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Batoul Tarhini

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**Oriented Paths in Digraphs  
and  
the  $S$ -packing Coloring of Subcubic Graphs**

PH.D THESIS

**BATOUL TARHINI**

**Advisor:** Amine El Sahili, Olivier Togni

**Co-advisor:** Maidoun Mortada

France - Lebanon, 2023

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Université de Bourgogne - Université Libanaise  
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Approved by

December 18, 2023.

*(Signature)*

*(Signature)*

*(Signature)*

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Amine El Sahili, Olivier Togni

.....  
Prof. Hervé Hocquard

.....  
Prof. Didem Gözüpek

France - Lebanon, 2023



# Abstract

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*“Everything is theoretically impossible, until it is done.”*

– ROBERT A. HEINLEIN –

This PhD thesis is structured into two principal parts: Part I delves into the existence of oriented paths in digraphs, aiming to establish a connection between a digraph’s chromatic number and the existence of specific oriented paths within it as subdigraphs. More specifically, our objective is to study the existence of an integer  $k$  such that any digraph with a chromatic number  $k$ , contains a copy of a given oriented path with three blocks as its subdigraph. Our journey towards achieving this objective relies significantly on fundamental concepts, including, induction on the order of a given digraph, final forests, leveling techniques, and strategic digraph decomposition methods. Part II takes us into the wide realm of  $S$ -packing coloring in graphs. Here, our focus is centered on an intriguing conjecture proposed by Brešar et al., which pertains to the packing coloring of subdivisions of subcubic graphs. Our desired aim is to provide a confirmation of this conjecture for a specific class of subcubic graphs, and to address the unresolved issues raised within this subject matter.

Before proceeding into our research topics of interest, we commence this thesis with Chapter 1, where we present fundamental definitions, standard notations, and preliminary results well-established in the field of graph theory. These concepts are closely aligned with our research subjects, and the purpose of this chapter is to provide the reader with a foundational understanding of the terminology and principles that will be employed throughout this research report. As a preliminary exploration in Part I, Chapter 2 provides a brief historical overview of the research conducted by graph theorists concerning the existence of oriented trees and in particular paths in digraphs. This chapter encompasses numerous influential findings that have served to inspire not only our own work but also a multitude of mathematicians. These contributions have stimulated new ideas and approaches while presenting us with intriguing problems to address. The problem that we delve into in Part I, which is considered a prominent problem in graph theory, can be reframed as a fundamental question: Does there exist an integer  $k$  such that every member of a specified family  $\mathcal{H}$  of digraphs can be found as a subgraph in every  $k$ -chromatic digraph? In this context, a ‘copy’ of a digraph  $H$  in another digraph  $D$  is defined by an injective function  $c : V(H) \rightarrow V(D)$  such that for every  $(x, y) \in E(H)$ , the pair  $(c(x), c(y))$  belongs to  $E(D)$ . This concept of copy provides a way to examine the existence of a given digraph

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within a larger one. Digraphs contained in any  $n$ -chromatic digraph are called  $n$ -universal. It is important to highlight that the concept of  $n$ -universal graphs will be central to the discussions in Part I of this thesis. Our focus in this part is on studying the universality of oriented paths which are special cases of oriented trees aiming for an exploration of partial solutions for the previously posed problem. Specifically, we aim to determine the smallest integer  $k$  such that any  $k$ -chromatic digraph contains a copy of a given oriented path of order  $n$ . To embark on this quest, it is essential to start by revisiting a fundamental concept: any path of order  $n$  can be fundamentally regarded as a tree of order  $n$ . Consequently, it can be embedded within any digraph with a chromatic number of at least  $(n - 1)^2$ , as originally established by Burr [20]. As exhibited throughout Chapter 2, we consider in the first topic, two main conjectures. Burr's conjecture concerning the universality of oriented trees, and El Sahili's conjecture concerning the universality of oriented paths. Burr [20] conjectured that an oriented tree of order  $n$  is  $(2n - 2)$ -universal. The most well-known general bound for the chromatic number of an oriented tree of order  $n$ , which encompasses oriented paths, was formulated by Addario-Berry et al. [1], resulting in  $\frac{n^2}{2} - \frac{n}{2} + 1$ . However, for specific oriented paths, more favorable bounds for the chromatic number have been achieved. The most famous result in this context is the Gallai-Hasse-Roy-Vitaver theorem [37, 47, 61, 69], which specifically addresses directed paths. It states that every directed path of order  $n$  is  $n$ -universal. Another well-explored category of oriented trees is that of two-block paths, where a block is defined as a maximal directed sub-path. Addario et al. [2] have demonstrated that any path of order  $n$  with precisely two blocks is  $n$ -universal. Inspired by this research on the chromatic number of digraphs containing a directed path or path with two blocks, El Sahili [26] conjectured that any oriented path of order  $n$  is  $n$ -universal. In Chapter 3, we study the existence of the path  $P(k, 1, l)$  in digraphs. Using the concept of final forests, and considering a partition of the digraph into two subdigraphs of equal chromatic numbers, we were able to prove the existence of the path  $P(k, 1, l)$  in any  $(2n + 2)$ -chromatic digraph. Introducing a specific condition to the digraph reduces the complexity of the problem, making the results more specific and interesting. Fortunately, when dealing with Hamiltonian digraphs, we have successfully confirmed El Sahili's conjecture for this particular class of digraphs. More clearly, we have established the existence of any path  $P(k, 1, l)$  of order  $n$  in any  $n$ -chromatic Hamiltonian digraph.

Furthermore, depending on the result we achieved on this specific class of digraphs, we proved the correctness of El Sahili's conjecture on a more general class of digraphs which is digraphs containing a Hamiltonian directed path.

In Chapter 4, we move forward to consider the general case of oriented paths with three blocks. We introduce a new technique which is represented by a decomposition of the digraph into subdigraphs defined by a series of successive operations applied to the digraph relying on the famous theorem of Roy [61] which establishes the existence of a directed path of order  $n$  in any  $n$ -chromatic digraph. This technique has proven to be instrumental in establishing a new linear bound for the chromatic number of digraphs that lack an oriented path with three blocks. By categorizing the problem into three cases based on the lengths of the three blocks, we established a linear bound for each case. It is worth noting that the reached bounds are the best achieved,

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so that improving all the previous bounds. Even more intriguing is that this marks the first occurrence of the bound reaching  $3n$ . Moreover, the new technique has served us to improve the chromatic number's bound of digraphs containing any path  $P(k, 1, l)$  of order  $n$ . In fact, we improved the bound from  $2n + 2$  into  $2n - 3$ . What makes the new achieved bound interesting is the ability to confirm Burr's conjecture for this special type of paths which is a particular case of oriented trees.

Moving on to Part II, we begin with Chapter 5 in which we provide a modest panoramic view on the problem of  $S$ -packing coloring and the approaches that have contributed over the years to this problem and the conjectures proposed in this topic. The concept of the packing chromatic number, initially referred to as the "broadcast chromatic number," was pioneered by Goddard, S.M. Hedetniemi, S.T. Hedetniemi, Harris, and Rall [42]. This concept emerged from the potential applications it held in addressing broadcast assignment problems. Subsequently, in all the works that followed the seminal paper, the terms "packing coloring" and "packing chromatic number" became the used terminology, as originally coined by Brešar, Klavžar, and Rall [16]. Given the extensive body of literature dedicated to the packing chromatic number, and its sustained significance, we believe that it is necessary to compile the main results and elucidate unresolved questions related to our work in this field in the form of a comprehensive survey in Chapter 5. We shed light on the conjecture proposed by Brešar et al. [18] concerning the packing coloring of subcubic graphs which states for any subcubic graph  $G$ , the packing chromatic number of its subdivision graph  $S(G)$  is bounded from above by 5. We present some of the results that establish connections between  $S$ -packing colorings of a graph  $G$  and those of its subdivision graph  $S(G)$  and the approaches done by graph theorists in their examination for this conjecture. Notably, all the attempts to confirm the conjecture have been confined to specific subclasses of subcubic graphs.

In the end, Chapter 6 involves our contribution to the studied conjecture, in which it is affirmed for a class of subcubic graphs. We deal with cubic Halin graphs. We will mention in Chapter 5 an observation for Gastineau and Togni which states that if a graph  $G$  is  $(1, 1, 2, 2)$ -packing colorable, then the chromatic number of its subdivision graph  $S(G)$  is at most 5, and hence it satisfies the conjecture. Depending on this observation, and in order to prove the correctness of the conjecture for the class we worked on, we studied its  $S$ -packing coloring aiming to prove that it admits a  $(1, 1, 2, 2)$ - packing coloring. We proved that a cubic Halin graph is  $(1, 1, 2, 3)$ -packing colorable, which is stronger than  $(1, 1, 2, 2)$ -packing coloring, and so we confirm the conjecture for this class. Moreover, Gastineau and Togni [40], after proving that every subcubic graph is  $(1, 2, 2, 2, 2, 2)$ -packing colorable, have posed an open problem on whether every subcubic graph is  $(1, 2, 2, 2, 2, 2)$ -packing colorable. We answer this question in affirmative in the particular class we worked on; we proved that cubic Halin graphs are  $(1, 2, 2, 2, 2, 2)$ -packing colorable.

**Keywords.** Subdivision, oriented cycle, oriented path, three-blocks path, chromatic number, final out-tree,  $S$ -packing coloring, packing chromatic number, subcubic graph, Halin graph.





# Résumé

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*“Tout est théoriquement impossible jusqu’à ce que ce soit fait.”*

– ROBERT A. HEINLEIN –

Cette thèse de doctorat est structurée en deux parties principales : La Partie I se penche sur l’existence de chemins orientés dans les digraphes, dans le but d’établir un lien entre le nombre chromatique d’un digraphe et l’existence de chemins orientés spécifiques en tant que sous-digraphes. Plus précisément, notre objectif est d’étudier l’existence d’un entier  $k$  tel que tout digraphe avec un nombre chromatique de  $k$  contient une copie d’un chemin orienté donné avec trois blocs en tant que sous-digraphe. Notre démarche pour atteindre cet objectif repose fortement sur des concepts fondamentaux, notamment l’induction sur l’ordre d’un digraphe donné, les forêts finales, les techniques de nivellement et les méthodes de décomposition stratégique de digraphes.

La Partie II nous emmène dans le vaste domaine de la coloration par  $S$ -packing dans les graphes. Ici, notre attention est centrée sur une conjecture intrigante proposée par Brešar et al., qui concerne la coloration par packing des subdivisions de graphes subcubiques. Notre objectif est de confirmer cette conjecture pour une classe spécifique de graphes subcubiques, et de traiter les questions non résolues soulevées dans ce domaine.

Avant de nous plonger dans nos sujets de recherche, nous commençons cette thèse par le Chapitre 1, où nous présentons des définitions fondamentales, des notations standard et des résultats préliminaires bien établis dans le domaine de la théorie des graphes. Ces concepts sont étroitement liés à nos sujets de recherche, et le but de ce chapitre est de fournir au lecteur une compréhension fondamentale de la terminologie et des principes qui seront utilisés tout au long de ce rapport de recherche.

En tant qu’exploration préliminaire dans la Partie I, le Chapitre 2 offre un bref aperçu historique des recherches menées par les théoriciens des graphes concernant l’existence d’arbres orientés et en particulier de chemins orientés dans les graphes orientés. Ce chapitre englobe de nombreuses découvertes influentes qui ont non seulement inspiré notre propre travail, mais aussi de nombreux mathématiciens. Ces contributions ont stimulé de nouvelles idées et approches tout en nous présentant des problèmes intrigants à résoudre.

Le problème que nous abordons dans la Partie I, considéré comme un problème important en théorie des graphes, peut être reformulé en une question fondamentale : existe-t-il un entier  $k$

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tel que chaque membre d'une famille spécifiée  $\mathcal{H}$  de graphes orientés puisse être trouvé comme sous-graphe dans tout graphe orienté  $k$ -chromatique ? Dans ce contexte, une 'copie' d'un graphe orienté  $H$  dans un autre graphe orienté  $D$  est définie par une fonction injective  $c : V(H) \rightarrow V(D)$  telle que pour tout  $(x, y) \in E(H)$ , la paire  $(c(x), c(y))$  appartienne à  $E(D)$ . Ce concept de copie offre une manière d'examiner l'existence d'un graphe donné au sein d'un autre plus grand. Les graphes orientés contenus dans tout graphe  $n$ -chromatique sont appelés  $n$ -universels. Il est important de souligner que le concept de graphes  $n$ -universels sera central dans les discussions de la Partie I de cette thèse. Notre objectif dans cette partie est d'étudier l'universalité des chemins orientés, qui sont des cas particuliers d'arbres orientés, dans le but d'explorer des solutions partielles au problème précédemment posé. Plus précisément, nous visons à déterminer le plus petit entier  $k$  tel que tout graphe orienté  $k$ -chromatique contienne une copie d'un chemin orienté donné d'ordre  $n$ . Pour entreprendre cette quête, il est essentiel de commencer par revisiter un concept fondamental : tout chemin d'ordre  $n$  peut être fondamentalement considéré comme un arbre d'ordre  $n$ . Par conséquent, il peut être incorporé dans tout graphe orienté avec un nombre chromatique d'au moins  $(n - 1)^2$ , comme l'a établi Burr [20]. Comme le montre le Chapitre 2, nous considérons dans le premier sujet deux conjectures principales. La conjecture de Burr concernant l'universalité des arbres orientés, et la conjecture d'El Sahili concernant l'universalité des chemins orientés. Burr [20] a conjecturé qu'un arbre orienté d'ordre  $n$  est  $(2n - 2)$ -universel. La borne générale la plus connue pour le nombre chromatique d'un arbre orienté d'ordre  $n$ , qui englobe les chemins orientés, a été formulée par Addario-Berry et al. [1], et vaut  $\frac{n^2}{2} - \frac{n}{2} + 1$ . Cependant, pour des chemins orientés spécifiques, des bornes plus favorables pour le nombre chromatique ont été obtenues. Le résultat le plus célèbre dans ce contexte est le théorème de Gallai-Hasse-Roy-Vitaver [37, 47, 61, 69], qui concerne spécifiquement les chemins dirigés. Il énonce que tout chemin dirigé d'ordre  $n$  est  $n$ -universel. Une autre catégorie bien explorée d'arbres orientés est celle des chemins à deux blocs, où un bloc est défini comme un sous-chemin dirigé maximal. Addario et al. [2] ont démontré que tout chemin d'ordre  $n$  avec précisément deux blocs est  $n$ - universel. Inspirée par ces recherches sur le nombre chromatique des graphes orientés contenant un chemin dirigé ou un chemin à deux blocs, El Sahili [26] a conjecturé que tout chemin orienté d'ordre  $n$  est  $n$ - universel.

Dans le Chapitre 3, nous étudions l'existence du chemin  $P(k, 1, l)$  dans les graphes orientés. En utilisant le concept de forêts finales, et en considérant une partition du graphe orienté en deux sous-graphes de même nombre chromatique, nous avons pu prouver l'existence du chemin  $P(k, 1, l)$  dans tout graphe orienté  $(2n + 2)$ -chromatique. En introduisant une condition spécifique au graphe orienté, nous réduisons la complexité du problème, rendant les résultats plus intéressants. Heureusement, lorsqu'il s'agit de graphes orientés hamiltoniens, nous avons confirmé avec succès la conjecture d'El Sahili pour cette classe particulière de graphes orientés. Plus précisément, nous avons établi l'existence de tout chemin  $P(k, 1, l)$  d'ordre  $n$  dans tout graphe orienté hamiltonien  $n$ -chromatique.

De plus, en fonction du résultat que nous avons obtenu sur cette classe spécifique de graphes orientés, nous avons prouvé la véracité de la conjecture d'El Sahili pour une classe de graphes orientés plus générale, à savoir les graphes orientés contenant un chemin dirigé hamiltonien.

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Dans le Chapitre 4, nous passons à l'étude du cas général des chemins orientés avec trois blocs. Nous introduisons une nouvelle technique qui consiste en une décomposition du graphe orienté en sous-graphes définis par une série d'opérations successives appliquées au graphe orienté, en se basant sur le célèbre théorème de Roy [61] qui établit l'existence d'un chemin dirigé d'ordre  $n$  dans tout graphe orienté  $n$ -chromatique. Cette technique s'est révélée essentielle pour établir une nouvelle borne linéaire pour le nombre chromatique des graphes orientés qui ne contiennent pas de chemin orienté avec trois blocs. En catégorisant le problème en trois cas en fonction de l'ordre des longueurs des trois blocs, nous établissons une borne linéaire pour chaque cas. Il est à noter que les bornes obtenues sont les meilleures réalisées, surpassant ainsi toutes les bornes précédentes. Plus intrigant encore, il s'agit de la première occurrence de la borne atteignant  $3n$ . De plus, la nouvelle technique nous a permis d'améliorer la borne du nombre chromatique des graphes orientés contenant un chemin  $P(k, 1, l)$  quelconque d'ordre  $n$ . En effet, nous avons amélioré la borne de  $2n + 2$  à  $2n - 3$ . Ce qui rend intéressante cette nouvelle borne c'est qu'elle permet de confirmer la conjecture de Burr pour ce type de chemins, qui est un cas particulier des arbres orientés.

En passant à la Partie II, nous commençons par le Chapitre 5 dans lequel nous présentons une vue d'ensemble modeste du problème de la coloration  $S$ -packing et des approches qui ont contribué au fil des ans à ce problème ainsi qu'aux conjectures proposées dans ce domaine. Le concept du nombre chromatique de packing, initialement appelé "nombre chromatique de diffusion", a été introduit par Goddard, S.M. Hedetniemi, S.T. Hedetniemi, Harris et Rall [42]. Ce concept est né des applications potentielles qu'il pouvait avoir pour résoudre des problèmes d'attribution de diffusion. Par la suite, dans tous les travaux qui ont suivi l'article fondateur, les termes "coloration de packing" et "nombre chromatique de packing" sont devenus la terminologie utilisée, telle qu'initialement formulée par Brešar, Klavžar et Rall [16]. Compte tenu de l'ampleur de la littérature consacrée au nombre chromatique de packing et de son importance constante, nous estimons nécessaire de compiler les principaux résultats et d'élucider les questions non résolues liées à notre travail dans ce domaine sous la forme d'une enquête complète au Chapitre 5. Nous mettons en lumière la conjecture proposée par Brešar et al. [18] concernant la coloration de packing des graphes subcubiques, qui affirme que pour tout graphe subcubique  $G$ , le nombre chromatique de packing de son graphe de subdivision  $S(G)$  est au plus 5. Nous présentons certains des résultats qui établissent des liens entre les colorations  $S$ -packing d'un graphe  $G$  et celles de son graphe de subdivision  $S(G)$ , ainsi que les approches menées par les théoriciens des graphes dans leur examen de cette conjecture. Notamment, toutes les tentatives pour confirmer la conjecture ont été limitées à des sous-classes spécifiques de graphes subcubiques. En fin de compte, le Chapitre 6 concerne notre contribution à la conjecture étudiée, par la confirmation de celle-ci affirmée pour une classe de graphes subcubiques. Nous traitons des graphes de Halin cubiques. Nous mentionnerons au Chapitre 5 une observation de Gastineau et Togni qui énonce que si un graphe  $G$  possède une  $(1, 1, 2, 2)$ -coloration de packing, alors le nombre chromatique de son graphe de subdivision  $S(G)$  est au plus égal à 5, et donc elle satisfait la conjecture. En fonction de cette observation, et afin de prouver la véracité de la conjecture pour la classe sur laquelle nous avons travaillé, nous avons étudié sa  $S$ -coloration visant à prouver qu'elle

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admet une coloration en  $(1, 1, 2, 2)$ -coloration de packing. Nous avons prouvé qu'un graphe de Halin cubique admet une  $(1, 1, 2, 3)$ -coloration de packing, ce qui est plus fort que la  $(1, 1, 2, 2)$ -coloration, et nous confirmons ainsi la conjecture pour cette classe. De plus, Gastineau et Togni [40], après avoir prouvé que tout graphe subcubique est  $(1, 2, 2, 2, 2, 2)$ -colorable, ont posé une question ouverte sur le fait de savoir si tout graphe subcubique est  $(1, 2, 2, 2, 2, 2)$ -colorable. Nous répondons à cette question de manière affirmative pour la classe particulière sur laquelle nous avons travaillé ; nous avons prouvé que les graphes de Halin cubiques possèdent une  $(1, 2, 2, 2, 2, 2)$ -coloration de packing.

**Mots-clés.** Subdivision, cycle orienté, chemin orienté, chemin à trois blocs, nombre chromatique, arbre final,  $S$ -coloration de packing, nombre chromatique de packing, graphe subcubique, graphe de Halin.

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*“Gratitude makes sense of our past, brings peace for today, and creates a vision for tomorrow.”*

– MELODY BEATTIE –

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# Contents

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<b>Abstract</b>	<b>1</b>
<b>Résumé</b>	<b>5</b>
<b>1 Graph theory basics</b>	<b>15</b>
1.1 Graphs . . . . .	15
1.1.1 Definitions and Fundamental Properties . . . . .	15
1.1.2 Vertex degree . . . . .	16
1.1.3 Subgraphs . . . . .	17
1.1.4 Types of graphs . . . . .	18
1.1.5 Petersen graph . . . . .	19
1.1.6 Graph generalization . . . . .	20
1.2 Digraphs . . . . .	20
1.2.1 Definitions and basic properties . . . . .	20
1.2.2 In-degree and out-degree of a vertex . . . . .	21
1.2.3 Types of digraphs . . . . .	22
1.3 Graph and digraph operations . . . . .	24
1.3.1 Unary operations . . . . .	24
1.3.2 Binary operations . . . . .	25
1.4 Connectivity . . . . .	25
1.5 Oriented and unoriented forests . . . . .	25
1.5.1 Forests and trees . . . . .	25
1.5.2 Oriented forests and trees . . . . .	26
1.6 Median order . . . . .	28
1.7 Coloring . . . . .	28
1.7.1 Graph coloring . . . . .	28
1.7.2 Digraph coloring . . . . .	31
1.8 $S$ -Packing coloring . . . . .	31
<b>I Oriented paths in digraphs</b>	<b>33</b>
<b>2 History and new approaches for paths with two blocks</b>	<b>35</b>
2.1 $k$ -universal graphs . . . . .	35

2.2	About the existence of oriented cycles . . . . .	36
2.3	About the existence of oriented trees . . . . .	38
2.4	About the existence of oriented paths . . . . .	39
2.5	New approaches concerning paths with two blocks . . . . .	42
<b>3</b>	<b>The existence of the path <math>P(k, 1, l)</math></b>	<b>47</b>
3.1	The existence of the path $P(k, 1, l)$ in Hamiltonian digraphs . . . . .	47
3.2	The existence of $P(k, 1, l)$ in digraphs containing a Hamiltonian directed path . .	51
3.3	The existence of $P(k, 1, l)$ in a $(2n + 2)$ -chromatic digraph . . . . .	51
<b>4</b>	<b>The existence of paths with three blocks in digraphs</b>	<b>53</b>
4.1	An overview of the new technique . . . . .	53
4.2	The existence of the path $P(k, 1, l)$ . . . . .	54
4.3	The existence of the path $P(k, l, r)$ in digraphs with chromatic number bounded from above . . . . .	57
<b>II</b>	<b><math>S</math>-packing coloring of subcubic graphs</b>	<b>61</b>
<b>5</b>	<b>Survey on <math>S</math>-packing coloring</b>	<b>63</b>
5.1	Recall of necessary definitions . . . . .	63
5.2	The effect of some local operations on the packing chromatic number . . . . .	64
5.3	Packing chromatic number of specific classes . . . . .	65
5.4	A Comprehensive Survey of $S$ -Packing Coloring of subcubic graphs . . . . .	68
<b>6</b>	<b><math>S</math>-packing coloring of cubic Halin graphs</b>	<b>75</b>
6.1	$(1, 1, 2, 3)$ -packing coloring of cubic Halin graphs. . . . .	75
6.2	$(1, 2, 2, 2, 2, 2)$ -packing coloring of cubic Halin graphs . . . . .	80
<b>7</b>	<b>Directions for further work</b>	<b>83</b>

# 1

## Graph theory basics

---

*“The cosmos is within us. We are made of star-stuff. We are a way for the universe to know itself.”*

– CARL SAGAN –

The fundamental principles of graph theory possess remarkable simplicity and have widespread applications in various disciplines. The primary aim of this introductory chapter is to acquaint the reader with the terminology and notation that will be employed throughout this thesis. To facilitate the reader’s understanding and reinforce comprehension, we provide proofs of several well-known results that predominantly pertain to our specific area of research. By doing so, we aim to guide the reader on the path of exploration within our field of study.

### 1.1 Graphs

In the world of discrete mathematics, particularly in graph theory, a graph is a mathematical construct which is a collection of points that are called vertices, and lines called edges that establish connections between certain pairs of these vertices. The presence of an edge between two vertices indicates some form of relationship or association between them. For instance, in the context of representing countries on a map, the existence of a commercial exchange between two countries can be depicted by an edge connecting them. The mentioned example is just one of many scenarios which graph theory provides mathematical model for. For instance, electrical circuits are effectively described using graphs, as demonstrated by the pioneering work of the German physicist Gustav Kirchoff. Notably, the specific manner in which the points are linked does not hold significance in graphs.

#### 1.1.1 Definitions and Fundamental Properties

A *graph*  $G$  is represented as an ordered pair of disjoint sets  $(V, E)$ , where  $E$  is a subset of the set  $[V]^2 := \{\{x, y\}; x, y \in V \text{ and } x \neq y\}$ , which represents the set of unordered distinct pairs

of vertices of  $V$ . The elements of  $V$  are referred to as *vertices* of  $G$ , and the elements of  $E$  are known as the *edges* of  $G$ . For clarity, we use  $V(G)$  and  $E(G)$  instead of  $V$  and  $E$  to indicate that  $V$  and  $E$  serve as the vertex-set and edge-set of  $G$ , respectively.

Throughout this thesis, we deal only with finite graphs, ensuring that  $V(G)$  and  $E(G)$  are always finite. It is important to note that  $V(G)$  is commonly assumed to be non-empty, but  $E(G)$  is allowed to be empty.

The *order* of a graph  $G$ , denoted by  $v(G)$ , refers to the cardinality of  $V(G)$ , while the *size* of  $G$ , denoted by  $e(G)$ , denotes the number of its edges.

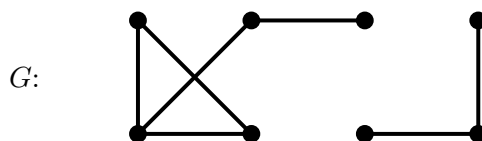


Figure 1.1: An example of a graph  $G$  with 8 vertices and 7 edges.

