Stability of shocks and their viscous approximations

Paul Blochas

COLLEGE MATHS, TELECOMS
DOCTORAL INFORMATIQUE, SIGNAL
BRETAGNE SYSTEMES, ELECTRONIQUE

Université de Rennes

THÈSE DE DOCTORAT DE

L'UNIVERSITÉ DE RENNES

ÉCOLE DOCTORALE N° 601

Mathématiques, Télécommunications, Informatique, Signal, Systèmes, Électronique

Spécialité : *Mathématiques et leurs intéractions*

Par

Paul BLOCHAS

Stabilité des chocs et de leurs approximations visqueuses

Thèse présentée et soutenue à Rennes, le date
Unité de recherche : UMR CNRS 6625 Institut de Recherche Mahématiques de Rennes (IRMAR)

Rapporteurs avant soutenance :

Franck Sueur — Professeur, Université de Bordeaux
Frédéric Rousset — Professeur, Université Paris-Sud.

Composition du Jury :

Examinateurs :	Boris ANDREÏANOV	Professeur des Universités à l'Université de Tours
	Sylvie BENZONI-GAVAGE	Professoresse des Universités
	Benjamin BOUTIN	Maître de Conférences à l'Université de Rennes
	Anne-Laure DALIBARD	Professoresse des Universités à Sorbonne Université
	Vincent DUCHÊNE	Chargé de Recherche CNRS à l'Université de Rennes
	Luis Miguel RODRIGUES	Professeur des Universités à l'Université de Rennes
	Frédéric ROUSSET	Professeur des Universités à l'Université Paris-Saclay
	Franck SUEUR	Professeur des Universités à l'Université de Bordeaux
Dir. de thèse :	Luis Miguel RODRIGUES	Professeur, Université de Rennes

REMERCIEMENTS

Table of Contents

RÉSUMÉ SUBSTANTIEL EN FRANÇAIS

Cette thèse s'inscrit dans la continuité du projet de recherche initié par les articles [DR20, DR22], et, plus généralement, dans l'étude la stabilité orbitale asymptotique d'ondes solutions d'équations aux dérivées partielles non linéaires.

0.0.1 Rappels généraux

Ondes progressives

Un ouvrage introductif à ce sujet est [KP13]. Rappelons juste que la notion de stabilité généralement utilisée dans le domaine des ondes, bien que proche de la notion de stabilité d'un point d'équilibre d'une équation différentielle, est plus complexe. Il s'agit de la stabilité orbitale. Elle diffère de la stabilité classique en ce qu'elle autorise une perturbation de l'onde à ne pas rester proche de l'onde elle-même, mais seulement de l'ensemble de ses translatés.

Équations de lois d'équilibre

En particulier, la problématique générale qu'il s'agit ici d'étudier est la stabilité asymptotique de chocs dans des lois d'équilibre.

Rappelons qu'unsystème de lois d'équilibre est système d'équations aux dérivées partielles de la forme

$$\partial_t u + \mathrm{div}_x(f(u)) = g(u),$$

où f et g sont des fonctions indéfiniment différentiables, de $\mathbf{R}^n$ dans $\mathbf{R}^{n\times m}$ et $\mathbf{R}^n$ respectivement, et u, l'inconnue, est une fonction de la variable temporelle ($t \in \mathbf{R}$) et de la variable spatiale ($x \in \mathbf{R}^m$).

De plus, des conditions dites d'hyperbolicité sont requises sur f pour obtenir le caractère bien posé de l'équation dans des espaces usuels. La théorie classique d'existence et d'unicité des solutions à ces équations dans le cadre d'un problème dans l'espace tout entier et dans un domaine à bord est décrite dans [BGS].

La formation de singularités en temps fini est un phénomène habituel parmi les solu-

tions lisses de telles équations. Pour s'en coinvaincre, le cas de l'équation de Burgers sans viscosité

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0,$$

pour laquelle une donnée initiale lisse à support compact a une unique solution maximale lisse qui reste bornée dans L^∞ mais la norme L^∞ de sa dérivée spatiale qui n'est pas bornée sur cet intervalle. Pour des équations de lois de conservation (c'est-à-dire $g \equiv 0$) scalaires, ce phénomène est systématique si tant est que $f''(0) \neq 0$. Dans le cas d'un système, de tels résultats ont également été prouvés en dimension 1 suite aux travaux pionniers de [Joh74]. Des résultats généralisant ce travail, en particulier en travaillant en dimension supérieure, ont depuis été obtenu. De plus, des limites de ces phénomènes dans des espaces de dimensions assez grandes ont également été établies, en particulier dans le cas de l'équation des ondes. Le lecteur intéressé peut se référé aux livres [Ali95, Spe16].

Un aspect crucial de la formation de ces singularité est que la norme L^∞ de la solution elle-même reste bornée. Il est même possible, dans le cas scalaire, d'obtenir une limite de la solution dans un sens faible au temps de la singularité. De ce fait, il est possible de tenter de prolonger cette solution en considérant une notion plus faible de solutions.

Différents travaux ont apporté des réponses partielles à ces questions, et sont discutés plus loin dans cette thèse.

Chocs

L'étude des chocs commencent par l'étude du cas scalaire, et des conditions de Rankine-Hugoniot et d'Oleinik qui caractérisent les chocs admissibles.

Pour tenter de développer une théorie permettant d'inclure les chocs parmi les solutions admissibles de ces équations, des travaux de Majda [Maj83b, Maj83a] et de Métivier [Mét01], ont permis d'obtenir le caractère bien posé de perturbations des chocs.

Approximations visqueuses

Une notion importante concernant les équations hyperboliques est leur approximation visqueuse. En effet, pour différents systèmes déquations hyperboliques, dont les équations d'Euler compressible, le système d'équations peut être plongé dans une famille de systèmes visqueux dont il est le cas particulier associé à la viscosité nulle. Évidemment, la limite formellement associée est singulière. En particulier, les problèmes de couches-limitesen mécanique des fluides compressibles sont liés à ces questions.

Un objectif naturel est donc de montrer que les solutions des équations paraboliques

approchent, dans la limite de faible viscosité, la solution de l'équation hyperbolique. Dans le cas scalaire et en temps court, c'est l'un des accomplissements du travail de Kružkov [Kru70]. Ici, l'un des objectifs est d'obtenir une version de ce résultat valable globalement en temps autour d'objets stables. En particulier, le cas d'un choc est d'intérêt car il s'agit d'un des blocs principaux. La présente thèse se focalise principalement sur le cas scalaire en dimension 1.

0.0.2 Travaux de la thèse

Problèmes de détonation

Dans le premier chapitre, un travail [BW22] réalisé avec Aric Wheeler , doctorant sous la direction de Kevin Zumbrun, est présenté. Il s'agit d'étudier la formation de singularités dans le cadre de modèles de combustion.

