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Disordered semiclassical pseudo-differential operators: Spectral asymptotics, statistics and eigenvector (de-)localization

Martin Vogel

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Habilitation à diriger des recherches

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AVANCÉE

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**Disordered semiclassical pseudo-differential
operators**

—
**Spectral asymptotics, statistics and eigenvector
(de-)localization**

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devant la commission d'examen

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Université

de Strasbourg

Disordered semiclassical pseudo-differential operators:
Spectral asymptotics, statistics and eigenvector (de-)localization

Martin Vogel

September 30, 2024

A Laura

Preface

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Chapter 1

Outline

Spectral theory of linear operators is an old, sophisticated and highly developed subject. It not only finds its application in a vast variety of modern problems in mathematics and physics, but it is also still a highly active field of research posing many open problems and enigmas.

Spectral theory, as we see it today, has been initiated by D. Hilbert [106–111] who was strongly motivated by early works of I. Fredholm [79] on integral equations and the famous « alternative de Fredholm ». In hindsight, however, modern spectral theory’s early forerunners can be traced back to various subjects. For instance the *principal axes theorem* of analytic geometry, see [180, sec. 1], contained in works of P. de Fermat and R. Descartes.

Hilbert’s initial work sparked a veritable plethora of subsequent works by E. Schmidt, E. Hellinger, H. Weyl, J. von Neumann and F. Riesz, to name but a few major contributors. These works lead to an operator theoretic point of view of spectral theory and an abstract definition of Hilbert spaces and, by intertwining Hilbert’s original formulation of spectral theory with the then newly developed integration theory of Lebesgue, to the spectral theorem for bounded symmetric linear operators in its integral form.

The passage from spectral theory for bounded linear operators to unbounded linear operators (in the language of modern spectral theory) came with the advent of quantum mechanics in 1925. As formulated by W. Heisenberg and E. Schrödinger it describes physical observables using operators and their spectra. It became quickly apparent that Hilbert’s spectral theory provided a solid mathematical framework for quantum mechanics. However, the back then restriction to bounded linear operators was too restrictive from a physics perspective. The reason being that many fundamental observables in physics involve unbounded operators such as the position operator, which multiplies an L^2 function by its position variable, or the momentum operator, which takes the differential of an L^2 function.

Since these early days, spectral theory has enjoyed many developments which often have come parallel and closely intertwined with advances in physics. A detailed overview would be beyond the scope of this text, although, further below we will make some relevant connections.

1.1 Scope of this habilitation thesis

The main focus of this habilitation thesis are the spectral properties of disordered semiclassical pseudo-differential operators and Toeplitz matrices. The disorder will predominantly be given by small random perturbations.

In Part I we will discuss the spectral distribution on a macroscopic level, in the form of a probabilistic Weyl law, and on the microscopic level, in the form of spectral statistics, of non-selfadjoint semiclassical pseudo-differential operators subject to small random perturbations. The relevant publications are

- S. Becker, I. Oltman, and M. Vogel, *Absence of small magic angles for disordered tunneling*

potentials in twisted bilayer graphene, arXiv:2402.12799, (2024).

- S. Nonnenmacher and M. Vogel, *Local eigenvalue statistics of one dimensional random non-selfadjoint pseudodifferential operators*, J. Eur. Math. Soc. **23** (2021), no. 5, 1521–1612.
- J. Sjöstrand and M. Vogel, *Tunneling for the $\bar{\partial}$ -bar operator*, accepted for publication in Vietnam Journal of Mathematics, arXiv:2303.06096 (2023).
- M. Vogel, *Spectral statistics of non-selfadjoint operators subject to small random perturbations*, Séminaire Laurent Schwartz - EDP et applications **19** (2016–2017), 24 p.
- M. Vogel, *Almost sure Weyl law for quantized tori*, Comm. Math. Phys **378** (2020), no. 2, 1539–1585.

In Part II we will first discuss probabilistic Weyl laws for noisy non-selfadjoint Toeplitz matrices and related eigenvector localization. These noisy Toeplitz matrices are excellent toy models for noisy pseudo-differential operators. The relevant publications are

- A. Basak, M. Vogel, and O. Zeitouni, *Localization of eigenvectors of non-Hermitian banded noisy Toeplitz matrices*, Probability and Mathematical Physics **4** (2023), no. 3, 477–607.
- J. Sjöstrand and M. Vogel, *Large bi-diagonal matrices and random perturbations*, J. of Spectral Theory **6** (2016), no. 4, 977–1020.
- J. Sjöstrand and M. Vogel, *Interior eigenvalue density of large bi-diagonal matrices subject to random perturbations*, Microlocal analysis and singular perturbation theory, RIMS Kôkyûroku Bessatsu, **B61** (2017), 201–227.
- J. Sjöstrand and M. Vogel, *General toeplitz matrices subject to gaussian perturbations*, Ann. Henri Poincaré **22** (2021), no. 1, 49–81.
- J. Sjöstrand and M. Vogel, *Toeplitz band matrices with small random perturbations*, Indagationes Mathematicae **32** (2021), no. 1, 275–322.
- M. Vogel and O. Zeitouni, *Deterministic equivalence for noisy perturbations*, Proc. Amer. Math. Soc. **149** (2021), 3905–3911.

In Part III we will consider a form of eigenvector delocalization in a selfadjoint setting. More precisely, we will discuss Berry’s conjecture for Langrangian states which have been propagated for a long time under the Schrödinger evolution semi-group induced by a random Schrödinger type operator. The relevant publications are

- M. Ingremeau and M. Vogel, *Emergence of gaussian fields in noisy quantum chaotic dynamics*, arXiv:2306.11617, (2023)
- M. Ingremeau and M. Vogel, *Improved L^∞ bounds for eigenfunctions under random perturbations in negative curvature*, arXiv:2403.13739, (2024).

In the Appendix A we review some basic notions of semiclassical analysis.

1.2 Notation

We frequently use the following notation: when we write $a \ll b$, we mean that $Ca \leq b$ for some sufficiently large constant $C > 0$. The notation $f = \mathcal{O}(N)$ means that there exists a constant $C > 0$ (independent of N) such that $|f| \leq CN$. When we want to emphasize that the constant $C > 0$ depends on some parameter k , then we write C_k , or with the above big-O notation $\mathcal{O}_k(N)$.

When we write $f = \mathcal{O}(N^{-\infty})$, then we mean that for every $M \in \mathbb{N}$, there exists a constant $C_M > 0$, depending on M , such that $|f| \leq C_M N^{-M}$. Similarly we will also use the notation $f = \mathcal{O}(h^\infty)$, $f = \mathcal{O}_k(h)$, with $h \in]0, 1]$.

When we write $f = o(1)$, as $N \rightarrow \infty$, then we mean that $f \rightarrow 0$ as $N \rightarrow \infty$. Moreover, writing $f = o(N)$, as $N \rightarrow \infty$, means that $N^{-1}f \rightarrow 0$ as $N \rightarrow \infty$. Similarly, we will also use the notation $f = o(h)$, $f = o(1)$, when $h \rightarrow 0$.

Writing $a \asymp b$ means that there exists a constant $C > 1$ such that $C^{-1}a \leq b \leq Ca$. We will denote by $[N]$, $N \in \mathbb{N}^*$, the discrete interval $[1, N] \cap \mathbb{N}$.

Part I

Non-selfadjoint disordered pseudo-differential operators

Chapter 2

Spectral instability of non-selfadjoint operators

2.1 Introduction

The spectral analysis of linear operators acting on a Hilbert space is much developed in the case of selfadjoint operators: one can then use powerful tools, like the spectral theorem, or variational methods. This fact has been very useful in mathematical physics, for example in quantum mechanics, where the natural operators (quantum observables, Hamiltonian) are selfadjoint. However, non-selfadjoint operators also appear in mathematical physics, and deserve to be investigated. For instance, in quantum mechanics, the study of scattering systems naturally leads to the concept of quantum resonances, which appear as the (complex valued) poles of the analytic continuation of the scattering matrix (or of the resolvent of the Hamiltonian) into the so-called nonphysical sheet of the complex energy plane. These resonances may also be obtained as *bona fide* eigenvalues of a non-selfadjoint operator, obtained from the initial selfadjoint Hamiltonian through a complex dilation procedure [2, 11]. Still in quantum mechanics, when considering the evolution of a "small system" in contact with an "environment", one can be lead to express the effective dynamics of the small system through a non-selfadjoint *Lindblad operator* [134]. In statistical mechanics, the evolution of the system may be described by a linear operator, which is often non-selfadjoint: the Fokker-Planck, or the linearized Boltzmann equation typically contain convective as well as dissipative terms, leading to non-selfadjoint operators. In hydrodynamics, the operators appearing when linearizing the Navier-Stokes equation in the vicinity of some specific solution are generally not selfadjoint [83].

When studying evolution problems generated by linear operators, one is naturally lead to analyze the spectrum of that operator. Yet, in the non-selfadjoint case, the connection between the long time evolution and a spectrum of complex eigenvalues is not so obvious as in the selfadjoint case, since eigenstates do not form an orthonormal family. This difficulty of relating spectrum and dynamics is linked with a characteristics of non-selfadjoint operators, namely the possible strong *instability* of their spectrum with respect to small perturbations, a phenomenon nowadays commonly called *pseudospectral effect*. Traditionally this spectral instability was considered as a drawback, since it can be at the source of immense numerical errors, see [70]. However, as we will see below, analyzing this instability can also exhibit interesting phenomena. Numerical analysis studies, e.g. by L.N. Trefethen [188], somewhat changed the perspective of this instability problem: they showed that considering the *pseudospectrum* of the (non-selfadjoint) operator — that is the region where the norm of the resolvent operator exceeds some (large) threshold, see Definition 2.2.1 below — is often more relevant than considering its spectrum, and can reveal important dynamical information. As an example, when studying a certain class of nonlinear diffusion equations, Sandsteede-Scheel [158], Raphael-Zworski [151] and Galkowski [82] showed that the pseudospectrum of the (non-selfadjoint) linearization of the equation can explain the

finite time blow-up of the solutions to the full nonlinear equation, while the mere study of its spectrum would suggest a stable evolution.

2.2 Spectral instability of non-selfadjoint operators

We begin by recalling the definition of the *pseudospectrum* of a linear operator, an important notion which quantifies its spectral instability. This notion seems to have originated in the second half of the 20th century in various contexts, see [188] for a historic overview. It quickly became an important notion in numerical analysis as it allows to quantify how much eigenvalues can *spread out* under the influence of small perturbations, see [187, 188] and the book [70]. We follow here the latter reference.

Let \mathcal{H} be a complex Hilbert space (assumed separable for simplicity) with norm $\|\cdot\|$ and scalar product $(\cdot|\cdot)$. Let $P : \mathcal{H} \rightarrow \mathcal{H}$ be a closed densely defined linear operator, with resolvent set $\rho(P)$ and spectrum $\text{Spec}(P) = \mathbb{C} \setminus \rho(P)$.

Definition 2.2.1. *For any $\varepsilon > 0$, we define the ε -pseudospectrum of P by*

$$\text{Spec}_\varepsilon(P) := \text{Spec}(P) \cup \{z \in \rho(P); \|(P - z)^{-1}\| > \varepsilon^{-1}\}. \quad (2.2.1)$$

We remark that some authors define the ε -pseudospectrum with a \geq rather than a $>$. We, however, follow here [70]. Note that with this choice of non-strict inequality the $\text{Spec}_\varepsilon(P)$ is an open set in \mathbb{C} .

For P selfadjoint (or even normal), the spectral theorem implies that

$$\text{Spec}_\varepsilon(P) \subset \text{Spec}(P) \cup D(0, \varepsilon). \quad (2.2.2)$$

For P non-selfadjoint, the pseudospectrum of P can be much larger, as illustrated by the following example.

Example 2.2.2. *For $N \gg 1$ consider the Jordan block matrix*

$$P_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \dots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} : \mathbb{C}^N \rightarrow \mathbb{C}^N. \quad (2.2.3)$$

The spectrum of P_N is given by $\{0\}$. However, Theorem 6.4.2 below shows that for any $\varepsilon > 0$ and any $r \in]0, 1[$ we have that for $N > 1$ sufficiently large

$$D(0, r) \subset \text{Spec}_\varepsilon(P_N).$$

An immediate consequence of (2.2.1) is the property that pseudospectra are nested. More precisely,

$$\text{Spec}_{\varepsilon_2}(P) \subset \text{Spec}_{\varepsilon_1}(P), \quad \varepsilon_1 > \varepsilon_2 > 0. \quad (2.2.4)$$

The set (2.2.1) describes a region of spectral instability of the operator P , since any point in the ε -pseudospectrum of P lies in the spectrum of a certain ε -perturbation of P [70].

Theorem 2.2.3. *Let $\varepsilon > 0$. Then*

$$\text{Spec}_\varepsilon(P) = \bigcup_{\substack{Q \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \\ \|Q\| < 1}} \text{Spec}(P + \varepsilon Q). \quad (2.2.5)$$

Proof. See [70, p. 31]. \square

A third, equivalent definition of the ε -pseudospectrum of P is via the existence of approximate solutions to the eigenvalue problem $(P - z)u = 0$.

Theorem 2.2.4. *Let $\varepsilon > 0$ and $z \in \mathbb{C}$. Then the following statements are equivalent:*

1. $z \in \text{Spec}_\varepsilon(P)$;
2. $z \in \text{Spec}(P)$ or there exists a $u_z \in \mathcal{D}(P)$ such that $\|(P - z)u_z\| < \varepsilon\|u_z\|$, where $\mathcal{D}(P)$ denotes the domain of P .

Proof. See [70, p. 31]. \square

Such a state u_z is called an ε -quasimode, or simply a *quasimode* of $P - z$.

2.3 Spectral instability of semiclassical pseudo-differential operators

Although the notion of ε -pseudospectrum defined in Definition 2.2.1 is valid in the setting of semiclassical pseudo-differential operators, we present here a somewhat different, yet still related notion, which is more adapted to semiclassical setting. Here “semiclassical” means that our operators depend on a parameter $h \in]0, 1]$ (often referred to as “Planck’s parameter”), and that we will be interested in the asymptotic (*semiclassical*) regime $h \searrow 0$. This small parameter will provide us with a natural threshold to define the pseudospectrum, and thereby to measure the spectral instability. The following discussion is based on the works by Davies [58] and Dencker, Sjöstrand and Zworski [62].

Let the symbol $p \in S(T^*\mathbb{R}^d, m) = S(m)$, $m \geq 1$, see Appendix A.1.1 for a definition of this symbol class, be “classical”, namely it satisfies an asymptotic expansion in the limit $h \rightarrow 0$:

$$p(\rho; h) \sim p_0(\rho) + hp_1(\rho) + \dots \quad \text{in } S(m), \quad (2.3.1)$$

where each $p_j \in S(m)$ is independent of h . We assume that there exists a $z_0 \in \mathbb{C}$ and a $C_0 > 0$ such that

$$|p_0(\rho) - z_0| \geq m(\rho)/C_0, \quad \rho \in T^*\mathbb{R}^d. \quad (2.3.2)$$

In this case we call p_0 the (semiclassical) principal symbol of p . We then define two subsets of \mathbb{C} associated with p_0 :

$$\Sigma := \Sigma(p_0) := \overline{p_0(T^*\mathbb{R}^d)}, \quad \Sigma_\infty := \{z \in \Sigma; \exists(\rho_j)_{j \geq 1} \text{ s.t. } |\rho_j| \rightarrow \infty, p_0(\rho_j) \rightarrow z\}. \quad (2.3.3)$$

The set Σ is the *classical spectrum*, and Σ_∞ can be called the *classical spectrum at infinity* of the Weyl quantization of $P_h = p^w(x, hDx; h)$, $h \in]0, 1]$, defined in (A.1.3).

2.3.1 Semiclassical pseudospectrum

Similar to [62], we define for a symbol $p \in S(m)$ as in (2.3.1).

$$\Lambda_\pm(p) := \left\{ p(\rho); \pm \frac{1}{2i} \{\bar{p}, p\}(\rho) < 0 \right\} \subset \Sigma \subset \mathbb{C}, \quad (2.3.4)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. Note that the condition $\frac{1}{2i} \{\bar{p}, p\} \neq 0$ is the classical analogue of the $[P_h^*, P_h] \neq 0$. As in [62] we call the set

$$\Lambda(p) := \overline{\Lambda_- \cup \Lambda_+} \quad (2.3.5)$$

the *semiclassical pseudospectrum*.

Theorem 2.3.1 ([62]). *Suppose that $n \geq 2$, $C_b^\infty(T^*\mathbb{R}^d) \ni p \sim p_0 + hp_1 + \dots$, and $p_0^{-1}(z)$ is compact for a dense set of values $z \in \mathbb{C}$. If $P_h = p^w(x, hD_x)$, then*

$$\Lambda(p_0) \setminus \Sigma_\infty \subset \overline{\Lambda_+(p_0)}$$

and for every $z \in \Lambda_+(p_0)$ and every $\rho_0 \in T^*\mathbb{R}^d$ with

$$p_0(\rho_0) = z, \quad \frac{1}{2i} \{\bar{p}_0, p_0\}(\rho_0) < 0,$$

there exists $0 \neq e_+ \in L^2(\mathbb{R}^d)$ such that

$$\|(P_h - z)e_+\| = \mathcal{O}(h^\infty)\|e_+\|, \quad \text{WF}_h(e_+)^1 = \{\rho_0\}. \quad (2.3.6)$$

If, in addition, p has a bounded holomorphic continuation to $\{\rho \in \mathbb{C}^{2d}, |\text{Im } \rho| \leq 1/C\}$, then (2.3.6) holds with the h^∞ replaced by $\exp(-1/(Ch))$.

If $n = 1$, then the same conclusion holds, provide that in addition to the general assumptions, each component of $\mathbb{C} \setminus \Sigma_\infty$ has a nonempty intersection with $\mathbb{C}\Lambda(p)$.

This result can be extended to unbounded symbols $p \in S(T^*\mathbb{R}^d, m)$, as in (2.3.1), and the corresponding operators P_h with principal symbol p_0 , by applying Theorem 2.3.1 to $\tilde{P}_h = (P_h - z_0)^{-1}(P_h - z_1)$, with principal symbol $\tilde{p}_0 \in C_b^\infty(T^*\mathbb{R}^d)$ and z_0 as in (2.3.2) and $z_0 \neq z_1$. Indeed, note that $z \in \Sigma(p_0)$ if and only if $0 \in \Sigma(\tilde{p}_0)$, and that $\rho \in p_0^{-1}(z)$ with $\pm\{\text{Re } p_0, \text{Im } p_0\}(\rho) < 0$ is equivalent to $\rho \in \tilde{p}_0^{-1}(0)$ with $\pm\{\text{Re } \tilde{p}_0, \text{Im } \tilde{p}_0\}(\rho) < 0$. Furthermore, a quasimode u as Theorem 2.3.1 for \tilde{P}_h then provides, after a possible truncation, a quasimode for $P_h - z$ in the same sense.

By replacing P_h with its formal adjoint P_h^* , and thus p with \bar{p} , Theorem 2.3.1 yields that for every $z \in \Lambda_-(p)$ and every $\rho_0 \in T^*\mathbb{R}^d$ with

$$p_0(\rho_0) = z, \quad \frac{1}{2i} \{\bar{p}_0, p_0\}(\rho_0) > 0,$$

there exists $0 \neq e_- \in L^2(\mathbb{R}^d)$ such that

$$\|(P_h - z)^*e_-\| = \mathcal{O}(h^\infty)\|e_-\|, \quad \text{WF}_h(e_-) = \{\rho_0\}.$$

The additional statements of Theorem 2.3.1 about symbols admitting a holomorphic extension to a complex neighborhood of \mathbb{R}^{2d} , and the case when $n = 1$ hold as well.

Example 2.3.2. *The guiding example to keep in mind is the case of the non-selfadjoint Harmonic oscillator*

$$P_h = (hD_x)^2 + ix^2$$

seen as an unbounded operator $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$. The principal symbol of P_h is given by $p(x, \xi) = \xi^2 + ix^2 \in S(T^*\mathbb{R}, m)$, with weight function $m(x, \xi) = 1 + \xi^2 + x^2$. We equip P_h with the domain $H(m) := (P_h + 1)^{-1}L^2(\mathbb{R})$, where the operator on the right is the pseudo-differential inverse of $P_h + 1$. This choice of domain makes P_h a closed densely defined operator. Using, for instance, the method of complex scaling we see that the spectrum of P_h is given by

$$\text{Spec}(P_h) = \{e^{i\pi/4}(2n+1)h; n \in \mathbb{N}\}. \quad (2.3.7)$$

Furthermore, Σ is the closed 1st quadrant in the complex plane, whereas $\Sigma_\infty = \emptyset$. For $\rho = (x, \xi) \in T^*\mathbb{R}$, we find that

$$\frac{1}{2i} \{\bar{p}, p\}(x, \xi) = 2\xi \cdot x. \quad (2.3.8)$$

¹This is to say that the semiclassical wavefront set of e_+ is given by ρ_0 . In other words, the state e_+ is concentrated in position and frequency near the point ρ_0 . See (A.2.1) for a definition.

Thus, for every $z \in \overset{\circ}{\Sigma}$ there exist points

$$\rho_+^j(z) = (-1)^j(-\sqrt{|\operatorname{Re} z|}, \sqrt{|\operatorname{Im} z|}), \quad \rho_-^j(z) = (-1)^j(-\sqrt{|\operatorname{Re} z|}, -\sqrt{|\operatorname{Im} z|}), \quad j = 1, 2,$$

such that

$$\pm \frac{1}{2i} \{\bar{p}, p\}(\rho_{\pm}^j(z)) < 0, \quad j = 1, 2.$$

Using the WKB method, we can construct quasimodes of the form $e_+^j(x; h) = a_+^j(x; h)e^{i\phi_+^j(x)/h}$ with $a_+^j(x; h) \in C_c^\infty(\mathbb{R})$ admitting an asymptotic expansion $a_+^j(x; h) \sim a_{+,0}^j(x) + ha_{+,1}^j(x) + \dots$ with $\operatorname{WF}_h(e_+^j) = \{\rho_+^j(z)\}$ and

$$\|(P_h - z)e_+^j\| = \mathcal{O}(e^{-1/Ch}), \quad (2.3.9)$$

see [58, 59] for an explicit computation, and [62] for a more general construction.

In fact the works of Davies [58, 59] provide an explicit WKB construction for a quasimode u for one-dimensional non-selfadjoint Schrödinger operators $P_h - z = (hD_x)^2 + V(x) - z$ on $L^2(\mathbb{R})$ with $V \in C^\infty(\mathbb{R})$ complex-valued and $z = V(a) + \eta^2$, for some $a \in \mathbb{R}$, $\eta > 0$. Furthermore, one assumes that $\operatorname{Im} V'(a) \neq 0$. These works were the starting point for the quasimode construction for non-selfadjoint (pseudo-)differential operators. Zworski [202] linked Davies' quasimode construction under the condition on the gradient of $\operatorname{Im} V$ to a quasimode construction under a non-vanishing condition of the Poisson bracket $\frac{1}{2i} \{\bar{p}, p\}$. Furthermore, Zworski [202] established the link to the famous commutator condition of Hörmander [116, 117]. A full generalization of the quasimode construction under a non-vanishing condition of the poisson bracket, see Theorem 2.3.1 above, was then achieved by Dencker, Sjöstrand and Zworski [62]. Finally, Pravda-Starov [148–150] improved these results by modifying a quasimode construction by Moyer and Hörmander, see [119, Lemma 26.4.14], for adjoints of operators that do not satisfy the Nirenberg-Trèves condition (Ψ) for local solvability.

Notice, that (2.3.6) (or (2.3.9) in the example above) implies that if the resolvent $(P_h - z)^{-1}$ exists then it is larger than any power of h when $h \rightarrow 0$, or even larger than $e^{1/Ch}$ in the analytic case. We call each family $(e_+^j(z, h))$ an h^∞ -quasimode of $P_h - z$, or for short a quasimode of $P_h - z$.

From the quasimode equation (2.3.6) it is easy to exhibit an operator Q of norm 1 and a parameter $\delta = \mathcal{O}(h^\infty)$, such that the perturbed operator $P_h + \delta Q$ has an eigenvalue at z . For instance, if we call the error $r_+ = (P_h - z)e_+$, we may take the rank 1 operator $\delta Q = -r_+ \otimes (e_+)^*$. By Theorem 2.3.1 we see that the interior of the set $\Lambda(p)$, away from the set Σ_∞ , is a zone of strong spectral instability for P_h . For this reason we may refer to the semiclassical pseudospectrum $\Lambda(p)$ also as the (h^∞) -pseudospectrum of P_h .

2.3.2 Outside the semiclassical pseudospectrum.

When

$$z \in \mathbb{C} \setminus \Sigma(p),$$

then by condition (2.3.2) we have that $(p_0(\rho) - z) \geq m(\rho)/C$ for some sufficiently large $C > 0$ and so we know that the inverse $(P_h - z)^{-1}$ is a pseudo-differential operator with principal symbol $(p_0 - z)^{-1} \in S(1/m) \subset S(1)$. Hence, $(P_h - z)^{-1}$ maps $L^2 \rightarrow L^2$ and

$$\|(P_h - z)^{-1}\| = \mathcal{O}(1) \quad (2.3.10)$$

uniformly in $h > 0$. Hence, from the semiclassical point of view we may consider $\mathbb{C} \setminus \Sigma$ as a *zone of spectral stability*.

2.3.3 At the boundary of the semiclassical pseudospectrum

At the boundary of the semiclassical pseudospectrum we find a transition between the zone of strong spectral instability and stability. Indeed at the boundary we find an improvement on the resolvent bounds, assuming some additional non-degeneracy:

Splitting a symbol $p \in C_b^\infty(T^*\mathbb{R}^d)$ into real and imaginary part, $p = p_1 + ip_2$, we consider the iterated Poisson bracket

$$p_I := \{p_{i_1}, \{p_{i_2}, \{\dots, \{p_{i_{k-1}}, p_{i_k}\}\}\dots\}\}$$

where $I \in \{1, 2\}^k$, and $|I| = k$ is called the *order* of the Poisson bracket. The *order* of p at $\rho \in T^*\mathbb{R}^d$ is given by

$$k(\rho) := \max\{j \in \mathbb{N}; p_I(\rho) = 0, 1 < |I| \leq j\}.$$

The *order* of $z_0 \in \Sigma \setminus \Sigma_\infty$ is the maximum of $k(\rho)$ for $\rho \in p^{-1}(z_0)$.

Theorem 2.3.3. [62, 170] *Assume that $C_b^\infty(T^*\mathbb{R}^d) \ni p \sim p_0 + hp_1 + \dots$. Let $P_h = p^w(x, hD_x)$, and let $z_0 \in \partial\Sigma(p_0) \setminus \Sigma_\infty(p_0)$. Suppose that $dp_0 \neq 0$ at every point in $p_0^{-1}(z_0)$, that z_0 is of finite order $k \geq 1$ for p . Then, k is even and for $h > 0$ small enough*

$$\|(P_h - z)^{-1}\| \leq Ch^{-\frac{k}{k+1}}.$$

In particular, there exists a $c_0 > 0$, such that for $h > 0$ small enough

$$\{z \in \mathbb{C}; |z - z_0| \leq c_0 h^{\frac{k}{k+1}}\} \cap \text{Spec}(P_h) = \emptyset.$$

This result was proven in dimension 1 by Zworski [203], and in certain cases by Boulton [40]. Further refinements have been obtained in [170]. Similar to the discussion after Theorem 2.3.1, we can extend Theorem 2.3.3 to unbounded symbols $p \in S(T^*\mathbb{R}^d, m)$ and their corresponding quantizations.

Example 2.3.4. *Recall the non-selfadjoint Harmonic oscillator $P_h = (hD_x)^2 + ix^2$ from Example 2.3.2. Here $\partial\Sigma = \mathbb{R}_+ \cup i\mathbb{R}_+$, so we see by (2.3.8) that for $0 \neq z_0 \in \Sigma$*

$$\frac{1}{2i} \{\bar{p}, p\}(\rho) = \{\text{Re } p, \text{Im } p\}(\rho) = 0, \quad \rho \in p^{-1}(z_0).$$

However, at such points we have

$$\text{either } \{\text{Re } p, \{\text{Re } p, \text{Im } p\}\}(\rho) = 4\xi^2 \neq 0, \quad \text{or } \{\text{Im } p, \{\text{Re } p, \text{Im } p\}\}(\rho) = -4x^2 \neq 0,$$

so z_0 is of order 2 for $p = \xi^2 + ix^2$, and Theorem 2.3.3 tells us that

$$\|(P_h - z_0)^{-1}\| \leq Ch^{-\frac{2}{3}}.$$

We see that for a the ε -pseudospectrum of P_h to reach the boundary of Σ , we require $\varepsilon > h^{2/3}/C$.

2.4 Singular values

In this section we review some basic notions from the spectral theory of non-selfadjoint operators. We refer the reader to the book by Gohberg and Kreĭn [89] for an excellent overview.

2.4.1 Singular values

For a compact operator $A : \mathcal{H} \rightarrow \mathcal{H}$ on a complex separable Hilbert space we define, following [89], the *singular values* of A to be the decreasing sequence

$$s_1(A) \geq s_2(A) \geq \dots \searrow 0, \quad (2.4.1)$$

of all eigenvalues of the compact selfadjoint operator $(A^*A)^{1/2}$. The intertwining relations

$$A(A^*A) = (AA^*)A, \quad (A^*A)A^* = A^*(AA^*)$$

imply that the non-vanishing singular values of A and A^* coincide. Furthermore, we have min-max characterization, see for example [89, p. 25], of the singular values

$$s_j(A) = \inf_{L \subset \mathcal{H}} \sup_{u \in L \setminus \{0\}} \frac{((A^*A)^{1/2}u|u)}{(u|u)},$$

where the infimum is taken over all closed subspaces $L \subset \mathcal{H}$ of codimension $= j - 1$.

The following theorem is due to Ky Fan [89, Corollary 2.2].

Theorem 2.4.1. *Let $A, B : \mathcal{H} \rightarrow \mathcal{H}$ be compact operators. Then, for any $n, m \geq 1$*

$$\begin{aligned} s_{n+m-1}(A+B) &\leq s_n(A) + s_m(B), \\ s_{n+m-1}(AB) &\leq s_n(A)s_m(B). \end{aligned} \quad (2.4.2)$$

In the case when $P : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded Fredholm operator of index 0 we introduce the increasing sequence

$$0 \leq t_1(P) \leq t_2(P) \leq \dots \quad (2.4.3)$$

consisting first of eigenvalues of $(P^*P)^{1/2}$, counting multiplicities, below the infimum of the essential spectrum (should there be any) and then, if there are only finitely many such eigenvalues, we repeat indefinitely that infimum. Again we have that $t_j(P) = t_j(P^*)$. Indeed the number of 0 eigenvalues coincides as P is of index 0. Furthermore, given a normalized eigenvector e such that $P^*Pe = t^2e$, $t > 0$, we see that $PP^*f = t^2f$, with $f = t^{-1}Pe$, and vice versa.

Notice that if $\dim \mathcal{H} =: N < \infty$, then

$$t_j(P) = s_{N+1-j}(P). \quad (2.4.4)$$

When $P : \mathcal{H} \rightarrow \mathcal{H}$ is a closed densely defined unbounded Fredholm operator of index 0 we know by a classical result due to von Neumann [129, Theorem 3.24, p. 275] that the operator P^*P is selfadjoint equipped with its natural domain

$$\mathcal{D}(P^*P) := \{u \in \mathcal{D}(P); Pu \in \mathcal{D}(P^*)\}. \quad (2.4.5)$$

Furthermore, $\mathcal{D}(P^*P)$ is a core of P in the sense that the set $\{(u, Pu); u \in \mathcal{D}(P^*P)\}$ is dense in the graph(P). We then define the sequence (2.4.3) as above.

We will also call the eigenvalues in (2.4.3) the *singular values* of P .

2.5 Pseudospectra of random matrices

Let $M \in \mathbb{C}^{N \times N}$ be a complex $N \times N$ matrix. It follows from the definition of singular values presented in Section 2.4.1, that if $M - z$ is bijective for some $z \in \mathbb{C}$, then

$$\|(M - z)^{-1}\| = s_N(M - z)^{-1}.$$

In view of (2.2.1), the ε -pseudospectrum of M is then characterized by the condition that $z \in \text{Spec}_\varepsilon(M)$

$$z \in \text{Spec}_\varepsilon(M) \iff s_N(M - z) < \varepsilon.$$

A classical result due to Sankar, Spielmann and Teng [159, Lemma 3.2] (stated there for real Gaussian random matrices) tells us that with high probability the smallest singular value of a deformed random matrix is not too small.

Theorem 2.5.1 ([159]). *There exists a constant $C > 0$ such that the following holds. Let $N \geq 2$, let X_0 be an arbitrary complex $N \times N$ matrix, and let Q be an $N \times N$ complex Gaussian random matrix whose entries are all independent copies of a complex Gaussian random variable $q \sim \mathcal{N}_{\mathbb{C}}(0, 1)$. Then, for any $\delta > 0$*

$$\mathbf{P}(s_N(X_0 + \delta Q) < \delta t) \leq CNt^2.$$

Proof. For real matrices the proof can be found in [159, Lemma 3.2], see also [183, Theorem 2.2]. For complex matrices a proof is presented for instance in [193, Appendix A]. \square

Theorem 2.5.1 tells us that any fixed $z \in \mathbb{C}$ is not in the ε -pseudospectrum of $X + \delta Q$ with probability $\geq 1 - CN\varepsilon^2\delta^{-2}$. We can interpret this result as saying that the pseudospectrum of random matrices is typically *not too large*. Theorem 2.5.1 has enjoyed many extensions. For instance Rudelson and Vershynin [153] consider the case random matrices with iid sub-Gaussian entries. Tao and Vu [181] consider iid entries of non-zero variance. Cook [56] consider the case of random matrices whose entries have an inhomogeneous variance profile under appropriate assumptions. We end this section by noting the following, quantitative result due to Tao and Vu.

Theorem 2.5.2 ([183]). *Let q be a random variable with mean zero and bounded second moment, and let $\gamma \geq 1/2$, $A \geq 0$ be constants. Then, there exists a constant $C > 0$, depending on q, γ, A such that the following holds. Let Q be the random matrix of size N whose entries are independent and identically distributed copies of q , let X_0 be a deterministic matrix satisfying $\|X_0\| \leq N^\gamma$. Then,*

$$\mathbf{P}\left(s_n(X_0 + Q) \leq n^{-\gamma(2A+2)+1/2}\right) \leq C\left(n^{-A+o(1)} + \mathbf{P}(\|Q\| \geq n^\gamma)\right). \quad (2.5.1)$$

Example 2.5.3. *Consider the case where q is a random variable satisfying the moment conditions*

$$\mathbf{E}[q] = 0, \quad \mathbf{E}[|q|^2] = 1, \quad \mathbf{E}[|q|^4] < +\infty. \quad (2.5.2)$$

From [131] we know that (2.5.2) implies that $\mathbf{E}[\|Q\|] \leq CN^{1/2}$, which, using Markov's inequality, yields that for any $\varepsilon > 0$

$$\mathbf{P}\left[\|Q\| \geq CN^{1/2+\varepsilon}\right] \leq C^{-1}N^{-1/2-\varepsilon}\mathbf{E}[\|Q\|] \leq N^{-\varepsilon}. \quad (2.5.3)$$

In this case (2.5.1) becomes

$$\mathbf{P}\left(s_n(X_0 + Q) \leq n^{-(\varepsilon+1/2)(2A+2)+1/2}\right) \leq C\left(n^{-A+o(1)} + N^{-\varepsilon}\right). \quad (2.5.4)$$

2.6 Grushin problem

We review a fundamental technique: the well-posed Grushin problem for a linear densely defined family of operator $P(z) : \mathcal{H} \rightarrow \mathcal{H}$, $z \in \mathbb{C}$, on a complex (separable) Hilbert space \mathcal{H} with domain $\mathcal{D} \subset \mathcal{H}$. The general idea of setting up a Grushin problem is to extend this operator to an operator of the form

$$\mathcal{P}(z) \stackrel{\text{def}}{=} \begin{pmatrix} P(z) & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : \mathcal{D} \oplus \mathcal{H}_- \longrightarrow \mathcal{H} \oplus \mathcal{H}_+,$$

where \mathcal{H}_\pm (resp. $R_\pm(z)$) are well chosen auxiliary spaces (resp. operators). The Grushin problem is said to be *well-posed* if the extended operator $\mathcal{P}(z)$ is bijective for the range of z under study, with a good control on its inverse. Roughly speaking, the role of $R_+(z)$ is to map the quasi-kernel of $P(z)$ to the auxiliary space, while $R_-(z)$ maps the latter to the quasi-cokernel of $P(z)$; both actions finally make $\mathcal{P}(z)$ invertible.

In the case where $\dim \mathcal{H}_- = \dim \mathcal{H}_+ < \infty$, one decomposes the inverse operator blockwise as

$$\begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} \stackrel{\text{def}}{=} \mathcal{E}(z).$$

From now on we assume that $P(z)$ is Fredholm of index 0. We then identify $\mathcal{H}_\pm \simeq \mathbb{C}^N$.

Historically, Grushin Problems go back to Grushin [92] in the theory of linear partial differential equations where it was used to study hypoelliptic operators. In a different setting, such an enlarged system was used by Sjöstrand [165], whose notation we use. Grushin problems have proven useful in bifurcation theory, numerical analysis, and for treatments of spectral problems arising in electromagnetism and quantum mechanics. For further details we refer the reader to the review paper [177].

Now, given a well-posed Grushin Problem, the key observation, going back to Schur's complement formula, is the following: the initial operator $P(z)$ is invertible if and only if the finite rank operator $E_{-+}(z) : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is invertible, in which case both inverses are related by:

$$P^{-1} = E - E_+ E_{-+}^{-1} E, \quad E_{-+}^{-1} = -R_+ P^{-1} R_-. \quad (2.6.1)$$

The finite rank operator $E_{-+}(z)$ is often called an *effective Hamiltonian* for the original problem $P(z)$. It depends in a nonlinear way on the spectral parameter z , but it has the advantage of being finite dimensional. In a sense, $E_{-+}(z)$ encapsulates, in a minimal way, the spectral properties of P . If the spectrum of P is discrete in the z -range under study, its eigenvalues can be obtained as the zeros of $\det E_{-+}(z)$ (with multiplicities).

We will present an explicit Grushin problem in Section 7.6.1 below.

Chapter 3

Spectra of disordered non-selfadjoint pseudo-differential operators

3.1 Introduction

In physical situations, an “ideal” evolution operator can be perturbed by many different sources, most of them uncontrolled by the experimentalist. Hence, it seems relevant to set up a model of *random* perturbations, and to investigate how the spectrum of our initial operator reacts upon the addition of such perturbations. The spectrum of the perturbed operator thereby becomes random which can be investigated by probabilistic methods. In this chapter we will discuss results describing the *macroscopic* spectral distribution. More precisely we are interested in counting eigenvalues in sets of diameter of order 1, for a particular class of non-selfadjoint operators. Namely, we will focus on *semiclassical pseudo-differential operators with complex valued symbols*, and with some ellipticity assumption ensuring that the spectrum is discrete (at least in some region of the complex plane). As discussed in Section 2.3.1, the spectrum of these operators is in general very sensitive to perturbations. In many examples the spectrum of the initial operator is localized along 1-dimensional curves in the complex plane, while the spectrum of the perturbed operator fills up the classical spectrum Σ , see (2.3.3), defined by the symbol of our unperturbed operator. This filling up of the classical spectrum through perturbation has been studied in a series of works by Hager [94, 95], Sjöstrand [96, 168, 169], Bordeaux-Montrieux [33] (see also [50] for a similar phenomenon in the framework of Toeplitz operators on the 2-dimensional torus). These authors show that the spectrum of the randomly perturbed operator satisfies, with high probability, a complex valued version of Weyl’s law: the density of eigenvalues near a given “complex energy” z_0 inside the classical spectrum, is approximately given by $(2\pi h)^{-d} D(z_0)$, where $D(z_0) > 0$ is the *classical density* at the energy z_0 , associated with the symbol of our initial operator.

In this chapter we will discuss these *probabilistic Weyl laws*, and in particular two extensions obtained in [19, 193].

3.2 Probabilistic Weyl law for non-selfadjoint operators

Consider the operator $P_h = p^w(x, hD_x)$, with p as in (2.3.1), seen as an unbounded operator $L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$. We equip P_h with the domain $H(m) := (P_h - z_0)^{-1}L^2(\mathbb{R}^d)$. Note that $(P_h - z_0)^{-1}$ exists for $h > 0$ small enough by the ellipticity condition (2.3.2). We will denote by $\|u\|_m := \|(P_h - z_0)u\|$ the associated norm on $H(m)$. Although this norm depends on the choice of the symbol $p_0 - z_0$, it is equivalent to the norm defined by any operator with elliptic principal symbol $q \in S(m)$, so that the space $H(m)$ only depends on the order function m . Since $H(m)$ contains the Schwartz functions $\mathcal{S}(\mathbb{R}^d)$ it is dense in $L^2(\mathbb{R}^d)$.

Let us check that P_h equipped with domain $H(m)$ is closed. Let $(P_h - z_0)u_j \rightarrow v$ and $u_j \rightarrow u$ in L^2 . Since $(P_h - z_0) : H(m) \rightarrow L^2$ is bijective, it follows that $u_j \rightarrow (P_h - z_0)^{-1}v$ in $H(m)$ and

also in L^2 . So $u = (P_h - z_0)^{-1}v$. Summing up, P_h equipped with domain $H(m)$ is a densely defined closed linear operator.

Recall (2.3.3), and let

$$\Omega \Subset \mathbb{C} \setminus \Sigma_\infty \quad (3.2.1)$$

be not entirely contained in Σ . Using the ellipticity assumption (2.3.2) it was proven in [96, Section 3] that

- $\text{Spec}(P_h) \cap \Omega$ is discrete for $h > 0$ small enough,
- For all $\varepsilon > 0$ there exists an $h(\varepsilon) > 0$ such that

$$\text{Spec}(P_h) \cap \Omega \subset \Sigma + D(0, \varepsilon), \quad 0 < h \leq h(\varepsilon),$$

where $D(0, \varepsilon)$ denotes the disc in \mathbb{C} of radius ε and centered at 0.

3.2.1 The selfadjoint setting

If P_h above is selfadjoint, which implies in particular that p is real-valued, we have the classical Weyl asymptotics. We follow here [63] for a brief review.

Theorem 3.2.1. *Let Ω be as in (3.2.1). For every h -independent interval $I \subset \Omega \cap \mathbb{R}$ with $\text{Vol}_{\mathbb{R}^{2d}}(\partial I) = 0$,*

$$\#(\text{Spec}(P_h) \cap I) = \frac{1}{(2\pi h)^d} \left(\int_{p_0^{-1}(I)} dx d\xi + o(1) \right), \quad h \rightarrow 0. \quad (3.2.2)$$

This result is, in increasing generality, due to Chazarain [49], Helffer and Robert [99, 100], Petkov and Robert [147] and Ivrii [125]. See also [63] for an overview. We highlight two special cases: when $I = [a, b]$, $a < b$, and a, b are not critical points of p_0 , then the error term becomes $\mathcal{O}(h)$, see Chazarain [49], Helffer-Robert [99] and Ivrii [125]. When additionally the unions of periodic H_{p_0} trajectories¹ in the energy shell $p_0^{-1}(a)$ and $p_0^{-1}(b)$ are of Liouville measure 0, then the error term is of the form

$$h \left(\int_{p_0=a} p_1(\rho) L_a(d\rho) - \int_{p_0=b} p_1(\rho) L_b(d\rho) \right) + o(h), \quad (3.2.3)$$

where L_λ denotes the Liouville measure on $p_0^{-1}(\lambda)$. See Petkov and Robert [147] and Ivrii [125], as well as [63], for details. Let us also highlight that similar results to Theorem 3.2.1 also hold on compact smooth manifolds, see for instance [90, Chapter 12] and the references therein.

The corresponding results in the setting of selfadjoint partial differential operators in the high energy limit go back to the seminal work of Weyl [198] and have a long and very rich history. These are however beyond the scope of this review.

Example 3.2.2. *The guiding example to keep in mind is the selfadjoint Harmonic oscillator*

$$P_h = (hD_x)^2 + x^2 : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$$

seen as an unbounded operator. The principal symbol of P_h is given by $p(x, \xi) = \xi^2 + x^2 \in S(T^\mathbb{R}, m)$, with weight function $m(x, \xi) = 1 + \xi^2 + x^2$. We equip P_h with the domain $H(m) := (P_h + 1)^{-1}L^2(\mathbb{R})$, where the operator on the right is the pseudo-differential inverse of $P_h + 1$. This choice of domain makes P_h a closed densely defined operator. It is well-known (see for instance [204, Theorem 6.2]) that the spectrum of P_h is given by*

$$\text{Spec}(P_h) = \{(2n + 1)h; n \in \mathbb{N}\}.$$

¹ H_{p_0} denotes the Hamilton vector field induced by p_0 .

Counting the points $(2n + 1)h$ contained in an interval $[a, b]$, $0 \leq a < b < \infty$, gives

$$\#(\text{Spec}(P_h) \cap [a, b]) = \frac{b - a}{2h} + \mathcal{O}(1).$$

Since $\text{Vol}_{\mathbb{R}^2}(\{a \leq \xi^2 + x^2 \leq b\}) = \pi(b - a)$, we confirm Theorem 3.2.1 for the Harmonic oscillator.

3.2.2 The non-selfadjoint setting

The natural counterpart of Theorem 3.2.1 for non-selfadjoint operators would be eigenvalue asymptotics in a complex domain $\Omega \Subset \mathbb{C}$ as in (3.2.1). Recall the non-selfadjoint Harmonic oscillator P_h from Example 2.3.2 with principal symbol $p(x, \xi) = \xi^2 + ix^2$. In this case $\Sigma = \{z \in \mathbb{C}; \text{Re } z, \text{Im } z \geq 0\}$ and $\Sigma_\infty = \emptyset$. Any $\emptyset \neq \Omega \Subset \Sigma$ away from the line $e^{i\pi/4}\mathbb{R}_+$ gives in view of (2.3.7) that

$$\#(\text{Spec}(P_h) \cap \Omega) = 0.$$

On the other hand

$$\frac{1}{2\pi h} \int_{p^{-1}(\Omega)} dx d\xi > 0.$$

This example suggests that a direct generalization of Theorem 3.2.1 to non-selfadjoint operators with complex valued principal symbol cannot hold.

Let us comment on two settings where a form of Weyl asymptotics is known to hold: Upon assuming analyticity, one may recover a sort of Weyl asymptotics. More precisely, as shown in the works of Melin and Sjöstrand [142], Sjöstrand [167], Hitrik and Sjöstrand [112–114], Hitrik, Sjöstrand and Vũ Ngọc [115] and Rouby [152], the discrete spectrum of certain analytic non-selfadjoint pseudo-differential operators is confined to curves in Σ . Moreover, one may recover eigenvalue asymptotics via Bohr-Sommerfeld quantization conditions.

The second setting is when the non-selfadjointness of the operator P_h comes not from the principal symbol p_0 (assumed to be real-valued) but from the subprincipal symbol p_1 . For instance, when studying the damped wave equation on a compact Riemannian manifold X one is led to study the eigenvalues of the corresponding stationary operator

$$P_h(z) = -h^2\Delta + 2ih\sqrt{a(x)}\sqrt{z}, \quad a \in C^\infty(X; \mathbb{R}).$$

Here, Δ denotes the Laplace-Beltrami operator on X and we call $z \in \mathbb{C}$ an eigenvalue of $P_h(z)$ if there exists a corresponding L^2 function u contained in the kernel of $P_h(z) - z$. Actually, such a u is smooth by elliptic regularity. Using Fredholm theory one can show that these eigenvalues form a discrete set in \mathbb{C} .

The principal part of $P_h = P_h(z)$ is given by $-h^2\Delta$, and thus is self-adjoint, with principal symbol is $p_0(x, \xi) = |\xi|_x^2$ (the norm here is with respect to the Riemannian metric on X). However, the subprincipal part is complex valued and non-selfadjoint.

Lebeau [132] established that there exist $a_\pm \in \mathbb{R}$ such that for every $\varepsilon > 0$ there are only finitely many eigenvalues such that

$$\frac{\text{Im } z}{h} \notin [a_- - \varepsilon, a_+ + \varepsilon].$$

Remark 3.2.3. In fact Lebeau provided precise expressions for a_\pm in terms of infimum and the supremum over the co-sphere bundle S^*X of the long time average of the damping function a evolved via the geodesic flow. Further refinements have been obtained by Sjöstrand [166], and when X is negatively curved by Anantharaman [5] and Jin [127].

Additionally Markus and Matsaev [139] and Sjöstrand [166] proved the following analogue of the Weyl law. For $0 < E_1 < E_2 < \infty$ and for $C > 0$ sufficiently large

$$\#(\text{Spec}(P_h) \cap ([E_1, E_2] + i[-Ch, Ch])) = \frac{1}{(2\pi h)^d} \left(\iint_{p_0^{-1}([E_1, E_2])} dx d\xi + \mathcal{O}(h) \right). \quad (3.2.4)$$

Finer results have been obtained by Anantharaman [5] and Jin [127] when X is negatively curved.

3.2.3 Probabilistic Weyl asymptotics

In a series of works by Hager [94–96] and Sjöstrand [168, 169], the authors proved a Weyl law, with overwhelming probability, for the eigenvalues in a compact set $\Omega \Subset \mathbb{C}$ as in (3.2.1) for randomly perturbed operators

$$P^\delta = P_h + \delta Q_\omega, \quad 0 < \delta = \delta(h) \ll 1, \quad (3.2.5)$$

where P_h is as in Section 2.3 and the random perturbation Q_ω is one of the following two types.

Random Matrix. Let $N(h) \rightarrow \infty$ sufficiently fast as $h \rightarrow 0$. Let $q_{j,k}$, $0 \leq j, k < N(h)$ be independent copies of a complex Gaussian random variable $\alpha \sim \mathcal{N}_{\mathbb{C}}(0, 1)$. We consider the random matrix

$$Q_\omega = \sum_{0 \leq j, k < N(h)} q_{j,k} e_j \otimes e_k^*, \quad (3.2.6)$$

where $\{e_j\}_{j \in \mathbb{N}} \subset L^2(\mathbb{R}^d)$ is an orthonormal basis and $e_j \otimes e_k^* u = (u|e_k)e_j$ for $u \in L^2(\mathbb{R})$. The condition on $N(h)$ is determined by the requirement that the microsupport of the vectors in the orthonormal system $\{e_j\}_{j < N(h)}$ “covers” the compact set $p_0^{-1}(\Omega) \subset T^*\mathbb{R}^d$, where p_0 is the principal symbol of P_h . For instance, we could take the first $N(h)$ eigenfunctions (ordered according to increasing eigenvalues) of the Harmonic oscillator $P_h = -h^2\Delta + x^2$ on \mathbb{R}^d . The number $N(h)$ is then determined by the condition that the semiclassical wavefront sets of e_j , $j \geq N(h)$, are disjoint from $p_0^{-1}(\Omega)$. Alternatively, as in [96], one may take $N(h) = \infty$, however then one needs to conjugate Q_ω by suitable elliptic Hilbert-Schmidt operators. We refer to [96] for more details.

Random Potential. Similar to the above, we take $N(h)$ and an orthonormal family $(e_k)_{k \in \mathbb{N}}$. Let v be real or complex random vector in $\mathbb{R}^{N(h)}$ or $\mathbb{C}^{N(h)}$, respectively, with joint probability law

$$v_*(d\mathbf{P}) = Z_h^{-1} \mathbf{1}_{B(0,R)}(v) e^{\phi(v)} L(dv), \quad (3.2.7)$$

where $Z_h > 0$ is a normalization constant, $B(0, R)$ is either the real ball $\Subset \mathbb{R}^{N(h)}$ or the complex ball $\Subset \mathbb{C}^{N(h)}$ of radius $R = R(h) \gg 1$, and centered at 0, $L(dv)$ denotes the Lebesgue measure on either $\mathbb{R}^{N(h)}$ or $\mathbb{C}^{N(h)}$ and $\phi \in C^1$ with

$$\|\nabla_v \phi\| = \mathcal{O}(h^{-\kappa_4}) \quad (3.2.8)$$

uniformly, for an arbitrary but fixed $\kappa_4 \geq 0$. In [95] the case of non-compactly supported probability law was considered. More precisely, the entries of the random vector v were supposed to be independent and identically distributed (iid) complex Gaussian random variables $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$. In [168, 169], the law (3.2.7) was considered. For the sake of simplicity we will not detail here the precise conditions on the e_k , $R(h)$, and $N(h)$, in this case but refer the reader to [168, 169]. However, one example of a random vector v with law (3.2.8) is a truncated complex or real Gaussian random variables with expectation 0, and uniformly bounded covariances. In fact, the methods in [168, 169] can be extended to non-compactly supported probability distributions, provided sufficient decay conditions at infinity are assumed. For instance iid complex Gaussian random variables, as in the one dimensional case [95], are permissible. Finally, we remark that the methods in [168, 169] can probably also be modified to allow for the case of more general independent and identically distributed random variables.

We define the random function

$$V_\omega = \sum_{0 \leq j < N(h)} v_j e_j. \quad (3.2.9)$$

We call this perturbation a “random potential”, even though V_ω is complex valued. When we consider this type of perturbation, we will make the additional symmetry assumption:

$$p(x, \xi; h) = p(x, -\xi; h). \quad (3.2.10)$$

Let $\Omega \Subset \mathbb{C}$ be an open simply connected set as in (3.2.1). For $z \in \Omega$ and $0 \leq t \ll 1$ we set

$$V_z(t) = \text{Vol}\{\rho \in T^*\mathbb{R}^d; |p_0(\rho) - z|^2 \leq t\}. \quad (3.2.11)$$

Let $\Gamma \Subset \Omega$ be open with \mathcal{C}^2 boundary and make the following non-flatness assumption

$$\exists \kappa \in]0, 1], \text{ such that } V_z(t) = \mathcal{O}(t^\kappa), \text{ uniformly for } z \in \text{neigh}(\partial\Gamma), \ 0 \leq t \ll 1. \quad (3.2.12)$$

The above mentioned works show the following result.

Theorem 3.2.4 (Probabilistic Weyl’s law). *Let Ω be as in (3.2.1). Let $\Gamma \Subset \Omega$ be open with \mathcal{C}^2 boundary. Let P_h^δ be a randomly perturbed operator as in (3.2.5) with $e^{-1/Ch} \ll \delta \leq h^\theta$ with $\theta > 0$ sufficiently large. Then, in the limit $h \rightarrow 0$,*

$$\#(\text{Spec}(P_h^\delta) \cap \Gamma) = \frac{1}{2\pi h} \left(\iint_{p_0^{-1}(\Gamma)} dx d\xi + o(1) \right) \quad \text{with probability } \geq 1 - Ch^\eta, \quad (3.2.13)$$

for some fixed $\eta > 0$.

The works [94–96, 168, 169] also provide an explicit control over θ , the error term in Weyl’s law, and the error term in the probability estimate. Theorem 3.2.4 has also been extended to case of elliptic semiclassical differential operators on compact manifolds [169] and to Toeplitz quantization of the torus in [50]. We illustrate Theorem 3.2.4 with a numerical simulation in Figure 3.1 below.

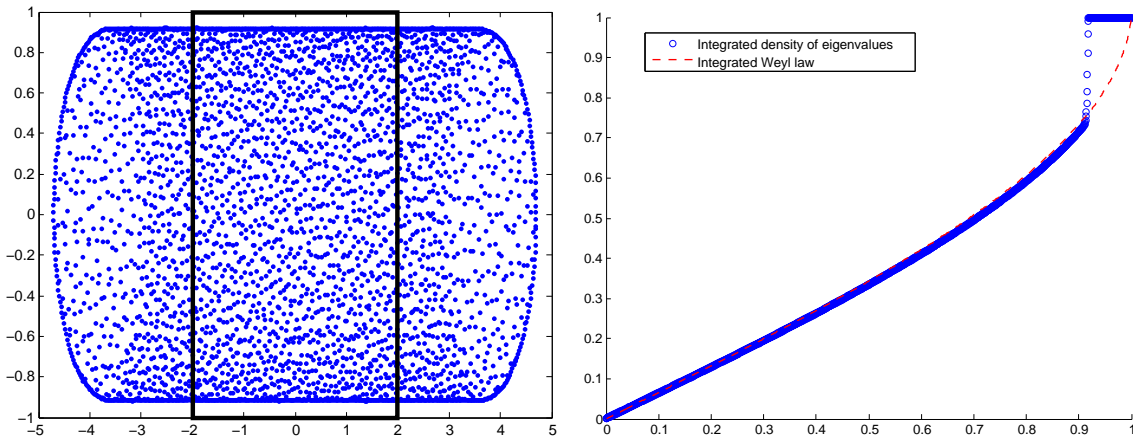


Figure 3.1: The left hand side shows the spectrum of a discretization of $P_h = hD_x + \exp(-ix)$, $x \in S^1$, (approximated by a $N \times N$ -matrix, $N = 3999$) perturbed with a random Gaussian matrix δQ_ω with $h = N^{-1}$ and $\delta = N^{-4}$. The spectrum of the perturbed operator is in stark contrast to the spectrum of the unperturbed operator P_h which is given by $\{hk; k \in \mathbb{Z}\}$. Counting the eigenvalues in the black box gives the integrated experimental density of eigenvalues (as a function of the imaginary part), averaged over 400 realizations of random Gaussian matrices, depicted as blue circles. Comparing it with the integrated Weyl law (red line), we can see that the two densities coincide in the interior of the classical spectrum Σ , whereas they differ close to boundary $\partial\Sigma$. This figure stems from [192].