For an edge  $e = x, y$  connecting two vertices  $x$  and  $y$  in a graph  $G$ , we refer to  $e$  as being *incident* with  $x$  (or  $y$ ), and vice versa, stating that  $x$  is incident with  $e$ . In this situation,  $x$  and  $y$  are denoted as the *endvertices* of  $e$ , and we say that they are *adjacent* vertices in  $G$ . To simplify notation, it is common to represent an edge  $\{x, y\}$  as  $xy$ .

Two edges  $e$  and  $e'$  of  $G$  are considered *adjacent* if they share a common endvertex. If they do not share an endvertex, they are called *independent* edges

### 1.1.2 Vertex degree

Given a graph  $G$ , the *neighborhood* of a vertex  $x$  in  $G$ , denoted by  $N_G(x)$ , is defined as the set of vertices adjacent to  $x$  in  $G$ . The set formed by combining a vertex  $x$  with its neighborhood is referred to as the *closed neighborhood* of  $x$  in  $G$ , denoted by  $N_G[x]$ . Symbolically,  $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$ , and  $N_G[x] = N_G(x) \cup \{x\}$ . The *degree* of a vertex  $x$  in  $G$ , denoted by  $d_G(x)$ , is the number of vertices in its neighborhood  $N_G(x)$ . When the context makes the graph clear, we may omit the subscript and refer to  $N_G(x)$  as  $N(x)$  and to  $d_G(x)$  as  $d(x)$ .

A vertex  $v$  of a graph  $G$  is said to be a *whole* vertex if  $d_G(v) = v(G) - 1$ , that is, if  $v$  is adjacent to all other vertices of  $G$ . Otherwise, we say that  $v$  is a *non-whole* vertex of  $G$ .

The *maximum degree* of vertices in a graph  $G$  is denoted by  $\Delta(G)$  and the *minimum degree* by  $\delta(G)$ . In other words,  $\Delta(G) = \max \{d_G(x); x \in V(G)\}$  and  $\delta(G) = \min \{d_G(x); x \in V(G)\}$ . If  $\delta(G) = \Delta(G) = k$ , meaning that every vertex of  $G$  has a degree equal to  $k$ , then  $G$  is classified as a *k-regular* graph.

In this context, it is essential to mention a famous theorem of Euler [35] which establishes a fundamental identity relating the sum of the degrees of vertices in a graph and the number of its edges:

**Theorem 1.1.** (*Euler, 1736*) For any graph  $G$ ,

$$\sum_{x \in V(G)} d(x) = 2e(G). \quad (1.1)$$

Consequently, as a prompt corollary, Equality 1.1 asserts that the sum of degrees of vertices is even. Equivalently, the count of vertices of odd degree is an even number.

### 1.1.3 Subgraphs

A *subgraph*  $H$  of graph  $G$  is a graph such that its vertex-set is a subset of the vertex-set of  $G$ , and its edge-set is a subset of the edge-set of  $G$ , that is,  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . If  $H$  includes all the edges of  $G$  that have both endpoints in  $V(H)$ , then  $H$  is said to be an *induced* subgraph of  $G$ , or equivalently,  $H$  is the subgraph of  $G$  induced by  $V(H)$ . In this case, we represent  $H$  as  $G[V(H)]$ . Thus, an induced subgraph keeps both adjacency and non-adjacency of the inducing vertices, unlike to an ordinary subgraph that preserves only non-adjacency. When  $V(H) = V(G)$ , we say that  $H$  is a *spanning* subgraph of  $G$ .

Concerning two graphs  $G$  and  $H$ , if  $G$  does not contain a subgraph that is a copy of  $H$ , we define  $G$  as an  *$H$ -free* graph. In this situation, we classify  $H$  as a *forbidden* subgraph in  $G$ . We shall see in the first part of this thesis the substantial role that forbidden subgraphs play in giving information about the graph.

New graphs can be constructed from old ones by removing or adding some vertices or edges. Let  $G$  be a graph,  $U$  a subset of  $V(G)$  and  $W$  a subset of  $E(G)$ . Then  $G - U$  represents the subgraph of  $G$  induced by  $V(G) \setminus U$ , meaning that it is the subgraph of  $G$  obtained by removing the vertices in  $U$  and all edges incident with them. Similarly,  $G - W$  corresponds to the spanning subgraph of  $G$  whose edge-set is  $E(G) \setminus W$ . When  $U = \{u\}$  and  $W = \{xy\}$ , this notation is usually simplified to  $G - u$  and  $G - xy$ . Identically, if  $W \subseteq [V(G)]^2$ , then  $G + W$  is the graph constructed by adding all the elements of  $W \setminus E(G)$ .

A *matching*  $M$  of a graph  $G$  is a subgraph of  $G$  whose edges are pairwise independent, that is, no two edges of  $M$  share a common vertex. We say that  $M$  is a *perfect* matching of  $G$  if  $V(M) = V(G)$ , in other words, if every vertex of  $G$  is incident to exactly one edge of the matching  $M$ .

As previously mentioned, the subgraph of graph  $G$  induced by a subset  $U$  of its vertex-set is denoted as  $G[U]$ . When the edge-set of  $G[U]$  is empty, we classify  $U$  as a *stable* set in  $G$ . The *stability* of graph  $G$ , denoted by  $\alpha(G)$ , corresponds to the cardinality of a maximum stable set. On the other hand, a subset  $U$  of vertices in graph  $G$  is called a *clique* if every two distinct vertices of  $U$  are adjacent.

Given two subgraphs  $G_1$  and  $G_2$  of a graph  $G$ , we denote by  $G_1 \cup G_2$  the subgraph of  $G$  with the vertex-set being  $V(G_1) \cup V(G_2)$  and edge-set being  $E(G_1) \cup E(G_2)$ .

### 1.1.4 Types of graphs

□ **Complete graphs.** A *complete graph* is a graph in which every pair of its vertices is joined by an edge. In other words, for a complete graph  $G$ , its edge-set is defined as  $E(G) = \{xy; x, y \in V(G) \text{ and } x \neq y\}$ . Conventionally, the complete graph of order  $n$  is represented as  $K_n$ .

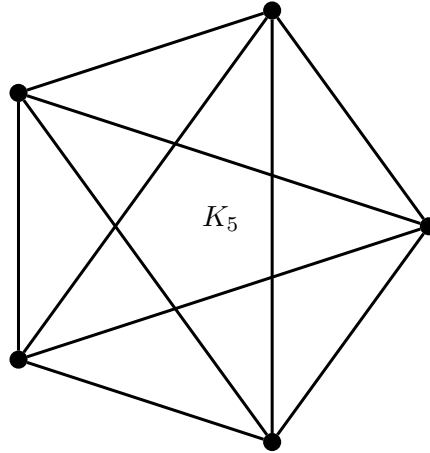


Figure 1.2: A complete graph with 5 vertices.

□ **Paths.** A *path*  $P$  is a graph with a vertex-set defined as  $V(P) = \{x_1, x_2, \dots, x_n\}$  and an edge-set defined as  $E(P) = \{x_i x_{i+1} | 1 \leq i \leq n - 1\}$ . This means that the edges of the path connect consecutive vertices in the given order. For simplicity, we represent the path as  $P = x_1, x_2, \dots, x_n$ . In this case, the vertices  $x_1$  and  $x_n$  are called the *endvertices* of  $P$ , and  $P$  is described as an  $x_1 x_n$ -path. The *length* of a path  $P$ , denoted by  $l(P)$ , corresponds to the number of its edges. A *subpath*  $P'$  of a path  $P$  is a subgraph of  $P$  which forms itself a path. Assuming that  $x_i$  and  $x_j$  are the endvertices of the subpath  $P'$  for some  $1 \leq i < j \leq n$ , we write  $P' = P[x_i, x_j]$ . Throughout this thesis, we denote by  $P[x_i, x_j[$  the subpath  $P[x_i, x_j] - x_j$ ,  $P]x_i, x_j]$  the subpath  $P[x_i, x_j] - x_i$ , and  $P]x_i, x_j[$  the subpath  $P[x_i, x_j] - \{x_i, x_j\}$ .

Given a graph  $G$ , a path  $P$  in  $G$  is said to be a *Hamiltonian path* of  $G$  if  $V(P) = V(G)$ . The *distance* between two vertices  $x$  and  $y$  in  $G$ , denoted by  $d_G(x, y)$ , is defined to be the minimal length of an  $xy$ -path, or  $+\infty$  if such path does not exist. An  $xy$ -path  $P$  for which  $d_G(x, y) = l(P)$  is referred to as an *xy-geodesic*. The *diameter* of a graph  $G$ , denoted by  $d(G)$ , represents the maximum distance between two vertices in  $G$ .

□ **Cycles.** A *cycle*  $C$  is a graph with a vertex-set defined by  $V(C) = \{x_1, x_2, \dots, x_n\}$ , and an edge-set defined by  $E(C) = \{x_i x_{i+1} | 1 \leq i \leq n - 1\} \cup \{x_n x_1\}$ . For simplicity, we represent the cycle as  $C = x_1, x_2, \dots, x_n, x_1$ . The *length* of a cycle  $C$ , denoted by  $l(C)$ , is defined to be the number of its edges, that is,  $l(C) = e(C)$ . If  $l(C)$  is even, then  $C$  is said to be an *even cycle*. Otherwise, it is called an *odd cycle*. A *subpath*  $P$  of a cycle  $C$  is a path which is a subgraph of  $C$ . If  $P = x_i, x_{i+1}, \dots, x_j$ , then we write  $P = C[x_i, x_j]$ .

Given a graph  $G$ , a cycle  $C$  in  $G$  is said to be a *Hamiltonian cycle* of  $G$  if it passes through all the

vertices of  $G$ , that is,  $V(C) = V(G)$ . In this case,  $G$  is said to be a *Hamiltonian graph*. By simply deleting any edge of a Hamiltonian cycle we get a Hamiltonian path. However, it's important to note that a non-Hamiltonian graph can still possess a Hamiltonian path, i.e., Hamiltonian paths cannot always be used to form Hamiltonian cycles. For example, as shown in Figure 1.3,  $G_1$  has no Hamiltonian path (and so no Hamiltonian cycle),  $G_2$  has a Hamiltonian path  $P = x_1x_2x_3x_4$  but lacks a Hamiltonian cycle, whereas  $G_3$  has the Hamiltonian cycle  $C = x_1x_2x_4x_3x_1$ .

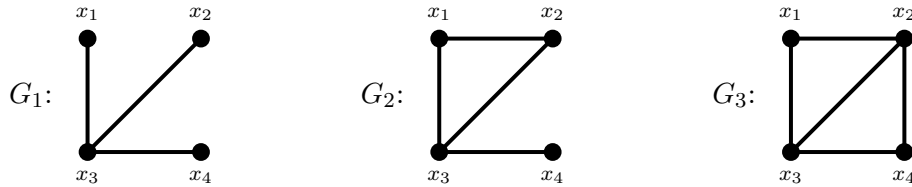


Figure 1.3:  $G_1$  has no Hamiltonian path,  $G_2$  has a Hamiltonian path but no Hamiltonian cycle, whereas  $G_3$  has a Hamiltonian cycle.

If  $G$  contains no cycles, then it is called an *acyclic graph*. The *girth* of  $G$ , denoted by  $g(G)$ , is the length of a shortest cycle in  $G$  if there is one; otherwise it is defined to be zero.

□ **Bipartite graphs.** A *bipartite graph* refers to a graph  $G$  whose vertex-set,  $V(G)$ , can be partitioned into two disjoint sets  $A$  and  $B$  such that each edge in  $E(G)$  joins a vertex in  $A$  to a vertex from  $B$ , i.e.,  $A$  and  $B$  are two stable sets in  $G$  such that  $V(G) = A \cup B$ .

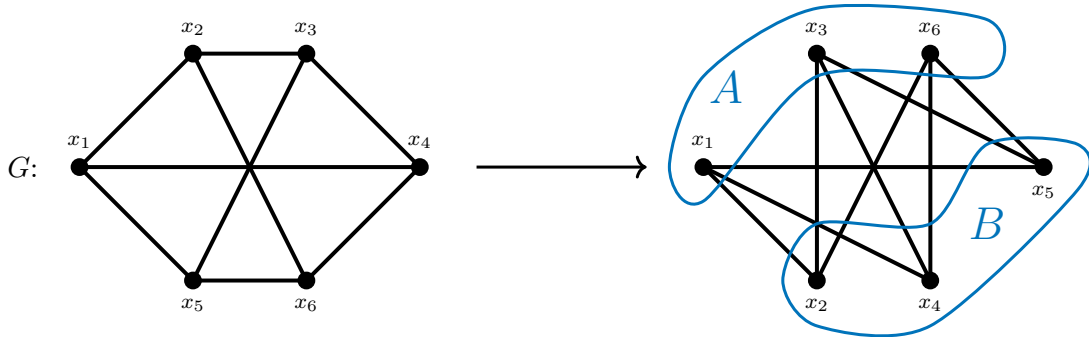


Figure 1.4: An example of a bipartite graph on 6 vertices.

□ **Empty graphs.** An *empty graph* is a graph with an empty edge-set, meaning that any empty graph  $G$  of arbitrary order satisfies  $e(G) = 0$ .

### 1.1.5 Petersen graph

In the realm of mathematical graph theory, the Petersen graph stands out as an undirected graph comprising 10 vertices and 15 edges. This small graph serves a dual purpose, serving as both an example and a counterexample in various graph theory problems. The name "Petersen



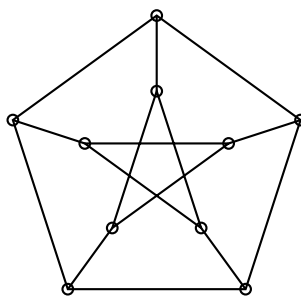


Figure 1.5: Petersen graph

graph" refers to Julius Petersen, who, in 1898, constructed it to represent the smallest bridgeless cubic graph incapable of being three-edge-colored.

### 1.1.6 Graph generalization

A *hypergraph* is a generalization of a graph that allows an edge to join any number of vertices. Formally, a hypergraph  $\mathcal{H}$  is a pair  $(X, E)$  where  $X$  is a set of elements referred to as *vertices*, and  $E$  is a set of non-empty subsets of  $X$  called *hyperedges*. It is often desirable to study *k-uniform* hypergraphs which are hypergraphs in which each hyperedge connects precisely  $k$  vertices.

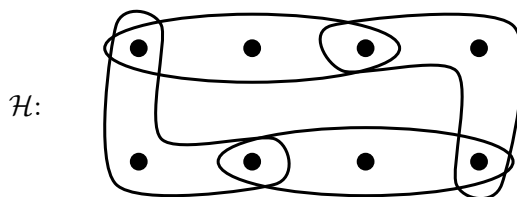


Figure 1.6: An example of a 3-uniform hypergraph on 8 vertices.

For instance, a graph is considered as a 2-uniform hypergraph.

## 1.2 Digraphs

Digraphs are mathematical structures consisting of vertices and edges joining them, in which, unlike graphs, we are interested in the orientations of edges. Returning to the example presented in the introduction of §1.1, consider any two countries represented by points  $A$  and  $B$ . In this context, a line is directed from  $A$  towards  $B$  if and only if  $A$  exports goods and services to  $B$ .

### 1.2.1 Definitions and basic properties

A *digraph*  $D$  is defined as an ordered pair  $D = (V(D), E(D))$  where  $V(D)$  is a non-empty set of elements called the *vertices* of  $D$ , and  $E(D)$  is a subset of the set of ordered pairs of  $V(D)$ , that is  $E(D) \subseteq \{(x, y); x, y \in V(D) \text{ and } x \neq y\}$ . The elements of  $E(D)$  are said to be the *arcs*

of  $D$ . Given an arc  $a = (x, y)$  of a digraph  $D$ , the direction of  $a$  is from  $x$  to  $y$ ,  $x$  is called the *tail* of  $a$  and  $y$  is its *head*. Two arcs  $a$  and  $a'$  are said to be *bidirected* arcs if the tail of  $a$  is the head of  $a'$  and vice versa.



Figure 1.7: Two bidirected arcs.

In this thesis, all digraphs under consideration are assumed to be without bidirected arcs, unless explicitly stated otherwise.

The *underlying graph* of a digraph  $D$ , denoted by  $G[D]$ , is the graph obtained from  $D$  by ignoring the orientations of its arcs.

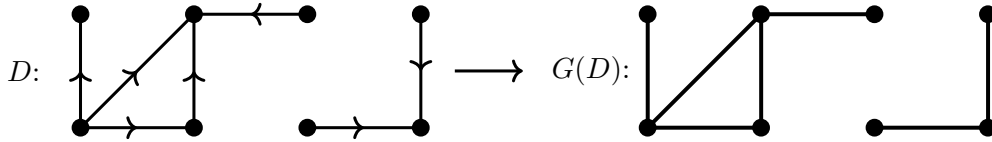


Figure 1.8: An example showing a digraph  $D$  and its underlying graph  $G[D]$ .

The concepts introduced for graphs may be extended to digraph by considering their underlying graphs. To avoid unnecessary repetition, all the notions defined for graphs in Subsection 1.1.1 and Subsection 1.1.3 are similarly defined to digraphs.

### 1.2.2 In-degree and out-degree of a vertex

Consider a digraph  $D$  and let  $e = (x, y)$  be an arc of  $D$ . We say that  $x$  is an *in-neighbor* of  $y$ , and  $y$  is an *out-neighbor* of  $x$ . The (*first*) *out-neighborhood* of  $x$  in  $D$ , denoted by  $N_D^+(x)$ , is defined to be the set of all out-neighbors of  $x$  in  $D$ . The (*first*) *in-neighborhood* of  $x$  in  $D$ , denoted by  $N_D^-(x)$ , is defined to be the set of all in-neighbors of  $x$  in  $D$ . The *out-degree* of  $x$  in  $D$ , denoted by  $d_D^+(x)$ , is the cardinality of  $N_D^+(x)$ . Similarly, the *in-degree* of  $x$  in  $D$ , denoted by  $d_D^-(x)$ , is the cardinality of  $N_D^-(x)$ . A vertex  $x$  is said to be a *sink* if  $d_D^+(x) = 0$ . If  $d_D^-(x) = 0$ , then  $x$  is said to be a *source*. Comparing a digraph with its underlying graph, it becomes evident that  $N_{G[D]}(x) = N_D^+(x) \cup N_D^-(x)$  and  $d_{G[D]}(x) = d_D^+(x) + d_D^-(x)$ .

In accordance with the last observation, Identity 1.1 which establishes a relation between the number of edges of a graph and the degrees of its vertices, may be generalized to digraphs in the following manner:

**Proposition 1.1.** For any digraph  $D$ ,

$$\sum_{x \in V(D)} d^+(x) = e(D) = \sum_{x \in V(D)} d^-(x). \quad (1.2)$$

For a digraph  $D$ , we define the *maximum out-degree*  $\Delta^+(D)$ , the *minimum out-degree*  $\delta^+(D)$ , the *maximum in-degree*  $\Delta^-(D)$  and the *minimum in-degree*  $\delta^-(D)$  as follows:

$$\begin{aligned} \Delta^+(D) &= \max \{d_D^+(x); x \in V(D)\}; \\ \delta^+(D) &= \min \{d_D^+(x); x \in V(D)\}; \\ \Delta^-(D) &= \max \{d_D^-(x); x \in V(D)\}; \\ \delta^-(D) &= \min \{d_D^-(x); x \in V(D)\}. \end{aligned}$$

On particularizing the out-neighborhoods of  $x$ , the *second out-neighborhood* of  $x$  in  $D$  is defined by  $N_D^{++}(x) = \{z \in N_D^+(y) \setminus N_D^+(x); y \in N_D^+(x)\}$ . In other words,  $N_D^{++}(x)$  is the set of vertices that are at distance 2 from  $x$ . The *second out-degree* of  $x$  in  $D$ , denoted by  $d_D^{++}(x)$ , is the number of elements in  $N_D^{++}(x)$ . Note that we may omit the subscript if there is no confusion in the context.

### 1.2.3 Types of digraphs

□ **Tournaments.** A *tournament* is a digraph whose underlying graph is a complete graph. A tournament of order  $n$  is usually denoted by  $T_n$ .

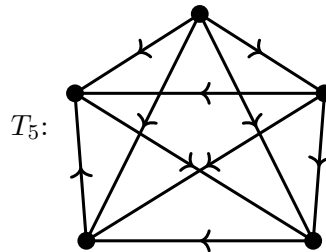


Figure 1.9: A tournament on 5 vertices.

□ **Oriented paths.** An *oriented path* is a digraph whose underlying graph is a path. A *directed path*, or simply a *dipath*, is an oriented path in which all the arcs have the same direction. More formally, an oriented path  $P$  having its vertex-set  $V(P) = \{x_1, x_2, \dots, x_n\}$  and its edge-set  $E(P) = \{(x_i, x_{i+1}); 1 \leq i \leq n - 1\}$  is called a directed path, or simply a *dipath*. In this case, it is represented by  $P = x_1, x_2, \dots, x_n$ . We say that  $P$  is an  $x_1x_n$ -directed path, where  $x_1$  is its *initial vertex (origin)* and  $x_n$  is its *terminal vertex (terminus)*. The *length* of an oriented path  $P$  is denoted by  $l(P)$  and it is the number of its arcs. In what follows,  $P_k^+$  denotes a directed path  $P$  of length  $k$ .

Let  $P = x_1, x_2, \dots, x_n$  be an oriented path. We denote by  $P[x_i, x_j]$  the *oriented subpath*  $x_i, x_{i+1}, \dots, x_j$ . We say that an arc  $a$  of  $P$  is *forward* if  $a = (x_i, x_{i+1})$ , otherwise  $a$  is said to be *backward*. A *block* of  $P$  is a maximal directed subpath of  $P$ . A block is said to be *forward* (resp. *backward*) if it consists only of forward (resp. backward) arcs. If  $P$  has  $l$  blocks of consecutive lengths  $k_1, k_2, \dots, k_l$ , then we write  $P = P^+(k_1, k_2, \dots, k_l)$  if the block of length  $k_1$  is forward, otherwise it is written  $P = P^-(k_1, k_2, \dots, k_l)$ . In both cases, we say that  $P$  is an  *$l$ -blocks path*.



Figure 1.10: This figure shows the difference between  $P^+(2, 1, 1)$  and  $P^-(2, 1, 1)$ .

A oriented path  $P$  of a digraph  $D$  is said to be *Hamiltonian* if its underlying graph is a Hamiltonian path. The *distance* between two vertices  $x$  and  $y$  in a digraph  $D$ , denoted by  $d_D(x, y)$ , is the minimal length of an  $xy$ -directed path, or  $+\infty$  if such a directed path does not exist.

□ **Oriented Cycles.** The notions and notations used for oriented cycles will be identical to those of oriented paths. An *oriented cycle* is a digraph whose underlying graph is a cycle. A *directed cycle*, or simply a *circuit*, is an oriented cycle in which all its arcs have the same orientation. This means that the vertex-set and the edge-set of a directed cycle  $C$  are defined by  $V(C) = \{x_1, x_2, \dots, x_n\}$  and  $E(C) = \{(x_i, x_{i+1}); 1 \leq i \leq n - 1\} \cup \{(x_n, x_1)\}$  respectively. The length of an oriented cycle  $C$ , denoted by  $l(C)$ , is the number of its arcs.

Given an oriented cycle  $C = x_1x_2, \dots, x_nx_1$ , we denote by  $C[x_i, x_j]$ , the *oriented subpath* of  $C$  induced by the subset  $\{x_i, x_{i+1}, \dots, x_j\}$  of  $V(C)$ . A *block* of  $C$  is a maximal directed subpath of  $C$ . If  $C$  has  $l$  blocks of consecutive lengths  $k_1, k_2, \dots, k_l$ , then we say that  $C$  is an  *$l$ -blocks cycle* and we write  $C = C(k_1, k_2, \dots, k_l)$ . According to the definition of a block, it is straightforward to see that  $l$  must be an even integer. If all the blocks of a cycle  $C$  are of length 1, then  $C$  is called an *antidirected cycle*.

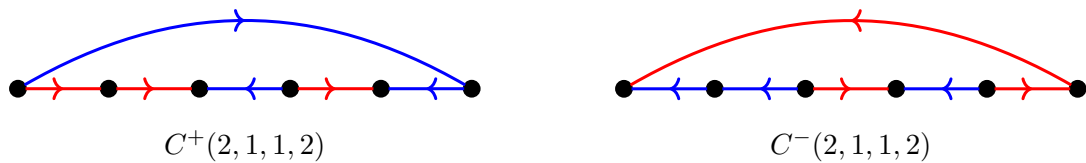


Figure 1.11: This figure shows the difference between  $C^+(2, 1, 1, 2)$  and  $C^-(2, 1, 1, 2)$ .

Given a digraph  $D$ , a cycle  $C$  of  $D$  is said to be a *Hamiltonian cycle* in  $D$  if  $C$  passes through all the vertices of  $D$ , that is  $V(C) = V(D)$ . In this case, we say that  $D$  is a *Hamiltonian digraph*.



Figure 1.12: An anti-directed cycle on 4 vertices.

In our work, we are interested in studying Hamiltonian digraphs, this will be elaborated in Chapter 3. If  $D$  does not contain any directed cycle, then  $D$  is said to be *acyclic*.

## 1.3 Graph and digraph operations

Graph operations generate new graphs from original ones. These operations can be categorized into two main groups: unary operations and binary operations. Note that all the operations defined below on graphs can be extended to digraphs. However, in case an operation is exclusive to digraphs only, this will be mentioned explicitly in its definition.

### 1.3.1 Unary operations

*Unary* operations produce a new graph from a single initial one. Some of these operations, like addition or deletion of a vertex or an edge, are done by a simple local change. These operations are referred to as *elementary* operations or *editing* operations. Many others that require more complex changes are called *advanced* operations. The operations that are essential for the coming chapters will be explicitly described in complete details below.

□ **Vertex Contraction.** The *contraction* of a subset  $S$  of the vertex-set of a graph  $G$  produces a graph  $G'$  in which the vertices of  $S$  are replaced with a single vertex  $v_S$ , such that  $v_S$  is adjacent in  $G'$  to the union of the neighborhoods of the elements of  $S$  in  $G$ . More formally, given a graph  $G = (V, E)$  and a subset  $S$  of  $V$ , the contraction of  $S$  into a new vertex  $v_S$  results in a new graph  $G' = (V', E')$  whose vertex-set is  $V' = (V \setminus S) \cup \{v_S\}$ , and whose edge set  $E'$  is defined as follows: Let  $x$  be a vertex in  $V \setminus S$  and  $y$  be a vertex in  $V'$ . If  $y = v_S$ , then  $xv_S \in E'$  if and only if there exists a vertex  $z$  in  $S$  such that  $xz \in E$ . Else if  $y \in V' \setminus \{v_S\}$ , then  $xy \in E'$  if and only if  $xy \in E$ . To avoid multiple edges, if two vertices of  $S$  have a common neighborhood  $x$  in  $V \setminus S$ , then the edge  $v_Sx$  is added just one time to  $G'$ .