Les modèles étudiées sont le modèle de Majda et le modèle Zel'dovich-Neumann-Döring (ZND). Certaines solutions de ces modèles-ci sont des ondes représentant une détonation. Le premier objectif était de montrer que l'on peut étendre le résultat de [Joh74] à ce cadre en considérant une topologie sans poids. Le second objectif était d'en étudier les limites en présence de poids augmentant la localisation à l'infini.

En particulier, dans le modèle de Majda, il est possible, pour les ondes de faible amplitude, de montrer leur stabilité non linéaire en introduisant des poids adaptés. Pour des ondes arbitraires, l'absence de phénomène de formation de singularité de type Burgers est prouvé pour de petites perturbations. À l'opposé, dans le cas du modèle ZND, il est montré qu'il y a instabilité, y compris pour des espaces à poids stabilisant les détonations du modèle de Majda.

Stabilité uniforme d'approximations visqueuses d'un choc

Dans le second chapitre, un travail [BR23] réalisé avec mon directeur de thèse, Miguel Rodrigues, est présenté. Il s'agit d'un analogue visqueux de [DR20]. L'équation considérée est

$$\partial_t u + \partial_x(f(u)) = g(u) + \varepsilon \partial_x^2 u$$

où $\varepsilon > 0$ est un paramètre de viscosité.

L'objectif est de prouver une version uniforme en le paramètre de viscosité de stabilité orbitale asymptotique de [DR20]. La difficulté principale ici est qu'en utilisant la méthode classique de [Sat76], le traitement dans la formulation de Duhamel de la partie nonlinéaire du terme $\partial_x(f(u))$ repose sur les effets régularisants de l'équation de la chaleur de diffusion ε et donc induise une perte en $\varepsilon^{-1/2}$ incopatible avec l'uniformité visée. Ce problème reflète un véritable obstacle, lié à la présence de couches-limites visqueuses, raides par définition. Pour contourner ce problème, nous avons mis en place une stratégie originale, dont les deux principaux points-clés sont les suivants. D'une part, nous travaillons avec des normes multi-échelles

$$\|v\|_{W^{1,\infty}_{\varepsilon,\theta}} := \|v\|_\infty + \left\| \frac{\varepsilon \partial_x v}{\varepsilon + e^{-\theta \varepsilon^{-1}|\cdot|}} \right\|_\infty$$

où $\theta > 0$ est un paramètre de localisation indépendant de ε, et nous bouclons en régularité par une estimation vraiment non linéaire (plutôt qu'avec la formulation de Duhamel seule). D'autre part, pour pouvoir injecter ces estimations multi-échelles dans la formulation de Duhamel, nous revisitons, en variable rapide, la méthode de fonctions de Green, développée principalement par Kevin Zumbrun et ses collaborateurs, notamment dans [ZH98], la quête d'uniformité nous conduisant à une choix radicalement nouveau d'opérateur de phase.

Influence des poids sur la stabilité uniforme d'approximations visqueuses

Le dernier chapitre discute d'un travail effectué seul, pas encore exploité sous forme de (pré)publication. Il s'agit d'une variation visqueuse du travail [GR] qui cherche lui-même à étendre l'analyse de [DR20, DR22] par l'introduction de poids pour stabiliser les fronts qui sont par ailleurs instables dans des normes invariantes par translation.

L'on commence par étudier des chocs de Riemann dont une des valeurs est associé à un problème spectralement instable en s'intéressant d'abord à des poids stabilisant le problème spectral en créant un trous spectral, et ensuite au cas limite d'une stabilisation marginale. Cette fois, des ondes lisses solutions du problème hyperboliques sont aussi étudiées dans la limite de faible viscosité . Pour les deux types d'ondes, on s'intéresse à la fois au cas où la stabilisation est suffisamment forte pour créer un trou spectral et au cas critique où, au niveau hyperbolique, le spectre à poids contient l'axe imaginaire pur.

Le problème principal pour le choc de Riemann est celui de la stabilisation marginale, qui nécessite l'introduction de poids polynomiaux en plus des poids exponentiels

initialement introduits. Plus précisément, les poids sont de la forme

$$x \mapsto \frac{e^{\theta_\varepsilon x_+}}{(1+x^2)^{\frac{\alpha}{2}}},$$

où l'instabilité est en x proche de $+\infty$, θ_ε est le coefficient associé au poids assurant une stabilisation marginale, $\alpha > 1$ un coefficient permettant d'améliorer la localisation de la donnée initiale sans changer le caractère critique du spectre.

Dans le cas du front hyperbolique lisse, la construction du profil repose sur une procédure de type perturbation singulière, et est elle-même non triviale. La stabilité non linéaire est en revanche montrée par la conception, relativement élémentaire, d'un principe du maximum adapté au problème non linéaire.

Chapter 1

Introduction

1.1 Hyperbolic systems

1.1.1 Local existence theory

This section is there to remind classical results of the resolution of the Cauchy problem for hyperbolic problems.

In particular, classical solutions are discussed, and, then, the formation of singularities, and, afterwards, a few results on the Cauchy problem for weak solutions are presented : the Kružkhov theory for scalar equations and the BV theory for one-dimensional systems.

Smooth solutions

The functional framework will be based on the spaces BUC^k for the scalar. The notations here will be that $W^{\infty,\infty}(\mathbf{R}^d) := \bigcap_{k\in\mathbf{N}} W^{k,\infty}$ and for any $k \in \mathbf{N}$, $BUC^k(\mathbf{R}^d) = \overline{W^{\infty,\infty}(\mathbf{R}^d)}^{W^{k,\infty}}$.

First, the classical results for scalar equations are presented. Fix d, a positive integer, f and g, smooth functions from $\mathbf{R}$ with value in $\mathbf{R}^d$ and $\mathbf{R}$ respectively.

Theorem 1. *The Cauchy problem associated to the equation $\partial_t u + \mathrm{div}_x(f(u)) = g(u)$ is locally well-posed in the functional space $BUC^1(\mathbf{R}^d)$, with blow-up criteria in $BUC^1(\mathbf{R}^d)$ and continuous dependence on the initial data.*

Now, for multi-dimensional system, this presentation will follow the results from [BGS] (to state the theorem 10.2 from this reference).

Let d and n be positive integers, as well as smooth functions $f^0 : \mathbf{R}^n \to \mathbf{R}^n$, $f := (f^1, ..., f^d) : \mathbf{R}^n \to (\mathbf{R}^n)^d$ and $g : \mathbf{R}^n \to \mathbf{R}^n$. Consider the equation $\partial_t(f^0(u)) + \sum_{k=1}^d \partial_k(f^k(u)) = g(u)$.