Theorem 3.2.4 is remarkable because such Weyl laws are typically a feature of selfadjoint operator, whereas in the non-selfadjoint case they generally fail. Indeed, as laid out in Section 3.2.2, the discrete spectrum of the (unperturbed) non-selfadjoint operator P_h is usually localized to curves in the pseudospectrum Σ , see [112–115, 142, 152]. In contrast, Theorem 3.2.4 shows that a “generic” perturbation of size $\mathcal{O}(h^\infty)$ is sufficient for the spectrum to “fill out” Σ .

To illustrate this phenomenon recall the non-selfadjoint harmonic oscillator $P_h = -h^2\partial_x^2 + ix^2$ on \mathbb{R} from Example 2.3.2. Its spectrum is given by $\{e^{i\pi/4}(2n+1)h; n \in \mathbb{N}\}$ [59] on the line $e^{i\pi/4}\mathbb{R}_+ \subset \mathbb{C}$. The Theorem 3.2.4 shows that a “generic” perturbation of arbitrarily small size is sufficient to produce spectrum roughly equidistributed in its classical spectrum Σ , which is in this case the upper right quadrant of \mathbb{C} .

As observed in [50] for real analytic p condition (3.2.12) always holds for some $\kappa > 0$. Similarly, when p is real analytic and such that $\Sigma \subset \mathbb{C}$ has non-empty interior, then

$$\forall z \in \partial\Omega : dp|_{p^{-1}(z)} \neq 0 \implies (3.2.12) \text{ holds with } \kappa > 1/2. \quad (3.2.14)$$

For smooth p we have that when for every $z \in \partial\Omega$

$$\begin{aligned} dp, d\bar{p} \text{ are linearly independent at every point of } p^{-1}(z), \\ \text{then (3.2.12) holds with } \kappa = 1. \end{aligned} \quad (3.2.15)$$

Observe that dp and $d\bar{p}$ are linearly independent at ρ when $\{p, \bar{p}\}(\rho) \neq 0$, where $\{a, b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b$ denotes the Poisson bracket. Moreover, in dimension $d = 1$ the condition $\{p, \bar{p}\} \neq 0$ on $p^{-1}(z)$ is equivalent to $dp, d\bar{p}$ being linearly independent at every point of $p^{-1}(z)$. However, in dimension $d > 1$ this cannot hold in general as the integral of $\{p, \bar{p}\}$ with respect to the Liouville measure on $p^{-1}(z)$ vanishes on every compact connected component of $p^{-1}(z)$, see [141, Lemma 8.1]. Furthermore, condition (3.2.15) cannot hold when $z \in \partial\Sigma$. However, some iterated Poisson bracket may not be zero there. For example, it was observed in [96, Example 12.1] that if

$$\forall \rho \in p^{-1}(\partial\Omega) : \{p, \bar{p}\}(\rho) \neq 0 \text{ or } \{p, \{p, \bar{p}\}\}(\rho) \neq 0, \text{ then (3.3.9) holds with } \kappa = \frac{3}{4}. \quad (3.2.16)$$

3.3 Probabilistic Weyl law for Quantized tori

In this section we present an extension of Theorem 3.2.4 for perturbations given by more general random matrix ensembles presented in [193]. Furthermore, we will consider perturbations up to the *limiting strength* such that $\delta\|Q_\omega\| = o(1)$ with overwhelming probability. This regime was not covered by the results presented in Theorem 3.2.4 as the perturbation there is supposed to be sufficiently small: typically, this puts us in the regime where $\delta\|Q_\omega\| = \mathcal{O}(h^\mathfrak{d})$ with overwhelming probability for some $\mathfrak{d} > 0$ large enough.

We consider Toeplitz quantizations of complex-valued functions on the $2d$ -dimensional dimensional torus $\mathbb{T}^{2d} = \mathbb{R}^{2d}/\mathbb{Z}^{2d}$. This quantization maps smooth functions to $N^d \times N^d$ matrices (in general non-selfadjoint),

$$C^\infty(\mathbb{T}^{2d}) \ni p \mapsto p_N \in \mathcal{L}(\mathbb{C}^{N^d}, \mathbb{C}^{N^d}). \quad (3.3.1)$$

We will describe this procedure in Section 3.3.1, for more details see [193]. However, first we observe that when $d = 1$, then $\mathbb{T}^2 = S_x^1 \times S_\xi^1$ and

$$\begin{aligned} f = f(x) &\mapsto f_N = \text{diag}(f(l/N); l = 0, \dots, N-1) \\ g = g(\xi) &\mapsto g_N = \mathcal{F}_N^* \text{diag}(g(l/N); l = 0, \dots, N-1) \mathcal{F}_N, \end{aligned} \quad (3.3.2)$$

where $\mathcal{F}_N^* = N^{-1/2}(\exp(2\pi i k \ell / N))_{0 \leq k, \ell \leq N-1}$ is the discrete Fourier transform. In the case of \mathbb{T}^2 , the operators p_N are also referred to as *twisted Toeplitz matrices*, see [70, 189].

Example 3.3.1. Consider the Scottish flag operator [50, 70] given by the symbol

$$p(x, \xi) = \cos(2\pi x) + i \cos(2\pi \xi), \quad (x, \xi) \in \mathbb{T}^2. \quad (3.3.3)$$

From (3.3.2) we get that

$$p_N = \begin{pmatrix} \cos x_1 & i/2 & 0 & 0 & \dots & i/2 \\ i/2 & \cos x_2 & i/2 & 0 & \dots & 0 \\ 0 & i/2 & \cos x_3 & i/2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & i/2 & \cos x_{N-1} & i/2 \\ i/2 & 0 & \dots & 0 & i/2 & \cos x_N \end{pmatrix} \quad (3.3.4)$$

where $x_j = 2\pi j/N$, $j = 1, \dots, N$.

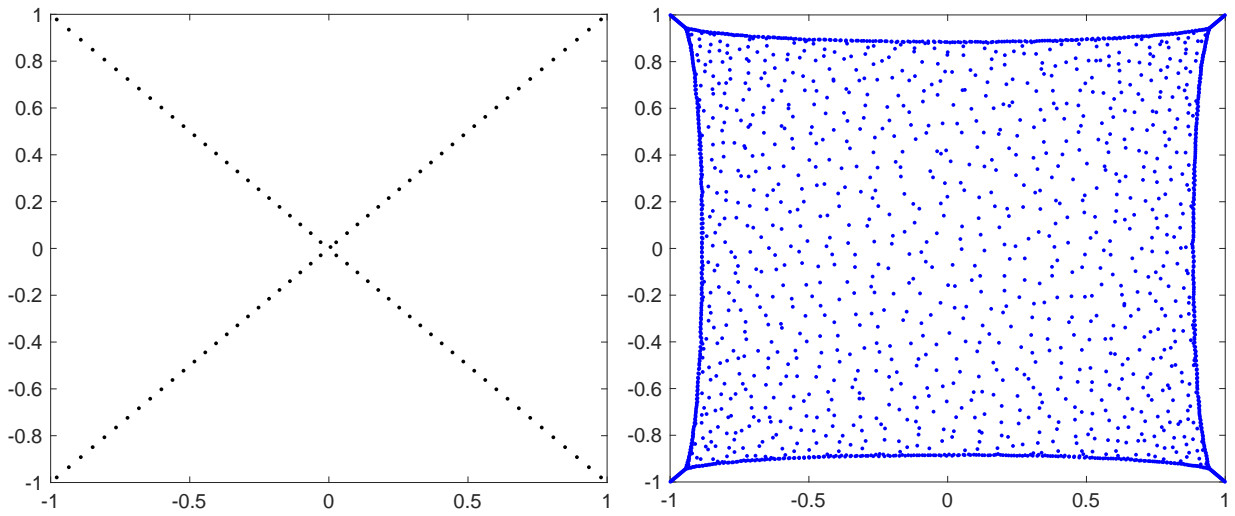


Figure 3.2: The left hand side shows the spectrum of the unperturbed Scottish flag operator p_N (3.3.3), and the right hand side shows the spectrum of a $p_N + \delta Q_N$, for $N = 1000$, Q_N a complex Gaussian random matrix and $\delta = 10^{-12}$.

In [50] Christiansen and Zworski established a Weyl law for the expected number of eigenvalues of small Gaussian random perturbations of p_N . They proved

Theorem 3.3.2 ([50]). Suppose that $f \in C^\infty(\mathbb{T}^{2d})$, and that Ω is a simply connected open set with a smooth boundary, $\partial\Omega$, such that for all z in a neighbourhood of $\partial\Omega$,

$$\text{vol}_{\mathbb{T}^{2d}}(\{w : |f(w) - z|^2 \leq t\}) = \mathcal{O}(t^\kappa), \quad 0 \leq t \ll 1, \quad (3.3.5)$$

with $1/2 < \kappa \leq 1$. Let Q_N be a complex Gaussian random $N^d \times N^d$ -matrix with independent and identically distributed entries $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$. Then for any $p \geq p_0 > d + 1/2$

$$\mathbb{E}(|\text{Spec}(f_N + N^{-p}Q_N) \cap \Omega|) = N^d \text{vol}_{\mathbb{T}^{2d}}(f^{-1}(\Omega)) + \mathcal{O}(N^{d-\beta}), \quad (3.3.6)$$

for any $\beta < (\kappa - 1)(\kappa + 1)$.

Let us remark that the original result of [50] is presented with $|f(w) - z|$ in (3.3.5) instead of $|f(w) - z|^2$, which then leads to $1 < \kappa \leq 2$. We modified the notation to be more easily comparable with the results that follow.

In Theorem 3.3.4 and 3.3.9 below we present a stronger result, estimating the probability that this asymptotic holds and providing more precise error estimates. Moreover, we remove the lower

bound on κ and simply demand it to be > 0 . Furthermore, we allow for a universal probability distribution in the perturbation, see Theorem 3.3.9. Finally, we remark that in our results we allow for coupling constants which may go up to the critical case of N^{-p} with $p > d/2$ and down to being sub-exponentially small in N .

In [50] the authors state the following

Conjecture 3.3.3 ([50]). *Suppose that (3.3.5) holds for all $z \in \mathbb{C}$ with a fixed $0 < \kappa \leq 1$. Define random probability measures*

$$\mu_N = N^{-d} \sum_{\lambda \in \text{Spec}(f_N + N^{-p}Q_N)} \delta_\lambda,$$

with $p \geq p_0 > d + 1/2$. Then, almost surely

$$\mu_N \rightarrow f_*(\sigma^n/n!), \quad N \rightarrow \infty,$$

where $\sigma = \sum_1^d d\xi_k \wedge dx_k$, $(x, \xi) \in \mathbb{T}^{2d}$, is the symplectic form in \mathbb{T}^{2d} .

We prove this conjecture, see Corollary 3.4.3 below, for general random matrix ensembles, and coupling constants $\delta = N^{-p}$, $p > d/2 + 1$. When $d/2 + 1 \geq p > d/2$ we show that the convergence still holds in probability.

3.3.1 Almost sure Weyl law

We are interested in the Toeplitz quantization of smooth functions on the $2d$ -dimensional torus $\mathbb{T}^{2d} = \mathbb{R}^{2d}/\mathbb{Z}^d$. This is related to the more general Berezin-Toeplitz quantization of compact symplectic Kähler manifolds, see [37] or for instance [78] for an introduction. A symbol $p \in C^\infty(\mathbb{T}^{2d})$ can be identified with a smooth periodic function on \mathbb{R}^{2d} . Hence p is in the symbol class $S(1)$, i.e. the class of smooth functions $a \in C^\infty(\mathbb{R}^{2d})$ such that for any $\beta \in \mathbb{N}^{2d}$ there exists a constant $C_\beta > 0$ such that

$$|\partial^\beta a(\rho)| \leq C_\beta. \quad (3.3.7)$$

We let $h \in]0, 1]$ denote the semiclassical parameter. A symbol $a \in S(1) = S(T^*\mathbb{R}^d, 1)$ may depend on h , in which case we demand that the constants in the estimates (3.3.7) are uniform with respect to h . The h -Weyl quantization $a^w(x, hD_x)$ of such a symbol a is defined as in (A.1.3). The operator $a^w(x, hD_x)$ is a continuous linear map $\mathcal{S} \rightarrow \mathcal{S}$, $\mathcal{S}' \rightarrow \mathcal{S}'$ and a bounded linear map $L^2 \rightarrow L^2$, see for instance in [63, 140, 204].

We denote by \mathcal{H}_h^d the space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ which are \mathbb{Z}^d -translation invariant in position and in frequency, more precisely

$$u(x+n) = u(x), \quad \mathcal{F}_h(u)(\xi+n) = \mathcal{F}_h(u)(\xi), \quad \forall n \in \mathbb{Z}^d.$$

Here \mathcal{F}_h denotes the semiclassical Fourier transform, defined by

$$(\mathcal{F}_h u)(\xi) := \frac{1}{(2\pi h)^{d/2}} \int e^{\frac{i}{h}x\xi} u(x) dx, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

which maps $\mathcal{S} \rightarrow \mathcal{S}$, $\mathcal{S}' \rightarrow \mathcal{S}'$ by duality, and $L^2 \rightarrow L^2$ unitarily. The space \mathcal{H}_h^d is $\neq \{0\}$ if and only if $h = 1/(2\pi N)$, for some $\mathbb{N} \ni N > 0$, in which case $\dim \mathcal{H}_h^d = N^d$, and we can identify $\mathcal{H}_h^d \simeq \ell^2(\mathbb{Z}^d/N\mathbb{Z}^d) \simeq \mathbb{C}^{N^d}$.

When $p \in C^\infty(\mathbb{T}^{2d})$, possibly h dependent in the above sense, then $p^w(x, hD_x)$ maps \mathcal{H}_h^d into itself, see [50], and the restriction

$$p_N \stackrel{\text{def}}{=} p^w(x, hD_x) \upharpoonright_{\mathcal{H}_h^d}: \mathcal{H}_h^d \longrightarrow \mathcal{H}_h^d, \quad h = \frac{1}{2\pi N},$$

defines a quantization

$$C^\infty(\mathbb{T}^{2d}) \ni p \mapsto p_N \in \mathcal{L}(\mathbb{C}^{N^d}, \mathbb{C}^{N^d}).$$

Using the Fourier transform, we can show that the matrix elements of p_N are given by

$$(p_N)_{m,j} = \sum_{n,r \in \mathbb{Z}^d} \widehat{p}(n, j - m - rN) e^{\frac{in}{N}(j+m) \cdot n} (-1)^{n \cdot r}, \quad m, j \in (\mathbb{Z}/N\mathbb{Z})^d.$$

where \widehat{p} is the Fourier transform of p .

Let $h = 1/(2\pi N)$, $\mathbb{N} \ni N > 0$, and suppose that for $p \in C^\infty(\mathbb{T}^{2d})$ there exist $p_\nu \in C^\infty(\mathbb{T}^{2d})$, $\nu \in \mathbb{N}$, so that

$$p(\rho; h) \sim p_0(\rho) + hp_1(\rho) + \dots \text{ in } S(1), \quad (3.3.8)$$

meaning that $p - \sum_0^M h^\nu p_\nu \in h^{M+1}S(1)$ for all $M \in \mathbb{N}$. We call p_0 the *principal symbol* of p .

We are interested in studying the eigenvalue distribution of $p_N + \delta Q_\omega$ for δ in a suitable range and for Q_ω in a suitable ensemble of $N^d \times N^d$ random matrices.

Let $\Omega \Subset \mathbb{C}$ be an open relatively compact simply connected set with a uniformly Lipschitz boundary $\partial\Omega$. For z in a neighbourhood of $\partial\Omega$ (denoted by $\text{neigh}(\partial\Omega)$) and $0 \leq t \ll 1$ we set

$$V_z(t) = \text{Vol}\{\rho \in \mathbb{T}^{2d}; |p_0(\rho) - z|^2 \leq t\}. \quad (3.3.9)$$

We make the following non-flatness assumption

$$\exists \kappa \in]0, 1], \text{ such that } V_z(t) = \mathcal{O}(t^\kappa), \text{ uniformly for } z \in \text{neigh}(\partial\Omega), 0 \leq t \ll 1. \quad (3.3.10)$$

See in and around (3.2.14) for concrete examples when this non-flatness assumption holds. The first result concerns the case of a perturbation by a complex Gaussian random matrix.

Theorem 3.3.4. *Let $p \in C^\infty(\mathbb{T}^{2d})$ satisfy (3.3.8) and let $N \geq 2$. Let $\Omega \Subset \mathbb{C}$ be an open relatively compact simply connected set with a uniformly Lipschitz boundary $\partial\Omega$, so that (3.3.10) holds. Let Q_ω be a complex Gaussian random $N^d \times N^d$ -matrix with independent and identically distributed entries, i.e.*

$$Q_\omega = (q_{i,j}(\omega))_{1 \leq i,j \leq N^d}, \quad q_{i,j}(\omega) \sim \mathcal{N}_{\mathbb{C}}(0, 1) \text{ (iid)}. \quad (3.3.11)$$

Let $N^{-1} \ll \alpha \ll 1$, let $C > 0$ be sufficiently large, and let

$$\varepsilon \gg \alpha^\kappa \log \frac{CN^{d/2}}{\delta\alpha^2} + \delta N^{\frac{d}{2}} \alpha^{-1/2}, \quad 0 < \delta \ll N^{-d/2} \alpha^{1/2}. \quad (3.3.12)$$

Then,

$$\left| \#(\text{Spec}(p_N + \delta Q_\omega) \cap \Omega) - N^d \int_{p_0^{-1}(\Omega)} d\rho \right| \leq \mathcal{O}(N^d) \left(\int_{p_0^{-1}(\partial\Omega + D(0,r))} d\rho + \frac{\varepsilon}{r} + r^\kappa \right)$$

for $0 < r \ll 1$, with probability

$$\geq 1 - \mathcal{O}(r^{-1}) \left(N^{d/2} \alpha^{-2} \delta^{-1} \exp\left(\alpha^{-\kappa} (2\delta N^{\frac{d}{2}} \alpha^{-1/2} - \varepsilon/C)\right) + e^{-N^d} \right).$$

This Weyl law shows that the eigenvalues of the small random perturbations of p_N roughly equidistribute in $\Sigma = p_0(\mathbb{T}^{2d})$ the numerical range of the principal symbol of the operator p_N . This is illustrated in Figure 3.2. In Corollaries 3.3.5, 3.3.6, 3.3.7, 3.3.8, below we provide some special cases of Theorem 3.3.4.

Before we turn to the case of more general perturbations, let us discuss some special cases of Theorem 3.3.4. Notice that when $\kappa > 1/2$ then (3.3.10) implies that

$$\int_{p_0^{-1}(\partial\Omega + D(0,r))} d\rho = \mathcal{O}(r^{2\kappa-1}). \quad (3.3.13)$$

One can easily see that $r = \varepsilon^{\frac{1}{2\kappa}}$ minimizes (up to a constant) the error term in Theorem 3.3.4, and it becomes

$$\mathcal{O}(N^d \varepsilon^{\frac{2\kappa-1}{2\kappa}}).$$

Taking $\alpha = CN^{-1}$ and $\varepsilon = C_0 N^{-\kappa} (\log N)^2$, for some sufficiently large constants $C, C_0 > 1$, one obtains from Theorem 3.3.4 the following

Corollary 3.3.5. *Under the assumptions of Theorem 3.3.4, we let $\kappa \in]1/2, 1]$ and set for $p \geq (d+1)/2 + \kappa$*

$$\delta = \frac{1}{C} N^{-p},$$

for some sufficiently large $C > 0$. Then,

$$\left| \#(\text{Spec}(p_N + \delta Q_\omega) \cap \Omega) - N^d \int_{p_0^{-1}(\Omega)} d\rho \right| \leq \mathcal{O}\left(N^{d-\kappa+1/2} (\log N)^{(2\kappa-1)/\kappa}\right)$$

with probability $\geq 1 - \mathcal{O}(N^{-\infty})$.

Notice that at the price of increasing the error term of the eigenvalue counting estimate by a factor N^β , with $\beta \in]0, 1[$, one can obtain the above result with probability $\geq 1 - e^{-N^{\beta'}/C}$, for some $\beta' \in]0, 1[$.

For $p_0 \in]0, \kappa]$, we set $\alpha = CN^{-p_0/\kappa}$ and $\varepsilon = C_0 N^{-p_0} (\log N)^2$, for some sufficiently large constants $C, C_0 > 1$. Then, one gets from Theorem 3.3.4 the following

Corollary 3.3.6. *Under the assumptions of Theorem 3.3.4, we let $\kappa \in]1/2, 1]$ and for $p_0 \in]0, \kappa]$ set*

$$\delta = \frac{1}{C} N^{-(d+1)/2-p_0},$$

for some sufficiently large $C > 0$. Then,

$$\left| \#(\text{Spec}(p_N + \delta Q_\omega) \cap \Omega) - N^d \int_{p_0^{-1}(\Omega)} d\rho \right| \leq \mathcal{O}\left(N^{d-p_0(2\kappa-1)/(2\kappa)} (\log N)^{(2\kappa-1)/\kappa}\right)$$

with probability $\geq 1 - \mathcal{O}(N^{-\infty})$.

Taking $\alpha = CN^{-1}$ and $\varepsilon = C_0 N^{\beta-\kappa}$, for some sufficiently large constants $C, C_0 > 1$, and $\beta \in]0, \kappa[$, one obtains from Theorem 3.3.4 the following

Corollary 3.3.7. *Under the assumptions of Theorem 3.3.4, we let $\kappa \in]1/2, 1]$ and $\beta \in]0, \kappa[$. Set*

$$\delta = e^{-N^\beta},$$

then,

$$\left| \#(\text{Spec}(p_N + \delta Q_\omega) \cap \Omega) - N^d \int_{p_0^{-1}(\Omega)} d\rho \right| \leq \mathcal{O}\left(N^{d+(\beta-\kappa)(2\kappa-1)/(2\kappa)}\right)$$

with probability

$$\geq 1 - \mathcal{O}\left(e^{-N^\beta/C}\right).$$

When $0 < \kappa \leq 1/2$, then condition (3.3.13) still holds, however it does not provide us anymore with decay. However, in many situations it is still reasonable to assume that

$$\int_{p_0^{-1}(\partial\Omega + D(0,r))} d\rho = o(1), \quad r \rightarrow 0,$$

In this situation, the best we can have is an error term of order

$$o(N^d).$$

Similarly to Corollary 3.3.5, 3.3.7, we get that

Corollary 3.3.8. *Under the assumptions of Theorem 3.3.4, we let $\kappa \in]0, 1]$ and for $p > d/2$ set*

$$\delta = \frac{1}{C} N^{-p},$$

for some sufficiently large $C > 0$. Then,

$$\left| \#(\text{Spec}(p_N + \delta Q_\omega) \cap \Omega) - N^d \int_{p_0^{-1}(\Omega)} d\rho \right| \leq o(N^d) \quad (3.3.14)$$

with probability $\geq 1 - \mathcal{O}(N^{-\infty})$. Moreover, when

$$\delta = e^{-N^\beta}, \quad \text{for some } \beta \in]0, \kappa[,$$

then (3.3.14) holds with probability $\geq 1 - \mathcal{O}(e^{-N^\beta/C})$.

The next result concerns the case of a perturbation by an iid matrix.

Theorem 3.3.9. *Let $p \in C^\infty(\mathbb{T}^{2d})$ satisfy (3.3.8). Let $\Omega \Subset \mathbb{C}$ be an open relatively compact simply connected set with a uniformly Lipschitz boundary $\partial\Omega$, so that (3.3.10) holds. Let Q_ω be a random $N^d \times N^d$ -matrix whose entries are independent copies of a random variable q satisfying*

$$\mathbb{E}[q] = 0, \quad \mathbb{E}[|q|^2] = 1, \quad \mathbb{E}[|q|^4] < +\infty.$$

For $\delta_0 > 0$ and some sufficiently large $C > 0$, let

$$\delta = \frac{1}{C} N^{-d/2 - \delta_0},$$

and for $\tau \in [0, 1]$, set

$$\varepsilon = N^{-\min(\delta_0, 1)\tau\kappa} \log N + N^{-\tau\delta_0/2},$$

Then,

$$\left| \#(\text{Spec}(p_N + \delta Q_\omega) \cap \Omega) - N^d \int_{p_0^{-1}(\Omega)} d\rho \right| \leq \mathcal{O}(N^d) \left(\int_{p_0^{-1}(\partial\Omega + D(0,r))} d\rho + \frac{\varepsilon}{r} + r^\kappa \right),$$

for $0 < r \ll 1$, with probability

$$\geq 1 - \mathcal{O}(r^{-1}) N^{-(1-\tau)\delta_0}.$$

Similarly to Corollaries 3.3.5, 3.3.6, 3.3.7, 3.3.8, one can use Theorem 3.3.9 to get precise error estimates in the various situations.

As a consequence of the proofs of Theorem 3.3.9, or of Theorem 3.3.4 in the Gaussian case, we obtain the following result providing a positive response to Conjecture 3.3.3 by [50].

Theorem 3.3.10. *Let Q_ω and $\delta > 0$ be as in Theorem 3.3.9 and assume that (3.3.10) holds uniformly for all $z \in \mathbb{C}$. Set*

$$\mu_N = N^{-d} \sum_{\lambda \in \text{Spec}(p_N + \delta Q_\omega)} \delta_\lambda.$$

Then, for $\delta_0 > 1$

$$\mu_N \rightharpoonup (p_0)_*(d\rho), \quad \text{almost surely,}$$

and for $\delta_0 \in]0, 1]$,

$$\mu_N \rightharpoonup (p_0)_*(d\rho), \quad \text{in probability.}$$

We remark that in the case of \mathbb{T}^{2d} the measure induced by the symplectic volume form $\sigma^n/n!$ given in Conjecture 3.3.3 is equal to the Lebesgue measure $d\rho$ on \mathbb{T}^{2d} .

3.3.2 Related results

The case of Toeplitz matrices given by symbols on \mathbb{T}^2 of the form $\sum_{n \in \mathbb{Z}} a_n e^{in\xi}$, $(x, \xi) \in \mathbb{T}^2$, was studied in a series of recent works by Davies and Hager [60], Guionnet, Wood and Zeitouni [93], Basak, Paquette and Zeitouni [13, 14], Sjöstrand and the author of this text [174, 175]. Such symbols amount to the case of symbols which are constant in the x variable. In these works the non-selfadjointness of the problem does however not come from the symbol itself but from boundary conditions destroying the periodicity of the symbol in x by allowing for a discontinuity. Nevertheless, these works show that by adding some small random noise the limit of the empirical eigenvalues measure μ_N of the perturbed operator converges in probability (or even almost surely in some cases) to $p_*(d\rho)$.

In [14] the authors treated in particular the special case of upper triangular banded twisted Toeplitz matrices given by symbols of the form

$$\tilde{p}(x, \xi) = \sum_{n=0}^{N_+} f_n(2\pi x) e^{-2\pi i n \xi}, \quad (x, \xi) \in \mathbb{T}^2$$

where f_n is only assumed to be a Hölder continuous function and can have a discontinuity. They showed through quite different methods from ours that the μ_N converges weakly in probability to the measure

$$\tilde{\mu} = \tilde{p}_*(d\rho).$$

Thus we recover this result of [14] (at least in the smooth periodic setting) with Theorem 3.3.10. This suggests that the results of Theorem 3.3.10 also hold in the case of general twisted Toeplitz matrices with band entries defined by C^1 functions which are defined on a compact interval with non-periodic boundary conditions.

Let us remark that Theorem 3.3.10 has been extended to general Berezin-Toeplitz quantizations on compact Kähler manifolds by Oltman [146] in the case of complex Gaussian noise.

3.3.3 Ideas of the proof

In the sequel we will identify $\mathcal{H}_h^d \simeq \mathbb{C}^{N^d}$. We see $P = p_N$ as a bounded linear operator $\ell^2 \rightarrow \ell^2$. For $z \in \mathbb{C}$ let

$$0 \leq t_1^2 \leq \dots \leq t_{N^d}^2 \tag{3.3.15}$$

denote the eigenvalues of $Q = (P - z)^*(P - z)$ with an associated orthonormal basis of eigenfunctions $e_1, \dots, e_{N^d} \in \mathcal{H}_h^d$.

Using methods from semiclassical analysis and functional calculus we can show that for $0 < N \in \mathbb{N}$, $N^{-1} \ll \alpha \ll 1$, $\kappa \in]0, 1]$ as in (3.3.10), and for $\psi \in C_c^\infty(\mathbb{R})$

$$\text{tr} \psi \left(\frac{Q}{\alpha} \right) = N^d \left(\int \psi \left(\frac{q_0}{\alpha} \right) dV_z(q_0) + \mathcal{O}(N\alpha)^{-1} \alpha^\kappa \right). \tag{3.3.16}$$

Moreover, for $\chi \in C_c^\infty([0, \infty[, [0, \infty[)$ with $\chi(0) > 0$,

$$\begin{aligned} \log \det \left(Q + \alpha \chi \left(\frac{Q}{\alpha} \right) \right) \\ = N^d \left(\int_{\mathbb{T}^{2d}} \log |p_0(\rho) - z|^2 d\rho + \mathcal{O} \left(\alpha^\kappa \log \frac{1}{\alpha} \right) \right). \end{aligned} \quad (3.3.17)$$

The trace formula (3.3.16) implies that the number $N(\alpha)$ of eigenvalues of $(p - z)_N^*(p - z)_N$ in the interval $[0, \alpha]$ is

$$N(\alpha) = \mathcal{O}(N^d \alpha^\kappa). \quad (3.3.18)$$

Grushin Problem for the unperturbed operator Since P is Fredholm of index 0, the spectra of Q and $Q' = (P - z)(P - z)^*$ are equal, and we can find an orthonormal basis f_1, \dots, f_{N^d} of \mathcal{H}_h^d comprised of eigenfunctions of Q' associated with the eigenvalues (3.3.15), such that

$$(P - z)^* f_i = t_i e_i, \quad (P - z) e_i = t_i f_i, \quad i = 1, \dots, N^d. \quad (3.3.19)$$

Let $M > 0$ be so that $0 \leq t_1^2 \leq \dots \leq t_M^2 \leq \alpha$, and let δ_i , $1 \leq i \leq M$, denote an orthonormal basis of \mathbb{C}^M . Then, we know from (3.3.18) that

$$M = \mathcal{O}(N^d \alpha^\kappa). \quad (3.3.20)$$

We put

$$R_+ : \mathcal{H}_h^d \longrightarrow \mathbb{C}^M : u \longmapsto \sum_1^M (u | e_i) \delta_i, \quad (3.3.21)$$

and

$$R_- : \mathbb{C}^M \longrightarrow \mathcal{H}_h^d : u_- \longmapsto \sum_1^M u_-(i) f_i, \quad (3.3.22)$$

where $u_-(i) = (u_- | \delta_i)$. One can easily show that the Grushin problem

$$\mathcal{P}(z) := \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_h^d \times \mathbb{C}^M \longrightarrow \mathcal{H}_h^d \times \mathbb{C}^M, \quad (3.3.23)$$

is bijective with inverse

$$\mathcal{P}^{-1}(z) = \mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} \quad (3.3.24)$$

where

$$\begin{aligned} E(z) &= \sum_{M+1}^{N^d} \frac{1}{t_i} e_i \circ f_i^*, & E_+(z) &= \sum_1^M e_i \circ \delta_i^*, \\ E_-(z) &= \sum_1^M \delta_i \circ f_i^*, & E_{-+}(z) &= - \sum_1^M t_j \delta_j \circ \delta_j^*. \end{aligned} \quad (3.3.25)$$

By construction

$$\|E(z)\| \leq \frac{1}{\sqrt{\alpha}}, \quad \|E_\pm\| = 1, \quad \|E_{-+}\| \leq \sqrt{\alpha}. \quad (3.3.26)$$

It follows from (3.3.23) that

$$|\det \mathcal{P}(z)|^2 = \prod_{M+1}^{N^d} t_i^2 = \alpha^{-M} \det \mathbf{1}_\alpha(Q), \quad \mathbf{1}_\alpha(x) = \max(x, \alpha). \quad (3.3.27)$$

Since $\log |\det \mathcal{P}(z)|^2 = \log \det \mathbf{1}_\alpha(Q) + M \log \frac{1}{\alpha}$, we approximate $\mathbf{1}_\alpha(x)$ by $x + \alpha \chi \left(\frac{x}{\alpha} \right)$, with $\chi \in C_c^\infty(\mathbb{R})$ a suitable cut-off function, apply (3.3.17) and get

$$\log |\det \mathcal{P}(z)|^2 = N^d \left(\int_{\mathbb{T}^{2d}} \log |p_0(\rho) - z|^2 d\rho + \mathcal{O} \left(\alpha^\kappa \log \frac{1}{\alpha} \right) \right). \quad (3.3.28)$$

Grushin Problem for the perturbed operator Let $Q_\omega : \mathcal{H}_h^d \rightarrow \mathcal{H}_h^d$ be a bounded linear operator and replace P by $P^\delta = P + \delta Q_\omega$, $0 \leq \delta \ll 1$ in (3.3.23). This yields a Grushin problem for the perturbed operator

$$\mathcal{P}^\delta(z) \stackrel{\text{def}}{=} \begin{pmatrix} P^\delta - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_h^d \times \mathbb{C}^M \longrightarrow \mathcal{H}_h^d \times \mathbb{C}^M. \quad (3.3.29)$$

Using a Neumann series argument, $\mathcal{P}^\delta(z)$ remains bijective, provided that $\delta \|Q_\omega\| \alpha^{-1/2} \leq \frac{1}{2}$, with inverse

$$\mathcal{E}^\delta(z) = \begin{pmatrix} E^\delta(z) & E_+^\delta(z) \\ E_-^\delta(z) & E_{-+}^\delta(z) \end{pmatrix}, \quad (3.3.30)$$

where $\|E^\delta\| \leq 2\alpha^{-1/2}$, $\|E_\pm^\delta\| \leq 2$, and $\|E_{-+}^\delta - E_{-+}\| \leq 2\|\delta Q_\omega\| \ll 1$. The Schur complement formula applied to \mathcal{P}^δ and \mathcal{E}^δ , gives

$$\log |\det(P^\delta - z)| = \log |\det \mathcal{P}^\delta(z)| + \log |\det E_{-+}^\delta(z)|. \quad (3.3.31)$$

Since $\frac{d}{d\delta} \log \det \mathcal{P}^\delta = \text{tr} \mathcal{E}^\delta \frac{d}{d\delta} \mathcal{P}^\delta$, we have

$$\left| \log |\det \mathcal{P}^\delta| - \log |\det \mathcal{P}| \right| = \left| \text{Re} \int_0^\delta \text{tr} (E^\tau Q_\omega) d\tau \right| \leq \mathcal{O}(\delta \alpha^{-1/2} \|Q\|_{\text{tr}}). \quad (3.3.32)$$

So combining this with (3.3.28) and (3.3.31) gives

$$\begin{aligned} \log |\det(P^\delta - z)| &= N^d \left(\int_{\mathbb{T}^{2d}} \log |p_0(\rho) - z| d\rho + \mathcal{O}(\alpha^\kappa \log \frac{1}{\alpha}) + \mathcal{O}(\delta N^{-d/2} \alpha^{-1/2} \|Q\|_{\text{HS}}) \right) \\ &\quad + \log |\det E_{-+}^\delta(z)|. \end{aligned} \quad (3.3.33)$$

From the above estimates we can extract the deterministic upper bound

$$\log |\det E_{-+}^\delta(z)| \leq \mathcal{O}(N^d \alpha^\kappa) |\log \alpha|. \quad (3.3.34)$$

Probabilistic upper bound Consider the random matrix

$$Q_\omega = (q_{i,j}(\omega))_{1 \leq i,j \leq N^d} \quad (3.3.35)$$

whose entries $q_{i,j}(\omega)$ are independent copies of a random variable q satisfying the moment conditions

$$\mathbb{E}[q] = 0, \quad \mathbb{E}[|q|^2] = 1, \quad \mathbb{E}[|q|^4] < +\infty. \quad (3.3.36)$$

Using a classical result due to Latala [131] and Markov's inequality shows that for $\delta_0 > 0$ and $\tau \in]0, 1[$

$$\mathbf{P} \left[\|Q_\omega\| \geq CN^{d/2+(1-\tau)\delta_0} \right] \leq N^{-(1-\tau)\delta_0}. \quad (3.3.37)$$

Taking $\delta = \frac{1}{C} N^{-d/2-\delta_0}$ and $\alpha = N^{-\min(\delta_0, 1)\tau}$, the relations (3.3.33) and (3.3.34) give that

$$\log |\det(P^\delta - z)| \leq N^d \left(\int_{\mathbb{T}^{2d}} \log |p_0(\rho) - z| d\rho + \mathcal{O}(N^{-\min(\delta_0, 1)\tau\kappa} \log N) + \mathcal{O}(N^{-\tau\delta_0/2}) \right). \quad (3.3.38)$$

with probability $\geq 1 - N^{-(1-\tau)\delta_0}$.

Probabilistic lower bound Using Theorem 2.5.2, we can prove the following

Proposition 3.3.11. *Let $K \Subset \mathbb{C}$ be an open connected relatively compact set and let $z \in K$. Then, there exist a positive constants $\beta, C > 0$ such that for all $\tau \in]0, 1[$ and for each $z \in K$*

$$P\left(\log |\det E_{-+}^\delta(z)| \geq -\beta N^{d-\kappa\tau \min(\delta_0, 1)} \log N \text{ and } \|Q_\omega\| \leq CN^{d/2+(1-\tau)\delta_0}\right) \geq 1 - N^{-(1-\tau)\delta_0}. \quad (3.3.39)$$

Applying the Ky Fan inequalities (2.4.2) to the right hand side of (2.6.1), one can show that

$$\frac{1}{8}t_n(E_{-+}^\delta) \leq t_n(P^\delta - z) \leq t_n(E_{-+}^\delta), \quad 1 \leq n \leq M.$$

So, with the same probability as in (3.3.39), we have that $t_1(P^\delta - z) > 0$ and so $(P^\delta - z)$ is bijective.

Using again (3.3.33), it follows from Proposition 3.3.11 that there exists a constant $C > 0$ such that for each $z \in K$

$$\log |\det(P^\delta - z)| \geq N^d \left(\int_{\mathbb{T}^{2d}} \log |p_0(\rho) - z| d\rho - CN^{-\min(\delta_0, 1)\tau\kappa} \log N - CN^{-\tau\delta_0/2} \right). \quad (3.3.40)$$

with probability $\geq 1 - N^{-(1-\tau)\delta_0}$.

Combining the probabilistic upper bound (3.3.38) on $\log |\det(p_N + \delta Q_\omega - z)|$ and the probabilistic lower bound (3.3.40), with Theorem 1.2 from [170] which provides estimates on the number of zeros of holomorphic functions with exponential growth in Lipschitz domains, gives Theorem 3.3.9. Theorem 3.3.4 can be proven similarly.

Direct proof of Theorem 3.3.10 We begin by recalling some basic facts concerning the weak convergence of measures. Let $\mathcal{P}(\mathbb{C})$ denote the space of probability measures μ on \mathbb{C} , integrating the logarithm at infinity

$$\int \log(1 + |x|) \mu(dx) < +\infty. \quad (3.3.41)$$

We define the *logarithmic potential* of μ by

$$U_\mu(z) \stackrel{\text{def}}{=} - \int \log |z - x| \mu(dx). \quad (3.3.42)$$

Since $U_\mu \in L^1_{\text{loc}}(\mathbb{C}, L(dz))$, it follows that $U_\mu(z) < +\infty$ for Lebesgue almost every (a.e.) $z \in \mathbb{C}$.

One property of the logarithmic potential is that for a given sequence of probability measures $\{\mu_n\}_n \in \mathcal{P}(\mathbb{C})$, satisfying some suitable uniform integrability assumption, a.e. convergence of the associated logarithmic potentials $U_{\mu_n}(z) \rightarrow U_\mu(z)$, for some $\mu \in \mathcal{P}(\mathbb{C})$, implies the weak convergence $\mu_n \rightarrow \mu$. We refer the reader to [35] for a survey of related results and techniques.

There are various versions of the above observation known in the case of random measures, see for instance [182, Theorem 2.8.3] or [35, 36]. In the following we describe a slightly modified version of [182, Theorem 2.8.3].

Theorem 3.3.12. *Let $K, K' \Subset \mathbb{C}$ be open relatively compact sets with $\overline{K} \subset K'$, and let $\{\mu_n\}_{n \in \mathbb{N}} \in \mathcal{P}(\mathbb{C})$ be a sequence of random measures so that for n large enough, almost surely (resp., with probability $\geq 1 - o(1)$)*

$$\text{supp } \mu_n \subset K. \quad (3.3.43)$$

Suppose that for a.e. $z \in K'$ almost surely (resp., in probability)

$$U_{\mu_n}(z) \rightarrow U_\mu(z), \quad n \rightarrow \infty, \quad (3.3.44)$$

where $\mu \in \mathcal{P}(\mathbb{C})$ is some probability measure with $\text{supp } \mu \subset K$. Then, almost surely (resp., in probability),

$$\mu_n \rightarrow \mu, \quad n \rightarrow \infty, \quad \text{weakly.} \quad (3.3.45)$$

Let μ_N be the empirical eigenvalue measure of P^δ , and let $\mu = (p_0)_*(d\rho)$, be as in Theorem 3.3.10. Since $p \in S(1)$, standard estimates from semiclassical analysis show that $\|p_N\| \leq \mathcal{O}(1)$, uniformly in N . So (3.3.37), in combination with a Borel-Cantelli argument when $\delta_0 > 1$, show that the condition on the supports of μ_N and μ is satisfied.

For $z \notin \text{Spec}(P^\delta)$ we have that

$$U_{\mu_N}(z) = -\frac{1}{N^d} \log |\det(P^\delta - z)|.$$

Furthermore, $U_\mu(z) = -\int_{\mathbb{T}^{2d}} \log |p_0(\rho) - z| d\rho$ for any $z \in \mathbb{C}$. The existence of this integral follows from the fact that (3.3.10) is assumed to hold uniformly for all $z \in \mathbb{C}$. Thus, fixing $z \in \mathbb{C}$, it follows from (3.3.40), (3.3.38) that

$$|U_{\mu_N}(z) - U_\mu(z)| = \mathcal{O}(N^{-\min(\delta_0, 1)\tau\kappa} \log N) + \mathcal{O}(N^{-\tau\delta_0/2}).$$

with probability $\geq 1 - CN^{-(1-\tau)\delta_0}$. Hence, for any $\varepsilon > 0$ we find that $\mathbf{P}[|U_{\mu_N}(z) - U_\mu(z)| \geq \varepsilon] \leq CN^{-(1-\tau)\delta_0}$ for $N > 1$ sufficiently large. Hence, $U_{\mu_N}(z) \rightarrow U_\mu(z)$ in probability when $N \rightarrow \infty$. Additionally, when $\delta_0 > 1$ we can choose $\tau \in]0, 1[$ such that the probabilities of the above events are summable and so, by the Borel-Cantelli lemma, we have that $U_{\mu_N}(z) \rightarrow U_\mu(z)$ almost surely, when $N \rightarrow \infty$. Theorem 3.3.12 then readily implies Theorem 3.3.10.

3.4 Probabilistic Weyl law for random tunneling potentials

In this section we present another extension of Theorem 3.2.4 to a different *type* of random perturbations: *random tunneling potentials*. In the proof of Theorem 3.3.4 and 3.3.9 it is crucial to show that the smallest singular value of the perturbed operator $P - z$, at some fixed $z \in \mathbb{C}$, is not *too small* with high probability. In this section, we will show this property in the context of an operator appearing in a physical model for twisted bilayer graphene. Furthermore, as a byproduct, we obtain a probabilistic Weyl law for such operators subject to a small perturbation by a random tunneling potential. The results presented here come from [19].

3.4.1 Twisted bilayer graphene

Twisted bilayer graphene is a stacked and twisted two-dimensional carbon material that exhibits a variety of strongly correlated electron phenomena such as superconducting phases [48]. In the one-particle band structure, the existence of strongly correlated phases is indicated by the occurrence of flat bands. The purpose of this section is to study the stability of flat bands under small random perturbations of the tunneling potentials described in Section 3.4.2. Such perturbations adequately reflect material impurities e.g. due to internal strain effects (lattice relaxations) [143].

The stability of flat bands under random perturbations depends sensitively on the nature of the disorder. If the disorder is signed, then classical Wegner-type estimates rule out the presence of flat bands under disorder, since the integrated density of states does not exhibit any jump discontinuities. This has been implemented for magnetic Schrödinger operators [55, 84–86] and for twisted bilayer graphene Becker, Oltman and the author in [18].

Here, we deal with a more realistic scenario, where we study random perturbations of the standard high-symmetry tunneling potentials in the Bistritzer-MacDonald Hamiltonian [25] for twisted bilayer graphene, see [44, 197]. Since the potential perturbations are not signed, it is not immediate that they can sufficiently perturb the spectrum to destroy the flat band. Theorem 3.4.2 below, shows that for random tunneling potential perturbations, there are no flat bands with overwhelming probability.

Let us start by introducing the Hamiltonian. Thus, let $h \in]0, 1]$ be proportional to the physical twisting angle, then the Bistritzer-MacDonald Hamiltonian is a semiclassical matrix-valued first

order differential operator of the form

$$H_{\text{BM}}(w, h) = \begin{pmatrix} wC & D_h \\ D_h^* & wC \end{pmatrix} \quad (3.4.1)$$

acting on $L^2(\mathbb{C}; \mathbb{C}^4)$ with domain $H^1(\mathbb{C}; \mathbb{C}^4)$.

Letting $D_x = \frac{1}{2}(D_{x_1} - iD_{x_2})$ and $D_{x_j} = -i\partial_{x_j}$, the matrix-valued entries in (3.4.1) are

$$D_h := \begin{pmatrix} 2hD_{\bar{x}} & U(x) \\ U(-x) & 2hD_{\bar{x}} \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & V(x) \\ \overline{V(x)} & 0 \end{pmatrix}, \quad (3.4.2)$$

where the tunnelling potentials U, V are smooth functions which are characterized, for $a_j = \frac{4}{3}\pi i\omega^j$ and $\omega = \exp(2\pi i/3)$, by

$$\begin{aligned} V(x + \mathbf{a}_j) &= \bar{\omega}V(x), & V(\omega x) &= V(x), & \overline{V(\bar{x})} &= V(-x), & V(\bar{x}) &= V(-x), \\ U(x + \mathbf{a}_j) &= \bar{\omega}U(x), & U(\omega x) &= \omega U(x), & \overline{U(\bar{x})} &= U(x). \end{aligned} \quad (3.4.3)$$

In this article, we focus on the chiral limit [186]. This limit is obtained by setting $w_0 \equiv 0$ in the Hamiltonian (3.4.1)

$$H_{\text{chiral}} = \begin{pmatrix} 0 & D_h \\ D_h^* & 0 \end{pmatrix}.$$

Since H_{chiral} is periodic with respect to the lattice

$$\Gamma := 4\pi(i\omega\mathbb{Z} \oplus i\omega^2\mathbb{Z}), \quad (3.4.4)$$

we can apply the Bloch-Floquet transform, see [17, Sec.2.3], to obtain an equivalent family of operators parametrized by $k \in \mathbb{C}$ on $L^2(\mathbb{C}/\Gamma; \mathbb{C}^4)$ with domain $H_h^1(\mathbb{C}/\Gamma; \mathbb{C}^4)$

$$H_{\text{chiral}}(k) = \begin{pmatrix} 0 & D_h + hk \\ D_h^* + h\bar{k} & 0 \end{pmatrix}. \quad (3.4.5)$$

We see \mathbb{C}/Γ as a smooth compact manifold equipped with a smooth positive density of integration dx . A natural choice would be the Riemannian volume density inherited from \mathbb{C} .

The operator (3.4.5) satisfies the chiral symmetry

$$\text{diag}(1, -1)H_{\text{chiral}}(k)\text{diag}(1, -1) = -H_{\text{chiral}}(k).$$

This implies that eigenvalues of $H_{\text{chiral}}(k)$ come in pairs $E_{-n}(k) = -E_n(k)$ with

$$\dots \leq E_{-2}(k) \leq E_{-1}(k) \leq 0 \leq E_1(k) \leq E_2(k) \leq \dots$$

When $E_1(k) \equiv 0$ we say that H_{chiral} exhibits a flat band at energy zero. Since D_h is a Fredholm operator of index 0, see e.g. [17, Proposition 2.3], one concludes that

$$E_1(k) \equiv 0 \iff \text{Spec}(D_h) = \mathbb{C}. \quad (3.4.6)$$

A twisting angle proportional to h at which (3.4.6) holds is referred to as a *magic angle* [17]. In the present article, we shall use (3.4.6) to study the equivalent magic angle condition $\text{Spec}(D_h) = \mathbb{C}$ under random perturbations by off-diagonal potentials of D_h . In Theorem 3.4.2, we show that the lowest singular value of a suitable random perturbation $D_h - z$ being at least an exponentially small distance away from zero is overwhelmingly high. Thus, z is not in the spectrum of the perturbation of D_h with great probability and thus there cannot be any flat band, or equivalently, a magic angle.

One might object that an exponentially flat band would in practise still look fairly flat. But in the case of the chiral model, it is known [17] that there are $\geq \mathcal{O}(1/h)$ many bands that are $\mathcal{O}(e^{-c/h})$ close to zero energy. Thus, the bound in Theorem 3.4.2 implies that the designated flat band may mix with the other exponentially small bands which do not carry a non-zero Chern number.

3.4.2 Random tunneling perturbation

Let \tilde{P}_1, \tilde{P}_2 be h -independent elliptic positive second order differential operators on \mathbb{C}/Γ with smooth coefficients. Put $P_j = h^2 \tilde{P}_j$, $j = 1, 2$, and let $\{\psi_n^1\}_{n \in \mathbb{N}}, \{\psi_n^2\}_{n \in \mathbb{N}} \subset L^2(\mathbb{C}/\Gamma; \mathbb{C})$ be two orthonormal bases composed of eigenfunctions of P_1 and P_2 , respectively, so that

$$P_j \psi_n^j = \mu_{n,j}^2 \psi_n^j, \quad \mu_{n,j} \geq 0. \quad (3.4.7)$$

For $L \gg 1$ and $D_j = D_j(L) > 0$ we consider the potentials

$$q_\alpha^j(x) = \sum_{0 \leq \mu_n^j \leq L} \alpha_n \psi_n^j(x), \quad \alpha \in \mathbb{C}^{D_j}. \quad (3.4.8)$$

The Weyl law, see e.g. [90, Theorem 12.7] for the eigenvalues of elliptic self-adjoint second order differential operators yields that

$$D_j \asymp L^2 h^{-2}. \quad (3.4.9)$$

We then know from standard Sobolev estimates that for $j = 1, 2$ and $s > 1$

$$\|q_\alpha^j\|_{H_h^s(\mathbb{C}/\Gamma)} \leq \mathcal{O}_s(1) L^s \|\alpha\|_{\mathbb{C}^{D_j}}, \quad \|q_\alpha^j\|_{L^\infty(\mathbb{C}/\Gamma)} \leq \mathcal{O}_s(1) h^{-1} L^s \|\alpha\|_{\mathbb{C}^{D_j}}. \quad (3.4.10)$$

Here, $H_h^s(\mathbb{C}/\Gamma)$ is a semiclassical Sobolev space. Constraints on L and $\|\alpha\|_{\mathbb{C}^{D_j}}$ will be specified later on. The potentials (3.4.8) give rise to a tunneling potential

$$Q_\gamma = \begin{pmatrix} 0 & q_\alpha^1(x) \\ -q_\beta^2(x) & 0 \end{pmatrix} : L^2(\mathbb{C}/\Gamma; \mathbb{C}^2) \rightarrow L^2(\mathbb{C}/\Gamma; \mathbb{C}^2), \quad (3.4.11)$$

with $\gamma = (\alpha, \beta) \in \mathbb{C}^{D_1} \times \mathbb{C}^{D_2} \simeq \mathbb{C}^D$, $D = D_1 + D_2$. By (3.4.10)

$$\|Q_\gamma\|_{L^2(\mathbb{C}/\Gamma; \mathbb{C}^2) \rightarrow L^2(\mathbb{C}/\Gamma; \mathbb{C}^2)} \leq \|Q_\gamma\|_{L^\infty(\mathbb{C}/\Gamma; \mathbb{C}^{2 \times 2})} = \mathcal{O}_s(1) h^{-1} L^s \|\gamma\|_{\mathbb{C}^D}.$$

Let γ be real or complex random vector in \mathbb{R}^D or \mathbb{C}^D , respectively, with joint probability law

$$\gamma_*(d\mathbf{P}) = Z_h^{-1} \mathbf{1}_{B(0,R)}(\gamma) e^{\phi(\gamma)} L(d\gamma), \quad (3.4.12)$$

where $Z_h > 0$ is a normalization constant, $B(0, R)$ is either the real ball $\Subset \mathbb{R}^D$ or the complex ball $\Subset \mathbb{C}^D$ of radius $R \gg 1$, and centered at 0, $L(d\gamma)$ denotes the Lebesgue measure on either \mathbb{R}^D or \mathbb{C}^D and $\phi \in C^1$ with

$$\|\nabla_\gamma \phi\| = \mathcal{O}(h^{-\kappa_4}) \quad (3.4.13)$$

uniformly, for an arbitrary but fixed $\kappa_4 \geq 0$.

Fix $s > 1$ and $\varepsilon \in]0, s - 1[$, and for $C > 0$ large enough, we fix

$$L = Ch^{\frac{5}{s-1-\varepsilon}}, \quad Ch^{-2-\frac{5s}{s-1-\varepsilon}} \leq R \leq Ch^{-\kappa_3}, \quad \kappa_3 \geq 2 + \frac{5(1+\varepsilon)}{s-1-\varepsilon}, \quad (3.4.14)$$

and

$$\kappa_1 = 1 + \frac{5s}{s-1-\varepsilon} + \kappa_3, \quad \kappa_5 = \kappa_3 + \kappa_4 + 2 + \frac{10}{s-1-\varepsilon}. \quad (3.4.15)$$

Remark 3.4.1. One example for a random vector γ with law (3.4.12) is a truncated complex or real Gaussian random variables with expectation 0, and uniformly bounded covariances. The covariance matrix $\Sigma \in \mathbb{C}^{D \times D}$ then satisfies $\Sigma > 0$ and $\|\Sigma\| = \mathcal{O}(\sqrt{D})$. In this case $\phi(\gamma) = -\langle \Sigma^{-1} \gamma | \gamma \rangle$ and (3.4.13) holds with $\kappa_4 = 1 + \frac{5}{s-1-\varepsilon} + \kappa_3$.

3.4.3 Absence of small magic angels with overwhelming probability

Let $h \in]0, 1]$ and consider the unbounded semiclassical differential operator

$$D_h := \begin{pmatrix} 2hD_{\bar{x}} & U(x) \\ U(-x) & 2hD_{\bar{x}} \end{pmatrix} : L^2(\mathbb{C}/\Gamma; \mathbb{C}^2) \rightarrow L^2(\mathbb{C}/\Gamma; \mathbb{C}^2), \quad (3.4.16)$$

with $U \in C^\infty(\mathbb{C}/\Gamma; \mathbb{C})$ as in (3.4.3) below and equipped with the domain $H_h^1(\mathbb{C}/\Gamma; \mathbb{C}^2)$, making it a closed densely defined unbounded operator. The semiclassical principal symbol of D_h is

$$d(x, \xi) = \begin{pmatrix} \xi_1 + i\xi_2 & U(x) \\ U(-x) & \xi_1 + i\xi_2 \end{pmatrix} \in S^1(T^*(\mathbb{C}/\Gamma); \text{Hom}(\mathbb{C}^2, \mathbb{C}^2)), \quad (3.4.17)$$

see [19, Appendix A] for a review of matrix valued semiclassical pseudo-differential calculus. Using the elliptic of d for $|\xi| \gg 1$, one can easily check that D_h is Fredholm operator of index 0. See for instance [17, Proposition 2.3].

