k), \widehat{B}_{Q_2}^{\text{WE}}(\omega_p^k)) \right)_{k=1}^p \right) U_p^* \quad (3.4.10)$$

and

$$S_{\text{WE}}(B_{Q_1}^{\text{SN},q}, B_{Q_2}^{\text{SN},q}) = U_q \left(\text{Diag} \left(S_{\text{WE}}(\widehat{B}_{Q_1}^{\text{SN}}(\omega_q^k), \widehat{B}_{Q_2}^{\text{SN}}(\omega_q^k)) \right)_{k=1}^q \right) U_q^*. \quad (3.4.11)$$

Then,

$$\text{Diag} \left(S_{\text{SN}}(\widehat{B}_{Q_1}^{\text{WE}}(\omega_p^k), \widehat{B}_{Q_2}^{\text{WE}}(\omega_p^k)) \right)_{k=1}^p = \text{Diag} \left(\widehat{B}_{S_{\text{SN}}(Q_1, Q_2)}^{\text{WE}}(\omega_p^k) \right)_{k=1}^p \quad (3.4.12)$$

and

$$\text{Diag} \left(S_{\text{WE}}(\widehat{B}_{Q_1}^{\text{SN}}(\omega_q^k), \widehat{B}_{Q_2}^{\text{SN}}(\omega_q^k)) \right)_{k=1}^q = \text{Diag} \left(\widehat{B}_{S_{\text{WE}}(Q_1, Q_2)}^{\text{SN}}(\omega_q^k) \right)_{k=1}^q. \quad (3.4.13)$$

Thanks to this lemma, for $B_Q^{\text{WE},p}$ for instance, up to a change of basis one can work with p independent one-dimensional Gaussian Markov processes with values in \mathcal{H}_1 rather than with one Gaussian Markov process with values in \mathcal{H}_1^p . By doing so, we reduce the Schur product involving matrices of dimension $(2p \dim \mathcal{H}_1)$ -by- $(2p \dim \mathcal{H}_1)$ in the direct space to the computation of p Schur products of dimension $(2 \dim \mathcal{H}_1)$ -by- $(2 \dim \mathcal{H}_1)$ in the space of Fourier. These Schur products are performed, mode per mode, on the spectrum of $B_Q^{\text{WE},p}$, given by $\widehat{B}_Q^{\text{WE}}(\omega_p^k)$ for $k \in \{1, \dots, p\}$. As we will see later, for doubly-infinite strips, infinite dimensional Schur product reduces to finite dimensional on the spectral density $\theta \in S_1 \mapsto \widehat{B}_Q^{\text{WE}}(e^{i\theta})$ with a bit more of analysis involved.

3.4.2.2 Doubly-infinite strips

Quadratic forms living on doubly infinite strips are expected to act on infinite-dimensional spaces. It is a priori not clear how to extend Schur products of section 3.3 to such operators since we are dealing with infinite-dimensional Gaussian measures. This subsection is devoted to the definition of doubly-infinite strips together with their products based on the heuristics provided by the transfer matrix method. We will see that these infinite dimensional Schur products are in fact automatically well defined and can even be reduced to elementary finite dimensional ones, already described in detail in [Bod23].

Schur complement. Here is a brief recall of Schur complement formulas, useful later in this section. See [Zha06] for additional details. Let M be the following block matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix} \text{ and } (M/M_{2,2}) = M_{1,1} - M_{1,2}M_{2,2}^{-1}M_{2,1} \quad (3.4.14)$$

its Schur complement, provided that $M_{2,2}$ is invertible. Then,

$$\det(M) = \det(M_{2,2}) \det(M/M_{2,2})$$

and, if $(M/M_{2,2})$ is invertible so is M and

$$M^{-1} = \begin{pmatrix} (M/M_{2,2})^{-1} & -(M/M_{2,2})^{-1}M_{1,2}M_{2,2}^{-1} \\ -M_{2,2}^{-1}M_{2,1}(M/M_{2,2})^{-1} & M_{2,2}^{-1} + M_{2,2}^{-1}M_{2,1}(M/M_{2,2})^{-1}M_{1,2}M_{2,2}^{-1} \end{pmatrix}. \quad (3.4.15)$$

thanks to a LDU decomposition of M (as the product of a lower triangular matrix, a diagonal matrix and an upper triangular matrix). Moreover, thanks to the same LDU decomposition, we have the following useful lemma.

Lemma 3.4.3. *Let M and $(M/M_{2,2})$ as in (3.4.14). Suppose that M is self-adjoint. Then, M is positive definite if and only if $M_{2,2}$ and $(M/M_{2,2})$ are positive definite.*

Based on the previous section, we define

$$\begin{aligned} \widehat{B}_Q^{\text{WE}} : S^1 &\rightarrow \text{End}(\mathcal{H}_1 \oplus \mathcal{H}_1) \\ z &\mapsto Q_{[\text{SN}], [\text{SN}]} - (Q_{[\text{SN}], [\text{W}]} + Q_{[\text{SN}], [\text{E}]}z)\phi_Q^{\text{WE}}(z)^{-1}(Q_{[\text{W}], [\text{SN}]} + Q_{[\text{E}], [\text{SN}]}z^{-1}) \end{aligned} \quad (3.4.16)$$

and

$$\begin{aligned} \widehat{B}_Q^{\text{SN}} : S^1 &\rightarrow \text{End}(\mathcal{H}_2 \oplus \mathcal{H}_2) \\ z &\mapsto Q_{[\text{WE}],[\text{WE}]} - (Q_{[\text{WE}],[\text{S}]} + Q_{[\text{WE}],[\text{N}]}z)\phi_Q^{\text{SN}}(z)^{-1}(Q_{[\text{S}],[\text{WE}]} + Q_{[\text{N}],[\text{WE}]}z^{-1}). \end{aligned} \quad (3.4.17)$$

We remark that these functions are given by Schur complement of, respectively, Ψ_Q^h and Ψ_Q^v defined in (3.2.8). Indeed, as we will see, they give the quadratic form for the random fields after integrating on every vertical (resp. horizontal) edges after a suitable Fourier transform on these edges. Similarly to the face operator Q , we have the following block decomposition

$$\widehat{B}_Q^{\text{WE}} = \begin{pmatrix} \widehat{B}_{Q,\text{SS}}^{\text{WE}} & \widehat{B}_{Q,\text{SN}}^{\text{WE}} \\ \widehat{B}_{Q,\text{NS}}^{\text{WE}} & \widehat{B}_{Q,\text{NN}}^{\text{WE}} \end{pmatrix} \text{ and } \widehat{B}_Q^{\text{SN}} = \begin{pmatrix} \widehat{B}_{Q,\text{WW}}^{\text{SN}} & \widehat{B}_{Q,\text{WE}}^{\text{SN}} \\ \widehat{B}_{Q,\text{EW}}^{\text{SN}} & \widehat{B}_{Q,\text{EE}}^{\text{SN}} \end{pmatrix}. \quad (3.4.18)$$

We have the following lemma.

Lemma 3.4.4. *For all $\theta \in [0, 2\pi]$, $\widehat{B}_Q^{\text{WE}}(e^{i\theta})$ and $\widehat{B}_Q^{\text{SN}}(e^{i\theta})$ are self-adjoint, positive definite and therefore invertible.*

Proof. Positive definite property is a direct consequence of positive definite property of Q . For all $\theta \in [0, 2\pi]$, $\Psi_Q^h(e^{i\theta})$ and $\Psi_Q^v(e^{i\theta})$ are positive definite thanks to lemma 3.2.3. Now, we only have to apply lemma 3.4.3 to $\widehat{B}_Q^{\text{WE}}(e^{i\theta})$ and $\widehat{B}_Q^{\text{SN}}(e^{i\theta})$ respectively obtained from a Schur complement of $\Psi_Q^h(e^{i\theta})$ and $\Psi_Q^v(e^{i\theta})$. \square

Some useful operators. Let \mathcal{H} some finite dimensional Hilbert space and $\widehat{a} : S^1 \rightarrow \text{End}(\mathcal{H})$ such that

$$\|\widehat{a}\|_\infty := \sup_{\theta \in [0, 2\pi]} \|\widehat{a}(e^{i\theta})\|_{\text{End}(\mathcal{H})} < \infty.$$

Let $M(\widehat{a})$ be the multiplication operator defined as

$$\begin{aligned} M(\widehat{a}) : L^2(S^1 \rightarrow \mathcal{H}) &\rightarrow L^2(S^1 \rightarrow \mathcal{H}) \\ f &\mapsto \widehat{a}f. \end{aligned} \quad (3.4.19)$$

obtained by pointwise multiplication of $f(\theta)$ by $\widehat{a}(\theta)$. This naturally defines a convolution operator $C(\widehat{a}) : l^2(\mathbb{Z} \rightarrow \mathcal{H}) \rightarrow l^2(\mathbb{Z} \rightarrow \mathcal{H})$ represented as a doubly infinite matrix with entries given by

$$C(\widehat{a})_{k,l} = C_{k-l}(\widehat{a}) \quad (3.4.20)$$

the $(k-l)$ -th Fourier coefficient of \widehat{a} for all k and $l \in \mathbb{Z}$, see (3.2.12). We have the following lemma.

Lemma 3.4.5. *If for all $\theta \in [0, 2\pi]$, $\widehat{a}(e^{i\theta}) \in \text{End}(\mathcal{H})$ is self-adjoint and positive definite, then, $M(\widehat{a})$ and $C(\widehat{a})$ are self-adjoint and positive definite.*

Proof. The result is clear for $M(\widehat{a})$ from which we deduce the one for $C(\widehat{a})$ by Fourier identification. \square

Another easy and interesting property from Fourier analysis is

$$C(\widehat{a})C(\widehat{b}) = C(\widehat{ab}). \quad (3.4.21)$$

We deduce that $M(\widehat{a})$ and $C(\widehat{a})$ are invertible if and only if \widehat{a} is invertible with bounded inverse on S^1 .

Doubly-infinite line spaces and strips. We are now ready to use the previous subsection as a heuristic to guess the doubly-infinite strips and their associated spaces. Doubly-infinite spaces associated to each direction are given by the direct spaces

$$\mathcal{W}_{WE} = l^2(\mathbb{Z} \rightarrow \mathcal{H}_1) \quad \mathcal{W}_{SN} = l^2(\mathbb{Z} \rightarrow \mathcal{H}_2). \quad (3.4.22)$$

These spaces will of course be identified with Fourier L^2 spaces, especially for infinite-dimensional gluing between doubly-infinite strips. As we will see, doubly infinite strips are operators in the following spaces

$$\begin{aligned} \mathcal{Q}_{\infty WE} &= \{B \in \text{End}(\mathcal{W}_{WE} \oplus \mathcal{W}_{WE}); B^* = B \text{ and } B \text{ positive definite} \} \\ \mathcal{Q}_{\infty SN} &= \{B \in \text{End}(\mathcal{W}_{SN} \oplus \mathcal{W}_{SN}); B^* = B \text{ and } B \text{ positive definite} \}. \end{aligned} \quad (3.4.23)$$

We are now able to define them doubly-infinite strips operators.

Definition 3.4.1 (Operators on doubly-infinite strips). We call doubly-infinite West-East strip associated to a self-adjoint and positive definite matrix Q the operator

$$B_Q^{WE,\infty} = \mathbf{C}^{WE}(\widehat{B}_Q^{WE}) := \begin{pmatrix} \mathbf{C}(\widehat{B}_{Q,SS}^{WE}) & \mathbf{C}(\widehat{B}_{Q,SN}^{WE}) \\ \mathbf{C}(\widehat{B}_{Q,NS}^{WE}) & \mathbf{C}(\widehat{B}_{Q,NN}^{WE}) \end{pmatrix} \in \text{End}(\mathcal{W}_{WE} \oplus \mathcal{W}_{WE}) \quad (3.4.24)$$

and for all $a, b \in \{S, N\}$, we denote $B_{Q,ab}^{WE,\infty} := \mathbf{C}(\widehat{B}_{Q,ab}^{WE})$. Similarly the doubly-infinite South-North strip operator associated to Q is the operator

$$B_Q^{SN,\infty} = \mathbf{C}^{SN}(\widehat{B}_Q^{SN}) := \begin{pmatrix} \mathbf{C}(\widehat{B}_{Q,WW}^{SN}) & \mathbf{C}(\widehat{B}_{Q,WE}^{SN}) \\ \mathbf{C}(\widehat{B}_{Q,EW}^{SN}) & \mathbf{C}(\widehat{B}_{Q,EE}^{SN}) \end{pmatrix} \in \text{End}(\mathcal{W}_{SN} \oplus \mathcal{W}_{SN}) \quad (3.4.25)$$

and for all $a, b \in \{W, E\}$, we denote $B_{Q,ab}^{SN,\infty} := \mathbf{C}(\widehat{B}_{Q,ab}^{SN})$.

The following lemma ensures that these doubly-infinite strips operators are elements of the sets $\mathcal{Q}_{\infty WE}$ and $\mathcal{Q}_{\infty SN}$ previously introduced in (3.4.23).

Lemma 3.4.6. *Under the previous definition 3.4.1, we have $B_Q^{WE,\infty} \in \mathcal{Q}_{\infty WE}$ and $B_Q^{SN,\infty} \in \mathcal{Q}_{\infty SN}$.*

Proof. We only have to check that both operators are self-adjoint and positive definite. Because Q is positive definite, this is a direct consequence of lemmas 3.4.4 and 3.4.5. \square

Schur product on doubly-infinite strips. Doubly-infinite strips come with a transversal products. For instance we define

$$\begin{aligned} \begin{array}{|c|} \hline B_{Q_2}^{WE,\infty} \\ \hline B_{Q_1}^{WE,\infty} \\ \hline \end{array} &:= S_{SN}(B_{Q_1}^{WE,\infty}, B_{Q_2}^{WE,\infty}) \\ &:= \begin{pmatrix} B_{Q_1,SS}^{WE,\infty} - B_{Q_1,SN}^{WE,\infty} K^{-1} B_{Q_1,NS}^{WE,\infty} & -B_{Q_1,SN}^{WE,\infty} K^{-1} B_{Q_2,SN}^{WE,\infty} \\ -B_{Q_2,NS}^{WE,\infty} K^{-1} B_{Q_1,NS}^{WE,\infty} & B_{Q_2,NN}^{WE,\infty} - B_{Q_2,NS}^{WE,\infty} K^{-1} B_{Q_2,SN}^{WE,\infty} \end{pmatrix} \end{aligned}$$

where $K = (B_{Q_2,S,S}^{WE,\infty} + B_{Q_1,N,N}^{WE,\infty})$. There is an analog West-East Schur product for doubly-infinite South-North strips denoted S_{WE} . These Schur products involve infinite dimensional operators and their inverses. We then have to show that they are well defined. This is the purpose of the following proposition.

Property 3.4.1. *Schur products on doubly-infinite strips are well-defined for positive-definite quadratic forms Q_1 and Q_2 and we have*

$$S_{\text{SN}}(B_{Q_1}^{\text{WE},\infty}, B_{Q_2}^{\text{WE},\infty}) = \mathbf{C}^{\text{WE}}(S_{\text{SN}}(\widehat{B}_{Q_1}^{\text{WE}}, \widehat{B}_{Q_2}^{\text{WE}})) = \mathbf{C}^{\text{WE}}(\widehat{B}_{S_{\text{SN}}(Q_1, Q_2)}^{\text{WE}}) \in \mathcal{Q}_{\infty\text{WE}} \quad (3.4.26)$$

$$S_{\text{WE}}(B_{Q_1}^{\text{SN},\infty}, B_{Q_2}^{\text{SN},\infty}) = \mathbf{C}^{\text{SN}}(S_{\text{WE}}(\widehat{B}_{Q_1}^{\text{SN}}, \widehat{B}_{Q_2}^{\text{SN}})) = \mathbf{C}^{\text{SN}}(\widehat{B}_{S_{\text{WE}}(Q_1, Q_2)}^{\text{SN}}) \in \mathcal{Q}_{\infty\text{SN}} \quad (3.4.27)$$

with the following pointwise definition of the Schur products of functions:

$$\begin{aligned} S_{\text{SN}}(\widehat{B}_{Q_1}^{\text{WE}}, \widehat{B}_{Q_2}^{\text{WE}}) : S^1 &\rightarrow \text{End}(\mathcal{H}_1 \oplus \mathcal{H}_1) \\ e^{i\theta} &\mapsto S_{\text{SN}}(\widehat{B}_{Q_1}^{\text{WE}}(e^{i\theta}), \widehat{B}_{Q_2}^{\text{WE}}(e^{i\theta})) \\ S_{\text{WE}}(\widehat{B}_{Q_1}^{\text{WE}}, \widehat{B}_{Q_2}^{\text{SN}}) : S^1 &\rightarrow \text{End}(\mathcal{H}_2 \oplus \mathcal{H}_2) \\ e^{i\theta} &\mapsto S_{\text{WE}}(\widehat{B}_{Q_1}^{\text{WE}}(e^{i\theta}), \widehat{B}_{Q_2}^{\text{SN}}(e^{i\theta})). \end{aligned}$$

Proof. It is sufficient to consider $S_{\text{SN}}(B_{Q_1}^{\text{WE},\infty}, B_{Q_2}^{\text{WE},\infty})$. We check that $\theta \mapsto \widehat{K}(e^{i\theta}) = (\widehat{B}_{Q_1, \text{SS}}^{\text{WE}}(e^{i\theta}) + \widehat{B}_{Q_2, \text{NN}}^{\text{WE}}(e^{i\theta}))$ is invertible with bounded inverse as a consequence of lemma 3.4.4. Then, the first equality is a direct consequence of (3.4.21) and the definition of S_{SN} . Moreover,

$$S_{\text{SN}}(\widehat{B}_{Q_1}^{\text{WE}}, \widehat{B}_{Q_2}^{\text{WE}}) = \widehat{B}_{S_{\text{SN}}(Q_1, Q_2)}^{\text{WE}}$$

which is true on every root of unity from lemma 3.4.2 and can be extended to all $\theta \in [0, 2\pi]$ by continuity of $\widehat{B}_Q^{\text{WE}}$ for all Q positive definite. Finally, we have $\mathbf{C}^{\text{WE}}(\widehat{B}_{S_{\text{SN}}(Q_1, Q_2)}^{\text{WE}}) \in \mathcal{Q}_{\infty\text{WE}}$. Indeed, the self-adjoint and positive definite properties are deduced from lemmas 3.4.4 and 3.4.5 by using the fact that $S_{\text{SN}}(Q_1, Q_2)$ is positive definite, see (3.3.7). \square

This proposition shows that infinite dimensional Schur products on doubly-infinite strips are automatically well defined. More importantly, they can be computed with a finite-dimensional Schur products in the Fourier space: " θ by θ ".

3.4.3 Half-planes, algebraic curve and Markov property for horizontal infinite lines

In this section we give and study half-planes operators of [Sim23]. As we will see, half-planes operators are elements of the following spaces

$$\mathcal{Q}_{\infty\text{WE}, \infty\text{S}} = \mathcal{Q}_{\infty\text{WE}, \infty\text{N}} = \{B \in \text{End}(\mathcal{W}_{\text{WE}}); B^* = B \text{ and } B \text{ positive definite}\} \quad (3.4.28)$$

$$\mathcal{Q}_{\infty\text{W}, \infty\text{SN}} = \mathcal{Q}_{\infty\text{E}, \infty\text{SN}} = \{B \in \text{End}(\mathcal{W}_{\text{SN}}); B^* = B \text{ and } B \text{ positive definite}\}.$$

where the doubly-infinite line spaces \mathcal{W}_{WE} and \mathcal{W}_{SN} are defined in (3.4.22). The spaces $\mathcal{Q}_{\infty\text{WE}, \infty\text{S}}$ and $\mathcal{Q}_{\infty\text{WE}, \infty\text{N}}$ (resp. $\mathcal{Q}_{\infty\text{W}, \infty\text{SN}}$ and $\mathcal{Q}_{\infty\text{E}, \infty\text{SN}}$) correspond to South and North half-planes spaces (resp. West and East half-plane spaces). For seek of simplicity, we will focus on South and North half-planes operators. Similar arguments work for West and East half-planes.

The sequence of doubly infinite West-East strips on top of each other defines a one-dimensional Gaussian Markov field with values in an infinite dimensional space, either in the direct space $l^2(\mathbb{Z} \rightarrow \mathcal{H}_1)$ or in the Fourier space $L^2(S^1 \rightarrow \mathcal{H}_1)$. In this latter Fourier space, this random field can be reduced to a collection of infinitely many one-dimensional Gaussian Markov processes with values in \mathcal{H}_1 with dynamics given by the spectral density $\widehat{B}_Q^{\text{WE}}(e^{i\theta})$, i.e. " θ by θ ". We thus fall into the scope of application of [Bod23]. We will start by having a look at South and North invariant boundary conditions to this dynamic, which correspond to the pattern shapes \mathbf{hp}_S and \mathbf{hp}_N of South and North half-planes of [Sim23]. We will also

be interested in giving remarkable identities behind this one-dimensional dynamic: link with the algebraic curve \mathcal{S}_Q , Markov property, and others.

Based on [Bod23] and its associated reminder in section 3.2.1.5, the strip quadratic form $G_S \in \text{End}(l^2(\mathbb{Z} \rightarrow \mathcal{H}_1))$ is said to be South invariant for $B_Q^{\text{WE},\infty}$ if it satisfies

$$G_S = B_{Q,\text{NN}}^{\text{WE},\infty} - B_{Q,\text{NS}}^{\text{WE},\infty} (B_{Q,\text{SS}}^{\text{WE},\infty} + G_S)^{-1} B_{Q,\text{SN}}^{\text{WE},\infty} \quad (3.4.29)$$

which corresponds, after a lift to quadratic forms with Schur complements, to an invariant Gaussian weight under the action of the Gaussian weight associated to $B_Q^{\text{WE},\infty}$. In other words, G_S is invariant under the action of the doubly-infinite West-East strip. In that case, G_S corresponds to the South half-plane operator of [Sim23]. Once again, assuming that $G_S = C(\widehat{G}_S)$ for some continuous and positive definite function $\widehat{G}_S : S^1 \rightarrow \text{End}(\mathcal{H}_1)$, we can restrict ourselves to the study of the pointwise equation:

$$\widehat{G}_S(e^{i\theta}) = \widehat{B}_{Q,\text{NN}}^{\text{WE}}(e^{i\theta}) - \widehat{B}_{Q,\text{NS}}^{\text{WE}}(e^{i\theta}) (\widehat{B}_{Q,\text{SS}}^{\text{WE}}(e^{i\theta}) + \widehat{G}_S(e^{i\theta}))^{-1} \widehat{B}_{Q,\text{SN}}^{\text{WE}}(e^{i\theta}) \in \text{End}(\mathcal{H}_1 \oplus \mathcal{H}_1). \quad (3.4.30)$$

for all $\theta \in [0, 2\pi]$. Then, we are in the situation of [Bod23] and the problem can be solved by considering the meromorphic function

$$\phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z) := \widehat{B}_{Q,\text{SS}}^{\text{WE}}(e^{i\theta}) + \widehat{B}_{Q,\text{NN}}^{\text{WE}}(e^{i\theta}) + \widehat{B}_{Q,\text{SN}}^{\text{WE}}(e^{i\theta})z + \widehat{B}_{Q,\text{NS}}^{\text{WE}}(e^{i\theta})z^{-1} \quad (3.4.31)$$

for all $z \in \mathbb{C}^*$, together with its zeros

$$\mathcal{S}_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}^{1D} = \{z \in \mathbb{C}^*; \det \phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z) = 0\}, \quad (3.4.32)$$

and their associated bases provided by the ones of $\ker \phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z)$ for all $z \in \mathcal{S}_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}^{1D}$, see section 3.2.1.5 for a reminder concerning 1D results of [Bod23]. These quantities are expected to be related to Ψ_Q , from the construction of the functions Ψ in (3.2.8). The following lemma clarifies this point by providing the link between $\phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}$, its zeros $\mathcal{S}_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}^{1D}$ and Ψ_Q and its associated algebraic curve \mathcal{S}_Q . We first introduce the following notations

$$\mathcal{S}_Q(e^{i\theta}, \bullet) := \{z \in \mathbb{C}^*; (e^{i\theta}, z) \in \mathcal{S}_Q\} \quad (3.4.33)$$

for all $\theta \in [0, 2\pi]$ (section of $e^{i\theta}$ in \mathcal{S}_Q) and $\text{mult}_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z)$ (resp. $\text{mult}_{\mathcal{S}_Q(e^{i\theta}, \bullet)}(z)$) for the multiplicity of the root z with respect to the polynomial $\det(X\phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(X))$ (resp. $\det(X\Psi_Q(e^{i\theta}, X))$).

Lemma 3.4.7. *For all $\theta \in [0, 2\pi]$,*

$$\mathcal{S}_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}^{1D} = \mathcal{S}_Q(e^{i\theta}, \bullet) \text{ and } \text{mult}_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z) = \text{mult}_{\mathcal{S}_Q(e^{i\theta}, \bullet)}(z) \quad (3.4.34)$$

for all $z \in \mathcal{S}_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}^{1D}$. Moreover, for such a $z \in \mathcal{S}_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}^{1D}$,

$$\begin{aligned} \ker \Psi_Q(e^{i\theta}, z) &\rightarrow \ker \phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z) \\ (u^{(1)}, u^{(2)}) &\mapsto u^{(1)} \end{aligned}$$

is invertible with inverse map given by $u \mapsto (u, (\Psi_Q(e^{i\theta}, z)_{2,2})^{-1}(\Psi_Q(e^{i\theta}, z))_{2,1}u)$.

Proof. We use the notation introduced in (3.2.9) for the blocks of Ψ_Q . For all $\theta \in [0, 2\pi]$, $\det(\Psi_Q(e^{i\theta}, z)_{2,2}) = \det(\phi_Q^{\text{WE}}(e^{i\theta})) > 0$ due to lemma 3.2.3 and we remark that

$$\phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z) = (\Psi_Q(e^{i\theta}, z))_{1,1} - (\Psi_Q(e^{i\theta}, z))_{1,2}(\Psi_Q(e^{i\theta}, z)_{2,2})^{-1}(\Psi_Q(e^{i\theta}, z))_{2,1}$$

so that $\phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z) = (\Psi_Q(e^{i\theta}, z)/\Psi_Q(e^{i\theta}, z)_{2,2})$ the Schur complement of $\Psi_Q(e^{i\theta}, z)$, see (3.4.14). Then, the Schur complement formula for determinants gives

$$\det \Psi_Q(e^{i\theta}, z) = \det(\phi_Q^{\text{WE}}(e^{i\theta})) \det(\phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z)).$$

Therefore, for all $z \in \mathbb{C}^*$, $\det \Psi_Q(e^{i\theta}, z) = 0$ if and only if $\det \phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z) = 0$ which implies (3.4.34). Still using a Schur complement formula, $\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} \in \ker \Psi_Q(e^{i\theta}, z)$ if and only if $u^{(1)} \in \ker \phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}$ and $u^{(2)} = (\Psi_Q(e^{i\theta}, z)_{2,2})^{-1}(\Psi_Q(e^{i\theta}, z))_{2,1}u^{(1)}$. This proves the second assertion. \square

Remark that, for all $z \notin \mathcal{S}_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}^{1D}$,

$$\phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z)^{-1} = \Psi_Q^{-1}(e^{i\theta}, z)_{1,1}. \quad (3.4.35)$$

due to the Schur complement inverse formula (3.4.15). This equation is not surprising as we already saw in 3.4 that in thermodynamic limit, the covariance matrix of horizontal edges is given by Fourier coefficients of $(\Psi_Q^{-1})_{1,1}$.

We are now ready to build South invariant boundary condition, i.e South half-plane, from Ψ_Q and its algebraic curve \mathcal{S}_Q . We recall that $C_{\bullet,k}(e^{i\theta})$ stands for the partial k -th Fourier coefficient of Ψ_Q , see (3.2.15). We have the following result.

Theorem 3.4.1. *Assume that for all $\theta \in [0, 2\pi]$, and all $z \in \mathcal{S}_Q(e^{i\theta}, \bullet)$, $\dim \ker \Psi_Q(e^{i\theta}, z) = \text{mult}_{\mathcal{S}_Q(e^{i\theta}, \bullet)}(z) = 1$ so that $\ker \Psi_Q(e^{i\theta}, z) = \text{Vect} \begin{pmatrix} u_z^{(1)}(\theta) \\ u_z^{(2)}(\theta) \end{pmatrix}$. Letting*

$$\mathcal{S}_Q^{>1}(e^{i\theta}, \bullet) = \{z \in \mathbb{C}^*; z \in \mathcal{S}_Q(e^{i\theta}, \bullet) \text{ and } |z| > 1\},$$

the family $(u_z^{(1)}(\theta))_{z \in \mathcal{S}_Q^{>1}(e^{i\theta}, \bullet)}$ is a basis of \mathcal{H}_1 . Let $\widehat{W}_S(e^{i\theta}) \in \text{End}(\mathcal{H}_1)$ diagonal in this basis with $\widehat{W}_S(e^{i\theta})u_z^{(1)}(\theta) = (1/z)u_z^{(1)}(\theta)$ for all $z \in \mathcal{S}_Q^{>1}(e^{i\theta}, \bullet)$. Then for all $\theta \in [0, 2\pi]$,

$$\widehat{G}_S(e^{i\theta}) = \widehat{B}_{Q,\text{NN}}^{\text{WE}}(e^{i\theta}) + \widehat{B}_{Q,\text{NS}}^{\text{WE}}(e^{i\theta})\widehat{W}_S(e^{i\theta}) \quad (3.4.36)$$

is self-adjoint and positive definite and solves the equation (3.4.30). Moreover, we have the following expression

$$\widehat{W}_S(e^{i\theta}) = C_{\bullet,1}^{1,1}(e^{i\theta})(C_{\bullet,0}^{1,1}(e^{i\theta}))^{-1} \quad (3.4.37)$$

and $\widehat{G}_S : S^1 \rightarrow \text{End}(\mathcal{H}_1)$ is continuous. Finally $G_S = \mathcal{C}(\widehat{G}_S) \in \mathcal{Q}_{\infty\text{WE}, \infty\text{S}}$ solves the equation (3.4.29), we call it the South half-plane operator.