Define $A_k(u) := (df^0(u))^{-1} \circ df^k(u)$ (as usual, identified with the matrices in the canonical basis) for $k \in \{1, ..., d\}$ and $u \in \mathbf{R}^n$.

Furthermore, consider the symbols $A(u, \xi) := \sum_{k=1}^d \xi_k A_k(u)$. To ensure well-posedness in some Sobolev spaces, it is enough to require the existence of a symbolic symmetrizer of

the quasilinear system, that is the existence of a smooth mapping $S : \mathbf{R}^n \times (\mathbf{R}^d \backslash \{0\}) \to \mathcal{M}_n(\mathbf{C})$ homogeneous of degree 0 in ξ, with value in the subset of hermitian definite positive matrices and such that, for every u and ξ, $S(u,\xi)A(u,\xi) = A(u,\xi)^* S(u,\xi)$.

Theorem 2. *Assume that $g(0) = 0$. Then, the Cauchy problem in H^s for $s > 1 + \frac{d}{2}$ is locally well-posed, and the blow-up criteria is the explosion of the $W^{1,\infty}$ norm of the solution.*

The continuous dependence is not stated in their theorem, but can be obtained for $H^{s'}$ norm for $s' < s$ by showing it for the L^2 norm and applying interpolation by the local boundedness of the solution in H^s (the result in L^2 using the boundedness in H^s and interpolation inequalities).

Formation of singularities

An important subject in the study of hyperbolic equations is the formation of singularity, and especially of wavebreaking type. In particular, a well-known striking feature of such equations is that, even for one of the simplest nonlinear models, the Burgers' equation (given by $\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0$), the solution induced by an initially smooth and compactly supported nonzero initial data will blow up in $W^{1,\infty}(\mathbf{R})$, but remain bounded in $L^\infty(\mathbf{R})$. Such a phenomena gave raise to the theory of Kružkhov in the scalar case (that will be discussed afterwards), and, in the system case, to several extensions of this blow-up result, as well as study to give a well-suited notion of weak solutions to extend these solutions after the blow-up of the L^∞ norm of the gradient of the solution.

Regarding the formation of discontinuity, the study of systems needed the introduction of a particular class of hyperbolic pdes, to guarantee that the nonlinearity is of Burgers type, and not a simple linear equation. A landmark paper on that subject, [Joh74], proved the blow-up of solutions associated to arbitrarily small smooth initial data for one-dimensional problems containing only genuinely nonlinear characteristic fields.

As they will be discussed in the first chapter, the definition of linearly degenerate and genuinely nonlinear characteristic fields in such a context are given here. These notions are also presented in [Bre00].

First fix $A : \mathbf{R}^n \to \mathcal{M}_n(\mathbf{R})$, smooth, such that for every $u \in \mathbf{R}^n$, $A(u)$ is diagonizable with n distinct real eigenvalues. The system is then called strictly hyperbolic. Under these assumptions, the said eigenvalues $\lambda_1(u) < ... < \lambda_n(u)$ are smooth. Furthermore, there exist (right) eigenvectors $(r_k(u))_k$ such that, for every k and u, $A(u)r_k(u) = \lambda_k(u)r_k(u)$. Automatically, the r_k are also smooth.

Definition 1. *The k-characteristic field of A is called genuinely nonlinear if for every $u \in \mathbf{R}^n$, $d\lambda_k(u)(r_k(u)) \neq 0$.*

It is called linearly degenerate if for every $u \in \mathbf{R}^n$, $d\lambda_k(u)(r_k(u)) = 0$.

In fact, this notion can be adapted in the multi-dimensional setting, as in [Maj84]. Furthermore, the strictly hyperbolic assumption can be relaxed. In particular, assuming only that one has access to d families of smooth left and right eigenvectors to $A(u)$, $(l_k(u))_{1\leqslant k\leqslant d}$ and $(r_k(u))_{1\leqslant k\leqslant d}$ is enough to give sense to for the previous definition, despite A not being strictly hyperbolic.

The result from John implies the following :

Theorem 3. *Assume that every characteristic field of A is genuinely nonlinear. Then, there exists $\delta > 0$ such that any $v_0 \in C^2(\mathbf{R}, \mathbf{R}^n)$ with support in $[0, 1]$ satisfying $\|v_0\|_{W^{2,\infty}} \leqslant \delta$ generates a solution to $\partial_t u + A(u)\partial_x u = 0$ that remain bounded in L^∞ but blows up in $W^{1,\infty}$ in finite time.*

Since then, tremendous progresses have been achieved to generalize this result in various directions, and the interested reader can start by looking to the following books for the conservation laws [Lax73, Joh74, Liu79, Ali95, Spe16].
In the opposite direction, there are various types of hyperbolic equations for which there is not any formation of such singularities. In particular, the use of dispersive effects can allow one to prove that such a blow-up cannot happen in high-dimension, as discussed in [Ḧ97]. Furthermore, the study of the totally linearly degenerate case, that is, when every characteristic field is linearly degenerate, is also of importance. It is conjectured in [Maj84] that in that case, classical solutions are global.

Scalar case

Consider n, an integer, and smooth functions $f : \mathbf{R} \to \mathbf{R}^n$ and $g : \mathbf{R} \to \mathbf{R}$.

The challenge was until the 70's, to understand what was the right notion of solutions for such equations. In fact, even though some Cauchy theory of classical solutions was available for smooth enough initial data, a major issue remained to be understood. In fact, the blow-up criteria (which was known to be optimal) was that either the L^∞ norm of the solution blows up at time T_{max} or the L^∞ norm of the derivative blows up at time T_{max}. In particular, in some cases, both happen, and, in other cases, only one of them occurs (and cases were known for each one of them). Furthermore, in the context of bounded initial data, the lack of uniqueness of distributional solutions was known. Kružkhov thus needed to define a notion of solutions that allowed both the restoration of the existence and uniqueness of solutions at least in a class containing data coming from the blow-up

of the gradient of an initially smooth classical solutions to the problem.

Since the seminal work from [Kru70] , the notion of entropy solutions to equations of the form

$$\partial_t u + \mathrm{div}_x(f(u)) = g(u)$$

has been considered to be the right notion of solutions in this context (that is, for the scalar problem). They are defined through

Definition 2. *A function $u \in C^0([0,T[, L^\infty(\mathbf{R}^d)) \cap C^0([0,T[, L^1_{loc}(\mathbf{R}^d))$ is an entropy solution to the equation $\partial_t u + \partial_x(f(u)) = g(u)$ if it satisfies the inequalities, for all convex and C^2 functions η, and with F_η such that $F'_\eta = \eta' f'$*

$$\partial_t(\eta(u)) + \mathrm{div}_x(F_\eta(u)) \leqslant \eta'(u)g(u)$$

in the distributional sense.