Let $\tau_0 \in]0, \sqrt{h}]$, let $C_0 > 0$ be large enough, and consider the perturbed operator

$$D_h^\delta := D_h + \delta h^{\kappa_1} Q_\gamma, \quad \delta = \frac{\tau_0 h^{\kappa_1 + 2}}{C_0}, \quad (3.4.18)$$

which, equipped with the domain $H_h^1(\mathbb{C}/\Gamma; \mathbb{C}^2)$, is a closed densely defined operator $L^2(\mathbb{C}/\Gamma; \mathbb{C}^2) \rightarrow L^2(\mathbb{C}/\Gamma; \mathbb{C}^2)$.

Fix $z \in \mathbb{C}$. Then

$$S := (D_h - z)^*(D_h - z)$$

is selfadjoint on $H_h^2(\mathbb{C}/\Gamma; \mathbb{C}^2)$. Since S is a positive selfadjoint operator with domain that injects compactly into $L^2(\mathbb{C}/\Gamma; \mathbb{C}^2)$, it follows that S has compact resolvent, and therefore its spectrum contains only isolated eigenvalues of finite multiplicity. Let $N := N(\tau_0^2)$ be the number of eigenvalues t_j^2 of S in $[0, \tau_0^2]$, i.e.

$$0 \leq t_1^2 \leq \dots \leq t_N^2 \leq \tau_0^2 < t_{N+1}^2 \leq \dots,$$

with $t_j(D_h - z) = t_j \geq 0$. In [19, (4.13)] we prove an upper bound on N , using semiclassical calculus similar to (3.3.16), which in combination with the corresponding lower bound on N proven in [17, Theorem 5] gives

$$\frac{1}{C} h^{-1} \leq N(\tau_0^2) \leq C h^{-1}, \quad (3.4.19)$$

for some $C > 0$, provided that $\tau_0 \geq h^M$, for some arbitrary but fixed $M > 0$. When the potential U in (5.1.2) is analytic, we have the same conclusion when $\tau_0 \geq e^{-1/Ch}$, for some $C > 0$ sufficiently large.

We have the following probabilistic estimate on the smallest singular value of D_h^δ in (3.4.19).

Theorem 3.4.2. *Let D_h^δ be as in (3.4.19), let $0 < \varepsilon < \exp(-C_1 h^{-2} \varepsilon_0(h))$, $C_1 > 1$ sufficiently large, let $N = N(\tau_0^2)$ be as in (3.4.18), and let*

$$\varepsilon_0(h) := C(\log \tau_0^{-1} + (\log h^{-1})^2)(h + h^2 \log h^{-1}).$$

with $C > 0$ large enough. Then, for $h > 0$ small enough

$$\mathbf{P} \left(t_1(D_h^\delta - z) \geq \frac{\varepsilon}{8(C\tau_0)^{N-1}} \right) \geq 1 - C h^{-\kappa_5} \varepsilon_0(h) \exp \left(-\frac{h^2}{C\varepsilon_0(h)} \log \frac{1}{\varepsilon} \right).$$

Since D_h^δ (3.4.18) is Fredholm of index 0, we immediately deduce from Theorem 3.4.2 the following

Corollary 3.4.3. *Under the assumptions of Theorem 3.4.2, and $h > 0$ small enough, the spectrum of D_h^δ is discrete with probability*

$$\geq 1 - Ch^{-\kappa_5} \varepsilon_0(h) \exp\left(-\frac{h^2}{C\varepsilon_0(h)} \log \frac{1}{\varepsilon}\right).$$

Consider now the perturbed chiral operator

$$H_{\text{chiral}}^\delta = \begin{pmatrix} 0 & D_h^\delta \\ (D_h^\delta)^* & 0 \end{pmatrix}. \quad (3.4.20)$$

In view of (3.4.6) and Corollary 3.4.3 we get the the following

Theorem 3.4.4. *Under the assumptions of Theorem 3.4.2, and $h > 0$ small enough, H_{chiral}^δ does not exhibit a flat band at energy 0, or equivalently $h > 0$ small enough is not a magic angle, with probability*

$$\geq 1 - Ch^{-\kappa_5} \varepsilon_0(h) \exp\left(-\frac{h^2}{C\varepsilon_0(h)} \log \frac{1}{\varepsilon}\right).$$

We may take for instance $\varepsilon = \exp(-1/(Ch) - C_1 h^{-2} \varepsilon_0(h))$, with $C > 0$ large enough, so that $h > 0$ small enough is not a magic angle with probability $\geq 1 - C \exp(-1/(Ch))$.

Theorem 3.4.4 can be interpreted as an instability or absence of magic angles due to small noise since for each $h > 0$ small enough, with overwhelming probability the perturbed chiral model does not exhibit at flat band at energy 0.

3.4.4 Eigenvalue asymptotics – a probabilistic Weyl law

As a byproduct of Theorem 3.4.2 we obtain the following probabilistic Weyl law.

Theorem 3.4.5. *Let $\Omega \Subset \mathbb{C}$ be relatively compact with Lipschitz boundary. Let $\lambda_j(x, \xi)$, $(x, \xi) \in T^*(\mathbb{C}/\Gamma)$, $j = 1, 2$ denote the two eigenvalues of $d(x, \xi)$. Then, under the assumptions of Theorem 3.4.2, and for $h > 0$ small enough,*

$$\#(\sigma(D_h^\delta) \cap \Omega) = \frac{1}{(2\pi h)^2} \sum_1^2 \iint_{(\lambda_j)^{-1}(\Omega)} dx d\xi + \mathcal{O}(h^{-1}) \quad (3.4.21)$$

with probability

$$\geq 1 - Ch^{-\kappa_5} \varepsilon_0(h) \exp\left(-\frac{h^2}{C\varepsilon_0(h)} \log \frac{1}{\varepsilon}\right).$$

A direct calculation shows that

$$\sum_1^2 \iint_{(\lambda_j)^{-1}(\Omega)} dx d\xi = 2|\Omega| \cdot |\mathbb{C}/\Gamma|.$$

Furthermore, since the error term is independent of $\varepsilon > 0$, we may take for instance $\varepsilon = \exp(-1/(Ch) - C_1 h^{-2} \varepsilon_0(h))$, with $C > 0$ large enough, so that for $h > 0$ small enough (3.4.21) holds with probability $\geq 1 - C \exp(-1/(Ch))$.

The arguments presented in [19, Section 6] show that (3.4.21) holds for $\delta = 0$ for $h \in]0, 1[\setminus \mathfrak{h}$ where $\mathfrak{h} \subset]0, 1[$ is a countable set. For the exceptional values $h \in \mathfrak{h}$ we have that $\text{Spec}(D_h) = \mathbb{C}$. These exceptional h are proportional to the *magic angles* in twisted bilayer graphene. See Section 3.4.1 for a discussion. The probabilistic Weyl law Theorem (3.4.5) extends the Weyl law to *all* $h \in]0, 1[$ for $h > 0$ small enough. This is numerically illustrated in Figure 3.3.

In [95, 168, 169] Hager and Sjöstrand proved a result similar to Theorem 3.4.2 for semiclassical elliptic differential operators $P(x, hD_x)$ with principal symbol $p(x, \xi)$ subject to small perturbations of the form (3.4.8). However, in their work one requires fundamentally that the principal

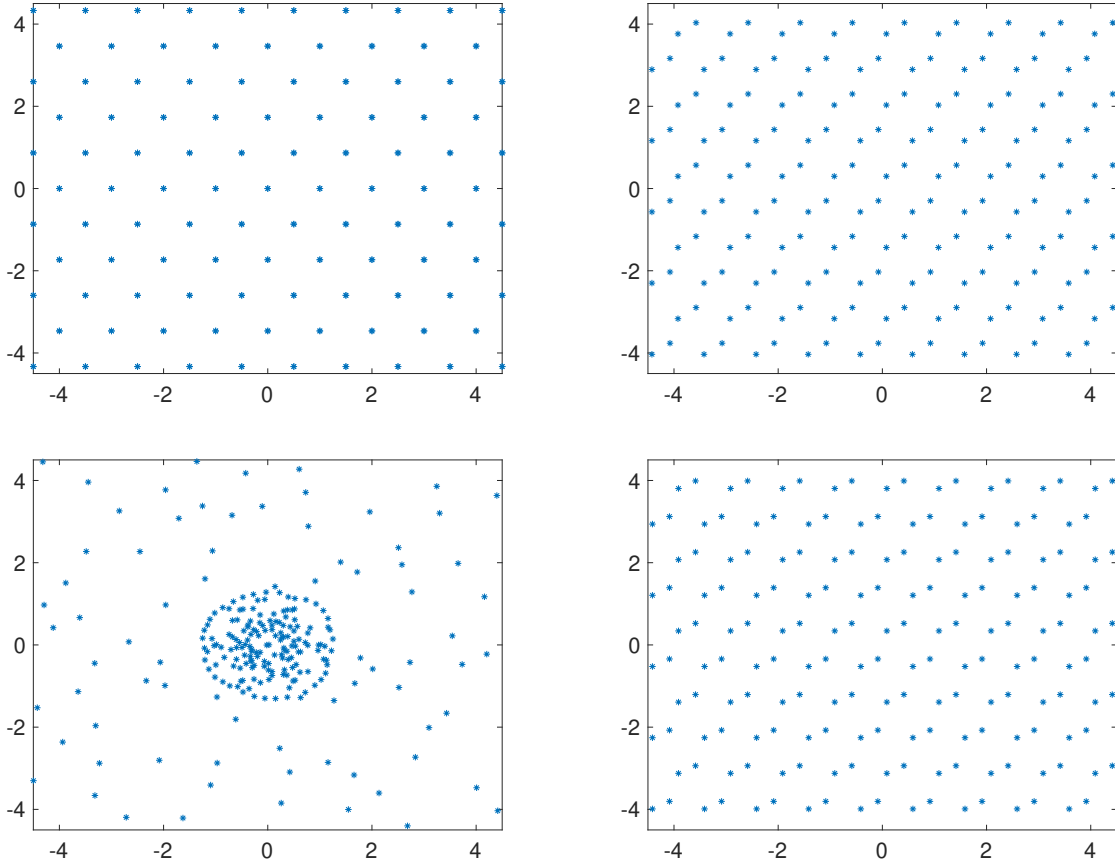


Figure 3.3: Top left: Away from magic angles h , $\text{Spec}(D_h) = h\Gamma^*$ with two-fold multiplicity. Top right: Spectrum of randomly perturbed operator D_h away from magic angles h . Bottom left: Spectrum of D_h numerically computed at largest magic angle h . In reality the spectrum is the entire complex plane. Bottom right: Spectrum of random perturbation of D_h for h the largest magic angle.

symbol satisfies the symmetry $p(x, \xi) = p(x, -\xi)$. In the present work we can circumvent this assumption due to the 2×2 matrix structure of D_h which allows for the use of random tunneling potentials of the form (3.4.11).

Another stark difference to these previous works is that the operator D_h is somewhat pathological from a spectral theory point of view. Indeed, for the discrete values of $h \in \mathfrak{h}$ the spectrum of operator D_h can be the entire complex plane \mathbb{C} (precisely when h is proportional to a magic angle). This possibility has been excluded in previous works by [95, 168, 169]. However, the example of D_h shows that operators exhibiting such behavior are physically relevant. See Section 3.4.1 for a discussion and application.

3.4.5 Ideas of the proof

The strategy of proof is similar to that described in Section 3.3.3 with, however, two key differences. The first difference is the probabilistic estimate on the smallest singular value described in Theorem 3.4.2. To prove this we use a modified version of a method developed Sjöstrand [168], adapted to suit the operator D_h , and construct a tunneling potential Q such that the smallest singular value of $D_h + \delta Q$ is not too small. Then we show that a good lower bound holds in fact with high probability. This shows, in particular, that the spectrum of D_h^δ is purely discrete for $h > 0$ small enough with high probability.

To prove Theorem 3.4.5, we first show that the spectrum D_h^δ is translation invariant under the action of the dual lattice $h\Gamma^*$. Then, using a deformation method we show that number of

eigenvalues of D_h^δ in a primitive cell of $h\Gamma^*$ is constant under small suitable deformations.

Chapter 4

Spectral statistics

In this section we discuss the works obtained in [145, 191].

4.1 Introduction

The probabilistic Weyl's law, Theorem 3.2.4, counts the eigenvalues in any regions, independent of h , of discrete spectrum. In other words, it describes the spectrum at the *macroscopic scale*. We focus in this section on pseudo-differential operators P_h , as in the beginning of Section 2.3, on \mathbb{R} . By Theorem 3.2.4, the density of eigenvalues of a randomly perturbed operators P_h^δ near a given "complex energy" z_0 in the region of discrete spectrum inside the classical spectrum, is approximately given by $(2\pi h)^{-1} D(z_0)$, where $D(z_0) > 0$ is the *classical density* at the energy z_0 , associated with the symbol of our initial operator. Since the spectral density is of order h^{-1} , it is reasonable to think that the typical distance between nearest eigenvalues should be of order $h^{1/2}$, which we will call the *microscopic scale*. Our aim is to investigate the distribution of eigenvalues at this microscopic scale, from a statistical point of view; in other words, we aim at studying the *local spectral statistics*, for our family of randomly perturbed operators, in particular the type of statistical correlations between nearby eigenvalues. A first result on these correlations has been obtained by the author in [191], where we computed the 2-point correlation between the eigenvalues of our randomly perturbed operator, in case of Gaussian perturbations.

We will give a full description of these local statistics, expressed in terms of a certain *Gaussian analytic functions*. In particular, we will show a partial form of *universality* with respect to the law of the random perturbation.

Before stating our results more precisely, and to provide some motivation, let us recall some background on the topic of spectral statistics, from a mathematical physics perspective. In the 1950s Wigner had the idea, when studying the spectra of complicated Hamiltonian operators in nuclear physics, to replace these (very structured) operators by large (non-structured) random matrices [199]. Those random matrices could not reproduce the large scale density fluctuations of the nuclear spectra, which depend on specific features of the system, but they could (empirically) reproduce the local statistical properties of the spectra, at the scale of the mean spacing between eigenvalues. Wigner and Dyson understood that these *local* statistical properties only depend on certain global symmetries of the Hamiltonian, like time reversal invariance, but not on the fine details of the Hamiltonian: these statistical properties were thus said to be *universal* [68]. In the 1980s, this universality conjecture was extended to simpler Hamiltonians, namely Laplacians on Euclidean domains with specific shapes: Bohigas-Giannoni-Schmidt observed that if the billiard flow in the domain is "chaotic", then the local spectral statistics of the corresponding Laplacian correspond to Dyson's Gaussian Orthogonal ensemble of random matrices [29]. In parallel, a large variety of non-Gaussian random Hermitian matrix ensembles were developed and studied, notably the Wigner random matrices (all entries are i.i.d., up to Hermitian symmetry), for which the local spectral statistics was recently shown to be identical with that of the Gaussian ensembles [72],

another manifestation of universality.

How about non-selfadjoint operators? Various random ensembles of nonhermitian matrices have also been introduced in the theoretical physics literature. The main objective has been to understand the distribution of quantum resonances for various types of scattering or dissipative systems, see for instance [81, 88, 130, 205] (a short recent review can be found in [80]). For most of these models, the focus has been to derive the mean spectral density, without investigating the correlations between the eigenvalues. The “historical” nonhermitian random matrix model, for which the full eigenvalue statistics has been derived in closed form, is the complex Ginibre ensemble [87], where all entries are i.i.d. complex Gaussian; the nearby eigenvalues then exhibit a statistical repulsion between themselves, similar to the case of Dyson’s Gaussian Unitary Ensemble of hermitian matrices. For certain non-Gaussian ensembles, recent results [42, 184] have been obtained on the eigenvalue distribution at the microscopic scale, including some partial universality results.

Let us also mention a model studied recently by Capitaine and Bordenave [34] (see also [60]), namely the case of a large Jordan block perturbed by a Ginibre random matrix: the authors prove that most eigenvalues of the perturbed matrix lie close to the unit circle, but they also show that the “outliers” (the relatively few eigenvalues away from the unit circle) are statistically distributed like the zeros of a “hyperbolic” Gaussian analytic function (GAF). A similar result was proved by Sjöstrand and the author [173] in the case of a non-selfadjoint bi-diagonal matrix, perturbed by a small Ginibre matrix. In these two models, GAFs appear because the perturbation is chosen to be Gaussian.

Our results will also involve Gaussian analytic functions, but of “Euclidean” type. In our case, these GAFs will describe the bulk of the spectrum, as opposed to a few outliers; also, in our case Gaussian functions appear in the limit, even though the perturbation operator or potential is not necessarily Gaussian distributed.

4.2 The non-selfadjoint Harmonic oscillator revisited

Before stating our results in full generality, we will illustrate them by first focussing on a simple case: the non-selfadjoint Harmonic oscillator $P_h = -h^2 \partial_x^2 + ix^2$, see Example 2.3.2.

We begin by constructing two types of *random* perturbations, following [96]. Let $\{e_k\}_{k \in \mathbb{N}}$ denote an orthonormal basis of $L^2(\mathbb{R})$ comprised of the eigenfunctions of the *nonsemiclassical* harmonic oscillator $H = -\partial_x^2 + x^2$, and let $\{q_{jk}\}_{j,k \in \mathbb{N}}$, $\{v_j\}_{j \in \mathbb{N}}$ be independent and identically distributed (i.i.d.) complex Gaussian random variables with expectation 0 and variance 1 (that is, with distribution $\mathcal{N}_{\mathbb{C}}(0, 1)$). Call $N(h) = C_1/h^2$, with $C_1 > 0$ large enough. Using these data, we define two types of random operators Q :

1. A random, Ginibre-type matrix

$$Q = M_\omega = \frac{1}{N(h)} \sum_{0 \leq j, k < N(h)} q_{j,k} e_j \otimes e_k^*$$

2. A random (complex valued) potential

$$Q = V_\omega = \frac{1}{N(h)} \sum_{0 \leq j < N(h)} v_j e_j$$

(more precisely, Q is the operator of multiplication by the potential V_ω).

The coupling parameter $\delta = \delta(h)$ will be assumed to be in the range

$$h^M \leq \delta \leq h^\kappa, \tag{4.2.1}$$

where $\kappa > 3$, and $M > \kappa$ is an arbitrarily large but fixed constant. Although the random operator Q and δ depend on h , we will omit this dependence in our notations. We are interested in the spectrum of the perturbed operator

$$P_h^\delta = -h^2 \partial_x^2 + ix^2 + \delta Q = P_h + \delta Q, \quad (4.2.2)$$

where the random operator Q is either M_ω or V_ω . Note that, since the operator Q is bounded on L^2 , the spectrum of P_h^δ remains purely discrete. More quantitatively, with probability exponentially close to 1 as $h \rightarrow 0$, we have the bounds $\|M_\omega\|_{\text{HS}} \leq Ch^{-1}$, $\|V_\omega\|_\infty \leq Ch^{-1}$ [94, 96].

Our objective will be to study the spectrum of P_h^δ in a microscopic neighbourhood of some given point $z_0 \in \Omega$. As explained in the previous section, the probabilistic Weyl's law [94, 96] shows that the typical density of eigenvalues near z_0 is of order h^{-1} , so we expect nearby eigenvalues to be at distances $\sim h^{1/2}$ from one another. In order to test the statistical correlations between nearby eigenvalues, we zoom to the scale $h^{1/2}$ at the point z_0 , by defining the *rescaled spectral point process*:

$$\mathcal{Z}_{h,z_0}^Q \stackrel{\text{def}}{=} \sum_{z \in \text{Spec}(P_h + \delta Q)} \delta_{(z-z_0)h^{-1/2}}.$$

Our main result is that, in the semiclassical limit, this rescaled point process converges in distribution to the point process formed by the zeros of a certain *random analytic function*. The building block of this random functions is the (Euclidean) Gaussian analytic function (GAF), which we now review.

4.2.1 The euclidean Gaussian analytic function

Let $(\alpha_n)_{n \in \mathbb{N}}$ be i.i.d. normal complex Gaussian random variables, i.e. $\alpha_n \sim \mathcal{N}_{\mathbb{C}}(0, 1)$. For a given $\sigma > 0$, we consider the random entire series

$$g_\sigma(w) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \alpha_n \frac{\sigma^{n/2} w^n}{\sqrt{n!}}, \quad w \in \mathbb{C}. \quad (4.2.3)$$

With probability one, this series converges absolutely on the full plane, and defines a Gaussian analytic function (GAF) on \mathbb{C} : g_σ is a random entire function, so that for any $n \in \mathbb{N}$ and any $w_1, \dots, w_n \in \mathbb{C}$ the random vector $(g_\sigma(w_1), \dots, g_\sigma(w_n))$ is a centred complex Gaussian

$$(g_\sigma(w_1), \dots, g_\sigma(w_n)) \sim \mathcal{N}_{\mathbb{C}}(0, \Gamma), \quad (4.2.4)$$

where the *covariance matrix* $\Gamma \in GL_n(\mathbb{C})$ has the entries

$$\Gamma_{i,j} = \mathbb{E} \left[g_\sigma(w_i) \overline{g_\sigma(w_j)} \right] \stackrel{\text{def}}{=} K_\sigma(w_i, \bar{w}_j) = \exp(\sigma w_i \bar{w}_j). \quad (4.2.5)$$

The function $\mathbb{C}^2 \ni (u, v) \mapsto K_\sigma(u, \bar{v})$ is called the *covariance kernel* of the GAF g_σ , it completely determines its distribution. As a result, K_σ also completely determines the distribution of

$$\mathcal{Z}_{g_\sigma} \stackrel{\text{def}}{=} \sum_{w \in g_\sigma^{-1}(0)} \delta_w,$$

the random point process defined by the zeros of the GAF g_σ , see for instance [120].

The GAF zero process \mathcal{Z}_{g_σ} has interesting geometric properties. Its covariance kernel shows that for any $w_0 \in \mathbb{C}$, the translated function $g_\sigma(w + w_0)$ is equal in distribution to the function $e^{\sigma(w\bar{w}_0 + |w_0|^2)} g_\sigma(w)$, which has the same zeros as $g_\sigma(w)$: hence the zero process \mathcal{Z}_{g_σ} is translation invariant on \mathbb{C} . The average density (1-point function) of \mathcal{Z}_{g_σ} is thus constant over the plane, it is equal to σ/π (see section 4.7). The linear dependence in σ is coherent with the scaling covariance $g_\sigma(w) \stackrel{d}{=} g_1(\sqrt{\sigma}w)$: dilating the zero process \mathcal{Z}_{g_1} by $1/\sqrt{\sigma}$ multiplies the average density by σ .

Let us give a short historical background of the GAF. It has appeared in the context of holomorphic representations of quantum mechanics, when investigating the properties of *random states*. In the framework of Toeplitz quantization on a compact Kähler manifold M , one defines a positive holomorphic line bundle L over M , and for any integer $N \geq 1$ a "quantum" Hilbert space \mathcal{H}_N is formed by the holomorphic sections of the bundle $L^{\otimes N}$; the limit $N \rightarrow \infty$ is interpreted as a semiclassical limit. In the case of the 1-dimensional projective space $M = \mathbb{C}P^1$, which is the phase space of the spin, Hannay [97] defined a natural ensemble of random holomorphic sections in \mathcal{H}_N , and studied the point process formed by their zeros (topological constraints impose that any section has exactly N zeros). He explained how to compute the k -point correlation function of this process, and explicitly computed the limit (after microscopic rescaling) of the 2-point correlation function, which coincides with the 2-point function of the GAF. A few years later, Bleher-Schiffman-Zelditch [27] proved that, for a general Toeplitz quantization (M, L) , the zeros of random holomorphic sections converge, when $N \rightarrow \infty$, to a *universal* process depending only on the dimension of M . In dimension 1, this process is given by the zero process of the GAF.

4.2.2 Spectral statistics for the noisy non-selfadjoint Harmonic oscillator

We are now equipped to state our Theorem concerning the spectrum of P_h^δ (4.2.2).

Theorem 4.2.1 (Complex harmonic oscillator). *Fix $z_0 = X_0 + iY_0 \in \mathring{\Sigma}$, and define the classical density for the symbol $p(x, \xi) = \xi^2 + ix^2$ at the points $\rho_\pm^j \in p^{-1}(z_0)$:*

$$\sigma(z_0) \stackrel{\text{def}}{=} \frac{1}{|\{\text{Re } p, \text{Im } p\}(\rho_\pm^j(z_0))|} = \frac{1}{4\sqrt{X_0 Y_0}}, \quad j = 1, 2.$$

For $h \in]0, 1]$, let the random perturbation Q be either M_ω or V_ω , and take δ in the interval (4.2.1). Then, for any domain $O \Subset \mathbb{C}$, the rescaled spectral point process at z_0 converges in distribution as follows:

$$\mathcal{Z}_{h,z_0}^Q \xrightarrow{d} \mathcal{Z}_{G_{z_0,Q}} \quad \text{on } O, \quad \text{as } h \rightarrow 0.$$

Here $\mathcal{Z}_{G_{z_0,Q}}$ is the zero point process for the random entire function $G_{z_0,Q}$ described below:

1. if the perturbation $Q = V_\omega$ then

$$G_{z_0,V}(w) = g_{z_0}^1(w)g_{z_0}^2(w), \quad w \in \mathbb{C},$$

where $g_{z_0}^1, g_{z_0}^2$ are two independent copies of the GAF $g_{\sigma(z_0)}$.

2. if $Q = M_\omega$ then

$$G_{z_0,M}(w) = \det(g_{z_0}^{i,j}(w))_{1 \leq i,j \leq 2}, \quad w \in \mathbb{C},$$

where $g_{z_0}^{i,j}$, $1 \leq i, j \leq 2$, are 4 independent copies of the GAF $g_{\sigma(z_0)}$.

The convergence in distribution of point processes is described more explicitly in Thm 4.7.1. As we will explain in section 4.7, this convergence implies that all k -point measures converge as well to the limiting ones.

4.3 Spectral statistics in a more general framework

The above theorem can be generalized to a large class of 1-dimensional non-selfadjoint h -pseudo-differential operators, and with random perturbations which are not necessarily Gaussian. We first present the class of unperturbed operators we will be dealing with.

Let us fix the type of unperturbed operators we will consider in this section. We will use the notation $\rho = (x, \xi) \in \mathbb{R}^2 \simeq T^*\mathbb{R}$ for phase space points.

We consider a “classical” symbol $p \in S(\mathbb{R}^2, m)$, as in (A.1.2), namely it satisfies an asymptotic expansion in the limit $h \rightarrow 0$:

$$p(\rho; h) \sim p_0(\rho) + hp_1(\rho) + \dots \quad \text{in } S(\mathbb{R}^2, m), \quad (4.3.1)$$

where each $p_j \in S(\mathbb{R}^2, m)$ is independent of h . In this case we call p_0 the (semiclassical) principal symbol of p . We then define two subsets of \mathbb{C} associated with p_0 :

$$\Sigma \stackrel{\text{def}}{=} \overline{p_0(\mathbb{R}^2)}, \quad \Sigma_\infty \stackrel{\text{def}}{=} \{z \in \Sigma; \exists(\rho_j)_{j \geq 1} \text{ s.t. } |\rho_j| \rightarrow \infty, p_0(\rho_j) \rightarrow z\}. \quad (4.3.2)$$

Σ is the classical spectrum of the operator P_h defined below, while Σ_∞ can be called the classical spectrum at infinity. Furthermore, we suppose that the principal symbol p_0 is *elliptic* at some “energy” $z_{out} \in \mathbb{C} \setminus \Sigma$:

$$\exists C_0 > 0, \quad |p_0(\rho) - z_{out}| \geq m(\rho)/C_0, \quad \forall \rho \in \mathbb{R}^2. \quad (4.3.3)$$

For $h \in]0, 1]$ we let $P_h = p^w(x, hD_x; h)$ denote the h -Weyl quantization of the symbol p , defined in (A.1.3). The closure of P_h as an unbounded operator on L^2 , has the dense domain $H(m) \stackrel{\text{def}}{=} (P_h - z_{out})^{-1}(L^2(\mathbb{R})) \subset L^2(\mathbb{R})$; we will still denote this closed operator by P_h . Moreover, we will denote by $\|u\|_m \stackrel{\text{def}}{=} \|(P_h - z_{out})u\|$ the associated norm on $H(m)$ ¹.

Let $\tilde{\Omega} \subset \mathbb{C}$ be open simply connected, not entirely contained in Σ , and such that $\tilde{\Omega} \cap \Sigma_\infty = \emptyset$. Then, the spectrum of P_h inside $\tilde{\Omega}$ satisfies the following properties in the semiclassical limit [94, 96]:

- for $h > 0$ small enough, $\text{Spec}(P_h) \cap \tilde{\Omega}$ is discrete
- for all $\varepsilon > 0$, $\exists h(\varepsilon) > 0$ such that

$$\text{Spec}(P_h) \cap \tilde{\Omega} \subset \Sigma + D(0, \varepsilon), \quad \forall 0 < h < h(\varepsilon). \quad (4.3.4)$$

Here, $D(0, \varepsilon) \subset \mathbb{C}$ denotes the open disc of radius $\varepsilon > 0$ centred at 0.

In this work we will study the spectrum of small random perturbations of P_h , in the semiclassical limit $h \rightarrow 0$, in the interior of $\Sigma \cap \tilde{\Omega}$.

4.4 Pseudospectrum and the energy shell

Let $\tilde{\Omega}$ be as above and let

$$\Omega \Subset \tilde{\Omega} \cap \mathring{\Sigma} \text{ be open, simply connected.} \quad (4.4.1)$$

Recall that p_0 is the principal symbol of p , see (4.3.1). We assume that:

$$\text{for every } \rho \in p_0^{-1}(\overline{\Omega}), \text{ the 1-forms } dp_0, d\overline{p_0} \text{ are linearly independent.} \quad (4.4.2)$$

Since the dimension $d = 1$, this condition is equivalent to:

$$\text{for every } \rho \in p_0^{-1}(\overline{\Omega}), \{\text{Re } p_0, \text{Im } p_0\} \neq 0, \quad (4.4.3)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket:

$$\{p, q\}(\rho) \stackrel{\text{def}}{=} \partial_\xi p(\rho) \partial_x q(\rho) - \partial_\xi q(\rho) \partial_x p(\rho), \quad \rho = (x, \xi) \in \mathbb{R}^2.$$

¹Although this norm depends on the choice of the symbol $p - z_{out}$, it is equivalent to the norm defined from any elliptic operator in $q \in S(m)$, so that the space $H(m)$ only depends on the order function m .

It was observed by Dencker, Sjöstrand and Zworski [62], and Sjöstrand [171] (see Theorem 2.3.1) that since Ω is relatively compact and simply connected, (4.4.2), or equivalently (4.4.3), implies that there exists $J \in \mathbb{N}^*$ depending only on Ω , so that for any $z \in \Omega$, the “energy shell” $p_0^{-1}(z)$ consists of exactly $2J$ points:

$$p_0^{-1}(z) = \{\rho_{\pm}^j(z); j = 1, \dots, J\}, \text{ with } \pm \{\operatorname{Re} p, \operatorname{Im} p\}(\rho_{\pm}^j(z)) < 0, \quad (\text{HYP})$$

$$\rho_{\pm}^i(z) \neq \rho_{\pm}^j(z) \text{ if } i \neq j,$$

and the points $\rho_{\pm}^j(z) = (x_{\pm}^j(z), \xi_{\pm}^j(z))$ depend smoothly on z .

We shall make the further (generic) assumption

$$\forall z \in \Omega, \quad x_{\pm}^i(z) \neq x_{\pm}^j(z) \text{ if } i \neq j, \quad (\text{HYP-x})$$

which will play a role when studying the perturbation by a random potential.

Theorem 2.3.1 shows that (HYP) implies, for each $z \in \Omega$ and each $j = 1, \dots, J$, the existence of an h^∞ -quasimode for $P_h - z$ (resp. $(P_h - z)^*$), microlocalized on $\rho_{+}^j(z)$ (resp. $\rho_{-}^j(z)$). We will denote those modes by $e_{\pm}^j = e_{\pm}^j(z; h) \in L^2(\mathbb{R})$, normalize them as $\|e_{\pm}^j\| = 1$, and they satisfy

$$\|(P_h - z)e_{+}^j\| = \mathcal{O}(h^\infty) \quad \text{and} \quad \operatorname{WF}_h(e_{+}^j) = \{\rho_{+}^j(z)\}, \quad (4.4.4)$$

respectively

$$\|(P_h - z)^*e_{-}^j\| = \mathcal{O}(h^\infty) \quad \text{and} \quad \operatorname{WF}_h(e_{-}^j) = \{\rho_{-}^j(z)\}. \quad (4.4.5)$$

The notion of semiclassical wavefront set $\operatorname{WF}_h(u)$ is recalled in (A.2.1).

In view of the characterisation given in Theorem 2.2.4 of the pseudospectrum, we see that the assumption (4.4.2) implies that Ω is contained in the h^∞ -pseudospectrum of P_h , a spectrally highly unstable region.

4.5 Adding a random perturbation

We will now consider random perturbations of the operator P_h which are given by either a random matrix or a random potential, generalizing a little the constructions made in section 4.2. As in that section, we consider $\{e_k\}_{k \in \mathbb{N}}$ the orthonormal eigenbasis of the (nonsemiclassical) harmonic oscillator $H = -\partial_x^2 + x^2$.

Remark 4.5.1. *This choice of orthonormal basis is convenient for us, but it is far from unique. In fact what we need is a family of states (not necessarily orthonormal) such that the first $N(h)$ states microlocally cover a sufficiently large part of phase space, namely a neighbourhood of $p_0^{-1}(\Omega)$. We also need to avoid states which would have a large overlap with some of the quasimodes e_{\pm}^j , cf. (4.4.4), (4.4.5).*

Let α be a complex valued random variable defined on some probability space $(\mathcal{M}, \mathcal{F}, \mathbf{P})$, with the properties

$$\mathbb{E}[\alpha] = 0, \quad \mathbb{E}[\alpha^2] = 0, \quad \mathbb{E}[|\alpha|^2] = 1, \quad \mathbb{E}[|\alpha|^{4+\varepsilon_0}] < +\infty, \quad (4.5.1)$$

where $\varepsilon_0 > 0$ is an arbitrarily small but fixed constant. Here, $\mathbb{E}[\cdot]$ denotes the expectation with respect to the probability measure \mathbf{P} . The Markov inequality implies the following tail estimate: there exists a constant $\kappa_\alpha > 0$ such that

$$\mathbf{P}[|\alpha| \geq \gamma] \leq \kappa_\alpha \gamma^{-(4+\varepsilon_0)}, \quad \forall \gamma > 0. \quad (4.5.2)$$

Remark 4.5.2. *For instance, the complex centred Gaussian random variable of eq. (4.2.3) satisfies the above assumptions.*

Random Matrix. Let $N(h) = C_1/h^2$, $C_1 > 0$ large enough (we will be more precise about this condition later). Let $q_{j,k}$, $0 \leq j, k < N(h)$ be independent copies of the random variable α satisfying the conditions (4.5.1). We consider the random matrix

$$M_\omega = \frac{1}{N(h)} \sum_{0 \leq j, k < N(h)} q_{j,k} e_j \otimes e_k^*, \quad (\text{RM})$$

where $e_j \otimes e_k^* u = (u|e_k)e_j$ for $u \in L^2(\mathbb{R})$. For some coupling parameter $0 < \delta \ll 1$, we define the randomly perturbed operator

$$P_M^\delta = P_h + \delta M_\omega. \quad (4.5.3)$$

Random Potential. Take $N(h) = C_1/h^2$, $C_1 > 0$ as above. Let v_j , $0 \leq j < N(h)$ be independent copies of the random variable α . Still using the same orthonormal family $(e_k)_{k \in \mathbb{N}}$, we define the random function

$$V_\omega = \frac{1}{N(h)} \sum_{0 \leq j < N(h)} v_j e_j. \quad (\text{RP})$$

For $0 < \delta \ll 1$, write the perturbed operator

$$P_V^\delta = P_h + \delta V_\omega. \quad (4.5.4)$$

We call this perturbation a “random potential”, even though V_ω is complex valued. When we consider this type of perturbation, we will make the additional symmetry assumption:

$$p(x, \xi; h) = p(x, -\xi; h). \quad (\text{SYM})$$

This hypothesis implies that we can group the points forming $p_0^{-1}(z)$ (see (HYP)) in pairs, such that $\rho_\pm^j = (x^j, \pm\xi^j)$. As a result, the centres of microlocalization of the quasimodes e_+^j and e_-^j are located on the same fibre $T_{x^j}^* \mathbb{R} = \{(x^j, \xi), \xi \in \mathbb{R}\}$.

Remark 4.5.3. *We could relax the assumption (SYM) into requiring this symmetry only at the level of the principal symbol, i.e. $p_0(x, \xi) = p_0(x, -\xi)$. However, for the simplicity of the presentation we prefer to make the above stronger hypothesis.*

Restricting to bounded perturbations. For both types of perturbations, it will be easier for us to restrict the random variables to large discs $D(0, C/h)$, i.e. assume that

$$|v_i|, |q_{i,j}| \leq C/h, \quad 0 \leq i, j < N(h), \quad \text{for some } C > 0 \text{ sufficiently large.} \quad (4.5.5)$$

This restriction induces the boundedness of the perturbations M_ω, V_ω . Indeed on this restricted probability space we have the bound,

$$\|M_\omega\|_{HS} \leq Ch^{-1}, \quad (4.5.6)$$

where $\|M_\omega\|_{HS}$ denotes the Hilbert-Schmidt norm of M_ω . Respectively, in the case of the random potential,

$$\|V_\omega\|_\infty \leq Ch^{-1}. \quad (4.5.7)$$

We note that even for unrestricted random variables, these bounds on the perturbations hold with high probability. Indeed, using (4.5.2) to estimate the probability that (4.5.5) holds, we deduce that (4.5.6) holds with probability $\geq 1 - C_2 h^{\varepsilon_0}$, and that (4.5.7) occurs with probability $\geq 1 - C_2 h^{2+\varepsilon_0}$, for some $C_2 > 0$.

Finally, we will take the coupling parameter $\delta = \delta(h)$ in the same interval as in (4.2.1).

As discussed after (3.2.1), the spectra of P_M^δ and P_V^δ in Ω are purely discrete. The principal aim of this section is to show that the statistical properties of these spectra, in a microscopic neighbourhood of any $z_0 \in \Omega$, are universal, in a sense that we will specify later on.

Since $p_0 - z$ is elliptic for every $z \in \mathbb{C} \setminus \Sigma$, the resolvent norm $\|(P_h - z)^{-1}\| = \mathcal{O}(1)$, uniformly for z in compact subset of $\mathbb{C} \setminus \Sigma$, as $h \rightarrow 0$. In view of (4.5.6), (4.5.7) and (4.2.1), we are considering random perturbations of size $\|\delta M_\omega\|_{HS} \ll h^2$ and $\|\delta V_\omega\|_\infty \ll h^2$. Therefore, in view of the characterisation (2.2.5) of the pseudospectrum, the spectra of P_M^δ and P_V^δ are contained in $\Sigma + D(0, \epsilon)$, for any given $\epsilon > 0$ and $h > 0$ small enough. Moreover, since $\Omega \Subset \dot{\Sigma}$, we will not feel the effects of the boundary of Σ ; we will simply say that Ω lies in the *bulk* of the spectrum of the perturbed operator.

4.6 Probabilistic Weyl's law and local statistics

Recall the probabilistic Weyl law stated in Theorem 3.2.4. It shows that, with probability close to 1, the number of eigenvalues of the perturbed operator P_h^δ in any *fixed* subset of Ω is of order $\asymp h^{-1}$. Hence, the spectrum of P_h^δ will spread across Ω , with an average spacing between nearby eigenvalues of order $h^{1/2}$.

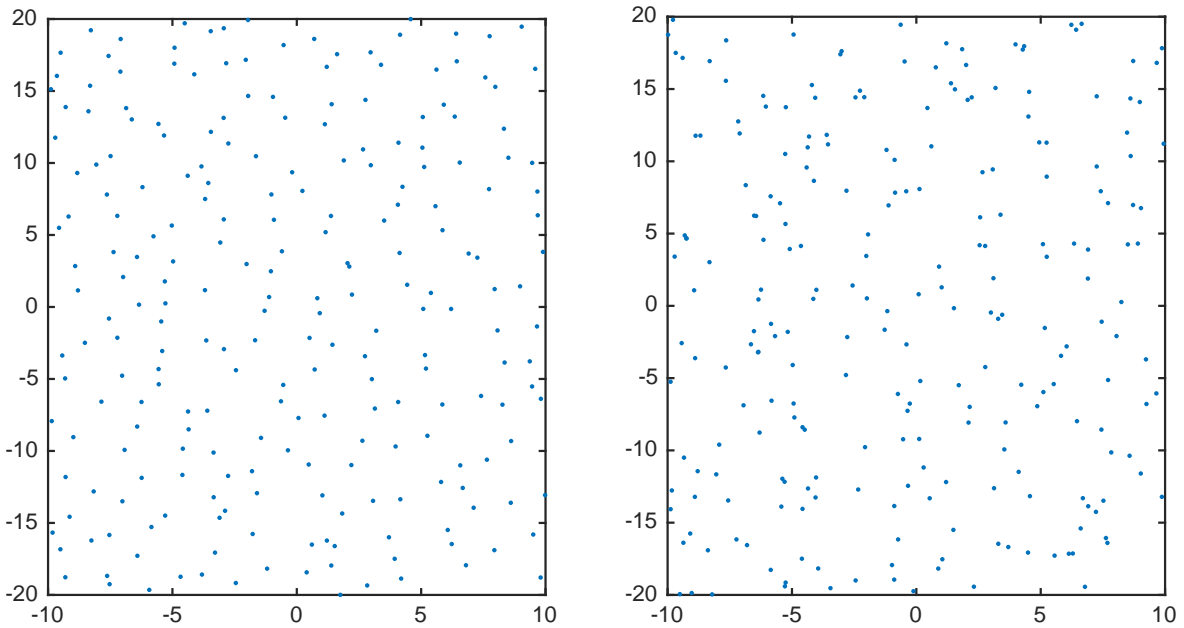


Figure 4.1: Numerically computed spectra of the operators $-h^2 \partial_x^2 + e^{3ix} + \delta Q$ acting on $L^2(\mathbb{T})$, with $h = 10^{-3}$ and $\delta = 10^{-12}$. The perturbation Q is either a Gaussian random matrix M_ω (left), or a Gaussian random potential V_ω (right). The region plotted is the same, it is part of the bulk (the units on the axes are arbitrary). In the region the number of quasimodes is $2J$, $J = 6$.

Figure 4.1 illustrates this behaviour for a choice of operator $P_h = -h^2 \partial_x^2 + e^{3ix}$ acting on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We draw random perturbations M_ω , V_ω and plot some region of the spectra of P_M^δ and P_V^δ , in the interior of Σ . Both spectra are grossly uniform over the plotted region, yet in the case of P_V^δ (right plot) the distribution of the eigenvalues seems a bit “less uniform” than in the case of P_M^δ (left plot), in particular it allows the presence of small clusters of very near eigenvalues.

To quantify this difference of uniformity between the spectra of P_M^δ and P_V^δ , we study the local statistics of the eigenvalues, that is the statistics of the eigenvalues on the scale of their mean level spacing. For this purpose, we fix a point $z_0 \in \Omega$. In both cases $Q = M_\omega$ and $Q = V_\omega$, we view the

rescaled spectrum of the randomly perturbed operator P_Q^δ as a random point process

$$\mathcal{Z}_{h,z_0}^Q \stackrel{\text{def}}{=} \sum_{z \in \text{Spec}(P_Q^\delta)} \delta_{(z-z_0)h^{-1/2}}, \quad Q = M_\omega \text{ or } Q = V_\omega, \quad (4.6.1)$$

where the eigenvalues are counted according to their algebraic multiplicities.

Notice that the rescaled eigenvalues $(z_j - z_0)h^{-1/2}$ have a mean spacing of order $\asymp 1$. Our principal aim is to show that, under the assumption (4.5.1) on the random coefficients, in the limit $h \rightarrow 0$ the correlation functions of the processes \mathcal{Z}_{h,z_0}^M and \mathcal{Z}_{h,z_0}^V are *universal*, in the sense that they

- *depend* only on the structure of the energy shell $p_0^{-1}(z)$ and on the type of random perturbation used, either M_ω or V_ω ;
- *are independent* of the law of the random variable α used to define the random perturbations, as long as satisfies (4.5.1).

Finally, let us stress that our results concern solely the eigenvalues in the bulk of the spectrum of P_h^δ , that is in the interior of the h^∞ -pseudospectrum of P_h . Near the boundary of that pseudospectrum, we expect the statistical properties of the eigenvalues to change drastically. It has been shown by the author in [192] in the case of a model operator, that the probabilistic Weyl's law breaks down in the vicinity of $\partial\Sigma$, in fact, the density of eigenvalues explodes near that boundary.

4.7 Perturbation by a random potential

We begin with the case of a perturbation by a random potential V_ω , eq. (4.5.4). In Weyl's law of Thm 3.2.4, the main term on the right hand side can be easily expressed in terms of the *classical spectral density*, pull-back of the symplectic measure on $T^*\mathbb{R}$ through the symbol p_0 :

$$\int_{p_0^{-1}(\Gamma)} dx d\xi = \int_{\Gamma} (p_0)_*(dx d\xi)$$

(the Lebesgue measure $dx d\xi$ on \mathbb{R}^2 is also the measure induced by the symplectic form on $T^*\mathbb{R} \cong \mathbb{R}^2$).

From the structure (HYP) of the energy shell $p_0^{-1}(z)$, the classical spectral density at the energy z can be expressed as follows:

$$(p_0)_*(dx d\xi) = \sum_{j=1}^J (\sigma_+^j(z) + \sigma_-^j(z)) L(dz), \quad \sigma_\pm^j(z) = \frac{1}{\mp \{\text{Re } p_0, \text{Im } p_0\}(\rho_\pm^j(z))}. \quad (4.7.1)$$

Here L denotes the Lebesgue measure on \mathbb{C} . In other words, each point ρ_\pm^j of the energy shell provides a density component $\sigma_\pm^j(z) > 0$, which depends smoothly on $z \in \Omega$.

If we additionally assume the symmetry (SYM) and group the points such that $\rho_\pm^j = (x^j, \pm \xi^j)$, we find that $\sigma_+^j(z) = \sigma_-^j(z)$ for all $j = 1, \dots, J$.

Universal limiting point process

Let us now state our main theorem for the perturbed operators P_V^δ . It provides the asymptotic behaviour of the rescaled spectral point processes \mathcal{Z}_{h,z_0}^V in the semiclassical limit.

Theorem 4.7.1. *Let p be as in (4.3.1) satisfying (4.4.2) and (SYM). Let $\Omega \Subset \mathring{\Sigma}$ be as in (4.4.1), and choose $z_0 \in \Omega$. Then, for any bounded open set $O \Subset \mathbb{C}$, the rescaled spectral point processes at z_0 converge in distribution:*

$$\mathcal{Z}_{h,z_0}^V \xrightarrow{d} \mathcal{Z}_{G_{z_0}} \quad \text{in } O, \quad \text{as } h \rightarrow 0.$$

This convergence means that for any test function $\phi \in \mathcal{C}_c(O, \mathbb{R})$,

$$\langle \mathcal{Z}_{h,z_0}^V, \phi \rangle = \sum_{z \in \text{Spec}(P_h^\delta)} \phi((z - z_0)h^{-1/2}) \xrightarrow{d} \langle \mathcal{Z}_{G_{z_0}}, \phi \rangle = \sum_{z \in G_{z_0}^{-1}(0)} \phi(z), \quad \text{as } h \rightarrow 0.$$

Here $\mathcal{Z}_{G_{z_0}}$ is the zero point process for the random analytic function

$$G_{z_0}(z) = \prod_{j=1}^J g_{z_0}^j(z), \quad z \in \mathbb{C},$$

where J is as in (HYP), the $g_{z_0}^j$ are J independent GAFs $g_{z_0}^j \sim g_{\sigma_+^j(z_0)}$ (see section 4.2.1), with $\sigma_+^i(z_0)$ the local spectral densities given in (4.7.1).

The definition and basic properties of the GAFs have been presented in section 4.2.1. We refer the reader to [120] for an in depth review.

This theorem tells us that at any given point $z_0 \in \Omega$ in the bulk of the pseudospectrum, the rescaled spectral point process converges, in the limit $h \rightarrow 0$, to the point process given by the zeros of the product of J independent GAFs. This limiting point process is the superposition of J independent processes, each one generated by a GAF $g_{z_0}^j$. The latter only depends on the part of the classical spectral density coming from the pair of points $\rho_\pm^j = (x^j, \pm \xi^j)$. In particular, this limiting process is independent of the precise probability distribution of the coefficients (v_j) , as long as it satisfies (4.5.1), or of the orthonormal family (e_j) used to generate the random potential V_ω ; this process only depends on the cardinal $2J$ of the energy shell $p_0^{-1}(z)$ and of the local spectral densities $\{\sigma_+^j(z_0), j = 1, \dots, J\}$.

It is known that the zero process of a single GAF exhibits a local repulsion between the nearby points (see the section 4.7). On the opposite, as a superposition of J independent point processes, the limiting process $\mathcal{Z}_{G_{z_0}}$ authorizes the presence of *clusters* of at most J points very close to one another, confirming our observations in the right plot of Fig. 4.1 (for the considered operator and plotted region, we have $J = 6$). In the next section we will analyze this clustering by computing the correlation functions between the points of the process.

Scaling limit of the k -point measures

An explicit way to quantify the statistical correlations between k nearby eigenvalues of P_V^δ consists in defining the k -point measures of the point process \mathcal{Z}_{h,z_0}^V . These are positive measures μ_h^{k,V,z_0} on O^k , where O is the open domain as in Theorem 4.7.1. These measures are defined through their action on an arbitrary test function $\phi \in \mathcal{C}_c(O^k, \mathbb{R}_+)$ as follows:

$$\begin{aligned} \mathbb{E} \left[(\mathcal{Z}_{h,z_0}^V)^{\otimes k}(\phi) \right] &= \mathbb{E} \left[\sum_{z_1, \dots, z_k \in \text{Spec}(P_h^\delta)} \phi \left((z_1 - z_0)h^{-1/2}, \dots, (z_k - z_0)h^{-1/2} \right) \right] \\ &\stackrel{\text{def}}{=} \int_{O^k} \phi(w) \mu_{h,z_0}^{k,V}(dw). \end{aligned} \quad (4.7.2)$$

In practice, one often studies these measures away from the generalized diagonal $\Delta = \{z \in \mathbb{C}^k; \exists i \neq j \text{ s.t. } z_i = z_j\}$, in order to avoid trivial self-correlations. Hence the test functions we will use below will be chosen in $\mathcal{C}_c(O^k \setminus \Delta, \mathbb{R}_+)$.

When these k -point measures are absolutely continuous with respect to the Lebesgue measure on \mathbb{C}^k , we call their densities the k -point functions.

Theorem 4.7.2. *Let $\mu_{h,z_0}^{k,V}$ be the k -point measure of \mathcal{Z}_{h,z_0}^V , defined in (4.7.2), and let $\mu_{z_0}^{k,V}$ be the k -point measure of the point process $\mathcal{Z}_{G_{z_0}}$, given in Theorem 4.7.1. Then, for any domain $O \Subset \mathbb{C}$ and for all $\phi \in \mathcal{C}_c(O^k \setminus \Delta, \mathbb{R}_+)$,*

$$\int_{O^k \setminus \Delta} \phi(w) \mu_{h,z_0}^{k,V}(dw) \longrightarrow \int_{O^k \setminus \Delta} \phi(w) \mu_{z_0}^{k,V}(dw), \quad h \rightarrow 0.$$

Moreover, $\mu_{z_0}^{k,V}$ is absolutely continuous with respect to the Lebesgue measure on \mathbb{C}^k . Its density $d_{z_0}^{k,V}$ is given by the following formula:

$$d_{z_0}^{k,V}(w_1, \dots, w_k) = \sum_{\substack{\alpha \in \mathbb{N}^J \\ \sum_j \alpha_j = k}} \sum_{\tau \in \mathfrak{S}_k} \frac{1}{\alpha!} \prod_{j=1}^J d_{g_j}^{\alpha_j}(w_{\tau(\alpha_1 + \dots + \alpha_{j-1} + 1)}, \dots, w_{\tau(\alpha_1 + \dots + \alpha_j)}) \quad (4.7.3)$$

where \mathfrak{S}_k is the symmetric group on k elements, and for all $1 \leq j \leq J$ and all $r \in \mathbb{N}^*$,

$$d_{g_j}^r(w) = \frac{\text{perm}[C_j^r(w) - B_j^r(w)(A_j^r)^{-1}(w)(B_j^r)^*(w)]}{\det \pi A_j^r(w)}, \quad \text{while } d_{g_j}^0(w) \equiv 1. \quad (4.7.4)$$

Here, perm denotes the permanent of a matrix; A_j^r, B_j^r, C_j^r are complex $r \times r$ -matrices given by

$$(A_j^r)_{n,m} = K^j(w_n, \bar{w}_m), \quad (B_j^r)_{n,m} = (\partial_w K^j)(w_n, \bar{w}_m), \quad (C_j^r)_{n,m} = (\partial_{w\bar{w}}^2 K^j)(w_n, \bar{w}_m),$$

where $K^j(w, \bar{w}) = \exp(\sigma_+^j(z_0)w\bar{w})$ is the covariance function of the GAFs $g_{z_0}^j$ appearing in Thm 4.7.1.

The function $d_{g_j}^r(z)$ in (4.7.4) is the r -point function for the zero process of the Gaussian analytic function g^j . The limiting k -point functions are thus obtained by concatenating the r -point functions ($1 \leq r \leq k$) of the J GAFs g^j associated with the points ρ_{\pm}^j of the energy shell. The zeros associated with different points ρ_{\pm}^j are uncorrelated with one another.

A result by Nazarov and Sodin [77, Theorem 1.1] implies the following estimate for the r -point densities of a single GAF.

Proposition 4.7.3. [77] *Let $O \Subset \mathbb{C}$ be a bounded domain. Let $(g^j = g_{z_0}^j)_{1 \leq j \leq J}$ be the GAFs appearing in Thm 4.7.1, and let $d_{g^j}^r(w)$, $1 \leq r \leq k$, be the corresponding r -point functions as in (4.7.4).*

Then there exists a constant $C = C(r, g^j, O) > 1$ such that, for any configuration of pairwise distinct points $w_1, \dots, w_k \in O$,

$$C^{-1} \prod_{i < j} |w_i - w_j|^2 \leq d_{g^j}^r(w_1, \dots, w_k) \leq C \prod_{i < j} |w_i - w_j|^2.$$

This estimate shows that the zeros of a GAF enjoy a *statistical (quadratic) repulsion* at short distance, namely they are very unlikely to approach one another much more than the mean distance.

In formula (4.7.3) we see that if $k > J$, each summand has at least one factor $d_{g^j}^{\alpha_j}$ with $\alpha_j \geq 2$. Hence, Thm 4.7.2 and Prop. 4.7.3 lead to the following

Corollary 4.7.4. *Let $O \Subset \mathbb{C}$ be a bounded domain, let $k > J$, and let $d_{z_0}^{k,V}(w)$ be as in (4.7.3). Then there exists a positive constant $C = C(r, O)$ such that, for any configuration of pairwise distinct points $w_1, \dots, w_k \in O$,*

$$d_{z_0}^{k,V}(w_1, \dots, w_k) \leq C \sum_{i < j} |w_i - w_j|^2.$$

We have seen by Theorem 4.7.2 that the limiting point process of the rescaled eigenvalues is given by the superposition of J independent processes given by the zeros of independent Gaussian analytic functions. Due to this independence, k points, each originating from a different GAF process, may approach each other without any statistical repulsion: this authorizes the formation of clusters of at most J points. As a result, for $k \leq J$ the limiting k -point functions do not decay to zero as the distances between the k points gets smaller: this authorizes the presence of clusters of at most J points. This behaviour is made more explicit in the next section in the case $k = 2$.

