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Field theories on quantum space-times: towards the phenomenology of quantum gravity

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Field Theories on Quantum Space-Times: Towards the Phenomenology of Quantum Gravity

*Théories de Champs en Espace-Temps Quantiques :
vers la Phénoménologie de la Gravité Quantique*

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Title: Field Theories on Quantum Space-Times: Towards the Phenomenology of Quantum Gravity

Keywords: noncommutative geometry, quantum space-times, quantum field theory, gauge theory, phenomenology of quantum gravity.

Abstract: Noncommutative geometry is a mathematical framework that expresses the structure of space-time in terms of operator algebras. By using the tools of quantum mechanics to describe the geometry, noncommutative space-times are expected to give rise to quantum gravity effects, at least in some regime. This manuscript focuses on the physical aspects of these so-called quantum space-times, in particular through the formalism of field and gauge theories. Scalar field theories are shown to possibly trigger mixed divergences in the infra-red and ultra-violet for the 2-point function at one loop. This phenomenon is generically called UV/IR mixing and stems from a diverging behaviour of the propagator. The analysis of such divergences differs from the commutative case because the momentum space is now also noncommutative. From another perspective, a gauge theory on κ -MINKOWSKI, a quantum deformation of the MINKOWSKI space-time, is derived. A first perturbative computation is shown to break the gauge invariance, a pathological behaviour common to other quantum space-times. A causality toy model is also developed on κ -MINKOWSKI, in which an analogue of the speed-of-light limit emerges. The phenomenology of quantum gravity arising from quantum space-times is discussed, together with the actual constraints it imposes. Finally, a toy model for noncommutative gravity is tackled, using the former κ -MINKOWSKI space-time to describe the tangent space. It necessitates the notion of noncommutative partition of unity specifically defined there.

Titre : Théories de Champs en Espace-Temps Quantiques : vers la Phénoménologie de la Gravité Quantique

Mots-clefs : géométrie noncommutative, espace-temps quantiques, théorie quantique des champs, théorie de jauge, phénoménologie de la gravité quantique

Résumé : La géométrie noncommutative est un formalisme mathématique qui exprime la structure de l'espace-temps avec des algèbres d'opérateurs. On s'attend à ce que les espace-temps noncommutatifs fassent émerger des effets de gravité quantiques, au moins dans un certain régime, notamment parce qu'ils utilisent les outils de la mécanique quantique pour décrire la géométrie. Ce manuscrit se concentre sur les aspects physiques de ces espace-temps quantiques, tout particulièrement à travers le formalisme des théories de champs et de jauge. Il est montré que les théories de champs scalaires engendrent possiblement des divergences dans l'infra-rouge et l'ultra-violet pour la fonction 2-point à une boucle. Ce phénomène s'appelle génériquement le mélange UV/IR et découle de la divergence du propagateur. L'analyse de ces divergences diffère du cas commutatif car l'espace des moments y est noncommutatif. D'autre part, une théorie de jauge sur κ -MINKOWSKI, une déformation quantique de l'espace de MINKOWSKI, est construite. Un premier calcul perturbatif produit une brisure de l'invariance de jauge, un comportement pathologique commun à d'autres espace-temps quantiques. Un modèle-jouet de causalité est aussi développé sur κ -MINKOWSKI, dans lequel apparaît un analogue de la vitesse de lumière comme vitesse limite. La phénoménologie de la gravité quantique émergeant des espace-temps quantiques est abordée, avec les contraintes qu'elle impose. Finalement, un modèle-jouet de gravité noncommutative, utilisant κ -MINKOWSKI pour décrire l'espace tangent, est traité. Il nécessite le concept de partition de l'unité noncommutative spécialement défini dans ce contexte.

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Résumé long

Il existe une incompatibilité forte entre la théorie de la gravité, correspondant à la relativité générale, et le modèle standard de la physique des particules, qui se base sur une description quantique des autres forces fondamentales. Cette incompatibilité porte sur des différences de traitement, comme la description du temps ou la nature de l'espace-temps à l'échelle de PLANCK, ou bien des différences de prédictions, comme la non renormalisabilité de la relativité générale, lorsque quantifiée par une intégrale de chemin, ou le paradoxe de l'information des trous noirs. Ces problèmes seraient résolus par une théorie de gravité quantique dont l'établissement est toujours en cours. D'un point de vue expérimental, aucun effet de gravité quantique n'a été observé à ce jour. Néanmoins, une forte communauté autour de la phénoménologie de la gravité quantique a émergé et essaye d'atteindre des régimes observationnels proches de l'échelle de PLANCK.

La phénoménologie de la gravité quantique postule des propriétés qu'une théorie de gravité quantique complète devrait avoir et qui mènerait à des prédictions observables. Dans cette logique, l'étude des espaces-temps noncommutatifs, aussi dits espace-temps quantiques, a mis en avant un certain nombre de phénomènes utilisés pour des modèles phénoménologiques. L'introduction d'une longueur indépendante de l'observateur, qui s'ajoute à la vitesse indépendante de l'observateur, dans la théorie de relativité doublement restreinte possède une formulation naturelle en terme de géométrie noncommutative. La noncommutativité peut également engendrer une courbure dans l'espace des moments qui est à la base des géométries de BORN. De plus, des relations d'incertitude généralisées apparaissent dans certains espace-temps quantiques. D'autres modèles phénoménologiques, comme la violation de la symétrie de LORENTZ ou bien la violation de CPT, découlent des déformations de symétries des espace-temps quantiques.

L'apport des théories physiques sur des espace-temps quantiques à la phénoménologie de la gravité quantique est donc important, mais il pourrait le devenir d'avantage si ces espaces étaient mieux compris. La physique des espace-temps quantiques reste, en effet, peu explorée. Son étude a été complexifiée par l'émergence de nouveaux phénomènes lors de l'étude de théorie de champs sur ces espaces. Cette thèse essaye de caractériser et de s'attaquer à l'étude de ces phénomènes.

Si l'on s'intéresse à un espace-temps \mathcal{M} , l'espace des observables sur \mathcal{M} correspond à l'ensemble de ses fonctions lisses $\mathbb{A} = \mathcal{C}^\infty(\mathcal{M})$, qui forment une algèbre pour le produit point à point. On construit un espace-temps quantique en changeant ce dernier produit pour un nouveau produit noncommutatif, aussi appelé star-produit, noté usuellement \star . L'hypothèse que la donnée (\mathbb{A}, \star) détermine un espace-temps (quantique) provient du théorème de GEL'FAND-NAÏMARK commutatif (voir le [Théorème B.2.3](#)) qui dit que la donnée (\mathbb{A}, \star) pour \star un produit commutatif correspond à un ensemble de fonctions lisses sur un espace-temps. La mécanique quantique s'écrit comme une géométrie noncommutative, avec un espace quantique nommé MOYAL qui consiste en une déformation de l'espace \mathbb{R}^{2n} . La méthode de construction des star-produits s'appelle la quantification par déformation et peut être produite de plusieurs manières. Les deux méthodes les plus utilisées dans un contexte physique passent soit par des techniques d'algèbre de convolution, où l'on considère le produit de convolution sur l'espace des moments, ou bien par un twist de DRINFEL'D qui permet de d'abord déformer les symétries classiques et de propager cette déformation sur l'espace lui-même.

De là, la géométrie différentielle (sur \mathcal{M}) se généralise en géométrie différentielle noncommutative (sur \mathbb{A}) en considérant une manière de définir la structure différentielle dans le cas noncommutatif. L'approche par les triplets spectraux considère l'opérateur de DIRAC comme un élément central de la géométrie et se base sur lui pour définir un calcul différentiel dit "quantifié". Une autre méthode, basée sur les groupes quantiques, utilise les symétries de l'espace-temps quantique pour définir une structure différentielle. Enfin, les champs de vecteur d'un espace-temps classique correspondent aux dérivations de son espace de fonction, de tel sorte que l'on peut se baser sur les dérivations de l'algèbre des observables pour définir un calcul différentiel. Ces trois approches ne sont pas nécessairement incompatibles car elles possèdent des similarités.

La structure de fibré, nécessaire à la définition de connexion (ou champs de jauge A) et de courbure (ou tenseur d'intensité de champs F), le théorème de SERRE-SWAN donne une correspondance entre les sections du fibré et la structure de module. Comme il est possible de définir un module dans le cadre algébrique de la géométrie noncommutative, cette dernière solution est privilégiée pour définir un fibré.

En utilisant les précédents éléments, il est possible d'implémenter une théorie de jauge sur \mathbb{A} qui s'appuie sur des définitions algébriques de la connexion et de sa courbure. Le groupe de jauge s'écrit également dans ce formalisme, le tout étant déformé afin de s'adapter à la structure noncommutative. Il est alors possible de définir une action de type YANG-MILLS, invariante de jauge. L'étude de ces théories de jauge noncommutative ont été florissantes notamment sur l'espace

de MOYAL, car elles sont en liens avec la théorie des cordes. Néanmoins, cette dernière théorie possède une structure de vide très complexe qui empêche une analyse plus poussée.

Pour une théorie de champs d'un espace-temps noncommutatif de type algèbre de LIE, c'est à dire dont les coordonnées vérifient $[x^\mu, x^\nu]_\star = C^{\mu\nu}_\rho x^\rho$, il est toujours possible de définir un espace des moments. Cette dernière assertion découle de la formule de BAKER-CAMPBELL-HAUSSDORFF et donne une loi d'addition des moments noncommutative, notée ici $p \boxplus q$, pour deux moments p et q . La caractérisation de l'espace des moments permet d'utiliser des notions d'intégration sur des groupes pour réaliser le développement perturbatif de la fonction génératrice d'une action de théorie de champs. Cette construction a été suivi pour quantifiée une théorie de champs scalaire noncommutative de type ϕ^4 [76].

L'étude de la fonction 2-point à une boucle montre que le phénomène du mélange infrarouge-ultraviolet (UV/IR), déjà apparu dans des espace-temps quantiques comme MOYAL ou des déformations de MINKOWSKI, se généralise à une plus large classe d'espace-temps quantiques. Ce dernier est définit dans [76] selon trois caractéristiques : les diagrammes planaires divergent dans l'UV, les diagrammes non-planaires sont singuliers dans l'IR, et les digrammes non-planaires convergent dans l'UV. Ce phénomène, propre au noncommutatif, empêcherait la renormalisation perturbative de ces théories de champs due au fait que la singularité infrarouge, aux ordres supérieurs, rend imprévisible la forme des contre-termes. Il est montré que la divergence, ou non, de l'intégrale du propagateur est un critère permettant de juger de la présence, ou non, des deux premières caractéristiques. L'analyse de la troisième caractéristique est rendu plus complexe par l'apparition d'une loi de conservation des moments non-triviale s'écrivant $p \boxplus k \boxminus q \boxminus k = 0$ pour p et q les moments externes et k le moment interne. Cette dernière équation possède, dans le cas noncommutatif uniquement, une solution en k , dont la forme n'a pas été déterminée pour une loi \boxplus générale.

L'étude des théories de champ noncommutatives reste donc à des ordres en boucles et en nombre de point petits. L'analyse au-delà de ses petits ordres, et des possibles nouveaux phénomènes qu'elle engendre, reste à explorer. Le formalisme précédent pourrait également être généralisé, en définissant l'espace des moments comme le dual, au sens des algèbres de HOPF, de l'espace-temps quantique. Finalement, les théories de jauge pourrait être analysées selon la même perspective, car la présence du mélange a déjà été observée sur quelques exemples.

L'espace-temps de κ -MINKOWSKI, correspondant à une déformation quantique de l'espace de MINKOWSKI, est considéré comme un bon candidat pour faire émerger des effets de gravité quantique, au moins dans un certain régime. Il a été beaucoup étudié en ce sens, notamment du point

de vue de ses symétries. Elles correspondent à κ -POINCARÉ, une déformation au sens des groupes quantiques, du groupe de POINCARÉ. En suivant les étapes présentées précédemment, on peut construire une théorie de jauge sur κ -MINKOWSKI. Cependant, une action de type YANG-MILLS ne sera pas invariante de jauge à cause de la non-cyclicité de l'intégrale, *i.e.* $\int dx f \star g = \int dx \mathcal{E}^d(g) \star f$, où f et g sont des éléments de κ -MINKOWSKI, $\mathcal{E} = e^{-P_0/\kappa}$ correspond à une translation (complexe) dans le temps et d est la dimension spatiale considérée pour κ -MINKOWSKI. Une théorie de jauge a été construite en se basant sur des dérivations dites tordues, qui font apparaître l'élément \mathcal{E} . L'usage de ces dérivations, invisible à la limite commutative, permettent de modifier la transformation de jauge du tenseur d'intensité de champs F et donc de rendre l'action de type YANG-MILLS invariante de jauge. Cette invariance impose tout de même la contrainte que l'on soit en $4 + 1$ dimensions, due au fait que la dimension d apparaisse dans la non-cyclicité de l'intégrale.

Ce modèle a ensuite été quantifié et l'étude perturbative de la fonction 1-point à une boucle conduite. Elle montre que la fonction 1-point ne s'annule pas, induisant une instabilité du vide pour cette théorie. La symétrie de jauge est même brisée par la procédure de quantification, ce qui n'est pas le cas de la théorie commutative. L'apparition d'une anomalie de jauge à été observée dans d'autres espace-temps noncommutatifs, comme MOYAL ou \mathbb{R}_λ^3 et reste un mystère. Une étude plus approfondie de la quantification des théorie de jauge noncommutative serait nécessaire pour comprendre la présence de cette anomalie. Peut-être même que cette quantification devrait être revue dans un cadre noncommutative, car les études faites en théorie de jauge noncommutative utilisent la même procédure de BECCHI-ROUET-STORA-TYUTIN que dans le cas commutatif.

D'autre part, un modèle jouet de théorie de causalité sur κ -MINKOWSKI a été obtenu. Il consiste en l'application du modèle de causalité sur les triplets spectraux Lorentzien appliqué au cas de κ -MINKOWSKI. Ce dernier se base sur la notion de fonctions causales, dont le graphe correspond aux points de simultanéité causale, et la notion de relation causale entre deux points de l'espace-temps, qui sont toutes deux généralisées au cas noncommutatif. Les fonctions causales correspondent à des fonctions de \mathbb{A} qui satisfont la contrainte (3.3.4b). Elles permettent de relier causalement deux états (pures).

Pour l'exemple de κ -MINKOWSKI, l'opérateur de DIRAC est pris comme un analogue noncommutatif du DIRAC de MINKOWSKI. L'espace de HILBERT, permettant dans la représentation, est pris selon la représentation de SCHRÖDINGER de l'espace-temps κ -MINKOWSKI. De là, l'évolution causale entre deux états pures complète n'a pas pu être calculée car trop complexe. Cependant, en écrivant la relation de causalité entre deux états pour une fonction causale de la forme $f = x^0 + vx^1$, avec $v \in [-1, 1]$, on obtient une inégalité entre l'évolution causale de l'opérateur temps x^0 et

l'évolution causale de l'opérateur spatial x^1 . Cette inégalité peut être interprétée comme l'analogie quantique de la vitesse de la lumière comme vitesse limite. Les conséquences de cette vitesse limite seulement en moyenne n'ont pas été explorées mais pourraient mener à des considérations phénoménologiques importantes. De plus, le devenir de cette relation pour des fonctions causales plus complexes pourrait également mener à des études intéressantes.

Finalement, les conséquences physiques des modèles de gravité noncommutative ont été principalement étudiées dans le cadre des trous noirs, dit trous noirs noncommutatifs. Cependant, les intuitions que le noncommutatif pourrait apporter à l'étude de la gravité quantique vont au-delà des trous noirs. Dans cette mesure, un modèle jouet de gravité noncommutative a été entrepris afin d'obtenir une version modifiée de la relativité générale. Celui-ci se base sur l'étude d'espaces-temps noncommutatifs ayant pour espace tangent local κ -MINKOWSKI. Cette hypothèse se base sur le cas commutatif, dans lequel tout espace-temps possède pour espace tangent local MINKOWSKI. De plus, elle vise à quantifier l'apport que peut avoir κ -MINKOWSKI sur la physique de l'espace-temps quantique global.

La local trivialité d'un espace-temps commutatif s'écrit au niveau des champs de vecteurs comme $\Gamma(U_\alpha) \simeq \mathcal{C}^\infty(U_\alpha) \otimes \mathbb{R}^{1,d}$, où U_α désigne un ouvert (local) de l'espace-temps, $\Gamma(U_\alpha)$ l'ensemble des champs de vecteurs sur U_α et $\mathbb{R}^{1,d}$ l'espace-temps de MINKOWSKI. On cherche alors à exprimer les dérivations sur un espace-temps quantique local $\text{Der}(\mathbb{A}_\alpha)$ comme un analogue de la relation précédente. Pour ce faire, on fait l'hypothèse que l'espace-temps quantique local \mathbb{A}_α est une $\mathbb{R}_\kappa^{1,d}$ -algèbre de module, c'est à dire que $\mathbb{R}_\kappa^{1,d}$ correspond aux symétries de \mathbb{A}_α . L'analyse tensorielle de \mathbb{A}_α (*i.e.* les champs de vecteurs et les formes) sont alors définies aisément grâce à κ -MINKOWSKI. L'exportation de ces données à l'espace-temps quantique global \mathbb{A} se fait alors via l'introduction d'une partition de l'unité noncommutative. Cette partition est définie de manière analogue à la partition de l'unité pour des espace-temps commutatifs. On peut alors définir globalement des champs tensoriels définis à partir de champ tensoriels locaux.

Le précédent modèle pose une structure mathématique sur laquelle peut reposer un modèle physique, encore à définir. Ce dernier peut être énoncé sous la forme d'une action qui nécessite une notion d'intégrale (ou de trace). La partition de l'unité pourrait permettre d'exporter une intégrale définie localement à une intégrale globale. On pourrait alors mesurer l'impact de κ -MINKOWSKI comme espace tangent local.

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Introduction

Suppose you want to drink a can. You would first open the can and then drink its content. To do the opposite, that is to try to drink the can first and then open it, makes few sense. In the first case, you have drunk something and in the second you have not. Two processes that cannot be performed interchangeably are said to be *noncommutative*¹. Let us try to understand where this noncommutativity comes from. We have just compared the ordering importance of two *actions* (“trying to drink” and “opening”) applied to the same *system* (the can). One could think of another action that could be performed interchangeably with “trying to drink” the can, such as “measuring the length” of the can. The can has the same length whether it is full or empty. In this case, these two processes are said to be commutative.

The noncommutativity of the first scenario lies in the fact that the action of “opening” has changed the can from closed to open: it has affected our system. We say that the can has changed of *state*. Drinking while in the state “open” is not the same thing as drinking while in the state “closed”. The second scenario is fully commutative because the action of measuring is assumed not to change the state of the system. However, one of the radical changes quantum mechanics brought, is precisely that performing a measure on a system changes it. At the scale of the can, the change produced by the measurement process are negligible, but not for a quantum system. Therefore, the actions of “trying to drink” and “measuring the length” could be noncommutative for a would-be quantum can. This noncommutativity has been implemented in the model of quantum mechanics precisely through the use of observables and [operator algebras](#).

In this thesis, we study noncommutative (also called quantum) space-times, which are defined as [operator algebras](#). In the same spirit as above, the introduction of noncommutativity on the space-time structure expresses the fact that some processes can change the space-time itself. In particular, for the quantum space-times studied in this manuscript, the measure of space and time distances cannot be performed independently: one necessarily affects the other.

The previous hypothesis on space-time measurements follows from the idea that space-time should loose its smooth structure at the PLANCK scale. The puzzle regarding the fate of space-time at small scales arises from the theoretical inconsistencies between the frameworks of quantum mechanics and general relativity. In this sense, [quantum gravity](#) corresponds to a would-be theory of gravity that accommodates the geometrical interpretation of gravity with the probabilistic viewpoint of the quantum world. Despite many efforts and the development of many theoretical frameworks for [quantum gravity](#), there is no observed effects due to [quantum gravity](#).

The lack of a consistent theoretical framework urged some authors to opt for a bottom-up approach. In order to grasp more insights, [quantum gravity](#) phenomenology postulates properties which would be carried by a complete theory of [quantum gravity](#) and that makes testable predictions.

¹This illustration to explain noncommutativity with drinks is actually due to CONNES in a [communiqué from CNRS](#) [in French].

The study of the symmetries of quantum space-times, often called quantum or deformed symmetries, has triggered a promising pool of [quantum gravity](#) phenomenological frameworks. Some focus have been made on deformations of the POINCARÉ group that imposes both an observer independent speed and an observer independent length. Those are called doubly special relativity theories. The space-time carrying such a doubly special relativity cannot be smooth and naturally writes as a quantum space-time. The possible noncommutativity, induced by such theories, also spread to the momentum space, which is therefore now curved. The framework of curved momentum space and BORN geometries is heavily studied for [quantum gravity](#) phenomenology. The implementation of a minimal length can also be performed by generalising the HEISENBERG uncertainties that stems from the noncommutativity of the position and momentum operators. This minimal length turns field theories with a generalised uncertainty principle to be ultra-violet finite and therefore with strong predictive power. Note that generalised uncertainty principles may appear on some quantum space-times. From another perspective, the deformation of a classical symmetry implies that this classical symmetry is broken at some scale. The deformations of POINCARÉ symmetry may thus induce LORENTZ invariance violation or CPT violation. Both phenomenological frameworks have already attracted lot of attention, especially in the context of standard model extensions.

As all these considerations on deformed symmetries suggest, the physics of quantum space-times could provide even more insights into the nature of [quantum gravity](#) effects. Yet, the physics of quantum space-times remains poorly known.

The field and gauge theories on such space-times have been explored up to the one-loop level. Already in this loop order, these theories have been shown to exhibit a behaviour called the [UV/IR mixing](#). This behaviour corresponds to the appearance of mixed infra-red and ultra-violet divergences and is thought to spoil the usual perturbative renormalisation. The mixing was first experienced on the [MOYAL](#) space and later in other quantum space-times, but there is, so far, no general notion of when and how such a mixing occurs. The latter problem is tackled in [76]. The possible impact of the [UV/IR mixing](#) on particle physics or gravity on quantum space-times has been poorly studied. The appearance of higher loops phenomenons, as well as the fate of renormalisability, has yet not been deepened. In the case of (YANG-MILLS-like) gauge theories, it was shown that the gauge invariance is broken after quantisation, already at the level of the one-point function. This “gauge anomaly” has been noticed but not characterised. The unitarity of these fields theories with higher derivatives is also an open question.

Furthermore, there is no consensual notion of vacuum states and vacuum energy. The formalism of quantum groups allows one to perform non-linear change of coordinates of momentum so that expressions of the form p^2 , for p is a momentum, is not covariant anymore. One even struggles to define multi-particle states in the noncommutative framework [77].

Finally, the classical notion of causality is thought break at the PLANCK scale, since superimposed states of mass implies superimposed space-time geometries and so competing causalities. Accordingly, the first studies of causality on quantum space-times points to a conceptual change for a would-be quantum causality. Yet, the noncommutative causality frameworks already developed mainly evolve around “flat” causality since they were defined on deformations of MINKOWSKI. Moreover, they lack of global coherence and common theoretical grounds.

The aim of this manuscript is threefold. First, it introduces and motivates the use of [noncommutative geometry](#) in the context of physics study. Second, it summarises the state of the art in field and gauge theories on quantum space-times, focusing on the more recent developments

proposed by the author. Finally, it underlines the importance of [noncommutative geometry](#) for [quantum gravity](#) phenomenology and advocates that the study of quantum space-times could shed even more light on this topic.

The [Chapter 1](#) focuses on defining what [noncommutative geometry](#) is, and the different mathematical frameworks it encompasses. The physical motivations for such geometries are also discussed. The way in which field and gauge theories are implemented on quantum space-times is reviewed in [Chapter 2](#) together with the recent approach [76] based on momentum space analysis. The promising deformations of MINKOWSKI are discussed in [Chapter 3](#). The focus is made on the κ -MINKOWSKI deformation for which the first quantum properties of a gauge theory are analysed. Considerations of (quantum) gravity appear in [Chapter 4](#), where the phenomenology is introduced together with the theoretical motivations for studying [quantum gravity](#). A toy model of noncommutative gravity, based on a κ -MINKOWSKI tangent space, is tackled. Finally, more mathematical content is gathered in the Appendices. The [Appendix A](#) deals with quantum groups formalism and the [Appendix B](#) presents the necessary notions of C^* -algebras.

In all the manuscript, we apply EINSTEIN summation convention of repeated indices. This convention is only broken for the labelling of open covers and algebra covers in [Section 4.3](#), as specified in the latter Section. We also work in natural units, *i.e.* $c = \hbar = G = 1$, where c is the speed of light, \hbar the PLANCK constant and G the NEWTON constant.

Contribution of the author

The present manuscript tries to have a global viewpoint of the physics of and on noncommutative space-times. Therefore, it gathers well-established results as well as new contributions due, fully or partly, to the author. These new contributions were obtained during a 3-year Ph.D. study from October 2021 to October 2024, under the supervision of Jean-Christophe WALLEY, at IJCLab in Orsay, France.

The new material contained in this manuscript, in which the author has taken an active part, corresponds to

- [104] K. HERSENT, P. MATHIEU and J.-C. WALLEY, “*Quantum instability of gauge theories on κ -Minkowski space*”, *Phys. Rev. D* **105** (2022) 106013, [arXiv:2107.14462](#).
- [103] K. HERSENT, P. MATHIEU and J.-C. WALLEY, “*Algebraic structures in κ -Poincaré invariant gauge theories*”, *Int. Jour. Geo. Meth. Mod. Phys.* **19** (2022) 2250078, [arXiv:2110.10763](#).
- [135] K. HERSENT and J.-C. WALLEY, “*Gauge theory models on κ -Minkowski space: Results and prospects*”, *PoS CORFU2021* **406** (2022) 286, [arXiv:2203.12706](#).
- [23] K. HERSENT, P. MATHIEU and J.-C. WALLEY, “*Gauge theories on quantum spaces*”, *Phys. Rep.* **1014** (2023) 1-83, [arXiv:2210.11890](#).
- [106] K. HERSENT, “*Quantum properties of $U(1)$ -like gauge theory on κ -Minkowski*”, *PoS CORFU2022* (2023) 328, [arXiv:2302.03998](#).
- [113] N. FRANCO, K. HERSENT, V. MARIS and J.-C. WALLEY, “*Quantum causality in κ -Minkowski and related constraints*”, *Class. Quant. Grav.* **40** (2023) 164001, [arXiv:2302.10734](#).
- [121] K. HERSENT and J.-C. WALLEY, “*Field theories on ρ -deformed Minkowski space-time*”, *JHEP* **07** (2023) 031, [arXiv:2304.05787](#).

- [76] K. HERSENT, “*On the UV/IR mixing of Lie algebra-type noncommutative ϕ^4 -theories*”, *JHEP* **2024** (2024) 23, [arXiv:2309.08917](#).
- [137] K. HERSENT and J.-C. WALLET, “ *κ -Minkowski as tangent space I: quantum partition of unity*”, arXiv preprint (2024), [arXiv:2311.12584](#).

The review [23] gathers general gauge theory constructions on quantum space-times as well as detailed construction for MOYAL, \mathbb{R}_λ^3 and κ -MINKOWSKI. The general setting for gauge theories as been put in Chapter 2 and the gauge theory on κ -MINKOWSKI is presented in Section 3.2.

The paper tackling ϕ^4 -theory on arbitrary LIE algebra-type noncommutative space-times [76] has been summarised in Section 2.6. It also proposes an unambiguous definition of the UV/IR mixing and hints toward a possible criterion for ϕ^4 -theory to trigger the mixing or not.

In [103], the space-time dimension constraint obtained via gauge invariance of the noncommutative YANG-MILLS theory action on κ -MINKOWSKI has been shown to be strong. The gauge theory, from which this action is derived, is reviewed in [135] and is elucidated in Subsection 3.2.1.

Given the latter gauge theory on κ -MINKOWSKI, the computation of the one-loop one-point function (tadpole) has been performed in [104] and discussed more closely in [106]. The results are grouped in Subsection 3.2.2.

The causality toy model on κ -MINKOWSKI space-time is constructed in Section 3.3 and has been computed in [113].

A ϕ^4 -theory on the ϱ -MINKOWSKI space-time is studied in [121], as briefly mentioned in Section 3.4.

A toy model for gravity on quantum space-times having κ -MINKOWSKI as tangent space is constructed in [137]. It necessitates the notion of noncommutative partition of unity which is defined there. The latter model is detailed in Section 4.3.

Chapter 1

The principle of noncommutative geometry

Noncommutative geometry is a broad field of mathematics and physics that tries to have an algebraic point of view on geometry. Its main goal is to use the language of quantum mechanics (expressed by C^* -algebras) to encode the geometry. Therefore, there is a strong motivation to use the framework of **noncommutative geometry** to develop toy models of **quantum gravity**. But the motivations for studying “noncommutative space-times” capture other aspects of new physics, such as beyond the standard model studies.

As we will see in the following, the **noncommutative geometry** framework is too wide to be tackled in a single manuscript chapter. Therefore, this part of the thesis is mainly aimed at understanding the basic principles and the physical motivations for studying this field.

The most detailed aspects of **noncommutative geometry** that this manuscript deals with mostly evolves around noncommutative field theory. Other aspects of **noncommutative geometry** are still mentioned in this **Chapter 1**. A curious reader can find more details in the references.

1.1 Foundations of **noncommutative geometry**

All of modern theoretical physics is governed by two main models: general relativity and quantum field theory. Both general relativity and classical field theory are expressed in the same mathematical language of differential geometry. The goal of **noncommutative geometry** is to generalise differential geometry into a global framework that allows for more complex geometries, such as discrete geometries or fractals. As noncommutativity is a key component of quantum mechanics, the **noncommutative geometries** are also thought to encode quantum fields and “quantum space-times” into only geometrical notions.

Here we develop the main features of non-commutative geometry and the different paths it has taken, from a very physical point of view.

In order to generalise differential geometry, one has to start with differential geometry. The three main components of a differential geometry one has to generalise are the topology, the differential structure and the fiber bundles.

The topology corresponds to the part of space you can continuously deform, *i.e.* without closing holes, opening holes, tearing, gluing, or passing through itself. It gives information about the

space in terms of its number of “pieces” (connected components) or holes and allows to define distances between objects. Still, it does not give information about how the space is curved.

The differential structure comes with the smoothness of the space and allows to define derivatives along curves, or vector fields. A (smooth) field can be derived only if a differential structure is defined on the space. This implies that one can construct the set of forms on the space and also to define its curvature.

Finally, the fiber bundles correspond to an additional structure one sets on the space and that allows to define fermionic (spinor) fields or gauge fields.

First, let us generalise the topology. Let \mathcal{M} be a space-time and $\mathfrak{A} = \mathcal{C}^\infty(\mathcal{M})$ the space of observables. Then, an observable $f \in \mathfrak{A}$ attaches to each point $x \in \mathcal{M}$ a number $f(x) \in \mathbb{R}$ or \mathbb{C} . The observables \mathfrak{A} form an algebra (see [Definition A.1.1](#)) using the product of \mathbb{R} or \mathbb{C} through the point-wise product

$$(f \cdot g)(x) = f(x) g(x). \tag{1.1.1}$$

This algebra is further commutative, that is $f \cdot g = g \cdot f$, because the products in \mathbb{R} and \mathbb{C} are commutative.

Now, consider the space of states, noted $\Psi_{\mathfrak{A}}$, as the functions over the space of observables \mathfrak{A} . Explicitly, it corresponds to the set of elements ψ which, to a given observable f , associates a number $\psi(f) \in \mathbb{R}$ or \mathbb{C} . Then, the commutative [GEL’FAND-NAÏMARK theorem](#) states that the space of states $\Psi_{\mathfrak{A}}$ is in one-to-one correspondence with the space-time \mathcal{M} itself. Explicitly, given a point $x \in \mathcal{M}$, there exists one and only one state ψ_x such that $\psi_x(f) = f(x)$. One can underline the correspondence with the “bracket” notation. If one writes

$$\langle f | x \rangle = f(x), \qquad \langle f | \psi \rangle = \psi(f), \tag{1.1.2}$$

then the correspondence is

$$|\psi_x\rangle = |x\rangle. \tag{1.1.3}$$

This picture shows how the [GEL’FAND-NAÏMARK theorem](#) relates the notion of space-time point to the notion of state.

From there, the point of view of [noncommutative geometry](#) is to consider the quantum version of this picture, where f is a quantum observable and ψ is a quantum state. In order to do so, one turns \mathfrak{A} into a noncommutative algebra. In this case, the noncommutative space-time (the would-be set of point x) is pictured as being the (pure) states ψ through an extrapolated correspondence of the form [\(1.1.3\)](#).

Thus, the way to generalise the topological picture is to consider an algebra, which is not necessarily commutative¹ and which stands as the algebra of smooth function over a “noncommutative space-time”. Understanding the properties and characteristics of this kind of space-time is a central question in [noncommutative geometry](#).

¹In fact, the algebraic generalisation of geometry was undertaken before [noncommutative geometry](#) came into play, with algebraic geometry. However, this setting requires commutativity and cannot account for some geometric spaces which are noncommutative (like the quantum mechanics space, see [Section 1.2](#)). We refer to CONNES’ book [1] for more details. It should be mentioned that there are many attempts into constructing noncommutative algebraic geometry, but this goes far beyond the scope of this thesis and of the author’s personal knowledge.

More quantitatively, one can construct a noncommutative algebra of functions by starting with a commutative algebra of functions over some space-time, say the one of (1.1.1), and deform the latter with a new product that is noncommutative, generically called the star product. This procedure is called deformation quantisation.

Deformation quantisation

The deformation quantisation setting aims at finding an unambiguous quantisation procedure and thus at classifying the different quantisation methods. Inspired by the formalism of quantum mechanics, one can deform the smooth functions of a (POISSON) manifold into an algebra of functions with a noncommutative product, called the star product, generically denoted by \star in this manuscript. This product expands over a small parameter, called the deformation parameter, similar to \hbar for quantum mechanics. In this sense, and as it is made explicit in Section 1.2 through equation (1.2.2), the star product formalism is linked to canonical quantisation. The standard definition of deformation quantisation and some ways of constructing star products will be given in Section 1.6.

The second step is to generalise the differential structure, and thus the geometry, in the sense that one defines the metric thanks to this differential structure.

One can hint that this task will be much more difficult. Indeed, the topology gathers “global” properties of a space and so functions defined on the whole space allow to recover the latter properties. However, the differential structure is local and even defined point by point. But the algebra of functions cannot render points by itself, as it does not contain the DIRAC delta function. This smearing out of the noncommutative space-time, due to its definition through its algebra of functions, is the reason noncommutative space-times are sometimes called “fuzzy” space-times in the physics community. This also echoes some physical interpretation of fuzziness discussed in Subsection 1.2.3.

There are several ways to generalise the differential structure which are presented below: the spectral triple with its quantised calculus, the quantum group approach and the derivation-based differential calculus.

Spectral triple

The most commonly acknowledged approach toward the generalisation of the differential structure is the CONNES’ spectral triple approach to noncommutative geometry. Its starting point is to consider the DIRAC operator as containing information on the geometry. Indeed, one has, in a local coordinate chart,

$$\mathcal{D} = -i\gamma^\mu(\partial_\mu + \omega_\mu), \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_{d+1}, \quad (1.1.4)$$

where g is the space-time metric and ω_μ is the spin connection. Therefore, the DIRAC operator gathers the differential and the metric (through the gamma matrices).

At first, there was hope that the Laplacian and its spectrum could bring all the needed information. To this end, KAC asked the question “Can one hear the shape of a drum?” [2]. But it turns out we cannot [3]. The missing information of the Laplace operator can be recovered when considering a HILBERT space on which we (faithfully) represent the noncommutative algebra. We then need a triplet for our noncommutative geometry which is called the spectral triple: the algebra, the HILBERT space and the DIRAC operator. More details about this structure are gathered in Section 1.3.

From there, using the so-called quantised calculus, one can define a differential calculus on the [spectral triple](#). The main idea is to represent the function of the algebra as operators on the [HILBERT space](#) and then introduce the differential given by the commutator of operators. The master piece of this setting then resides in [CONNES' reconstruction theorem](#) which states that for any compact oriented Riemannian manifold, one can recover the usual differential calculus thanks to the triple.

Quantum groups

Quantum groups are generalisations of the usual notion of group and their applications are broader than only [noncommutative geometry](#). Their definition may also vary since different approaches to quantum groups were considered. In our case, and throughout this manuscript, the notion of quantum group will be associated with the structure of [HOPF algebra](#) and both terms will be used equivalently.

In the context of [noncommutative geometry](#), quantum groups are mainly associated with quantum symmetries, in a similar way as groups are associated with symmetries for commutative geometries. Therefore, one can define from the universal differential calculus a calculus that satisfies this quantum symmetry. From there, a full picture analogue to Riemannian geometry with quantum groups can be constructed [4].

However, the link between quantum groups and geometry is deeper thanks to their structure: the dual of a [HOPF algebra](#) is also a [HOPF algebra](#). Thus, the noncommutative space-time, seen as a dual to its quantum symmetries, also have a [HOPF algebra](#) structure. In other words, the correspondence (1.1.3) associates similar structure to the space-time and its observables via a [HOPF algebra](#) formalism. In the context of noncommutative space-times, this situation is actually occurring. More details will be given in [Section 1.4](#) and the mathematical definitions are postponed to [Appendix A](#).

Derivation-based differential calculus

Another approach to noncommutative differential geometry focuses on the tangent space. Indeed, the differential calculus over a space-time is based on the notion of vector field. If one were to generalise the vector fields to a noncommutative setting, then all the usual definitions of tensors, connections or curvature could be algebraically copy-pasted from their differential geometry expressions. The key point is to observe that vector fields are in one-to-one correspondence with the derivations of the algebra of smooth functions. Then, one could define the analogues of noncommutative vector fields to be the derivations of the noncommutative algebra.

This construction is called the derivation-based differential calculus [5]. It is the main topic of [Section 1.5](#).

Remark 1.1.1

An important remark to point out is that these different approaches to noncommutative geometry are not necessarily in competition with one another. One can find bridges linking all these settings, but there is not, at this date, a coherent language to gather all. We here make some links explicit.

First, the deformation quantisation setting allows one to go from any smooth space-time to a noncommutative algebra of functions with a star product. Thus, quantum space-time can be constructed through deformation quantisation and then combined with any of the other three formalisms by adding more structure, explicitly a [spectral triple](#), a [HOPF algebra](#) of symmetries, or the derivations of the algebra.

From any algebra, one can build a differential calculus called the universal differential calculus. It is called universal in the sense that one can recover any graded differential calculus over the algebra from the universal one. Therefore, the quantised differential calculus and the derivation-based differential calculus can both be seen as specific restrictions of the same universal differential calculus.

In the case of a noncommutative space-time on which acts a [HOPF algebra](#) of symmetries, one can read from the [HOPF algebra](#) structure some derivations over the space-time algebra. Indeed, as it is made clear in the following, a trivial symmetry element with a trivial coproduct acts as a derivation over the algebra. This makes a link between derivation-based differential calculus and [HOPF algebras](#) as it will be used in [Chapter 2](#) and [3](#).

The spectral structure of quantum groups was thought not to exist but it was exhibited a [spectral triple](#) for the quantum group $SU_q(2)$ [6].

Now that the differential part has been set up, there remains the question on how to build a connection and its corresponding curvature on a “noncommutative fiber bundle”. This question is central in both gravity theory and quantum field theory², as connection and curvature encode physical quantities. The key ingredient to generalise the notion of fiber bundle to the noncommutative setting is the SERRE-SWAN theorem. This theorem relates the geometric notion of vector bundles to the algebraic concept of projective modules (see [Subsection 1.4.3](#) for more details on modules and how it arises in [noncommutative geometry](#)), so that one can consider the modules as a generalisation of vector bundles in a noncommutative geometry. The definitions of connections and their curvature will then follow from their usual expressions exported to the case of the module.

This observation is the starting point of noncommutative field theory and gauge theory as detailed in [Chapter 2](#).

Considering the coordinate function, that is the function $x^\mu(x) = x^\mu$, on a noncommutative space-time, one has that $x^\mu \star x^\nu \neq x^\nu \star x^\mu$ by definition of noncommutativity. Here, the \star denotes the noncommutative product of the space, be it generated by deformation quantisation or not. The previous statement will be most of the time written under the form

$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu \neq 0 \tag{1.1.5}$$

as the bracket $[\cdot, \cdot]_\star$ precisely measures the amount of “noncommutativity”.

In terms of quantum mechanical interpretation, the relation (1.1.5) states that coordinates cannot be measured simultaneously and that the measurement of x^μ affects the measurement of x^ν ,

²In quantum field theory, the connection corresponds to the gauge field A and its curvature to the field strength F .

and *vice versa*. For this precise reason, the noncommutative space-times are often called quantum space-times and both terminology will be used in an interchangeable fashion in this manuscript.

1.2 Physical motivations of [noncommutative geometry](#)

Beyond the mathematical purpose of a generalised setting for geometric objects, the [noncommutative geometry](#) framework has several motivations coming from physics. The primary motivation and first example of such a [noncommutative geometry](#) is the phase space of quantum mechanics.

The geometry of quantum mechanics

One of the cornerstones of quantum mechanics is that the measurement of the system affects the post-measurement system by projecting it to a certain state. Therefore, the order of a series of measurements becomes of primordial importance since this order may change the latter measurements. In other words, measuring a quantity \hat{A} and then a quantity \hat{B} might not amount to measuring \hat{B} and then \hat{A} . We say that the measurement process is noncommutative. Furthermore, one can quantify the difference between the two measurements by computing the quantity $[\hat{A}, \hat{B}]$, which is non-zero by definition. In quantum mechanics, the noncommutativity is applied to the phase space coordinates x and p , that is why, roughly speaking, quantum mechanics is a noncommutative geometry.

Let us be a bit more explicit. Consider the phase space of classical mechanics in one space dimension, that is (a subset of) \mathbb{R}^2 , with coordinates x and p . The following discussion can be extended to any dimension, that is on (a subset of) \mathbb{R}^{2n} , straightforwardly. From there, x and p are numbers that stand for the position and momentum of a certain particle, respectively. If one wants to impose commutation relation such as $[x, p] \neq 0$, it cannot be done with only numbers, since any numbers commute. Therefore, we need to implement another structure.

Taking the observable point of view, like in [Section 1.1](#), we can consider the functions $x, p \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that $x(x, p) = x$ and $p(x, p) = p$. The space $\mathcal{C}^\infty(\mathbb{R}^2)$ has now the structure of an algebra, thanks to the product [\(1.1.1\)](#), but it is still commutative. Therefore, one cannot implement the commutation relation $[x, p] \neq 0$ even at the level of functions.

The procedure, to obtain the previous commutation relations, is to perform a quantisation. To do so, one needs a quantisation map [\[7\]](#) $\mathcal{Q} : \mathcal{C}^\infty(\mathbb{R}^2) \rightarrow \mathfrak{A}$, where \mathfrak{A} is called the [operator algebra](#) or [C*-algebra](#). For example, the position function x is turned into the position operator^a $\mathcal{Q}(x) = \hat{x}$. There are several ways to construct such a quantisation map, see [Section 1.6](#) for more details. One usually represents \mathfrak{A} on a [HILBERT space](#) \mathcal{H} through a (representation) map $\pi : \mathfrak{A} \rightarrow \mathcal{L}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ is the set of linear operators on \mathcal{H} . To exemplify the previous objects, let us take a physical example. Given a state $|\psi\rangle \in \mathcal{H}$, the position and momentum operators act on $|\psi\rangle$ through^b

$$\pi(\hat{x}) |\psi(x)\rangle = x |\psi(x)\rangle, \quad \pi(\hat{p}) |\psi(x)\rangle = -i \frac{d}{dx} |\psi(x)\rangle \quad (1.2.1)$$

where the π is generally omitted in both physics and mathematical literature. The [operator algebra](#) \mathfrak{A} can be viewed as the “[noncommutative geometry](#)” of quantum mechanics.

In the deformation quantisation formalism, one can go back from the **operator algebra** \mathfrak{A} to a space of functions, noted \mathbb{A} , using \mathcal{Q}^{-1} . An operator \hat{f} is thus transformed back into a function on \mathbb{R}^2 , f , and the product of \mathfrak{A} is also brought back as a new (star) product \star_{\hbar} via

$$f \star_{\hbar} g = \mathcal{Q}^{-1}(\mathcal{Q}(f)\mathcal{Q}(g)). \quad (1.2.2a)$$

The space \mathbb{A} together with \star_{\hbar} forms an algebra. Moreover, this star product is, by construction, noncommutative. One can obtain an explicit expression for the previous star product [8, 9, 10], called the **MOYAL** product in **WEYL** ordering,

$$(f \star_{\hbar} g)(x, p) = \frac{1}{(\pi\hbar)^2} \int dydqdzds f(x+y, p+q) g(x+z, p+s) e^{\frac{2i}{\hbar}(qz-ys)}, \quad (1.2.2b)$$

where f and g are functions of \mathbb{R}^2 and x and p the coordinates of \mathbb{R}^2 . This star product transforms the functions on \mathbb{R}^2 into a noncommutative algebra of functions, denoted generically by $\mathbb{A} = \mathcal{C}^{\infty}(\mathbb{R}^2)[[\hbar]]$. This notation means that any such functions can be written formally as a power series expansion in \hbar . Defining the position function $x(x, p) = x$ and the momentum function $p(x, p) = p$ as above, one obtains

$$[x, p]_{\star_{\hbar}} = x \star_{\hbar} p - p \star_{\hbar} x = i\hbar. \quad (1.2.2c)$$

In this sense, \mathbb{A} can be viewed as the “**noncommutative geometry**” of quantum mechanics, and the star product formalism as a straightforward way to implement canonical quantisation.

^aWe adopt here the usual physics hatted notation for operators, but we drop the hat outside this discussion.

^bNote that we chose $\mathcal{H} = L^2(\mathbb{R})$ here, that is the space of square integrable functions on \mathbb{R} . It corresponds to the usual **SCHRÖDINGER** representation of quantum mechanics, where the “sum” of probabilities corresponds to the integral.

One can find more examples of **noncommutative geometries** in [11].

The star product formalism structurally generates semi-classical and phenomenological frameworks. Indeed, the star product deformation is usually expressed as power series expansions of the deformation parameter. The latter parameter is called \bar{k} in a general context. The product of two functions write, in generic form,

$$f \star_{\bar{k}} g = f \cdot g + \sum_{n=1}^{\infty} \bar{k}^n C_n(f, g) \quad (1.2.3)$$

where C_n is a bilinear differential operator. The appearance of the usual commutative product of functions \cdot (1.1.1) in the star product $\star_{\bar{k}}$ (1.2.3) precisely account for the fact that $\star_{\bar{k}}$ is a deformation of \cdot , a deformation which is controlled by the deformation parameter \bar{k} . In other words, the product \cdot correspond to the zeroth order in \bar{k} of the product $\star_{\bar{k}}$.

One can consider to expand $\star_{\bar{k}}$ only to first order in \bar{k} and thus have a linear correction to the usual commutative theory. This would give a semi-classical theory, in which \bar{k} is a free parameter to be constrained by experiments. As we will see in the following, the deformation parameter is most of the time dimensionful (like \hbar) and would thus give a scale below which the commutative theory is valid.

Another general argument on how [noncommutative geometry](#) could hint new physics concerns deformations of symmetries. As the space-time structure is deformed, the corresponding symmetries will also be deformed. Most of the deformed symmetries encountered in this manuscript takes the structure of a [HOPF algebra](#), therefore underlying that these symmetries are quantum. In the context of star product deformation, the deformation of the symmetry is also controlled by the deformation parameter \hbar . This implies that the classical symmetries are broken and replaced by quantum ones. However, the classical symmetries can be recovered at some scale, corresponding to the limit $\hbar \rightarrow 0$.

The violations of some symmetries are intensely studied in the context of (high energy) new physics, like LORENTZ invariance violation, POINCARÉ deformation or CPT violation [12]. These phenomenological frameworks can naturally arise in the context of [noncommutative geometry](#) because of symmetry deformations. This is more discussed in [Section 4.2](#).

Beyond the broad arguments above for considering [noncommutative geometry](#) as an inspiration for new physics, more specific arguments can be made for aspects of physics that relies on quantum geometries like the standard model of particle physics or some model of [quantum gravity](#).

1.2.1 The geometry of particle physics

The main achievement of [noncommutative geometry](#) concerning particle physics is the noncommutative standard model [13]. All the standard model fields, minimally coupled to gravity, are encoded in a single (noncommutative) geometry expressed as a [spectral triple](#). The axioms of the [spectral triple](#) yield the gauge fields and the HIGGS field. Both emerge and are treated on the same footing. The HIGGS potential as well as the EINSTEIN-HILBERT action and the standard model action in curved space appear when computing the so-called spectral action. One of the main asset of this model is that it predicts the HIGGS mass.

Noncommutative standard model

Let \mathcal{M} be a Riemannian spin geometry (a space without time) and let $\mathbb{A} = \mathcal{C}^\infty(\mathcal{M}) \otimes (\mathbb{C} \oplus \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_3(\mathbb{C}))$. This algebra encodes the functions on a [noncommutative geometry](#) having four different parts: $\mathcal{C}^\infty(\mathcal{M})$ only encodes the functions over the space \mathcal{M} and thus the geometry of \mathcal{M} , \mathbb{C} imposes that functions are complex-valued and so takes into account the electric charge, $\mathbb{M}_2(\mathbb{C})$ accounts for the isospin doublets and $\mathbb{M}_3(\mathbb{C})$ stands for the three colors of quarks. \mathbb{A} can be viewed as the set of functions which associates to a point of space \mathcal{M} a complex number and two complex matrices of size 2 and 3. Those are called “almost commutative” spaces.

One represents this algebra on $\mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^{96}$, where $L^2(\mathcal{M}, \mathcal{S})$ stands for square integrable functions on the spinor bundle \mathcal{S} , which encodes the fermionic (spinor) fields, and \mathbb{C}^{96} describes the latter fermions counting the isospin doublet (up/down or electron/neutrino, with 2 choices), the 3 generations or flavours, the 2 chiralities (left or right), if it encodes a particle or an anti-particle (2 choices) and if the fermion is a quark or a lepton (the quarks have 3 colors and the lepton do not, so 3+1 choices): therefore, there are $2 \times 2 \times 3 \times 2 \times 2 \times (3+1) = 96$

fields. Finally, the DIRAC operator writes

$$\mathcal{D} = \not{\partial} \otimes \mathbf{1}_{96} + \gamma^5 \otimes \begin{pmatrix} 0 & M & M_R & 0 \\ M^\dagger & 0 & 0 & 0 \\ M_R^\dagger & 0 & 0 & \overline{M} \\ 0 & 0 & {}^t M & 0 \end{pmatrix} \quad (1.2.4)$$

where $\not{\partial}$ is the DIRAC operator on the space \mathcal{M} , M contains the quarks, leptons, neutrinos YUKAWA couplings, as well as the quarks and neutrinos mixing parameters and M_R contains the MAJORANA neutrinos mass.

The **spectral triple** $(\mathbb{A}, \mathcal{H}, \mathcal{D})$ encodes the geometry of the standard model of particle physics minimally coupled to gravity.

From there, one can compute the gauge group, the gauge bosons and the full action associated with that geometry. The gauge group corresponds to the unitary elements of the algebra \mathbb{A} , that are here $\text{Diff}(\mathcal{M}) \times \text{Map}(\mathcal{M}, U(1) \times SU(2) \times SU(3))$, where $\text{Diff}(\mathcal{M})$ stands for the diffeomorphism group of the space \mathcal{M} , that is the group of coordinate change, and $\text{Map}(\mathcal{M}, G)$ stands for the function from \mathcal{M} to G . The previous notation means that each point of the space \mathcal{M} has its own $U(1) \times SU(2) \times SU(3)$ gauge symmetry, which adapts to the choice of coordinates thanks to $\text{Diff}(\mathcal{M})$.

The gauge bosons can be read in the fluctuation of the DIRAC operator, corresponding to (1.3.6), and their gauge transformation through the first order axiom of the **spectral triple** formalism (1.3.1d). More explicitly, if the DIRAC operator (1.2.4) is noted $\mathcal{D} = \not{\partial} \otimes \mathbf{1} + \gamma^5 \otimes \mathcal{D}_F$, then one computes

$$\begin{aligned} f[\mathcal{D}, g] &= -i\gamma^\mu \otimes f\partial_\mu g + \gamma_5 \otimes f[\mathcal{D}_F, g] \\ &= \gamma^\mu \otimes A_\mu + \gamma_5 \otimes \phi \end{aligned} \quad (1.2.5)$$

where A_μ have all properties of the gauge fields and ϕ corresponds to the HIGGS boson.

Finally, the action is computed thanks to the spectral action for the DIRAC operator (1.2.4). The space part of the DIRAC operator gives rise to the EINSTEIN-HILBERT action, which is summed with the standard model action in curved space with a spontaneously broken HIGGS ϕ .

The noncommutative standard model may give new phenomenological insights on particle physics. First, the HIGGS mechanism, that is the HIGGS boson and its potential, is implemented naturally. Then, the model imposes mass relations between the fermions and predicts the HIGGS mass. Finally, the see-saw mechanism can be implemented within the framework. The phenomenology of this model is still under study.

This manuscript does not expand too much on the noncommutative standard model as it was not part of the author's work. Moreover, one should note that this model is still under construction, as their actual formulation are purely classical (in the sense that there was no second/path integral quantisation done) and Riemannian, therefore without a causal structure³. Nevertheless, the **Remark 1.1.1** is to be underlined as some part of what is contained in this thesis may, one day, be

³For Lorentzian extensions of the noncommutative standard model, see for example [14, 15].

linked to this model. Relevant textbooks on the noncommutative standard model are, for example, [1, 16].

The study of field theories on noncommutative space-time, called generically noncommutative field theory, have shown a peculiar behaviour which might be of physical interest, called the ultra-violet/infra-red mixing, or **UV/IR mixing**.

The first idea of studying field theory on quantum space-time, goes back to SNYDER [17] who expected that the noncommutativity would regularize the ultra-violet divergences occurring in quantum field theory, before renormalisation came into play. The hope of SNYDER was that the deformation parameter could play the role of a ultra-violet cut-off. However, this phenomenon happens only to very specific cases and noncommutative field theory also experiences divergence, some similar to the commutative case and some new ones. For example, in the **MOYAL** space a ϕ^4 -theory gives rise to the usual UV divergence for so-called planar diagrams (1.2.8a), but exhibits UV finiteness and IR singularity for non-planar diagrams (1.2.8b). The presence or absence of divergences has not been characterised yet for general noncommutative field theory.

UV/IR mixing

The first appearance of the **UV/IR mixing** occurred on the **MOYAL** space [18] (see Section 2.5 for more details on the **MOYAL** space). It starts by considering a deformed ϕ^4 -theory, with an action given by

$$S = \int d^4x \frac{1}{2} (\partial_\mu \phi \star_\theta \partial^\mu \phi)(x) + \frac{m^2}{2} (\phi \star_\theta \phi)(x) + \frac{g^2}{4!} (\phi \star_\theta \phi \star_\theta \phi \star_\theta \phi)(x) \quad (1.2.6)$$

where \star_θ is the **MOYAL** star product (2.5.3), m is the mass of the field ϕ and g the coupling constant. By performing path integral quantisation, one finds that the 2-point function at one-loop has two contributions, one coming from planar diagrams and one from non-planar diagrams (see Figure 1).

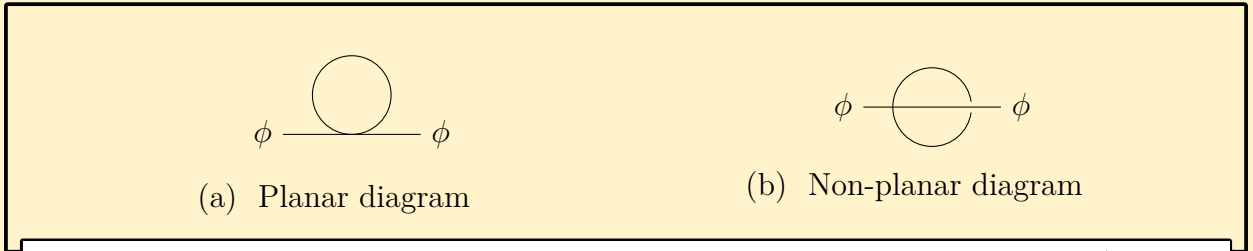


Figure 1 – The two contributions to the 2-point function at one-loop of the ϕ^4 -theory (1.2.6) on **MOYAL** space.

which reads

$$\langle \phi(p)\phi(q) \rangle_{1\text{-loop}}^{\text{p}} = \frac{g^2}{3} \delta(p+q) \int d^4k \frac{1}{k^2 + m^2} \quad (1.2.7a)$$

$$\langle \phi(p)\phi(q) \rangle_{1\text{-loop}}^{\text{np}} = \frac{g^2}{6} \delta(p+q) \int d^4k \frac{e^{ik_\mu \Theta^{\mu\nu} p_\nu}}{k^2 + m^2} \quad (1.2.7b)$$

where subscripts p and np stands for “planar” and “non-planar” respectively and p and q are the external (incoming and outgoing) momenta. The constant matrix $\Theta^{\mu\nu}$ corresponds to the deformation of the commuting coordinates, *i.e.* $[x^\mu, x^\nu]_{\star_\theta} = i\Theta^{\mu\nu}$.

One can analyse the behaviour of these integrals by first going into SCHWINGER parametrization $\frac{1}{k^2+m^2} = \int_0^{+\infty} d\alpha e^{-\alpha(k^2+m^2)}$ which turns (1.2.7) into Gaussian integrals. Then, by multiplying by a regulator $e^{-1/(\alpha\Lambda^2)}$, where Λ stands as a UV scale, one obtains

$$\langle \phi(p)\phi(q) \rangle_{1\text{-loop}}^{\text{p}} = \frac{g^2}{3} \delta(p+q) \left(\Lambda^2 - m^2 \log \left(\frac{\Lambda^2}{m^2} \right) + \mathcal{O}(1) \right) \quad (1.2.8a)$$

$$\langle \phi(p)\phi(q) \rangle_{1\text{-loop}}^{\text{np}} = \frac{g^2}{6} \delta(p+q) \left(\Lambda_{\text{eff}}^2 - m^2 \log \left(\frac{\Lambda_{\text{eff}}^2}{m^2} \right) + \mathcal{O}(1) \right) \quad (1.2.8b)$$

with

$$\Lambda_{\text{eff}}^2 = \frac{1}{1/\Lambda^2 + (p\Theta)^2/4} \quad (1.2.8c)$$

Here, $(p\Theta)^2 = p_\mu \Theta^{\mu\nu} p_\rho \Theta^{\rho\sigma} g_{\nu\sigma}$. From there, one observes that the planar contribution behaves as the commutative ϕ^4 -theory and is UV divergent. However, for non-vanishing external momentum p , the non-planar contribution now has a “noncommutative” regulator $\Lambda_{\text{eff}} = 4/(p\Theta)^2$, which makes it UV finite. Whenever $p \rightarrow 0$, this regulator diverges and the non-planar contribution also diverges in a similar fashion as the planar contribution. The fact that the IR limit $p \rightarrow 0$ makes the non-planar contribution diverge in the UV is precisely what is called the [UV/IR mixing](#).