Proof. For all $\theta \in [0, 2\pi]$, $\widehat{B}^{\text{WE}}(e^{i\theta})$ is positive definite, see lemma 3.4.4. Moreover, using lemma 3.4.7, every $z \in \mathcal{S}_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}^{1D}$ is of multiplicity 1 and $\ker \phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})} = \text{Vect}(u_z^{(1)}(\theta))$. Assumptions (1D) are fulfilled and we can therefore apply the results of [Bod23] for every $\theta \in [0, 2\pi]$ to the one-dimensional Gaussian Markov process generated by $\widehat{B}_Q^{\text{WE}}(e^{i\theta})$, see

theorem 3.2.1 and equation (3.2.20). We check that \widehat{W}_S is continuous on S^1 thanks to (3.4.37), which expresses \widehat{W}_S in terms of continuous partial Fourier coefficients of Ψ_Q . Then, \widehat{G}_S is continuous thanks to continuity of $\widehat{B}_Q^{\text{WE}}$. Finally, since $\widehat{G}_S(e^{i\theta})$ is self-adjoint and positive definite for all $\theta \in [0, 2\pi]$, $\mathbb{C}(\widehat{G}_S)$ is self-adjoint and positive definite thanks to lemma 3.4.5 and therefore $\mathbb{C}(\widehat{G}_S) \in \mathcal{Q}_{\infty\text{WE}, \infty\text{S}}$ (defined in (3.4.28)). A rewriting of (3.4.30) in the direct space shows that this operators solves (3.4.29). \square

The assumptions of the last theorem are only made to match with the results of [Bod23], see section 3.2.1.5 for more details. Similarly, one can give the North invariant boundary condition thanks to [Bod23]. This corresponds to the North half-plane operator.

When considering invariant boundary conditions for one-dimensional Gaussian Markov process generated by the doubly infinite West-East strip, we only see a portion of the algebraic curve \mathcal{S}_Q given by

$$\mathcal{S}_Q \cap (S^1 \times \mathbb{C}^*). \quad (3.4.38)$$

The unit circle S^1 parametrizes the solutions: there is a family of zeros $(\mathcal{S}_Q^{>1}(e^{i\theta}, \bullet))_{e^{i\theta} \in S^1}$ together with a family of bases of \mathcal{H}_1 , $((u_z^{(1)}(\theta))_{z \in \mathcal{S}_Q^{>1}(e^{i\theta}, \bullet)})_{e^{i\theta} \in S^1}$, which lead to the construction of the function $\widehat{W}_S : S^1 \rightarrow \text{End}(\mathcal{H}_1)$ and the South invariant boundary condition. This parametrization is reminiscent of the West-East periodicity imposed by the transfer matrix approach and may be obtained by a suitable construction of the infinite volume Gibbs measure by a suitable control of the convergence of the Riemann sums over the roots of unity of order p as $p \rightarrow \infty$.

A slight rewriting of the last theorems leads to the following result, which links \widehat{G}_S with harmonic computations of section 3.3.

Corollary 3.4.1. *South half-plane operator is given by*

$$\widehat{G}_S(e^{i\theta}) = Q_{\text{NN}} + \left(Q_{\text{NS}} C_{\bullet,1}^{1,1}(e^{i\theta}) + (Q_{\text{NW}} + Q_{\text{NE}} e^{i\theta}) C_{\bullet,1}^{2,1}(e^{i\theta}) \right) (C_{\bullet,0}^{1,1}(e^{i\theta}))^{-1}. \quad (3.4.39)$$

Proof. Using the previous theorem and by definition of $\widehat{B}_Q^{\text{WE}}$ and (3.4.37) for replacing \widehat{W}_S , we get

$$\begin{aligned} \widehat{G}_S(e^{i\theta}) &= Q_{\text{NN}} + Q_{\text{NS}} C_{\bullet,1}^{1,1}(e^{i\theta}) (C_{\bullet,0}^{1,1}(e^{i\theta}))^{-1} \\ &\quad - (Q_{\text{NW}} + Q_{\text{NE}} e^{i\theta}) \phi_Q^{\text{WE}}(e^{i\theta})^{-1} ((Q_{\text{WN}} + Q_{\text{EN}} e^{-i\theta}) C_{\bullet,0}^{1,1} \\ &\quad + (Q_{\text{WS}} + Q_{\text{ES}} e^{-i\theta}) C_{\bullet,1}^{1,1}) (C_{\bullet,0}^{1,1}(e^{i\theta}))^{-1}. \end{aligned}$$

Then, after writing Fourier coefficients as integrals, we get

$$\begin{aligned} \widehat{G}_S(e^{i\theta}) &= Q_{\text{NN}} + Q_{\text{NS}} C_{\bullet,1}^{1,1}(e^{i\theta}) (C_{\bullet,0}^{1,1}(e^{i\theta}))^{-1} \\ &\quad - (Q_{\text{NW}} + Q_{\text{NE}} e^{i\theta}) \int_0^{2\pi} \frac{d\alpha}{2\pi} \phi_Q^{\text{WE}}(e^{i\theta})^{-1} (\Psi_Q(e^{i\theta}, e^{i\alpha}))_{2,1} (\Psi_Q(e^{i\theta}, e^{i\alpha})^{-1})_{1,1} e^{-i\alpha} (C_{\bullet,0}^{1,1}(e^{i\theta}))^{-1}. \end{aligned}$$

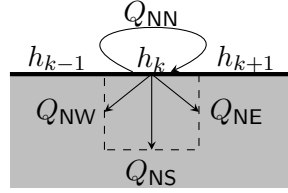
Schur complement formula for the inverse (3.4.15) gives

$$\phi_Q^{\text{WE}}(e^{i\theta})^{-1} (\Psi_Q(e^{i\theta}, e^{i\alpha}))_{2,1} (\Psi_Q(e^{i\theta}, e^{i\alpha})^{-1})_{1,1} = (\Psi_Q(e^{i\theta}, e^{i\alpha})^{-1})_{2,1}$$

for all $\theta, \alpha \in [0, 2\pi]$. We conclude by definition of partial Fourier coefficient $C_{\bullet,1}^{2,1}$, see (3.2.15). \square

In the previous expression we recognize, in the Fourier space, the operator $D_{e,\text{N}}^{(1)}(x)$ from (3.3.13). Indeed, in the direct space, every edge h_k , for all $k \in \mathbb{Z}$, is linked with the rest only

through its nearest neighbors at the South (included itself) from which it is connected with Q_{NN} , Q_{NS} , Q_{NW} , Q_{NE} , represented in the following drawing



The previous corollary therefore provides another expression for harmonic, in the South half-plane, in term of Fourier coefficients of Ψ_Q .

We have the following useful property concerning partial Fourier coefficients of Ψ_Q .

Lemma 3.4.8. *Under assumptions of theorem 3.4.1, for all $\theta \in [0, 2\pi]$, and all $k, l \in \mathbb{Z}$ such that $kl \geq 0$, we have*

$$C_{\bullet, k}^{1,1}(e^{i\theta})(C_{\bullet, 0}^{1,1}(e^{i\theta}))^{-1}C_{\bullet, l}^{1,1}(e^{i\theta}) = C_{\bullet, k+l}^{1,1}(e^{i\theta}). \quad (3.4.40)$$

Proof. This is (3.2.23) applied to $\phi_{\widehat{B}_Q^{\text{WE}}(e^{i\theta})}(z)^{-1} = \Psi_Q(e^{i\theta}, z)_{1,1}^{-1}$, thanks to (3.4.35). \square

As recalled in section 3.2.1.5, this is, θ being fixed, the Markov property of the one-dimensional Gaussian Markov process generated by $\widehat{B}_Q^{\text{WE}}(e^{i\theta})$ in thermodynamic limit, see (3.2.23). Morally, in the direct space, (3.4.40) gives the Markov property between doubly-infinite horizontal lines in infinite volume Gibbs measure. Indeed, we recall that, in infinite-volume Gibbs measure, the field restricted to horizontal edges is a centered Gaussian with covariance matrix built with the Fourier coefficients $(C_{n,m}^{1,1})_{n,m \in \mathbb{Z}}$, see (3.4.3). In the direct space, the previous equation becomes

$$C(C_{\bullet, k+l}^{1,1}) - C(C_{\bullet, k}^{1,1})C(C_{\bullet, 0}^{1,1})^{-1}C(C_{\bullet, l}^{1,1}) = 0$$

where $C(\cdot)$ is the convolution operator introduced in (3.4.20). This is the conditional covariance of two infinite horizontal lines given one in their middle, at distance $|k|$ from the first one and $|l|$ from the second one. Conditionally to this middle infinite line, the two others are independent. In other words, a doubly-infinite horizontal line cuts \mathbb{Z}^2 into two independent components.

A quick word concerning the other half-planes. As already mentioned, we could have treated the case of North, West or East half-planes respectively given by the symbols \widehat{G}_N , \widehat{G}_W or \widehat{G}_E . They follow very similar properties. For instance, under analogous assumptions of Theorem 3.4.1, the West half-plane is given by, for all $\theta \in [0, 2\pi]$,

$$\widehat{G}_W(e^{i\theta}) = \widehat{B}_{Q,EE}^{\text{SN}}(e^{i\theta}) + \widehat{B}_{Q,WE}^{\text{WE}}(e^{i\theta})\widehat{W}_W(e^{i\theta}) \quad (3.4.41)$$

which is self-adjoint, positive definite and Schur-invariant with respect to the strip $\widehat{B}_Q^{\text{SN}}$ in the manner of equation (3.4.30). Moreover, we have the following expression

$$\widehat{W}_W(e^{i\theta}) = C_{1,\bullet}^{2,2}(e^{i\theta})(C_{0,\bullet}^{2,2}(e^{i\theta}))^{-1}. \quad (3.4.42)$$

The operator attached to the West half-plane is therefore given by $C(\widehat{G}_W) \in \mathcal{Q}_{\infty W, \infty \text{SN}}$. When working with half-strips in section 3.5.1.2 or corners in section 3.5.2, Q is assumed to be dihedral-invariant, see (3.2.6). In that case, we have $\widehat{G}_S = \widehat{G}_W = \widehat{G}_N = \widehat{G}_E$, the four half-planes are equal.

3.5 Boundary weights, half-strips and corners: construction of new spaces

In this section, we construct new spaces and objects related to [Sim23]. These objects, such as half-strips and corners, are quadratic forms on the well-chosen Hilbert spaces. They can be glued together thanks to Schur products. We explicit and give a sense of these algebraic relations which combinatorics is given by the Guill_2 operad.

3.5.1 Half-strips

We are now interested in giving and studying quadratic forms associated to the half-strip shapes of [Sim23] in our case. Here again, we will be mainly focusing on West and East half-strips. South and North half-strips follow the exact same arguments. As we saw, the transfer matrix approach provides quadratic forms associated to doubly-infinite strips. Similar but more sophisticated computations with additional observables allow to separate the doubly-infinite West-East strip objects into a half West strip object and a half East strip object. We won't present these computations here as they are heuristics and they don't give much more insights than what the previous computations gave. Harmonic computations of section 3.3.3 also lead to the same definition of half-strips. Before defining half-strip quadratic forms, we will briefly introduce Toeplitz, Hankel and Fold operators. These operators, which act on half-infinite line spaces, are building blocks of half-strips. Once half-strips defined, we will highlight some interesting computations. In particular, we will show that West and East half-strip objects glued together give back the doubly-infinite West-East strip quadratic forms previously defined. Moreover, we will see that the face weight Q acts on half-strips by lengthening them. This lengthening is precisely the morphisms introduced in [Sim23] to generalize the notion of eigenvalues and eigenvectors, at the level of quadratic forms.

Because the situation is much nicer under full symmetry assumption and since our main purpose is not the classification of all the possible behaviours but rather the Gaussian incarnation of the objects in [Sim23], we will now restrict ourselves to the dihedral-invariant case, introduced in (3.2.6),

$$Q = \begin{pmatrix} T & A & U & U \\ A & T & U & U \\ U & U & T & A \\ U & U & A & T \end{pmatrix} \quad (3.5.1)$$

which already contains all the relevant ingredients. In particular, with the help of the "fold operator" (3.5.10), this will drastically simplify the analysis behind the infinite dimensional Schur product responsible of stacking half-strips, see Proposition 3.5.2. In that case, $\widehat{B}_Q^{\text{WE}} = \widehat{B}_Q^{\text{SN}}$ are even functions, see (3.4.16) and (3.4.17) for their definitions. Moreover, following [Bod23] and its reminder in section 3.2.1.5, the one-dimensional Gaussian Markov process with weight given by $Q_{[\text{SN}],[\text{SN}]} = Q_{[\text{WE}],[\text{WE}]}$ can be studied thanks to the function

$$\phi_Q(z) = 2T + (z + z^{-1})A$$

for all $z \in \mathbb{C}^*$ and the Fourier coefficients of its inverse, $(C_k(\phi_Q^{-1}))_{k \in \mathbb{Z}}$. We remark that, thanks to symmetry assumption,

$$\left(C_k(\phi_Q^{-1})\right)^* = C_{-k}(\phi_Q^{-1}) = C_k(\phi_Q^{-1}) \quad (3.5.2)$$

for all $k \in \mathbb{Z}$. If $Q_{[\text{SN}],[\text{SN}]} = Q_{[\text{WE}],[\text{WE}]}$ satisfies the assumptions (1D), we recall that

$$C_k(\phi_Q^{-1})(C_0(\phi_Q^{-1}))^{-1}C_l(\phi_Q^{-1}) = C_{k+l}(\phi_Q^{-1}) \quad (3.5.3)$$

for all $kl \geq 0$, see (3.2.23). Finally, we denote

$$W_Q := C_1(\phi_Q^{-1})(C_0(\phi_Q^{-1}))^{-1} \quad (3.5.4)$$

which, in [Bod23], is also built from zeros of ϕ_Q in the associated basis, see Theorem 3.2.1. To be able to use all these formulas, we will only work under the following assumption from now.

Assumption (\star). The self-adjoint and positive definite face operator $Q \in \mathcal{Q}_{1,1}$ is dihedral-invariant (see (3.2.6)) and $Q_{[\text{SN}],[\text{SN}]} = Q_{[\text{WE}],[\text{WE}]}$ satisfies the assumptions (1D).

Under this last assumption, the dihedral-invariant property imposes $\mathcal{H}_1 = \mathcal{H}_2$ and many other equalities. For instance, $\mathcal{W}_S = \mathcal{W}_W$, the South and West half-line spaces, later defined in (3.5.15). Nevertheless, we will keep the indices for a better distinction between horizontal and vertical edges and between half-line spaces.

3.5.1.1 Hardy-like spaces, Toeplitz, Hankel and fold operators.

We introduce some useful spaces and operators. Let \mathcal{H} be a finite dimensional Hilbert space and $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{\leq 0}$) the set of strictly positive (resp. non-positive) integers. We introduce the following l^2 spaces, identified with their associated Hardy spaces through Fourier transform,

$$l^2(\mathbb{Z}_{>0} \rightarrow \mathcal{H}) \cong \left\{ \hat{f} \in L^2(S^1 \rightarrow \mathcal{H}); C_k(\hat{f}) = 0 \text{ for all } k \leq 0 \right\} \quad (3.5.5)$$

and

$$l^2(\mathbb{Z}_{\leq 0} \rightarrow \mathcal{H}) \cong \left\{ \hat{f} \in L^2(S^1 \rightarrow \mathcal{H}); C_k(\hat{f}) = 0 \text{ for all } k > 0 \right\}. \quad (3.5.6)$$

Note that these spaces do not correspond to the standard definition of Hardy spaces because of the 0-order term. This slight difference does not play any special role here, it is just a consequence of the way we originally labeled the edges of the lattice \mathbb{Z}^2 . For a fixed $k > 0$, we go from $l^2(\mathbb{Z}_{>0} \rightarrow \mathcal{H})$ to $l^2(\mathbb{Z}_{>k} \rightarrow \mathcal{H})$ either by shifting the argument of f by k in the direct space or multiplying $\hat{f}(\theta)$ by $e^{i\theta k}$ in the Fourier space. We define the following projector operator.

$$P^W : l^2(\mathbb{Z} \rightarrow \mathcal{H}) \rightarrow l^2(\mathbb{Z}_{>0} \rightarrow \mathcal{H}) \quad (3.5.7)$$

$$(f_k)_{k \in \mathbb{Z}} \mapsto (f_k)_{k > 0}$$

and $P^E = I - P^W$, the projectors onto each Hardy-like spaces. We also define J the flip operator as

$$J : l^2(\mathbb{Z} \rightarrow \mathcal{H}) \rightarrow l^2(\mathbb{Z} \rightarrow \mathcal{H}) \quad (3.5.8)$$

$$(f_k)_{k \in \mathbb{Z}} \mapsto (f_{1-k})_{k \in \mathbb{Z}}$$

which maps a Hardy-like space onto the other one. We check that P^W , P^E and J are self-adjoint operators. We define the Toeplitz-like and Hankel-like operators associated to a symbol $\hat{a} : S^1 \rightarrow \text{End } \mathcal{H}$, with $\|\hat{a}\|_\infty < \infty$ respectively as

$$\begin{aligned} \mathbb{T}^W(\hat{a}) &= P^W \mathbb{C}(\hat{a}) P^W & \mathbb{T}^E(\hat{a}) &= P^E \mathbb{C}(\hat{a}) P^E, \\ \mathbb{H}^W(\hat{a}) &= P^W \mathbb{C}(\hat{a}) J P^W & \mathbb{H}^E(\hat{a}) &= P^E \mathbb{C}(\hat{a}) J P^E. \end{aligned}$$

where $\mathbb{C}(\hat{a})$ is the convolution operator on l^2 introduced in (3.4.20). Even if these operators are defined on $l^2(\mathbb{Z} \rightarrow \mathcal{H})$, they are naturally identified with their restriction to $l^2(\mathbb{Z}_{>0} \rightarrow \mathcal{H})$ and $l^2(\mathbb{Z}_{\leq 0} \rightarrow \mathcal{H})$ respectively. In terms of infinite matrices we verify that

$$\begin{aligned} \mathbb{T}^W(\hat{a}) &= (C_{k-l}(\hat{a}))_{k,l=1}^\infty & \mathbb{T}^E(\hat{a}) &= (C_{k-l}(\hat{a}))_{k,l=0}^\infty \\ \mathbb{H}^W(\hat{a}) &= (C_{k+l-1}(\hat{a}))_{k,l=1}^\infty & \mathbb{H}^E(\hat{a}) &= (C_{k+l-1}(\hat{a}))_{k,l=0}^\infty. \end{aligned}$$

Shift operators. Denote $e^{i\bullet} : \theta \in [0, 2\pi] \mapsto e^{i\theta}$. The operators $\mathsf{T}^{\mathsf{W}}(e^{i\bullet})$ and $\mathsf{T}^{\mathsf{E}}(e^{-i\bullet})$ are the so-called shift operators. Their adjoint operators satisfy

$$\mathsf{T}^{\mathsf{W}}(e^{i\bullet})^* = \mathsf{T}^{\mathsf{W}}(e^{-i\bullet}) \quad \mathsf{T}^{\mathsf{E}}(e^{-i\bullet})^* = \mathsf{T}^{\mathsf{E}}(e^{i\bullet}). \quad (3.5.9)$$

These operators will be particularly helpful to take in count the lengthening of half-line spaces when acting with a face operator on half-strips. They are linked with the morphism introduced in [Sim23], see theorem 3.5.1.

Fold operators. Very useful operators are the fold operators defined as

$$\begin{aligned} \mathsf{F}^{\mathsf{W}}(\hat{a}) &= \mathsf{T}^{\mathsf{W}}(\hat{a}) - \mathsf{H}^{\mathsf{W}}(\hat{a}) \in \text{End}(l^2(\mathbb{Z}_{>0}) \rightarrow \mathcal{H}) \\ \mathsf{F}^{\mathsf{E}}(\hat{a}) &= \mathsf{T}^{\mathsf{E}}(\hat{a}) - \mathsf{H}^{\mathsf{E}}(\hat{a}) \in \text{End}(l^2(\mathbb{Z}_{\leq 0}) \rightarrow \mathcal{H}). \end{aligned} \quad (3.5.10)$$

where the Toeplitz-like and Hankel-like operators have been identified with their respective restrictions to $l^2(\mathbb{Z}_{>0}) \rightarrow \mathcal{H}$ and $l^2(\mathbb{Z}_{\leq 0}) \rightarrow \mathcal{H}$. These operators are a certain class of what is called "Toeplitz + Hankel operators" in the literature, see [BE99] for instance. As we will see later, in our context, these operators allow to fold doubly-infinite strips and half-planes into half-strips and corners. Just as the multiplication operator, fold operators are multiplicative when symbols are even function. Remarking that $\mathsf{F}^{\mathsf{W}}(\hat{a}) = P^{\mathsf{W}}\mathsf{C}(\hat{a})(I - J)P^{\mathsf{W}}$, and $\mathsf{F}^{\mathsf{E}}(\hat{a}) = P^{\mathsf{E}}\mathsf{C}(\hat{a})(I - J)P^{\mathsf{E}}$, we deduce that the operator norms are controlled by $\|\hat{a}\|_{\infty}$:

$$\|\mathsf{F}^{\mathsf{W}}(\hat{a})\| \leq \text{cst}\|\hat{a}\|_{\infty} \quad \|\mathsf{F}^{\mathsf{E}}(\hat{a})\| \leq \text{cst}'\|\hat{a}\|_{\infty}. \quad (3.5.11)$$

This is the purpose of the (standard) following lemma.

Lemma 3.5.1. *For all $\hat{a}, \hat{b} : S^1 \rightarrow \text{End}(\mathcal{H})$ bounded and even functions,*

$$\mathsf{F}^{\mathsf{W}}(\hat{a}\hat{b}) = \mathsf{F}^{\mathsf{W}}(\hat{a})\mathsf{F}^{\mathsf{W}}(\hat{b}) \quad \mathsf{F}^{\mathsf{E}}(\hat{a}\hat{b}) = \mathsf{F}^{\mathsf{E}}(\hat{a})\mathsf{F}^{\mathsf{E}}(\hat{b}). \quad (3.5.12)$$

In particular, if \hat{a} is invertible with bounded inverse, $\mathsf{F}^{\mathsf{W}}(\hat{a})$ and $\mathsf{F}^{\mathsf{E}}(\hat{a})$ are invertible with $\mathsf{F}^{\mathsf{W}}(\hat{a}^{-1}) = \mathsf{F}^{\mathsf{W}}(\hat{a})^{-1}$ and $\mathsf{F}^{\mathsf{E}}(\hat{a}^{-1}) = \mathsf{F}^{\mathsf{E}}(\hat{a})^{-1}$.

Proof. We restrict ourselves to the case of F^{W} (the proof F^{E} follows from similar arguments). The following relations can directly be verified from the definition of Toeplitz and Hankel operators

$$\begin{aligned} \mathsf{T}^{\mathsf{W}}(\hat{a}\hat{b}) &= \mathsf{T}^{\mathsf{W}}(\hat{a})\mathsf{T}^{\mathsf{W}}(\hat{b}) + \mathsf{H}^{\mathsf{W}}(\hat{a})\mathsf{H}^{\mathsf{W}}(R(\hat{b})) \\ \mathsf{H}^{\mathsf{W}}(\hat{a}\hat{b}) &= \mathsf{T}^{\mathsf{W}}(\hat{a})\mathsf{H}^{\mathsf{W}}(\hat{b}) + \mathsf{H}^{\mathsf{W}}(\hat{a})\mathsf{T}^{\mathsf{W}}(R(\hat{b})) \end{aligned}$$

with $R(\hat{b})(e^{i\theta}) = \hat{b}(e^{-i\theta})$. Then, if \hat{b} is even, using these formulas, we get the desired result. \square

The following lemma, that will be useful for proposition 3.5.1, gives a positive definite criterion for the Fold operator and the "Toeplitz+Hankel" operator.

Lemma 3.5.2. *Let $\hat{a} : S^1 \rightarrow \text{End}(\mathcal{H})$ a bounded and even function such that $\hat{a}(e^{i\theta}) \in \text{End}(\mathcal{H})$ is self-adjoint and positive definite for all $\theta \in [0, 2\pi]$. Then, the operators $\mathsf{F}^{\mathsf{W}}(\hat{a})$, $\mathsf{F}^{\mathsf{E}}(\hat{a})$ and*

$$\tilde{\mathsf{F}}^{\mathsf{W}} := \mathsf{T}^{\mathsf{W}}(\hat{a}) + \mathsf{H}^{\mathsf{W}}(\hat{a}) \quad \tilde{\mathsf{F}}^{\mathsf{E}} := \mathsf{T}^{\mathsf{E}}(\hat{a}) + \mathsf{H}^{\mathsf{E}}(\hat{a})$$

are self-adjoint positive definite.

Proof. We restrict ourselves to the case of F^W and \tilde{F}^W (the proof for F^E and \tilde{F}^E follows similar arguments.) Let $\hat{a} : S^1 \rightarrow \text{End}(\mathcal{H})$ a bounded and even function such that $\hat{a}(e^{i\theta}) \in \text{End}(\mathcal{H})$ is positive definite for all $\theta \in [0, 2\pi]$. Remark that, due to this last property, $C(\hat{a})$ is self-adjoint and positive definite thanks to lemma 3.4.5. Moreover, because \hat{a} is even, we check that

$$F^W(\hat{a}) = \frac{1}{2}P^W(I - J)C(\hat{a})(I - J)P^W \quad \tilde{F}^W(\hat{a}) = \frac{1}{2}P^W(I + J)C(\hat{a})(I + J)P^W.$$

We conclude using self-adjoint property of P^W and J . \square

South and North Toeplitz, Hankel and Fold operators. As we will see, South and North half-line spaces are quite similar to West and East half-line spaces respectively. Therefore, we denote

$$T^S(\hat{a}) := T^W(\hat{a}) \quad H^S(\hat{a}) := H^W(\hat{a}) \quad F^S(\hat{a}) := F^W(\hat{a}) \quad (3.5.13)$$

$$T^N(\hat{a}) := T^E(\hat{a}) \quad H^N(\hat{a}) := H^E(\hat{a}) \quad F^N(\hat{a}) := F^E(\hat{a}). \quad (3.5.14)$$

Half-line spaces. We now introduce the half-lines spaces associated to each direction:

$$\mathcal{W}_W = l^2(\mathbb{Z}_{>0} \rightarrow \mathcal{H}_1) \quad \mathcal{W}_E = l^2(\mathbb{Z}_{\leq 0} \rightarrow \mathcal{H}_1), \quad (3.5.15a)$$

$$\mathcal{W}_S = l^2(\mathbb{Z}_{>0} \rightarrow \mathcal{H}_2) \quad \mathcal{W}_N = l^2(\mathbb{Z}_{\leq 0} \rightarrow \mathcal{H}_2). \quad (3.5.15b)$$

Geometrically, these are the Hilbert spaces attached to half-lines that form the boundaries of half-strips and corners, see definitions 3.5.1 and 3.5.5 for instance. For all $a \in \{S, N, W, E\}$, the Toeplitz, Hankel and Fold operators $T^a, H^a, F^a : \mathcal{W}_a \rightarrow \mathcal{W}_a$ will be operators on the associated half-line spaces.

3.5.1.2 Half-strips operators

We are now ready to define half-strips operators of [Sim23] in our context. These operators live in the following coloured spaces

$$\mathcal{Q}_{\infty W, q} = \{B \in \text{End}(\mathcal{W}_W \oplus \mathcal{W}_W \oplus \mathcal{H}_2^q); B^* = B \text{ and } B \text{ positive definite} \}$$

$$\mathcal{Q}_{\infty E, q} = \{B \in \text{End}(\mathcal{W}_E \oplus \mathcal{W}_E \oplus \mathcal{H}_2^q); B^* = B \text{ and } B \text{ positive definite} \}$$

$$\mathcal{Q}_{\infty S, p} = \{B \in \text{End}(\mathcal{W}_S \oplus \mathcal{W}_S \oplus \mathcal{H}_1^p); B^* = B \text{ and } B \text{ positive definite} \}$$

$$\mathcal{Q}_{\infty N, p} = \{B \in \text{End}(\mathcal{W}_N \oplus \mathcal{W}_N \oplus \mathcal{H}_1^p); B^* = B \text{ and } B \text{ positive definite} \}$$

for $p, q \in \mathbb{N}^*$. We recall that such spaces are associated with half-strip pattern shapes in [Sim23], respectively given by $(-\infty, 0] \times [0, q]$, $[0, +\infty) \times [0, q]$, $[0, p] \times (-\infty, 0]$ and $[0, p] \times [0, +\infty)$. In the extended guillotine operad described in [Sim23], various products are associated to these shapes. We list them here and introduce the references to the corresponding sections and results of the present paper where these operadic products are implemented through Schur complements on the quadratic forms:

1. gluing of neighbouring half-strips, here on the west

$$\begin{array}{|c|} \hline B_2 \\ \hline B_1 \\ \hline \end{array} \Big|_{\mathcal{Q}}$$

with a Schur product $\mathcal{Q}_{\infty W, q_1} \times \mathcal{Q}_{\infty W, q_2} \rightarrow \mathcal{Q}_{\infty W, q_1 + q_2}$, see section 3.5.1.4 for the precise definitions and properties);

2. gluing of a rectangle at the end of a half-strip, like

$$\boxed{B_Q^{W,\infty} \mid Q} \quad (3.5.16)$$

with a Schur product $\mathcal{Q}_{\infty W,q} \times \mathcal{Q}_{p,q} \rightarrow \mathcal{Q}_{\infty W,q}$, performed in two stages, firstly with a Schur product $\mathcal{Q}_{\infty W,q} \times \mathcal{Q}_{p,q} \rightarrow \mathcal{Q}_{\infty W,q}^{(1)}$ with values in an auxiliary space $\mathcal{Q}_{\infty W,q}^{(1)}$ constructed from $\mathcal{Q}_{\infty W,q}$ and \mathcal{H}_1 , and later shown to be isomorphic to $\mathcal{Q}_{\infty W,q}$ itself. See section 3.5.1.5 for the precise definitions and properties;

3. gluing of opposite half-strips, like

$$\boxed{B_Q^{W,\infty} \mid B_Q^{E,\infty}}$$

with a product $\mathcal{Q}_{\infty W,q} \times \mathcal{Q}_{\infty E,q} \rightarrow \mathcal{Q}_{\infty WE}$ with $\mathcal{Q}_{\infty WE}$ defined in (3.4.23), see section 3.5.1.3 for the precise definitions and properties.