On the opposite, if $k > J$ then at least two points must originate from the same GAF process, and therefore statistically repel each other when approaching each other. This is exactly what Corollary 4.7.4 tells us: the probability to find more than J points close together decays at least quadratically with the distance. As a result, finding small clusters containing more than J eigenvalues is very unlikely.

2-point correlation function

The 2-point *correlation function* of a point process is defined by the 2-point function, renormalized by the local 1-point functions (or local average densities):

$$K_{z_0}^{2,Q}(w_1, w_2) = \frac{d_{z_0}^{2,Q}(w_1, w_2)}{d_{z_0}^{1,Q}(w_1)d_{z_0}^{1,Q}(w_2)}, \quad w_1 \neq w_2 \in O, \quad Q = V_\omega, M_\omega.$$

By Theorem 4.7.2, the limiting local 1-point function $d_{z_0}^{1,V}(w)$ is a constant function, given by

$$d_{z_0}^{1,V}(w) = \sum_{j=1}^J \frac{\sigma_+^j(z_0)}{\pi}, \quad \forall w \in O.$$

This average density of eigenvalues (at the *microscopic* scale near z_0) exactly corresponds to the *macroscopic* density predicted by the probabilistic Weyl's law in Theorem 3.2.4, see also (4.7.1).

The limiting 1-point and 2-point functions of the zero process generated by a single GAF g_σ (see section 4.2.1) are given by

$$d_{g_\sigma}^1(w_1) = \frac{\sigma}{\pi}, \quad \text{respectively} \quad d_{g_\sigma}^2(w_1, w_2) = \left(\frac{\sigma}{\pi}\right)^2 \kappa\left(\frac{\sigma|w_1 - w_2|^2}{2}\right),$$

with the scaling function

$$\kappa(t) \stackrel{\text{def}}{=} \frac{(\sinh^2 t + t^2) \cosh t - 2t \sinh t}{\sinh^3 t}, \quad \forall t \geq 0. \quad (4.7.5)$$

The function $\kappa(\sigma|w_1 - w_2|^2/2)$ describes the 2-point correlation function of the zeros of the GAF

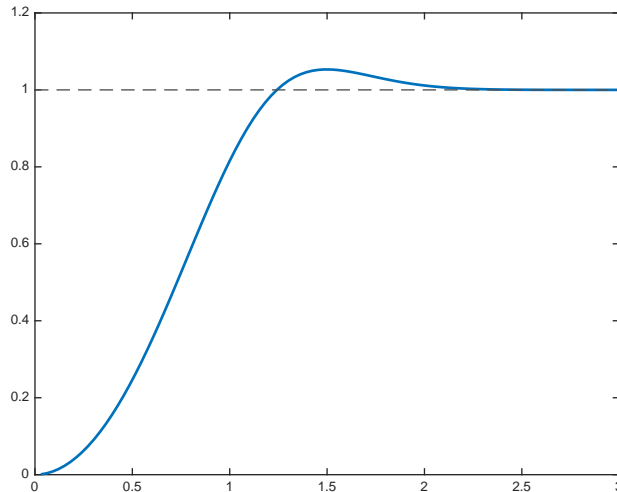


Figure 4.2: Plot of the function $t \mapsto \kappa(t^2)$, see (4.7.5).

g_σ . A remarkable property of this function is its isotropy: it only depends on the distance between the points w_1, w_2 . In Figure 4.2 we plot the function $t \mapsto \kappa(t^2)$; it behaves as $\kappa(t^2) = t^2(1 + \mathcal{O}(t^4))$ when $t \rightarrow 0$, which reflects the quadratic repulsion between the nearby zeros of g_σ . On the opposite, when $t \gg 1$ it converges exponentially fast to unity, showing a fast decorrelation between the zeros at large distances.

To our knowledge, the function κ was first computed by Hannay [97], as the scaling limit 2-point correlation function for the zeros of certain ensembles of random polynomials. In the work by Bleher, Shiffman and Zelditch [27], κ describes the scaling limit 2-point correlation function for the zeros of random holomorphic sections of large powers of a positive Hermitian line bundle over a compact complex Kähler surface.

In the present work, κ appears as a building block for the limit 2-point correlation function of the eigenvalues of P_V^δ :

$$K_{z_0}^{2,V}(w_1, w_2) = 1 + \sum_{j=1}^J \frac{(\sigma_+^j(z_0))^2}{\left(\sum_{j=1}^J \sigma_+^j(z_0)\right)^2} \left[\kappa \left(\frac{\sigma_+^j(z_0)|w_1 - w_2|^2}{2} \right) - 1 \right]. \quad (4.7.6)$$

Let us study more closely this 2-point correlation function:

Long range decorrelation: For $|w_1 - w_2| \gg 1$, in the limit $h \rightarrow 0$, the 2-point correlation function converges exponentially fast to unity

$$K_{z_0}^{2,V}(w_1, w_2) = 1 + \mathcal{O} \left(e^{-\min_j \sigma_+^j(z_0)|w_1 - w_2|^2} \right).$$

This shows that two points at distances $|w_1 - w_2| \gg 1$ are statistically uncorrelated.

A weak form of repulsion: When $|w_1 - w_2| \ll 1$, in the limit $h \rightarrow 0$, there is a weak form of repulsion between two nearby eigenvalues,

$$K_{z_0}^{2,V}(w_1, w_2) = 1 - \sum_{j=1}^J \frac{\sigma_+^j(z_0)^2}{\left(\sum_{l=1}^J \sigma_+^l(z_0)\right)^2} \left[1 - \frac{\sigma_+^j(z_0)|w_1 - w_2|^2}{2} + \mathcal{O}(|w_1 - w_2|^4) \right]. \quad (4.7.7)$$

This formula shows that the probability to find two rescaled eigenvalues w_1, w_2 at distance $\ll 1$ is smaller than the one to find them at large distances: pairs of rescaled eigenvalues show a weak repulsion at short distance. However, the correlation function does not converge to zero when $|w_1 - w_2| \rightarrow 0$, but to the positive value $1 - \frac{\sum_{j=1}^J \sigma_+^j(z_0)^2}{\left(\sum_{l=1}^J \sigma_+^l(z_0)\right)^2}$. This weak repulsion can be explained by the fact that the random function G_{z_0} is the product of J independent GAFs: two zeros w_1, w_2 will not repel each other if they originate from different GAFs, while they will repel quadratically if they come from the same GAF. The net result is this weak form of repulsion. The larger the number of quasimodes J , the weaker this repulsion becomes, since two zeros w_1, w_2 chosen at random will have a smaller chance to come from the same GAF.

In Figure 4.3 we compare the limiting 2-point correlation functions $K_{z_0}^{2,V}$ with the one obtained from numerical spectra of two operators on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$:

$$P_{h,q}^\delta = -h^2 \partial_x^2 + e^{iqx} + \delta V_\omega, \quad q = 1, 3, \quad x \in \mathbb{T}. \quad (4.7.8)$$

We took the parameters $h = 10^{-3}$, $\delta = 10^{-12}$, and the Gaussian random potential V_ω as in section 4.2. We use operators defined on \mathbb{T} because they are numerically easier to diagonalize than operators on \mathbb{R} . For each operator $P_{h,q}$, we drew 1000 samples of the random potential V_ω , and computed the corresponding spectra of P_V^δ , then extracted from these spectra the correlation function.

The analysis of the principal symbols $p_{q,0}$ shows that the classical spectrum is, in both cases, given by $\Sigma = \mathbb{R}_+ + U(1)$. At the energy $z_0 = 1.6$ (clearly located in the ‘‘bulk’’), the operator $(P_{h,q} - z_0)$ admits $J = 2q$ quasimodes. Figure 4.3 compares the numerically obtained 2-point correlation functions (shown as blue dots) of the operators $P_{h,1}^\delta$ (left) and $P_{h,3}^\delta$ (right), with the theoretical scaling limit described in (4.7.7). For the two operators, the theoretical curve fits quite well the numerical points, including at short distances $|w_1 - w_2| \ll 1$.

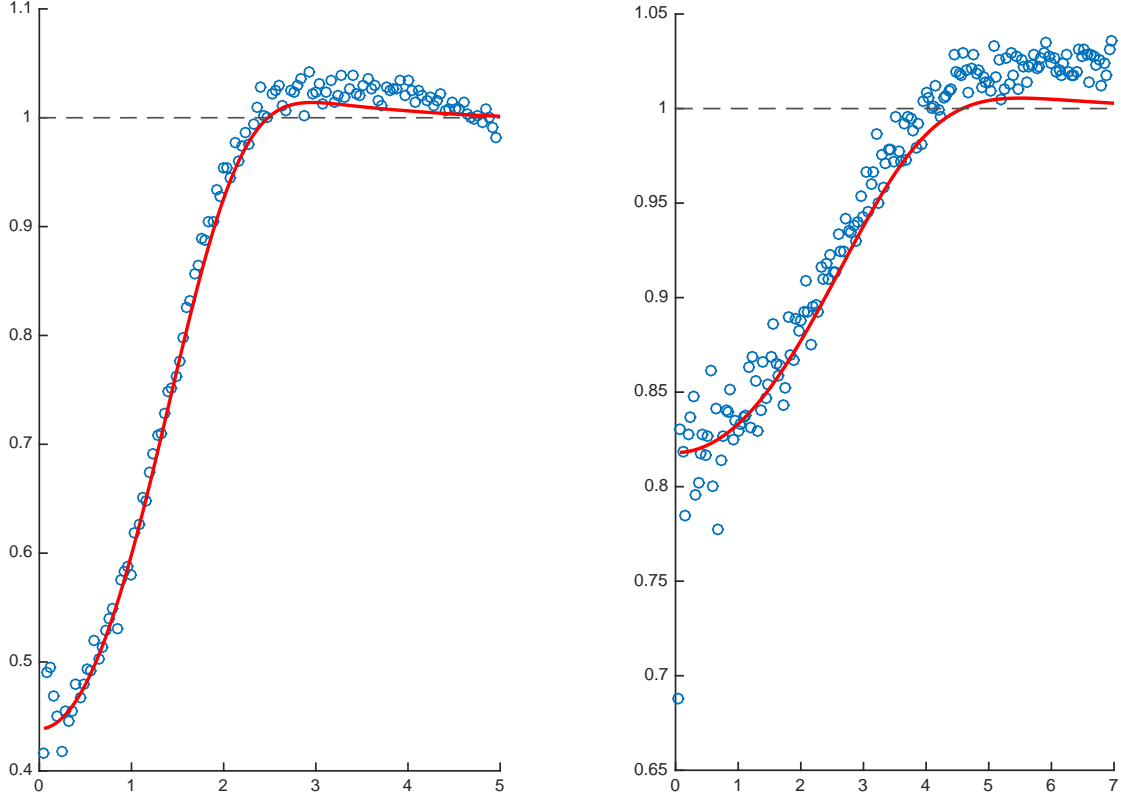


Figure 4.3: Blue points: rescaled 2-point correlation functions near the energy $z_0 = 1.6$, obtained by numerically computing the spectra of the operators $P_{h,1}$ (left) and $P_{h,3}$ (right) perturbed by a Gaussian random potential δV_ω . Red curves: scaling limit 2-point correlation functions $K_{z_0}^{2,V}$ for both operators, as given in (4.7.6); the horizontal coordinate is the rescaled square distance $|w_1 - w_2|^2$.

4.8 Perturbation by a random matrix

We now describe the situation where the operator P_h is perturbed by a small random matrix δM_ω , as described in (RM,4.5.3). In this case we do not need to assume the symmetry property (SYM) for the symbol p_0 .

Universal limiting point process

Here as well, we can prove a convergence of the rescaled spectral point process \mathcal{Z}_{h,z_0}^M (see (4.6.1)) towards a limiting zero process when $h \rightarrow 0$.

Theorem 4.8.1. *Let p be as in (4.3.1) satisfying (4.4.2). Let $\Omega \Subset \hat{\Sigma}$ be as in (4.4.1). Choose $z_0 \in \Omega$. Then, for any bounded open set $O \Subset \mathbb{C}$, the rescaled spectral point process \mathcal{Z}_{h,z_0}^M converges in distribution towards the zero point process associated with a random analytic function \tilde{G}_{z_0} described below:*

$$\mathcal{Z}_{h,z_0}^M \xrightarrow{d} \mathcal{Z}_{\tilde{G}_{z_0}} \text{ on } O \quad \text{as } h \rightarrow 0. \quad (4.8.1)$$

The random function \tilde{G}_{z_0} is defined as

$$\tilde{G}_{z_0}(w) \stackrel{\text{def}}{=} \det(g_{z_0}^{i,j}(w))_{1 \leq i,j \leq J}, \quad w \in \mathbb{C},$$

where $g_{z_0}^{i,j}$, for $1 \leq i,j \leq J$, are J^2 independent GAFs $g_{z_0}^{i,j} \sim g_{\sigma_{z_0}^{i,j}}$, for J is as in (HYP) and for the parameters

$$\sigma_{z_0}^{i,j} = \frac{1}{2}(\sigma_+^i(z_0) + \sigma_-^j(z_0)). \quad (4.8.2)$$

The local classical densities $\sigma_\pm^i(z_0)$ associated with the points $\rho_\pm^i(z_0)$, were defined in (4.7.1).

Theorem 4.8.1 tells us that at any given point $z_0 \in \tilde{\Omega} \cap \mathring{\Sigma}$ in the bulk of the pseudospectrum, the local rescaled point process of the eigenvalues of P_M^δ is given, in the limit $h \rightarrow 0$, by the zero process associated with the determinant of a $J \times J$ matrix, whose entries are independent GAFs. The GAF situated at the entry i, j of the matrix only depends on the local classical densities of the points $\rho_+^i(z_0)$ and $\rho_-^j(z_0)$.

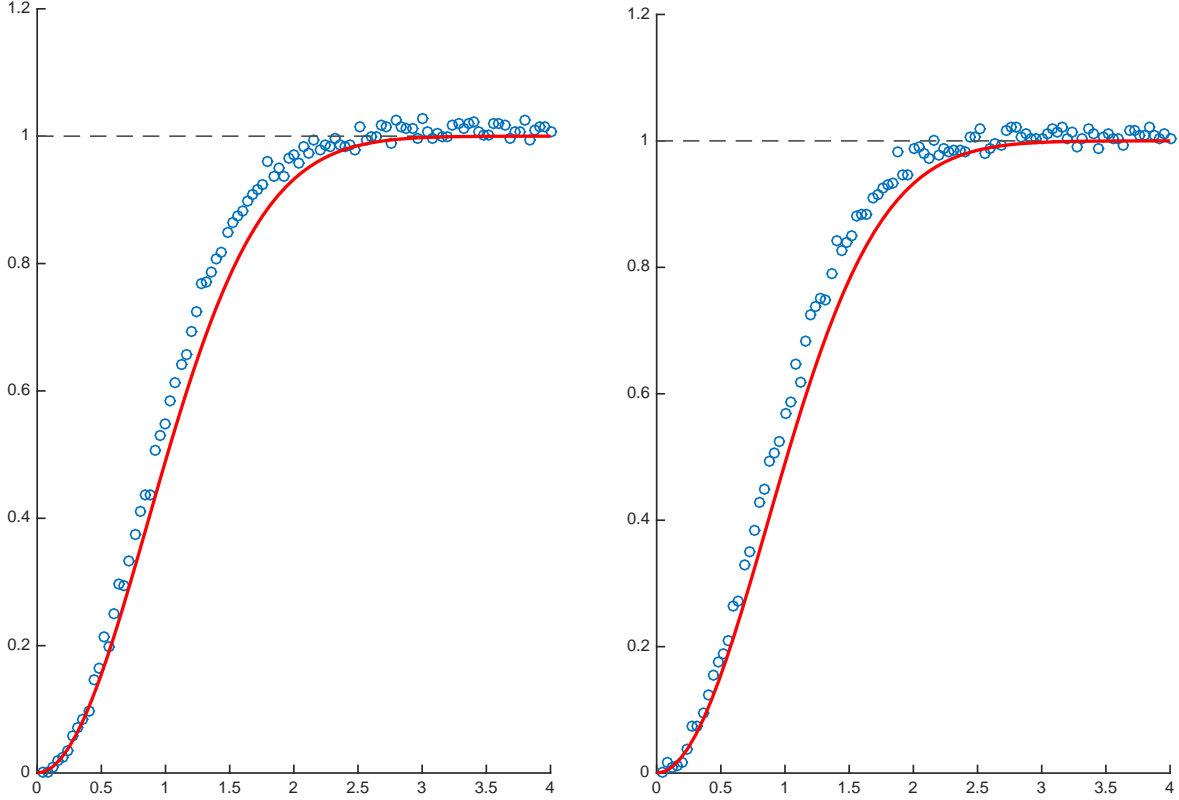


Figure 4.4: Blue points: values of the 2-point correlation functions, obtained by numerically computing the spectra of the operators $P_{h,1}$ (left) and $P_{h,3}$ (right) perturbed by a Gaussian random matrix δM_ω . The parameters z_0, h, δ are as in Figure 4.3. Red curves: the 2-point correlation function for the Ginibre ensemble, $K_{z_0}^{2, Gin}$, as given in (4.8.5). The horizontal coordinate is the rescaled square distance $|w_1 - w_2|^2$.

The limiting point process $\mathcal{Z}_{\tilde{G}_{z_0}}$ features some partial form of universality: it is independent of the precise law on the entries of the perturbation M_ω (4.5.1), but only depends on the cardinal $2J$ of the energy shell $p_0^{-1}(z)$, and of the local classical densities $\{\sigma_\pm^j(z_0); j = 1, \dots, J\}$ (notice that in absence of the symmetry (SYM), the densities $\sigma_+^j(z_0)$ and $\sigma_-^j(z_0)$ are a priori unrelated).

The limiting process $\mathcal{Z}_{\tilde{G}_{z_0}}$ is different from the universal limit $\mathcal{Z}_{G_{z_0}}$ studied in the previous section. In particular the function \tilde{G}_{z_0} is not given by a simple product of GAFs, but by a more complicated expression, namely a determinant. As we will see below, we expect the zeros of \tilde{G}_{z_0} to exhibit a quadratic repulsion between nearby points, as opposed to the zeros of the function G_{z_0} in Thm 4.7.1.

Scaling limit k -point measures

A direct consequence of the convergence of the zero processes \mathcal{Z}_{h, z_0}^M is the convergence of their k -point measures to those of the limiting point process.

Corollary 4.8.2. *Let $\mu_{h, z_0}^{k, M}$ be the k -point measure of \mathcal{Z}_{h, z_0}^M , defined as in (4.7.2), and let $\mu_{z_0}^{k, M}$ be the k -point measure of the point process $\mathcal{Z}_{\tilde{G}_{z_0}}$ described in Thm 4.8.1. Then, for any $O \in \mathbb{C}$ open*

connected domain and for all $\phi \in \mathcal{C}_c(O^k \setminus \Delta, \mathbb{R}_+)$,

$$\int_{O^k \setminus \Delta} \phi(w) \mu_{h,z_0}^{k,M}(dw) \longrightarrow \int_{O^k \setminus \Delta} \phi(w) \mu_{z_0}^{k,M}(dw), \quad h \rightarrow 0.$$

One can calculate the densities of the limiting 1-point measures $\mu_{z_0}^{1,M}$:

$$d_{z_0}^{1,M}(w) = \sum_{i=1}^J \frac{\sigma_+^i(z_0) + \sigma_-^i(z_0)}{2\pi}, \quad \forall w \in O. \quad (4.8.3)$$

Not surprisingly, this microscopic density is exactly the rescaling of the macroscopic spectral density at z_0 predicted by the probabilistic Weyl's law in Thm 3.2.4, see also (4.7.1). In the case of an operator P_h satisfying the symmetry assumption (SYM), this microscopic density is equal to the one obtained for the operator perturbed by a random potential: for such symmetric symbols, the microscopic densities $d_{z_0}^{1,V}$ and $d_{z_0}^{1,M}$ coincide, and therefore cannot distinguish between the type of perturbation imposed on P_h .

On the other hand, we believe that for $k > 1$, the k -point densities $d_{z_0}^{k,Q}$ (equivalently, the k -point correlation functions $K_{z_0}^{k,Q}$) can distinguish between the two types of perturbation (still assuming the symmetry (SYM) for the symbol). We have not been able to compute in closed form the densities of the limiting k -point measures $d_{z_0}^{k,M}$ associated with the random function \tilde{G}_{z_0} ; however, the numerical experiments presented in Figure 4.4, as well as the Proposition 4.7.3, lead us to the following

Conjecture 4.8.3. *The k -point densities $d_{z_0}^{k,M}$ of the zero point process of the random function \tilde{G}_{z_0} described in Theorem 4.8.1 exhibit a quadratic repulsion at short distance. Namely, for any open set $O \Subset \mathbb{C}$, there exists a constant $C > 1$ depending only on O and k such that, for all pairwise distinct points $w_1, \dots, w_k \in O$,*

$$C^{-1} \prod_{i < j} |w_i - w_j|^2 \leq d_{z_0}^{k,M}(w_1, \dots, w_k) \leq C \prod_{i < j} |w_i - w_j|^2.$$

In Figure 4.4 we compare numerical values of the 2-point correlation function with the 2-point correlation function of a well-known spectral point process on \mathbb{C} , namely the spectrum of large Ginibre random matrices. This ensemble corresponds to random matrices M_ω alone, when the entries are i.i.d. Gaussian $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$, in the limit $h \rightarrow 0$, or equivalently the limit of large matrices. It has been known since the work of Ginibre [87] that the eigenvalues of these matrices repel each other quadratically at short distance. When the eigenvalues are rescaled such that the mean local density is given by $d^1(w) = \sigma/\pi$, the 2-point correlation function takes the simple form

$$K_\sigma^{2,Gin}(w_1, w_2) = 1 - \exp(-\sigma|w_1 - w_2|^2). \quad (4.8.4)$$

Hence, in view of our local density (4.8.3), we draw on Fig. 4.4 the 2-point function

$$K_{z_0}^{2,Gin}(w_1, w_2) = 1 - \exp \left[-\frac{1}{2} \left(\sum_{i=1}^J \sigma_+^i(z_0) + \sigma_-^i(z_0) \right) |w_1 - w_2|^2 \right]. \quad (4.8.5)$$

This function is markedly different from the scaling function $\kappa(t^2)$ corresponding to the zero GAF process (4.7.5). It seems rather close to our experimental data of Fig. 4.4, even though we observe a deviation for values $|w_1 - w_2| \sim 1$. Is this deviation due to the finite value of h used in our numerical experiment? Or does the deviation persist when $h \rightarrow 0$, that is in the limiting correlation function $K_{z_0}^{2,M}$? We conjecture that the latter correlation function $K_{z_0}^{2,M}$ differs from the (appropriately rescaled) Ginibre function $K_{z_0}^{2,Gin}$, but that it becomes closer and closer to it when the number of quasimodes J increases (a property which purely concerns the classical symbol p_0). Indeed, when $J \gg 1$ the function \tilde{G}_{z_0} is the determinant of a large matrix of independent GAFs.

In any case, computing the k -point densities for the process $\mathcal{Z}_{G_{z_0}}$ seems to us to be an interesting open problem.

We also mention the recent universality result by A. Maltsev and M. Osman [138] who proved universality of the local Ginibre statistics for complex random matrices with iid entries with finite moments and independent real and imaginary parts. Compared to our result the situation is however quite different as the randomness is not small. In our setting the randomness is effectively small compared to the unperturbed operator which, at least for finite J , can produce a deviation from the universal Ginibre statistics, see Theorem 4.8.1.

Translation invariance

One easy property of the limiting point processes obtained in Theorems 4.7.2 and 4.8.1, is that they are homogeneous and isotropic. This property is naturally inherited from the translation invariance of the zero process of individual GAFs, as mentioned in section 4.2.1.

Proposition 4.8.4. *The limiting point processes $\mathcal{Z}_{G_{z_0}}$, resp. $\mathcal{Z}_{\tilde{G}_{z_0}}$ obtained in Theorems 4.7.2 and 4.8.1 are invariant in distribution under the action of the group of translations and rotations on \mathbb{C} . More precisely, for arbitrary $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = 1$ let us define the plane isometry $\tau(w) = \alpha w + \beta$, $w \in \mathbb{C}$.*

Then, the zero processes of G_{z_0} and \tilde{G}_{z_0} satisfy

$$\mathcal{Z}_{G_{z_0}} \stackrel{d}{=} \mathcal{Z}_{G_{z_0 \circ \tau}}, \quad \text{resp.} \quad \mathcal{Z}_{\tilde{G}_{z_0}} \stackrel{d}{=} \mathcal{Z}_{\tilde{G}_{z_0 \circ \tau}}.$$

4.9 Sketch and key ideas of the proof

The proof of the main results is built on two distinct parts. The first part uses linear algebra and semiclassical methods to reduce the eigenvalue problem of the infinite dimensional operator P_h , to a nonlinear spectral problem expressed in terms of a finite dimensional matrix (called the *effective Hamiltonian*), which depends non-linearly on the spectral parameter. This reduction will be applied to the randomly perturbed operators P_Q^δ as well. The reduction is based on the construction of quasimodes of the unperturbed operator. We use these quasimodes to construct a *well-posed Grushin problem* for the operators P_h or P_Q^δ , which leads to the effective Hamiltonian. The spectrum of the random operator P_Q^δ is now obtained as the zero locus of the determinant of the effective Hamiltonian; in the case of the randomly perturbed operator P_Q^δ , this determinant is a certain type of random analytic function.

In the second part of the argument, we analyze the statistical properties of this random analytic function. First, we rescale the spectral parameter near a given point z_0 , to the scale of the average spacing between eigenvalues. Then, we show that, for each type of perturbation, the determinant of the effective Hamiltonian (after some “change of gauge”) converges in distribution to the universal random analytic function stated in Thm 4.7.1, resp. Thm 4.8.1.

Let us now give some more details on the successive steps.

Part I - Reduction to an effective Hamiltonian

By (4.4.2) for all $z \in \Omega$ and for each point $\rho_\pm^j(z)$, $j = 1, \dots, J$, in the energy shell $p^{-1}(z)$, we can construct an h^∞ -quasimode e_\pm^j for the problem $P_h - z$ resp. $(P_h - z)^*$ as in (4.4.4, 4.4.5). These quasimodes are microlocalized on ρ_\pm^j , i.e. $\text{WF}_h(e_\pm^j) = \{\rho_\pm^j\}$. The quasimodes e_+^j (resp. e_-^j) essentially span the space of singular values of $(P_h - z)$ (resp. $(P_h - z)^*$) smaller than h/C . This property will be used later to extend the operator $P_h - z$ into a well-posed Grushin problem.

Almost holomorphic quasimodes For the quasimode construction, near each point $\rho_+ = (x_+, \xi_+) \in p^{-1}(z)$ we use the Malgrange preparation theorem to factorize our operator $P - z$, microlocally near ρ_+ , into a simple form $\tilde{P} = hD_x + g^+(x, z)$. The WKB-method, then allows to construct a state

$$e_+^{hol}(x, z; h) = a_+(x, z; h) e^{\frac{i}{h}\phi_+(x, z)}, \quad (4.9.1)$$

satisfying the quasimode equation

$$\|(P_h - z)e_+^{hol}\| = \mathcal{O}(h^\infty)\|e_+^{hol}\|. \quad (4.9.2)$$

For a fixed z this WKB construction is standard [63], but we also need to control how the quasimodes depend on the parameter z . In [95] the author treated the case where the symbol p (4.3.1) is analytic. Here we are only assuming the symbol to be smooth, and we take particular care to construct quasimodes depending on an *almost holomorphic* way of z , at least near the reference point z_0 where we study the spectrum. The almost holomorphicity near z_0 takes the form

$$e^{-\frac{1}{h}\Phi_+(z; h)} \|\partial_{\bar{z}} e_+^{hol}(z; h)\| = \mathcal{O}(h^{-1}|z - z_0|^\infty + h^\infty). \quad (4.9.3)$$

Because we will eventually focus on z in an $\mathcal{O}(h^{1/2})$ neighbourhood of z_0 , the right hand side will effectively be $\mathcal{O}(h^\infty)$.

This holomorphicity implies that the states $e_+^{hol}(z; h)$ are not L^2 normalized for all z . Indeed, we show that for z in a neighbourhood of z_0 , their norm takes the form

$$\|e_+^{hol}(z; h)\| = e^{\frac{1}{h}\Phi_+(z; h)}, \quad (4.9.4)$$

with a phase function

$$\Phi_+(z; h) = \Phi_{+,0}(z) + \mathcal{O}(h \log h), \quad \Phi_{+,0}(z) := -\text{Im} \phi_+(x_+(z), z).$$

The normalized quasimode for $(P_h - z)$, microlocalized on ρ_+ , as in (4.4.4), can then be defined as $e_+(z; h) = e^{-\frac{1}{h}\Phi_+(z; h)} e_+^{hol}(z; h)$.

Similarly, for the adjoint problem $(P_h - z)^*$, we construct WKB states $e_-^{hol}(z; h)$ which are almost anti-holomorphic w.r.t. z and their normalized version $e_-(z; h) = e^{-\frac{1}{h}\Phi_-(z; h)} e_-^{hol}(z; h)$.

Interaction between the quasimodes Next we analyse the overlaps (“interactions”) between nearby quasimodes. These interactions will be relevant when computing the covariance of the components of the effective Hamiltonian. Since two points $\rho_+^j(z)$, $\rho_+^k(z)$ ($j \neq k$) remain at finite distance, the corresponding quasimodes are essentially orthogonal to each other. On the other hand, we will need to control the interactions between the quasimodes microlocalized on $\rho_+^j(z)$ and $\rho_+^j(w)$ when the energies z, w are close to one another other (in practice, $|z - w| = \mathcal{O}(h^{1/2})$). Exploiting the almost (anti-)holomorphy of $e_\pm^{hol, j}$ to show that

$$(e_+^{j, hol}(z) | e_+^{j, hol}(w)) = e^{\frac{2}{h}\Psi_+^j(z, w; h)} + \text{small}, \quad \Psi_+^j(z, w; h) = \Psi_{+,0}^j(z, w) + \mathcal{O}(h \log h). \quad (4.9.5)$$

Here $\Psi_{+,0}^j(z, w)$ is a polarization of the phase function $\Phi_{+,0}(z)$, such as to be almost z -holomorphic and almost w -antiholomorphic near $\{(z_0, z_0)\}$. In particular it satisfies the second-order Taylor expansion

$$\Psi_{+,0}(z_0 + \zeta_1, z_0 + \zeta_2) = \sum_{|\alpha| \leq 2} (\partial_z^{\alpha_1} \partial_{\bar{z}}^{\alpha_2} \Phi_{+,0})(z_0) \frac{\zeta_1^{\alpha_1} \bar{\zeta}_2^{\alpha_2}}{\alpha!} + \mathcal{O}(|\zeta|^3). \quad (4.9.6)$$

Remark 4.9.1. *In the case of perturbation by a random potential δV_ω , we will also need to compute the interactions between the squared functions $(e_-^j(x, z))^2$, namely estimate scalar products of the form $((e_-^j(z))^2 | (e_+^j(w))^2$.*

Grushin Problem for the perturbed operator P_h^δ The next step in the proof is to use the quasimodes $e_\pm(z; h)$ to construct a well-posed Grushin problem $\mathcal{P}(z)$ for the operator $P_h - z$, see Section 2.6.

In our case we construct our Grushin problem using the normalized quasimodes: we take

$$\begin{aligned} R_+(z) : H(m) &\rightarrow \mathbb{C}^J & (R_+(z)u)_j &= (u|e_+^j(z)), \quad j = 1, \dots, J, \\ R_-(z) : \mathbb{C}^J &\rightarrow L^2, & R_-(z)u_- &= \sum_{j=1}^J u_-(j)e_-^j(z). \end{aligned}$$

The roles of $R_\pm(z)$ are quite transparent: $R_+(z)$ indeed maps the quasi-kernel of $(P - z)$ (the quasimodes $e_+^j(z)$) to the auxiliary space, while $R_-(z)$ constructs the quasi-cokernel of $(P - z)^*$. We thus obtain a well-posed Grushin problem $\mathcal{P}(z)$. Note that this construction was already performed in [95].

After restricting our random variables to discs of radius Ch^{-1} , the perturbations δQ of P_h are small in norm, i.e. $\|\delta Q\| \ll 1$. As a result, the Grushin problem is still well-posed if we replace P_h by the perturbed operator P_Q^δ . The eigenvalues of $P_Q^\delta - z$ are then given by the zeros of $\det(E_{-+}^\delta(z))$, where $E_{-+}^\delta(z)$ is the perturbed effective Hamiltonian. We compute this effective Hamiltonian:

$$\delta^{-J} \det(E_{-+}^\delta(z)) = (-1)^J \det \left[(Qe_+^j(z)|e_-^i(z))_{i,j \leq J} + \mathcal{O}(\delta h^{-5/2}) \right].$$

A crucial feature of this expression is the fact that the effective Hamiltonian is dominated by the random perturbation, in spite of the fact that the latter is of size $\propto \delta$, which is a small parameter. However, the unperturbed effective Hamiltonian $E_{-+}(z)$ is actually of size $\mathcal{O}(h^\infty)$, allowing a perturbation of size h^N to be comparatively large.

The quasimodes used in the definition of the effective Hamiltonian were normalized, hence $E_{-+}^\delta(z)$ is not holomorphic. Multiplying $\det(E_{-+}^\delta(z))$ by an appropriate nonvanishing function of z , we obtain a holomorphic function, of the following form:

$$G^\delta(z; h) = (1 + R_1) \det \left[h^{-1/2} (Qe_+^{j,hol}(z)|e_-^{i,hol}(z))_{i,j \leq J} + R_2 \right], \quad (4.9.7)$$

with R_1, R_2 some small (z -dependent) error terms. Not surprisingly, the normalized quasimodes have been replaced by their almost (anti-)holomorphic counterparts in the expression. The entries of the matrix on the right hand side are dominated by scalar products $(Qe_+^{j,hol}(z)|e_-^{i,hol}(z))$, which represent the *coupling* between the quasimodes through the perturbation operator Q . Remember that these quasimodes are microlocalized at different phase space points $\rho_+^j(z), \rho_-^i(z)$; hence, the coupling will be nonnegligible only if the perturbation operator Q is able to “transport mass” from one point to the other.

Remark 4.9.2. Here a major difference occurs between the two types of perturbations. The random operator M_ω (eq. (RM)) will typically be able to couple any pair of quasimodes $(e_+^{j,hol}, e_-^{i,hol})$, leading to a full $J \times J$ matrix. On the opposite, the multiplication by a random potential V_ω will not be able to couple quasimodes localized at different positions $x_+^j \neq x_-^i$. This is the reason why, in this case, we need to assume the symmetry property (SYM), which ensures that each quasimode e_+^j admits a dual quasimode e_-^j with $x_+^j = x_-^j$. The further assumption (HYP-x) ensures that no other quasimode will be localized at x_+^j : this property makes the matrix $(V_\omega e_+^{j,hol}(z)|e_-^{i,hol}(z))_{i,j \leq J}$ approximately diagonal. This diagonal structure, will lead to a limiting random determinant given by the product of J independent GAFs, each one corresponding to one of the diagonal entries. On the opposite, for a random matrix perturbation, the full matrix $(M_\omega e_+^{j,hol}(z)|e_-^{i,hol}(z))_{i,j \leq J}$ will lead to a full matrix of GAFs.

Part II - Convergence to Gaussian analytic functions

In the second part we study the point process consisting in the zeros of the random analytic function $G^\delta(z; h)$ of eq. (4.9.7). Performing the rescaling $z = z_w := qz_0 + h^{1/2}w$, with w contained in some bounded open set $O \Subset \mathbb{C}$, we are led to study the zeros of the rescaled random function:

$$F_h^\delta(w) \stackrel{\text{def}}{=} G^\delta(z_0 + h^{1/2}w; h) = (1 + R_1) \det \left[(f_{i,j}^{\delta,h}(w))_{i,j \leq J} + R_2 \right]. \quad (4.9.8)$$

The terms R_1 and R_2 are small, and they converge to 0 in probability sufficiently quickly. Hence the expression is dominated by $\det(f_{i,j}^{\delta,h}(w))_{i,j \leq J}$, where $f_{i,j}^{\delta,h}(w) = h^{-1/2}(Qe_+^{j,\text{hol}}(z_w)|e_-^{i,\text{hol}}(z_w))$.

A key observation [162, Proposition 2.3] concerning random holomorphic functions and the associated zero processes is that if a sequence of random holomorphic functions f_n converges in distribution to a random holomorphic function f (which is almost surely $\neq 0$), then the zero point processes of f_n converges in distribution to the zero point process of f .

Therefore, we need to show that the function F_h^δ (actually, after multiplication by appropriate “gauge” factors) converges in distribution to the random analytic function G_{z_0} in the case of Thm 4.7.1, resp. to \tilde{G}_{z_0} in the case of Thm 4.8.1. In this sketch we mostly describe the second case $Q = M_\omega$, and highlight the differences with the perturbation by a random potential.

Covariances To show the convergence of F_h^δ , we will need to show that the entries $f_{i,j}^{\delta,h}(w)$ converge in distribution to J^2 independent GAFs.

The assumptions (4.5.1) on the coefficients of the random matrix M_ω imply that at each point w the random variable $f_{i,j}^{\delta,h}(w)$ is centered. The second step is to compute the covariances

$$\mathbb{E}[f_{i,j}^{\delta,h}(v)\overline{f_{l,k}^{\delta,h}(w)}] = h^{-1}\mathbb{E}[(M_\omega e_+^{j,\text{hol}}(z_v)|e_-^{i,\text{hol}}(z_v))(e_-^{k,\text{hol}}(z_w)|M_\omega e_+^{l,\text{hol}}(z_w))].$$

Expanding the random operator M_ω in the orthonormal family $(e_m)_{m < N(h)}$ leads to the following expression:

$$\mathbb{E}[f_{i,j}^{\delta,h}(v)\overline{f_{l,k}^{\delta,h}(w)}] = h^{-1}(e_+^{j,\text{hol}}(z_v)|\Pi_{N(h)}e_+^{k,\text{hol}}(z_w))(e_-^{l,\text{hol}}(z_w)|\Pi_{N(h)}e_-^{i,\text{hol}}(z_v)), \quad (4.9.9)$$

where $\Pi_{N(h)}$ is the orthogonal projector on the space spanned by $(e_m)_{m < N(h)}$. From our assumption on this orthonormal basis, this projector is equivalent with the identity microlocally near Ω ; since all our quasimodes are microlocalized inside Ω , the projectors $\Pi_{N(h)}$ may be removed from the scalar products, up to negligible errors. The covariance is hence expressed in terms of the interactions between neighbouring quasimodes.

From our analysis of these interactions (4.9.5), (4.9.6), we deduce the following expressions for the covariances:

$$\begin{aligned} \mathbb{E}[f_{i,j}^{\delta,h}(v)\overline{f_{l,k}^{\delta,h}(w)}] &\approx h^{-1}(e_+^{j,\text{hol}}(z_v)|e_+^{k,\text{hol}}(z_w))(e_-^{l,\text{hol}}(z_w)|e_-^{i,\text{hol}}(z_v)) \\ &\approx \delta_{i,l}\delta_{j,k}e^{2(\partial_{z\bar{z}}^2\Phi_{+,0}^j(z_0)+\partial_{z\bar{z}}^2\Phi_{-,0}^i(z_0))v\bar{w}}e^{F_{i,j}(v)+\overline{F_{i,j}(w)}} \\ &= \delta_{i,l}\delta_{j,k}e^{\frac{1}{2}(\sigma_+^j(z_0)+\sigma_-^i(z_0))v\bar{w}}e^{F_{i,j}(v)+\overline{F_{i,j}(w)}} \\ &=: K^{i,j}(v, \bar{w})e^{F_{i,j}(v)+\overline{F_{i,j}(w)}}. \end{aligned} \quad (4.9.10)$$

The Kronecker factors already hint at the fact that the random functions $f_{i,j}^{\delta,h}$ and $f_{l,k}^{\delta,h}$ are statistically independent if $(i, j) \neq (l, k)$. To obtain the second line, we have expanded the phase function $\Psi_{\pm,0}$ describing the interaction kept up to second order (see the Taylor expansion (4.9.6)), and have separated the mixed term $v\bar{w}$ from the separated terms $|v|^2, |w|^2$ which we grouped in the functions $F_{i,j}(\bullet)$.

To obtain the third line of (4.9.10) we used the relation between the phase function $\Phi_{\pm,0}^j$ describing the L^2 norm of the quasimode, and the local classical density σ_{\pm}^j , see (4.7.1). A direct computation shows that

$$\partial_{z\bar{z}}^2\Phi_{\pm,0}^j(z_0) = \frac{1}{4}\sigma_{\pm}^j(z_0).$$

In the mixed term in (4.9.10) we recognize the covariance of the GAF $g_{\sigma^{ij}}$ with parameter $\sigma^{ij} = \frac{1}{2}(\sigma_+^j + \sigma_-^i)$ (see (4.2.5)). Hence, the whole expression corresponds to the covariance of the modified GAF

$$f_{i,j}^{GAF} \stackrel{\text{def}}{=} e^{F_{i,j}} g_{\sigma^{ij}}.$$

Remark 4.9.3. *In the case of a random potential, we perform a similar computation. In that case the covariance will involve scalar products of the type $((e_-^{j,\text{hol}}(z_v))^2 | (e_-^{k,\text{hol}}(z_w))^2)$, which will be nonnegligible only if $j = k$; we also recover the covariance of a GAF multiplied by a gauge factor.*

Convergence to a Gaussian functions Computing the covariances is not sufficient to prove the convergence in distribution of the random functions $f_{i,j}^{\delta,h}$ towards the modified GAFs $f_{i,j}^{GAF}$. The entries of M_ω are in general not Gaussian, so neither are the functions $f_{i,j}^{\delta,h}$. Prohorov's Theorem shows that to prove the convergence of random functions, it is enough to prove the convergence in the sense of finite dimensional distributions: namely, for any $n \in \mathbb{N}^*$ and any set of points $(w_1, \dots, w_n) \in O^n$, we need to show that the random complex vector

$$(f_{i,j}^{\delta,h}(w_1), \dots, f_{i,j}^{\delta,h}(w_n)) \text{ converges in distribution to } (f_{i,j}^{GAF}(w_1), \dots, f_{i,j}^{GAF}(w_n)).$$

This type of convergence will be denoted by $f_{i,j}^{\delta,h} \xrightarrow{fd} f_{i,j}^{GAF}$. We actually need to show that the various functions $f_{i,j}^{\delta,h}$ are asymptotically independent from one another, and converge towards independent GAFs $f_{i,j}^{GAF}$.

The random operator M_ω is in general not Gaussian, so to obtain convergences to Gaussian vectors, we need to apply a suitable version of the central limit theorem. The application of the CLT resides on the fact that each quasimode e_\pm^j has nonnegligible overlaps with many of the basis states e_m used to construct M_ω . Thanks to this property, the higher moments of the $f_{i,j}^{\delta,h}$ will involve sums over many i.i.d. random variables α , and hence to a Gaussian law.

Taking into account the small error terms and applying Prohorov's theorem, this leads to the convergence in distribution of the full determinant

$$F_h^\delta(\bullet) \xrightarrow{d} \det((f_{i,j}^{GAF}(\bullet))_{1 \leq i,j \leq J}) \quad \text{when } h \rightarrow 0.$$

Finally, we use the fact that the ‘‘gauge’’ functions split into $F_{i,j}(v) = \phi_-^i(v) + \phi_+^j(v)$. This splitting allows to extract the gauge factors $e^{F_{i,j}}$ from the random matrix as follows:

$$(f_{i,j}^{GAF}(v))_{i,j} = \text{diag}(e^{\phi_-^i(v)}) (g_{\sigma^{ij}}(v))_{i,j} \text{diag}(e^{\phi_+^j(v)}).$$

The determinant of the diagonal matrices never vanishes, so the zero process is that of the determinant of matrix of GAFs $g_{\sigma^{ij}}$, as in Thm 4.8.1.

Remark 4.9.4. *In the case of a random potential, the major difference lies in the fact that the off-diagonal entries in (4.9.8) are negligible; this leads to a product of J independent functions, each one converging to a GAF.*

k -correlation functions The convergence of the point processes implies the convergence of the k -point correlation measures. There remains to compute the latter, as given in Theorem 4.7.2, Corollary 4.8.2 and Proposition 4.8.4. In the case of a perturbation by a random matrix type operator, we can only compute the 1-point function, whereas in the case of a random potential we obtain explicit formulas for the limiting k -point correlation functions.

Chapter 5

Weyl law for exponentially small singular values

In this section we discuss the results obtained in [176].

5.1 Introduction

In this section we study the small singular values of a non-selfadjoint semiclassical differential operator P_h , of Fredholm index 0, on X being either \mathbb{R}^d or a d -dimensional smooth compact manifold, with semiclassical parameter $h > 0$ and symbol p . The motivation behind this question is that if we wish to extend the result of Chapter 4 and Chapter 7 to the case of pseudo-differential operators in dimension $d > 1$, then we need a precise control on the number of small singular values. The standard Weyl law for self-adjoint operators would provide us with information in intervals of the form $[0, Ch]$ at best. The aim of this section is to gain information in much smaller intervals near 0. We believe that such a result is as well of independent interest.

A natural sufficient condition for the non-selfadjointness of P_h is that $\{p, \bar{p}\} \neq 0$ on T^*X – here $\{\cdot, \cdot\}$ denotes the Poisson bracket – since this is the principal symbol of $\frac{i}{h}[P_h, P_h^*]$. It was shown in the string of works [57–59, 62, 202] (reviewed in Section 2.3) that if there is a point $\rho_+ \in T^*X \cap p^{-1}(z)$, $z \in \mathbb{C}$, such that $\frac{1}{i}\{p, \bar{p}\}(\rho_+) > 0$ then there exists a smooth function e_+ on X , microlocalized to ρ_+ , such that

$$(P - z)e_+ = O(h^\infty). \quad (5.1.1)$$

The existence of such a quasimode implies the existence of a small singular value of $(P - z)$ of size $O(h^\infty)$. Moreover, when p is analytic, then we may replace the above error estimates by $\mathcal{O}(e^{-1/Ch})$.

On the other hand, near regions where $\frac{1}{i}\{p, \bar{p}\} < 0$ on $p^{-1}(z)$ microlocally $|P - z| \geq h/C$, where we set $|P - z| := ((P - z)^*(P - z))^{1/2}$. This indicates that the eigenfunctions of $|P - z|$, i.e. the singular vectors of $P - z$ should be microlocalized near the regions in $p^{-1}(z)$ where $\frac{1}{i}\{p, \bar{p}\} > 0$. Similar observations, with a factor of (-1) in front of the Poisson bracket, hold for the singular vectors of $(P - z)^*$.

In dimension $d = 1$, when $p^{-1}(z)$ is compact and $\{p, \bar{p}\} \neq 0$ on $p^{-1}(z)$, we can show that there are $\#p^{-1}(z) =: n_0$ many singular values of $(P - z)$ and $(P - z)^*$ which are of size $\mathcal{O}(h^\infty)$ (or $\mathcal{O}(e^{-1/Ch})$ when p is analytic), see for instance [62]. Here $\#A$ denotes the number of elements in a set A . However, the $n_0 + 1$ -st singular value will typically be of size \sqrt{h} .

In higher dimensions the situation is more complicated. We will focus on the situation where $\{p, \bar{p}\} \neq 0$ and separated by regions in $p^{-1}(z)$ where $\{p, \bar{p}\} = 0$ but $\{p, \{p, \bar{p}\}\} \neq 0$. For related results in more degenerate cases, see [148–150].

The aim of this chapter is to study the number of small singular values for certain 2-dimensional model operators P_h with the symbols p such that $\{p, \bar{p}\} \neq 0$ or $\{p, \{p, \bar{p}\}\} \neq 0$. We refer the reader also to [12], where a related problem for the Dirichlet-Pauli operator was studied.

The model. We will work on the complex 1-dimensional manifold $X + iY$, where $X = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and Y is equal to \mathbb{R} or S^1 . We frequently identify $X + iY$ with the real 2 dimensional manifold $X \times Y$, by writing $z = x + iy$, $x \in X$, $y \in Y$.

Let $h \in]0, h_0]$, $h_0 > 0$, and let $\phi(z)$ be a real-valued smooth function on $X + iY$ and consider the operator

$$P = 2e^{-\phi/h} \circ hD_{\bar{z}} \circ e^{\phi/h} = 2(hD_{\bar{z}} + (D_{\bar{z}}\phi)) = hD_x + h\partial_y + \partial_y\phi, \quad (5.1.2)$$

where $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ and $D_{\bar{z}} = \frac{1}{i}\partial_{\bar{z}} = \frac{1}{2}(D_x + iD_y)$. With a slight abuse of notation we write $\phi(z) = \phi(y)$, for $z = x + iy \in X + iY$, and we choose the following two model cases

$$\phi(y) = \begin{cases} \frac{1}{3}y^3, & \text{when } Y = \mathbb{R}, \\ \sin y, & \text{when } Y = S^1. \end{cases} \quad (5.1.3)$$

We equip P with the domain

$$\mathcal{D}(P) = \begin{cases} \{u \in L^2(S_x^1 \times Y); (1 + y^2)u, hD_x u, hD_y u \in L^2(S_x^1 \times Y)\}, & Y = \mathbb{R}_y, \\ H_h^1(S_x^1 \times Y) = \{u \in L^2(S_x^1 \times Y); hD_x u, hD_y u \in L^2(S_x^1 \times Y)\}, & Y = S_y^1. \end{cases} \quad (5.1.4)$$

The operator P has the symbol

$$p(x, y; \xi, \eta) := \xi + i\eta + \partial_y\phi =: p_\xi(y, \eta), \quad (5.1.5)$$

defined on $S_x^1 \times Y \times \mathbb{R}_\xi \times \mathbb{R}_\eta$. The characteristic set of P is given by

$$p^{-1}(0) = \{(x, y; \xi, \eta) \in S^1 \times Y \times \mathbb{R}_\xi \times \mathbb{R}_\eta; \eta = 0, \partial_y\phi = -\xi\}. \quad (5.1.6)$$

When $Y = \mathbb{R}$ we have $\xi \leq 0$ on $p^{-1}(0)$ and when $Y = S^1$ we have $\xi \in [-1, 1]$. We exclude the limiting values $\xi = 0$ and $\xi \in \{-1, 1\}$ respectively and put

$$\Sigma = \{(x, y; \xi, \eta) \in p^{-1}(0); \xi < 0 \text{ when } Y = \mathbb{R}, -1 < \xi < 1 \text{ when } Y = S^1\}. \quad (5.1.7)$$

We recall that that the Hamilton vector field of a C^1 function $f(x, y, \xi, \eta)$ is given by

$$H_f = \partial_\xi f \partial_x + \partial_\eta f \partial_y - \partial_x f \partial_\xi - \partial_y f \partial_\eta.$$

Then we get

$$\frac{1}{2i}\{p, \bar{p}\} = \frac{1}{2i}H_p(\bar{p}) = \frac{1}{2i}(\partial_\xi p \partial_x \bar{p} + \partial_\eta p \partial_y \bar{p} - \partial_x p \partial_\xi \bar{p} - \partial_y p \partial_\eta \bar{p}) = \partial_y^2 \phi \quad (5.1.8)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. We split $\Sigma = \Sigma_+ \cup \Sigma_-$, where

$$\Sigma_\pm = \{(x, y; \xi, \eta) \in p^{-1}(0); y = y_\pm(\xi), \xi \in]-\infty, 0[\text{ when } Y = \mathbb{R}, \\ \xi \in]-1, 1[\text{ when } Y = S^1\}, \quad (5.1.9)$$

where we recall from (5.1.6) that $\eta = 0$ on $p^{-1}(0)$ and $y_\pm(\xi)$ are the solutions to $\partial_y\phi(y_\pm(\xi)) = -\xi$ with $\pm\partial_y^2\phi(y_\pm(\xi)) > 0$. Observe that $\frac{1}{2i}\{p_\xi, \bar{p}_\xi\} = \partial_y^2\phi$ and that $\{p, \bar{p}\}(x, y; \xi, \eta) = \{p_\xi, \bar{p}_\xi\}(y, \eta)$, so

$$\Sigma_\pm = \{(x, y; \xi, \eta) \in p^{-1}(0); \pm\frac{1}{i}\{p, \bar{p}\}(x, y; \xi, \eta) > 0\}. \quad (5.1.10)$$

The submanifolds Σ_\pm are symplectic. Indeed let $\sigma = d\xi \wedge dx + d\eta \wedge dy$ be the symplectic form on $T^*(X \times Y)$. Then,

$$\sigma|_{\Sigma_\pm} \simeq d\xi \wedge dx, \quad (5.1.11)$$

since $\eta = 0$ on Σ_\pm .

5.2 Reduction to a one-dimensional model

We are interested in the singular values of P which are $\ll h^{2/3}$. We recall that the singular spectrum of P is defined as the square root of the spectrum of P^*P which is a positive essentially self-adjoint operator and we equip it with its natural domain. Indeed, when $Y = S^1$, then by the Rellich theorem we know that the inclusion $H_h^2(S^1 \times S^1) \hookrightarrow L^2(S^1 \times S^1)$ is compact which implies that P^*P has a compact resolvent and so the spectrum of P^*P is purely discrete and $\subset [0, \infty[$. When $Y = \mathbb{R}$, we may perform a partial Fourier transformation in x to see that the resulting operator is a Schrödinger operator with confining potential. So, also in this case the spectrum of P^*P being contained in $[0, \infty[$, is discrete away from 0. The *singular values* of P are then defined as the square roots of the eigenvalues of P^*P .

We define the partial Fourier transform of a function $u(x, y)$ on $X \times Y$ by

$$\mathcal{F}u(\xi, y) = \widehat{u}(\xi, y) = \int_X e^{-ix\xi/h} u(x, y) dx, \quad \xi \in \widehat{X} = h\mathbb{Z}.$$

Then $\mathcal{F} : L^2(X \times Y) \rightarrow L^2(\widehat{X} \times Y)$ is unitary when X and Y are equipped with the Lebesgue measure, and $\widehat{X} = h\mathbb{Z}$ is equipped with the $1/(2\pi)$ times the counting measure on $h\mathbb{Z}$.

After applying \mathcal{F} , the equation $Pu = v$ becomes

$$P_\xi \widehat{u}(\xi, y) = \widehat{v}(\xi, y), \quad \text{where } P_\xi = h\partial_y + \xi + \partial_y\phi, \quad \xi \in \widehat{X}, \quad y \in Y, \quad h \in]0, h_0]. \quad (5.2.1)$$

Correspondingly P has the orthogonal direct sum decomposition

$$P = \bigoplus_{\xi \in h\mathbb{Z}} P_\xi, \quad P_\xi = \xi + h\partial_y + \partial_y\phi \text{ on } Y, \quad h \in]0, h_0]. \quad (5.2.2)$$

We equip the operator P_ξ with its natural domain given by the semiclassical Sobolev space

$$\mathcal{D}(P_\xi) = \{u \in L^2(\mathbb{R}); (1 + y^4)^{1/2}u, hD_y u \in L^2(\mathbb{R})\} \quad (5.2.3)$$

when $Y = \mathbb{R}$, and

$$\mathcal{D}(P_\xi) = H_h^1(S^1) = \{u \in L^2(S^1); hD_y u \in L^2(S^1)\},$$

when $Y = S^1$. In fact P_ξ is an elliptic pseudo-differential operator in its natural class and it will follow that $P_\xi : \mathcal{D}(P_\xi) \rightarrow L^2(Y)$ is a Fredholm operator of index 0, (can also be verified with direct ODE arguments). This extends to the adjoint

$$P_\xi^* = -h\partial_y + \xi + \partial_y\phi, \quad (5.2.4)$$

which has the same domain as P_ξ . The principal symbol p_ξ of P_ξ is given by

$$p_\xi(y, \eta) = i\eta + \xi + \partial_y\phi(y), \quad (y, \eta) \in T^*Y, \quad (5.2.5)$$

see also (5.1.5), and the one of P_ξ^* is equal to \bar{p}_ξ . Both operators are of principal type in the sense that $dp_\xi \neq 0$ and $d\bar{p}_\xi \neq 0$. The common characteristic set $p_\xi^{-1}(0)$ of P_ξ and P_ξ^* in (5.2.1), (5.2.4) is given by

$$\eta = 0, \quad \xi + \partial_y\phi = 0. \quad (5.2.6)$$

This is illustrated in Figure 5.1 below.