More details about how the mixing arises and how it is linked to a noncommutative context will be given in [Section 2.6](#).

Some authors have studied how the [UV/IR mixing](#), specific to the noncommutative case, could hint for new physics.

First, a deep insight and physical reasoning on the [UV/IR mixing](#) was proposed in [19]. The previous authors pointed out that the [UV/IR mixing](#) phenomenon could not be accounted for in an effective field theory, as the effective field theory cannot generate IR contributions from UV regularization. In some toy models, they showed how a low mass scale could arise from the regularization of a UV divergence. This could hint for a testable solution to the hierarchy problem.

Second, in the context of noncommutative YANG-MILLS theory, the [UV/IR mixing](#) also arises for the gauge fields on the [MOYAL](#) space. The theory of noncommutative YANG-MILLS theory is taken to be a $U(n)$ gauge theory rather than $SU(n)$ since the [LIE algebra](#) $\mathfrak{su}(n)$ is not closed in the noncommutative theory, but $\mathfrak{u}(n)$ is. Then, the gauge group can be split into $U(n) \simeq SU(n) \times U(1)$ and one can show that the [UV/IR mixing](#) is actually generated by the extra $U(1)$ part. The basis of emergent gravity [20] in this context is to interpret this extra $U(1)$ part as a coupling between the $SU(n)$ gauge field and an emerging gravitational field.

1.2.2 Quantum gravity and noncommutativity

Most of the work done on noncommutative space-time arises in the context of [quantum gravity](#), since [noncommutative geometry](#) could be a hint for the formulation of [quantum gravity](#) or at least for some [quantum gravity](#) effects. The distinction that noncommutative space-time could be used to tackle beyond general relativity experiments and not necessarily emerge as a full model of [quantum gravity](#), is the very motivation for this thesis. Indeed, most of the noncommutative

space-times discussed in this manuscript are not necessarily thought of as the one space-time of nature, but are rather studied for their new properties and their phenomenology.

The recent interests in quantum space-times come from its appearance in string theory [21]. Noncommutative space-time has been shown to emerge in limiting regime of string theory and matrix (M) theory with magnetic background. More precisely, the noncommutative setting was used to compute VENEZIANO amplitude of bosonic open strings. Furthermore, a noncommutative version of YANG–MILLS theory arises naturally when studying closed strings in some limit natural for M theory.

However, the link to other [quantum gravity](#) models has been made. The main example is the relationship between the noncommutative field theory on a deformation of \mathbb{R}^3 , called \mathbb{R}_λ^3 , and the group field theory. It has first been pointed out by considering matter coupled to 3-dimensional quantum gravity [22], which, upon integrating out the gravity fluctuations, yields a noncommutative field theory on \mathbb{R}_λ^3 , interpreted as the effective theory of the dynamics of matter.

More details about the link between noncommutative field theory and these frameworks can be found in [23].

There is a well-known argument as to why the space-time structure should lose its continuous property close to the PLANCK scale. In other words, in a theory of quantum gravity, the space-time should not be a manifold.

Suppose we want to measure a space-time distance Δx as precisely as possible. The HEISENBERG uncertainty principle, coming from quantum mechanics, specifies the minimal amount of energy to deliver to a region of volume $(\Delta x)^4$, explicitly $\Delta E \geq 1/\Delta x$, in order to have such a precision. Decreasing the distance Δx will thus make the minimal energy input grow. In a continuous space-time, Δx can be arbitrarily small. Therefore, one can reach a Δx where the minimal energy corresponds to the PLANCK mass M_P . For this distance $1/M_P$, a black hole is created, which hampers one to probe smaller distances. Therefore, by mixing quantum mechanics and general relativity, one ends up in a space-time with a smallest measurable distance⁴.

To avoid generating a black hole when doing high resolution microscopy, DOPLICHER, FREDENHAGEN and ROBERTS [24] developed a space-time framework which generates space-time uncertainties of the form

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2} |\Theta^{\mu\nu}|. \quad (1.2.9)$$

This implies that Δx^μ and Δx^ν cannot be arbitrarily small independently⁵. There is a minimal area, modelled by Θ , that one can probe in such a space-time. This relation could be enforced by

⁴One should note that this argument involves classical and quantum properties of space-time. Therefore, a black hole is not expected to pop up *in reality*. This argument is just here to put forward inconsistencies when merging both models.

Besides, one could also argue that such a black hole evaporates. However, evaporation would make the energy flow away and precision is lost. A more striking argument would be to consider the evaporating time, which writes in this case $t_e = 5120\pi \frac{G^2}{c^4 \hbar} M_P^3 = 1280\pi \frac{\ell_P}{c} \sim 10^{-40}$ s, where ℓ_P is the PLANCK length and c is the speed of light. This consists thus also in a smallest scale of time.

⁵*A priori* this does not prevent one from having an arbitrarily small distance, only to have arbitrarily small area. Indeed, one can take Δx^μ to be arbitrarily small if Δx^ν is very large. The set-up for minimal length does rather appear in the framework of generalised uncertainty principle [26]. We refer to [Section 4.2](#) for more details on [quantum gravity](#) phenomenology.

considering a quantum space-time satisfying

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu} \tag{1.2.10}$$

which corresponds to the **MOYAL** space (see [Section 2.5](#) for details).

The previous framework showed how a deformation parameter (here θ) can be introduced as a physical limit for a coherent approach of both quantum mechanics and general relativity. The deformation parameter plays the role of a new dimensionful physical constant the space-time has to accommodate for. This approach features new possibilities for a space-time “beyond general relativity”, but is rather arbitrary in the sense that this new physical quantity is magically imposed.

However, one could also introduce the deformation parameter directly through the symmetries of the space-time by deforming the classical symmetries, via quantum groups. In doing so, the deformation parameter becomes the scale at which these deformed symmetries should replace the usual symmetries. For example, the κ -MINKOWSKI space-time, a deformation of the MINKOWSKI space-time, has κ -POINCARÉ symmetries, which correspond to a deformation of the POINCARÉ group. In this space-time the deformation parameter is κ and has the dimension of an energy. Therefore, considering a noncommutative field theory on κ -MINKOWSKI only makes sense only at energies similar to κ , or above. For energies far below κ , one can consider a field theory on the MINKOWSKI space-time.

The κ -MINKOWSKI spaces and the κ -POINCARÉ algebra are presented in detail in [Section 3.1](#). Still, we want to emphasise here that the deformation parameter actually indicates whether the space-time is to be regarded as noncommutative or not, and the symmetries as deformed or not. This is exactly the behaviour expected for a theory of quantum gravity that does stand at the PLANCK scale, but from which one can recover general relativity at low energies.

Finally, the physics of noncommutative space-time has sparked a tremendous amount of phenomenological framework, most of them handling around [quantum gravity](#) [12]. Here, we cite the main examples as a motivation, but more details about [quantum gravity](#) phenomenology will be given in [Section 4.2](#).

First, if one considers a quantum space-time with deformed symmetries, then the classical symmetries would no more stand and thus be violated above some scale. Therefore, the quantum group framework allows one to consider deformations of the LORENTZ group and leads to LORENTZ invariance violation. The phenomenological framework of LORENTZ invariance violation is used both in the search of [quantum gravity](#) and in some standard model extensions. The main advantage of using deformations through quantum groups, in this context, is that one can recover the usual LORENTZ symmetry of the standard model and of local general relativity by considering a scale far below the scale induced by the deformation parameter.

The noncommutativity of the space-time can also affect the dispersion relation of a particle. The propagation of a single particle is given by the d’Alembertian operator⁶ on the considered space-time. From the point of view of infinitesimal quantities (that is the point of view of the [LIE algebra](#) rather than the LIE group), the d’Alembertian is given by the Casimir operator. Moreover, in a physical context, this d’Alembertian can be written in terms of the space-time symmetries. Therefore, when considering deformed symmetries via the formalism of quantum

⁶In a mathematical context, the d’Alembertian corresponds to the Laplacian in a pseudo-Riemannian context (that is on a manifold having a time coordinate). The d’Alembertian is thus sometimes called Laplacian even when the space considered is pseudo-Riemannian, *i.e.* when the considered object is a space-time.

groups, the Casimir operator is also deformed, which means that the particles propagate differently in a quantum space-time. The deformation of the particle propagation is studied in the context of modified dispersion relations.

The LORENTZ group symmetry was first implemented in special relativity in order to have invariance under the change of inertial frame and conservation of the speed of light. If one wants further that an energy scale is conserved under the change of “inertial frame”, where inertial has now a new meaning, one should also consider deformed symmetries. The framework, in which two observer-independent upper scale, one of speed, one of energy, are implemented, is called doubly special relativity⁷ [25]. It was first set up to ease a mismatch between the contraction and dilatations of length in a special relativity context and the hypothetical minimal PLANCK length of [quantum gravity](#). In this context, one cannot find an alternative LIE group to the LORENTZ group in order to have such a frame transformation. Still, one can realise a doubly special relativity via the HOPF algebra deformation of the LORENTZ group or of the POINCARÉ group, as for example κ -POINCARÉ.

The minimal length scenario was also not compatible with the HEISENBERG uncertainty principle and therefore led to consider generalised uncertainty principles. A generalised uncertainty principle takes the form

$$\Delta x \Delta p \geq \frac{\hbar}{2} (1 + f(p^2)), \quad (1.2.11a)$$

where f is some smooth function vanishing at zero. This relation can follow from a deformation of the commutator

$$[x, p] = i\hbar (1 + f(p^2)), \quad (1.2.11b)$$

which, in turn, can be realised in a [noncommutative geometry](#).

Finally, other phenomenological frameworks can be inherited from a [noncommutative geometry](#). Most of the time, the noncommutativity of coordinates (1.1.5) also implies the noncommutativity of the law for momentum addition. This is made explicit in the [Section 2.6](#). This noncommutativity of momenta implies that the momentum space is curved. It is also believed that CPT violation arises in a noncommutative momentum framework, but this claim is still debated. Indeed, in such a phase space, there are ambiguities in defining the discrete symmetries, *i.e.* the charge conjugation C, the parity P and the time reversal T.

1.2.3 When objects are fuzzy

The [noncommutative geometry](#) framework handles both continuous and discrete geometries so that a noncommutative space-time can be continuous, discrete or both. This “undetermined shape” of a [noncommutative geometry](#) pushed the literature toward calling the quantum space-times “fuzzy space-times”. This denomination can actually make sense physically and mathematically. But before addressing these points let us discuss the example of \mathbb{R}_λ^3 .

⁷It was also named deformed special relativity.

Description of \mathbb{R}_λ^3

One can define the space \mathbb{R}_λ^3 to be a pile of matrices of different sizes, that is

$$\mathbb{R}_\lambda^3 = \bigoplus_{n=0}^{\infty} \mathbb{M}_n(\mathbb{C}) \quad (1.2.12)$$

where $\mathbb{M}_n(\mathbb{C})$ corresponds to the space of $n \times n$ matrices with complex entries. In terms of [noncommutative geometry](#), the space of matrices $\mathbb{M}_n(\mathbb{C})$ is more often called the fuzzy sphere. Therefore, \mathbb{R}_λ^3 is a pile of fuzzy sphere and was sometimes dubbed the “fuzzy onion”.

As a deformation of \mathbb{R}^3 , the latter space has another definition in terms of [LIE algebra](#) of coordinates

$$[x^j, x^k]_{\star_\lambda} = i\lambda \varepsilon^{jk}_l x^l \quad (1.2.13)$$

where λ is the deformation parameter and ε is the LEVI-CIVITA fully anti-symmetric tensor. The [LIE algebra](#) (1.2.13) corresponds to $\mathfrak{su}(2)$. Considering one of the position operators x^j , its spectrum is made of the integers $n \in \mathbb{N}$, where the different n correspond to the sizes of the fuzzy spheres that x^j crosses. However, each sphere has its own continuous $SO_3(\mathbb{R})$ symmetry stemming from the action of $SU(2)$ on $\mathfrak{su}(2)$. Therefore, this space contains both discrete and continuous structures.

In view of the discussion of [Subsection 1.2.2](#), one can note that the deformation parameter λ has a length dimension.

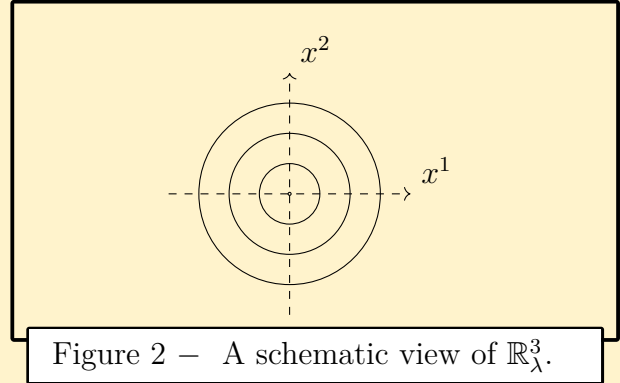


Figure 2 – A schematic view of \mathbb{R}_λ^3 .

Beyond continuity and discreteness, the space-time uncertainty relations (1.2.9) implies that one cannot measure a space-time point as accurately as one wants. Therefore, at small scales, the quantum space-time becomes blurry because it cannot be resolved with infinite precision. This is roughly the idea of the fuzziness of a quantum space-time.

1.3 Spectral triple

Given a space with topology, as depicted in [Section 1.1](#), we further want to have a differential structure to generate essential elements of geometry, like the metric or the curvature. KAC’s drum (see [Section 1.1](#)) asked whether one could build the differential structure using only the (spectrum of the) Laplacian⁸. It turns out not to be possible, but, by considering the “square root” of the Laplacian, *i.e.* the DIRAC operator, CONNES was able to reconstruct the geometry. However, in order to define the DIRAC operator, one needs additionally a [HILBERT space](#) on which the algebra is represented. Therefore, the geometry is gathered in three objects forming a [spectral triple](#): the algebra (of smooth functions), the DIRAC operator and the [HILBERT space](#).

The [spectral triple](#) formalism is presented briefly here. A curious reader can find much more details in the following references. The main writing on [noncommutative geometry](#) through [spectral](#)

⁸The Laplacian corresponds to the d’Alembertian in a pseudo-Riemannian context, that is considering the geometry of a space-time.

triple is CONNES' book [1] which gives a full overview of the philosophy of this construction. Other physics oriented textbooks can be consulted like [27, 28, 29].

Here, we define the spectral triple and gather a more extensive discussion on the motivations after.

Definition 1.3.1 – Spectral triple

A spectral triple is a set of three data $(\mathbb{A}, \mathcal{H}, \mathcal{D})$, where

- ❖ \mathbb{A} is a (unital) C^* -algebra,
- ❖ \mathcal{H} is a HILBERT space, such that the algebra \mathbb{A} represents faithfully on $\mathcal{B}(\mathcal{H})$, the set of bounded operators on \mathcal{H} ,
- ❖ \mathcal{D} is a self-adjoint operator on \mathcal{H} , with compact resolvent and satisfying $[\mathcal{D}, f] \in \mathcal{B}(\mathcal{H})$ for any $f \in \mathbb{A}$.

The latter spectral triple is said *odd* as such and is further said *even* given that there exists a \mathbb{Z}_2 -grading on \mathcal{H} , that is a map $\gamma : \mathcal{H} \rightarrow \mathcal{H}$ satisfying $\gamma^\dagger = \gamma$ and $\gamma^2 = 1$, such that

$$[\gamma, f] = 0, \quad \{\gamma, \mathcal{D}\} = 0, \quad (1.3.1a)$$

for any $f \in \mathbb{A}$.

Furthermore, we say that the spectral triple is *real* if there exists an anti-linear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$J^2 = \pm 1, \quad J\mathcal{D} = \pm \mathcal{D}J, \quad J\gamma = \pm \gamma J, \quad (1.3.1b)$$

where the $+$ and $-$ are determined by the so-called KO-dimension of the triple.

Finally, the spectral triple should satisfy the zeroth-order condition

$$[f, Jg^\dagger J^{-1}] = 0, \quad (1.3.1c)$$

and the first-order condition

$$[[\mathcal{D}, f], Jg^\dagger J^{-1}] = 0, \quad (1.3.1d)$$

for any $f, g \in \mathbb{A}$.

There are many comments to be made about this definition, which are summarised below.

- ❖ There is *a priori* no need of further structure for \mathbb{A} than being an operator algebra, *i.e.* a C^* -algebra or a VON NEUMANN algebra. This requirement is made in order that \mathbb{A} can be represented on a HILBERT space \mathcal{H} , and therefore for \mathcal{H} to exist. However, in some noncommutative space-time \mathbb{A} is not a C^* -algebra itself but close to it, like a pre- C^* -algebra or a dense $*$ -subalgebra of a C^* -algebra.
- ❖ The spectral triple formalism was designed first for compact manifold, which corresponds to unital algebras. CONNES' reconstruction theorem, discussed below, was only proved in the compact case. Therefore, one can ask for \mathbb{A} to have a unit, since unital algebras corresponds

to compact spaces. The **spectral triple** formulation above is the one used in the non-compact case.

- ❖ The **HILBERT space** \mathcal{H} is necessary to define the **DIRAC** operator and so the differential structure. Therefore, it is defined as a representation of our **noncommutative geometry** \mathbb{A} via $\pi : \mathbb{A} \rightarrow \mathcal{B}(\mathcal{H})$. As already mentioned, π is always implied in notations.
- ❖ The representation π above is required to be faithful so that information stored in \mathbb{A} are not lost when working with only operators on \mathcal{H} .
- ❖ Self-adjointness (or Hermiticity) is required in a physical context as observables correspond to self-adjoint operators in quantum mechanics. As the **DIRAC** operator contains the gauge fields, it is required to be self-adjoint. For example, in particle physics, the **DIRAC** operator corresponds to the covariant derivative $\mathcal{D} = i\gamma^\mu(\partial_\mu + A_\mu)$, where A_μ is the gauge field.
- ❖ The compact resolvent requirement states that the operator $(\mathcal{D} - z)^{-1}$ is compact, for any $z \notin \mathbb{R}$, in the case of a unital algebra \mathbb{A} (and so a compact geometry). This condition is changed to $f(\mathcal{D} - z)^{-1}$ been compact, for any $z \notin \mathbb{R}$ and $f \in \mathbb{A}$, in the non-unital case.
- ❖ In the case of a spin manifold, *i.e.* a space on which one can define fermionic (spinor) fields, one can write down

$$[\mathcal{D}, f] = -i c(df) \tag{1.3.2}$$

where c is the **CLIFFORD** action. As the action is merely a left multiplication, equation (1.3.2) motivates both the requirement of boundedness for $[\mathcal{D}, f]$ and the basic property of the quantised calculus that the differential d can be expressed through $[\mathcal{D}, \cdot]$ in (1.3.3). Moreover, in the commutative case, the condition (1.3.2) is also linked to the smoothness of f .

- ❖ The grading map γ is actually quite common for particle physicists since it plays the exact same role as γ^5 . The chirality of a particle is actually a \mathbb{Z}_2 -grading that splits the space into positive and negative chiralities. Here, γ plays a similar role (and so is often called the chirality) as one can define \mathcal{H}^\pm depending if an element of \mathcal{H} will have eigenvalue $+1$ or -1 for γ . The grading map is here to ensure the orientability of underlying manifold.
- ❖ Similarly that bosons commute (with γ^5) and fermions anti-commute (with γ^5), the condition (1.3.1a) states that space elements do not change the chirality but the **DIRAC** operator does.
- ❖ The element J , often called the **TOMITA** involution, can be viewed as a generalisation of the complex i to operators on a **HILBERT space**. Therefore, it is used to disjoint “real” and “imaginary” quantities. It corresponds to the isometry part (in the polar decomposition) of the operator induced by the involution \dagger of \mathbb{A} . Note that there is a way to construct such a J operator, satisfying (1.3.1c), through **TOMITA**’s theorem.
- ❖ In the commutative case, one only requires that $g = Jg^\dagger J^{-1}$ so that the zeroth-order condition (1.3.1c) is automatically satisfied.
- ❖ The first-order condition (1.3.1d) states that the **DIRAC** operator is a first order differential operator⁹. This expression also allows for the quantised calculus expressions like $f_1[\mathcal{D}, f_2] \cdots [\mathcal{D}, f_{n+1}]$ to be coherent (as a representation of $\mathbb{A}^{\otimes(n+1)}$) and so to form a differential calculus over \mathbb{A} .

⁹This can be compared to an equivalent relation with the Laplacian Δ , which is of second order: $[[\Delta, f], g] = \mathbf{g}^{-1}(df, dg)$, where \mathbf{g} is the metric.

The main motivation to study a [spectral triple](#) as a “[noncommutative geometry](#)” is given by the following [30]

Theorem 1.3.2 – CONNES’ reconstruction theorem

Let $(\mathbb{A}, \mathcal{H}, \not{D})$ be a [spectral triple](#) satisfying

- (i) \mathbb{A} is commutative.
- (ii) Let $\lambda_0 \geq \lambda_1 \geq \dots$ denotes the eigenvalues of $|\not{D}|^{-1}$, then it exists $n \in \mathbb{N}$, such that $\lambda_k = \mathcal{O}_{k \rightarrow +\infty}(k^{-1/n})$.
- (iii) For any $f \in \mathbb{A}$, $f, [\not{D}, f] \in \text{Dom}(\delta^j)$, for $j \in \mathbb{N}_1$ and $\delta(T) = [|\not{D}|, T]$.
- (iv) The triple is either odd or even (and if so γ is defined).
- (v) The space of smooth vectors $\mathcal{H}^\infty = \bigcap_{j \in \mathbb{N}_1} \text{Dom}(\not{D}^j)$ is a finitely generated projected \mathbb{A} -module with a hermitian pairing, given by the noncommutative integral.

Then, there exists a compact oriented smooth Riemannian manifold \mathcal{M} such that $\mathbb{A} = \mathcal{C}^\infty(\mathcal{M})$.

Beyond the very mathematical requirements that we try to clarify below, [CONNES’ reconstruction theorem](#) states that a commutative [spectral triple](#) corresponds to a classical geometry. Therefore, a [noncommutative geometry](#) would be given by a noncommutative [spectral triple](#).

In the above requirements, (ii) states that the eigenvalues of the inverse modulus of the DIRAC operator $|\not{D}|^{-1}$ are decreasing to zero as $k^{-1/n}$ for some n . As explained below with the distance formula (1.3.8), $|\not{D}|^{-1}$ can be interpreted as a length element “ dx ”. Therefore, $|\not{D}|^{-n}$, corresponding to a volume element in n -dimension $(dx)^n$, has eigenvalues decreasing as $1/k$. The n thus corresponds to the usual space dimension in commutative geometry¹⁰ and is called the metric dimension of the [spectral triple](#).

The condition (iii) requires that f and $[\not{D}, f]$ are smooth, since one can derivate them an arbitrary number of times (corresponding to j in the theorem) with the derivative δ .

Finally, the requirement (v) imposes that quantities such as $f|\not{D}|^{-n}$ are measurable and so that the noncommutative integral is well-defined. Indeed, $f|\not{D}|^{-n} = f(dx)^n$ represents the volume element of the space on which we want to sum. But, considering (ii), the sum of eigenvalues are logarithmically divergent $\sum_{k < N} \lambda_k \sim \sum_{k < N} \frac{1}{k} \sim \log(N)$, and so $\int f|\not{D}|^{-n}$ should also diverge logarithmically. Convergence of this quantity is ensured precisely on \mathcal{H}^∞ , which plays a similar role that of the test functions for the distributions.

Note that [CONNES’ reconstruction theorem](#) can adapt to the case of a spin manifold.

Now that we have defined the [spectral triple](#) and explained how it relates to the usual notion of geometry, we can go to the missing part of what makes this a [noncommutative geometry](#), that is the differential structure. In order to do so, one needs a differential operator d that sends a n -form to a $n + 1$ -form, and the latter set of forms.

Inspired by the relation (1.3.2), one can define the differential d , by

$$df = [\not{D}, f] \tag{1.3.3}$$

¹⁰One notices that, in a more general context, this n needs not be an integer. Part of the motivation for studying [noncommutative geometry](#), was that it could depict non-integer dimensional space, like fractals.

for any $f \in \mathbb{A}$. The formula (1.3.3) can be viewed as the equation of motion in classical Hamiltonian mechanics, where the POISSON bracket has been replaced by the commutator $[\cdot, \cdot]$. From there one defines a generic one-form $\omega \in \Omega^1(\mathbb{A})$ by

$$\omega = f[\not{D}, g] \quad (1.3.4)$$

for some $f, g \in \mathbb{A}$. By applying the differential d to this one-form several times, one obtains that a generic n -form $\omega \in \Omega^n(\mathbb{A})$ writes

$$\omega = f_1[\not{D}, f_2] \cdots [\not{D}, f_{n+1}] \quad (1.3.5)$$

for any $f_1, \dots, f_{n+1} \in \mathbb{A}$.

The fact that a one-form writes $\omega_u = u[\not{D}, u^\dagger]$, states that the failure of invariance of the DIRAC operator with respect to the gauge transformation $u \in \mathbb{A}$, satisfying $u \star u^\dagger = u^\dagger \star u = 1$, is a one-form:

$$u\not{D}u^\dagger = \not{D} + \omega_u, \quad (1.3.6)$$

which corresponds to the formalism of gauge theory.

From there, one can define the noncommutative integral (defined thanks to the DIXMIER trace) and check that it matches the usual integration on space in the context of CONNES' reconstruction theorem. Using the noncommutative integral (or noncommutative trace) tr , one can define the so-called spectral action for a fermionic field $\psi \in \mathcal{H}$ as

$$S(\not{D}) = \langle \psi, \not{D}\psi \rangle + \text{tr} \left(\alpha \left(\frac{\not{D}^2}{\Lambda^2} \right) \right), \quad (1.3.7)$$

where $\langle \cdot, \cdot \rangle$ is the inner product of \mathcal{H} , α is a positive functional and Λ is an energy scale. The tracial part of the action can be expanded by heat kernel methods (which generates an expansion over $1/\Lambda$), and gives rise to geometric invariants in the commutative setting. In other words, the spectral action displays all the gauge and diffeomorphic (*i.e.* change of coordinate) invariant quantities that one can build within the geometry. Moreover, the expansion over the energy scale Λ implies that one can treat (1.3.7) as an effective field theory.

One of the key feature of the spectral triple formalism, is that one can lift the notion of geodesic distance to the noncommutative case. Indeed, by dualising the geodesic distance formula, one obtains

$$\text{dist}(\psi_1, \psi_2) = \sup \left\{ |\psi_1(f) - \psi_2(f)|, f \in \mathbb{A}, \left\| [\not{D}, f] \right\| \leq 1 \right\} \quad (1.3.8)$$

which gives the distance between two arbitrary states $\psi_1, \psi_2 \in \Psi_{\mathbb{A}}$. In the case of smooth functions, the expression $\left\| [\not{D}, f] \right\|$ corresponds to the maximal value of the (first) derivative of f . This is called the LIPSCHITZ norm and can be defined for non-differentiable functions also. A direct consequence is that one can interpret the inverse modulus of \not{D} , noted $|\not{D}|^{-1}$, as a length element dx .

1.4 Quantum groups

There are several definitions of quantum groups coming from different approaches. DRINFEL'D and JIMBO developed the construction of deformation of universal enveloping algebras of any semi-simple LIE algebra. At about the same time, WORONOWICZ made his theory of compact matrix quantum groups. Moreover, an algebraic approach to quantised coordinate algebra was introduced by MANIN. Even at this date, there is no global definition for what quantum groups are, but as pointed out in [33]: “Instead of searching for a rigorous definition of a quantum group it seems to be more fruitful to look for classes of Hopf algebras that give rise to a rich theory with important applications and contain enough interesting examples.” Therefore, we will rather explain here the philosophy of HOPF algebras and how they can be related to physics.

The usual textbooks on HOPF algebra and quantum groups are [31, 32, 33]. The basic elements and examples of HOPF algebra needed for this manuscript are presented here. However, more mathematical details are postponed to Appendix A.

1.4.1 The emergence of HOPF algebra

In the same spirit as in Section 1.1, we want to characterise a group G through its dual, that is the space of functions over it. We denote the latter by G' . Physically, G may be thought of as a group of symmetries. In a very similar fashion to the product (1.1.1), one can define a product, and so an algebra structure, on G' through (1.4.1b). The group structure of G is however not exported to G' as such. As shown in Example 1.4.1, one needs to equip G' with a HOPF algebra structure in order to read the group structure of G in G' .

Example 1.4.1 – The algebra of functions over a group

Let G be a group and let G' be the algebra of complex-valued functions on G , *i.e.* $f \in G'$ is a function from G to \mathbb{C} . G' is called the dual of G .

The space G' is canonically an algebra (see Definition A.1.1) if we consider the point-wise multiplication in \mathbb{C} . Explicitly, for any $z_1, z_2 \in \mathbb{C}$, $f_1, f_2 \in G'$ and $u \in G$,

$$\left. \begin{aligned} z_1 f_1 + z_2 f_2 : G &\rightarrow \mathbb{C} \\ u &\mapsto z_1 f_1(u) + z_2 f_2(u) \end{aligned} \right] \quad \text{i.e.} \quad z_1 f_1 + z_2 f_2 \in G', \quad (1.4.1a)$$

therefore, G' is a vector space. Considering the product,

$$(f_1 \cdot f_2)(u) = f_1(u) f_2(u), \quad (1.4.1b)$$

where the right-hand-side product corresponds to the product in \mathbb{C} , G' satisfies the definition of the algebra. The associativity property of the product is directly inherited from the associativity of the product of \mathbb{C} . Note that this algebra is unital since $1(u) = 1$ belongs to G' . This algebra structure, however, does not allow one to characterise G from G' as it does not involve the product of G . In other words, only knowing the algebra structure of G' one does not have all the information needed to reconstruct G .

Let $\Delta : G' \rightarrow (G \times G)'$ be the function defined as, for any $f \in G'$ and $u_1, u_2 \in G$,

$$\Delta(f)(u_1, u_2) = f(u_1 u_2). \quad (1.4.2a)$$

This map is called the coproduct and allows one to get the missing information of the product of G . Indeed, it characterises how the function f should split in $(G \times G)'$ to match the product of G through (1.4.2a). Considering 1 to be the unit of G , one can also define a counit $\varepsilon : G' \rightarrow \mathbb{C}$ through

$$\varepsilon(f) = f(1). \quad (1.4.2b)$$

Again, the unit of G' only involves $1 \in \mathbb{C}$ and does not resolve the unit of G , whereas the counit ε is precisely defined to see how functions behave on $1 \in G$. As additional constraints, these two maps should satisfy, in some way, the associativity of the product in G and the unit property $u1 = 1u = u$ of $1 \in G$, respectively. The way to do this is to see that the coproduct is coassociative (1.4.4) and the counit property (1.4.5) is satisfied. Those properties are shown in the proof below. This will form what is called the coalgebra sector of G' .

But every group element $u \in G$ has, by definition, an inverse u^{-1} . Therefore, we need another structure on G' to account for the inverse. The latter map is denoted $S : G' \rightarrow G'$ and is called the antipode (or coinverse). It is defined as

$$(S(f))(u) = f(u^{-1}), \quad (1.4.3)$$

and once again it should match the inverse property $uu^{-1} = u^{-1}u = 1$ of G . This is implemented by requiring the S satisfies the coinverse property (1.4.6), as detailed in the proof.

Finally, one can check that the coalgebra sector is adapted to the algebra structure (1.4.1) of G' through (1.4.7). This will be detailed in the proof below.

Proof. We here show that some properties of Example 1.4.1 are satisfied.

First, let us check the coassociativity of the function Δ defined in (1.4.2a). In algebraic terms it writes

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta. \quad (1.4.4)$$

One computes, for any $f \in G'$, any $u_1, u_2, u_3 \in G$,

$$\begin{aligned} (((\Delta \otimes \text{id}) \circ \Delta)(f))(u_1, u_2, u_3) &= f((u_1 u_2) u_3) \\ (((\text{id} \otimes \Delta) \circ \Delta)(f))(u_1, u_2, u_3) &= f(u_1 (u_2 u_3)). \end{aligned}$$

Therefore, the equality is fulfilled thanks to the associativity of the product in G . Now, the counit property writes

$$(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}, \quad (1.4.5)$$

and it is satisfied thanks to the unit property $1u = u1 = u$ of $1 \in G$. Indeed,

$$\begin{aligned} (((\varepsilon \otimes \text{id}) \circ \Delta)(f))(u) &= f(1u) \\ (((\text{id} \otimes \varepsilon) \circ \Delta)(f))(u) &= f(u1) \\ (\text{id}(f))(u) &= f(u). \end{aligned}$$

In a similar fashion, the coinverse property writes

$$m \circ (S \otimes \text{id}) \circ \Delta = m \circ (\text{id} \otimes S) \circ \Delta = \eta \circ \varepsilon \quad (1.4.6)$$

where $m : (G \times G)' \rightarrow G'$ is the product of G' (1.4.1b), and $\eta : \mathbb{C} \rightarrow G'$ is the unit of G' , with $\eta(z) = z1$ for any $z \in \mathbb{C}$. It is satisfied, thanks to the inverse property $uu^{-1} = u^{-1}u = 1$ in G , through

$$\begin{aligned} ((m \circ (S \otimes \text{id}) \circ \Delta)(f))(u) &= f(u^{-1}u) \\ ((m \circ (\text{id} \otimes S) \circ \Delta)(f))(u) &= f(uu^{-1}) \\ ((\eta \circ \varepsilon)(f))(u) &= f(1). \end{aligned}$$

Second, the consistency relations for Δ and ε with the algebra G' state that Δ and ε are algebra homomorphisms, that is, for any $f_1, f_2 \in G'$

$$\Delta(f_1 f_2) = \Delta(f_1) \Delta(f_2), \quad \varepsilon(f_1 f_2) = \varepsilon(f_1) \varepsilon(f_2). \quad (1.4.7)$$

Considering any $u_1, u_2 \in G$, one has, using the product definition (1.4.1b),

$$\begin{aligned} (\Delta(f_1 f_2))(u_1, u_2) &= (f_1 f_2)(u_1 u_2) = f_1(u_1 u_2) f_2(u_1 u_2) = \Delta(f_1)(u_1, u_2) \Delta(f_2)(u_1, u_2), \\ \varepsilon(f_1 f_2) &= (f_1 f_2)(1) = f_1(1) f_2(1) = \varepsilon(f_1) \varepsilon(f_2). \quad \square \end{aligned}$$

Remark 1.4.2

If one wants to have a fully algebraic version of a group G as depicted in Example 1.4.1, one should have an abstract definition of G' without the mention of G . This is done in the previous example, through the structure of **HOPF algebra**, to the exception of the coproduct. In (1.4.2a), the coproduct Δ has an image in $(G \times G)'$, but from the point of view of the **HOPF algebra**, its image should be in $G' \otimes G'$. This distinction is also present for the product of G' but it was not made explicit in the example.

In general, one has the inclusion

$$G' \otimes G' \subset (G \times G)', \quad (1.4.8)$$

so that the **HOPF algebra** structure is more restrictive than the algebraic version we are looking for. Even if the amount of missing information is not quantified, it is not considered to be pathological in the literature.

As a side remark, let us observe that the equality in (1.4.8) stands when G is finite.

1.4.2 The structure of HOPF algebra

Considering the construction of G' above, the quantum group formalism studies **HOPF algebra** and considers it mimics G' as a space of functions over some, at this point undetermined, deformed version of some group G .

In this context, a quantum group is therefore a **HOPF algebra** $(\mathcal{H}, \cdot, 1, \Delta, \varepsilon, S)$, generically noted only \mathcal{H} . The product \cdot makes \mathcal{H} an algebra, with a unit 1. The coproduct $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ decomposes an element into its product constituents, as expressed by the SWEEDLER notations

$$\Delta(X) = \sum X_{(1)} \otimes X_{(2)} \quad (1.4.9)$$

for any $X \in \mathcal{H}$. In (1.4.9), $X_{(1)}, X_{(2)} \in \mathcal{H}$ are the constituents of X . The sum symbol is here to express that several such decompositions may exist and the coproduct corresponds to the sum of all¹¹. In the context of symmetries, one can interpret the coproduct of a symmetry as its action on a multi-particle state. For example, if one considers the angular momentum operator J_z , acting on a two-particle state $|\psi_1\rangle \otimes |\psi_2\rangle$, one has that

$$J_z(|\psi_1\rangle \otimes |\psi_2\rangle) = J_z(|\psi_1\rangle) \otimes |\psi_2\rangle + |\psi_1\rangle \otimes J_z(|\psi_2\rangle), \quad (1.4.10a)$$

which can be recovered from

$$\Delta(J_z) = J_z \otimes 1 + 1 \otimes J_z. \quad (1.4.10b)$$

The counit $\varepsilon : \mathcal{H} \rightarrow \mathbb{C}$ is, in some sense, the “unit” of the coproduct. From the multi-particle state interpretation above, the counit is “suppressing” a particle in the sense that one goes from a symmetry in a n -particle state to a symmetry in a $(n - 1)$ -particle state by applying the counit.

The coassociativity property (1.4.4) states that, whenever Δ is applied to (1.4.9), the choice of applying it on $X_{(1)}$ (*i.e.* to the left) or on $X_{(2)}$ (*i.e.* to the right) does not matter. In terms of multi-particle state interpretation, one can construct the action of a symmetry on a n -particle state via its action on a $(n - 1)$ -particle state. For example, from (1.4.10b), one has

$$\begin{aligned} \Delta^2(J_z) &= \Delta(J_z) \otimes 1 + \Delta(1) \otimes J_z = J_z \otimes \Delta(1) + 1 \otimes \Delta(J_z) \\ &= J_z \otimes 1 \otimes 1 + 1 \otimes J_z \otimes 1 + 1 \otimes 1 \otimes J_z. \end{aligned}$$

The counit property (1.4.5) states also that applying ε to the left or right of (1.4.9) gives the same result. When considering multi-particle states, this means that one can “suppress” any of the particle in a n -particle state.

Finally, the antipode $S : \mathcal{H} \rightarrow \mathcal{H}$ corresponds to a generalised inverse. The coinverse property (1.4.6) imposes that S is actually a generalised version of the inverse, be it a right or left inverse. More precisely, with the SWEEDLER notation (1.4.9), one has

$$\sum S(X_{(1)})X_{(2)} = \sum X_{(1)}S(X_{(2)}) = \varepsilon(X)1. \quad (1.4.11)$$

We refer to [Definition A.1.13](#) for a more detailed definition of the **HOPF algebra** structure.

We now give a useful example of the construction of a **HOPF algebra**.

Example 1.4.3 – The HOPF algebra of $U(\mathfrak{g})$

Consider \mathfrak{g} to be a semi-simple **LIE algebra**. This construction is made for any semi-simple **LIE algebra**, but one can consider \mathfrak{g} to be the **LIE algebra** of vector fields in the context

¹¹This vague sentence is to be thought in a broad sense. We present below the multi-particle interpretation of the coproduct, because of this manuscript interest. However, other physical interpretations can be made of the coproduct, like in statistical physics. We refer to [32] for more insights on this topic.

of general relativity, or as infinitesimal symmetry transformations associated to some LIE group. We will study the enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} . The latter corresponds to the algebra such that, for any $X, Y \in U(\mathfrak{g})$,

$$[X, Y] = XY - YX$$

where $[\cdot, \cdot]$ is the bracket product of \mathfrak{g} . In general, $U(\mathfrak{g})$ is wider than \mathfrak{g} . As an example, the product of two vector fields is not a vector field because it does not obey the LEIBNIZ rule, only the commutator is. The enveloping algebra of vector fields then corresponds to the set of all objects generated when multiplying vector fields.

From the algebra $U(\mathfrak{g})$, one can construct a HOPF algebra by defining the following maps, for any $X \in U(\mathfrak{g})$,

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \varepsilon(X) = 0, \quad S(X) = -X. \quad (1.4.12)$$

From there, one can check that the HOPF algebra axioms are satisfied.

1.4.3 The quantum space-time

If the above quantum group stands as the quantum symmetries, then what is the quantum space-time having such symmetries? The answer lies again in the classical case and from which all the algebraic structures are exported. For a symmetry $u \in G$, one goes from a space-time point $x \in \mathcal{M}$ to its symmetric $y \in \mathcal{M}$ by acting with the transformation: $y = u \triangleright x$. Then, any space-time function $f \in \mathcal{C}^\infty(\mathcal{M})$ transform under G via $(u \triangleright f)(x) = f(u \triangleright x)$. This corresponds to a module structure. We say that $\mathcal{C}^\infty(\mathcal{M})$ is a G -module. In addition, $\mathcal{C}^\infty(\mathcal{M})$ has an algebra structure thanks to the product (1.1.1). This makes $\mathcal{C}^\infty(\mathcal{M})$ a G -module algebra.

More explicitly, given a HOPF algebra \mathcal{H} , a \mathcal{H} -module \mathbb{A} is a vector space with a linear action $\triangleright : \mathcal{H} \otimes \mathbb{A} \rightarrow \mathbb{A}$ satisfying¹²

$$(XY) \triangleright f = X \triangleright (Y \triangleright f), \quad 1 \triangleright f = f \quad (1.4.13)$$

for any $X, Y \in \mathcal{H}$ and $f \in \mathbb{A}$. Physically, the action \triangleright can be viewed as the action of a symmetry, like in Example 1.4.4. In this picture, (1.4.13) states that applying the transformation X and then Y amounts to apply directly XY , and that applying the identity amounts not to transform f .

Example 1.4.4 – The module of rotations on the Euclidean space

Let $\mathbb{A} = \mathcal{C}^\infty(\mathcal{M})$ with $\mathcal{M} = \mathbb{R}^3$, the 3-dimensional Euclidean space, and $X = R_z(\theta) \in SO_3(\mathbb{R})$ be the rotation of angle θ around the z -axis. A space point, noted $x = \vec{x} = (x_1, x_2, x_3) \in \mathbb{M}_{3,1}(\mathbb{R}) \simeq \mathbb{R}^3$, is transformed to $y = R_z(\theta)\vec{x}$ by matrix multiplication. Here, y corresponds to x rotated of angle θ with respect to the origin.

Let us consider observables of \mathbb{R}^3 , that is $f \in \mathcal{C}^\infty(\mathbb{R}^3)$. f could be any measurable quantity in space, like temperature, electric charge, etc... Considering a rotation of angle θ around the z -axis $X = R_z(\theta)$, we define the action $(R_z(\theta) \triangleright f)(x) = f(R_z(\theta)x)$, which

¹²Note that we define here a left action, but one can define similarly a right action. See Definition A.2.1.

corresponds to the evaluation of f at the point $R_z(\theta)x$. The algebraic structure linking the space $\mathcal{C}^\infty(\mathbb{R}^3)$ and the symmetries $SO_3(\mathbb{R})$ via the action \triangleright is called a module, *i.e.* $\mathcal{C}^\infty(\mathbb{R}^3)$ is said to be a $SO_3(\mathbb{R})$ -module.

In this example, left hand side of (1.4.13) states that rotating of angle θ and then of $\tilde{\theta}$ amounts to rotating of angle $\theta + \tilde{\theta}$, *i.e.*

$$R_z(\tilde{\theta}) \triangleright (R_z(\theta) \triangleright f) = (R_z(\tilde{\theta})R_z(\theta)) \triangleright f = R_z(\tilde{\theta} + \theta) \triangleright f.$$

In the same spirit, the right hand side of (1.4.13) imposes that a rotation of angle 0 does not change the function f , *i.e.*

$$R_z(0) \triangleright f = 1 \triangleright f = f.$$

A \mathcal{H} -module algebra \mathbb{A} is a \mathcal{H} -module and an algebra, such that both structures are compatible. More quantitatively, if $(\mathcal{H}, \cdot, 1_{\mathcal{H}}, \Delta, \varepsilon, S)$ is a **HOPF algebra**, then $(\mathbb{A}, \star, 1_{\mathbb{A}})$ is a \mathcal{H} -module algebra if and only if

$$X \triangleright (f \star g) = \sum (X_{(1)} \triangleright f) \star (X_{(2)} \triangleright g), \quad X \triangleright 1_{\mathbb{A}} = \varepsilon(X)1_{\mathbb{A}} \quad (1.4.14)$$

for any $f, g \in \mathbb{A}$ and $X \in \mathcal{H}$, where we use SWEEDLER notations (1.4.9).

Example 1.4.5 – The module algebra of rotations on the Euclidean space

We consider the same situation as in Example 1.4.4. In order to do so, one needs to complete $SO_3(\mathbb{R})$ to have a **HOPF algebra** structure. This is done by considering $U(\mathfrak{so}_3)$ as in Example 1.4.3. Therefore, an infinitesimal rotation R_z has a coproduct and counit given by (1.4.12), and applying left hand side of (1.4.14), one obtains

$$R_z \triangleright (fg) = (R_z \triangleright f)g + f(R_z \triangleright g).$$

R_z acts like a derivation, which is in agreement with the usual association of R_z with a vector field in \mathbb{R}^3 : $R_z = x^1\partial_2 - x^2\partial_1$. The right hand side, states that

$$R_z \triangleright 1_{\mathbb{A}} = 0.$$

This can be understood as rotating $1_{\mathbb{A}}(x) = 1$ does not change $1_{\mathbb{A}}$, *i.e.* $1_{\mathbb{A}}(R_z(\theta)x) = 1_{\mathbb{A}}(x) = 1$, therefore any infinitesimal rotation R_z of $1_{\mathbb{A}}$ vanishes.

Therefore, a quantum space-time \mathbb{A} having a **HOPF algebra** \mathcal{H} as its space of symmetries is defined as a \mathcal{H} -module algebra. In order to make contact with Subsection 1.4.2, one can check that (1.4.10a) is recovered from (1.4.10b) using (1.4.14).

Two main comments are in order.

- ❖ One can construct a \mathcal{H} -module algebra canonically by considering the dual **HOPF algebra** of \mathcal{H} , see (A.2.5). This is used in the context of the κ -MINKOWSKI space-time in Section 3.1.
- ❖ In (1.4.14), the algebra sector of \mathbb{A} (that is the product \star and the unit $1_{\mathbb{A}}$) is linked to the coalgebra sector of \mathcal{H} (that is the coproduct Δ and the counit ε). Therefore, the noncommutativity of \mathbb{A} can be linked with the noncocommutativity of \mathcal{H} . The cocommutativity of a coproduct is defined as the interchangeability of the constituents: $\sum X_{(1)} \otimes X_{(2)} = \sum X_{(2)} \otimes X_{(1)}$.

Particular types of **HOPF algebra** have been studied in the physics literature, because of their link to physical systems or their relative simplicity.

Braided geometry

The (quasi-)triangular **HOPF algebra** has an additional element called the \mathcal{R} -matrix, generically denoted $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$. It is invertible and must satisfy some consistency equations. When considering a quantum space-time \mathbb{A} with symmetries \mathcal{H} , *i.e.* a \mathcal{H} -module algebra \mathbb{A} , one can consider braided commutativity [34, 35], that is when noncommutativity of \mathbb{A} is controlled by the \mathcal{R} -matrix through

$$f \star g = \sum (\mathcal{R}_1^{-1} \triangleright g) \star (\mathcal{R}_2^{-1} \triangleright f), \quad (1.4.15)$$

where we noted $\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2 \in \mathcal{H} \otimes \mathcal{H}$ in the spirit of the SWEEDLER notation. The main point of this braided construction is that one can export many more commutative structures just by adapting them to be braided commutative. More details are given in Section 2.2.

Bicrossproduct structure

One can export internal structures of groups to the quantum group setting, especially the semi-direct product. Let G be a group with semi-direct structure, *i.e.* $G = G_1 \ltimes G_2$. This means that transformations with G_2 also involve elements of G_1 . As an example, the POINCARÉ group can be decomposed as $\mathcal{P}^{1,3} = \mathcal{T}^{1,3} \ltimes SO_{1,3}(\mathbb{R})$, where $\mathcal{T}^{1,3}$ denotes the group of translations and $SO_{1,3}(\mathbb{R})$ the group of rotations and boosts. Using the **HOPF algebra** formalism, the semi-direct product can be generalised as the bicrossproduct [32] $\mathcal{H} = \mathcal{H}_1 \blacktriangleleft \mathcal{H}_2$ via the introduction of an action \triangleright of \mathcal{H}_1 on \mathcal{H}_2 and a coaction \blacktriangleleft of \mathcal{H}_2 on \mathcal{H}_1 . As an example, the κ -POINCARÉ **HOPF algebra** is constructed as a quantum deformation of the POINCARÉ group thanks to this bicrossproduct structure (see Section 3.1 for more details).

The quantum space-time can emerge as the dual of \mathcal{H} . But MAJID [32] has developed another idea on how geometry could emerge from this bicrossproduct structure, as the latter can be seen as the quantisation of homogeneous spaces. The semi-direct product actually appears when considering a space-time \mathcal{M} with a LIE group of symmetries G . As noted above, the symmetries G acts on the space-time so that $\mathcal{M} \ltimes G$ and taking the dual point of view $\mathcal{C}^\infty(\mathcal{M}) \ltimes \mathbb{C}G$, where $\mathbb{C}G$ denotes the vector space of formal \mathbb{C} -linear combinations of elements of G . At this point, $\mathcal{C}^\infty(\mathcal{M})$ can be turned into a noncommutative algebra \mathbb{A} and $\mathbb{C}G$ into a (noncocommutative) **HOPF algebra** \mathcal{H} , giving $\mathbb{A} \ltimes \mathcal{H}$. One further forms a bicrossproduct by requiring that the space-time “reacts” on the momentum space $\mathcal{C}^\infty(\mathcal{M}) \blacktriangleright \mathbb{C}G$ and one exports that to the quantum setting $\mathbb{A} \blacktriangleright \mathcal{H}$.

DRINFEL'D twist

Starting from a **HOPF algebra** \mathcal{H} , one can construct another **HOPF algebra** $\tilde{\mathcal{H}}$ via a **DRINFEL'D twist**. The **DRINFEL'D twist** is an invertible element $\mathcal{F} = \sum \mathcal{F}_1 \otimes \mathcal{F}_2 \in \mathcal{H} \otimes \mathcal{H}$ satisfying the so-called 2-cocycle condition (1.6.14a) and the normalisation (1.6.14b). If we define

$$\Delta^{\mathcal{F}} = \mathcal{F} \Delta \mathcal{F}^{-1}, \quad S^{\mathcal{F}} = \chi S \chi^{-1} \quad (1.4.16)$$

where $\chi = \mathcal{F}_1 S(\mathcal{F}_2) \in \mathcal{H}$, then $\tilde{\mathcal{H}} = (\mathcal{H}, \cdot, 1, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ forms a new **HOPF algebra**. This is convenient in two ways. First, one can start with a very simple **HOPF algebra** structure, like the one in [Example 1.4.3](#), and construct a more complex **HOPF algebra** using the **DRINFEL'D twist**. Second, if we consider a \mathcal{H} -module algebra \mathbb{A} , then by adapting its product it can be turned into a $\tilde{\mathcal{H}}$ -module algebra. Explicitly, the new product of \mathbb{A} is

$$f \star_{\mathcal{F}} g = \sum (\mathcal{F}_1^{-1} \triangleright f) \star (\mathcal{F}_2^{-1} \triangleright g) \quad (1.4.17)$$

where \star was the previous product of \mathbb{A} . Thus, one can consider a commutative algebra of functions over a classical space-time $\mathbb{A} = \mathcal{C}^\infty(\mathcal{M})$ with some symmetry group \mathcal{H} (for example the **LIE algebra** of vector fields) and construct a quantum space-time with quantum symmetries by simply introducing a **DRINFEL'D twist** $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$. See [Subsection 1.6.2](#) for more insights.

1.5 Derivation-based differential calculus

When considering a space-time \mathcal{M} , one can define quantities through smooth functions $\mathcal{C}^\infty(\mathcal{M})$, like position. But, one needs also their derivatives, like in the computation of speed. When the space-time is curved, the speed vector does not “belong” to the space-time but to its tangent space. Moreover, comparing vectors attached to distant points is no longer straightforward. This lack of consistency generated by the geometry is in fact not a curse since it is precisely how we quantify and characterise the geometry. In other words, all the geometric information goes into vector fields from which we construct quantitative geometrical objects like the metric, the connection or the curvature.

Considering that we want to generalise the previous setting into a purely algebraic one, we should start by generalising the vector fields, from which all other quantities would follow.

The generalisation of vector fields to the non-commutative setting has been considered in several ways: via **HOPF algebra** (see section 14.1 of [\[33\]](#)), or via quantum principal fiber bundles [\[36\]](#), among others. The point of view taken here [\[5\]](#) is to see that there is a one-to-one correspondence between the vector fields and the derivations over $\mathcal{C}^\infty(\mathcal{M})$, noted $\text{Der}(\mathcal{C}^\infty(\mathcal{M}))$.

Given an algebra \mathbb{A} , a derivation $X \in \text{Der}(\mathbb{A})$ over \mathbb{A} is defined as a linear mapping satisfying the **LEIBNIZ rule**, *i.e.* for any $f, g \in \mathbb{A}$,

$$X(f \star g) = X(f) \star g + f \star X(g) \quad (1.5.1)$$

where \star denotes the product of \mathbb{A} . The derivations form a **LIE algebra**, in the sense that given two derivations $X, Y \in \text{Der}(\mathbb{A})$, then $[X, Y] = XY - YX$ is also a derivation.

Proof. First, $[X, Y]$ is linear as a composition and sum of linear maps. One then need to

check that it follows the LEIBNIZ rule (1.5.1). Let $f, g \in \mathbb{A}$,

$$\begin{aligned}
 [X, Y](f \star g) &= X(Y(f \star g)) - Y(X(f \star g)) \\
 &= X(Y(f) \star g + f \star Y(g)) - Y(X(f) \star g + f \star X(g)) \\
 &= XY(f) \star g + \cancel{Y(f) \star X(g)} + \cancel{X(f) \star Y(g)} + f \star XY(g) \\
 &\quad - YX(f) \star g - \cancel{X(f) \star Y(g)} - \cancel{Y(f) \star X(g)} - f \star YX(g) \\
 &= [X, Y](f) \star g + f \star [X, Y](g) \quad \square
 \end{aligned}$$

Furthermore, if we define the action of \mathbb{A} on $\text{Der}(\mathbb{A})$, by $f \triangleright X = f \star X$, then $\text{Der}(\mathbb{A})$ is not an \mathbb{A} -module, since $f \star X$ does not follow the LEIBNIZ rule. But, as one can notice in performing this computation (done in the proof below), $f \star X$ would be a derivation if f was commuting with any element of \mathbb{A} . Even if we work in a noncommutative context, it can exist elements of the algebra that commutes with all others. The set of those elements is called the center of the algebra and is denoted $\mathcal{Z}(\mathbb{A})$. Explicitly, f is an element of the center $\mathcal{Z}(\mathbb{A})$ if and only if $f \star g = g \star f$ for any $g \in \mathbb{A}$. Thus, $\text{Der}(\mathbb{A})$ is a $\mathcal{Z}(\mathbb{A})$ -module.

Proof. One needs to check that $f \star X \in \text{Der}(\mathbb{A})$ for any $f \in \mathcal{Z}(\mathbb{A})$ and $X \in \text{Der}(\mathbb{A})$. As above, $f \star X$ is a linear map because X is. Now let us check the LEIBNIZ rule. Given any $g, h \in \mathbb{A}$,

$$\begin{aligned}
 (f \star X)(g \star h) &= f \star (X(g) \star h + g \star X(h)) \\
 &= f \star X(g) \star h + f \star g \star X(h) \\
 &= f \star X(g) \star h + g \star f \star X(h) \\
 &= (f \star X)(g) \star h + g \star (f \star X)(h) \quad \square
 \end{aligned}$$

One should note that, when considering the algebra of smooth functions $\mathbb{A} = \mathcal{C}^\infty(\mathcal{M})$, one has $\mathcal{Z}(\mathbb{A}) = \mathcal{C}^\infty(\mathcal{M}) = \mathbb{A}$ since this algebra is commutative.

The vector fields are thus generalised, in the noncommutative context, as the derivations over the algebra. This observation is a cornerstone for the definition of some physical models developed in this manuscript. In order to build the derivation-based differential calculus, one has to define the set of forms. Inspired by the commutative case, one defines a n -forms $\omega \in \Omega^n(\mathbb{A})$ as a $\mathcal{Z}(\mathbb{A})$ -multilinear antisymmetric map from $\text{Der}(\mathbb{A})^n \rightarrow \mathbb{A}$. This means that ω takes n entries from $\text{Der}(\mathbb{A})$, like $X_1, \dots, X_n \in \text{Der}(\mathbb{A})$, and sends it to \mathbb{A} :

$$\omega(X_1, \dots, X_n) \in \mathbb{A}. \quad (1.5.2a)$$

The antisymmetry states that swapping two of the n entries (here X_j and X_k) generates a minus sign

$$\begin{aligned}
 \omega(X_1, \dots, X_{j-1}, X_j, X_{j+1}, \dots, X_{k-1}, X_k, X_{k+1}, \dots, X_n) \\
 = -\omega(X_1, \dots, X_{j-1}, X_k, X_{j+1}, \dots, X_{k-1}, X_j, X_{k+1}, \dots, X_n) \quad (1.5.2b)
 \end{aligned}$$

for any $k, j = 1, \dots, n$. Finally, $\mathcal{Z}(\mathbb{A})$ -multilinearity states that any of the n entries (here X_j) is linear for the scalars and for the action of $\mathcal{Z}(\mathbb{A})$:

$$\begin{aligned} & \omega(X_1, \dots, X_{j-1}, f \star X_j + g \star Y_j, X_{j+1}, \dots, X_n) \\ &= f \star \omega(X_1, \dots, X_{j-1}, X_j, X_{j+1}, \dots, X_n) + g \star \omega(X_1, \dots, X_{j-1}, Y_j, X_{j+1}, \dots, X_n) \end{aligned} \quad (1.5.2c)$$

for any $f, g \in \mathcal{Z}(\mathbb{A})$ and $Y_j \in \text{Der}(\mathbb{A})$.