□ **Complement graph.** The *complement* of a graph  $G$  is a graph  $H$  obtained from  $G$  which is constructed by adding to  $G$  all the missing edges required to form a complete graph, and then removing all the edges that were previously there. In formal terms, the complement of a graph  $G$ , denoted by  $\bar{G}$ , is a graph with vertex-set  $V(G)$ , such that two distinct vertices are adjacent in  $\bar{G}$  if and only if they are not adjacent in  $G$ , that means, the edge-set of  $\bar{G}$  is defined by the set  $E(K_{v(G)}) - E(G)$ . For example, a stable set in a graph  $G$  is a clique in the complement graph of  $G$  and vice versa.

□ **Digraph subdivision.** A *subdivision* of a digraph  $H$  is a digraph  $H'$  obtained from  $H$  by replacing each arc  $(x, y)$  by an  $xy$ -directed path of length at least 1. For example, it can be

observed that any directed cycle of length at least  $k$  is a subdivision of  $C_k^+$ . A digraph  $D$  is said to be *H-subdivision-free* if it does not contain any subdivision of  $H$ .

□ **Reverse digraph.** The *reverse* of a digraph  $D$  is a digraph  $D^c$  on the same vertex-set of  $D$  with all the arcs are of reversed orientations compared to the corresponding arcs in  $D$ . In formal terms, if  $(x, y)$  is an arc of  $D$  then the reverse digraph  $D^c$  of  $D$  contains the arc  $(y, x)$  and vice versa. For giving a notation for reverse digraphs, there is no general agreement on a preferred one.

### 1.3.2 Binary operations

*Binary* operations are tools used to produce a new graph from two initial graphs. One of the most used binary operations throughout this thesis is the *graph union*. Conventionally, the union of two graphs  $G_1$  and  $G_2$  is denoted by  $G_1 \cup G_2$ , and it is the graph whose vertex-set is  $V(G_1) \cup V(G_2)$  and whose edge-set is  $E(G_1) \cup E(G_2)$ . Note that it is not necessarily for  $V(G_1)$  and  $V(G_2)$  to be disjoint.

## 1.4 Connectivity

A graph  $G$  is said to be *connected* if any two vertices in  $G$  are joined by a path. Otherwise,  $G$  is said to be *disconnected*. A *connected component* of  $G$  refers to an inclusion-wise maximal connected subgraph of  $G$ . A *cut-vertex* of  $G$  is a vertex whose removal increases the number of components of  $G$ . More generally, a *cut-set*  $U$  of a connected graph  $G$  is defined as a subset of its vertex-set such that  $G - U$  is disconnected.

The above notions can be extended to digraphs by applying them to their underlying graphs. A digraph  $D$  is said to be *strongly connected*, or simply *strong*, if there exists an  $xy$ -directed path for every pair  $\{x, y\}$  of distinct vertices of  $D$ . A *strong component* of  $D$  refers to a strong induced subdigraph of  $D$  which is maximal with respect to inclusion.

## 1.5 Oriented and unoriented forests

Forests can be considered the simplest class of graphs. Despite their simplicity, they hold significant importance and play a fundamental role in many fields due to their rich structural properties. In Part I, we heavily rely on the trees properties as a key tool to construct some of our proofs and achieve our desired results.

### 1.5.1 Forests and trees

A *forest* is an acyclic graph. A *tree* is a connected forest. The relation of a tree to a forest sounds less absurd if we note that a forest is a disjoint union of trees, in other words, a forest is a graph whose every connected component is a tree.

One illustrious feature of forests is that every graph contains a forest as a spanning subgraph. This significant property is an easy consequence of the following observation on trees:

**Proposition 1.2.** *Every connected graph  $G$  contains a spanning tree  $T$ .*

*Proof.* The proof follows directly by taking a tree  $T$  of maximal order in  $G$  and supposing that  $T$  does not span  $G$ . ■

Furthermore, it can be straightforwardly remarked that any tree  $T$  contains at least  $\Delta(T)$  vertices of degree 1. Such vertices are called the *leaves* of  $T$ . Another prominent attribute of trees is that any two vertices of a tree are joined by a uniquely determined path.

A *rooted tree* is a tree in which one vertex is distinguished and designated to be its starting point. This vertex is called a *root*. The latter qualification assigned to a vertex of a tree  $T$  is of great importance, because it induces a partial ordering  $(V(T), \leq_T)$  on the vertices of a tree associated with its root  $r$ , called the *tree-order*, defined by:  $x \leq_T y$  if and only if the unique  $ry$ -path in  $T$  passes through  $x$ . A rooted spanning tree of a graph  $G$  is said to be *normal*, if the endvertices of every edge of  $G$  are comparable in this tree-order. Normal spanning trees of finite graphs are more commonly known as *depth-first search trees*, and they are widely used as a structural tool in both algorithmic and pure graph theory. It is well-known that every connected graph contains such a tree:

**Proposition 1.3.** *Every connected graph has a normal spanning tree with any preassigned root.*

## 1.5.2 Oriented forests and trees

An *oriented tree* is an orientation of a tree. An *out-tree*, also called an *out-branching*, is an oriented tree in which all the vertices have in-degree at most 1. An *in-tree*, called an *in-branching*, is an oriented tree in which all the vertices have out-degree at most 1. This implies that in an out-tree (respectively, in-tree) there exists exactly one vertex of indegree (respectively, outdegree) zero, and it is called the *root*. The root of an out-branching is called a *source*, and that of an in-branching is said to be a *sink*. The *leaves* of an out-branching (resp. in-branching) are the vertices of out-degree (resp. in-degree) zero. In the language of digraphs, an *oriented forest* is a digraph whose all connected components are oriented trees.

According to the previous terminology, an *outforest* (respectively, *inforest*) is an oriented forest in which its connected components are out-trees (respectively, in-trees).

It is worth mentioning that the class of digraphs having a spanning out-tree is more vast than that of strongly connected digraphs:

**Proposition 1.4.** *Every strongly connected digraph  $D$  contains a spanning out-tree  $T$ .*

*Proof.* Consider an out-tree  $T$  in  $D$  of maximum order. Assume that  $T$  is not spanning, then  $D - T$  is non-empty. Combining the last observation with the fact that  $D$  is strong induces the existence of an arc  $a$  in  $D$  whose tail is a vertex in  $T$  and whose head is in  $D - T$ . Then  $T + a$  is an out-tree in  $D$  of order strictly greater than that of  $T$ , which is a contradiction to the maximality of  $T$  in  $D$ . Hence,  $T$  is a spanning out-branching of  $D$ . ■

As a generalization of the structural property presented in Proposition 1.4 on strong digraphs, we get the following on general digraphs:

**Proposition 1.5.** *Every digraph contains a spanning out-forest.*

*Proof.* The result is established simply by considering the strong components of the digraph and applying Proposition 1.4 to each of them. ■

Let  $F$  be a spanning out-forest of a digraph  $D$ , having  $F_1, F_2, \dots, F_k$  its out-branchings, rooted at  $r_1, r_2, \dots, r_k$  respectively.

The unique directed path in  $F$  joining  $x$  with the root of the out-branching it belongs to, is denoted by  $P_F(x)$ .

The *level* of a vertex  $x$  in  $D$  with respect to  $F$ , denoted by  $l_F(x)$ , is the order of the path  $P_F(x)$ . For  $i \geq 1$ ,  $L_i(F)$  is defined to be the set containing all the vertices  $x \in V(F)$  such that  $l_F(x) = i$ . We say that an arc  $(x, y) \in E(D)$  is a *forward* arc with respect to an outforest (respectively, inforest)  $F$  of  $D$  if  $l_F(x) < l_F(y)$  (respectively,  $l_F(x) > l_F(y)$ ), otherwise we call it a *backward* arc.

A *spanning outforest* (respectively, *spanning inforest*)  $F$  of a digraph  $D$  is said to be *final out-forest* (respectively, *final inforest*) of  $D$  if and only if for every backward arc  $(x, y)$  with respect to  $F$ ,  $F$  contains a  $yx$  (respectively,  $xy$ )-directed path.

It is clearly noticed that in a final outforest (respectively, inforest)  $F$ ,  $L_i(F)$  is stable for every  $i$ .

A spanning out-forest of a digraph  $D$  is called a *maximal forest* if  $\sum_{x \in V(D)} l_F(x)$  is maximal.

For short,  $l(F)$  is commonly used to refer to  $\sum_{x \in V(D)} l_F(x)$ .

The notion of a maximal forest was first introduced in 2007 by El Sahili and Kouider [24]. They proved that a maximal forest of a digraph  $D$  is a final forest in it. Furthermore, they pointed the existence of a maximal out-forest in each digraph:

**Proposition 1.6.** *Every digraph  $D$  contains a maximal out-forest  $F$ .*

*Proof.* Let  $F$  be a spanning out-forest of  $D$ . The existence of  $F$  is guaranteed due to Proposition 1.5. Assume that  $F$  is chosen so that  $l(F)$  is maximal. Initially, set  $F_0 = F$ . If  $F_0$  is maximal, there is nothing to do. Otherwise there is an arc  $(x, y)$  of  $D$  which is backward with respect to  $F_0$ , such that  $F_0$  contains no  $yx$ -directed path. Let  $F_1$  be the out-forest obtained from  $F_0$  by adding  $(x, y)$  to  $F_0$ , and deleting the arc of head  $y$  in  $F_0$ . We can easily see that the level of each vertex in  $F_1$  is at least its level in  $F_0$ , and that the level of  $y$  strictly increases. Consequently,  $F_1$  is a spanning out-forest of  $D$  with  $l(F_1)$  is strictly greater than  $l(F_0)$ , a contradiction. This yields the hoped result. ■



## 1.6 Median order

Given an enumeration  $L = x_1x_2\dots x_n$  of the vertices of a digraph  $D$ , an arc  $(x_i, x_j) \in E(D)$  is said to be *forward* with respect to  $L$  if  $i < j$ . Otherwise, it is called *backward* with respect to  $L$ . The enumeration  $L$  is called a *median order* of  $D$  when the set of forward arcs of  $D$  with respect to  $L$  is maximum. Let  $f(L)$  denotes the number of forward arcs of  $D$  with respect to  $L$ . The concept of a median order can be restated as follows:  $L$  is a median order of  $D$  if  $f(L)$  is maximum. The last vertex in a median order is said to be the *feed* vertex. In fact, the use of median orders is a powerful tool as demonstrated in many studies built upon this concept, such as Sumner's conjecture.

## 1.7 Coloring

The concept of graph coloring originates from coloring the countries of a map, where it has been postulated that four colors are sufficient to color any map in a way that regions sharing a common border must receive distinct colors. Subsequently, the problem of coloring a graph arises in many theoretical challenges and finds application in various practical scenarios such as scheduling, exam timetabling and solving Sudoku puzzles. Nowadays, graph coloring remains a highly active field of research that catches the curiosity and interest of a great number of adventurous graph theorists. As we shall see throughout this thesis, the focus in Part I of our work is to discover the relation between the existence of a given oriented path in a digraph as its subdigraph with a measure of digraph coloring. In what follows, we shortly refer to the set  $\{1, 2, \dots, k\}$  by  $[k]$ , for every positive integer  $k$ .

### 1.7.1 Graph coloring

A  $k$ -coloring of a graph  $G$  can be defined as a mapping  $c : V(G) \rightarrow \{1, 2, \dots, k\}$  that assigns to each vertex  $x$  of  $G$  a color  $i$  from the set  $[k]$ . Usually, we write  $c(x)$  to refer to the *color* assigned to the vertex  $x$ . The *color class*  $i$  of  $G$  with respect to  $c$  is defined by the set  $C_i = \{x \in V(G); c(x) = i\}$ . By the definition of  $C_i$ , a  $k$ -coloring of  $G$  can be redefined as a partition of the vertex-set into  $k$  color classes. A coloring  $c$  is said to be a *proper* coloring of  $G$  if any two adjacent vertices of  $G$  are differently colored by  $c$ , that means, if the endvertices of each edge in  $G$  belong to two distinct color classes of  $G$  with respect to  $c$ . The latter definition implies that a  $k$ -coloring of  $G$  is considered proper if and only if the vertex-set of  $G$  can be partitioned into  $k$  stable sets induced by the  $k$  color classes of  $G$  with respect to  $c$ . A graph  $G$  is said to be  *$k$ -colorable* if it admits a proper  $k$ -coloring.

The *chromatic number* of a graph  $G$ , denoted by  $\chi(G)$ , is the minimum number of colors needed to color  $G$  by a proper coloring. It can also be seen as the minimum number of sets required to form a partition of  $V(G)$  into stable subsets. This implies that in case  $G$  is  $k$ -colorable, it follows that  $\chi(G) \leq k$ . If  $\chi(G) = k$ , then we say that  $G$  is a  *$k$ -chromatic* graph.

The next proposition shows a basic inequality that relates the chromatic number of a graph and its maximum degree:

**Proposition 1.7.** *For every graph  $G$ ,*

$$\chi(G) \leq \Delta(G) + 1. \tag{1.3}$$

*Proof.* Suppose that  $\chi(G) = k$ . Given that  $c$  is a proper  $k$ -coloring of  $G$ , assume that  $V(G)$  is partitioned into  $\chi(G)$  stable subsets  $\{C_1, \dots, C_k\}$ , where  $C_i$  is the color class  $i$  of  $G$  with respect to  $c$ . Assume that for every  $v \in C_1$ ,  $d(v) < k - 1$ . Then, for every  $v \in C_1$ , there exists  $2 \leq i \leq k$  such that  $C_i \cup \{v\}$  is a stable set. This implies that  $V(G)$  may be decomposed into  $k - 1$  stable subsets, which is a contradiction with the minimality of the chromatic number  $k$ . Thus there is a vertex  $v \in C_1$  such that  $d(v) \geq k - 1$  and therefore  $\Delta(G) \geq k - 1$ , which proves the required inequality. ■

A sharper upper bound was given by Brooks [19] in 1941 for connected graphs, and it is a very useful theorem:

**Theorem 1.2.** *If  $G$  is a connected graph of maximum degree  $n$ , then it holds that*

$$\chi(G) \leq \Delta(G); \tag{1.4}$$

*unless  $G$  is the complete graph on  $n + 1$  vertices or an odd cycle.*

A graph  $G$  is said to be a *critical* graph if the deletion of any vertex decreases its chromatic number. Obviously, such a decrease cannot exceed 1. More specifically, a graph  $G$  is said to be  *$k$ -critical* if  $\chi(G) = k$  and  $\chi(G - v) = k - 1$  for every vertex  $v$  of  $G$ . Proceeding by induction on the number of vertices of a  $k$ -chromatic graph  $G$ , it becomes easy to see that  $G$  contains a  $k$ -critical subgraph  $G'$ . Note that there's a possibility that  $G' = G$ . This observation highlights the fact that critical graphs are the minimal members in terms of the chromatic number, which is a very important measure in the world of graph theory.

The following proposition shows a relation between the chromatic number of a critical graph and its minimum degree:

**Proposition 1.8.** *For every  $\chi(G)$ -critical graph  $G$ ,*

$$\delta(G) \geq \chi(G) - 1. \tag{1.5}$$

*Proof.* Let  $G$  be a  $k$ -critical graph and suppose to the contrary that  $\delta(G) < k - 1$ . Then there exists a vertex  $x$  of  $G$  such that  $d_G(x) \leq k - 2$ . Let  $G'$  be the graph obtained from  $G$  by removing the vertex  $x$ . According to the fact that  $G$  is a  $k$ -critical graph, it follows that  $G'$  admits a proper  $(k - 1)$ -coloring, say  $c'$ . As  $d_G(x) \leq k - 2$ , then the neighbors of  $x$  are colored properly by the coloring  $c'$  using at most  $k - 2$  colors, say  $c'(N_G(x)) \in \{1, 2, \dots, k - 2\}$ . Thus, we can extend this coloring to a proper  $(k - 1)$ -coloring of  $G$  by coloring  $x$  with the color  $k - 1$ .

This contradicts the minimality of the chromatic number  $k$ , and so it indicates that our initial supposition is wrong. Therefore, the wanted inequality holds. ■

A graph  $G$  is said to be *d-degenerate*, if any subgraph of  $G$  contains a vertex having at most  $d$  neighbors. One way to prove that a graph  $G$  is  $(d+1)$ -colorable is to show that  $G$  is  $d$ -degenerate:

**Lemma 1.1.** *Every  $d$ -degenerate graph  $G$  is  $(d+1)$ -colorable.*

*Proof.* We argue by induction on  $n = v(G)$ , say  $n$ . For  $n = 1$ , the result holds trivially. Suppose now that every  $d$ -degenerate graph of order  $n-1$  is  $(d+1)$ -colorable, and let  $G$  be a  $d$ -degenerate graph on  $n$  vertices. By definition,  $G$  contains a vertex  $x$  whose degree is at most  $d$ . Let  $G'$  be the graph obtained from  $G$  by removing  $x$ . Since  $G'$  is a subgraph of  $G$  and any subgraph of  $G'$  is also a subgraph of  $G$ , it follows that  $G'$  is  $d$ -degenerate. Applying the induction hypothesis to  $G'$ , we obtain a proper coloring  $c'$  of  $G'$  using  $d+1$  colors. Since  $d_G(x) \leq d$ , we may assume that  $c'(N_G(x)) \in \{1, 2, \dots, d\}$ . Assigning the color  $d+1$  to  $x$ , we get a proper  $(d+1)$ -coloring of  $G$ , and so the hoped result follows. ■

The following well-known lemma, which bounds from above the chromatic number of the union of graphs, will be essential for the coming proofs:

**Lemma 1.2.** *For any two graphs  $G_1$  and  $G_2$ ,*

$$\chi(G_1 \cup G_2) \leq \chi(G_1) \times \chi(G_2). \quad (1.6)$$

*Proof.* For  $i \in \{1, 2\}$ , let  $c_i : V(G_i) \rightarrow \{1, 2, \dots, \chi(G_i)\}$  be a proper  $\chi(G_i)$ -coloring of  $G_i$ . Define  $\psi$ , the coloring of  $V(G_1 \cup G_2)$ , as follows:

$$\psi(x) = \begin{cases} (c_1(x), 1) & x \in V(G_1) \setminus V(G_2); \\ (c_1(x), c_2(x)) & x \in V(G_1) \cap V(G_2); \\ (1, c_2(x)) & x \in V(G_2) \setminus V(G_1). \end{cases}$$

We may easily verify that  $\psi$  is a proper coloring of  $G_1 \cup G_2$  with color-set  $\{1, 2, \dots, \chi(G_1)\} \times \{1, 2, \dots, \chi(G_2)\}$ . Consequently, it follows that  $\chi(G_1 \cup G_2) \leq \chi(G_1) \times \chi(G_2)$ . ■

A consequence of the previous lemma is that, if we partition the edge-set of a graph  $G$  into  $E_1, E_2, \dots, E_k$ , then bounding the chromatic number of each spanning subgraph  $G_i$  of  $G$  induced by the edge-set  $E_i$  gives an upper bound for the chromatic number of  $G$ .

The following lemma gives the analogous result for the case where the vertex-set of a graph is decomposed into several subsets:

**Lemma 1.3.** *Let  $G$  be a graph whose vertex-set is decomposed into two subsets  $V_1$  and  $V_2$ , then*

$$\chi(G) \leq \chi(G_1) + \chi(G_2), \quad (1.7)$$

where  $G_1$  and  $G_2$  are defined to be the subgraphs of  $G$  induced by  $V_1$  and  $V_2$  respectively.

We say graph  $G$  is  $k$ -universal if it is contained in every  $k$ -chromatic graph. Indeed, the literature shows that  $k$ -universal graphs are amongst the highly attractive fields of inquiry. A brief historical overview of the research carried out in this area will be presented in Chapter 2.

### 1.7.2 Digraph coloring

All the above notions and notations for graph coloring are extended to digraphs by applying them to their underlying graphs.

Let  $\mathcal{D}$  be a class of digraphs. The *chromatic number* of  $\mathcal{D}$ , denoted by  $\chi(\mathcal{D})$ , is the smallest integer  $k$  such that  $\chi(D) \leq k$  for all  $D \in \mathcal{D}$ , or  $+\infty$  if no such  $k$  exists. By convention, if  $\mathcal{D} = \emptyset$  then  $\chi(\mathcal{D}) = 0$ . If  $\chi(\mathcal{D}) \neq +\infty$ , we say that  $\mathcal{D}$  has a *bounded* chromatic number.

## 1.8 $S$ -Packing coloring

Let  $G$  be a graph with vertex-set  $V(G)$  and edge-set  $E(G)$ . For a positive integer  $i$ , an  $i$ -packing in  $G$  refers to a subset  $X$  of  $V(G)$  such that the distance  $d_G(u, v)$  between any two distinct vertices  $u$  and  $v$  in  $X$  is greater than  $i$ .

The *packing chromatic number* of  $G$ , denoted by  $\chi_p(G)$ , is defined as the smallest positive integer  $k$  for which the vertex set of  $G$  can be divided into sets  $X_1, \dots, X_k$ , where each  $X_i$  is an  $i$ -packing. This partitioning corresponds to a mapping  $c : V(G) \rightarrow [k]$ , where  $X_i$  is formed by the vertices  $u$  in  $V(G)$  such that  $c(u) = i$ . Moreover, this mapping  $c$  ensures the property that if  $c(u) = c(v) = i$  and  $u \neq v$ , then the distance  $d_G(u, v)$  between  $u$  and  $v$  is greater than  $i$ . This mapping  $c$  is called a *packing  $k$ -coloring*.

Throughout this thesis, it will be useful to have the following generalization of the packing chromatic number in hand.

Given that  $S = (a_1, a_2, \dots)$  represents an infinite, non-decreasing sequence of positive integers. An  $S$ -packing coloring of  $G$  denotes a mapping  $c : V(G) \rightarrow \mathbb{N}$  where the pre-image  $c^{-1}(i)$  is an  $a_i$ -packing for every positive integer  $i$ . If such a mapping  $c$  can be established for a given graph  $G$ , it is said that  $G$  is  *$S$ -packing colorable*.

Thus, for an integer  $k$ , we say that  $G$  is  $(a_1, \dots, a_k)$ -packing colorable if  $V(G)$  can be partitioned into sets  $V_1, \dots, V_k$  such that for each  $1 \leq i \leq k$  and  $u \neq v \in V_i$ , we have  $d(u, v) \geq a_i + 1$ .

The  *$S$ -packing chromatic number* of  $G$ , denoted by  $\chi_S(G)$ , is defined to be the smallest  $k$  for which  $G$  is  $(a_1, a_2, \dots, a_k)$ -packing colorable. However, if  $G$  does not admit an  $S$ -packing  $k$ -coloring for any positive integer  $k$ , then  $\chi_S(G)$  is set to  $\infty$ .



# Part I

## Oriented paths in digraphs

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### Contents

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<b>2</b>	<b>History and new approaches for paths with two blocks</b>	<b>35</b>
2.1	$k$ -universal graphs . . . . .	35
2.2	About the existence of oriented cycles . . . . .	36
2.3	About the existence of oriented trees . . . . .	38
2.4	About the existence of oriented paths . . . . .	39
2.5	New approaches concerning paths with two blocks . . . . .	42
<b>3</b>	<b>The existence of the path <math>P(k, 1, l)</math></b>	<b>47</b>
3.1	The existence of the path $P(k, 1, l)$ in Hamiltonian digraphs . . . . .	47
3.2	The existence of $P(k, 1, l)$ in digraphs containing a Hamiltonian directed path . . . . .	51
3.3	The existence of $P(k, 1, l)$ in a $(2n + 2)$ -chromatic digraph . . . . .	51
<b>4</b>	<b>The existence of paths with three blocks in digraphs</b>	<b>53</b>
4.1	An overview of the new technique . . . . .	53
4.2	The existence of the path $P(k, 1, l)$ . . . . .	54
4.3	The existence of the path $P(k, l, r)$ in digraphs with chromatic number bounded from above . . . . .	57

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## History and new approaches for paths with two blocks

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*“ He who sees things grow from the beginning will have the clearest view of them.”*

– ARISTOTLE –

Paths and cycles (as well as oriented paths and cycles) are undoubtedly among the most extensively studied classes in the realm of graphs (and digraphs). An essential facet of studying these categories involves delving into the connection between a graph’s chromatic number  $\chi(G)$  (or a digraph’s chromatic number  $\chi(D)$ ) and the presence of paths and cycles (or oriented paths and cycles) within  $G$  (or  $D$ ) as subgraphs (or subdigraphs). To be more precise, a thought-provoking question arises regarding the chromatic number of a graph that does not contain a particular path as a subgraph. Furthermore, we might wonder whether a digraph devoid of a particular oriented cycle or path has a bounded chromatic number. These inquiries, alongside numerous others, have been explored by a considerable number of graph theorists over the years, and this chapter provides a brief discussion of these problems. In the upcoming sections, we present a range of important issues linked to these topics, some of which we are investigating in the Part I of this thesis. Following that, we provide a brief overview of the research efforts aimed to solve these problems to varying degrees. Moreover, we provide, in the last section, some new approaches concerning paths with two blocks.

### 2.1 $k$ -universal graphs

The concept of  $k$ -universal graphs has a rich history in graph theory, and it has captured the interest of numerous mathematicians and graph theorists over the years.

The celebrated Gallai-Roy Theorem [61, 37] states that every  $k$ -chromatic digraph contains a directed path of order  $k$ . Inspired by this, one can ask of which digraphs are contained in any  $k$ -chromatic digraph. Such digraphs are called  $k$ -universal.

Obviously, a graph can obtain a large chromatic number by containing a large complete sub-



graph. However, this is not the only scenario, since we may affirm that there exist graphs with large chromatic number but containing no large complete subgraphs. This fact is guaranteed due to a famous theorem of Erdős [32] which states that:

**Theorem 2.3.** (Erdős, 1959) *For any two positive integers  $g$  and  $k$ , there exists a graph  $G$  with girth larger than  $g$  and chromatic number larger than  $k$ .*

In deed, this theorem guarantees the existence of graphs with an arbitrarily high chromatic number and an arbitrarily high girth. It is clear that for a graph containing a complete subgraph of order at least 3, the girth is 3, so this theorem proves the mentioned fact.