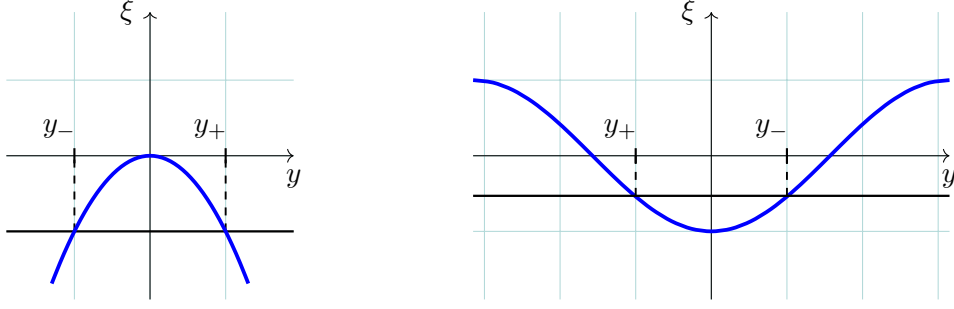


Figure 5.1: The characteristic set $p_\xi^{-1}(0) = \{y_\pm(\xi), 0\}$ depicted at $\eta = 0$: On the left hand side $\phi(y) = y^3/3$, $y \in \mathbb{R}$, and on the right hand side $\phi(y) = \sin y$, $y \in S^1 \simeq [-\pi, \pi[$, see also (5.1.3).

5.3 Weyl asymptotics via tunneling estimates

We are interested in the small singular values of P_ξ and P_ξ^* , see (5.2.1). More precisely we shall study the bottom of the spectrum of the semiclassical differential operators $P_\xi^*P_\xi$ and $P_\xi P_\xi^*$ which are elliptic away from the set (5.2.6). Both operators are essentially selfadjoint with domain

$$\begin{aligned} \mathcal{D}(P_\xi P_\xi^*) &= \mathcal{D}(P_\xi^* P_\xi) \\ &= \begin{cases} \{u \in L^2(Y); hD_y u, (hD_y)^2 u, (1+y^2)^2 u \in L^2(Y)\}, & \text{when } Y = \mathbb{R}; \\ H^2(S^1) = \{u \in L^2(S^1); hD_y u, (hD_y)^2 u \in L^2(S^1)\}, & \text{when } Y = S^1. \end{cases} \end{aligned} \quad (5.3.1)$$

Notice that $P_\xi^*P_\xi$ and $P_\xi P_\xi^*$ have purely discrete spectrum $\subset [0, +\infty[$. Indeed, when $Y = S^1$, then by selfadjointness the operator $(P_\xi^*P_\xi - z)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, is bounded $L^2(S^1) \rightarrow H^2(S^1)$. By Rellich's theorem the inclusion $H^2(S^1) \hookrightarrow L^2(S^1)$ is compact, so $P_\xi^*P_\xi$ has compact resolvent $L^2(S^1) \rightarrow L^2(S^1)$, and therefore purely discrete spectrum. When $Y = \mathbb{R}$, then $P_\xi^*P_\xi = -h^2\partial_y^2 + |\xi + \partial_y\phi|^2 - h\partial_y^2\phi$ is a semiclassical Schrödinger operator on $L^2(\mathbb{R})$ with potential $|\xi + \partial_y\phi|^2 - h\partial_y^2\phi \rightarrow +\infty$ when $|y| \rightarrow +\infty$. Hence, the spectrum of $P_\xi^*P_\xi$ is purely discrete spectrum.

Moreover, the spectra $\sigma(P_\xi^*P_\xi)$ and $\sigma(P_\xi P_\xi^*)$ coincide away from 0. To see this we argue in the spirit of [89]. First notice that $\mathcal{N}(P_\xi^*P_\xi) = \mathcal{N}(P_\xi)$ and $\mathcal{N}(P_\xi P_\xi^*) = \mathcal{N}(P_\xi^*)$. Since P_ξ is a Fredholm operator of index 0, the dimensions of $\mathcal{N}(P_\xi)$ and $\mathcal{N}(P_\xi^*)$ coincide and are finite. Hence we know that $P_\xi^*P_\xi$ and $P_\xi P_\xi^*$ have the same number, say N_0 of 0 eigenvalues. Let $e_1, e_2, \dots \in \mathcal{D}(P_\xi^*P_\xi)$ be orthonormal basis of $L^2(Y)$ of eigenvectors of $P_\xi^*P_\xi$ and let $0 \leq t_1^2 \leq t_2^2 \leq \dots$ be the corresponding eigenvalues. For $j > N_0$, we have

$$P_\xi^*P_\xi e_j = t_j^2 e_j.$$

By elliptic regularity $e_j \in \mathcal{S}(\mathbb{R})$ when $Y = \mathbb{R}$ and $e_j \in C^\infty(S^1)$ when $Y = S^1$. So we can apply P_ξ to the above equation and deduce that $P_\xi e_j$ is an eigenvector of $P_\xi P_\xi^*$ with eigenvalue t_j^2 . So $\sigma(P_\xi^*P_\xi) \subset \sigma(P_\xi P_\xi^*)$. Applying the same argument to $P_\xi P_\xi^*$ shows that converse inclusion.

Summing up what we have shown so far, the spectra $\text{Spec}(P_\xi^*P_\xi)$ and $\text{Spec}(P_\xi P_\xi^*)$ coincide for all $\xi \in \mathbb{R}$. Denoting the eigenvalues of $P_\xi^*P_\xi$ and $P_\xi P_\xi^*$ by $t_j^2(P_\xi)$ and $t_j^2(P_\xi^*)$ respectively, arranged in increasing order with $t_j(P_\xi) \geq 0$, $t_j(P_\xi^*) \geq 0$. We will denote the *singular values* of P_ξ by $t_k(\xi)$ (or by $t_k(P_\xi)$ when needed), $k \in \mathbb{N}$, counting with their multiplicity and ordered in an increasing way, i.e.

$$0 \leq t_0(\xi) \leq t_1(\xi) \leq \dots \rightarrow +\infty. \quad (5.3.2)$$

The elliptic region Until further notice we restrict ξ such that P_ξ is elliptic or close to elliptic. More precisely, in view of Figure 5.1 and (5.2.6), we let we let $-C_0 h^{2/3} \leq \xi$ when $Y = \mathbb{R}$, and $1 - C_0 h^{2/3} \leq |\xi|$ when $Y = S^1$. Through resolvent estimates for P_ξ we obtain the following

Theorem 5.3.1. *We define ϕ as in (5.1.3) and $P_\xi = h\partial_y + \xi + \partial_y\phi$ as in (5.2.1).*

1. Let $Y = \mathbb{R}$. For every $C_0 > 0$, there exists a constant $C > 0$ such that if $-C_0 h^{2/3} \leq \xi$ and $h_0 > 0$ is small enough, then the smallest singular value $t_0(\xi)$ of P_ξ satisfies

$$t_0(P_\xi) \geq \frac{1}{C}(\xi_+ + h^{2/3}),$$

where $x_+ := \max\{x, 0\}$, $x \in \mathbb{R}$.

2. Let $Y = \mathbb{R}/2\pi\mathbb{Z}$. For every $C_0 > 0$, there exists a constant $C > 0$ such that if $1 - C_0 h^{2/3} \leq |\xi|$ and $h_0 > 0$ is small enough, then the smallest singular value $t_0(\xi)$ of P_ξ satisfies

$$t_0(P_\xi) \geq \frac{1}{C}((|\xi| - 1)_+ + h^{2/3}).$$

From a microlocal perspective, the above result is based on the fact that when $\xi > 0$ when $Y = \mathbb{R}$ or $|\xi| > 1$ when $Y = S^1$, the principal symbol of P_ξ is in modulus > 0 and the result follows from a suitable version of Gårding's inequality. The resulting estimates can be extended to the region $|\xi| \leq \mathcal{O}(h^{2/3})$ when $Y = \mathbb{R}$ and to $||\xi| - 1| \leq \mathcal{O}(h^{2/3})$ by conjugation with a suitable weight function.

Remark 5.3.2. When $\xi = 0$, we have that 0 is a point in the boundary of $\overline{p_\xi(T^*Y)}$ and $\{p, \bar{p}\}(0, 0) = 0$. The symbol p_ξ satisfies the following subellipticity condition

$$H_{\text{Im } p_\xi}^2 \text{Re } p_\xi(0, 0) = \partial_y^3 \phi(0) \neq 0$$

where $H_{\text{Im } p_\xi}$ denotes the Hamilton vector field induced by $\text{Im } p_\xi$. This, by [62, Theorem 1.4], reviewed in Section 2.3.3, and [170, Theorem 1.1] (see also the references therein), implies that

$$\|P_0^{-1}\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} = \mathcal{O}(h^{-2/3}). \quad (5.3.3)$$

Using a Neumann series argument (5.3.3) immediately implies that

$$\|P_\xi^{-1}\|_{L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})} = \mathcal{O}(h^{-2/3}), \quad \text{for } |\xi| \ll h^{2/3}. \quad (5.3.4)$$

For our model operators we will use more direct arguments.

The non-elliptic region Next, we study the “non-elliptic” region, that is to say that we study the region where $-\xi \gg h^{2/3}$ when $Y = \mathbb{R}$ and $h^{2/3} \ll 1 - |\xi|$ when $Y = S^1$. Linking $P_\xi^* P_\xi$ to a one-dimensional Witten Laplacian, we can employ the techniques from [101–104] to obtain the following

Theorem 5.3.3. We define ϕ as in (5.1.3) and $P_\xi = h\partial_y + \xi + \partial_y \phi$ as in (5.2.1) with $h_0 > 0$ small enough. For $\xi < 0$, when $Y = \mathbb{R}$, and $\xi \in]-1, 1[$ when $Y = S^1$, let $y_+, y_- \in Y$ be the two solutions of the equation $\partial_y \phi(y) = -\xi$, labelled so that $\pm \partial_y^2 \phi(y_\pm) > 0$. Let d denote the Lithner-Agmon distance on Y for the metric $(\xi + \partial_y \phi(y))^2 dy^2$ and define the action

$$S_0 : \mathcal{D}(S_0) := \begin{cases}]-\infty, 0[, & Y = \mathbb{R}, \\]-1, 0[, & Y = S^1 \end{cases} \rightarrow]0, +\infty[, \quad \xi \mapsto S_0(\xi) = d(y_+(\xi), y_-(\xi)).$$

Then, uniformly with respect to ξ varying in a compact h -independent subset of $] -\infty, 0[$ when $Y = \mathbb{R}$, and of $] -1, 1[$ when $Y = S^1$, the smallest singular value of P_ξ has the following asymptotic expansion

$$t_0(\xi) = h^{\frac{1}{2}} \left(\left(\frac{|\{p_\xi, \bar{p}_\xi\}(y_+, 0)|}{4\pi} \right)^{1/4} \left(\frac{|\{p_\xi, \bar{p}_\xi\}(y_-, 0)|}{4\pi} \right)^{1/4} + \mathcal{O}(h) \right) \\ \times \begin{cases} e^{-S_0/h}, & \text{when } Y = \mathbb{R} \\ \left| e^{-\frac{1}{h}d_J(y_+, y_- - 2\pi)} - e^{-\frac{1}{h}d_J(y_+, y_-)} \right|, & \text{when } Y = S^1, \end{cases}$$

for any $\varepsilon > 0$. In the latter case we identify y_+ , y_- with points in $] - \pi, 0[$ and $]0, \pi[$ respectively and let d_J denote the Lithner-Agmon distance on $J =]y_- - 2\pi, y_-[$.

Furthermore, for $C_0 > 0$ large enough and $h_0 > 0$ small enough, if $-C_0 h^{2/3} \geq \xi \geq -1/C_0$ or $-C_0 \geq \xi$ when $Y = \mathbb{R}$, or if $C_0 h^{2/3} \leq 1 \pm \xi \leq 1/C_0$ when $Y = S^1$, then

$$\begin{aligned} t_0(\xi) &= h^{\frac{1}{2}} \left(\left(\frac{|\{p_\xi, \bar{p}_\xi\}(y_+, 0)|}{4\pi} \right)^{1/4} \left(\frac{|\{p_\xi, \bar{p}_\xi\}(y_-, 0)|}{4\pi} \right)^{1/4} + \mathcal{O}(h|\xi|^{-5/4}) \right) e^{-S_0/h} \\ &= \frac{\sqrt{h}|\xi|^{1/4}}{\sqrt{\pi}} \left(1 + \mathcal{O}(h|\xi|^{-3/2}) \right) e^{-4|\xi|^{3/2}/3h}. \end{aligned}$$

We note here that the Lithner-Agmon distance d on Y for the metric $(\xi + \partial_y \phi(y))^2 dy^2$ satisfies

$$d(y_\pm, y) = \pm(f(y) - f(y_\pm)), \quad y \in \text{neigh}(y_\pm),$$

with $f(y, \xi) = y\xi + \phi(y)$.

On the other, we have the following control on the second smallest singular value of P_ξ .

Theorem 5.3.4. *Under the assumption of Theorem 5.3.1, when ξ varies in a compact h -independent subset of $] - \infty, 0[$ when $Y = \mathbb{R}$, and of $] - 1, 1[$ when $Y = S^1$, there exists a constant $C > 0$ such that for $h_0 > 0$ small enough $t_1(P_\xi) \geq h^{1/2}/C$ uniformly in ξ .*

For every $C_0 > 0$ the following holds: if $C_0 h^{2/3} \leq -\xi \leq 1/C_0$ or $-\xi \geq C_0$ when $Y = \mathbb{R}$, there exists a constant $C > 0$ such that for $h_0 > 0$ small enough $t_1(P_\xi) \geq h^{1/2}|\xi|^{1/4}/C$. If $C_0 h^{2/3} \leq 1 + \xi \leq 1/C_0$ or $C_0 h^{2/3} \leq 1 - \xi \leq 1/C_0$ when $Y = S^1$, there exists a constant $C > 0$ such that for $h_0 > 0$ small enough $t_1(P_\xi) \geq h^{1/2}(1 - |\xi|)^{1/4}/C$.

Notice that this result only treats the “non-elliptic” region, i.e. $-\xi \gg h^{2/3}$ when $Y = \mathbb{R}$ and $h^{2/3} \ll 1 - |\xi|$ when $Y = S^1$. However, in the remaining parameter range of ξ we already have a lower bound on the smallest singular value thanks to Theorem 5.3.1. Putting both results together, we deduce that the second smallest singular value of P_ξ satisfies $t_1(P_\xi) \geq h^{2/3}/C$ for all $\xi \in \mathbb{R}$.

From the orthogonal decomposition (5.2.2) we see that the singular values of P are of the form $t_k(\xi)$, $k \in \mathbb{N}$, $\xi \in h\mathbb{Z}$, where $t_k(\xi)$ denote the singular values of P_ξ . By Theorems 5.3.1, 5.3.3 and 5.3.4, the singular values of P in $[e^{-1/C_h}, h^{2/3}/C]$ when $Y = \mathbb{R}$ and in $[0, h^{2/3}/C]$ when $Y = S^1$, for $C > 0$ sufficiently large and $h_0 > 0$ small enough, are of the form $t_0(\xi)$ where

$$\xi \in h\mathbb{Z} \text{ satisfies } \begin{cases} C_0 h^{2/3} \leq -\xi \leq C_0, & Y = \mathbb{R}, \\ C_0 h^{2/3} \leq 1 - |\xi|, & Y = S^1, \end{cases}$$

for $C_0 > 0$ sufficiently large.

Recall (5.1.9), (5.1.10). The action S_0 defined in Theorem 5.3.3 can be seen as an application

$$S_0 : \Sigma_+ \rightarrow [0, +\infty[$$

Our problem is then to study the distribution of the values $t_0(\xi)$. Here the counting measure $\sum_{\xi \in h\mathbb{Z}} \delta_\xi$ appears naturally as well as its approximation $\frac{1}{h} d\xi$. Because of the translation invariance in x , we can use the measure $\frac{1}{2\pi h} dx \otimes d\xi$ on $S^1 \times \mathbb{R}_\xi$ and $dx \otimes d\xi$ can be identified with the symplectic volume element $\sigma|_{\Sigma_+}$ on Σ_+ , see (5.1.10), (5.1.11). Using the Theorems 5.3.4 and 5.3.3, we then can obtain the following Weyl-type asymptotics.

Theorem 5.3.5. *Let P be (5.1.2), let Σ_+ be as in (5.1.10) and recall (5.1.11). Let S_0 be as in Theorem 5.3.3. Let $C_0 > 0$ be large enough and let*

$$C_0 h \leq \frac{\delta^{3/2}}{\log \delta^{-1}}, \quad \delta > 0.$$

Then, for $0 < a < b$ with $b \asymp 1$ and $a \asymp \delta^{3/2}$,

$$\left| \# \left(\text{Spec}(\sqrt{P^*P}) \cap [e^{-b/h}, e^{-a/h}] \right) - \frac{1}{2\pi\hbar} \int_{S_0^{-1}([a,b])} d\sigma|_{\Sigma_+} \right| = \mathcal{O}(1) \frac{\log \delta^{-1}}{\sqrt{\delta}}.$$

5.3.1 Possible generalizations

It is a natural and interesting question whether we can extend Theorem 5.3.5 to more general two-dimensional (or even d -dimensional) operators satisfying similar Poisson bracket conditions. However, it will require different techniques, as the above considered example is quite special allowing not only for separation of variables but also for a link to one-dimensional Witten Laplacian which is fundamental in the proof of the above discussed results.

Part II

Non-selfadjoint disordered Toeplitz matrices

Chapter 6

Spectra of disordered NSA Toeplitz matrices

Spectra of disordered non-selfadjoint Toeplitz matrices

6.1 Introduction

The problems and results which we have discussed in Chapter 3 can also be studied in the context of randomly perturbed $N \times N$ Toeplitz matrices in the large dimensional limit $N \rightarrow \infty$ (playing the role of the semiclassical limit $h \rightarrow \infty$). Indeed, in Theorems 6.5.1, 6.5.2 and 6.5.4 we will show probabilistic Weyl asymptotics for such randomly perturbed Toeplitz matrices. To an extent we have already done this in Section 3.3. However, here we consider the somewhat easier case of genuine Toeplitz matrices, which will also allow for a lower regularity of the considered symbol.

The principal motivation for us to study such randomly perturbed Toeplitz matrices is that they present a good toy model for the more intricate case of randomly perturbed pseudo-differential operators. In this chapter we present the results from [172–175].

6.2 Toeplitz matrices

Let $a_\nu \in \mathbf{C}$, for $\nu \in \mathbf{Z}$ and assume that

$$|a_\nu| \leq \mathcal{O}(1)m(\nu), \quad (6.2.1)$$

where $m : \mathbf{Z} \rightarrow]0, +\infty[$ satisfies

$$(1 + |\nu|)m(\nu) \in \ell^1, \quad (6.2.2)$$

and

$$m(-\nu) = m(\nu), \quad \forall \nu \in \mathbf{Z}. \quad (6.2.3)$$

Let

$$p(\tau) = \sum_{-\infty}^{+\infty} a_\nu \tau^\nu, \quad (6.2.4)$$

act on complex valued functions on \mathbf{Z} . Here τ denotes translation by 1 unit to the right: $\tau u(j) = u(j - 1)$, $j \in \mathbf{Z}$. By (6.2.2) we know that $p(\tau) = \mathcal{O}(1) : \ell^2(\mathbf{Z}) \rightarrow \ell^2(\mathbf{Z})$. Indeed, for the corresponding operator norm, we have

$$\|p(\tau)\| \leq \sum |a_j| \|\tau^j\| = \|a\|_{\ell^1} \leq \mathcal{O}(1)\|m\|_{\ell^1}. \quad (6.2.5)$$

From the identity, $\tau(e^{ik\xi}) = e^{-i\xi}e^{ik\xi}$, we define the symbol of $p(\tau)$ by

$$p(e^{-i\xi}) = \sum_{-\infty}^{\infty} a_\nu e^{-i\nu\xi}. \quad (6.2.6)$$

It is an element of the Wiener algebra [39] and by (6.2.2) in $C^1(S^1)$.

Definition 6.2.1. Let $p(\tau)$ be as in (6.2.4). We call

$$P_N := 1_{[0,N]} p(\tau) 1_{[0,N]}, \quad (6.2.7)$$

acting on $\mathbb{C}^N \simeq \ell^2([0, N])$, for $1 \geq N < \infty$, a Toeplitz matrix.

In matrix form we have

$$P_N = \begin{pmatrix} a_0 & a_{-1} & \dots & a_{-N_-} & \dots \\ a_1 & a_0 & a_{-1} & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{N_+} & \dots & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & a_{N_+} & \dots & a_0 \end{pmatrix}. \quad (6.2.8)$$

We frequently identify $\ell^2([0, N])$ with the space $\ell^2_{[0,N]}(\mathbf{Z})$ of functions $u \in \ell^2(\mathbf{Z})$ with support in $[0, N]$.

6.3 Toeplitz band matrices

We will also consider the special case when only finitely many coefficients a_ν in (6.2.4) are non-zero. In this case we let $N_\pm \geq 0$ be in \mathbb{N} , such that either $N_+ \neq 0$ or $N_- \neq 0$, and we consider the operator

$$p(\tau) := \sum_{j=-N_-}^{N_+} a_j \tau^j, \quad a_{-N_-}, a_{-N_-+1}, \dots, a_{N_+} \in \mathbb{C}, \quad a_{\pm N_\pm} \neq 0, \quad (6.3.1)$$

with symbol

$$p(e^{-i\xi}) = \sum_{j=-N_-}^{N_+} a_j e^{-i\nu\xi}, \quad \text{or with } \zeta = e^{i\xi}, \quad \mathbb{C} \ni \zeta \mapsto p(1/\zeta) = \sum_{j=-N_-}^{N_+} a_j \zeta^{-j}. \quad (6.3.2)$$

The latter is a meromorphic function.

Definition 6.3.1. Let $p(\tau)$ be as in (6.3.1). We call

$$P_N := 1_{[0,N]} p(\tau) 1_{[0,N]}, \quad (6.3.3)$$

acting on $\mathbb{C}^N \simeq \ell^2([0, N])$, for $1 \geq N < \infty$, a Toeplitz band matrix.

In matrix form a banded Toeplitz matrix P_N looks like (6.2.8), however with all entries $a_\nu = 0$ whenever $\nu > N_+$ or $\nu < -N_-$.

6.4 Spectra and Pseudospectra of Toeplitz matrices

The study of the spectra of such Toeplitz matrices P_N (6.2.7) has a long and rich history, see [39] for an overview. Note that $p(\tau)$ (6.2.4) as an operator $\ell^2(\mathbf{Z}) \rightarrow \ell^2(\mathbf{Z})$ is a normal operator, sometimes called the *Laurent operator*, and by Fourier series expansions, we see that its spectrum is given by

$$\text{Spec}(p(\tau)) = p(S^1). \quad (6.4.1)$$

The restriction $P_N = p(\tau)|_{\ell^2(\mathbb{N})}$ of $p(\tau)$ to $\ell^2(\mathbb{N})$, usually called a *Toeplitz operator*, is in general no longer normal, except for specific choices of the coefficients a_ν . The essential spectrum of the

Toeplitz operator $P_{\mathbb{N}}$ is given by $p(S^1)$, and we have point-spectrum in all loops of $p(S^1)$ with non-zero winding number, see [39, Theorem 1.17], i.e.

$$\text{Spec}(P_{\mathbb{N}}) = p(S^1) \cup \{z \in \mathbb{C}; \text{ind}_{p(S^1)}(z) \neq 0\}. \quad (6.4.2)$$

By a result of Krein [39, Theorem 1.15] (or [175, Proposition 3.11] for a direct proof) the winding number of $p(S^1)$ around the point $z \notin p(S^1)$ is related to the Fredholm index of $P_{\mathbb{N}} - z$:

$$\text{Ind}(P_{\mathbb{N}} - z) = -\text{ind}_{p(S^1)}(z). \quad (6.4.3)$$

The spectrum of the Toeplitz matrix P_N is contained in a small neighborhood of the spectrum of $P_{\mathbb{N}}$. More precisely, for every $\epsilon > 0$,

$$\text{Spec}(P_N) \subset \text{Spec}(P_{\mathbb{N}}) + D(0, \epsilon) \quad (6.4.4)$$

for $N > 0$ sufficiently large, where $D(z, r)$ denotes the open disc of radius r , centered at z . Moreover, the limit of $\sigma(P_N)$ as $N \rightarrow \infty$ is contained in a union of analytic arcs inside $\sigma(P_{\mathbb{N}})$, see [39, Theorem 5.28]. This phenomenon can also be observed in the numerical simulations presented on the left hand side in Figure 6.1. In general, we know that asymptotically the ϵ -

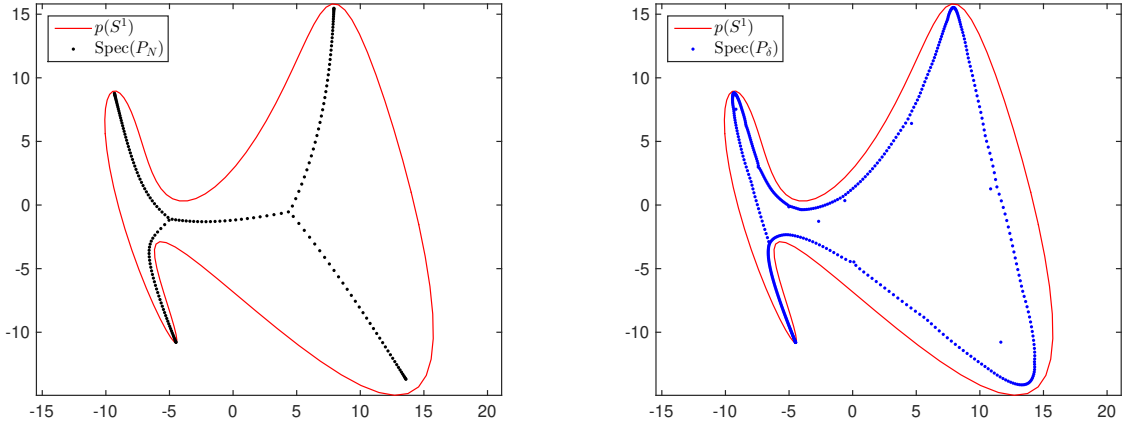


Figure 6.1: The pictures on the left hand side shows the spectrum of the Toeplitz matrix P_N given by the symbol $p(1/\zeta) = 2i\zeta^{-1} + \zeta^2 + \frac{7}{10}\zeta^3$ and the right hand side shows the spectrum of a random perturbation $P_N + \delta Q$, Q a $N \times N$ Ginibre matrix, with $N = 10^3$ and coupling constant $\delta = 10^{-5}$. The red line shows the symbol curve $p(S^1)$.

pseudospectrum of a Toeplitz matrix P_N is given by the ϵ -pseudospectrum of the corresponding Toeplitz operator $P_{\mathbb{N}}$.

Theorem 6.4.1 ([70, Theorem 7.3]). *Let $p(\tau)$ be as in (6.2.4), $P_{\mathbb{N}} = p(\tau)|_{\ell^2(\mathbb{N})}$ and P_N as in Definition 6.2.1. Then, for any $\epsilon > 0$ we have*

$$\lim_{N \rightarrow \infty} \text{Spec}_{\epsilon}(P_N) = \text{Spec}_{\epsilon}(P_{\mathbb{N}}),$$

where the limit is with respect to the Hausdorff metric.

Since pseudospectra are nested, see (2.2.4), Theorem 6.4.1 and (6.4.2) already suggest that every point $z \in \mathbb{C}$ enclosed by the curve $p(S^1)$ with non-zero winding is in the region of strong spectral instability. This is made concrete for banded Toeplitz matrices by the following

Theorem 6.4.2 ([15, 70, 175]). *Let $p(\tau)$ be as in (6.3.1) and P_N as in Definition 6.3.1. Then, for any $z \in \{z \in \mathbb{C}; \text{ind}_{p(S^1)}(z) \neq 0\}$ there exists a normalized $u_N \in \ell^2([0, N])$ such that*

$$\|(P_N - z)u_N\| = \mathcal{O}(N^{-\infty}).$$

In fact the error term can be shown to be of order $\mathcal{O}(e^{-N/C})$, for some $C > 0$, as proven for instance in [15, 175]. Indeed, as shown in these works and in view of (6.4.3) we have that

- if $\text{ind}_{p(S^1)}(z) < 0$, then the Fredholm index of $P_{\mathbb{N}} - z$ satisfies

$$\text{Ind}(P_{\mathbb{N}} - z) = \dim \ker(P_{\mathbb{N}} - z) = -\text{ind}_{p(S^1)}(z);$$

- if $\text{ind}_{p(S^1)}(z) > 0$,

$$\text{Ind}(P_{\mathbb{N}} - z) = -\dim \ker(P_{\mathbb{N}} - z)^* = -\text{ind}_{p(S^1)}(z).$$

Moreover, these kernels are spanned by exponentially decaying functions. In the first case, restricting such a function $u \in \ker(P_{\mathbb{N}} - z)$ to the interval $[0, N - 1]$ yields an approximate solution to the equation $(P_N - z)u = 0$. More precisely, setting $e_+ = \|\mathbf{1}_{[0, N-1]}u\|^{-1}\mathbf{1}_{[0, N-1]}u$, we get that

$$(P_N - z)e_+ = \mathcal{O}(e^{-N/C}).$$

Similarly, we get in the second case a $e_- \in \ell^2([0, N - 1])$, $\|e_-\| = 1$, with

$$(P_N - z)^*e_- = \mathcal{O}(e^{-N/C}).$$

In either case, it follows that $z \in \sigma(P_N)$ or $\|(P_N - z)^{-1}\| \geq Ce^{-N/C}$. So, these exponentially precise quasimodes show that any fixed z with $\text{ind}_{p(S^1)}(z) \neq 0$, is contained in the $e^{-N/C}/C$ -pseudospectrum of P_N .

6.5 Probabilistic Weyl law for noisy Toeplitz matrices

In this section we discuss the works obtained in [173–175, 194]. We show in Theorem 3.3.1 below that after adding a small random perturbation to P_N , most of its eigenvalues will be close to the curve $p(S^1)$ with probability very close to 1. See Figure 6.1 and 6.2 for a numerical illustration.

6.5.1 Small Gaussian perturbation

Consider the random matrix

$$Q_\omega := Q_\omega(N) := (q_{j,k}(\omega))_{1 \leq j, k \leq N} \quad (6.5.1)$$

with complex Gaussian law

$$(Q_\omega)_*(d\mathbb{P}) = \pi^{-N^2} e^{-\|Q\|_{\text{Hs}}^2} L(dQ),$$

where L denotes the Lebesgue measure on $\mathbb{C}^{N \times N}$. In other words the entries $q_{j,k}$ of Q_ω are independent copies of a complex Gaussian random variable $\alpha \sim \mathcal{N}_{\mathbb{C}}(0, 1)$ with expectation 0, and variance 1. We recall that the probability distribution of such an α is given by

$$\alpha_*(d\mathbb{P}) = \pi^{-1} e^{-|\alpha|^2} L(d\alpha),$$

where $L(d\alpha)$ denotes the Lebesgue measure on \mathbb{C} . We are interested in studying the spectrum of the random perturbations of the matrix $P_N^0 = P_N$:

$$P_N^\delta := P_N^0 + \delta Q_\omega, \quad 0 \leq \delta \ll 1. \quad (6.5.2)$$

6.5.2 Eigenvalue asymptotics in fixed smooth domains

Let $\Omega \Subset \mathbf{C}$ be an open simply connected set with smooth boundary $\partial\Omega$, which is independent of N , satisfying

1. $\partial\Omega$ intersects $p(S^1)$ in at most finitely many points;
2. $p(S^1)$ does not self-intersect at these points of intersection;
3. these points of intersection are non-critical, i.e.

$$dp \neq 0 \text{ on } p^{-1}(\partial\Omega \cap p(S^1));$$

4. $\partial\Omega$ and $p(S^1)$ are transversal at every point of the intersection.

The first result concerns the eigenvalue asymptotics of randomly perturbed *general* Toeplitz matrices.

Theorem 6.5.1. *Let p be as in (6.2.6) and let P_N^δ be as in (6.2.7) and (6.5.2). Let Ω be as above, satisfying conditions (1) - (4), pick a $\delta_0 \in]0, 1[$ and let $\delta_1 > 3$. If*

$$e^{-N^{\delta_0}} \leq \delta \ll N^{-\delta_1}, \quad (6.5.3)$$

then there exists $\varepsilon_N = o(1)$, as $N \rightarrow \infty$, such that

$$\left| \#(\sigma(P_N^\delta) \cap \Omega) - \frac{N}{2\pi} \int_{S^1 \cap p^{-1}(\Omega)} L_{S^1}(d\theta) \right| \leq \varepsilon_N N, \quad (6.5.4)$$

with probability

$$\geq 1 - e^{-N^{\delta_0}}. \quad (6.5.5)$$

In (6.5.4) we view p as a map from S^1 to \mathbf{C} . Theorem 3.3.1 shows that most eigenvalues of P_N^δ can be found close to the curve $p(S^1)$ with probability subexponentially close to 1. This is illustrated in Figure 6.2 for two different symbols. The left hand side of Figure 6.2 shows the

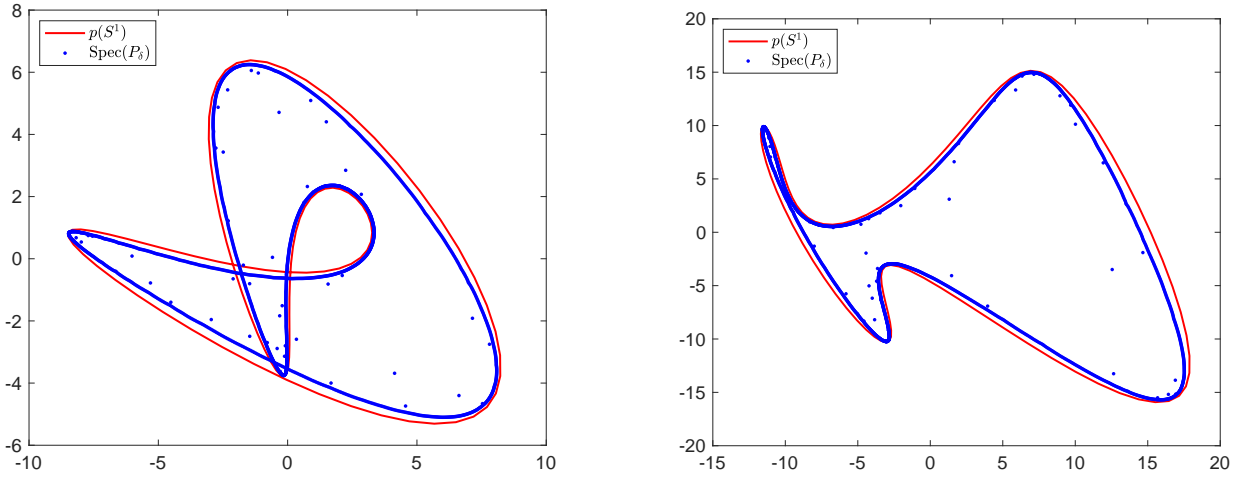


Figure 6.2: The left hand side shows the spectrum of the perturbed Toeplitz matrix with symbol defined in (6.5.6), (6.5.7) and the right hand side shows the spectrum of the perturbed Toeplitz matrix with symbol defined in (6.5.8), (6.5.7). The red line shows the symbol curve $p(S^1)$.

spectrum of a perturbed Toeplitz matrix with $N = 2000$ and $\delta = 10^{-14}$, given by the symbol $p = p_0 + p_1$ where

$$p_0(1/\zeta) = -\zeta^{-4} - (3 + 2i)\zeta^{-3} + i\zeta^{-2} + \zeta^{-1} + 10\zeta + (3 + i)\zeta^2 + 4\zeta^3 + i\zeta^4 \quad (6.5.6)$$

and

$$p_1(1/\zeta) = \sum_{\nu \in \mathbb{Z}} a_\nu \zeta^\nu, \quad a_0 = 0, \quad a_{-\nu} = 0.7|\nu|^{-5} + i|\nu|^{-9}, \quad a_\nu = -2i\nu^{-5} + 0.5\nu^{-9} \quad \nu \in \mathbb{N}. \quad (6.5.7)$$

The red line shows the curve $p(S^1)$. The right hand side of Figure 6.2 similarly shows the spectrum of the perturbed Toeplitz matrix given by $p = p_0 + p_1$ where p_1 is as above and

$$p_0(1/\zeta) = -4\zeta^1 - 2i\zeta^2 + 2i\zeta^{-1} - \zeta^{-2} + 2\zeta^{-3}. \quad (6.5.8)$$

When we restrict to banded Toeplitz matrices, we have a more precise remainder estimate

Theorem 6.5.2. *Let p be as in (6.3.1), set $M = N_+ + N_-$ and let P_N^δ be as in (6.3.3) and (6.5.2). Let Ω be as above, satisfying conditions (1) - (4) and pick a $\delta_0 \in]0, 1[$. There exists a constant $C > 0$, such that, for $N > 1$ sufficiently large, if*

$$C e^{-N^{\delta_0}/(2M)} \leq \delta \leq \frac{N^{-4}}{C},$$

then we have that

$$\left| \#(\text{Spec}(P_N^\delta) \cap \Omega) - \frac{N}{2\pi} \int_{p^{-1}(\Omega \cap p(S^1))} L_{S^1}(d\theta) \right| \leq \mathcal{O}(N^{\delta_0} \log N). \quad (6.5.9)$$

with probability

$$\geq 1 - \mathcal{O}(\log N) \left(e^{-N^2} + \delta^{-M} e^{-\frac{1}{2}N^{\delta_0}} \right). \quad (6.5.10)$$

6.5.3 Eigenvalue asymptotics in thin smooth domains

In Section 6.5.2 we saw that most eigenvalues of P_N^δ lie “near” the curve $p(S^1)$. Now we want to give a quantitative estimate on *how close* these eigenvalues are to the $p(S^1)$. For this purpose let $\Omega \Subset \mathbb{C}$ be an open simply connected set with smooth boundary $\partial\Omega$ which is independent of N and satisfies properties (1) - (4), as in Section 6.5.2.

Remark 6.5.3. *We recall the notion of domains with associated Lipschitz weight: Let $N \geq 1$ be a large parameter, and let $\Omega \Subset \mathbb{C}$ be an open simply connected set with Lipschitz boundary $\omega = \partial\Omega$ which may depend on N . More precisely, we assume that $\partial\Omega$ is Lipschitz with an associated Lipschitz weight $r : \omega \rightarrow]0, +\infty[$, which is a Lipschitz function of modulus $\leq 1/2$, in the following way :*

There exists a constant $C_0 > 0$ such that for every $x \in \omega$ there exist new affine coordinates $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$ of the form $\tilde{y} = U(y - x)$, $y \in \mathbb{C} \simeq \mathbb{R}^2$ being the old coordinates, where $U = U_x$ is orthogonal, such that the intersection of Ω and the rectangle $R_x := \{y \in \mathbb{C}; |\tilde{y}_1| < r(x), |\tilde{y}_2| < C_0 r(x)\}$ takes the form

$$\{y \in R_x; \tilde{y}_2 > f_x(\tilde{y}_1), |\tilde{y}_1| < r(x)\}, \quad (6.5.11)$$

where $f_x(\tilde{y}_1)$ is Lipschitz on $[-r(x), r(x)]$, with Lipschitz modulus $\leq C_0$.

Notice that (6.5.11) remains valid if we shrink the weight function r .

We consider an open simply connected N -dependent set Ω_N , with a uniformly Lipschitz boundary $\partial\Omega_N$, which coincides with Ω in small tube around $p(S^1)$. More precisely, let

$$\frac{C}{N} \leq \tau \leq \mathcal{O}(1), \quad (6.5.12)$$

and suppose that

$$\Omega_N \cap \{z \in \mathbb{C}; \text{dist}(z, p(S^1)) < \tau\} = \Omega \cap \{z \in \mathbb{C}; \text{dist}(z, p(S^1)) < \tau\}, \quad (6.5.13)$$

and that $\partial\Omega_N$ is uniformly Lipschitz with weight function

$$r(x) := \frac{1}{C} \left(\text{dist}(x, p(S^1)) + \frac{1}{N} \right), \quad x \in \partial\Omega_N \cap \{z \in \mathbb{C}; \text{dist}(z, p(S^1)) < \tau\}, \quad (6.5.14)$$

inside $\{z \in \mathbb{C}; \text{dist}(z, p(S^1)) < \tau\}$ and with constant weight function

$$r(x) := \tau, \quad x \in \partial\Omega_N \cap \{z \in \mathbb{C}; \text{dist}(z, p(S^1)) \geq \tau\} \quad (6.5.15)$$

outside. Let

$$0 < \ell(N) \quad (6.5.16)$$

be the length of $\partial\Omega_N \cap \{z \in \mathbb{C}; \text{dist}(z, p(S^1)) \geq \tau\}$.

Theorem 6.5.4. *Let p be as in (6.3.1), set $M = N_+ + N_-$ and let P_N^δ be as in (6.3.3) and (6.5.2). Let τ be as in (6.5.12) and let $\Omega_N \Subset \mathbb{C}$ be a relatively compact open simply connected set satisfying (6.5.13)–(6.5.16). Pick a $\delta_0 \in]0, 1[$.*

There exists a constant $C > 0$ such that for $N > 1$ sufficiently large, if

$$C e^{-N^{\delta_0}/(2M)} \leq \delta \leq \frac{N^{-4}}{C},$$

then,

$$\left| \#(\text{Spec}(P_N^\delta) \cap \Omega) - \frac{N}{2\pi} \int_{p^{-1}(\Omega \cap p(S^1))} L_{S^1}(d\theta) \right| \leq \mathcal{O}(N^{\delta_0} \ell(N) \tau^{-1} + N^{\delta_0} \log(\tau N)). \quad (6.5.17)$$

with probability

$$\geq 1 - \mathcal{O}(\ell(N) \tau^{-1} + \log(\tau N)) \left(e^{-N^2} + C_2 \delta^{-|J|} e^{-\frac{1}{2} N^{\delta_0}} \right). \quad (6.5.18)$$

Remark 6.5.5. *In the assumption 6.5.13 on Ω_N we assumed that it coincides with an Ω with smooth boundary, which is independent of N , inside a tube of radius τ around $p(S^1)$. Therefore, assumption 6.5.13 implies that $\ell(N) \geq 1/C > 0$. However, the proof of Theorems 6.5.2 and 6.5.4 shows that we can allow for Ω to be N dependent as long as its boundary $\partial\Omega$ remains uniformly Lipschitz in the sense discussed above, and satisfies conditions (1)–(4) at the beginning of Section 6.5.2. Hence, Theorem 6.5.4 holds as well for sets Ω_N , satisfying (6.5.12)–(6.5.15) with*

$$\frac{C}{N} \leq \ell(N). \quad (6.5.19)$$

The upper bound on the coupling constant δ in Theorems 6.5.1, 6.5.2 and 6.5.4 is somewhat artificial. We believe that following the same approach it can be extended to $\delta \leq N^{-\kappa}$, $\kappa > 1/2$, however, at the price of worse error and probability estimates. Furthermore, one can allow for much more general perturbations which only need satisfy Assumption (6.6.3) and (6.6.4) below. One such example of perturbations is given by random matrices whose entries are iid copies of a centred random variables with bounded fourth moment, similar to the perturbations considered in Section 3.3. However, we will not go into further details here.

6.6 Convergence of the empirical measure and related results

As already discussed in the context of Theorem 3.3.10, an alternative way to study the limiting distribution of the eigenvalues of P_N^δ , up to errors of $o(N)$, is to study the *empirical measure* of eigenvalues, defined by

$$\xi_N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{\lambda \in \text{Spec}(P_N^\delta)} \delta_\lambda \quad (6.6.1)$$

where the eigenvalues are counted including multiplicity and δ_λ denotes the Dirac measure at $\lambda \in \mathbb{C}$. We can show that, almost surely, ξ_N converges weakly to the push-forward of the uniform measure on S^1 by the symbol p .

Theorem 6.6.1. *Let $\delta_0 \in]0, 1[$, let $\delta_1 > 3$ and let p be as in (6.2.4). If (6.5.3) holds, i.e.*

$$e^{-N^{\delta_0}} \leq \delta \ll N^{-\delta_1}$$

then, almost surely,

$$\xi_N \rightharpoonup p_* \left(\frac{1}{2\pi} L_{S^1} \right), \quad N \rightarrow \infty, \quad (6.6.2)$$

weakly, where L_{S^1} denotes the Lebesgue measure on S^1 .

Related results and extensions Similar results to Theorem 6.6.1 have been proven in various settings. In [13, 14], the authors consider the special case of band Toeplitz matrices, i.e. P_N with p as in (6.3.1). They allow for perturbations by a quite general class of $N \times N$ random matrices Q_ω satisfying

1. **Norm bound** Assume that

$$\mathbb{E}[\|Q_\omega\|_{\text{HS}}^2] = \mathcal{O}(N^2). \quad (6.6.3)$$

2. **Anti-concentration bound** For each $\theta > 0$ there exist a $\beta > 0$ and an $\varepsilon_N(\theta) = o(1)$, as $N \rightarrow \infty$, such that for any fixed deterministic complex $N \times N$ matrix D with $\|D\| = \mathcal{O}(N^\theta)$, we have that

$$\mathbf{P}(s_N(D + Q_\omega) \leq N^{-\beta}) = \varepsilon_N(\theta). \quad (6.6.4)$$

Remark 6.6.2. *These assumptions hold for a large class of noise matrices, including those with iid entries of zero mean and finite variance. We refer to [13, Remark 1.3] for details and references.*

In this case they show that the convergence (6.6.2) holds weakly in probability for a coupling constant $\delta = N^{-\gamma}$, with $\gamma > 1/2$. Furthermore, they prove a version of this theorem for Toeplitz matrices with non-constant coefficients in the bands, see [14, Theorem 1.3, Theorem 4.1]. They follow a different approach than we do: They compute directly $\log |\det P_N^\delta - z|$ by relating it to $\log |\det \tilde{P}_N(z)|$, where $\tilde{P}_N(z)$ is a truncation of $P_N - z$, where the smallest singular values of $M_N - z$ have been sufficiently lifted. Our approach, leading to the above results, is based on a Grushin problem approach similar (although using a different Grushin problem) to what was discussed in Section 3.3.3. This approach essentially also replaces $\det(P_N^\delta - z)$ by a deterministic version where the small singular values have been sufficiently lifted similar to the approach in [13, 14]. However, the Grushin problem approach is robust enough to pass from finitely banded Toeplitz matrices to general Toeplitz matrices. The link between the two methods, showing a deterministic equivalence for noisy perturbations principle, has been made precise in [194] by Zeitouni and the author. Also, as discussed there, we can weaken assumption (6.6.3) to assuming that there exists a $\kappa_1 > 0$ such that such that $\mathbb{E}[\|Q_\omega\|] = \mathcal{O}(N^{\kappa_1})$. However, we will not go into further details here.

Theorem 6.6.1 also has notable shortcomings: We do not allow for a coupling constant up to limiting strength of $\delta_1 > 1/2$, nor do we allow for more general random matrix ensembles as perturbation. The proofs, can however be modified to allow for these extensions.

In the earlier work [93, 178], the authors prove that the convergence (6.6.2) holds weakly in probability for the Jordan block matrix P_N with $p(\tau) = \tau^{-1}$ (6.2.4) and a perturbation given by a complex Gaussian random matrix whose entries are independent complex Gaussian random variables whose variances vanish (not necessarily at the same speed) polynomially fast, with minimal decay of order $N^{-1/2+}$. See also [60] for a related result.

In [200], using a replacement principle developed in [185], it was shown that the result of [93] holds for perturbations given by complex random matrices whose entries are independent and identically distributed random complex random variables with expectation 0 and variance 1 and a coupling constant $\delta = N^{-\gamma}$, with $\gamma > 2$.

Chapter 7

Eigenvector localization of disordered non-selfadjoint Toeplitz matrices

In this section we discuss the works obtained in [15].

7.1 Introduction

In this chapter we focus on the eigenvectors of large deterministic non-Hermitian $N \times N$ Toeplitz matrices P_N with small additive random perturbations. As discussed in chapter 6, the spectra of such matrices, apart from finitely many fluctuating outliers [13, 14, 16, 174, 175], mimic the absolutely continuous spectra of the associated infinite dimensional Laurent operator on \mathbb{Z} . This is particularly striking since a perturbation of size $\mathcal{O}(N^{-\infty})$ is sufficient to produce this effect, whereas the spectrum of the unperturbed matrix is far from the spectrum of the Laurent operator.

The aim of this chapter is to discuss the eigenvectors associated with the eigenvalues of such perturbed Toeplitz matrices. Are the eigenvectors *localized* or *delocalized*? The precise meanings of these notions vary over different subjects, however, they all serve to capture how much an ℓ^2 normalized eigenvectors *concentrates on* or *spreads out over* certain parts of its support.

In random matrix theory there are several ways of testing for localization or delocalization of ℓ^2 normalized eigenvectors ψ . One way is by comparing their ℓ^p norms for $2 < p \leq +\infty$ with $N^{1/p-1/2}$. Complete delocalization is said to occur when $\|\psi\|_p = \mathcal{O}(N^{1/p-1/2})$ (up to some logarithmic factors) since $N^{1/p-1/2}$ is the ℓ^p norm of the fully delocalized vector $(N^{-1/2}, \dots, N^{-1/2})$. Conversely localized eigenvectors have a large ℓ^p norm, as for instance the fully localized vector $(0, 0, 1, 0, \dots, 0)$ has ℓ^p norm equal to one. These notions were used for instance to prove delocalization via optimal ℓ^p bounds of the eigenvectors of Wigner matrices [73, 74], for non-Hermitian random matrices [154], and for the adjacency matrix of Erdős-Rényi graphs [71]. Recently, localization and delocalization of eigenvectors for the adjacency matrix of *critical* Erdős-Rényi graphs were established in [3].

There is a complementary notion of delocalization, known as *no-gaps delocalization*, which asserts that for any subset $I \subset [N]$, with $|I|$ reasonably large, one has $\|\psi\|_{\ell^2(I)} \gtrsim |I|$ (again allowing for logarithmic factors). Recently, such delocalizations have been established for Wigner matrices and matrices with independent and identically distributed (i.i.d.) entries (cf. [136, 137, 154]).

In the field of *quantum chaos* [160], in the setting of Hermitian pseudo-differential operators, localization and delocalization of normalized eigenvectors are studied via their associated *semi-classical defect measures*. Translated to the matrix setting [9], we note that $\sum_{x=1}^N |\psi(x)|^2 \delta_x$, where δ_x denotes the Dirac measure at x , defines a probability measure. One says that *quantum ergodicity* occurs when $\sum_{x=1}^N a(x) |\psi(x)|^2$ is close to $\frac{1}{N} \sum_{x=1}^N a(x)$ for most eigenvectors ψ , and *uniquely quantum ergodicity* occurs when this holds for all eigenvectors. In contrast, *scarring* occurs when we have concentration of the form $\sum_{x \in \Lambda} |\psi(x)|^2 \geq 1 - \varepsilon$, $\varepsilon > 0$, of the eigenvector on some small set Λ . On the other hand, if $\|\psi\|_p \asymp N^{f(p)}$ for some $f(p) \neq 1/2 - 1/p$, then the eigenstate ψ

is termed to be *non-ergodic* and *multi-fractal* [135]. These notions were recently applied to the study of the eigenfunctions of the discrete Laplacian on large regular graphs [6, 7], and to the proof of delocalization of eigenvectors of generalized Wigner matrices [41]. See also [20] for results on deformed Wigner matrices. For even stronger results concerning the delocalization of eigenvectors of Wigner matrices we refer the reader to the recent works on the *eigenstate thermalization hypothesis* in [51–54].

The aim of this chapter is to discuss recent results showing that the eigenvectors of non-selfadjoint Toeplitz matrices subject to small random perturbations localize on a set of cardinality $N/\log N$ in the sense that they scar on a set of size $N/\log N$ with probability close to one.

7.2 The setting and assumptions

We begin by recalling the setting, see also the discussion at the beginning of Chapter 6. Let $N_{\pm} \in \mathbb{Z}$ be such that $-N_- \leq N_+$ and either $N_+ \neq 0$ or $N_- \neq 0$. Let $a_i \in \mathbb{C}$, $i \in \mathbb{Z}$, be such that $a_{N_+} \neq 0$, $a_{-N_-} \neq 0$, and $a_i = 0$ for $i \notin [-N_-, N_+]$. Introduce the symbol $p(\zeta) = \sum_{-N_-}^{N_+} a_j \zeta^{-j}$ and the associated $N \times N$ Toeplitz matrix P_N with entries $P_N(i, j) = a_{i-j}$, that is

$$P_N = \begin{pmatrix} a_0 & a_{-1} & \dots & a_{-N_-} & \dots \\ a_1 & a_0 & a_{-1} & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{N_+} & \dots & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & a_{N_+} & \dots & a_0 \end{pmatrix}. \quad (7.2.1)$$

We consider here noisy perturbations of P_N of the form

$$P_{N,\gamma}^Q = P_N + N^{-\gamma} Q_N, \quad \gamma > 1, \quad (7.2.2)$$

with Q_N an $N \times N$ (random) matrix satisfying Assumptions 7.2.1 and 7.2.2 below. The first assumption is on the existence of finite moments.

Assumption 7.2.1. Let $\{Q_{i,j}\}_{i,j=1}^N$ be the entries of the $N \times N$ noise matrix $Q = Q_N$.

(i) The entries of Q are jointly independent and have zero mean.

(ii) For any $h \in \mathbb{N}$ there exists an absolute constant $\mathfrak{C}_h < \infty$ such that

$$\max_{i,j=1}^N \mathbb{E}[|Q_{i,j}|^{2h}] \leq \mathfrak{C}_h.$$

For notation convenience, we take the sequence \mathfrak{C}_h increasing in h . To introduce the second assumption, recall Lévy's concentration function, defined for any complex-valued random variable \mathbb{X} and $\varepsilon > 0$ by

$$\mathcal{L}(\mathbb{X}, \varepsilon) \stackrel{\text{def}}{=} \sup_{w \in \mathbb{C}} \mathbf{P}(|\mathbb{X} - w| \leq \varepsilon). \quad (7.2.3)$$

Assumption 7.2.2. Assume that there exist absolute constants $\eta \in (0, 1]$ and $C_{7.2.2} < \infty$, such that

$$\mathcal{L}(Q_{i,j}, \varepsilon) \leq C_{7.2.2} \varepsilon^{1+\eta}, \quad (7.2.4)$$

for all sufficiently small $\varepsilon > 0$, uniformly for all N and $i, j \in \{1, 2, \dots, N\}$.

(The standard example of a noise matrix satisfying Assumptions 7.2.1 and 7.2.2 is the complex Ginibre matrix, i.e. with i.i.d. entries that are standard complex Gaussian variables.)

As discussed after Theorem 6.6.1, we know that all but $o(N)$ of the eigenvalues $\{\lambda_i^N\}$ of $P_{N,\gamma}^Q$ lie in a small neighborhood of the curve $p(S^1)$, where $S^1 := \{z \in \mathbb{C} : |z| = 1\}$; in fact, it was shown in those references that the empirical measure of eigenvalues of $P_{N,\gamma}^Q$,

$$L_N := N^{-1} \sum_{i=1}^N \delta_{\lambda_i^N}, \quad (7.2.5)$$

converges weakly to the push forward of the uniform measure on S^1 by p . As part of our study, we will obtain more precise information, and show in Theorem 7.3.1 that most of the eigenvalues lie in certain neighborhoods of width of order $\log N/N$ that are separated from $p(S^1)$ by distance of the same order.

Our goal is to study the eigenvectors associated with the latter (random) eigenvalues. Roughly speaking, we will show that those eigenvalues \hat{z} away from certain isolated bad points of $p(S^1)$ have corresponding eigenvectors which are close to a random linear combination of the eigenvectors e_j of $(P_N - \hat{z}I)^*(P_N - \hat{z}I)$ associated with its smallest eigenvalues. In particular we will show that this random linear combination of vectors localizes at scale $N/\log N$. To state our results precisely requires the introduction of some machinery, which we now do.

Sometimes, the symbol p possesses a natural contraction, defined as follows. Set

$$\mathbf{g}(p) := \gcd\{|j| : j \neq 0 \text{ and } a_j \neq 0\}, \quad (7.2.6)$$

where ‘‘gcd’’ denotes the greatest common divisor. If $\mathbf{g}_0 := \mathbf{g}(p) > 1$ then $p(\zeta) = q_p(\zeta^{\mathbf{g}_0})$ for some Laurent polynomial q_p . If $\mathbf{g}_0 = 1$ then $q_p = p$. For $\varepsilon > 0$ and a set $\mathcal{B} \subset \mathbb{C}$, \mathcal{B}^ε denotes the ε -blow up of \mathcal{B} , that is the Minkowski sum of the sets \mathcal{B} and $D(0, \varepsilon)$, the open disc of radius ε centered at zero.