In particular, we will see below that the fold operators defined in (3.5.10) partially define a reciprocal of this last product: they will provide some sub-blocks (South-North interactions) of the operators in $\mathcal{Q}_{\infty W,q}$ and $\mathcal{Q}_{\infty E,q}$ from the knowledge of the operator in $\mathcal{Q}_{\infty WE}$. The other sub-blocks that couple different directions together (South-North with East or West interactions) are not easily accessible. We will see how they can be recovered from an additional method, see the very end of section 3.5.1.5.

Some precisions concerning definition and associativity of Schur products. In Proposition 3.3.1 and 3.3.1 we saw how Gaussian weights $(\mathcal{D}_{p,q})_{(p,q) \in C}$ is equipped with a Guill_2 structure under their gluings (marginalization). Indeed, from a probabilistic point of view, as marginalizations, these Gaussian densities are automatically stable under gluing products which are of course associative. This structure can be lifted to the set quadratic forms making $(\mathcal{Q}_{p,q})_{(p,q) \in C}$ a Guill_2 -algebra under Schur products. Stability and associativity are direct consequences of stability and associativity at the level of Gaussian weights. In fact, from an algebraic point of view, it is possible to show that $(\mathcal{Q}_{p,q})_{(p,q) \in C}$ is stable under Schur products, which are associative, without relying on any results concerning Gaussian weights $(\mathcal{D}_{p,q})_{(p,q) \in C}$. Indeed, the Schur product of positive definite matrices is always well-defined. It is given by the Schur complement, therefore well-defined, of the well-chosen matrix, see Property 4.1 of [Bod23] for instance. Then, we always have a decomposition of this matrix as the product of a lower triangular matrix, a diagonal matrix and an upper triangular matrix, called LDU decomposition, from which it is possible to show that the Schur product of two positive definite matrices produces a positive definite matrix. Hence positive definite are stable under Schur products. Similarly, the LDU decomposition ensures products to be associative. More generally, being given a stable set of matrices under well-defined Schur products, Schur complements are also well-defined and so are LDU decompositions. We thus automatically always have associativity when Schur products are well-defined. In sections 3.5.1.2 and 3.5.2, as stated in items 1, 2 and 3, we are interested in Schur products on the extended set of positive definite operators $\mathcal{Q}_\bullet := (\mathcal{Q}_{p,q})_{(p,q) \in C_{LR}}$ where C_{LR} is the extended set of colours, see (3.2.24). In reality, because now the dimension is infinite, these Schur products might be ill-defined. Therefore, Schur products of items items 1, 2 and 3 (and the ones in section 3.5.2) will in fact be defined on subsets of \mathcal{Q}_\bullet , only composed of the elements we are interested in: half-strips and corners operators associated to the face weight Q . On these subsets we will show that Schur products are well-defined. Then, the associated infinite-dimensional Schur complements and LDU decompositions are also well-defined,

ensuring that all the products of section 3.5.1.2 and 3.5.2 are associative. Moreover, because half-strips and corners operators will be shown to be positive definite, see Propositions 3.5.1 and 3.5.3, LDU decompositions ensure that the Schur products of such operators are still positive definite.

As explained in the beginning of the section, we work under assumption (\star) . In particular, we recall that the face operator Q will always be dihedral-invariant, with block denoted

$$Q = \begin{pmatrix} T & A & U & U \\ A & T & U & U \\ U & U & T & A \\ U & U & A & T \end{pmatrix}.$$

As usual, we will skip the definitions of South and North strips as they are analogous to the ones of West and East half-strips. We start by introducing West half-strips.

Definition 3.5.1 (West half-strip). Under assumption (\star) , we call West half-strip operator associated to Q , the self-adjoint operator $B_Q^{W,\infty} \in \text{End}(\mathcal{W}_W \oplus \mathcal{W}_W \oplus \mathcal{H}_2)$ defined as

$$B_Q^{W,\infty} = \begin{pmatrix} F^W(\widehat{B}_{Q,SS}^{WE}) & F^W(\widehat{B}_{Q,SN}^{WE}) & (\mathcal{P}_Q^W)^* \\ F^W(\widehat{B}_{Q,NS}^{WE}) & F^W(\widehat{B}_{Q,NN}^{WE}) & (\mathcal{P}_Q^W)^* \\ \mathcal{P}_Q^W & \mathcal{P}_Q^W & T + AW_Q \end{pmatrix} =: \begin{pmatrix} B_{Q,SS}^{W,\infty} & B_{Q,SN}^{W,\infty} & B_{Q,SE}^{W,\infty} \\ B_{Q,NS}^{W,\infty} & B_{Q,NN}^{W,\infty} & B_{Q,NE}^{W,\infty} \\ B_{Q,ES}^{W,\infty} & B_{Q,EN}^{W,\infty} & B_{Q,EE}^{W,\infty} \end{pmatrix} \quad (3.5.17)$$

where the operator \mathcal{P}_Q^W is defined as

$$\begin{aligned} \mathcal{P}_Q^W : \mathcal{W}_W &\rightarrow \mathcal{H}_2 \\ h &\mapsto \sum_{k>0} (W_Q^*)^{k-1} (I_{\mathcal{H}_2} + W_Q^*) U h_k. \end{aligned} \quad (3.5.18)$$

We remark that \mathcal{P}_Q is well defined. Indeed, we recall that, from [Bod23], $\|W_Q\| < 1$, where $\|W_Q\|$ is the spectral norm of the matrix W_Q , previously defined in (3.5.4). This is due to the fact that its eigenvalues are contained in the (open) unit disk, see Theorem 3.2.1. This makes the operator \mathcal{P}_Q well defined.

We give a few comments concerning the blocks of $B_Q^{W,\infty}$, already motivated by section 3.3.3. From a geometrical perspective following [Sim23], the objects $B_Q^{W,\infty}$ is associated to a pattern shape \mathbf{hs}_W and appears in operadic computations as:

$$\boxed{B_Q^{W,\infty}}$$

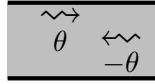
The space \mathcal{W}_W is then associated to the two half-lines on the top and bottom sides and the space \mathcal{H}_2 to the right side of length 1. It is shown in Proposition 3.5.1 that $B_Q^{W,\infty}$ is self-adjoint and positive definite.

South-North block. The South-North block is given by a fold operator because of full-symmetry assumption. In general settings, transfer matrix computations do not lead to such a self-coupling between South and North. Indeed, without full-symmetry assumption, the Toeplitz part of the fold operator is preserved whereas the Hankel is replaced by something more complicated. This makes the analysis of future gluings harder. On the other hand, as we will see later, due to lemma 3.5.1, the fold operator allows easy transversal gluing between half-strips and, especially, between half-strips and corners. This is the main reason why we will only work under full-symmetry assumption from now. Moreover, the fold operator

$F^W(\widehat{B}_{Q,ab}^{WE})$ is, by definition, a difference of the associated Toeplitz and Hankel operators. Roughly speaking, considering the oriented process from right to left composed of the variables attached to South and North infinite half-lines, the stationary part of this process is the Toeplitz contribution, expressed in terms of the Fourier coefficients $C_{k-l}(\widehat{B}_{Q,ab}^{WE})$, which is "penalized" by the Hankel contribution, expressed in terms of the Fourier coefficients $C_{k+l-1}(\widehat{B}_{Q,ab}^{WE})$. This last term is due to the interaction with the cut at the East of the half-strip. From the definition (3.4.16) of $\widehat{B}_{Q,ab}^{WE}$, the coefficients of this Hankel operator can in fact be rewritten in terms of $C_{k+l-1}((\phi_Q^{WE})^{-1})$, the Fourier coefficients of $(\phi_Q^{WE})^{-1}$, decreasing to zero with exponential rate, see [Bod23]. It means that, far away from the cut, k and l are large and this penalization term is exponentially small. Hence, we don't see the cut, we only see the stationary contribution. A more physical representation is the following. As mentioned in the proof of lemma 3.5.2, the fold operator can be written

$$F^W(\widehat{B}_{Q,ab}^{WE}) = \frac{1}{2} P^W (I - J) C(\widehat{B}_{Q,ab}^{WE}) (I - J) P^W$$

because $\widehat{B}_{Q,ab}^{WE}$ is even. In L^2 , the presence of the flip operator, acting as $J(f)(e^{i\theta}) = e^{i\theta} f(-\theta)$ (defined in l^2 in equation (3.5.8)), makes the fold operator coupling θ with $-\theta$. As in the following drawing, this represents the interaction with the cut:



In this drawing, a wave θ propagating from left to right in the half-strip encounters a mirror (the cut at the East) and reflects with a wave $-\theta$ propagating in the opposite direction: the two are coexisting and interacting inside the strip. This is reminiscent of sines of section 3.3 which also imply interactions between θ and $-\theta$.

East and (South/North)-East blocks. These blocks have already been motivated by harmonic constructions in section 3.3.3. Here again, because $\|W_Q\| < 1$, we observe an exponential decay in the coupling between the East cut and far horizontal South or North edges. As we will see in theorem 3.5.1, (South/North)-East blocks behave particularly well under the East action of the face operator Q on $B_Q^{W,\infty}$, i.e. we will show that the element

$$\left[\begin{array}{|c|c|} \hline B_Q^{W,\infty} & Q \\ \hline \end{array} \right]_Q$$

(where the product will involve Schur complement on the \mathcal{H}_2 block) can be further simplified, due to the following lemma. It states that the action of W_Q^* on \mathcal{P}_Q^W is just \mathcal{P}_Q^W itself restricted to the shifted space by the operator $T^W(e^{i\bullet})$ previously introduced in section 3.5.1.1.

Lemma 3.5.3. *Under assumption (\star) , we have the following properties*

$$W_Q^* \mathcal{P}_Q^W = \mathcal{P}_Q^W T^W(e^{i\bullet}) \quad (\mathcal{P}_Q^W)^* W_Q = T^W(e^{-i\bullet}) (\mathcal{P}_Q^W)^*. \quad (3.5.19)$$

Proof. We prove the first identity and the second one will follow taking the adjoint. For all $h \in \mathcal{W}_W$, we verify that

$$\mathcal{P}_Q^W T^W(e^{i\bullet}) h = \sum_{k>1} (W_Q^*)^{k-1} (I_{\mathcal{H}_2} + W_Q^*) U h_{k-1} = \sum_{k>0} (W_Q^*)^k (I_{\mathcal{H}_2} + W_Q^*) U h_k = W_Q^* \mathcal{P}_Q^W h.$$

□

The (very) similar definition of East half-strips is the following.

Definition 3.5.2 (East half-strip). Under assumption (\star) , we call East half-strip operator associated to Q the self-adjoint operator $B_Q^{E,\infty} \in \text{End}(\mathcal{W}_E \oplus \mathcal{W}_E \oplus \mathcal{H}_2)$ defined as

$$B_Q^{E,\infty} = \begin{pmatrix} \mathbb{F}^E(\widehat{B}_{Q,SS}^{WE}) & \mathbb{F}^E(\widehat{B}_{Q,SN}^{WE}) & (\mathcal{P}_Q^E)^* \\ \mathbb{F}^E(\widehat{B}_{Q,NS}^{WE}) & \mathbb{F}^E(\widehat{B}_{Q,NN}^{WE}) & (\mathcal{P}_Q^E)^* \\ \mathcal{P}_Q^E & \mathcal{P}_Q^E & T + AW_Q \end{pmatrix} =: \begin{pmatrix} B_{Q,SS}^{E,\infty} & B_{Q,SN}^{E,\infty} & B_{Q,SW}^{E,\infty} \\ B_{Q,NS}^{E,\infty} & B_{Q,NN}^{E,\infty} & B_{Q,NW}^{E,\infty} \\ B_{Q,WS}^{E,\infty} & B_{Q,WN}^{E,\infty} & B_{Q,WW}^{E,\infty} \end{pmatrix} \quad (3.5.20)$$

where the operators between transverse directions are given by:

$$\begin{aligned} \mathcal{P}_Q^E : \mathcal{W}_E &\rightarrow \mathcal{H}_2 \\ h &\mapsto \sum_{k \leq 0} (W_Q^*)^{-k} (I_{\mathcal{H}_2} + W_Q^*) U h_k. \end{aligned} \quad (3.5.21)$$

From a geometrical perspective following [Sim23], the objects $B_Q^{E,\infty}$ is associated to a pattern shape \mathbf{hs}_E and appears in operadic computations as:

$$B_Q^{E,\infty}$$

The space \mathcal{W}_E is then associated to the two half-lines on the top and bottom sides and the space \mathcal{H}_2 to the right side of length 1.

We remark that $B_Q^{E,\infty}$ and $B_Q^{W,\infty}$ are quite similar. This is a consequence of dihedral-invariance property of Q , see assumption (\star) . Nevertheless, we observe a shift of order 1 between their coefficients, only due to the labeling convention. For instance, for $k > 0$, in the West half-strip definition, the edge h_k is coupled with the East cut by $(W_Q^*)^{k-1} (I_{\mathcal{H}_2} + W_Q^*) U$. Similarly, h_{-k+1} is coupled with the West cut by the exact same quantity in the East half-strip definition. This is because the labeling of East horizontal edges in \mathcal{W}_E starts at 0 whereas the ones in \mathcal{W}_W starts at 1. This emphasizes the fact that the distance from the cut is what matters in half-strips. We may also use half-integer x -coordinates for the horizontal edges to restore the left-right symmetry but it would require additional discussions about square roots of operators.

The next lemma will be very useful for proving that half-strips operators are positive definite, see Proposition 3.5.1 and also for multiple proofs concerning future gluings, see Proposition 3.5.2 or Theorem 3.5.1 for instance.

Lemma 3.5.4. Under assumption (\star) , let $a, b \in \{\mathbf{S}, \mathbf{N}\}$. We have

$$-(\mathcal{P}_Q^W)^* C_0(\phi_Q^{-1}) \mathcal{P}_Q^W = \mathbb{H}^W(\widehat{B}_{Q,ab}^{WE}) \quad -(\mathcal{P}_Q^E)^* C_0(\phi_Q^{-1}) \mathcal{P}_Q^E = \mathbb{H}^E(\widehat{B}_{Q,ab}^{WE}).$$

Moreover

$$-(\mathcal{P}_Q^W)^* C_0(\phi_Q^{-1}) \mathcal{P}_Q^E = P^W C(\widehat{B}_{Q,ab}^{WE}) P^E. \quad (3.5.22)$$

The first equation of this lemma makes precise what has already been pointed out right after the definition 3.5.1 concerning the Hankel penalization in South-North block of $B_Q^{W,\infty}$. As we see here, this penalization is exactly given by the interaction of South and North horizontal edges with the East cut (represented by the operator \mathcal{P}_Q^W). A similar remark holds for East half-strips.

Proof. We first remark that non-zero order Fourier coefficients of $\widehat{B}_{Q,ab}^{WE}$ do not depend on a and b thanks to full symmetry assumption. Now fix $a, b \in \{\mathbf{S}, \mathbf{N}\}$. Using definition of W_Q , see (3.5.4), and equations (3.5.3) and (3.5.2), we get

$$(W_Q^*)^n = \left(C_0(\phi_Q^{-1}) \right)^{-1} C_n(\phi_Q^{-1}),$$

for all $n > 0$. Then, for all $k, l > 0$, we have

$$(I_{\mathcal{H}_1} + W_Q)W_Q^{k-1}C_0(\phi^{-1})(W_Q^*)^{l-1}(I_{\mathcal{H}_1} + W_Q^*) = C_{k+l-2}(\phi_Q^{-1}) + 2C_{k+l-1}(\phi_Q^{-1}) + C_{k+l}(\phi_Q^{-1})$$

thanks to (3.5.3). Hence by the definition of \mathcal{P}_Q^W in 3.5.1 and the definition of \widehat{B}_Q^{WE} in (3.4.16),

$$-((\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})\mathcal{P}_Q^W)_{k,l} = C_{k+l-1}(\widehat{B}_{Q,ab}^{WE})$$

for all k and $l > 0$. The results follows. The second identity is proved with similar arguments. For all $k > 0$ and $l \leq 0$,

$$(I_{\mathcal{H}_1} + W_Q)W_Q^{k-1}C_0(\phi_Q^{-1})(W_Q^*)^l(I_{\mathcal{H}_1} + W_Q^*) = C_{k-l-1}(\phi_Q^{-1}) + 2C_{k-l}(\phi_Q^{-1}) + C_{k-l+1}(\phi_Q^{-1}),$$

so that,

$$-((\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})\mathcal{P}_Q^E)_{k,l} = C_{k-l}(\widehat{B}_{Q,ab}^{WE})$$

and the last result follows. \square

Positive-definite property for half-strips. Now that we introduced West and East half-strips operators, we show that they are positive definite, just as the original face operator Q . This is the purpose of the following important proposition.

Property 3.5.1. *Under assumption (\star) , the West and East half-strips operators are self-adjoint and positive definite, i.e $B_Q^{W,\infty} \in \mathcal{Q}_{\infty_W,1}$ and $B_Q^{E,\infty} \in \mathcal{Q}_{\infty_E,1}$.*

Proof. As usual, we only treat the case of the West half-strip. We first check self-adjoint property. The term $B_{Q,EE}^{W,\infty} = T + AW_Q$ is self-adjoint due to results of [Bod23], reminded Theorem 3.2.1, and the previous definition of W_Q in (3.5.4). Moreover, $F(\widehat{B}_Q^{WE})$ is also self-adjoint thanks to lemma 3.5.2, recalling that for all $\theta \in [0, 2\pi]$, $\widehat{B}_Q^{WE}(e^{i\theta})$ is self-adjoint by lemma 3.4.4. Then, $B_Q^{W,\infty}$ is self-adjoint. We now show it is positive definite. A Schur LDU decomposition, as for lemma 3.4.3, implies that $B_Q^{W,\infty} \in \mathcal{Q}_{\infty_W,1}$ is positive definite if and only if $B_{Q,EE}^{W,\infty}$ and the Schur complement $(B_Q^{W,\infty}/B_{Q,EE}^{W,\infty})$ (see (3.4.14) for definition) are positive definite. Because Q is positive definite, the term $B_{Q,EE}^{W,\infty} = T + AW_Q$ is indeed positive definite from Theorem 3.2.1. We thus have to verify that $(B_Q^{W,\infty}/B_{Q,EE}^{W,\infty})$ is positive definite. We recall, from lemma 5.5 of [Bod23], that $2(T + AW_Q) = C_0(\phi_Q^{-1})^{-1}$, see (3.2.19). Therefore, we have

$$\begin{aligned} (B_Q^{W,\infty}/B_{Q,EE}^{W,\infty}) &= F^W(\widehat{B}_Q^{WE}) - \begin{pmatrix} (\mathcal{P}_Q)^* \\ (\mathcal{P}_Q)^* \end{pmatrix} (T + AW_Q)^{-1} \begin{pmatrix} \mathcal{P}_Q & \mathcal{P}_Q \end{pmatrix} \\ &= F^W(\widehat{B}_Q^{WE}) - 2 \begin{pmatrix} (\mathcal{P}_Q)^* \\ (\mathcal{P}_Q)^* \end{pmatrix} C_0(\phi_Q^{-1}) \begin{pmatrix} \mathcal{P}_Q & \mathcal{P}_Q \end{pmatrix} \\ &= T^W(\widehat{B}_Q^{WE}) + H^W(\widehat{B}_Q^{WE}) \end{aligned}$$

where we used lemma 3.5.4 for the last line together with the definition of fold operator (3.5.10). Now, under dihedral-invariant property provided by assumption (\star) , \widehat{B}_Q^{WE} is an even function. Moreover, for all $\theta \in [0, 2\pi]$, $\widehat{B}_Q^{WE}(e^{i\theta})$ is positive definite thanks to lemma 3.4.4 and we can apply lemma 3.5.2 to show that the operator $(T^W(\widehat{B}_Q^{WE}) + H^W(\widehat{B}_Q^{WE}))$ is positive definite. We get the desired result. \square

3.5.1.3 Longitudinal gluing

In this section we study the longitudinal gluing between half-strips whose geometrical interpretation is given by the item 3 of section 3.5.1.2. This is a product at the level of West and East half-strips $S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty}) \in \text{End}(\mathcal{W}_W \oplus \mathcal{W}_E \oplus \mathcal{W}_W \oplus \mathcal{W}_E)$ defined as

$$\begin{aligned} \boxed{B_Q^{W,\infty} \mid B_Q^{E,\infty}} &= S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty}) \\ &:= \begin{pmatrix} S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty})_{SS} & S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty})_{SN} \\ S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty})_{NS} & S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty})_{NN} \end{pmatrix} \end{aligned} \quad (3.5.23)$$

with blocks are given by

$$\begin{aligned} S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty})_{ab} &= \begin{pmatrix} B_{Q,ab}^{W,\infty} - B_{Q,aE}^{W,\infty}(B_{Q,E}^{W,\infty} + B_{Q,W}^{E,\infty})^{-1}B_{Q,Eb}^{W,\infty} & -B_{Q,aE}^{W,\infty}(B_{Q,E}^{W,\infty} + B_{Q,W}^{E,\infty})^{-1}B_{Q,Wb}^{E,\infty} \\ -B_{Q,aW}^{E,\infty}(B_{Q,E}^{W,\infty} + B_{Q,W}^{E,\infty})^{-1}B_{Q,Eb}^{W,\infty} & B_{Q,ab}^{E,\infty} - B_{Q,aW}^{E,\infty}(B_{Q,E}^{W,\infty} + B_{Q,W}^{E,\infty})^{-1}B_{Q,Wb}^{E,\infty} \end{pmatrix} \end{aligned}$$

for all $a, b \in \{S, N\}$. This product traduces a gluing at the level of half-line spaces formally given by the direct sum: $(\mathcal{W}_W, \mathcal{W}_E) \mapsto \mathcal{W}_W \oplus \mathcal{W}_E = \mathcal{W}_{WE}$. In other words, we recover the doubly-infinite line space introduced in (3.4.22) by gluing the two half-line spaces and $S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty}) \in \text{End}(\mathcal{W}_{WE} \oplus \mathcal{W}_{WE})$. This trivial identification hides a more interesting result. Indeed, the Schur product of West and East half-strips is equal to the doubly-infinite strip $B_Q^{WE,\infty}$ introduced in (3.4.24). This is the purpose of the following proposition.

Property 3.5.2. *Under assumption (\star) , we have*

$$\boxed{B_Q^{W,\infty} \mid B_Q^{E,\infty}} = S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty}) = B_Q^{WE,\infty} \in \mathcal{Q}_{\infty WE}. \quad (3.5.24)$$

Proof. We have $(B_{Q,E}^{W,\infty} + B_{Q,W}^{E,\infty})^{-1} = C_0(\phi_Q^{-1})$, see (3.2.19). We first remark that, for all $a, b \in \{S, N\}$,

$$S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty})_{ab} = \begin{pmatrix} F^W(\widehat{B}_{Q,ab}^{WE}) - (\mathcal{P}_Q^W)^* C_0(\phi_Q^{-1}) \mathcal{P}_Q^W & -(\mathcal{P}_Q^W)^* C_0(\phi_Q^{-1}) \mathcal{P}_Q^E \\ -(\mathcal{P}_Q^E)^* C_0(\phi_Q^{-1}) \mathcal{P}_Q^W & F^E(\widehat{B}_{Q,ab}^{WE}) - (\mathcal{P}_Q^E)^* C_0(\phi_Q^{-1}) \mathcal{P}_Q^E \end{pmatrix}.$$

From lemma 3.5.4 and definition of F^W and F^E , we deduce

$$S_{WE}(B_Q^{W,\infty}, B_Q^{E,\infty})_{ab} = \begin{pmatrix} T^W(\widehat{B}_{Q,ab}^{WE}) & P^W C(\widehat{B}_{Q,ab}^{WE}) P^E \\ P^E C(\widehat{B}_{Q,ab}^{WE}) P^W & T^E(\widehat{B}_{Q,ab}^{WE}) \end{pmatrix} = C(\widehat{B}_{Q,ab}^{WE}) = B_{Q,ab}^{WE,\infty}.$$

□

This last proposition, together with the fold operators, closes the relationship between half-strips operators and strip transfer matrices. We will now concentrate on the additional internal operadic properties of the half-strip operators.

3.5.1.4 Transversal gluing

As announced in the begining of the section 3.5.1.2, there is a transversal gluing between half-strips which "stacks" them to form wider half-strips, see item 1. As we will see, this product defines half-strips LengthAss-algebras. This is a novelty compared to boundary objects of one-dimensional Gaussian Markov processes. Indeed, in 1D, there is no transversal

product between left (resp. right) boundary conditions, see [Bod23]. For West half-strips, provided that Q_1 and Q_2 verify assumption (\star) , this product is given by the following Schur product $S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty}) \in \text{End}(\mathcal{W}_{\text{W}} \oplus \mathcal{W}_{\text{W}} \oplus \mathcal{H}_2 \oplus \mathcal{H}_2)$ defined as

$$\begin{array}{|c|} \hline B_{Q_2}^{\text{W},\infty} \\ \hline B_{Q_1}^{\text{W},\infty} \\ \hline \end{array} \Big|_{\text{Q}} = S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty}) \quad (3.5.25)$$

$$:= \begin{pmatrix} S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{SS}} & S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{SN}} & S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{SE}} \\ S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{NS}} & S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{NN}} & S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{NE}} \\ S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{ES}} & S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{EN}} & S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{EE}} \end{pmatrix}$$

where, for instance,

$$S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{SS}} = B_{Q_1, \text{SS}}^{\text{W},\infty} - B_{Q_1, \text{SN}}^{\text{W},\infty} K^{-1} B_{Q_1, \text{NS}}^{\text{W},\infty}$$

and

$$S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty})_{\text{ES}} = \begin{pmatrix} B_{Q_1, \text{ES}}^{\text{W},\infty} - B_{Q_1, \text{EN}}^{\text{W},\infty} K^{-1} B_{Q_1, \text{NS}}^{\text{W},\infty} \\ -B_{Q_2, \text{ES}}^{\text{W},\infty} K^{-1} B_{Q_1, \text{NS}}^{\text{W},\infty} \end{pmatrix}$$

where $K = (B_{Q_1, \text{NN}}^{\text{W},\infty} + B_{Q_1, \text{SS}}^{\text{W},\infty})$. We leave to the reader the writing of the other blocks. Remark that, at the East, the dimension of the vector space has doubled after the gluing. The two East edges of each half-strips have been glued together to form a lengthened one. There is of course a similar product for East half-strips also denoted $S_{\text{SN}}(B_{Q_1}^{\text{E},\infty}, B_{Q_2}^{\text{E},\infty})$. The following lemma ensures that these products make sense.

Lemma 3.5.5. *For all $q \geq 2$ and Q_1, Q_2, \dots, Q_q face operators satisfying assumption (\star) , the recursive Schur product $S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty}, \dots, B_{Q_q}^{\text{W},\infty}) \in \mathcal{Q}_{\infty, \text{W}, q}$, given by*

$$\begin{array}{|c|} \hline B_{Q_q}^{\text{W},\infty} \\ \hline \vdots \\ \hline B_{Q_1}^{\text{W},\infty} \\ \hline \end{array} \Big|_{\text{Q}} = S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty}, \dots, B_{Q_q}^{\text{W},\infty})$$

$$:= S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, S_{\text{SN}}(B_{Q_2}^{\text{W},\infty}, \dots, S_{\text{SN}}(B_{Q_{q-1}}^{\text{W},\infty}, B_{Q_q}^{\text{W},\infty}))),$$

is well-defined and associative. Moreover, for all $a, b \in \{\text{S}, \text{N}\}$,

$$S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty}, \dots, B_{Q_q}^{\text{W},\infty})_{ab} = \text{F}^{\text{W}}(S_{\text{SN}}(\widehat{B}_{Q_1}^{\text{WE}}, \widehat{B}_{Q_2}^{\text{WE}}, \dots, \widehat{B}_{Q_q}^{\text{WE}})_{ab}) \quad (3.5.26)$$

where

$$S_{\text{SN}}(\widehat{B}_{Q_1}^{\text{W}}, \widehat{B}_{Q_2}^{\text{W}}, \dots, \widehat{B}_{Q_q}^{\text{W}}) := S_{\text{SN}}(\widehat{B}_{Q_1}^{\text{W}}, S_{\text{SN}}(\widehat{B}_{Q_2}^{\text{W}}, \dots, S_{\text{SN}}(\widehat{B}_{Q_{q-1}}^{\text{W}}, \widehat{B}_{Q_q}^{\text{W}})))$$

Similar result holds for East half-strips.