Hence, for a family of digraphs  $\mathcal{D}$  where one of its elements contains an oriented cycle of a specific length, it is impossible to find an integer  $k$  such that any  $k$ -chromatic digraph contains every element of  $\mathcal{D}$ . This leads mathematicians to reduce the research area into studying the following two interesting main categories of digraphs: oriented trees ( since they contain no cycles) and oriented cycles with no upper bound on their length.

## 2.2 About the existence of oriented cycles

### 2.2.1 Oriented cycles in digraphs

On one hand, Bondy [8] demonstrated that every strong digraph contains a circuit whose length is greater than its chromatic number. This assertion holds under the condition of the digraph being strong, since for example, any acyclic  $k$ -chromatic digraph is defined to be devoid of cycles and, therefore, incapable of including any circuit, regardless of its length. Remarkably, this condition of strength might also be necessary when studying non-directed cycles. This hypothesis finds partial validation in the work of Gyárfás and Thomassen [2], who proved the existence of acyclic digraphs with arbitrarily large chromatic numbers, yet containing no oriented cycles with two blocks.

This result was extended into a broader context by Cohen et al. [22] to any number of blocks. More specifically, they established a stronger theorem, depending on a construction introduced by Erdős and Lovász [34], which proves the existence of hypergraphs having a high girth and a large chromatic number. To be precise, for any positive integers  $b$  and  $c$ , they demonstrated the existence of an acyclic digraph  $D$  with  $\chi(D) \geq c$ , where all oriented cycles within this digraph consist of more than  $b$  blocks.

Hence, it becomes evident that merely working within the general form of the digraph is insufficient; additional properties, such as being strong or Hamiltonian, need to be incorporated before working on the quest to establish the existence of subdivisions of non-circuit cycles. Along these lines, Cohen et al. [22] conjectured in 2018 that Bondy's theorem can be extended to encompass all oriented cycles:

**Conjecture 1.** *For each oriented cycle  $C$ , there exists a constant  $f(C)$  such that any strongly connected digraph with a chromatic number of at least  $f(C)$  includes a subdivision of  $C$ .*

In their paper, Cohen et al. [22] solved this conjecture for the scenario of cycles with two blocks. To be more precise, they demonstrated that the chromatic number of strong digraphs containing no subdivisions of two-block cycles  $C(k_1, k_2)$  is upper-bounded by  $O((k_1 + k_2)^4)$ :

**Theorem 2.4.** *Consider two positive integers  $k_1$  and  $k_2$  satisfying the conditions  $k_1 \geq k_2 \geq 2$  and  $k_1 \geq 3$ . Let  $D$  be a strong digraph with no subdivisions of  $C(k_1, k_2)$  cycles. Then, the chromatic number of  $D$  does not exceed  $(k_1 + k_2 - 2)(k_1 + k_2 - 3)(2k_2 + 2)(k_1 + k_2 + 1)$ .*

In their proof, Cohen et al. heavily depend on the strategies of leveling and digraph decomposition : Given that a strong digraph  $D$  has an out-branching  $T$ , as clarified in Proposition 1.4, they proceeded to divide the arc set of  $D$  into four distinct sets,  $A_0, A_1, A_2, A_3$ , based on the levels of  $T$ . Consequently, they properly colored each subdigraph  $D_i$  of  $D$ , defined with an arc set  $A_i$ . Through this process, their proof attains completion, guided by the recognition that  $\chi(H) \leq \chi(H_1) \times \chi(H_2)$  for any pair of subdigraphs  $H_1$  and  $H_2$  such that  $H = H_1 \cup H_2$ . Kim et al. [52] recently improved this bound, progressing from  $O((k_1 + k_2)^4)$  to  $O((k_1 + k_2)^2)$  based on a distinct methodology rooted in the concept of cycle-tree ordering and the utilization of contracting and un-contracting strategies.

**Theorem 2.5.** *(Kim et al., 2018) Consider two positive integers  $k_1$  and  $k_2$  satisfying the condition that  $k_1 \geq k_2 \geq 1$  and  $k_1 \geq 2$ . Given a strong digraph  $D$  with no subdivisions of  $C(k_1, k_2)$  cycles, then the chromatic number of  $D$  does not exceed  $2(2k_1 - 3)(k_1 + 2k_2 - 1)$ .*

## 2.2.2 Oriented cycles in tournaments

Regarding cycles in tournaments, Camion [21] showed early that a tournament contains a Hamiltonian directed cycle if and only if it is strong. Shifting the focus to non-directed cycles, Thomassen [68] in 1973 demonstrated that for any  $n \geq 50$ , a tournament of order  $n$  includes a Hamiltonian antidirected cycle. Subsequently, Rosenfeld [62], in 1974, improved Thomassen's result to  $n \geq 28$  and proposed a conjecture which states that there exists an integer  $N \geq 9$  such that any tournament of order  $n \geq N$  contains any non-directed Hamiltonian cycle. This conjecture was approached through various methods, leading to several noteworthy results.

Initially, Grünbaum [44] established the existence of cycles with a block of length  $n - 1$ . In 1983, Benhocine and Wojda [6] proved the existence of cycles with two blocks. Concurrently, Petrové [58] improved Thomassen's and Rosenfeld's bounds by demonstrating the same result for  $n \geq 16$ .

The first comprehensive proof of Rosenfeld's conjecture was provided by Thomason [67], covering tournaments of order  $n \geq 2^{128}$ . Thomason, however, suggested that the conjecture likely holds for tournaments of order 9 and above. Subsequently, Havet [50] improved this result by proving the correctness of the conjecture for tournaments of order  $n \geq 68$ . Then, Ayman El-Zein, in [31], solved Rosenfeld's cycles conjecture, establishing that any tournament contains every Hamiltonian non-directed cycle with only 30 exceptions, all of which have orders less than 9.

## 2.3 About the existence of oriented trees

In this section, we deal with oriented trees in digraphs and in tournaments, excluding paths which we will separately discuss in the next section.

### 2.3.1 Oriented trees in tournaments

In the realm of trees within tournaments, Sumner (as referenced in [70]) introduced a conjecture in the early 1970s, which reads as follows:

**Conjecture 2.** *Every tournament of order  $2n - 2$  contains any oriented tree of order  $n$ .*

The sharpness of this bound can be easily shown since a regular tournament of order  $2k - 3$  has no vertex of out-degree at least  $k - 1$ , and thus, it does not contain the oriented tree  $S_k^+$  consisting of a vertex dominating  $k - 1$  leaves. Notably, Reid and Wormald [59] confirmed Sumner's conjecture for near-regular tournaments. Here, a  $k$ -tournament is called near-regular if  $k$  is even, and each vertex is of an out-degree either  $\frac{k}{2}$  or  $\frac{k}{2} - 1$ . However, the conjecture in its general form remains unresolved. Over time, this conjecture has been the subject of a series of research, each notable for its contribution in progressively reducing the previously established bounds.

Wormald [70] proved that any tournament with  $n \log_2(2n/e)$  vertices contains every oriented tree on  $n$  vertices. Häggkvist and Thomason [46] established the first linear bound,  $12n$ , and Havet, in [49], further decreased it to  $7.6n$ . It's worth noting that these two results shared common proving techniques, specifically defined as the concept of a " $k$ -heart" of a tree.

Havet and Thomassé [51] reduced this bound to  $4n - 6$  and then, through complicated technical arguments, further reduced it to  $(7n - 5)/2$ . Additionally, Havet and Thomassé extended their confirmation of Sumner's conjecture to cover out-branchings and in-branchings. These achievements were reached through innovative applications of median orders.

Utilizing similar principles, El Sahili [23] achieved the most recent and optimal bound, demonstrating that any  $(3n - 3)$ -tournament includes any tree of order  $n$ .

### 2.3.2 Oriented trees in digraphs

Burr [20] considered the function  $f(k)$  such that every oriented tree of order  $k$  is  $f(k)$ -universal. In his work, he provided a proof for  $f(k) \leq (k-1)^2$  and conjectured that  $f(k) = 2k - 2$ ,

**Conjecture 3.** *(Burr, 1980)  $f(k) = 2k - 2$ , i.e., every oriented tree of order  $k$  is  $(2k - 2)$ -universal.*

This conjecture is a generalization of Sumner's conjecture which we mentioned previously in Subsection 2.3.1. It holds a significant position in this thesis, being among the main conjectures we investigate.

In 2013, Addario-Berry et al. [1] proved several results which improves the one of Burr. To be more specific, they showed that  $f(k) \leq k^2/2 - k/2 + 1$ . This is the best reached upper bound

for a general oriented tree. Moreover, they studied the universality of antidirected trees, which are oriented trees such that every vertex has an in-degree of 0 or out-degree of 0. Burr [20] showed that every digraph  $D$  with at least  $4(k-1)|V(D)|$  arcs includes all antidirected trees of order  $k$ . He deduced that any antidirected tree of order  $k$  is  $(8k-7)$ -universal. Addario-Berry et al. [1] improved this bound to  $(5k-9)$  (for  $k \geq 2$ ), and subsequently proved Conjecture 3 for antidirected trees of diameter 3. They gave some sufficient conditions for a  $k$ -chromatic digraph to contain some oriented tree. In particular, they proved that every acyclic  $k$ -chromatic digraph contains any oriented tree of order  $k$ .

Finally, they conjectured that if  $|E(D)| > (k-2)|V(D)|$ , then the digraph  $D$  contains every antidirected tree of order  $k$ .

## 2.4 About the existence of oriented paths

Oriented paths, as a class of oriented trees, have been the interest of study of several graph theorists. This section is divided into two parts: the first part discusses research on the existence of oriented paths in tournaments, while the second part provides a historical overview of successive contributions regarding the existence of oriented paths in digraphs. This historical context is particularly relevant as it forms the basis for the discussion and contributions presented in Part I of our thesis.

### 2.4.1 Oriented paths in tournaments

For tournaments, it is well known that every tournament contains a Hamiltonian directed path. This result was first achieved by Rédei in [60]. In addition, he proved in the same paper that the number of Hamiltonian directed paths in every tournament is odd. In 1972, Rosenfeld [62] conjectured that there exists an integer  $K \geq 8$  such that any  $n$ -tournament,  $n \geq K$ , contains any Hamiltonian oriented path. Alspach, Rosenfeld [4] and Straight [63] proved the correctness of Rosenfeld's conjecture for Hamiltonian paths with two blocks. Forcade [36] proved Rosenfeld's conjecture for any tournament of order  $2^k$  in 1973. Thomason [67], in 1986, was the first in providing a comprehensive solution by demonstrating that there exists  $n_0 < 2^{128}$  such that any tournament with an order of  $n \geq n_0$ , includes every possible Hamiltonian oriented path. Havet and Thomassé [48] conclusively solved the problem by providing a proof that there exist only three exceptions identified by Grünbaum that do not satisfy Rosenfeld's conjecture. Havet and Thomassé's proof involves refining a fundamental concept initially introduced by Thomason. It asserts that any set of  $b_1 + 1$  vertices in an  $n$ -tournament includes an origin of any  $(n-1)$ -path with its first block having a length of  $b_1$ . They established that when  $s^+(x, y) \geq b_1 + 1$ , either vertex  $x$  or  $y$  serves as the origin of a copy of such a path, where  $s^+(x, y)$  is defined as the cardinality of the set  $\{z \in T : z \text{ can be reached from } x \text{ or } y \text{ by a directed path}\}$ . Their enhanced approach enabled them to conclude that demonstrating the existence of an oriented path with an order of  $(n-1)$  within any  $n$ -tournament  $T$  is equivalent to confirming the existence of any Hamiltonian path  $P$  in the tournament. This equivalence holds true unless the pair  $(T, P)$  falls

into one of the 69 exceptional cases, each of which was verified individually. In [9], Bou Hanna gives a new simple proof of the result without the need to treat all the mentioned exceptions individually.

## 2.4.2 Oriented paths in digraphs

When studying the existence of oriented paths, which are special forms of oriented trees, we are able to achieve better results than those established for the general form of an oriented tree, as mentioned in the previous section. When presenting the research efforts focused on the existence of paths in digraphs, we will progressively explore the results in an increasing manner of the blocks number in the path.

Starting with the case of an oriented path with one block, that is the directed path, the most celebrated result is known as Gallai-Hasse-Roy-Vitaver theorem [37, 47, 61, 69]. It confirms that any  $n$ -chromatic digraph contains a directed path of length  $n - 1$ :

**Theorem 2.6.** *If  $D$  is an  $n$ -chromatic digraph with  $n \geq 2$ , then  $D$  contains a directed path of length  $n - 1$ .*

This result, which marked the initial exploration for the existence of a directed path in an  $n$ -chromatic digraph  $D$ , can be easily established by considering a final forest  $F$  of  $D$ , and looking at the directed path joining the root with any vertex located on the maximum level of  $F$ . It is noteworthy that  $L_i(F)$  forms a stable set in  $D$  for all  $i \geq 1$ , which implies that the number of levels in  $F$  should be at least equal to the chromatic number of  $D$  which is  $n$ .

Transitioning gradually to oriented paths with two blocks, which is a well studied class of oriented trees, we consider El-Sahili's conjecture [25] proposed in 2004. It states that every  $n$ -chromatic digraph with  $n \geq 4$ , contains any oriented path with two blocks.

Following this, El-Sahili and Kouider [24], who introduced the concept of maximum forest, made a significant progress toward verifying El-Sahili's conjecture. They proved that every  $(n + 1)$ -chromatic digraph contains any path of order  $n$  with two blocks. But, it's important to bear in mind that in their argument, they did not utilize paths consisting of more than four vertices.

**Theorem 2.7.** *Every  $(n + 1)$ -chromatic digraph,  $n \geq 3$ , contains any path of length  $n - 1$  with two blocks.*

In 2007, Addario-Berry, Havet and Thomassé [2] used strongly connected digraphs and a theorem of Bondy [8] to prove the correctness of El-Sahili's conjecture:

**Theorem 2.8.** *Every  $n$ -chromatic digraph with  $n \geq 4$ , contains any path with two blocks of length  $n - 1$ .*

Then, El-Sahili et al. [27] gave a new elementary proof without using strongly connected digraph.

After that, El-Sahili [26] conjectured the following :

**Conjecture 4.** *Any  $n$ -chromatic digraph with  $n \geq 8$  contains any oriented path of length  $n - 1$ .*

After the case of two blocks was solved, the case of three blocks remained open. Concerning the general case of paths with three blocks, the first linear bound was given by El Joubbeh [29] who proved that any path  $P(k, r, l)$  of length  $n - 1$  is contained in any  $(4.6n)$ -chromatic digraph. We notice by observing the studies done in this direction that this bound can be improved when considering special types of paths with three blocks by adding some conditions on the length of each block.

Mortada et al. [56] proved that the path  $P(n - 3, 1, 1)$  is contained in any  $(n + 1)$ -chromatic digraph. In addition, Mortada et al. [57] proved that any  $(2k + 1)$ -chromatic digraph contains a  $P(1, k, 1)$ .

El Joubbeh et al. [30] recently improved slightly the bound in the case of three and four blocks paths and established a bound for the general case of Conjecture 4; they proved that any oriented path of order  $n$  with  $t$  blocks is contained in every  $(4^{r(t)}n + q(t))$ -chromatic digraph where  $r(t) = \lfloor \log_2(t - 1) \rfloor$  and  $q(t) = -\frac{20}{7} \cdot 8^{r(t)-1} + \frac{6}{7}$ .

In Chapter 3, we deal with the existence of any  $P(k, 1, l)$ . We prove that any  $(2n + 2)$ -chromatic digraph contains any path  $P(k, 1, l)$  of length  $n - 1$ . Moreover, by putting an additional condition on the  $n$ -chromatic digraph  $D$  which is that it contains a Hamiltonian directed path, then we prove that it contains any  $P(k, 1, l)$  of length  $n - 1$ . By this, we prove the correctness of Conjecture 4 for in this special case.

In Chapter 4 We show an improvement of the bound reached in [65] presented in Chapter 3 concerning the path  $P(k, 1, l)$  from  $2n + 2$  into  $2n - 3$ . By achieving this bound, we assert that Conjecture 3 holds for this special type of oriented trees.

Furthermore, concerning the general case of three blocks paths  $P(k, l, r)$ , we show an improvement of the bound reached by El Joubbeh [29]. We partition the problem into three cases according to the values of  $k$ ,  $l$  and  $r$ . We establish a new linear bound for the chromatic number for digraphs lacking a  $P(k, l, r)$  which is  $2(n - 1) + r$ ,  $2(n - 1) + l + r - k$  and  $2(n + l - 1) - k$  in the three cases of the problem respectively.

The bound that we established is at most  $3n - 6$  for two cases of the problem and  $4n - 13$  for the remaining case in its worst scenario, and so it is the first linear bound that hits  $3n$ .

## 2.5 New approaches concerning paths with two blocks

The proofs that we present in this section have served as the bedrock of our thinking to the achievements in this thesis, marking the evolution of our thinking and paving the way for the new contributions that unfold in the pages ahead. They have not only illuminated key aspects of graph theory but have also provided the spark of inspiration that fueled our curiosity and guided our quest for new results. As we mentioned in Section 2.4, the existence of paths with two blocks in  $n$ -chromatic digraphs was conjectured by El Sahili and solved first by Addario-Berry et al. . It is worthy to note that before proving the conjecture, El Sahili and Bondy [25] had given an elementary proof of this conjecture in a special case in which one of the blocks is of length 1. We explain their proof and provide another one. Concerning the general case  $P(k, l)$ , we give an elementary proof of the existence of this path in  $n$ -chromatic digraphs containing a Hamiltonian directed path.

### 2.5.1 Bondy-El Sahili Theorem about path with two blocks

**Theorem 2.9.** (*J.A.Bondy-A.El Sahili, [25]*) *Let  $n$  be an integer greater than three. Then, any  $n$ -chromatic digraph contains a path of length  $n - 1$  formed by two blocks, one of which is of length 1.*

*Proof.* Let  $D$  be an  $n$ -chromatic digraph, and suppose without loss of generality that  $D$  is  $n$ -critical. It is required to prove that  $D$  contains a path  $P(n - 2, 1)$ .

let  $Q = v_1v_2\dots v_t$  be the longest directed path in  $D$ . By Gallai-Roy theorem [37], we conclude that  $t \geq n$ . If  $v_t$  has a neighbor  $w \in V(D) - V(Q)$ , then  $w$  is an inneighbor of  $v_t$ . Since otherwise  $l(Q \cup (v_t, w)) > l(Q)$ , a contradiction. In this case  $Q \cup (w, v_t)$  contains  $P(n - 2, 1)$ .

Now suppose that  $N(v_t) \subseteq Q$ . Let  $v_s$  be the neighbor of  $v_t$  of minimal index. Since  $D$  is  $n$ -critical, then  $d_{G[D]}(v_t) \geq n - 1$ , and so  $t - s \geq n - 1$ . If  $v_s$  is an inneighbor of  $v_t$  then the path  $v_{t-n+2}\dots v_tv_s$  is a  $P(n - 2, 1)$ . Otherwise:

- If  $s > 1$ , then  $v_{s+1}\dots v_tv_sv_{s-1}$  contains a path  $P(n - 2, 1)$ .
- If  $s = 1$ , then  $v_1v_2\dots v_tv_1$  is a directed cycle, and we have two cases:

Case 1:  $V(D) \neq \{v_1, v_2, \dots, v_t\}$ . Let  $w$  be a vertex of  $D$  such that  $wv_i$  is an edge of  $G[D]$  which is arbitrary oriented with  $1 \leq i \leq t$ . Then  $D$  contains a directed path of length  $t$  which is a contradiction.

Case 2:  $V(D) = \{v_1, v_2, \dots, v_t\}$ . Since  $D$  is not a tournament then  $t \geq n + 1$ . Let  $v_iv_j$  be a chord intercepting a path of minimal length of the Hamiltonian cycle. Without loss of generality suppose that  $i = 1$  and  $v_1\dots v_j$  is the path of minimal length intercepted by  $v_iv_j$ .

We claim that  $t - j + 2 \geq n$ . This is obvious for  $j = 3$ .

If  $j > 3$ , then we have  $N(v_2) \setminus \{v_1, v_3\} \subseteq \{v_{j+1}, \dots, v_{t-j+3}\}$ . Since otherwise we get a chord intercepting a path of length less than  $j - 1$ . Now since  $D$  is  $n$ -critical, then  $N(v_2)$  contains at least  $n - 1$  vertices, so:

$$t - j + 2 \geq d(v_2) + 1 \geq n.$$

It follows that  $v_{j+1}\dots v_tv_1v_jv_{j-1}$  contains a path  $P(n - 2, 1)$ . ■

### 2.5.2 Another proof of Bondy-El Sahili theorem

The new proof that we provide in this section, is similar to the previous one in its first part which is dealing with the longest directed path. The difference appears when they consider the chord intercepting a path of minimal length of the Hamiltonian cycle and have benefit from  $D$  being not a tournament. However, we study some orientations of the arcs reaching the end vertex  $v$ . We consider the predecessor of  $v$  on the path and proceed by induction to prove that all its neighbors are inneighbors.

*Proof.* Suppose to the contrary that  $D$  contains no  $P(n-2, 1)$ . Suppose without loss of generality that  $D$  is  $n$ -critical. Let  $P$  be the longest directed path in  $D$ , then by Gallai-Roy theorem [37], we get  $l(P) \geq n-1$ . Let  $u$  be the origin of  $P$  and  $v$  its end.

- $v$  has no outneighbor  $w \notin P$ , since otherwise  $l(P \cup (v, w)) \geq l(P)$ , a contradiction.
- $v$  has no inneighbor  $w \notin P$  since otherwise  $P \cup (w, v)$  contains a  $P(n-2, 1)$ , a contradiction. Then,  $N_{G[D]}(v) \subseteq V(P)$ . Let  $N_{G[D]} = \{u_i, 1 \leq i \leq s\}$ . Since  $D$  is considered  $n$ -critical, then  $d_{G[D]}(v) \geq n-1$  and so  $s \geq n-1$ . Suppose that the neighbors of  $v$  are located on  $P$  in an increasing order of the index  $i$ .
- $(v, u_1) \in E(D)$  since otherwise,  $P_{[u_2, v]} \cup (u_1, v)$  contains  $P(n-2, 1)$ .
- $(u_2, v) \in E(D)$  since otherwise,  $P_{[u_3, v]} \cup (v, u_2) \cup (u'_2, u_2)$  contains a  $P(n-2, 1)$  where  $u'_2$  is the predecessor of  $u_2$  on  $P$ .
- $(v, u_{n-2}) \in E(D)$  since otherwise,  $P_{[u_1, u_{n-2}]} \cup (u_{n-2}, v) \cup (u_s, v)$  contains a  $P(n-2, 1)$ . Consequently we get that  $u = u_1$  since otherwise,  $l(P_{[u, u_{n-2}]}) \geq n-2$  and so  $P_{[u, u_{n-2}]} \cup (v, u_{n-2})$  contains a  $P(n-2, 1)$ .

Consider the vertex  $u_s$ :

$u_s$  has no neighbor outside  $P$  since otherwise, we get a contradiction with the fact that  $P$  is the longest directed path in  $D$ . So,  $N_{G[D]}(u_s) \subseteq V(P)$ .

- $N(u_s) = \{u_1, u_2, \dots, u_{n-2}, v\}$  since, suppose  $u_s$  has a neighbor  $w \notin \{u_1, u_2, \dots, u_{n-2}, v\}$ , then:

If  $w \in P_{[u_1, u_{n-2}]}$  then  $P_{[u_1, u_{n-2}]} \cup (v, u_{n-2})$  contains a  $P(n-2, 1)$ .

If  $w \in P_{[u_{n-2}, v]}$  then  $P_{[u_3, v]} \cup (u_2, v)$  contains  $P(n-2, 1)$ .

Studying some orientations :

- $(u_1, u_s) \in E(D)$  since otherwise  $(u_s, u_1) \cup P_{[u_1, u_{n-2}]} \cup (v, u_{n-2})$  contains a  $P(n-2, 1)$ .
- $(u_2, u_s) \in E(D)$  since otherwise  $(u_1, u_s) \cup (u_s, u_2) \cup P_{[u_2, u_{n-2}]} \cup (v, u_{n-2})$  contains a  $P(n-2, 1)$ .

Suppose  $(u_i, u_s) \in E(D)$  and let us prove that  $(u_{i+1}, u_s) \in E(D)$ :

Otherwise,  $P_{[u_1, u_i]} \cup (u_i, u_s) \cup (u_s, u_{i+1}) \cup P_{[u_{i+1}, u_{n-2}]} \cup (v, u_{n-2})$  contains a  $P(n-2, 1)$ .

So,  $(v, u_1) \cup P_{[u_1, u_{n-3}]} \cup (u_{n-3}, u_s) \cup (u_{n-2}, u_s)$  contains a  $P(n-2, 1)$ , see Fig.2.1, a contradiction.

This completes the proof. ■



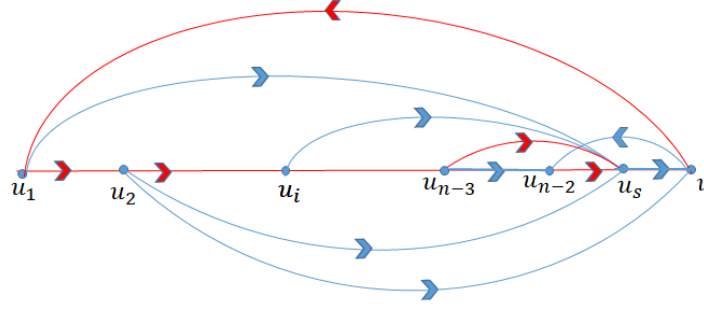


Figure 2.1:  $N(u_s) = N_P(u_s) = N_P^+(u_s)$  and the red marked path is  $P(n - 2, 1)$

### 2.5.3 The existence of the path $P(k, l)$ in a particular class of digraphs

Here we give a simple proof showing the correctness of El Sahili's conjecture for two blocks paths in a specific class of digraphs which is that containing a Hamiltonian directed path. The procedure that we followed in the proof, has inspired us to deal with the case of paths with three blocks in such special classes of digraphs.

**Lemma 2.4.** *Let  $D$  be an  $n$ -chromatic Hamiltonian digraph then  $D$  contains any path of length  $n - 1$  with two blocks  $P(k, l)$  such that  $k + l = n - 1$ .*

*Proof.* If  $D$  is a tournament then  $D$  contains any  $P(k, l)$  such that  $k + l = n - 1$ .

So, it is sufficient to study the case when  $D$  is not a tournament.

Suppose to the contrary that there exists  $k, l$  such that  $k + l = n - 1$  and  $D$  contains no  $P(k, l)$ .

Let  $m = v(D)$  and  $C = v_1 v_2 \dots v_m$  be a Hamiltonian circuit in  $D$ .