In particular, such solution have been constructed in this article as limits the solutions to

$$\partial_t u + \mathrm{div}_x(f(u)) = g(u) + \varepsilon\Delta_x u$$

where ε is a positive parameter which goes to 0.

In particular, this shows that in the short time theory, solutions to the hyperbolic problem are actually the limits of the solutions to the parabolic equations in the vanishing viscosity limit. There have been work done to understand the speed of the convergence, as in [Kuz76], in which study of the speed of the convergence at time t in the case of a conservation law is of size $\sqrt{\varepsilon t}$.

In our setting, the main theorem proved in that paper reads as follows :

Theorem 4. *For f and g smooth from $\mathbf{R}$ to $\mathbf{R}^d$ (respectively $\mathbf{R}$) and any $u_0 \in L^\infty(\mathbf{R}^d)$, there exists $u \in C^0([0,T[, L^\infty(\mathbf{R}^d) - w^*)$ for some $T \in]0, +\infty]$ such that u is a maximal entropy solution to the equation $\partial_t u + \mathrm{div}_x(f(u)) = g(u)$ and for which $u(0,\cdot) = u_0$. Furthermore, any entropy solution v such that $T' \in]0, +\infty]$ $u \in C^0([0,T'[, L^\infty(\mathbf{R}^d - w^*))$ and $u(0,\cdot) = u_0$ is such that $T' \leqslant T$ and $v = u_{|[0,T'[\times\mathbf{R}^d}$.*

Finally, the following $L^1_{loc}(\mathbf{R}^d)$ continuity results holds true : for any entropy solutions u and v, defined on $[0,T]$, and $M \geqslant 0$, if the bounds $\|u(t,x)\| \leqslant M$ and $\|v(t,x)\| \leqslant M$, we have, for any $t \in [0,T]$, $R > 0$ and $x_0 \in \mathbf{R}^d$, as well as $N := \max\{\|f'(u)\|_2 \,|\, u \in [-M,M]\}$,

$\gamma := \|g\|_{\infty,[-M,M]}$ *and* $S_t := \{y \in \mathbf{R}^d \,|\, \|y - x_0\|_2 \leqslant R - tN\}$, *that*

$$\int_{S_t} |u(t,y) - v(t,y)|dy \leqslant e^{\gamma t} \int_{S_0} |u_0(y) - v_0(y)|dy.$$

System case

Even for systems, such entropy-entropy flux pairs (that is, such a couple (η, F_η)) have analogous, which are also important, but have the huge inconvenient that, quite often, there are not that many of them. For further discussions on that subject, see [Daf00, BGS], in particular the section 4, chapter VII of the first reference, and, for the second one, the section 2 of the chapter 10.

More generally, obtaining for general hyperbolic systems a good local theory for a suitable subclass of weak solutions is still an open problem, discussed in the aforementioned books.

Still, results were obtained for particular cases.

Let us mention the case of one-dimensional systems for which there is a theory of solutions initially small in BV. To do so, consider a smooth function $F : \mathbf{R}^n \to \mathbf{R}^n$, and the equation $\partial_t u + \partial_x(F(u)) = 0$. In that context, the following result is proved in [Daf00] (in the chapter XIII).

Theorem 5. *Under the assumptions that $dF(u)$ is a diagnalizable operator on $\mathbf{R}^n$ with distinct real eigenvalues for every $u \in \mathbf{R}^n$.*

There exist $\delta > 0$ and $C > 0$ such that for every $u_0 \in L^1(\mathbf{R}, \mathbf{R}^n) \cap BV(\mathbf{R}, \mathbf{R}^n)$ satisfying $\|u_0\|_{L^1} < \delta$ and $TV(u_0) < \delta$, there exists u, locally of bounded variation on $\mathbf{R}_+ \times \mathbf{R}$, such that, for every $t \in \mathbf{R}$ $u(t,\cdot) \in BV(\mathbf{R}, \mathbf{R}^n)$, with $TV(u(t,\cdot)) \leqslant CTV(u_0)$, $\|u(t,\cdot)\|_{L^\infty} \leqslant C\|u_0\|_{L^\infty}$, and, for every $(t,t') \in (\mathbf{R}_+)^2$ $\|u(t,\cdot) - u(t',\cdot)\|_{L^1} \leqslant C|t - t'|$, $u(0,\cdot) = u_0$.

Furthermore, it is a solution in the weak sense.

Discussion on the uniqueness is more delicate, and the interested reader can refer to [Bre00, Daf00] for such results : see the chapter for the first one, and the section 10 of the chapter XIV for the second one. It can be noted that a particular uniqueness result will be reminded later on. In fact, the solution constructed in the aforementioned books is constructed by a particular method, and it is actually also given as the solution obtained in the sense of the vanishing viscosity limit, as in [BB05].

1.1.2 Shocks

Scalar case

The short time existence theory in the scalar case has been given in the paper by Kružkov.

To study the asymptotics stability of shocks, one first needs to have some local in time persistence result. This is not given by Kružkov's result. To know whether an initial data containing a shock will give raise to a solution that still contains a shock for small time needs to take into account the conditions of Oleinik, that is, in the one-dimensional case, a function $u :]0,T[\times\mathbf{R} \to \mathbf{R}$ $C^1(]0,T[\times\mathbf{R}\backslash\{(t,\psi(t)) \,|\, t \in]0,T[\}, \mathbf{R})$, $u(t,\cdot)$ discontinuous at $\psi(t)$ for every t and bounded as well its spatial derivative on $]0,T[\times\mathbf{R}$ where ψ is a C^1 function on $[0,T[$ is an entropy solution if it is a solution on $]0,T[\times\mathbf{R}\backslash\{(t,\psi(t)) \,|\, t \in \mathbf{R}\}$ and that it satisfies the Rankine-Hugoniot condition $\psi'(t) = \frac{f(u(t,\psi(t)^+))-f(u(t,\psi(t)^-))}{u(t,\psi(t)^+)-u(t,\psi(t)^-)}$.

A multi-dimensional analogous exists. Both are presented in [DR20].

One-dimensional systems

In the case of one-dimensional systems, shocks, and more genrally, solutions to the Riemann problem (that is, finding the solution, in an adequate sense to the initial value problem $\partial_t u + \partial_x(F(u)) = 0$ with $u(0,x) = \underline{u}_+$ for $x > 0$ and $u(0,x) = \underline{u}_-$ for $x < 0$) has been understood in the context of small BV initial data, as described in [Bre00]. In particular, (approximately) solving this kind of problem is the building block of the Glimm scheme.

Persistence theory for multidimensional systems

In this part, we will focus on the conservative case. The case of balance laws is similar but the theorems are more cumbersome to write, as for the theory of Kružkov. Thus, for the sake of simplicity, we restrict the description to that case, as the references described do.