From there, we can define the (wedge) product between forms of different degrees (here n and m) via

$$\begin{aligned} & (\omega \wedge \eta)(X_1, \dots, X_{n+m}) \\ &= \frac{1}{n!m!} \sum_{\sigma \in \mathfrak{S}_{n+m}} (-1)^{\text{sign}(\sigma)} \omega(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \star \eta(X_{\sigma(n+1)}, \dots, X_{\sigma(n+m)}), \end{aligned} \quad (1.5.3)$$

where $\omega \in \Omega^n(\mathbb{A})$, $\eta \in \Omega^m(\mathbb{A})$, \mathfrak{S}_{n+m} denotes the set of permutations of $1, \dots, n+m$ and sign stands for the signature of such a permutation. This definition is similar to the commutative one. One can notice however, the presence of the algebra product \star in the expression (1.5.3). If \star is noncommutative, then \wedge will not be graded commutative, that is

$$\omega \wedge \eta \neq (-1)^{|\omega||\eta|} \eta \wedge \omega \quad (1.5.4)$$

where $|\cdot|$ here denotes the degree of the form.

This observation is more general than the derivation-based differential calculus formalism since it also occurs in other formalism of [noncommutative geometry](#). One of the main consequence, like in any noncommutativity context, is that one has to pay attention to the order of the terms. Especially, when considering local coordinates in $(d+1)$ -dimensional space-time (x^0, \dots, x^d) , $d+1$ -forms are expressed in the commutative case as

$$\omega = f dx^0 \wedge \dots \wedge dx^d \quad (1.5.5)$$

where f is a smooth function. But since $dx^j \wedge dx^k$ and $dx^k \wedge dx^j$ are a priori unrelated if \mathbb{A} is noncommutative, then one has to take into account any permutation of the dx^j in the expression (1.5.5).

This implies that one can generate forms of arbitrary degrees. Indeed, in a commutative setting $dx^j \wedge dx^j = 0$ because of antisymmetry of 2-forms due to graded commutativity. Therefore, if one wants to build a form of degrees higher than the space-time dimension, that is higher than $d+1$, one should add at least one dx^j in expression (1.5.5). However, this dx^j is already present, and from the previous argument, such a form should necessarily vanish. In the presence of a noncommutative product, the previous demonstration falls short because $dx^j \wedge dx^j$ can be non-zero due to (1.5.4). Thus, there is no upper bound on the degrees of non-vanishing forms.

Finally, one can jump to higher degrees by using the differential $d : \Omega^n(\mathbb{A}) \rightarrow \Omega^{n+1}(\mathbb{A})$ which is defined, as in the commutative setting, via the KOSZUL formula

$$\begin{aligned} d\omega(X_1, \dots, X_{n+1}) &= \sum_{j=1}^{n+1} (-1)^{j+1} X_j \left(\omega(X_1, \dots, \overset{j}{\check{V}}, \dots, X_{n+1}) \right) \\ &+ \sum_{1 \leq j < k \leq n+1} (-1)^{j+k} \omega([X_j, X_k], X_1, \dots, \overset{j}{\check{V}}, \dots, \overset{k}{\check{V}}, \dots, X_{n+1}), \end{aligned} \quad (1.5.6)$$

where $\overset{j}{\check{V}}$ denotes the omission of the element X_j . One can see that this definition does not involve the product of \mathbb{A} and therefore, does not see the noncommutativity.

From the previous definition, two main properties arise

$$d^2 = 0, \tag{1.5.7a}$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|} \omega \wedge d\eta. \tag{1.5.7b}$$

From (1.5.7b), one reads that d is a graded derivation (satisfying a graded LEIBNIZ rule) and, from (1.5.7a), that it squares to zero. Therefore, d is a differential operator.

Finally, one defines the set of all forms as the sum of all sets of forms of any degrees

$$\Omega^\bullet(\mathbb{A}) = \bigoplus_{n=0}^{\infty} \Omega^n(\mathbb{A}), \tag{1.5.8}$$

where $\Omega^0(\mathbb{A}) = \mathbb{A}$. The triplet

$$(\Omega^\bullet(\mathbb{A}), \wedge, d) \tag{1.5.9}$$

is the differential algebra defining the derivation-based differential calculus.

1.6 Deformation quantisation

General relativity and the standard model of particle physics are the two main models of theoretical physics that give very accurate predictions of the nature we observe. They are expressed as field theories on a classical space-time. If we were to generalise these two pictures to field theories on a quantum space-time, then this new theory would need to account for the results given by the two former models, in some way. This is the precise point of deformation quantisation: considering a classical geometry, how can one deform it into a **noncommutative geometry**?

The way the classical geometry is “kept track of” relies on a parameter, called the deformation parameter, that expresses if the geometry is to be considered classical or quantum. In most of our physical models, the deformation parameter corresponds to an energy scale (often associated to the PLANCK mass). Therefore, for energies comparable or above the deformation parameter, the noncommutative nature of the geometry gives relevant contributions, but for energies far below the deformation parameter, the geometry can be considered classical. The previous way of thinking is akin to quantum mechanics, where the deformation parameter is \hbar . Deformation quantisation was indeed first developed as a mathematical model of quantum mechanics [8, 9, 10], as detailed in Section 1.2.

We here give a brief introduction to deformation quantisation and refer to [37, 38] for more historical and theoretical aspects.

After WEYL [8] and VON NEUMANN [9] works, mathematicians have gone on and tried to deform more complex structures. The existence of a star-product on symplectic manifolds was proven by DE WILDE and LECOMPTE [39] and, in parallel, by BAYEN and collaborators [40]. The existence of the star-product on general POISSON manifold was established by KONTSEVICH [41].

In general terms, let \mathcal{M} be a POISSON manifold, that is a space-time with phase-space structure. A star product \star_{\hbar} (1.2.3) is an associative product of function expressed formally as a power series expansion in the deformation parameter \hbar . That is for two functions f and g

$$f \star_{\hbar} g = f \cdot g + \sum_{n=1}^{\infty} \hbar^n C_n(f, g) \tag{1.6.1}$$

where \cdot is the classical product of functions (1.1.1), C_n is a bilinear differential operator and \hbar is a constant. A priori f and g are taken to be smooth functions, *i.e.* $f, g \in \mathcal{C}^\infty(\mathcal{M})$. But, since the operation \star_\hbar (1.6.1) may not be convergent for any smooth functions, the set of considered functions might be smaller than $\mathcal{C}^\infty(\mathcal{M})$, and is generically called the multiplier space. In the context of formal deformation, we simply state that f and g can be written as formal power series in \hbar , that is

$$f = f_0 + \sum_{n=1}^{\infty} \hbar^n f_n \quad (1.6.2)$$

where f_0 is the classical smooth function. We usually note that f is a formal power expansion in \hbar of the form (1.6.2) as $f \in \mathcal{C}^\infty(\mathcal{M})[[\hbar]]$. Note that with expression (1.6.2), the operator C_n in (1.6.1) writes

$$C_n(f, g) = \sum_{m=0}^n f_m g_{n-m}. \quad (1.6.3)$$

The main feature of this star-product formalism is that the limit of a vanishing deformation parameter, corresponding here to $\hbar \rightarrow 0$, makes one recover (formally) all the classical structure of \mathcal{M} . More explicitly, $f \star_\hbar g \rightarrow f \cdot g$ and

$$[f, g]_{\star_\hbar} = f \star_\hbar g - g \star_\hbar f \rightarrow i\hbar \{f, g\}, \quad (1.6.4)$$

where $\{\cdot, \cdot\}$ correspond to the POISSON bracket of \mathcal{M} .

In a physical context, the limit $\hbar \rightarrow 0$ can correspond to a low energy limit, if $1/\hbar$ is an energy scale. Furthermore, the formula (1.6.4) corresponds mathematically to the usual sentence “quantising a physical theory is replacing the brackets with commutators”. It is the precise reason of the word quantisation in “deformation quantisation”. In this sense, the star product formalism is a generalisation of canonical quantisation of quantum mechanics.

In order to perform computations, an explicit formula of \star_\hbar is needed. In the mathematical literature, this field of research is called strict deformation quantisation. There are several ways to generate explicit star product. In this manuscript, we follow the procedure of GUTT with deformation of the universal enveloping algebra [42] and REIFFEL through convolution algebra techniques [43]. We also present the DRINFEL’D twist method.

1.6.1 Convolution algebra techniques

The procedure presented here is the one of [42, 43]. Let \mathcal{M} be a space-time with a linear POISSON structure, that is, for any $f, g \in \mathcal{C}^\infty(\mathcal{M})$,

$$\{f, g\}(x) = C^{\nu\rho}{}_\mu x^\mu \frac{\partial f}{\partial x^\nu} \frac{\partial g}{\partial x^\rho} \quad (1.6.5)$$

where x^μ are (local) coordinates on \mathcal{M} and $C^{\nu\rho}{}_\mu$ is a constant. Then, \mathcal{M} naturally identifies to the dual \mathfrak{g}' of a LIE algebra \mathfrak{g} with KIRILLOV-KOSTANT-SOURIAU structure, where C is the structure constant of \mathfrak{g} . Those are called linear POISSON or LIE-POISSON structures. In this context, $\{x^\mu\}_{\mu=0,\dots,d}$ corresponds to coordinate functions of \mathfrak{g}' that satisfies

$$\{x^\mu, x^\nu\} = C^{\mu\nu}{}_\rho x^\rho. \quad (1.6.6)$$

The LIE algebra \mathfrak{g} is associated to a LIE group \mathcal{G} , which can be interpreted as the momentum space. The first step of this construction is to build the convolution algebra of \mathcal{G} . We refer to textbooks like [44] for more mathematical details on harmonic analysis and group theory. This introduction will also be repeated in the more explicit case of the deformed group of momentum in Subsection 2.6.1.

Convolution algebra

In order to define a convolution, we need a notion of “sum” (or “integral”) over the group \mathcal{G} . In other words, we need a measure $d\mu$ over \mathcal{G} . In the case of locally compact groups (which we suppose always to be the case), there exists a unique left-invariant (resp. right-invariant) measure called the left (resp. right) HAAR measure, denoted $d\mu$ (resp. $d\mu_R$). The invariance states that

$$d\mu(u_1 u_2) = d\mu(u_2), \quad d\mu_R(u_1 u_2) = d\mu_R(u_1), \quad (1.6.7)$$

for any $u_1, u_2 \in \mathcal{G}$. The uniqueness of both measures imposes that there exists a positive function $\Delta : \mathcal{G} \rightarrow \mathbb{R}_0^+$ linking the two, *i.e.*

$$d\mu_R(u) = \Delta(u^{-1}) d\mu(u) \quad (1.6.8)$$

for any $u \in \mathcal{G}$. The function Δ is called the modular function. It is a group homomorphism, meaning it satisfies

$$\Delta(u_1 u_2) = \Delta(u_1) \Delta(u_2), \quad \Delta(u^{-1}) = \Delta(u)^{-1}, \quad \Delta(1) = 1. \quad (1.6.9)$$

In the case where $\Delta = 1$, the group \mathcal{G} is said to be unimodular and one has $d\mu = d\mu_R$.

We can now integrate functions on the group like $F : \mathcal{G} \rightarrow \mathbb{R}$ as

$$\int_{\mathcal{G}} d\mu(u) F(u). \quad (1.6.10)$$

The set of functions for which (1.6.10) does not diverge is noted $L^1(\mathcal{G})$ and corresponds to integrable functions on the group. The choice of left or right HAAR measure in (1.6.10) has no impact, since one can go from one to the other via a change of variable.

Finally, given two functions $F_1, F_2 \in L^1(\mathcal{G})$, one can define the convolution product $\hat{\circ}$ through

$$(F_1 \hat{\circ} F_2)(u) = \int_{\mathcal{G}} d\mu(u_0) F_1(u_0) F_2(u u_0^{-1}). \quad (1.6.11)$$

The space $L^1(\mathcal{G})$ together with the product $\hat{\circ}$ forms an algebra that is called the convolution algebra of \mathcal{G} . One can show that $\hat{\circ}$ is commutative if and only if the group \mathcal{G} is Abelian.

This new product $\hat{\circ}$ is a key ingredient to define our star product. One then just needs to make the link between the function over \mathfrak{g}' , that are functions on the space-time \mathcal{M} , and functions of \mathcal{G} , that are functions on the momentum space. This link is made, as in quantum mechanics, through the FOURIER transform:

$$\mathcal{F}(f)(u) = \int_{\mathfrak{g}'} e^{i(\log(u), x)} f(x) dx, \quad \mathcal{F}^{-1}(F)(x) = \int_{\mathcal{G}} e^{-i(\log(u), x)} F(u) d\mu(u), \quad (1.6.12)$$

where dx is the LEBESGUE measure on \mathfrak{g}' , $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g}' \rightarrow \mathbb{R}$ is the dual pairing between \mathfrak{g} and \mathfrak{g}' , and $\log : \mathcal{G} \rightarrow \mathfrak{g}$ is simply here to make sense of the expression $\langle \log(u), x \rangle$ via the correspondence of the **LIE algebra** \mathfrak{g} and the LIE group \mathcal{G} . Finally, we define the star product on \mathcal{M} through

$$f \star g = \mathcal{F}^{-1}(\mathcal{F}(f) \hat{\circ} \mathcal{F}(g)). \quad (1.6.13)$$

One can make several comments of the expression (1.6.13).

An explicit expression of the star product can be obtained from (1.6.12), (1.6.13) together with the expression of the HAAR measure. The latter is derived via the group law, which itself comes from the **LIE algebra** structure (1.6.6) via the BAKER-CAMPBELL-HAUSSDORFF formula [45, 46]. A derivation of the star product (1.6.13) is done for a deformation of MINKOWSKI space-time in Section 3.1.

This star product is non-local because of the convolution product (1.6.11). Indeed, if one interprets u_0 as a momentum, then the integration over u_0 imposes that $F_1 \hat{\circ} F_2$ at momentum u is constituted of the (convoluted) sum of F_1 and F_2 for all momenta. Therefore, given an energy, the value of $F_1 \hat{\circ} F_2$ at this energy depends of the value of F_1 and F_2 at all energies. Going back to position space x , this will impose that the value of $f \star g$ at x depends of the value of f and g everywhere in space and time.

The deformation parameter is not explicitly shown here, but is present in the structure constant C of (1.6.6) and so in the group law via the BAKER-CAMPBELL-HAUSSDORFF formula. The commutative limit thus corresponds to $C \rightarrow 0$ and so to an Abelian group law. In this case, one can show that the successive integrations of (1.6.13) involve DIRAC delta functions and gives $f \star g = f \cdot g$, where \cdot is the commutative product of functions (1.1.1).

Finally, to make contact with the quantisation of mechanics presented in Section 1.2, one can consider a (faithful \ast -) representation of the group algebra on a **HILBERT space** $\pi : L^1(\mathcal{G}) \rightarrow \mathcal{B}(\mathcal{H})$. Thus, the star product (1.6.13) can be expressed via a quantisation map, similarly to (1.2.2a), with $\mathcal{Q} = \pi \circ \mathcal{F}$.

1.6.2 Deformations through DRINFEL'D twist

The other mainly used approach to an explicit star product construction is given by **DRINFEL'D twist** deformation of product [47, 48]. The main idea of this construction is to use Example 1.4.3 in order to form a commutative **HOPF algebra** out of a **LIE algebra** \mathfrak{g} and then to deform it to a noncommutative algebra via the **DRINFEL'D twist** through (1.4.17). We explicit this below. Note that this Subsection 1.6.2 uses quantum groups notions, so we refer the reader to Section 1.4 or Appendix A for more details on **HOPF algebras**.

Let \mathfrak{g} be a **LIE algebra**, that can be the **LIE algebra** of vector fields in the context of general relativity, or as infinitesimal symmetry transformations associated to some LIE group. We consider the universal enveloping algebra $U(\mathfrak{g})$ which corresponds to the algebra such that the commutator of the product of $U(\mathfrak{g})$ corresponds to the bracket $[\cdot, \cdot]$ of \mathfrak{g} . This algebra can be endowed with a trivial **HOPF algebra** structure, as developed in Example 1.4.3.

Let $\mathcal{F} = \sum \mathcal{F}_1 \otimes \mathcal{F}_2 \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be an invertible element. We say that \mathcal{F} is a **DRINFEL'D twist** if it further satisfies

$$(\mathcal{F} \otimes 1)(\Delta \otimes \text{id})(\mathcal{F}) = (1 \otimes \mathcal{F})(\text{id} \otimes \Delta)(\mathcal{F}), \quad (2\text{-cocycle condition}) \quad (1.6.14a)$$

$$(\text{id} \otimes \varepsilon)(\mathcal{F}) = (\varepsilon \otimes \text{id})(\mathcal{F}) = 1, \quad (\text{normalisation}) \quad (1.6.14b)$$

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\hbar), \quad (\text{semi-classical limit}) \quad (1.6.14c)$$

where Δ and ε are the coproduct and counit of $U(\mathfrak{g})$ respectively and \hbar is the deformation parameter. The **DRINFEL'D twist** \mathcal{F} can be viewed as a function of the deformation parameter \hbar , and the condition (1.6.14c) ensures that the twist vanish at the commutative limit $\hbar \rightarrow 0$.

Let us define

$$\Delta^{\mathcal{F}} = \mathcal{F} \Delta \mathcal{F}^{-1}, \quad S^{\mathcal{F}} = \chi S \chi^{-1}, \quad (1.6.15)$$

where S is the antipode of $U(\mathfrak{g})$ and we noted $\chi = \mathcal{F}_1 S(\mathcal{F}_2)$. Then, the set $U(\mathfrak{g})^{\mathcal{F}} = (U(\mathfrak{g}), \cdot, 1, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ is a **HOPF algebra** (see [Theorem A.1.20](#)). Note that it is often denoted $U(\mathfrak{g})_{\star}$ in the physics literature, in reference to the star product \star (1.6.16).

Now, let \mathcal{M} be a space-time that we want to quantise and \mathcal{G} a **LIE group** of symmetries that acts on \mathcal{M} . We apply the previous procedure to \mathfrak{g} the **LIE algebra** of \mathcal{G} . In this context, $U(\mathfrak{g})$ corresponds to the (infinitesimal) symmetries of $\mathcal{C}^{\infty}(\mathcal{M})$ because $\mathcal{C}^{\infty}(\mathcal{M})$ is a $U(\mathfrak{g})$ -module algebra (see details in [Subsection 1.4.3](#)).

We want to deform this picture by starting with the symmetries. As detailed above, one can consider a **DRINFEL'D twist** \mathcal{F} , to deform $U(\mathfrak{g})$ in a “non-trivial” quantum group $U(\mathfrak{g})^{\mathcal{F}}$. The quantum space-time \mathbb{A} , corresponding to a deformation of $\mathcal{C}^{\infty}(\mathcal{M})$, which has $U(\mathfrak{g})^{\mathcal{F}}$ as its algebra of quantum symmetries, is determined by the new product

$$f \star g = \sum (\mathcal{F}_1^{-1} \triangleright f) (\mathcal{F}_2^{-1} \triangleright g) \quad (1.6.16)$$

for any $f, g \in \mathbb{A}$.

At this point several comments are in order.

First, the noncommutativity of (1.6.16), which is linked to the “quantum” trait of \mathbb{A} , is linked to the noncocommutativity of $\Delta^{\mathcal{F}}$. The undeformed coproduct (1.4.12) is cocommutative, and it is linked to the commutativity of $\mathcal{C}^{\infty}(\mathcal{M})$. Therefore, one pictures that the noncommutativity of \mathbb{A} comes entirely from the **DRINFEL'D twist** \mathcal{F} .

The deformation parameter is not shown explicitly here. However, it is usually contained in the **DRINFEL'D twist** expression, so that the star product (1.6.16) is indeed parametrized by the deformation parameter. The commutative limit of \mathcal{F} is given by the requirement (1.6.14c), and combined with (1.6.16) one can verify that the star product of \mathbb{A} corresponds to the commutative product of $\mathcal{C}^{\infty}(\mathcal{M})$ in the commutative limit.

Example 1.6.1 – Abelian DRINFEL'D twist

Let X, Y be two commuting elements of $U(\mathfrak{g})$. For example, one could consider X and Y to be some generators of translations in the case where \mathfrak{g} is the **LIE algebra** of the **POINCARÉ group**. Let

$$\mathcal{F} = \exp(i\hbar X \otimes Y) \in U(\mathfrak{g}) \otimes U(\mathfrak{g}),$$

then one can show that \mathcal{F} is a **DRINFEL'D twist** (see proof below). From there, the star product (1.6.16) writes

$$f \star g = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} (X^n \triangleright f) (Y^n \triangleright g).$$

Note that, one could have equivalently considered, instead of X and Y , a family of

commuting elements $\{X_\mu\}_{\mu=0,\dots,d}$ of $U(\mathfrak{g})$. In this case, a **DRINFEL'D twist** could write

$$\mathcal{F} = \exp\left(i\hbar \Theta^{\mu\nu} X_\mu \otimes X_\nu\right) \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

where Θ is matrix of constant coefficients. If we further consider that X_μ acts on functions as a derivation on the μ -th coordinate, one has $X_\mu \triangleright f = \partial_\mu f$. Therefore, the star product (1.6.16) writes

$$f \star g = \sum_{n=0}^{\infty} \frac{(i\hbar)^n}{n!} \Theta^{\mu_1\nu_1} \dots \Theta^{\mu_n\nu_n} \partial_{\mu_1} \dots \partial_{\mu_n} f \partial_{\nu_1} \dots \partial_{\nu_n} g.$$

Proof. Let $\mathcal{F} = \exp(i\hbar X \otimes Y)$ as in [Example 1.6.1](#). It is invertible straightforwardly as it is expressed via an exponential. Thus, in order to show that \mathcal{F} is a **DRINFEL'D twist**, one needs to verify (1.6.14).

Considering first the 2-cocycle condition, one uses the fact that Δ is an algebra homomorphism to obtain that $\Delta(\exp(X)) = \exp(\Delta(X))$ for any $X \in U(\mathfrak{g})$. Then, using the coproduct expression (1.4.12), one computes

$$\begin{aligned} (\mathcal{F} \otimes 1)(\Delta \otimes \text{id})(\mathcal{F}) &= \exp(i\hbar X \otimes Y \otimes 1) \exp\left(i\hbar (X \otimes 1 + 1 \otimes X) \otimes Y\right) \\ &= \exp\left(i\hbar (X \otimes Y \otimes 1 + X \otimes 1 \otimes Y + 1 \otimes X \otimes Y)\right), \\ (1 \otimes \mathcal{F})(\text{id} \otimes \Delta)(\mathcal{F}) &= \exp(i\hbar 1 \otimes X \otimes Y) \exp\left(i\hbar X \otimes (Y \otimes 1 + 1 \otimes Y)\right) \\ &= \exp\left(i\hbar (1 \otimes X \otimes Y + X \otimes Y \otimes 1 + X \otimes 1 \otimes Y)\right), \end{aligned}$$

so that (1.6.14a) is satisfied.

Again, using the fact that the counit is an algebra homomorphism, one obtains that $\varepsilon(\exp(X)) = \exp(\varepsilon(X))$. From the expression (1.4.12), one computes

$$\begin{aligned} (\text{id} \otimes \varepsilon)(\mathcal{F}) &= \exp(i\hbar X \otimes \varepsilon(Y)) = \exp(0) = 1 \\ (\varepsilon \otimes \text{id})(\mathcal{F}) &= \exp(i\hbar \varepsilon(X) \otimes Y) = \exp(0) = 1, \end{aligned}$$

so that (1.6.14b) is satisfied.

Finally, one obtains (1.6.14c) by simply expanding the exponential into an infinite sum

$$\mathcal{F} = 1 \otimes 1 + \hbar \sum_{n=1}^{\infty} \frac{i^n \hbar^{n-1}}{n!} X^n \otimes Y^n = 1 \otimes 1 + \mathcal{O}(\hbar). \quad \square$$

We have gathered all the ingredients to construct a quantum space-time out of a classical space-time. The next step is to look at the behaviour of fields on this quantum space-time. Therefore, the next Chapters look at a toy model of ϕ^4 -theory and try to capture the deformation that noncommutativity induces on $U(1)$ gauge theory.

Chapter 2

Noncommutative field and gauge theories

Over classical space-times, gauge theories are expressed through a (principal) fiber bundle, on which a gauge group acts. This fiber bundle corresponds to the geometric space where the connection (or the gauge fields) and the curvature (or the field strength) live.

For example, in the context of general relativity, the considered bundle is the tangent bundle. A vector field is expressed as a section of the tangent bundle, *i.e.* to any point in space-time $x \in \mathcal{M}$, a vector field $X \in \Gamma(\mathcal{M})$, associate a vector in the tangent space at x , $X(x) \in T_x\mathcal{M}$. Given two distant points $x, y \in \mathcal{M}$, the vectors $X(x)$ and $X(y)$ cannot be compared as they live in different vector spaces. Therefore, one needs the notion of a parallel transport, or equivalently a connection (or even a covariant derivative), to transport $X(x)$ in the tangent space at y , *i.e.* in $T_y\mathcal{M}$. The covariant derivative is defined, in the way of KOSZUL [49], as $\nabla : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{M})$ and writes in local coordinates

$$\nabla_\mu(\partial_\nu) = \Gamma_{\mu\nu}^\rho \partial_\rho \quad (2.0.1)$$

where $\Gamma_{\mu\nu}^\rho$ is the connection. Finally, the curvature $R : \Gamma(\mathcal{M})^2 \times \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{M})$ is defined as the mismatch of the starting and ending point when performing a loop of parallel transports (see [Figure 3](#)). In local coordinates, this writes

$$R_{\mu\nu}(\partial_\rho) = \nabla_\mu(\nabla_\nu(\partial_\rho)) - \nabla_\nu(\nabla_\mu(\partial_\rho)) = R_{\mu\nu\rho}^\sigma \partial_\sigma. \quad (2.0.2)$$

One can generalise this picture to the case of a principal fiber bundle \mathcal{P} over a space-time \mathcal{M} , with a structure group given by a LIE group \mathcal{G} . In such a bundle \mathcal{G} acts on \mathcal{P} via a right action. One then considers (V, ρ) a representation of \mathcal{G} and considers the associated vector bundle $\mathcal{X} = \mathcal{P} \times_\rho V$. The notation \times_ρ denotes here the fact that \mathcal{G} acts on the V part of \mathcal{X} via ρ . The sections $s : \mathcal{M} \rightarrow \mathcal{X}$, noted $s \in \Gamma(\mathcal{X})$, on such a bundle form a $\mathcal{C}^\infty(\mathcal{M})$ -module for the point-wise product. As above, one can compare distant regions of this bundle thanks to a covariant derivative $\nabla : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{X}) \rightarrow \Gamma(\mathcal{X})$, and can assess the loop mismatch via the curvature $R : \Gamma(\mathcal{M})^2 \times \Gamma(\mathcal{X}) \rightarrow \Gamma(\mathcal{X})$ (see [Figure 3](#)).

In this context, the other main example is the case where $\mathcal{G} = U(1)$ and $V = \mathbb{R}$, which corresponds to $\Gamma(\mathcal{X}) = \mathcal{C}^\infty(\mathcal{M}) \otimes \mathbb{R}$. From the definition of ∇ , one has in local coordinates (see [\(2.1.9\)](#))

$$\nabla_\mu f = \partial_\mu f - iA_\mu f \quad (2.0.3)$$

for any $f \in \mathcal{C}^\infty(\mathcal{M})$. Note that for simplicity we have abbreviated $f = f \otimes 1 \in \mathcal{C}^\infty(\mathcal{M}) \otimes \mathbb{R}$, by using $\mathcal{C}^\infty(\mathcal{M}) \otimes \mathbb{R} \simeq \mathcal{C}^\infty(\mathcal{M})$. One recognizes in (2.0.3) the particle physics expression of the covariant derivative together with the gauge field A_μ . The curvature then writes

$$R_{\mu\nu}(f) = i(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu])f = iF_{\mu\nu}f \quad (2.0.4)$$

where one recognizes the field strength F . Therefore, (classical) $SU(n)$ gauge theories can also be written in this general framework.

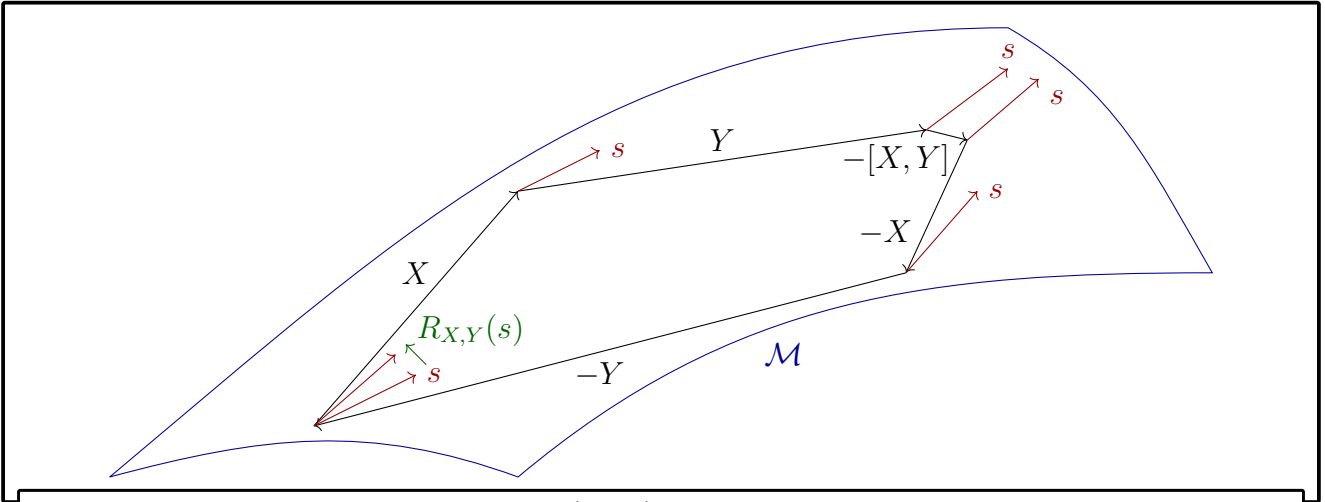


Figure 3 – Illustration of the formula (2.1.3), consisting in a loop of parallel transports, for s a vector field.

To generalise gauge theories to the algebraic context of **noncommutative geometry**, we first need a generalisation of the notion of (principal) fiber bundle. The algebraic analogue of sections of the bundle is, in fact, directly given by the SERRE-SWAN theorem, which states that fiber bundles are in one-to-one correspondence with the notion of (projective) module (see the **Definition A.2.1** and the **Example 1.4.4** for an explicit example). Therefore, the first element to build a gauge theory is a module, generically denoted by \mathbb{X} here, that is supposed to stand as the noncommutative counterpart of $\Gamma(\mathcal{X})$. The covariant derivative and the curvature definitions are then exported from the commutative case as such. Finally, one can also implement the gauge transformations in this setting.

To summarise, we consider here a quantum space-time \mathbb{A} as introduced in **Chapter 1**. The gauge theory over this quantum space-time is defined on a module \mathbb{X} over the algebra \mathbb{A} in which we implement the notions of covariant derivative, associated curvature and gauge transformations. From there, one recovers the usual physical quantities (gauge fields, field strength, *etc...*) on the quantum space-time.

Note that these notions depend on how one generalises the vector fields $\Gamma(\mathcal{M})$ and so how one generalises the differential calculus. As already expressed in **Chapter 1**, there are three main ways of doing so, leading thus to different formulations of gauge theory on quantum space-times. Here, we detail two of such constructions: gauge theories coming from derivation-based differential calculus and the one coming from **DRINFEL'D twist** construction. The way gauge theories are implemented in the **spectral triple** formalism was roughly introduced by (1.2.5) and (1.3.6) and is not treated further here.

The previous scheme of gauge theory has been applied to several quantum space-times, like to κ -MINKOWSKI (see Section 3.2). For a complete review of gauge theories on quantum space-time, see [23]. It should also be noted that the following constructions are made for right modules, but could be equally well made for left modules or bimodules [23].

Note that building a gauge theory on a noncommutative space is *a priori* not straightforward. Indeed, consider \mathfrak{g} to be the LIE algebra of infinitesimal gauge transformations, with associated gauge (LIE) group G , with a matrix representation. The gauge field A is a \mathfrak{g} -valued connection. Then, the noncommutative analogue of A_μ is an element of $\mathfrak{g} \otimes \mathbb{A}$, which, in the matrix representation, corresponds to a matrix with coefficients in \mathbb{A} . However, since \mathbb{A} is noncommutative, the LIE algebra closure rules are likely to be broken. Explicitly, for $\alpha \otimes f, \beta \otimes g \in \mathfrak{g} \otimes \mathbb{A}$, one has

$$[\alpha \otimes f, \beta \otimes g] = [\alpha, \beta] \otimes (f \star g) + \alpha\beta \otimes [f, g]_\star, \quad (2.0.5)$$

which would correspond to a term of the form $[A_\mu, A_\nu]$ in the field strength. The first term of (2.0.5) is stable in $\mathfrak{g} \otimes \mathbb{A}$ but the second is not, since in general $\alpha\beta \notin \mathfrak{g}$, and does not vanish as the star product is not commutative. The usual solution to this issue is to consider that the connection takes values in $U(\mathfrak{g}) \otimes \mathbb{A}$, where $U(\mathfrak{g})$ is the universal enveloping algebra of \mathfrak{g} . Then, some conditions may be imposed on A to recover a connection that takes values in $\mathfrak{g} \otimes \mathbb{A}$. For the most used case of $G = U(n)$, which corresponds to $\mathfrak{g} = \mathfrak{u}(n)$, one has that $U(\mathfrak{u}(n)) \simeq \mathbb{M}_n(\mathbb{C})$, so that A takes values in $\mathbb{M}_n(\mathbb{C}) \otimes \mathbb{A}$. To recover $\mathfrak{u}(n)$ from $\mathbb{M}_n(\mathbb{C})$, a hermiticity condition is imposed on A using the involution of \mathbb{A} .

Each of the following proposals eliminates the previous problem, mainly by considering different types of deformation. The gauge theory schemes, based on derivations, developed in Section 2.1 bypass this problem by considering a deformed gauge group. In the case of the DRINFEL'D twist formulation of Section 2.2, another solution was pointed out. One can deform the gauge transformation thanks to the DRINFEL'D twist in such a way that all the deformations go into the LEIBNIZ rule which arises when one gauge transforms a product. The LEIBNIZ rule of the gauge transformation is seen, in this formalism, in its coproduct. For its part, the SEIBERG-WITTEN map of Section 2.3 requires the gauge parameter to depend on the undeformed gauge field, leading to a modified gauge transformation.

2.1 Derivation based theories

In this Section 2.1, we introduce the gauge theory built on derivation-based differential calculus, as it was set up in [50, 51]. For a review, see [5]. We consider \mathbb{A} to be a quantum space-time and \mathbb{X} to be a \mathbb{A} -module. The vector fields of \mathbb{A} are considered to be the derivations of the algebra $\text{Der}(\mathbb{A})$, following the construction of Section 1.5.

Given a derivation $X \in \text{Der}(\mathbb{A})$, we define a connection *à la* KOSZUL [49] $\nabla_X : \mathbb{X} \rightarrow \mathbb{X}$ as

$$\nabla_X(s \triangleleft f) = \nabla_X(s) \triangleleft f + s \triangleleft X(f), \quad (\text{LEIBNIZ rule}) \quad (2.1.1a)$$

$$\nabla_{X+zY}(s) = \nabla_X(s) + \nabla_Y(s) \triangleleft z, \quad (\mathcal{Z}(\mathbb{A})\text{-linearity}) \quad (2.1.1b)$$

for any $Y \in \text{Der}(\mathbb{A})$, $s \in \mathbb{X}$, $f \in \mathbb{A}$ and $z \in \mathcal{Z}(\mathbb{A})$. Considering the differential calculus constructed in Section 1.5, the definition (2.1.1) is equivalent to

$$\nabla : \mathbb{X} \rightarrow \mathbb{X} \otimes_{\mathbb{A}} \Omega^1(\mathbb{A}), \quad \nabla(s \triangleleft f) = \nabla(s) \triangleleft f + s \otimes df, \quad (2.1.2)$$

where $d : \mathbb{A} \rightarrow \Omega^1(\mathbb{A})$ is the differential. It straightforwardly extends to the covariant derivative $\nabla : \mathbb{X} \rightarrow \mathbb{X} \otimes_{\mathbb{A}} \Omega^\bullet(\mathbb{A})$.

Performing a loop of parallel transports $0 \rightarrow X \rightarrow XY \rightarrow XY - Y \rightarrow XY - YX$ (see Figure 3), one does not necessarily come back to the same point. This mismatch is measured by $R_{X,Y} : \mathbb{X} \rightarrow \mathbb{X}$, the curvature associated to ∇ , thus defined as

$$\begin{aligned} R_{X,Y}(s) &= [\nabla_X, \nabla_Y](s) - \nabla_{[X,Y]}(s) \\ &= \nabla_X(\nabla_Y(s)) - \nabla_Y(\nabla_X(s)) - \nabla_{[X,Y]}(s). \end{aligned} \quad (2.1.3)$$

One can show, using (2.1.1a), that the curvature is a module homomorphism, that is $R_{X,Y}(s \triangleleft f) = R_{X,Y}(s) \triangleleft f$. This condition actually justifies that one can only consider the components $F_{\mu\nu}$ in (2.0.4) or $R_{\mu\nu}{}^\sigma$ in (2.0.2) instead of the full curvature.

Proof.

$$\begin{aligned} R_{X,Y}(s \triangleleft f) &= \nabla_X \nabla_Y (s \triangleleft f) - \nabla_Y \nabla_X (s \triangleleft f) - \nabla_{[X,Y]}(s \triangleleft f) \\ &= \nabla_X (\nabla_Y (s) \triangleleft f + s \triangleleft Y(f)) - \nabla_Y (\nabla_X (s) \triangleleft f + s \triangleleft X(f)) \\ &\quad - \nabla_{[X,Y]}(s) \triangleleft f - s \triangleleft [X, Y](f) \\ &= \nabla_X \nabla_Y (s) \triangleleft f + \nabla_Y (s) \triangleleft X(f) + \nabla_X (s) \triangleleft Y(f) + s \triangleleft XY(f) \\ &\quad - \nabla_Y \nabla_X (s) \triangleleft f - \nabla_X (s) \triangleleft Y(f) - \nabla_Y (s) \triangleleft X(f) - s \triangleleft YX(f) \\ &\quad - \nabla_{[X,Y]}(s) \triangleleft f - s \triangleleft [X, Y](f) \\ &= R_{X,Y}(s) \triangleleft f \end{aligned} \quad \square$$

The gauge group is defined as the group of automorphisms of \mathbb{X} , that is the invertible linear maps $\varphi : \mathbb{X} \rightarrow \mathbb{X}$ such that $\varphi(s \triangleleft f) = \varphi(s) \triangleleft f$. The gauge transformation of ∇ and consequently of the curvature are given by

$$\nabla_X^\varphi = \varphi^{-1} \circ \nabla_X \circ \varphi, \quad R_{X,Y}^\varphi = \varphi^{-1} \circ R_{X,Y} \circ \varphi. \quad (2.1.4)$$

One can actually check that the gauge transformed connection ∇^φ is indeed a connection as it satisfies (2.1.1).

In the context of unitary gauge groups, as in YANG-MILLS theory, one further needs a Hermitian structure $(\cdot, \cdot) : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{A}$, *i.e.* a sesquilinear map (see (B.1.2a) and (B.1.2b)) which satisfies

$$(s_1, s_2)^\dagger = (s_2, s_1), \quad (s_1 \triangleleft f_1, s_2 \triangleleft f_2) = f_1^\dagger \star (s_1, s_2) \star f_2 \quad (2.1.5)$$

for any $s_1, s_2 \in \mathbb{X}$ and $f_1, f_2 \in \mathbb{A}$. In this context, the connection is Hermitian if and only if

$$X((s_1, s_2)) = (\nabla_X(s_1), s_2) + (s_1, \nabla_X(s_2)) \quad (2.1.6)$$

for any $X \in \text{Der}(\mathbb{A})$ such that $X(f^\dagger) = X(f)^\dagger$. One says that X is a real derivative when the latter condition holds. When considering deformed YANG-MILLS theories, the condition (2.1.6) is equivalent to $A_\mu^\dagger = A_\mu$, which corresponds to A_μ being real-valued in the commutative limit.

Finally, a gauge transformation φ is said to be unitary if

$$(\varphi(s_1), \varphi(s_2)) = (s_1, s_2). \quad (2.1.7)$$

We denote the set of unitary gauge transformations as $\mathcal{U}(\mathbb{X})$.

All the previous definitions make more sense when applied to the specific case of $\mathbb{X} = \mathbb{A}$, as in [Example A.2.2](#).

Noncommutative electrodynamics

Consider^a $\mathbb{X} = \mathbb{A}$, with action $\triangleleft = \star$ and Hermitian structure $(f, g)_{\mathbb{A}} = f^\dagger \star g$. One can start by actually checking that \star is indeed an action and that $(\cdot, \cdot)_{\mathbb{A}}$ satisfies (2.1.5). The equation (2.1.1) now states how the connection ∇ behaves on products, *i.e.*

$$\nabla_X(g \star f) = \nabla_X(g) \star f + g \star X(f) \quad (2.1.8)$$

and considering $g = 1$, this gives

$$\nabla_X(f) = X(f) - iA_X \star f, \quad (2.1.9)$$

where we noted $\nabla_X(1) = -iA_X$. If one considers local coordinates, then (2.1.9) is akin to (2.0.3) for $X = \partial_\mu$. Our ∇ thus corresponds to the deformed covariant derivative associated to some gauge field A . Using the curvature definition (2.1.3), one computes with (2.1.9) that

$$\begin{aligned} R_{X,Y}(f) &= i\left(X(A_Y) - Y(A_X) - i[A_X, A_Y]_\star + A_{[X,Y]}\right) \star f \\ &= F_{X,Y} \star f, \end{aligned} \quad (2.1.10)$$

which again gives the same expression as (2.0.4) if $X = \partial_\mu$ and $Y = \partial_\nu$. Furthermore, if one requires that ∇ is Hermitian, then, using (2.1.6) and (2.1.9), one computes that

$$A_X^\dagger = A_X \quad (2.1.11)$$

for X a real derivation.

A gauge transformation φ is fully determined by its value at 1 since $\varphi(f) = \varphi(1) \star f$. We denote $u = \varphi(1)$ in the following. Through (2.1.4), the gauge transform of the gauge field and the field strength are calculated to be

$$A_X^u = u^\dagger \star A_X \star u - iu^\dagger \star X(u), \quad (2.1.12a)$$

$$F_{X,Y}^u = u^\dagger \star F_{X,Y} \star u. \quad (2.1.12b)$$

The unitary gauge group, given by (2.1.7), writes

$$\mathcal{U}(1) = \left\{ u \in \mathbb{A}, u^\dagger \star u = u \star u^\dagger = 1 \right\}. \quad (2.1.13)$$

^aIn accordance with the commutative case described in (2.0.3), it would be more appropriate to consider that $\mathbb{X} = \mathbb{A} \otimes \mathbb{C}$. In this decomposition, \mathbb{C} is as the 1-dimensional vector space to which \mathbb{A} is associated to form a vector bundle. However, one can use that $\mathbb{A} \otimes \mathbb{C} \simeq \mathbb{A}$ to simplify the notations.

Several comments are in order.

The deformed ‘‘Abelian’’ gauge theory developed above actually is in fact more like a non-Abelian one, because the product \star is noncommutative. Indeed, the bracket term $[A_X, A_Y]_\star$ is present in (2.1.10) even if electrodynamics has been considered. However, in Abelian gauge theories, the field strength contains directly measurable quantities, that are the electric and magnetic fields. In this

sense, it must be gauge invariant, as the measured fields should not depend on the chosen gauge. From (2.1.12b) it follows that, due to the noncommutativity of \star , the deformed field strength F is not gauge invariant, but rather gauge covariant.

Accordingly, the field strength may not be a physical quantity in the noncommutative theory and one should build the relevant fields out of F . This is already done in non-Abelian gauge theory when one considers the gauge bosons of the electroweak interaction not to be the hypercharge and isospin fields B , W^1 , W^2 and W^3 (which would all correspond to different copies of A here) but rather $W^\pm = \frac{W^1 \mp iW^2}{\sqrt{2}}$, $\gamma = \cos(\theta_W)B + \sin(\theta_W)W^3$ and $Z^0 = \cos(\theta_W)W^3 - \sin(\theta_W)B$. In the case of MOYAL space, it is actually more convenient to work with the so-called covariant coordinate or invariant connection, $\mathcal{A}_\mu = i(A_\mu + \Theta_{\mu\nu}^{-1}x^\nu)$, rather than A_μ . However, there are *a priori* an infinite number of possible candidate that would have the suitable properties to be a gauge field and the good commutative limit. The question of which one is the physically relevant quantity has not been settled yet.

The gauge theory considered here already has some phenomenological consequences on the physical model under study. Indeed, if the noncommutative space \mathbb{A} is constructed as the deformation of some commutative space, as in Section 1.6, one can consider the expansion of F up to a given order m in the deformation parameter. This would give an effective YANG-MILLS action with supplementary operators of (mass) dimension 5 to $4 + m$, for which the deformation parameter is the scale of “new physics”. From another perspective, one could also look into the deformation induced by the symmetries. Indeed, the gauge group $\mathcal{U}(1)$ is a deformed version of the commutative $U(1)$ gauge group but is not $U(1)$. Therefore, the usual gauge group should be broken at scales close to the deformation parameter. A more extended discussion on phenomenology of quantum space-times is given in Section 4.2.

One can construct similarly $\mathcal{U}(n)$ gauge theories, generically called noncommutative YANG-MILLS theories.

Noncommutative YANG-MILLS theory

Consider^a $\mathbb{X} = \mathbb{A}^{\otimes n}$, for some $n \in \mathbb{N}_1$, with action $\triangleleft = \star \otimes \cdots \otimes \star$ (see Example A.2.2). For convenience, one writes

$$\mathbf{e}_j = 0 \otimes \cdots \otimes 0 \otimes \overset{(j)}{1} \otimes 0 \otimes \cdots \otimes 0, \quad (2.1.14)$$

where the only non-zero entry is at the j -th place, for any $j = 1, \dots, n$. Thus, for any $s \in \mathbb{X}$, one can decompose $s = \sum_{j=1}^n \mathbf{e}_j \star s^j$, where $s^j \in \mathbb{A}$. From there, the action writes $s \triangleleft f = \sum_{j=1}^n \mathbf{e}_j \star s^j \star f$, for any $f \in \mathbb{A}$. The Hermitian structure considered is

$$(s_1, s_2) = \sum_{j=1}^n (s_1^j)^\dagger \star s_2^j. \quad (2.1.15)$$

Thanks to the LEIBNIZ rule (2.1.1a), one has

$$\nabla_X(s) = \sum_{j=1}^n \nabla_X(\mathbf{e}_j \star s^j) = \sum_{j=1}^n \mathbf{e}_j \star X(s^j) + \nabla_X(\mathbf{e}_j) \star s^j. \quad (2.1.16)$$

Therefore, the connection is fully determined by its values on the basis $\{e_j\}_j$, which are written in components as $\nabla_X(e_j) = -i \sum_{k=1}^n e_k \star (A_X)_j^k$. If one associates the module with column vectors of elements in \mathbb{A} , that is $\mathbb{X} = \mathbb{M}_{1,n}(\mathbb{A}) = \mathbb{M}_{1,n}(\mathbb{C}) \otimes \mathbb{A}$, one can write the connection A_X under a matrix form

$$A_X = \sum_{j,k=1}^n e^j \star e_k \star (A_X)_j^k = \begin{pmatrix} (A_X)_1^1 & \cdots & (A_X)_n^1 \\ \vdots & \ddots & \vdots \\ (A_X)_1^n & \cdots & (A_X)_n^n \end{pmatrix} \in \mathbb{M}_n(\mathbb{A}) = \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{A} \quad (2.1.17)$$

where $\{e^j\}_j$ are the basis elements of $\mathbb{M}_{n,1}(\mathbb{A})$. It is also quite convenient to combine the matrix transpose t and the involution † of \mathbb{A} to get a generalisation of the adjoint matrix ${}^\ddagger = {}^t \otimes {}^\dagger$. In these notations $e_j^\ddagger = e^j$, and (2.1.16) writes $(s_1, s_2) = s_1^\ddagger \star s_2$, where \star here denotes the matrix product with \star .

Equipped with these notations, the requirement that ∇ is Hermitian (2.1.6) now writes

$$A_X^\ddagger = A_X. \quad (2.1.18)$$

Furthermore, the gauge transformations are fully determined by their action on the basis $\{e_j\}_j$, since $\varphi(s) = \sum_{j=1}^n \varphi(e_j) \star s^j$. Indeed, one computes that $\varphi(e_j) = \sum_{k=1}^n u_j^k \star e_k$ with $u_j^k \in \mathbb{A}$. The latter element can be written in matrix notations as $u = (u_j^k)_{j,k} \in \mathbb{M}_n(\mathbb{A})$. Such a gauge transformation is said unitary if $u^\ddagger \star u = \mathbb{1}_n$, as one computes from (2.1.7). Therefore, the (unitary) gauge group is

$$\mathcal{U}(n) = \left\{ u \in \mathbb{M}_{1,n}(\mathbb{A}), u^\ddagger \star u = u \star u^\ddagger = \mathbb{1}_n \right\}. \quad (2.1.19)$$

Finally, the connection A and the curvature F are computed to gauge transform as

$$A_X^u = u^\ddagger \star A_X \star u - i u^\ddagger \star X(u), \quad (2.1.20a)$$

$$F_{X,Y}^u = u^\ddagger \star F_{X,Y} \star u. \quad (2.1.20b)$$

^aIn accordance with the footnote *a* in the example of noncommutative electrodynamics, one should rather consider $\mathbb{X} = \mathbb{A} \otimes \mathbb{C}^n$. The notations are simplified by using $\mathbb{A} \otimes \mathbb{C}^n \simeq \mathbb{A}^{\otimes n}$.

Proof. We here derive some equations stated above, in the case of noncommutative electrodynamics $\mathcal{U}(1)$. The $\mathcal{U}(n)$ case can be computed similarly.

First let us prove (2.1.10). To do so, one mainly uses (2.1.8) and (2.1.9).

$$\begin{aligned} R_{X,Y}(f) &= \nabla_X(\nabla_Y(f)) - \nabla_Y(\nabla_X(f)) - \nabla_{[X,Y]}(f) \\ &= \nabla_X(Y(f) - iA_Y \star f) - \nabla_Y(X(f) - iA_X \star f) - [X,Y](f) + iA_{[X,Y]} \star f \\ &= \cancel{XY(f)} - \cancel{iA_X \star Y(f)} - i\nabla_X(A_Y) \star f - \cancel{iA_Y \star X(f)} \\ &\quad - \cancel{YX(f)} + \cancel{iA_Y \star X(f)} + i\nabla_Y(A_X) \star f + \cancel{iA_X \star Y(f)} \\ &\quad - \cancel{[X,Y](f)} + iA_{[X,Y]} \star f \\ &= -i(X(A_Y) - Y(A_X) - i[A_X, A_Y] - A_{[X,Y]}) \star f. \end{aligned}$$

The hermitian structure considered together with (2.1.6), imposes that for any real derivation X , one has

$$\begin{aligned} X((f, g)) &= X(f^\dagger \star g) = X(f)^\dagger \star g + f^\dagger \star X(g) \\ &= (\nabla_X(f), g) + (f, \nabla_X(g)) = (X(f) - iA_X \star f)^\dagger \star g + f^\dagger \star (X(g) - iA_X \star g) \\ &= X(f)^\dagger \star g + f^\dagger \star X(g) + if^\dagger \star (A_X^\dagger - A_X) \star g, \end{aligned}$$

from which one obtains (2.1.11).

The unitary elements of the gauge group given by (2.1.7) reads

$$\begin{aligned} (\varphi(f), \varphi(g)) &= (u \star f, u \star g) = f^\dagger \star u^\dagger \star u \star g \\ &= (f, g) = f^\dagger \star g, \end{aligned}$$

which implies (2.1.13).

Finally, given the expressions for the gauge transformations (2.1.4), one computes

$$\begin{aligned} \nabla_X^\varphi(f) &= \varphi^{-1}(\nabla_X(\varphi(f))) = u^\dagger \star (\nabla_X(u \star f)) \\ &= u^\dagger \star (\nabla_X(u) \star f + u \star X(f)) \\ &= u^\dagger \star (X(u) - iA_X \star u) \star f + X(f) \\ &= X(f) - iA_X^u \star f \end{aligned}$$

where $\varphi^{-1}(f) = u^{-1} \star f = u^\dagger \star f$, for $u \in \mathcal{U}(1)$. One directly reads (2.1.12a) and deduces (2.1.12b) from standard computations. \square

Finally, suppose that there exists an integral over \mathbb{A} , potentially inherited from the integral over the space-time \mathcal{M} , for which \mathbb{A} is the deformed space of smooth functions. In the context of noncommutative YANG-MILLS theory, let us consider the action

$$S = \int d^{d+1}x \operatorname{tr}(F^{\mu\nu} \star F_{\mu\nu}^\dagger) \quad (2.1.21)$$

where tr stands for the matrix trace. If one considers that the integral is cyclic for \star , that is

$$\int d^{d+1}x f \star g = \int d^{d+1}x g \star f \quad (2.1.22)$$

then, the action (2.1.21) is gauge invariant for the $\mathcal{U}(n)$ gauge group. Indeed, for any $u \in \mathcal{U}(n)$, one has

$$\begin{aligned} S^u &= \int d^{d+1}x \operatorname{tr}((F^u)^{\mu\nu} \star (F^u)_{\mu\nu}^\dagger) \\ &= \int d^{d+1}x \operatorname{tr}(u^\dagger \star F^{\mu\nu} \star u \star (u \star F_{\mu\nu} \star u^\dagger)^\dagger) \\ &= \int d^{d+1}x \operatorname{tr}(u^\dagger \star F^{\mu\nu} \star \mathcal{X} \star \mathcal{X}^\dagger \star F_{\mu\nu}^\dagger \star u) \\ &= \int d^{d+1}x \operatorname{tr}(\mathcal{X} \star \mathcal{X}^\dagger \star F^{\mu\nu} \star F_{\mu\nu}^\dagger) = S \end{aligned} \quad (2.1.23)$$

By construction, the action (2.1.21) gives the YANG-MILLS action at the commutative limit, and thus consists of a coherent deformation of the YANG-MILLS theory. In the context of the [MOYAL](#)

space, this model is discussed in [Section 2.5](#). When one builds a gauge theory similar to (2.1.21) on the κ -MINKOWSKI space-time, the lack of cyclicity (2.1.22) prevents the action (2.1.21) to be straightforwardly gauge invariant. The latter obstacle and the possible ways around it are discussed in [Section 3.2](#).

2.2 DRINFEL'D twist based theories

Another way of defining the differential calculus in a consistent manner consists of using the twist deformation of a classical differential calculus [52, 53]. As detailed in [Subsection 1.6.2](#), one starts with a space-time \mathcal{M} and its **LIE algebra** of vector fields $\Gamma(\mathcal{M})$. Then, a **DRINFEL'D twist**, defined on the enveloping algebra, is used to deform the vector fields and derive the corresponding quantum space-time \mathbb{A} . Finally, one defines the bundle \mathbb{X} as a \mathbb{A} -module.

Within this scheme, deformed $U(n)$ gauge theories have first been considered in [54, 55, 56] and adapted to gravity in [57, 58].

Given a space-time \mathcal{M} , its set of vector fields forms a **LIE algebra**. Thus, one can deform its universal enveloping algebra, that we note¹ $\Xi = U(\Gamma(\mathcal{M}))$ for simplicity, via a **DRINFEL'D twist** (see [Subsection 1.6.2](#) for more details). Considering a **DRINFEL'D twist** $\mathcal{F} \in \Xi \otimes \Xi$, one defines the **HOPF algebra** $\Xi^{\mathcal{F}}$ of deformed vector fields. The associated quantum space-time \mathbb{A} is a $\Xi^{\mathcal{F}}$ -module algebra and corresponds to a deformation of the smooth function algebra $\mathcal{C}^\infty(\mathcal{M})$. The new (noncommutative) product on \mathbb{A} is given by

$$f \star g = \sum \left(\mathcal{F}_1^{-1} \triangleright f \right) \cdot \left(\mathcal{F}_2^{-1} \triangleright g \right) \quad (2.2.1)$$

for any $f, g \in \mathbb{A}$, where we noted $\mathcal{F} = \sum \mathcal{F}_1 \otimes \mathcal{F}_2$. This product corresponds to (1.6.16).

One can then take the usual differential structure of \mathcal{M} and twist it all the way to the end. Let us begin by the tensor product \otimes , which can be twisted through

$$X \otimes_{\mathcal{F}} Y = \sum \left(\mathcal{F}_1^{-1} \triangleright X \right) \otimes \left(\mathcal{F}_2^{-1} \triangleright Y \right) \quad (2.2.2)$$

for any $X, Y \in \Xi^{\mathcal{F}}$, where $\mathcal{F}_j^{-1} \triangleright X = [\mathcal{F}_j^{-1}, X]$ stands for the LIE derivative. The product $\otimes_{\mathcal{F}}$ is often called the star tensor product and denoted \otimes_{\star} . The definition (2.2.2) can be exported to tensors of any rank. Then, the wedge product of forms is twisted in a similar way to

$$\omega \wedge_{\mathcal{F}} \eta = \sum \left(\mathcal{F}_1^{-1} \triangleright \omega \right) \wedge \left(\mathcal{F}_2^{-1} \triangleright \eta \right) \quad (2.2.3)$$

for any $\omega, \eta \in \Omega^1(\mathbb{A})$, *i.e.* they are linear functional from $\Xi^{\mathcal{F}}$ to \mathbb{A} . The algebra of forms $\Omega^\bullet(\mathbb{A})$ is built from the star wedge product (2.2.3) and actually corresponds to multilinear braided antisymmetric maps from $(\Xi^{\mathcal{F}})^n$ to \mathbb{A} .

The fact that the star wedge is braided antisymmetric, that is it satisfies (2.2.5), is not innocuous. The **HOPF algebra** Ξ is a triangular **HOPF algebra** (see [Definition A.1.18](#)) with a \mathcal{R} -matrix being trivial, *i.e.* $\mathcal{R} = 1 \otimes 1$. The **DRINFEL'D twist** deformation actually conserves this triangularity property (see [Theorem A.1.21](#)) with a new \mathcal{R} -matrix given by $\mathcal{R} = \mathcal{F}_{21} \mathcal{F}^{-1}$. The triangularity property thus transforms commutativity into braided commutativity

$$f \star g = \sum \left(\mathcal{R}_1^{-1} \triangleright g \right) \star \left(\mathcal{R}_2^{-1} \triangleright f \right), \quad (2.2.4)$$

¹In [58], $\Xi = \Gamma(\mathcal{M})$ which makes our notations differ a bit.

and antisymmetry into braided antisymmetry

$$\omega \wedge_{\mathcal{F}} \eta = - \sum \left(\mathcal{R}_1^{-1} \triangleright \eta \right) \wedge_{\mathcal{F}} \left(\mathcal{R}_2^{-1} \triangleright \omega \right). \quad (2.2.5)$$

Beyond the invocation of the triangular structure, one can check that (2.2.4) and (2.2.5) are satisfied, as done in the proof below.