We may consider, without loss of generality, that  $k \leq l$ .

Suppose that  $v_1$  is of maximum degree  $\Delta$ . Note that since  $D$  is not a tournament, we get  $m \geq n + 1 = k + l + 2$  and consequently,  $m - l + 1 \neq k + 2$ .

If  $D$  contains any chord joining  $v_1$  and  $v_j$  such that :

$$C_{[v_1, v_j]} \geq k + 1 \text{ and } C_{[v_j, v_1]} \geq l + 1 \text{ or}$$

$$C_{[v_j, v_1]} \geq k + 1 \text{ and } C_{[v_1, v_j]} \geq l + 1,$$

then we get:  $v_2 v_3 \dots v_j v_1 v_m \dots v_{m-l+1}$  contains a  $P(k, l)$  whatever the orientation of the edge  $v_1 v_j$  is .

So such an arc does not exist, and  $v_1 v_j \notin E(G[D])$  for all  $j \in \{k + 2, \dots, m - l\}$ . Thus,  $N_{G[D]}(v_1) \subseteq \{v_2, \dots, v_{k+1}\} \cup \{v_{m-l+1}, \dots, v_m\}$  so  $|N(v_1)| \leq k + l = n - 1$ .

Since  $D$  is neither a tournament nor an odd cycle, then using [19] we get :  $\chi(D) \leq \Delta \Rightarrow \chi(D) \leq d_{G[D]}(v_1) \leq n - 1$ , a contradiction. Therefore,  $D$  contains any path  $P(k, l)$ . ■

Based on this lemma, and following a similar approach to that used in Section 3.2, we were able to prove the following:

**Theorem 2.10.** *Let  $D$  be an  $n$ -chromatic digraph containing a Hamiltonian directed path. Then,  $D$  contains any  $P(k, l)$  such that  $k + l = n - 1$ .*

## 2. History and new approaches for paths with two blocks

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We will omit the proof because a similar procedure will be explained in Chapter 3 (in Section 3.2).



### 3

## The existence of the path $P(k, 1, l)$

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*“Success is a journey, not a destination. The doing is often more important than the outcome.”*

– ARTHUR ASHE –

In this chapter, we show some contributions for studying conjecture 4. As we have mentioned in Chapter 2, Conjecture 4 is affirmed by Addario-Berry et al. [2] for the case of two blocks paths. Interests are then directed towards studying the existence of paths with three blocks. Before delving into the general case of paths with three blocks, a question arises: Under what constraints on the length of each block, or on the class of digraphs we can affirm the conjecture? When studying particular cases of paths with three blocks, several bounds are established. First, Mortada et al. [56] proved that the path  $P(n - 3, 1, 1)$  is contained in any  $(n + 1)$ -chromatic digraph. In addition, Mortada et al. [57] proved that any  $(2k + 1)$ -chromatic digraph contains a  $P(k, 1, 1)$ . In this chapter, we deal with the path with three blocks with adding a condition on the length of one of its blocks, which is the path  $P(k, 1, l)$ . We first study its existence in a special class of digraphs which is the class of Hamiltonian digraphs. In this case, we prove [65] the correctness of Conjecture 4 for the path  $P(k, 1, l)$ . Then, we go further into a more general class which is digraphs containing a Hamiltonian directed path. To establish the correctness of the conjecture within this class, we rely on our previously proven result concerning Hamiltonian digraphs.

Following that, we proceed to eliminate any restrictions imposed on the class of digraphs. Thus, we deal with the general case of digraphs. We prove [65] that  $(2n + 2)$ -chromatic digraphs contain any path  $P(k, 1, l)$  of length  $n - 1$ . This result will be improved using a new technique which we explain in details in Chapter 4.

### 3.1 The existence of the path $P(k, 1, l)$ in Hamiltonian digraphs

In [65] we are able to prove the existence of any  $P(k, 1, l)$  of length  $n - 1$  in any  $n$ -chromatic Hamiltonian digraph. In our proof, we take advantage of Brooks' theorem [19] which states

that:

**Theorem 3.11.** *For every connected graph  $G$  that is neither an odd cycle nor a complete graph,  $\chi(G) \leq \Delta(G)$ .*

**Remark:** Since in tournaments with  $n$  vertices, the existence of any oriented path of length  $n - 1$  is already proved [48], in particular the path  $P(k, 1, l)$  of length  $n - 1$ , it follows that all the digraphs considered in this section are not tournaments.

In order to reach the proofs of our main results, we prove the following lemma:

**Lemma 3.5.** *Let  $D$  be a Hamiltonian digraph with  $\chi(D) \geq n$  such that  $D$  contains no  $P(k, 1, l)$  of length  $n - 1$  for some  $k, l \in \mathbb{N}^*$ . Then,  $\Delta(D) = n$ .*

*Proof.* Let  $D$  be a Hamiltonian digraph with  $\chi(D) \geq n$  such that  $D$  contains no  $P(k, 1, l)$  for some  $k, l \in \mathbb{N}^*$ . Let  $v(D) = m$  and  $C = v_1 v_2 \dots v_m$  be a Hamiltonian circuit in  $D$ . Since  $n = k + 1 + l + 1 > 3$ , we get that  $D$  is not an odd cycle. Besides, as  $D$  is not a tournament, then by Theorem 3.11,  $\chi(D) \leq \Delta(D)$ .

We still have to prove that  $\Delta(D) \leq n$ .

For every  $v_t \in V(D)$ , we define the vertices  $a_t$  and  $a'_t$  and the set  $A_t$  such that:

- $l(C_{[v_t, a_t]}) = k + 1$ .
- $l(C_{[a'_t, v_t]}) = l + 1$ .
- $A_t = V(C_{[a_t, a'_t]})$ .

To avoid any confusion, set  $v_{t+1} = v_1$  for  $t = m$ , and  $v_{t-1} = v_m$  for  $t = 1$ .

We are going to show that  $|N(v_t) \cap A_t| \leq 2$ :

If  $v_t$  has two inneighbors in the set  $A_t$ , say  $v_i$  and  $v_j$  with  $i < j$ , then:

$C_{[v_{t+1}, v_i]} \cup (v_i, v_t) \cup (v_j, v_t) \cup C_{[v_j, v_{t-1}]}$  contains a path  $P(k, 1, l)$ , a contradiction. So,  $v_t$  has at most one inneighbor in the set  $A_t$ .

If  $v_t$  has two outneighbors in the set  $A_t$ , say  $v_i$  and  $v_j$  with  $i < j$ , then:

$C_{[v_{t+1}, v_i]} \cup (v_t, v_i) \cup (v_t, v_j) \cup C_{[v_j, v_{t-1}]}$  contains a path  $P(k, 1, l)$ , a contradiction. Thus,  $v_t$  has at most one outneighbor in the set  $A_t$ . Hence, we have

$$|N(v_t)| \leq |N(v_t) \cap A_t| + |V(C) - A_t| \leq 2 + k + l = n \quad (*)$$

Therefore,  $\Delta(D) \leq n$ , and the required equation holds. ■

As a direct conclusion from Lemma 3.5:

Every  $(n + 1)$ -chromatic Hamiltonian digraph  $D$  contains any path  $P(k, 1, l)$  of length  $n - 1$ , since otherwise we have  $\Delta(D) = n$ , and so  $\chi(D) \leq n$ , a contradiction.

### 3. The existence of the path $P(k, 1, l)$

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One can easily notice that a cycle of type  $C(1, r)$  such that  $1 + r \geq n$ , contains any path  $P(k, 1, l)$  of length  $n - 1$ .

Before proving the main theorem in this section, we are going to draw attention to the following remark and prove it in order to use it in what follows.

**Remark 3.1.**

If  $D$  is an  $n$ -chromatic digraph with  $v(D) = n + 1$ , then  $D$  contains an  $n$ -tournament.

*Proof.* Let  $S_1, \dots, S_n$  be  $n$  stable sets covering  $V(D)$ . Since  $v(D) = n + 1$ , so there exists  $i_0 \in \{1, \dots, n\}$  such that  $|S_{i_0}| = 2$  and  $|S_i| = 1$  for every  $i \neq i_0$ . We have  $D[\bigcup_{i \neq i_0} S_i]$  is an  $(n - 1)$ -tournament  $T_{n-1}$  since if there exists  $\{i, j\} \subset \{1, \dots, n\}$  such that  $S_i \cup S_j$  is stable in  $D$ , then  $\chi(D) \leq n - 1$  which is a contradiction. There exists  $x \in S_{i_0}$  such that  $x$  is adjacent to all the vertices in  $\bigcup_{i \neq i_0} S_i$  since otherwise the vertices of  $S_{i_0}$  can be added to  $\bigcup_{i \neq i_0} S_i$  forming  $n - 1$  stable sets in  $D$  covering it, a contradiction with  $\chi(D) = n$ . Therefore  $D[T_{n-1} \cup \{x\}]$  is an  $n$ -tournament. ■

**Theorem 3.12.** *Let  $D$  be an  $n$ -chromatic Hamiltonian digraph. Then  $D$  contains any  $P(k, 1, l)$  of length  $n - 1$ .*

*Proof.* Let  $C = v_1 v_2 \dots v_m$  be a Hamiltonian circuit in  $D$ . Suppose, without loss of generality, that  $d(v_1) = \Delta(D)$ . We proceed the proof by induction on  $v(D) = m \geq n + 1$ .

It is true for  $m = n + 1$ . In this case, according to Remark 3.1,  $D$  contains a tournament  $T$  of order  $n$  which contains any  $P(k, 1, l)$ .

Let us prove it for  $m \geq n + 2$ , assuming that it is true up to  $m - 1$ .

Suppose to the contrary that  $D$  contains no  $P(k, 1, l)$  for some  $k, l \in \mathbb{N}^*$ .

Using Lemma 3.5, we have  $d(v_1) = n$  and so by (\*), we get:

- $V(C_{[v_2, v_{k+1}]}) \cup V(C_{[v_{m-l+1}, v_m]}) \subset N(v_1)$ .
- $v_1$  has one inneighbor and another outneighbor in the set  $A_1$ .

To continue our proof, we need to consider two cases concerning the values of  $k$  and  $l$ .

- Case 1:  $k > 2$ .

We have  $(v_1, v_3) \in E(D)$ , since otherwise  $(v_3, v_1) \cup C_{[v_3, v_1]}$  is a  $C(1, m - 2)$ , a contradiction.

Now consider the vertex  $v_2$ , using (\*), we have:

$$|N(v_2)| \leq |V(C_{[v_3, v_{k+2}]}) \cup V(C_{[v_{m-l+2}, v_1]})| + 2.$$

We are going to prove that  $v_4 \notin N(v_2)$ . Indeed:

If  $(v_4, v_2) \in E(D)$ , then  $(v_4, v_2) \cup C_{[v_4, v_2]}$  is a  $C(1, m - 2)$ .

If  $(v_2, v_4) \in E(D)$ , then  $(v_1, v_3) \cup (v_2, v_3) \cup (v_2, v_4) \cup C_{[v_4, v_1]}$  is a  $C(1, m - 1)$ .

So, in both cases,  $D$  contains a  $P(k, 1, l)$ , a contradiction.

Also,  $v_5 \notin N(v_2)$ , since otherwise:

If  $(v_5, v_2) \in E(D)$ , then  $(v_5, v_2) \cup C_{[v_5, v_2]}$  is a  $C(1, m - 3)$ .

If  $(v_2, v_5) \in E(D)$ , then  $(v_1, v_3) \cup (v_2, v_3) \cup (v_2, v_5) \cup C_{[v_5, v_1]}$  is a  $C(1, m - 2)$ .

So, in both cases,  $D$  contains a  $P(k, 1, l)$ , a contradiction.

Thus,  $d(v_2) \leq n - 2$ .

Now consider the digraph  $D' = D - \{v_2\}$ . Note that  $\chi(D') = \chi(D) = n$  and  $C_{[v_3, v_1]} \cup (v_1, v_3)$  is a Hamiltonian circuit in  $D'$ . Thus, by induction,  $D'$  contains a  $P(k, 1, l)$ , a contradiction.

We omit the case when  $l > 2$ , since it is done analogously by proving that  $d(v_m) \leq n - 2$ , and then applying the induction hypothesis on  $D' = D - \{v_m\}$ .

• Case 2:  $k \leq 2$  and  $l \leq 2$ .

In this case,  $P(k, 1, l) \in \{P(1, 1, 1), P(1, 1, 2), P(2, 1, 1), P(2, 1, 2)\}$ .

We are going to deal with the existence of each type in  $D$ .

1. The existence of a  $P(1, 1, 1)$ :

Let  $v_j$  be the outneighbor of  $v_1$  in  $A_1 = V(C_{[v_3, v_{m-1}]})$ . We have  $j = 3$ , since otherwise  $(v_{j-1}, v_j) \cup (v_1, v_j) \cup (v_1, v_2)$  is a  $P(1, 1, 1)$ , a contradiction. Moreover,  $N^-(v_2) = \{v_1\}$ , since otherwise  $(w, v_2) \cup (v_1, v_2) \cup (v_1, v_3)$  is a  $P(1, 1, 1)$ , where  $w$  is an inneighbor of  $v_2$  other than  $v_1$ , a contradiction. We have  $N^+(v_2) = \{v_3\}$ , since otherwise  $(v_1, v_3) \cup (v_2, v_3) \cup (v_2, w)$  is a  $P(1, 1, 1)$ , where  $w$  is an outneighbor of  $v_2$  other than  $v_3$ , a contradiction. Consequently,  $d(v_2) = 2 = n - 2$ . Let  $D' = D - \{v_2\}$ , then  $D'$  is an  $n$ -chromatic Hamiltonian digraph and  $C_{[v_3, v_1]} \cup (v_1, v_3)$  is a Hamiltonian circuit in  $D'$ . Thus, by induction,  $D'$  contains a  $P(1, 1, 1)$ , a contradiction.

2. The existence of a  $P(1, 1, 2)$ :

$(v_{m-1}, v_1) \in E(D)$ , since otherwise a  $C(1, m - 2)$  appears in  $D$ , a contradiction. We have  $v_{m-2} \notin N(v_m)$ , since otherwise either  $(v_{m-2}, v_m) \cup (v_{m-1}, v_m) \cup (v_{m-1}, v_1) \cup (v_1, v_2)$  is a  $P(1, 1, 2)$  or  $(v_m, v_{m-2}) \cup C_{[v_m, v_{m-2}]}$  is a  $C(1, m - 2)$ , a contradiction.

No outneighbor of  $v_m$  exists in  $A_m = V(C_{[v_2, v_{m-3}]})$ , since otherwise  $(v_{m-1}, v_1) \cup (v_m, v_1) \cup (v_m, v_i) \cup (v_i, v_{i+1})$  is a  $P(1, 1, 2)$  with  $v_i$  is an outneighbor of  $v_m$  in  $A_m$ , a contradiction.

Using (\*), we get  $d(v_m) \leq |\{v_1\} \cup \{v_{m-1}\}| + 1 = 3 = n - 2$ . Let  $D' = D - \{v_m\}$ , then  $D'$  is an  $n$ -chromatic Hamiltonian digraph with  $C_{[v_1, v_{m-1}]} \cup (v_{m-1}, v_1)$  is a Hamiltonian circuit in  $D'$ . Thus, by induction,  $D'$  contains a  $P(1, 1, 2)$ , a contradiction.

3. The existence of a  $P(2, 1, 1)$ :

We proved that  $D$  contains a  $P(1, 1, 2)$ . Applying this result on  $D^c$ , which is an  $n$ -chromatic Hamiltonian digraph, we get that  $D^c$  contains a  $P(1, 1, 2)$ , and so  $D$  contains a  $P(2, 1, 1)$ , a contradiction.

4. The existence of a  $P(2, 1, 2)$ :

By (\*), we have  $|N(v_2)| \leq |\{v_3, v_4\} \cup \{v_m, v_1\}| + 2$ . We use a similar argument to that used in the beginning of Case 1, to prove that neither  $v_4$  nor  $v_m$  is a neighbor of  $v_2$ , so  $d(v_2) \leq 4 = n - 2$ . Let  $D' = D - \{v_2\}$ , then, as in the previous cases,  $D'$  contains a  $P(2, 1, 2)$ .

■

### 3.2 The existence of $P(k, 1, l)$ in digraphs containing a Hamiltonian directed path

In this section, we are able, in Theorem 3.13, to show an improvement of Theorem 3.12, so that we prove the correctness of Conjecture 1 on a more general class than Hamiltonian digraphs, which is that containing a Hamiltonian directed path.

**Theorem 3.13.** *Let  $D$  be an  $n$ -chromatic digraph containing a Hamiltonian directed path. Then  $D$  contains any  $P(k, 1, l)$  of length  $n - 1$  with  $n \geq 5$  and  $k, l \in \mathbb{N}^*$ .*

*Proof.* We proceed by induction on  $v(D)$ . For  $m = v(D) = n + 1$ ,  $D$  contains an  $n$ -tournament which contains any  $P(k, 1, l)$ . Let us prove it for  $m \geq n + 2$ , assuming that it is true up to  $m - 1$ . Let  $P = v_1 \dots v_m$  be a Hamiltonian directed path in  $D$ . Suppose that  $D$  contains no  $P(k, 1, l)$  for some  $k, l \in \mathbb{N}^*$  and, without loss of generality, suppose that  $k \geq l$ . We have  $k + l = n - 2$  with  $n \geq 5$ , and  $k \geq l$  so it follows that  $k \geq 2$ .

Note that  $v_1 v_m \notin E(G[D])$ , since otherwise either  $D$  is a Hamiltonian digraph if  $(v_m, v_1) \in E(D)$ , or  $(v_1, v_m) \cup P$  is a  $C(1, m - 1)$  if  $(v_1, v_m) \in E(D)$ , and so, in both cases,  $D$  contains a  $P(k, 1, l)$ , a contradiction.

We are going to prove now that  $v_2 v_m \notin E(G[D])$ . If  $(v_2, v_m) \in E(D)$ , then  $(v_2, v_m) \cup P_{[v_2, v_m]}$  is a  $C(1, m - 2)$ , a contradiction. Suppose that  $(v_m, v_2) \in E(D)$ . Then  $v_1$  has no outneighbors in  $P_{[v_3, v_{m-n+3}]}$ , since otherwise  $(v_m, v_2) \cup (v_1, v_2) \cup (v_1, w) \cup P_{[w, v_m]}$  is a  $C(1, r)$  with  $r \geq n - 1$ , where  $w \in N^+(v_1) \cap P_{[v_3, v_{m-n+3}]}$ , a contradiction. Besides,  $v_1$  has at most one inneighbor in  $P_{[v_3, v_{m-n+3}]}$ , since otherwise  $(v_m, v_2) \cup P_{[v_2, v_i]} \cup (v_i, v_1) \cup (v_j, v_1) \cup P_{[v_j, v_m]}$  is a  $C(1, r)$  with  $r \geq n$ , where  $v_i$  and  $v_j, i < j$ , are two inneighbors of  $v_1$  in  $P_{[v_3, v_{m-n+3}]}$ , a contradiction. Thus,  $d(v_1) \leq n - 2$ . Hence,  $D - \{v_1\}$  is an  $n$ -chromatic digraph containing a Hamiltonian circuit  $C = (v_m, v_2) \cup P_{[v_2, v_m]}$ , then using Theorem 3.12,  $D - \{v_1\}$  contains a  $P(k, 1, l)$ , a contradiction.

Set  $A = V(P_{[v_{k+1}, v_{m-l-1}]})$ . Then, by following a similar argument to that introduced in Lemma 3.5, we get that  $v_m$  has at most one inneighbor and at most one outneighbor in the set  $A$ . Thus,  $|N(v_m) \cap A| \leq 2$ , and so  $d(v_m) \leq |N_{P_{[v_1, v_k]}}(v_m)| + |N_{P_{[v_{m-l}, v_{m-1}]}(v_m)}| + 2 \leq (k - 2) + l + 2 = n - 2$ . Therefore  $\chi(D - \{v_m\}) = n$  and  $P_{[v_1, v_{m-1}]}$  is a Hamiltonian directed path in  $D - \{v_m\}$ . Then, by induction,  $D - \{v_m\}$  contains a  $P(k, 1, l)$ , a contradiction. ■

### 3.3 The existence of $P(k, 1, l)$ in a $(2n + 2)$ -chromatic digraph

In this section, we study the chromatic number of digraphs containing any  $P(k, 1, l)$  of length  $n - 1$  and we get the following result:

**Theorem 3.14.** *Let  $D$  be a  $(2n + 2)$ -chromatic digraph; then  $D$  contains any  $P(k, 1, l)$  of length  $n - 1$  with  $k, l \in \mathbb{N}^*$  where  $\mathbb{N}^*$  is the set of positive integers.*

*Proof.* Let  $D$  be a  $(2n + 2)$ -chromatic digraph and suppose to the contrary that there exist  $k, l \in \mathbb{N}^*$  such that  $D$  contains no  $P(k, 1, l)$  of length  $n - 1$ . Divide  $D$  into two induced subdigraphs  $D_1$  and  $D_2$  such that  $\chi(D_1) = \chi(D_2) = n + 1$ .



Let  $F_1$  be a final spanning outforest of  $D_1$  and  $F_2$  be a final spanning inforest of  $D_2$ . Let  $x \in D_1$  and  $y \in D_2$  with  $l_{F_1}(x) \geq k + 1$  and  $l_{F_2}(y) \geq l + 1$ ; then  $(x, y) \in E(D)$  whenever  $x$  and  $y$  are neighbors in  $G[D]$ . In fact, if  $(y, x) \in E(D)$ , then  $P_{F_1}(x) \cup (y, x) \cup P_{F_2}(y)$  contains a  $P(k, 1, l)$ , a contradiction.

Set  $B = \{y \in \bigcup_{i \geq l+1} L_i(F_2); y \text{ has an inneighbor } x \in \bigcup_{i \geq k+1} L_i(F_1)\}$ . There exists no backward arc

in  $D[B]$  with respect to  $F_2$ . Actually, if  $(y_1, y_2)$  is a backward arc with respect to  $F_2$  in  $D[B]$ , then  $P_{F_1}(x) \cup (x, y_2) \cup (y_1, y_2) \cup P_{F_2}(y_1)$  contains a  $P(k, 1, l)$ , where  $x$  is an inneighbor of  $y_2$  in

$\bigcup_{i \geq k+1} L_i(F_1)$ , a contradiction. It follows that  $D[B]$  contains no circuits. Moreover,  $d_{D[B]}^+(y) \leq 1$

for every  $y \in B$ , since if there exists  $y \in B$  such that  $d_{D[B]}^+(y) \geq 2$ , let  $y_1$  and  $y_2$  be two outneighbors of  $y$  in  $B$  with  $l_{F_2}(y_1) \geq l_{F_2}(y_2)$ ; then  $P_{F_1}(x) \cup (x, y_1) \cup (y, y_1) \cup (y, y_2) \cup P_{F_2}(y_2)$  contains a  $P(k, 1, l)$ , where  $x$  is an inneighbor of  $y_1$  in  $\bigcup_{i \geq k+1} L_i(F_1)$ , a contradiction. Thus,

$D[B]$  contains no cycles and so it is a bipartite graph. Color the first  $k$  levels of  $F_1$  and the first  $l$  levels of  $F_2$  by  $k + l$  colors by giving each level a color distinct from the others. Color  $(\bigcup_{i \geq k+1} L_i(F_1)) \cup (\bigcup_{i \geq l+1} L_i(F_2) - B)$  by  $n + 1$  new colors and  $B$  by 2 colors. All colorings are done properly, then the obtained coloring is a  $(2n + 1)$ -proper coloring, a contradiction. ■

## The existence of paths with three blocks in digraphs

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*“The best solutions are often simple, yet unexpected.”*

– JULIAN CASABLANCAS –

In this chapter, we explain the new technique that we introduced in [64] and used to study the existence of oriented paths in digraphs. This technique has served us in improving the chromatic number bounds of digraphs when dealing with the existence of several types of oriented paths. More precisely, while focusing on three blocks paths, we first study the path  $P(k, 1, l)$  and then we deal with the general case which is the paths  $P(k, l, r)$ . By using the new technique, we improved the bound reached in [65] for digraphs containing  $P(k, 1, l)$  which we presented in Chapter 3, so that the new bound that we established,  $2n - 3$ , proves the correctness of Burr’s conjecture for this types of paths which is clearly a special type of oriented trees. Concerning the general case, we partitioned the problem into three cases according to the values of  $k, l$  and  $r$  and in each case we improved the latest established linear bound for the chromatic number. We shed light in this chapter on this technique, explain it and show its importance in dealing with the two conjectures in hand.

### 4.1 An overview of the new technique

In this section we give the reader the intuition how the technique works, what are its most important steps and how it is used in this setting. In the cases under study, the followed technique has served us in improving the latest previous bounds of the chromatic number. We also believe that it may be useful in dealing with different types of oriented paths. This technique is followed in the two theorems that we prove, but with some differences that have advantages when the second block is of length 1 (Theorem 4.15). First, we start by defining a sequence  $\{D_i, i \in I\}$  of subdigraphs of the digraph  $D$ , by taking out at each step a directed path  $P_i$  with all its neighbors  $N_i$  where the length of the path is precised according to the case of the path we are studying its existence. Let  $P$  and  $N$  be the union of the paths  $P_i$  and their neighbors  $N_i$  respectively. The next step is partitioning the set of neighbors  $N$  into subsets according to the

localization of their neighbors on the paths  $P_i$ . We suppose that the digraph is  $P(k, l, r)$ -free and start assigning the colors to its vertices. In the coloring process, we have benefit that there are no edges between one path and another, and that we have an upper bound for the chromatic number of the remaining subdigraph since it contains no directed path of the specified length. When dealing with the subsets forming the set of neighbors, we use the concept of final outforest and inforest, and proceed the coloring of each subset according to its definition based on the localization of its neighbors. At the end, we reach a contradiction with the given chromatic number and this completes the proof.

More specifically, this technique is used in what follows:

- In Section 4.2, we deal with the path  $P(k, 1, l)$ .
- In Section 4.3, we deal with the general case of paths with three blocks.

As we mentioned above, we were able to improve the bound shown in Chapter 2, so that we proved the following:

Let  $D$  be a  $(2n - 3)$ -chromatic digraph; then  $D$  contains any path  $P(k, 1, l)$  of length  $n - 1$ , with  $n \geq 7$ .

This verifies that the path  $P(k, 1, l)$  satisfies Conjecture 4.

Concerning the general case of three blocks paths  $P(k, l, r)$ , we partition the problem into three cases according to the values of  $k$ ,  $l$  and  $r$ . We establish a linear bound for the chromatic number which is  $2(n - 1) + r$ ,  $2(n - 1) + l + r - k$  and  $2(n + l - 1) - k$  in the three cases of the problem respectively. The reached bound is at most  $3n - 6$  for two cases of the problem and  $4n - 13$  for the remaining case, and so it is the first linear bound that hits  $3n$ .