Now, it is important to consider the problem of proving the existence of perturbations of Riemann shocks under certain stability conditions.

Here, the presentation will be restricted to equations of the form (for u with value

in $\mathbf{R}^n$)

$$\partial_t u + \sum_{k=1}^{d} \partial_k (f^k(u)) = g(u)$$

where $f : \mathbf{R}^n \to (\mathbf{R}^n)^d$ (with $f = (f^1, ..., f^d)$, each $f^j : \mathbf{R}^n \to \mathbf{R}^n$) is a smooth function.

In various references, the equations are generally given in the more general form

$$\partial_t (f^0(u)) + \mathrm{div}_x (f(u)) = 0$$

with $df^0(u)$ being invertible for every u. This choice is made due to the form of various equations, such as the compressible Euler equations which will be discussed later on. However, the changes needed to adapt to that case are not consequential (mainly, one needs to replace the operators $df^k(u)$ by $df^0(u)^{-1} \circ df^k(u)$, and, in the Rankine-Hugoniot conditions, $U_+ - U_-$ by $f^0(U_+) - f^0(U_-)$.

An other important building block is the local in time existence of perturbation of shocks in the system case. prove well-posedness results in that context. Such work was done in the most comprehensive way yet realized for Lax shocks by A. Majda (in [Maj83a, Maj83b]), and then generalized by G. Métivier (in [Met]). Furthermore, these progresses are presented in [BGS] (especially in the chapter 12 for the general theory, and the chapter 15 for the Euler equations).

First fix $\underline{U}_\pm \in (\mathbf{R}^n)^2$, distinct, such that there exists $\sigma \in \mathbf{R}$ such that $f(\underline{U}_+) - f(\underline{U}_-) = \sigma(\underline{U}_+ - \underline{U}_-)$.

Here, the choice of a shock propagating in the direction x_n with jump initially on the hypersurface $\{x_d = 0\}$ and propagating with speed σ. In particular, the function constant by part taking the values $\underline{U}_+$ if $x_d > t\sigma$ and $\underline{U}_-$ if $x_d < t\sigma$ is the shock studied. It must satisfies an other condition. The number of eigenvalues of $A_d(\underline{U}_+)$ greater than the shock speed σ must be the number of eigenvalues greater than the shock speed at $A_d(\underline{U}_)$− -1. The context under study is associated to functions solutions to the system of equations outside of some hypersurface, and, such that, the jumps of the solution at each point of the said hypersurface satisfy the Rankine-Hugoniot condition which are (in a suitable sense) perturbations of the reference shock.

For other types of shocks, such as undercompressive shocks, works have been done by Jean-François Coulombel in [Cou03]. For overcompressive shocks, work have been done to study the viscous analogous. However, some work by Majda seems to prohibit the existence of such shocks at the inviscid level due to issues at the linear level. In fact, as in [BGS], it is proven that such shocks have the property that their associated linearization is not surjective (see the chapter 12, section 2).

Coming back to the question of Lax shocks, the idea is to first rewrite the problem as a problem which is called a Free Boundary Problem (FBP). It takes the form of two initial boundary values hyperbolic pdes (corresponding to the hyperbolic problems on each side of the positions of the jumps) with the boundary conditions being non trivial as they arise through the Rankine-Hugoniot conditions, and actually form some lower dimensional evolution system relating the localization of the shock with the boundary values. In particular, it reads, for a hyperbolic conservation law given by

$$\partial_t U + \mathrm{div}_x(F(U)) = 0$$

as

$$\partial_t U(t,y,x_d) + \sum_{j=1}^{d-1} \partial_j f^j(U(t,y,x_d)) = 0$$

for all $(t,y,x_d) \in]0,T[\times\mathbf{R}^d$ such that $x_d \neq \chi(t,y)$

$$\begin{aligned}
&(U(t,y,\chi(t,y)^+) - U(t,y,\chi(t,y)^-))\partial_t\chi(t,y) \\
&+ \sum_{j=1}^{d-1}(f^j(U(t,y,\chi(t,y)^+) - f^j(U(t,y,\chi(t,y)^-))\partial_y\chi(t,y) \\
&- (f^d(U(t,y,\chi(t,y)^+) - f^d(U(t,y,\chi(t,y)^-)) = 0
\end{aligned}$$

on $]0,T[\times\mathbf{R}^{d-1}$.

Here, u and χ are regular enough functions on

$$\{(t,y,x_d) \in [0,T[\times\mathbf{R}^d \,|\, x_d \neq \chi(t,y,x_d)\}$$

for u, and on

$$[0,T[\times\mathbf{R}^{d-1}$$

for χ. More precisely, solutions shall be functions in $C^0([0,T[,H^s)\cap C^1(]0,T[,H^{s-1})$ and $C^0([0,T[,H^{s'})\cap C^1(]0,T[H^{s'-1})$, both small enough in L^∞, and where s, and s', are big enough. In particular, it is natural, and seems necessary, to at least require to have the regularity threshold of the embedding in $W^{1,\infty}$, that is of $s > 1 + \frac{d}{2}$ as it is needed for the proof of existence of smooth solutions.

As said before, an issue that needs to be solved is the determination of compatibility conditions : given a small perturbation v_0, even in one-dimensional systems, in general, the solution may fail to remain a shock, as intermediate shocks or rarefaction waves may form, as, in particular, the Rankine-Hugoniot condition will not remain satisfied (even

by changing the speed σ). Compatibility conditions are there to ensure that this holds true initially, and, also, that it will continue to hold true at least formally, which imposes conditions on the time derivatives initially (which are given by the equation) to enforce the persistence of the solutions.

Compatibility conditions are designed to ensure that the time derivatives of the Rankine-Hugoniot condition hold initially to be able to rewrite the time derivatives at the boundary and to replace them by the (boundary) expression given by the equation, and to apply it iteratively for some $k \leqslant n$ (for some fixed integer n to be chosen later on). Details are given in [BGS, Met] (chapter 12 again for the first one).