Definition 7.2.3 (Set of bad points). *Let \mathcal{B}_1 be the collection of self intersection points of $q_p(S^1)$, and let \mathcal{B}_2 be the set of branch points, i.e. points z where the Laurent polynomial $p(\cdot) - z$ has double roots. Set $\mathcal{B}_p := \mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{G}_{p,\varepsilon} := p(S^1) \setminus \mathcal{B}_p^\varepsilon$.*

In Definition 7.2.3, a point $w \in \mathbb{C}$ is a self intersection point of $q_p(S^1)$ if there exist $\zeta_1 \neq \zeta_2 \in S^1$ so that $q_p(\zeta_1) = q_p(\zeta_2) = w$.

Throughout this chapter, we make the following assumption on the symbol p .

Assumption 7.2.4. *The symbol p satisfies $a_{-N_-}, a_{N_+} \neq 0$, and \mathcal{B}_1 is a finite set.*

7.3 Placement of most eigenvalues

Under Assumption 7.2.4, \mathcal{B}_p is a finite set. Indeed, \mathcal{B}_2 is precisely the set of all z 's such that the discriminant of the polynomial $\zeta \mapsto \zeta^{N_+} p(\zeta) - z$ vanishes, and [38, Lemma 11.4] yields that \mathcal{B}_2 is a finite set. We note that by [128], unless $N_- = N_+$ and $|a_{-N_-}| = |a_{N_+}|$, \mathcal{B}_1 has cardinality bounded above by $(N_+ + N_- - 1)^2$, so symbols avoiding this situation satisfy Assumption 7.2.4. For $z \in \mathbb{C}$ let

$$d(z) = \text{ind}_{p(S^1)}(z)$$

denote the winding number of the curve $p(S^1)$ around z . We now describe the collection of eigenvalues of interest to us. For $0 < \varepsilon, C < \infty$ and N large enough so that $2C \log N/N < \varepsilon$, set

$$\Omega(\varepsilon, C, N) := \{z \in \mathbb{C} : C^{-1} \log N/N < \text{dist}(z, \mathcal{G}_{p,\varepsilon}) < C \log N/N, d(z) \neq 0\}, \quad (7.3.1)$$

where for a set $\mathcal{B} \subset \mathbb{C}$ and $w \in \mathbb{C}$ we denote $\text{dist}(w, \mathcal{B}) := \inf_{w' \in \mathcal{B}} |w - w'|$. Let

$$\mathcal{N}_{\Omega(\varepsilon, C, N), N, \gamma} := |\{\lambda_i^N \in \Omega(\varepsilon, C, N)\}|$$

denote the number of eigenvalues of $P_{N,\gamma}^Q$ that lie in $\Omega(\varepsilon, C, N)$.

The following theorem shows that *most* eigenvalues of $P_{N,\gamma}^Q$ lie in $\Omega(\varepsilon, C, N)$ for appropriate ε, C .

Theorem 7.3.1. *Let Assumptions 7.2.1, 7.2.2, and 7.2.4 hold. Fix $\mu > 0$ and $\gamma > 1$. Then there exist $0 < \varepsilon_{7.3.1}, C_{7.3.1} < \infty$ (depending on γ, μ and p only) so that*

$$\mathbf{P}(\mathcal{N}_{\Omega(\varepsilon_{7.3.1}, C_{7.3.1}, N), N, \gamma} < (1 - \mu)N) \rightarrow_{N \rightarrow \infty} 0. \quad (7.3.2)$$

Theorem 7.3.1 may be of independent interest since it improves upon previous results [13, 14, 174, 175] by providing a much sharper estimate on the position of the eigenvalues of $P_{N,\gamma}^Q$. See also the discussion after Theorem 6.6.1. In what follows, we fix $\mu > 0$, $\gamma > 1$ and consider the $\varepsilon_{7.3.1}$ and $C_{7.3.1}$ determined by Theorem 7.3.1. We then consider eigenvalues $\lambda_i^N \in \Omega(\varepsilon_{7.3.1}, C_{7.3.1}, N)$. By Theorem 7.3.1, *most* eigenvalues are of this type. Notice also that for any $z \in \Omega(\varepsilon_{7.3.1}, C_{7.3.1}, N)$, we have that $d = d(z) \neq 0$.

7.4 Eigenvector localization

The main result is the following description of the (right) eigenvectors of $P_{N,\gamma}^Q$.

Theorem 7.4.1. *Fix $\varepsilon_{7.3.1}, C_{7.3.1}$ and the notation as above.*

1. *The following occurs with probability approaching one as $N \rightarrow \infty$. For each $\hat{z} \in \Omega(\varepsilon_{7.3.1}, C_{7.3.1}, N)$ which is an eigenvalue of $P_{N,\gamma}^Q$, let $v = v(\hat{z})$ denote the corresponding (right) eigenvector, normalized so that $\|v\|_2 = 1$. Then there exists a vector w with $\|w\|_2 = 1$ such that*

$$\|v - w\|_2 = o(1), \quad (7.4.1)$$

and a constant $c > 0$, depending on γ , so that for any $\ell = \ell(N) \in [N]$,

$$\begin{aligned} \|w\|_{\ell^2([\ell, N])} &\leq e^{-c\ell \log N/N}/c, & \text{if } d > 0, \\ \|w\|_{\ell^2([1, N-\ell])} &\leq e^{-c\ell \log N/N}/c, & \text{if } d < 0. \end{aligned} \quad (7.4.2)$$

The vector w can be taken as a (random) linear combination of the $|d|$ eigenvectors of $(P_N - \hat{z}I)^*(P_N - \hat{z}I)$ corresponding to the $|d|$ smallest eigenvalues.

2. *Fix $z_0 = z_0(N) \in \Omega(\varepsilon_{7.3.1}, C_{7.3.1}, N)$ deterministic, C_0, \tilde{C}_0 large, and $\eta > 0$ small. Then, there exist constants $c_1 = c_1(\eta, C_0, \tilde{C}_0)$ and $c_0 = c_0(\gamma) \in (0, 1)$, with $c_0 \rightarrow 1$ as $\gamma \rightarrow 1$ and $c_0 \rightarrow 0$ as $\gamma \rightarrow \infty$, so that, with probability at least $1 - \eta$, for every $\hat{z} = \lambda_i^N \in D(z_0, C_0 \log N/N)$, any $0 < \ell \leq \ell' \leq \tilde{C}_0 N / \log N$ satisfying $\ell' - \ell > N^{c_0}$ and all large N ,*

$$\begin{aligned} \|w\|_{\ell^2([\ell, \ell'])}^2 &\geq c_1(\ell' - \ell) \log N/N, & \text{if } d > 0, \\ \|w\|_{\ell^2([N-\ell', N-\ell])}^2 &\geq c_1(\ell' - \ell) \log N/N, & \text{if } d < 0. \end{aligned} \quad (7.4.3)$$

Further, for any $0 < c' \leq \tilde{C}_0$,

$$\begin{aligned} \|v\|_{\ell^2([1, c'N/\log N])}^2 &\geq c'c_1/2, & \text{if } d > 0, \\ \|v\|_{\ell^2([N-c'N/\log N, N])}^2 &\geq c'c_1/2, & \text{if } d < 0. \end{aligned} \quad (7.4.4)$$

Theorem 7.4.1 shows a *localization* phenomenon, numerically illustrated in the examples of Figure 7.1: for all eigenvalues in the good regions, the corresponding eigenvectors localize at scale $N/\log N$, and for most eigenvalues, this is the scale at which the eigenvector is ‘‘spread out’’. (Contrast the situation with the regime $\gamma < 1$, where delocalization is observed in simulations, see Figure 7.2; we discuss predictions for that regime in Section 7.6.4, after we introduce relevant notions and in particular the relevant *Grushin problem*.)

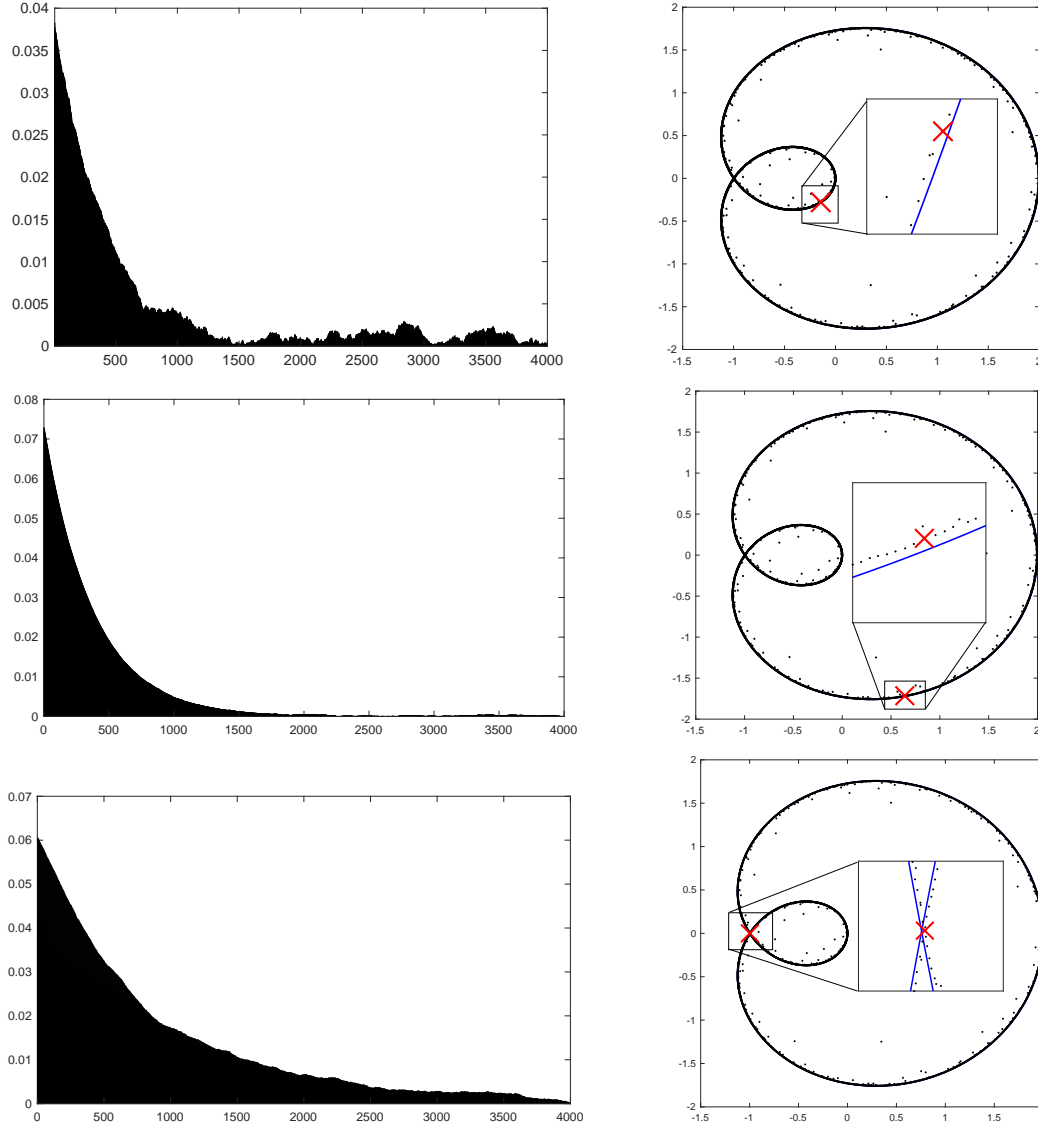


Figure 7.1: Eigenvectors (left panel) and eigenvalues (right panel) for $N = 4000$, $\gamma = 1.2$ and symbol $\zeta + \zeta^2$. Plotted are the moduli of the entries of the eigenvector that corresponds to the eigenvalue marked with a red \times . The top two rows correspond to situations covered by Theorem 7.4.1; note the localization, which occurs at scale $N/\log N$. The bottom row is not covered by Theorem 7.4.1, because the chosen eigenvalue is at vanishing distance from \mathcal{B}_1 .

Building on Theorem 7.4.1, equipped with a local estimate on the number of eigenvalues λ_i^N in regions of diameters $O(\log N/N)$ (a element used in the proof of Theorem 7.3.1, see [15, Theorem 7.4]) and applying a Fubini type argument, one can show that except for an arbitrarily small fraction of the eigenvalues, the corresponding eigenvectors $v(\lambda_i^N)$ localize at scale $N/\log N$. In particular, we prove the following result.

Corollary 7.4.2. *Let Assumptions 7.2.1, 7.2.2, and 7.2.4 hold. Then, for any $\mu > 0$ there exists $\mu_1, \mu_2 > 0$ so that with $|\text{supp}_{\mu_1}(v)| := \min\{|I| : \|v\|_{\ell^2(I)} > 1 - \mu_1\}$,*

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \mathbf{E} \#\{i : \text{supp}_{\mu_1}(v(\lambda_i^N)) < \mu_2 N / \log N\} \leq \mu.$$

Theorem 7.4.1 states that the eigenvectors of $P_{N,\gamma}^Q$ corresponding to most eigenvalues \hat{z} can be approximated by a random linear combination of the eigenvectors e_j of $(P_N - \hat{z}I)^*(P_N - \hat{z}I)$ associated with its $|d|$ smallest eigenvalues. These are precisely the right singular vectors of $P_N - \hat{z}I$

associated with its $|d|$ smallest singular values. However, these singular vectors e_j are (in general) difficult objects to study and do not admit an easy description. Therefore, we approximate these singular vectors with certain *quasimodes* of the operator P_N . Recall that the term quasimode for P_N and a (quasi-)eigenvalue z refers to an approximate ℓ^2 -normalized eigenvectors $\psi \in \mathbb{C}^N$ of $P_N - z$ in the sense that

$$\|(P_N - z)\psi\| \rightarrow 0, \quad N \rightarrow \infty.$$

As explained in and after Theorem 6.4.2, when $|d| = |\text{ind}_{p(S^1)}(z)|$, then have a $|d|$ -dimensional space of quasimodes associated with $P_N - zI$. We can show that the eigenvectors e_j of $(P_N - zI)^*(P_N - zI)$ associated with its $|d|$ smallest eigenvalues are close to these quasimodes. We avoid giving more details here, however, we emphasize that the *construction* of the e_j -s depends on p , N and z only and not on Q (even if eventually the value of z to which it will be applied will depend on Q).

The upper bound (7.4.2) is due to the decaying nature of these $|d|$ linearly independent quasimodes either to the left (when $d < 0$) or to the right (when $d > 0$). These quasimodes decay exponentially quickly, $|u(n)| \asymp e^{-rn}$ or $|u(n)| \asymp e^{-r(N-n)}$, however, at different rates $r > 0$. Out of these $|d|$ quasimodes the first $(|d| - g_0)$ (recall (7.2.6)) quasimodes decay at a constant rate $r > 0$, resulting in them being completely localized to a point, i.e. either on the left or right hand side of the interval $[1, N]$. In contrast, the rest decay at a rate $r \asymp \log N/N$, which implies that they localize at a scale $N/\log N$. In contrast, the lower bound (7.4.3) follows upon showing that w has a non-negligible projection (in ℓ^2) onto \mathcal{S} , the subspace spanned by the last g_0 quasimodes that decay precisely at rate $\log N/N$.

Note that Theorem 7.4.1 and Corollary 7.4.2 establish absences of quantum ergodicity and no-gaps delocalization, and show that the semiclassical defect measure in this setting is the Dirac measure at zero or one, depending on whether d is positive or negative.

One may wonder whether the assumption $\hat{z} \in D(z_0, C_0 \log N/N)$ in the second part of Theorem 7.4.1 is optimal. It is clear from the proof that to derive (7.4.3) one needs to control the supremum of the random field $z \mapsto \tilde{f}(z)^t Q \tilde{e}(z)$ for some $\tilde{e}(z), \tilde{f}(z) \in \mathbb{C}^N$ such that the Lipschitz norms of the functions $z \mapsto \tilde{e}(z)$ and $z \mapsto \tilde{f}(z)$ are $\mathcal{O}(N/\log N)$ and $\mathcal{O}(1)$, respectively. It is then standard to check that the supremum of the field $\{\tilde{e}(z)^t Q \tilde{f}(z)\}$ can only be bounded by an $\mathcal{O}(1)$ quantity in discs of radius $\mathcal{O}(\log N/N)$. The boundedness of this random field is crucial in deriving that w has non-negligible projection onto \mathcal{S} . Repeating the same reasoning one can also observe that for all \hat{z} simultaneously in the good region, with probability approaching one, the ℓ^2 norm of the projection of w onto \mathcal{S} is at least of the order $\log N/N$, and we believe that this is the correct picture.

7.5 Extensions

A natural extension of our results would be to the region $\gamma \in (1/2, 1]$, see Section 7.6.4 for a discussion. We mention a couple of additional potentially interesting extensions, of broad interest.

Sup-norm delocalization versus no-gaps delocalization. As already mentioned in Section 7.1, there are two complementary notions of delocalizations in the random matrix literature. It is shown in [22, 155] that both these notions of delocalization hold for Wigner matrices under various assumptions on its entries. There is no reason to believe that these two notions of delocalizations should hold simultaneously in any given setting.

Indeed, from the proof of Theorem 7.4.1 it follows that when P_N is a Jordan block, i.e. its symbol is $p(\zeta) = \zeta$, and $\gamma > 3/2$ we have that $\|v\|_\infty \asymp \sqrt{\log N/N}$ for an eigenvector v corresponding to a bulk eigenvalue. *Thus, in this simple setting, the eigenvectors corresponding to most of the eigenvalues are completely delocalized according to the sup-norm criterion. However, they do not satisfy no-gaps delocalization.* It is worth investigating whether one indeed has that $\|v\|_\infty \asymp \sqrt{\log N/N}$ for all $\gamma > 1$ and any finitely banded P_N .

Localization for the outlier eigenvalues and multi-fractal structure. Based on simulations and some heuristic arguments, we predict that the eigenvector ψ corresponding to an eigenvalue residing at a distance of order $N^{\alpha-1}$, $\alpha \in (0, 1]$, from the spectral curve localizes at scale $N^{1-\alpha}$. This shows in particular that for such a ψ one has $\|\psi\|_p \asymp N^{(\alpha-1)\cdot(\frac{1}{2}-\frac{1}{p})}$, establishing that such eigenvectors are multi-fractal. The same reasoning shows that the eigenvectors corresponding to outlier eigenvalues, i.e. those are at a distance of order one from $p(S^1)$, would be completely localized. That is, most of their mass is carried by finitely many entries. It seems plausible that the methods of the proof of Theorem 7.4.1 in [15] could be adapted to prove these results.

7.6 Ideas of the proof of Theorem 7.4.1

We begin by setting up, in some generality, a well-posed Grushin problem, based on [96, 193], see also Seciton 2.6. It, and its behavior under perturbations, plays a crucial role in our analysis.

7.6.1 Grushin problem for the unperturbed operator

Let P be a complex $N \times N$ -matrix. (In our application, we will often take $P = P_N - zI$ where P_N is the (deterministic) Toeplitz matrix with symbol p and z is a *random* parameter close to the spectral curve $p(S^1)$. Then, all objects implicitly depend on z , and we suppress this dependence in notation when not needed.) Let

$$0 \leq t_1^2 \leq \dots \leq t_N^2 \quad (7.6.1)$$

denote the eigenvalues of P^*P with associated orthonormal basis of eigenvectors $e_1, \dots, e_N \in \mathbb{C}^N$. The spectra of P^*P and PP^* are equal and we can find an orthonormal basis $f_1, \dots, f_N \in \mathbb{C}^N$ of eigenvectors of PP^* associated with the eigenvalues (7.6.1) such that

$$P^*f_i = t_i e_i, \quad P e_i = t_i f_i, \quad i = 1, \dots, N. \quad (7.6.2)$$

Let $0 < \alpha \ll 1$ and let $M > 0$ be the number of singular values $t_i \in [0, \alpha]$, i.e.

$$0 \leq t_1 \leq \dots \leq t_M \leq \alpha < t_{M+1} \leq \dots \leq t_N. \quad (7.6.3)$$

Let δ_i , $1 \leq i \leq M$, denote an orthonormal basis of \mathbb{C}^M . Put

$$R_+ := \sum_{i=1}^M \delta_i \circ e_i^*, \quad R_- := \sum_{i=1}^M f_i \circ \delta_i^*, \quad (7.6.4)$$

Then the Grushin problem

$$\mathcal{P} := \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : \mathbb{C}^N \times \mathbb{C}^M \longrightarrow \mathbb{C}^N \times \mathbb{C}^M \quad (7.6.5)$$

is bijective. To see this we take $(v, v_+) \in \mathbb{C}^N \times \mathbb{C}^M$ and proceed to solve

$$\mathcal{P} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}. \quad (7.6.6)$$

We write $u = \sum_1^N u(j)e_j$ and $v = \sum_1^N v(j)f_j$. Similarly, we express u_-, v_+ in the basis $\delta_1, \dots, \delta_M$. The relation (7.6.2) then shows that (7.6.6) is equivalent to

$$\begin{cases} \sum_1^N t_i u(i) f_i + \sum_1^M u_-(j) f_j = \sum_1^N v(j) f_j, \\ u(j) = v_+(j), \quad j = 1, \dots, M, \end{cases}$$

which can be written as

$$\begin{cases} t_i u(i) = v(i), & i = M+1, \dots, N, \\ \begin{pmatrix} t_i & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u(i) \\ u_-(i) \end{pmatrix} = \begin{pmatrix} v(i) \\ v_+(i) \end{pmatrix}, & i = 1, \dots, M. \end{cases} \quad (7.6.7)$$

Since

$$\begin{pmatrix} t_i & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -t_i \end{pmatrix},$$

we see that

$$\mathcal{P}^{-1} =: \mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \quad (7.6.8)$$

with

$$E = \sum_{M+1}^N \frac{1}{t_i} e_i \circ f_i, \quad E_+ = \sum_1^M e_i \circ \delta_i^*, \quad E_- = \sum_1^M \delta_i \circ f_i^*, \quad \text{and} \quad E_{-+} = - \sum_1^M t_j \delta_j \circ \delta_j^*. \quad (7.6.9)$$

From (7.6.3) and (7.6.9) it follows that we have the following norm estimates

$$\|E(z)\| \leq \frac{1}{\alpha}, \quad \|E_{\pm}\| = 1, \quad \|E_{-+}\| \leq \alpha. \quad (7.6.10)$$

Next, we recall a general fact on well-posed Grushin problems.

Lemma 7.6.1. *Let \mathcal{H} be an N -dimensional complex Hilbert space, and let $N \geq M > 0$. Suppose that*

$$\mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H} \times \mathbb{C}^M \longrightarrow \mathcal{H} \times \mathbb{C}^M$$

is a bijective matrix of linear operators, with inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}.$$

Then, $E_+ : \mathcal{N}(E_{-+}) \rightarrow \mathcal{N}(P)$ is bijective with inverse $R_+ \upharpoonright_{\mathcal{N}(P)}$, and $E_-^ : \mathcal{N}(E_{-+}^*) \rightarrow \mathcal{N}(P^*)$ is bijective with inverse $R_-^* \upharpoonright_{\mathcal{N}(P^*)}$.*

Proof. From $\mathcal{P}\mathcal{E} = 1$, we get that $PE_+ + R_-E_{-+} = 0$ and so

$$E_+ : \mathcal{N}(E_{-+}) \rightarrow \mathcal{N}(P). \quad (7.6.11)$$

Similarly, we get from $\mathcal{E}\mathcal{P} = 1$ the equation $E_-P + E_{-+}R_+ = 0$, and hence

$$R_+ : \mathcal{N}(P) \rightarrow \mathcal{N}(E_{-+}). \quad (7.6.12)$$

The identity $EP + E_+R_+ = 1$ yields that $E_+R_+ = 1$ on $\mathcal{N}(P)$, which, together with $R_+E_+ = 1$, shows that (7.6.11), (7.6.12), are bijective and inverser to each other. The proof of the second claim is similar, one can follow the same arguments applied to $\mathcal{P}^*\mathcal{E}^* = \mathcal{E}^*\mathcal{P}^* = 1$. \square

7.6.2 Grushin problem for the perturbed operator

Now we turn to the perturbed operator

$$P^\delta := P + \delta Q, \quad 0 \leq \delta \ll 1. \quad (7.6.13)$$

where Q is a complex $N \times N$ -matrix (eventually, random). Let R_\pm be as in (7.6.4), and put

$$\mathcal{P}^\delta := \begin{pmatrix} P^\delta & R_- \\ R_+ & 0 \end{pmatrix} : \mathbb{C}^N \times \mathbb{C}^M \longrightarrow \mathbb{C}^N \times \mathbb{C}^M \quad (7.6.14)$$

with $\mathcal{P} = \mathcal{P}^0$. Applying \mathcal{E} (see (7.6.8)) from the right to (7.6.14) yields

$$\mathcal{P}^\delta \mathcal{E} = I_{N+M} + \begin{pmatrix} \delta QE & \delta QE_+ \\ 0 & 0 \end{pmatrix}. \quad (7.6.15)$$

Suppose that $(I + \delta QE)$ is invertible. It is then straightforward to check that \mathcal{P}^δ is invertible, with inverse

$$(\mathcal{P}^\delta)^{-1} =: \mathcal{E}^\delta = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix}, \quad (7.6.16)$$

where

$$E^\delta = E(I + \delta QE)^{-1}, \quad E_-^\delta = E_-(I + \delta QE)^{-1}, \quad (7.6.17)$$

$$E_{-+}^\delta = E_{-+} - E_-(I + \delta QE)^{-1} \delta QE_+, \quad (7.6.18)$$

and

$$E_+^\delta = E_+ - E(I + \delta QE)^{-1} \delta QE_+. \quad (7.6.19)$$

We note that if one takes $P = P_N - zI_N$ with z an eigenvalue of $P_N + \delta Q$, then Lemma 7.6.1 applied to P^δ gives a convenient description of the null-space of P , which is precisely the eigenspace of $P_N + \delta Q$ corresponding to the eigenvalue z . This observation will be a crucial part of our analysis, see (7.6.25) below.

Remark 7.6.2. *Under the additional assumption that*

$$2\delta \|Q\| \alpha^{-1} \leq 1, \quad (7.6.20)$$

which will occur in our setup of Q as in Assumption 7.2.1 if $\delta = N^{-\gamma}$ with $\gamma > 3/2$ (and with $\alpha \asymp N^{-1}$), we obtain by a Neumann series argument that

$$\mathcal{E}^\delta = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix} = \mathcal{E} + \sum_{n=1}^{\infty} (-\delta)^n \begin{pmatrix} E(QE)^n & (EQ)^n E_+ \\ E_-(QE)^n & E_-(QE)^{n-1} QE_+ \end{pmatrix}, \quad (7.6.21)$$

where by (7.6.20), (7.6.10),

$$\begin{aligned} \|E^\delta\| &= \|E(1 + \delta QE)^{-1}\| \leq 2\|E\| \leq 2\alpha^{-1}, \\ \|E_+^\delta\| &= \|(1 + \delta QE)^{-1} E_+\| \leq 2\|E_+\| \leq 2, \\ \|E_-^\delta\| &= \|E_-(1 + \delta QE)^{-1}\| \leq 2\|E_-\| \leq 2, \\ \|E_{-+}^\delta - E_{-+}\| &= \|E_-(1 + \delta QE)^{-1} \delta QE_+\| \leq 2\|\delta Q\| \leq \alpha. \end{aligned} \quad (7.6.22)$$

In particular, in that case,

$$E_+^\delta = E_+ - \delta QE(1 + O(\delta \|Q\| \alpha^{-1})) E_+ \quad (7.6.23)$$

and

$$E_{-+}^\delta = E_{-+} - \delta E_-(1 + O(\delta \|Q\| \alpha^{-1})) QE_+. \quad (7.6.24)$$

7.6.3 Structure of the proof of Theorem 7.4.1

A key ingredient for the proof Theorem 7.4.1 is Theorem 7.3.1. The proof of the latter result splits into two parts: In the first part we show that all eigenvalues must be separated from $p(S^1)$ by a distance of the order $\log N/N$. At a very high level it involves an expansion of the determinant of $P_{N,\gamma}^Q - zI_N$, with $z \in \mathbb{C}$, identifying the dominant term in that expansion, and showing that the dominant cannot be equal to zero (with probability approaching one) when z is in the vicinity of the spectral curve.

The second part of Theorem 7.3.1 requires us to show that most of the eigenvalues must be within a distance $O(\log N/N)$ from spectral curve, again with probability approaching one. This is achieved by an application of Jensen's formula together with upper and lower bounds on the log-potential of L_N (see (7.2.5)).

In the remainder of this section we describe the structure of the proof of Theorem 7.4.1, taking for granted Theorem 7.3.1 and various technical estimates. The proof of Theorem 7.4.1 begins with the Grushin problem for $P_z^\delta = P_{N,\gamma}^Q - zI$, see (7.6.16), for $\delta = N^{-\gamma}$, z which is roughly an eigenvalue, and $M = |\text{ind}_{p(S^1)}(z)|$ (this will lead to $t_{M+1} \gtrsim \log N/N$ and α bounded below by a constant multiple of $\log N/N$). To keep track of the dependence on z , throughout this section we write $E(z), E_+(z)$, etc. To relate the null-space of P_z^δ with the null space of $E_{-+}^\delta(z)$ we will use Lemma 7.6.1 in an indirect manner: As in its proof note that from (7.6.14) and (7.6.16),

$$E^\delta(z)P_z^\delta + E_+^\delta(z)R_+(z) = I \quad \text{and} \quad E_-^\delta(z)P_z^\delta + E_{-+}^\delta(z)R_+(z) = 0. \quad (7.6.25)$$

If $z = \hat{z}$ were an eigenvalue of $P_{N,\gamma}^Q$ with corresponding normalized eigenvector v then, with notation as in Section 7.6 and recalling the definition of $E_+^\delta(z)$, we would obtain from (7.6.25) that since $\{e_i(\hat{z})\}$ forms an orthonormal basis of \mathbb{C}^N ,

$$\begin{aligned} \sum_{i=M+1}^N (e_i(\hat{z})^*v) \cdot e_i(\hat{z}) &= (I - E_+(\hat{z})R_+(\hat{z}))v \\ &= (I - E_+^\delta(\hat{z})R_+(\hat{z}))v - E(\hat{z})(I + \delta QE(\hat{z}))^{-1}\delta QE_+(\hat{z})R_+(\hat{z})v \\ &= E(\hat{z})(I + \delta QE(\hat{z}))^{-1}\delta QE_+(\hat{z})R_+(\hat{z})v. \end{aligned} \quad (7.6.26)$$

Consider first the case where γ is large ($\gamma > 3/2$ will do). Since $\|Q\| = O(N^{1/2+\epsilon})$, for any $\epsilon > 0$, with high probability, we obtain that $N^{-\gamma}\|Q\|\alpha^{-1} = o(1)$ and therefore (7.6.20) holds. Using then (7.6.23)-(7.6.24), the projection of v on $\text{span}(e_i(z), i \geq M+1)$ is negligible, which yields the first part of Theorem 7.4.1.

To see the second part, still in the case of large γ (here we will need $\gamma > 2$) and $z = \hat{z}$, we obtain from (7.6.25) that

$$\begin{aligned} 0 &= -E_{-+}^\delta(\hat{z})R_+(\hat{z})v = -E_{-+}(\hat{z})R_+(\hat{z})v + \delta E_-(\hat{z})QE_+(\hat{z})R_+(\hat{z})v \\ &\quad - \delta^2 E_-(\hat{z})(I + \delta QE(\hat{z}))^{-1}QE(\hat{z})QE_+(\hat{z})R_+(\hat{z})v, \end{aligned} \quad (7.6.27)$$

where we also have used the resolvent expansion. By the same reasoning as above, the third term in (7.6.27) turns out to be of order $o(\delta)$, hence negligible compared to the first two terms. Therefore, recalling the definitions of $E_\pm(z), E_{-+}(z)$, and $R_+(z)$ we obtain that, with $a_j = (e_j(\hat{z})^*v)$,

$$\begin{aligned} &-E_{-+}(\hat{z})R_+(\hat{z})v + \delta E_-(\hat{z})QE_+(\hat{z})R_+(\hat{z})v \\ &= \sum_{i=1}^M a_i t_i \delta_i + \delta \sum_{i=1}^M \left[\sum_{j=1}^M a_j (f_i(\hat{z})^* Q e_j(\hat{z})) \right] \delta_i = o(\delta). \end{aligned} \quad (7.6.28)$$

Now, we know that there exists $M > M_0 \geq 0$ so that t_j decay exponentially in N for $j \in [M_0]$. Thus, we obtain from (7.6.28) that for γ large,

$$\left\| \sum_{i=1}^{M_0} \left[\sum_{j=1}^M a_j \cdot (f_i(\hat{z})^* Q e_j(\hat{z})) \right] \delta_i \right\| = o(1).$$

Assume now that $a_j = o(1)$ for $j = M_0 + 1, \dots, M$. Using a basic chaining argument we would then conclude that

$$\left\| \sum_{i=1}^{M_0} \left[\sum_{j=1}^{M_0} a_j \cdot (f_i(\hat{z})^* Q e_j(\hat{z})) \right] \delta_i \right\| = \|a^\top A\| = o(1),$$

where A is the $M_0 \times M_0$ matrix with entries $A_{i,j} = f_i(\hat{z})^* Q e_j(\hat{z})$. If \hat{z} were deterministic, we would have that the smallest singular value of A is $o(1)$ and this would lead to a contradiction. Since \hat{z} is actually random, we will proceed by using the fact that the functions f_i, e_i are localized, which makes the minimal singular value of A continuous in z .

When $\gamma \in (1, 2]$, the analysis, although following similar steps, is much more involved and necessitates a fair amount of auxiliary results. Therefore, we do not discuss them here, but refer the reader to [15].

7.6.4 The case $\gamma < 1$ - discussion and speculations

We end this section with some brief remarks concerning $\gamma < 1$. In that regime, the single entries of $\delta Q = N^{-\gamma} Q$ are larger than N^{-1} , and in particular are asymptotically larger than the distance of the eigenvalue from the spectral curve. In particular, when writing the Grushin problem (7.6.14), one is forced to take M growing with N (in fact, essentially $M \asymp N^{2(1-\gamma)}$); This is forced by the requirement that $\|E_-(\delta Q)E_+\| < M/N$. The resulting eigenvectors of P^δ are expected to be given by a combination of the M bottom quasimodes, with random coefficients. Since the quasimodes oscillate at scale $N/M = N^{2\gamma-1}$, the combination is expected to converge to a γ -dependent Gaussian process with correlation length of that scale. The simulations in Figure 7.2 are in line with this picture, although proving it require ideas going beyond the methods of the present work.

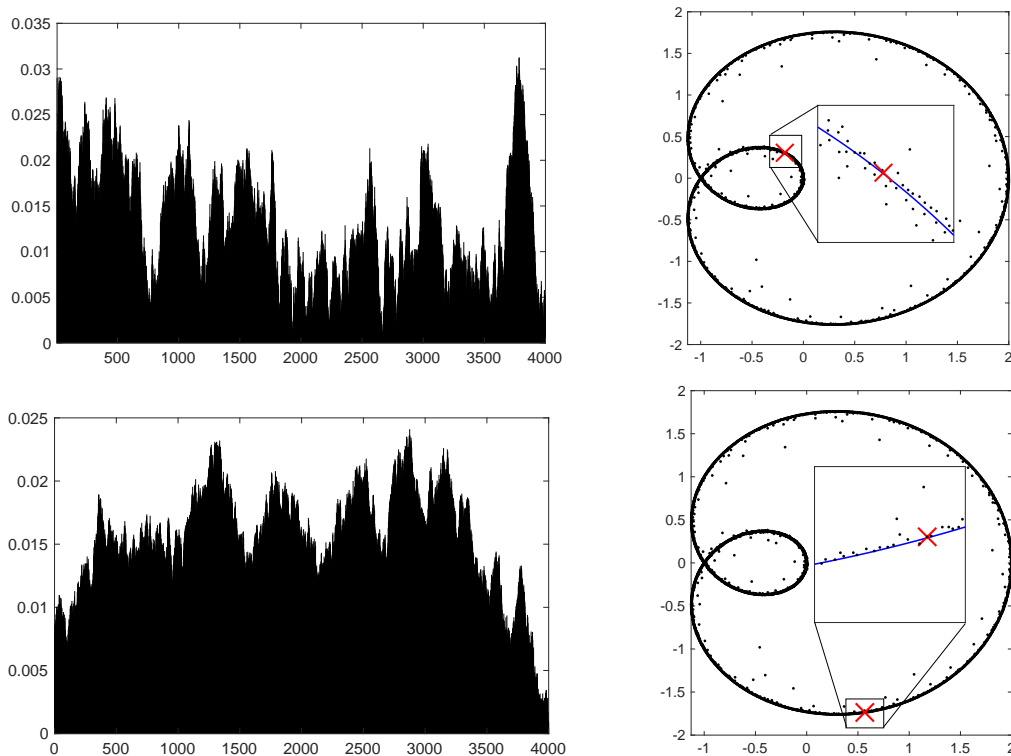


Figure 7.2: Eigenvectors (left panel) and eigenvalues (right panel) for $N = 4000$, $\gamma = 0.8$ and symbol $\zeta + \zeta^2$. These cases are not covered by Theorem 7.4.1. Note the stark difference with the corresponding situations in Figure 7.1, in both the location of eigenvalues relative to $p(S^1)$ and in the localization properties of eigenvectors.

Part III

Selfadjoint disordered operators in quantum chaos

Chapter 8

Eigenvector delocalization for a noisy chaotic quantum Hamiltonian

In this section we discuss the works obtained in [123, 124].

8.1 Introduction

Background The theory of quantum chaos aims at understanding the nature of a quantum system when its associated classical Hamiltonian system is chaotic. A guiding example is the Laplace-Beltrami operator on a negatively curved smooth Riemannian manifold X . There the geodesic flow has the Anosov property [69] which, in a sense, is the ideal chaotic behavior. The corresponding quantum dynamics is given, in the high-energy or semiclassical limit, by the unitary group generated by the Laplace-Beltrami operator Δ_g on $L^2(X)$. The chaotic nature of the geodesic flow is conjectured (and indeed proven in some cases) to have equidistributing influence on the spectral properties of the Laplacian. For instance, the random matrix conjecture by Bohigas-Giannoni-Schmit [28, 30, 31] states that the fluctuations of the high-lying eigenvalues should resemble those of large Wigner random matrices. The corresponding eigenfunctions are conjectured by Rudnick-Sarnack to be *uniquely quantum ergodic* [156] (see also [195]). More precisely, it is conjectured that the family of eigenfunctions $\{\psi_\lambda\}_\lambda$ of Δ_g indexed by their corresponding eigenvalue satisfies

$$\langle \text{Op}_{\lambda^{-1/2}}(a)\psi_\lambda | \psi_\lambda \rangle \rightarrow \int_{S^*X} a d\rho, \quad \lambda \rightarrow \infty, \quad (8.1.1)$$

for any $a \in C^\infty(S^*X)$. This conjecture is motivated by the quantum ergodicity theorem of Šnirel'man [196], Zelditch [201] and Colin de Verdière [61], claiming that (8.1.1) holds for a density one sequence of eigenvalues λ . We refer the reader to [64] for an account of recent advances regarding the quantum unique ergodicity conjecture.

Berry's random wave conjecture Another way of understanding the delocalization properties of the Laplacian's eigenfunctions is covered by *Berry's random wave conjecture* [24]. It claims that, in the high energy limit, quantum chaotic eigenfunctions should resemble at a local scale a random superposition of plane waves. For decades, this statement, comparing a sequence of deterministic objects with a random object, was considered as a heuristic rather than a precise mathematical statement. However, motivated by the Benjamini-Schramm convergence in the theory of large random graphs, and by work by Bourgain [43] on the torus, it was recently suggested in [1, 121] that to make sense of the randomness in Berry's heuristic one should look at the eigenfunctions near a random point. More precisely, when ψ_λ – an eigenfunction of the Laplacian at energy λ – is rescaled to the scale of the wavelength $\lambda^{-\frac{1}{2}}$ around a randomly chosen point on the manifold, it defines a family of random functions, whose law should converge weakly to that of an isotropic

stationary monochromatic Gaussian random field. See section 8.4 for definitions and for a more precise statement. Note that this interpretation of Berry's conjecture implies the quantum unique ergodicity conjecture, as is proven in [121].

For related recent advances on Berry's random wave conjecture in the context of Wigner matrices we refer the reader to [21].

The setting: Quantum chaotic propagation Here, work we will adopt a semiclassical point of view. Rescaling the eigenvalue equation $(-\Delta_g - \lambda)\psi_\lambda = 0$ by the eigenvalue $\lambda = h^{-2}$ we get the semiclassical equation $(-h^2\Delta_g - 1)\psi_h = 0$ where $h > 0$ denotes the semiclassical parameter. Moreover, here, we will not be concerned with genuine eigenfunctions of the Laplacian but rather with another important question in quantum chaos: understanding the long-time behaviour of the Schrödinger equation for highly oscillating initial data. We wish to study the long-time evolution of highly oscillatory initial data under the Schrödinger evolution semigroup $e^{-iht_h\Delta_g}f_h$ which is the solution to

$$\begin{cases} ih\partial_t u = -h^2\Delta_g u, \\ u|_{t=0} = f_h = ae^{\frac{i}{h}\phi}. \end{cases} \quad (8.1.2)$$

For such long-time propagated quantum objects, one can sometimes prove properties analogous to those of genuine eigenfunctions. For instance, in [161], Schubert considered Lagrangian states f_h associated to Lagrangian manifolds that are transverse to the stable directions of the dynamics (see section 8.2), on a manifold of negative sectional curvature. He could show that the analogue of (8.1.1) holds, with ψ_λ replaced with $e^{-iht_h\Delta_g}f_h$, where t_h goes to infinity as $h \rightarrow 0$, while remaining smaller than some constant times $|\log h|$. Hence, the large-time evolution of Lagrangian states under the semiclassical Schrödinger equation (8.1.2) satisfies quantum unique ergodicity. It is thus natural to wonder if such functions do also satisfy an analogue of Berry's conjecture.

This questions was first raised in [122], and was given a partial positive answer. Namely, in [122], the authors considered Lagrangian states with a *generic* phase, and first took the limit $h \rightarrow 0$, and then $t \rightarrow \infty$ to obtain convergence to a Gaussian field. This result is probably not optimal, and it seems natural to conjecture that the family of functions $e^{-iht_h\Delta_g}f_h$ satisfy Berry's conjecture as soon as f_h is a Lagrangian state associated to Lagrangian manifold that is transverse to the stable directions of the dynamics, and as soon as $t_h \rightarrow \infty$ with $t_h \leq c|\log h|$, for some small enough c .

The result: noisy quantum chaotic propagation Our aim is to prove a result of this kind, not for the genuine semiclassical Laplacian $-\frac{h^2}{2}\Delta_g$, but for generic small perturbations of the form

$$P_{h,\omega} = -\frac{h^2}{2}\Delta_g + h^\alpha Q_\omega, \quad \alpha > 0,$$

where Q_ω is either a bounded random potential or a bounded semiclassical random pseudo-differential operator obtained from the quantization of a random symbol oscillating at scale h^β , $\beta \in]0, 1/2[$. The presence of a small noise term can be motivated by the fact that in genuine physical situations an "ideal" evolution operator can be perturbed by many different sources, many of which are uncontrolled by the experimentalist. It therefore seems relevant on its own to study the propagation of initial data under the Schrödinger evolution semi-group induced by $P_{h,\omega}$.

Our main aim is to study the family of functions

$$e^{it_h P_{h,\omega}} f_h$$

where f_h is a Lagrangian state associated to a Lagrangian manifold that is close enough to the unstable directions of the dynamics. Following our interpretation of Berry's conjecture, we rescale the propagated Lagrangian state to the microscale h around a uniformly at random chosen x

point on X . This rescaling around x makes $e^{it_h P_{h,\omega}} f_h$ a random smooth function that depends on the *additional* random parameter ω

The first main result (Theorem 8.5.2 below) states that, whenever $t_h \rightarrow \infty$ with $t_h \ll |\log h|$ and under appropriate conditions on α and β , the law of the randomly rescaled smooth function $(e^{it_h P_{h,\omega}} f_h)$ converges in probability (with respect to ω), to an isotropic stationary monochromatic Gaussian field – the Berry Gaussian field.

The quantitative nature of our result shows (Corollary 8.5.4) that, for any sufficiently fast decaying subsequence $h_j \rightarrow 0$, the randomly rescaled smooth function $(e^{it_h P_{h,\omega}} f_h)$ satisfies Berry’s conjecture ω -almost surely.

We can use these results (or more precisely elements of the proofs leading to these results) on the random propagation of Lagrangian states to obtain

the second main result (Theorem 8.6.1 below), which states that with overwhelming probability, we have a polynomial improvement over the Avakumović–Hörmander–Levitan bound (8.6.1) for the eigenfunctions of the the Laplace-Beltrami operator subject to small random perturbations on negatively curved compact manifolds.

The idea of adding a small generic perturbation to the semiclassical Laplace-Beltrami operator to obtain additional properties on the propagator is not new. For instance, in [47, 76], the authors propagate eigenfunctions by the Schrödinger equation perturbed by a random perturbation, and obtain improved L^p bounds by averaging over the perturbation; however, these results do not give information about the eigenfunctions (or propagated eigenfunctions) of a genuine Schrödinger operator. In a similar spirit, in [75], the authors perturbed the Laplace-Beltrami operator on a manifold of negative sectional curvature, by adding to it a small random potential of size $\gg h^{1/2}$. They show that, for any initial data which is microlocalized near the energy layer there is a high probability that its propagation up to time $O(|\log h|)$ by the perturbed Schrödinger equation satisfies some form of quantum ergodicity.

Note that the kind of perturbations we consider is quite different from those of [47, 75, 76]. The perturbations imposed in these papers are always large enough to modify the underlying classical dynamics: a wave packet microlocalized around (x_0, ξ_0) , when propagated by the perturbed Schrödinger equation in the time scales under consideration in these papers, is not microlocalized around the image of (x_0, ξ_0) by the corresponding geodesic flow. In contrast, we permit much smaller perturbations, which do not affect the classical dynamics, but which will only modify the phases of wave packets.

This allows for some delicate phase cancellations between wave packets, which we believe to be a good toy model for quantum chaos. It should thus be easier to prove quantum chaotic properties for eigenfunctions of $P_{h,\omega}$ with a generic ω than for the genuine Laplacian; such considerations will be pursued elsewhere.

8.2 Lagrangian states

Let (X, g) be a smooth compact connected Riemannian manifold without boundary and of negative sectional curvature. A *Lagrangian state* on X is a family of functions $f_h \in C^\infty(X)$ indexed by $h \in]0, 1]$, defined by

$$f_h(x) = a(x)e^{i\phi(x)/h}, \quad (8.2.1)$$

where $\phi \in C^\infty(\mathcal{O})$ for some connected and simply connected open subset $\mathcal{O} \subset X$ and $a \in C_c^\infty(\mathcal{O})$. To a Lagrangian state we can associate a *Lagrangian manifold*

$$\Lambda_\phi := \{(x, d_x \phi); x \in \mathcal{O}\} \subset T^*X.$$

A Lagrangian state is called *monochromatic* if $\Lambda_\phi \subset S^*X := \{(x, \xi) \in T^*X; |\xi|_x = 1\}$, i.e. if

$$|d_x\phi| = 1 \quad \text{for all } x \in \mathcal{O}. \quad (8.2.2)$$

The dynamics of the geodesic flow is hyperbolic on S^*X , so for any $\rho \in S^*X$ we may decompose the tangent spaces $T_\rho S^*X$ into *unstable*, *neutral* and *stable directions*

$$T_\rho S^*X = E_\rho^+ \oplus E_\rho^0 \oplus E_\rho^-.$$

Definition 8.2.1. *For every $\eta > 0$, we say that a Lagrangian manifold $\Lambda_\phi \subset S^*X$ is η -unstable if, for every $\rho \in \Lambda_\phi$ and for every $v \in T_\rho\Lambda$, writing $v = (v_+, v_-, v_0) \in E_\rho^+ \oplus E_\rho^- \oplus E_\rho^0$, we have*

$$|(0, v_-, 0)|_\rho \leq \eta|v|_\rho.$$

Recall that the intrinsic distance $\text{dist}_\Lambda(\rho_1, \rho_2)$ between two points $\rho_1, \rho_2 \in \Lambda$ is the minimal length of curves in Λ joining ρ_1 and ρ_2 , the length being computed using an arbitrary metric on T^*X . We define the *distortion* of Λ as

$$\text{distortion}(\Lambda) := \sup_{\rho_1, \rho_2 \in \Lambda} \frac{\text{dist}_\Lambda(\rho_1, \rho_2)}{\text{dist}_{T^*X}(\rho_1, \rho_2)}. \quad (8.2.3)$$

8.3 Noisy propagation of Lagrangian states

Let $h > 0$ and let $0 \leq \delta = \delta(h) \ll 1$. Consider the Schrödinger-type operator

$$P_h^\delta := -\frac{h^2}{2}\Delta_g + \delta Q_\omega, \quad (8.3.1)$$

where Δ_g denotes the Laplace-Beltrami operator on (X, g) and where Q_ω is a random perturbation described in detail below. Our aim is to study the large-time evolution of monochromatic Lagrangian states f_h on X under the Schrödinger equation

$$\begin{cases} ih\partial_t u = P_h^\delta u, \\ u|_{t=0} = f_h. \end{cases}$$

In other word we are interested in the propagated Lagrangian state

$$u = e^{i\frac{t}{h}P_h^\delta} f_h, \quad \text{for } t \gg 1. \quad (8.3.2)$$

We will consider the two types of random perturbations Q_ω :

1. The case where Q_ω is the operator of multiplication by a random real-valued function

$$q_\omega : X \longrightarrow \mathbb{R}. \quad (\text{Random Potential case})$$

2. The case where Q_ω is a pseudo-differential operator given by the quantization

$$Q_\omega := \text{Op}_h(q_\omega) \quad (\text{Random } \Psi\text{DO case})$$

of a random real-valued function

$$q_\omega : T^*X \longrightarrow \mathbb{R}$$

belonging to the symbol class $S_\beta^{-\infty}(T^*X)$, see the Appendix for a definition of this notion.

Let us now describe what models of random functions q_ω we consider. Fix a parameter $\beta \in]0, 1/2[$, let $J_h \subset \mathbb{N}$ be a set of indices of cardinality $|J_h| = O(h^{-M})$, for some $M > 0$, and let $\{q_j\}_{j \in J_h}$ be a family of possibly h -dependent smooth compactly supported functions on X , when we are in the [Random Potential case](#), or on T^*X when we are in the [Random \$\Psi\$ DO case](#). To construct a random function on X (resp. on the phase space T^*X) from the single-site potentials q_j , we let $\omega = \{\omega_j\}_{j \in J_h}$ be a sequence of independent and identically distributed (iid) random variables (the precise assumptions we make on the $\{\omega_j\}$ will be described in Hypothesis [8.3.5](#) below) and we set

$$\begin{aligned} q_\omega(\rho) &:= \sum_{j \in J_h} \omega_j q_j(\rho), & \text{in the [Random } \Psi\text{DO case,} \\ q_\omega\(x\) &:= \sum_{j \in J_h} \omega_j q_j\(x\), & \text{in the \[Random Potential case.} \end{aligned} \tag{8.3.3}\]\(#\)](#)$$

We make the following additional assumptions.

Hypothesis 8.3.1 (Hypotheses on the single-site potential).

- i. Each q_j is compactly supported, with a support of diameter $O(h^\beta)$ uniformly in $j \in J_h$.
- ii. There exists $C > 0$, independent of h , such that for all $\rho \in X$ (resp. $\rho \in T^*X$), ρ belongs to the support of at most C functions q_j .
- iii. For any $k \in \mathbb{N}$, there exists $C_k > 0$ such that

$$\|q_j\|_{C^k} \leq C_k h^{-\beta k} \quad \forall j \in J_h. \tag{8.3.4}$$

- iv. There exists $c_0 > 0$ such that, for any $T > 0$ and any $\rho \in S^*X$, we have

$$\sum_{j \in J_h} \int_0^T q_j(\Phi^t(\rho)) dt \geq c_0 T, \tag{8.3.5}$$

in the [Random \$\Psi\$ DO case](#). Here, $\Phi^t : T^*X \rightarrow T^*X$ denotes the geodesic flow. In the [Random Potential case](#), we work with the same assumption but with $q_j(\Phi^t(\rho))$ replaced by $q_j(\pi_X \circ \Phi^t(\rho))$, where $\pi_X : T^*X \rightarrow X$ denotes the projection on the base manifold X .

Example 8.3.2. To build such a family of single-site potentials, one may for instance cover X (resp. S^*X) by geodesic balls $B(\rho_j, h^\beta)$ of radius h^β and centred at ρ_j , such that each point belongs to at most C balls. We may then take

$$q_j = \chi\left(h^{-\beta} \text{dist}(\rho_{j,h}, \rho)\right),$$

where $\chi \in C_c^\infty([0, \infty); [0, 1])$ takes value 1 on $[0, 1]$, and where dist means either dist_X or dist_{T^*X} .

Hypothesis 8.3.3. We suppose that there exists $0 < \varepsilon_0 < \frac{1}{4}$ and $h_0 > 0$ such that for all $h \leq h_0$, we have

$$\delta h^{-2\beta - \varepsilon_0} \leq 1, \tag{8.3.6}$$

$$\delta^2 h^{\beta - 2} \geq h^{-\varepsilon_0}. \tag{8.3.7}$$

In the [Random Potential case](#), we will also need to assume that

$$\delta h^{\beta - 1} \leq h^{\varepsilon_0}. \tag{8.3.8}$$

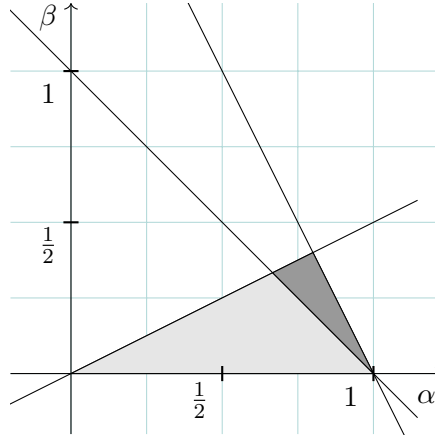


Figure 8.1: Admissible parameters α and β , see Hypothesis 8.3.3. The dark grey region is admissible for the [Random Potential case](#) and the [Random \$\Psi\$ DO case](#), while the light grey region is only admissible in the [Random \$\Psi\$ DO case](#).

Remark 8.3.4. *It is natural to consider the case $\delta = h^\alpha$ with $2\beta \leq \alpha$. However, we will stick with a coupling constant δ for the sake of generality.*

Note that when $\delta = h^\alpha$, conditions (8.3.6) and (8.3.7) rewrite

$$0 < \beta < \min\left(\frac{\alpha}{2}, 2 - 2\alpha\right),$$

while (8.3.8) rewrites

$$\beta > 1 - \alpha.$$

These conditions are plotted on Figure 8.1.

Finally, we need some assumption on the probability distributions ω_j .

Hypothesis 8.3.5. *We suppose that the iid random variables $(\omega_j)_{j \in J_h}$ are real-valued and satisfy the following assumptions:*

1. *The ω_j are bounded.*
2. *$\text{Var}(\omega_j) > 0$.*
3. *We suppose that the random variables $(\omega_j)_{j \in J_h}$ have a common distribution with a compactly supported density $m \in C_c^2(\mathbb{R}; [0, +\infty])$ with respect to the Lebesgue measure.*

8.4 Randomization, local weak limits and the Berry Gaussian field

We aim to compare a noisily propagated Lagrangian state $u = e^{i\frac{t}{h}P_h^\delta} f_h$ (8.3.2) locally near a *randomly chosen point* on X with a stationary isotropic smooth monochromatic Gaussian stochastic process. To do this we will – roughly speaking – pick a point x_0 of X uniformly at random, rescale u near x_0 to the microscopic scale h in local geodesic coordinates, and then compare this now probabilistic rescaled version of u with a Gaussian stochastic process. To make this precise we will recall notions introduced in [122].

Random smooth functions.