**Remark:** If we prove that an  $f(n)$ -chromatic digraph  $D$  contains any path  $P(k_1, k_2, k_3)$  for  $k_1 \leq k_3$ , then it is trivial that  $D^c$  contains the path  $P(k_1, k_2, k_3)$  because it has the same chromatic number as  $D$ . Moreover, the existence of a path  $P(k_1, k_2, k_3)$ , for  $k_1 \leq k_3$ , in the digraph  $D^c$  results in the existence of the path  $P(k_3, k_2, k_1)$  in  $D$ . So to prove the existence of any path  $P(k_1, k_2, k_3)$  in a digraph  $D$ , it is sufficient to prove it for  $k_1 \leq k_3$  and the case when  $k_1 > k_3$  is deduced by using the digraph  $D^c$ . Therefore, in what follows it is sufficient to deal only with the case when  $k_1 \leq k_3$ .

## 4.2 The existence of the path $P(k, 1, l)$

In this section, we deal with the path  $P(k, 1, l)$ . We show its existence in any  $(2n - 3)$ -chromatic digraph.

**Theorem 4.15.** *Let  $D$  be a  $(2n - 3)$ -chromatic digraph; then  $D$  contains any path  $P(k, 1, l)$  of length  $n - 1$ , with  $n \geq 7$ .*

*Proof.* First, it is worthy to note that the case when  $k = 1$  or  $l = 1$  is treated in [56], in which

#### 4. The existence of paths with three blocks in digraphs

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they proved that any  $(n + 1)$ -chromatic digraph contains the path  $P(n - 3, 1, 1)$ . Thus, we assume in what follows that  $k \neq 1$  and  $l \neq 1$ . Suppose that  $D$  is  $P(k, 1, l)$ -free and, without loss of generality, suppose that  $l \geq k$ , and so  $l \geq 3$ . Let:

- $D_1 = D$
- $D_{i+1} = D_i - (P_i \cup N(P_i))$  for every  $1 \leq i \leq r - 1$ , where :
  - $P_i$  is a directed path of length  $k + l - 2$  in  $D_i$ .
  - $N(P_i) = \{v \in D_i - P_i; v \text{ has a neighbor in } P_i\}$ .
  - $r$  is the minimal value satisfying that  $D_r$  contains no directed path of length  $k + l - 2$ .

Let  $D' = D[\bigcup_{i=1}^{r-1} V(P_i) \cup D_r]$  and  $N = \bigcup_{i=1}^{r-1} N(P_i)$ .

By definition, we have:

- $v(P_i) = k + l - 1$ , then  $\chi(D[V(P_i)]) \leq k + l - 1 = n - 3$  for every  $1 \leq i \leq r - 1$ .
- $D_r$  contains no directed path of length  $k + l - 2 = n - 4$ , then  $\chi(D_r) \leq n - 4$ .
- $P_{i+1} \in D_{i+1} = D_i - (P_i \cup N(P_i))$ , then  $uv \notin E(G[D])$  for every  $u \in P_i$  and  $v \in P_j$  with  $i \neq j$  for all  $1 \leq i, j \leq r - 1$ , similarly for every  $u \in D_r$  and  $v \in P_i$ .

So,  $\chi(D') \leq n - 3$ . We color  $D'$  by the set  $\{1, 2, \dots, k + l - 1\}$ .

Let  $P_i = v_1^i \dots v_{k+l-1}^i$ , and let  $A$  and  $B$  be two subsets of  $N$  such that:

$B = \bigcup_{i=1}^{r-1} N^+(v_k^i) \cup N(P_{i_{[v_{k+1}^i, v_{k+l-1}^i]}})$  and  $A = N - B$ , i.e.  $A \subseteq \bigcup_{i=1}^{r-1} N(P_{i_{[v_1^i, v_{k-1}^i]}}) \cup N^-(v_k^i)$ .

**Claim 4.15.1.** :  $\chi(D[A]) \leq k + 2$ .

*Proof.* Let  $F_A$  be a final spanning outforest of  $D[A]$ .

Let  $A' = D[\bigcup_{i \geq k+1} L_i(F_A)]$  and  $F'_A$  be the sub-outforest of  $F_A$  spanning the vertices of  $A'$ . For

every  $v \in A'$ ,  $l(P_{F_A}(v)) \geq k$ , then for all  $1 \leq i \leq r - 1$ ,  $v$  is not an outneighbor of a vertex in  $P_{i_{[v_1^i, v_{k-1}^i]}}$ , say  $v_j$ , otherwise  $P_{F_A}(v) \cup (v_j, v) \cup P_{i_{[v_j, v_{k+l-1}^i]}}$  contains a  $P(k, 1, l)$ , a contradiction.

So, for every  $v \in A'$ ,  $v$  is an inneighbor of a vertex in  $P_{i_{[v_1^i, v_k^i]}}$  for some  $1 \leq i \leq r - 1$ .

$A'$  is a bipartite digraph, otherwise there exists  $e \in E(A') \setminus E(F'_A)$ , so we study two cases:

Case 1:  $e = (x, y)$  is backward with respect to  $F_A$ .

We have  $x \in A'$ , so  $x$  an inneighbor of a vertex in  $P_{i_{[v_1^i, v_k^i]}}$ , say  $v_j$ . Thus,  $P_{F_A}(y) \cup (x, y) \cup (x, v_j) \cup P_{i_{[v_j, v_{k+l-1}^i]}}$  contains a  $P(k, 1, l)$ , a contradiction.

Case 2 :  $e = (x, y)$  is forward with respect to  $F_A$ .

Let  $z = N_{F_A}^-(y)$ , we have  $z \in A'$  and so it is an inneighbor of a vertex in  $P_{i_{[v_1^i, v_k^i]}}$ , say  $v_j$ . Thus,  $P_{F_A}(x) \cup (x, y) \cup (z, y) \cup (z, v_j) \cup P_{i_{[v_j, v_{k+l-1}^i]}}$  contains a  $P(k, 1, l)$ , a contradiction.

Therefore,  $E(A') = E(F'_A)$ . Thus,  $A'$  is an outforest and consequently, a bipartite digraph which we partition into two stable sets  $A_1$  and  $A_2$ . So  $\chi(D[A]) \leq k + 2$ .  $\blacksquare$

**Claim 4.15.2.**  $\chi(D[B]) \leq l + 2$ .

*Proof.* Let  $F_B$  be a final spanning inforest of  $D[B]$ .

Let  $B' = D[\bigcup_{i \geq l+1} L_i(F_B)]$  and  $F'_B$  be the sub-inforest of  $F_B$  spanning the vertices of  $B'$ . In a similar way to the proof of claim 4.15.1, we prove that  $E(B') = E(F'_B)$ . Consequently,  $B'$  is an inforest, which we partition into two stable sets  $B_1$  and  $B_2$ .  $\blacksquare$

**Claim 4.15.3.**  $\chi(D[A_1 \cup B']) \leq 2$ .

*Proof.* We have  $\chi(B') \leq 2$ . Let  $v \in A_1, (u, v) \notin E(D)$  for every  $u \in B'$ , since otherwise  $P_{F_A}(v) \cup (u, v) \cup P_{F_B}(u)$  contains a  $P(k, 1, l)$ , a contradiction.

Now we are going to prove that  $v$  has at most one outneighbor in  $B'$ . Suppose that  $v$  has two outneighbors  $u_1$  and  $u_2$  in  $B'$ . Without loss of generality, suppose that  $l_{F_B}(u_1) \leq l_{F_B}(u_2)$ . We have  $u_2 \in B'$ , then  $u_2$  is not an inneighbor of a vertex in  $P_{i_{[v_{k+1}^i, v_{k+l-1}^i]}}$ , say  $v_j$ , since otherwise  $P_{i_{[v_1^i, v_j]}} \cup (u_2, v_j) \cup P_{F_B}(u_2)$  contains a  $P(k, 1, l)$ , a contradiction. Thus,  $u_2$  is an outneighbor of a vertex in  $P_{i_{[v_k^i, v_{k+l-1}^i]}}$ , say  $v_j$ . But  $P_{i_{[v_1^i, v_j]}} \cup (v_j, u_2) \cup (v, u_2) \cup (v, u_1) \cup P_{F_B}(u_1)$  contains a  $P(k, 1, l)$ , a contradiction.

So  $v$  has at most one neighbor in  $B'$  and consequently,  $\chi(D[A_1 \cup B']) \leq 2$ .  $\blacksquare$

**Claim 4.15.4.** Let  $F_r$  be a final spanning outforest of  $D_r$ . If  $L_{k+l-2}(D_r) \neq \emptyset$ , then  $\chi(D[B' \cup A_1 \cup L_{k+l-2}(D_r)]) \leq 2$

*Proof.* Let  $v \in L_{k+l-2}(D_r)$ . We have  $l \geq 3 \Rightarrow l_{F_{D_r}}(v) \geq k + 1$ , then:

- $v$  has no inneighbor in  $B'$  since otherwise,  $P_{F_{D_r}}(v) \cup (u, v) \cup P_{F_B}(u)$  contains a  $P(k, 1, l)$  where  $u$  is an inneighbor of  $v$  in  $B'$ , a contradiction.
- $v$  has no inneighbor in  $A'$ , say  $u$  since otherwise,  $P_{F_{D_r}}(v) \cup (u, v) \cup (u, v_j) \cup P_{i_{[v_j, v_{k+l-1}^i]}}$  contains a  $P(k, 1, l)$  where  $v_j$  is an outneighbor of  $u$  in  $P_{i_{[v_1^i, v_k^i]}}$ , a contradiction.

Now we are going to prove that  $v$  has at most one outneighbor in  $B' \cup A_1$ . Suppose that  $v$  has two outneighbors  $x$  and  $y$  in  $B' \cup A_1$ :

If  $x$  and  $y$  belong to  $B'$ , then suppose, without loss of generality, that  $l_{F_B}(x) \leq l_{F_B}(y)$ .

We have  $y \in B'$ , then  $y$  is an outneighbor of a vertex in  $P_{i_{[v_k^i, v_{k+l-1}^i]}}$ , say  $v_j$ . Thus,  $P_{i_{[v_1^i, v_j]}} \cup (v_j, y) \cup (v, y) \cup (v, x) \cup P_{F_B}(x)$  contains a  $P(k, 1, l)$ , a contradiction.

If  $x$  and  $y$  belong to  $A_1$ , then suppose, without loss of generality, that  $l_{F_A}(x) \leq l_{F_A}(y)$ . We have  $y \in A_1$ , then  $y$  is an inneighbor of a vertex in  $P_{i_{[v_1^i, v_k^i]}}$ , say  $v_j$ . Thus,  $P_{F_A}(x) \cup (v, x) \cup (v, y) \cup (y, v_j) \cup P_{i_{[v_j, v_{k+l-1}^i]}}$  contains a  $P(k, 1, l)$ , a contradiction.

If  $x \in A_1$  and  $y \in B'$ , then we have  $P_{F_A}(x) \cup (v, x) \cup (v, y) \cup P_{F_B}(y)$  contains a  $P(k, 1, l)$ , a

contradiction.

Therefore  $v$  has at most one neighbor in  $B' \cup A_1$ . By Claim 4.15.3, we have  $\chi(B' \cup A_1) \leq 2$ , so  $\chi(D[B' \cup A_1 \cup L_{k+l-2}(D_r)]) \leq 2$ . ■

Now we are ready to continue the proof of Theorem 4.15:

By definition,  $A$  has neighbors only in  $\bigcup_{i=1}^{r-1} P_{i[v_1^i, v_k^i]}$ . So, we choose the coloring of  $D'$  such that the

colors of  $V(\bigcup_{i=1}^{r-1} P_{i[v_1^i, v_k^i]})$  are picked from the set  $\{1, 2, \dots, k\}$ , then the colors  $\{k+1, \dots, k+l-1\}$

are not used in the coloring of  $N_{\{P_1 \cup \dots \cup P_{r-1}\}}(A)$ .

$D_r$  contains no path of length  $k+l-2$ , so  $\chi(D_r) \leq k+l-2$ . Recall that  $F_r$  is a final spanning outforest of  $D_r$ , we color  $L_i(F_{D_r})$  by the color  $i$  for every  $i$ .

If  $\chi(D_r) \leq k+l-3$ , then we color  $D_r$  by the set  $\{1, \dots, k+l-3\}$ .

Otherwise,  $L_{k+l-2}(D_r) \neq \emptyset$ . We have by Claim 4.15.4,  $\chi(D[B' \cup A_1 \cup L_{k+l-2}(D_r)]) \leq 2$ , so we can recolor  $L_{k+l-2}(D_r)$  by the set of colors given to  $B'$ .

Thus, in both cases, the colors  $\{k+l-2, k+l-1\}$  are not used in the coloring of  $D_r$ .

Now we conclude that the colors  $\{k+l-2, k+l-1\}$  are not used in the coloring of  $N_{D'}(A)$ , so we color  $L_1(F_A)$  and  $A_2$  by these two colors.

Finally,  $\chi(D) \leq \chi(L_1(F_A) \cup A_2 \cup D' - (L_{k+l-2}(D_r))) + \chi(B \cup A_1 \cup L_{k+l-2}(D_r)) + \chi(A - (A_1 \cup A_2 \cup L_1(F_A)))$

$\leq (n-3) + (l+2) + (k-1) = n-3+n-1 = 2n-4$ , a contradiction. ■

### 4.3 The existence of the path $P(k, l, r)$ in digraphs with chromatic number bounded from above

When studying the existence of a path  $P(k, l, r)$ , it is worthy to note that the case when  $k=1$  or  $r=1$  is treated in [3], in which Al Mniny proved that any  $(2n-2)$ -chromatic digraph contains the path  $P(k, l, 1)$  of length  $n-1$ . Moreover, the case when  $l=1$  is treated in Theorem 4.15 in this paper. Thus, in what follows, we suppose that  $1 \notin \{k, l, r\}$ .

**Theorem 4.16.** *Let  $D$  be a  $P(k, l, r)$ -free digraph with  $k+l+r = n-1$  and  $k \leq r$ , then*

$$\chi(D) \leq \begin{cases} 2(n-1) + r & \text{if } l \leq k \leq r \\ 2(n-1) + l + r - k & \text{if } k \leq l \leq r \\ 2(n+l-1) - k & \text{if } k \leq r \leq l \end{cases} \quad (4.1)$$

*Proof.* Let:

- $D_1 = D$
- $D_{i+1} = D_i - (P_i \cup N(P_i))$  for every  $1 \leq i \leq h-1$ , where :

- $P_i$  is a directed path of length  $k + r - 2$  in  $D_i$ .
- $N(P_i) = \{v \in D_i - P_i; v \text{ has a neighbor in } P_i\}$ .
- $h$  is the minimal value satisfying the fact that  $D_h$  contains no directed path of length  $k + r - 2$ .

Let  $D' = D[\bigcup_{i=1}^{h-1} V(P_i) \cup D_h]$  and  $N = \bigcup_{i=1}^{h-1} N(P_i)$ .

By definition, we have:

- $v(P_i) = k + r - 1$ , then  $\chi(D[V(P_i)]) \leq k + r - 1$  for every  $1 \leq i \leq h - 1$ .
- $D_h$  contains no directed path of length  $k + r - 2$ , then  $\chi(D_h) \leq k + r - 2$ .
- $P_{i+1} \in D_{i+1} = D_i - (P_i \cup N(P_i))$ , then  $uv \notin E(G[D])$  for every  $u \in P_i$  and  $v \in P_j$  with  $i \neq j$  for all  $1 \leq i, j \leq h - 1$ , similarly for every  $u \in D_h$  and  $v \in P_i$ .

So,  $\chi(D') \leq k + r - 1$ . Color  $D'$  by  $\{1, 2, \dots, k + r - 1\}$ .

Let  $P_i = v_1^i \dots v_{k+r-1}^i$ , and let  $A, B, C$ , and  $H$  be four subsets covering  $N$  such that:

$$H = \bigcup_i N^+(P_i[v_k^i, v_{k+r-1}^i])$$

$$C = \bigcup_i N^-(P_i[v_{k+1}^i, v_{k+r-1}^i]) \cap (N - H)$$

$$B = \bigcup_i N^+(P_i[v_1^i, v_{k-1}^i]) \cap (N - (H \cup C))$$

$$A = N - (H \cup C \cup B), \text{ i.e. } A \subseteq \bigcup_i N^-(P_i[v_1^i, v_k^i]).$$

Now let  $F_A$  and  $F_B$  be final spanning outforests of  $A$  and  $B$  respectively. Let  $F_C$  and  $F_H$  be final spanning inforests of  $C$  and  $H$  respectively. Suppose, without loss of generality, that  $k \leq r$ . Then it is enough to study the following three cases:

Case 1 :  $l \leq k \leq r$ .

Let  $A' = \bigcup_{i \geq k+1} L_i(F_A)$  and  $B' = \bigcup_{i \geq l} L_i(F_B)$ .

We have  $uv \notin E(G[D])$  for every  $u \in A'$  and  $v \in B'$  since otherwise, suppose without loss of generality, that  $(u, v) \in E(D)$ , then  $P_{F_A}(u) \cup (u, v) \cup P_{F_B}(v)_{[z, v]} \cup (v_j, z) \cup P_i[v_j, v_{k+r-1}^i]$  contains a  $P(k, l, r)$  where  $P_{F_B}(v)_{[z, v]}$  is the subpath of  $P_v(F_B)$  of length  $l - 1$  and  $v_j$  is an inneighbor of  $z$  on  $P_i[v_1^i, v_{k-1}^i]$  for some  $i$ .

**Claim 4.16.1.**  $\chi(A') \leq l + 1$ .

*Proof.* We have  $A_i = \{\bigcup_{j \geq 0} L_{i+j(l+1)}\}$  is stable for every  $i \geq k + 1$ , since otherwise let  $uv \in E(G[D])$  such that  $L_{F_A}(u) = i_0$  and  $L_{F_A}(v) = i_0 + j_0(l + 1)$  for some  $i_0, j_0 \in \mathbb{N}^*$  with  $i_0 \geq k + 1$ . Suppose, without loss of generality, that  $(u, v) \in E(D)$ . We have  $P_{F_A}(u) \cup (u, v) \cup P_{F_A}(v)_{[z, v]} \cup (z, v_j) \cup P_i[v_j, v_{k+r-1}^i]$  contains a  $P(k, l, r)$  where  $P_{F_A}(v)_{[z, v]}$  is the subpath of  $P_{F_A}(v)$  of length  $l$  and  $v_j$  is an outneighbor of  $z$  in  $P_i[v_1^i, v_k^i]$  for some  $i$ . ■

**Claim 4.16.2.**  $\chi(B') \leq k + 1$ .

*Proof.* We define  $B_i = \{\bigcup_{j \geq 0} L_{i+j(k+1)}\}$  for every  $i \geq l$ . Similarly to the proof of Claim 4.16.1, we prove that  $\chi(B') \leq k + 1$ . ■

We conclude, from claim 4.16.1 and Claim 4.16.2, that  $\chi(A' \cup B') \leq k + 1$ . Therefore,  $\chi(A \cup B) \leq \chi(A' \cup B') + \chi(A - A') + \chi(B - B') \leq k + 1 + k + l - 1 = k + k + l$ .

**Claim 4.16.3.** *Let  $F_h$  be a final spanning outforest of  $D_h$ . Then,  $uv \notin E(G[D])$  for all  $u \in A' \cup B'$  and  $v \in \bigcup_{i \geq k+1} L_i(F_h)$ .*

*Proof.* Let  $uv \in E(G[D])$  with  $u \in A' \cup B'$  and  $v \in \bigcup_{i \geq k+1} L_i(F_h)$ . Suppose, without loss of generality, that  $(u, v) \in E(D)$  and  $u \in A'$ . We have  $P_{F_h}(v) \cup (u, v) \cup P_{F_A}(u)_{[z, u]} \cup (z, v_j) \cup P_{i[v_j, v_{k+r-1}^i]}$  contains a  $P(k, l, r)$  where  $z$  and  $v_j$  are defined as above, a contradiction. ■

Now we are ready to continue the proof of Theorem 4.16:

We have, by definition,  $A$  and  $B$  have neighbors only in the set  $V(\bigcup_{i \geq 1} P_{i[v_1^i, v_k^i]})$ . We choose the

coloring of  $D'$  such that the colors of  $V(\bigcup_{i \geq 1} P_{i[v_1^i, v_k^i]})$  are picked from the set of colors  $\{1, \dots, k\}$ .

Then the colors  $\{k + 1, \dots, k + r - 1\}$  are not used in coloring of  $N_{\{P_1 \cup \dots \cup P_{h-1}\}}(A \cup B)$ .

Concerning  $D_h$ , we give  $L_i(F_h)$  the color  $i$  for every  $i$ . By Claim 4.16.3, we can give the colors  $\{k + 1, \dots, k + r - 1\}$  to  $A' \cup B'$ . We get that  $\chi(D' \cup A \cup B) \leq \chi(D' \cup A' \cup B') + \chi(A - A') + \chi(B - B') \leq k + r - 1 + 2 + k + l - 1 = k + k + r + l$ .

Concerning  $C$  and  $H$ , let  $C' = \bigcup_{i \geq l} L_i(F_C)$  and  $H' = \bigcup_{i \geq r+1} L_i(F_H)$ . Similarly we prove that  $uv \notin E(G[D])$  for all  $u \in C'$ , and  $v \in H'$  and that  $\chi(C' \cup H') \leq r + 1$ . Thus,  $\chi(C \cup H) \leq l - 1 + r + r + 1 = l + r + r$ .

Therefore,  $\chi(D) \leq k + k + r + l + l + r + r = n - 1 + n - 1 + r = 2n - 2 + r = 2(n - 1) + r$ .

Case 2:  $k \leq l \leq r$ . Let  $A' = \bigcup_{i \geq l+1} L_i(F_A)$ ,  $B' = \bigcup_{i \geq l} L_i(F_B)$ ,  $C' = \bigcup_{i \geq l} L_i(F_C)$ ,  $H' = \bigcup_{i \geq r+1} L_i(F_H)$ , and let  $F_h$  be a final spanning outforest of  $D_h$ . We follow a similar procedure to that followed in the previous case to obtain the following:  $\chi(A' \cup B') \leq k + 1$ ,  $\chi(A \cup B) \leq k + 1 + l + l - 1 = k + l + l$ , and  $\chi(D' \cup A \cup B) \leq k + r - 1 + 2 + l + l - 1 = k + r + l + l$ .

$\chi(C' \cup H') \leq r + 1$ , and  $\chi(C \cup H) \leq l + r + r$ . So,  $\chi(D) \leq k + r + l + l + r + r + l = n - 1 + n - 1 + r - k + 1 = 2n - 2 + l + r - k = 2n - 2 + l + r - k = 2(n - 1) + l + r - k$ .

Case 3:  $k \leq r \leq l$ . Let  $A' = \bigcup_{i \geq l+1} L_i(F_A)$ ,  $B' = \bigcup_{i \geq l} L_i(F_B)$ ,  $C' = \bigcup_{i \geq l} L_i(F_C)$ ,  $H' = \bigcup_{i \geq r+1} L_i(F_H)$ , and let  $F_h$  be a final spanning outforest of  $D_h$ . We follow a similar procedure to that followed in Case 1 and we get the following:  $\chi(A' \cup B') \leq k + 1$ ,  $\chi(A \cup B) \leq k + 1 + l + l - 1 = k + l + l$ ,



and  $\chi(D' \cup A \cup B) \leq k + r - 1 + 2 + l + l - 1 = k + r + l + l$ .

$\chi(C' \cup H') \leq l + 1$ , and  $\chi(C \cup H) \leq l + l + r$ . So,  $\chi(D) \leq k + r + l + l + l + l + r = n - 1 + n - 1 - k + l + l = 2(n + l - 1) - k$ . ■

## Part **II**

# *S*-packing coloring of subcubic graphs

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### Contents

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<b>5</b>	<b>Survey on <i>S</i>-packing coloring</b>	<b>63</b>
5.1	Recall of necessary definitions . . . . .	63
5.2	The effect of some local operations on the packing chromatic number . . . . .	64
5.3	Packing chromatic number of specific classes . . . . .	65
5.4	A Comprehensive Survey of S-Packing Coloring of subcubic graphs . . . . .	68
<b>6</b>	<b><i>S</i>-packing coloring of cubic Halin graphs</b>	<b>75</b>
6.1	(1, 1, 2, 3)-packing coloring of cubic Halin graphs. . . . .	75
6.2	(1, 2, 2, 2, 2)-packing coloring of cubic Halin graphs . . . . .	80
<b>7</b>	<b>Directions for further work</b>	<b>83</b>

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## Survey on $S$ -packing coloring

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“The purpose of life is to conjecture and prove.”

– PAUL ERDÖS –

In this chapter, we embark on a concise exploration of the extensive research conducted in the realm of  $S$ -packing coloring. Our primary focus is directed towards the endeavors made to investigate the  $S$ -packing coloring of subcubic graphs.

### 5.1 Recall of necessary definitions

#### 5.1.1 $i$ -packing

In the context of a graph  $G = (V(G), E(G))$  and a positive integer  $i$ , a subset  $X$  of the vertex set  $V(G)$  is said to be an  $i$ -packing in  $G$  if the distance  $d_G(u, v)$ , is required to be greater than  $i$  for every  $\{u, v\} \subseteq X$ .

#### 5.1.2 Packing chromatic number

The *packing chromatic number* of a graph  $G$ , denoted by  $\chi_\rho(G)$ , is defined to be the smallest positive integer  $k$  such that the vertices of  $G$  can be partitioned into sets  $X_1, \dots, X_k$ , where each  $X_i$  is an  $i$ -packing for all  $i$  in the range  $[k] = 1, \dots, k$ . This partition corresponds to a mapping  $c : V(G) \rightarrow [k]$  where  $X_i = \{u \in V(G) : c(u) = i\}$ . Furthermore, this mapping satisfies the following condition:

If  $c(u) = c(v) = i$  and  $u \neq v$ , then the distance  $d_G(u, v)$  must be greater than  $i$ . This function  $c$  is referred to as a  $k$ -packing coloring.