Introduce a few notations to enounce the results below : $\mathcal{A}_j := \begin{pmatrix} A_j(\underline{U}_+) & 0 \\ 0 & A_j(\underline{U}_-) \end{pmatrix}$ for $j \in \{1, ..., d-1\}$ and $\mathcal{A}_n := \begin{pmatrix} A_n(\underline{U}_+) - \sigma I_n & 0 \\ 0 & \sigma I_n - A_n(\underline{U}_-) \end{pmatrix}$, as well as (for $\tau \in \mathbf{C}$ and $\eta \in \mathbf{R}^{d-1}$) $b(\tau, \eta) := \tau(\underline{U}_+ - \underline{U}_-) + \sum_{k=1}^{d-1} \eta_k (f^k(\underline{U}_+) - f^k(\underline{U}_-))$ and $\mathbb{E}_+(\tau, \eta) := \bigoplus_{\Im(\xi_n)>0} \mathbb{E}_+(\xi, \tau, \eta)$, with $\mathbb{E}_+(\xi, \tau, \eta) := \{h(0) \,|\, \text{ with } \tilde{h} := h(\cdot)e^{i\xi_n \cdot}$, h a polynomial, such that $\tilde{h}$ solves $\mathcal{A}_n \partial_n \tilde{h} + i(\tau I_{2n} + \sum_{j=1}^{n-1} \eta_j \mathcal{A}_j)\tilde{h} = 0\}$.

Let us now consider the condition that will be enforced to obtain bounds that will give to have a quantification on the hyperbolic boundary problems associated. They are called "uniform stability conditions" and take the following form : (in what follows, the symmetric hyperbolicity in the sense that of the problem is assumed at every point on the neighborhood of $\underline{U}_+$)

Definition 3. *The Uniform-Kreiss Lopatinski condition of the shock $U(t,x) = U_-$ for $x_d < t\sigma$ and $U(t,x) = U_+$ for $x_d > t\sigma$ is defined as : there exists $c > 0$ such that for every $\tau \in \mathbf{C}$ such that $\Im(\tau) < 0$ and $\eta \in \mathbf{R}^{d-1}$, if $|\tau|^2 + \|\eta\|^2 = 1$ for every $(k,h) \in \mathbf{C} \times \mathcal{E}(\tau, \eta)$ $\|ib(\tau,\eta)k + Mh\| \geqslant c(\|k\| + \|h\|)$.*

To define the regularity of the solutions, consider the sets $\omega_T :=]0, T[\times \mathbf{R}^{d-1}$ and $\Omega_T := \omega_T \times \mathbf{R}^*$

Definition 4.

$$CH^s(\Omega_T) := \{u \in C^0([0,T], H^s(\mathbf{R}^{d-1} \times \mathbf{R}^*)) \,|\, \forall j \in \{0, ..., s\}, \\ \partial_t^j u \in C^0([0,T], H^{s-j}(\mathbf{R}^{d-1} \times \mathbf{R}^*))\}.$$

Finally, we note $\underline{u}(t,x) = \underline{U}_{(sgn)(x_d)}$.

Remark : The uniform stability conditions imply that the shock is actually a Lax shock, and more precisely that it satisfies

The following theorem can be found as an adaptation of the theorem obtained in [Met] :

Theorem 6. *Let $s > \frac{d+2}{2}$, an integer and with f satisfying the hyperbolic conditions, and assume also the uniform stability conditions at the jump associated with the Riemann shock.*

Then, there exists $\rho > 0$ and $T > 0$ such that, for any $u_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^d \backslash \{x_d = 0\})$, such that $\|u_0\|_{L^\infty} \leqslant \rho$ and u_0 satisfy the compatibility conditions up to order $s-1$, there exists a unique solution (u, ψ) with $u : [0, T[\to \mathbf{R}^n$ and $\psi : [0, T[\times \mathbf{R}^{d-1} \to \mathbf{R}$ with $u - \underline{u} \in CH^s(\Omega_T)$ and $\psi \in H^{s+1}(\omega_T)$ to the problem with initial data in the sense that $u(0, \cdot) = u_0 + \underline{u}$, $\psi(0, \cdot) = 0$ and u solves the equation in the classical sense on $\{(t, \tilde{x}, x_d) \in]0, T[\times \mathbf{R}^{d-1} \times \mathbf{R} \,|\, x_d \neq \psi(t, \tilde{x})\}$, and (u, ψ) satisfies the Rankine-Hugoniot conditions at $x_d = 0$ for every $t \in [0, T[$. Furthermore, every solution with the same initial data coincides with that solution on the intersection of their domain of existence.

Furthermore, a criteria of the maximal time of existence can be derived. It involves that either $\|\nabla_x u\|_{L^\infty(\Omega_T)}$ is infinite, or either $u - \underline{u}$ or $\nabla_y \psi$ exits some small enough neighborhood of 0. The last condition is there to ensure that the uniform Kreiss-Lopatinskii conditions will remain true at time T_{max}.

Actually, optimal regularity (avoiding the loss of half a derivative) has been recently proved for initial boundary value problems in [Aud22] and we expect that it could also be obtained here.

Note that the compatibility conditions are automatically satisfied for an initial perturbation localized outside of some neighborhood of the shock.

In particular, for the easier case of a one-dimensional system, as, in that case, the estimates easier to derive with the fact that $\mathbf{R}^{d-1}$ is of dimension 0, and one just needs to check the invertibility of a matrix, the uniform Lopatinsky condition can be reduced to algebraic conditions.

From now on, $(\lambda_k(u))_{1 \leqslant k \leqslant n}$ will denote the increasing sequence of eigenvalues of $A_1(u)$ for $u \in \mathbf{R}^n$, which are assumed to be such that $\lambda_1(u) < ... < \lambda_n(u)$ (the system is strictly hyperbolic) and all distinct from σ. Note $k_\pm$, the integers in $\{0, ..., n\}$ (with the conventions $\lambda_0 = -\infty$ and $\lambda_{n+1} = +\infty$) such that $\lambda_{k_-}(\underline{U}_-) < \sigma < \lambda k_- + 1(\underline{U}_-)$ and $\lambda_{k_+}(\underline{U}_+) < \sigma < \lambda_{k_+ + 1}(\underline{U}_+)$.

Proposition 1. *Here, $d = 1$. The (uniform) Kreiss Lopatinskii conditions described above are equivalent to the following conditions : the shock satisfies the Lax condition and $\mathbf{C}^N = \mathbf{C}b_0 \oplus E_+ \oplus E_-$ where $b_0 = \underline{U}_+ - \underline{U}_-$, as before, $E_+ := \bigoplus_{k \geqslant k_+ + 1} Ker(A_1(\underline{U}_+) - \sigma - \lambda_k(\underline{U}_+))$, and $E_- := \bigoplus_{k \leqslant k_-} Ker(A_1(\underline{U}_+) - \sigma - \lambda_k(\underline{U}_-))$.*

In the above mentioned references, there is no reaction term G. However, it is possible to adapt the proofs to that case as this term does not change the argument : in fact, it

induces a perturbation of order 0. Furthermore, it can be noted that for other types of solutions containing a shock, such as solutions not constant on one side but constant on the hyperplane $\{x_d = 0\}$ on both sides, there can be adaptations made although one needs to be careful about the compatibility conditions.

In fact, to adapt the result to that case, one can use the finite speed of propagation of the hyperbolic equation.