The main remark to make here is that the \mathcal{R} -matrix parametrises the noncommutativity, which could be expected since it parametrizes the noncocommutativity of the coproduct of $\Xi^{\mathcal{F}}$. The triangularity condition $\mathcal{R}_{21}\mathcal{R} = 1 \otimes 1$ allows the braiding to be a symmetry². Indeed, if one applies (2.2.4) two times, one gets $f \star g = f \star g$. Finally, in the context of deformation quantisation, the semi-classical condition on the DRINFEL'D twist (1.6.14c), explicitly $\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\hbar)$, imposes a similar condition on the \mathcal{R} -matrix, *i.e.* $\mathcal{R} = 1 \otimes 1 + \mathcal{O}(\hbar)$. Therefore, in the commutative limit $\hbar \rightarrow 0$, the \mathcal{R} -matrix becomes trivial and the product \star equals the commutative product of functions.

Proof. We verify here the equality (2.2.4). The computation leading to (2.2.5) is very similar. The main thing we need is $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$, *i.e.* $\mathcal{R}^{-1} = \mathcal{F}\mathcal{F}_{21}^{-1}$.

$$\begin{aligned} \sum \left(\mathcal{R}_1^{-1} \triangleright g \right) \star \left(\mathcal{R}_2^{-1} \triangleright f \right) &= \sum \left(\left(\mathcal{F}_1^{-1} \mathcal{R}_1^{-1} \right) \triangleright g \right) \left(\left(\mathcal{F}_2^{-1} \mathcal{R}_2^{-1} \right) \triangleright f \right) \\ &= \sum \left(\left(\mathcal{F}_1^{-1} \mathcal{F}_1 \mathcal{F}_2^{-1} \right) \triangleright g \right) \left(\left(\mathcal{F}_2^{-1} \mathcal{F}_2 \mathcal{F}_1^{-1} \right) \triangleright f \right) \\ &= \sum \left(\mathcal{F}_2^{-1} \triangleright g \right) \left(\mathcal{F}_1^{-1} \triangleright f \right) = f \star g \quad \square \end{aligned}$$

The construction of a gauge theory on the quantum space-time \mathbb{A} can follow the same steps as in Section 2.1 by replacing $\text{Der}(\mathbb{A})$ by $\Xi^{\mathcal{F}}$. One has to be careful about the actions though. Explicitly, consider a \mathbb{A} -module \mathbb{X} , as the generalised fiber bundle. A connection $\nabla_X : \mathbb{X} \rightarrow \mathbb{X}$, for any $X \in \Xi^{\mathcal{F}}$, is defined as

$$\nabla_X(s \triangleleft_{\mathbb{X}} f) = \nabla_X(s) \triangleleft_{\mathbb{X}} f + s \triangleleft_{\mathbb{X}} (X \triangleright f), \quad (\text{LEIBNIZ rule}) \quad (2.2.6a)$$

$$\nabla_{X+Z}(s) = \nabla_X(s) + \nabla_Z(s) \triangleleft_{\mathbb{X}} z, \quad (\mathcal{Z}(\mathbb{A})\text{-linearity}) \quad (2.2.6b)$$

for any $Y \in \Xi^{\mathcal{F}}$, $s \in \mathbb{X}$ and $z \in \mathbb{A}$. In the previous expression, $\triangleleft_{\mathbb{X}}$ corresponds to the action of \mathbb{A} on \mathbb{X} , which is to be thought as a “scalar” product on the bundle. Still, it is actually very different from \triangleright introduced in (2.2.1), which corresponds here to the action of $\Xi^{\mathcal{F}}$ on \mathbb{A} and should be thought as an action of derivation. Explicitly, $X \in \Xi^{\mathcal{F}}$ is a generalised vector field and $f \in \mathbb{A}$ a generalised function, so that $X \triangleright f$ corresponds to the derivative of f “along” X .

Beyond this subtlety, one can construct the noncommutative YANG-MILLS theory with $\mathcal{U}(n)$ gauge group, for any n , as in Section 2.1. Rather than repeating this construction, we make two important remarks.

In the gauge theory as originally developed in [54, 55, 56], the gauge group is undeformed, and the papers mainly evolve around matching the deformed field theory with the undeformed gauge. Besides, these studies were mainly done on the MOYAL space.

²At this point, one should stop calling \mathcal{R} a braiding since it is a symmetry through the triangular condition. In other words, a braiding, contrary to a symmetry, never brings one back to the original position, as when one braids hairs. However, the triangular condition specifically states that “braiding” two times is akin to doing nothing and thus does not correspond to a braiding properly speaking. Here, we stick to the denomination of [35] in which everything is called “braided” even in the triangular case.

The formulation of the latter authors may look different from ours as they work with infinitesimal gauge transformations, but one can relate the two. Explicitly, considering a $U(n)$ gauge theory, a field ϕ transforms under the gauge $u \in U(n)$ as $\phi^u = u\phi$ and the gauge field A as in (2.1.20a). If now one considers $u = \exp(i\alpha_a T^a)$, where T^a are the generators of the **LIE algebra** $\mathfrak{u}(n)$, then u stands for the full transformation and $\alpha = \alpha_a T^a$ for the infinitesimal one. Therefore, taking the expansion to first order in α , one computes the infinitesimal gauge transformations of ϕ and A to be

$$\delta_\alpha \phi = \phi^\alpha - \phi = i\alpha\phi, \quad \delta_\alpha A_\mu = A_\mu^\alpha - A_\mu = \partial_\mu \alpha + i[\alpha, A_\mu]. \quad (2.2.7)$$

Note that here, we can consider gauge transformation to be either local or global, by considering u and α to be x -dependant or not. This can be enlarged to any **LIE algebra** \mathfrak{g} of any gauge group G .

In their early works, the latter authors considered deformed transformations δ_α^* to act as in the commutative case, that is through (2.2.7). The deformation appears when one makes δ_α^* act on a product, like $\delta_\alpha^*(\phi_1 \star \phi_2)$. Equivalently, the deformation is contained in the coproduct of the gauge differential, that is in $\Delta(\delta_\alpha^*)$, which can be computed with the **DRINFEL'D twist** \mathcal{F} . One can check that the usual gauge transformation of the field strength $F_{\mu\nu}$ is recovered and that the algebra of gauge transformation indeed closes through $\delta_\alpha^* \delta_\beta^* - \delta_\beta^* \delta_\alpha^* = \delta_{-i[\alpha, \beta]}^*$. Finally, one obtains that an action of the form

$$S = \int d^{d+1}x \operatorname{tr}(F^{\mu\nu} \star F_{\mu\nu}), \quad (2.2.8)$$

is gauge invariant, upon cyclicity of the integral with respect to \star (2.1.22).

The second remark concerns the notion of right or left modules. In the commutative theory, there is no ordering problem because every function commutes. When constructing a noncommutative theory, the ordering has a primordial importance, all the more that the ordering choice is not seen in the commutative limit. In a sense, the ordering is a new symmetry of the noncommutative theories as discussed in **Section 2.6**.

Within the context of braided geometry [34, 35] discussed above, the braided commutativity property (2.2.4) is reducing the importance of the ordering. For example, the braided commutativity implies that left module \mathbb{X} is also a braided right module, that is, given the left action $\triangleright_{\mathbb{X}}$, one can define a right action $\triangleleft_{\mathbb{X}}$ through³

$$f \triangleright_{\mathbb{X}} s = \sum (\mathcal{R}_1^{-1} \triangleright s) \triangleleft_{\mathbb{X}} (\mathcal{R}_2^{-1} \triangleright f). \quad (2.2.9)$$

Many other structures can be braided in a similar fashion [59], like derivations or connections. The braided derivations $\operatorname{Der}_{\mathcal{R}}(\mathbb{A})$, defined as the linear functional satisfying a braided **LEIBNIZ rule**

$$X(f \star g) = X(f) \star g + \sum (\mathcal{R}_1^{-1} \triangleright f) \star ((\mathcal{R}_2^{-1} \triangleright X)(g)), \quad (2.2.10)$$

form a \mathbb{A} -bimodule for the braided symmetric actions of (2.2.9), whereas the derivations $\operatorname{Der}(\mathbb{A})$ only form a $\mathcal{Z}(\mathbb{A})$ -module.

³One could equivalently start with a bimodule structure and require that the two actions are braided symmetric in the sense that they satisfy (2.2.9).

2.3 SEIBERG-WITTEN map

In the context of the **MOYAL** space, for deformed $U(n)$ gauge theories, **SEIBERG** and **WITTEN** [21] (see also [60]) found a correspondence between noncommutative gauge fields, noted \hat{A} in this Section 2.3, and the ordinary (commutative) one, noted A . The relation (2.3.2) found in [21] was first considered for open strings with a magnetic field, but was also applied in fully noncommutative contexts thanks to its fairly general form.

The **SEIBERG-WITTEN** map is defined by

$$\hat{A}_\mu(A_\nu) + \hat{\delta}_{\hat{\alpha}}\hat{A}_\mu(A_\nu) = \hat{A}_\mu(A_\nu + \delta_\alpha A_\nu) \quad (2.3.1)$$

where $\hat{\delta}_{\hat{\alpha}}$ is the infinitesimal noncommutative gauge transformation with parameter $\hat{\alpha}$ given by

$$\hat{\delta}_{\hat{\alpha}}\hat{A}_\mu = \partial_\mu\hat{\alpha} + i[\hat{\alpha}, \hat{A}_\mu]_\theta. \quad (2.3.2)$$

A strong hypothesis of this construction is the so-called ‘‘gauge equivalence’’. The main idea being that if A and A' are related by a gauge transformation α , *i.e.* $A' = A + \delta_\alpha A$, then \hat{A} and \hat{A}' should relate by a deformed gauge transformation $\hat{\alpha}$, *i.e.* $\hat{A}' = \hat{A} + \hat{\delta}_{\hat{\alpha}}\hat{A}$. This implies that the deformed gauge transformation depends on the undeformed one and on the gauge field. In other words, the quantity $\hat{\alpha}$ depends on α and A , namely $\hat{\alpha} = \hat{\alpha}(A, \alpha)$. In this sense, (2.3.1) is a mathematical formulation of the gauge equivalence.

The full expression of (2.3.2) is obtained by expansion in powers of θ . For instance, up to the second order, one obtains

$$\hat{\delta}_{\hat{\alpha}}\hat{A}_\mu = \partial_\mu\hat{\alpha} - \Theta^{\rho\sigma}\partial_\rho\alpha\partial_\sigma A_\mu + \mathcal{O}(\theta^2), \quad (2.3.3)$$

which, combined with (2.3.1), yields

$$\hat{A}_\mu(A_\nu) = A_\mu - \frac{1}{2}\Theta^{\rho\sigma}A_\rho(\partial_\sigma A_\mu + F_{\sigma\mu}) + \mathcal{O}(\theta^2), \quad (2.3.4)$$

where $F_{\mu\nu}$ is the ordinary field strength. Accordingly, the noncommutative field strength takes the form

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \Theta^{\rho\sigma}(F_{\mu\rho}F_{\nu\sigma} - A_\rho\partial_\sigma F_{\mu\nu}) + \mathcal{O}(\theta^2). \quad (2.3.5)$$

One would proceed similarly whenever a (fermionic) matter field is included so as to obtain a θ -expanded noncommutative field theory.

This construction suffers from two main caveats despite its explicit formulation. First, the map (2.3.1) can only be computed as an infinite power expansion over θ . This is highly sufficient when one wants to characterise the semi-classical behaviour of the theory, but is hampering a full treatment. Moreover, it was shown [61] that the matter field part of the theory was not renormalisable, at least in the standard way.

The **SEIBERG-WITTEN** map has attracted lot of attention concerning the study of quantum properties of these gauge theories, or their phenomenological traits. One can find the relevant references in [23].

2.4 New approaches

More recent approaches to gauge theories on quantum space-time have been proposed and are gathered here.

2.4.1 Gauge theories with L_∞ -algebras

The algebraic framework of L_∞ -algebras was shown to be able to render field theory dynamics and gauge theories, at least at the classical level [62]. An L_∞ -algebra is a sort of infinite extension of a **LIE algebra**, where the bracket $[\cdot, \cdot]$ for two elements has counterpart brackets for 3, 4, ... up to infinity elements. The **JACOBI** identity of the bracket $[\cdot, \cdot]$ has now also counterparts for the other brackets. Explicitly, a L_∞ -algebra is a graded vector space $V = \bigoplus_{k \in \mathbb{Z}} V_k$ with graded antisymmetric multilinear maps $\ell_n : \bigotimes_{n \in \mathbb{N}} V \rightarrow V$ called n -brackets. Thus, the 2-bracket ℓ_2 is the analogue of $[\cdot, \cdot]$. However, classical gauge theories can be recovered by only using 4 degrees, *i.e.* $V = V_0 \oplus \dots \oplus V_3$, where degree 0 fields are gauge parameters, degree 1 fields are gauge fields, degree 2 fields encode the equations of motion and degree 3 fields encode the **NOETHER** identities.

The authors of [63, 64] advocate that the L_∞ -algebra framework is natural in the context of deformation quantisation. Furthermore, the L_∞ -algebras are known to encode both noncommutative and non-associative algebras, a case that may occur when deforming the so-called “quasi-**POISSON** structures”, according to [63]. Therefore, the latter authors generalised the L_∞ -algebra gauge theory to noncommutative gauge theory in the context of deformed **CHERN-SIMONS** and **YANG-MILLS** actions.

As a brief summary, we detail how the noncommutative **YANG-MILLS** theory fits into the L_∞ -algebra setting. If one requires the general gauge transformation of a vector field $A \in V_1$, defined by

$$\delta_\alpha A = \sum_{n=0}^{+\infty} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_{n+1}(\alpha, A, \dots, A), \quad (2.4.1)$$

to correspond to the straightforward deformed gauge transformation $\delta_\alpha A = \partial\alpha + i[\alpha, A]_\star$, then one has

$$\ell_1(f) = \partial f, \quad \ell_2(f, A) = i[f, A]_\star. \quad (2.4.2)$$

The higher n -brackets are determined by the equations of motion for A . Moreover, one can check that the **JACOBI** identities are satisfied. For example, the first one states that $\epsilon_\mu^{\nu\rho} \partial_\nu \partial_\rho f = 0$, which is always true and the second one requires that the bracket $[f, \cdot]_\star$ satisfies the **LEIBNIZ** rule.

Another version of L_∞ -algebras encoding **noncommutative geometry** was constructed in [65] and presented below. Given a classical space-time \mathcal{M} , one considers the universal enveloping algebra of vector fields $\Xi = U(\Gamma(\mathcal{M}))$. The classical L_∞ -algebra associated to \mathcal{M} corresponds to a \mathbb{Z} -graded Ξ -module for which the n -bracket ℓ_n commutes with the action of Ξ . Since the elements of Ξ are polynomials of derivatives, it simply means that the derivatives act straightforwardly on the n -bracket, as the generalised analogue of the relation $\partial[f, g] = [\partial f, g] + [f, \partial g]$.

The noncommutative version of the latter classical picture is obtained via a **DRINFEL'D twist** deformation, similarly as described in **Section 2.2**. The noncommutative L_∞ -algebra is defined to be a $\Xi^{\mathcal{F}}$ -module, with $\mathcal{F} \in \Xi \otimes \Xi$ the **DRINFEL'D twist**. The deformed n -bracket $\ell_n^{\mathcal{F}}$ are defined as the twisted versions of ℓ_n , that is

$$\ell_n^{\mathcal{F}}(f_1, \dots, f_n) = \ell_n(f_1 \otimes_{\mathcal{F}} \dots \otimes_{\mathcal{F}} f_n) \quad (2.4.3)$$

with $\ell_1^{\mathcal{F}} = \ell_1$, where $\otimes_{\mathcal{F}}$ is defined in (2.2.2). One can show that the deformed n -bracket is braided graded antisymmetric, *i.e.*

$$\ell_n^{\mathcal{F}}(f_1, \dots, f_j, \dots, f_k, \dots, f_n) = (-1)^{|f_j||f_k|} \ell_n^{\mathcal{F}}(f_1, \dots, \mathcal{R}_1^{-1} \triangleright f_k, \dots, \mathcal{R}_2^{-1} \triangleright f_j, \dots, f_n) \quad (2.4.4)$$

where $|f_j|$ is the degree of f_j , and $\mathcal{R} = \mathcal{F}_{21} \mathcal{F}^{-1}$ the \mathcal{R} -matrix associated to \mathcal{F} . The gauge theory is then defined by twisting the usual (undeformed) gauge theory on L_∞ -algebra. For example, the deformed gauge transformation of A (2.4.1) becomes

$$\delta_\alpha^* A = \sum_{n=0}^{+\infty} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_{n+1}^{\mathcal{F}}(\alpha, A, \dots, A). \quad (2.4.5)$$

The deformed action, defined as the twisted classical action, can be shown to be gauge invariant under (2.4.5).

For the case of the **MOYAL** space, deformed with the twist (2.5.4), the study of the deformed $U(1)$ gauge theory is undertaken with L_∞ -algebra [65]. The quantisation method uses the **BATALIN–VILKOVISKY** formalism, which had previously been formulated in the L_∞ -algebra setting. The photon propagator is unchanged compared to the undeformed case and there are no three-photon or four-photon interactions, contrary to an action of the form (2.1.21). However, the fermion-photon vertex is non-trivially deformed. Therefore, the first deformed diagram would be the photon self-energy one. It appears that this diagram is UV-divergent, triggering a **UV/IR mixing** in the theory. We refer to **Section 2.6** for more details on noncommutative field theory and how the **UV/IR mixing** arises. Contrary to (2.6.20), the 2-point function, at one loop, has no non-planar diagrams and still triggers a mixing. The authors of [65] advocate that this **UV/IR mixing** comes from an unadapted quantisation procedure for braided theories.

2.4.2 POISSON gauge theories

The **POISSON** gauge theories correspond to noncommutative gauge theories in which the gauge transformations of the gauge field A and the field strength F are tuned so as to gauge transform classically and therefore as to obtain a gauge invariant action in the usual manner.

One starts with an *ansatz* for the gauge transformation of the gauge field A and the expression of the curvature F of the form

$$\delta_\alpha A_\mu = \gamma_\mu^\nu(A) \partial_\nu(f) + \{A_\mu, \alpha\}, \quad F_{\mu\nu} = P_{\mu\nu}{}^{\rho\sigma}(A) \partial_\rho A_\sigma + R_{\mu\nu}{}^{\rho\sigma}(A) \{A_\rho, A_\sigma\}, \quad (2.4.6)$$

where γ , P and R are to be determined. To match with the usual commutative limit, these new fields have to satisfy the following condition

$$\gamma_\mu^\nu = \delta_\mu^\nu + \mathcal{O}(\hbar), \quad P_{\mu\nu}{}^{\rho\sigma} = \delta_\mu^\rho \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\rho + \mathcal{O}(\hbar), \quad R_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2} (\delta_\mu^\rho \delta_\nu^\sigma - \delta_\mu^\sigma \delta_\nu^\rho) + \mathcal{O}(\hbar). \quad (2.4.7)$$

One imposes additional conditions corresponding to the closure of the gauge transformation and the covariance of the field strength, which writes respectively

$$[\delta_\alpha, \delta_\beta] A = \delta_{\{\alpha, \beta\}} A, \quad \delta_\alpha F = \{F, \alpha\}. \quad (2.4.8)$$

This whole set of conditions imposes constraints on γ , P , R , to which a solution, if it exists, can give rise to a gauge invariant action. Indeed, considering a **YANG-MILLS-like** action with integrand

$F^{\mu\nu} \star F_{\mu\nu}$, the gauge transformation (2.4.8) implies that the Lagrangian transforms covariantly, which leads to the gauge invariant of this action upon integral cyclicity (2.1.22).

Note that this *ansatz* was made in the semi-classical limit, *i.e.* approximating the \star -commutator of functions by the POISSON bracket $\{\cdot, \cdot\}$. This is the reason why the latter POISSON bracket appears in (2.4.8).

The fact that the expression for the noncommutative field strength considered here in (2.4.6) is different from other previous proposals, say (2.1.10), can be related to the discussion of Section 2.1. Indeed, the noncommutative formulations of gauge theories work through analogues and lack of physical intuition when defining the “physical” fields. Therefore, there is no argument for the “noncommutative electromagnetic field” (if such a thing makes sense) to be (2.1.10) rather than (2.4.6).

This framework was first developed in [66] for the MOYAL space and extended to LIE algebra-type noncommutativity in [67]. A LIE algebra-type noncommutativity corresponds to a quantum space-time in which the bracket coordinates is linear, that is

$$[x^\mu, x^\nu]_\star = C^{\mu\nu}{}_\rho x^\rho \quad (2.4.9)$$

with C a constant. As examples, \mathbb{R}_λ^3 (1.2.13), κ -MINKOWSKI (3.1.3a) and ϱ -MINKOWSKI (3.4.2) are of this type.

Furthermore, it was shown [68] that the POISSON gauge theory can be cast into a L_∞ -algebra formalism. In the slowly varying field approximation, the starred bracket $[\cdot, \cdot]_\star$, corresponding to ℓ_2 as in (2.4.2), is replaced by the POISSON bracket $\{\cdot, \cdot\}$. The higher degree n -brackets are determined by the so-called L_∞ -bootstrap, using the generalised JACOBI identities.

2.5 The example of the MOYAL space

In this Section 2.5, the MOYAL space is briefly introduced together with its noncommutative gauge theory. This gauge theory relies on methods introduced in Section 2.1 and Section 2.2. The UV/IR mixing popping out of the photon propagator correction (2-point function), at one-loop, is discussed. Some other gauge theories, mainly developed to cure the mixing, are slightly discussed. We refer to [23] for more details and an extensive list of references.

The MOYAL space, generically denoted \mathbb{R}_θ^4 , corresponds to a deformation quantisation of the (symplectic) space \mathbb{R}^4 . One can deform in a similar fashion the space \mathbb{R}^{2n} for any $n \in \mathbb{N}_1$, but we stick to 4 dimensions here. Its coordinates satisfy

$$[x^\mu, x^\nu]_{\star_\theta} = i\Theta^{\mu\nu}, \quad \text{with } \Theta = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad (2.5.1)$$

where θ is the deformation parameter and \star_θ is the star product of \mathbb{R}_θ^4 .

The expression of the star product \star_θ can be obtained either through convolution algebra techniques (see Subsection 1.6.1), or by DRINFEL'D twist deformation (see Subsection 1.6.2).

In the first case, one considers the **LIE algebra** \mathfrak{g} to be the **HEISENBERG** algebra (that is the algebra of (2.5.1)) in 5 dimensions (or $2n + 1$ in the general case). This has to do with the fact that one needs to add 1 as a generator in order for the **LIE** bracket (2.5.1) to close. The **HAAR** measure of the **HEISENBERG** group is the usual **LEBESGUE** measure. One still needs to go down from 5 dimensions to 4 and this is done via the map

$$f^\#(x_1, \dots, x_4) = \int_{\mathbb{R}} dz f(z, x_1, \dots, x_4) e^{-2\pi i \theta z} \quad (2.5.2)$$

The expression of the **FOURIER** transform is then given by the usual expression of the **FOURIER** transform on \mathbb{R}^4 , and one computes from expression (1.6.13) that

$$(f \star_\theta g)(x) = \frac{1}{(\pi\theta)^4} \int_{\mathbb{R}^4 \times \mathbb{R}^4} d^4 y d^4 z f(x + y) g(x + z) e^{-2i y^\mu \Theta_{\mu\nu}^{-1} z^\nu}. \quad (2.5.3)$$

The second case relies on the **DRINFEL'D twist**

$$\mathcal{F} = \exp\left(-\frac{i}{2} \Theta^{\mu\nu} \partial_\mu \otimes \partial_\nu\right) \quad (2.5.4)$$

which corresponds to an Abelian **DRINFEL'D twist** (see **Example 1.6.1**), for which the considered vector fields correspond to the derivative $\{\partial_\mu = \frac{\partial}{\partial x^\mu}\}_\mu$. The corresponding star product expression is

$$(f \star_\theta g)(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \Theta^{\mu_1 \nu_1} \dots \Theta^{\mu_n \nu_n} \partial_{\mu_1} \dots \partial_{\mu_n} f(x) \partial_{\nu_1} \dots \partial_{\nu_n} g(x). \quad (2.5.5)$$

The expression (2.5.5) can be made equal to (2.5.3) by writing the (infinite) **TAYLOR** expansion of f and g in (2.5.3). One can also check that both previous expression of \star_θ satisfies (2.5.1). The corresponding involution \dagger is the complex conjugation.

Note that the integral of \mathbb{R}^4 defines an integral over \mathbb{R}_θ^4 and satisfies

$$\int_{\mathbb{R}^4} d^4 x (f \star_\theta g)(x) = \int_{\mathbb{R}^4} d^4 x (g \star_\theta f)(x) = \int_{\mathbb{R}^4} d^4 x f(x) g(x). \quad (2.5.6)$$

The first equality states that the integral is cyclic and the second one that it is closed.

The deformation of the space \mathbb{R}^4 induces deformation in the symmetries of the space [69, 57, 70]. These deformations are more easily obtained in the **DRINFEL'D twist** formalism (2.5.4). One interprets the derivations ∂_μ as the action of the usual translation P_μ on functions, so that (2.5.4) is actually a **DRINFEL'D twist** for the **POINCARÉ algebra** \mathfrak{P}^4 of \mathbb{R}^4 . The twist thus deforms $U(\mathfrak{P}^4)$ into $\mathfrak{P}_\theta^4 = U(\mathfrak{P}^4)^\mathcal{F}$, the latter being the new symmetries of the **MOYAL** space.

The symmetries of the field theory one builds on the **MOYAL** space are therefore deformed. It induces phenomenological constraints on the theory as discussed in **Section 4.2**. Through **POINCARÉ** deformation, the field theory may exhibit a **LORENTZ** invariance violation.

In order to build the differential calculus on the **MOYAL** space, one can consider the Abelian **LIE algebra** \mathfrak{D}_1 of derivations generated by the ∂_μ 's. It is a **LIE** subalgebra of the full set of derivations, $\mathfrak{D}_1 \subset \text{Der}(\mathbb{R}_\theta^4)$, so that one can trade it for $\text{Der}(\mathbb{R}_\theta^4)$ in the construction of **Section 2.1**. Therefore, the gauge invariant action of noncommutative **YANG-MILLS** theory (2.1.21) can be built through the connection and its curvature defined on \mathfrak{D}_1 .

Note that if one were to consider the full set of derivations $\text{Der}(\mathbb{R}_\theta^4)$, then there would be an infinite number of components in A , instead of 4, thus leading to an infinite number of degrees of freedom in the theory. Therefore, one usually chooses to work with a restricted amount of derivations to avoid this inconvenience. One should remark, however, that both cases lead to the same commutative limit with *a priori* 4 degrees of freedom, since in this case any (real) derivation $X \in \text{Der}(\mathbb{R}^4)$ can be written as a linear combination of the ∂_μ 's, *i.e.* $X = X^\mu \partial_\mu$, with $X^\mu \in \mathbb{R}$.

The simplest action for a noncommutative YANG-MILLS theory corresponds to (2.1.21). Even in the deformed $U(1)$ case, this action has 3-vertex and 4-vertex interactions for A . The vertices are quite similar to the commutative (non-Abelian) one with extra phase factors of the form $\sin\left(\frac{p \otimes q}{2}\right)$ with external momenta p and q . Whenever one uses a BECCHI-ROUET-STORA-TYUTIN quantisation procedure on this action, one obtains a vanishing 1-point function (contrary to κ -MINKOWSKI, see Section 3.2) and a vacuum polarisation tensor (2-point function) which, at one-loop, triggers a UV/IR mixing due to the appearance of a new infra-red singularity.

The latter behaviour, described in Section 2.6, distinguishes some diagrams, called planar, that diverge at large momenta (UV) and some others, called non-planar, that are singular at vanishing momenta (IR). In the mainstream meaning of renormalisability, a theory with a UV/IR mixing is not perturbatively renormalisable due to the presence of quadratic and linear IR divergences. Indeed, this IR singularity induces an uncontrolled UV divergence for higher loop orders for which one cannot predict the counter term expression. Note that in this case, the IR limit expression of the polarisation tensor is gauge invariant, so that the UV/IR mixing of this gauge theory is not a gauge artefact.

Some methods have succeeded to get rid of the UV/IR mixing and have shown to be fully renormalisable to all orders, as discussed in Section 2.6. However, these methods are bound to the MOYAL space and involves a modification of the action (2.1.21).

Another heavily studied gauge theory on the MOYAL space is the so-called induced gauge theory [71]. The term ‘‘induced’’ stems from the fact that the study of counter-term suggested to work with powers of the covariant coordinate \mathcal{A} instead of A .

The gauge invariant connection

In the case of the MOYAL space, the derivations ∂_μ are inner derivations, meaning that it exists $\xi_\mu \in \mathbb{R}_\theta^4$ such that

$$\partial_\mu f = [\xi_\mu, f]_{\star_\theta} \quad (2.5.7)$$

for any $f \in \mathbb{R}_\theta^4$. One can even explicitly compute $\xi_\mu = -i\Theta_{\mu\nu}^{-1}x^\nu$. In this case, one can build a so-called gauge invariant connection ∇^{inv} on \mathbb{R}_θ^4 . Indeed, if one considers the expression (2.1.9) where the gauge field A is replaced by ξ above, then one has

$$\nabla_\mu^{\text{inv}}(f) = \partial_\mu(f) - i\Theta_{\mu\nu}^{-1}x^\nu \star_\theta f = i f \star_\theta \Theta_{\mu\nu}^{-1}x^\nu, \quad (2.5.8)$$

where we used (2.5.7). Given a connection ∇ on \mathbb{R}_θ^4 , it follows that $i(\nabla - \nabla^{\text{inv}})$ is a 1-form tensor given by $\mathcal{A} = A + \xi$, which gauge transforms as $\mathcal{A}^u = u^\dagger \star_\theta \mathcal{A} \star_\theta u$, for any $u \in \mathcal{U}(1)$. Finally, one computes from (2.1.10), that the field strength writes

$$F_{\mu\nu} = [\mathcal{A}_\mu, \mathcal{A}_\nu]_{\star_\theta} - i\Theta_{\mu\nu}^{-1}. \quad (2.5.9)$$

One can note that the curvature associated to ∇^{inv} is $-i\Theta_{\mu\nu}^{-1}$, which corresponds to the last term of (2.5.9).

$\mathcal{A}_\mu = A_\mu - i\Theta_{\mu\nu}^{-1}x^\nu$ is sometimes called the covariant coordinate. It can be straightforwardly generalised to the case of $\mathcal{U}(n)$ gauge theory.

In view of the gauge transformation of \mathcal{A} , a term of the form $\int \mathcal{A}_\mu \star_\theta \mathcal{A}^\mu$ is gauge invariant, so that a general gauge invariant action writes

$$S(\mathcal{A}) = \int d^4x \left(\frac{1}{4} F_{\mu\nu} \star_\theta F^{\mu\nu} + \frac{\Omega^2}{4} \{ \mathcal{A}_\mu, \mathcal{A}_\nu \}_\theta^2 + \kappa \mathcal{A}_\mu \star_\theta \mathcal{A}^\mu \right). \quad (2.5.10)$$

In the previous expression, F is defined in terms of \mathcal{A} through (2.5.9), while Ω and κ are constants. The latter action actually corresponds to a type IIB (IKKT) matrix model [72]. It was also shown to have a **spectral triple** formulation [73].

This theory exhibits a very complex vacuum configuration [74], which heavily depends on Ω and κ . For specific values of the latter constants, tedious expressions were found. This consists in the main obstacle in studying the renormalisability of such models. Note also that the previous model *a priori* suffers from a non-vanishing tadpole (1-point function).

Note that the previous computations are done in the matrix basis of **MOYAL**, which consists of a convenient tool for computation.

MOYAL matrix basis

The matrix basis [75] $\{f_{mn}\}_{m,n \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{S}(\mathbb{R}^4) \subset \mathbb{R}_\theta^4$, the set of SCHWARZ function of \mathbb{R}^4 , *i.e.*

$$f_{mn} \star_\theta f_{kl} = \delta_{nk} f_{ml}, \quad f_{mn}^\dagger = f_{nm}, \quad (2.5.11a)$$

$$\int d^4x \left(f_{mn}^\dagger \star_\theta f_{kl} \right)(x) = 2\pi \theta \delta_{mk} \delta_{nl}. \quad (2.5.11b)$$

Thus any elements $g, h \in \mathbb{R}_\theta^4$ can be decomposed as

$$g = \sum_{m,n=0}^{+\infty} g_{mn} f_{mn}, \quad h = \sum_{m,n=0}^{+\infty} h_{mn} f_{mn}, \quad (2.5.12)$$

where $g_{mn}, h_{mn} \in \mathbb{C}$. It is called the “matrix” basis since the decomposition (2.5.12) implies that g is fully determined by the infinite size matrix $\{g_{mn}\}_{mn}$. Furthermore, the product of functions looks similar to a matrix product as one can compute

$$(g \star_\theta h)(x) = \sum_{m,n=0}^{+\infty} \left(\sum_{k=0}^{+\infty} g_{mk} h_{kn} \right) f_{mn}(x), \quad \text{i.e. } (g \star_\theta h)_{mn} = \sum_{k=0}^{+\infty} g_{mk} h_{kn}. \quad (2.5.13)$$

2.6 LIE algebra-type noncommutative ϕ^4 -theory

In this section, we consider \mathbb{A} to be a quantum space-time corresponding to the deformation of \mathcal{M} a $d+1$ -dimensional space-time with star product \star and involution \dagger . The (local) coordinate functions of \mathcal{M} , $\{x^\mu\}_\mu$ are assumed to satisfy

$$[x^\mu, x^\nu]_\star = C^{\mu\nu}{}_\rho x^\rho \quad (2.6.1)$$

with $C^{\mu\nu}_\rho \in \mathbb{C}$. The relation (2.6.1) is sometimes called a **LIE algebra**-type noncommutativity. As examples, \mathbb{R}_λ^3 (1.2.13), κ -MINKOWSKI (3.1.3a) and ϱ -MINKOWSKI (3.4.2) are of this type. One can also write the **MOYAL** space (2.5.1) under this form, as detailed in Subsection 2.6.3.

We want to study a ϕ^4 -theory on such a quantum space-time \mathbb{A} of the form

$$S(\phi) = \int d^{d+1}x K[\phi](x) + \frac{g^2}{4!}(\phi \star \phi \star \phi \star \phi)(x) \quad (2.6.2)$$

where g is the coupling constant and K is a kinetic term (differential operator). Note that the integral in (2.6.2) is possibly inherited from the integral over \mathcal{M} . A usual expression for K , in the context of noncommutative ϕ^4 -theory, is

$$K[\phi] = \mathbf{g}^{\mu\nu} \partial_\mu \phi \star \partial_\nu \phi + m^2 \phi \star \phi \quad (2.6.3)$$

where m is a mass term and \mathbf{g} is the metric of \mathcal{M} .

We want here to capture the simplest quantum properties of the action (2.6.2) by computing the 2-point function at one-loop thanks to the generating functional of the connected GREEN functions. The latter study was performed in [76].

2.6.1 Deformed momentum space

Similarly to most textbook detailing the computation of a quantum ϕ^4 -theory, one handles more easily the action (2.6.2) by first going to momentum space. The usual way of describing a momentum in noncommutative field theories is by considering wave packets, that are functions of the form $e^{ip_\mu x^\mu}$ where p is the momentum. When one multiplies two wave packets in the commutative theory, the momenta add up through

$$e^{ip_\mu x^\mu} \cdot e^{iq_\mu x^\mu} = e^{i(p_\mu + q_\mu)x^\mu}. \quad (2.6.4)$$

However, in the noncommutative case, this no longer holds in general. The main reason to assume **LIE algebra**-type noncommutativity (2.6.1) is that a similar equation to (2.6.4) can be written thanks to the **BAKER-CAMPBELL-HAUSSDORFF** formula. In other words, one has

$$e^{ip_\mu x^\mu} \star e^{iq_\mu x^\mu} = e^{i(p_\mu \boxplus q_\mu)x^\mu}. \quad (2.6.5)$$

where \boxplus is a deformed version of the addition of momenta⁴ (see Subsection 2.6.3 for examples). Its explicit expression, given by the **BAKER-CAMPBELL-HAUSSDORFF** formula, is in general very cumbersome. Therefore, we consider a general \boxplus law in the following. Note, however, that the commutative limit of \boxplus is straightforwardly $+$ since \star goes to \cdot and so (2.6.5) boils down to (2.6.4). From (2.6.5), one can show that \boxplus is associative, because \star is, and that it is commutative if and only if \star is. In our case, we consider that \boxplus is noncommutative, despite its additive notation.

The involution \dagger allows one to define the deformed inverse of momenta \boxminus through $(e^{ip_\mu x^\mu})^\dagger = e^{i(\boxminus p_\mu)x^\mu}$. One then can check that the usual group rules stand

$$p_\mu \boxplus (\boxminus p_\mu) = 0, \quad \boxminus(\boxminus p_\mu) = p_\mu, \quad \boxminus(p_\mu \boxplus q_\mu) = (\boxminus q_\mu) \boxplus (\boxminus p_\mu). \quad (2.6.6)$$

⁴The latter is sometimes noted $+\star$, \oplus or \oplus_\star in the literature.

The wave packet is not the only way to define a particle. One could equivalently go through particle state definition by using the deformed symmetries, *i.e.* deformed POINCARÉ. Yet, in this formalism, the multi-particle state seems rather complex to define, specifically due the deformed + law for momenta [77]. Note that a definition of a multi-particle state in (light-like⁵) κ -MINKOWSKI, based on wave packet study, was proposed [78]. However, the deformed + law is unbraided in this precise context.

It is of major importance to note that the wave packet definition suffers from an ordering ambiguity.

The ordering ambiguity of noncommutative wave packets

In noncommutative field theories, wave packet expressions of the form $e^{ip_\mu x^\mu}$ may be written in many distinct ways. A non-exhaustive list of possibilities could be

$$e^{ip_\mu x^\mu} = e^{ip_0 x^0} \dots e^{ip_d x^d}, \quad e^{ip_\mu x^\mu} = e^{i(p_0 x^0 + \dots + p_d x^d)}, \quad e^{ip_\mu x^\mu} = e^{ip_d x^d} \dots e^{ip_0 x^0}. \quad (2.6.7)$$

All the different expressions (mentioned above or not) have the same commutative limit $e^{ip_\mu x^\mu}$. The major issue at stake here is that the \boxplus law expression depends on the chosen ordering so that the ordering prescription affects the rest of the computation. This would mean either that the theory *needs* a specific “physical” ordering to match observations, or that the physical observables need to be invariant under the ordering choice. However, it is not clear that “physical observables” computed from (2.6.2) are not changed when considering a different ordering.

The latter assumption of invariance has however a nice interpretation in view of the κ -MINKOWSKI case. Indeed, when studying the momentum space of κ -MINKOWSKI, it has been shown [79] that all orderings of (2.6.7) are linked by a coordinate transformation of the momenta. In this view, requiring that the theory is ordering independent would mean that one imposes general covariance on the (curved) momentum space. To summarise, the ordering ambiguity (2.6.7) only arises at the noncommutative level and therefore could stand as a new physical symmetry of the model.

The latter question is still open in the noncommutative literature and will not affect much the computation, since it is performed for a generic \boxplus law.

In order to write (2.6.2) in momentum space, one first needs to define the (deformed) FOURIER transform of ϕ , and its inverse. The latter requires to know how to integrate (or sum) over momenta, which is no more straightforward. Through the wave packet formalism, the momentum space is defined by exponentiating the LIE algebra of coordinates (2.6.1), and in this sense forms a LIE group. For the sake of computability, we assume that this LIE group is locally compact. This allows us to define a HAAR measure on the LIE group. See [44] for more details on harmonic analysis and group theory.

⁵The light-like κ -MINKOWSKI corresponds to a deformation of the MINKOWSKI space-time satisfying (3.1.5) with a^μ being light-like, *i.e.* $a^\mu a_\mu = 0$.

Integrations on locally compact groups

We consider here our (noncommutative) group of momenta, which is supposed to be locally compact. Then, there exists a unique left-invariant (resp. right-invariant) measure called the left (resp. right) HAAR measure, denoted $d\mu$ (resp. $d\mu_R$), *i.e.* it satisfies

$$d\mu(p \boxplus q) = d\mu(q), \quad (\text{resp. } d\mu_R(p \boxplus q) = d\mu_R(p)) \quad (2.6.8)$$

for any momenta p and q . If one further defines $d\mu_p(q) = d\mu(p \boxplus q)$, then it is itself left-invariant because

$$d\mu_p(q^1 \boxplus q^2) = d\mu(q^1 \boxplus q^2 \boxplus p) = d\mu(q^2 \boxplus p) = d\mu_p(q^2).$$

By uniqueness, $d\mu$ and $d\mu_p$ must be proportional to each other, so that one can write $d\mu_p = \Delta(p) d\mu$, where the map Δ is called the modular function of the group. One can show that Δ is a continuous group homomorphism, that is that

$$\Delta(p \boxplus q) = \Delta(p)\Delta(q), \quad \Delta(\boxminus p) = \Delta(p)^{-1}, \quad \Delta(0) = 1 \quad (2.6.9)$$

with $\Delta(p) \in \mathbb{R}_0^+$.

The modular function actually quantifies the difference between the left and the right HAAR measure. Indeed, if one considers $d\mu_{\boxminus}(p) = d\mu(\boxminus p) = \Delta(\boxminus p)d\mu(p)$ then it is right invariant through

$$d\mu_{\boxminus}(p \boxplus q) = d\mu(\boxminus(p \boxplus q)) = d\mu((\boxminus q) \boxplus (\boxminus p)) = d\mu(\boxminus p) = d\mu_{\boxminus}(p).$$

Therefore, up to an irrelevant positive constant, $d\mu_R(p) = \Delta(\boxminus p)d\mu(p)$. The case of a unimodular group corresponds to $\Delta = 1$, thus implying that the left and right HAAR measure are equal.

In the computation below, we consider the left HAAR measure, but one could equivalently consider the right HAAR measure as they are related by the modular function.

In the context of group integration, we require the analogue of the DIRAC delta function δ to be defined as

$$\int d\mu(p) f(p) \delta(p \boxplus q) = f(\boxminus q), \quad (2.6.10a)$$

$$\int d\mu(p) f(p) \delta(q \boxplus p) = \Delta(\boxminus q) f(\boxminus q) \quad (2.6.10b)$$

for any function f . Note that this definition (2.6.10) should be slightly changed in the case of the right HAAR measure. The latter definition is purely algebraic in flavour since it solves a possible mismatch when computing $\int d\mu(p) f(p) \delta(p \boxplus q)$ by integrating the δ directly, or by performing the change of variable $p \rightarrow p \boxminus q$ first. From the definition (2.6.10), one can show the following rules hold

$$\delta(p \boxplus q) = \Delta(\boxminus q) \delta(q \boxplus p), \quad (2.6.11a)$$

$$\delta(\boxminus p) = \delta(p), \quad (2.6.11b)$$

$$\int d\mu(p) f(p) \delta(p \boxminus q) = \int d\mu(p) f(p) \delta(q \boxminus p) = f(q). \quad (2.6.11c)$$

While (2.6.11b) and (2.6.11c) are expected noncommutative versions of the usual DIRAC delta function properties, the deformed cyclicity (2.6.11a) is remarkable and can actually be linked with the cyclicity of the integral through

$$\begin{aligned} \int d^{d+1}x (f \star g)(x) &= \int d\mu(p)d\mu(q) f(p) g(q) \delta(p \boxplus q) \\ &= \int d\mu(p)d\mu(q) \Delta(\boxminus q) f(p) g(q) \delta(q \boxplus p) \\ &= \int d^{d+1}x (\Delta(g) \star f)(x) \end{aligned} \quad (2.6.12)$$

where Δ , in the last line, denotes the FOURIER transform operator of the (inverse) modular function. Therefore, the (non-)cyclicity property of the integral over \mathbb{A} is governed by the modular function of the momentum space. In the case of a unimodular space, like the MOYAL space, the integral is necessarily cyclic, while in a non-unimodular case, like κ -MINKOWSKI, the integral is not cyclic. In the latter case, the loss of cyclicity can be shown to correspond to the κ -POINCARÉ generator \mathcal{E} , which indeed corresponds to the FOURIER transform operator of the (inverse) modular function.

Note that the previous analysis was made regardless of any star product explicit expression. The importance of the cyclicity of the integral in the construction of noncommutative gauge theory was made clear in (2.1.23). This is why the loss of cyclicity in some noncommutative theories has been seen as the main obstacle to the construction of a gauge theory. From (2.6.12), one can check if the integral is cyclic by deriving the modular function of the momentum space, for any (LIE algebra-type) quantum space-time. Two main paths have been followed to construct gauge invariant actions without the cyclicity of the integral, either by restoring the cyclicity through a non-trivial measure on the position space, or by trying to deform the gauge transformation of the field strength F . This will be discussed in more details in Section 3.2.

With the previous tools in hand, we define the FOURIER transform and its inverse⁶ as

$$\phi(p) = \int d^{d+1}x e^{i(\boxminus p)_\mu x^\mu} \star \phi(x), \quad \phi(x) = \frac{1}{(2\pi)^{d+1}} \int d\mu(p) \phi(p) e^{ip_\mu x^\mu}. \quad (2.6.13)$$

The fact that the field ϕ is left \star -multiplied by the exponential in (2.6.13) is linked to the choice of the left HAAR measure. A right-invariant measure choice would require a right \star -multiplication.

Using this FOURIER transform, one writes the action (2.6.2) as

$$S(\phi) = \int d\mu(k) \phi(k) K(k) \phi(\boxminus k) + \frac{g^2}{4!} \int d\mu(k^1) \cdots d\mu(k^4) \delta(k^1 \boxplus \cdots \boxplus k^4) \phi(k^1) \cdots \phi(k^4) \quad (2.6.14)$$

where K is the FOURIER transform operator of the kinetic operator K in (2.6.2). For example, the kinetic term (2.6.3) has a FOURIER transform operator given by

$$K(k) = g^{\mu\nu} k_\mu (\boxminus k)_\nu + m^2 \quad (2.6.15)$$

In order for K to only depend on k , the kinetic term needs to be translation invariant, a property which is assumed in the following. The expression (2.6.3) generically is translation invariant. Moreover, we require that the kinetic operator satisfies

$$K(\boxminus k) = K(k) \quad (2.6.16)$$

which is satisfied by (2.6.15).

⁶The FOURIER transform of any functions f is also denoted f in accordance with the usual quantum field theory notations.

2.6.2 UV/IR mixing

In order to quantise (2.6.14), we consider the generating functional of the connected GREEN functions

$$Z(J) = \int d\phi \exp \left(-S(\phi) + \frac{1}{2} \int d^{d+1}x (J \star \phi)(x) + \frac{1}{2} \int d^{d+1}x (\phi \star J)(x) \right) \quad (2.6.17)$$

where J is a source term. The symmetrized⁷ star product $\frac{1}{2}(J \star \phi + \phi \star J)$ is here to ensure that, when considering the change of variable $\phi \rightarrow \phi + K^{-1}J$, the usual simplification

$$\begin{aligned} & -\frac{1}{2} \int d\mu(k) \phi(k)K(k)\phi(\boxplus k) + \frac{1}{2} \int d^{d+1}x (J \star \phi)(x) + (\phi \star J)(x) \\ &= -\frac{1}{2} \int d\mu(k) \phi(k)K(k)\phi(\boxplus k) - \frac{1}{2} \int d\mu(k) J(k)K^{-1}(k)J(\boxplus k) \end{aligned}$$

occurs, where (2.6.16) has been used.

Then, the perturbative expansion of the generating function writes

$$\begin{aligned} Z(J) &= Z(0) \exp \left(-S_{\text{int}} \left(\frac{\partial}{\partial J} \right) \right) \exp \left(-\frac{1}{2} \int d\mu(k) J(k)K^{-1}(k)J(\boxplus k) \right) \\ &= Z(0) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} S_{\text{int}}^n \left(\frac{\partial}{\partial J} \right) \exp \left(-\frac{1}{2} \int d\mu(k) J(k)K^{-1}(k)J(\boxplus k) \right). \end{aligned} \quad (2.6.18)$$

The 2-point function at one-loop order corresponds to

$$\langle \phi(p)\phi(q) \rangle_{1\text{-loop}} = - \frac{\partial}{\partial J(p)} \frac{\partial}{\partial J(q)} S_{\text{int}} \left(\frac{\partial}{\partial J} \right) \exp \left(-\frac{1}{2} \int d\mu(k) J(k)K^{-1}(k)J(\boxplus k) \right) \Big|_{J=0}. \quad (2.6.19)$$

where one-loop means that only the second term ($n = 1$) in the exponential expansion of (2.6.18) is considered. Note that we have $\frac{\partial J(p)}{\partial J(q)} = \delta(p \boxplus q)$. After the calculation and the removal of the disconnected components, one derives the relevant diagrams. Eight of these diagrams are planar, as pictured in Figure 4, and four are non-planar, see Figure 5.

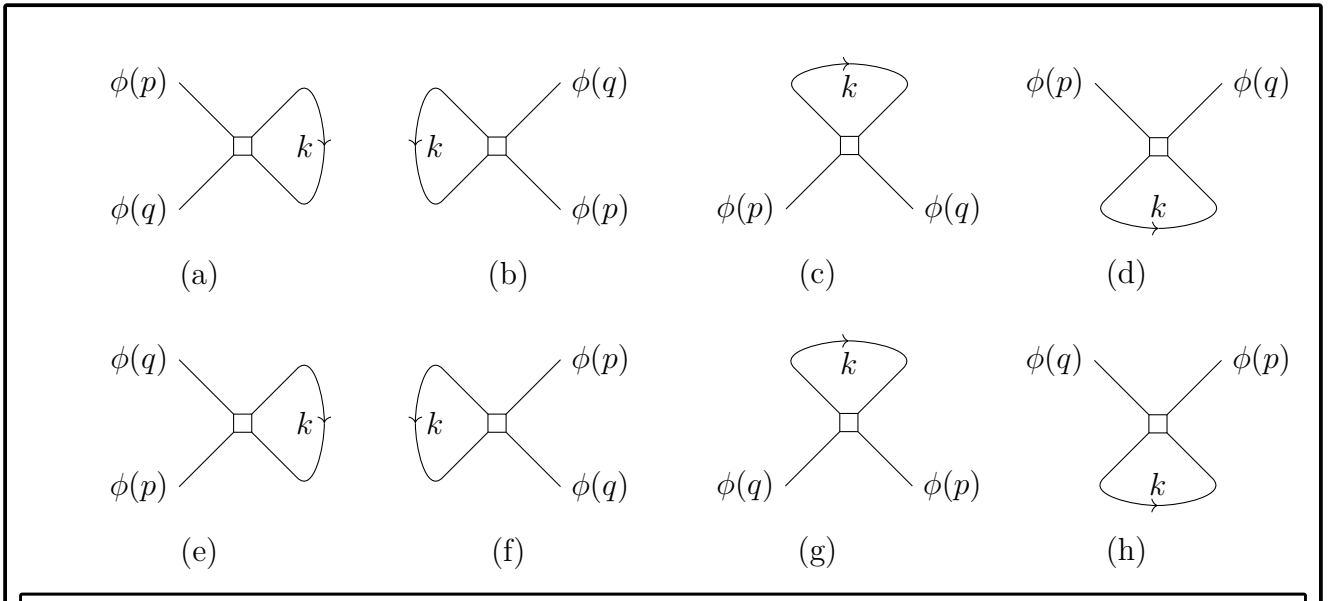


Figure 4 – Planar contributions to the 2-point function (2.6.19).

⁷In the unimodular case, the integral is cyclic, thanks to (2.6.12), and the symmetrize star product corresponds to the star product.

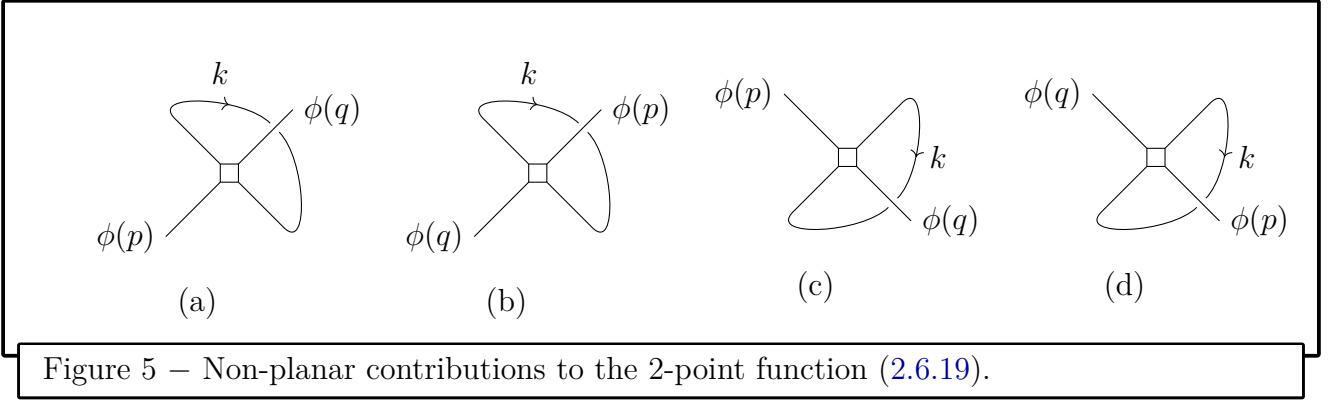


Figure 5 – Non-planar contributions to the 2-point function (2.6.19).

All the diagrams under study are not equal, but they factorise so that the 2-point function has two contributions, one planar (first term of (2.6.20)) and one non-planar (second term of (2.6.20)). Thus, it writes

$$\begin{aligned} \langle \phi(p)\phi(q) \rangle_{1\text{-loop}} &= \frac{g^2}{4!} \delta(p \boxplus q) (1 + \Delta(q)) \int d\mu(k) K^{-1}(k) (3 + \Delta(k)) \\ &+ \frac{g^2}{4!} \int d\mu(k) K^{-1}(k) \left(1 + \Delta(k)^{-1}\right) \left(1 + \Delta(q)\Delta(k)^{-2}\right) \delta(p \boxplus k \boxplus q \boxplus k). \end{aligned} \quad (2.6.20)$$

The commutative limit of this 2-point function is done by considering $\boxplus = +$, $\boxminus = -$, $\Delta = 1$ and $d\mu(k) = d^{d+1}k$ (the LEBESGUE measure). Therefore, it yields

$$\langle \phi(p)\phi(q) \rangle_{1\text{-loop}} = \frac{g^2}{2} \delta(p + q) \int d^{d+1}k K^{-1}(k), \quad (2.6.21)$$

which corresponds to the usual formula for a ϕ^4 -theory.

The planar contribution of (2.6.20), which consists of the first term, is very similar to a deformed version of the commutative 2-point function (2.6.21). The conservation of momenta, stored in the DIRAC delta function, is now deformed and the measure is the noncommutative one. However, the non-planar contribution is far less casual, and possesses a deformed conservation of momenta that involves the loop momentum k .

By studying the UV/IR mixing on the MOYAL space, as in (1.2.7), or in other quantum space-time (see references in [76]), one realises that this phenomenon, even if not well defined, always has the same structure. One of the goals of [76] was to capture this structure in an unambiguous description of the UV/IR mixing. The first criterion is given by the ultra-violet divergence of the planar diagram. The second one appears already in MOYAL and corresponds to the divergence of the non-planar contribution when the external momenta vanishes (infra-red), the latter divergence being due to the ultra-violet divergent behaviour of the planar contribution. Finally, if one only includes this two criterion then the commutative contribution (2.6.21) would satisfy the UV/IR mixing. Therefore, one needs to add a third point that requires the non-planar contribution to be ultra-violet finite for non-vanishing external momenta. In other words, [76] defines the UV/IR mixing as

- (i) The planar contribution is diverging in the UV.
- (ii) The non-planar contribution is singular in the IR, due to the UV divergence of (i).

(iii) The non-planar contribution is UV finite.

The analysis carried out in [76] showed that a relevant criterion for the latter definition is the (UV) divergence of

$$\int d\mu(k) K^{-1}(k). \quad (2.6.22)$$

The modular function (or its inverse) can be accommodated to be bounded so that (2.6.20) is mainly governed by a contribution akin to (2.6.22). In this sense, the divergence of (2.6.22) can be shown to be equivalent to both (i) and (ii).

Its link with the criterion (iii) is more tricky since it necessitates to integrate out the $\delta(p \boxplus k \boxplus q \boxminus k)$ contribution. The equation $p \boxplus k \boxplus q \boxminus k = 0$ has no trivial solution for k when $p, q \neq 0$. Besides, one has to distinguish between “commutative” components of \boxplus , that is the set of index μ such that $p_\mu \boxplus q_\mu = p_\mu + q_\mu$, and the “purely noncommutative” components, which consists of the rest, *i.e.* $k = (k_{\text{nc}}, k_c)$. By doing so, the δ s with commutative components can be taken out of the integral, and the noncommutative ones give rise to a solution $k_{\text{nc}}^* \neq 0$ such that $p_{\text{nc}} \boxplus k_{\text{nc}}^* \boxplus q_{\text{nc}} \boxminus k_{\text{nc}}^* = 0$. What remains is the integral of $K^{-1}(k_{\text{nc}}^*, k_c)$ over the commutative components k_c . Since the explicit formula for $k_{\text{nc}}^*(k_c, p, q)$ and the behaviour of the propagator K^{-1} with respect to k_c are not known, it is quite hard to quantify the would-be UV finiteness of (2.6.22).

The **UV/IR mixing** is thought to spoil the perturbative renormalisability of the theory. The renormalisation is not present in this discussion. It appears that in the case where (2.6.22) is finite, no perturbative renormalisation is needed, and the **UV/IR mixing** is not present. However, the link between the renormalisability of the theory and the badness of the divergence of (2.6.22) have not been carried out yet.

Most studies on the **UV/IR mixing** has been done on the **MOYAL** space. There were two main outcome of such studies: either the author wanted to get rid of it, or they wanted to use its properties for physical purpose.