#### 5.1.3 $S$ -packing chromatic number

Along the way, we will utilize the following generalization of the packing chromatic number in hand. Given an infinite, non-decreasing sequence  $S = (s_1, s_2, \dots)$  of positive integers, an  $S$ -packing coloring of a graph  $G$  is a partition of the vertex set of  $G$  into  $k$  subsets  $\{V_1, V_2, \dots, V_k\}$

such that for each  $1 \leq i \leq k$ , the distance between any two distinct vertices  $u$  and  $v$  in  $V_i$  is at least  $s_i + 1$ . In other words, it is a mapping  $c : V(G) \rightarrow \mathbb{N}$  such that  $c^{-1}(i)$  is an  $s_i$ -packing for every  $i \in \mathbb{N}$ . If  $k$  is a positive integer, and for every vertex  $u$  in the graph  $G$ ,  $c(u)$  belongs to the range  $[k]$ , we refer to this coloring  $c$  as an  $S$ -packing  $k$ -coloring or an  $(s_1, \dots, s_k)$ -packing coloring of  $G$ .

The  $S$ -packing chromatic number of  $G$ , denoted by  $\chi_S(G)$ , is the smallest positive integer  $k$  for which there exists an  $S$ -packing  $k$ -coloring of the graph  $G$ . It is worthy to note that  $((1)_{i \in \mathbb{N}})$ -packing colorings correspond to the standard proper vertex colorings of a graph, while  $((i)_{i \in \mathbb{N}})$ -packing colorings are just the packing colorings. In addition, for a positive integer  $\ell$ ,  $((\ell)_{i \in \mathbb{N}})$ -packing colorings are referred to in the literature as either  $\ell$ -distance colorings or colorings of the  $\ell^{\text{th}}$  power of a graph. Remark : For every subgraph  $H$  of  $G$ , we have  $\chi_\rho(H) \leq \chi_\rho(G)$ .

## 5.2 The effect of some local operations on the packing chromatic number

### 5.2.1 Vertex deletion

In this section, we delve into the concept of vertex deletion. Depending on the previous remark, for any vertex  $v \in V(G)$ , we can conclude that,  $\chi_\rho(G - v) \leq \chi_\rho(G)$ . However, it is important to note that the difference  $\chi_\rho(G) - \chi_\rho(G - v)$  can exhibit considerable variability, as demonstrated in reference [17]. While for leaves, the situation is different.

The following lemma sheds light on the behavior of leaf vertices in this context:

**Lemma 5.6** (Lemma 4.1, [53]). *For a leaf  $x$  in a graph  $G$ , we have  $\chi_\rho(G) - 1 \leq \chi_\rho(G - x) \leq \chi_\rho(G)$ .*

### 5.2.2 Edge deletion

By revisiting the above remark, we conclude that the result  $\chi_\rho(G - e) \leq \chi_\rho(G)$  holds true for each edge  $e \in E(G)$ . Moreover, we have:

**Theorem 5.17** (Theorem 4.4, [14]). *For an edge  $e$  in a graph  $G$ , it is established that  $\chi_\rho(G - e) \geq \frac{\chi_\rho(G) + 1}{2}$ .*

It's worth noting that this bound is not just a theoretical limit; it is achievable in practice.

Furthermore, Brešar and Ferme's work introduces a realization result demonstrating that  $\chi_\rho(G - e)$  can achieve any integer value within the range defined by the two bounds mentioned in Theorem 5.17.

### 5.2.3 Edge subdivision

In the study presented in [42], it was established that the packing chromatic number  $\chi_\rho(S(G))$  is consistently equal to 3 for every connected bipartite graph  $G$  containing at least two edges.

The impact of edge subdivision on the packing chromatic number was further explored by Brešar, Klavžar, and Rall [16].

A noteworthy result is summarized in the following proposition:

**Proposition 5.9** (Proposition 4.11, [16]). *For  $n \geq 3$ , the packing chromatic number of the subdivided complete graph  $\chi_\rho(S(K_n))$  equals  $n + 1$ .*

Additionally, an insightful theorem regarding the relationship between the clique number, the packing chromatic number after edge subdivision, and the original one is presented:

**Theorem 5.18** (Theorem 4.12, [16]). *For a connected graph  $G$  with an order of at least 3, it holds that:*

$$\omega(G) + 1 \leq \chi_\rho(S(G)) \leq \chi_\rho(G) + 1,$$

*and these bounds represent the best achievable values.*

Let  $G$  be a graph and  $e$  an edge in  $G$ . The graph obtained from  $G$  by subdividing its edge  $e$  is denoted by  $Se(G)$ .

Brešar, Klavžar, Rall, and Wash established the following result for a single subdivided edge:

**Theorem 5.19** (Theorem 4.13, [17]). *Given a graph  $G$  with  $\chi_\rho(G) = j$ , it holds that  $\lfloor \frac{j}{2} \rfloor + 1 \leq \chi_\rho(Se(G)) \leq j + 1$ . Furthermore, for any  $k \geq 2$ , there exists a graph  $G$  with an edge  $e$  such that  $k = \chi_\rho(G) = \chi_\rho(Se(G)) - 1$ .*

### 5.3 Packing chromatic number of specific classes

The following inequality represents a trivial relation between the chromatic number and the packing chromatic number of a graph  $G$ :

$$\chi(G) \leq \chi_\rho(G) \tag{5.1}$$

#### 5.3.1 Packing coloring of trees

The packing chromatic number of a tree with a diameter of 2 (or simply a star) is 2, and for a tree with a diameter of 3, it is 3. However, determining the packing chromatic number becomes more intricate when dealing with a tree of diameter 4.

**Proposition 5.10** ([42]). *Consider a tree  $T$  with a diameter of 4 and a central vertex  $v$ . For each  $i$  in the range  $[3]$ , let  $n_i$  represent the number of neighbors of  $v$  having degree  $i$ . Furthermore, let  $L$  be the number of neighbors of  $v$  having degrees of at least 4. In the case where  $L = 0$ , we have the following:*

$$\chi_\rho(T) = \begin{cases} 4, & \text{if } n_3 \geq 2 \text{ and } n_1 + n_2 + n_3 \geq 3, \\ 3, & \text{otherwise.} \end{cases}$$

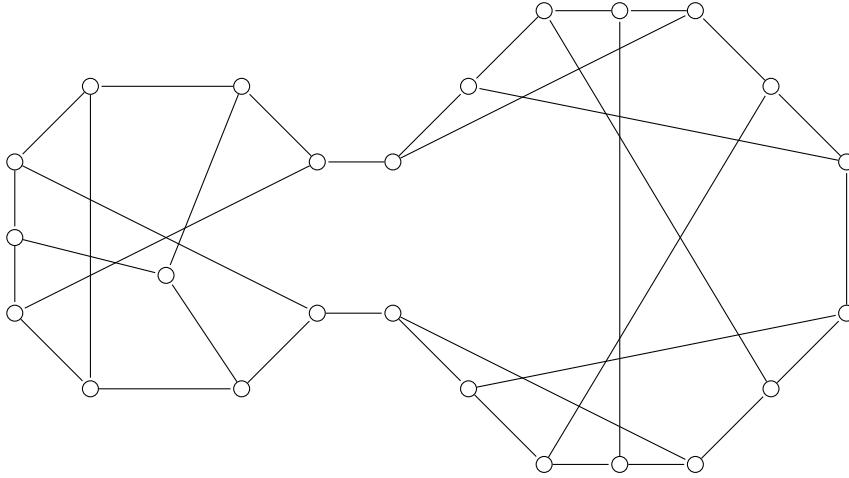


Figure 5.1: A cubic graph of order 24 and packing chromatic number 11

If  $L > 0$ , we have

$$\chi_\rho(T) = \begin{cases} L + 3, & \text{if } n_3 \geq 1 \text{ and } n_1 + n_2 + n_3 \geq 2, \\ L + 1, & \text{if } n_1 = n_2 = n_3 = 0, \\ L + 2, & \text{otherwise.} \end{cases}$$

The following theorem provides an upper bound of the packing chromatic number of a tree in terms of its order.

**Theorem 5.20.** *In the case of a tree  $T$  with  $v(T) = n$ , it holds that  $\chi_\rho(T) \leq \frac{n+7}{4}$ , except for the cases when  $n$  equals either 4 or 8. In these exceptional cases, the bound increases by  $1/4$ . These bounds represent sharp values.*

### 5.3.2 Packing coloring of subcubic graphs

The question of whether subcubic graphs have bounded packing chromatic number remained open for a long time. Gastineau and Togni [40] presented an example of a cubic graph with 38 vertices, having a packing chromatic number of 13 (see Figure 5.2).

They also provided another example of a cubic graph with 24 vertices and a packing chromatic number of 11 (see Figure 5.1).

In addition to these findings, they posed the question of whether a (sub)cubic graph with a larger packing chromatic number exists. The affirmation to this question was subsequently provided by Brešar et al. in [17], where they constructed a cubic graph on 78 vertices, demonstrating a packing chromatic number of at least 14.

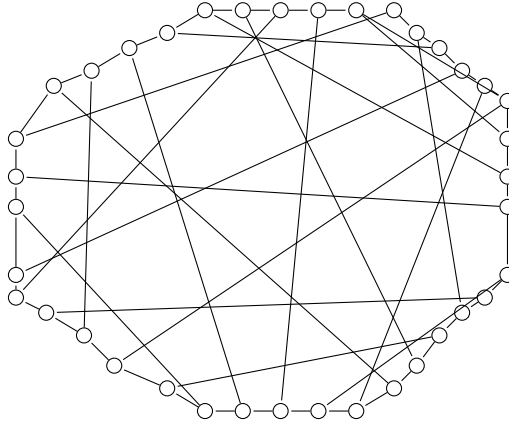


Figure 5.2: A cubic graph of order 38 and diameter 4 and packing chromatic number 13

Following these developments, Balogh, Kostochka, and Liu [10] made a significant breakthrough, utilizing a probabilistic approach.

**Theorem 5.21.** *For any fixed integer  $k \geq 12$ , almost every cubic graph  $G$  with an order of  $n$  and a girth of at least  $2k + 2$  has a packing chromatic number greater than  $k$ .*

The proof of Theorem 5.21 is intricate and relies on the application of the configuration model technique. However, it does not offer a specific construction for a group of subcubic graphs that exhibit an unlimited packing chromatic number. Independently, such a group of graphs was introduced by Brešar and Ferme [13].

Considering Theorem 5.21 as a starting point, Brešar and Ferme raised the question of whether the packing chromatic number remains bounded within the class of subcubic planar graphs [13]. This inquiry has yet to receive a definitive resolution, even when further constrained to subcubic outerplanar graphs. In the forthcoming theorem, we state various results concerning the packing chromatic number of subcubic outerplanar graphs, which is collected from the work of Gastineau, Holub, and Togni. It is worth noting that when a 2-connected outerplanar graph  $G$  is presented in a plane drawing, there exists a cycle  $C$  encompassing all the vertices of  $G$ , with non-crossing chords of  $C$  that partition the interior of  $C$  into faces. Among these faces, some are categorized as internal, implying that they are bounded by more than two chords of  $C$ . Importantly, several of these results offer a positive response to Brešar and Ferme's problem within specific subclasses.

**Theorem 5.22** ([39]). *1. If  $G$  is a 2-connected subcubic outerplanar graph with no internal face, then  $\chi_\rho(G) \leq 15$ .*

*2. If  $G$  is a 2-connected subcubic outerplanar graph with  $r$  internal faces, then  $\chi_\rho(G) \leq 17 \cdot 6^{3r-2}$ .*

*3. If  $G$  is a connected subcubic outerplanar graph with  $r$  (non-external) faces, then  $\chi_\rho(G) \leq 9 \cdot 6^{r-2}$ .*



4. If  $G$  is a 2-connected subcubic outerplanar graph with exactly one internal face, then  $\chi_\rho(G) \leq 51$ .
5. If  $G$  is a connected subcubic outerplanar graph with no internal face such that the block graph of  $G$  is a path, then  $\chi_\rho(G) \leq 305$ .
6. There exists an infinite family of 2-connected subcubic outerplanar graphs without internal faces and with a packing chromatic number of 5.

In the particular case of bipartite, 2-connected, subcubic outerplanar graphs, the work of Brešar, Gastineau, and Togni [15] established the following:

**Theorem 5.23.** *Let  $G$  be a 2-connected, bipartite, subcubic outerplanar graph, it holds that  $\chi_\rho(G) \leq 7$ . It is worth noting that this bound is sharp.*

## 5.4 A Comprehensive Survey of S-Packing Coloring of subcubic graphs

The study of  $S$ -packing colorings of graphs is a well developed area in graph theory, and so covering all the achieved results in this survey is a little bit hard. In this section, we present a historical overview of some of the previous work done in this field.

In the following subsection, we delve into a pivotal conjecture proposed by Brešar et al. and outline the various approaches that have been explored in attempts to address and resolve this conjecture. This conjecture captures a primary focus of our research and investigations in Chapter 6.

### 5.4.1 Brešar et al. conjecture on the packing coloring of subdivision of subcubic graphs

The question of whether the subdivision of any subcubic graph is packing 5-colorable was originally posed by Gastineau and Togni [40]. Subsequently, Brešar, Klavžar, Rall, and Wash [18] transformed this inquiry into a conjecture:

**Conjecture 5.** *For any subcubic graph  $G$ ,  $\chi_\rho(S(G)) \leq 5$ .*

Given that  $\chi_\rho(S(G)) = 3$  for any connected bipartite graph  $G$  with at least 3 vertices, and knowing that every subcubic graph can be considered a subgraph of a cubic graph, our focus narrows to cubic non-bipartite graphs. Moreover, the fact that  $\chi_\rho(S(K_4)) = 5$  directs our attention primarily to 3-chromatic cubic graphs.

However, as  $\chi_\rho(S(K_{n,n,n}))$  approaches infinity as  $n$  tends to infinity (as demonstrated in [18], Proposition 2.2), it becomes evident that Conjecture 5 cannot universally apply to all 3-chromatic graphs.

The most notable contribution to the understanding of Conjecture 5 comes from the work of Balogh, Kostochka, and Liu.

**Theorem 5.24** (Theorem 4.15, [11]). *For any connected subcubic graph  $G$ , the graph  $S(G)$  admits a packing 8-coloring in which the color 8 is used at most once.*

Moreover, Gastineau and Togni delved into the study of  $S$ -packing colorings of graph subdivisions [40]. Their subsequent observation establishes a direct relationship between  $S$ -packing colorings for a given graph  $G$  and those for its corresponding subdivision graph  $S(G)$ . This connection, as we will see further, offers a strategy for investigating Conjecture 5.

Their observation is shown in the following proposition:

**Proposition 5.11** ( Proposition 7.20, [40]). *Let  $S = (s_1, \dots, s_k)$ . If a graph  $G$  is  $S$ -packing colorable, then its subdivision graph  $S(G)$  is  $(1, 2s_1 + 1, \dots, 2s_k + 1)$ -packing colorable.*

By this observation, if one can prove that a subcubic graph  $G$  is  $(1, 1, 2, 2)$ - packing colorable, then the graph  $G$  satisfies the conjecture.

Brooks' theorem guarantees that every subcubic graph, with the exception of  $K_4$ , can be colored as a  $(1, 1, 1)$ -packing. Gastineau and Togni, building upon this result, derived the subsequent corollary from Proposition 5.11 by providing a specific  $(1, 3, 3, 3)$ -packing coloring for  $S(K_4)$ .

This corollary is stated as follows:

**Corollary 5.1** (Corollary 7.21, [40]). *For any subcubic graph  $G$ , the corresponding subdivision graph  $S(G)$  admits a  $(1, 3, 3, 3)$ -packing coloring.*

Goddard and Xu [43] established the NP-hardness of determining whether a graph  $G$  can be colored with a  $(1, 1, a_3)$ -packing coloring for all values of  $a_3$ . However, when it comes to subcubic graphs, the following result holds:

**Theorem 5.25** (Theorem 7.26, [40]). *Every subcubic graph admits a  $(1, 1, 2, 2, 2)$ -packing coloring.*

Furthermore, the subsequent proposition elucidates that, due to the presence of the Petersen graph, the bound established in Theorem 5.25 is, in some sense, the best attainable:

**Proposition 5.12** (Proposition 7.27, [40]). *For cases where  $a_3 \geq 2$ , the Petersen graph can not be colored with a  $(1, 1, a_3, a_4)$ -packing coloring.*

The subsequent lemma played a valuable role in illustrating that various classes of cubic graphs can be assigned  $(1, 1, 2, 2)$ -packing colorings. The concept of the  $2^{nd}$  power of a graph  $G$  refers to the square of  $G$ .

**Lemma 5.7** (Lemma 7.28, [18]). *A graph  $G$  can be colored with a  $(1, 1, 2, 2)$ -packing coloring if and only if there exists a partition  $\{V_1, V_2, V_3\}$  of  $V(G)$  such that  $V_2$  and  $V_3$  form independent sets in  $G$ , and the vertices in  $V_1$  induce a bipartite graph when considering the square of  $G$ .*

A graph  $G$  having  $2n$  vertices is called a generalized prism of a cycle if it is constructed by adding a perfect matching  $M$  to the disjoint union of two cycles  $C$  and  $C'$ , each of order  $n$ , in such a way that every edge in  $M$  joins one vertex from  $C$  to one vertex from  $C'$ .

Brešar, Klavžar, Rall, and Wash [18] provided the subsequent characterization of the Petersen graph, which is linked to Proposition 5.12:

**Theorem 5.26** (Theorem 7.29, [18]). *If  $G$  is a generalized prism of a cycle, then  $G$  can be colored with a  $(1, 1, 2, 2)$ -packing coloring if and only if  $G$  is not the Petersen graph.*

For clarity, we mention the following definition: A graph  $G$  is said to be  $d$ -irregular if it does not contain adjacent vertices, both having degree  $d$ .

In [40], Gastineau and Togni identified a rare example of a cubic graph that does not admit a  $(1, 1, 3, 3, 3)$ -packing coloring. However, on a more encouraging note, the subsequent theorem has been proven:

**Theorem 5.27** (Theorem 7.30, [40]). *Any 3-irregular subcubic graph admits a  $(1, 1, 2)$ -packing coloring.*

They posed a problem: Is every 3-irregular subcubic graph is  $(1, 1, 3)$ -packing colorable? Yang and Wu [71] recently proved that it is true. In the subsequent subsection, we state various applications of  $S$ -packing coloring to offer valuable insights into Conjecture 5. The following proposition establishes a direct link between the conjecture and  $(1, 1, 2, 2)$ -packing colorings.

**Proposition 5.13** (Proposition 7.31, [18]). *If  $G$  admits a  $(1, 1, 2, 2)$ -packing coloring, then  $\chi_\rho(S(G)) \leq 5$ .*

The combination of Theorem 5.26 and Proposition 5.13 implies that Conjecture 5 holds true for all generalized prisms of cycles, except for the Petersen graph. A packing 5-coloring of the subdivision of the Petersen graph was detailed in [18]. As a result, we deduce the subsequent corollary.

**Corollary 5.2** (Corollary 7.32, [18]). *If  $G$  is a generalized prism of a cycle, then  $\chi_\rho(S(G)) \leq 5$ .*

The subsequent theorem extends the scope of Theorem 5.26.

**Theorem 5.28** (Theorem 7.33, [18]). *Consider a connected, cubic graph  $G$  of order  $2n$  equipped with a 2-factor  $F$  and a perfect matching  $M$ . If  $F$  has a cycle  $C$  of length  $n$  where none of the edges in  $M$  connect both vertices in  $C$ , and if  $F$  contains at most one 5-cycle, then  $G$  admits a  $(1, 1, 2, 2)$ -packing coloring.*

In addition, the following result further gives more support for the validity of Conjecture 5. Here,  $P(n, k)$  denotes a generalized Petersen graph.

**Corollary 5.3** (Corollary 7.34, [18]). *For positive integers  $n$  and  $k$  such that  $k < n/2$ , and provided that  $n$  is not a multiple of 5,  $P(n, k)$  admits a  $(1, 1, 2, 2)$ -packing coloring, thus ensuring  $\chi_\rho(S(P(n, k))) \leq 5$ .*

While intuition might lead one to assume that any optimal packing coloring of  $S(G)$  assigns color 1 to all the newly introduced vertices in the subdivision, the subsequent proposition contradicts this notion.

**Proposition 5.14** (Proposition 7.35, [18]). *In the case that  $G$  is not  $(1, 1, 2, 2)$ -packing colorable, and  $\chi_\rho(S(G)) \leq 5$ , then within any packing 5-coloring of  $S(G)$ , at least one of the vertices in  $S(G) - V(G)$  will be assigned a color greater than 1.*

A more refined upper bound than the one provided in Theorem 4.15 is attainable when dealing with subcubic graphs that are 2-degenerate.

**Theorem 5.29** (Theorem 7.36, [11]). *If  $G$  is a subcubic graph in which each subgraph contains a vertex with a degree of at most 2, then  $G$  can be colored with a  $(1, 1, 2, 2, 3, 3)$ -packing coloring. As a consequence,  $\chi_\rho(S(G))$  is bounded above by 7.*

Liu, Rolek, and Yu [55] investigated Conjecture 5 in the context of subcubic graphs characterized by a bounded maximum average degree, denoted as  $\text{mad}(G)$ , defined as follows:

$$\text{mad}(G) = \max_{H \text{ subgraph of } G} \frac{2|E(H)|}{|V(H)|}.$$

Their findings can be summarized as follows:

**Theorem 5.30** (Theorem 7.37, [55]). *If  $G$  is a subcubic graph with  $\text{mad}(G) < \frac{30}{11}$ , then  $G$  can be colored with a  $(1, 1, 2, 2)$ -packing coloring, implying that  $\chi_\rho(S(G)) \leq 5$ .*

For planar graphs with a girth of at least  $g$ , it has been established that the maximum average degree is bounded by  $\frac{2g}{g-2}$ , as detailed in [12, Observation 1]. This leads to the following immediate conclusion:

**Corollary 5.4** (Corollary 7.38, [55]). *If  $G$  is a subcubic planar graph with a girth of at least 8, then  $G$  can be colored with a  $(1, 1, 2, 2)$ -packing coloring, resulting in  $\chi_\rho(S(G)) \leq 5$ .*

#### 5.4.2 $S$ -packing coloring with 1 appearing once in $S$

Let  $S = (1, a_2, \dots, a_k)$ , where  $a_2 \geq 2$ . If  $G$  is a graph that can be colored with an  $S$ -packing coloring  $c$ , and there exists a vertex  $v \in V(G)$  with  $c(v) = 1$ , then it follows that  $\deg(v) \leq k - 1$  since no two neighbors of  $v$  can be assigned the same color according to  $c$ . Consequently, this implies that no cubic graph can possess a  $(1, 2, 2)$ -packing coloring.

However, it is worth noting that there exist subcubic graphs that can be  $(1, 2, 2)$ -packing colored (for instance, any path), and Gastineau [38] demonstrated that determining whether a subcubic bipartite graph can be  $(1, 2, 2)$ -packing colored is NP-complete.

By permitting a partition of  $V(G)$  into an independent set and incorporating at least six 2-packings, we arrive at the following conclusion:

**Theorem 5.31** (Theorem 7.22, [40]). *Every subcubic graph can be colored with a  $(1, 2, 2, 2, 2, 2)$ -packing coloring.*

The result is sharp, as the Petersen graph, a cubic graph, cannot be colored with a  $(1, 2, 2, 2, 2, 2)$ -packing coloring.

Computational evidence leans towards the possibility that the Petersen graph might be the only subcubic exception, as indicated in (Table 1, [40]). Moreover, in the same paper, Gastineau and Togni posed an open question which we consider in Chapter 6:

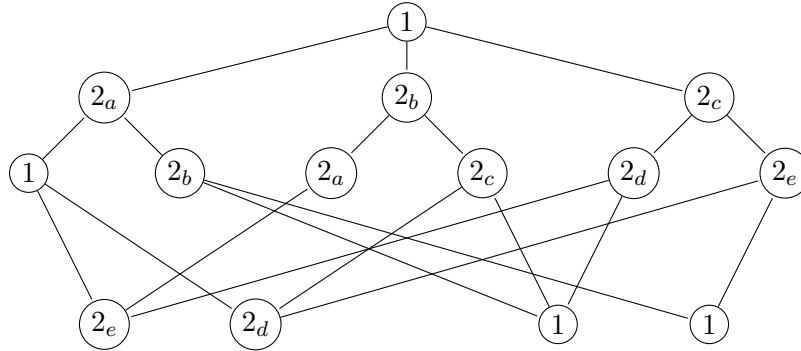


Figure 5.3: A cubic bipartite graph of order 14 that is not  $(1, 2, 2, 2, 2, 3)$ -packing coloring

**Is any subcubic graph  $(1, 2, 2, 2, 2, 2)$ -packing colorable?**

However, the following proposition shows the sensitivity of Theorem 5.31 :

**Proposition 5.15** (Proposition 7.23, [40]). *There exist bipartite cubic graphs that do not admit a  $(1, 2, 2, 2, 2, 3)$ -packing coloring. (see Figure 5.3)*

The following theorem shows that by adding specific restrictions on the adjacencies of a subcubic graph, we can strengthen the conclusions drawn from Theorem 5.31 and Proposition 5.15.

**Theorem 5.32** ([40]). *Let  $G$  be a 3-irregular subcubic graph, then  $G$  admits a  $(1, 2, 2, 2)$ -packing coloring.*

The cycle of order 5 serves as a demonstration, highlighting the necessity of three 2-packings, as mentioned in Theorem 5.32. In this specific sense, the conclusion stands as the most suitable option for 3-irregular subcubic graphs. On the other hand, it is important to mention that there exist 3-irregular subcubic graphs that admit  $(1, 2, 2, 2)$ -packing coloring  $c$ , with the condition that  $|c^{-1}(4)| = 1$ . In such cases, the initial coloring  $c$  can be adjusted to give a  $(1, 2, 2, 3)$ -packing coloring, as exemplified in [40]. It is clear, by definition, that any bipartite graph  $G$  can be colored using a  $(1, 1)$ -packing coloring. Proposition 7.20 establishes that as a consequence, the graph  $S(G)$  can be colored with a  $(1, 3, 3)$ -packing coloring. Consequently,  $S(G)$  also admits both a  $(1, 2, 3)$ -packing coloring and a  $(1, 2, 2)$ -packing coloring. Surprisingly, Gastineau and Togni demonstrated that if a graph possesses a minimum degree of at least 3, all of the following are equivalent.

**Proposition 5.16** (Proposition 7.25, [40]). *If  $G$  is a graph with  $\delta(G) \geq 3$ , then the following statements are equivalent :*

- $S(G)$  is  $(1, 2, 2)$ -packing colorable.
- $S(G)$  is  $(1, 2, 3)$ -packing colorable.

- $S(G)$  is  $(1, 3, 3)$ -packing colorable.
- $G$  is bipartite.