Proof. We start by showing that recursive product are well-defined and we do it recursively. If $q = 2$, the Schur product is defined in terms of products between blocks of each half-strip and $K^{-1} = (B_{Q_1, \text{NN}}^{\text{W},\infty} + B_{Q_2, \text{SS}}^{\text{W},\infty})^{-1}$. Every block is a bounded operator, one only need to verify that K is indeed invertible. We have $K = \text{F}^{\text{W}}(\widehat{B}_{Q_1, \text{NN}}^{\text{W}}) + \text{F}^{\text{W}}(\widehat{B}_{Q_1, \text{SS}}^{\text{W}}) = \text{F}^{\text{W}}(\widehat{B}_{Q_1, \text{NN}}^{\text{W}} + \widehat{B}_{Q_1, \text{SS}}^{\text{W}})$ and $\widehat{B}_{Q_1, \text{NN}}^{\text{W}} + \widehat{B}_{Q_1, \text{SS}}^{\text{W}}$ is even and invertible with bounded inverse on S^1 thanks assumption (\star) and to lemma 3.4.4. Therefore, K is invertible thanks to lemma 3.5.1 with $K^{-1} = \text{F}^{\text{W}}((\widehat{B}_{Q_1, \text{NN}}^{\text{W}} + \widehat{B}_{Q_1, \text{SS}}^{\text{W}})^{-1})$ and the Schur product is well-defined. Because $B_{Q_1}^{\text{W},\infty}$ and $B_{Q_2}^{\text{W},\infty}$

are self-adjoint and positive definite thanks to lemma 3.5.1, a LDU decomposition ensures that $S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty}) \in \mathcal{Q}_{\infty\text{W},2}$, see the discussion in subsection 3.5.1.2. Moreover the identity (3.5.26) is also a direct consequence of lemma 3.5.1 when $q = 2$. Now, assume that the recursive product of order $q \geq 2$ is well defined and (3.5.26) is fulfilled. Then, South-North self-coupling of the q -order Schur product is again given by the fold operator of a continuous and even symbol with positive definite values. Therefore, thanks to (3.5.26), and using similar arguments, the Schur product of order $(q + 1)$ is well-defined, with value in $\mathcal{Q}_{\infty\text{W},q+1}$ and (3.5.26) is verified. The associativity is discussed in subsection 3.5.1.2. \square

The equation (3.5.26) states that the fold operator preserves the Schur product restricted to South-North blocks. In other words, the product of half-strips in $\mathcal{Q}_{\infty\text{W},q}$ is partially deduced from the "unfold" product of doubly-infinite strips in $\mathcal{Q}_{\infty\text{WE}}$ defined in (3.4.23). This is the main interesting property of the fold operator.

In general, the Schur product of multiple West half-strips with dihedral-invariant symbols Q_i is not a West half-strip with a dihedral-invariant symbol. This is not a problem as, thanks to (3.5.26), its South-North block is always given by the fold operator of an even function, which is what we need to be able to stack half-strips together.

LengthAss algebra generated by West half-strips. Denote by $\mathcal{Q}_{1,1}^{(\star)} \subset \mathcal{Q}_{1,1}$ the subset of face operator satisfying the assumption (\star) . The previous lemma implies that West half-strips with dihedral-invariant parameters generate a LengthAss algebra under the product S_{SN} , denoted $\mathcal{B}_{\infty\text{W},\bullet} = (\mathcal{B}_{\infty\text{W},q})_{q \in \mathbb{N}^*}$ where

$$\mathcal{B}_{\infty\text{W},q} = \{S_{\text{SN}}(B_{Q_1}^{\text{W},\infty}, B_{Q_2}^{\text{W},\infty}, \dots, B_{Q_q}^{\text{W},\infty}); (Q_1, \dots, Q_q) \in (\mathcal{Q}_{1,1}^{(\star)})^q\} \subset \mathcal{Q}_{\infty\text{W},q} \quad (3.5.27)$$

for all $q \in \mathbb{N}^*$ with $S_{\text{SN}}(B_Q^{\text{W},\infty}) = B_Q^{\text{W},\infty}$ for all Q satisfying the assumption (\star) by convention. The associative product of this LengthAss algebra is of course given by

$$\begin{aligned} S_{\text{SN}} : \mathcal{B}_{\infty\text{W},q_1} \times \mathcal{B}_{\infty\text{W},q_2} &\rightarrow \mathcal{B}_{\infty\text{W},q_1+q_2} \\ (B_1, B_2) &\mapsto S_{\text{SN}}(B_1, B_2). \end{aligned} \quad (3.5.28)$$

for all $q_1, q_2 \in \mathbb{N}^*$. We insist on the fact that this product takes an element $B_1 \in \text{End}(\mathcal{W}_{\text{W}} \oplus \mathcal{W}_{\text{W}} \oplus \mathcal{H}_2^{q_1})$ together with an element $B_2 \in \text{End}(\mathcal{W}_{\text{W}} \oplus \mathcal{W}_{\text{W}} \oplus \mathcal{H}_2^{q_2})$ and produces one in $\text{End}(\mathcal{W}_{\text{W}} \oplus \mathcal{W}_{\text{W}} \oplus \mathcal{H}_2^{q_1+q_2})$. In other words, the lengthening of the East edge formally given $(\mathcal{H}_2^{q_1}, \mathcal{H}_2^{q_2}) \rightarrow \mathcal{H}_2^{q_1+q_2}$ after the gluing is associated to the addition of colours: $q_1 + q_2$ in the operadic world.

In practice, we will be more interested in the sub-LengthAss-algebra $\mathcal{B}_{Q,\infty\text{W},\bullet}$ generated by a fixed West half-strip $B_Q^{\text{W},\infty}$ for describing the boundary conditions of a homogeneous Gaussian-Markov field of face weight Q .

3.5.1.5 Acting on half-strips with a face weight Q

In this section, we study the "action" of the face weight Q on half-strips, see item 2 of section 3.5.1.2 for geometrical interpretation. As we will see, this action naturally comes with a shift on half-line spaces and, later, with a morphism preserving gluings between half strips, see Theorem 3.5.2. This is the heart of the definition of generalized eigenvalue up to morphisms in [Sim23] which is one of the key ingredient to access the infinite volume Gibbs measure, together with the corners described below. This action of Q on the West half-strip is given by a Schur product $S_{\text{WE}}(B_Q^{\text{W},\infty}, Q) \in \text{End}((\mathcal{W}_{\text{W}} \oplus \mathcal{H}_1) \oplus (\mathcal{W}_{\text{W}} \oplus \mathcal{H}_1) \oplus \mathcal{H}_2) =: \mathcal{Q}_{\infty\text{W},1}^{(1)}$: each space $\mathcal{W}_{\text{W}} \oplus \mathcal{H}_1$ corresponds in the gluing (3.5.16) to the lengthening of the horizontal half-lines (associated to \mathcal{W}_{W} by a horizontal segment of the rectangle (associated to \mathcal{H}_1);

the disappearing of the inner vertical edge corresponds to the Schur product on its edge and there remains only the right vertical edge of the rectangle (associated to \mathcal{H}_2). The Schur product is precisely defined as

$$\begin{aligned} \boxed{B_Q^{W,\infty} \mid Q} \Big|_Q &= S_{WE}(B_Q^{W,\infty}, Q) \\ &:= \begin{pmatrix} S_{WE}(B_Q^{W,\infty}, Q)_{SS} & S_{WE}(B_Q^{W,\infty}, Q)_{SN} & S_{WE}(B_Q^{W,\infty}, Q)_{SE} \\ S_{WE}(B_Q^{W,\infty}, Q)_{NS} & S_{WE}(B_Q^{W,\infty}, Q)_{NN} & S_{WE}(B_Q^{W,\infty}, Q)_{NE} \\ S_{WE}(B_Q^{W,\infty}, Q)_{ES} & S_{WE}(B_Q^{W,\infty}, Q)_{EN} & S_{WE}(B_Q^{W,\infty}, Q)_{EE} \end{pmatrix} \end{aligned} \quad (3.5.29)$$

with blocks given by $S_{WE}(B_Q^{W,\infty}, Q)_{EE} = Q_{EE} - Q_{EW}(B_{Q,E}^{W,\infty} + Q_{WW})^{-1}Q_{WE}$,

$$\begin{aligned} S_{WE}(B_Q^{W,\infty}, Q)_{ab} &= \begin{pmatrix} B_{Q,ab}^{W,\infty} - B_{Q,aE}^{W,\infty}(B_{Q,E}^{W,\infty} + Q_{WW})^{-1}B_{Q,Eb}^{W,\infty} & -B_{Q,aE}^{W,\infty}(B_{Q,E}^{W,\infty} + Q_{WW})^{-1}Q_{Wb} \\ -Q_{aW}(B_{Q,E}^{W,\infty} + Q_{WW})^{-1}B_{Q,Eb}^{W,\infty} & Q_{ab} - Q_{aW}(B_{Q,E}^{W,\infty} + Q_{WW})^{-1}Q_{Wb} \end{pmatrix} \end{aligned}$$

for all $a, b \in \{\mathbf{S}, \mathbf{N}\}$ and

$$S_{WE}(B_Q^{W,\infty}, Q)_{aE} = \begin{pmatrix} -B_{Q,aE}^{W,\infty}(B_{Q,E}^{W,\infty} + Q_{WW})^{-1}Q_{WE} \\ Q_{aE} - Q_{aW}(B_{Q,E}^{W,\infty} + Q_{WW})^{-1}Q_{WE} \end{pmatrix}$$

and $S_{WE}(B_Q^{W,\infty}, Q)_{Ea} = (S_{WE}(B_Q^{W,\infty}, Q)_{aE})^*$ for all $a \in \{\mathbf{S}, \mathbf{N}\}$.

The term $(B_{Q,EE}^{W,\infty} + Q_{WW})^{-1}$ corresponds to the disappearing interior edge of the guillotine partition (3.5.16), which has two contributions: a East-East one coming from the half-strip and a West-West one coming from the quadratic form Q associated to the rectangle. Because Q is positive definite, the term $B_{Q,EE}^{W,\infty} = T + AW_Q$ is also positive definite from Theorem 1.2 of [Bod23] and the previous definition of W_Q in (3.5.4). Therefore, we check that this product is well-defined. Moreover, it satisfies a one-dimensional associativity, see the discussion in subsection 3.5.1.2, and the recursive action of Q of order $p \in \mathbb{N}^*$ is thus given by

$$\boxed{B_Q^{W,\infty} \mid Q^{[p,1]}} \Big|_Q := S_{WE}(B_Q^{W,\infty}, Q^{[p,1]}) \in \text{End}(\mathcal{W}_W \oplus \mathcal{H}_1^p \oplus \mathcal{W}_W \oplus \mathcal{H}_1^p \oplus \mathcal{H}_2) := \mathcal{Q}_{\infty W, 1}^{(p)}. \quad (3.5.30)$$

An operator associated with the lengthening. When acting recursively $p \in \mathbb{N}^*$ times on the West half-strip with a single face weight, it lengthens the half-line space \mathcal{W}_W into $\mathcal{W}_W \oplus \mathcal{H}_1^p$. We therefore introduce the operator D_p^W which brings this lengthened space back to the original half-line space (and its analogous operator D_q^S with $q \in \mathbb{N}^*$, useful in section 3.5.2, in the same time):

$$\begin{aligned} D_p^W : \mathcal{W}_W \oplus \mathcal{H}_1^p &\rightarrow \mathcal{W}_W & D_q^S : \mathcal{W}_S \oplus \mathcal{H}_2^q &\rightarrow \mathcal{W}_S \\ (h, x) &\mapsto \mathsf{T}^W(e^{ip\bullet})h + \iota_p^W(x) & (h, x) &\mapsto \mathsf{T}^S(e^{iq\bullet})h + \iota_q^S(x) \end{aligned} \quad (3.5.31)$$

where $\mathsf{T}^W(e^{ip\bullet})$ and $\mathsf{T}^S(e^{iq\bullet})$ are the p -th and q -th power of shift operators on \mathcal{W}_W and \mathcal{W}_S respectively, introduced in section 3.5.1.1, $\iota_p^W : \mathcal{H}_1^p \rightarrow \mathcal{W}_W$, $(x_1, \dots, x_p) \mapsto (x_p, x_{p-1}, \dots, x_1, 0, \dots)$ and ι_q^S similar but defined on \mathcal{H}_2^q . In Fourier space, this for instance corresponds to $D_p^W(\hat{h}, x)(e^{i\theta}) = e^{ip\theta}\hat{h}(e^{i\theta}) + e^{ip\theta}x_1 + e^{i(p-1)\theta}x_2 + \dots + e^{i\theta}x_p$. These operators are invertible with inverses given by

$$(D_p^W)^{-1}h^W = (D_p^W)^*h^W = (\mathsf{T}^W(e^{-ip\bullet})h, (\iota_p^W)^*h^W)$$

$$(D_q^S)^{-1}h^S = (D_q^S)^*h^W = (\mathbb{T}^S(e^{-iq\bullet})h, (\iota_q^S)^*h^S)$$

for all $h^W \in \mathcal{W}_W$ and $h^S \in \mathcal{W}_S$. As for half-strips, both bottom and top half-lines are lengthened through the action of Q , the following operator will naturally appear

$$D_p^W \oplus D_p^W : (\mathcal{W}_W \oplus \mathcal{H}_1^p) \oplus (\mathcal{W}_W \oplus \mathcal{H}_1^p) \rightarrow \mathcal{W}_W \oplus \mathcal{W}_W. \quad (3.5.32)$$

Thanks to these operators, half-line spaces lengthened by one or multiple segments after a gluing are brought back to a same base space. This allows the comparison between operators before and after gluings.

Before computing (3.5.29), we need the following lemma.

Lemma 3.5.6. *Under assumption (\star) , we have*

$$W_Q C_0(\phi_Q^{-1}) = C_0(\phi_Q^{-1}) W_Q^* \quad (3.5.33)$$

$$W_Q A^{-1} = (W_Q^2 - I_{\mathcal{H}_2}) C_0(\phi_Q^{-1}). \quad (3.5.34)$$

where we recall that A is a block of Q under (\star) , see (3.2.6).

Proof. Recalling that $W_Q = C_1(\phi_Q^{-1})(C_0(\phi_Q^{-1}))^{-1}$, we have $W_Q C_0(\phi_Q^{-1}) = C_1(\phi_Q^{-1}) = C_0(\phi_Q^{-1}) W_Q^*$ using (3.5.2). Moreover, from the recursive relation (3.2.22), we get

$$I_{\mathcal{H}_2} = C_1(\phi_Q^{-1})A + C_0(\phi_Q^{-1})2T + C_1(\phi_Q^{-1})A$$

so that, multiplying by $W_Q = C_1(\phi_Q^{-1})(C_0(\phi_Q^{-1}))^{-1}$ and identifying W_Q^2 on the first term of the r.h.s and using (3.5.3) on the last two terms of the r.h.s, we get

$$W_Q = W_Q^2 C_0(\phi_Q^{-1})A + C_1(\phi_Q^{-1})2T + C_2(\phi_Q^{-1})A = (W_Q^2 - I_{\mathcal{H}_2})C_0(\phi_Q^{-1})A. \quad \square$$

We are now able to compute explicitly the action of Q on $B_Q^{W,\infty}$. This is the purpose of the following theorem.

Theorem 3.5.1. *Under assumption (\star) ,*

$$\Phi_{1,1}^W(S_{WE}(B_Q^{W,\infty}, Q)) := \left(D_1^W \oplus D_1^W \oplus I_{\mathcal{H}_2} \right) S_{WE}(B_Q^{W,\infty}, Q) \left(D_1^W \oplus D_1^W \oplus I_{\mathcal{H}_2} \right)^{-1} = B_Q^{W,\infty}. \quad (3.5.35)$$

And, by associativity, for all $p \in \mathbb{N}^*$,

$$\begin{aligned} \Phi_{p,1}^W(S_{WE}(B_Q^{W,\infty}, Q^{[p,1]})) &:= \left(D_p^W \oplus D_p^W \oplus I_{\mathcal{H}_2} \right) S_{WE}(B_Q^{W,\infty}, Q^{[p,1]}) \left(D_p^W \oplus D_p^W \oplus I_{\mathcal{H}_2} \right)^{-1} \\ &= B_Q^{W,\infty}. \end{aligned} \quad (3.5.36)$$

Before proving this theorem we briefly comment its very interesting statement. The Schur product $S_{WE}(B_Q^{W,\infty}, Q)$ is, a priori, complicated. It changes half-line spaces and acts on the operators \mathcal{P}_Q^W and $F^W(\widehat{B}_Q^{WE})$ in a non-trivial way. Nevertheless, this last theorem states that this Schur action is simply conjugate to the identity through the "change of basis" given by $(D_1^W \oplus D_1^W \oplus I_{\mathcal{H}_2})$. We will see, in section 3.5.1.7, that this change of basis is linked with the notion of *invariant boundary up to morphism* introduced in [Sim23].

Proof of Theorem 3.5.1. From (3.2.18), we have $2T + AW_Q = -AW_Q^{-1}$ so that

$$(B_{Q,E}^{W,\infty} + Q_{WW})^{-1} = -W_Q A^{-1}. \quad (3.5.37)$$

We use this formula in every following computation. We rather show that $S(B_Q^{W,\infty}, Q) = (D_1^W \oplus D_1^W \oplus I_{\mathcal{H}_2})^{-1} B_Q^{W,\infty} (D_1^W \oplus D_1^W \oplus I_{\mathcal{H}_2})$. This proof is divided in three part, one for each block.

The block $S_{WE}(B_Q^{W,\infty}, Q)_{EE}$. We have $S_{WE}(B_Q^{W,\infty}, Q)_{EE} = T + AW_Q = B_{Q,EE}^{W,\infty}$.

The blocks $S_{WE}(B_Q^{W,\infty}, Q)_{aE}$. Let $a \in \{S, N\}$. We have

$$\begin{aligned} S_{WE}(B_Q^{W,\infty}, Q)_{aE} &= \begin{pmatrix} -B_{Q,aE}^{W,\infty}(B_{Q,E}^{W,\infty} + Q_{WW})^{-1}Q_{WE} \\ Q_{aE} - Q_{aW}(B_{Q,E}^{W,\infty} + Q_{WW})^{-1}Q_{WE} \end{pmatrix} \\ &= \begin{pmatrix} (\mathcal{P}_Q^W)^*W_Q \\ U(I_{\mathcal{H}_2} + W_Q) \end{pmatrix} = (D_1^W)^{-1}(\mathcal{P}_Q^W)^* \end{aligned}$$

using equation (3.5.19). Thus, $S_{WE}(B_Q^{W,\infty}, Q)_{aE} = (D_1^W)^{-1}B_{Q,aE}^{W,\infty}$. Taking the adjoint we also have $S_{WE}(B_Q^{W,\infty}, Q)_{Ea} = B_{Q,Ea}^{W,\infty}D_1^W$.

The blocks $S_{WE}(B_Q^{W,\infty}, Q)_{ab}$. Let $a, b \in \{S, N\}$. We have, after using (3.5.37),

$$S_{WE}(B_Q^{W,\infty}, Q)_{ab} = \begin{pmatrix} F^W(\widehat{B}_{Q,ab}^{WE}) + (\mathcal{P}_Q^W)^*W_Q A^{-1}\mathcal{P}_Q^W & (\mathcal{P}_Q^W)^*W_Q A^{-1}U \\ UW_Q A^{-1}\mathcal{P}_Q^W & Q_{ab} + UW_Q A^{-1}U \end{pmatrix}.$$

We treat every block on its own. Firstly, we have, using (3.5.34) and (3.5.33),

$$\begin{aligned} (\mathcal{P}_Q^W)^*W_Q A^{-1}\mathcal{P}_Q^W &= (\mathcal{P}_Q^W)^*(W_Q^2 - I_{\mathcal{H}_2})C_0(\phi_Q^{-1})\mathcal{P}_Q^W \\ &= (W_Q^*\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})W_Q^*\mathcal{P}_Q^W - (\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})\mathcal{P}_Q^W \\ &= -\mathsf{T}^W(e^{-i\bullet})\mathsf{H}^W(\widehat{B}_{Q,ab}^{WE})\mathsf{T}^W(e^{i\bullet}) + \mathsf{H}^W(\widehat{B}_{Q,ab}^{WE}) \end{aligned}$$

where, we used (3.5.19) together with lemma 3.5.4 for the last line. Now, remarking that $\mathsf{T}^W(e^{-i\bullet})\mathsf{T}^W(\widehat{B}_{Q,ab}^{WE})\mathsf{T}^W(e^{i\bullet}) = \mathsf{T}^W(\widehat{B}_{Q,ab}^{WE})$, we get

$$F^W(\widehat{B}_{Q,ab}^{WE}) + (\mathcal{P}_Q^W)^*W_Q A^{-1}\mathcal{P}_Q^W = \mathsf{T}^W(e^{-i\bullet})F^W(\widehat{B}_{Q,ab}^{WE})\mathsf{T}^W(e^{i\bullet}) = \mathsf{T}^W(e^{-i\bullet})B_{Q,ab}^{W,\infty}\mathsf{T}^W(e^{i\bullet}).$$

Similarly, with (3.5.34) and (3.5.33) we get

$$\begin{aligned} (\mathcal{P}_Q^W)^*W_Q A^{-1}U &= (W_Q^*\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})W_Q^*U - (\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})U \\ &= (W_Q^*\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})(I_{\mathcal{H}_2} + W_Q^*)U - (\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})(I_{\mathcal{H}_2} + W_Q^*)U. \end{aligned}$$

Thus, equation (3.5.19) and the definition of \mathcal{P}_Q^W lead to

$$(\mathcal{P}_Q^W)^*W_Q A^{-1}U = \mathsf{T}^W(e^{-i\bullet})(\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})\mathcal{P}_Q^W \iota_1^W - (\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})\mathcal{P}_Q^W \iota_1^W$$

so that, with lemma 3.5.4,

$$(\mathcal{P}_Q^W)^*W_Q A^{-1}U = -\mathsf{T}^W(e^{-i\bullet})\mathsf{H}^W(\widehat{B}_{Q,ab}^{WE})\iota_1^W + \mathsf{H}^W(\widehat{B}_{Q,ab}^{WE})\iota_1^W = \mathsf{T}^W(e^{-i\bullet})F^W(\widehat{B}_{Q,ab}^{WE})\iota_1^W$$

where, for the last equality, we noticed that $\mathsf{H}^W(\widehat{B}_{Q,ab}^{WE})\iota_1^W = \mathsf{T}^W(e^{-i\bullet})\mathsf{T}^W(\widehat{B}_{Q,ab}^{WE})\iota_1^W$. Finally, with exact same arguments,

$$\begin{aligned} Q_{ab} + UW_Q A^{-1}U &= Q_{ab} + U(W_Q^2 - I_{\mathcal{H}_2})C_0(\phi_Q^{-1})U \\ &= Q_{ab} - UC_0(\phi_Q^{-1})U + (W_Q^*U)C_0(\phi_Q^{-1})W_Q^*U. \end{aligned}$$

We have

$$\begin{aligned} (W_Q^*U)C_0(\phi_Q^{-1})W_Q^*U &= \left((I_{\mathcal{H}_2} + W_Q^*)U\right)^*C_0(\phi_Q^{-1})(I_{\mathcal{H}_2} + W_Q^*)U \\ &\quad - UC_0(\phi_Q^{-1})U - 2UC_0(\phi_Q^{-1})W_Q^*U \\ &= (\iota_1^W)^*(\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})\mathcal{P}_Q^W \iota_1^W - UC_0(\phi_Q^{-1})U - 2UC_0(\phi_Q^{-1})W_Q^*U. \end{aligned}$$

Thus,

$$\begin{aligned} Q_{ab} + UW_Q A^{-1}U &= Q_{ab} - 2UC_0(\phi_Q^{-1})U - 2UC_0(\phi_Q^{-1})W_Q^*U + (\iota_1^W)^*(\mathcal{P}_Q^W)^*C_0(\phi_Q^{-1})\mathcal{P}_Q^W\iota_1^W \\ &= (\iota_1^W)^*F^W(\widehat{B}_{Q,ab}^{WE})\iota_1^W. \end{aligned}$$

using lemma 3.5.4 and remarking that

$$Q_{ab} - 2UC_0(\phi_Q^{-1})U - 2UC_0(\phi_Q^{-1})W_Q^*U = Q_{ab} - 2UC_0(\phi_Q^{-1})U - 2UC_1(\phi_Q^{-1})U = \mathbb{T}^W(\widehat{B}_{Q,ab}^{WE})_{1,1}$$

by a direct computation from the definition of $\widehat{B}_{Q,ab}^{WE}$, see (3.4.16). We conclude that

$$S_{WE}(B_Q^{W,\infty}, Q)_{ab} = (D_1^W)^{-1}B_{Q,ab}^{W,\infty}D_1^W$$

and the result follows. The associativity is a direct consequence of the discussion in subsection 3.5.1.2. \square

Invariant boundary conditions up to morphism. Remark that the situation is different to case of one-dimensional Gaussian Markov processes. In [Bod23] an invariant boundary condition is a boundary weight which is invariant under one-dimensional Schur product, see (3.2.17). This is exactly the equation treated for the block $S_{WE}(B_Q^{W,\infty}, Q)_{EE}$ alone in the last proof. Here, it is not possible to stipulate that $B_Q^{W,\infty}$ is invariant under $S_{WE}(\cdot, Q)$. Indeed, $S_{WE}(B_Q^{W,\infty}, Q)$ acts on lengthened half-line spaces which prevents any direct comparison with $B_Q^{W,\infty}$ itself. Nevertheless, this last theorem states that the West half-strip is invariant under the action of Q up to a shift provided by $\Phi_{1,1}^W$ built with D_1^W . As we will see, this is exactly the notion of eigen-boundary conditions up to morphism introduced in [Sim23]. We make this precise in section 3.5.1.7. However, there is no eigenvalue involved in the last theorem yet. It is due to the fact that we dropped the parameter space $\mathbb{R} \times \mathcal{Q}_\bullet$ to focus on quadratic forms \mathcal{Q}_\bullet only. Quadratic forms on their own are not "weighted" anymore and they can't produce any eigenvalue nor partition function. To make the eigenvalue appear we will have to work in the parameter space and take into account the cocycles γ_{WE} . This requires a precise study of a renormalization procedure involving the asymptotic of certain determinants. We do this at a non-rigorous level in section 3.5.6. Another option could be to work directly on the spaces of Gaussian densities $\mathcal{D}_{p,q}$ defined in (3.3.3) for segments with products defined by integration but, as it will be seen below in section 3.5.3, the infinite-dimensional nature of the half-line spaces makes such integration too difficult to define rigorously.

A method for guessing the blocks $B_{Q,SE}^{W,\infty}$ and $B_{Q,NE}^{W,\infty}$. Remark that, besides harmonic or transfer matrix method computations, the coupling of transverse directions $B_{Q,SE}^{W,\infty} = B_{Q,NE}^{W,\infty} = (\mathcal{P}_Q^W)^*$ can entirely be recovered with the only knowledge of one-dimensional Gaussian Markov processes. We recall that the East block $B_{Q,EE}^{W,\infty}$ is obtained as the one-dimensional Schur invariant boundary of [Bod23]. Then, for $a \in \{S, N\}$, if the block $B_{Q,aE}^{W,\infty}$ was unknown, one would only have to compute the action of Q on $B_Q^{W,\infty}$ to guess it. Indeed, following the computations in the previous proof of Theorem 3.5.1, we get

$$S_{WE}(B_Q^{W,\infty}, Q)_{aE} = \begin{pmatrix} ? \\ Q_{aE} - Q_{aW}(B_{Q,E}^{W,\infty} + Q_{WW})^{-1}Q_{WE} \end{pmatrix} = \begin{pmatrix} ? \\ U(I_{\mathcal{H}_2} + W_Q) \end{pmatrix}$$

where ? is the unknown part. Now, recursively $p \in \mathbb{N}^*$ times, we get

$$S_{\text{WE}}(B_Q^{\text{W},\infty}, Q^{[p,1]})_{\text{aE}} = \begin{pmatrix} ? \\ U(I_{\mathcal{H}_2} + W_Q)W_Q^{p-1} \\ \vdots \\ U(I_{\mathcal{H}_2} + W_Q) \end{pmatrix} = \begin{pmatrix} ? \\ (\mathcal{P}_Q^{\text{W}})_p^* \\ \vdots \\ (\mathcal{P}_Q^{\text{W}})_1^* \end{pmatrix}$$

and we see $(\mathcal{P}_Q^{\text{W}})^*$ appear as $p \rightarrow \infty$. It makes $B_{Q,\text{aE}}^{\text{W},\infty} = (\mathcal{P}_Q^{\text{W}})^*$ the unique choice for expecting Theorem 3.5.1 to be true. We use this method for guessing off-diagonal blocks for corner operators, see definition 3.5.5 below for instance.