Two main models are curing the field theory of the mixing: the **GROSSE-WULKENHAAR** model [80, 81, 82] and the **IR damping model** [83]. The two are considering the action (2.6.2) with a kinetic term of the form (2.6.3), and an extra-term. In the **GROSSE-WULKENHAAR** model, the extra-term correspond to a harmonic oscillator of the form $x^2 \phi \star_\theta \phi$. The latter term introduces an exponential decrease of the propagator in the IR, thus getting rid of the singularity. It was shown to be renormalisable to all orders, but breaks translation invariance due to the presence of x^2 in the action. The **IR damping model** has an extra term of the form $1/k^2$ in the kinetic term. This imposes a better convergence of the integral of the propagator in the IR and thus get rid of the IR singularity. This model was also shown to be renormalisable to all orders, but exhibits a propagator with unusual behaviour for small momenta. Note that other solutions were proposed in the literature [84, 85, 86].

Still, some authors [19, 20] have used the **UV/IR mixing** as a physical trait of **noncommutative geometry** and studied some insight it could bring on new physics. See **Subsection 1.2.1** for an extended introduction to these models.

2.6.3 Examples on known quantum space-times

Finally, we apply the previous results to the **MOYAL** space and the κ -**MINKOWSKI** space-time.

The **MOYAL** space⁸, described more specifically in [Section 2.5](#), has a **LIE algebra** of coordinates given by (2.5.1), which is not, strictly speaking, of **LIE algebra**-type. However, if one adds the coordinate $x^5 = 1$, one obtains that

$$[x^\mu, x^\nu]_{\star_\theta} = i\Theta^{\mu\nu}x^5, \quad [x^\mu, x^5]_{\star_\theta} = 0 \quad (2.6.23)$$

which is of **LIE algebra**-type (2.6.1) with $C^{ab}_c = i\Theta^{ab}\delta_c^5(1 - \delta_c^a)(1 - \delta_c^b)$, where $a, b, c = 0, \dots, 5$. The corresponding momentum space can be computing to be

$$p_\mu \boxplus q_\mu = p_\mu + q_\mu, \quad p_5 \boxplus q_5 = p_5 + q_5 + i p_\mu \Theta^{\mu\nu} q_\nu, \quad (2.6.24a)$$

$$\boxminus p_\mu = -p_\mu, \quad \boxminus p_5 = -p_5 \quad (2.6.24b)$$

As it is associated to $x^5 = 1$, the p_5 “momenta” actually corresponds to a pure phase term $e^{ip_5x^5} = e^{ip_5}$. This group is further unimodular, so that $\Delta = 1$ and has the **LEBESGUE** measure as its **HAAR** measure, *i.e.* $d\mu(p) = d^4p$.

For the 2-point function at one-loop, one computes then that the planar diagrams are equal to $\delta(p_\mu + q_\mu) \delta(p_5 + q_5)$ and the non-planar diagrams to $\delta(p_\mu + q_\mu) \delta(p_5 + q_5 + 2p_\mu \Theta^{\mu\nu} k_\nu)$, where p and q are external moment and k is the internal one. The 2-point function then writes

$$\langle \phi(p)\phi(q) \rangle_{1\text{-loop}} = \frac{g^2}{6} \delta(p+q) \int d^4k K^{-1}(k) \left(2 + e^{2ip_\mu \Theta^{\mu\nu} k_\nu} \right) \quad (2.6.25)$$

which exactly corresponds to the expression (1.2.7) with the kinetic operator (2.6.15). By taking the latter kinetic operator, the same analysis of the **UV/IR mixing** as in [Subsection 1.2.1](#) can be done. Therefore, (i), (ii) and (iii) are fulfilled. On the other hand, the integral of the propagator (2.6.22) for the kinetic operator (2.6.15) with the laws (2.6.24) simplifies to

$$\int d\mu(k) K^{-1}(k) = \int d^4k \frac{1}{-k^2 + m^2} \quad (2.6.26)$$

which is indeed (UV) divergent. Note that as **MOYAL** is a deformation of \mathbb{R}^4 , we took the Euclidean metric $\mathbf{g} = (+ \dots +)$ in (2.6.15).

The κ -**MINKOWSKI** space-time $\mathbb{R}_\kappa^{1,d}$ corresponds to a deformation of the **MINKOWSKI** space-time $\mathbb{R}^{1,d}$, with a **LIE algebra** of coordinates satisfying

$$[x^0, x^j]_{\star_\kappa} = \frac{i}{\kappa} x^j, \quad [x^j, x^k]_{\star_\kappa} = 0, \quad (2.6.27)$$

where $j, k = 1, \dots, d$ and κ is the deformation parameter. How this quantum space-time is constructed and how (2.6.27) arises is explained in more detail in the [Chapter 3](#). One can write (2.6.27) under the form (2.6.1) with $C^{\mu\nu}_\rho = \frac{i}{\kappa}(\delta_0^\mu \delta_\rho^\nu - \delta_0^\nu \delta_\rho^\mu)$. One computes the corresponding momentum space to be

$$p_0 \boxplus q_0 = p_0 + q_0, \quad p_j \boxplus q_j = p_j + e^{-p_0/\kappa} q_j, \quad (2.6.28a)$$

$$\boxminus p_0 = -p_0, \quad \boxminus p_j = -e^{p_0/\kappa} p_j \quad (2.6.28b)$$

This group is not unimodular with a left **HAAR** measure $d\mu(p) = e^{dp_0/\kappa} d^{d+1}p$ and a right **HAAR** measure $d\mu_R(p) = d^{d+1}p$. Therefore, the modular function is $\Delta(p) = e^{dp_0/\kappa}$.

⁸This paragraph is written for 4-dimensional **MOYAL** space, but can be straightforwardly generalised to any dimension, see [76].

The 2-point function at one-loop writes

$$\begin{aligned} \langle \phi(p)\phi(q) \rangle_{1\text{-loop}} &= \frac{g^2}{4!} \delta(p_\mu \boxplus q_\mu) \left(1 + e^{dq_0/\kappa}\right) \int d^{d+1}k K^{-1}(k) e^{dk_0/\kappa} \left(3 + e^{dk_0/\kappa}\right) \\ &+ \frac{g^2}{4!} \delta(p_0 + q_0) \int d^{d+1}k K^{-1}(k) \left(1 + e^{dk_0/\kappa}\right) \left(1 + e^{-d(p_0+2k_0)/\kappa}\right) \\ &\times \delta\left(p_j + e^{-(p_0+k_0)/\kappa} q_j - \left(1 - e^{-p_0/\kappa}\right) k_j\right). \end{aligned} \quad (2.6.29)$$

It was shown in [76], that for $d > 1$ and a kinetic term of the form (2.6.15), the 2-point function (2.6.29) is neither diverging in the UV nor IR singular. Therefore, (i) and (ii) are not fulfilled. The requirement (iii) is satisfied because the 2-point function at one-loop is finite. On the other hand, if one studies the integral of the propagator, one has

$$\int d\mu(k) K^{-1}(k) = 4\pi \left(\frac{4\pi\kappa m}{d}\right)^{\frac{d-1}{2}} \mathcal{K}_{\frac{d-1}{2}}\left(\frac{md}{2\kappa}\right) \quad (2.6.30)$$

where \mathcal{K} is a BESSEL function and m is the mass of the scalar field ϕ . Therefore, (2.6.30) is finite, even in the massless case (see [76]), which is in accordance with the fact that (i) and (ii) are not fulfilled. Note that, in (2.6.30), κ is playing the role of a UV cut-off, as expected by SNYDER (see Subsection 1.2.1).

Note that other scalar field theory on κ -MINKOWSKI suffers from a UV/IR mixing [87]. This enlightens the fact that the choice of the quantum space-time is not linked to the presence of the mixing, as this study was made on a generic space. The source of the UV/IR mixing seems rather to come from the (integral of the) propagator.

Finally, let us see the impact of the ordering ambiguity (2.6.7) on the 2-point function computation. Due to cumbersomeness, the 2-point function could not be explicitly computed for two different orderings. However, one can already compare the expressions of the integral of the propagators.

The deformed additive laws (2.6.28) were obtained for time-to-the-right ordering, corresponding to the right term of (2.6.7). Therefore, we denote by \boxplus_{right} the law (2.6.28) in the following paragraph. One could consider the ‘‘sum’’ ordering, corresponding to the middle term of (2.6.7) and find that

$$p_\mu \boxplus_{\text{sum}} q_\mu = \frac{p_0 + q_0}{e^{-q_0/\kappa} - e^{p_0/\kappa}} \left(\frac{1 - e^{p_0/\kappa}}{p_0} p_\mu + \frac{e^{-q_0/\kappa} - 1}{q_0} q_\mu \right), \quad \boxminus_{\text{sum}} p_\mu = -p_\mu. \quad (2.6.31)$$

Note that the previous expression is finite in the limit $p_0, q_0 \rightarrow 0$. The modular function is unchanged $\Delta(p) = e^{dp_0/\kappa}$, and the left HAAR measure writes $d\mu_{\text{sum}}(p) = \left(\frac{1 - e^{p_0/\kappa}}{p_0}\right)^d d^{d+1}p$. Expressions (2.6.28) and (2.6.31) can be linked through

$$p_0 \boxplus_{\text{right}} q_0 = p_0 \boxplus_{\text{sum}} q_0, \quad p_j \boxplus_{\text{right}} q_j = \frac{1}{g\left(\frac{p_0+q_0}{\kappa}\right)} \left(g\left(\frac{p_0}{\kappa}\right) p_j \boxplus_{\text{sum}} g\left(\frac{q_0}{\kappa}\right) q_j \right) \quad (2.6.32)$$

where $g(x) = \frac{x}{1 - e^{-x}}$.

Considering the kinetic term (2.6.15), with the MINKOWSKI metric $\mathbf{g} = (+ - \dots -)$, one obtains

that

$$\text{Right ordering: } \int d\mu_{\text{right}}(k) K_{\text{right}}^{-1}(k) = \int d^{d+1}k \frac{e^{dk_0/2\kappa}}{-k_0^2 + k_j^2 + m^2}, \quad (2.6.33a)$$

$$\text{Sum ordering: } \int d\mu_{\text{sum}}(k) K_{\text{sum}}^{-1}(k) = \int d^{d+1}k \frac{e^{dk_0/2\kappa}}{-k_0^2 + k_j^2 + m^2} \left(-\frac{\sinh\left(\frac{k_0}{2\kappa}\right)}{k_0} \right)^d. \quad (2.6.33b)$$

The integral (2.6.33a) was computed in (2.6.30), however, the integral (2.6.33b) could not be computed. Their distinct expressions might be a computational artefact, but suggests that these two ordering have different 2-point functions. If the 2-point function is not ordering independent and one considers that the 2-point function is an observable [88] then, either there is a single physical ordering, or the result should be ordering invariant by a specific choice of propagator. This point needs to be more carefully analysed.

Chapter 3

The κ -MINKOWSKI space-time

The (quantum) deformations of the MINKOWSKI space-time have been studied extensively for the insights they might provide on the physics of quantum space-times. Indeed, the κ -MINKOWSKI space-time consists of the first (non-trivial) deformation of a space with time. Furthermore, its space of symmetries, the κ -POINCARÉ algebra, has a flourishing phenomenology, principally linked to [quantum gravity](#) (see [Section 4.2](#)). The deformation parameter κ has a mass dimension and is therefore sometimes associated with the PLANCK mass. κ -MINKOWSKI is considered as a good candidate to shed some light on possible [quantum gravity](#) effects, at least in some regime.

We here depict some recent topics concerning field and gauge theories on deformed MINKOWSKI space-times. Some older models are evoked.

3.1 From the deformed symmetries of κ -POINCARÉ to the κ -MINKOWSKI space-time

The κ -MINKOWSKI space-time, noted $\mathbb{R}_\kappa^{1,d}$, was first defined thirty years ago by MAJID and RUEGG [89] as the space having the κ -POINCARÉ HOPF algebra $\mathfrak{P}_\kappa^{1,d}$ as its space of symmetries. In other words, κ -MINKOWSKI needed to be a $\mathfrak{P}_\kappa^{1,d}$ -module algebra, as explained in [Subsection 1.4.3](#). The latter deformation of the POINCARÉ algebra was introduced three years earlier by LUKIERSKI, NOWICKI, RUEGG and TOLSTOY [90, 91]. For an historical review of the construction of κ -MINKOWSKI and κ -POINCARÉ see [92].

Starting with the POINCARÉ algebra, one cannot perform a DRINFEL'D-JIMBO quantisation procedure, since the latter LIE algebra is not semi-simple. This quantisation consists of deforming the universal enveloping algebra of a semi-simple LIE algebra, by using its root system and the quantum deformation of $U(\mathfrak{sl}_2)$. The authors of [90] opted for a WIGNER-INONU contraction procedure, which consists in first building the DRINFEL'D-JIMBO quantum version of the DE SITTER algebra $\mathfrak{o}_{2,3}(\mathbb{R})$ and then taking the limit $q \rightarrow 0$ and $r \rightarrow +\infty$ with $ir \log(q) = \kappa^{-1}$ constant. Here q denotes the usual complex parameter in group quantisation, and r is the DE SITTER radius, that allows to go from $\mathfrak{o}_{2,3}(\mathbb{R})$ to $\mathfrak{P}^{1,3}$ when $r \rightarrow +\infty$. The computation for dimensions others than 4 was done shortly after [93].

From this computation, one obtains the κ -POINCARÉ algebra. The latter is given, in the so-called MAJID-RUEGG basis [89] by

$$[J_j, J_k] = i\epsilon_{jk}{}^l J_l, \quad [J_j, K_k] = i\epsilon_{jk}{}^l K_l, \quad [K_j, K_k] = -i\epsilon_{jk}{}^l J_l, \quad (3.1.1a)$$

$$[P_j, J_k] = -i\epsilon_{jk}{}^l P_l, \quad [P_j, \mathcal{E}] = [J_j, \mathcal{E}] = 0, \quad [P_j, P_k] = 0, \quad (3.1.1b)$$

$$[K_j, \mathcal{E}] = -\frac{i}{\kappa} P_j \mathcal{E}, \quad [P_j, K_k] = \frac{i}{2} \eta_{jk} \left(\kappa(1 - \mathcal{E}^2) + \frac{1}{\kappa} P_l P^l \right) + \frac{i}{\kappa} P_j P_k, \quad (3.1.1c)$$

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta P_j = P_j \otimes 1 + \mathcal{E} \otimes P_j, \quad (3.1.1d)$$

$$\Delta \mathcal{E} = \mathcal{E} \otimes \mathcal{E}, \quad \Delta J_j = J_j \otimes 1 + 1 \otimes J_j, \quad (3.1.1e)$$

$$\Delta K_j = K_j \otimes 1 + \mathcal{E} \otimes K_j - \frac{1}{\kappa} \epsilon_j{}^{kl} P_k \otimes J_l, \quad (3.1.1f)$$

$$\varepsilon(P_0) = \varepsilon(P_j) = \varepsilon(J_j) = \varepsilon(K_j) = 0, \quad \varepsilon(\mathcal{E}) = 1, \quad (3.1.1g)$$

$$S(P_0) = -P_0, \quad S(\mathcal{E}) = \mathcal{E}^{-1}, \quad S(P_j) = -\mathcal{E}^{-1} P_j, \quad (3.1.1h)$$

$$S(J_j) = -J_j, \quad S(K_j) = -\mathcal{E}^{-1} \left(K_j - \frac{1}{\kappa} \epsilon_j{}^{kl} P_k J_l \right). \quad (3.1.1i)$$

where $\{P_\mu\}_{\mu=0,\dots,d}$ are the generators of the deformed translations, $\{J_j\}_{j=1,\dots,d}$ the generators of the deformed rotations, $\{K_j\}_{j=1,\dots,d}$ the generators of the deformed boosts, and η is the MINKOWSKI metric. Note that we introduced $\mathcal{E} = e^{-P_0/\kappa}$ for convenience. κ is here the deformation parameter and already has a mass dimension.

The question of how to find a $\mathfrak{P}_\kappa^{1,d}$ -module algebra to form the κ -MINKOWSKI space-time is not trivial regarding the complex structure of (3.1.1). However, the authors of [89] observed that κ -POINCARÉ has a bicrossproduct structure stemming from the cross-product structure of the POINCARÉ algebra $\mathfrak{P}^{1,d} = \mathfrak{t}^{1,d} \rtimes \mathfrak{so}_{1,d}$, where $\mathfrak{t}^{1,d}$ denotes the algebra of translations and $\mathfrak{so}_{1,d}$ the algebra of rotations and boosts. Explicitly, if one defines the (HOPF) algebra of deformed translations $\mathcal{T}_\kappa^{1,d}$ as the set of $\{P_\mu\}_\mu$ and $U(\mathfrak{so}_{1,d})$ the universal enveloping algebra of the (deformed) rotations $\{J_j\}_j$ and boosts $\{K_j\}_j$, then one has

$$\mathfrak{P}_\kappa^{1,d} = \mathcal{T}_\kappa^{1,d} \blacktriangleright U(\mathfrak{so}_{1,d}). \quad (3.1.2)$$

Definitions and examples of (co)actions on (co)algebra can be found in Section A.2.

The κ -MINKOWSKI space-time is defined as the (HOPF algebra) dual of the deformed translations, $\mathbb{R}_\kappa^{1,d} = (\mathcal{T}_\kappa^{1,d})'$, and is fully determined as such. One can show, as done in the proof of Example A.1.17, that κ -MINKOWSKI is generated by $d+1$ elements $\{x^\mu\}_{\mu=0,\dots,d}$ which satisfy

$$[x^0, x^j] = \frac{i}{\kappa} x^j, \quad [x^j, x^k] = 0, \quad (3.1.3a)$$

$$\Delta(x^\mu) = x^\mu \otimes 1 + 1 \otimes x^\mu, \quad \varepsilon(x^\mu) = 0, \quad S(x^\mu) = -x^\mu. \quad (3.1.3b)$$

Thanks to this duality, one can define a (left) action of $\mathcal{T}_\kappa^{1,d}$ on $\mathbb{R}_\kappa^{1,d}$, via the action¹ (A.2.5), and dualise the coaction of $U(\mathfrak{so}_{1,d})$ on $\mathcal{T}_\kappa^{1,d}$ to an action of $U(\mathfrak{so}_{1,d})$ on $\mathbb{R}_\kappa^{1,d}$. One computes [89] that

$$(P_\mu \triangleright f)(x) = -i\partial_\mu f(x), \quad (\mathcal{E} \triangleright f)(x) = f(x^0 + \frac{i}{\kappa}, x^j), \quad (3.1.4a)$$

$$(J_j \triangleright f)(x) = \left(\epsilon_{jk}{}^l x^k P_l \triangleright f \right)(x), \quad (3.1.4b)$$

$$(K_j \triangleright f)(x) = \left(\left(\frac{1}{2} x^j \left(\kappa(1 - \mathcal{E}^2) + \frac{1}{\kappa} P_l P^l \right) + x^0 P_j - \frac{i}{\kappa} x^k P_k P_j \right) \triangleright f \right)(x). \quad (3.1.4c)$$

for any $f \in \mathbb{R}_\kappa^{1,d}$.

¹Note that $\mathbb{R}_\kappa^{1,d}$ is a (right) $\mathbb{R}_\kappa^{1,d}$ -comodule with the coaction given by the coproduct (3.1.3b).

With the action (3.1.4), κ -MINKOWSKI is a $\mathfrak{P}_\kappa^{1,d}$ -module algebra, thus allowing us to interpret κ -POINCARÉ as the (quantum) symmetries of the (quantum) deformed MINKOWSKI space-time. The connection between a quantum space-time and its symmetries has been discussed in Subsection 1.4.3. On the other hand, if one tries to determine the LIE group of momenta by exponentiating the LIE algebra of coordinates (3.1.3a), as done in Subsection 2.6.1 through the wave packet formalism, one finds that this group corresponds to the deformed translations $\mathcal{T}_\kappa^{1,d}$. The latter observation can be explained by the bicrossproduct structure (3.1.2), which give rise to a LIE algebra double dual to a LIE group double. Finally, κ -POINCARÉ gathers both the symmetries and the momentum space of our quantum space-time.

Generalised κ -deformations of MINKOWSKI were studied in [94], stemming from deformations of the POINCARÉ-WEYL algebra $\mathfrak{W}_{1,d}$, that is the POINCARÉ algebra with an extra generator of dilatation. The LIE algebra part (3.1.3a) is changed to

$$[x^\mu, x^\nu]_{\star_\kappa} = \frac{i}{\kappa}(a^\mu x^\nu - a^\nu x^\mu), \quad (3.1.5)$$

where $a^\mu \in \mathbb{R}^{1,d}$ is a constant vector. The κ -MINKOWSKI (3.1.3a) is recovered when $a^\mu = \delta_0^\mu = (1, 0, \dots, 0)$. The latter case is called “time-like” κ -MINKOWSKI since $a^\mu a_\mu = -1$ in this case, but one could also consider light-like ($a^\mu a_\mu = 0$) and space-like ($a^\mu a_\mu = +1$) deformations.

We now turn to the determination of a star product on κ -MINKOWSKI, which satisfies (3.1.3a). We present briefly here the two methods of Section 1.6.

The convolution algebra method was first carried out in [10, 95]. Starting from the LIE algebra of coordinates (3.1.3), called the affine algebra, one computes the associate LIE group to be the affine group, corresponding to (2.6.28). By computing the FOURIER transform of the convolution product and the involution on this group, one obtains

$$(f \star_\kappa g)(x) = \int \frac{dp_0}{2\pi} dy^0 e^{-iy^0 p_0} f(x^0 + y^0, x^j) g(x^0, e^{-p_0/\kappa} x^j), \quad (3.1.6a)$$

$$f^\dagger(x) = \int \frac{dp_0}{2\pi} dy^0 e^{-iy^0 p_0} \bar{f}(x^0 + y^0, e^{-p_0/\kappa} x^j) \quad (3.1.6b)$$

for any $f, g \in \mathbb{R}_\kappa^{1,d}$, where \bar{f} denotes the complex conjugation of f . The multiplier space of (3.1.6a), *i.e.* the set of (smooth) functions stable under the latter star product, was studied in [10]. It turns out that it contains at least functions whose derivatives to any order grow at most polynomially at infinity.

Proof. Here, we perform the computation of the star product (3.1.6a). The involution (3.1.6b) follows from a similar computation with

$$f^\dagger = \mathcal{F}^{-1} \left(\overline{\mathcal{F}(f)}(\boxplus \cdot) \Delta(\cdot) \right)$$

where \mathcal{F} is the FOURIER transform and Δ the modular function of the affine group (2.6.28).

The star product is computed through the formula (1.6.13). The right-invariant HAAR measure is considered on the group since it enables the construction of κ -POINCARÉ-invariant

actions. Thus, one has

$$\begin{aligned}
 (f \star_{\kappa} g)(x) &= \mathcal{F}^{-1}(\mathcal{F}(f) \hat{\circ} \mathcal{F}(g))(x) \\
 &= \int d\mu_R(p) e^{ip_{\mu}x^{\mu}} (\mathcal{F}(f) \hat{\circ} \mathcal{F}(g))(p) \\
 &= \int d^{d+1}p e^{ip_{\mu}x^{\mu}} \int d^{d+1}q \mathcal{F}(f)(p \boxplus q) \mathcal{F}(g)(q) \\
 &= \frac{1}{(2\pi)^{2d+2}} \int d^{d+1}p d^{d+1}q e^{ip_{\mu}x^{\mu}} \int d^{d+1}y d^{d+1}z e^{-i(p\boxplus q)_{\mu}y^{\mu}} e^{-iq_{\mu}z^{\mu}} f(y)g(z) \\
 &= \frac{1}{(2\pi)^{2d+2}} \int d^{d+1}p d^{d+1}q d^{d+1}y d^{d+1}z e^{ip_0(x^0-y^0)} e^{iq_0(y^0-z^0)} \\
 &\quad e^{ip_j(x^j-y^j)} e^{iq_j(e^{(q_0-p_0)/\kappa}y^j-z^j)} f(y)g(z) \\
 &= \frac{1}{(2\pi)^2} \int dp_0 dq_0 dy^0 dz^0 e^{ip_0(x^0-y^0)} e^{iq_0(y^0-z^0)} f(y^0, x^j) g(z^0, e^{(q_0-p_0)/\kappa}x^j) \\
 &\stackrel{(p_0 \rightarrow p_0+q_0)}{=} \frac{1}{(2\pi)^2} \int dp_0 dq_0 dy^0 dz^0 e^{ip_0(x^0-y^0)} e^{iq_0(x^0-z^0)} f(y^0, x^j) g(z^0, e^{-p_0/\kappa}x^j) \\
 &\stackrel{(y^0 \rightarrow x^0+y^0)}{=} \frac{1}{2\pi} \int dp_0 dy^0 e^{-ip_0y^0} f(x^0+y^0, x^j) g(x^0, e^{-p_0/\kappa}x^j). \quad \square
 \end{aligned}$$

The **DRINFEL'D twist** method for generating the star product is less straightforward here than for other quantum space-times. The first expressions were obtained in [96, 97], but shortly after a no-go theorem came out [98] stating that, in dimension 2 and 4, the κ -POINCARÉ algebra cannot be recovered by a twist deformation. In other words, there is no twist of $\mathfrak{P}^{1,3}$ that makes it possible to reconstruct the κ -MINKOWSKI space-time. The former authors actually deformed the POINCARÉ-WEYL algebra $\mathfrak{W}_{1,d}$, that is the **POINCARÉ algebra** with an extra generator corresponding to dilatations. The considered **DRINFEL'D twist** is Abelian and writes

$$\mathcal{F} = \exp\left(-\frac{i}{2\kappa}(P_0 \otimes D - D \otimes P_0)\right) = \exp\left(\frac{i}{2\kappa}(\partial_0 \otimes x^j \partial_j - x^j \partial_j \otimes \partial_0)\right), \quad (3.1.7)$$

where D is the dilatation generator, which can be shown to act on $\mathbb{R}_{\kappa}^{1,d}$ as $-ix^j \partial_j$. Some authors [99] alternatively considered the **DRINFEL'D twist** of JORDAN type

$$\mathcal{F} = \exp\left(-iD \otimes \ln\left(1 + \frac{1}{\kappa}P_0\right)\right). \quad (3.1.8)$$

Contrary to the **MOYAL** space, the three star products (3.1.6a), (3.1.7) and (3.1.8) have different expressions. It is not known if the physics of a quantum space-time described by different star products is the same or not. Studies on deformation quantisation methods have led to the notion of equivalent star products, *i.e.* two star products \star and $\tilde{\star}$ are said equivalent if there exists an invertible (formal power series of) differential operator T such that

$$f \tilde{\star} g = T^{-1}(T(f) \star T(g)). \quad (3.1.9)$$

This notion of equivalence has further been refined to the notion of MORITA equivalence, which basically compares the (\ast) -representations of two (\ast) -algebras. Therefore, two **operator algebras** which are MORITA equivalent will give rise to the same quantum states and observables. One can find a more extensive discussion and the relevant references in [38]. It is not clear that

this equivalence stands for more advanced notions, such as a n -point function of a field theory. Moreover, the different MORITA equivalence classes of κ -MINKOWSKI are not known up to date.

Note that the star product (3.1.6a) can be put under an exponential form with

$$\mathcal{F} = \exp\left(-\frac{i}{\kappa}\kappa(1 - \mathcal{E}) \otimes D\right) \quad (3.1.10)$$

but the latter expression does not satisfy the 2-cocycle condition (1.6.14a) and so is not a DRINFEL'D twist.

Proof. We factorise here the star product (3.1.6a) as the action of (3.1.10). In order to do so, we use the infinite regularity of functions on κ -MINKOWSKI to write infinite TAYLOR expansions of the form

$$f(x^0 + y^0, x^j) = \sum_{n=0}^{+\infty} \frac{(y^0)^n}{n!} \partial_0^n f(x), \quad g(x^0, e^{-p_0/\kappa} x^j) = \sum_{k=0}^{+\infty} \frac{(e^{-p_0/\kappa} - 1)^k}{k!} (x^j \partial_j)^k g(x),$$

where the decomposition $e^{-p_0/\kappa} x^j = x^j + (e^{-p_0/\kappa} - 1)x^j$ has been used for the second equality. Note that in the following computation integrals and infinite sums are swapped thanks to the convenient space in which f and g lives.

$$\begin{aligned} (f \star_{\kappa} g)(x) &= \int \frac{dp_0}{2\pi} dy^0 e^{-ip_0 y^0} f(x^0 + y^0, x^j) g(x^0, e^{-p_0/\kappa} x^j) \\ &= \int \frac{dp_0}{2\pi} dy^0 e^{-ip_0 y^0} \left(\sum_{n=0}^{+\infty} \frac{(y^0)^n}{n!} \partial_0^n f(x) \right) \left(\sum_{k=0}^{+\infty} \frac{(e^{-p_0/\kappa} - 1)^k}{k!} (x^j \partial_j)^k g(x) \right) \\ &= \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{\partial_0^n f(x)}{n!} \frac{(x^j \partial_j)^k g(x)}{k!} \int \frac{dp_0}{2\pi} dy^0 e^{-ip_0 y^0} (y^0)^n (e^{-p_0/\kappa} - 1)^k \\ &= \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{\partial_0^n f(x)}{n!} \frac{(x^j \partial_j)^k g(x)}{k!} \int \frac{dp_0}{2\pi} dy^0 e^{-ip_0 y^0} (y^0)^n \sum_{s=0}^k \binom{k}{s} e^{-sp_0/\kappa} (-1)^{k-s} \\ &= \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{\partial_0^n f(x)}{n!} \frac{(x^j \partial_j)^k g(x)}{k!} \sum_{s=0}^k \binom{k}{s} (-1)^{k-s} \int \frac{dp_0}{2\pi} dy^0 (y^0)^n e^{-ip_0(y^0 - is/\kappa)} \\ &= \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{\partial_0^n f(x)}{n!} \frac{(x^j \partial_j)^k g(x)}{k!} \sum_{s=0}^k \binom{k}{s} (-1)^{k-s} \left(\frac{is}{\kappa}\right)^n \\ &= \sum_{k=0}^{+\infty} \frac{(x^j \partial_j)^k g(x)}{k!} \sum_{s=0}^k \binom{k}{s} (-1)^{k-s} \sum_{n=0}^{+\infty} \frac{\partial_0^n f(x)}{n!} \left(\frac{is}{\kappa}\right)^n \\ &= \sum_{k=0}^{+\infty} \frac{(x^j \partial_j)^k g(x)}{k!} \sum_{s=0}^k \binom{k}{s} (-1)^{k-s} f\left(x^0 + \frac{is}{\kappa}, x^j\right) \\ &= \sum_{k=0}^{+\infty} \frac{(x^j \partial_j)^k g(x)}{k!} \sum_{s=0}^k \binom{k}{s} (-1)^{k-s} \mathcal{E}^s(f)(x) \\ &= \sum_{k=0}^{+\infty} \frac{(x^j \partial_j)^k g(x)}{k!} (\mathcal{E} - 1)^k(f)(x) \\ &= \cdot \circ \exp\left(-\frac{i}{\kappa}\kappa(1 - \mathcal{E}) \otimes D\right) \triangleright (f \otimes g) \quad \square \end{aligned}$$

If one considers the integral on MINKOWSKI space-time as an integral over κ -MINKOWSKI, then one can show that it is not cyclic. More precisely, one has

$$\int d^{d+1}x (f \star_{\kappa} g)(x) = \int d^{d+1}x \left((\mathcal{E}^d \triangleright g) \star_{\kappa} f \right)(x), \quad (3.1.11)$$

where $\mathcal{E} = e^{-P_0/\kappa}$ is a generator of κ -POINCARÉ, see (3.1.1). We say that the integral is a “twisted” trace, meaning that one of the factor gets transformed by the automorphism \mathcal{E}^d . Note that this formula could be obtained by (2.6.12), as one can compare $\mathcal{E}^d = e^{-dP_0/\kappa}$ and $\Delta(p)^{-1} = e^{-dp_0/\kappa}$. On the other hand, this integral enables the building of κ -POINCARÉ-invariant field theories since one can compute that

$$X \triangleright \int d^{d+1}x \mathcal{L}(\phi) = \varepsilon(X) \int d^{d+1}x \mathcal{L}(\phi), \quad (3.1.12)$$

for any $X \in \mathfrak{P}_{\kappa}^{1,d}$ and any Lagrangian density \mathcal{L} of any field ϕ .

The lost of cyclicity (3.1.11) was considered to be the main obstacle to construct a gauge theory on κ -MINKOWSKI, as already mentioned in Section 2.1. We now go to the construction of a gauge theory on κ -MINKOWSKI.

3.2 Gauge theory on κ -MINKOWSKI

As already reviewed in [23], several formulation of gauge theory on κ -MINKOWSKI have been considered. They are mainly based on the SEIBERG-WITTEN map and bypass the lost of cyclicity (3.1.11) by either taking a non-trivial integration measure (and so changing the notion of “integral” on κ -MINKOWSKI) or by considering a deformed version of the HODGE duality.

In this section, we mainly focus on the model developed by MATHIEU and WALLET [100, 101]. Its starting point is the observation that the cyclicity of the integral is not broken, but only twisted as shown in (3.1.11). Should the field strength F gauge transform in a “twisted” way accordingly, the integral of $F \star_{\kappa} F^{\dagger}$ would be gauge invariant. It appears that if one considers a set of natural (twisted) derivations and adapts the procedure of the derivation-based differential calculus (detailed in Section 2.1), one ends up with a gauge invariant action. However, the gauge invariance imposes a specific space-time dimension. The latter constraint stems from the fact that the twist in (3.1.11) depends on the spacial dimension d .

Let us first discuss how the straightforward noncommutative YANG-MILLS theory action (2.1.21) behaves with respect to the integral with twisted cyclicity (3.1.11). Performing the gauge transformation of the action similarly to (2.1.23), one ends up at the last step with a term $\mathcal{E}^d(u) \star_{\kappa} u^{\dagger}$ in the action. Recall that $\mathcal{E} = e^{-P_0/\kappa}$ is a generator of κ -POINCARÉ (3.1.1) appearing in (3.1.11). We have further shortened $\mathcal{E}^d \triangleright u$ to $\mathcal{E}^d(u)$. One could think of imposing that $\mathcal{E}^d(u) \star_{\kappa} u^{\dagger} = 1$ to restore gauge invariance. Therefore, the set of deformed gauge transformations $\mathcal{U}(n)$ should now satisfy

$$u^{\dagger} \star_{\kappa} u = u \star_{\kappa} u^{\dagger} = 1, \quad \mathcal{E}^{-d}(u)^{\dagger} \star_{\kappa} u = \mathcal{E}^d(u) \star_{\kappa} u^{\dagger} = 1. \quad (3.2.1)$$

for any $u \in \mathbb{R}_{\kappa}^{1,d}$.

The authors of [100] advocated that such a requirement (3.2.1) is too restrictive since it imposes that u is independent of time. From (3.2.1), one has $\mathcal{E}^d(u) = u$, which writes $u(x^0 + d_{\kappa}^i, x^j) =$

$u(x^0, x^j)$ so that u needs to be periodic in the imaginary direction along x^0 . If one considers that u needs to be entire, then the LIOUVILLE theorem states that an entire bounded complex function needs to be constant, thus u needs to be constant along x^0 . The latter requirement would also impact the commutative limit and be of less physical relevance.

3.2.1 The gauge theory on twisted derivation-based differential calculus of deformed translation

One starts by considering the generators of translations of the κ -POINCARÉ algebra P_μ . In commutative gauge theory, it consists of the usual set of derivations (*i.e.* the ∂_μ). One could think of using the P_μ 's as derivations to build the gauge theory along the lines of Section 2.1. In view of their coproduct (3.1.1d), the P_j 's are not derivations, but rather “twisted”² derivations, *i.e.*

$$P_j \triangleright (f \star_\kappa g) = (P_j \triangleright f) \star_\kappa g + (\mathcal{E} \triangleright f) \star_\kappa (P_j \triangleright g). \quad (3.2.2)$$

If one considers the set of functions on $\mathbb{R}_\kappa^{1,d}$ given by $X_0 = \kappa(1 - \mathcal{E}) \triangleright$ and $X_j = P_j \triangleright$, one obtains $d + 1$ twisted derivations

$$X_\mu(f \star_\kappa g) = X_\mu(f) \star_\kappa g + \mathcal{E}(f) \star_\kappa X_\mu(g). \quad (3.2.3)$$

In (3.2.3) and in the following, we abbreviate $\mathcal{E} \triangleright f$ as $\mathcal{E}(f)$. We denote the set of twisted derivations on κ -MINKOWSKI as $\text{Der}_\mathcal{E}(\mathbb{R}_\kappa^{1,d})$. The latter derivations consist of an Abelian LIE algebra and each P_μ boils down to ∂_μ at the commutative limit $\kappa \rightarrow +\infty$, up to an irrelevant $-i$ factor. One can indeed check that $\mathcal{E} \rightarrow 1$ in this limit and that a development of the exponential gives $X_0 \rightarrow P_0$.

One should note that twisted structures already appear in twisted spectral triples.

Twisted spectral triples

In the context of the noncommutative standard model, it was shown that some inconsistencies arise because the full DIRAC operator is not bounded as required by the spectral triple axioms (see Definition 1.3.1). That is, for $(\mathbb{A}, \mathcal{H}, \not{D})$ as spectral triple, the element $[\not{D}, f]$ is not a bounded operator in \mathcal{H} , for any $f \in \mathbb{A}$. CONNES and MOSCOVICI [102] cured the latter problem by considering instead a twisted spectral triple. The latter is called twisted since it introduces an automorphism of \mathbb{A} , $\rho \in \text{Aut}(\mathbb{A})$, such that

$$[\not{D}, f]_\rho = \not{D}f - \rho(f)\not{D} \quad (3.2.4)$$

is bounded. Correspondingly, the zeroth (1.3.1c) and first (1.3.1d) order conditions write now also with the twisted bracket $[\cdot, \cdot]_\rho$, so that the set of forms (1.3.5) (and so the differential calculus) is also twisted.

In the case of κ -MINKOWSKI, the twist ρ is considered to be $\mathcal{E} \in \text{Aut}(\mathbb{R}_\kappa^{1,d})$. To make contact even further with the twisted gauge theory under study, one can show that $[\not{D}, \cdot]_\rho$ is a twisted derivation, that is

$$[\not{D}, f \star g]_\rho = [\not{D}, f]_\rho \star g + \rho(f) \star [\not{D}, g]_\rho. \quad (3.2.5)$$

²The twist (\mathcal{E}) has here nothing to do with a DRINFEL'D twist. Despite the confusion that similar denomination can bring, the name “twist” for twisted derivations is coherent with respect to the name “twisted” spectral triple, the two sharing common grounds and properties.

The relation (3.2.5) has to be confronted with (3.2.3). Furthermore, ρ is required to be regular, that is $\rho^\dagger = \rho^{-1}$, a property satisfied by \mathcal{E} .

Due to the fact that we are considering twisted derivations instead of the usual derivations, the procedure of Section 2.1 must be adapted to the twisted setting, as described below. Indeed, if one wants to define a connection on a (right) $\mathbb{R}_\kappa^{1,d}$ -module \mathbb{X} similar to (2.1.1) with twisted derivations, then one is confronted with inconsistencies, which result from the equality of $\nabla_X(s \triangleleft (f \star_\kappa g))$ and $\nabla_X((s \triangleleft f) \triangleleft g)$. The problem lies in the LEIBNIZ rule (2.1.1a) and it can be shown [101] that the only possible twisted connection corresponds to

$$\nabla_\mu(s \triangleleft f) = \nabla_\mu(s) \triangleleft f + \mathcal{E}(s) \triangleleft X_\mu(f) \quad (3.2.6)$$

for $f, g \in \mathbb{R}_\kappa^{1,d}$ and $s \in \mathbb{X}$, where we noted $\nabla_{X_\mu} = \nabla_\mu$. In the previous expression (3.2.6) \mathcal{E} has been lifted to the module structure and one has $\mathcal{E}(s \triangleleft f) = \mathcal{E}(s) \triangleleft \mathcal{E}(f)$. In a similar fashion, the only twisted curvature, giving rise to a module homomorphism, writes

$$R_{\mu\nu} = \mathcal{E}^{-1}(\nabla_\mu \mathcal{E} \nabla_\nu - \nabla_\nu \mathcal{E} \nabla_\mu) \quad (3.2.7)$$

Recall that the module homomorphism property states that $R_{\mu\nu}(s \triangleleft f) = R_{\mu\nu}(s) \triangleleft f$, and is necessary in order to extract a field strength F out of the full curvature R through (2.1.10). It is important to note that there is no freedom in the choice of a (twisted) connection and its associated curvature, since expressions (3.2.6) and (3.2.7) are imposed by the choice of the twist.

Twisted gauge theory

The latter result can be generalised to any twisted derivations. Consider two algebra automorphisms^a $\rho_j : \mathbb{A} \rightarrow \mathbb{A}$, such that

$$X(f \star g) = X(f) \star \rho_1(g) + \rho_2(f) \star X(g), \quad (3.2.8)$$

for X a twisted derivation. Then, one can show that the connection and its associated curvature are given by

$$\nabla_X(s \triangleleft f) = \nabla_X(s) \triangleleft \rho_1(f) + \tau_2(s) \triangleleft X(f), \quad (3.2.9a)$$

$$R_{X,Y} = \tau_2^{-1} \nabla_X \tau_2 \nabla_Y - \tau_2^{-1} \nabla_Y \tau_2 \nabla_X - \nabla_{[X,Y]} \quad (3.2.9b)$$

where τ_2 is a twisted module automorphism $\tau_2(s \triangleleft f) = \tau_2(s) \triangleleft \rho_2(f)$ and ρ_1 and ρ_2 are assumed to commute with any (twisted) derivation X . In this case, the curvature R is a twisted module homomorphism

$$R_{X,Y}(s \triangleleft f) = R_{X,Y}(s) \triangleleft \rho_1^2(f). \quad (3.2.10)$$

One should note that the LIE bracket changes the twist, in the sense that two twisted derivations X and Y with twists ρ_1 and ρ_2 as in (3.2.8), one can show that $[X, Y]$ is a twisted derivation with twists ρ_1^2 and ρ_2^2 .

^aThe fact that the ρ_j needs to be algebra homomorphisms is imposed by the LEIBNIZ rule (3.2.8) applied to three or more elements.

Now consider the specific case of noncommutative electrodynamics $\mathbb{X} = \mathbb{R}_\kappa^{1,d}$. One follows the steps of [Section 2.1](#) in the twisted case. By defining $A_X = i\nabla_X(1)$, one computes

$$\nabla_\mu(f) = X_\mu(f) - iA_\mu \star_\kappa f, \quad (3.2.11a)$$

$$F_{\mu\nu} = X_\mu(A_\nu) - X_\nu(A_\mu) - i(\mathcal{E}(A_\mu) \star_\kappa A_\nu - \mathcal{E}(A_\nu) \star_\kappa A_\mu). \quad (3.2.11b)$$

Note that the X_μ 's are not real but twisted real³, *i.e.*

$$(X_\mu(f))^\dagger = -\mathcal{E}^{-1}X_\mu(f^\dagger). \quad (3.2.12)$$

The Hermiticity condition [\(2.1.6\)](#) has also to be twisted to [\[103\]](#)

$$X_\mu((s_1, s_2)) = (\mathcal{E}^{-1}\nabla_\mu(s_1), s_2) + (\mathcal{E}^{-1}(s_1), \nabla_\mu(s_2)) \quad (3.2.13)$$

for any $s_1, s_2 \in \mathbb{X}$, which imposes

$$A_\mu^\dagger = \mathcal{E}^{-1}(A_\mu). \quad (3.2.14)$$

The latter condition [\(3.2.14\)](#) has the same commutative limit then the usual (untwisted) Hermiticity condition $A_\mu^\dagger = A_\mu$, corresponding to a real valued gauge field A .

The gauge transformation of the connection, and so of the curvature, needs also to be twisted. As above, it can be shown [\[101\]](#) that it is imposed by the nature of the twist to be

$$A_\mu^u = \mathcal{E}(u^\dagger) \star_\kappa A_\mu \star_\kappa u + \mathcal{E}(u^\dagger) \star_\kappa X_\mu(u), \quad F_{\mu\nu}^u = \mathcal{E}^2(u^\dagger) \star_\kappa F_{\mu\nu} \star_\kappa u. \quad (3.2.15)$$

Therefore, if one considers an action of the form [\(2.1.21\)](#)

$$S = \int d^{d+1}x F^{\mu\nu} \star_\kappa F_{\mu\nu}^\dagger, \quad (3.2.16)$$

its gauge transform is computed quite similarly as in [\(2.1.23\)](#), except that the last prefactor is now twisted as

$$S^u = \int d^{d+1}x \mathcal{E}^{d-2}(u) \star_\kappa \mathcal{E}^2(u^\dagger) \star_\kappa F^{\mu\nu} \star_\kappa F_{\mu\nu}^\dagger. \quad (3.2.17)$$

In view of the $\mathcal{U}(1)$ gauge group, which imposes $u \star_\kappa u^\dagger = 1$, one obtains that the action S is gauge invariant, that is $S^u = S$, when the powers of the \mathcal{E} 's in the prefactor are equal. The latter equality imposes

$$d + 1 = 5. \quad (3.2.18)$$

Several comments of the latter result are in order.

In the context of κ -MINKOWSKI, the action [\(3.2.16\)](#) has a straightforward commutative limit, which correspond to the usual $U(1)$ YANG-MILLS action on the MINKOWSKI space-time. The previous statement holds for any dimension, but the gauge invariance analysis requires that the space-time dimension is fixed to 5 by [\(3.2.18\)](#). Therefore, the commutative limit of [\(3.2.16\)](#) corresponds to the 4 + 1-dimensional $U(1)$ YANG-MILLS action.

³The minus sign in [\(3.2.12\)](#) is matter of convention. Indeed, if one considers iX_μ instead of X_μ , that stems for a representation ∂_μ instead of $-i\partial_\mu$, then the minus sign disappears.

The action (3.2.16) can be shown to be κ -POINCARÉ invariant, such that it has even been named “ κ -POINCARÉ invariant gauge theory”. This implies that the POINCARÉ invariance is restored automatically at the commutative limit, but also that this $\mathcal{U}(1)$ gauge theory triggers a LORENTZ invariance violation via a POINCARÉ deformation. The previous observation sparks phenomenological considerations concerning this theory as discussed in Section 4.2.

The dimension constraint (3.2.18) can be traced back to the fact that the twist in the integral cyclicity (3.1.11) depends on the spacial dimensions d . Furthermore, one should note that the latter constraint is very strong. Indeed, a twisted module structure of the form $f \triangleleft g = f \star_\kappa \rho(g)$, with ρ an automorphism of $\mathbb{R}_\kappa^{1,d}$, does not affect the constraint (3.2.18) [103]. Moreover, there is actually a freedom in the gauge transformation (3.2.15), since one could also considered

$$F_{\mu\nu}^u = \mathcal{E}^2 \tilde{\rho}(u^\dagger) \star_\kappa F_{\mu\nu} \star_\kappa \tilde{\rho}(u) \quad (3.2.19)$$

where $\tilde{\rho} \in \text{Aut}(\mathbb{R}_\kappa^{1,d})$. However, the latter freedom leaves the requirement (3.2.18) unchanged, so that we have put $\tilde{\rho} = \text{id}$ previously.

3.2.2 Quantisation and tadpole computation

We now turn to the study of the quantisation and the perturbation theory of (3.2.16). Note that the commutative limit of (3.2.16) corresponds to the 5-dimensional (quantum) electrodynamics action, which is known to be non-renormalisable. Therefore, one should be careful when considering the commutative limit in the context of the perturbative study of (3.2.16).

We perform a BECCHI-ROUET-STORA-TYUTIN quantisation of the action (3.2.16), as in [104], for which the corresponding twisted symmetry has been studied in [105]. To do so, let us introduce the FADEEV-POPOV ghost field c , anti-ghost field \bar{c} and the NAKANISHI–LAUTRUP field b . The SLAVNOV operator s is then defined as

$$sA_\mu = X_\mu(c) - \mathcal{E}(c) \star_\kappa A_\mu + A_\mu \star_\kappa c, \quad sc = -c \star_\kappa c, \quad s\bar{c} = b, \quad sb = 0, \quad (3.2.20)$$

from which one computes

$$sF_{\mu\nu} = F_{\mu\nu} \star_\kappa c - \mathcal{E}^2(c) \star_\kappa F_{\mu\nu}, \quad s^2 = 0. \quad (3.2.21)$$

Then, one adds a gauge-fixing term to the action (3.2.16) of the form

$$s \int d^5x \bar{c} \star_\kappa \mathcal{E}^{-4} X_\mu(A^\mu), \quad s \int d^5x \bar{c} \star_\kappa \mathcal{E}^{-4} (A_0 - \lambda), \quad (3.2.22)$$

where the first one corresponds to the deformed LORENTZ gauge $X_\mu A^\mu = 0$ and the second one is the (parametrized) temporal gauge $A_0 = \lambda$, for $\lambda \in \mathbb{R}$. The full action consisting of (3.2.16) and (3.2.22) can be put under the form

$$S = S_{AA} + S_{c\bar{c}} + S_{AAA} + S_{AAAA} + S_{Ac\bar{c}} \quad (3.2.23)$$

where S_{AA} correspond to the gauge field kinetic term, $S_{c\bar{c}}$ the ghost kinetic term, S_{AAA} the gauge field 3-vertex, S_{AAAA} the 4-vertex and $S_{Ac\bar{c}}$ the gauge field-ghost interaction.

We then use usual functional methods to compute the tadpole diagram, that is the one-loop 1-point function.

Functional methods in quantum field theory

We describe here a textbook functional method to obtain the n -point function in quantum field theory. If one considers an action function $S(A, c, \bar{c})$ depending on three fields A (the gauge field) and c, \bar{c} (the ghost fields). The latter splits into a kinetic part $S_{\text{kin}}(A, c, \bar{c})$ and an interaction part $S_{\text{int}}(A, c, \bar{c})$. The kinetic term is put under the form

$$S_{\text{kin}}(A, c, \bar{c}) = \int \frac{1}{2} A_\mu K_A A^\mu + \bar{c} K_c c \quad (3.2.24)$$

where K_A and K_c are kinetic operators. We further introduce the source action $S_{\text{sou}} = \int A_\mu J^\mu + \bar{\eta} c + \bar{c} \eta$, where $J, \eta, \bar{\eta}$ are the source fields.

One defines the (resp. free) generating functional of the connected correlation functions^a (resp. $W_0(J, \bar{\eta}, \eta)$) $W(J, \bar{\eta}, \eta)$ as

$$e^{W(J, \bar{\eta}, \eta)} = \int dA dc d\bar{c} e^{-S_{\text{int}}(A, c, \bar{c}) - S_{\text{kin}}(A, c, \bar{c}) + S_{\text{sou}}}, \quad (3.2.25a)$$

$$W_0(J, \bar{\eta}, \eta) = \int \frac{1}{2} J_\mu K_A^{-1} J^\mu + \bar{\eta} K_c^{-1} \eta. \quad (3.2.25b)$$

In the integral of e^W , one can perform the change of variables $A_\mu \rightarrow A_\mu + K_A^{-1} J_\mu$, $c \rightarrow c + K_c^{-1} \eta$ and $\bar{c} \rightarrow \bar{c} + K_c^{-1} \bar{\eta}$. Then, upon infinite expansion of $e^{S_{\text{int}}}$, one can compute that

$$W(J, \bar{\eta}, \eta) = \ln \left(e^{W_0} \left(1 + e^{-W_0} \left(e^{-S_{\text{int}}\left(\frac{\partial}{\partial J}, \frac{\partial}{\partial \bar{\eta}}, \frac{\partial}{\partial \eta}\right)} - 1 \right) e^{W_0} \right) \right). \quad (3.2.26)$$

The first non-trivial term of the expansion of $e^{-S_{\text{int}}}$ yields

$$W^1(J, \bar{\eta}, \eta) = W_0(J, \bar{\eta}, \eta) - e^{-W_0(J, \bar{\eta}, \eta)} S_{\text{int}} \left(\frac{\partial}{\partial J}, \frac{\partial}{\partial \bar{\eta}}, \frac{\partial}{\partial \eta} \right) e^{W_0(J, \bar{\eta}, \eta)} \quad (3.2.27)$$

Finally, one computes the correlation functions thanks to the generating functional of proper vertices $\Gamma(A, c, \bar{c})$, that consists of the LEGENDRE transform of $W(J, \bar{\eta}, \eta)$. Explicitly,

$$\Gamma(A, c, \bar{c}) + W(J, \bar{\eta}, \eta) - \int A_\mu J^\mu + \bar{\eta} c + \bar{c} \eta = 0, \quad A_\mu = \frac{\partial W}{\partial J^\mu}, \quad J_\mu = \frac{\partial \Gamma}{\partial A^\mu}, \quad (3.2.28)$$

and similarly for c with η and \bar{c} with $\bar{\eta}$. At first order, the middle expressions of (3.2.28) boils down to $A_\mu = \frac{\partial W_0}{\partial J^\mu} = \int K_A^{-1} J_\mu$, with similar expressions for c and \bar{c} . The latter expression implies that

$$W_0(J, \bar{\eta}, \eta) = \int A_\mu J^\mu + \bar{\eta} c + \bar{c} \eta \quad (3.2.29)$$

Using (3.2.29), a simplification occurs in (3.2.28) so that the one-loop 1-point function writes

$$\langle A \rangle_{1\text{-loop}} = \Gamma^1(A, c, \bar{c}) = e^{-W_0(J, \bar{\eta}, \eta)} S_{\text{int}} \left(\frac{\partial}{\partial J}, \frac{\partial}{\partial \bar{\eta}}, \frac{\partial}{\partial \eta} \right) e^{W_0(J, \bar{\eta}, \eta)} \quad (3.2.30)$$

where $J_\mu = \int K_A A_\mu$, and similarly for η and $\bar{\eta}$.

^aNote that one has $W = e^Z$, where Z is the generating functional of the connected GREEN functions introduced in [Subsection 2.6.2](#).

One has all the ingredient to perform the computation of the tadpole (3.2.30), which writes

$$\langle A \rangle_{1\text{-loop}} = \int d^5x \mathcal{J}(\kappa) A_0(x), \quad (3.2.31)$$

where \mathcal{J} is a gauge dependent divergent integral, to be regularised. The fact that (3.2.31) is proportional to A_0 can be linked to the fact that the time component has a peculiar role in κ -MINKOWSKI. Moreover, it justifies the study of the tadpole in the temporal gauge $A_0 = \lambda$. The major result of this computation, beyond its non-vanishing, is the gauge dependence of the tadpole. This implies that the $\mathcal{U}(1)$ gauge symmetry has been broken in the quantisation process, as we started with a gauge invariant action. One should note however, that the commutative limit is correct since in any gauge, \mathcal{J} vanishes in the limit $\kappa \rightarrow +\infty$.

Several comments are in order. It gathers the main elements of discussions of [106].

First, one should know that non-zero tadpole has been experienced in other quantum space-time, like the 2-dimensional MOYAL space [107] and \mathbb{R}_λ^3 [108]. The two previous computations followed the same quantisation procedure as above. This may imply that the quantisation method used here cannot be applied in the context of noncommutative geometry, as suggested in [109].

Then, the notion of vacuum on quantum space-times has not reach consensus, so that the expression (3.2.31) may not be the physical vacuum expectation value of the quantum electrodynamical theory on κ -MINKOWSKI. On the one hand, the POINCARÉ symmetry is broken so that one cannot define particles as irreducible representation of the little group. This idea is known in quantum field theory on curved space-time, in which the vacuum state is only defined thanks to the asymptotic flatness hypothesis. Concerning κ -MINKOWSKI, there has been several attempts in defining a physical vacuum, either from a deformation of the little group study, or by defining its energy. The latter notion is based on the CASIMIR operator of the κ -POINCARÉ algebra, but is not well-defined as it is coordinate dependent.

Finally, as discussed in Section 2.1, one could consider other physical variables then A to encode the “noncommutative photon”, and so have a zero tadpole with this quantity.

3.3 Causality on κ -MINKOWSKI

Consider a single massive object made out of a spatial superposition of two states of different masses. The gravitational field will also be in a superposition of state thus leading to superposed space-time geometries. From there, one could argue that it is possible to superpose two geometries for which two events are causal (*i.e.* time-like) in one geometry, but non-causal (*i.e.* space-like) in the other. Therefore, the notion of classical causality beaks down at the quantum level. One could argue either that causality is not an intrinsic property of nature, but rather an emergent feature of some phenomenon, or that causality is deformed in the quantum setting. The latter consideration has pushed towards the search of the properties of the would be quantum causality. We refer to [110] for an early review on the topic.

In the context of quantum space-times, the question of what becomes of causality is interesting in several aspects. A notion of causality on a quantum space-time needs to give back the usual notion of causality at the commutative limit. In this sense, a causality on a quantum space-time should be a deformation of the usual causality that could lead to an effective behaviour when considering the first order correction. The phenomenology of deformed causality could be of primordial importance to have experimental tests or constraints on such models.

There is two algebraic formulation of causality which correspond to the causality on Lorentzian **spectral triple**, developed by FRANCO and ECKSTEIN [111] and the isocone-based approach of BESNARD [112]. We focus here on the first one that was applied to the κ -MINKOWSKI space in [113]. Note that the latter was also constructed on the “quantum MINKOWSKI” space (*i.e.* the MOYAL space with a MINKOWSKI metric), as well as the κ -MINKOWSKI with another DIRAC operator (see [113] for a short review).

One should note that models of causality have already been considered on the κ -MINKOWSKI space-time by MERCATI and SERGOLA [79, 114]. The latter study is based on the observation that commutative causality may be defined thanks to PAULI-JORDAN functions. Considering a scalar field theory with field ϕ , the PAULI-JORDAN function corresponds to $[\phi(x), \phi(y)]$ and encodes the light cone frontier. By implementing a noncommutative scalar field on κ -MINKOWSKI, the latter authors could derive the PAULI-JORDAN function of the scalar field and observe that the light cone was blurred: one does not go from time-like to space-like by an abrupt change, there is a smooth transition within which one is neither space nor time-like. By observing that the width of the blurred region increases with (space-time) distance, the authors advocate that this deformation of the light cone is close to the present measurement accuracy, if the effect is amplified by cosmological distance. Deformed light-cones could trigger time delays in photon travel from ultra-high energy cosmic rays toward Earth (see Section 4.2 for more details on quantum gravity phenomenology).

We introduce the formalism of Lorentzian **spectral triple** as well as the formulation of causality on it.