1.1.3 Inviscid limit

Vanishing-viscosity limit : system case

An important subject related to the inviscid limit is the prolongation of Kružkov's work towards system problem. As said before, the well-posedness theory does not directly transpose to the system case : in fact, convex entropies are lacking when it comes to systems (in general), and, in some cases, on cannot distinguish weak solutions from entropy solutions as all the entropies are trivial. However, several type of questions have been studied since then to answer these questions.

A first one is the study of the convergence of classical solutions of a viscous regularization towards some hyperbolic limit. See, for instance, the chapter [Mas07], in which the case of the convergence of the solution to some of the Navier-Stokes system are discussed.

An other important class of problem is the one-dimensional problem. In fact, in that setting, the celebrated Glimm scheme allowed to define a notion of solution for such problems in the framework of initial data BV, small enough. In that context, the importance of Riemann shocks, and, more generally, the solutions to Riemann problems need to be understood in the vanishing viscosity limit. However, compared to the work of this thesis, piecewise smoothness is not the context considered. It is rather a work done in the BV topology or the even rougher L^∞ one.

In particular, the question is whether or not solutions obtained through that mean can also be shown to be the limits of the solutions to parabolic equations with a vanishing viscosity.

The first breakthroughs in that program were obtained in 2×2 systems. The techniques used were, for example, based on compensated compactness (a classical reference being [Tar79]).

To obtain results for more general systems, new ideas emerged, and, in particular a program was developped by Bianchini and Bressan, which allowed them to obtain convergence for a constant coefficients viscosity $\varepsilon\partial_x^2$. The work was done in the following series of papers [BB00, BB02b, BB01, BB02a, BB05] and the programm is described in

the following article : [BB02a]. In fact, in the said articles, the hyperbolic systems studied are more general than conservation laws. However, to obtain that the limits constructed in the vanishing viscosity limit are the known solutions to the the limiting equation, one needs some well-posedness theory at the limit. Such a theory is only available in the case of conservation laws.

However, since that work, the generalization to real quasilinear viscosity is still open. For many important examples, such as the compressible Navier-Stokes equations and their limits (the Euler equations) being an inviscid limit of this form. It is still an considered as one of the main problems of the field. In fact, partial results were obtained, such as one close to our project here : the uniqueness and stability (locally in time) of Riemann shocks in the class of inviscid limits of solutions to the Navier-Stokes equations [KV21b]. Other articles related to that problem are [KV22] which discusses the case of two shocks, and [VW15] for an inviscid limit in the case of a contact discontinuity.

Boundary layers

A closely related topic to the work here, and related to the convergence of Navier-Stokes equations towards Euler equations for a bounded domain is the problem of boundary layers. Related techniques to those used in this thesis have been used in such a problem. A general introduction to the subject can be found in [Ḿ04]. In particular, one can mention [GR01] and [MR12] in which pointwise bounds on Green functions were used. This kind of techniques will be presented later on. However, in the second article, the problem at hands

In contrast to the present work, the goal is to study a problem of conservation laws. Furthermore, the point is to obtain bounds for a time-dependent problem but for finite time and to apply it to the nonlinear problem, while, here, the aim is to obtain some bounds on a time autonomous problem in the large time regime to apply it to a nonlinear problem.

Notably, in [MR12], to bound the nonlinear part, they used conormal spaces, a way to avoid as much as possible to consider derivatives in the normal direction to the boundary, that is, the one in which are appearing the layers. Finally, in [MZ05b], limitations of the Green functions are addressed, but, still, in [MR12], despite the higher dimension context, Green functions formalism is used for a part of the linear problem.

1.2 Traveling waves and their stability

1.2.1 Generalities about waves

First, remind that a traveling wave is a solution u to some evolution PDE with u of the form $u(t,x) = \underline{u}(x - t\sigma)$ for some speed $\sigma \in \mathbf{R}^d$. They are stationary solutions in the right frame. Thus, it is expected that they are important pieces to understand the asymptotic behavior of pdes. Here, the main discussions will remain at the level of abstract mathematical models, however, let us indicate [CH93] as a classical reference of the physical viewpoint.

On the theoretical viewpoint, they also appear in reaction diffusion due to theorems in the study of the asymptotic behavior of solutions to such a system. In fact, there has been recently progresses in the literature concerning the problem of the front selection. This problem is to ask, with a particular compactly supported initial data, which type of wave will be selected asymptotically.

There is a similar program in the field of dispersive pde ; the so called resolution conjecture. In [Tao06], a presentation of the basic conjecture and discussions about its solution in particular cases.

A plane wave is a particular type of wave which can be written as $(t,x) \mapsto \underline{v}(\langle \sigma, x\rangle - ct)$ for σ a unit vector and $c \in \mathbf{R}$). Waves are objects studied in various parts of the mathematical study of PDEs , and for different types of motivation.

Finally, similar type of statements is actually true in the framework of conservation laws and balance laws, under some assumptions. The Glimm scheme, which will be discussed a bit later, is actually some nonlinear procedure built through an (approximate) decomposition in waves, and the understanding of the interactions of the said waves. Furthermore, in the balance case, similar results actually hold in the scalar case, although the picture is, in general, broader [MS97].

To study their stability, the closely related notions of orbital stability and the associated asymptotic orbital stability are in fact better suited. They are defined by the following (for some Banach space X in which the Cauchy problem with initial data $\underline{u} + v_0$ for every $v_0 \in X$, and $\underline{u}$ the shape of the traveling wave).

— The wave is orbitally stable in X if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $v_0 \in X$ with $\|v_0\|_X \leqslant \delta$ the solution $w(t,\cdot)$ to the PDE with initial data $v_0 + \underline{u}$ satisfies that it is defined on $\mathbf{R}_+$ and for every $t \geqslant 0$

$$\inf_{\phi \in \mathbf{R}^d} \|w(t, \cdot + \phi) - \underline{u}\|_X \leqslant \epsilon,$$

— The wave is asymptotically orbitally stable if it is orbitally stable and if for some

$\delta > 0$ and every $v_0 \in X$ such that $\|v_0\|_X \leqslant \delta$,

$$\lim_{t \to +\infty} \left(\inf_{\phi \in \mathbf{R}^d} \|w(t, \cdot + \phi) - \underline{u}\|_X \right) = 0.$$

Furthermore, in the context of plane waves in the direction σ things can be modified slightly to take into accounts other kinds of effects. In particular, a choice is to consider actually that ϕ is actually a function from the subspace of vectors orthogonal to the direction of propagation of the wave with value in $\mathbf{R}$ (and in some space Z of such functions) and to look for $\inf_{\phi \in Z} \|w(t, \cdot) - \underline{v}(x + \phi(x - \langle x, \sigma \rangle \sigma) - t\sigma))\|_X)$. Finally, it is often also of interest to look for the regularity in time that can be obtained on a phase ϕ valued in Z such that $\|w(t, \cdot) - \underline{v}(\langle x, \sigma \rangle - ct + \phi(t, x - \langle x, \sigma \rangle \sigma))\|_X$ stays smaller than ε for every $t \geqslant 0$ or goes to 0 as $t \to +\infty$.