3.5.1.6 LengthAss generated by lengthened West half-strips.

Lengthened West half-strips, given by

$$\boxed{B_Q^{\text{W},\infty} \mid Q^{[p,1]}}_Q = S_{\text{WE}}(B_Q^{\text{W},\infty}, Q^{[p,1]}) \quad (3.5.38)$$

for $p \in \mathbb{N}^*$, can be glued together through a Schur product S_{SN} analogous to the one for West half-strips $B_Q^{\text{W},\infty}$, see (3.5.25). This product is just as well defined thanks to Theorem 3.5.1. This generates the following LengthAss algebra (associative product which respects the colours, see [Sim23]).

Definition 3.5.3 (Vertical LengthAss generated by lengthened West half-strips). Fix Q satisfying the assumption (\star) . For all $p \in \mathbb{N}^*$, the lengthened West half-strip $S_{\text{WE}}(B_Q^{\text{W},\infty}, Q^{[p,1]})$ generates a LengthAss algebra under the product S_{SN} , denoted $\mathcal{B}_{Q,\infty_{\text{W}},\bullet}^{(p)} = (\mathcal{B}_{Q,\infty_{\text{W}},q}^{(p)})_{q \in \mathbb{N}}$ with, for all $q \in \mathbb{N}^*$,

$$\mathcal{B}_{Q,\infty_{\text{W}},q}^{(p)} = \left\{ S_{\text{SN}} \left(S_{\text{WE}}(B_Q^{\text{W},\infty}, Q^{[p,1]}), \dots, S_{\text{WE}}(B_Q^{\text{W},\infty}, Q^{[p,1]}) \right) \right\} \quad (3.5.39)$$

where $S_{\text{SN}} \left(S_{\text{WE}}(B_Q^{\text{W},\infty}, Q^{[p,1]}), \dots, S_{\text{WE}}(B_Q^{\text{W},\infty}, Q^{[p,1]}) \right)$ is the q -times consecutive South-North product of $S_{\text{WE}}(B_Q^{\text{W},\infty}, Q^{[p,1]})$. Moreover, by convention, we let $\mathcal{B}_{Q,\infty_{\text{W}},\bullet}^{(0)} := \mathcal{B}_{Q,\infty_{\text{W}},\bullet}$ with $\mathcal{B}_{Q,\infty_{\text{W}},\bullet}$ the LengthAss algebra generated by $B_Q^{\text{W},\infty}$, see the end of section 3.5.1.4.

Remark that, compared to $\mathcal{B}_{\infty_{\text{W}},q}$ defined in (3.5.27), the set $\mathcal{B}_{Q,\infty_{\text{W}},q}^{(p)}$ is only composed of one element. This is because we are only interested in the LengthAss and Guill_2 algebras generated by Q and its associated boundary objects. We could of course enrich these sets by considering larger algebras with more half-strips to glue together (provided that it makes sense). Nevertheless this does not bring anything interesting concerning the main purpose of this paper: describing the boundary structure of homogeneous Gaussian fields with a fixed face-weight Q , as defined in 3.2.1. For this reason, we will only be considering the algebras generated by Q and its associated boundary objects.

3.5.1.7 Invariant strips up to morphism

In this section, in the continuity of Theorem 3.5.1 and motivated by [Sim23], we explore the notion of invariant half-strips operators up to morphism. As already mentioned, we will only work with quadratic forms. For simplicity, we stick to West half-strips but, as always, every argument is easily extendable to East, South, or North half-strips. For all p and $q \in \mathbb{N}^*$, we let

$$\begin{aligned} \Phi_{p,q}^{\text{W}} : \text{End}(\mathcal{W}_{\text{W}} \oplus \mathcal{H}_1^p \oplus \mathcal{W}_{\text{W}} \oplus \mathcal{H}_1^p \oplus \mathcal{H}_2^q) &\rightarrow \text{End}(\mathcal{W}_{\text{W}} \oplus \mathcal{W}_{\text{W}} \oplus \mathcal{H}_2^q) \\ M &\mapsto \left(D_p^{\text{W}} \oplus D_p^{\text{W}} \oplus I_{\mathcal{H}_2}^{\oplus q} \right) M \left(D_p^{\text{W}} \oplus D_p^{\text{W}} \oplus I_{\mathcal{H}_2}^{\oplus q} \right)^{-1} \end{aligned} \quad (3.5.40)$$

where D_p^W is defined in (3.5.31). Fix Q a face operator satisfying the assumption (\star) . For all $p \in \mathbb{N}$, we recall that $\mathcal{B}_{Q, \infty_W, \bullet}^{(p)} = (\mathcal{B}_{Q, \infty_W, q}^{(p)})_{q \in \mathbb{N}^*}$ is the LengthAss algebra generated by the lengthened West half-strip

$$S_{WE}(B_Q^{W, \infty}, Q^{[p, 1]}) \in \text{End}(\mathcal{W}_W \oplus \mathcal{H}_1^p \oplus \mathcal{W}_W \oplus \mathcal{H}_1^p \oplus \mathcal{H}_2)$$

under S_{SN} , see definition 3.5.3. A LengthAss morphism is a morphism which respects the colours of the LengthAss algebras, see [Sim23]. We have the following result.

Theorem 3.5.2. *Fix $p \in \mathbb{N}^*$. The map $\Phi_{p, \bullet}^W : \mathcal{B}_{Q, \infty_W, \bullet}^{(p)} \rightarrow \mathcal{B}_{Q, \infty_W, \bullet}$ is a LengthAss morphism, i.e a collection $\Phi_{p, q}^W : \mathcal{B}_{Q, \infty_W, q}^{(p)} \rightarrow \mathcal{B}_{Q, \infty_W, q}$ of LengthAss morphisms. Moreover, for all $q \in \mathbb{N}^*$,*

$$\Phi_{p, q}^W \left(S_{SN} \left(S_{WE}(B_Q^{W, \infty}, Q^{[p, 1]}), \dots, S_{WE}(B_Q^{W, \infty}, Q^{[p, 1]}) \right) \right) = S_{SN}(B_Q^{W, \infty}, B_Q^{W, \infty}, \dots, B_Q^{W, \infty}) \quad (3.5.41)$$

where the product S_{SN} is taken consecutively q times.

Proof. Let $p \in \mathbb{N}^*$. We start this proof by remarking that the linear map

$$\begin{aligned} \varphi_p^W : \text{End}(\mathcal{W}_W \oplus \mathcal{H}_1^p) &\rightarrow \text{End}(\mathcal{W}_W) \\ M &\mapsto D_p^W M (D_p^W)^{-1} \end{aligned}$$

is a morphism for the operator multiplication. We recall that S_{SN} between half-strips has been defined in (3.5.25). Now, let $B_1 \in \mathcal{B}_{Q, \infty_W, q_1}^{(p)}$, $B_2 \in \mathcal{B}_{Q, \infty_W, q_2}^{(p)}$ for $q_1, q_2 \in \mathbb{N}^*$. We first have to check that

$$\Phi_{p, q_1 + q_2}^W (S_{SN}(B_1, B_2)) = S_{SN}(\Phi_{p, q_1}^W(B_1), \Phi_{p, q_2}^W(B_2)). \quad (3.5.42)$$

To do so, we proceed block per block on $S_{SN}(\Phi_{p, q_1}^W(B_1), \Phi_{p, q_2}^W(B_2))$. We will only show the computations for South-North and East-South blocks, the others follow same arguments. Similarly to previous notations, we denote by $B_{1, ab}$ and $B_{2, ab}$ for $a, b \in \{\text{S}, \text{N}, \text{E}\}$, the blocks of B_1 and B_2 respectively.

South-North block. We have

$$\begin{aligned} S_{SN}(\Phi_{p, q_1}^W(B_1), \Phi_{p, q_2}^W(B_2))_{SN} &= -\varphi_p^W(B_{1, SN})(\varphi_p^W(B_{1, NN}) + \varphi_p^W(B_{2, SS}))^{-1} \varphi_p^W(B_{2, SN}) \\ &= \varphi_p \left(-B_{1, SN}(B_{1, NN} + B_{2, SS})^{-1} B_{2, SN} \right) \\ &= \Phi_{p, q_1 + q_2}^W (S_{SN}(B_1, B_2))_{SN} \end{aligned}$$

due to morphism property of φ_p^W .

East-South block. With similar arguments, we get

$$\begin{aligned} S_{SN}(\Phi_{p, q_1}^W(B_1), \Phi_{p, q_2}^W(B_2))_{ES} &= \left(B_{1, ES}(D_p^W)^{-1} - B_{1, EN}(D_p^W)^{-1}(\varphi_p^W(B_{1, NN}) + \varphi_p^W(B_{2, SS}))^{-1} \varphi_p^W(B_{1, NS}) \right. \\ &\quad \left. - B_{2, ES}(D_p^W)^{-1}(\varphi_p^W(B_{1, NN}) + \varphi_p^W(B_{2, SS}))^{-1} \varphi_p^W(B_{1, NS}) \right) \\ &= \left((B_{1, ES} - B_{1, EN}(B_{1, NN} + B_{2, SS})^{-1} B_{1, NS})(D_p^W)^{-1} \right. \\ &\quad \left. - B_{2, ES}(B_{1, NN} + B_{2, SS})^{-1} B_{1, NS} \right) (D_p^W)^{-1} \\ &= \Phi_{p, q_1 + q_2}^W (S_{SN}(B_1, B_2))_{ES}. \end{aligned}$$

We now prove the identity. We temporarily denote by $B_Q^{W,\infty,p} := S_{WE}(B_Q^{W,\infty}, Q^{[p,1]})$. We have, using the fact that $\Phi_{p,q}^W$ is a LengthAss morphism, for all $q \in \mathbb{N}^*$,

$$\begin{aligned} & \Phi_{p,q}^W(S_{SN}(B_Q^{W,\infty,p}, B_Q^{W,\infty,p}, \dots, B_Q^{W,\infty,p})) \\ &= S_{SN}(\Phi_{p,1}^W(B_Q^{W,\infty,p}), \Phi_{p,1}^W(B_Q^{W,\infty,p}), \dots, \Phi_{p,1}^W(B_Q^{W,\infty,p})) \\ &= S_{SN}(B_Q^{W,\infty}, B_Q^{W,\infty}, \dots, B_Q^{W,\infty}) \end{aligned}$$

using theorem 3.5.1. □

The maps $\Phi_{p,q}^W$ are precisely the morphisms of [Sim23] in the Gaussian setting lifted at the level of quadratic forms \mathcal{Q}_\bullet . We say that the half-strip operator $B_Q^{W,\infty}$ is invariant with respect to Q up to the morphisms $(\Phi_{p,q}^W)_{p,q \in \mathbb{N}}$. It is remarkable that these morphisms do not depend on Q at all. There are of course very similar morphisms for South, North and East half-strips. As already pointed out, compared to [Sim23], there is no eigenvalue involved yet. This is because we dropped the parameter space $\mathbb{R} \times \mathcal{Q}_\bullet$ for focusing on \mathcal{Q}_\bullet . Indeed, the eigenvalues are carried by the cocycles γ_{WE} and $\gamma_{SN} \in \mathbb{R}$, which are missing when considering \mathcal{Q}_\bullet only. To see eigenvalues appear, and therefore partition functions, we need a precise study of these cocycles, involving a renormalization procedure, see 3.5.6 for a non-rigorous exposition of this phenomenon.

3.5.2 Corners

In this section we give and study the quadratic forms associated to corners of [Sim23] in our case. Thanks to their geometry, gluing a half-strip on corners produces a corner. We therefore expect actions of half-strips on corners. Moreover, gluing two corners together produces a half-plane. In particular, we expect this gluing to be partially inverted by a fold operator in the good direction. Indeed, we saw that, at least for some blocks, doubly-infinite strips and half-strips are linked together with a fold operator. Similarly, we can guess diagonal blocks of the corner by folding half-planes. As already experienced in section 3.5.1.2, the fold operator preserves Schur products thanks to its multiplicative property 3.5.1. Therefore, many computations on the corners will be brought back to computations on half-planes, which are particularly easy and well understood thanks to 3.4.3. Off-diagonal blocks of the corners can be guessed by acting recursively on the corner with half-strips as explained in the end of section 3.5.1.5. Harmonic computations of section 3.3.3 also lead to the same result.

As for half-strips, we work under assumption (\star) so that Q will supposed to be dihedral-invariant for the rest of the section; see (3.2.6). As we will see, corner operators are elements of

$$\begin{aligned} \mathcal{Q}_{\infty W, \infty S} &:= \{C \in \text{End}(\mathcal{W}_W \oplus \mathcal{W}_S); C^* = C \text{ and } C \text{ positive definite}\} \\ \mathcal{Q}_{\infty E, \infty S} &:= \{C \in \text{End}(\mathcal{W}_E \oplus \mathcal{W}_S); C^* = C \text{ and } C \text{ positive definite}\} \\ \mathcal{Q}_{\infty W, \infty N} &:= \{C \in \text{End}(\mathcal{W}_W \oplus \mathcal{W}_N); C^* = C \text{ and } C \text{ positive definite}\} \\ \mathcal{Q}_{\infty E, \infty N} &:= \{C \in \text{End}(\mathcal{W}_E \oplus \mathcal{W}_N); C^* = C \text{ and } C \text{ positive definite}\}. \end{aligned} \tag{3.5.43}$$

We will only be considering the South-West corner but the situation is similar for South-East, North-West and North-East corners. We recall that such spaces are associated with corner pattern shapes in [Sim23], respectively given by $(-\infty, 0] \times (-\infty, 0]$, $[0, +\infty) \times (-\infty, 0]$, $(-\infty, 0] \times [0, +\infty)$ and $[0, +\infty) \times [0, +\infty)$. In the extended guillotine operad described in [Sim23], various products are associated to these shapes. We list them here and introduce the references to the corresponding sections and results of the present paper where these operadic products are implemented through Schur complements on the quadratic forms:

1. gluing of half-strips at one side of the corner, like

$$\begin{array}{c} B_Q^{W,\infty} \\ \hline C_Q^{SW} \end{array} \Big|_Q \quad (3.5.44)$$

with a Schur product $\mathcal{Q}_{\infty W, \infty S} \times \mathcal{Q}_{\infty W, q} \rightarrow \mathcal{Q}_{\infty W, \infty S}$ performed in two stages, firstly with a Schur product $\mathcal{Q}_{\infty W, \infty S} \times \mathcal{Q}_{\infty W, q} \rightarrow \mathcal{Q}_{\infty W, \infty S}^{(q)}$ with values in an auxiliary space $\mathcal{Q}_{\infty W, \infty S}^{(q)}$ constructed from $\mathcal{Q}_{\infty W, \infty S}$ and \mathcal{H}_2 , and later shown to be isomorphic to $\mathcal{Q}_{\infty W, \infty S}$ itself. See section 3.5.2.3 for the precise definitions and properties;

2. gluing of opposite corners, like

$$\begin{array}{c} C_Q^{NW} \\ \hline C_Q^{SW} \end{array} \Big|_Q$$

with a product $\mathcal{Q}_{\infty W, \infty S} \times \mathcal{Q}_{\infty W, \infty N} \rightarrow \mathcal{Q}_{\infty W, \infty SN}$, see section 3.5.2.2 for the precise definitions and properties.

Partial Toeplitz, Hankel and Fold. Let $\hat{a} : S^1 \times S^1 \rightarrow \text{End}(\mathcal{H})$ with $\|\hat{a}\|_\infty < \infty$. We introduce the partial Toeplitz, Hankel and Fold operators respectively as for all $\theta \in [0, 2\pi]$,

$$\mathbb{T}_1^W(\hat{a})(e^{i\theta}) = \mathbb{T}^W(\hat{a}(\bullet, e^{i\theta})) \quad \mathbb{H}_1^W(\hat{a})(e^{i\theta}) = \mathbb{H}^W(\hat{a}(\bullet, e^{i\theta})) \quad \mathbb{F}_1^W(\hat{a})(e^{i\theta}) = \mathbb{F}^W(\hat{a}(\bullet, e^{i\theta})) \quad (3.5.45a)$$

$$\mathbb{T}_2^W(\hat{a})(e^{i\theta}) = \mathbb{T}^W(\hat{a}(e^{i\theta}, \bullet)) \quad \mathbb{H}_2^W(\hat{a})(e^{i\theta}) = \mathbb{H}^W(\hat{a}(e^{i\theta}, \bullet)) \quad \mathbb{F}_2^W(\hat{a})(e^{i\theta}) = \mathbb{F}^W(\hat{a}(e^{i\theta}, \bullet)). \quad (3.5.45b)$$

Quick reminder of South and West half-plane properties. As already mentioned, half-planes will play an important role in many computations involving the corners. In section 3.4.3 we saw that the South-half plane operator is given by the self-adjoint and positive definite operator $\mathbb{C}(\hat{G}_S)$ where

$$\hat{G}_S(e^{i\theta}) = \hat{B}_{Q,NN}^{WE}(e^{i\theta}) + \hat{B}_{Q,NS}^{WE}(e^{i\theta})\hat{W}_S(e^{i\theta}) \in \text{End}(\mathcal{H}_1)$$

with, as stated in equation (3.4.37) of Theorem 3.4.1,

$$\hat{W}_S(e^{i\theta}) = C_{\bullet,1}^{1,1}(e^{i\theta}) \left(C_{\bullet,0}^{1,1}(e^{i\theta}) \right)^{-1}$$

where for all $k \in \mathbb{Z}$, $C_{\bullet,k}^{1,1}$ is the k -th partial Fourier coefficient of $(\Psi_Q^{-1})_{1,1}$, see (3.2.15). The operator $\mathbb{C}(\hat{G}_S)$ is one-dimensional Schur-invariant with respect to the West-East doubly-infinite strip $B_Q^{WE,\infty}$, see (3.4.29). Similar to \hat{G}_S , there are also North, West, and East half-planes given by the symbols \hat{G}_N , \hat{G}_W , and \hat{G}_E , associated with operators \hat{W}_N , \hat{W}_W , and \hat{W}_E respectively, see (3.4.41) for instance. Under dihedral-invariant property provided by assumption (\star) , which is our main interest, we have

$$\hat{G}_S = \hat{G}_N = \hat{G}_W = \hat{G}_E. \quad (3.5.46)$$

Nevertheless, we will keep the indices S, N, W and E in order to better keep track of their role in future computations. Moreover, under assumption (\star) , the partial Fourier coefficients of $(\Psi_Q^{-1})_{1,1}$, \widehat{W}_S , \widehat{G}_S and \widehat{G}_W are even functions and we also remark that

$$(C_{\bullet,k}^{1,1}(e^{i\theta}))^* = C_{\bullet,-k}^{1,1}(e^{i\theta}) = C_{\bullet,k}^{1,1}(e^{i\theta}) \tag{3.5.47}$$

for all $\theta \in [0, 2\pi]$ and $k \in \mathbb{Z}$. We also recall that, from (3.4.40) these Fourier coefficients satisfy

$$C_{\bullet,k}^{1,1}(e^{i\theta}) \left(C_{\bullet,0}^{1,1}(e^{i\theta}) \right)^{-1} C_{\bullet,l}^{1,1}(e^{i\theta}) = C_{\bullet,k+l}^{1,1}(e^{i\theta}),$$

for all $kl \geq 0$. To be able use the formulas just recalled, in addition of assumption (\star) , we will work under the assumptions of Theorem 3.4.1. We first recall the associated notations

$$\mathcal{S}_Q(e^{i\theta}, \bullet) := \{z \in \mathbb{C}^*; (e^{i\theta}, z) \in \mathcal{S}_Q\}$$

for all $\theta \in [0, 2\pi]$ where the spectral curve \mathcal{S}_Q is defined in (3.2.10) and $\text{mult}_{\mathcal{S}_Q(e^{i\theta}, \bullet)}(z)$ is the multiplicity of the root z with respect to the polynomial $\det(X\Psi_Q(e^{i\theta}, X))$. The assumptions of Theorem 3.4.1 are the following.

Assumption $(\star\star)$. Assume that for all $\theta \in [0, 2\pi]$, and all $z \in \mathcal{S}_Q(e^{i\theta}, \bullet)$,

$$\dim \ker \Psi_Q(e^{i\theta}, z) = \text{mult}_{\mathcal{S}_Q(e^{i\theta}, \bullet)}(z) = 1.$$

Various operadic decompositions of half-plane operators and their heuristic interpretations. As said above, \widehat{G}_W can be obtained as a fixed point of the transfer matrix (at the level of Schur products) $\widehat{B}_Q^{\text{SN}}$, represented graphically as

$$\left[\begin{array}{c|c} \widehat{G}_W & \widehat{B}_Q^{\text{SN}} \end{array} \right]_Q = \left[\widehat{G}_W \right]_Q \tag{3.5.48}$$

However, we have seen that $\widehat{B}_Q^{\text{SN}}$ can be obtained as a limit (in a suitable sense) of the vertical gluings:

$$\widehat{B}_Q^{\text{SN}} = \text{''lim''}_{q \rightarrow \infty} \left[\begin{array}{c} Q \\ \vdots \\ Q \\ Q \end{array} \right]_q \tag{3.5.49}$$

A major question is whether the element \widehat{G}_W can also be described as an infinite gluing of horizontal objects along left-infinite horizontal guillotine cuts. We have already seen that, up to morphisms Φ_{\bullet}^W , we have, for all $q \in \mathbb{N}^*$,

$$\left[\begin{array}{c|c} B_Q^{W,\infty} & Q \\ \vdots & \vdots \\ B_Q^{W,\infty} & Q \end{array} \right]_q \simeq_{\Phi_{\bullet}^W} \left[\begin{array}{c} B_Q^{W,\infty} \\ \vdots \\ B_Q^{W,\infty} \end{array} \right]_q$$

The major question, which goes far beyond the present particular case of Gaussian processes, is whether the element \widehat{G}_W can also be obtained as the following limit, which appears as a

natural candidate given the heuristic on $\widehat{B}_Q^{\text{SN}}$:

$$\widehat{G}_W \Big|_Q \stackrel{?}{=} \lim_{q \rightarrow \infty} q \begin{array}{c} \boxed{B_Q^{W,\infty}} \\ \vdots \\ \boxed{B_Q^{W,\infty}} \end{array} \Big|_Q \tag{3.5.50}$$

The limit is non trivial since it involves the morphisms Φ_\bullet^W and concerns infinite-dimensional operators and is left to a future work. In the present case of Gaussian processes in this paper, this heuristic limit is only used to guess a second representation of \widehat{G}_W and the equality between this guessed expression and the initial expression of \widehat{G}_W , which we can both write explicitly in terms of the initial matrix Q , is the content of the following lemma 3.5.8 at the heart of all subsequent operations. The following lemma 3.5.8 can then be seen as an incarnation of the square associativity of the operadic products pushed to the limit.

In a second time, this heuristic asymptotic approach can be pushed further. The r.h.s. of (3.5.50) is a limit of powers of a single element $B_Q^{W,\infty}$ (lifted at the level of Schur products of quadratic forms) and one may expect that it is equivalent to the Perron-Frobenius eigenvector of these objects (lifted again at the level of Schur products of quadratic forms). This is indeed almost the case where the Perron-Frobenius eigenvectors are given by the corner elements C_Q^{NW} and C_Q^{SW} , the only difference being that one must take care of the transversal directions, i.e. variables on the vertical edges; in particular these corner elements must also be related to the half-strip operators $B_Q^{S,\infty}$ and $B_Q^{N,\infty}$ from square associativity. Again, this heuristical approach provides the correct candidates for the corner elements C_Q^{NW} and C_Q^{SW} and the next sections are devoted to the detailed study of these expressions and, in particular, the proof that they are the correct ones using Gaussian calculus. In practice, they also heavily rely on the following lemma 3.5.8, in which the square associativity in the asymptotical regime as well as the morphisms are hidden.

An important property concerning half-planes. We now make precise the previous intuition by showing that the half-plane symbol \widehat{G}_W is not only obtained from a fixed point of the one-dimensional dynamic generated by the South-North doubly-infinite strip, as in (3.5.48), but also from the stacking (3.5.50) of infinitely many West half-strips in some suitable sense (given by vertical Fourier computations). We start by recalling and introducing some notations. Let \mathcal{H} , \mathcal{H}_W , and \mathcal{H}_E be Hilbert spaces and a face weight $R \in \text{End}(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}_W \oplus \mathcal{H}_E)$. Graphically, it is represented as in the following drawing

$$\mathcal{H}_W \begin{array}{c} \mathcal{H} \\ \boxed{R} \\ \mathcal{H} \end{array} \mathcal{H}_E .$$

Whenever it is well defined, a vertical Fourier change of basis leads to the operators, for all $\theta \in [0, 2\pi]$,

$$\Psi_R^v(e^{i\theta}) = \begin{pmatrix} \phi_R^{\text{SN}}(e^{i\theta}) & R_{\text{SW}} + e^{-i\theta} R_{\text{NW}} & R_{\text{SE}} + e^{-i\theta} R_{\text{NE}} \\ R_{\text{WS}} + e^{i\theta} R_{\text{WN}} & R_{\text{WW}} & R_{\text{WE}} \\ R_{\text{ES}} + e^{i\theta} R_{\text{EN}} & R_{\text{EW}} & R_{\text{EE}} \end{pmatrix} \in \text{End}(\mathcal{H} \oplus \mathcal{H}_W \oplus \mathcal{H}_E), \tag{3.5.51}$$

see (3.2.8), and

$$\psi_R^v(e^{i\theta}) = \left(\Psi_R^v(e^{i\theta}) / (\Psi_R^v(e^{i\theta}))_{\mathcal{H}\mathcal{H}} \right) =: \text{Schur}_{\mathcal{H}}(\Psi_R^v(e^{i\theta})) \in \text{End}(\mathcal{H}_W \oplus \mathcal{H}_E) \tag{3.5.52}$$

where $(\Psi_R^v(e^{i\theta}))_{\mathcal{H}\mathcal{H}} = \phi_R^{\text{SN}}(e^{i\theta})$ is the diagonal sub-block of $\Psi_R^v(e^{i\theta})$ associated with \mathcal{H} . These are the operators involved in the vertical transfer matrix method which leads to the expression (3.4.17) of $\widehat{B}_Q^{\text{SN}}(e^{i\theta})$. Indeed, we verify that, for all $\theta \in [0, 2\pi]$,

$$\psi_Q^v(e^{i\theta}) = \text{Schur}_{\mathcal{H}_1}(\Psi_Q^v(e^{i\theta})) = \widehat{B}_Q^{\text{SN}}(e^{i\theta}) \in \text{End}(\mathcal{H}_2 \oplus \mathcal{H}_2). \quad (3.5.53)$$

In that sense, we say that $\widehat{B}_Q^{\text{SN}}$ is obtained from the gluing of infinitely many face weights Q vertically, as in the drawing (3.5.49). Similarly, an alternative expression of \widehat{G}_W obtained from the gluing of infinitely many West half-strips $B_Q^{\text{W},\infty}$, as in the drawing (3.5.50), should be given by $\psi_{B_Q^{\text{W},\infty}}^v$. We now compute this quantity explicitly. We have, for all $\theta \in [0, 2\pi]$,

$$\Psi_{B_Q^{\text{W},\infty}}^v(e^{i\theta}) = \begin{pmatrix} \phi_{B_Q^{\text{W},\infty}}^{\text{SN}}(e^{i\theta}) & B_{Q,\text{SE}}^{\text{W},\infty} + e^{-i\theta} B_{Q,\text{NE}}^{\text{W},\infty} \\ B_{Q,\text{ES}}^{\text{W},\infty} + e^{i\theta} B_{Q,\text{EN}}^{\text{W},\infty} & B_{Q,\text{EE}}^{\text{W},\infty} \end{pmatrix} \in \text{End}(\mathcal{W}_W \oplus \mathcal{H}_2) \quad (3.5.54)$$

where $\mathcal{H}_W = \{0\}$ by convention. By the definition 3.5.1 of $B_Q^{\text{W},\infty}$ and the definition of $\phi_{\bullet}^{\text{SN}}$ (see section 3.2.1.4), we have for all $z \in S^1$,

$$\phi_{B_Q^{\text{W},\infty}}^{\text{SN}}(z)^{-1} = \mathbf{F}^W \left(\phi_{\widehat{B}_Q^{\text{WE}}}^{\text{SN}}(z)^{-1} \right)$$

but using (3.4.35), we obtain

$$\phi_{B_Q^{\text{W},\infty}}^{\text{SN}}(z)^{-1} = \mathbf{F}_1^W \left((\Psi_Q^{-1})_{1,1} \right) (z).$$

Therefore, by definition of $B_Q^{\text{W},\infty}$, we observe that

$$\begin{aligned} \psi_{B_Q^{\text{W},\infty}}^v(e^{i\theta}) &= \text{Schur}_{\mathcal{W}_W}(\Psi_Q^v(e^{i\theta})) \\ &= T + AW_Q - \mathcal{P}_Q^W(1 + e^{i\theta}) \left(\mathbf{F}_1^W \left((\Psi_Q^{-1})_{1,1} \right) (e^{i\theta}) \right) (1 + e^{-i\theta})(\mathcal{P}_Q^W)^* \end{aligned} \quad (3.5.55)$$

Before showing that this gives another formula for \widehat{G}_W , as motivated by the latter heuristic discussion, we give the following lemma which establishes a commutation relation between the vertical Fourier computation ψ^v and the horizontal Schur product S_{WE} . Heuristically, this property is inherited from the "square associativity" of the model.