Lorentzian spectral triple

The Lorentzian **spectral triple** is defined by the same set of data as the **spectral triple** of Definition 1.3.1, with \mathbb{A} a C^* -algebra which represents on a HILBERT space \mathcal{H} with inner product $\langle \cdot, \cdot \rangle$, and \mathcal{D} an operator on \mathcal{H} . The “Lorentzian” property is introduced through the so-called fundamental symmetry $I \in \mathcal{B}(\mathcal{H})$ which should satisfy

$$I^2 = 1, \quad I^\dagger = I, \quad [I, f] = 0, \quad (3.3.1a)$$

for any $f \in \mathbb{A}_1$, where \dagger is the adjoint for $\langle \cdot, \cdot \rangle$. For purely technical reasons, one needs to consider \mathbb{A}_1 , a unitalisation of \mathbb{A} . Note also that the representation $\pi : \mathbb{A} \rightarrow \mathcal{H}$ has been dropped in (3.3.1a). I transforms the positive form $\langle \cdot, \cdot \rangle$ of \mathcal{H} on an indefinite form $\langle \cdot, \cdot \rangle_I = \langle \cdot, I \cdot \rangle$, which is not necessarily positive. The latter I may be seen as a generalisation of the complex i to HILBERT spaces, and the form $\langle \cdot, \cdot \rangle_I$ as a WICK rotation of $\langle \cdot, \cdot \rangle$. The latter indefinite product is called a KREIN product, and \mathcal{H} equipped with it, a KREIN space.

From there, the Lorentzian **spectral triple** follows the axioms of the **spectral triple** adapted to the KREIN space. Explicitly, the self-adjointness of \mathcal{D} for $\langle \cdot, \cdot \rangle$ is replaced by a self-adjointness for $\langle \cdot, \cdot \rangle_I$, which writes

$$\mathcal{D}^\dagger I = -I \mathcal{D}. \quad (3.3.1b)$$

The compact resolvent condition writes

$$f(1 + \tilde{\mathcal{D}}^2)^{-\frac{1}{2}} \text{ is compact,} \quad (3.3.1c)$$

for any $f \in \mathbb{A}$, where $\tilde{\mathcal{D}}^2 = \frac{1}{2}(\mathcal{D}^\dagger \mathcal{D} + \mathcal{D} \mathcal{D}^\dagger)$. Finally, the DIRAC operator should satisfy

$$[\mathcal{D}, f] \in \mathcal{B}(\mathcal{H}), \quad (3.3.1d)$$

for any $f \in \mathbb{A}$, as in the usual [spectral triple](#) case.

Finally, there is an additional condition requiring that it exists a self-adjoint operator T and a positive element $N \in \mathbb{A}_1$ such that $\text{Dom}(T) \cap \text{Dom}(N)$ is dense in \mathcal{H} and satisfies

$$(1 + T)^{-\frac{1}{2}} \in \mathbb{A}_1, \quad I = -N[\mathcal{D}, T]. \quad (3.3.1e)$$

The operator T corresponds to the (noncommutative) generalisation of a global time function, while the right hand side of (3.3.1e) ensures a Lorentzian type signature. In this sense, note that $-I$ could satisfy all the axioms (3.3.1) if one changes T to $-T$. The latter symmetry can be directly linked to the choice of a signature.

The equation (3.3.1e) is not the only possible way to construct a fundamental symmetry I . Furthermore, one can define a Lorentzian [spectral triple](#) corresponding to a globally hyperbolic classical space-time. We refer to [111] for more details on these points.

Causality on Lorentzian spectral triples

If one considers a classical space-time \mathcal{M} , with metric \mathbf{g} , and one wants to know if two points $x, y \in \mathcal{M}$ are causally connected, then one can consider a causal curve relating the two. A causal curve is a smooth curve $\gamma : \mathbb{R} \rightarrow \mathcal{M}$ satisfying

$$\mathbf{g}(\gamma'(t), \gamma'(t)) \leq 0. \quad (3.3.2)$$

The previous relation states that, at any $t \in \mathbb{R}$, the tangent vector to the curve γ is future directed, *i.e.* time-like and pointing to increasing time (see [Figure 6](#)). Therefore, y is in the causal future of x if there exists a causal curve γ such that $\gamma(t_1) = x$ and $\gamma(t_2) = y$ for some $t_1 \leq t_2 \in \mathbb{R}$.

Within this setting the causal structure of \mathcal{M} is fully determined by its set of so-called causal functions. Explicitly, a causal function is a function $f : \mathcal{M} \rightarrow \mathbb{R}$ which is non-decreasing along every future directed causal curve γ . Therefore,

$$y \text{ is in the causal future of } x \text{ if and only if } f(x) \leq f(y) \quad (3.3.3)$$

for any causal function f [111]. The graph of a causal function f corresponds to the region of causal simultaneity.

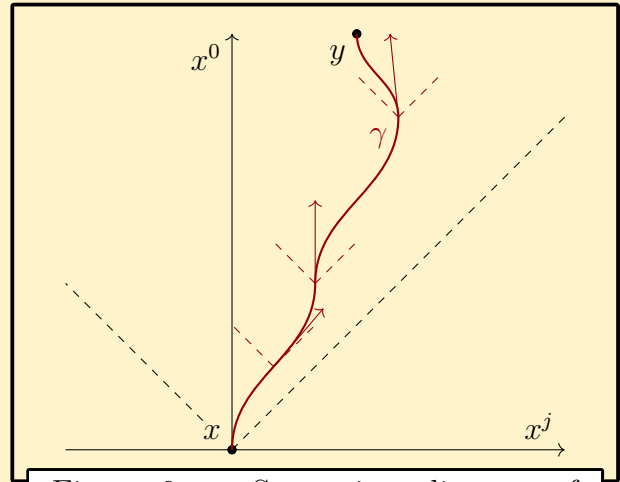


Figure 6 – Space-time diagram of a causal curve γ (in red) linking two points x and y of flat space-time. The derivative γ' along the curve is pictured to be always time-like.

To go to the (noncommutative) algebraic setting, one can replace space-time points by pure states of \mathbb{A} , $\Psi_{\mathbb{A}}^{\text{P}}$ (see [Definition B.2.5](#)). As \mathbb{A} contains the noncommutative analogue of the smooth functions, the causal curves will be generalised as a subset of \mathbb{A} , called the causal cone.

A causal cone $\mathcal{C} \subset \mathbb{A}_1$ is defined as

$$f^\dagger = f, \quad f + g \in \mathcal{C}, \quad \lambda f \in \mathcal{C}, \quad x1 \in \mathcal{C}, \quad (3.3.4a)$$

for any $f, g \in \mathcal{C}$, $\lambda \in \mathbb{R}^+$ and $x \in \mathbb{R}$. These stability conditions correspond to the mathematical definition of an (iso)cone. Furthermore, one should have that the linear span of \mathcal{C} form all \mathbb{A}_1 , so that there are no disconnected region. Finally, the ‘‘causal’’ feature of the causal cone is due to the requirement that^a

$$\langle \psi \mid [\mathcal{D}, f]\psi \rangle_I = \langle \psi \mid I[\mathcal{D}, f]\psi \rangle \leq 0 \quad (3.3.4b)$$

for any $f \in \mathcal{C}$ and $|\psi\rangle \in \mathcal{H}$. For commutative Lorentzian [spectral triple](#), it is possible to link [\(3.3.4b\)](#) directly to [\(3.3.2\)](#). Moreover, the sign of [\(3.3.4b\)](#) is a signature choice, similarly to [\(3.3.2\)](#). One could require it to be positive, if one changes I to $-I$.

To summarise, when considering a Lorentzian [spectral triple](#), an element $f \in \mathbb{A}$ is in the causal cone \mathcal{C} if and only if it satisfies [\(3.3.4b\)](#), for any $|\psi\rangle \in \mathcal{H}$. From there, we dispose of ‘‘causal time charts’’ f . Then, for any two (pure) states $\psi_1, \psi_2 \in \Psi_{\mathbb{A}}^{\text{P}}$

$$\psi_2 \text{ is in the causal future of } \psi_1 \text{ if and only if } \psi_1(f) \leq \psi_2(f), \quad (3.3.5)$$

for any $f \in \mathcal{C}$. The latter relation [\(3.3.5\)](#) is the noncommutative analogue of [\(3.3.3\)](#).

^aWe abbreviated the mathematical notation $\langle |\psi\rangle, |\psi\rangle \rangle$ to the physical one $\langle \psi \mid \psi \rangle$, in accordance with [Example B.2.6](#).

The previous mathematical framework was applied to 1 + 1-dimensional κ -MINKOWSKI $\mathbb{A} = \mathbb{R}_\kappa^{1,1}$ in [\[113\]](#). The considered [HILBERT space](#) is given by

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-, \quad \mathcal{H}_a = \mathbb{C}^2 \otimes L^2(\mathbb{R}) \quad (3.3.6)$$

where $a = 0, \pm$ and $L^2(\mathbb{R})$ consist of the square integrable functions on \mathbb{R} (with the LEBESGUE measure). On each summand \mathcal{H}_a , the representation π_a is inherited from the unitary irreducible representation of the affine group (the LIE group of momenta) on $L^2(\mathbb{R})$ given by

$$(\pi_a(f)\psi^{(a)})(p_0) = \int dq_0 f(q_0 - p_0, ae^{-p_0/\kappa}) \psi^{(a)}(q_0) \quad (3.3.7)$$

for any $f \in \mathbb{R}_\kappa^{1,1}$ and $\psi^{(a)} \in L^2(\mathbb{R})$. The fact that \mathbb{C}^2 appear in [\(3.3.6\)](#) comes from the fact that we use a 2-dimensional DIRAC operator below. The full representation on \mathcal{H} is given by $\pi = (\pi_+ \oplus \pi_0 \oplus \pi_-) \otimes \mathbb{1}_2$. In the following, the representations are clearly stated to avoid confusion. Considering two elements $\psi_1 = \bigoplus_a \psi_1^{(a)}, \psi_2 = \bigoplus_a \psi_2^{(a)} \in \mathcal{H}$, one defines the inner product as

$$\langle \psi_1, \psi_2 \rangle = \sum_{a=+,0,-} \int dp_0 (\psi_1^{(a)})^\dagger(p_0) \psi_2^{(a)}(p_0). \quad (3.3.8)$$

One can note that the representation (3.3.7) implies that $\pi_{\pm}(x^0) = -i\frac{d}{dp_0}$ and $\pi_{\pm}(x^1) = \pm e^{-p_0/\kappa}$. Therefore, one has that $x^0 = \hat{x}$ and $x^1 = \pm e^{-\hat{p}/\kappa}$, where \hat{x} , \hat{p} are the SCHRÖDINGER representation of the position and momentum operators respectively of the 1-dimensional quantum mechanical system.

The considered Lorentzian spectral triple writes

$$\not{D} = -i\gamma^{\mu}X_{\mu} \otimes \mathbb{1}_3 = \begin{pmatrix} 0 & X_- \\ X_+ & 0 \end{pmatrix} \otimes \mathbb{1}_3, \quad I = i\gamma^0 \otimes \mathbb{1}_3 \quad (3.3.9)$$

where $\gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $\gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ are the 2-dimensional DIRAC gamma matrices, X_{μ} corresponds to the twisted derivations (3.2.3), and $X_{\pm} = X_0 \pm X_1$. Note that $\mathbb{1}_3$ is here to express that the same DIRAC operator and fundamental symmetry are considered on each $+$, 0 and $-$ representations. Furthermore, the DIRAC operator expression (3.3.9) is the straightforward generalisation of the commutative DIRAC operator on the MINKOWSKI space-time $\not{D} = -i\gamma^{\mu}\partial_{\mu}$.

It is important to point out that the considered derivations in the DIRAC operator are twisted, so that we have to consider a twisted (Lorentzian) spectral triple, as introduced in Subsection 3.2.1. Therefore, one has to consider the twisted bracket $[\cdot, \cdot]_{\mathcal{E}}$ in the axioms (3.3.1), instead of the usual bracket. The twisted bracket expresses as

$$[\not{D}, f]_{\mathcal{E}} = \not{D}f - \mathcal{E}(f)\not{D} \quad (3.3.10)$$

in accordance with (3.2.4).

By considering

$$T = \bigoplus_a (\pi_a(x^0) \otimes \mathbb{1}_2), \quad N = 1, \quad (3.3.11)$$

one can compute, thanks to (3.3.7), that (3.3.1e) is satisfied. The equation (3.3.11) states that the global time T corresponds to the representation of x^0 . It seems quite straightforward to consider that indeed x^0 is a global time, even in the deformed theory.

It remains to consider how the conditions (3.3.3) and (3.3.4b) writes in this case.

One can show that a relevant set of pure states is given by

$$\psi^{\pm}(f) = \langle \psi, \pi_{\pm}(f)\psi \rangle \quad (3.3.12)$$

where $|\psi\rangle \in \mathcal{H}_{\pm}$. Therefore, the condition (3.3.3) imposes that $\psi_2 \in \Psi_{\mathbb{R}^{\kappa}}^{\mathbb{P}, 1, 1}$ is in the causal future of $\psi_1 \in \Psi_{\mathbb{R}^{\kappa}}^{\mathbb{P}, 1, 1}$ if and only if

$$\int dp_0 dq_0 f(q_0 - p_0, \pm e^{-p_0/\kappa}) \frac{d}{dt} (\bar{\psi}_t(p_0)\psi_t(q_0)) \geq 0 \quad (3.3.13)$$

for all $f \in \mathcal{C}$, where $t \in [1, 2]$ is a continuous parameter that interpolates between ψ_1 and ψ_2 .

The condition for a function f to be in the causal cone \mathcal{C} (3.3.4b) can be computed to be

$$\int dp_0 dq_0 \left(i(1 - e^{-(q_0 - p_0)/\kappa}) f(q_0 - p_0, ae^{-p_0/\kappa}) \pm \partial_1 f(q_0 - p_0, ae^{-p_0/\kappa}) \right) \bar{\psi}(p_0)\psi(q_0) \geq 0 \quad (3.3.14)$$

for all $\psi \in L^2(\mathbb{R})$ and $a = \pm, 0$, where ∂_1 denotes the derivative with respect to the second (spacial) variable. Note that any function $f = x^0 + vx^1$, with $v \in [-1, 1]$ satisfies (3.3.14) and therefore is in the causal cone, as one could expect. Merging the two condition (3.3.13) and (3.3.14) gives a non-trivial transport equation that was considered too tedious to be solved.

Even if the causality on $\mathbb{R}_\kappa^{1,1}$ has not been fully characterised, two comments are in order. First of all, the commutative limit of the former model has few physical interest since the considered representation (3.3.7) is 1-dimensional when $\kappa \rightarrow +\infty$. Thus, it raises the question of what happens when one considers a 2-dimensional representation with a good commutative limit, like the GEL'FAND-NAÏMARK-SEGAL representation (see Subsection B.2.2). One could also wonder how much this causality model depends on the considered HILBERT space.

Moreover, if one writes the condition that two states ψ_1, ψ_2 to be causally related (3.3.4b) with $f(p_0, \pm e^{-q_0/\kappa}) = p_0 \pm e^{-q_0/\kappa}$, one obtains

$$\langle \psi_2 | x^0 \psi_2 \rangle - \langle \psi_1 | x^0 \psi_1 \rangle \geq \left| \langle \psi_2 | x^1 \psi_2 \rangle - \langle \psi_1 | x^1 \psi_1 \rangle \right|. \quad (3.3.15)$$

Therefore, the equations that determine the causal evolution of states can be loosely written as $\langle \delta x^0 \rangle \geq |\langle \delta x^1 \rangle|$, which is the expectation value of the speed-of-light limit on MINKOWSKI: $\delta x^0 \geq |\delta x^1|$. The expression (3.3.15) was thus considered to be the analogue of the speed-of-light limit. The presence of expectation values suggests that the speed-of-light limit has to be satisfied on average but that it could be broken locally. This observation could lead to important phenomenological considerations. However, the expression (3.3.15) is not telling how much or how often such a violation could occur. This would require a deepened analysis. Furthermore, one could consider more complex causal functions and see what becomes of (3.3.15).

3.4 Other deformations of MINKOWSKI space-time

Despite that κ -MINKOWSKI has been the first and well-most studied deformation of the MINKOWSKI space-time, there exists other deformations of the POINCARÉ group [115, 116]. Mainly three have been studied as reviewed in [117]. First, the θ -deformation of MINKOWSKI corresponds to a MOYAL-like deformation (see Section 2.5) with a MINKOWSKI metric. Second, the main topic of this Chapter 3 correspond to the κ -deformation. Finally, there was recently a proposal for a new deformation of “angular” type [118] called ϱ -deformation. In this Section 3.4, we introduce the ϱ -MINKOWSKI space-time and discuss the first result of field and gauge theory on it.

Note that other deformations of MINKOWSKI have been considered in the literature with fewer manpower of research, like the generalisation of (3.1.5) [119] or the very recent T-MINKOWSKI [117].

The ϱ -MINKOWSKI space-time was first derived by a DRINFEL'D twist deformation (see Subsection 1.6.2) with the twist [118]

$$\begin{aligned} \mathcal{F} &= \exp \left(\frac{i\varrho}{2} (P_0 \otimes J_3 - J_3 \otimes P_0) \right) \\ &= \exp \left(-\frac{i\varrho}{2} (\partial_0 \otimes (x^1 \partial_2 - x^2 \partial_1) - (x^1 \partial_2 - x^2 \partial_1) \otimes \partial_0) \right). \end{aligned} \quad (3.4.1)$$

The latter is called “angular twist” and corresponds to a DRINFEL'D twist of the POINCARÉ algebra $\mathfrak{P}^{1,3}$: there is no need to extend the algebra as discussed in Section 3.1 for κ -deformation. One calculates the LIE algebra of coordinates to be

$$[x^0, x^1]_{\star_e} = i\varrho x^2, \quad [x^0, x^2]_{\star_e} = -i\varrho x^1, \quad [x^1, x^2] = 0 \quad (3.4.2)$$

with x^3 a central element, *i.e.* it commutes with the other coordinates. The deformation parameter ϱ has the dimension of a length. One can express the latter relations (3.4.2) in cylindrical coordinates $(x^0, x^r, x^\varphi, x^3)$ to obtain

$$[x^0, x^\varphi]_{\star_\varrho} = \varrho x^\varphi \quad (3.4.3)$$

and the other pairs of coordinates commute. One should note the similarity between (3.4.3) and a 1 + 1-dimensional κ -MINKOWSKI (3.1.3a) with two central coordinates when considering the change $\varrho \rightarrow \frac{i}{\kappa}$.

The ϱ -POINCARÉ, derived thanks to the DRINFEL'D twist (3.4.1), was shown to have a bi-crossproduct structure [120] quite similar to the one of κ -POINCARÉ.

A derivation of the star product through convolution algebra technique (see Subsection 1.6.1) has been done in [121]. The sketchy reasoning of the star product construction is made here with coordinates (3.4.2), but one could equivalently perform the construction with (3.4.3) and obtain a similar result.

The non-trivial part of (3.4.2) corresponds to the Euclidean LIE algebra, for which the associated LIE group is the Euclidean group. The latter can be derived to satisfy

$$p_0 \boxplus q_0 = p_0 + q_0, \quad \vec{p} \boxplus \vec{q} = \vec{p} + R(\varrho p_0) \vec{q}, \quad (3.4.4a)$$

$$\boxminus p_0 = -p_0, \quad \boxminus \vec{p} = -R(-\varrho p_0) \vec{p}, \quad (3.4.4b)$$

where we noted $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $R(\varrho p_0) = \begin{pmatrix} \cos(\varrho p_0) & -\sin(\varrho p_0) \\ \sin(\varrho p_0) & \cos(\varrho p_0) \end{pmatrix}$ corresponds to the rotation matrix of angle ϱp_0 around the x^3 axis. This group is unimodular $\Delta = 1$ and has a HAAR measure corresponding to the LEBESGUE measure, *i.e.* $d\mu(p) = d^4 p$. From there, one derives the star product and involution to be

$$(f \star_\varrho g)(x) = \int \frac{dp_0}{2\pi} dy^0 e^{-iy^0 p_0} f(x^0 + y^0, \vec{x}, x^3) g(x^0, R(\varrho p_0) \vec{x}, x^3), \quad (3.4.5a)$$

$$f^\dagger(x) = \int \frac{dp_0}{2\pi} dy^0 e^{-iy^0 p_0} \bar{f}(x^0 + y^0, R(\varrho p_0) \vec{x}, x^3), \quad (3.4.5b)$$

where we again used the shorthand notation $\vec{x} = (x^1, x^2)$.

From the unimodularity of the group (3.4.4), one can deduce (see (2.6.12)) that the integral is cyclic

$$\int d^4 x (f \star_\varrho g)(x) = \int d^4 x (g \star_\varrho f)(x). \quad (3.4.6)$$

The latter property can also be derived by direct computation with (3.4.5a), or by the star product derived from the DRINFEL'D twist (3.4.1).

The analysis of a charged ϕ^4 -theory on ϱ -MINKOWSKI, based on the star product (3.4.5), has been studied in [121]. The “charged” property means that the considered action has ϕ^\dagger term, so that ϕ and $\bar{\phi}$ represents the field and its charge conjugated field respectively. The considered action is of the form

$$S(\phi, \bar{\phi}) = \int d^4 x (\partial_\mu \phi)^\dagger \star_\varrho \partial^\mu \phi + m^2 \phi^\dagger \star_\varrho \phi + \frac{g^2}{4!} V_{\text{int}}(\phi, \bar{\phi}) \quad (3.4.7)$$

where V_{int} is the interaction term considered to be

$$V_{\text{int}}(\phi, \bar{\phi}) = \phi^\dagger \star_\varrho \phi \star_\varrho \phi^\dagger \star_\varrho \phi, \quad V_{\text{int}}(\phi, \bar{\phi}) = \phi^\dagger \star_\varrho \phi^\dagger \star_\varrho \phi \star_\varrho \phi. \quad (3.4.8)$$

The first interaction term is referred to as ‘‘orientable’’ and the right one as ‘‘non-orientable’’. Thanks to the cyclicity of the integral (3.4.6), the two interactions (3.4.8) are the only 4-interaction one can write with two ϕ and two ϕ^\dagger fields.

In order to quantise the action (3.4.7), one can perform an analysis very similar to the one of Subsection 2.6.2. One should be careful, however, to the fact that ϕ and $\bar{\phi}$ have to be considered as distinct fields. Therefore, one need to introduce a source \bar{J} coupled to $\bar{\phi}$.

The 2-point function at one-loop is computed very similarly to (2.6.20), except that not all diagrams of Figure 4 and Figure 5 are accessible. This stems from the fact that a ϕ has to be linked to a $\bar{\phi}$ from the kinetic term of (3.4.7) together with the fact that ϕ and $\bar{\phi}$ cannot be interchanged. The latter observation makes the number of diagrams boils down from twelve to six, in each case. The former observation makes it go from six to four. Explicitly, the orientable interaction give rise to diagrams of Figure 4a, 4b, 4c and 4d, while the non-orientable interaction has diagrams of Figure 4a, 4b, 5a, 5c. From there, one computes that the orientable interaction has no UV/IR mixing in the 2-point function at one-loop, but the non-orientable has. This can be traced back to the fact that the orientable theory has no non-planar FEYNMAN diagrams, but the non-orientable has. This observation has already been made on κ -MINKOWSKI [95]. Moreover, the analysis of [121] shows that the 4-point function, at one-loop, of the orientable theory has a UV/IR mixing, stemming from the fact that now some non-planar diagram arises.

Finally, a deformed $U(1)$ gauge theory was considered on ϱ -MINKOWSKI [122]. As the integral is cyclic (3.4.6), one has *a priori* less troubles than for κ -MINKOWSKI (see Section 3.2). However, the coproduct of the deformed translations writes

$$\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta(P_3) = P_3 \otimes 1 + 1 \otimes P_3, \quad \Delta(P_\pm) = P_\pm \otimes 1 + \mathcal{E}^{\mp 1} \otimes P_\pm, \quad (3.4.9)$$

where $P_\pm = P_1 \pm iP_2$ and $\mathcal{E} = \exp(i\varrho P_0)$. Because of (3.4.9), twisted derivations are still considered, even if the integral is cyclic. We refer to Subsection 3.2.1 for a discussion on why the twisted cyclicity of the integral hinted for the use of twisted derivations. Note that this approach considered derivations with different twists so that one has to be careful when deriving the gauge theory.

The considered action is of the noncommutative YANG-MILLS theory type (2.1.21), with

$$F_{\mu\nu} = \mathcal{E}_\nu^{-1} \nabla_\mu \mathcal{E}_\nu \nabla_\nu - \mathcal{E}_\mu^{-1} \nabla_\nu \mathcal{E}_\mu \nabla_\mu \quad (3.4.10)$$

where $\mu, \nu = 0, \pm, 3$, $\mathcal{E}_\mu = (1, \mathcal{E}^{-1}, \mathcal{E}, 1)$ and ∇_μ is a twisted connection. The latter action has been shown to be ϱ -POINCARÉ invariant and $\mathcal{U}(1)$ gauge invariant.

Chapter 4

Quantum gravity and quantum space-times

As discussed in [Section 1.1](#) and [Section 1.2](#), [noncommutative geometry](#) has lots of motivations arising from [quantum gravity](#). The aim of this [Chapter 4](#) is threefold. First, we introduce the main issues and challenges of [quantum gravity](#) as a whole. Then, we depicts the main topics of [quantum gravity](#) phenomenology and the impact of the physics of quantum space-times in this field. Moreover, we also present the various constraints on the space-time deformations obtained by [quantum gravity](#) phenomenology and its experimental results. Finally, we depict the obstacles to build models of gravity on quantum space-times, models which are generically named “noncommutative gravity”. More specifically, we present a recent attempt in constructing an analogue of space-time for which the tangent space is κ -MINKOWSKI, rather than the MINKOWSKI space-time.

There has been many proposal for [quantum gravity](#) theories. For a historical note on [quantum gravity](#) research see for example [\[127\]](#). We wish to argue, in this [Chapter 4](#), that [noncommutative geometry](#), at least as a mathematical tool, is a serious candidate for expressing a [quantum gravity](#) theory, as it has inspired many promising phenomenological framework.

4.1 The motivations to study [quantum gravity](#)

4.1.1 Theoretical mismatches of quantum and gravity

The theory of general relativity has not been challenged by experiments yet, but there are theoretical reasons to think that general relativity is to be interpreted as an effective field theory of some more fundamental theory, valid below some energy scale. Indeed, when merged with quantum mechanics or quantum field theory, some inconsistencies arises.

The first one is the black hole information paradox. When considering a quantum field theory in SCHWARZSCHILD space-time, HAWKING [\[123\]](#) found that a black hole emits radiations, called HAWKING radiations, which causes the black hole to evaporate over time. The final state of the black hole is only determined by the total mass, charge and angular momentum of the initial state, so that different initial states may lead to the same final state. Therefore, by only knowing the information of the final state, one can reconstruct at most a class of initial states, but not the full

initial state: some information is lost. Therefore, there is a mismatch with quantum theories as information loss is prevented in quantum field theory via unitarity.

The second puzzle concerns the interpretation of time. In quantum mechanics, the time is considered as universal and absolute. It can be compared to a thermodynamical time or a parameter that controls the evolution of the quantum system. Whereas in general relativity, there is not a single notion of time since it is observer dependant. General covariance even requires that it is relative, in the sense that the evolution in space affects the evolution in time: one can think for example to time dilatation in special relativity. The interpretation of time in the two theories is so different that it is not even clear how a theory of quantum gravity should solve this so-called “problem of time”.

The third inconsistency lies in the computation of the vacuum energy (cosmological constant) of the Universe, if interpreted as the mean energy of all its elementary constituents described by quantum field theory. The discrepancy between the theoretical and experimental value is more then 56 orders of magnitude [124], highlighting the fact that the vacuum energy cannot be explained by quantum field theory alone.

The fourth discrepancy arises when one tries to apply quantum field theory methods on gravity. The value of the PLANCK length, corresponding to a SCHWARZSCHILD black hole having a radius of order of its COMPTON wavelength, suggests that quantum fluctuations of gravity could become relevant at small scale. Therefore, one would need a quantum (or at least semi-classical) formulation of general relativity to capture these effects. As detailed in Subsection 4.1.2, the path integral quantisation of general relativity is non-renormalisable so that we cannot make sense of the straightforward version of “quantum general relativity”. Therefore, gravity cannot be considered similarly as the other fundamental forces and needs to be quantised by other ways, if it needs to be quantised at all.

Many authors consider other puzzles that quantum gravity should solve.

- ❖ The nature of space-time at the PLANCK length is thought to be very different from a usual smooth manifold. The latter assertion takes its root in different aspects of quantum gravity be it loop quantum gravity, group field theory or the DOPLICHER, FREDENHAGEN and ROBERTS argument [24] (detailed in Subsection 1.2.2).
- ❖ The geodesic completeness, broken in singular space-times, is supposed to be restored in a quantum gravity theory thanks to the smearing out of the curvature singularity. Note that this singularity can correspond to a black hole singularity. In this sense, a quantum gravity theory is expected to grasp more insight on the black hole interior. This singularity can also be a primordial one, so that quantum gravity should resolve the constitution of the early Universe.
- ❖ The actual theoretical physics supposed to explain all what we know is made of two distinct theories: one for particle physics and the other for gravity. As the first is intrinsically quantum, a theory of quantum gravity is expected to shed light on some common ground for a unified theory.
- ❖ Some authors are hoping for a theory of quantum gravity to account for dark matter phenomena.

- ❖ As already discussed in [Section 3.3](#), the notion of causality is questioned when both quantum theory and (curved) space-time happen to coexist. A theory of [quantum gravity](#) should be able to settle if causality is an intrinsic property of nature, and if so, what quantum causality is.

4.1.2 The perturbative quantisation of gravity

We gather here the sketch of computation for the non-renormalisability of general relativity. We refer to [\[125\]](#) for more complete computations and an enhanced set of references.

The EINSTEIN-HILBERT action on a $d + 1$ -dimensional space-time \mathcal{M} is given by

$$S_{\text{EH}} = \frac{1}{16\pi} \int d^{d+1}x \sqrt{\mathbf{g}(x)} R(x), \quad (4.1.1)$$

where \mathbf{g} is the metric of \mathcal{M} and $R = \mathbf{g}^{\mu\nu} R_{\mu\nu}$ is the RICCI scalar. In perturbation theory, one consider small fluctuations \mathbf{h} of a fixed background metric η (usually considered to be the MINKOWSKI metric) as

$$\mathbf{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{16\pi} \mathbf{h}_{\mu\nu}(x). \quad (4.1.2)$$

It is important to note that this weak field expansion is not unique and that the perturbative expansion highly depends on it. Regardless of the gauge fixing, one can have different sets of FEYNMAN rules stemming from different weak field expansions. The integrand of the action can be shown to reduce, at lowest order in \mathbf{h} , to

$$-\frac{1}{4} \partial_\mu \mathbf{h}_{\nu\rho} \partial^\mu \mathbf{h}^{\nu\rho} + \frac{1}{8} (\partial_\mu \mathbf{h}^\nu{}_\nu)^2 + \frac{1}{2} \left(\partial_\nu \mathbf{h}^\nu{}_\mu - \frac{1}{2} \partial_\mu \mathbf{h}^\nu{}_\nu \right)^2, \quad (4.1.3)$$

where traces and inverse are taken with respect to the background metric. In order to perform the BECCHI-ROUET-STORA-TYUTIN quantisation, one has to introduce a gauge fixing. A convenient gauge fixing correspond to the DE DONDER gauge

$$\partial_\nu \mathbf{h}^{\nu\mu} - \frac{1}{2} \partial^\mu \mathbf{h}^\nu{}_\nu = 0. \quad (4.1.4)$$

The gauge [\(4.1.4\)](#) removes the zero modes $\mathbf{h}_{\mu\nu} \sim \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu$ which makes the propagator ill defined, in a similar fashion that the LORENZ gauge $\partial^\mu A_\mu = 0$ removes the zero modes and so the divergence in the YANG-MILLS propagator. The gauge fixing [\(4.1.4\)](#) cancels with the last term of [\(4.1.3\)](#) so that the full Lagrangian, at second order in \mathbf{h} , is given by

$$-\frac{1}{4} \partial_\mu \mathbf{h}_{\nu\rho} \partial^\mu \mathbf{h}^{\nu\rho} + \frac{1}{8} (\partial_\mu \mathbf{h}^\nu{}_\nu)^2 - \partial_\mu \bar{\mathbf{c}}_\nu \partial^\mu \mathbf{c}^\nu \quad (4.1.5)$$

where \mathbf{c} is the ghost field. One can read from [\(4.1.5\)](#) the propagator for the graviton and the ghost. The graviton 3-vertex and 4-vertex are given respectively by the expansion of [\(4.1.5\)](#) to third and fourth order in \mathbf{h} respectively. The FEYNMAN rules are quite tedious to obtain and we refer to [\[125\]](#) for their expressions.

The one-loop analysis in dimensional regularization shows that the needed counterterm for pure gravity writes

$$\frac{\sqrt{\mathbf{g}}}{8\pi^2(d+1-4)} \left(\frac{1}{120} R^2 + \frac{7}{20} R_{\mu\nu} R^{\mu\nu} \right), \quad (4.1.6)$$

where R is the RICCI scalar and $R_{\mu\nu}$ the RICCI tensor. Therefore, in $d + 1 = 4$ dimensions, general relativity is not perturbatively renormalisable, at least in the usual sense. One should note that adding minimally coupled matter to the action does not radically change the counterterm (4.1.6). The one-loop divergence can be removed either on-shell (*i.e.* when $R_{\mu\nu} = 0$), or by a field redefinition. However, the two-loop analysis shows that this magical behaviour does not occur for higher loops.

The later analysis could have been hinted by a dimensional analysis. Indeed, if one performs a power counting, the degree of divergence of a FEYNMAN diagram is given by

$$(d + 1)L + 2V - 2I \tag{4.1.7}$$

where L is the number of loop in the diagram, V the number of vertices and I the number of internal lines. The equation (4.1.8) stems from the fact that each loop involves an momentum integration $d^{d+1}p$, each vertex goes like p^2 and each internal line involves the propagator that goes as p^{-2} . For any diagram, one has the relation that $L = 1 + I - V$ which comes from a topological constraint on the graphs. Indeed, if one wants to have a single loop which goes through exactly V vertices, one need V internal lines. Each extra internal lines generates another loop, or needs another vertex not to form a loop. Therefore, there is a balance between $L - 1$ and $I - V$.

Merging this topological constraint with (4.1.7) imposes that the divergence degree of a diagram is

$$(d - 1)L + 2. \tag{4.1.8}$$

One should note that the latter number only depends on the number of loops L and that for dimensions $d + 1 > 2$, the divergence degree is growing with the number of loops. The higher the loop order, the worse the divergence. Therefore, general relativity is not perturbatively renormalisable.

One should note that the loop prefactor $(d - 1)$ in (4.1.8) is linked to the NEWTON's constant dimension ($1 - d$ length dimension).

The main pool of [quantum gravity](#) theories have tried to tackle the previous renormalisability problem by different ways. Concerning [noncommutative geometry](#), some authors are hoping that the deformation parameter would act as a ultra-violet cut-off in the spirit of SNYDER. However, the renormalisability (or sometimes finiteness) of a field theory is far more tricky on quantum space-times, even at the level of a scalar field theory, as exemplified by the [UV/IR mixing](#). The study of noncommutative field theories have some way to go before tackling the quantisation of a full noncommutative gravity theory.

Still, the main aspect we want to put forward in this manuscript is that, even if [noncommutative geometry](#) cannot give rise to a full theory of [quantum gravity](#) straightforwardly, it has been and still is an essential tool for [quantum gravity](#) phenomenology. In other words, the physics of quantum space-times could pave the way for early [quantum gravity](#) tests. In the following [Section 4.2](#), we discuss recent results in [quantum gravity](#) phenomenology and the constraint that one already has on quantum space-times.

4.2 Phenomenological considerations

4.2.1 Phenomenology of quantum gravity

At the time this manuscript is written, there is no probe of any quantum gravity effects, and even beyond general relativity ones¹. The usual cause pointed out for this experimental loophole is that energy scales involved are simply too high, the PLANCK mass being of order $M_{\text{P}} \sim 10^{19}$ GeV. However, the phenomenology of quantum gravity has already reached such energies by considering highly energetic astrophysical sources and possible effects that got amplified within cosmological distances.

We give here the main trends and results of quantum gravity phenomenology. Note that one can find many more details and an extensive set of references in the recent review of the COST Action CA18108 “Quantum Gravity Phenomenology in the multi-messenger approach” [12]. The collaboration gathered the most recent results of quantum gravity tests in their online catalogue [128].

The supposed quantum gravity effect that gathering the most hope for detection is the modification of particle dynamics in vacuum, called modified dispersion relations.

Modified dispersion relations

The modified kinematics is mostly thought to be energy dependant so that it writes

$$E = |\vec{p}| \left(1 + \epsilon \frac{n+1}{2} \frac{E^n}{E_{\text{QG}}^n} \right) \quad (4.2.1)$$

where E is the energy of the particle and \vec{p} its momentum, n is the order of deformation, E_{QG} is the energy scale at which the modification is relevant and $\epsilon = \pm 1$ parametrises if the modification is superluminal (+1) or subluminal (-1). Note that a modified dispersion relation can also trigger a modified interaction dynamics for particles. We refer to [12] for more details on this part.

As propagation is modified, two photons with two different energies would travel at different velocities². One can then think of a thought experiment of an astrophysical source emitting simultaneously two photons with different energies toward Earth. Since they travel differently, there will be a time delay in the detection of the two photons. The higher the energy discrepancy and the higher the distance to the source, the bigger the time delay. Note that one can perform the same thought experiment and change the cosmic messenger. In other words, one can consider gravitational waves or neutrinos instead of photons.

This has led to the search of time delays in highly energetic astrophysical events, mainly gamma ray bursts, active galactic nuclei and pulsars. For some events, the lower bound for E_{QG} for a first order modified dispersion relation (*i.e.* $n = 1$) has even reached the PLANCK scale. However, one has to handle a high systematic error coming from different sources: the expansion of the Universe, the “simultaneity” in time and energy measurements, the mechanism of emission at the source,

¹The flourishing phenomenology of modified gravity goes far beyond the scope of this manuscript.

²The term “velocity” here is pictorial but may be misleading. Some theoretical frameworks breaks the speed-of-light limit for particle so that the velocities of two photons are actually different. However, some other framework may require that the two photons travel at the same speed but in different (energy dependant here) backgrounds/space-times such that the arrival time of two simultaneously emitted photon is different.

scattering with the interstellar medium, *etc...* The latter errors are thought to be reduced by a statistical treatment, which would need more events.

There are other, possibly testable, effects that a modified dispersion relation triggers. First, if the messenger carries a polarisation, then a birefringence effect could occur. Specifically, the polarisation of light could be rotated of some tiny amount accumulating with distance, and gravitational wave could have different speed for $+$ and \times polarisations. Second, there could be modifications in the redshift due to the Universe expansion. Third, the gravitational lensing would become frequency dependent so that a black hole shadow would be affected by some rainbow effect. Fourth, the GREISEN-ZATSEPIN-KUZ'MIN cut-off of ultra-high-energy cosmic rays, due to their interaction with the cosmic microwave background, would be smaller as the cosmic microwave background could interact with less energetic photons. Fifth, the oscillation of neutrinos would be affected by a modification of propagation. One can find an extended list of effects and discussions in [12].

The modified dispersion relation has been experienced in several quantum gravity phenomenological frameworks, the most studied ones being doubly special relativity and LORENTZ invariance violation.

Doubly special relativity

There is an apparent contradiction between some quantum gravity framework claiming an observer-independent smallest length (of order of the PLANCK length) and the length dilatation and contraction of special relativity. The doubly special relativity framework aims at solving this issue by considering both an observer-independent upper limit for the speed (the speed of light) and an observer-independent lower limit for lengths, called ℓ_{DSR} . This implies that “inertial” observers are no more related by a POINCARÉ transformation but rather a deformation of it. Doubly special relativity models are thought to describe effectively some flat space-time limit of quantum gravity.

A doubly special relativity theory imposes a deformed momentum space via a deformation of the composition law of the momenta. Therefore, it may imply a modified dispersion relation for the photons, where the energy scale E_{QG} is to be linked with ℓ_{DSR}^{-1} .

LORENTZ invariance violation

LORENTZ invariance violation is a generic term for all effective theories that breaks LORENTZ invariance. It may be done by deforming the LORENTZ symmetry, as for example in doubly special relativity, or by imposing a preferred direction for field propagation. Indeed, one can add in the fermionic action a term of the form

$$iu_{\mu}u_{\nu}\bar{\psi}\gamma^{\mu}\nabla^{\nu}\psi \tag{4.2.2}$$

where ψ is the fermionic field, ∇ the covariant derivative and u the preferred direction of space-time encoded as a vector field. While some authors have considered u to be non-dynamical, it appears that nice results may be derived by a dynamical u coupled to gravity and having a non-zero expectation value.

LORENTZ invariance violation could generate time delays via non-covariant dispersion relations.

The mixing of dynamical tensors in the action leads to new gravitational wave polarisations. Finally, birefringence can appear for CPT-odd operators (in the context of CPT violation).

The LORENTZ invariance violation framework is vast and has already been tightly constrained by (minimal) standard model extensions, up to dimension five operators. Still, the constraints remain far below the PLANCK scale.

Other phenomenological settings arise in the context of [quantum gravity](#) phenomenology as detailed below.

Similarly to the Galilean time becoming relative in special relativity, it appears that locality becomes relative in doubly special relativity theories. More explicitly, if a process is local for a (close) observer, then for another observer (obtained after a doubly special relativity transformation) the process may be non-local. The notion of locality of, say, an interaction of particle becomes observer dependent. This is called the relative locality principle.

The notion of BORN geometry tries to settle a “covariant relative locality”. In the same spirit that space or time is observer dependent in special relativity but space-time is not, BORN geometries try to identify the global observer independent concept of causality that would lead to relative locality.

In doubly special relativity, the momentum space is curved due to a modification of the momentum addition law. It can be shown to be of constant curvature corresponding to ℓ_{DSR}^{-1} . The generalisation of doubly special relativity to a generally covariant framework has been to consider curved momentum spaces. One can ask the question of what becomes of the usual dispersion relation $p^2 + m^2 = 0$, since the latter is coordinate dependent and therefore not (momentum) covariant. In order to solve this issue, one introduces a covariant Hamiltonian for which the level sets $H(x, p) = \text{constant}$ correspond to dispersion relations. The Hamiltonian is defined on a curved phase space (x, p) , which is defined by a metric $g^{\mu\nu}(x, p)$.

As already formalised in [Subsection 1.2.2](#), the generalised uncertainty principle studies the effects of a momentum dependent HEISENBERG uncertainty principle. The HEISENBERG uncertainty principle does not impose one to have a minimal length, since one can have Δx as small as wanted if Δp is big enough. Therefore, even in its simplest form

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \ell_{\text{GUP}}^2 p^2 \right), \quad (4.2.3)$$

the generalised uncertainty principle imposes a smallest length ℓ_{GUP} . Constraints coming from gravitational wave detection, neutrino physics and cosmological studies affect the generalised uncertainty principle parameters but has not reached PLANCK scale yet. Besides, the generalised uncertainty principle imposes the black hole evaporation to stop at ℓ_{GUP} and so to leave a remnant behind. The latter consideration leads to a possible solution in the black hole information paradox discussed in [Subsection 4.1.1](#), as the “lost” information could be carried by this remnant.

4.2.2 The phenomenology of quantum space-times

Most of the phenomenological frameworks depicted above can emerge naturally in the context of [noncommutative geometry](#). We detail below how these frameworks appear on some example, as well as some constraints it imposes on the deformation parameter.

The κ -POINCARÉ algebra (3.1.1) realises a doubly special relativity with $\ell_{\text{DSR}} = \kappa^{-1}$. In other words, if one interpret the κ -POINCARÉ generators as frame transformations, then those transformations conserves both an observer independent speed and an observer independent length κ^{-1} .

LORENTZ invariance violation occurs via deformations of the POINCARÉ group. Therefore, a quantum space-time having a deformed POINCARÉ symmetry may break the usual LORENTZ invariance. Note that all forms of LORENTZ invariance violation are not necessarily linked to a symmetry deformation, since one could simply consider that the breaking of LORENTZ invariance does not give rise to another symmetry.

The ϱ -MINKOWSKI space (discussed in Section 3.4) was first considered for its close relation to relative locality [129]. The latter also has a deformed version of POINCARÉ algebra as its space of symmetries and, therefore, can be also linked with doubly special relativity and LORENTZ invariance violation.

Curved momentum space arises at least in LIE algebra-type noncommutative space-times, as developed in Section 2.6. Considering the momentum space to be the exponentiation of the coordinate space, a LIE algebra-type noncommutativity imposes a deformed composition law of momenta through the BAKER-CAMPBELL-HAUSDORFF formula. From the knowledge of this deformed law, one can compute a connection, and its associated curvature, of the momentum space.

Finally, generalised uncertainty principle are present in some quantum space-times, like the SNYDER space-time.

Note that CPT violation is thought to appear in quantum space-times. However, there are no unambiguous definitions of the discrete symmetries in a noncommutative context.

In view of all the physical considerations associated with quantum space-times, one could ask what kind of modified dispersion relation are been generated. However, when considering deformed symmetries, as in the case of κ -MINKOWSKI, the derived dispersion relation is highly dependent of the coordinate choice in the HOPF algebra of symmetries. For example, if one considers that the generalised derivative ∂_μ of the kinetic operator are P_μ , or X_μ (defined in Subsection 3.2.1), one ends up with two different dispersion relations which writes

$$P_\mu : \quad E^2 = |\vec{p}|^2, \quad X_\mu : \quad E^2 = |\vec{p}|^2 + \frac{1}{\kappa} E^3 + \mathcal{O}\left(\frac{1}{\kappa^2}\right). \quad (4.2.4)$$

Other expressions have been derived throughout the literature. If an action with a kinetic term involving the X_μ 's is considered, like in (3.2.16), the phenomenological constraints on time delays for high-energetic astrophysical photons imposes that $\kappa \gtrsim 10^{17} - 10^{19}$ GeV.

Choosing the appropriate kinetic term has been discussed extensively. It can be directly linked to the question of momentum coordinate invariance of Section 2.6.

Other physical constraints can be put on the deformation parameters, see [23] for a review. When considering a noncommutative YANG-MILLS theory (2.1.21), the expansion of the action with respect to the deformation parameter generates contributions from higher dimensional operator, which have already been studied in the context of standard model extensions. The strongest constraints comes from LORENTZ invariance violating operator which imposes $\tilde{k} \gtrsim 10^{17}$ GeV. The MOYAL space has been intensively studied in this sense, so that $\tilde{k} = \sqrt{\theta}$ in this context.

Another consideration one can have on the 5-dimensional model of κ -deformed YANG-MILLS theory developed in Section 3.2, is to relate the deformation parameter κ with the extra dimension

size. If one assumes a simple compactification scheme on the simple orbifold $\mathbb{S}^1/\mathbb{Z}_2$, then constraints from the Large Hadron Collider gives $\kappa \gtrsim 10^{13}$ GeV.

Other recent proposal for [noncommutative geometry](#) tests have been proposed, see for example [130, 131, 132, 133, 134].

4.3 Toy model of noncommutative gravity

There is a flourishing literature of formulations of gravity on quantum space-times. For a review see [23] and for an extensive set of references see [135]. Most of the studies focus on the noncommutative black hole, since this object is so compact that [quantum gravity](#) fluctuations are thought to critically change our understanding of black holes. For an early review of θ -deformed black holes see [136].

The question of what becomes of the metric is a major issue in noncommutative gravity. There have been many proposals for metric formulations on quantum space-times (bilinear or sesquilinear, symmetric or hermitian, invertible or nondegenerate, real or not, on derivations or on forms, *etc...*) and none has reached consensus.

A noncommutative metric faces other difficulties when one considers its commutative limit. Most of the noncommutative settings work with complex entries so that a noncommutative metric is necessarily complex. If no reality constraints is imposed, the corresponding commutative metric would also be complex. Furthermore, the noncommutativity of the star product \star generally imposes that the metric is not symmetric, which has also been shown to be pathological in a commutative setting.

Finally, it is not known if the “noncommutative metric” is the relevant object to study gravity on quantum space-times. For a classical space-time, the data of the metric fully determines the latter space-time and so the gravity behaviour. It is further obtained by solving the EINSTEIN equations. It is not know if such a property extends to the quantum space-times on which many metrics could be defined *a priori*. Besides, if noncommutative metrics would be in one to one correspondence with quantum space-times, then one would further need an analogue of the EINSTEIN equation, which up to date has not been derived.

We detail here the construction of the toy model of noncommutative gravity introduced in [137]. It consists of the first of a series and tries to put some mathematical grounds on the noncommutative analogue of a partition of unity. The main idea is to consider a space-time on which the MINKOWSKI tangent space is changed to be the κ -MINKOWSKI “tangent space”. If one imposes that κ -MINKOWSKI is a local tangent space, then one needs a notion of partition of unity to glue all the local pieces together and form global objects. In the following, the EINSTEIN summation convention applies for space-time indices (corresponding to the end of the Greek alphabet $\mu, \nu, \rho, \sigma, \tau$) and does not for the covering indices (corresponding to the beginning of the Greek alphabet α, β). In the latter case, the sum is always made explicit.

The previous idea takes its root in the commutative setting for which the MINKOWSKI space-time arise as a local tangent space. Indeed, if one is given a patch of local open sets $\{U_\alpha\}_\alpha$ covering the space-time \mathcal{M} , then

$$TU_\alpha \simeq U_\alpha \times \mathbb{R}^{1,d}, \quad (4.3.1)$$

where TU_α is the tangent space of U_α . The relation (4.3.1) gives rise to the vielbein formalism, where one can compare different frames of the tangent bundle at any point of U_α by a POINCARÉ transformation, meaning a transformation of $\mathbb{R}^{1,d}$. The property (4.3.1) is referred to as the local triviality of \mathcal{M} in the mathematical literature, stemming from the fact that locally the tangent bundle is only layers of copies of MINKOWSKI. Note that in the physical literature, the need of an open set U_α where local coordinates can be defined is frequently not mentioned. One usually exports objects defined on each U_α 's to the full space-time \mathcal{M} by the use of a partition of unity χ , also generally omitted in the physics literature.

Several questions now arise when trying to implement this framework. What is the analogue \mathbb{A}_α of the local set U_α in the **noncommutative geometry**? How can one implement that $\mathbb{R}_\kappa^{1,d}$ is the “tangent space” of \mathbb{A}_α , similarly to (4.3.1)? How does one glue elements of \mathbb{A}_α to the full \mathbb{A} , that is how can one define a noncommutative partition of unity? Possible answers to those questions are the specific topic of this [Section 4.3](#).

The question of generalising open covers $\{U_\alpha\}_\alpha$ of a space-time \mathcal{M} to the noncommutative setting was addressed in [138], through the notion of ideals and covering of algebras. If one considers \mathbb{A} to be the set of smooth functions $\mathcal{C}^\infty(\mathcal{M})$, then \mathbb{A}_α would correspond to $\mathcal{C}^\infty(U_\alpha)$. The two are related via the restriction on U_α , noted $|_\alpha : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathcal{C}^\infty(U_\alpha)$.

The kernel of the latter restriction $\text{Ker}(|_\alpha)$ corresponds to the set of (smooth) functions that vanish on U_α . It is an ideal of $\mathcal{C}^\infty(\mathcal{M})$ since a product of two functions, one from $\text{Ker}(|_\alpha)$ the other from $\mathcal{C}^\infty(\mathcal{M})$, vanishes necessarily on U_α . Furthermore, one has

$$\mathcal{C}^\infty(U_\alpha) = \mathcal{C}^\infty(\mathcal{M}) / \text{Ker}(|_\alpha), \quad (4.3.2)$$

in terms of quotient of ideals (see [Definition A.1.5](#)). The latter result is detailed in [Example A.1.6](#).

Furthermore, the property that $\{U_\alpha\}_\alpha$ forms a covering of \mathcal{M} translates, for the restrictions, to

$$\bigcap_\alpha \text{Ker}(|_\alpha) = \{0\}. \quad (4.3.3)$$

Indeed, if a (smooth) function is zero on every U_α , it is zero on the all \mathcal{M} as the U_α 's cover all \mathcal{M} .

Finally, concerning two open covers $\{U_\alpha\}_\alpha$ and $\{\tilde{U}_{\tilde{\beta}}\}_{\tilde{\beta}}$, the link between $\mathcal{C}^\infty(U_\alpha)$ and $\mathcal{C}^\infty(U_\alpha \cap \tilde{U}_{\tilde{\beta}})$ can also follow from (4.3.2) considering the restriction $|_{\tilde{\beta}}$ (and similarly for $\mathcal{C}^\infty(\tilde{U}_{\tilde{\beta}})$ with $|_\alpha$). On the other hand, one can go directly from $\mathcal{C}^\infty(\mathcal{M})$ to $\mathcal{C}^\infty(U_\alpha \cap \tilde{U}_{\tilde{\beta}})$ through

$$\mathcal{C}^\infty(U_\alpha \cap \tilde{U}_{\tilde{\beta}}) = \mathcal{C}^\infty(\mathcal{M}) / (\text{Ker}(|_\alpha) + \text{Ker}(|_{\tilde{\beta}})), \quad (4.3.4)$$

where $+$ denotes here the smallest closed $*$ -ideal containing both elements. The relation (4.3.4) holds for an intersection of two open sets, but it can be straightforwardly generalised to an intersection involving an arbitrary number of open sets.

The way one generalises the previous aspects to the noncommutative setting goes as follows.

Covering of algebras

Let \mathbb{A} be a **$*$ -algebra**, possibly obtained by deformation quantisation of a classical space-time. We define a family of $*$ -ideals $\{J_\alpha\}_\alpha$ to be a covering of \mathbb{A} if it satisfies

$$\bigcap_\alpha J_\alpha = \{0\}. \quad (4.3.5)$$

One can directly read that this is the noncommutative analogue of (4.3.3). Having defined

such a covering, one considers the “local” algebra \mathbb{A}_α to be the quotient of the “global” algebra \mathbb{A} by the ideal J_α , similarly to (4.3.2), *i.e.*

$$\mathbb{A}_\alpha = \mathbb{A}/J_\alpha. \quad (4.3.6)$$

Finally, the quotient of \mathbb{A} by several (here two) coverings $\{J_\alpha\}_\alpha$ and $\{\tilde{J}_\beta\}_\beta$ is given by the generalisation of (4.3.4)

$$\mathbb{A}_{\alpha\tilde{\beta}} = \mathbb{A}/(J_\alpha + \tilde{J}_\beta). \quad (4.3.7)$$

We say that a set $\{\mathbb{A}_\alpha\}_\alpha$ satisfying (4.3.5) and (4.3.6) is a covering of algebras, the ideals J_α being implied.

It is important to note that any \mathbb{A}_α can be made a ***-algebra** via the involution $[f]_\alpha^\dagger = [f^\dagger]_\alpha$, where $[f]_\alpha$ denotes the representative of the equivalent class of $f \in \mathbb{A}$. In the previous equation \dagger_α denotes the involution of \mathbb{A}_α and \dagger the involution on \mathbb{A} . Besides, we note the canonical projection $\pi_\alpha : \mathbb{A} \rightarrow \mathbb{A}_\alpha$, such that $\pi_\alpha(f) = [f]_\alpha$. The latter projection is surjective by definition. By considering the projection of \mathbb{A} on \mathbb{A}_α and then on $\mathbb{A}_{\alpha\tilde{\beta}}$, or directly from \mathbb{A} to $\mathbb{A}_{\alpha\tilde{\beta}}$, one can show that the diagram

$$\begin{array}{ccc} & \mathbb{A}_\alpha & \\ \pi_\alpha \nearrow & & \searrow \pi_\beta^\alpha \\ \mathbb{A} & \xrightarrow{\pi_{\alpha\tilde{\beta}}} & \mathbb{A}_{\alpha\tilde{\beta}} \\ \pi_\beta \searrow & & \nearrow \pi_\alpha^{\tilde{\beta}} \\ & \mathbb{A}_{\tilde{\beta}} & \end{array}$$

commutes, so that $\pi_\beta^\alpha \circ \pi_\alpha = \pi_\alpha^{\tilde{\beta}} \circ \pi_\beta = \pi_{\alpha\tilde{\beta}}$.

We now turn to the implementation of $\mathbb{R}_\kappa^{1,d}$ as the local tangent space. The idea we develop here shares some similarities with the quantum fiber bundle model developed by BRZEZIŃSKI and MAJID [36]. If one realises that the coordinates on κ -MINKOWSKI has trivial coproduct $\Delta(x_\mu) = x_\mu \otimes 1 + 1 \otimes x_\mu$ (see (3.1.3b)), then the x_μ 's behaves as derivations on any $\mathbb{R}_\kappa^{1,d}$ -module algebra. Explicitly, if one assumes that $(\mathbb{A}_\alpha, \star_\alpha)$ is a $\mathbb{R}_\kappa^{1,d}$ -module algebra, the coproduct yields

$$x_\mu \triangleright (f \star_\alpha g) = (x_\mu \triangleright f) \star_\alpha g + f \star_\alpha (x_\mu \triangleright g), \quad (4.3.8)$$

for any $f, g \in \mathbb{A}_\alpha$. Note that in the case of ***-algebras**, the relation

$$(x \triangleright f)^\dagger = S(x)^\dagger \triangleright f^\dagger \quad (4.3.9)$$

holds, for any $x \in \mathbb{R}_\kappa^{1,d}$ and $f \in \mathbb{A}_\alpha$, where S is the antipode of $\mathbb{R}_\kappa^{1,d}$, and \dagger its involution. As in [36], we require that

$$\mathbb{A}_\alpha \text{ is a } \mathcal{T}_\kappa^{1,d}\text{-comodule algebra} \quad (4.3.10)$$

(see Definition A.2.5), with coaction $\blacktriangleleft_\alpha : \mathbb{A}_\alpha \rightarrow \mathbb{A}_\alpha \otimes \mathcal{T}_\kappa^{1,d}$. From (4.3.10), one can derive that \mathbb{A}_α is a $\mathbb{R}_\kappa^{1,d}$ -module algebra with the action

$$x \triangleright f = \sum \langle f_{(1)}, x \rangle f_{(0)} \quad (4.3.11)$$

for any $x \in \mathbb{R}_\kappa^{1,d}$ and $f \in \mathbb{A}_\alpha$, where we used the SWEEDLER notations $\blacktriangleleft_\alpha = \sum f_{(0)} \otimes f_{(1)}$ and where $\langle \cdot, \cdot \rangle : \mathcal{T}_\kappa^{1,d} \times \mathbb{R}_\kappa^{1,d} \rightarrow \mathbb{C}$ denotes the dual pairing between $\mathcal{T}_\kappa^{1,d}$ and $\mathbb{R}_\kappa^{1,d}$.