Brešar, Gastineau, and Togni [15] contributed additional insights into  $S$ -packing colorings, particularly focusing on outerplanar graphs:

**Theorem 5.33** (Theorem 7.39, [15]). *Consider the sequence  $S = (1, 3, \dots, 3)$ , where 3 appears exactly  $k$  times, and assume that  $k \geq 3$ . If  $G$  is a bipartite outerplanar graph satisfying  $\Delta(G) \leq k$ , then  $G$  can be colored with an  $S$ -packing coloring.*

Brešar et al. presented an example demonstrating that when  $S = (1, 3, \dots, 3, 4)$  with 3 occurring  $\Delta(G) - 1$  times, there exists a bipartite outerplanar graph that does not admit an  $S$ -packing coloring. However, when dealing with subcubic outerplanar graphs free of triangles, they established the following result:

**Theorem 5.34** (Theorem 7.40, [15]). *If  $G$  is a subcubic, triangle-free, outerplanar graph, then  $G$  admits a  $(1, 2, 2, 2)$ -packing coloring.*

It is noteworthy that Theorem 5.34 is accompanied by two illustrative examples, highlighting that the theorem represents an optimal result for subcubic outerplanar graphs. The first example refers to a subcubic outerplanar graph containing triangles, which cannot be colored with a  $(1, 2, 2, 2)$ -packing coloring. The second example involves a subcubic, triangle-free, outerplanar graph that has no  $(1, 2, 2, 3)$ -packing coloring.

Kostochka and Liu [54] proved that every 2-connected subcubic outerplanar graph is  $(1, 1, 2)$ -packing colorable and every subcubic outerplanar graph is  $(1, 1, 2, 4)$ -packing colorable. Furthermore, they posed the following problem:

Is any subcubic 2-connected outerplane graph  $(1, 2, 2, 2)$ -colorable?

Recently, Yang and Wu [71] answered this problem in affirmative.

Furthermore, Gastineau and Togni [41] investigated the concept of edge-packing  $S$ -coloring in cubic graphs  $G$ , which aligns with  $S$ -packing colorings of the line graph  $L(G)$  derived from  $G$ . Their notable findings encompass several cases, including:

- Every line graph of a cubic graph with a 2-factor can be colored with a  $(1, 1, 1, 3, 3)$ -packing coloring, as well as a  $(1, 1, 1, 4, 4, 4, 4, 4)$ -packing coloring, in accordance with [41], Theorems 3.3 and 3.7.
- They also referenced a result, attributed to Payan and to Fouquet and Vanherpe, asserting that the line graph of a cubic graph is amenable to a  $(1, 1, 1, 2)$ -packing coloring, detailed in [41], Theorem 3.1.



## 6

### *S*-packing coloring of cubic Halin graphs

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*“A story has no beginning or end: arbitrarily one chooses that moment of experience from which to look back or from which to look ahead.”*

– GRAHAM GREENE –

On our way for proving Conjecture 5 for a subclass of subcubic graphs which are Halin, and depending on the observation of Gastineau and Togni, it would have been sufficient to prove that such graphs are  $(1, 1, 2, 2)$ -packing colorable, but we proved a stronger result [66] which is that cubic Halin graphs are  $(1, 1, 2, 3)$ -packing colorable.

Concerning the  $(1, 2, 2, \dots)$ -packing coloring of cubic Halin graphs, we proved [66] that cubic Halin graphs are  $(1, 2, 2, 2, 2, 2)$ -packing colorable, and this answers the question posed by Gastineau and Togni [40] for these class of graphs.

Before delving into the proofs, we will provide a brief reminder of the definition of Halin graphs. A Halin graph is a planar graph, constructed by connecting the leaves of a tree into a cycle such that the tree is of order at least four, in which the degree of each vertex is either one which is called a leaf, or of degree at least 3. For a Halin graph  $G$ , we write  $G = T \cup C$  where  $T$  is its characteristic tree and  $C$  its adjoint cycle

#### 6.1 $(1, 1, 2, 3)$ -packing coloring of cubic Halin graphs.

**Theorem 6.35.** *Every cubic Halin graph is  $(1, 1, 2, 3)$ -packing colorable.*

*Proof.* Let  $G = T \cup C$  be a cubic Halin graph. The set of colors that we are going to use is  $\{1, 1', 2, 3\}$ .

First, we color  $T$  by the coloring  $\phi_T$  in which we use the colors  $\{1, 1'\}$ , this is possible since  $T$  is a tree, hence 2-colorable. Going back to the graph  $G$ , by adding the edges of the cycle, conflicts may appear on the vertices of  $V(C)$  such that two consecutive vertices on  $C$  may be receiving a same color. So we are going to recolor the vertices of the cycle by benefiting from the remaining



colors.

Let  $n = |V(C)|$  and  $C = a_1a_2 \dots a_n$  arranged in an increasing order according to a clockwise direction.

We have two main cases to study:

**Case 1:** There exist two consecutive vertices  $a_{i_0}$  and  $a_{i_0+1}$  on  $C$  such that  $\phi_T(a_{i_0}) = 1'$  and  $\phi_T(a_{i_0+1}) = 1$ .

Let  $u$  and  $v$  be two vertices on the cycle  $C$ , we define  $d_C(u, v)$ , the distance between  $u$  and  $v$  given by the cycle, as  $\min\{l(C_{[u,v]}), l(C_{[v,u]})\}$ . As a first step, we are going to recolor  $V(C)$  by a (1, 1, 2, 3)-packing coloring  $\phi_C$  such that this coloring considers  $d_C(u, v)$  (we mean by this that when giving two vertices same color  $i$ , we must make sure that the distance  $d_C$  between these two vertices is at least  $i + 1$ ). Then we will deal with the conflicts that appear due to the distance given by  $G$  since we possibly have  $d_C(u, v) > d_G(u, v)$  for some  $u, v \in C$ .

- If  $n \equiv 0 \pmod 4$  then we color  $V(C)$  by repeating the sequence  $i2i3$  where  $i \in \{1, 1'\}$  depending on  $\phi_T$ .
- If  $n \equiv 1 \pmod 4$ , then we color  $V(C)$  starting from  $a_{i_0+1}$  and repeating the sequence  $i2i3$  where  $i \in \{1, 1'\}$  depending on  $\phi_T$ , this is possible since  $\phi_T(a_{i_0}) = 1'$ .
- If  $n \equiv 2 \pmod 4$ , then we color  $V(C_{[a_{i_0+1}, a_{i_0-2}]})$  starting from  $a_{i_0+1}$  and repeating the sequence  $i2i3$  where  $i \in \{1, 1'\}$  depending on  $\phi_T$ . If  $\phi_T(a_{i_0-1}) = 1$  then we are done, otherwise we give  $a_{i_0-1}$  the color 2.
- If  $n \equiv 3 \pmod 4$ , then we color  $V(C)$  starting from  $a_{i_0+1}$  and repeating the sequence  $i2i3$  where  $i \in \{1, 1'\}$  depending on  $\phi_T$ , this is possible since  $\phi_T(a_{i_0}) = 1'$ .

Now we are going to deal with the conflicts that appear due to the distance given by  $G$ . We have two types of conflicts:

Conflict of type 1: There exists  $x \in T \setminus C$  and  $\{a_i, a_j\} \subset C$  such that  $\{xa_i, xa_j\} \subset E(G)$ ,

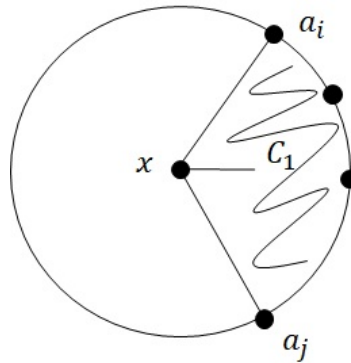


Figure 6.1: Conflict of type 1.

and both  $a_i$  and  $a_j$  receive the color 2 or 3 by the coloring  $\phi_C$  as illustrated in Fig.6.1.

We have  $d(x) = 3$ , so we may suppose, without loss of generality, that the third neighbor of  $x$  is in the region  $C_1 = C_{[a_i, a_j]} \cup xa_i \cup xa_j$ . Since  $G$  is connected and planar, then the characteristic tree  $T$  of  $G$  is contained in the region  $C_1$ . Thus,  $a_j$  and  $a_i$  are consecutive on  $C$ , and this

contradicts the fact that the coloring  $\phi_C$  is a  $(1, 1, 2, 3)$ -packing coloring of  $V(C)$ .

Conflict of type 2: There exist  $\{x, y\} \subset T \setminus C$ ,  $\{a_i, a_j\} \subset C$  with  $j < i$  such that  $\{xa_i, xy, ya_j\} \subset C$

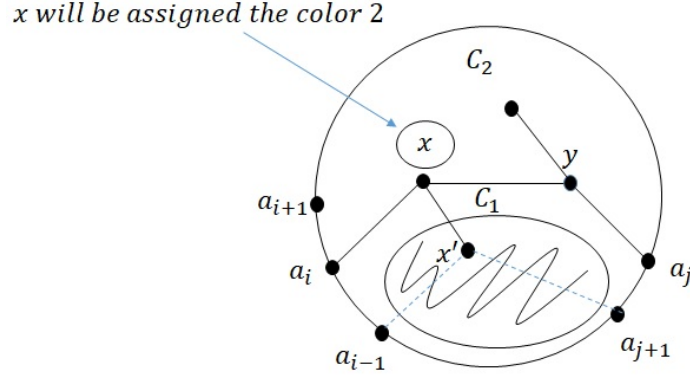


Figure 6.2: Conflict of type 2.

$E(G)$ , and both  $a_i$  and  $a_j$  receive the color 3 by the coloring  $\phi_C$ , see Fig.6.2.

In what follows, we are going to deal with such type of conflict. Since the vertices of  $C$  are colored by the packing coloring  $\phi_C$ , and both  $a_i$  and  $a_j$  receive the color 3 according to  $\phi_C$ , then  $d_C(a_j, a_i) \geq 4$ . Thus clearly they are not consecutive on  $C$  and so the third neighbor of  $x$  and that of  $y$  must exist in the two different regions of  $C$  which are  $C_1 = C_{[a_j, a_i]} \cup xa_i \cup xy \cup ya_j$  and  $C_2 = C_{[a_i, a_j]} \cup xa_i \cup xy \cup ya_j$  since otherwise the characteristic tree will be contained in one of the two regions and  $a_i$  and  $a_j$  will be consecutive which is a contradiction. We may suppose, without loss of generality, that the third neighbor of  $x$ , say  $x'$ , is in the region  $C_1$ .

Now we assign to  $x$  the color 2, and then we recolor  $a_i$  by the color  $\alpha \in \{1, 1'\}$ , chosen such that  $\alpha \neq \phi_C(a_{i+1})$ .

If  $\alpha = \phi_C(a_{i-1})$  then we switch the colors 1 and  $1'$  of the vertices in the region  $C_1$  except the vertex  $y$ . This switch of the colors is possible since  $x$  is now colored 2.

We have  $x' \notin C$ , since otherwise  $C_{[a_j, a_i]} = a_j x' a_i$  and so  $d_C(a_i, a_j) = 2$ , a contradiction with the packing coloring  $\phi_C$ . Note that if  $x'$  has a neighbor on  $C$  then this neighbor must be either  $a_{j+1}$  or  $a_{i-1}$ , since otherwise we get a contradiction with the fact that  $G$  is planar and connected.

Now we are going to deal with conflicts appearing due to giving  $x$  the color 2.

(i) A conflict may appear if  $x'$  is adjacent to  $a_{j+1}$  and  $\phi_C(a_{j+1}) = 2$ . According to sequences of colors that we followed, the color 3 is followed by the color 2 only in the case when  $n \equiv 2 \pmod 4$  in which the pattern of colors is  $(i2i3)^* 21'$  where this notation  $(i2i3)^*$  denotes the repetition of the sequence  $i2i3$ . So  $C_{[a_j, a_i]}$  is colored as follows :  $321'(i2i3)^* i2i\alpha$ .

Now we assign to the vertex  $a_{j+1}$  the color  $\phi_T(a_{j+1})$ , assign to the vertex  $a_{j+2}$  the color 2, switch the colors 2 and 3 in  $C_{[a_{j+3}, a_{i-1}]}$ .

Thus  $C_{[a_j, a_i]}$  will have the following pattern:  $3\phi_T(a_{j+1})2(i3i2)^* i3i\alpha$ .

(ii) A conflict may appear if  $\phi_C(a_{i+1}) = 2$ . As before, we notice that this pattern, which is color 3 followed by color 2, only appears in the case when  $n \equiv 2 \pmod{4}$ . Then  $C_{[a_i, a_j]}$  is colored as follows :  $\alpha 21'(i2i3)^*$ . So we assign to  $a_{i+1}$  the color 3.

Note that the conflicts (i) and (ii) do not occur at the same time, because this pattern exists once in the coloring of  $V(C)$ . So the coloring of  $a_{i+1}$  in (ii) does not make conflict with the switch of the colors 2 and 3 mentioned in (i).

(iii) A conflict may appear if  $a_{i-1}$  receives the color 2. This is not possible since according to the sequence of colors used to color  $V(C)$ , the color 2 is not followed by 3 in any of the cases. After dealing with the conflict of type 2, by giving  $x$  the color 2, we want to make sure that if there exists two or more vertices in  $T \setminus C$  having the same situation as  $x$  and so receiving the color 2, then they do not make a conflict with each other.

Let  $S = \{x_i, i \in I\}$  be the set of vertices in  $T \setminus C$  receiving color 2 by treating conflicts of type 2. For every  $x_i \in S$ , set  $\{a_{i_1}, a_{i_2}\} \subset C$  with  $i_1 < i_2$ , and  $y_i \in T \setminus C$  to be the vertices such that  $x_i$  and  $y_i$  are adjacent in which one of them is adjacent to  $a_{i_1}$  and the other is adjacent to  $a_{i_2}$  where  $\{a_{i_1}, a_{i_2}\}$  receive the color 3. Also for every  $x_i$ , let  $C_1^i$  and  $C_2^i$  be the two regions of the cycle defined as  $C_1$  and  $C_2$  are defined previously. Let  $x_i$  and  $x_j$  be two vertices in  $S$ , the new color assigned to these vertices is the color 2.

Note that:

$$N(x_i) \subseteq C_1^i \text{ and } N(x_j) \subseteq C_1^j \quad (*)$$

(since as we mentioned before, the vertices in  $S$ , that are assigned the new color 2, are chosen according to this location of their neighbors).

In what follows, we are going to study three cases.

Subcase 1:  $[a_{i_1}, a_{i_2}] \cap [a_{j_1}, a_{j_2}] = \emptyset$ . By using (\*), we get that neither  $x_i y_j$  nor  $x_j y_i$  belongs to  $E(G)$ . So  $d_G(x_i, x_j) \geq 3$  and there exists no conflict.

Subcase 2:  $a_{i_1} = a_{j_2}$ . By using (\*), we conclude that  $C = C_{[a_{j_1}, a_{i_2}]} \cup a_{i_2} a_{j_1}$ , and so  $a_{i_2}$  and  $a_{j_1}$  are consecutive on  $C$  and both receiving a color 3 according to the coloring  $\phi_C$ , which is a contradiction.

Subcase 3:  $[a_{j_1}, a_{j_2}] \subset [a_{i_1}, a_{i_2}]$ . Conflict may appear only if  $x_i$  and  $y_j$  are adjacent, but if such an edge exists then we get that  $a_{i_1}$  and  $a_{j_1}$  are consecutive on  $C$ , a contradiction with the coloring  $\phi_C$ .

**Case 2:** All the vertices of  $C$  receive the same color according to  $\phi_T$ .

We may suppose, without loss of generality, that all the vertices of  $C$  receive the color 1.

First note that if  $n \equiv 0 \pmod{4}$ , then  $\chi_p(C) = 3$  and so we can color  $C$  by the colors  $\{1, 2, 3\}$  by repeating the sequence 1213.

In what follows, we suppose that  $n \pmod{4} \in \{1, 2, 3\}$ .

Our aim is to choose a vertex from  $T \setminus C$  to give it the color 2, and give its neighbor on  $C$  the color 1', and then deal with the problem similarly to the first case.

Since all the vertices of  $C$  receive the same color according to  $\phi_T$ , then there exists no path  $P$  of odd length joining two vertices  $a_i$  and  $a_j$  on  $C$  such that  $P \cap C = \{a_i, a_j\}$ .

Now we suppose that there exists a vertex  $w \in T \setminus C$  such that  $w$  has exactly one neighbor on  $C$ .

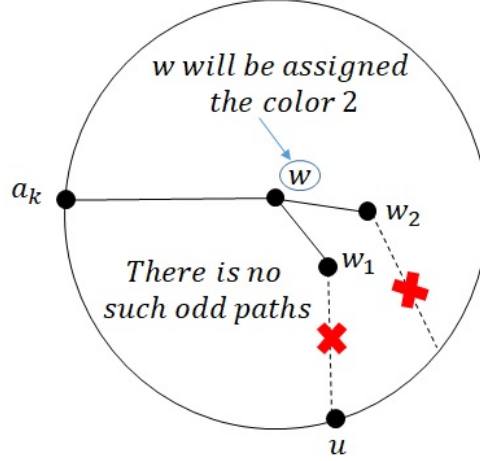


Figure 6.3:  $w$  has one neighbor on  $C$  and paths with odd length are not allowed.

Assign to  $w$  the color 2 and give its neighbor on  $C$ , say  $a_k$ , the color  $1'$ .

If  $n \equiv 1 \pmod{4}$ , then we color  $V(C \setminus \{a_k\})$  starting from  $a_{k+1}$  by repeating the sequence 1213. Vertex  $w$  has no neighbors on  $C$  other than  $a_k$ , so let  $w_1$  and  $w_2$  be the neighbors of  $w$  in  $T \setminus C$ . A conflict may appear if  $w_i$  for some  $i \in \{1, 2\}$  is adjacent to a vertex  $u$  on  $C$  receiving the color 2, but  $w_i$  has no neighbors on  $C$  since otherwise  $a_k w w_i u$  is an odd length path, see Fig.6.3.

If  $n = 2 \pmod{4}$ . Color  $V(C \setminus \{a_k\})$  starting from  $a_{k+1}$  by repeating the sequence 1213. This coloring assigns to  $a_{k-1}$  the color 1, so it has no conflict with  $w$ . Also as the previous case the neighbors of  $w$  make no conflicts.

If  $n = 3 \pmod{4}$ , then color  $V(C \setminus \{a_k, a_{k-1}, a_{k-2}\})$  starting from  $a_{k+1}$  by repeating the sequence 1213, and assign to  $a_{k-1}$  the color 1 and to  $a_{k-2}$  the color 2.

Now assume there exists no vertex in  $T \setminus C$  which has exactly one neighbor on  $C$ . So all the vertices in  $T \setminus C$  that are adjacent to  $C$ , have two neighbors on  $C$ . It is clear that any vertex on  $C$  is adjacent to one vertex in  $T \setminus C$ , therefore we get that  $n$  is even and we are in the case  $n = 2 \pmod{4}$ . Let  $w \in T \setminus C$ , and let  $a_i$  and  $a_j$  be the neighbors of  $w$  on  $C$  with  $i < j$ . Since  $G$  is planar and connected, we conclude that  $a_i$  and  $a_j$  are consecutive. We assign to  $w$  the color 2 and to  $a_j$  the color  $1'$  and color  $V(C \setminus \{a_j\})$  starting from  $a_{j+1}$  till reaching  $a_i$  by repeating the sequence 1213.

This completes the proof. ■

### 6.1.1 Sharpness of the proved result

First, it is worthy to note that without the condition that the graph is cubic, it is easy to find examples proving that a Halin graph is not  $(1, 1, 2, 3)$ -packing colorable in general. Moreover, the result of Theorem 6.35 is sharp in the sense that there exists a cubic Halin graph

that is not  $(1, 1, 3, 3)$ -packing colorable.

We claim that the graph  $G_1$  shown in Fig.6.4, which is clearly cubic Halin, is not  $(1, 1, 3, 3)$ -packing colorable.

*Proof.* Suppose that  $G_1$  is  $(1, 1, 3, 3)$ -packing colorable and let  $\{1_a, 1_b, 3_a, 3_b\}$  be the set of colors that we will use. For every subgraph  $T_i$  of  $G$ , with  $i \in \{1, 2, 3\}$ , there exists a vertex  $v_i \in T_i$  such that the coloring of  $v_i$  must be chosen from the set  $\{3_a, 3_b\}$ . So we get that two vertices from  $\{v_1, v_2, v_3\}$  must receive the same color, without loss of generality, say  $3_a$ . But the distance between any two of them is at most 3, which is a contradiction. ■

So this graph can not be colored with such a packing coloring. And consequently, it is not colored by  $(1, 2, 3, 4)$ . Therefore, cubic Halin graphs are neither  $(1, 1, 3, 3)$ -packing colorable nor 4-packing colorable in general.

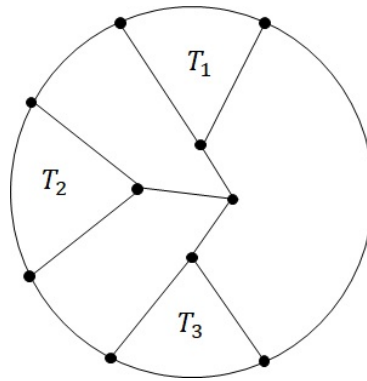


Figure 6.4: A non  $(1,1,3,3)$ -packing colorable cubic Halin graph  $G_1$

## 6.2 $(1, 2, 2, 2, 2)$ -packing coloring of cubic Halin graphs

Remark first that any tree  $T$  of order at least 4 such that all its leaves are either of degree 1 or 3, contains a vertex that is adjacent to two leaves. This is easy to be proved, so we will omit its proof.

Moreover, as a conclusion of the previous remark and by using the induction, we get that such type of tree has even number of vertices.

**Lemma 6.8.** *Let  $T$  be a tree of order at least 4, such that all its vertices are either of degree 1 or 3. Then  $T$  admits a  $(1, 2, 2, 2)$ -packing coloring in which all the leaves are given the color 1.*

*Proof.* We will proceed the proof by induction on  $v(T) = |V(T)|$ .

For  $v(T) = 4$ , it is clearly true. Let us prove it for  $v(T) = m$ , knowing that it is true up to  $m - 2$ . Let  $T$  be a tree of order  $m$  satisfying the above conditions. Let  $x$  and  $y$  be two leaves which are adjacent to the same vertex  $z$ . Let  $T' = T \setminus \{x, y\}$ . By applying the induction hypothesis

on  $T'$ , we get that  $T'$  admits a  $(1, 2, 2, 2)$ -packing coloring such that all its leaves are colored by 1. Going back to  $T$ , by adding the vertices  $x$  and  $y$ , we have all the leaves of  $T$  are colored by 1 except  $x$  and  $y$ . The color assigned to  $z$  is 1 because it is a leaf in  $T'$ . Now we are going to give both vertices  $x$  and  $y$  the color 1, and then recolor  $z$ . Note that the set of colors that we are using is  $\{1, 2_a, 2_b, 2_c\}$ . Let  $z'$  be the third neighbor of  $z$  and  $N^2$  be the set of neighbors of  $z'$  other than  $z$ . If there exists a color  $i \in \{2_a, 2_b, 2_c\}$  such that  $i$  is not given to any of the vertices of  $N^2 \cup \{z'\}$ , then we assign to  $z$  the color  $i$ . Otherwise, we give  $z'$  the color 1 and assign to  $z$  the color that was previously given to  $z'$  from the set  $\{2_a, 2_b, 2_c\}$ . Now  $T$  admits a  $(1, 2, 2, 2)$ -packing coloring in which all its leaves are given the color 1, and this completes the proof. ■

**Theorem 6.36.** *If  $G$  is a cubic Halin graph, then  $G$  is  $(1, 2, 2, 2, 2)$ -packing colorable.*

*Proof.* Let  $\{1, 2_a, 2_b, 2_c, 2_d, 2_e\}$  be the set of colors that we are going to color  $G$  with. First, by using Lemma 6.8, we use the set of colors  $\{1, 2_a, 2_b, 2_c\}$  to color the vertices by a  $(1, 2, 2, 2)$ -packing coloring such that all the vertices of the cycle, which are the leaves of the tree, are colored by 1. Then, we are going to recolor the vertices of the cycle by combining the sequences  $12_d2_e$  and  $12_d12_e$  depending on the order of the cycle. This completes the proof. ■

Notice that the unique cubic Halin graph on 6 vertices is not  $(1, 2, 2, 2)$ -packing colorable, however we are not able to find a cubic Halin graph that is not  $(1, 2, 2, 2, 2)$ -packing colorable, hence the result of the previous theorem is maybe not sharp.



## 7

### Directions for further work

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*“I hope we’ll be able to solve these problems before we leave.”*

– PAUL ERDÖS –

In this thesis we showed several results concerning the existence of oriented paths with three blocks in digraphs, and the  $S$ -packing coloring of subcubic graphs. Let us give some directions for further work.

Problems on Part I:

We begin with the main problem that we study in this part which is El Sahili’s conjecture:

**Problem 1.** *Does every  $n$ -chromatic digraph contain any oriented path of order  $n$ ?*

Moreover, we consider Burr’s conjecture:

**Problem 2.** *Does every  $(2n - 2)$ -chromatic digraph contain any oriented tree of order  $n$ ?*

Knowing that oriented paths are special type of oriented tree, we aim to address the challenge posed by the intersection of these two conjectures. Our goal is to simplify the problem, making it more amenable to resolution.

**Problem 3.** *Does every  $(2n - 2)$ -chromatic digraph contain any oriented path of order  $n$ ?*

By adding some conditions on the class of a digraph, we have the following problem:

**Problem 4.** *In the case of  $n$ -chromatic Hamiltonian digraphs, can we generalize the result obtained on  $P(k, 1, l)$  into  $P(k, l, r)$ ? into any oriented path with any number of blocks?*

By ignoring the condition being on the chromatic number, and adding different conditions, we have the following problem:

**Problem 5.** *What about studying the existence of oriented paths in digraphs having conditions on their diameter? on their girth?*



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Problems on Part II:

**Problem 6.** *Is it true that  $\chi_\rho(S(G)) \leq 5$  for every  $G$  subcubic graph?*

**Problem 7.** *Is every subcubic graph except the Petersen graph  $(1, 1, 2, 2)$ -packing colorable?*

**Problem 8.** *Is it true that every subcubic graph is  $(1, 2, 2, 2, 2, 2)$ -packing colorable ?*

**Problem 9.** *Is every cubic Halin graph  $(1, 1, 2, 4)$ -packing colorable?  $(1, 1, 2, 5)$ -packing colorable?*

**Problem 10.** *Is every cubic Halin graph  $(1, 2, 2, 2, 2)$ -packing colorable?*

**Problem 11.** *Can we have benefit from the procedure we followed to prove our result on Halin graphs to prove it on generalized Halin graphs?*

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