Lemma 3.5.7. *For all $\theta \in [0, 2\pi]$, we have*

$$\psi_{S_{\text{WE}}(B_Q^{\text{W},\infty}, Q)}^v(e^{i\theta}) = S_{\text{WE}} \left(\psi_{B_Q^{\text{W},\infty}}^v(e^{i\theta}), \psi_Q^v(e^{i\theta}) \right). \quad (3.5.56)$$

Proof. The quantities $\psi_{B_Q^{\text{W},\infty}}^v(e^{i\theta})$ and $\psi_Q^v(e^{i\theta})$ and their West-East Schur product are well defined. Moreover, thanks to Theorem 3.5.1, $S_{\text{WE}}(B_Q^{\text{W},\infty}, Q)$ and $\psi_{S_{\text{WE}}(B_Q^{\text{W},\infty}, Q)}^v(e^{i\theta})$ also make sense. We graphically represent the Hilbert spaces attached to the half-lines and edges involved in the Schur product $S_{\text{WE}}(B_Q^{\text{W},\infty}, Q)$ in the following drawing

$$\begin{array}{ccc} \mathcal{W}_W & \mathcal{H}_1 & \\ \hline \mathcal{H}_W & \mathcal{H}_1 & \mathcal{H}_2 \\ \hline \mathcal{W}_W & \mathcal{H}_1 & \end{array} \cdot \quad (3.5.57)$$

Remark that we temporary renamed the Hilbert space attached to the edge along the one we glue into \mathcal{H}_U only to avoid any confusion in the following proof. By definition, the Schur product $S_{WE}(B_Q^{W,\infty}, Q)$ is a Schur complement of the operator $j_{WE}(B_Q^{W,\infty}, Q) \in \text{End}(\mathcal{H}_U \oplus \mathcal{W}_W \oplus \mathcal{H}_1 \oplus \mathcal{W}_W \oplus \mathcal{H}_1 \oplus \mathcal{H}_2)$ defined through the following block decomposition

$$j_{WE}(B_Q^{W,\infty}, Q) := \begin{pmatrix} B_{Q,EE}^{W,\infty} + Q_{WW} & B_{Q,ES}^{W,\infty} & Q_{WS} & B_{Q,EN}^{W,\infty} & Q_{WN} & Q_{WE} \\ B_{Q,SE}^{W,\infty} & B_{Q,SS}^{W,\infty} & 0 & B_{Q,SN}^{W,\infty} & 0 & 0 \\ Q_{SW} & 0 & Q_{SS} & 0 & Q_{SN} & Q_{SE} \\ B_{Q,NE}^{W,\infty} & B_{Q,NS}^{W,\infty} & 0 & B_{Q,NN}^{W,\infty} & 0 & 0 \\ Q_{NW} & 0 & Q_{NS} & 0 & Q_{NN} & Q_{NE} \\ Q_{EW} & 0 & Q_{ES} & 0 & Q_{EN} & Q_{EE} \end{pmatrix}.$$

The operator $j_{WE}(B_Q^{W,\infty}, Q)$ is a weight which couples the edges of the drawing (3.5.57) together: the one at the South with values in $\mathcal{W}_W \oplus \mathcal{H}_1$, at the North with values in $\mathcal{W}_W \oplus \mathcal{H}_1$, at the East with values in \mathcal{H}_2 and the one at the middle with values in \mathcal{H}_U . As usual, we denote by $j_{WE}(B_Q^{W,\infty}, Q)_{ab}$ the sub-block of $j_{WE}(B_Q^{W,\infty}, Q)$ corresponding to the coupling between a and b where $a, b \in \{S, N, U, E\}$. For instance,

$$j_{WE}(B_Q^{W,\infty}, Q)_{US} = \begin{pmatrix} B_{Q,ES}^{W,\infty} & Q_{WS} \end{pmatrix}.$$

As we just said, the Schur product $S_{WE}(B_Q^{W,\infty}, Q)$ is obtained from $j_{WE}(B_Q^{W,\infty}, Q)$ as

$$S_{WE}(B_Q^{W,\infty}, Q) = (j_{WE}(B_Q^{W,\infty}, Q) / j_{WE}(B_Q^{W,\infty}, Q)_{UU}) =: \text{Schur}_{\mathcal{H}_U}(j_{WE}(B_Q^{W,\infty}, Q))$$

where $j_{WE}(B_Q^{W,\infty}, Q)_{UU} = B_{Q,EE}^{W,\infty} + Q_{WW}$ by definition. More generally, any West-East Schur product is the Schur complement of the associated j_{WE} . Under these notations, the l.h.s of (3.5.56) is

$$\psi_{S_{WE}(B_Q^{W,\infty}, Q)}^v(e^{i\theta}) = \text{Schur}_{\mathcal{W}_W \oplus \mathcal{H}_1} \circ \Psi_{\bullet}^v(e^{i\theta}) \circ \text{Schur}_{\mathcal{H}_U} \circ j_{WE}(B_Q^{W,\infty}, Q).$$

On the other hand, the r.h.s of (3.5.56) is

$$S_{WE} \left(\psi_{B_Q^{W,\infty}}^v(e^{i\theta}), \psi_Q^v(e^{i\theta}) \right) = \text{Schur}_{\mathcal{H}_U} \circ j_{WE} \circ (\text{Schur}_{\mathcal{W}_W} \times \text{Schur}_{\mathcal{H}_1})(\Psi_{B_Q^{W,\infty}}^v(e^{i\theta}), \Psi_Q^v(e^{i\theta})).$$

Direct computations from the definitions lead to

$$\begin{aligned} j_{WE} \circ (\text{Schur}_{\mathcal{W}_W} \times \text{Schur}_{\mathcal{H}_1})(\Psi_{B_Q^{W,\infty}}^v(e^{i\theta}), \Psi_Q^v(e^{i\theta})) &= \text{Schur}_{\mathcal{W}_W \oplus \mathcal{H}_1} \circ j'_{WE}(\Psi_{B_Q^{W,\infty}}^v(e^{i\theta}), \Psi_Q^v(e^{i\theta})) \\ &= \text{Schur}_{\mathcal{W}_W \oplus \mathcal{H}_1} \circ \tilde{\Psi}_{\bullet}^v(e^{i\theta}) \circ j_{WE}(B_Q^{W,\infty}, Q) \end{aligned}$$

where, just as for j_{WE} , $j'_{WE}(\Psi_{B_Q^{W,\infty}}^v(e^{i\theta}), \Psi_Q^v(e^{i\theta}))$ is the operator in $\text{End}(\mathcal{H}_U \oplus \mathcal{W}_W \oplus \mathcal{H}_1 \oplus \mathcal{H}_2)$ obtained from $\Psi_{B_Q^{W,\infty}}^v(e^{i\theta}) \in \text{End}(\mathcal{W}_W \oplus \mathcal{H}_U)$ and $\Psi_Q^v(e^{i\theta}) \in \text{End}(\mathcal{H}_1 \oplus \mathcal{H}_U \oplus \mathcal{H}_2)$ by summing the diagonal blocks associated to \mathcal{H}_U . The map $\tilde{\Psi}_{\bullet}^v(e^{i\theta})$ is similar to $\Psi_{\bullet}^v(e^{i\theta})$ as in (3.5.51) but W is simply replaced by U .

The equation (3.5.56) can therefore be rewritten in the following way

$$\text{Schur}_{\mathcal{W}_W \oplus \mathcal{H}_1} \circ \Psi_{\bullet}^v(e^{i\theta}) \circ \text{Schur}_{\mathcal{H}_U} \circ j_{WE}(B_Q^{W,\infty}, Q) = \text{Schur}_{\mathcal{H}_U} \circ \text{Schur}_{\mathcal{W}_W \oplus \mathcal{H}_1} \circ \tilde{\Psi}_{\bullet}^v(e^{i\theta}) \circ j_{WE}(B_Q^{W,\infty}, Q).$$

By associativity of Schur complements, see discussion 3.5.1.2, we only have to prove that

$$\Psi_{\bullet}^v(e^{i\theta}) \circ \text{Schur}_{\mathcal{H}_U} \circ j_{WE}(B_Q^{W,\infty}, Q) = \text{Schur}_{\mathcal{H}_U} \circ \tilde{\Psi}_{\bullet}^v(e^{i\theta}) \circ j_{WE}(B_Q^{W,\infty}, Q).$$

This follows from a direct computation of both quantities. \square

We are now ready to state and prove the following important lemma.

Lemma 3.5.8. *Under assumptions (\star) and $(\star\star)$, for all $\theta \in [0, 2\pi]$,*

$$\widehat{G}_W(e^{i\theta}) = T + AW_Q - \mathcal{P}_Q^W(1 + e^{i\theta}) \left(F_1^W \left((\Psi_Q^{-1})_{1,1} \right) (e^{i\theta}) \right) (1 + e^{-i\theta}) (\mathcal{P}_Q^W)^*. \quad (3.5.58)$$

Moreover, for all $k \in \mathbb{Z}$ its Fourier coefficients verify

$$C_k(\widehat{G}_W) = (T + AW_Q)\delta_{k,0} - \mathcal{P}_Q^W F^W \left(2C_{\bullet,k}^{1,1} + C_{\bullet,k-1}^{1,1} + C_{\bullet,k+1}^{1,1} \right) (\mathcal{P}_Q^W)^*. \quad (3.5.59)$$

Proof. We start by proving the equation (3.5.58). We let

$$\widetilde{G}_W(e^{i\theta}) := T + AW_Q - \mathcal{P}_Q^W(1 + e^{i\theta}) \left(F_1^W \left((\Psi_Q^{-1})_{1,1} \right) (e^{i\theta}) \right) (1 + e^{-i\theta}) (\mathcal{P}_Q^W)^* = \psi_{B_Q^{W,\infty}}^v(e^{i\theta})$$

for all $\theta \in [0, 2\pi]$ from (3.5.55). Remark that, because $B_Q^{W,\infty}$ is positive definite thanks to Property 3.5.1, $\Psi_{B_Q^{W,\infty}}^v(e^{i\theta})$ is also positive definite. Hence, $\widetilde{G}_W(e^{i\theta})$ is positive definite as the Schur complement of a positive definite operator. Then, we only need to prove that

$$\widetilde{G}_W(e^{i\theta}) = S_{WE} \left(\widetilde{G}_W(e^{i\theta}), \widehat{B}_Q^{SN}(e^{i\theta}) \right)$$

i.e $\widetilde{G}_W(e^{i\theta})$ is a "left" invariant boundary condition of the one-dimensional dynamic generated by $\widehat{B}_Q^{SN}(e^{i\theta})$. Indeed, by uniqueness of the infinite volume Gibbs measure generated by $\widehat{B}_Q^{SN}(e^{i\theta})$, see section 13.3 of [Geo11] and [Bod23], the invariant boundary conditions are unique and $\widetilde{G}_W(e^{i\theta}) = \widehat{G}_W(e^{i\theta})$ for all $\theta \in [0, 2\pi]$. Thanks to lemma 3.5.7 and equations (3.5.55) and (3.5.53), we have

$$\psi_{S_{WE}(B_Q^{W,\infty}, Q)}^v(e^{i\theta}) = S_{WE} \left(\widetilde{G}_W(e^{i\theta}), \widehat{B}_Q^{SN}(e^{i\theta}) \right).$$

Finally, thanks to Theorem 3.5.1, $S_{WE}(B_Q^{W,\infty}, Q)$ and $B_Q^{W,\infty}$ are linked through a conjugation and a direct computation shows that

$$\psi_{S_{WE}(B_Q^{W,\infty}, Q)}^v(e^{i\theta}) = \psi_{B_Q^{W,\infty}}^v(e^{i\theta}) = \widetilde{G}_W(e^{i\theta})$$

which proves (3.5.58). We now show the second identity. For all $\theta_1, \theta_2 \in [0, 2\pi]$, we denote

$$\Psi_Q(e^{i\theta_1}, e^{i\theta_2}) := (1 + e^{i\theta_2}) \left(\Psi_Q^{-1}(e^{i\theta_1}, e^{i\theta_2}) \right)_{1,1} (1 + e^{-i\theta_2}).$$

Then, for all $k \in \mathbb{Z}$, by the definition of partial fold operator in (3.5.45) we have

$$C_k \left(\mathcal{P}_Q^W(1 + e^{i\bullet}) F_1^W \left((\Psi_Q^{-1})_{1,1} \right) (1 + e^{-i\bullet}) (\mathcal{P}_Q^W)^* \right) = C_k \left(\mathcal{P}_Q^W F_1^W(\Psi_Q) (\mathcal{P}_Q^W)^* \right).$$

where $e^{\pm i\bullet} : [0, 2\pi] \rightarrow S^1, \theta \mapsto e^{\pm i\theta}$. Moreover,

$$\begin{aligned} C_k \left(\mathcal{P}_Q^W F_1^W(\Psi_Q) (\mathcal{P}_Q^W)^* \right) &= C_k \left(\sum_{n,m>0} (\mathcal{P}_Q^W)_n \left(F_1^W(\Psi_Q) \right)_{n,m} \left((\mathcal{P}_Q^W)^* \right)_m \right) \\ &= \sum_{n,m>0} (\mathcal{P}_Q^W)_n C_k \left(\left(F_1^W(\Psi_Q) \right)_{n,m} \right) \left((\mathcal{P}_Q^W)^* \right)_m \end{aligned}$$

where, in the last line, infinite sums and integral were inverted thanks to dominated convergence theorem by remarking that, for all $n, m > 0$,

$$\left\| \left(F_1^W \left((\Psi_Q^{-1})_{1,1} \right) \right)_{n,m} \right\| = \left\| C_{n-m,\bullet}^{1,1} - C_{n+m-1,\bullet}^{1,1} \right\| \leq 2 \left\| (\Psi_Q^{-1})_{1,1} \right\|_\infty.$$

and recalling that $(\mathcal{P}_Q^W)_n = (W_Q^*)^{n-1}(I_{\mathcal{H}_2} + W_Q)U$ with $\|W_Q\| < 1$ (spectral norm), see definition 3.5.1. Now, remark that for all k, n and $m > 0$,

$$C_k \left(F_1^W(\Psi_Q)_{n,m} \right) = \left(F^W(C_{\bullet,k}(\Psi_Q)) \right)_{n,m}$$

where $C_{\bullet,k}(\Psi_Q)$ is the partial Fourier coefficient Ψ_Q , see (3.2.13). Then,

$$C_k \left(\mathcal{P}_Q^W F_1^W(\Psi_Q) (\mathcal{P}_Q^W)^* \right) = \mathcal{P}_Q^W F^W(C_{\bullet,k}(\Psi_Q)) (\mathcal{P}_Q^W)^*$$

and the result follows from

$$C_{\bullet,k}(\Psi_Q) = 2C_{\bullet,k}^{1,1} + C_{\bullet,k-1}^{1,1} + C_{\bullet,k+1}^{1,1}.$$

□

3.5.2.1 Corner operators, definition and properties

We recall that, in definition 3.5.1 of West half-strips, the operator \mathcal{P}_Q^W describes the couplings between horizontal half-lines of the strips and the East vertical edge. We are now ready to give the definition of South-West corner.

Definition 3.5.4 (South-West corner). Under assumptions (\star) and $(\star\star)$, we call South-West corner operator associated to Q , the operator $C_Q^{\text{SW}} \in \text{End}(\mathcal{W}_W \oplus \mathcal{W}_S)$ defined as

$$C_Q^{\text{SW}} = \begin{pmatrix} F^W(\widehat{G}_S) & \mathcal{P}_Q^{\text{SW}} \\ (\mathcal{P}_Q^{\text{SW}})^* & F^S(\widehat{G}_W) \end{pmatrix} =: \begin{pmatrix} C_{Q,\text{NN}}^{\text{SW}} & C_{Q,\text{NE}}^{\text{SW}} \\ C_{Q,\text{EN}}^{\text{SW}} & C_{Q,\text{EE}}^{\text{SW}} \end{pmatrix} \quad (3.5.60)$$

where the operator $\mathcal{P}_Q^{\text{SW}}$ is defined as

$$\begin{aligned} (\mathcal{P}_Q^{\text{SW}})^* : \mathcal{W}_W &\rightarrow \mathcal{W}_S \\ h &\mapsto \left(\mathcal{P}_Q^W \left(I_{\mathcal{W}_W} + F^W(\widehat{W}_S) \right) F^W(\widehat{W}_S)^{k-1} h \right)_{k>0}. \end{aligned} \quad (3.5.61)$$

In the correspondence with [Sim23], a corner element C_Q^{SW} is associated to a pattern shape \mathbf{c}_{SW} , which corresponds to a quarter of plane $] - \infty, 0] \times] - \infty, 0]$; it is represented in operadic computations as

$$\boxed{C_Q^{\text{SW}}}$$

and can be combined with elements associated to half-strips \mathbf{hs}_W and \mathbf{hs}_S through horizontal and vertical products. Since corner shapes have a boundary made of two half-lines, one horizontal and one vertical, the corner element C_Q^{SW} is thus given by a quadratic form on $\mathcal{W}_W \oplus \mathcal{W}_S$ as seen in the previous definition 3.5.4. We now briefly comment this definition, block per block.

Diagonal blocks. By analogy with doubly-infinite strips, folded into half-strips, the diagonal blocks of the corner are given by the fold operator of half-planes. Because the multiplicative property of the fold operator preserves Schur products, when acting with a half-strip on the corner, many computations will transfer to the computation of the action of a doubly-infinite strip on the associated half-plane, which is particularly well understood thanks to section 3.4.3. We recall that $F^S(\widehat{G}_W) = T^S(\widehat{G}_W) - H^S(\widehat{G}_W)$. Just as for half-strips, the stationary part given by $T^S(\widehat{G}_W)$ is penalized by the contribution of $H^S(\widehat{G}_W)$. This penalization is precisely given by the interaction between the vertical half-line of the corner and the horizontal one, see lemma lemma 3.5.10. As it will be discussed in the following paragraph, it gets exponentially low as we are far away from to horizontal half-line.

Off-diagonal blocks. We first quickly check that $\mathcal{P}_Q^{\text{SW}}$ is well defined. We have $\|\widehat{W}_S\|_\infty := \sup_{\theta \in [0, 2\pi]} \|\widehat{W}_S(e^{i\theta})\| < 1$, where $\|\widehat{W}_S(e^{i\theta})\|$ is the spectral norm of the matrix $\widehat{W}_S(e^{i\theta})$. Indeed, its eigenvalues are contained in the (open) unit disk for all $\theta \in [0, 2\pi]$ and \widehat{W}_S is continuous, see Theorem 3.4.1. Now, using the multiplicative property of the fold operator and the control of its norm by the norm of its symbol (3.5.11), we get

$$\|F^W(\widehat{W}_S)^{k-1}\| = \|F^W(\widehat{W}_S^{k-1})\| \leq \text{cst} \|\widehat{W}_S\|_\infty^{k-1}$$

for all $k > 0$. Thus,

$$\|(I_{\mathcal{W}_W} + F^W(\widehat{W}_S)) F^W(\widehat{W}_S)^{k-1} h\|_{\mathcal{H}_2} \leq \text{cst}' \|\widehat{W}_S\|_\infty^{k-1} \|h\|_{\mathcal{W}_W}.$$

This makes the operator $\mathcal{P}_Q^{\text{SW}}$ well defined. Now remark that the East vertical edge, at vertical distance $k > 0$ from the North horizontal half-line of the corner, is coupled with this horizontal half-line through

$$\mathcal{P}_Q^W (I_{\mathcal{W}_W} + F^W(\widehat{W}_S)) F^W(\widehat{W}_S)^{k-1}.$$

This coupling can be guessed by acting recursively on the corner with a West half-strip as explained in the end of section 3.5.1.5. Thanks to what has just been said, it converges to zero with exponential speed. In other words, when k is great, the vertical edge is far away from the North horizontal half-line of the corner, and its coupling is negligible. Now, just as in lemma 3.5.3, the action of $(F^W(\widehat{W}_S))^*$ on $\mathcal{P}_Q^{\text{SW}}$ is just $\mathcal{P}_Q^{\text{SW}}$ itself, restricted to the shifted space by the operator $T^S(e^{i\bullet})$ previously introduced in section 3.5.1.1. The role of this lemma is analogous to the one the lemma 3.5.3, it ensures that off-diagonal blocks of the corner behave particularly well under the action of a half-strip, see Theorem 3.5.3.

Lemma 3.5.9. *Under assumptions (\star) and $(\star\star)$, we have the following properties*

$$\begin{aligned} (F^W(\widehat{W}_S))^* \mathcal{P}_Q^{\text{SW}} &= \mathcal{P}_Q^{\text{SW}} T^S(e^{i\bullet}) \\ (\mathcal{P}_Q^{\text{SW}})^* F^W(\widehat{W}_S) &= T^S(e^{-i\bullet}) (\mathcal{P}_Q^{\text{SW}})^*. \end{aligned} \quad (3.5.62)$$

Proof. We prove the second identity and the first one will follow taking the adjoint. Under assumption (\star) , \widehat{W}_S is an even function. For all $h \in \mathcal{W}_W$, we verify that, using multiplicative property of the Fold operator of lemma (3.5.1),

$$(\mathcal{P}_Q^{\text{SW}})^* F^W(\widehat{W}_S) h = \left(\mathcal{P}_Q^W (I_{\mathcal{W}_W} + F^W(\widehat{W}_S)) F^W(\widehat{W}_S)^k h \right)_{k>0} = T^S(e^{-i\bullet}) (\mathcal{P}_Q^{\text{SW}})^* h.$$

□

Because of dihedral-invariant property of Q provided by assumption (\star) , the four corner operators are quite similar. Therefore, we are not going to define all of them. Nevertheless, we give the definition of the North-West corner, useful later see Proposition 3.5.4 for the gluing of adjacent corners. We recall that the fold operator F^N has been defined in (3.5.14).

Definition 3.5.5 (North-West corner). Under assumptions (\star) and $(\star\star)$, we call North-West corner operator associated to Q , the operator $C_Q^{\text{NW}} \in \text{End}(\mathcal{W}_W \oplus \mathcal{W}_N)$ defined as

$$C_Q^{\text{NW}} = \begin{pmatrix} F^W(\widehat{G}_N) & \mathcal{P}_Q^{\text{NW}} \\ (\mathcal{P}_Q^{\text{NW}})^* & F^N(\widehat{G}_W) \end{pmatrix} =: \begin{pmatrix} C_{Q,SS}^{\text{NW}} & C_{Q,SE}^{\text{NW}} \\ C_{Q,ES}^{\text{NW}} & C_{Q,EE}^{\text{NW}} \end{pmatrix} \quad (3.5.63)$$

where the operator $\mathcal{P}_Q^{\text{NW}}$ is defined as

$$\begin{aligned} (\mathcal{P}_Q^{\text{NW}})^* : \mathcal{W}_W &\rightarrow \mathcal{W}_N \\ h &\mapsto \left(\mathcal{P}_Q^W (I_{\mathcal{W}_W} + F^W(\widehat{W}_N)) F^W(\widehat{W}_N)^{-k} h \right)_{k \leq 0}. \end{aligned} \quad (3.5.64)$$

We recall that, under (\star) , $\widehat{G}_N = \widehat{G}_S$, $F^N(\widehat{G}_N) = F^S(\widehat{G}_S)$ and $\widehat{W}_N = \widehat{W}_S$ but the indices are kept for a clarity, see the discussion in the beginning of this section 3.5.2.

The next lemma is similar to lemma 3.5.4. Similarly, it will be useful for proving that the corner operators are positive definite, see Proposition 3.5.3, and also for multiple proofs concerning several gluings, see Proposition 3.5.4 or Theorem 3.5.3 for instance. Moreover, it gives the good interpretation of the Hankel counterpart in the diagonal block $F^S(\widehat{G}_W)$ of the corner as the interaction of the East vertical half-line with the North horizontal one. We recall that $C(\cdot)$ is the convolution operator defined in (3.4.20).

Lemma 3.5.10. *Under assumptions (\star) and $(\star\star)$, let $a, b \in \{S, N\}$. We have*

$$-(\mathcal{P}_Q^{SW})^* F^W(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{SW} = H^S(\widehat{G}_W) \quad -(\mathcal{P}_Q^{NW})^* F^W(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{NW} = H^N(\widehat{G}_W).$$

and

$$-(\mathcal{P}_Q^{SW})^* F^W(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{NW} = P^S C(\widehat{G}_W) P^N. \quad (3.5.65)$$

Proof. The beginning of the proof is quite similar to the one of lemma 3.5.4. Using the explicit formula of \widehat{W}_S , see (3.4.37), and equations (3.4.40) and (3.5.2), we get

$$(\widehat{W}_S^*(e^{i\theta}))^n = (C_{\bullet,0}^{1,1}(e^{i\theta}))^{-1} C_{\bullet,n}^{1,1}(e^{i\theta}),$$

for all $\theta \in [0, 2\pi]$ and all $n > 0$. Using this fact just after the multiplicative property of the fold operator, given by lemma 3.5.1, gives for all $k, l > 0$,

$$\begin{aligned} & -((\mathcal{P}_Q^{SW})^* F^W(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{SW})_{k,l} \\ &= -\mathcal{P}_Q^W \left(I_{W_W} + F^W(\widehat{W}_S) \right) F^W(\widehat{W}_S)^{k-1} F^W(C_{\bullet,0}^{1,1}) (F^W(\widehat{W}_S)^*)^{l-1} \left(I_{W_W} + F^W(\widehat{W}_S)^* \right) (\mathcal{P}_Q^W)^* \\ &= -\mathcal{P}_Q^{SW} F^W(C_{\bullet,k+l-2}^{1,1} + 2C_{\bullet,k+l-1}^{1,1} + C_{\bullet,k+l}^{1,1}) (\mathcal{P}_Q^{SW})^* = C_{k+l-1}(\widehat{G}_W) = H^S(\widehat{G}_W)_{k,l} \end{aligned}$$

thanks to lemma 3.5.8. The second and the third identities are proved with similar arguments. \square

Positive-definite property for the South-West corner. We now prove that the South-West and the North-West corner operators are elements of $\mathcal{Q}_{\infty_W, \infty_S}$ and $\mathcal{Q}_{\infty_W, \infty_N}$ respectively, i.e they are self-adjoint and positive definite, just as Q and its associated half-strips.

Property 3.5.3. *Under assumptions (\star) and $(\star\star)$, we have $C_Q^{SW} \in \mathcal{Q}_{\infty_W, \infty_S}$ and $C_Q^{NW} \in \mathcal{Q}_{\infty_W, \infty_N}$.*

Proof. We only treat the case of the South-West corner operator. The one for North-West is similar. The proof is similar to the one of Proposition 3.5.1. The blocks $F(\widehat{G}_S)$ and $F(\widehat{G}_W)$ are clearly self-adjoint and positive definite because $\widehat{G}_S(e^{i\theta})$ and $\widehat{G}_W(e^{i\theta})$ are self-adjoint and positive definite for all $\theta \in [0, 2\pi]$, see lemma 3.5.2. Thus, C_Q^{SW} is self-adjoint. Moreover, from a LDU decomposition, as in lemma 3.4.3, it is positive definite if the Schur complement $(C_Q^{SW}/C_{Q,NN}^{SW})$ is positive definite. Under dihedral-invariant property, $2\widehat{G}_S = (C_{\bullet,0}^{1,1})^{-1}$ thanks to lemma 5.5 of [Bod23], see (3.2.19). Thus, from definition of the Schur complement (3.4.14), we get

$$\begin{aligned} (C_Q^{SW}/C_{Q,NN}^{SW}) &= F^S(\widehat{G}_W) - (\mathcal{P}_Q^{WE})^* F^W(\widehat{G}_S^{-1}) \mathcal{P}_Q^{WE} \\ &= F^S(\widehat{G}_W) - 2(\mathcal{P}_Q^{SW})^* F^W(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{SW} \\ &= T^S(\widehat{G}_W) + H^S(\widehat{G}_W) \end{aligned}$$

thanks to lemma 3.5.10. This operator is positive definite thanks to lemma 3.5.2. \square

3.5.2.2 Gluing two corners

We recall that half-lines spaces \mathcal{W}_S and \mathcal{W}_N are defined in (3.5.15). As for half-strips, there are opposite pairings which glue corners together whose geometrical interpretation is given by item 2 of section 3.5.2. At the level of South-West and North-West corner operators is given by a Schur product $S_{SN}(C_Q^{SW}, C_Q^{NW}) \in \text{End}(\mathcal{W}_S \oplus \mathcal{W}_N)$ where

$$\begin{array}{c|c} C_Q^{NW} \\ \hline C_Q^{SW} \\ \hline \end{array}_Q = S_{SN}(C_Q^{SW}, C_Q^{NW})$$

$$:= \begin{pmatrix} C_{Q,EE}^{SW} - C_{Q,EN}^{SW}(C_{Q,NN}^{SW} + C_{Q,SS}^{NW})^{-1}C_{Q,NE}^{SW} & -C_{Q,EN}^{SW}(C_{Q,NN}^{SW} + C_{Q,SS}^{NW})^{-1}C_{Q,SE}^{NW} \\ -C_{Q,ES}^{NW}(C_{Q,NN}^{SW} + C_{Q,SS}^{NW})^{-1}C_{Q,NE}^{SW} & C_{Q,EE}^{NW} - C_{Q,ES}^{NW}(C_{Q,NN}^{SW} + C_{Q,SS}^{NW})^{-1}C_{Q,SE}^{NW} \end{pmatrix}.$$

We remark that this product traduces a gluing at the level of half-line spaces formally given by $(\mathcal{W}_S, \mathcal{W}_N) \mapsto \mathcal{W}_S \oplus \mathcal{W}_N = \mathcal{W}_{SN}$ the doubly-infinite line space defined in (3.4.22). In fact, by gluing the two corners, we recover the west half-plane operator $C(\widehat{G}_W)$ and it shows once again how objects with boundaries in both dimensions are richer than the ones with boundaries in only one dimension, such as strips and half-planes. This is the purpose of the following proposition.