In what follows we equip the space $C^\infty(\mathbb{R}^d)$ with the topology of the convergence of all derivatives over all compact sets, i.e. for the topology induced by the family of seminorms

$$\|f\|_n := \max_{x \in B(0, n)} \max_{|\alpha| \leq n} |\partial^\alpha f(x)|, \quad n \in \mathbb{N}.$$

Notice that $C^\infty(\mathbb{R}^d)$ is a separable Fréchet, and therefore a Polish, space. The above topology can metrized with the distance

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min(\|f - g\|_n, 1). \quad (8.4.1)$$

We equip $C^\infty(\mathbb{R}^d)$ with the Borel σ -algebra $\mathcal{B}(C^\infty(\mathbb{R}^d))$. A *random smooth function on \mathbb{R}^d* is a random variable with values in $C^\infty(\mathbb{R}^d)$. We refer the reader to the review [144, Appendix A] for more details on this notion.

We highlight the notion of *convergence in law*. A sequence of random smooth functions $\{\mathfrak{f}_n\}_{n \in \mathbb{N}}$ on \mathbb{R}^d is said to converge in law to a random smooth function \mathfrak{f} on \mathbb{R}^d , i.e.

$$\mathfrak{f}_n \xrightarrow{d} \mathfrak{f}, \quad n \rightarrow \infty,$$

if the laws of the random functions converge weakly. More explicitly, this means that for all bounded continuous functions $F \in C_b(C^\infty(\mathbb{R}^d))$ we have that

$$\mathbb{E}[F(\mathfrak{f}_n)] \longrightarrow \mathbb{E}[F(\mathfrak{f})], \quad n \rightarrow \infty.$$

Randomization and local weak limits.

Our aim is to study the convergence of a sequence of deterministic smooth functions on X near a randomly chosen point at the scale $h > 0$. To avoid any topological difficulties, we define this convergence locally, though all of our results will hold regardless of the choice of localization.

Let $\mathcal{U} \subset X$ be a small enough open set so that we can define an orthonormal frame $V = (V_1, \dots, V_d)$ on it, that is to say a family of smooth sections $(V_i)_{i=1, \dots, d} : \mathcal{U} \rightarrow TX$ such that, for each $x \in \mathcal{U}$, $(V_1(x), \dots, V_d(x))$ is an orthonormal basis of $T_x X$. If $x \in \mathcal{U}$ and $\mathbf{y} \in \mathbb{R}^d$, we will write $\mathbf{y}V(x) := \mathbf{y}_1 V_1(x) + \dots + \mathbf{y}_d V_d(x) \in T_x X$, and

$$\exp_x(\mathbf{y}) := \exp_x(\mathbf{y}V(x)). \quad (8.4.2)$$

Here \exp_x denotes the exponential map restricted to $T_x X$. Note that the map (8.4.2) is well defined for all $\mathbf{y} \in \mathbb{R}^d$ since the underlying Riemannian manifold is complete. All the constructions in this section will depend on the choice of the local frame V , and will hence not be intrinsic.

With the above quantities and definitions in mind, we can define our notion of *local weak limit*.

Definition 8.4.1. (*Local weak limit*) Let (X, g) be a compact smooth Riemannian manifold. Let $\mathcal{U} \subset X$ be an open set and V an orthonormal frame on \mathcal{U} as in (8.4.2). Let $\{f_h\}_{h>0}$ be a family of functions in $C^\infty(X)$, and let \mathfrak{f} be a smooth random function on $C^\infty(\mathbb{R}^d)$. Let x be a random variable with values in \mathcal{U} uniformly distributed with respect to the Riemannian volume measure on \mathcal{U} .

Then, we say that \mathfrak{f} is the local weak limit of $\{f_h\}_h$ in the frame V if the random smooth function $\mathfrak{f}_{x,h}(\mathbf{y}) := f_h(\exp_x(h\mathbf{y}))$ on \mathbb{R}^d converges in law to \mathfrak{f} as $h \rightarrow 0$, i.e. if

$$\mathfrak{f}_{x,h} \xrightarrow{d} \mathfrak{f}, \quad h \rightarrow 0.$$

Let us give some remarks on that definition: $\mathfrak{f}_{x,h}$ is a well defined $C^\infty(\mathbb{R}^d)$ random function \mathbb{R}^d and, by definition, saying that \mathfrak{f} is the local weak limit of $\{\mathfrak{f}_{x,h}\}_h$ in the frame V means that, for any bounded continuous functional $F : C^\infty(\mathbb{R}^d) \rightarrow \mathbb{R}$, we have

$$\mathbb{E}_x[F(\mathfrak{f}_{x,h})] := \frac{1}{\text{Vol}(\mathcal{U})} \int_{\mathcal{U}} F(\mathfrak{f}_{x,h}) dv_g(x) \longrightarrow \mathbb{E}[F(\mathfrak{f})] \quad \text{as } h \rightarrow 0, \quad (8.4.3)$$

where dv_g denotes the Riemannian volume measure on X .

The Berry Gaussian field.

An almost surely (or a.s.) C^∞ (centered) Gaussian field on \mathbb{R}^d is a random variable f taking, up to a set of probability 0, values in $C^\infty(\mathbb{R}^d)$ such that for any finite collection of points $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^d$, the random vector $(f(\mathbf{x}_1), \dots, f(\mathbf{x}_k)) \in \mathbb{C}^d$ is (centered) Gaussian. We say that two fields f_1 and f_2 are *equivalent* if they have the same law. In the sequel, unless otherwise stated, we will always identify fields which are equivalent.

Let f be an a.s. C^∞ , centered Gaussian field on \mathbb{R}^d . Then, its *covariance kernel* $K : (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{E}[f(\mathbf{x})\overline{f(\mathbf{y})}]$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ is *positive definite*, meaning that for each k -tuple $(\mathbf{x}_1, \dots, \mathbf{x}_k) \in (\mathbb{R}^d)^k$, the matrix $K(\mathbf{x}_i, \mathbf{x}_j)_{i,j}$ is positive. As explained for instance in Appendix A.11 of [144], the function K belongs to $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and there is actually a one-to-one correspondence (up to equivalence) between smooth covariance kernels and a.s. C^∞ centred Gaussian fields on \mathbb{R}^d .

Definition 8.4.2. *The Berry Gaussian field with normalization constant $\lambda \in \mathbb{R}$, denoted by BGF_λ , is the unique (up to equivalence) a.s. C^∞ stationary Gaussian field on \mathbb{R}^d whose covariance kernel is $\lambda \int_{\mathbb{S}^{d-1}} e^{i(x-y)\cdot\xi} d\sigma(\xi)$, where σ is the uniform probability measure on \mathbb{S}^{d-1} .*

If $F : C^\infty(\mathbb{R}^d) \rightarrow \mathbb{R}$ is a bounded continuous functional, its expectation with respect to the BGF_λ will be denoted by $\mathbf{E}_{\text{BGF}_\lambda}[F]$.

We remark that the Berry Gaussian field BGF_λ is the unique (up to equivalence) *normalized monochromatic stationary isotropic Gaussian field* with normalization $\mathbf{E}[|\text{BGF}_\lambda(0)|^2] = \lambda$. Indeed, stationary means that its covariance kernel depends only on the difference $(x - y)$. Isotropic means that the covariance kernel is invariant under (the same) rotation of x and y , so it only depends on $|x - y|$. Monochromatic means that the covariance kernel satisfies $-\Delta K = K$ with respect to both variables \mathbf{x} and \mathbf{y} , corresponding to the fact that a realization f of the Berry Gaussian field satisfies $-\Delta f = f$ a.s.

8.5 Emergence of the BGF in noisy long-time propagation of Lagrangian states

Let f_h be a monochromatic Lagrangian state whose associated Lagrangian manifold is η -unstable for η small enough. We study the local weak limit of the propagated Lagrangian state $u_h = e^{i\frac{t}{h}P_h^\delta} f_h$ (8.3.2). Following Definition 8.4.1 we are interested in studying the limiting law of the random function

$$\left(e^{i\frac{t}{h}P_h^\delta} f_h \right) (\exp_{\mathbf{x}}(h\mathbf{y})) \quad (8.5.1)$$

where \mathbf{x} is a uniformly distributed random variable in $\mathcal{U} \subset X$. Notice that in this expression we have two different sources of randomness: one coming from the perturbation Q_ω and one coming from \mathbf{x} . To make this distinction clear we will denote the expectation with respect to \mathbf{x} by $\mathbf{E}_{\mathbf{x}}$, see (8.4.3), and the probability with respect to the law of ω by \mathbf{P}_ω . Accordingly we will denote the associated expectation by \mathbf{E}_ω .

Our first result shows that when fixing \mathbf{x} , away from a set of asymptotically negligible measure, the random function (8.5.1) converges in law to a BGF.

Theorem 8.5.1. *Let X be a compact connected Riemannian manifold with negative sectional curvature and without boundary. Let P_h^δ be as in (8.3.1) and suppose that Hypotheses 8.3.1, 8.3.3 and 8.3.5 are satisfied. Let $D > 0$. There exists $\eta = \eta(D) > 0$ such that the following holds.*

Let $f_h = ae^{\frac{i}{h}\phi}$ be a monochromatic Lagrangian state associated to a Lagrangian manifold which is η -unstable, has distortion $\leq D$ and satisfies $\|\phi\|_{C^3} \leq D$. There exists $X_h^0 \subset X$, with $\text{Vol}(X_h^0) \rightarrow 0$ as $h \rightarrow 0$, such that the following holds: Let $\mathcal{U} \subset X$ be an open set, and V be an orthonormal frame on \mathcal{U} . Let $(t_h)_{h>0}$ be such that $t_h \rightarrow +\infty$, as $h \rightarrow 0$, and $|t_h| = o_{h \rightarrow 0}(|\log h|)$.

Then, for every $x \in \mathcal{U} \setminus X_h^0$

$$\left(e^{\frac{i}{h} t_h P_h^\delta} f_h \right) (\exp_x(h \cdot)) \xrightarrow{d} \text{BGF}_{\lambda_a}$$

with $\lambda_a = \frac{\|a\|^2}{\text{Vol}(X)}$.

The assumption on the distortion of Λ (as defined in (8.2.3)) is purely technical, and it automatically follows from bounds on $\|\phi\|_{C^2}$ if ϕ is defined on a convex set. We insist here on the fact that the convergence in law stated in Theorem 8.5.1 is with respect to the ω random variables coming from the random perturbation and for a fixed x .

Our second main result concerns the random smooth function

$$\left(e^{i \frac{t}{h} P_h^\delta} f_h \right) (\exp_x(h \mathbf{y})) \quad (8.5.2)$$

where \mathbf{x} is a uniformly distributed random variable in $\mathcal{U} \subset X$. The random variable \mathbf{x} generates the law of (8.5.2) which depends on ω . Indeed, this law is, with respect to ω a random probability measure. The result below states that this random law converges weakly in probability (with respect to ω) to the law of the BGF.

Theorem 8.5.2. *Let X be a compact connected Riemannian manifold with negative sectional curvature and without boundary. Let P_h^δ be as in (8.3.1) and suppose that Hypotheses 8.3.1, 8.3.3 and 8.3.5 are satisfied. Let $D > 0$. There exists $\eta = \eta(D) > 0$ such that the following holds:*

Let $\mathcal{U} \subset X$ be an open set, and V be an orthonormal frame on \mathcal{U} . Let $f_h = a e^{\frac{i}{h} \phi}$ be a monochromatic Lagrangian state associated to a Lagrangian manifold which is η -unstable, has distortion $\leq D$ and satisfies $\|\phi\|_{C^3} \leq D$. Let $(t_h)_{h>0}$ be such that $t_h \xrightarrow{h \rightarrow 0} +\infty$ and $|t_h| = o_{h \rightarrow 0}(|\log h|)$. Then, for any $\varepsilon > 0$ and every $F \in C_b(C^\infty(\mathbb{R}^d))$ we have that for $h > 0$ small enough

$$\mathbf{P}_\omega \left[\left| \mathbf{E}_x [F(e^{\frac{i}{h} t_h P_h^\delta} f_h(\exp_x(h \cdot)))] - \mathbf{E}_{\text{BGF}_{\lambda_a}} [F] \right| \geq \varepsilon \right] = \mathcal{O}(h^\infty) \quad (8.5.3)$$

where $\lambda_a = \frac{\|a\|^2}{\text{Vol}(X)}$.

Remark 8.5.3. *The assumption $|t_h| = o_{h \rightarrow 0}(|\log h|)$ can be slightly weakened. Indeed, the proof of Theorem 8.5.2 actually shows that for every $L \in \mathbb{N}$, there exists $c_L > 0$ such that, if $t_h \leq c_L |\log h|$, we have that for any $\varepsilon > 0$ and every $F \in C_b(C^L(\mathbb{R}^d))$ we have that for $h > 0$ small enough*

$$\mathbf{P}_\omega \left[\left| \mathbf{E}_x [F(e^{\frac{i}{h} t_h P_h^\delta} f_h(\exp_x(h \cdot)))] - \mathbf{E}_{\text{BGF}_{\lambda_a}} [F] \right| \geq \varepsilon \right] = \mathcal{O}(h^\infty).$$

Here $F : C^L(\mathbb{R}^d) \rightarrow \mathbb{R}$ is a bounded functional which is continuous for the topology of convergence of derivatives over compact sets.

The possibility of extending our results to longer time scales will be explored in future works.

Corollary 8.5.4. *Under the assumptions of Theorem 8.5.2 we have that for any sequence $h_j \rightarrow 0$, $j \rightarrow \infty$, such that there exists an $M > 0$ such that $(h_j^M)_{j \in \mathbb{N}} \in \ell^1(\mathbb{N})$, we have that for every $F \in C_b(C^\infty(\mathbb{R}^d))$*

$$\mathbf{E}_x [F(e^{\frac{i}{h_j} t_{h_j} P_{h_j}^\delta} f_{h_j}(\exp_x(h_j \cdot)))] \rightarrow \mathbf{E}_{\text{BGF}_{\lambda_a}} [F], \quad j \rightarrow \infty, \quad (8.5.4)$$

ω -almost surely. In particular, ω -almost surely, BGF_{λ_a} is the local weak limit of $\{e^{\frac{i}{h_j} t_{h_j} P_{h_j}^\delta} f_{h_j}\}_{h_j}$ in the frame V .

While the first part of Corollary 8.5.4 follows readily from Theorem 8.5.2 and the Borel-Cantelli lemma, the second part (about the local weak limit) is slightly more involved.

8.6 An application: Improved L^∞ bounds for eigenfunctions

Background There are several ways of describing the delocalization properties of eigenfunctions ψ_λ of the Laplace-Beltrami operator: one is through the use of semiclassical measures, i.e. weak limits of the probability measures $|\psi_\lambda(x)|^2 dx$ (and their lifts to S^*X); another is by studying the L^p norms of ψ_λ , and in particular, their L^∞ norms.

It has been known since the work of Avakumović, Hörmander and Levitan [10, 118, 133] that, on any compact smooth Riemannian manifold, if $-\Delta_g \psi_\lambda = \lambda \psi_\lambda$, $\psi_\lambda \in L^2$, then

$$\|\psi_\lambda\|_{L^\infty} \leq C \lambda^{\frac{d-1}{4}} \|\psi_\lambda\|_{L^2}, \quad (8.6.1)$$

and these bounds are saturated by zonal spherical harmonics. More generally, sharp bounds for the other L^p norms of eigenfunctions were obtained by Sogge in [179].

It is believed that, on manifolds of negative curvature, such bounds can be largely improved, thus reflecting the aforementioned delocalization of eigenfunctions. For instance, on surfaces of negative curvature, it was conjectured by Sarnak [160] that $\|\psi_\lambda\|_{L^\infty} = \mathcal{O}(\lambda^\varepsilon)$ for any $\varepsilon > 0$. However, on negatively curved manifolds of higher dimension, such a bound cannot hold, since [157] gave examples of eigenfunctions whose L^∞ norm grows polynomially.

While the study of semiclassical measures of Laplace eigenfunctions on manifolds of negative curvature has enjoyed major advances in the past years, see for instance [4, 8, 65, 66], much less is known for the L^∞ norms. Bérard [23] proved a logarithmic improvement on (8.6.1) on manifolds of negative curvature, i.e.

$$\|\psi_\lambda\|_{L^\infty} \leq C \frac{\lambda^{\frac{d-1}{4}}}{\sqrt{\log \lambda}} \|\psi_\lambda\|_{L^2}. \quad (8.6.2)$$

In the special setting of certain arithmetic surfaces Iwaniec and Sarnak [126] obtained polynomial improvements on the estimate (8.6.1).

Recently, the estimate (8.6.2) was generalized to milder dynamical conditions than negative curvature by Bonthonneau [32] and Canzani and Galkowski [46]. Similar logarithmic improvements were obtained by several authors for other L^p norms, see [26, 45, 98, 105]. However, no polynomial improvements over (8.6.1) were obtained in non-arithmetic settings.

It is believed that estimates as (8.6.1) or (8.6.2) could more easily be improved in a *generic* setting, for instance by changing generically the metric or adding a generic potential. This is the approach of the works by Eswarathasan and Toth [76] and Canzani, Jakobson and Toth [47]. Yet, in these papers, the authors obtain bounds on the quantity $\psi_\lambda(x)$ averaged over the perturbation, hence, such results do not give information about genuine eigenfunctions of a generically perturbed operator.

Here, we are interested in L^∞ norms of eigenfunctions under generic perturbation. Namely, we consider a manifold of negative curvature and build random perturbations of the Laplace-Beltrami operator by adding a small random pseudo-differential operator with a symbol rapidly oscillating on a mesoscopic scale. Roughly speaking, our main result given in Theorem 8.6.1 below states that

With overwhelming probability, we have a polynomial improvement over (8.6.1) for the eigenfunctions of the perturbed operator.

The setting Let (X, g) be a compact smooth Riemannian manifold of negative sectional curvature, connected and without boundary, and let $0 < \lambda_1 < \lambda_2$. Let Δ_g denote the Laplace-Beltrami operator on X .

In the sequel, we will use semiclassical notations: let $h > 0$ be a small parameter (corresponding to a multiple of $\lambda^{-1/2}$ in the introduction), and $\psi_h \in L^2(\mathbb{R}^d)$ will be a normalized eigenfunction

of $-h^2\Delta_g$, satisfying

$$-h^2\Delta_g\psi_h = E_h\psi_h, \quad (8.6.3)$$

for some $E_h \in (\lambda_1, \lambda_2)$. The bound (8.6.1) can be rephrased as

$$\|\psi_h\|_{L^\infty} = \mathcal{O}(h^{\frac{1-d}{2}}). \quad (8.6.4)$$

Our aim is to show that (8.6.4) can be improved if a small generic perturbation is added to $-h^2\Delta_g$.

8.6.1 Conjugating with noisy quantum dynamics

We work here under the same assumptions as in Section 8.3. The operator whose eigenfunctions we will study in this section is not P_h^δ , see (8.3.1), itself, but rather the operator

$$\tilde{P}_h^\delta := e^{-i\frac{t}{h}P_h^\delta}(-h^2\Delta_g)e^{i\frac{t}{h}P_h^\delta}, \quad (8.6.5)$$

for some fixed $t > 0$, independent of h . By a slightly exotic version of Egorov's theorem A.4, we have

$$\begin{aligned} \tilde{P}_h^\delta &= e^{-i\frac{t}{h}P_h^\delta}P_h^\delta e^{i\frac{t}{h}P_h^\delta} - \delta e^{-i\frac{t}{h}P_h^\delta}Q_\omega e^{i\frac{t}{h}P_h^\delta} \\ &= -h^2\Delta_g + \delta Q_\omega - \delta e^{-i\frac{t}{h}P_h^\delta}Q_\omega e^{i\frac{t}{h}P_h^\delta} \\ &=: -h^2\Delta_g + \delta\tilde{Q}_\omega. \end{aligned}$$

Here, $\tilde{Q}_\omega \in \Psi_\beta^{\text{comp}}(X)$ with principal symbol

$$\tilde{q}_\omega = q_\omega - (\Phi_\delta^t)^*q_\omega + \mathcal{O}(h^{1-2\beta}) \in S_\beta^{\text{comp}}. \quad (8.6.6)$$

The operator \tilde{P}_h^δ may thus be seen as a perturbation of $-h^2\Delta_g$ by a random pseudo-differential operator $\delta\tilde{Q}_\omega$ given in leading order by a pseudo-differential operator with a random symbol in $\delta S_\beta^{\text{comp}}(T^*X)$. The operator \tilde{P}_h^δ has the same eigenvalues as $-h^2\Delta_g$, and if ψ_h is an eigenfunction of $-h^2\Delta_g$, then $e^{-i\frac{t}{h}P_h^\delta}\psi_h$ is an eigenfunction of \tilde{P}_h^δ . Our aim will thus be to obtain improved bounds on

$$\|e^{-i\frac{t}{h}P_h^\delta}\psi_h\|_{L^\infty}$$

with high probability (under appropriate conditions on δ and β).

8.6.2 Improved L^∞ bounds

Recall Hypothesis 8.3.3 and Remark 8.3.4. Let us take $\Gamma > 0$ such that, for h small enough,

$$\begin{aligned} \delta h^{-2\beta} &\leq h^\Gamma \\ h^{1-\Gamma} &\leq \delta h^{\beta/2}, \end{aligned} \quad (8.6.7)$$

which is possible thanks to Hypothesis 8.3.3. In particular, when $\delta = h^\alpha$, we may take

$$\Gamma < \min\left(1 - \alpha - \frac{\beta}{2}, \alpha - 2\beta\right). \quad (8.6.8)$$

We then set

$$\Gamma' := \min\left(\Gamma, \frac{(d-1)\beta}{2}\right). \quad (8.6.9)$$

Theorem 8.6.1. *Let $0 < \lambda_1 < \lambda_2$, let $t > 0$, and let \tilde{P}_h^δ be of the form (8.6.5). There exists $\Omega_h \subset \Omega$ with $\mathbf{P}(\Omega_h) = 1 - \mathcal{O}(h^\infty)$ such that, for all $\omega \in \Omega_h$, the following holds.*

If ψ_h satisfies $\tilde{P}_h^\delta\psi_h = E_h\psi_h$ with $E_h \in (\lambda_1, \lambda_2)$, then, for any Γ as in (8.6.9), we have

$$\|\psi_h\|_{L^\infty} \leq Ch^{\frac{1-d}{2} + \Gamma' - \varepsilon} \|\psi_h\|_{L^2}. \quad (8.6.10)$$

Actually, the same result holds with the same proof for spectral clusters of the form

$$\psi_h = \sum_{\lambda_h \in [E_h, E_h + \mathcal{O}(h)]} \psi_{\lambda_h},$$

where $\tilde{P}_h^\delta \psi_{\lambda_h} = \lambda_h \psi_{\lambda_h}$ and $E_h \in (\lambda_1, \lambda_2)$.

Remark 8.6.2. *The general strategy of the proof of Theorem 8.6.1 is that using the standard normal form for operators of principal type and a suitable microlocal partition of unity, we can show that an eigenfunction ψ_h of the Laplacian can be decomposed as a sum of $\mathcal{O}(h^{1-d-\varepsilon})$ many Lagrangian states. We can then apply our results (or more precisely elements of the proofs leading to these results) on noisy propagation of Lagrangian states to obtain the improved L^∞ bounds. For more details we refer the reader to [124].*

Remark 8.6.3. *When $d = 2$, we may take $\alpha = \frac{5}{7}$ and $\beta = \frac{2}{7} - \varepsilon$, leading to $\Gamma' = \frac{1}{7} - \varepsilon$ for an arbitrarily small $\varepsilon > 0$, i.e.*

$$\|\psi_h\|_{L^\infty} \leq Ch^{-\frac{5}{14}-\varepsilon} \|\psi_h\|_{L^2}. \quad (8.6.11)$$

When $d = 3$, the optimal value of Γ' we can obtain from (8.6.9) is $\Gamma' = \frac{2}{9} - \varepsilon$.

Remark 8.6.4. *By interpolation, one can obtain bounds for the other L^p norms. However, the value of Γ' we can obtain (at least for $d = 2$ or $d = 3$) does not improve Sogge's estimates on the smallest L^p norms, see [204, Theorem 10.10].*

Remark 8.6.5. *In the above, we make the assumption that X is negatively curved, so as to use directly the technical results of the proofs in [123]. However, it is likely that the present argument could be adapted without the negative curvature assumption, possibly with further restrictions on α and β , and on the time t in (8.6.5). This will be pursued elsewhere.*

8.7 Ideas of the proof of Theorems 8.5.1 and 8.5.2

The central tool to obtain the results of the previous paragraph is the WKB method, which gives a precise description of the evolution of a Lagrangian state by the semiclassical Schrödinger equation. Namely, when working on the universal cover \tilde{X} of a manifold of negative curvature, it is standard that the function $e^{i\frac{t}{h}\tilde{P}_h^\delta} \tilde{f}_h$ can be well-approximated by another Lagrangian state:

$$\tilde{a}(\tilde{x}; t, h, \delta) e^{i\frac{t}{h}\tilde{\phi}(\tilde{x}; t, h, \delta)}. \quad (8.7.1)$$

We can show that for the perturbations described in subsection 8.3, we may actually write

$$e^{i\frac{t}{h}\tilde{P}_h^\delta} \tilde{f}_h(\tilde{x}) \approx a(\tilde{x}; t) e^{i\frac{t}{h}\phi(\tilde{x}; t)} e^{i\frac{\delta}{h}\tilde{\Theta}(\tilde{x}; t, h, \delta)}, \quad (8.7.2)$$

so that the randomness of P_h^δ appears only through the random phase $\tilde{\Theta}$. Actually, $\tilde{\Theta}$ can be written as the integral of q_ω over a geodesic going from $\tilde{\Lambda}_\phi$ to \tilde{x} .

When working on the initial manifold X , we need to sum contributions coming from different sheets in the universal cover, so that we get

$$e^{i\frac{t}{h}P_h^\delta} f_h(x) \approx \sum_k a_k(x; t) e^{i\frac{t}{h}\phi_k(x; t)} e^{i\frac{\delta}{h}\Theta_k(x; t, h, \delta)}, \quad (8.7.3)$$

where the number of terms grows exponentially with k . When performing a rescaling at scale h , we get

$$\left(e^{i\frac{t}{h}P_h^\delta} f_h \right) (\exp_x(h\mathbf{y})) \approx \sum_k a_k(x; t) e^{i\frac{t}{h}\phi_k(x; t)} e^{i\frac{\delta}{h}\Theta_k(x; t, h, \delta)} e^{i\mathbf{y} \cdot \nabla \phi_k(x; t)}. \quad (8.7.4)$$

This is thus the sum of a large number of plane waves (in the \mathbf{y} variable) with random phases. We can show that the phases can be made independent by excluding a set of points x of small measure, so that Theorem 8.5.1 will follow from the Central Limit Theorem.

To obtain Theorem 8.5.2, we show that, if x and x' are at a distance $h^{\beta-\varepsilon}$ from each other, then the phases $\Theta_k(x; t, h, \delta)$ and $\Theta_k(x'; t, h, \delta)$ are independent from each other, for most choices of x and x' . This will allow us to transfer the randomness coming from P_h^δ to spatial randomness, obtained by picking the point x at random.

Part IV
Appendix

Appendix A

A brief review of Semiclassical Analysis

In this appendix we review a few notions of semiclassical analysis following the books [63, 67, 90, 140, 164, 204].

A.1 Semiclassical Pseudo-differential Calculus

Let X be a smooth d -dimensional Riemannian manifold without boundary. Let T^*X denote the cotangent bundle and let $\overline{T^*X}$ denote the fiber-radial compactified cotangent bundle, see [190, Section 2] and the book [67, Section E.1.3]. This bundle is a manifold with interior T^*X and boundary diffeomorphic to $S^*X \simeq \partial\overline{T^*X}$.

A.1.1 Symbol classes and quantization on \mathbb{R}^d

Let $d \geq 1$ and note that $T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$. An *order function* $m \in C^\infty(\mathbb{R}^{2d}; [1, \infty[)$, namely a function satisfying the following growth conditions:

$$\exists C_0 \geq 1, \exists N_0 > 0 : \quad m(\rho) \leq C_0 \langle \rho - \mu \rangle^{N_0} m(\mu), \quad \forall \rho, \mu \in \mathbb{R}^{2d}, \quad (\text{A.1.1})$$

with the usual ‘‘Japanese brackets’’ notation $\langle \rho - \mu \rangle := \sqrt{1 + |\rho - \mu|^2}$. We will also sometimes write $(x, \xi) = \rho \in \mathbb{R}^{2d}$, so that $\xi \in \mathbb{R}^d$. To this order function is associated a semiclassical symbol class [63, 204]. For $\eta \in [0, 1/2]$ let

$$S_\eta(m) := S_\eta(\mathbb{R}^{2d}, m) := \left\{ q \in C^\infty(\mathbb{R}_\rho^{2d},]0, 1]_h); \forall \alpha \in \mathbb{N}^{2d}, \exists C_\alpha > 0 : \right. \\ \left. |\partial_\rho^\alpha q(\rho; h)| \leq C_\alpha h^{-\eta|\alpha|} m(\rho), \forall \rho \in \mathbb{R}^{2d}, \forall h \in]0, 1] \right\}. \quad (\text{A.1.2})$$

When $\eta = 0$ we will simply write $S(m) = S_0(m)$. For $h \in]0, 1]$ we let P_h denote the h -Weyl quantization of the symbol $p \in S_\eta(\mathbb{R}^{2d}, m)$,

$$P_h u(x) = p^w(x, hD_x; h)u(x) = \frac{1}{(2\pi h)^d} \iint_{\mathbb{R}^{2d}} e^{\frac{i}{h}(x-y)\cdot\xi} p\left(\frac{x+y}{2}, \xi; h\right) u(y) dy d\xi, \quad (\text{A.1.3})$$

for $u \in \mathcal{S}(\mathbb{R}^d)$, seen as an oscillatory integral. Note that $P_h : \mathcal{S} \rightarrow \mathcal{S}$, and by duality $P_h : \mathcal{S}' \rightarrow \mathcal{S}'$, continuously.

A.1.2 Symbol classes and quantization on general manifolds

For $k \in \mathbb{R}$ and $\eta \in [0, 1/2[$ we say that a smooth function $a(x, \xi; h) \in C^\infty(T^*X \times (0, 1])$ lies in the *symbol class* $S_\eta^k(X)$ if and only if for all compact sets $K \subset X$ and all multiindices $\alpha, \beta \in \mathbb{N}^d$ there exists a constant $C_{\alpha, \beta, K} > 0$ such that

$$\sup_{x \in K} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha, \beta, K} h^{-\eta(|\alpha| + |\beta|)} \langle \xi \rangle^{k - |\beta|}. \quad (\text{A.1.4})$$

Here $\langle \xi \rangle := (1 + |\xi|_g^2)^{1/2}$. Note that this symbol class is independent of the choice of local coordinates. We define a residual class of symbols by $S_\eta^{-\infty}(X) := \bigcap_{k \in \mathbb{R}} S_\eta^k(X)$. Furthermore, $h^\infty S_\eta^{-\infty}(X)$ is defined as follows

$$a \in h^\infty S_\eta^{-\infty}(X) \iff \forall \alpha, \beta \in \mathbb{N}^d, K \Subset X, N > 0: \partial_x^\alpha \partial_\xi^\beta a(x, \xi; h) = \mathcal{O}_{\alpha, \beta, K, N}(h^N \langle \xi \rangle^{-N}),$$

uniformly when x varies in K .

We will also consider the symbol classes $S_\eta^{\text{comp}}(X)$ consisting of compactly supported functions $a(x, \xi; h) \in C^\infty(T^*X \times (0, 1])$ satisfying (A.1.4), with a support bounded independently of h . Hence, $S_\eta^{\text{comp}}(X) \subset S_\eta^k(X)$ for all $k \in \mathbb{R}$. When $\eta = 0$, then we write $S^{\text{comp}}(X) = S_0^{\text{comp}}(X)$ and $S^k(X) = S_0^k(X)$ for simplicity. If U is an open set of T^*X , we will sometimes write $S_\eta^{\text{comp}}(U)$ for the set of functions a in $S_\eta^{\text{comp}}(X)$ such that for any $h \in]0, 1]$, a has its support in U .

Pseudodifferential operators and quantization. A linear continuous map $R = R_h : \mathcal{E}'(X) \rightarrow C^\infty(X)$ is called *negligible*, or in the *residual class* $h^\infty \Psi^{-\infty}$, if its distribution kernel K_R is smooth and each of its $C^\infty(X \times X)$ seminorms is $\mathcal{O}(h^\infty)$, i.e. it satisfies

$$\partial_x^\alpha \partial_y^\beta K_R(x, y) = \mathcal{O}(h^\infty),$$

for all $\alpha, \beta \in \mathbb{N}^d$, when expressed in local coordinates.

A linear continuous map $P_h : C_c^\infty(X) \rightarrow \mathcal{D}'(X)$ is called a *semiclassical pseudo-differential operator* belonging to the space $\Psi_{h, \eta}^m$ if and only if the following two conditions hold:

1. $\phi P_h \psi$ is negligible for all $\phi, \psi \in C_c^\infty(X)$ with $\text{supp } \phi \cap \text{supp } \psi = \emptyset$;
2. for every cut-off chart (κ, χ) there exists a symbol $p_\kappa \in S_\eta^m(\mathbb{R}^d)$ such that

$$\chi P_h \chi = \chi \kappa^* p_\kappa^w(x, hD_x; h) (\kappa^{-1})^* \chi. \quad (\text{A.1.5})$$

Here, $\kappa : X \ni U \rightarrow V \subset \mathbb{R}^d$ is a diffeomorphism between open sets and $\chi \in C_c^\infty(U)$. We refer to the pair (κ, χ) as a *cut-off chart*. In (A.1.5) we use the semiclassical Weyl quantization of the symbols p_κ defined in (A.1.3).

We have the surjective semiclassical principal symbol map

$$\sigma : \Psi_{h, \eta}^m(X) \rightarrow S_\eta^m(X) / h^{1-2\eta} S_\eta^{m-1}(X) \quad (\text{A.1.6})$$

whose kernel is given by $h^{1-2\eta} \Psi_{h, \eta}^{m-1}$ and its right inverse is given by a non-canonical quantization map

$$\text{Op}_h : S_\eta^m(X) \rightarrow \Psi_{h, \eta}^m(X). \quad (\text{A.1.7})$$

Such a (not intrinsic) quantization map can be defined for example as follows: let $(\phi_j, \chi_j)_j$ be a countable family of cut-off charts whose domains U_j cover X and such that $\sum_j \chi_j = 1$ on X . Let $\chi'_j \in C_c^\infty(U_j; [0, 1])$ be equal to 1 near $\text{supp } \chi_j$. Then, given $a \in S_\eta^m(X)$ we define

$$\text{Op}_h(a) := \sum_j \chi'_j \phi_j^* ((\tilde{\phi}_j^{-1})^* (\chi_j a))^w(x, hD_x; h) (\phi_j^{-1})^* \chi'_j. \quad (\text{A.1.8})$$

where $\tilde{\phi}_j : T^*U_j \rightarrow T^*V_j$ is lift of the chart ϕ_j defined by $\tilde{\phi}_j(x, \xi) = (x, (d\phi(x))^{-T} \xi)$. Notice that $\text{Op}_h(a)$ is in particular properly supported. Furthermore, we have that

$$\Psi_{h, \eta}^m(X) = \text{Op}_h(S_\eta^m(X)) + h^\infty \Psi^{-\infty}. \quad (\text{A.1.9})$$

We see that any operator in $\Psi_{h,\eta}^m(X)$ can be represented by a properly supported operator in $\Psi_{h,\eta}^m(X)$ up to a negligible term in $h^\infty\Psi^{-\infty}$.

Compositions of pseudo-differential operators are pseudo-differential operators and we have for properly supported $A \in \Psi_\eta^m(X)$, $B \in \Psi_\eta^{m'}(X)$ that $A \circ B \in \Psi_\eta^{m+m'}(X)$ and

$$\begin{aligned} \sigma_h(A \circ B) &= \sigma_h(A)\sigma_h(B), \\ \sigma_h([A, B]) &= -ih\{\sigma_h(A), \sigma_h(B)\}, \end{aligned} \tag{A.1.10}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket with respect to the natural symplectic structure on T^*X .

A.1.3 Semiclassical Sobolev spaces

We now recall the definition of *semiclassical Sobolev spaces* on X . First of all, when $X = \mathbb{R}^d$, we define for $s \in \mathbb{R}$ the *semiclassical Sobolev space* $H_h^s(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ as the space of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|u\|_{H_h^s(\mathbb{R}^d)} = \|\text{Op}_h(\langle \xi \rangle^s)u\|_{L^2(\mathbb{R}^d)} < \infty.$$

When X is a smooth manifold, we define for $s \in \mathbb{R}$ the *local semiclassical Sobolev space* $H_{h,\text{loc}}^s(X) \subset \mathcal{D}'(X)$ as the set of all distributions $u \in \mathcal{D}'(X)$ such that

$$(\kappa^{-1})^*\chi u \in H_h^s(\mathbb{R}^d)$$

for all local coordinate charts $\kappa : X \supset U \rightarrow V \subset \mathbb{R}^d$ and cut-off functions $\chi \in C_c^\infty(U)$. We can turn $H_{h,\text{loc}}^s(X)$ into a Fréchet space by equipping it with the countable family of seminorms $\{\|(\kappa_k^{-1})^*\chi_k u\|_{H_h^s(\mathbb{R}^d)}\}_{k \in K}$, for any fixed, open and locally finite countable covering of X with coordinate charts $\{\kappa_k : X \supset U_k \rightarrow V_k \subset \mathbb{R}^d\}_{k \in K}$ and subordinate partition of unity $\{\chi_k\}_{k \in K}$, $\chi_k \in C_c^\infty(U_k)$. The topology on $H_{h,\text{loc}}^s(X)$ induced by such a family of seminorms is independent of the choice of open locally finite covering, coordinate charts and partition of unity.

We denote by $H_{h,\text{comp}}^s(X)$ the space of all elements of $H_{h,\text{loc}}^s(X)$ which are supported inside some h -independent compact subset of X . When we are dealing with the case $h = 1$ in the local Sobolev norms, we will simply write $H_{\text{comp}}^s(X)$ and $H_{\text{loc}}^s(X)$ for the corresponding spaces. We note the following regularity result: each $A \in \Psi_{h,\eta}^m(X)$ is bounded uniformly in h on compact sets as an operator

$$A : H_{h,\text{comp}}^s(X) \longrightarrow H_{h,\text{loc}}^{s-m}(X). \tag{A.1.11}$$

When X is compact then

$$H_{h,\text{comp}}^s(X) = H_{h,\text{loc}}^s(X) =: H_h^s(X)$$

and we can equip it with the norm

$$\|u\|_{H_h^s(X)}^2 = \sum_{k \in K} \|(\kappa_k^{-1})^*\chi_k u\|_{H_h^s(\mathbb{R}^d)}^2, \quad \text{Card}(K) < +\infty, \tag{A.1.12}$$

where $\{\kappa_k\}$ is a finite collection of coordinate charts with a subordinate partition of unity $1 = \sum_k \chi_k$ as above. This norm is not intrinsically defined, but taking different coordinate patches and cut-off functions in (A.1.12) yields an equivalent norm.

Similarly we define, for every $L \in \mathbb{N}$, the C^L norms on X , by

$$\|u\|_{C^L(X)} = \max_{k \in K} \|(\kappa_k^{-1})^*\chi_k u\|_{C^L(\mathbb{R}^d)}. \tag{A.1.13}$$

Since X is compact it follows that taking different coordinate patches and cut-off functions in (A.1.13) yields an equivalent norm. By standard arguments one then gets the Sobolev inequalities

$$\|u\|_{C^L(X)} = O_{d,s,L}(1)h^{-d/2-L}\|u\|_{H_h^s(X)} \quad \text{for } s > L + d/2. \quad (\text{A.1.14})$$

On non-compact manifolds it is more delicate to obtain well-defined Sobolev norms. For the following discussion we refer the reader to [163, Appendix A].

A smooth Riemannian manifold X is called a *manifold of bounded geometry* if it has a strictly positive injectivity radius $r_{\text{inj}} > 0$ and every covariant derivative of the Riemann curvature tensor R is bounded, i.e. for every $m = 0, 1, \dots$ there exists a $C_m > 0$ such that $|\nabla^m R| \leq C_m$.

Let X be a manifold of bounded geometry and denote by dv_g the Riemannian density on X . We define the semiclassical Sobolev norm $\|\cdot\|_{H_h^s(X)}$, $s \in \mathbb{N}$, on $C_c^\infty(X)$ by

$$\|u\|_{H_h^s(X)}^2 = \sum_{m=0}^s \int_X |(h\nabla)^m u|^2 dv_g, \quad (\text{A.1.15})$$

where $|\cdot|$ is understood as the norm on tensors induced by the Riemannian metric g .

We then define the Sobolev space $H_h^s(X)$ to be the completion of $C_c^\infty(X)$ with respect to the norm (A.1.15). The space $H_h^s(X)$ has a natural structure of a Hilbert space, and it is naturally included in the space of distributions $\mathcal{D}'(X)$. In particular $H_h^0(X) = L^2(X, dv_g)$, where the latter is defined via the L^2 norm with respect to the integration measure dv_g . The usual embedding theorems hold, i.e. $H_h^s(X) \subset C_b^k(X)$ if $s > k + d/2$.

Since by assumption X has a strictly positive injectivity radius, we have the following result essentially due to M. Gromov [91], see also [163, Lemma A.1.2] for a proof.

Lemma A.1.1. *Let X be a smooth Riemannian manifold of bounded geometry and put $\varepsilon_0 = r_{\text{inj}}/3$. Then, for every $\varepsilon \in (0, \varepsilon_0)$ there exists a countable covering of X by balls of radius ε such that $X = \bigcup_{k \in K} B(x_k, \varepsilon)$ and such that the covering of X by balls $B(x_k, 2\varepsilon)$ with double radius and the same centres satisfies that the maximal number of the balls with non-empty intersection is finite.*

This result implies the existence of a ‘‘uniform’’ partition of unity of X subordinate to the covering by balls from the above Lemma. Indeed, for every $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 > 0$ as in the Lemma above, there exists a partition of unity $1 = \sum_k \chi_k$ on X such that $\chi_k \in C_c^\infty(X; [0, \infty[)$ with $\text{supp } \chi_k \subset B(x_k, 2\varepsilon)$ (the points x_k as in the above Lemma) and such that for every $\alpha \in \mathbb{N}_d$ there exists a constant $C_\alpha > 0$ such that $|\partial_y^\alpha \chi_k(y)| \leq C_\alpha$ in geodesic normal coordinates and uniformly with respect to k .

Using this partition of unity we can give an alternative definition of the semiclassical Sobolev norm $\|u\|_{h,s}$. Indeed, let $\kappa_k : B(x_k, 2\varepsilon) =: U_k \rightarrow V_k \subset \mathbb{R}^d$ be the local geodesic coordinate chart in $B(x_k, 2\varepsilon)$, then we can define for $s \in \mathbb{R}$

$$\|u\|_{H_h^s(X)}^2 = \sum_{k \in K} \|(\kappa_k^{-1})^* \chi_k u\|_{H_h^s(\mathbb{R}^d)}^2. \quad (\text{A.1.16})$$

The norms (A.1.15) and (A.1.16) are equivalent for $s \in \mathbb{N}$.

In chapter 8 we are working with a compact smooth Riemannian manifold X of negative sectional curvature and with its universal cover \tilde{X} . Let $\kappa_k : U_k \rightarrow V_k \subset \mathbb{R}^d$, $k = 1, \dots, M$, be the local geodesic coordinates on X with $U_k = B(x_k, 2\varepsilon)$ as in the discussion above (A.1.16), such that the coordinate patches U_k , $k = 1, \dots, M$, form a finite open covering of X . Furthermore, let $\chi_k \in C_c^\infty(X; [0, 1])$, $k = 1, \dots, M$, be a finite partition of unity of X subordinate to this open covering. Using that the covering map is a local isometry, we find that the lifted coordinate charts $\tilde{\kappa}_{k,\iota} = \tilde{\pi}_{k,\iota}^* \kappa_k$, where $\tilde{\pi}_{k,\iota} := \tilde{\pi}|_{U_{k,\iota}} : \tilde{U}_{k,\iota} \rightarrow U_k$, as in the beginning of Section A.1.4, are local geodesic coordinates on \tilde{X} , the universal covering of X , with coordinate patches $\tilde{U}_{k,\iota} =$

$B(\tilde{\pi}_{k,\ell}^{-1}(x_k), 2\epsilon)$. Furthermore, the coordinate patches form a locally finite open covering of \tilde{X} . The lifted cut-off functions $\chi_{k,\ell}$ form a locally finite partition of unity on \tilde{X} , i.e. $\sum_{k,\ell} \tilde{\chi}_{k,\ell} = 1$ on \tilde{X} . We will be working mostly with the following Sobolev and C^L norms: let $\tilde{U} \subset X$ be an open set, then

$$\|u\|_{H_h^s(\tilde{X})}^2 = \sum_{k,\ell} \|(\kappa_{k,\ell}^{-1})^* \chi_{k,\ell} u\|_{H_h^s(\mathbb{R}^d)}^2, \quad \|u\|_{C^L(\tilde{U})} := \max_{k,\ell} \|(\kappa_{k,\ell}^{-1})^* \chi_{k,\ell} u\|_{C^L(\mathbb{R}^d \cap \kappa_{k,\ell}(\tilde{U} \cap \tilde{U}_{k,\ell}))}. \tag{A.1.17}$$

A.1.4 Lifting pseudodifferential operators to the universal cover

In the sequel, (X, g) will be a smooth connected Riemannian manifold of negative sectional curvature without boundary. We denote by r_I its injectivity radius, which is a finite positive number as soon as X is compact.

We write $\chi_1 \succ \chi_2$ if $\chi_1, \chi_2 \in C_c^\infty$ take values in $[0, 1]$ and $\text{supp } \chi_2 \subset \mathcal{C} \text{supp } (1 - \chi_1)$. Similarly, we write for an open relatively compact set K that $\chi \succ \mathbf{1}_K$ and $\mathbf{1}_K \succ \chi$, if $\overline{K} \subset \mathcal{C} \text{supp } (1 - \chi)$ and $\text{supp } \chi \subset K$, respectively.

If M is a matrix, its transpose will be denoted by M^\dagger . If A is a measurable subset of \mathbb{R}^d or of a Riemannian manifold, its volume will be denoted either by $\text{Vol}(A)$ or by $|A|$. If A is a finite set, we will denote its cardinality by $\text{Card}(A)$ or $|A|$. Writing $a \asymp b$ means that there exists a constant $C > 1$ such that $C^{-1}a \leq b \leq Ca$.

Cotangent space We denote by dist_X the geodesic distance on X . We denote by T^*X the cotangent bundle of X , and by $\pi_X : T^*X \rightarrow X$ the canonical projection. We recall that the cotangent space T^*X can be equipped in a canonical way with a symplectic form σ .

By $|\cdot|_x$ and by $\langle \cdot, \cdot \rangle_x$ we denote the norm and scalar product on T_x^*X (respectively on T_xX whenever convenient) induced by the metric g . Furthermore, we equip the cotangent bundle T^*X with an arbitrary metric g_0 such that the induced geodesic distance dist_{T^*X} on T^*X is so that $\text{dist}_{T^*X}(\rho_1, \rho_2) \geq c \text{dist}_X(\pi_X(\rho_1), \pi_X(\rho_2))$ for some fixed constant $c > 0$. This is for instance the case when we take g_0 to be the Sasaki metric on T^*X induced by g .

We will denote by $S^*X \subset T^*X$ the unit cotangent bundle, and by $\Phi^t : T^*X \rightarrow T^*X$ the geodesic flow. We will denote by the same letter its restriction $\Phi^t : S^*X \rightarrow S^*X$.

Universal cover We will denote the universal cover of X by $\tilde{\pi} : \tilde{X} \rightarrow X$. Since X is a connected Riemannian manifold of negative sectional curvature, \tilde{X} is a simply-connected manifold of negative sectional curvature. We equip \tilde{X} and $T^*\tilde{X}$ with the lifted Riemannian metrics \tilde{g} and \tilde{g}_0 , respectively. We denote by $\hat{\pi} : T^*\tilde{X} \rightarrow T^*X$ the local diffeomorphism given by $\hat{\pi}(\tilde{x}, \tilde{\xi}) = (\tilde{\pi}(\tilde{x}), (d_{\tilde{x}}\tilde{\pi})^{-T}\tilde{\xi})$. We will be working not only on a compact connected smooth Riemannian manifold X of negative sectional curvature, but also on its universal cover \tilde{X} . It will be useful to discuss lifting a pseudo-differential operator P on X to a pseudo-differential operator \tilde{P} on \tilde{X} .

Recall the covering map $\tilde{\pi} : \tilde{X} \rightarrow X$ and the map $\hat{\pi} : T^*\tilde{X} \rightarrow T^*X$ defined above. By the covering property we have that if $U \subset X$ is a sufficiently small open set, then $\tilde{\pi}^{-1}(U) = \bigsqcup_{\ell \in I} \tilde{U}_\ell$ is a countable union of disjoint open sets. Furthermore, the restriction $\tilde{\pi}_\ell := \tilde{\pi}|_{\tilde{U}_\ell} : \tilde{U}_\ell \rightarrow U$ is a diffeomorphism. Hence, we may lift a chart $\phi : U \rightarrow V \subset \mathbb{R}^d$ to a chart $\tilde{\phi}_\ell := (\tilde{\pi}_\ell^{-1})^* \phi : \tilde{U}_\ell \rightarrow V$ on \tilde{X} . Similarly, we can lift a function $\chi \in C_c^\infty(U)$ to $\tilde{\chi}_\ell := (\tilde{\pi}_\ell^{-1})^* \chi \in C_c^\infty(\tilde{U}_\ell)$. Slightly abusing notation, we will also denote by $\tilde{\chi}_\ell$ its extension by 0 outside its support to a smooth compactly supported function on \tilde{X} .

Lifting a pseudo-differential operator. Given a semiclassical pseudo-differential operator $P \in \Psi_{h,\eta}^m(X)$ with principal symbol $\sigma(P) = p \in S_\eta^m(T^*X)$, we say that $\tilde{P} \in \Psi_{h,\eta}^m(\tilde{X})$ is a lift of P to the universal cover \tilde{X} if the following three conditions hold:

1. the principal symbol of \tilde{P} is given by lifted principal symbol of P , i.e.

$$\sigma(\tilde{P}) = \hat{\pi}^* \sigma(P) \in S_\beta^m(T^* \tilde{X}); \quad (\text{A.1.18})$$

2. for every cut-off chart (κ, χ) on X such that (A.4.20) holds with $p_\kappa \in S_\eta^m(T^* \mathbb{R}^d)$, we have that

$$(\tilde{\kappa}_\iota^{-1})^* \tilde{\chi}_\iota \tilde{P}_h \tilde{\chi}_\iota \tilde{\kappa}_\iota^* = \chi \circ \kappa^{-1} \text{Op}_h(p_\kappa) \chi \circ \kappa^{-1}. \quad (\text{A.1.19})$$

3. for every $\phi, \psi \in C_c^\infty(X)$ with $\text{supp } \phi \cap \text{supp } \psi = \emptyset$ which may depend on h as in the paragraph after (A.4.20), we have that $\tilde{\phi}_\iota \tilde{P}_h \tilde{\psi}_\iota$ is negligible and that for every $N \in \mathbb{N}$

$$\tilde{\phi}_\iota \tilde{P}_h \tilde{\psi}_\iota = O_{\phi, \psi, N}(h^\infty) : H_h^{-N}(\tilde{X}) \rightarrow H_h^N(\tilde{X}). \quad (\text{A.1.20})$$

In other words $\tilde{\phi}_\iota \tilde{P}_h \tilde{\psi}_\iota$ is a bounded operator $H_h^{-N}(\tilde{X}) \rightarrow H_h^N(\tilde{X})$ with operator norm $= O_{\phi, \psi, N}(h^\infty)$ which is independent of ι, ι' .

Given a $P \in \Psi_{h, \eta}^m(X)$, such a lift \tilde{P} always exists. For instance, we may construct \tilde{P} from the non-canonical quantization (A.4.21) by lifting the cut-off charts. More precisely

$$\tilde{P}_h = \sum_{k \in K, \iota \in I} \psi_{k, \iota} \kappa_{k, \iota}^* \text{Op}_h(p_{\kappa_k}) (\kappa_{k, \iota}^{-1})^* \psi_{k, \iota} \in \Psi_{h, \eta}^m(\tilde{X}) \quad (\text{A.1.21})$$

satisfies (A.1.18–A.1.20).

A.2 Wavefront set

Wavefront set of a family of L^2 functions For $u = (u(h))_{h \in]0, 1]}$ a bounded family in $L^2(\mathbb{R}^d)$, its semiclassical wavefront set $\text{WF}_h(u)$ denotes the phase space region where u is h -microlocalized:

$$\text{WF}_h(u) \stackrel{\text{def}}{=} \mathfrak{C} \left\{ (x, \xi) \in \mathbb{R}^{2d}; \exists a \in C_c^\infty(\mathbb{R}^{2d}), a(x, \xi) = 1, \|a^w(x, hD_x)u(h)\|_{L^2} = \mathcal{O}(h^\infty), \right\} \quad (\text{A.2.1})$$

where a^w denotes the Weyl quantization of a . This notion can readily be generalized to h -tempered distributions on smooth manifolds. However, we do not need this here.

Wavefront sets of pseudo-differential operators For $a \in S_\eta^k(T^*X)$ we define its essential support $\text{ess-supp } a \subset \overline{T^*X}$ as follows: a point $\rho \in \overline{T^*X}$ is not contained in $\text{ess-supp } a$ if there exists a neighborhood U of ρ in $\overline{T^*X}$ such that for all $\alpha, \beta \in \mathbb{N}^d$, $N > 0$ there exists a constant $C_{\alpha, \beta, N} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha, \eta, N} h^N \langle \xi \rangle^{-N}, \quad (x, \xi) \in U \cap T^*X.$$

For $P_h \in \Psi_{h, \eta}^m$ we define the *wavefront set* $\text{WF}_h(P_h) \subset \overline{T^*X}$ as follows: a point $(x, \xi) \in \overline{T^*X}$ does not lie in $\text{WF}_h(P_h)$ if and only if for each cut-off chart (ϕ, χ) , such that x is contained in the domain of ϕ , we have that $(\phi(x), (d\phi(x))^{-T} \xi) \notin \text{ess-supp } p_\kappa$, with p_κ defined as in (A.1.5).

If $A \in \Psi_{h, \eta}^m$, $B_h \in \Psi_{h, \eta'}^{m'}$ are properly supported, then

$$\text{WF}_h(AB) \subset \text{WF}_h(A) \cap \text{WF}_h(B). \quad (\text{A.2.2})$$

In particular, if $b \equiv 1$ on $\text{WF}_h(A)$, we have

$$A = \text{Op}_h(b)A + h^\infty \Psi^{-\infty}. \quad (\text{A.2.3})$$

Compactly supported pseudo-differential operators in $\Psi_\eta^m(X)$ with compact wavefront sets in T^*X are called *compactly microlocalized* and we will denote this class by $\Psi_\eta^{\text{comp}}(X)$. Notice that $\Psi_\eta^{\text{comp}}(X) \subset \Psi_\eta^m(X)$ for all $m \in \mathbb{R}$. Furthermore,

$$\Psi_{h,\eta}^{\text{comp}}(X) = \text{Op}_h(S_\eta^{\text{comp}}(X)) + h^\infty \Psi^{-\infty}. \quad (\text{A.2.4})$$

When we make use of this identity for an $A \in \Psi_{h,\eta}^{\text{comp}}(X)$ then we call an $a \in S_\eta^{\text{comp}}(X)$, such that $A = \text{Op}_h(a) + h^\infty \Psi^{-\infty}$, a *full symbol* of A (though this notion is not intrinsic and depends on a choice of quantization). For $A = \text{Op}_h(a) \in \Psi_h^{\text{comp}}(X)$, we have

$$\text{WF}_h(A) = \text{ess-supp } a, \quad (\text{A.2.5})$$

noting that this does not depend on the choice of the quantization. When K is a compact subset of T^*X and $\text{WF}_h(A) \subset K$, we will sometimes say that A is *microsupported* inside K .