Considering the set

$$\mathfrak{D}_\kappa = \text{Span}\left(\{x_\mu\}_{\mu=0,\dots,d}\right), \quad (4.3.12)$$

consisting of the linear span of the x_μ 's, one has from (4.3.8), stemming from (4.3.10) and (4.3.11), that \mathfrak{D}_κ is a sub-LIE algebra of $\text{Der}(\mathbb{A}_\alpha)$. In the spirit of the restricted derivation-based differential calculus, we consider the set of restricted derivations (see discussion of Section 2.5)

$$\text{Der}_R(\mathbb{A}_\alpha) = \mathcal{Z}(\mathbb{A}_\alpha) \otimes \mathfrak{D}_\kappa \quad (4.3.13)$$

on \mathbb{A}_α . It can be shown that $\text{Der}_R(\mathbb{A}_\alpha)$ is a sub-LIE algebra of $\text{Der}(\mathbb{A}_\alpha)$ and a $\mathcal{Z}(\mathbb{A}_\alpha)$ -module. One can further build a (restricted) differential calculus $(\Omega_R^\bullet(\mathbb{A}_\alpha), d_\alpha, \wedge_\alpha)$ by following the steps of Section 1.5. It can also be shown that $\Omega_R^1(\mathbb{A}_\alpha) \simeq \mathbb{A}_\alpha \otimes \mathfrak{D}'_\kappa$. Here \mathfrak{D}'_κ stands as the dual of \mathfrak{D}_κ and corresponds to the linear span of the deformed translations $P^\mu \in \mathcal{T}_\kappa^{1,d}$.

The situation above corresponds quite closely to the commutative setting (4.3.1). Indeed, the derivations correspond to the sections of the tangent bundle, that is $\text{Der}(\mathcal{C}^\infty(U_\alpha)) = \Gamma(U_\alpha)$, but from (4.3.1), one has

$$\Gamma(U_\alpha) \simeq \mathcal{C}^\infty(U_\alpha) \otimes \mathbb{R}^{1,d} \quad (4.3.14)$$

which exactly match (4.3.13) at the commutative limit. The previous statement is usually written, in the physics literature, as $\zeta = \zeta^\mu \partial_\mu$ for any vector field ζ (the tensor product being implied). Following these notation, we denote below $\zeta = \zeta^\mu x_\mu$, for any $\zeta \in \text{Der}_R(\mathbb{A}_\alpha)$, where $\zeta^\mu \in \mathcal{Z}(\mathbb{A}_\alpha)$. Correspondingly, the set of one-forms writes $\Omega^1(U_\alpha) \simeq \mathcal{C}^\infty(U_\alpha) \otimes \mathbb{R}_\kappa^{1,d}$, which is the commutative limit of $\Omega_R^1(\mathbb{A}_\alpha) \simeq \mathcal{C}^\infty(U_\alpha) \otimes \mathfrak{D}'_\kappa$.

The latter construction of local derivations, because of its close relation to the commutative setting, has some similarities with the general relativity setting for gravity. Indeed, if one defines a connection ∇ , as in Section 2.1, on the module $\mathbb{X} = \text{Der}_R(\mathbb{A}_\alpha)$, then it is fully determined by its components on \mathfrak{D}_κ , that is

$$\nabla_{x_\mu}(x_\nu) = \Gamma_{\mu\nu}^\rho x_\rho \quad (4.3.15)$$

with $\Gamma_{\mu\nu}^\rho \in \mathcal{Z}(\mathbb{A}_\alpha)$. If one requires that ∇ is hermitian, that is $\nabla_{\zeta^\dagger \alpha}(\xi^\dagger \alpha) = (\nabla_\zeta(\xi))^\dagger \alpha$ for any $\zeta, \xi \in \text{Der}_R(\mathbb{A}_\alpha)$, then one has

$$(\Gamma_{\mu\nu}^\rho)^\dagger \alpha = -\Gamma_{\mu\nu}^\rho \quad (4.3.16)$$

by making use of (3.1.3b) together with (4.3.9). Moreover, the associated curvature, defined in (2.1.3), writes

$$R_{\mu\nu\rho}{}^\sigma = (x_\mu \triangleright \Gamma_{\nu\rho}^\sigma) - (x_\nu \triangleright \Gamma_{\mu\rho}^\sigma) + \Gamma_{\nu\rho}^\tau \Gamma_{\mu\tau}^\sigma - \Gamma_{\mu\rho}^\tau \Gamma_{\nu\tau}^\sigma - C_{\mu\nu}{}^\tau \Gamma_{\tau\rho}^\sigma \quad (4.3.17)$$

where $C_{\mu\nu}{}^\tau = \frac{i}{\kappa}(\delta_\mu^0 \delta_\nu^\tau - \delta_\nu^0 \delta_\mu^\tau)$ is the structure constant of the \mathfrak{D}_κ algebra, that one can read from (3.1.3a). The expression (4.3.17) is very akin to the usual expression for the space-time curvature.

Finally, we tackle the last question on how to define a coherent noncommutative partition of unity. It is quite instructive to start with the commutative definition of a partition of unity

Definition 4.3.1 – Partition of unity

Let \mathcal{M} be a space-time. A partition of unity on \mathcal{M} is a set of functions $\{\chi_\alpha\}_\alpha$ satisfying

- (i) $\chi_\alpha \in \mathcal{C}^\infty(\mathcal{M})$,
- (ii) $\{\text{Supp}(\chi_\alpha)\}_\alpha$ is locally finite,
- (iii) $\chi_\alpha \geq 0$,
- (iv) $\sum_\alpha \chi_\alpha = 1$,

where Supp denotes the support. The partition is further said to be subordinate to an open cover $\{U_\alpha\}_\alpha$ of \mathcal{M} , if for every β there exists α such that $\text{Supp}(\chi_\beta) \subset U_\alpha$. We say that the partition is adapted to the cover whenever $\beta = \alpha$. In this case, the cover and the partition of unity share the same set of indices.

A partition of unity is manufactured to glue pieces of objects, defined only locally, to have a globally defined object. Requirement (i) ensures that the partition of unity does not deteriorate the smoothness of the objects we work with. (ii) allows for χ_α to be only non-zero inside U_α (in the case of an adapted partition) and thus makes χ_α “select” the desired U_α . Finally, (iii) together with (iv) ensure that no information is lost: gluing all the pieces together gives a coherent object. As an example, if one considers $f \in \mathcal{C}^\infty(\mathcal{M})$, then the identity

$$f = \sum_\alpha \chi_\alpha f|_\alpha \quad (4.3.18)$$

holds. The different local pieces $f|_\alpha$ are reconstructed to be f via the partition of unity χ_α . The previous statement (4.3.18) is at the very basis of what we want to achieve with a noncommutative partition of unity, that is to define objects on the local algebras \mathbb{A}_α and to glue local pieces together to have a global object on \mathbb{A} . Note that the latter gluing should not depend on the choice of the cover or the partition of unity. The independence relies on the property that the product of a partition of unity is still a partition of unity. Therefore, we need to export this property to the noncommutative setting also. More explicitly, if $\{U_\alpha\}_\alpha$ and $\{\tilde{U}_\beta\}_\beta$ are open covers of \mathcal{M} with adapted partition of unity $\{\chi_\alpha\}_\alpha$ and $\{\tilde{\chi}_\beta\}_\beta$ respectively, then $\{\chi_\alpha \tilde{\chi}_\beta\}_{\alpha, \beta}$ is an adapted partition of unity for $\{U_\alpha \cap \tilde{U}_\beta\}_{\alpha, \beta}$.

We define the noncommutative partition of unity as

Definition 4.3.2 – Noncommutative partition of unity

Let \mathbb{A} be a **-algebra*. A partition of unity on \mathbb{A} is a set of elements $\{\chi_\alpha\}_\alpha$ satisfying

- (i) $\chi_\alpha \in \mathbb{A}$,
- (ii) $\{\text{Supp}(\chi_\alpha)\}_\alpha$ is locally finite,
- (iii) $\chi_\alpha \geq 0$, i.e. it exists $\varsigma_\alpha \in \mathbb{A}$ such that $\chi_\alpha = \varsigma_\alpha \star \varsigma_\alpha^\dagger$,
- (iv) For any $f \in \mathbb{A}$, $\sum_\alpha \chi_\alpha \star f = f$,

where $\text{Supp}(f) = \{\varphi \in \Phi_\mathbb{A}, \varphi(f) \neq 0\}$ and $\Phi_\mathbb{A}$ is the space of characters of \mathbb{A} (see [Subsection B.2.1](#)). The partition of unity is further said to be subordinate to an algebra cover $\{\mathbb{A}_\alpha\}_\alpha$ of \mathbb{A} , if for every β there exists α such that

$$\text{Supp}(\chi_\alpha) \subset \text{Ker}(J_\beta) = \{\varphi \in \Phi_\mathbb{A}, \varphi(f) = 0 \forall f \in J_\beta\} \quad (4.3.19)$$

We say that the partition is adapted to the cover whenever $\beta = \alpha$.

Several comments are in order.

First, the [Definition 4.3.2](#) perfectly match the [Definition 4.3.1](#) at the commutative limit. The latter statement mainly holds thanks to the [GEL'FAND-NAÏMARK theorem](#), which gives a one to one correspondence between the space of characters and the set of points of the space. Note that the requirement [\(iv\)](#) has been changed to match the case of a non-unital algebra. However, one could work with an approached unit or a unitalisation of the algebra and define [\(iv\)](#) as in [Definition 4.3.1](#).

The condition of subordinate partition [\(4.3.19\)](#) is akin to the one of [Definition 4.3.1](#) in the commutative case. Indeed, in this case, one has that $J_\beta = \text{Ker}(|_\beta)$ so that $\text{Ker}(J_\beta) = \{x \in \mathcal{M}, f(x) = 0 \forall f \in \text{Ker}(|_\beta)\} = U_\beta$.

Furthermore, one has to specify a topology in order to define the notion of local finiteness in [\(ii\)](#). Indeed, the local finiteness imposes that there are a finite number of indices α for which $\text{Supp}(\chi_\alpha) \cap V \neq \emptyset$, given any neighbourhood V . Therefore, one needs to specify what are the neighbourhoods in $\Phi_{\mathbb{A}}$. We take the latter to be given by the weak* topology.

One can show that given two partitions of unity $\{\chi_\alpha\}_\alpha$ and $\{\tilde{\chi}_\beta\}_\beta$, adapted to some covering of algebras $\{\mathbb{A}_\alpha\}_\alpha$ and $\{\tilde{\mathbb{A}}_\beta\}_\beta$ of \mathbb{A} , then $\{\chi_\alpha \star \tilde{\chi}_\beta\}_{\alpha,\beta}$ is a partition of unity adapted to the covering $\{\mathbb{A}_{\alpha\beta}\}_{\alpha,\beta}$. This can be shown using that

$$\chi_\alpha \star \tilde{\chi}_\beta = \tilde{\chi}_\beta \star \chi_\alpha = \chi_\alpha \bullet \tilde{\chi}_\beta = \tilde{\chi}_\beta \bullet \chi_\alpha, \quad (4.3.20)$$

where $\chi_\alpha \bullet f = \varsigma_\alpha \star f \star \varsigma_\alpha^\dagger$, for any $f \in \mathbb{A}$ and where ς_α is defined by [\(iii\)](#).

Finally, one can show that the diagonal elements of the matrix basis on the [MOYAL](#) space $\{f_{mm}\}_{m \in \mathbb{N}}$ (see [Section 2.5](#)), forms a partition of unity.

Having built the noncommutative partition of unity, we know how to export the local derivations and the local forms to define global objects. Thus, if one considers

$$\text{Der}_R(\mathbb{A}) = \left\{ \sum_\alpha \chi_\alpha \star (\zeta_\alpha \circ \pi_\alpha), \zeta_\alpha \in \text{Der}_R(\mathbb{A}_\alpha) \right\}, \quad (4.3.21)$$

then it consists of a $\mathcal{Z}(\mathbb{A})$ -submodule of $\text{Der}(\mathbb{A})$. Using the explicit expression of $\text{Der}_R(\mathbb{A}_\alpha)$ [\(4.3.13\)](#), one has that any $\zeta \in \text{Der}_R(\mathbb{A})$ is of the form

$$\zeta = \sum_\alpha (\chi_\alpha \star \zeta_\alpha^\mu) x_\mu^\alpha \quad (4.3.22)$$

where $\zeta_\alpha^\mu \in \mathcal{Z}(\mathbb{A}_\alpha)$ and $x_\mu^\alpha = x_\mu \circ \pi_\alpha$. The latter form [\(4.3.22\)](#) corresponds to the usual formula for vector fields on a differential manifold.

Similarly as above, one can define

$$\Omega_R^n(\mathbb{A}) = \left\{ \sum_\alpha \chi_\alpha \star \omega^\alpha \left(\pi_\alpha(\cdot \star \chi_\alpha), \dots, \pi_\alpha(\cdot \star \chi_\alpha) \right), \omega^\alpha \in \Omega_R^n(\mathbb{A}_\alpha) \right\} \quad (4.3.23)$$

and show that it is a \mathbb{A} -submodule of $\Omega^n(\mathbb{A})$. This noncommutative analogue seems also to conserve the ‘‘local’’ property of forms. This follows from

$$\omega^\alpha \wedge_\alpha \eta^\alpha = \pi_\alpha(\omega \wedge \eta), \quad (d\omega)|_\alpha = d_\alpha \omega|_\alpha, \quad (4.3.24)$$

where in the first equality $\omega^\alpha \in \Omega^n(\mathbb{A}_\alpha)$ (resp. $\eta^\alpha \in \Omega^m(\mathbb{A}_\alpha)$) denotes a convenient projection of $\omega \in \Omega^n(\mathbb{A})$ (resp. $\eta \in \Omega^m(\mathbb{A})$), and in the second equality $\omega|_\alpha \in \Omega^n(\mathbb{A}_\alpha)$ denotes an appropriate

notion of restriction of $\omega \in \Omega^n(\mathbb{A})$. Using the explicit structure of $\Omega_R^1(\mathbb{A}_\alpha) \simeq \mathbb{A}_\alpha \otimes \mathfrak{D}'_\kappa$, any $\omega \in \Omega_R^1(\mathbb{A})$ is of the form

$$\omega = \sum_\alpha (\chi_\alpha \star \omega_\mu^\alpha) P_\alpha^\mu, \quad (4.3.25)$$

where $\omega_\mu^\alpha \in \mathbb{A}_\alpha$ and $P_\alpha^\mu = P^\mu(\pi_\alpha(\cdot \star \chi_\alpha))$. Again, the component form (4.3.25) corresponds to the usual formula for 1-forms on a differential manifold.

The previous formalism offers a convenient generalisation of locally trivial tangent bundle with κ -MINKOWSKI arising as the space of derivations. Furthermore, it brings a formalism close to the tensorial formalism on commutative manifolds. However, a gravity theory would need further an action functional to be defined. In order to define an action, one needs a notion of integral (or trace) that has not been defined in this context. The partition of unity could be of use to export an integral on a local algebra (\mathbb{A}_α) to an integral on the global algebra (\mathbb{A}), as in the commutative case. One should then explore the impact of this local triviality on the theory defined on \mathbb{A}_α .

Conclusion

The physics of quantum space-times is very promising for the study of [quantum gravity](#) phenomenology and has not yet delivered all its secrets. Most of the puzzles of quantum space-times arise when considering field theories and particle states. The possible alternatives or redefinitions of such objects could pave the way for new physics and maybe experimental tests. Even if [noncommutative geometry](#) is not the fitted formalism for a full theory of [quantum gravity](#), it remains one of the best framework that may describe [quantum gravity](#) effects, at least in some regime.

As the [Chapter 1](#) illustrates, “[noncommutative geometry](#)” is a generic word which designates intertwined mathematical frameworks that tries to quantise the geometry. The field and gauge formalisms described in [Chapter 2](#) also convey the actual lack of a global framework or at least some criterion to distinguish between all the possible choices. Constructing noncommutative analogues can be performed by several ways and introduces a huge, and for now unconstrained, freedom to write noncommutative field theories.

Beyond the plural nature of quantum space-times, some common features arise. Indeed, in [Section 2.6](#) the [UV/IR mixing](#) phenomenon is tackled. Vaguely described in the literature as the appearance of mixed ultra-violet and infra-red divergences in field and gauge theories, an unambiguous definition [\[76\]](#) of the [UV/IR mixing](#) is proposed in this manuscript. It consists of three propositions: the UV divergence of the planar diagrams, the IR singularity of the non-planar diagrams and the UV finiteness of the non-planar diagrams. It was shown that the two first points is equivalent to the divergence of the propagator, thus designated as a criterion for [UV/IR mixing](#). However, the third point has not been addressed since the non-planar diagrams involve a non-trivial conservation of momenta that mixes the external momenta with the internal momenta. The study of the UV finiteness of the non-planar diagrams requires both the behaviour of the propagator and the solution of this non-trivial momentum conservation. The previous computation works for any [LIE algebra](#)-type noncommutative space-times and so consists of the first attempt to generalise [UV/IR mixing](#) studies.

The path to the understanding of the mixing has still some way to go. First, it consists of a one-loop analysis of the 2-point function. One could ask what become of the mixing to higher loop orders and for generic n -point functions. The presence of other new phenomenon for higher loops and higher correlation functions needs to be better characterised. The [UV/IR mixing](#) has been experienced mostly for [LIE algebra](#)-type noncommutative space-times, but may be generalised to other quantum space-times. The [LIE algebra](#)-type allows one to define properly a momentum space and therefore to work with it. In the case of [HOPF algebra](#) formalism, one could also consider a momentum space through a HOPF duality with the quantum space-time. This idea needs to be further explored. Moreover, the analysis of [Section 2.6](#) is performed for scalar field theory, but the mixing has also been experienced in noncommutative YANG-MILLS theories on

MOYAL. Therefore, one could think of broadening this study to gauge theories. Eventually, the ordering ambiguity defining the momentum space may be of primordial importance in the study of noncommutative field theories. Indeed, it is not clear whether physical predictions depend or not on the chosen ordering prescription, which is only present in the noncommutative setting. The question of choosing a “physical” ordering or working with an ordering invariant theory³ has not been settled and could have a dramatic impact on the physical interpretation of these theories.

The study of gauge theories on quantum space-times has been explored through several examples. The case of κ -MINKOWSKI is discussed in [Section 3.2](#). In order to solve the loss of cyclicity of the integral, this gauge theory relies on the notion of twisted derivations. Still, the gauge invariance of the κ -POINCARÉ-invariant YANG-MILLS-like action imposes that the space-time dimension is 5. This can be directly traced back to the fact that the deformed cyclicity of the integral involves the space-time dimension. The 1-point function (tadpole) of the latter action is computed at one-loop order. It is shown to be both non-vanishing and gauge dependant, as experienced on other quantum space-times, like [MOYAL](#) and \mathbb{R}_λ^3 . The gauge dependence of the tadpole is very puzzling since the considered action is gauge invariant and since the gauge dependence is not present in the commutative theory. Note that the tadpole still vanishes at the commutative limit in any gauge.

The origin of this noncommutative gauge anomaly has not been traced back and could stem from different sources. First, it is not clear that the A variable corresponds to the physical (or at least mathematically convenient) variable to encode the “noncommutative photon”. One of the promising alternative variable on [MOYAL](#) was the so-called covariant coordinate \mathcal{A} . Alternatives to A have not been explored outside [MOYAL](#). Second, the gauge anomaly could be due to an unsuitable quantisation procedure. The path followed in this manuscript consists of the usual BECCHI-ROUET-STORA-TYUTIN quantisation method. The latter may need to be adapted to the noncommutative case for a consistent treatment.

In [Section 3.3](#), a toy model of causality on κ -MINKOWSKI is constructed. It relies on the formulation of Lorentzian [spectral triple](#) and establishes the causality relation of two (pure) states, as inspired by the commutative setting. Even if this model relies on a too small HILBERT space, an analogue of the speed-of-light limit has been derived. It corresponds to an inequality constraint between the expectation value of the time and space operators for causally related states.

This model consists of one of the first established causality on quantum space-times and has therefore room for improvements. One of the most phenomenologically relevant investigation consists of deriving this speed-of-light limit for more complex operators. This could lead to a full set of constraints.

Finally, a toy model for noncommutative gravity is derived in [Section 4.3](#). It aims at characterising the quantum space-times through κ -MINKOWSKI which arises as the local tangent space. The notion of locality is here defined in the mathematical sense of open sets. Once objects are defined on the local algebras via the use of κ -MINKOWSKI, they are glued together via a noncommutative partition of unity to define global objects. One can thus build the tensorial formalism on the global algebra by only making use of properties of κ -MINKOWSKI.

³The link between ordering invariance and momentum diffeomorphism invariance has only been made for the κ -MINKOWSKI space-time. Whether this correspondence can be exported to other quantum space-times is still to be determined.

This rather mathematical framework needs a physical follow up in which an action is implemented. One has first to define an integral (or trace) on the global algebra, which could express through the integral over κ -MINKOWSKI.

Appendix A

HOPF algebras and quantum groups

There is no well-posed definition of what quantum groups are. But it is commonly acknowledged that **HOPF algebras** give a common structure to most of the quantum groups encountered. In this sense, **HOPF algebras** seems to be an interesting mathematical structure that we focus on in this Appendix. More insights on quantum groups are given in [Section 1.4](#), and we refer to [\[31, 32, 33\]](#) for more details on these algebraic structures.

A.1 Algebraic structures

We here introduce the algebraic structure needed in the context of quantum groups. They are defined mainly as vector space over \mathbb{C} , but one could also consider \mathbb{R} or other generic fields.

A.1.1 Algebras

In **noncommutative geometry**, the classical notion of space-time is replaced by the algebraic notion of algebra, as motivated by [Example A.1.2](#) and the commutative **GEL'FAND-NAÏMARK theorem**.

Definition A.1.1 – Algebra

An associative algebra is a vector space \mathbb{A} over \mathbb{C} with a multiplicative intern law $\star : \mathbb{A} \otimes \mathbb{A} \rightarrow \mathbb{A}$, which satisfies

$$f \star (g + h) = f \star g + f \star h, \quad (\text{distributive over the sum}) \quad (\text{A.1.1a})$$

$$(zf) \star g = f \star (zg) = z(f \star g), \quad (\text{compatible with the scalar product}) \quad (\text{A.1.1b})$$

$$f \star (g \star h) = (f \star g) \star h, \quad (\text{associativity}) \quad (\text{A.1.1c})$$

for any $f, g, h \in \mathbb{A}$ and $z \in \mathbb{C}$.

The algebra \mathbb{A} is said unital if it exists $1_{\mathbb{A}} \in \mathbb{A}$ such that

$$f \star 1_{\mathbb{A}} = 1_{\mathbb{A}} \star f = f. \quad (\text{A.1.1d})$$

The algebra \mathbb{A} is further said commutative if \star satisfies

$$f \star g = g \star f. \quad (\text{A.1.1e})$$

As straightforward examples, $(\mathbb{R}, \cdot, 1)$ and $(\mathbb{C}, \cdot, 1)$ are unital commutative algebras. Despite, this genuine remark, one can build an algebra out of any set \mathcal{X} through the space of functions over \mathcal{X} going into \mathbb{C} , as detailed [Example A.1.2](#).

The most simple and known example of a noncommutative algebra is the space of $n \times n$ matrices $\mathbb{M}_n(\mathbb{C})$ (with $n > 1$), equipped with the matrix product and a unit given by the identity matrix $\mathbb{1}_n$. In the context of [noncommutative geometry](#), the space of matrices was quite studied as it allows one to mimic a $SU(n)$ gauge theory. See the noncommutative YANG-MILLS theory of [Section 2.1](#).

Example A.1.2 – Algebra of functions

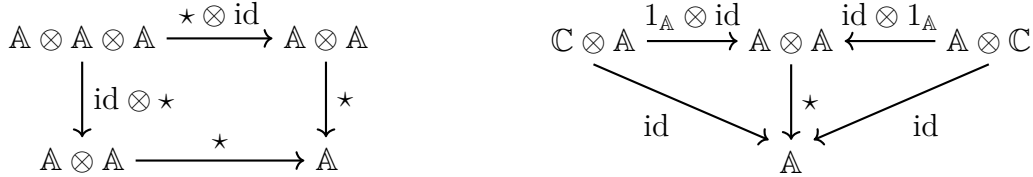
Consider \mathcal{X} to be any set of any kind of objects. Now consider the space of functions $f : \mathcal{X} \rightarrow \mathbb{C}$, noted \mathcal{X}' , called the dual of \mathcal{X} . From the algebra structure of \mathbb{C} , one can endow \mathcal{X}' with an algebra structure with product and sum given by

$$(zf + g)(x) = zf(x) + g(x), \quad (f \cdot g)(x) = f(x)g(x)$$

for any $f, g \in \mathcal{X}'$, $x \in \mathcal{X}$ and $z \in \mathbb{C}$. The product of \mathcal{X}' is called the “commutative product” throughout the manuscript and corresponds to the product (1.1.1).

Note that the [Example A.1.2](#) could be done also considering functions on \mathbb{R} instead of \mathbb{C} .

One could express the [Definition A.1.1](#) by considering the maps $\star : \mathbb{A} \otimes \mathbb{A} \rightarrow \mathbb{A}$ and $1_{\mathbb{A}} : \mathbb{C} \rightarrow \mathbb{A}$ and require that the following diagrams commute



In this sense, one can have the correspondence

$$\begin{aligned} (\star \otimes \text{id}) \circ \star &= (\text{id} \otimes \star) \circ \star && \iff && \text{(A.1.1c)} \\ (1_{\mathbb{A}} \otimes \text{id}) \circ \star &= (\text{id} \otimes 1_{\mathbb{A}}) \circ \star = \text{id} && \iff && \text{(A.1.1d)} \\ \star \circ \tau &= \star && \iff && \text{(A.1.1e)} \end{aligned}$$

where $\tau : \mathbb{A} \otimes \mathbb{A} \rightarrow \mathbb{A} \otimes \mathbb{A}$ is the flip map, *i.e.* $\tau(f \otimes g) = g \otimes f$. Recall that (A.1.1c) corresponds to the associativity of \star , (A.1.1d) expresses that $1_{\mathbb{A}}$ is a unit for \star and (A.1.1e) that \star is commutative.

The elements of \mathbb{A} commuting with any other elements of \mathbb{A} corresponds to the center of \mathbb{A} , denoted $\mathcal{Z}(\mathbb{A})$, *i.e.* $z \in \mathcal{Z}(\mathbb{A})$ if and only if

$$z \star f = f \star z, \tag{A.1.2}$$

for any $f \in \mathbb{A}$.

The notions of the charge conjugation of a particle or of adjoint operators are defined by involutions. This is implemented via the notion of \star -algebra.

Definition A.1.3 – *-algebra

An algebra \mathbb{A} is said to be a ***-algebra** if it has an antihomomorphism involution $\dagger : \mathbb{A} \rightarrow \mathbb{A}$, *i.e.*

$$(z_1 f + z_2 g)^\dagger = \bar{z}_1 f^\dagger + \bar{z}_2 g^\dagger, \quad (\text{antilinear}) \quad (\text{A.1.3a})$$

$$(f \star g)^\dagger = g^\dagger \star f^\dagger, \quad (\text{antimorphism}) \quad (\text{A.1.3b})$$

$$(f^\dagger)^\dagger = f, \quad (\text{idempotent}) \quad (\text{A.1.3c})$$

for any $f, g \in \mathbb{A}$ and $z_1, z_2 \in \mathbb{C}$, where \bar{z} denotes the complex conjugate of z .

If \mathbb{A} is unital, then one has $1^\dagger = 1$.

The algebra of [Example A.1.2](#) can be turned into a ***-algebra** thanks to the complex conjugation $f^\dagger(x) = \overline{f(x)}$. In this example, it is important to note that the algebra structure of \mathcal{X}' does not bring any information on the elements of \mathcal{X} , since it is fully exported from \mathbb{C} . Similarly, the ***-structure** does not bring additional information on the space \mathcal{X} , since it also comes from \mathbb{C} . In the context of deformation quantisation, one can deform the product of \mathcal{X}' into a noncommutative one to obtain a (noncommutative) algebra of functions noted $\mathcal{X}'[[\hbar]]$. Here, \hbar denotes the deformation parameter. One can also deform the involution \dagger such that $\mathcal{X}'[[\hbar]]$ becomes a ***-algebra**.

The structure of algebra carries also a notion of ideals, that is of great use in [Section 4.3](#). An ideal is a subset of elements that is stable under multiplication. It can be seen as a set of elements sharing a property and propagating this property to other elements. Its importance relies in the notion of quotient algebra. The quotient algebra correspond to the packing of the algebra elements, which are grouped with respect to the latter property. A simple example corresponds to $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ which correspond to packing integers to two “groups”: odd and even.

Definition A.1.4 – Ideal of algebra

Let \mathbb{A} be an algebra. A sub-vector space J is a left (resp. right) ideal if it is stable by left (resp. right) multiplication, *i.e.* for any $f \in \mathbb{A}$ and $g \in J$, one has $f \star g \in J$ (resp. $g \star f \in J$). It is often written as $\mathbb{A}J \subset J$ (resp. $J\mathbb{A} \subset J$).

A two-sided ideal is both a left and right ideal.

Definition A.1.5 – Quotient algebra

Let J be a two-sided ideal of an algebra \mathbb{A} . We define the relation $f \sim g$ if and only if $f - g \in J$, so that its equivalent class are written $[f] = f + J$. The quotient space, that is the space of equivalent classes, \mathbb{A}/J forms an algebra with product $[f \star g] = [f] \star [g]$.

We further define the so-called canonical projection $\pi : \mathbb{A} \rightarrow \mathbb{A}/J$ by $\pi(f) = [f]$. It is surjective by definition.

Proof. We need to verify that the product in \mathbb{A}/J is well-defined:

$$[f] \star [g] = (f + J) \star (g + J) = f \star g + f \star J + J \star g + J^2 = f \star g + J = [f \star g] \quad \square$$

An example of algebra quotient is given by the universal enveloping algebra, in [Definition A.1.10](#). We give another example with some importance for this manuscript.

Example A.1.6 – Local functions as a quotient algebra

Let \mathcal{M} be a space-time and $\mathcal{C}^\infty(\mathcal{M})$ the algebra of smooth functions over \mathcal{M} . Consider a covering of open set $\{U_\alpha\}_\alpha$ of \mathcal{M} , that is $U_\alpha \subset \mathcal{M}$ for any α and $\bigcup_\alpha U_\alpha = \mathcal{M}$. In the context of differential manifolds, this patch of open sets $\{U_\alpha\}_\alpha$ always exist and U_α can further be equipped with coordinates in $\mathbb{R}^{1,d}$, where d is the spacial dimension of \mathcal{M} . In other words, U_α corresponds to a region of space that is “local” and that we can describe with coordinates.

We want to characterise algebraically the space of local functions, that is $\mathcal{C}^\infty(U_\alpha)$, by only knowing $\mathcal{C}^\infty(\mathcal{M})$. This is done by considering some quotient algebra. Indeed, let $|\alpha : \mathcal{C}^\infty(\mathcal{M}) \rightarrow \mathcal{C}^\infty(U_\alpha)$ denote the restriction from \mathcal{M} to U_α , i.e. $f|_\alpha(x) = f(x)$ for any $x \in U_\alpha$. Then, $\text{Ker}(|\alpha)$ is a (two-sided) ideal of $\mathcal{C}^\infty(\mathcal{M})$ and we have

$$\mathcal{C}^\infty(U_\alpha) = \mathcal{C}^\infty(\mathcal{M})/\text{Ker}(|\alpha). \quad (\text{A.1.4})$$

Finally, the equivalent classes corresponds to the further restriction, that is $[f]_\alpha = f|_\alpha$, for any $f \in \mathcal{C}^\infty(\mathcal{M})$.

Proof. We prove here that

$$\text{Ker}(|\alpha) = \{f \in \mathcal{C}^\infty(\mathcal{M}), f|_\alpha = 0\} = \{f \in \mathcal{C}^\infty(\mathcal{M}), f(x) = 0 \text{ for any } x \in U_\alpha\}$$

is a two-sided ideal. Indeed, let $g \in \mathcal{C}^\infty(\mathcal{M})$ and $f \in \text{Ker}(|\alpha)$, then for any $x \in U_\alpha$,

$$(f \cdot g)(x) = f(x)g(x) = 0, \quad (g \cdot f)(x) = g(x)f(x) = 0,$$

so that $f \cdot g \in \text{Ker}(|\alpha)$ and $g \cdot f \in \text{Ker}(|\alpha)$. □

A.1.2 LIE algebras

The notion of group of symmetries is rather mathematically expressed as elements of a LIE group. Any LIE group is in one-to-one correspondence with a [LIE algebra](#). The latter corresponds physically to the infinitesimal transformations associated with the former symmetries.

Definition A.1.7 – LIE algebra

A [LIE algebra](#) is a vector space \mathfrak{g} over \mathbb{C} equipped with a bracket $[\cdot, \cdot]_\mathfrak{g} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$[z_1X + z_2Y, Z]_\mathfrak{g} = z_1[X, Z]_\mathfrak{g} + z_2[Y, Z]_\mathfrak{g}, \quad (\text{bilinearity}) \quad (\text{A.1.5a})$$

$$[X, Y]_\mathfrak{g} = -[Y, X]_\mathfrak{g}, \quad (\text{antisymmetric}) \quad (\text{A.1.5b})$$

$$[X, [Y, Z]_\mathfrak{g}]_\mathfrak{g} + [Y, [Z, X]_\mathfrak{g}]_\mathfrak{g} + [Z, [X, Y]_\mathfrak{g}]_\mathfrak{g} = 0, \quad (\text{JACOBI identity}) \quad (\text{A.1.5c})$$

for any $X, Y, Z \in \mathfrak{g}$ and $z_1, z_2 \in \mathbb{C}$.

Note that the subscript \mathfrak{g} is often omitted when writing the bracket.

A simple physically motivated example of a LIE algebra is the HEISENBERG algebra. It corresponds to the algebra generated by 3 elements \hat{x} , \hat{p} and 1 such that

$$[\hat{x}, \hat{p}] = i\hbar 1, \quad [\hat{x}, 1] = 0, \quad [\hat{p}, 1] = 0.$$

We give another physically motivated example.

Example A.1.8 – POINCARÉ algebra

The POINCARÉ algebra $\mathfrak{P}^{1,d}$ in d spacial dimensions corresponds to a LIE algebra generated by the translations $\{P_\mu\}_{\mu=0,\dots,d}$, the rotations $\{J_j\}_{j=1,\dots,d}$ and the boosts $\{K_j\}_{j=1,\dots,d}$ via

$$\begin{aligned} [P_0, J_j] &= 0, & [P_j, J_k] &= -i\epsilon_{jk}^l P_l, & [P_0, K_j] &= iP_j, & [P_j, K_k] &= i\eta_{jk} P_0, \\ [P_\mu, P_\nu] &= 0, & [J_j, J_k] &= i\epsilon_{jk}^l J_l, & [J_j, K_k] &= i\epsilon_{jk}^l K_l, & [K_j, K_k] &= -i\epsilon_{jk}^l K_l, \end{aligned}$$

where η is the MINKOWSKI metric with signature $(- + \dots +)$, and ϵ is the LEVI-CIVITA fully antisymmetric tensor.

Remark A.1.9 – Algebras and LIE algebras

An algebra is always a LIE algebra. This is done by considering the bracket $[f, g]_\star = f \star g - g \star f$, where \star is the product of the algebra. One can show that such a bracket (often without subscripts) satisfies (A.1.5).

The converse is not necessarily true. Therefore, the LIE algebra structure is broader than the algebra one in a sense. However, from any LIE algebra \mathfrak{g} , one can construct an algebra called the universal enveloping algebra $U(\mathfrak{g})$ as detailed in Definition A.1.10.

From a LIE algebra¹ \mathfrak{g} , one can consider tensor powers of \mathfrak{g} , that is $\mathfrak{g}^{\otimes n} = \mathfrak{g} \otimes \dots \otimes \mathfrak{g}$ (n times), for any $n \in \mathbb{N}$, with the convention $\mathfrak{g}^{\otimes 0} = \mathbb{C}$. Then, the sum $T(\mathfrak{g}) = \bigoplus_{n=0}^{+\infty} \mathfrak{g}^{\otimes n}$ is an algebra, called the tensor algebra, with the product $XY = X \otimes Y \in \mathfrak{g}^{\otimes(j+k)}$, for any $X \in \mathfrak{g}^{\otimes j}$ and $Y \in \mathfrak{g}^{\otimes k}$.

Definition A.1.10 – Universal enveloping algebra

Given a LIE algebra $(\mathfrak{g}, [\cdot, \cdot])$, one defines

$$U(\mathfrak{g}) = T(\mathfrak{g}) / \{X \otimes Y - Y \otimes X - [X, Y], X, Y \in \mathfrak{g}\}$$

The definition above corresponds to the quotient of $T(\mathfrak{g})$ by the two-sided ideal $J = \{X \otimes Y - Y \otimes X - [X, Y], X, Y \in \mathfrak{g}\}$. For simplicity, the definition of $U(\mathfrak{g})$ can be understood as the tensor algebra, *i.e.* formal polynomial of elements of \mathfrak{g} , for which the bracket of \mathfrak{g} is realised by the product, *i.e.* $[X, Y] = XY - YX$.

A.1.3 Coalgebras

Motivated by the structure of the dual of a group (see Example 1.4.1 for the motivations and detailed explanations), we define here the notion of coalgebra.

¹Actually, this construction can be made for any vector space.

Definition A.1.11 – Coalgebra

A coalgebra is a vector space \mathbb{A} over \mathbb{C} with two linear mappings called the coproduct $\Delta : \mathbb{A} \rightarrow \mathbb{A} \otimes \mathbb{A}$ and the counit $\varepsilon : \mathbb{A} \rightarrow \mathbb{C}$, satisfying

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta, \quad (\text{coassociativity}) \quad (\text{A.1.6a})$$

$$(\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}. \quad (\text{counit property}) \quad (\text{A.1.6b})$$

It is further said cocommutative if Δ satisfies

$$\tau \circ \Delta = \Delta. \quad (\text{A.1.6c})$$

The coproduct Δ is often written in terms of SWEEDLER notations

$$\Delta(f) = \sum f_{(1)} \otimes f_{(2)} \quad (\text{A.1.7})$$

for any $f \in \mathbb{A}$. In these notations $f_{(1)}, f_{(2)} \in \mathbb{A}$ are the “components” of f with respect to Δ and the sum sign \sum is here to remind that Δ corresponds to the sum of all the components.

Using these notations, the axioms (A.1.6a) and (A.1.6b) becomes respectively

$$\sum f_{(1)(1)} \otimes f_{(1)(2)} \otimes f_{(2)} = \sum f_{(1)} \otimes f_{(2)(1)} \otimes f_{(2)(2)}, \quad (\text{A.1.8a})$$

$$\sum \varepsilon(f_{(1)}) f_{(2)} = \sum \varepsilon(f_{(2)}) f_{(1)} = f, \quad (\text{A.1.8b})$$

for any $f \in \mathbb{A}$. The equation (A.1.8a) states that the element on which we take subscripts does not matter, only the order does. Therefore, one usually also writes (A.1.8a) as

$$\sum f_{(1)} \otimes f_{(2)} \otimes f_{(3)}. \quad (\text{A.1.8a})$$

The requirements (A.1.6a) and (A.1.6b) can be put under the form of commutative diagrams (respectively)

$$\begin{array}{ccc} \mathbb{A} \otimes \mathbb{A} \otimes \mathbb{A} & \xleftarrow{\Delta \otimes \text{id}} & \mathbb{A} \otimes \mathbb{A} \\ \uparrow \text{id} \otimes \Delta & & \uparrow \Delta \\ \mathbb{A} \otimes \mathbb{A} & \xleftarrow{\Delta} & \mathbb{A} \end{array} \quad \begin{array}{ccc} \mathbb{C} \otimes \mathbb{A} & \xleftarrow{\varepsilon \otimes \text{id}} & \mathbb{A} \otimes \mathbb{A} & \xrightarrow{\text{id} \otimes \varepsilon} & \mathbb{A} \otimes \mathbb{C} \\ & \searrow \text{id} & \uparrow \Delta & \nearrow \text{id} & \\ & & \mathbb{A} & & \end{array}$$

Note that these commutative diagrams are very similar to the ones of the algebra definition, to the exception that the coproduct and counit are replaced by the product and the unit respectively and that the arrow are reversed. This is due to the fact that algebras and coalgebras are dual structures, as depicted in Figure 7.

The previous statement can be made more general. Let $(\mathbb{A}, \Delta_{\mathbb{A}})$ be a coalgebra and $(\mathbb{B}, \star_{\mathbb{B}})$ be an algebra. Let $\mathcal{L}(\mathbb{A}, \mathbb{B})$ denote the set of linear functions from \mathbb{A} to \mathbb{B} . Then, $\mathcal{L}(\mathbb{A}, \mathbb{B})$ is an algebra with the product

$$(f \star g)(a) = \star_{\mathbb{B}} \circ (f \otimes g)(\Delta_{\mathbb{A}}(a)) = \sum f(a_{(1)}) \star_{\mathbb{B}} g(a_{(2)}) \quad (\text{A.1.9})$$

for any $f, g \in \mathcal{L}(\mathbb{A}, \mathbb{B})$ and $a \in \mathbb{A}$. The case of the dual of \mathbb{A} is obtained when considering $\mathbb{B} = \mathbb{C}$.

Finally, a $*$ -coalgebra is a colagebra \mathbb{A} equipped with an involution $\dagger : \mathbb{A} \rightarrow \mathbb{A}$ satisfying (A.1.3a), (A.1.3c) and

$$\Delta(f^\dagger) = \Delta(f)^\dagger \quad (\text{A.1.10})$$

where $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$. In this case, one has $\varepsilon(f^\dagger) = \overline{\varepsilon(f)}$.

A.1.4 HOPF algebra

Algebras and coalgebras can be merged into one single structure called the bialgebras. The latter requires that the algebra sector (product, unit) and the coalgebra sector (coproduct, counit) are merged coherently.

Definition A.1.12 – Bialgebra

A bialgebra \mathbb{A} is both an algebra and a coalgebra such that the product and unit are coalgebra homomorphisms, or equivalently, such that the coproduct and the counit are algebra homomorphisms, *i.e.*

$$\Delta(f \star g) = \Delta(f) \star \Delta(g), \quad \varepsilon(f \star g) = \varepsilon(f)\varepsilon(g), \quad \Delta(1) = 1 \otimes 1, \quad \varepsilon(1) = 1. \quad (\text{A.1.11})$$

It is further said to be a $*$ -bialgebra if it is both a $*$ -algebra and a $*$ -colagebra.

In the context of a group dual of [Example 1.4.1](#), one reads that the bialgebra structure is not sufficient to render all the group structure since it does not contain the information about the inverse of the group. In order to do so, one can make the structure of bialgebra grow to the one of [HOPF algebra](#) by defining a generalised notion of inverse S , called antipode or coinverse.

Definition A.1.13 – HOPF algebra

A bialgebra^a $(\mathcal{H}, \cdot, 1, \Delta, \varepsilon)$ is called a [HOPF algebra](#), if there exists a linear map $S : \mathcal{H} \rightarrow \mathcal{H}$ satisfying

$$\cdot \circ (S \otimes \text{id}) \circ \Delta = \cdot \circ (\text{id} \otimes S) \circ \Delta = 1 \circ \varepsilon \quad (\text{A.1.12})$$

If \mathcal{H} is further a $*$ -bialgebra, then it is said to be a HOPF $*$ -algebra.

^aNote that we changed notations between [Definition A.1.12](#) and [Definition A.1.13](#). The one from [Definition A.1.12](#) was chosen so that it matches with the star product formalism of the deformed algebra of functions. However, the notations of [Definition A.1.13](#) were chosen to be coherent with [Section 1.4](#).

In SWEEDLER notations ([A.1.7](#)), the condition ([A.1.12](#)) writes

$$\sum S(X_{(1)})X_{(2)} = \sum X_{(1)}S(X_{(2)}) = \varepsilon(X)1 \quad (\text{A.1.13})$$

for any $X \in \mathcal{H}$.

The structure of [HOPF algebra](#) has many physical applications, part of which are discussed in [Section 1.4](#). We refer to [\[32\]](#) for more details. Note that more contemporary use of quantum groups are made in machine learning or quantum computing [\[139\]](#).

One can derive several properties from this definition.

First, the antipode S is an algebra anti-homomorphism and a coalgebra anti-homomorphism, that is

$$S(XY) = S(Y)S(X), \quad S(1) = 1, \quad (\text{A.1.14a})$$

$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta, \quad \varepsilon \circ S = \varepsilon, \quad (\text{A.1.14b})$$

for any $X, Y \in \mathcal{H}$. If \mathcal{H} is further a HOPF $*$ -algebra then

$$S \circ \dagger \circ S \circ \dagger = \text{id}, \quad (\text{A.1.14c})$$

which implies that S is invertible and $S^{-1} = \dagger \circ S \circ \dagger$. Note that, in SWEEDLER notations, left hand side of (A.1.14b) becomes

$$\sum S(X)_{(1)} \otimes S(X)_{(2)} = \sum S(X_{(2)}) \otimes S(X_{(1)}), \quad (\text{A.1.14b})$$

which states that applying S to a decomposition reverts the elements of the latter decomposition. This property is the reason why S is sometimes called a braiding.

Proof. First, given $X \in \mathcal{H}$, applying (A.1.13) to X^\dagger gives

$$\sum X_{(1)}^\dagger S(X_{(2)}^\dagger) = \sum S(X_{(1)}^\dagger) X_{(2)}^\dagger = \overline{\varepsilon(X)} 1$$

and applying \dagger again gives

$$\sum S(X_{(2)}^\dagger)^\dagger X_{(1)} = \sum X_{(2)} S(X_{(1)}^\dagger)^\dagger = \varepsilon(X) 1. \quad (\text{A.1.15})$$

Thus, we have

$$\begin{aligned} S(S(X^\dagger)^\dagger) &= S\left(S\left(\left(\sum \varepsilon(X_{(1)}) X_{(2)}\right)^\dagger\right)^\dagger\right) && (\text{A.1.8b}) \\ &= S\left(S\left(\sum \overline{\varepsilon(X_{(1)})} X_{(2)}^\dagger\right)^\dagger\right) \\ &= S\left(\left(\sum \overline{\varepsilon(X_{(1)})} S(X_{(2)}^\dagger)\right)^\dagger\right) && (\text{linearity of } S) \\ &= S\left(\sum \varepsilon(X_{(1)}) S(X_{(2)}^\dagger)^\dagger\right) \\ &= \sum \varepsilon(X_{(1)}) S(S(X_{(2)}^\dagger)^\dagger) && (\text{linearity of } S) \\ &= \sum X_{(1)} S(X_{(2)}) S(S(X_{(3)}^\dagger)^\dagger) && ((\text{A.1.13}) \text{ on } X_{(1)}) \\ &= \sum X_{(1)} S(S(X_{(3)}^\dagger)^\dagger X_{(2)}) && (\text{left equation of (A.1.14a)}) \\ &= \sum X_{(1)} S(\varepsilon(X_{(2)}) 1) && (\text{A.1.15}) \\ &= \sum X_{(1)} \varepsilon(X_{(2)}) && (\text{linearity of } S \text{ and (A.1.14a)}) \\ &= X && (\text{A.1.13}) \end{aligned}$$

which proves (A.1.14c).

Then, given $X, Y \in \mathcal{H}$,

$$\begin{aligned}
 S(Y)S(X) &= \sum S(Y_{(1)}\varepsilon(Y_{(2)}))S(X_{(1)}\varepsilon(X_{(2)})) && \text{(A.1.8b)} \\
 &= \sum S(Y_{(1)})S(X_{(1)})\varepsilon(X_{(2)}Y_{(2)}) && \text{((A.1.11) and linearity of } S) \\
 &= \sum S(Y_{(1)})S(X_{(1)})(X_{(2)}Y_{(2)})_{(1)}S((X_{(2)}Y_{(2)})_{(2)}) && \text{(A.1.13)} \\
 &= \sum S(Y_{(1)})S(X_{(1)})X_{(2)}Y_{(2)}S(X_{(3)}Y_{(3)}) && \text{(A.1.8a)} \\
 &= \sum S(Y_{(1)})(\varepsilon(X_{(1)})1)Y_{(2)}S(X_{(2)}Y_{(3)}) && \text{(A.1.13)} \\
 &= \sum \varepsilon(X_{(1)})\varepsilon(Y_{(1)})S(X_{(2)}Y_{(2)}) && \text{(A.1.13)} \\
 &= S(XY) && \text{(following the reverse two first steps)}
 \end{aligned}$$

This shows the left identity of (A.1.14a). The right one is obtained by applying (A.1.12) to 1, combined with $\Delta(1) = 1 \otimes 1$ and $\varepsilon(1) = 1$ (A.1.11).

Similarly, for $X \in \mathcal{H}$,

$$\begin{aligned}
 \sum S(X_{(2)}) \otimes S(X_{(1)}) &= \sum S(X_{(2)}\varepsilon(X_{(3)})) \otimes S(X_{(1)}) && \text{((A.1.8b) on } X_{(2)}) \\
 &= \sum (S(X_{(2)}) \otimes S(X_{(1)}))(\varepsilon(X_{(3)})1 \otimes 1) && \text{(tensor product definition)} \\
 &= \sum (S(X_{(2)}) \otimes S(X_{(1)}))(\Delta(X_{(3)}S(X_{(4)}))) && \text{(A.1.13)} \\
 &= \sum (S(X_{(2)}) \otimes S(X_{(1)}))(X_{(3)} \otimes X_{(4)})\Delta(S(X_{(5)})) && \text{(A.1.11)} \\
 &= \sum (S(X_{(2)})X_{(3)} \otimes S(X_{(1)})X_{(4)})(\Delta(S(X_{(5)}))) && \text{(tensor product definition)} \\
 &= \sum (\varepsilon(X_{(2)})1 \otimes S(X_{(1)})X_{(3)})(\Delta(S(X_{(4)}))) && \text{(A.1.13)} \\
 &= \sum (1 \otimes S(X_{(1)})X_{(2)})(\Delta(S(X_{(3)}))) && \text{((A.1.8b) applied to } \varepsilon(X_{(2)})X_{(3)}) \\
 &= \sum (1 \otimes \varepsilon(X_{(1)})1)(\Delta(S(X_{(2)}))) && \text{(A.1.13)} \\
 &= \Delta(S(X)) && \text{(using (A.1.8b) applied to } \varepsilon(X_{(1)})X_{(2)})
 \end{aligned}$$

which proves the left relation of (A.1.14b). The right relation comes from

$$\varepsilon(S(X)) = \varepsilon\left(S\left(\sum X_{(1)}\varepsilon(X_{(2)})\right)\right) = \varepsilon\left(\sum S(X_{(1)}X_{(2)})\right) = \varepsilon(\varepsilon(X)1) = \varepsilon(X). \quad \square$$

Given a LIE algebra \mathfrak{g} , one can endow $U(\mathfrak{g})$ with a HOPF algebra structure as done in Example 1.4.3. It is the most simple example of HOPF algebra. It is used to generate non-trivial HOPF algebra via a DRINFEL'D twist deformation (see Subsection A.1.5).

Another example, that is relevant for this manuscript, is the HOPF algebra of κ -POINCARÉ. More details on this algebra are given in Section 3.1.

Example A.1.14 – The HOPF algebra of κ -POINCARÉ

In [90, 89], the quantum deformation of the POINCARÉ algebra is constructed and is given

by the following structure:

$$[J_j, J_k] = i\epsilon_{jk}^l J_l, \quad [J_j, K_k] = i\epsilon_{jk}^l K_l, \quad [K_j, K_k] = -i\epsilon_{jk}^l J_l, \quad (\text{A.1.16a})$$

$$[P_j, J_k] = -i\epsilon_{jk}^l P_l, \quad [P_j, \mathcal{E}] = [J_j, \mathcal{E}] = 0, \quad [P_j, P_k] = 0, \quad (\text{A.1.16b})$$

$$[K_j, \mathcal{E}] = -\frac{i}{\kappa} P_j \mathcal{E}, \quad [P_j, K_k] = \frac{i}{2} \eta_{jk} \left(\kappa(1 - \mathcal{E}^2) + \frac{1}{\kappa} P_l P^l \right) + \frac{i}{\kappa} P_j P_k, \quad (\text{A.1.16c})$$

$$\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta P_j = P_j \otimes 1 + \mathcal{E} \otimes P_j, \quad (\text{A.1.16d})$$

$$\Delta \mathcal{E} = \mathcal{E} \otimes \mathcal{E}, \quad \Delta J_j = J_j \otimes 1 + 1 \otimes J_j, \quad (\text{A.1.16e})$$

$$\Delta K_j = K_j \otimes 1 + \mathcal{E} \otimes K_j - \frac{1}{\kappa} \epsilon_j^{kl} P_k \otimes J_l, \quad (\text{A.1.16f})$$

$$\varepsilon(P_0) = \varepsilon(P_j) = \varepsilon(J_j) = \varepsilon(K_j) = 0, \quad \varepsilon(\mathcal{E}) = 1, \quad (\text{A.1.16g})$$

$$S(P_0) = -P_0, \quad S(\mathcal{E}) = \mathcal{E}^{-1}, \quad S(P_j) = -\mathcal{E}^{-1} P_j, \quad (\text{A.1.16h})$$

$$S(J_j) = -J_j, \quad S(K_j) = -\mathcal{E}^{-1} \left(K_j - \frac{1}{\kappa} \epsilon_j^{kl} P_k J_l \right). \quad (\text{A.1.16i})$$

where $\mathcal{E} = e^{-P_0/\kappa}$ and κ is the deformation parameter.

Example A.1.15 – The HOPF algebra of deformed translation

We here consider the algebra of deformed translations, noted $\mathcal{T}_\kappa^{1,d}$, generated by $\{P_\mu\}_{\mu=0,\dots,d}$. One can read from (A.1.16) that it is a HOPF subalgebra of κ -POINCARÉ.

We now go back to the statement of Section 1.1 that HOPF algebras are extensions of algebras in a “self-dual” form. As coalgebras and algebras are dual to one another, the HOPF algebra is “self-dual” in the sense that it contains the two in a single structure. But this duality goes a bit beyond this vague argument, since one can define the dual of a HOPF algebra \mathcal{H} , noted \mathcal{H}' .

Definition A.1.16 – Dual HOPF algebra

Let \mathcal{H} be a HOPF algebra. One defined the dual HOPF algebra $(\mathcal{H}', \cdot, 1_{\mathcal{H}'}, \Delta_{\mathcal{H}'}, \varepsilon_{\mathcal{H}'}, S_{\mathcal{H}'})$ of \mathcal{H} by

$$(fg)(X) = (f \otimes g)(\Delta(X)) = \sum f(X_{(1)})g(X_{(2)}), \quad 1_{\mathcal{H}'}(X) = \varepsilon(X), \quad (\text{A.1.17a})$$

$$\Delta_{\mathcal{H}'}(f)(X \otimes Y) = f(XY), \quad \varepsilon_{\mathcal{H}'}(f) = f(1), \quad (\text{A.1.17b})$$

$$(S_{\mathcal{H}'}(f))(X) = f(S(X)), \quad (\text{A.1.17c})$$

for any $f, g \in \mathcal{H}'$ and $X, Y \in \mathcal{H}$. One can read the similarities with Example 1.4.1.

In Definition A.1.16, the algebra sector of \mathcal{H}' corresponds to the dual of the coalgebra sector of \mathcal{H} . Conversely, the coalgebra sector of \mathcal{H}' corresponds to the dual of the algebra sector of \mathcal{H} , as pictured in Figure 7.

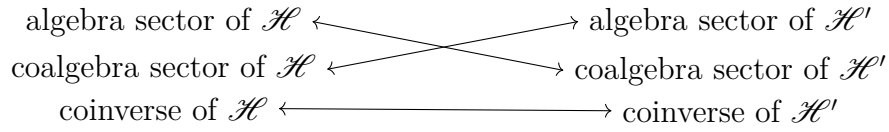


Figure 7 – Schematic representation of the correspondence between a HOPF algebra \mathcal{H} and its dual \mathcal{H}' .

Note that the relations (A.1.17) and the correspondence depicted in Figure 7 are even clearer when using a dual pairing notation. Let us define $\langle \cdot, \cdot \rangle : \mathcal{H}' \otimes \mathcal{H} \rightarrow \mathbb{C}$, defined by

$$\langle f, X \rangle = f(X)$$

Then, one has

$$\langle fg, X \rangle = \langle f \otimes g, \Delta(X) \rangle, \quad \langle 1_{\mathcal{H}'}, X \rangle = \varepsilon(X), \quad (\text{A.1.17a})$$

$$\langle \Delta(f), X \otimes Y \rangle = \langle f, XY \rangle, \quad \varepsilon_{\mathcal{H}'}(f) = \langle f, 1 \rangle, \quad (\text{A.1.17b})$$

$$\langle S_{\mathcal{H}'}(f), X \rangle = \langle f, S(X) \rangle. \quad (\text{A.1.17c})$$

Example A.1.17 – The κ -MINKOWSKI space

We here define the κ -MINKOWSKI space as the dual HOPF algebra of $\mathcal{T}_{\kappa}^{1,d}$, similarly to [89]. One can show that κ -MINKOWSKI, noted $\mathbb{R}_{\kappa}^{1,d}$ has $d + 1$ generators noted $\{x^{\mu}\}_{\mu=0,\dots,d}$ satisfying

$$[x^0, x^j] = \frac{i}{\kappa} x^j, \quad [x^j, x^k] = 0, \quad (\text{A.1.18a})$$

$$\Delta(x^{\mu}) = x^{\mu} \otimes 1 + 1 \otimes x^{\mu}, \quad \varepsilon(x^{\mu}) = 0, \quad S(x^{\mu}) = -x^{\mu}. \quad (\text{A.1.18b})$$

Proof. Let $\{x^{\mu}\}_{\mu=0,\dots,d}$ be the set of generators dual to $\{P_{\mu}\}_{\mu=0,\dots,d}$, that is

$$\langle P_{\mu}, x^{\nu} \rangle = i\delta_{\mu}^{\nu}$$

where $\langle \cdot, \cdot \rangle : \mathcal{T}_{\kappa}^{1,d} \times \mathbb{R}_{\kappa}^{1,d} \rightarrow \mathbb{C}$ is the dual pairing. Recall from (A.1.16b) that the P_{μ} 's are commutative, *i.e.* $[P_{\mu}, P_{\nu}] = 0$. This implies that the x^{μ} are cocommutative. Explicitly,

$$\begin{aligned} 0 &= \langle [P_{\mu}, P_{\nu}], x^{\rho} \rangle \\ &= \langle P_{\mu} \otimes P_{\nu}, \Delta(x^{\rho}) \rangle - [\mu \leftrightarrow \nu] \\ &= \langle P_{\mu} \otimes P_{\nu}, \Delta(x^{\rho}) - \tau \circ \Delta(x^{\rho}) \rangle, \end{aligned} \quad (\text{A.1.17a})$$

so that $\Delta = \tau \circ \Delta$. One considers for simplicity that x^{μ} has a trivial coproduct, *i.e.* $\Delta x^{\mu} = x^{\mu} \otimes 1 + 1 \otimes x^{\mu}$. This takes root first because this space is seen as some deformed universal enveloping algebra of the MINKOWSKI space-time (with commuting coordinates) and also because the trivial coproduct is cocommutative.