Property 3.5.4. *Under assumptions (\star) and $(\star\star)$, we have*

$$S_{SN}(C_Q^{SW}, C_Q^{NW}) = C(\widehat{G}_W). \quad (3.5.66)$$

Proof. We recall that for all $\theta \in [0, 2\pi]$, $\widehat{G}_S(e^{i\theta})$ and $\widehat{G}_N(e^{i\theta})$ are the one-dimensional Schur-invariant boundary conditions with respect to the doubly-infinite strip symbol $\widehat{B}_Q^{WE}(e^{i\theta})$, see Theorem 3.4.1 for instance. Therefore, we can use results of [Bod23] recalled in section 3.2.1.5. Thus we have, using the multiplicative property of the fold operator given by lemma 3.5.1 together with (3.2.19) of section 3.2.1.5,

$$(C_{Q,NN}^{SW} + C_{Q,NN}^{NW})^{-1} = F^W((\widehat{G}_S + \widehat{G}_N)^{-1}) = F^W(C_{\bullet,0}^{1,1}).$$

Then, by definitions 3.5.4 and 3.5.5 of South-West and North-West corner operators, we have

$$\begin{aligned} & S_{SN}(C_Q^{SW}, C_Q^{NW}) \\ &= \begin{pmatrix} F^S(\widehat{G}_W) - (\mathcal{P}_Q^{SW})^* F^W(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{SW} & -(\mathcal{P}_Q^{SW})^* F^W(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{NW} \\ -(\mathcal{P}_Q^{NW})^* F^W(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{SW} & F^N(\widehat{G}_W) - (\mathcal{P}_Q^{NW})^* F^W(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{NW} \end{pmatrix} \\ &= \begin{pmatrix} T^S(\widehat{G}_W) & P^S C(\widehat{G}_W) P^N \\ P^N C(\widehat{G}_W) P^S & T^N(\widehat{G}_W) \end{pmatrix} = C(\widehat{G}_W) \end{aligned}$$

where we used lemma 3.5.10 and definition of the convolution operator, see (3.4.20). \square

3.5.2.3 Acting on the corner with half-strips $B_Q^{W,\infty}$ and $B_Q^{E,\infty}$

In this section, we study the "action" of the half-strips operators $B_Q^{W,\infty}$ and $B_Q^{E,\infty}$ on the corner C_Q^{SW} , see item 1 of section 3.5.2 for geometrical interpretation. Just as the action of Q on half-strips, see section 3.5.1.5, this action naturally comes with a shift on half-line spaces and later, with a morphism preserving the gluings see (3.5.75).

We focus on the action of the West half-strip $B_Q^{W,\infty}$. It is given by the a Schur product $S_{SN}(C_Q^{SW}, B_Q^{W,\infty}) \in \text{End}(\mathcal{W}_W \oplus (\mathcal{W}_S \oplus \mathcal{H}_2))$: the space $(\mathcal{W}_S \oplus \mathcal{H}_2)$ corresponds, in the gluing

(3.5.44), to the lengthening of the vertical half-line (associated to \mathcal{W}_S by the vertical segment at the end of the half-strip (associated to \mathcal{H}_2); after the Schur product, the inner horizontal half-line disappears and only the top horizontal half-line of the half-strip (associated to \mathcal{W}_W) remains. The Schur product is precisely defined as

$$\begin{array}{|c|} \hline B_Q^{W,\infty} \\ \hline C_Q^{SW} \\ \hline \end{array} \Big|_Q = S_{SN}(C_Q^{SW}, B_Q^{W,\infty}) := \begin{pmatrix} S_{SN}(C_Q^{SW}, B_Q^{W,\infty})_{NN} & S_{SN}(C_Q^{SW}, B_Q^{W,\infty})_{NE} \\ S_{SN}(C_Q^{SW}, B_Q^{W,\infty})_{EN} & S_{SN}(C_Q^{SW}, B_Q^{W,\infty})_{EE} \end{pmatrix} \quad (3.5.67)$$

with blocks given by $S_{SN}(C_Q^{SW}, B_Q^{W,\infty})_{NN} = B_{Q,NN}^{W,\infty} - B_{Q,NS}^{W,\infty}(C_{Q,NN}^{SW} + B_{Q,SS}^{W,\infty})^{-1}B_{Q,SN}^{W,\infty}$,

$$S_{SN}(C_Q^{SW}, B_Q^{W,\infty})_{EE} = \begin{pmatrix} C_{Q,EE}^{SW} - C_{Q,EN}^{SW}(C_{Q,NN}^{SW} + B_{Q,SS}^{W,\infty})^{-1}C_{Q,NE}^{SW} & -C_{Q,EN}^{SW}(C_{Q,NN}^{SW} + B_{Q,SS}^{W,\infty})^{-1}B_{Q,SE}^{W,\infty} \\ -B_{Q,ES}^{W,\infty}(C_{Q,NN}^{SW} + B_{Q,SS}^{W,\infty})^{-1}C_{Q,NE}^{SW} & B_{Q,EE}^{W,\infty} - B_{Q,ES}^{W,\infty}(C_{Q,NN}^{SW} + B_{Q,SS}^{W,\infty})^{-1}B_{Q,SE}^{W,\infty} \end{pmatrix},$$

$$S_{SN}(C_Q^{SW}, B_Q^{W,\infty})_{EN} = \begin{pmatrix} -C_{Q,EN}^{SW}(C_{Q,NN}^{SW} + B_{Q,SS}^{W,\infty})^{-1}B_{Q,SN}^{W,\infty} \\ B_{Q,EN}^{W,\infty} - B_{Q,ES}^{W,\infty}(C_{Q,NN}^{SW} + B_{Q,SS}^{W,\infty})^{-1}B_{Q,SN}^{W,\infty} \end{pmatrix}$$

and $S_{SN}(C_Q^{SW}, B_Q^{W,\infty})_{NE} = (S_{SN}(C_Q^{SW}, B_Q^{W,\infty})_{EN})^*$.

The term $(C_{Q,NN}^{SW} + B_{Q,SS}^{W,\infty})^{-1}$ corresponds to the disappearing interior horizontal half-line of the guillotine partition (3.5.44), which has two contributions: a North-North one coming from the corner and a South-South one coming from the half-strip. Thanks to the multiplicative property of the fold operator provided by 3.5.1, we get

$$(C_{Q,NN}^{SW} + B_{Q,SS}^{W,\infty})^{-1} = \left(F^W(\widehat{G}_S) + F^W(\widehat{B}_{Q,SS}^{WE}) \right)^{-1} = F^W((\widehat{G}_S + \widehat{B}_{Q,SS}^{WE})^{-1}) \quad (3.5.68)$$

and $(C_{Q,NN}^{SW} + B_{Q,SS}^{W,\infty})^{-1}$ is well-defined and, therefore, the Schur product $S_{SN}(C_Q^{SW}, B_Q^{W,\infty})$ as well. Moreover, it satisfies a one-dimensional associativity (see the discussion in subsection 3.5.1.2) and the recursive action of $B_Q^{W,\infty}$ of order $q \in \mathbb{N}^*$ is thus given by

$$S_{SN}(C_Q^{SW}, B_Q^{W,\infty}, \dots, B_Q^{W,\infty}) \in \text{End}(\mathcal{W}_W \oplus (\mathcal{W}_S \oplus \mathcal{H}_2^q)) \quad (3.5.69)$$

where $B_Q^{W,\infty}$ appears q times. As for the recursive action of Q on the West half-strip, see (3.5.30), the action of the West half-strip on the corner lengthens its vertical half-line from \mathcal{W}_S to $\mathcal{W}_S \oplus \mathcal{H}_2^q$. Thus, just as for half-strips, we will need the operator $D_q^S : \mathcal{W}_S \oplus \mathcal{H}_2^q \rightarrow \mathcal{W}_S$ already defined in (3.5.31).

Before computing (3.5.67), we first need the following lemma.

Lemma 3.5.11. *Under assumptions (\star) and $(\star\star)$, we have*

$$\widehat{W}_S C_{\bullet,0}^{1,1} = (C_{\bullet,0}^{1,1})(\widehat{W}_S)^* \quad (3.5.70)$$

$$\widehat{W}_S(\widehat{B}_{Q,SN}^{WE})^{-1} = (\widehat{W}_S^2 - I_{\mathcal{H}_1})C_{\bullet,0}^{1,1}. \quad (3.5.71)$$

Proof. We work point-wise θ by θ and the proof follows the exact same arguments of the ones used to prove (3.5.33) and (3.5.34). \square

We have the following theorem.

Theorem 3.5.3. *Under assumptions (\star) and $(\star\star)$,*

$$\Phi_{0,1}^{\text{SW}}(S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty})) := (I_{\mathcal{W}\mathcal{W}} \oplus D_1^{\text{S}}) S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty}) (I_{\mathcal{W}\mathcal{W}} \oplus D_1^{\text{S}})^{-1} = C_Q^{\text{SW}}. \quad (3.5.72)$$

And, by associativity, for all $q \in \mathbb{N}^*$,

$$\begin{aligned} \Phi_{0,q}^{\text{SW}}(S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty}, \dots, B_Q^{\text{W},\infty})) &:= (I_{\mathcal{W}\mathcal{W}} \oplus D_q^{\text{S}}) S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty}) (I_{\mathcal{W}\mathcal{W}} \oplus D_q^{\text{S}})^{-1} \\ &= C_Q^{\text{SW}} \end{aligned} \quad (3.5.73)$$

where $B_Q^{\text{W},\infty}$ appears q times in the l.h.s.

This theorem is similar to Theorem 3.5.1 for half-strips, except now, the action $S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty})$ is even more complicated. It now relies on an infinite-dimension Schur product. Nevertheless, this last theorem states that this Schur action is again simply conjugate to the identity through the "change of basis" given by $(I_{\mathcal{W}\mathcal{W}} \oplus D_1^{\text{S}})$. This conjugation brings back the lengthened corner the original space in the exact same fashion as the conjugation in theorem 3.5.1 for half-strips.

Proof of Theorem 3.5.3. The proof is quite similar to the one of Theorem 3.5.1. From the expression of \widehat{G}_{S} given by Theorem 3.4.1 together with (3.2.18), we get

$$\widehat{B}_{Q,\text{SS}}^{\text{WE}} + \widehat{G}_{\text{S}} = \widehat{B}_{Q,\text{SS}}^{\text{WE}} + \widehat{B}_{Q,\text{NN}}^{\text{WE}} + \widehat{B}_{Q,\text{NS}}^{\text{WE}} \widehat{W}_{\text{S}} = -\widehat{B}_{Q,\text{SN}}^{\text{WE}} \widehat{W}_{\text{S}}^{-1}.$$

Then, thanks to the multiplicative property of the fold operator provided by 3.5.1, we get

$$(C_{Q,\text{NN}}^{\text{SW}} + B_{Q,\text{SS}}^{\text{W},\infty})^{-1} = (\text{F}^{\text{W}}(\widehat{G}_{\text{S}}) + \text{F}^{\text{W}}(\widehat{B}_{Q,\text{SS}}^{\text{WE}}))^{-1} = -\text{F}^{\text{W}}(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1}) \quad (3.5.74)$$

We rather show that $S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty}) = (I_{\mathcal{W}\mathcal{W}} \oplus D_1^{\text{S}})^{-1} C_Q^{\text{SW}} (I_{\mathcal{W}\mathcal{W}} \oplus D_1^{\text{S}})$ and we proceed block per block.

The block $S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty})_{\text{NN}}$. We have, using the multiplicative property of the fold operator, see lemma 3.5.1,

$$\begin{aligned} S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty})_{\text{NN}} &= B_{Q,\text{NN}}^{\text{W},\infty} - B_{Q,\text{NS}}^{\text{W},\infty} (C_{Q,\text{NN}}^{\text{SW}} + B_{Q,\text{SS}}^{\text{W},\infty})^{-1} B_{Q,\text{SN}}^{\text{W},\infty} \\ &= \text{F}^{\text{W}}(\widehat{B}_{Q,\text{NN}}^{\text{WE}}) - \text{F}^{\text{W}}(\widehat{B}_{Q,\text{NS}}^{\text{WE}}) (\text{F}^{\text{W}}(\widehat{G}_{\text{S}}) + \text{F}^{\text{W}}(\widehat{B}_{Q,\text{SS}}^{\text{WE}}))^{-1} \text{F}^{\text{W}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}}) \\ &= \text{F}^{\text{W}}(\widehat{B}_{Q,\text{NN}}^{\text{WE}} - \widehat{B}_{Q,\text{NS}}^{\text{WE}} (\widehat{G}_{\text{S}} + \widehat{B}_{Q,\text{SS}}^{\text{WE}})^{-1} \widehat{B}_{Q,\text{SN}}^{\text{WE}}) = \text{F}^{\text{W}}(\widehat{G}_{\text{S}}) = C_{Q,\text{NN}}^{\text{SW}} \end{aligned}$$

using Schur invariant property of \widehat{G}_{S} with respect to $\widehat{B}_{Q,\text{NS}}^{\text{WE}}$, see (3.4.30).

The block $S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty})_{\text{EN}}$. We recall that

$$\begin{aligned} S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty})_{\text{EN}} &= \begin{pmatrix} -C_{Q,\text{EN}}^{\text{SW}} (C_{Q,\text{NN}}^{\text{SW}} + B_{Q,\text{SS}}^{\text{W},\infty})^{-1} B_{Q,\text{SN}}^{\text{W},\infty} \\ B_{Q,\text{EN}}^{\text{W},\infty} - B_{Q,\text{ES}}^{\text{W},\infty} (C_{Q,\text{NN}}^{\text{SW}} + B_{Q,\text{SS}}^{\text{W},\infty})^{-1} B_{Q,\text{SN}}^{\text{W},\infty} \end{pmatrix} \\ &= \begin{pmatrix} -(\mathcal{P}_Q^{\text{SW}})^* (\text{F}^{\text{W}}(\widehat{G}_{\text{S}}) + \text{F}^{\text{W}}(\widehat{B}_{Q,\text{SS}}^{\text{WE}}))^{-1} \text{F}^{\text{W}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}}) \\ \mathcal{P}_Q^{\text{W}} - \mathcal{P}_Q^{\text{W}} (\text{F}^{\text{W}}(\widehat{G}_{\text{S}}) + \text{F}^{\text{W}}(\widehat{B}_{Q,\text{SS}}^{\text{WE}}))^{-1} \text{F}^{\text{W}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}}) \end{pmatrix} \end{aligned}$$

So that, using equation (3.5.74) together with the multiplicative property of the fold operator,

$$S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty})_{\text{EN}} = \begin{pmatrix} (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}}(\widehat{W}_{\text{S}}) \\ \mathcal{P}_Q^{\text{W}} (I_{\mathcal{W}\mathcal{W}} + \text{F}^{\text{W}}(\widehat{W}_{\text{S}})) \end{pmatrix} = (D_1^{\text{W}})^{-1} (\mathcal{P}_Q^{\text{SW}})^* = (D_1^{\text{S}})^{-1} C_{Q,\text{EN}}^{\text{SW}}$$

using equation (3.5.62). Taking the adjoint we also have $S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty})_{\text{NE}} = C_{Q,\text{NE}}^{\text{SW}} D_1^{\text{S}}$.

The block $S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty})_{\text{EE}}$.

$$\begin{aligned} & S_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty})_{\text{EE}} \\ &= \begin{pmatrix} C_{Q,\text{EE}}^{\text{SW}} - C_{Q,\text{EN}}^{\text{SW}}(C_{Q,\text{NN}}^{\text{SW}} + B_{Q,\text{SS}}^{\text{W},\infty})^{-1}C_{Q,\text{NE}}^{\text{SW}} & -C_{Q,\text{EN}}^{\text{SW}}(C_{Q,\text{NN}}^{\text{SW}} + B_{Q,\text{SS}}^{\text{W},\infty})^{-1}B_{Q,\text{SE}}^{\text{W},\infty} \\ -B_{Q,\text{ES}}^{\text{W},\infty}(C_{Q,\text{NN}}^{\text{SW}} + B_{Q,\text{SS}}^{\text{W},\infty})^{-1}C_{Q,\text{NE}}^{\text{SW}} & B_{Q,\text{EE}}^{\text{W},\infty} - B_{Q,\text{ES}}^{\text{W},\infty}(C_{Q,\text{NN}}^{\text{SW}} + B_{Q,\text{SS}}^{\text{W},\infty})^{-1}B_{Q,\text{SE}}^{\text{W},\infty} \end{pmatrix} \\ &= \begin{pmatrix} \text{F}^{\text{S}}(\widehat{G}_{\text{W}}) + (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) \mathcal{P}_Q^{\text{SW}} & (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) (\mathcal{P}_Q^{\text{W}})^* \\ \mathcal{P}_Q^{\text{W}} \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) \mathcal{P}_Q^{\text{SW}} & T + A W_Q + \mathcal{P}_Q^{\text{W}} \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) (\mathcal{P}_Q^{\text{W}})^* \end{pmatrix} \end{aligned}$$

We treat every block on its own. Firstly, we have, using (3.5.71)

$$(\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) \mathcal{P}_Q^{\text{SW}} = (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}} \left((\widehat{W}_{\text{S}}^2 - I_{\mathcal{H}_1}) C_{\bullet,0}^{1,1} \right) \mathcal{P}_Q^{\text{SW}}$$

so that, using multiplicative property of the fold operator of lemma 3.5.1 and (3.5.70),

$$\begin{aligned} (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) \mathcal{P}_Q^{\text{SW}} &= \left(\text{F}^{\text{W}}(\widehat{W}_{\text{S}})^* \mathcal{P}_Q^{\text{SW}} \right)^* \text{F}^{\text{W}}(C_{\bullet,0}^{1,1}) \left(\text{F}^{\text{W}}(\widehat{W}_{\text{S}})^* \mathcal{P}_Q^{\text{SW}} \right) \\ &\quad - (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}}(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{\text{SW}} \\ &= -\text{T}^{\text{S}}(e^{-i\bullet}) \text{H}^{\text{S}}(\widehat{G}_{\text{W}}) \text{T}^{\text{S}}(e^{i\bullet}) + \text{H}^{\text{S}}(\widehat{G}_{\text{W}}) \end{aligned}$$

where the last line is a consequence of (3.5.62) together with lemma 3.5.10. Now, remarking that $\text{T}^{\text{S}}(e^{-i\bullet}) \text{T}^{\text{S}}(\widehat{G}_{\text{W}}) \text{T}^{\text{S}}(e^{i\bullet}) = \text{T}^{\text{S}}(\widehat{G}_{\text{W}})$, we get

$$\text{F}^{\text{S}}(\widehat{G}_{\text{W}}) + (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) \mathcal{P}_Q^{\text{SW}} = \text{T}^{\text{S}}(e^{-i\bullet}) C_{Q,\text{EE}}^{\text{SW}} \text{T}^{\text{S}}(e^{i\bullet}).$$

Similarly, with (3.5.71) and (3.5.70) we get

$$\begin{aligned} & (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) (\mathcal{P}_Q^{\text{W}})^* \\ &= \left(\text{F}^{\text{W}}(\widehat{W}_{\text{S}})^* \mathcal{P}_Q^{\text{SW}} \right)^* \text{F}^{\text{W}}(C_{\bullet,0}^{1,1}) \left(\text{F}(\widehat{W}_{\text{S}})^* (\mathcal{P}_Q^{\text{W}})^* \right) - (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}}(C_{\bullet,0}^{1,1}) (\mathcal{P}_Q^{\text{W}})^* \\ &= \left(\text{F}^{\text{W}}(\widehat{W}_{\text{S}})^* \mathcal{P}_Q^{\text{SW}} \right)^* \text{F}^{\text{W}}(C_{\bullet,0}^{1,1}) \left(I_{\mathcal{W}_{\text{W}}} + \text{F}(\widehat{W}_{\text{S}})^* \right) (\mathcal{P}_Q^{\text{W}})^* \\ &\quad - (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}}(C_{\bullet,0}^{1,1}) \left(I_{\mathcal{W}_{\text{W}}} + \text{F}^{\text{W}}(\widehat{W}_{\text{S}})^* \right) (\mathcal{P}_Q^{\text{W}})^* \end{aligned}$$

Thus, equation (3.5.62) and the definition of \mathcal{P}_Q^{W} lead to

$$\begin{aligned} (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) (\mathcal{P}_Q^{\text{W}})^* &= \text{T}^{\text{S}}(e^{-i\bullet}) (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}}(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{\text{SW}} \iota_1^{\text{S}} \\ &\quad - (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}}(C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{\text{SW}} \iota_1^{\text{S}} \end{aligned}$$

so that, with lemma 3.5.10,

$$\begin{aligned} (\mathcal{P}_Q^{\text{SW}})^* \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) (\mathcal{P}_Q^{\text{W}})^* &= -\text{T}^{\text{S}}(e^{-i\bullet}) \text{H}^{\text{S}}(\widehat{G}_{\text{W}}) \iota_1^{\text{S}} + \text{H}^{\text{S}}(\widehat{G}_{\text{S}}) \iota_1^{\text{S}} \\ &= \text{T}^{\text{S}}(e^{-i\bullet}) \text{F}^{\text{S}}(\widehat{G}_{\text{S}}) \iota_1^{\text{S}} = \text{T}^{\text{S}}(e^{-i\bullet}) C_{Q,\text{EE}}^{\text{SW}} \iota_1^{\text{S}} \end{aligned}$$

where, for the last equality, we noticed that $\text{H}^{\text{S}}(\widehat{G}_{\text{W}}) \iota_1^{\text{S}} = \text{T}^{\text{W}}(e^{-i\bullet}) \text{T}^{\text{W}}(\widehat{G}_{\text{W}}) \iota_1^{\text{S}}$. Taking the adjoint we also get

$$\mathcal{P}_Q^{\text{W}} \text{F}^{\text{W}} \left(\widehat{W}_{\text{S}}(\widehat{B}_{Q,\text{SN}}^{\text{WE}})^{-1} \right) \mathcal{P}_Q^{\text{SW}} = (\iota_1^{\text{S}})^* C_{Q,\text{EE}}^{\text{SW}} \text{T}^{\text{S}}(e^{i\bullet}).$$

Finally, with exact same arguments,

$$\begin{aligned} T + AW_Q + \mathcal{P}_Q^W F^W \left(\widehat{W}_S (\widehat{B}_{Q,SN}^{WE})^{-1} \right) (\mathcal{P}_Q^W)^* \\ = T + AW_Q + \mathcal{P}_Q^W F^W \left((\widehat{W}_S^2 - I_{\mathcal{H}_1}) C_{\bullet,0}^{1,1} \right) (\mathcal{P}_Q^W)^* \\ = T + AW_Q - \mathcal{P}_Q^W F^W (C_{\bullet,0}^{1,1}) (\mathcal{P}_Q^W)^* + \left(F^W (\widehat{W}_S)^* (\mathcal{P}_Q^W)^* \right)^* F^W (C_{\bullet,0}^{1,1}) \left(F^W (\widehat{W}_S)^* (\mathcal{P}_Q^W)^* \right)^*. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \left(F^W (\widehat{W}_S)^* (\mathcal{P}_Q^W)^* \right)^* F^W (C_{\bullet,0}^{1,1}) \left(F^W (\widehat{W}_S)^* (\mathcal{P}_Q^W)^* \right) \\ &= \left((I_{\mathcal{W}_W} + F^W (\widehat{W}_S)^*) (\mathcal{P}_Q^W)^* \right)^* F^W (C_{\bullet,0}^{1,1}) \left((I_{\mathcal{W}_W} + F^W (\widehat{W}_S)^*) (\mathcal{P}_Q^W)^* \right) \\ & \quad - \mathcal{P}_Q^W F^W (C_{\bullet,0}^{1,1}) (\mathcal{P}_Q^W)^* - 2\mathcal{P}_Q^W F^W (C_{\bullet,0}^{1,1}) F^W (\widehat{W}_S)^* (\mathcal{P}_Q^W)^* \\ &= (\iota_1^S)^* (\mathcal{P}_Q^{SW})^* F^W (C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{SW} \iota_1^S - \mathcal{P}_Q^W F^W (C_{\bullet,0}^{1,1}) (\mathcal{P}_Q^W)^* - 2\mathcal{P}_Q^W F^W (C_{\bullet,0}^{1,1}) F^W (\widehat{W}_S)^* (\mathcal{P}_Q^W)^*. \end{aligned}$$

But, thanks to lemma 3.5.10, $(\iota_1^S)^* (\mathcal{P}_Q^{SW})^* F^W (C_{\bullet,0}^{1,1}) \mathcal{P}_Q^{SW} \iota_1^S = (\iota_1^S)^* H^S(\widehat{G}_W) \iota_1^S$ and

$$\begin{aligned} T + AW_Q + \mathcal{P}_Q^W F^W \left(\widehat{W}_S (\widehat{B}_{Q,SN}^{WE})^{-1} \right) (\mathcal{P}_Q^W)^* \\ = T + AW_Q - 2\mathcal{P}_Q^W F^W (C_{\bullet,0}^{1,1}) (\mathcal{P}_Q^W)^* - 2\mathcal{P}_Q^W F^W (C_{\bullet,0}^{1,1}) F^W (\widehat{W}_S)^* (\mathcal{P}_Q^W)^* - (\iota_1^S)^* H^S(\widehat{G}_W) \iota_1^S \\ = (\iota_1^S)^* F^S(\widehat{G}_S) \iota_1^S \end{aligned}$$

where we used lemma 3.5.8, together with (3.5.2) and the expression $\widehat{W}_S = C_{\bullet,1}^{1,1} (C_{\bullet,0}^{1,1})^{-1}$ of Theorem 3.4.1:

$$\begin{aligned} \mathbb{T}(\widehat{G}_S)_{1,1} = C_0(\widehat{G}_W) &= T + AW_Q - 2\mathcal{P}_Q^W F^W (C_{\bullet,0}^{1,1}) (\mathcal{P}_Q^W)^* - 2\mathcal{P}_Q^W F^W (C_{\bullet,1}^{1,1}) (\mathcal{P}_Q^W)^* \\ &= T + AW_Q - 2\mathcal{P}_Q^W F^W (C_{\bullet,0}^{1,1}) (\mathcal{P}_Q^W)^* - 2\mathcal{P}_Q^W F^W (C_{\bullet,0}^{1,1}) F^W (\widehat{W}_S)^* (\mathcal{P}_Q^W)^* \end{aligned}$$

We conclude that

$$S_{SN}(C_Q^{SW}, B_Q^{W,\infty})_{EE} = (D_1^S)^{-1} C_{Q,EE}^{SW} D_1^S.$$

and the result follows. The associativity is a direct consequence of the discussion in subsection 3.5.1.2. \square

A similar result holds for the action of the South half-strip associated to Q , denoted $B_Q^{S,\infty}$, on the South-West corner C_Q^{SW} (by gluing the half-strip at the East boundary of the corner). In the continuity of the notion of invariant half-strip up to morphism of section 3.5.1.7, these are the invariant property up to morphism of the corner C_Q^{SW} . This invariance property is purely two-dimensional: C_Q^{SW} is invariant (up to a morphism) under the actions in both directions, one by $B_Q^{W,\infty}$ and the other by $B_Q^{S,\infty}$. This is different from the one-dimensional invariant property of the half-strip under the "one-directional" action of Q , see Theorem 3.5.2. The morphism involved here is $\Phi^{SW} = (\Phi_{p,q}^{SW})_{(p,q) \in \mathbb{N}^2}$ where, for all p and $q \in \mathbb{N}^*$,

$$\begin{aligned} \Phi_{p,q}^{SW} : \text{End}(\mathcal{W}_W \oplus \mathcal{H}_1^p \oplus \mathcal{W}_S \oplus \mathcal{H}_2^q) &\rightarrow \text{End}(\mathcal{W}_W \oplus \mathcal{W}_S) \\ M &\mapsto \left(D_p^W \oplus D_q^S \right) M \left(D_p^W \oplus D_q^S \right)^{-1} \end{aligned} \quad (3.5.75)$$

where D_p^W and D_q^S are defined in (3.5.31). By convention we let $D_0^a = I_{\mathcal{W}_a}$ for $a \in \{S, W\}$. As it is given by conjugations, which respect Schur products, a similar proof than the one of Theorem 3.5.2 shows that Φ^{SW} is a South-West morphism in the sense of [Sim23]. Roughly speaking, it means that it respects the gluing products.

3.5.3 No Gaussian reference measure on half-lines for ROPE products

Various scalar products on the same spaces. The operadic structure of Gaussian weights on rectangles can be lifted to Schur products on quadratic forms in proposition 3.3.1 and, backwards, these quadratic forms define Gaussian densities by using the Lebesgue measure as a reference measure for all the products (among all the non-degenerate weights and regardless of the transverse size of the rectangles). In particular, all the South-North and West-East products can be interpreted as integration over the same σ -finite measured spaces \mathcal{H}_i^r where r is the length of the cut.

All the computations in sections 3.3 and 3.4 have then been done at the level of quadratic forms through Schur complement, without considering the associated Gaussian weights. There is a deep reason for this approach, which we explain now.

Remark 3.5.1 (no reference measure behind the ROPE products). There is no reference σ -finite measure ν_\bullet on half-line spaces \mathcal{W}_\bullet such that all the products between half-strips of arbitrary width can be written as an integral over this measure such as

$$\frac{\bar{\mathbf{e}}_{\beta, Q^{(E, q_2)}}}{\bar{\mathbf{e}}_{\beta, Q^{(E, q_1)}}} = \int_{\mathcal{W}_\bullet} \bar{\mathbf{e}}'_{\beta, Q^{(E, q_1)}}(x_S, u, x_{W_1}) \bar{\mathbf{e}}'_{\beta, Q^{(E, q_2)}}(u, x_N, x_{W_2}) d\nu(u)$$

where the $\bar{\mathbf{e}}'$ denote a suitable Radon-Nikodym change of Gaussian density.