The principal symbol map

$$\sigma_h : \Psi_{h,\eta}^{\text{comp}}(X) \rightarrow S_\eta^{\text{comp}}(X) / (h^{1-2\eta} S_\eta^{\text{comp}}(X) + h^\infty S^{-\infty}(X))$$

is surjective with kernel given by $h^{1-2\eta} \Psi_{h,\eta}^{\text{comp}}(X)$. Thanks to (A.2.2), the composition of two compactly microlocalized operators is still compactly microlocalized, i.e.

$$A, B \in \Psi_\eta^{\text{comp}}(X) \quad \Rightarrow \quad A \circ B \in \Psi_\eta^{\text{comp}}(X),$$

and (A.1.10) is still valid. Moreover, the standard composition formulas of symbols in \mathbb{R}^{2d} , see for instance [63, Proposition 7.6] show also that when $A \in \Psi_0^k(X)$ and $B \in \Psi_\eta^{\text{comp}}(X)$ then $A \circ B \in \Psi_\eta^{\text{comp}}(X)$ and

$$\sigma_h(AB) \equiv \sigma_h(A)\sigma_h(B) \quad \text{mod } h^{1-\eta} S_\eta^{\text{comp}}(X) + h^{-\infty} S^\infty(X). \quad (\text{A.2.6})$$

A.3 Microlocalization

Given the notion of semiclassical wavefront set, it is natural to consider operators and their properties *microlocally*. Let X_1, X_2 be two smooth d -dimensional manifolds and let $h \in]0, h_0]$. We say that a family of distributions $u_h \in \mathcal{D}'(X_1)$ is h -tempered if for each $\chi \in C_c^\infty(X_1)$ there exist constants $N \geq 0$ and $C > 0$ such that

$$\|\chi u_h\|_{H_h^{-N}(X_1)} \leq Ch^{-N},$$

where $\|\cdot\|_{H_h^{-N}(X_1)}$, $s \in \mathbb{R}$, denotes the semiclassical Sobolev norm of order s , see e.g. [67, Definition E.19]. We say that a family of operators $T_h : C_c^\infty(X_2) \rightarrow \mathcal{D}'(X_1)$ is h -tempered if the family of associated Schwartz kernels $K_{T_h} \in \mathcal{D}'(X_1 \times X_2)$ is h -tempered.

For open sets $V \Subset T^*X_1$ and $U \Subset T^*X_2$, the operators *defined microlocally* near $V \times U$ are given by the equivalence classes of tempered operators defined by the following relation: $T \sim T'$ if and only if there exists open sets $\tilde{V} \Subset T^*X_1$, $\tilde{U} \Subset T^*X_2$ with $U \Subset \tilde{U}$, $V \Subset \tilde{V}$ such that, for any $A \in \Psi_h^{\text{comp}}(X_1)$, $B \in \Psi_h^{\text{comp}}(X_2)$ with $\text{WF}_h(A) \subset \tilde{V}$, $\text{WF}_h(B) \subset \tilde{U}$, we have

$$A(T - T')B = \mathcal{O}(h^\infty)_{\Psi^{-\infty}} \quad (\text{A.3.1})$$

For two such operators we say that $T = T'$ *microlocally* near $V \times U$. Similarly, we say that $S = T^{-1}$ *microlocally* near $V \times V$ (and thus, that S is *microlocally invertible* near $V \times V$), if $ST = Id$ *microlocally* near $U \times U$, and $TS = Id$ *microlocally* near $V \times V$.

A.4 A slightly exotic version of Egorov's Theorem

To finish, we present a version of Egorov's theorem for symbols in an exotic symbol class. This theorem, which relates the evolution of quantum and classical observables, is usually stated when the Hamiltonian generating the classical evolution is independent of h , see for instance [204, Theorem 11.1]. In our case the Hamiltonian $p = |\xi|_x^2 + \delta q_\omega$ induced by (8.3.1), depends on h , and so we need this more general version.

Let X be a compact smooth Riemannian manifold, and recall that the operator P_h^δ given in (8.3.1) is of the form

$$P_h^\delta := -\frac{h^2}{2}\Delta + \delta Q, \quad 0 \leq \delta \ll 1,$$

with $Q \in \Psi_\beta^{-\infty}(X)$ self-adjoint. The operator P_h^δ does thus have with full symbol $p(x, \xi; \delta) = -\frac{1}{2}|\xi|_g^2 + \delta q$, $q \in S_\beta^{-\infty}(T^*X)$.

Recall from (8.3.6) that we assume that there exists $0 < \varepsilon_0 < \frac{1}{4}$ and $h_0 > 0$ such that for all $h \leq h_0$, we have

$$\delta h^{-2\beta-\varepsilon_0} \leq 1. \quad (\text{A.4.1})$$

By the Kato-Rellich theorem we know that the operator P_h^δ is selfadjoint with domain $H_h^2(X)$. By Stone's Theorem (see for instance [204, Theorem C.13]), we then know that it induces a strongly continuous unitary group

$$U_\delta(t) = e^{-i\frac{t}{h}P_h^\delta} = \mathcal{O}(1) : L^2(X) \rightarrow L^2(X), \quad t \in \mathbb{R}. \quad (\text{A.4.2})$$

Since the Laplacian $-\Delta_g$ is a positive elliptic second order differential operator on X , we can equip the Sobolev space $H_h^k(X)$ with the norm $\|(1 - h^2\Delta_g)^{k/2}f\|_{H_h^0}$. Since P_h^δ and $U_\delta(t)$ commute, we see that

$$\begin{aligned} \|(1 - h^2\Delta_g)U_\delta(t)f\|_{H_h^k} &\leq \|U_\delta(t)(1 + P_h^\delta)f\|_{H_h^k} + \|\delta Qf\|_{H_h^k} \\ &\leq \|U_\delta(t)(1 + P_h^\delta)f\|_{H_h^k} + \mathcal{O}_k(\delta)\|f\|_{H_h^0} \\ &\leq \|U_\delta(t)(1 - h^2\Delta_g)f\|_{H_h^k} + \|U_\delta(t)\delta Qf\|_{H_h^k} + \mathcal{O}_k(\delta)\|f\|_{H_h^0}, \end{aligned} \quad (\text{A.4.3})$$

where in the last line we used that since Q is compactly microlocalized, it follows that $Q = \mathcal{O}_N(1) : H_h^{-N} \rightarrow H_h^N$. When $k = 0$, then (A.4.2) and (A.4.3) yield that $U_\delta(t) = \mathcal{O}(1) : H_h^2(X) \rightarrow H_h^2(X)$. Iterating this argument, shows that $U_\delta(t) = \mathcal{O}_n(1) : H_h^{2n}(X) \rightarrow H_h^{2n}(X)$, and we deduce by duality and interpolation that

$$U_\delta(t) = \mathcal{O}_s(1) : H_h^s(X) \rightarrow H_h^s(X), \quad s \in \mathbb{R}. \quad (\text{A.4.4})$$

Let H_p denote the Hamilton vector flow induced by p , and let Φ_δ^t denote the associated Hamilton flow. Let $K \subset T^*X$ be a compact set and let $T > 0$, then there exists a compact set $K_T \subset T^*X$, independent of h and δ , such that $\Phi_\delta^t(K) \subset K_T$ for all $t \in [-T, T]$. For each k , fix a norm $\|\cdot\|_{C^k(U; K_T)}$, by covering U and K_T in finitely many local coordinate charts, for the space $C^k(U; K_T)$ of k times continuously differentiable functions on U with values in K_T . Then, by mimicking the proof of [123, Lemma 4.1] for finite times, we find that for each $k \in \mathbb{N}$

$$\|\Phi_\delta^t\|_{C^k(U; K_T)} \leq \mathcal{O}_{k,T}(1)(1 + \delta h^{-\beta(k+1)}) \quad (\text{A.4.5})$$

uniformly in $t \in [-T, T]$.

Proposition A.4.1. *Let $\beta \in [0, 1/2[$. Then for each $t \in \mathbb{R}$ and each $A \in \Psi_\beta^{\text{comp}}(X)$ there exists a $A_\delta^t \in \Psi_\beta^{\text{comp}}(X)$, such that*

$$U_\delta(-t)AU_\delta(t) = A_\delta^t + h^\infty\Psi^{-\infty}. \quad (\text{A.4.6})$$

Moreover, $\text{WF}_h(A_\delta^t) \subset \Phi_\delta^t(\text{WF}_h(A))$ and $\sigma(A_\delta^t) = \sigma(A) \circ \Phi_\delta^t + \mathcal{O}(h^{1-2\beta}) \in S_\beta^{\text{comp}}(X)$.

Proof. We will mimic the proof of [204, Theorem 11.1].

1. Let $a \in S_\beta^{\text{comp}}$ be the full symbol of A and define

$$a^t := a \circ \Phi_\delta^t. \quad (\text{A.4.7})$$

Let $t \in [-T, T]$, and notice that a^t is compactly supported in $\Phi_\delta^{-t}(\text{supp } a)$. Since Φ_δ^t is a family of diffeomorphisms depending smoothly on t , there exists a compact set $K \subset T^*X$ such that $\Phi_\delta^{-t}(\text{supp } a) \subset K$ for all $t \in [-T, T]$.

Recall that the derivatives of the composition of a smooth functions $f \in C^\infty(\mathbb{R}^d)$ and $g \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ are given by the generalized Fàa di Bruno formula

$$\partial^\alpha(f \circ g) = \sum_{\alpha_\ell, j} c_{\alpha, j} (\partial_{x_{j_1} \dots x_{j_m}} f) \circ g \cdot \prod_{\ell=1}^m \partial^{\alpha_\ell} \psi_{j_\ell}, \quad \alpha \in \mathbb{N}^d, \quad (\text{A.4.8})$$

where $c_{\alpha, j}$ are constants $j_\ell \in \{1, \dots, d\}$ and $\alpha_1, \dots, \alpha_m$ are multiindices whose sum is equal to α . Since $a \in S_\beta^{\text{comp}}(X)$, it follows from (A.4.5) and (A.4.8), that $a^t \in S_\beta^{\text{comp}}(X)$.

2. Set $B(t) := U_\delta(-t)AU_\delta(t)$ and define a family of operators $B_k(t)$, $k \in \mathbb{N}$, iteratively, as follows: First, put $B_0(t) = \text{Op}_h(a^t) \in \Psi_\beta^{\text{comp}}$. Then,

$$[P_\delta, B_0(t)] = [P_0, B_0(t)] + \delta[Q, B_0(t)] = \frac{h}{i} \text{Op}_h(\{p, a^t\}) - \widehat{E}_0(t) - \widetilde{E}_0(t), \quad (\text{A.4.9})$$

where $\widehat{E}_0(t) = [P_0, B_0(t)] - \frac{h}{i} \text{Op}_h(\{p, a^t\}) \in h^{2(1-\beta)} \Psi_\beta^{\text{comp}}$ and $\widetilde{E}_0(t) = \delta[Q, B_0(t)] \in \delta h^{2(1-2\beta)} \Psi_\beta^{\text{comp}}$ by (A.1.10) and (A.2.6). By (A.4.1), it follows that $E_0(t) := \widehat{E}_0(t) + \widetilde{E}_0(t) \in h^{2(1-\beta)} \Psi_\beta^{\text{comp}}$. Notice that $E_0(t)$ depends smoothly on t since this is the case for $B_0(t)$ and a^t . Let $e_0(t) \in h^{2(1-\beta)} S_\beta^{\text{comp}}$ be the full symbol of $E_0(t)$. Since $\partial_t a^t = H_p a^t$, it follows from (A.4.9) that

$$hD_t B_0(t) = [P_h^\delta, B_0(t)] + E_0(t). \quad (\text{A.4.10})$$

Next define a sequence of symbols $e_k(t) \in h^{k(1-2\beta)+2(1-\beta)} S_\beta^{\text{comp}}$, $c_{k+1}(t) \in h^{(k+1)(1-2\beta)} S_\beta^{\text{comp}}$, $k \in \mathbb{N}$, depending smoothly on t , with $e_0(t) \in h^{2(1-\beta)} S_\beta^{\text{comp}}$ as above and

$$c_{k+1}(t) = \frac{i}{h} \int_0^t (\Phi_\delta^{t-s})^* e_k(s) ds, \quad c_{k+1}(0) = 0, \quad (\text{A.4.11})$$

and $e_{k+1}(t)$ being the full symbol of $E_{k+1}(t) \in h^{(k+1)(1-2\beta)+2(1-\beta)} \Psi_\beta^{\text{comp}}$ defined via the relation

$$\begin{aligned} hD_t \text{Op}_h(c_{k+1}(t)) &= \frac{h}{i} \text{Op}_h(\{p, c_{k+1}(t)\}) + \text{Op}_h(e_k(t)) \\ &= [P_h^\delta, \text{Op}_h(c_{k+1}(t))] - E_{k+1}(t) + \text{Op}_h(e_k(t)). \end{aligned} \quad (\text{A.4.12})$$

Here, we used the fact that $hD_t c_{k+1}(t) = H_p c_{k+1}(t) + e_k(t)$ in the first line, and (A.1.10), (A.2.6) in the last line. Put

$$B_{k+1}(t) = B_k(t) - \text{Op}_h(c_{k+1}(t)). \quad (\text{A.4.13})$$

Then, for each $k \in \mathbb{N}^*$

$$hD_t B_k(t) = [P_h^\delta, B_k(t)] + E_k(t) + h^\infty \Psi^{-\infty}. \quad (\text{A.4.14})$$

3. By Borel summation, there exists a $b(t) \in S_\beta^{-\infty}(X)$ such that

$$b(t) \sim \sum_{k \in \mathbb{N}} b_k(t), \quad b_0(t) = a(t) \in S_\beta^{\text{comp}}, \quad b_k(t) = -c_k(t) \in h^{k(1-2\beta)} S_\beta^{\text{comp}}, \quad k \geq 1.$$

Hence, $\text{Op}_h(b(0)) = A + A_1$, with $A_1 \in h^\infty\Psi^{-\infty}$, and

$$hD_t\text{Op}_h(b(t)) = [P_h^\delta, \text{Op}_h(b(t))] + R(t), \quad R(t) \in h^\infty\Psi^{-\infty}$$

By integration, we get that

$$\begin{aligned} U_\delta(-t)AU_\delta(t) - \text{Op}_h(b(t)) &= U_\delta(-t)(A - U_\delta(t)\text{Op}_h(b(t))U_\delta(-t))U_\delta(t) \\ &= -U_\delta(-t) \left[\int_0^t hD_s(U_\delta(s)\text{Op}_h(b(s))U_\delta(-s)) ds \right] U_\delta(t) - U_\delta(-t)A_1U_\delta(t) \\ &= - \int_0^t U_\delta(s-t)R(s)U_\delta(t-s)ds - U_\delta(-t)A_1U_\delta(t). \end{aligned} \tag{A.4.15}$$

Since $R(s) \in h^\infty\Psi^{-\infty}$, we see by (A.4.4) that $U_\delta(-t)AU_\delta(t) - \text{Op}_h(b(t)) \in h^\infty\Psi^{-\infty}$. Applying the same argument to the term containing A_1 , we conclude (A.4.6).

The statement on the wave front set follows from (A.2.5) and the observation that $\text{supp } c_{k+1}(t) \subset \Phi_\delta^{-t}(\text{supp } a)$, see (A.4.11), as is true for $a(t)$. \square

A.4.1 Semiclassical pseudo-differential operators

Let $m \in \mathbb{R}$ and let $\eta \in [0, 1/2[$. We consider the class of symbols

$$S_\eta^m(T^*X) = \left\{ a(\cdot; h) \in C^\infty(T^*X); h \in]0, 1], |\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha, \beta} h^{-\eta(|\alpha| + |\beta|)} \langle \xi \rangle^{m - |\beta|} \right\}, \tag{A.4.16}$$

where $\langle \xi \rangle = (1 + |\xi|_x^2)$.

We will define the symbol space of order $-\infty$ by $S_\eta^{-\infty}(T^*X) := \bigcap_m S_\eta^m(T^*X)$. A linear continuous map $R = R_h : \mathcal{E}'(X) \rightarrow C^\infty(X)$ is called *negligible* if its distribution kernel K_R is smooth and each of its $C^\infty(X \times X)$ seminorms is $O(h^\infty)$, i.e. it satisfies

$$\partial_x^\alpha \partial_y^\beta K_R(x, y) = O(h^\infty), \tag{A.4.17}$$

for all $\alpha, \beta \in \mathbb{N}^d$, when expressed in local coordinates.

A linear continuous map $P_h : C_c^\infty(X) \rightarrow \mathcal{D}'(X)$ is called a *semiclassical pseudo-differential operator* belonging to the space $\Psi_{h, \eta}^m$ if and only if we can express P_h as

$$P_h = \sum_{k \in K} \chi_k \kappa_k^* \text{Op}_h(p_k) (\kappa_k^{-1})^* \chi_k + K_h, \tag{A.4.18}$$

where $p_k \in S_\eta^m(T^*\mathbb{R}^d)$, K_h is negligible, the $\kappa_k : U_k \rightarrow V_k$ are a collection of diffeomorphisms between open sets $U_k \subset X$ and $V_k \subset \mathbb{R}^d$ with the collection of U_k being locally finite, and $\chi_k \in C_c^\infty(U_k)$. We will refer to the induced family $(\kappa_k, \chi_k)_k$ as *cut-off charts*. In (A.4.18) we use the standard semiclassical quantization of the symbols p_k

$$\text{Op}_h(p_k)u(x) = \frac{1}{(2h\pi)^d} \iint_{\mathbb{R}^{2d}} e^{\frac{i}{h} \cdot (x-y) \cdot \xi} p_k(x, \xi; h) u(y) dy d\xi, \quad u \in C_c^\infty(V_k), \tag{A.4.19}$$

seen as an oscillatory integral. Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on \mathbb{R}^d .

Equivalently, a linear continuous map $P_h : C_c^\infty(X) \rightarrow \mathcal{D}'(X)$ is in $\Psi_{h, \eta}^m$ if and only if the following two conditions hold:

1. $\phi P_h \psi$ is negligible for all $\phi, \psi \in C_c^\infty(X)$ with $\text{supp } \phi \cap \text{supp } \psi = \emptyset$ (*pseudolocality*);
2. for every cut-off chart (κ, χ) there exists a symbol $p_\kappa \in S_\eta^m(T^*\mathbb{R}^d)$ such that

$$\chi P_h \chi = \chi \kappa^* \text{Op}_h(p_\kappa) (\kappa^{-1})^* \chi. \tag{A.4.20}$$

The property of pseudolocality can be extended to h -dependent cut-off functions $\phi, \psi \in C_c^\infty$ with support contained in some h -independent compact set, with $|\partial^\alpha \phi(x)|, |\partial^\alpha \psi(x)| \leq O_\alpha(h^{-\varepsilon|\alpha|})$, for some $0 \leq \varepsilon < 1/2$ and $\text{dist}(\text{supp } \phi, \text{supp } \psi) \geq h^{\varepsilon_0}/C$, $0 \leq \varepsilon_0 < 1/2$, $C > 0$.

Given a symbol $p \in S_\eta^m(T^*X)$ one can obtain an operator $P_h \in \Psi_{h,\eta}^m$, for instance, in the following way: Take a partition of unity $\{\psi_k\}_{k \in K}$ subordinate to a locally finite covering of X by coordinate charts $\{\kappa_k : X \supset U_k \rightarrow V_k \subset \mathbb{R}^d\}_{k \in K}$ such that $\sum \psi_k^2 = 1$. Then

$$P_h = \sum_{k \in K} \psi_k \kappa_k^* \text{Op}_h(p_{\kappa_k}) (\kappa_k^{-1})^* \psi_k \in \Psi_{h,\eta}^m, \quad (\text{A.4.21})$$

where $p_{\kappa_k} = p \circ \widehat{\kappa}_k^{-1}$ is the pullback of p to T^*V_k via the symplectomorphism $\widehat{\kappa}_k^{-1} : T^*V_k \rightarrow T^*U_k$ defined by $\widehat{\kappa}_k^{-1}(x, \xi) = (\kappa_k^{-1}(x), (d_x \kappa_k^{-1})^{-T} \xi)$. Given a symbol $p \in \Psi_{h,\eta}^m$, we will often write

$$P_h = \text{Op}_h(p), \quad (\text{A.4.22})$$

for a pseudo-differential operator P_h with principal symbol $\sigma(P_h) = p$.

The correspondence $P_h \mapsto p$ is not globally well-defined, but it gives rise to a bijection

$$\Psi_{h,\eta}^m / h^{1-2\eta} \Psi_{h,\eta}^{m-1} \longrightarrow S_\eta^m(T^*X) / h^{1-2\eta} S_\eta^{m-1}(T^*X). \quad (\text{A.4.23})$$

The image σ_P of P under the map (A.4.23) is called *principal symbol* of P .

Bibliography

- [1] M. Abert, N. Bergeron, and E. Le Masson, *Eigenfunctions and random waves in the Benjamini-Schramm limit*, arXiv preprint arXiv:1810.05601 (2018).
- [2] J. Aguilar and J.-M. Combes, *A class of analytic perturbations for one-body schrödinger hamiltonians*, Commun. Math. Phys. **22** (1971), 269–279.
- [3] J. Alt, R. Ducatez, and A. Knowles, *Delocalization transition for critical erdős rényi graphs*, Comm. Math. Phys. **388** (2021), 507–579.
- [4] N. Anantharaman, *Entropy and the localization of eigenfunctions*, Ann. of Math. **168** (2008), no. 2, 435–475.
- [5] N. Anantharaman, *Spectral deviations for the damped wave equation*, Geom. Funct. Anal. **20** (2010), no. 3.
- [6] N. Anantharaman, *Quantum ergodicity on regular graphs*, Comm. Math. Phys. **535** (2017), 633–690.
- [7] N. Anantharaman and E. Le Masson, *Quantum ergodicity on large regular graphs*, Duke Math. J. **164** (2015), no. 4, 723–765.
- [8] N. Anantharaman and S. Nonnenmacher, *Half-delocalization of eigenfunctions for the laplacian on an anosov manifold*, Ann. Inst. Fourier **57** (2007), no. 7, 2465–2523.
- [9] N. Anantharaman and M. Sabri, *Quantum ergodicity on graphs: From spectral to spatial delocalization*, Annals of Math. **189** (2019), 753–835.
- [10] G. V. Avakumović, *Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten*, Math. Z. **65** (1956), 327–344.
- [11] E. Balslev and J.-M. Combes, *Spectral properties of many-body Schrödinger operators with dilation analytic interactions*, Comm. Math. Phys. **22** (1971), 280–294.
- [12] J.-M. Barbaroux, L. Le Treust, N. Raymond, and E. Stockmeyer, *On the semiclassical spectrum of the dirichlet-pauli operator*, J. Eur. Math. Soc. **23** (2021), no. 10, 3279–3321.
- [13] A. Basak, E. Paquette, and O. Zeitouni, *Regularization of non-normal matrices by gaussian noise - the banded toeplitz and twisted toeplitz cases*, Forum of Mathematics, Sigma 7, paper e3 (2019).
- [14] ———, *Spectrum of random perturbations of Toeplitz matrices with finite symbols*, Trans. Amer. Math. Soc. **373** (2020), no. 7, 4999–5023.
- [15] A. Basak, M. Vogel, and O. Zeitouni, *Localization of eigenvectors of non-Hermitian banded noisy Toeplitz matrices*, Probability and Mathematical Physics **4** (2023), no. 3, 477–607.
- [16] A. Basak and O. Zeitouni, *Outliers of random perturbation of toeplitz matrices with finite symbols*, Probability Theory and Related Fields **178** (2020), 771–826.

- [17] S. Becker, M. Embree, J. Wittsten, and M. Zworski, *Mathematics of magic angles in a model of twisted bilayer graphene*, *Probab. Math. Phys.* **3** (2022), 69–103.
- [18] S. Becker, I. Oltman, and M. Vogel, *Magic angle (in)stability and mobility edges in disordered chern insulators*, preprint <https://arxiv.org/abs/2309.02701> (2023).
- [19] ———, *Absence of small magic angles for disordered tunneling potentials in twisted bilayer graphene*, preprint <https://arxiv.org/abs/2402.12799> (2024).
- [20] L. Benigni, *Eigenvectors distribution and quantum unique ergodicity for deformed Wigner matrices*, *Ann. Inst. Henri Poincaré Probab. Stat.* **56** (2020).
- [21] L. Benigni and G. Cipolloni, *Fluctuations of eigenvector overlaps and the berry conjecture for wigner matrices*, arxiv: 2212.10694 (2022).
- [22] L. Benigni and P. Lopatto, *Optimal delocalization for generalized wigner matrices*, *Advances in Mathematics* **396** (2022), no. 108109.
- [23] P. H. Bérard, *On the wave equation on a compact riemannian manifold without conjugate points*, *Mathematische Zeitschrift* **155** (1977), no. 3, 249–276.
- [24] M. V. Berry, *Regular and irregular semiclassical wavefunctions*, *J. Phys. A* **10** (1977), no. 12, 2083.
- [25] R. Bistritzer and A. MacDonald, *Moiré bands in twisted double-layer graphene*, *PNAS* **108** (2011), 12233–12237.
- [26] M. D. Blair and C. D. Sogge, *Logarithmic improvements in L^p bounds for eigenfunctions at the critical exponent in the presence of nonpositive curvature*, *Invent. Math.* **217** (2019), no. 2, 703–748.
- [27] P. Bleher, B. Shiffman, and S. Zelditch, *Universality and scaling of correlations between zeros on complex manifolds*, *Inventiones Mathematicae* **142** (2000), 351–395, 10.1007/s002220000092.
- [28] O. Bohigas, *Random matrix theories and chaotic dynamics*, *Chaos et physique quantique* (Les Houches 1989), North-Holland, Amsterdam (1991), 87–199.
- [29] O. Bohigas, M.-J. Giannoni, and C. Schmit, *Characterization of chaotic quantum spectra and universality of level fluctuation laws*, *Phys. Rev. Lett.* **52** (1984), 1–4.
- [30] O. Bohigas, M. J. Giannoni, and C. Schmit, *Characterization of chaotic quantum spectra and universality of level fluctuation laws*, *Phys. Rev. Lett.* **52** (1984), no. 1.
- [31] ———, *Spectral properties of the laplacian and random matrix theories*, *Phys. Rev. Lett.* **45** (1984), 1015–1022.
- [32] Y. Bonthonneau, *The theta function and the weyl law on manifolds without conjugate points*, *Documenta Mathematica* **22** (2017), 1275–1283.
- [33] W. Bordeaux-Montrieux, *Loi de Weyl presque sûre et résolvente pour des opérateurs différentiels non-autoadjoints*, *Thèse*, pastel.archives-ouvertes.fr/docs/00/50/12/81/PDF/manuscrit.pdf (2008).
- [34] C. Bordenave and M. Capitaine, *Outlier eigenvalues for deformed i.i.d. random matrices*, *Comm. on Pure and Applied Math.* **69** (2016), no. 11, 2131–2194.
- [35] C. Bordenave and D. Chafaï, *Around the circular law.*, *Probab. Surveys* **9** (2012), 1–89.

- [36] C. Bordenave and D. Chafaï, *Lecture notes on the circular law*, Modern Aspects of Random Matrix Theory (V. H. Vu, ed.), vol. 72, Amer. Math. Soc., 2013, pp. 1–34.
- [37] D. Borthwick and A. Uribe, *On the pseudospectra of Berezin-Toeplitz operators*, Meth. Appl. Anal. **10** (2003), no. 1, 31–66.
- [38] A. Böttcher and S. M. Grudsky, *Spectral properties of banded toeplitz matrices*, SIAM, 2005.
- [39] A. Böttcher and B. Silbermann, *Introduction to large truncated toeplitz matrices*, Springer, 1999.
- [40] L. Boulton, *Non-self-adjoint harmonic oscillator, compact semigroups and pseudospectra*, J. Operator Theory **47** (2002), no. 2, 413–429.
- [41] P. Bourgade and H.-T. Yau, *The eigenvector moment flow and local quantum unique ergodicity*, Comm. Math. Phys. **350** (2017), no. 1, 231–278.
- [42] P. Bourgade, H.-T. Yau, and J. Yin, *Local circular law for random matrices*, Probab. Theory Related Fields **159** (2014), 545–595.
- [43] J. Bourgain, *On toral eigenfunctions and the random wave model*, Isr. J. Math. **201** (2014), no. 2, 611–630.
- [44] E. Cancès, L. Garrigue, and D. Gontier, *A simple derivation of moiré-scale continuous models for twisted bilayer graphene*, Phys. Rev. B **107** (2023), no. 155403.
- [45] Y. Canzani and J. Galkowski, *Growth of high l^p norms for eigenfunctions: an application of geodesic beams*, Analysis & PDE **16** (2023), no. 10, 2267–2325.
- [46] ———, *Improvements for eigenfunction averages: an application of geodesic beams*, Journal of Differential Geometry **124** (2023), no. 3, 443–522.
- [47] Y. Canzani, D. Jakobson, and J. A. Toth, *On the distribution of perturbations of propagated Schrödinger eigenfunctions*, J. Spectr. Theory **4** (2014), no. 2, 283–307.
- [48] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, *Unconventional superconductivity in magic-angle graphene superlattices*, Nature **556** (2018), no. 43–50.
- [49] J. Chazarain, *Spectre d’un hamiltonien quantique et mécanique classique*, Comm. in PDE **5** (1980), 595–644.
- [50] T.J. Christiansen and M. Zworski, *Probabilistic Weyl laws for quantized tori*, Comm. Math. Phys. **299** (2010), 305–334.
- [51] G. Cipolloni, L. Erdős, and J. Henheik, *Eigenstate thermalisation at the edge for wigner matrices*, arXiv: 2309.05488 (2023).
- [52] G. Cipolloni, L. Erdős, J. Henheik, and D. Schröder, *Optimal lower bound on eigenvector overlaps for non-hermitian random matrices*, arXiv: 2301.03549 (2023).
- [53] G. Cipolloni, L. Erdős, and D. Schröder, *Eigenstate thermalisation hypothesis for wigner matrices*, Comm. Math. Phys. **388** (2021), 1005–1048.
- [54] G. Cipolloni, L. Erdős, and D. Schröder, *Thermalisation for wigner matrices*, Journal of Functional Analysis **282** (2022), no. 109394.

- [55] J.-M. Combes, P. D. Hislop, and F. Klopp, *An optimal wegner estimate and its application to the global continuity of the integrated density of states for random schrödinger operators*, Duke Math J. **140** (2007), no. 3, 469–498.
- [56] N. Cook, *Lower bounds for the smallest singular value of structured random matrices*, Ann. Probab. **46** (2018), no. 6, 3442–3500.
- [57] E. B. Davies, *Pseudospectra of differential operators*, J. Oper. Th **43** (1997), 243–262.
- [58] ———, *Pseudo-spectra, the harmonic oscillator and complex resonances*, Proc. of the Royal Soc. of London A **455** (1999), no. 1982, 585–599.
- [59] ———, *Semi-classical states for non-self-adjoint Schrödinger operators*, Comm. Math. Phys (1999), no. 200, 35–41.
- [60] E. B. Davies and M. Hager, *Perturbations of Jordan matrices*, J. Approx. Theory **156** (2009), no. 1, 82–94.
- [61] Y. Colin de Verdière, *Ergodicité et fonctions propres du laplacien*, Comm. Math. Phys. **102** (1985), 497–502.
- [62] N. Dencker, J. Sjöstrand, and M. Zworski, *Pseudospectra of semiclassical (pseudo-) differential operators*, Communications on Pure and Applied Mathematics **57** (2004), no. 3, 384–415.
- [63] M. Dimassi and J. Sjöstrand, *Spectral Asymptotics in the Semi-Classical Limit*, London Mathematical Society Lecture Note Series 268, Cambridge University Press, 1999.
- [64] S. Dyatlov, *Macroscopic limits of chaotic eigenfunctions*, arXiv preprint arXiv:2109.09053 (2021).
- [65] S. Dyatlov and L. Jin, *Semiclassical measures on hyperbolic surfaces have full support*, Acta Math. **220** (2018), no. 2, 297–339.
- [66] S. Dyatlov, L. Jin, and S. Nonnenmacher, *Control of eigenfunctions on surfaces of variable curvature*, Journal of the American Mathematical Society **35** (2022), no. 2, 361–465.
- [67] S. Dyatlov and M. Zworski, *Mathematical Theory of Scattering Resonances*, American Mathematical Society, 2019.
- [68] F.J. Dyson, *Statistical theory of the energy levels of complex systems, i, ii, iii*, J. Math. Phys. **3** (1962), 140–175.
- [69] P. Eberlein, *Geodesic flows in manifolds of nonpositive curvature*, Proc. Sympos. Pure Math. **69** (2001), 525–572.
- [70] M. Embree and L. N. Trefethen, *Spectra and Pseudospectra: The behavior of nonnormal matrices and operators*, Princeton University Press, 2005.
- [71] L. Erdős, A. Knowles, H.-T. Yau, and J. Yin, *Spectral statistics of Erdős-Rényi graphs I: Local semicircle law*, Ann. Probab. **41** (2013), no. 3B, 2279–2375.
- [72] L. Erdős, J. Ramírez, B. Schlein, T. Tao, V. Vu, and H.-T. Yau, *Bulk universality for wigner hermitian matrices with subexponential decay*, Int. Math. Res. Not. **2010** (2010), 436–479.
- [73] L. Erdős, B. Schlein, and H.-T. Yau, *Local semicircle law and complete delocalization for Wigner random matrices*, Comm. Math. Phys. **287** (2009), 641–655.

- [74] ———, *Semicircle law on short scales and delocalization of eigenvectors for Wigner random matrices*, Ann. Probab. **37** (2009), no. 3, 815–852.
- [75] S. Eswarathasan and G. Rivière, *Perturbation of the semiclassical Schrödinger equation on negatively curved surfaces*, J. Inst. Math. Jussieu **16** (2017), no. 4, 787–835.
- [76] S. Eswarathasan and J. A. Toth, *Averaged pointwise bounds for deformations of Schrödinger eigenfunctions*, Ann. Henri Poincaré **14** (2013), no. 3, 611–637.
- [77] M. Sodin F. Nazarov, *Correlation functions for random complex zeroes: Strong clustering and local universality*, Comm. Math. Phys **310** (2012), no. 1, 75–98.
- [78] Y. Le Floch, *A brief introduction to berezin-toeplitz operators on compact kähler manifolds*, Springer, Cham, 2018.
- [79] I. Fredholm, *Sur une nouvelle méthode pour la résolution du problème de dirichlet*, Öfver. Vet. Akad. Förhand, Stockholm **57** (1900), 39–46.
- [80] Y.V. Fyodorov, *Random matrix theory of resonances: An overview*, URSI International Symposium on Electromagnetic Theory, EMTS 2016 (2016), 666–669.
- [81] Y.V. Fyodorov and H.-J. Sommers, *Statistics of resonance poles phase shifts and time delays in quantum chaotic scattering: Random matrix approach for systems with broken time reversal invariance*, J. Math. Phys. **38** (1997), 1918–1981.
- [82] J. Galkowski, *Pseudospectra of semiclassical boundary value problems*, J. of the Inst. of Math. of Jussieu (2014), no. 2, 405–449.
- [83] I. Gallagher, Th. Gallay, and F. Nier, *Spectral asymptotics for large skew-symmetric perturbations of the harmonic oscillator*, Int. Math. Res. Notices (2009), 2147–2199.
- [84] F. Germinet, A. Klein, and B. Mandy, *Delocalization for random landau hamiltonians with unbounded random variables*, Contemporary Mathematics **500** (2009).
- [85] F. Germinet, A. Klein, and J. Schenker, *Delocalization in random landau hamiltonians*, Annals of Mathematics **166** (2007), no. 215–244.
- [86] ———, *Quantization of the hall conductance and delocalization in ergodic landau hamiltonians*, Reviews in Mathematical Physics **21** (2009), no. 8, 1045–1080.
- [87] J. Ginibr, *Statistical ensembles of complex, quaternion, and real matrices*, J. Math. Phys **6** (1965), 440–449.
- [88] A. Goetschy and S.E. Skipetrov, *Non-hermitian euclidean random matrix theory*, Phys. Rev. **E 84** (2011), no. 011150.
- [89] I.C. Gohberg and M.G. Kreĭn, *Introduction to the theory of linear non-selfadjoint operators*, Translations of mathematical monographs, vol. 18, AMS, 1969.
- [90] A. Grigis and J. Sjöstrand, *Microlocal Analysis for Differential Operators*, London Mathematical Society Lecture Note Series 196, Cambridge University Press, 1994.
- [91] M. Gromov, *Curvature, diameter and betti numbers*, Comment Math. Helvetici **56** (1981), 179–195.
- [92] V. V. Grushin, *Les problèmes aux limites dégénérés et les opérateurs pseudodifférentiels*, Actes du Congrès International des Mathématiciens, Tome 2, Gauthier-Villars (1970), 737–743.

- [93] A. Guionnet, P. M. Wood, and O. Zeitouni, *Convergence of the spectral measure of non-normal matrices*, Proc. AMS **142** (2014), no. 2, 667–679.
- [94] M. Hager, *Instabilité Spectrale Semiclassique d’Opérateurs Non-Autoadjoints II*, Annales Henri Poincaré **7** (2006), 1035–1064.
- [95] ———, *Instabilité spectrale semiclassique pour des opérateurs non-autoadjoints I: un modèle*, Annales de la faculté des sciences de Toulouse Sé. 6 **15** (2006), no. 2, 243–280.
- [96] M. Hager and J. Sjöstrand, *Eigenvalue asymptotics for randomly perturbed non-selfadjoint operators*, Mathematische Annalen **342** (2008), 177–243.
- [97] J. H. Hannay, *Chaotic analytic zero points: exact statistics for those of a random spin state*, J. Phys. A: Math. Gen. (1996), no. 29, 101–105.
- [98] A. Hassell and M. Tacy, *Improvement of eigenfunction estimates on manifolds of nonpositive curvature*, Forum Math. **27** (2015), no. 3, 1435–1451.
- [99] B. Helffer and D. Robert, *Comportement semi-classique du spectre des hamiltoniens quantiques elliptiques*, Ann. Inst. Fourier, Grenoble **31** (1981), no. 3, 169–223.
- [100] ———, *Calcul fonctionnel par la transformation de mellin et opérateurs admissibles*, J. Funct. Anal. **53** (1983), no. 3, 246–268.
- [101] B. Helffer and J. Sjöstrand, *Multiple Wells in the Semi-Classical Limit I*, Comm. in PDE **9** (1984), no. 4, 337–408.
- [102] ———, *Multiple Wells in the Semi-Classical Limit III – Interaction Through Non-Resonant Wells*, Math. Nachr. **124** (1985), 263–313.
- [103] ———, *Puits multiples en mécanique semi-classique ii: Interaction moléculaire. symetries. perturbation*, Ann. Inst. Henri Poincaré **42** (1985), no. 2, 127–212.
- [104] ———, *Puits multiples en mécanique semi-classique iv: Etude du complexe de witten*, Comm. in PDE **10** (1985), no. 3, 245–340.
- [105] H. Hezari and G. Rivière, *l^p norms, nodal sets, and quantum ergodicity*, Adv. Math. **290** (2016), 938–966.
- [106] D. Hilbert, *Grundzüge einer allgemeinen theorie der linearen integralgleichungen, erste mitteilung*, Göttingen Nachrichten (1904), 49–91.
- [107] ———, *Grundzüge einer allgemeinen theorie der linearen integralgleichungen, zweite mitteilung*, Göttingen Nachrichten (1904), 213–259.
- [108] ———, *Grundzüge einer allgemeinen theorie der linearen integralgleichungen, dritte mitteilung*, Göttingen Nachrichten (1905), 307–338.
- [109] ———, *Grundzüge einer allgemeinen theorie der linearen integralgleichungen, fünfte mitteilung*, Göttingen Nachrichten (1906), 439–480.
- [110] ———, *Grundzüge einer allgemeinen theorie der linearen integralgleichungen, vierte mitteilung*, Göttingen Nachrichten (1906), 157–227.
- [111] ———, *Grundzüge einer allgemeinen theorie der linearen integralgleichungen, sechste mitteilung*, Göttingen Nachrichten (1910), 355–419.
- [112] M. Hitrik and J. Sjöstrand, *Non-selfadjoint perturbations of selfadjoint operators in 2 dimensions I*, Ann. Henri Poincaré **5** (2004), 1–73.

- [113] ———, *Non-selfadjoint perturbations of selfadjoint operators in 2 dimensions II. Vanishing averages*, Comm. Partial Differential Equations **30** (2005), 1065–1106.
- [114] ———, *Non-selfadjoint perturbations of selfadjoint operators in 2 dimensions III a. One branching point*, Canadian J. Math **60** (2008), 572–657.
- [115] M. Hitrik, J. Sjöstrand, and S. Vũ Ngọc, *Diophantine tori and spectral asymptotics for non-selfadjoint operators*, Amer. J. Math. **129** (2007), 105–182.
- [116] L. Hörmander, *Differential Equations without Solutions*, Math. Annalen (1960), no. 140, 169–173.
- [117] ———, *Differential operators of principal type*, Math. Annalen (1960), no. 140, 124–146.
- [118] L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), no. 1, 193–218.
- [119] L. Hörmander, *The Analysis of Linear Partial Differential Operators IV*, Grundlehren der mathematischen Wissenschaften, vol. 275, Springer-Verlag, 1985.
- [120] J.B. Hough, M. Krishnapur, Y. Peres, and B. Virág, *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*, American Mathematical Society, 2009.
- [121] M. Ingremeau, *Local weak limits of laplace eigenfunctions*, Tunis. J. Math. **3** (2021), no. 3, 481–515.
- [122] M. Ingremeau and A. Rivera, *How lagrangian states evolve into random waves*, Journal de l'École polytechnique—Mathématiques, **9** (2022), 177–212.
- [123] M. Ingremeau and M. Vogel, *Emergence of gaussian fields in noisy quantum chaotic dynamics*, arXiv preprint arXiv:2306.11617.
- [124] ———, *Improved L^∞ bounds for eigenfunctions under random perturbations in negative curvature*, arXiv preprint arXiv:2403.13739.
- [125] V. Ivrii, *Microlocal analysis and precise spectral asymptotics*, Monographs in Math., Springer, 1998.
- [126] H. Iwaniec and P. Sarnak, *l^∞ norms of eigenfunctions of arithmetic surfaces*, Ann. of Math. **41** (1995), no. 1, 301–320.
- [127] L. Jin, *Damped wave equations on compact hyperbolic surfaces*, Comm. Math. Phys. **373** (2020), 771–794.
- [128] S. Kalmykov and L. V. Kovalev, *Self intersections of Laurent polynomials and the density of Jordan curves*, Proc. Amer. Math. Soc. **151** (2023), no. 2, 547–554.
- [129] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Classics in Mathematics, Springer-Verlag Berlin Heidelberg, 1995.
- [130] J. P. Keating, M. Novaes, and H. Schomerus, *Model for chaotic dielectric microresonators*, Phys. Rev. A **77** (2008), no. 013834.
- [131] R. Latala, *Some estimates of norms of random matrices*, Proc. Amer. Math. Soc. **133** (2005), no. 5, 1273–1282.
- [132] G. Lebeau, *Équation des ondes amorties*, Algebraic and Geometric Methods in Mathematical Physics (Kaciveli, 1993), Math. Phys. Stud. 19, Kluwer Acad. Publ., Dordrecht (1996).

- [133] B. M. Levitan, *On the asymptotic behavior of the spectral function of a self-adjoint differential operator of second order*, *Isv. Akad. Nauk SSSR Ser. Mat.* **16** (1952), 325–352.
- [134] G. Lindblad, *On the generators of quantum dynamical semigroups*, *Commun. Math. Phys.* **48** (1976), 119–130.
- [135] A. De Luca, B. L. Altshuler, V. E. Kravtsov, and (2014). A. Scardicchio, , *Anderson localization on the bethe lattice: nonergodicity of extended states*, *Physical Review Letters* **113** (2014), no. 4, 046806.
- [136] K. Luh and S. O’Rourke, *Eigenvector delocalization for non-hermitian random matrices and applications*, *Random Structures & Algorithms* **571** (2020), no. 1, 169–210.
- [137] A. Lytova and K. Tikhomirov, *On delocalization of eigenvectors of random non-hermitian matrices*, *Probability Theory and Related Fields* **177** (2020), 465–524.
- [138] A. Maltsev and M. Osman, *Bulk universality for complex non-hermitian matrices with independent and identically distributed entries*, arxiv.org/pdf/2310.11429 (2023).
- [139] A.S. Markus and V.I. Matsaev, *Comparison theorems for spectra of linear operators and spectral asymptotics*, *Trudy Moskov. Mat. Obshch.* **45** (1982), 133–181.
- [140] A. Martinez, *An introduction to semiclassical and microlocal analysis*, Springer, 2002.
- [141] A. Melin and J. Sjöstrand, *Determinants of pseudodifferential operators and complex deformations of phase space*, *Methods Appl. Anal.* **9** (2002), no. 2, 177–237.
- [142] ———, *Bohr-Sommerfeld quantization condition for non-selfadjoint operators in dimension 2*, *Astérisque* **284** (2003), 181–244.
- [143] N. N. T. Nam and M. Koshino, *Lattice relaxation and energy band modulation in twisted bilayer graphene*, *Phys. Rev. B* **96** (2017), no. 075311.
- [144] F. Nazarov and M. Sodin, *Asymptotic laws for the spatial distribution and the number of connected components of zero sets of gaussian random functions*, *J. Math. Phys. Anal. Geom.* **12** (2016), no. 3, 205–278.
- [145] S. Nonnenmacher and M. Vogel, *Local eigenvalue statistics of one dimensional random non-selfadjoint pseudodifferential operators*, *J. Eur. Math. Soc.* **23** (2021), no. 5, 1521–1612.
- [146] I. Oltman, *A probabilistic weyl-law for berezin-toeplitz operators*, *Journal of Spectral Theory* **13** (2023), no. 2, 727–754.
- [147] V. Petkov and D. Robert, *Asymptotique semi-classique du spectre d’hamiltoniens quantiques et trajectoires classiques périodiques*, *Comm. in PDE* **10** (1985), no. 4, 365–390.
- [148] K. Pravda-Starov, *A general result about the pseudo-spectrum of Schrödinger operators*, *Proc. R. Soc. London Ser. A Math. Phys. Eng. Sci.* **460** (2004), 471–477.
- [149] ———, *Étude du pseudo-spectre d’opérateurs non auto-adjoints*, Ph.D. thesis, 2006.
- [150] ———, *Pseudo-spectrum for a class of semi-classical operators*, *Bull. Soc. Math. France* **136** (2008), no. 3, 329–372.
- [151] A. Raphael and M. Zworski, *Pseudospectral effects and basins of attraction*, unpublished notes (2005).
- [152] O. Rouby, *Bohr-Sommerfeld quantization conditions for non-selfadjoint perturbations of self-adjoint operators in dimension one*, *Int. Math. Res. Not. IMRN* **7** (2018), 2156–2207.

- [153] M. Rudelson and R. Vershynin, *The least singular value of a random square matrix is $O(n^{-1/2})$* , C. R. Math. Acad. Sci. Paris **346** (2008), 893–896.
- [154] ———, *Delocalization of eigenvectors of random matrices with independent entries*, Duke Math. J. **164** (2015), no. 13, 2507–2538.
- [155] ———, *No-gaps delocalization for general random matrices*, Geometric and Functional Analysis **26** (2016), 1716–1776.
- [156] Z. Rudnick and P. Sarnak, *The behaviour of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys. **161** (1994), no. 1, 195–213.
- [157] ———, *The behaviour of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys. **161** (1994), no. 1, 195–213.
- [158] B. Sandstede and A. Scheel, *Basin boundaries and bifurcations near convective instabilities: a case study*, J. of Diff. Equations (2005), no. 208, 176–193.
- [159] A. Sankar, D.A. Spielmann, and S.H. Teng, *Smoothed analysis of the condition numbers and growth factors of matrices*, SIAM J. Matrix Anal. Appl. **28** (2006), no. 2, 446–476.
- [160] P. Sarnak, *Arithmetic quantum chaos*, in The Schur Lectures (1992) (Tel Aviv), Israel Math. Conf. Proc **8** (1995), Bar-Ilan Univ., Ramat Gan, pp. 183–236.
- [161] R. Schubert, *Semiclassical behaviour of expectation values in time evolved lagrangian states for large times*, Comm. Math. Phys. **256** (2005), no. 1, 239–254.
- [162] T. Shirai, *Limit theorems for random analytic functions and their zeros*, RIMS Kôkyûroku Bessatsu **B34** (2012), 335–359.
- [163] M. A. Shubin, *Spectral theory of elliptic operators on non-compact manifolds*, Astérisque **207** (1992), 35–108.
- [164] ———, *Pseudodifferential operators and spectral theory*, Springer-Verlag Berlin Heidelberg, 2001.
- [165] J. Sjöstrand, *Operators of principle type with interior boundary conditions*, Acta Math. **130** (1973), no. 1–51.
- [166] J. Sjöstrand, *Asymptotic Distribution of Eigenfrequencies for Damped Wave Equations*, Publ. RIMS, Kyoto Univ. **36** (2000), 573–611.
- [167] J. Sjöstrand, *Perturbations of selfadjoint operators with periodic classical flow*, In: Wave Phenomena and Asymptotic Analysis. RIMS Kokyuroku 1315 (2003).
- [168] J. Sjöstrand, *Eigenvalue distribution for non-self-adjoint operators with small multiplicative random perturbations*, Annales Fac. Sci. Toulouse **18** (2009), no. 4, 739–795.
- [169] ———, *Eigenvalue distribution for non-self-adjoint operators on compact manifolds with small multiplicative random perturbations*, Ann. Fac. Toulouse **19** (2010), no. 2, 277–301.
- [170] ———, *Resolvent Estimates for Non-Selfadjoint Operators via Semigroups*, Around the Research of Vladimir Maz'ya III, International Mathematical Series, no. 13, Springer, 2010, pp. 359–384.
- [171] ———, *Non-self-adjoint differential operators, spectral asymptotics and random perturbations*, Pseudo-Differential Operators Theory and Applications, vol. 14, Birkhäuser Basel, 2019, preliminary version <http://sjostrand.perso.math.cnrs.fr/>.

- [172] J. Sjöstrand and M. Vogel, *Large bi-diagonal matrices and random perturbations*, J. of Spectral Theory **6** (2016), no. 4, 977–1020.
- [173] ———, *Interior eigenvalue density of large bi-diagonal matrices subject to random perturbations*, Microlocal analysis and singular perturbation theory, RIMS Kôkyûroku Bessatsu, **B61** (2017), 201–227.
- [174] J. Sjöstrand and M. Vogel, *General toeplitz matrices subject to gaussian perturbations*, Ann. Henri Poincaré **22** (2021), no. 1, 49–81.
- [175] ———, *Toeplitz band matrices with small random perturbations*, Indagationes Mathematicae **32** (2021), no. 1, 275–322.
- [176] J. Sjöstrand and M. Vogel, *Tunneling for the $\bar{\partial}$ -bar operator*, accepted for publication in Vietnam Journal of Mathematics, arXiv preprint arXiv:2303.06096 (2023).
- [177] J. Sjöstrand and M. Zworski, *Elementary linear algebra for advanced spectral problems*, Annales de l’Institut Fourier **57** (2007), 2095–2141.
- [178] P. Śniady, *Random Regularization of Brown Spectral Measure*, J. of Functional Analysis **193** (2002), 291–313.
- [179] C. D. Sogge, *Concerning the l^p norm of spectral clusters for second-order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), no. 1, 123–138.
- [180] L. A. Steen, *Highlights in the history of spectral theory*, The American Mathematical Monthly **80** (1973), no. 4, 359–381.
- [181] V. Vu T. Tao, *Random matrices: the circular law*, Commun. Contemp. Math. **10** (2008), no. 2, 261–307.
- [182] T. Tao, *Topics in Random Matrix Theory*, Graduate Studies in Mathematics, vol. 132, American Mathematical Society, 2012.
- [183] T. Tao and V. Vu, *Smooth analysis of the condition number and the least singular value*, Math. Comp. **79** (2010), no. 272, 2333–2352 (see also the Erratum arxiv.org/pdf/0805.3167v3.pdf).
- [184] T. Tao and V. Vu, *Random matrices: Universality of local spectral statistics of non-hermitian matrices*, The Annals of Probability **43** (2015), no. 2, 782–874.
- [185] T. Tao, V. Vu, and M. Krishnapur, *Random matrices: universality of esds and the circular law*, The Annals of Probability **38** (2010), no. 5, 2023–2065.
- [186] G. Tarnopolsky, A.J. Kruchkov, and A. Vishwanath, *Origin of magic angles in twisted bilayer graphene*, Phys. Rev. Lett. **122** (2019), no. 106405.
- [187] L.N. Trefethen, *Pseudospectra of matrices*, in Numerical Analysis 1991, D. F. Griffiths and G. A. Watson, eds., Longman Scientific and Technical, Harlow, UK, 1992, 234–266.
- [188] ———, *Pseudospectra of linear operators*, SIAM Rev. **39** (1997), no. 3, 383–406.
- [189] L.N. Trefethen and S.J. Chapman, *Wave packet pseudomodes of twisted Toeplitz matrices*, Comm. on Pure and Applied Mathematics **LVII** (2004), 1233–1264.
- [190] A. Vasy, *Microlocal analysis of asymptotically hyperbolic spaces and high-energy resolvent estimates*, CMath. Sci. Res. Inst. Publ. Cambridge University Press, Cambridge **60** (2013), 487–528.

- [191] M. Vogel, *Spectral statistics of non-selfadjoint operators subject to small random perturbations*, Séminaire Laurent Schwartz - EDP et applications **19** (2016–2017), 24 p.
- [192] M. Vogel, *The precise shape of the eigenvalue intensity for a class of non-selfadjoint operators under random perturbations*, Ann. Henri Poincaré **18** (2017), no. 2, 435–517.
- [193] M. Vogel, *Almost sure Weyl law for quantized tori*, Comm. Math. Phys **378** (2020), no. 2, 1539–1585.
- [194] M. Vogel and O. Zeitouni, *Deterministic equivalence for noisy perturbations*, Proc. Amer. Math. Soc. **149** (2021), 3905–3911.
- [195] A. Voros, *Semiclassical ergodicity of quantum eigenstates in the Wigner representation, Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, Proceedings of the Volta Memorial Conference, Italy, 1977 (eds. G. Casati, J. Ford, ed.), vol. 93, Lect. Notes Phys. Springer-Verlag, Berlin, 1979, pp. 326–333.
- [196] A. I. Šnirel'man, *Ergodic properties of eigenfunctions*, Uspehi Mat. Nauk **29** (1974), 181–182.
- [197] A. B. Watson, T. Kong, A. H. MacDonald, and M. Luskin, *Bistritzer-macdonald dynamics in twisted bilayer graphene*, J. Math. Phys. **64** (2023), no. 031502.
- [198] H. Weyl, *Das asymptotische verteilungsgesetz der eigenwerte linearer partieller differentialgleichungen (mit einer anwendung auf die theorie der hohlraumstrahlung)*, Math. Ann. **71** (1912), no. 4, 441–479.
- [199] E. Wigner, *Characteristic vectors of bordered matrices with infinite dimensions*, Ann. of Math. **62** (1955), 548–564.
- [200] P. M. Wood, *Universality of the esd for a fixed matrix plus small random noise: A stability approach*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **52** (2016), no. 4, 1877–1896.
- [201] S. Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. J. **55** (1987), no. 4, 919–941.
- [202] M. Zworski, *A remark on a paper of E.B. Davies*, Proc. A.M.S. (2001), no. 129, 2955–2957.
- [203] ———, *Pseudospectra of semi-classical operators.*, Unpublished lecture, King's College (2001).
- [204] ———, *Semiclassical Analysis*, Graduate Studies in Mathematics 138, American Mathematical Society, 2012.
- [205] K. Zyczkowski and H.-J. Sommers, *Truncations of random unitary matrices*, J. Phys. A: Math. Gen **33** (2000), 2045–2057.

Abstract The main focus of this habilitation thesis is on the spectral properties of disordered semiclassical pseudo-differential operators and Toeplitz matrices. The disorder will predominantly be given by small random perturbations.

This manuscript is organized in three main parts which each focus on a different aspect. In Part I we will discuss the spectral distribution of non-selfadjoint semiclassical pseudo-differential operators subject to small random perturbations on a macroscopic level, in the form of a probabilistic Weyl law, and on the microscopic level, in the form of spectral statistics.

In Part II we will first discuss recent results on eigenvector localization for large noisy Toeplitz matrices. These are excellent toy models for noisy semiclassical pseudo-differential operators. We will first present results concerning a probabilistic Weyl law for the eigenvalues of noisy non-selfadjoint Toeplitz matrices and then discuss a related eigenvector localization phenomenon.

In Part III we will consider a form of eigenvector delocalization in a selfadjoint quantum chaotic setting. More precisely, we will discuss recent results showing that on a compact negatively curved Riemannian manifold, Langrangian states which have been propagated for a long time under the Schrödinger evolution semi-group, induced by a random Schrödinger type operator, satisfy Berry's conjecture. Additionally, we discuss improved L^∞ bounds for the eigenfunctions of the semiclassical Laplacian subject to small random perturbations.

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