From there, one computes the counit via (A.1.6b) as

$$\begin{aligned} (\varepsilon \otimes \text{id}) \circ \Delta(x^{\mu}) &= \varepsilon(x^{\mu})1 + \varepsilon(1)x^{\mu} = \varepsilon(x^{\mu})1 + x^{\mu} \\ &= \text{id}(x^{\mu}) = x^{\mu} \end{aligned}$$

so $\varepsilon(x^\mu)1 = 0$ and $\varepsilon(x^\mu) = 0$.

Then, one computes

$$\begin{aligned} \langle P_k, [x^0, x^j] \rangle &= \langle P_k, x^0 x^j - x^j x^0 \rangle \\ &= \langle \Delta(P_k), x^0 \otimes x^j \rangle - [j \leftrightarrow 0] \end{aligned} \quad (\text{A.1.17b})$$

$$= \langle P_k \otimes 1 + \mathcal{E} \otimes P_k, x^0 \otimes x^j \rangle - [j \leftrightarrow 0] \quad (\text{A.1.16d})$$

$$= -\delta_k^0 \varepsilon(x^j) + \frac{1}{\kappa} \delta_k^j - [j \leftrightarrow 0]$$

$$= \frac{1}{\kappa} \delta_k^j \quad (k, j \neq 0)$$

where we have used $\langle \mathcal{E}, x^\nu \rangle = -\frac{i}{\kappa} \delta_0^\nu$. As $\Delta(P_0)$ does not involve \mathcal{E} , one computes similarly that $\langle P_0, [x^0, x^j] \rangle = 0$. With a similar computation, one shows that $\langle P_\mu, [x^j, x^k] \rangle = 0$ due to the fact that $\langle \mathcal{E}, x^j \rangle = 0$. Finally,

$$[x^0, x^j] = \langle P_\mu, [x^0, x^j] \rangle x^\mu = \frac{i}{\kappa} x^j$$

which is left hand side of (A.1.18a). The right hand side follows similarly. \square

A.1.5 DRINFEL'D twist and \mathcal{R} -matrix

In the [Example 1.4.1](#), the dual of the group is given a **HOPF algebra** structure which is cocommutative if and only if the group is commutative. There exists a class of **HOPF algebras**, called quasitriangular **HOPF algebras**, that are almost cocommutative. The term ‘‘almost’’ means here that their non-cocommutativity is measured by some element \mathcal{R} , called the \mathcal{R} -matrix. This induces a parametrized noncommutativity on the would-be group, dual to the **HOPF algebra**, and is the precise reason why some authors consider (strict) quantum groups to be quasitriangular **HOPF algebras** [32].

Definition A.1.18 – (Quasi)triangular HOPF algebra

Let \mathcal{H} be a **HOPF algebra**. It is said quasitriangular if it exists some invertible element $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$, called the \mathcal{R} -matrix, such that

$$(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}, \quad (\text{A.1.19a})$$

$$\tau \circ \Delta(X) = \mathcal{R} \Delta(X) \mathcal{R}^{-1}, \quad (\text{A.1.19b})$$

for any $X \in \mathcal{H}$, where τ is the flip map and

$$\mathcal{R}_{12} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes 1, \quad \mathcal{R}_{13} = \sum \mathcal{R}_1 \otimes 1 \otimes \mathcal{R}_2, \quad \mathcal{R}_{23} = \sum 1 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2.$$

Note that we used the notation $\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2$.

\mathcal{H} is further said triangular if

$$\mathcal{R}_{21} \mathcal{R} = 1 \otimes 1 \quad (\text{A.1.20})$$

where $\mathcal{R}_{21} = \sum \mathcal{R}_2 \otimes \mathcal{R}_1 = \tau(\mathcal{R})$.

The equation (A.1.19b) precisely states that \mathcal{R} deforms the cocommutativity condition $\tau \circ \Delta = \Delta$ (A.1.6c).

One can show that the \mathcal{R} -matrix satisfy the following identities

$$(\varepsilon \otimes \text{id})(\mathcal{R}) = (\text{id} \otimes \varepsilon)(\mathcal{R}) = 1 \quad (\text{A.1.21a})$$

$$(S \otimes \text{id})(\mathcal{R}) = \mathcal{R}^{-1}, \quad (\text{id} \otimes S)(\mathcal{R}^{-1}) = \mathcal{R}, \quad (\text{A.1.21b})$$

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \quad (\text{A.1.21c})$$

The equation (A.1.21c) is called the quantum YANG-BAXTER equation.

Another way to deform cocommutativity is to use a DRINFEL'D twist. The main interest of DRINFEL'D twist relies in the fact that one can build a new HOPF algebra by deforming a HOPF algebra. This new HOPF algebra has a deformed coproduct and antipode with respect to the former one. This is of great physical relevance since one can start with a commutative and cocommutative HOPF algebra corresponding to some symmetries over a classical space-time and deform the latter via a DRINFEL'D twist. The new (quantum) symmetries obtained will now act on a deformed (quantum) version of the latter space-time, for which all the deformation is controlled by the DRINFEL'D twist. This procedure for generating a quantum space-time is detailed in Subsection 1.4.3 and Subsection 1.6.2.

Definition A.1.19 – DRINFEL'D twist

Let \mathcal{H} be a HOPF algebra. A DRINFEL'D twist is an invertible element $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ that satisfies

$$(\mathcal{F} \otimes 1)(\Delta \otimes \text{id})(\mathcal{F}) = (1 \otimes \mathcal{F})(\text{id} \otimes \Delta)(\mathcal{F}), \quad (2\text{-cocycle condition}) \quad (\text{A.1.22a})$$

$$(\text{id} \otimes \varepsilon)(\mathcal{F}) = (\varepsilon \otimes \text{id})(\mathcal{F}) = 1, \quad (\text{normalisation}) \quad (\text{A.1.22b})$$

In the previous definition of the DRINFEL'D twist, we omitted the semi-classical condition (1.6.14c) since it more physically motivated, and therefore mainly do not appear in mathematical textbooks.

Theorem A.1.20 – Twisted HOPF algebra

Let \mathcal{H} be a HOPF algebra with a DRINFEL'D twist \mathcal{F} . If one defines

$$\Delta^{\mathcal{F}} = \mathcal{F} \Delta \mathcal{F}^{-1}, \quad S^{\mathcal{F}} = \chi S \chi^{-1}, \quad (\text{A.1.23})$$

where $\chi = \mathcal{F}_1 S(\mathcal{F}_2)$, then the set $\mathcal{H}^{\mathcal{F}} = (\mathcal{H}, \cdot, 1, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$ is a HOPF algebra.

We detail the proof here to show the importance of the 2-cocycle condition (A.1.22a) in this theorem.

Proof. One can check point by point that $\mathcal{H}^{\mathcal{F}}$ satisfies almost all the HOPF algebra axioms because \mathcal{H} is a HOPF algebra. The main requirement to check is that $\Delta^{\mathcal{F}}$ satisfies the

coassociativity property (A.1.6a). By using the expression (A.1.23), one can write

$$\begin{aligned} (\Delta^{\mathcal{F}} \otimes \text{id}) \circ \Delta^{\mathcal{F}}(X) &= \left((\mathcal{F} \otimes 1)(\Delta \otimes \text{id})(\mathcal{F}) \right) \left((\Delta \otimes \text{id}) \circ \Delta(X) \right) \left((\Delta \otimes \text{id})(\mathcal{F}^{-1})(\mathcal{F} \otimes 1)^{-1} \right) \\ (\text{id} \otimes \Delta^{\mathcal{F}}) \circ \Delta^{\mathcal{F}}(X) &= \left((1 \otimes \mathcal{F})(\text{id} \otimes \Delta)(\mathcal{F}) \right) \left((\text{id} \otimes \Delta) \circ \Delta(X) \right) \left((\text{id} \otimes \Delta)(\mathcal{F}^{-1})(1 \otimes \mathcal{F})^{-1} \right) \end{aligned}$$

for any $X \in \mathcal{H}$. Both lines of the previous equation are equal thanks to the coassociativity property of Δ , the fact that Δ is a homomorphism and the 2-cocycle condition of \mathcal{F} . \square

Finally, these two deformations of the coalgebra sector are linked to one another by the following theorem.

Theorem A.1.21 – Twisted (quasi)triangular HOPF algebra

Let \mathcal{H} a quasitriangular HOPF algebra with \mathcal{R} -matrix \mathcal{R} , and a DRINFEL'D twist \mathcal{F} . Then, the twisted HOPF algebra $\mathcal{H}^{\mathcal{F}}$ of Theorem A.1.20 is also quasitriangular with \mathcal{R} -matrix $\mathcal{R}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}$.

$\mathcal{H}^{\mathcal{F}}$ is further triangular if and only if \mathcal{H} is.

Proof. We refer to proof of Theorem 2.3.4 of [32]. \square

A.2 Representation theory

When working with a complicated structure, one would rather want to transform it into a simpler structure easier work with. This is precisely the goal of a representation. Instead of working with a group, one represents it on some set of matrices, for which there are known results. The main tool to have a representation is called a module. The module is a vector space on which our complicated structure will act. Taking back the example of the group, one can interpret it as a physical group of symmetries which acts on some space. The latter space is precisely the module.

In the present case, the study of modules for algebras and HOPF algebras is two-fold. First, it is the way to implement symmetries as discussed above. Therefore, one implements some (quantum) symmetries, gathered in a HOPF algebra, on some space precisely via the module structure. Second, the SERRE-SWAN theorem states that sections over a fiber bundle are in one-to-one correspondence with a module of the algebra of smooth functions. A gauge theory, relying on the notion of fiber bundle, is thus implemented in noncommutative geometry via the module structure. This is rather detailed in Chapter 2.

Definition A.2.1 – Module over an algebra

A left (resp. right) module \mathbb{X} over an algebra $(\mathbb{A}, \star, 1)$ is a vector space together with a linear action $\triangleright : \mathbb{A} \otimes \mathbb{X} \rightarrow \mathbb{X}$ (resp. $\triangleleft : \mathbb{X} \otimes \mathbb{A} \rightarrow \mathbb{X}$) such that

$$f \triangleright (g \triangleright s) = (f \star g) \triangleright s, \quad \left(\text{resp. } (s \triangleleft f) \triangleleft g = s \triangleleft (f \star g) \right) \quad (\text{A.2.1a})$$

$$1 \triangleright s = s, \quad \left(\text{resp. } s \triangleleft 1 = s \right) \quad (\text{A.2.1b})$$

for any $f, g \in \mathbb{A}$ and $s \in \mathbb{X}$. We say that \mathbb{X} is a left (resp. right) \mathbb{A} -module.

If \mathbb{X} is both a right and left \mathbb{A} -module such that

$$(f \triangleright s) \triangleleft g = f \triangleright (s \triangleleft g), \quad (\text{A.2.2})$$

then \mathbb{X} is said to be a \mathbb{A} -bimodule.

A physically motivated example of such a module structure can be found in [Example 1.4.4](#).

Another example is the module given by the algebra itself, that is $\mathbb{X} = \mathbb{A}$, with the action given by the product $\triangleright = \star$. In this case, [\(A.2.1a\)](#) is fulfilled thanks to the associativity property of \star [\(A.1.1c\)](#), and [\(A.2.1b\)](#) is satisfied by the definition of the unit 1 [\(A.1.1d\)](#).

One can generalise the previous example to

Example A.2.2 – The module of n copies of the algebra

Let $n \in \mathbb{N}$ and \mathbb{A} be an algebra. Consider the tensor product of n copies of the algebra: $\mathbb{A}^{\otimes n} = \mathbb{A} \otimes \cdots \otimes \mathbb{A}$. Then, $\mathbb{X} = \mathbb{A}^{\otimes n}$ is a (left) \mathbb{A} -module, with the action $\triangleright = \star \otimes \cdots \otimes \star$.

Indeed, let $f = f_1 \otimes \cdots \otimes f_n \in \mathbb{A}^{\otimes n}$. Then,

$$\begin{aligned} g_1 \triangleright (g_2 \triangleright f) &= g_1 \triangleright (g_2 \star f_1 \otimes \cdots \otimes g_2 \star f_n) \\ &= g_1 \star g_2 \star f_1 \otimes \cdots \otimes g_1 \star g_2 \star f_n = (g_1 \star g_2) \triangleright f \\ 1 \triangleright f &= 1 \star f_1 \otimes \cdots \otimes 1 \star f_n = f \end{aligned}$$

for any $g_1, g_2 \in \mathbb{A}$, where we used the associativity of \star [\(A.1.1c\)](#) and the unit property [\(A.1.1d\)](#). One proceeds similarly in the case of the right module.

One can also define modules over a coalgebra, that are called comodules.

Definition A.2.3 – Comodule over a coalgebra

A left (resp. right) comodule \mathbb{X} over a coalgebra $(\mathbb{A}, \Delta, \varepsilon)$ is a vector space together with a linear coaction $\blacktriangleright : \mathbb{X} \rightarrow \mathbb{A} \otimes \mathbb{X}$ (resp. $\blacktriangleleft : \mathbb{X} \rightarrow \mathbb{X} \otimes \mathbb{A}$) such that

$$(\text{id} \otimes \blacktriangleright) \circ \blacktriangleright = (\Delta \otimes \text{id}) \circ \blacktriangleright, \quad \left(\text{resp. } (\blacktriangleleft \otimes \text{id}) \circ \blacktriangleleft = (\text{id} \otimes \Delta) \circ \blacktriangleleft \right) \quad (\text{A.2.3a})$$

$$(\text{id} \otimes \varepsilon) \circ \blacktriangleright = \text{id}, \quad \left((\varepsilon \otimes \text{id}) \circ \blacktriangleleft = \text{id} \right) \quad (\text{A.2.3b})$$

We say that \mathbb{X} is a left (resp. right) \mathbb{A} -comodule.

A simple example of comodule is the coalgebra itself $\mathbb{X} = \mathbb{A}$, with the coaction given by the coproduct $\blacktriangleright = \Delta$.

Note that the coaction can be written in SWEEDLER notations through

$$\blacktriangleright s = \sum s_{(1)} \otimes s_{(0)}, \quad (\text{A.2.4})$$

where $s_{(0)} \in \mathbb{X}$ and $s_{(1)} \in \mathbb{A}$. Thanks to [\(A.2.3a\)](#), which states

$$\sum (s_{(1)})_{(1)} \otimes (s_{(1)})_{(2)} \otimes s_{(0)} = \sum s_{(1)} \otimes (s_{(0)})_{(1)} \otimes (s_{(0)})_{(0)}$$

denoted by $\sum s_{(2)} \otimes s_{(1)} \otimes s_{(0)}$ for simplicity, the SWEEDLER notation of the coaction is coherent with the SWEEDLER notation for the coproduct (A.1.7). It is also quite straightforward since the zeroth component is the one of the module and the others are the one of the algebra.

One can write commutative diagrams that stands for (A.2.3) and show that they are dual to the ones representing (A.2.1), in a similar fashion that the axioms of the coalgebra are dual to the ones of the algebra. One has to “reverse the arrows”.

This duality goes a bit beyond this observation. If one consider \mathbb{X} to be a left (resp. right) \mathcal{H} -comodule, for \mathcal{H} a HOPF algebra, then \mathbb{X} is a right (resp. left) \mathcal{H}' -module with the action

$$s \triangleleft f = \sum \langle f, s_{(1)} \rangle s_{(0)}, \quad \left(\text{resp. } f \triangleright s = \sum \langle f, s_{(1)} \rangle s_{(0)} \right) \quad (\text{A.2.5})$$

using SWEEDLER notations (A.2.4), for any $s \in \mathbb{X}$ and $f \in \mathcal{H}'$.

Proof. We do the proof for the case of a right comodule, the left case being similar. We have to verify the module axioms (A.2.1). Let $f, g \in \mathcal{H}'$ and $s \in \mathbb{X}$,

$$\begin{aligned} (s \triangleleft g) \triangleleft f &= \sum \langle g, s_{(1)} \rangle (s_{(0)} \triangleleft f) \\ &= \sum \langle g, s_{(1)} \rangle \langle f, s_{(2)} \rangle s_{(0)} \\ &= \left\langle g \otimes f, \sum s_{(1)} \otimes s_{(2)} \right\rangle s_{(0)} \\ &= \langle gf, s_{(1)} \rangle s_{(0)} \\ &= s \triangleleft (gf) \end{aligned}$$

where we used (A.2.3a) implied by the SWEEDLER notations and the property (A.1.17a) of the dual pairing.

The counit property goes as

$$s \triangleleft 1 = \sum \langle 1, s_{(1)} \rangle s_{(0)} = \sum \varepsilon(s_{(1)}) s_{(0)} = s$$

where the last equality corresponds to (A.2.3b) in SWEEDLER notations. \square

However, generically, a module structure does not give a comodule structure on the dual. This is only true in the finite dimensional case (see Lemma 1.6.3 of [31]).

In the context of **noncommutative geometry**, one studies an algebra \mathbb{A} that corresponds to the deformed functions over some space-time. We further want to implement that a HOPF algebra \mathcal{H} corresponds to the (quantum) symmetries of this space-time. This can be done in two ways, depending whether \mathcal{H} stands as the deformation of the group of symmetries or as the deformations of the algebra of infinitesimal transformations corresponding to these symmetries.

If we consider a group of symmetries G on a space-time \mathcal{M} , then one can build a HOPF algebra on G' as detailed in Example 1.4.1. From there, $\mathcal{C}^\infty(\mathcal{M})$ is a G -module (see Example 1.4.5 for an explicit case). By duality, this may be implemented by giving to $\mathcal{C}^\infty(\mathcal{M})$ a G' -comodule structure. As $\mathcal{C}^\infty(\mathcal{M})$ is also an algebra we say that it is a G' -comodule algebra. Therefore, when considering the noncommutative counterpart of this picture, we say that the quantum space-time \mathbb{A} has quantum symmetries \mathcal{H} if \mathbb{A} is a \mathcal{H} -comodule algebra.

Now if we consider a LIE algebra \mathfrak{g} of infinitesimal symmetries on a space-time \mathcal{M} , then one can endow $U(\mathfrak{g})$ with a HOPF algebra structure as in Example 1.4.3. The infinitesimal symmetries acts on functions, *i.e.* $\mathcal{C}^\infty(\mathcal{M})$ is a $U(\mathfrak{g})$ -module. As $\mathcal{C}^\infty(\mathcal{M})$ is also an algebra, we say that it is

a $U(\mathfrak{g})$ -module algebra. Thus, going noncommutative, the quantum space-time \mathbb{A} has quantum (infinitesimal) symmetries \mathcal{H} if \mathbb{A} is a \mathcal{H} -module algebra.

The latter meaning of quantum symmetry is the one we consider mainly in this manuscript, as in [Subsection 1.4.3](#). However, the two are coherent and actually can be seen as dual to one another.

We introduce below the two notions of module algebra and comodule algebra.

Definition A.2.4 – Module algebra

Let $(\mathbb{A}, \star, 1)$ be an algebra and \mathcal{H} be a HOPF algebra. \mathbb{A} is said to be a left (resp. right) \mathcal{H} -module algebra if \mathbb{A} is a left (resp. right) \mathcal{H} -module and the action \triangleright (resp. \triangleleft) links the coproduct of \mathcal{H} to the product of \mathbb{A} , *i.e.*

$$X \triangleright (f \star g) = \Delta(X) \triangleright (f \otimes g), \quad \left(\text{resp. } (f \star g) \triangleleft X = (f \otimes g) \triangleleft \Delta(X) \right) \quad (\text{A.2.6a})$$

$$X \triangleright 1 = \varepsilon(X)1, \quad \left(\text{resp. } 1 \triangleleft X = \varepsilon(X)1 \right) \quad (\text{A.2.6b})$$

for any $X \in \mathcal{H}$ and $f, g \in \mathbb{A}$.

This definition can be extended to the \ast -algebra case by enforcing the condition

$$(X \triangleright f)^\dagger = S(X)^\ddagger \triangleright f^\dagger \quad (\text{A.2.7})$$

where \dagger is the involution on \mathbb{A} , \ddagger the involution on \mathcal{H} and S the antipode of \mathcal{H} .

Definition A.2.5 – Comodule algebra

Let $(\mathbb{A}, \star, 1)$ be an algebra and \mathcal{H} be a HOPF algebra. \mathbb{A} is said to be a left (resp. right) \mathcal{H} -comodule algebra if \mathbb{A} is a left (resp. right) \mathcal{H} -comodule and the coaction \blacktriangleright (resp. \blacktriangleleft) links the product of \mathcal{H} and the product of \mathbb{A} , *i.e.*

$$\blacktriangleright (f \star g) = (\blacktriangleright f)(\star \otimes \cdot)(\blacktriangleright g), \quad \left(\text{resp. } \blacktriangleleft (f \star g) = (\blacktriangleleft f)(\cdot \otimes \star)(\blacktriangleleft g) \right) \quad (\text{A.2.8a})$$

$$\blacktriangleright 1 = 1 \otimes 1, \quad \left(\text{resp. } \blacktriangleleft 1 = 1 \otimes 1 \right) \quad (\text{A.2.8b})$$

for any $f, g \in \mathbb{A}$.

Again, comodule algebras can be extended to the \ast -algebra case, by requiring that the coaction is compatible with the involution \dagger of \mathbb{A} as

$$\blacktriangleright (f^\dagger) = \sum f_{(1)}^\dagger \otimes f_{(0)}^\dagger = (\blacktriangleright f)^\dagger \quad (\text{A.2.9})$$

where \ddagger is the involution of \mathcal{H} .

Note that one can define plenty of other algebraic structures in the same spirit, like module coalgebras, comodule coalgebras, module bialgebras, comodule bialgebras, *etc...*

Appendix B

Operator algebras

Operator algebras have been studied as a mathematical framework for quantum mechanics [7]. They have been inspired from early works of WEYL [8] and VON NEUMANN [9]. These works first evolved around **operator algebras** called VON NEUMANN algebras, formerly called W^* -algebras. A VON NEUMANN algebra corresponds to an algebra of bounded operator on a **HILBERT space**, as in [Example B.1.9](#), *i.e.* the operators are viewed as functions on some **HILBERT space** of states.

In this section, we focus on a more general class of **operator algebra** called the C^* -algebras. The C^* -algebras are broader in the sense that they do not specify a **HILBERT space** to be defined but can still be represented on **HILBERT spaces**. Therefore, it is further used for applications in physics as in algebraic quantum field theory [140] or in quantum field theory on curved space-time [141]. We refer to [Section 1.2](#) for an outline of the role of C^* -algebra in quantum physics.

One can go through textbooks like [28, 142, 143, 144, 145, 146] for more details on **operator algebras**.

In **noncommutative geometry**, the philosophy and use of C^* -algebras may be thought differently since the C^* -algebra is supposed to be the (functions on the) space-time itself, while the above mentioned approaches consider C^* -algebra on a specific (curved) classical space-time. Given a space \mathcal{M} , which at this point could be finite, discrete, fractal or infinite, the space of smooth functions $\mathcal{C}^\infty(\mathcal{M})$ can be endowed with a C^* -algebra structure. Moreover, the topology of \mathcal{M} is preserved in $\mathcal{C}^\infty(\mathcal{M})$ through its space of states. The latter statement corresponds to the (commutative) **GEL'FAND-NAĬMARK theorem** and is the principal motivation of studying C^* -algebra in **noncommutative geometry**.

Indeed, if one considers now a noncommutative C^* -algebra, from an extrapolation of the **GEL'FAND-NAĬMARK theorem**, this algebra can be considered as the space of functions over some geometry, given by its space of states. This geometry is the **noncommutative geometry**. We refer to [Section 1.1](#) for a more explicit guideline of what a **noncommutative geometry** is.

B.1 Topological algebraic structures

Some algebraic structure required in the study of **operator algebras** are gathered in this section.

B.1.1 HILBERT spaces

Considering some space V , the notion of “distance between objects” of V is introduced through the norm.

Definition B.1.1 – Normed vector space

Let V be a vector space. It is called a normed vector space if it is equipped with a function $\|\cdot\| : V \rightarrow \mathbb{R}^+$, called the norm, which satisfies

$$\|u + v\| \leq \|u\| + \|v\|, \quad (\text{triangle inequality}) \quad (\text{B.1.1a})$$

$$\|zu\| = |z| \|u\|, \quad (\text{homogeneity}) \quad (\text{B.1.1b})$$

$$\|u\| = 0 \Leftrightarrow u = 0, \quad (\text{non-degenerate}) \quad (\text{B.1.1c})$$

for any $u, v \in V$ and $z \in \mathbb{C}$.

Note that the previous notion of “distance between objects” is to be understood in a broad sense. For example, consider a spinless particle on the real line \mathbb{R} . Then, the particle has a probability amplitude given by a wave function $\psi(t, x)$, for $x \in \mathbb{R}$ a point on the line and $t \in \mathbb{R}$ a time. For a given region of the line, say $0 \leq x \leq L$, the integral

$$\|\psi(t)\|_L^2 = \int_0^L |\psi(t, x)|^2 dx$$

is interpreted as the probability that the particle is in $[0, L]$ at time t . Note that the previous statement requires the normalisation condition

$$\|\psi(t)\|^2 = \int_{-\infty}^{+\infty} |\psi(t, x)|^2 dx = 1$$

which can also express that the probability that the particle is on the line \mathbb{R} is 1, for any time t .

This follows from the fact that the considered (HILBERT) space is the one of square integrable functions on \mathbb{R} , noted $L^2(\mathbb{R}^2)$, and that $\|\cdot\|$ and $\|\cdot\|_L$ are norms for this space.

This notion of norm can also help us to make sense of other quantities, like sequential measurements.

Consider a particle in a space that moves along an infinite number of displacement x_n for $n \in \mathbb{N}$. The total distance travelled by the particle would correspond to $\sum_{n=0}^{\infty} x_n$. However, one needs a structure so that this sum has actually a meaning. This can be done in a normed vector spaces through absolute convergence, *i.e.* the convergence of $\sum \|x_n\|$. A space where the convergence of $\sum \|x_n\|$ is equivalent to the convergence of $\sum x_n$ is called a BANACH space and is said to be complete¹.

Definition B.1.2 – BANACH space

A complete normed vector space is called a BANACH space.

A convenient way to construct a norm is through a Hermitian sesquilinear form.

¹Completeness is more often defined by the convergence of CAUCHY sequences, but in the case of a BANACH space, this condition is equivalent to the absolute convergence implying the convergence.

Definition B.1.3 – Sesquilinear form

Let V be a vector space. A sesquilinear form on V is a map $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ such that

$$\langle z_1 u_1 + z_2 u_2, v \rangle = z_1 \langle u_1, v \rangle + z_2 \langle u_2, v \rangle, \quad (\text{linearity over the first variable}) \quad (\text{B.1.2a})$$

$$\langle u, z_1 v_1 + z_2 v_2 \rangle = \overline{z_1} \langle u, v_1 \rangle + \overline{z_2} \langle u, v_2 \rangle, \quad (\text{semilinearity over the second variable}) \quad (\text{B.1.2b})$$

for any $u, v, u_1, u_2, v_1, v_2 \in V$ and $z_1, z_2 \in \mathbb{C}$.

One can further add some properties to this form:

$$\langle u, v \rangle = \overline{\langle v, u \rangle}, \quad (\text{Hermitianity}) \quad (\text{B.1.2c})$$

$$\langle u, u \rangle \geq 0, \quad (\text{positivity or positive definite}) \quad (\text{B.1.2d})$$

$$\langle u, u \rangle = 0 \Leftrightarrow u = 0. \quad (\text{non-degenerate}) \quad (\text{B.1.2e})$$

If one considers a vector space equipped with a positive non-degenerate Hermitian sesquilinear form $(V, \langle \cdot, \cdot \rangle)$, then $\|u\| = \sqrt{\langle u, u \rangle}$, for any $u \in V$ defines a norm on V . This means $(V, \|\cdot\|)$ is a normed vector space as defined in [Definition B.1.1](#).

Note that the sesquilinear form satisfies the CAUCHY-SCHWARZ inequality

$$|\langle u, v \rangle| \leq \|u\| \|v\|. \quad (\text{B.1.3})$$

We are now ready to define a [HILBERT space](#).

Definition B.1.4 – HILBERT space

A vector space \mathcal{H} equipped with a positive non-degenerate Hermitian sesquilinear form $\langle \cdot, \cdot \rangle$, such that \mathcal{H} is separated (or HAUSDORFF) and complete for the norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, is called a [HILBERT space](#).

Note that, in this context, the form $\langle \cdot, \cdot \rangle$ is more commonly called scalar product or inner product. We now give the case in point example used in quantum mechanics.

Example B.1.5 – Quantum particle on the line

We consider a spinless quantum particle on a 1-dimensional space, that is on the line \mathbb{R} . It is described by square integrable functions $\psi \in L^2(\mathbb{R}^2)$, which are interpreted as probability amplitude wave functions. One then defines

$$\langle \psi_1(t), \psi_2(t) \rangle = \int_{-\infty}^{+\infty} \psi_1(t, x) \overline{\psi_2(t, x)} dx \quad (\text{B.1.4})$$

which can be shown to be an inner product on $L^2(\mathbb{R}^2)$. It has an associated norm which writes

$$\|\psi(t)\|^2 = \int_{-\infty}^{+\infty} |\psi(t, x)|^2 dx \quad (\text{B.1.5})$$

interpreted to be the probability of finding the particle on the line \mathbb{R} at time t . One can show that $L^2(\mathbb{R}^2)$ with the norm (B.1.5) is complete. Therefore, $L^2(\mathbb{R}^2)$ equipped with the inner product (B.1.4) is a [HILBERT space](#).

B.1.2 C^* -algebras

We now go to BANACH algebras which are the first step toward operator algebras. Indeed, one wants the structure of a BANACH space for the reasons stated in [Subsection B.1.1](#). Moreover, one wants the structure of algebra in order to consider functions (operators) over some space. Therefore, one ends up with a

Definition B.1.6 – BANACH algebra

A BANACH space \mathfrak{A} is a BANACH algebra, if it is an algebra and its norm satisfies

$$\|fg\| \leq \|f\| \|g\| \quad (\text{B.1.6})$$

for any $f, g \in \mathfrak{A}$. In this case, one has $\|1\| = 1$.

It is further said to be a BANACH $*$ -algebra, or B^* -algebra, if \mathfrak{A} is a $*$ -algebra and that the involution † is an isometry for the norm, *i.e.*

$$\|f^\dagger\| = \|f\| \quad (\text{B.1.7})$$

The notion of BANACH $*$ -algebra is all the more important for physics then physical observables are considered to be self-adjoint operators, that is operators f such that

$$f^\dagger = f. \quad (\text{B.1.8})$$

We give here an example of such a BANACH algebra.

Example B.1.7 – Bounded operators on a BANACH space

Let V be a BANACH space. Consider $\mathcal{B}(V)$ to be the bounded linear operators from V to V , that we equip with the norm

$$\|f\|_{\mathcal{B}(V)} = \sup_{u \neq 0} \frac{\|f(u)\|}{\|u\|} = \sup_{\|u\|=1} \|f(u)\|, \quad (\text{B.1.9})$$

for any $f \in \mathcal{B}(V)$, where $\|\cdot\|$ is the norm of V . This is often called the operator norm. It exists because we work with bounded operators. Then, $\mathcal{B}(V)$ equipped with the composition law \circ is a BANACH algebra.

Indeed, by definition (B.1.9), one has that for any $u \neq 0$, $\|f(u)\| \leq \|f\|_{\mathcal{B}(V)} \|u\|$. Therefore, if $f, g \in \mathcal{B}(V)$, for any $u \in V$, such that $u \neq 0$, one has

$$\frac{\|(f \circ g)(u)\|}{\|u\|} = \frac{\|f(g(u))\|}{\|g(u)\|} \frac{\|g(u)\|}{\|u\|} \leq \|f\|_{\mathcal{B}(V)} \|g\|_{\mathcal{B}(V)}.$$

Note that the previous equality can be written because $g(u) \neq 0$ thanks to $u \neq 0$ and the continuity property of g . Now, as the upper bounds found does not depend on u we can take the supremum and obtain that $\|f \circ g\|_{\mathcal{B}(V)} \leq \|f\|_{\mathcal{B}(V)} \|g\|_{\mathcal{B}(V)}$, which is (B.1.6).

In most cases, the subscript $\mathcal{B}(V)$ for the norm is omitted.

We now gather all these structure into a single one, the C^* -algebra.

Definition B.1.8 – C^* -algebra

A BANACH $*$ -algebra \mathfrak{A} is called a C^* -algebra if

$$\|f f^\dagger\| = \|f\|^2, \quad (\text{or equivalently } \|f^\dagger f\| = \|f\|^2) \quad (\text{B.1.10})$$

for any $f \in \mathfrak{A}$.

The previous definition is both motivated and explained by the important example of bounded operators on a **HILBERT space**. It is at the very basis of why **operator algebras** are considered in the context of quantum mechanics.

Example B.1.9 – Bounded operators on a HILBERT space

Let \mathcal{H} be a **HILBERT space** and $\mathcal{B}(\mathcal{H})$ denote the bounded linear operators on \mathcal{H} . \mathcal{H} has a norm associated to its inner product and, therefore, one can follow **Example B.1.7** to show that $\mathcal{B}(\mathcal{H})$ is a BANACH algebra.

We introduce the involution given by the adjoint operator. Given $f \in \mathcal{B}(\mathcal{H})$, one defines $f^\dagger \in \mathcal{B}(\mathcal{H})$ as the unique operator satisfying

$$\langle f(\psi_1), \psi_2 \rangle = \langle \psi_1, f^\dagger(\psi_2) \rangle, \quad (\text{B.1.11})$$

for any $\psi_1, \psi_2 \in \mathcal{H}$. It is an involution thanks to the uniqueness property and it preserves the norm. Indeed, for any $\psi \neq 0$, the CAUCHY-SCHWARZ inequality (**B.1.3**) writes

$$\begin{aligned} \|f(\psi)\|^2 &= \langle f(\psi), f(\psi) \rangle = \langle \psi, (f^\dagger \circ f)(\psi) \rangle \\ &\leq \|\psi\| \|(f^\dagger \circ f)(\psi)\| \end{aligned} \quad (\text{B.1.3})$$

$$\leq \|f^\dagger \circ f\| \|\psi\|^2 \quad (\text{B.1.9})$$

$$\leq \|f^\dagger\| \|f\| \|\psi\|^2. \quad (\text{B.1.6})$$

Dividing by $\|\psi\|^2$ and taking the supremum over ψ imposes that $\|f\| \leq \|f^\dagger\|$. One can then invert the role of f and f^\dagger to have the other inequality leading to $\|f\| = \|f^\dagger\|$ which corresponds to (**B.1.7**).

Finally, in the previous computation we have shown that $\|f\|^2 \leq \|f \circ f^\dagger\|$, using mainly (**B.1.3**). But using (**B.1.6**) and (**B.1.7**), one has that $\|f \circ f^\dagger\| \leq \|f\| \|f^\dagger\| = \|f\|^2$ so that (**B.1.10**) is satisfied. Therefore, $\mathcal{B}(\mathcal{H})$ is a C^* -algebra.

To make contact with usual notations of physics, $\mathcal{B}(\mathcal{H})$ denotes the quantum operators, but physical observables correspond to self-adjoint operators, *i.e.* satisfying (**B.1.8**). The elements of the **HILBERT space**, the states, are more often denoted with the “bra-ket” notation, *i.e.* $|\psi\rangle \in \mathcal{H}$. The bra-ket notation is actually here to render the inner product structure of \mathcal{H} since it is more often denoted $\langle \psi_1, \psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle$. Moreover, operators are often written in terms of hatted capital letters so that f should be \hat{A} .

The inner product and the norm are expressed as integrals like in (**B.1.4**) and (**B.1.5**) in the context of wave functions, and they are interpreted as probability amplitudes. In this sense, the operator norm $\|\hat{A}\|$, defined in (**B.1.9**), can be interpreted as the maximum probability of $\hat{A}|\psi\rangle$ for a normalised state $|\psi\rangle$, or equivalently the amount of definitely lost information when applying \hat{A} to the system.

Another important example, more motivated in the context of [noncommutative geometry](#), is of order. It corresponds to the algebra of complex-valued continuous functions, which vanish at infinity.

Example B.1.10 – Functions over a locally compact space

Let \mathcal{M} be a locally compact topological HAUSDORFF space. Let $\mathcal{C}_0(\mathcal{M})$ be the complex-valued continuous functions over \mathcal{M} that vanish at infinity. One can equip the functions with the sup-norm $\|f\| = \sup_{x \in \mathcal{M}} |f(x)|$, which is well defined thanks to the continuity of f and the vanishing of f at infinity together with the local compactness of \mathcal{M} . One can also define an involution using the complex conjugation through $f^\dagger(x) = \overline{f(x)}$.

The relation (B.1.6) can be proved using that $|f(x)g(x)| = |f(x)||g(x)|$, for any $f, g \in \mathcal{C}_0(\mathcal{M})$. And the presence of the complex modulus $|f(x)|$ implies that the norm does not distinguish f^\dagger from f , *i.e.* (B.1.7) and (B.1.10) are satisfied. This means that $\mathcal{C}_0(\mathcal{M})$ is a C^* -algebra.

Note that the [GEL'FAND-NAÏMARK theorem](#) can be thought as the converse statement of [Example B.1.10](#). Explicitly, if \mathfrak{A} is a commutative C^* -algebra, then the [GEL'FAND-NAÏMARK theorem](#) states that there exists a topological HAUSDORFF space such that $\mathfrak{A} \simeq \mathcal{C}_0(\mathcal{M})$.

B.2 Representation theory

We here introduce tools to understand and manipulate [operator algebras](#). First, the case of commutative C^* -algebra is quite well established, especially thanks to the [GEL'FAND-NAÏMARK theorem](#). In the noncommutative case, one can either go to [HILBERT spaces](#), by representing the C^* -algebra, or try to tackle the C^* -algebra by its own. The [HILBERT space](#) approach is always possible thanks to the [second GEL'FAND-NAÏMARK theorem](#) and somewhat well paved via the [GEL'FAND-NAÏMARK-SEGAL](#) construction. Note that part of the considered objects throughout the manuscript may have close links with the modular theory or [TOMITA-TAKESAKI](#) theory, but the latter is not presented here as it goes far beyond the scope of this thesis.

B.2.1 Commutative C^* -algebras and characters

The goal of this subsection is mainly to introduce the commutative [GEL'FAND-NAÏMARK theorem](#) which is at the very basis of [noncommutative geometry](#).

The first notion we need is the space of characters of a C^* -algebra \mathfrak{A} , denoted $\Phi_{\mathfrak{A}}$. It consists of continuous functions $\varphi : \mathfrak{A} \rightarrow \mathbb{C}$, such that $\varphi \neq 0$, and

$$\varphi(fg) = \varphi(f) \varphi(g) \tag{B.2.1}$$

for any $f, g \in \mathfrak{A}$. φ is called a character and $\Phi_{\mathfrak{A}}$ is sometimes named the structure space of \mathfrak{A} . If \mathfrak{A} is unital, then (B.2.1) and $\varphi \neq 0$ imposes that $\varphi(1) = 1$.

One can relate the algebra to the functions on the characters via the [GEL'FAND](#) transform.

Definition B.2.1 – GEL’FAND transform

For \mathfrak{A} a C^* -algebra, we define the GEL’FAND transform \tilde{f} of an element $f \in \mathfrak{A}$ as $\tilde{f}(\varphi) = \varphi(f)$.

The latter transform is an algebra homomorphism quite straightforwardly since $\tilde{f}g(\varphi) = \varphi(fg) = \varphi(f)\varphi(g) = \tilde{f}(\varphi)\tilde{g}(\varphi)$.

Theorem B.2.2

Let \mathfrak{A} be a commutative C^* -algebra, then the following hold true.

- ❖ $\Phi_{\mathfrak{A}}$ is a locally compact HAUSDORFF space.
- ❖ If \mathfrak{A} is unital, then $\Phi_{\mathfrak{A}}$ is compact.
- ❖ For any $f \in \mathfrak{A}$, the GEL’FAND transform \tilde{f} is continuous and vanishes at infinity, *i.e.* $\tilde{f} \in \mathcal{C}_0(\Phi_{\mathfrak{A}})$.
- ❖ For any $f \in \mathfrak{A}$, $\|\tilde{f}\|_{\Phi_{\mathfrak{A}}} \leq \|f\|$, so the GEL’FAND transform is continuous.

This theorem is quite important to understand the link between all the notions used in the **GEL’FAND-NAÏMARK theorem**. First, the space of characters is locally compact and HAUSDORFF so that it makes a suitable choice as a space for which \mathfrak{A} is the space of functions. This choice is made clear thanks to the third point. Then, unitality is linked to compactness. Finally, the continuity of the transform paves the way for it to be an isometry.

Theorem B.2.3 – GEL’FAND-NAÏMARK theorem

Let \mathfrak{A} be a commutative C^* -algebra. The GEL’FAND transform $\mathfrak{A} \rightarrow \mathcal{C}_0(\Phi_{\mathfrak{A}})$ is an isometric $*$ -isomorphism. In particular, $\|\tilde{f}\|_{\Phi_{\mathfrak{A}}} = \|f\|$ and $\varphi(f^\dagger) = \overline{\varphi(f)}$, for any $f \in \mathfrak{A}$ and $\varphi \in \Phi_{\mathfrak{A}}$. One further has that $\Phi_{\mathfrak{A}}$ is compact if and only if \mathfrak{A} is unital.

We refer to [142] for a complete proofs of these two theorems.

The **GEL’FAND-NAÏMARK theorem** states two important things.

First and foremost, any commutative C^* -algebra actually corresponds to functions over some topological space and allows alone to characterise this topology. Therefore, the knowledge of the set of points is not relevant since all the information is stored in the space of functions. This precise reason is at the foundation of **noncommutative geometry**, which characterise a geometry via the space of functions.

Consider \mathfrak{A}_{\hbar} to be a noncommutative C^* -algebra, where \hbar is a real parameter, such that \mathfrak{A}_0 (*i.e.* $\hbar \rightarrow 0$) is commutative. This is the usual framework of deformation quantisation as detailed in **Section 1.6**. Then, the **noncommutative geometry** defined by \mathfrak{A}_{\hbar} always have a “commutative limit” which links it to some classical space (corresponding here to $\Phi_{\mathfrak{A}_0}$).

Note that the points of the space underlying a commutative C^* -algebra are fully determined in the theorem and correspond to the characters. Therefore, the theorem implies that if \mathcal{M} is a (locally compact HAUSDORFF) topological space, then for any point $x \in \mathcal{M}$, there exist one and only character, that we note φ_x , such that $\varphi_x(f) = f(x)$ for all $f \in \mathcal{C}_0(\mathcal{M})$. Furthermore, no other character exists. In other words, there is no φ that is not a φ_x . This enlightens the correspondence (1.1.3).

B.2.2 States and representations

In a general context, a C^* -algebra can be associated with a HILBERT space in several ways. The first one is that we can represent a C^* -algebra on a HILBERT space, just as it is done for quantum mechanics with (1.2.1). The HILBERT space is seen as a convenient substrate on which the operator algebra is modelled and renders its properties. Given a specific state, that can be interpreted as a vacuum state, there is even an explicit construction of a representation, called the GEL'FAND-NAÏMARK-SEGAL construction, or GNS construction for short. This highlights the importance of the notion of states in the study of C^* -algebras.

Furthermore, we may be interested in how “accurate” this representation is. The faithfulness of a representation imposes that an element of the algebra $f \in \mathfrak{A}$ is associated to only one operator on the HILBERT space $\hat{f} \in \mathcal{B}(\mathcal{H})$. In the context of C^* -algebra the second GEL'FAND-NAÏMARK theorem precisely states that such a faithful representation always exists. One can also read in the proof the kind of states required for the representation to be faithful. Finally, the representation from the GEL'FAND-NAÏMARK-SEGAL construction can also have the property to be irreducible if the considered state is pure. The pure states correspond to points, in the commutative theory.

We here construct objects in the case of a unital C^* -algebra for simplicity, but all this construction adapts to the non-unital case. For the non-unital case, one can always consider an approximated unit.

Let us begin by defining a representation.

Definition B.2.4 – Representation of C^* -algebras

Given a C^* -algebra \mathfrak{A} and a HILBERT space \mathcal{H} , a representation $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ is a linear map satisfying

$$\pi(fg) = \pi(f) \circ \pi(g), \quad \pi(f^\dagger) = \pi(f)^\dagger \quad (\text{B.2.2})$$

for any $f, g \in \mathfrak{A}$. In (B.2.2), \circ denotes the composition of operators and $\pi(f)^\dagger$ corresponds to the adjoint operator of $\pi(f)$ as defined in Example B.1.9.

The representation is said faithful if π is injective. It is said irreducible if a (closed) subspace of \mathcal{H} which is stable under $\pi(\mathfrak{A})$ is either \mathcal{H} or $\{0\}$.

As expressed above, the notion of state is central in the representation theory of operator algebras. We define here the notion of state in the context of C^* -algebra.

An element f of a C^* -algebra \mathfrak{A} is said to be positive if it exists $g \in \mathfrak{A}$ such that $f = g^\dagger g$. We denote it by $f \geq 0$. Given a linear form $\psi : \mathfrak{A} \rightarrow \mathbb{C}$, it is said positive if $\psi(f) \geq 0$, for any $f \geq 0$, or equivalently, $\psi(g^\dagger g) \geq 0$, for any $g \in \mathfrak{A}$. Again we note $\psi \geq 0$.

Positive linear forms satisfy the following properties

- ❖ ψ is continuous with $\|\psi\| = \psi(1)$, where the norm is defined similarly as in (B.1.9).
- ❖ Conversely, any continuous linear form $\psi : \mathfrak{A} \rightarrow \mathbb{C}$ satisfying $\|\psi\| = \psi(1) = 1$ is positive.
- ❖ Given two positive linear forms ψ_1 and ψ_2 such that $\|\psi_1\| = \|\psi_2\|$ and $\psi_1 - \psi_2 \geq 0$, then $\psi_1 = \psi_2$.

Proof. ❖ For any $f \geq 0$, one has $0 \leq f \leq \|f\|1$, in the sense $f - \|f\|1 \leq 0$. Thus, applying ψ gives $0 \leq \psi(f) \leq \|f\|\psi(1)$. Therefore, $\|\psi\| \leq \psi(1)$ and the supremum is reached for $f = 1$ so that equality actually stands.

❖ Let $f \geq 0$. Let us first prove that $\psi(f) \in \mathbb{R}$. We write $\psi(f) = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. Let $g = f - \alpha 1$, then $\psi(g) = i\beta$, using that $\psi(1) = 1$. We compute for any $t \in \mathbb{R}$,

$$\begin{aligned} |\psi(g + it1)|^2 &= \beta^2 + 2t\beta + t^2 \\ &\leq \|g + it1\|^2 = \|(g + it1)^\dagger(g + it1)\| = \|g^2 + t^2\| \leq \|g\|^2 + t^2 \end{aligned}$$

where we used that $f = f^\dagger$ so that $g = g^\dagger$ and $\psi(h) \leq \|h\|\psi(1) = \|h\|$, for any $h \in \mathfrak{A}$, from the previous point. This implies that $\beta^2 + 2t\beta \leq \|g\|^2 + t^2$ for any $t \in \mathbb{R}$, which can only be fulfilled for $\beta \leq 0$. Repeating the argument for $-g$, one gets $\beta = 0$ and so $\psi(f) \in \mathbb{R}$.

Now, choose $\epsilon > 0$ small enough so that $\|1 - \epsilon f\| \leq 1$. Then, one computes

$$1 \geq \|1 - \epsilon f\| = \frac{\|\psi\|}{\psi(1)} \|1 - \epsilon f\| \geq \frac{|\psi(1) - \epsilon\psi(f)|}{\psi(1)}.$$

Thus, $|\psi(1) - \epsilon\psi(f)| \leq \psi(1)$ which can only be true if $\psi(f) \geq 0$.

❖ $\|\psi_1 - \psi_2\| = (\psi_1 - \psi_2)(1) = \|\psi_1\| - \|\psi_2\| = 0$. □

Definition B.2.5 – States of C^* -algebras

A positive linear form ψ of norm one, that is $\psi(1) = 1$, on a C^* -algebra \mathfrak{A} is called a state of \mathfrak{A} . We denote that space of states of \mathfrak{A} as $\Psi_{\mathfrak{A}}$.

A state $\psi \in \Psi_{\mathfrak{A}}$ is said to be faithful if, for any $f \geq 0$, $\psi(f) = 0$ implies $f = 0$.

$\Psi_{\mathfrak{A}}$ is a convex set, *i.e.* for any $\psi_1, \psi_2 \in \Psi_{\mathfrak{A}}$ and $t \in [0, 1]$, $(1 - t)\psi_1 + t\psi_2 \in \Psi_{\mathfrak{A}}$. The extreme point of this convex sets are called the pure states. The space of pure states is denoted $\Psi_{\mathfrak{A}}^p$.

Proof. Considering that ψ_1 and ψ_2 are positive linear functional, so is $(1 - t)\psi_1 + t\psi_2$. One need to check that it has norm 1, that is

$$\|(1 - t)\psi_1 + t\psi_2\| = (1 - t)\psi_1(1) + t\psi_2(1) = 1 - t + t = 1. \quad \square$$

The previous definition of state may be far from the physical definition of a state. However, one can link the two through the following example.

Example B.2.6 – States of operators on a HILBERT space

Let \mathfrak{A} be a C^* -algebra and \mathcal{H} a HILBERT space on which \mathfrak{A} is represented with inner product $\langle \cdot, \cdot \rangle$. Then, any normalised element $|\psi\rangle \in \mathcal{H}$, $\| |\psi\rangle \| = 1$ give rise to a state $\psi \in \Psi_{\mathfrak{A}}$ defined by^a

$$\psi(f) = \langle \psi | f\psi \rangle \quad (\text{B.2.3})$$

for any $f \in \mathfrak{A}$. We thereby justify our notation for ψ in both cases.

If $f = g^\dagger g \geq 0$, then $\psi(f) = \langle g\psi | g\psi \rangle = \|g|\psi\rangle\|^2 \geq 0$, so that the linear form ψ defined in (B.2.3) is positive. Moreover, it is of norm 1 since $|\psi\rangle$ is of norm 1, *i.e.* $\|\psi\| = \psi(1) = \langle \psi | \psi \rangle = \| |\psi\rangle \|^2 = 1$.

^aWe take, in (B.2.3), the physical notation for simplicity (as it was made for). Indeed, the associated mathematical equation would be $\psi(f) = \langle |\psi\rangle, \pi(f)|\psi\rangle$, for $|\psi\rangle \in \mathcal{H}$, where $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$ is the representation. π was also implied in (B.2.3).

However, the link goes even further.

Remark B.2.7 – States and probability

One can show that in the commutative case, *i.e.* $\mathfrak{A} = \mathcal{C}_0(\mathcal{M})$ for \mathcal{M} some topological space, the space of states $\Psi_{\mathfrak{A}}$ consist of all probability measures on \mathcal{M} (see Theorem 2.8.2 of [142], which mainly comes from RIESZ theorem of measure theory). This means that there is a direct link between the states we defined in Definition B.2.5 and the probability amplitude of a quantum particle on the space \mathcal{M} . This also means that the normalisation condition $\|\psi\| = 1$ can directly be interpreted as the conservation of probability, or equivalently stated, that the probability to find the particle associated to ψ anywhere on \mathcal{M} is 1.

Finally, one can show that, in the commutative theory, points are given by the pure states. In other words, if $\mathfrak{A} = \mathcal{C}_0(\mathcal{M})$ for \mathcal{M} some topological space, then $\Psi_{\mathfrak{A}}^p$ is homeomorphic to \mathcal{M} . This is done by showing that the space of pure states $\Psi_{\mathfrak{A}}^p$ is in one-to-one correspondence with the space of characters $\Phi_{\mathfrak{A}}$, when the algebra is commutative.

One of the main interest of states is that it allows to construct a representation through the GEL'FAND-NAÏMARK-SEGAL construction.

GEL'FAND-NAÏMARK-SEGAL construction

Let \mathfrak{A} be a C^* -algebra and $\psi \in \Psi_{\mathfrak{A}}$. First, observe that $\langle f, g \rangle_{\psi} = \psi(f^\dagger g)$ is a positive sesquilinear form on \mathfrak{A} . Indeed, it is sesquilinear thanks to linearity of ψ and positive thanks to positivity of ψ . It is further non-degenerate if ψ is faithful (by definition). Therefore, in a general case, one needs to remove all functions for which this would-be inner product vanishes in order to have a non-degenerate form. Explicitly, one considers

$$J_{\psi} = \{f \in \mathfrak{A}, \psi(f^\dagger f) = 0\} = \{f \in \mathfrak{A}, \psi(g^\dagger f) = 0 \text{ for any } g \in \mathfrak{A}\}$$

which corresponds to the set of positive elements on which ψ vanishes. The second equality is provided by the CAUCHY-SCHWARZ inequality $|\psi(g^\dagger f)|^2 \leq \psi(g^\dagger g) \psi(f^\dagger f)$. If ψ is faithful $J_{\psi} = \{0\}$.

J_{ψ} is a closed (left) ideal in \mathfrak{A} (see Definition A.1.4), indeed for any $g, h \in \mathfrak{A}$ and $f \in J_{\psi}$, $\psi(h^\dagger(gf)) = \psi((g^\dagger h)^\dagger f) = 0$, *i.e.* $gf \in J_{\psi}$. Thus, one can consider the quotient \mathfrak{A}/J_{ψ} (as vector spaces), such that all positive elements for which ψ vanishes are now grouped in a single equivalence class, which is $[0]$. Therefore, $\langle \cdot, \cdot \rangle_{\psi}$ is non-degenerate on \mathfrak{A}/J_{ψ} . We finally consider the completion^a of \mathfrak{A}/J_{ψ} , which is by definition a HILBERT space, denoted \mathcal{H}_{ψ} . If ψ is faithful, we can simply take $\mathcal{H}_{\psi} = \mathfrak{A}$.

In order to have a representation, we need a map $\pi_{\psi} : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_{\psi})$. One consider here simply the left multiplication map, *i.e.* $\pi_{\psi}(f) : [g] \mapsto [fg]$, for any $[g] \in \mathcal{H}_{\psi}$. First, for

any $f \in \mathfrak{A}$, one has $f^\dagger f \leq \|f^\dagger f\|1$. As conjugation preserves positivity, one obtains that $g^\dagger f^\dagger f g \leq \|f^\dagger f\|g^\dagger g$. Applying ψ , one obtains $\psi(g^\dagger f^\dagger f g) \leq \|f^\dagger f\|\psi(g^\dagger g)$. Therefore,

$$\|\pi_\psi(f)\| = \sup_{[g] \in \mathcal{H}_\psi} \frac{\|\pi_\psi(f)([g])\|_\psi}{\|[g]\|_\psi} = \sup_{[g] \in \mathcal{H}_\psi} \sqrt{\frac{\psi(g^\dagger f^\dagger f g)}{\psi(g^\dagger g)}} \leq \sup_{[g] \in \mathcal{H}_\psi} \sqrt{\frac{\|f^\dagger f\|\psi(g^\dagger g)}{\psi(g^\dagger g)}} = \|f\|$$

so that $\pi_\psi(f)$ is indeed a bounded operator with $\|\pi_\psi(f)\| \leq \|f\|$. Note that, in the previous computation, we used the associated norm $\|f\|_\psi = \sqrt{\langle f, f \rangle_\psi} = \sqrt{\psi(f^\dagger f)}$.

Moreover, one has for any $f, g, h \in \mathfrak{A}$,

$$\begin{aligned} \pi_\psi(fg)([h]) &= [fgh] = \pi_\psi(f)([gh]) = (\pi_\psi(f) \circ \pi_\psi(g))([h]), \\ \langle [g], \pi_\psi(f)([h]) \rangle_\psi &= \psi(g^\dagger fh) = \langle \pi_\psi(f^\dagger)([g]), [h] \rangle_\psi \end{aligned}$$

so that (B.2.2) is satisfied.

Finally, $(\mathcal{H}_\psi, \pi_\psi)$ is a representation of \mathfrak{A} .

^aSee discussion of Subsection B.1.1 about complete spaces. This completion may be physically irrelevant in this construction.

An important property of this construction is that the representation $(\mathcal{H}_\psi, \pi_\psi)$ is irreducible if and only if the state ψ is pure. More generally, any irreducible representation of a C^* -algebra is the GEL'FAND-NAĬMARK-SEGAL representation of a pure state.

Using this construction, one obtain the

Theorem B.2.8 – second GEL'FAND-NAĬMARK theorem

Any C^* -algebra has an isometric representation as a closed sub-algebra of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on some HILBERT space \mathcal{H} .

Proof. We follow here the sketch of proof given in [28]. First, one uses the HAHN-BANACH theorem to show that for any non-zero positive elements $g^\dagger g \in \mathfrak{A}$, there exist a state $\psi \in \Psi_{\mathfrak{A}}$ such that $\psi(g^\dagger g) = \|g^\dagger g\| = \|g\|^2$. Then, we construct, through GEL'FAND-NAĬMARK-SEGAL, the representation \mathcal{H}_ψ with inner product $\langle \cdot, \cdot \rangle_\psi$. We denote by $\|\cdot\|_\psi$ the associated norm, which relates to the norm of \mathfrak{A} via $\|\pi_\psi(g)([1])\|_\psi = \|g\|$.

The previous construction allows one to consider a family of state Ψ_0 , such that $\psi(g^\dagger g) = 0$ implies $g = 0$, for any $\psi \in \Psi_0$ and $g \in \mathfrak{A}$ (if necessary we can take all states of \mathfrak{A} , *i.e.* $\Psi_0 = \Psi_{\mathfrak{A}}$).

Finally, one consider the representation formed by the direct sum of the GEL'FAND-NAĬMARK-SEGAL representations of all states in Ψ_0 , that is $\pi = \bigoplus_{\psi \in \Psi_0} \pi_\psi$ and $\mathcal{H} = \bigoplus_{\psi \in \Psi_0} \mathcal{H}_\psi$.

In doing so, we have for any $f \in \mathfrak{A}$, $\|\pi(f)\| = \|f\|$, and so the representation is isometric. \square

The HILBERT space constructed in the proof of the second GEL'FAND-NAĬMARK theorem is mainly of mathematical interest, since it might be far too big for physical purpose. Still, this theorem helps us understanding the nature of C^* -algebras and the interplay it has with VON NEUMANN algebras.

The **second GEL'FAND-NAĬMARK theorem** underlines the interest of the C^* -algebras for physical purposes. It can render all the needed properties of quantum operators without necessarily specifying a **HILBERT space**. Still, if one wants to work in the **HILBERT space** formalism, there always exists nice representations of a C^* -algebra on some **HILBERT space** and such representations can be constructed explicitly.

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