The σ -finite requirement is needed in order to formulate associativities of these products in terms of Fubini's interversion theorem. There is a very simple justification for this remark. For half-strips and corners, all the vector spaces associated to half-line in the previous constructions are infinite-dimensional and of the form

$$\mathcal{W}_\bullet = \mathcal{H}_\bullet \otimes L^2([0, \pi], d\lambda) \tag{3.5.76}$$

due to the Fourier transform (the sine one or the folding of the exponential one); however it may be endowed with *various* scalar products depending on the surrounding geometric shapes. For example, on a single strip of width q , the quadratic parts SS and NN of $Q^{(E, q_i)}$ in (3.3.13) involves factors $(1 - \zeta_\theta^{-q})^{-1}$ in front of the quadratic terms $\hat{x}^S(\theta)$ ad $\hat{x}^N(\theta)$ (see (3.3.20)) and these factors normalize differently the scalar products on \mathcal{W}_E . If the half-line is the guillotine cut between two half-strips of width q_1 and q_2 , then the scalings $(1 - \zeta_\theta^{-q_i})^{-1}$ are added to form a new scalar product.

We are now left with a *single* vector space $\mathcal{W}_\bullet = \mathcal{H}_\bullet \otimes L^2([0, \pi], d\lambda)$ with *various* scalar products

$$\langle f|g \rangle_{Q_{q_1, q_2}} = \int_0^\pi \langle f(\theta)|Q_{q_1, q_2}(\theta)g(\theta) \rangle d\theta, \tag{3.5.77}$$

which depend on the guillotine partitions. From the standard construction of stochastic integral, it is easy to see that each space \mathcal{W}_\bullet endowed with a *given* scalar product of the previous type is the base Hilbert space for a Brownian motion and its stochastic integral through the Gaussian isometry induce by the stochastic Ito integrals

$$f \mapsto \int_0^\pi \langle dB_\theta|f(\theta) \rangle$$

where the covariance of B depends on the scalar product on \mathcal{W}_\bullet . The key point is now to observe that varying the scalar product Q_{q_1, q_2} — even just by scalar normalization factor — changes the value of the quadratic variation of the Brownian motion, which is an almost-sure constant quantity. Thus, there cannot be any reference measure behind the family of scalar products $Q_{q_1, q_2}(\cdot)$ since a Radon-Nykodym change of measure cannot affect the quadratic-variation.

Moreover, the absence of Lebesgue measure in infinite dimension cannot solve this problem by a infinite-mass but σ -finite reference measure.

Identities in law. A consequence of this remark is that the boundary weights g_R built out of the previous ROPE representation cannot be read out of suitable r.v. along half-lines outside the rectangles in the infinite-volume Gibbs measure. If this was the case, then g_R could be written as an integral over these variables of a suitable conditional expectation of the boundary variables w.r.t. to these half-line σ -algebras and hence the law of these half-line r.v. may provide a reference measure.

All the Gaussian weights built out of the quadratic forms in the ROPE on the various half-strip and corners thus correspond to identities in law for the boundary weights g_R and not to the identification of some relevant r.v. on the same probability space as the infinite-volume Gibbs measure. This situation is similar to the Central Limit Theorem, for which the convergence is only in law: the boundary weights g_R contains only some residual information on the laws of the boundary r.v. extracted asymptotically from the r.v. outside the rectangle R . This is intimately related to the scalings (3.3.26) chosen in the asymptotic regime of large rectangles at the end of section 3.3.

Clearly identifying the information outside the rectangle captured by the boundary weights g_R remains an open question, which goes beyond the single case of Gaussian processes and requires more examples.

3.5.4 Additional remarks on the probabilistic structure of the boundary weights g_R .

We present here both a description and an interpretation of the ROPE structure of boundary weights by the means of quadratic forms.

We have seen that there is no reference Gaussian measure due to the infinite dimension, however we can find a common Hilbert space in which we may embed all the quadratic forms. For each direction of the half-lines (South, West, North and East), we introduce the Hilbert spaces (3.5.76) with additional (measurable and bounded) families of Hermitian operators $(Q_\theta)_{\theta \in [0, \pi]}$ on the spaces \mathcal{H}_1 or \mathcal{H}_2 . We can then always define the square root operators $\sqrt{Q_\theta}$ acting on the spaces \mathcal{W}_a from (3.5.76), such that the inner product is given by:

$$\langle f|g \rangle_Q = \int_0^\pi \langle f(\theta)|Q_\theta g(\theta) \rangle_{\mathcal{H}} d\theta = \left\langle \sqrt{Q_\bullet} \cdot f \middle| \sqrt{Q_\bullet} \cdot g \right\rangle_{\mathcal{W}}$$

where the last inner product is the natural one on the spaces \mathcal{W} and the one in the integrand in the inner product of the corresponding space \mathcal{H}_i . The operators inside the integrand are defined as $(\sqrt{Q_\bullet} \cdot f)(\theta) = \sqrt{Q_\theta} f(\theta)$.

All the quadratic forms involved in half-strips present in the paper can then be embedded in a universal one associated to the Lebesgue measure on $[0, \pi]$ through the previous equation, which can be used as the universal structure in the ROPE representation of boundary weights; however this universality does not propagate to Gaussian weights by lack of Radon-Nikodym densities in infinite-dimension.

There may however exist an intermediate level between the Gaussian weights and the quadratic forms: the Fock space formalism. It encodes the L^2 spaces of Gaussian random variables by using only the initial Hilbert space structure and allows one to consider easily the $X_{\sqrt{Q_\bullet} \cdot f}$ by acting with quantized operators; moreover, it never needs the use of densities w.r.t. a common reference measures due to the non-commutative structure in which various distinct commutative structure can be embedded. The Fock space formalism seems better suited to encode the identities in law observed on the boundary weights.

This approach may deserve a deeper study and may have the advantage of applicability to determinantal or fermionic processes. We must then resort either to renormalization tricks or by a slight reinterpretation of the spaces and the products.

3.5.5 Conclusion

In this section we extended the Guill_2 algebra of quadratic forms \mathcal{Q}_\bullet to degenerated objects such as half-strips and corners. This algebraic structure is made of associative products given by Schur complements at the level of infinite dimensional quadratic forms. These new objects allow to define invariant boundary conditions for the Gaussian Markov field as defined in 3.2.1 on any domain R with the shape of a rectangle. Indeed, let

$$g(x_{\partial R}) = \exp\left(-\frac{1}{2}B_{\partial R}(x_{\partial R})\right)$$

be a non-normalized Gaussian boundary law with $B_{\partial R}$ a positive definite quadratic form on the boundary of R . With the help of $B_Q^{W,\infty}, B_Q^{E,\infty}, B_Q^{S,\infty}, B_Q^{N,\infty}$ respectively the West, East, South and North half-strip operators and $C_Q^{SW}, C_Q^{SE}, C_Q^{NE}, C_Q^{NW}$ respectively the South-West, South-East, North-East and North-West corner operators, we decompose the matrix $B_{\partial R}$ associated to the boundary quadratic form $B_{\partial R}$ in the following manner

$$B_{\partial R} = \begin{array}{|c|c|c|c|c|c|} \hline C_Q^{NW} & B_Q^{N,\infty} & B_Q^{N,\infty} & \dots & B_Q^{N,\infty} & C_Q^{NE} \\ \hline B_Q^{W,\infty} & \begin{array}{c} \vdots \\ R \\ \vdots \end{array} & & & & B_Q^{E,\infty} \\ \hline \vdots & & & & & \vdots \\ \hline B_Q^{W,\infty} & & & & & B_Q^{E,\infty} \\ \hline C_Q^{SW} & B_Q^{S,\infty} & B_Q^{S,\infty} & \dots & B_Q^{S,\infty} & C_Q^{SE} \\ \hline \end{array} \quad (3.5.78)$$

In this drawing, the operators we built are glued together, thanks to Schur products, all around the empty rectangle R . Moreover, the associativity and invariant (up to morphism) properties ensure that these boundary conditions are consistent under restriction. Since $B_{\partial R}$ characterizes the law of the boundary, the Gaussian boundary laws associated to the boundaries of any increasing sequence of rectangles are consistent. Thanks to Kolmogorov’s extension theorem we can build the infinite-volume Gibbs measure of the Gaussian field on the whole lattice \mathbb{Z}^2 , see Proposition 3.2.2. Nevertheless, by working in the space of quadratic forms, we lost any notion of partition function and eigenvalue.

How "minimal" are the objects constructed? As this boundary structure is only defined up to morphisms, we could legitimately ask ourselves whether the objects constructed in this manuscript are minimal. For instance, when $U = 0$ in (3.2.6), vertical and horizontal edges are not coupled anymore by the face operator Q . The Gaussian field is then only composed of independent horizontal and vertical one-dimensional Markov chains with same laws. As exposed in section 5.2.2 of [Sim23], and using the work of [Bod23] concerning one-dimensional Gaussian Markov processes, we can build the boundary quadratic form $B_{\partial R}$ only from finite dimensional objects: the invariant boundary conditions B_L and B_R associated to the one-dimensional dynamic generated by $Q_{[WE],[WE]} = Q_{[SN],[SN]}$. Compared to our construction, the dimension of boundary objects is drastically reduced in this latter. In general settings with $U \neq 0$, it would be interesting to study the minimality of the objects constructed in this manuscript.

3.5.6 Heuristic computations related to the eigenvalue

Following [Sim23], the partition function $Z_R^{\text{bw}}(Q; g_R)$ associated to the Gaussian model on a rectangle R , defined in 3.2.1, with invariant boundary conditions provided by half-strips and

corners built in this section, should be equal to $\Lambda_Q^{\text{Area}(R)}$. Thanks to section 3.4, we expect the eigenvalue Λ_Q to satisfy

$$\Lambda_Q = (2\pi)^{d_1+d_2} \exp \left(-\frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} \log \det \Psi_Q(e^{i\theta_1}, e^{i\theta_2}) d\theta_1 d\theta_2 \right). \quad (3.5.79)$$

see (3.4.5). As already mentioned, by dropping the space of Gaussian weights \mathcal{D}_\bullet and focusing on the quadratic forms \mathcal{Q}_\bullet in sections 3.4 and 3.5, we lost any notion of partition function. Indeed, quadratic forms on their own are not "weighted" anymore and they can't produce any eigenvalue nor partition function. This explains why, in theorems 3.5.1 and 3.5.3, no eigenvalue is involved in the invariant equation up to morphism, as opposed to definitions of [Sim23]. Nevertheless, thanks to section 3.5.3, we know that we cannot work in \mathcal{D}_\bullet as soon as we are dealing with degenerated objects such as half-strips and corners. This being said, to make the eigenvalue and, therefore, the partition function appear, we can work in the parameter space $\mathbb{R} \times \mathcal{Q}_\bullet$ rather than \mathcal{D}_\bullet . This requires to take into account the cocycles γ_{WE} and γ_{SN} as defined in the finite dimensional case in Proposition 3.3.1. These maps are related with the determinants of quadratic forms. To catch these values on infinite dimensional operators such as the half-strips, some precautions are needed. In this section, we show how the eigenvalue Λ_Q can be recovered under the non-rigorous computations of limit values of certain determinants.

A quick reminder of the one-dimensional case. Before catching the eigenvalue in the two-dimensional case, we start by looking at the one-dimensional case studied in [Bod23]. Under the notations introduced by the reminder in section 3.2.1.5, the left invariant equation for a one-dimensional Gaussian Markov field with weight given by Q^{1D} is

$$\int_{\mathcal{H}} \exp \left(-\frac{1}{2} x^* B_L x \right) \exp \left(-\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^* Q^{1D} \begin{pmatrix} x \\ y \end{pmatrix} \right) dx = \Lambda_{Q^{1D}} \exp \left(-\frac{1}{2} y^* B_L y \right)$$

where the eigenvalue $\Lambda_{Q^{1D}} = (2\pi)^d \det(B_L + Q_{LL}^{1D})^{-1}$. Roughly speaking, the number $\Lambda_{Q^{1D}}$ gives the "mass" of the Gaussian weight relatively to its invariant boundary condition. We denote this weight $e_{1,Q^{1D}}^{1D}$. If we wished to prevent $e_{1,Q^{1D}}^{1D}$ to contribute to the partition function of the system, we should renormalize this latter by dividing by $\Lambda_{Q^{1D}}$. It means that, relatively to its invariant boundary, we force the Gaussian weight $e_{1,Q^{1D}}^{1D}$ to be of mass 1 by considering the renormalized weight $e_{1,Q^{1D}}^{1D} / \Lambda_{Q^{1D}}$.

The two-dimensional case. We now make appear Λ_Q from the action of Q on the West half-strip $B_Q^{W,\infty}$ defined in (3.5.29)

$$\boxed{B_Q^{W,\infty} \quad Q}$$

by considering non-rigorous determinants limits emerging from cocycles γ_{SN} and γ_{WE} . The Schur formula for determinant, see (3.4.14), gives

$$\det \Psi_Q(e^{i\theta_1}, e^{i\theta_2}) = \det \phi_Q^{\text{WE}}(e^{i\theta_1}) \det \left(\Psi_Q(e^{i\theta_1}, e^{i\theta_2}) / \phi_Q^{\text{WE}}(e^{i\theta_1}) \right)$$

thanks to the definition of Ψ_Q , see (3.2.8). Then,

$$\Lambda_Q = \Lambda_{Q_{[\text{WE}],[\text{WE}]}}^{1D} \Lambda_{Q/\text{WE}} \quad (3.5.80)$$

where $\Lambda_{Q_{[\text{WE}],[\text{WE}]}}^{1D}$ is the one-dimensional eigenvalue associated to $Q_{[\text{WE}],[\text{WE}]}$ and

$$\Lambda_{Q/\text{WE}} = (2\pi)^{d_1} \exp \left(-\frac{1}{(2\pi)^2} \int_{[0,2\pi]^2} \log \det \left(\Psi_Q(e^{i\theta_1}, e^{i\theta_2}) / \phi_Q^{\text{WE}}(e^{i\theta_1}) \right) d\theta_1 d\theta_2 \right)$$

where we recall that

$$\left(\Psi_Q(e^{i\theta_1}, e^{i\theta_2}) / \phi_Q^{\text{WE}}(e^{i\theta_1}) \right) = \phi_Q^{\text{SN}}(e^{i\theta_2}) - (\Psi_Q(e^{i\theta_1}, e^{i\theta_2}))_{1,2} (\phi_Q^{\text{WE}}(e^{i\theta_1}))^{-1} (\Psi_Q(e^{i\theta_1}, e^{i\theta_2}))_{2,1}.$$

We check that $\Lambda_{Q_{[\text{WE}],[\text{WE}]}}^{1D} = (2\pi)^{d_2} \det(B_{Q,\text{EE}}^{\text{W},\infty} + Q_{\text{WW}})^{-1} = \gamma_{\text{WE}}(B_Q^{\text{W},\infty}, Q)$ is directly obtained after the gluing of Q . It comes from the erasing of the horizontal edge. Nevertheless, the term $\Lambda_{Q/\text{WE}}$ is still missing after the action of Q . We now show how it is related to the shift induced by the morphism $\Phi_{1,1}^{\text{W}}$ of theorem 3.5.1. To do so, we need to understand the "mass" carried by the half-strip. When computing the boundary condition as in (3.1.2), the half-strip will eventually be glued to a neighboring corner. Therefore, we expect this gluing to contribute in the partition function through

$$\gamma_{\text{SN}}(C_Q^{\text{SW}}, B_Q^{\text{W},\infty}) = (2\pi)^{d_1 \times \infty} \det(C_{Q,\text{NN}}^{\text{SW}} + B_{Q,\text{SS}}^{\text{W},\infty})^{-1}. \quad (3.5.81)$$

This is the "mass" of the West half-strip relatively to the South-West corner. This expression is of course ill-defined, firstly because of $(2\pi)^{d_1 \times \infty}$ and secondly because we take the determinant of an operator without showing that it is well-posed. Therefore, to prevent this term from appearing in the partition function, we wish to renormalize the half-strip by this latter. We cannot perform such a thing directly. Instead, we renormalize a truncated version of the West half-strip before letting the order of the truncation go to infinity. Consider a truncation at order $n \in \mathbb{N}^*$ of $B_Q^{\text{W},\infty}$. We forget about the Eastern edge of the half-strip and we focus on the one-dimensional dynamic generated by the truncation at order n of $B_{Q,[\text{SN}],[\text{SN}]}^{\text{W},\infty}$ given by

$$\mathbb{F}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}}) = \mathbb{T}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}}) - \mathbb{H}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})$$

where $\mathbb{T}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})$ and $\mathbb{H}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})$ are the truncated Toeplitz and Hankel operator at order n . The one-dimensional dynamic generated by this quadratic form is given by a one-dimensional Gaussian weight $e_{1,\mathbb{F}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D}$. From [Bod23], provided that the assumptions (1D) are fulfilled, we can compute explicitly its invariant boundary conditions and their associated eigenvalue is

$$\Lambda_{\mathbb{F}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D} = (2\pi)^{nd_1} \exp \left(-\frac{1}{2\pi} \int_0^{2\pi} \log \det \phi_{\mathbb{F}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D}(e^{i\theta}) d\theta \right),$$

where ϕ^{1D} is defined in section 3.2.1.5. This is the mass carried by the truncated West half-strip. For all n , we renormalize the truncated half-strip of order n by its mass $\Lambda_{\mathbb{F}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D}$.

We will study the asymptotic value of this renormalization. Thanks to linearity of ϕ^{1D} we remark that

$$\phi_{\mathbb{F}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D} = \phi_{\mathbb{T}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D} - \phi_{\mathbb{H}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D} = \phi_{\mathbb{T}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D} \left(I_{\mathcal{H}_1} - (\phi_{\mathbb{T}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D})^{-1} \phi_{\mathbb{H}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D} \right)$$

(on S^1 so we can take the inverse thanks to [Bod23]) so that,

$$\begin{aligned} \Lambda_{\mathbb{F}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D} &= \Lambda_{\mathbb{T}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D} \exp \left(-\frac{1}{2\pi} \int_0^{2\pi} \log \det \left(I_{\mathcal{H}_1} - (\phi_{\mathbb{T}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D}(e^{i\theta}))^{-1} \phi_{\mathbb{H}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D}(e^{i\theta}) \right) d\theta \right) \\ &= \Lambda_{\mathbb{T}_n^{\text{W}}(\widehat{B}_Q^{\text{WE}})}^{1D} \Lambda_n^{\text{perturbation}}. \end{aligned}$$

In this formula we observe the perturbation of the Hankel part on the "stationary" part given by the Toeplitz operator. The term $\det \left(I_{\mathcal{H}_1} - (\phi_{\mathbb{T}_n^W(\widehat{B}_Q^{\text{WE}})}^{1D}(e^{i\theta}))^{-1} \phi_{\mathbb{H}_n^W(\widehat{B}_Q^{\text{WE}})}^{1D}(e^{i\theta}) \right)$ makes us think of Fredholm determinant and we expect that it converges as $n \rightarrow \infty$. Therefore, under the proper justification, we expect $\Lambda_n^{\text{perturbation}}$ to converge as well. At the level of quadratic forms, when acting on the truncated half-strip of order n with Q , we obtain the truncated half-strip at order $(n+1)$ (thanks to exact same computations as the ones used for proving theorem 3.5.1). Now, if the truncated half-strips are renormalized by $\Lambda_{\mathbb{F}_n^W(\widehat{B}_Q^{\text{WE}})}^{1D}$ for all n , when acting on the renormalized truncated half-strip of order n with Q , we obtain the renormalized truncated half-strip of order $(n+1)$ times the following fraction

$$\frac{\Lambda_{\mathbb{F}_{n+1}^W(\widehat{B}_Q^{\text{WE}})}^{1D}}{\Lambda_{\mathbb{F}_n^W(\widehat{B}_Q^{\text{WE}})}^{1D}} = \frac{\Lambda_{\mathbb{T}_{n+1}^W(\widehat{B}_Q^{\text{WE}})}^{1D}}{\Lambda_{\mathbb{T}_n^W(\widehat{B}_Q^{\text{WE}})}^{1D}} \times \frac{\Lambda_{n+1}^{\text{perturbation}}}{\Lambda_n^{\text{perturbation}}}.$$

As $\Lambda_n^{\text{perturbation}}$ should converge from the previous discussion, this limit of this fraction is given by the limit of $(\Lambda_{\mathbb{T}_{n+1}^W(\widehat{B}_Q^{\text{WE}})}^{1D} / \Lambda_{\mathbb{T}_n^W(\widehat{B}_Q^{\text{WE}})}^{1D})$. Moreover, for all $n \in \mathbb{N}$, a direct computation from the definition of $\widehat{B}_Q^{\text{WE}}$ in (3.4.24) gives for all $\theta_2 \in [0, 2\pi]$,

$$\begin{aligned} \phi_{\mathbb{T}_n^W(\widehat{B}_Q^{\text{WE}})}^{1D}(e^{i\theta_2}) &= \mathbb{T}_{1,n}^W \left(\phi_Q^{\text{SN}}(e^{i\theta_2}) - (\Psi_Q(\bullet, e^{i\theta_2}))_{1,2} (\phi_Q^{\text{WE}}(\bullet))^{-1} (\Psi_Q(\bullet, e^{i\theta_2}))_{2,1} \right) \\ &= \mathbb{T}_{1,n}^W \left(\Psi_Q(\bullet, e^{i\theta_2}) / \phi_Q^{\text{WE}}(\bullet) \right) \end{aligned}$$

where $\mathbb{T}_{1,n}^W$ is the truncated partial Toeplitz operator at order n , see (3.5.45) for the definition of partial Toeplitz operator. Thus, a direct computation shows

$$\begin{aligned} \frac{\Lambda_{\mathbb{T}_{n+1}^W(\widehat{B}_Q^{\text{WE}})}^{1D}}{\Lambda_{\mathbb{T}_n^W(\widehat{B}_Q^{\text{WE}})}^{1D}} &= (2\pi)^{d_1} \exp \left(-\frac{1}{2\pi} \int_0^{2\pi} \log \frac{\det \phi_{\mathbb{T}_{n+1}^W(\widehat{B}_Q^{\text{WE}})}^{1D}(e^{i\theta_2})}{\det \phi_{\mathbb{T}_n^W(\widehat{B}_Q^{\text{WE}})}^{1D}(e^{i\theta_2})} d\theta_2 \right) \\ &= (2\pi)^{d_1} \exp \left(-\frac{1}{2\pi} \int_0^{2\pi} \log \frac{\det \mathbb{T}_{1,n+1}^W \left(\Psi_Q(\bullet, e^{i\theta_2}) / \phi_Q^{\text{WE}}(\bullet) \right)}{\det \mathbb{T}_{1,n}^W \left(\Psi_Q(\bullet, e^{i\theta_2}) / \phi_Q^{\text{WE}}(\bullet) \right)} d\theta_2 \right). \end{aligned}$$

Now, using Szegő's limit formula (under the good assumptions), we get for all $\theta_2 \in [0, 2\pi]$,

$$\lim_{n \rightarrow \infty} \frac{\det \mathbb{T}_{1,n+1}^W \left(\Psi_Q(\bullet, e^{i\theta_2}) / \phi_Q^{\text{WE}}(\bullet) \right)}{\det \mathbb{T}_{1,n}^W \left(\Psi_Q(\bullet, e^{i\theta_2}) / \phi_Q^{\text{WE}}(\bullet) \right)} = \exp \left(\int_0^{2\pi} \log \det \left(\Psi_Q(e^{i\theta_1}, e^{i\theta_2}) / \phi_Q^{\text{WE}}(e^{i\theta_1}) \right) \frac{d\theta_1}{2\pi} \right)$$

so that

$$\lim_{n \rightarrow \infty} \frac{\Lambda_{\mathbb{T}_{n+1}^W(\widehat{B}_Q^{\text{WE}})}^{1D}}{\Lambda_{\mathbb{T}_n^W(\widehat{B}_Q^{\text{WE}})}^{1D}} = \Lambda_{Q/\text{WE}}.$$

This non-rigorous approach shows how the eigenvalue Λ_Q comes as a limit from the proper renormalization of the truncated half-strips operators. From this computation we expect to build a morphism $\tilde{\Phi}^W$ on the extended parameter space $\mathbb{R} \times \mathcal{Q}_\bullet$ which acts like Φ^W in theorem 3.5.1 and 3.5.2 at the level of quadratic forms but also takes in count these mass phenomena by a rigorous analytical control of Fredholm determinants. Similarly, we could build similar morphisms for East, South and North half-strips (for corners as well). This would allow us to give the complete definition of invariant boundary up to morphism, as defined in [Sim23]. Under these boundary conditions we would therefore obtain the desired equality $Z_R^{\text{bw}}(Q; g_R) = \Lambda_Q^{\text{Area}(R)}$.

Bibliography

- [Bax16] R. J. Baxter. *Exactly solved models in statistical mechanics*. Elsevier, 2016.
- [BDTR17] C. Boutillier, B. De Tilière, and K. Raschel. “The Z-invariant massive Laplacian on isoradial graphs”. In: *Inventiones mathematicae* 208 (2017), pp. 109–189.
- [BE99] E. L. Basor and T. Ehrhardt. “On a class of Toeplitz+ Hankel operators”. In: *New York J. Math* 5.1 (1999), p. 16.
- [Bod23] E. Bodiot. “One-dimensional discrete Gaussian Markov processes: Harmonic decomposition of invariant boundary conditions”. In: *submitted* (2023). URL: <https://arxiv.org/abs/2305.18892>.
- [BP21] N. Berestycki and E. Powell. “Gaussian Free Field, Liouville Quantum Gravity and Gaussian multiplicative chaos”. In: *Lecture notes* (2021).
- [BS13] A. Böttcher and B. Silbermann. *Analysis of Toeplitz operators*. Springer Science & Business Media, 2013.
- [Car04] J. Cardy. “Boundary conformal field theory”. In: *arXiv preprint hep-th/0411189* (2004).
- [CG13] K. F. Clancey and I. Gohberg. *Factorization of matrix functions and singular integral operators*. Vol. 3. Birkhäuser, 2013.
- [DKRV16] F. David, A. Kupiainen, R. Rhodes, and V. Vargas. “Liouville quantum gravity on the Riemann sphere”. In: *Communications in Mathematical Physics* 342 (2016), pp. 869–907.
- [FH20] P. K. Friz and M. Hairer. *A course on rough paths*. Springer, 2020.
- [FV17] S. Friedli and Y. Velenik. *Statistical mechanics of lattice systems: a concrete mathematical introduction*. Cambridge University Press, 2017.
- [Geo11] H.-O. Georgii. “Gibbs measures and phase transitions”. In: *Gibbs Measures and Phase Transitions*. de Gruyter, 2011.
- [Gin15] G. Ginot. “Notes on factorization algebras, factorization homology and applications”. In: *Mathematical aspects of quantum field theories*. Springer, 2015, pp. 429–552.
- [GLNO22] C. Giusti, D. Lee, V. Nanda, and H. Oberhauser. “A topological approach to mapping space signatures”. In: *arXiv preprint arXiv:2202.00491* (2022).
- [GTZ10] G. Ginot, T. Tradler, and M. Zeinalian. “A Chen model for mapping spaces and the surface product”. In: *Annales scientifiques de l’Ecole normale supérieure*. Vol. 43. 5. 2010, pp. 811–881.
- [Jan97] S. Janson. *Gaussian hilbert spaces*. 129. Cambridge university press, 1997.
- [LV12] J.-L. Loday and B. Vallette. *Algebraic operads*. Vol. 346. Springer, 2012.
- [MSS02] M. Markl, S. Shnider, and J. D. Stasheff. *Operads in algebra, topology and physics*. Vol. 96. Mathematical Surveys and Monographs. American Mathematical Society, 2002.
- [MW17] J.-C. Mourrat and H. Weber. “The Dynamic Φ_3^4 Model Comes Down from Infinity”. In: *Communications in Mathematical Physics* 356 (2017), pp. 673–753.
- [Pes18] M. E. Peskin. *An introduction to quantum field theory*. CRC press, 2018.
- [RH05] H. Rue and L. Held. *Gaussian Markov random fields: theory and applications*. CRC press, 2005.

-
- [She07] S. Sheffield. “Gaussian free fields for mathematicians”. In: *Probability theory and related fields* 139.3-4 (2007), pp. 521–541.
- [Sim23] D. Simon. “Operadic approach to Markov processes with boundaries on the square lattice in dimension 2 and larger”. In: *submitted* (2023). URL: <https://arxiv.org/abs/2306.11126>.
- [Sze15] G. Szegő. “Ein Grenzwertsatz über die Toeplitzchen Determinanten einer reellen positiven Funktion”. In: *Mathematische Annalen* 76.4 (1915), pp. 490–503.
- [Tal22] M. Talagrand. *What Is a Quantum Field Theory?* Cambridge University Press, 2022.
- [Wid74] H. Widom. “Asymptotic behavior of block Toeplitz matrices and determinants”. In: *Advances in Mathematics* 13.3 (1974), pp. 284–322.
- [WP20] W. Werner and E. Powell. “Lecture notes on the Gaussian free field”. In: *arXiv preprint arXiv:2004.04720* (2020).
- [Yau16] D. Yau. *Colored Operads*. Graduate Studies in Mathematics. American Mathematical Society, 2016. URL: <https://books.google.fr/books?id=bd21CwAAQBAJ>.
- [Zha06] F. Zhang. *The Schur complement and its applications*. Vol. 4. Springer Science & Business Media, 2006.