Some classical techniques to study the stability of waves is the use of the so called semigroup of the linearization around the wave of the pde. It consists on studying the solutions of $\frac{du}{dt}(t) = A(u(t))$ on $\mathbf{R}_+$ for some elements u of a certain functional space (and A is a linear operator acting on a Banach space defined on $D(A) \subset X$ with values in X).

1.2.2 Reaction diffusion and smooth waves

Here are considered parabolic nonlinear partial differential equations (on $\mathbf{R}^d$ of the form

$$\partial_t u + \mathrm{div}_x(f(u)) = g(u) + \Delta_x u,$$

and the aim will be described the results obtained before for the stability of travelling waves solutions to such problems. Mainly, here, the points of interest will be the techniques used before the introduction of the pointwise estimates of Green functions (most cases presented here, will be restricted to $d = 1$, and f or g to be the function vanishing everywhere).

Here, the waves considered will be plane waves $\underline{U}$, solutions of the equation with a travelling speed $\sigma \in \mathbf{R}^d$, and restricted to the case in which the front $\underline{u}$ of the wave is a heteroclinic orbit (more precisely, the assumptions made will be that $\underline{u}$ converges exponentially fast towards some constant state u_- at $-\infty$ and towards u_+ at $+\infty$, and that its derivatives converge exponentially fast towards 0 at $\pm\infty$, and that $\underline{u}'$ does not vanish).

The focus on two kind of strategies used to study such problems :

— The first kind are both ways to obtain direct nonlinear results, namely the use of energy estimates and maximum principles,

— The second kind are techniques based on spectral estimates, followed by the use of specif ic properties of parabolic equations to derive linear estimates and then nonlinear stability.

Energy estimates and maximum principles

For scalar equations, one can use some maximum principles (also known as comparison principles). For a discussion of the maximum principles, one standard reference is [PW84]. Such an approach is discussed in [VVV94]. In particular, one can illustrate it through some particular model

$$\partial_t u = \partial_x^2 u + g(u)$$

with $g : \mathbf{R} \to \mathbf{R}$ smooth such that $g(0) = g(1) = 0$, $g'(0) < 0$, $g'(1) <$ and such that there exists a unique $\alpha \in (0,1)$ such that $g(\alpha) = 0$.
Now, some results from [FM77, Fif79] will be described.

First, the following comparison principle

Proposition 2. *Let $c \in \mathbf{R}$ be some constant. If $u, v : [0,T] \times \mathbf{R}$ be such that $u(0,\cdot) \leqslant v(0,\cdot)$ on $\mathbf{R}$, $\partial_t u - c\partial_x u - \partial_x^2 u - g(u) \leqslant 0 \leqslant \partial_t v - c\partial_x v - \partial_x^2 v - g(v)$ on $[0,T]\times\mathbf{R}$, $0 \leqslant u \leqslant 1$ and $0 \leqslant v \leqslant 1$ on $[0,T] \times \mathbf{R}$.*

Then $u \leqslant v$ on $[0,T] \times \mathbf{R}$.

Remark 1. *u and v are then called (respectively) a subsolution and a sursolution of the equation $\partial_t u - c\partial_x u = f(u) + \partial_x^2 u$.*

Theorem 7. *There exists a unique (up to translation) travelling wave of profile $\underline{u}$ solution (where $c \in \mathbf{R}$ denotes its speed) to $\partial_t u = \partial_x^2 u + g(u)$ such that $\underline{u}$ is valued in $(0,1)$, $\lim_{x\to+\infty} \underline{u}(x) = 1$ and $\lim_{x\to-\infty} \underline{u}(x) = 0$.*

Furthermore, for every $\phi \in C^\infty(\mathbf{R},(0,1))$, such that there exists $M > 0$ and $\varepsilon > 0$ such that, for every $x \geqslant M$, $\phi(x) \geqslant \alpha + \varepsilon$ and $\phi(-x) \leqslant \alpha - \varepsilon$, the unique classical maximal solution v with initial data ϕ is defined on $\mathbf{R}_+$, there exist $C, \omega > 0$ and $z \in \mathbf{R}$ such that, for every $t \geqslant 0$

$$\|v(t,\cdot) - \underline{u}(\cdot - ct - z)\|_\infty \leqslant Ce^{-t\omega}.$$

This theorem is proved by designing a subsolution and a sursolution v_- and v_+ to the equation, converging towards possible different translated versions of the wave and which satisfy $v_-(0,\cdot) \leqslant \phi \leqslant v_+(0,\cdot)$. Then by using the previous maximum principle, one can conclude that v is globally defined and that $v_- \leqslant v \leqslant v_+$ on $\mathbf{R}_+ \times \mathbf{R}$. This allows

to control from above and below the solution with translations of the wave. Then, by a compactness argument, a convergence without rate can be obtained. The rate is deduced by introducing a Lyapunov functional. It relies on the observation that the only thing to do is to prove it on some domain $\{(t,x) \in]T, +\infty[\times \mathbf{R} \,|\, |x| < \delta t\}$ for some δ, T positive (which can be respectively as small and big as needed). This last part is actually obtained through a weighted energy estimate.

For example, here, introducing the L^2 based space with norm

$$\|v\|_H = \left(\int_{\mathbf{R}} e^{-cx} |v(x)|^2 dx \right)^{\frac{1}{2}}$$

the basic goal is then to prove that there is some constants δ, ν positive such that, by assuming that initially $\|v_0\|_H < \delta$ then the solution induced by the initially data v_0 satisfies that the map $t \mapsto \|v(t)\|_H^2$ is differentiable and, as long as $\|v(t)\|_H^2$ remains small enough on $[0,T]$, the bound (on $[0,T]$)

$$\frac{d\left(\|v(\cdot)\|_H^2\right)}{dt}(t) \leqslant -\nu \|v(t)\|_H^2.$$

Obviously, various variations of this strategy may be used.

As presented in the example before, it is important to understand that such a strategy may need to be combine with other ideas to obtain the decay rate in the desired norm, or even just to be able to obtain the differentiable bound above.

Finally, one point about the example above that needs to be discussed is the lack of dissipation in the direction of the derivative of the profile of the front. As a consequence, to obtain the result, one needs to get rid of that zero mode. In the strategy discussed before, it was already taken care of through the first part of the argument.

To describe other strategies, one can refer to those used in various contexts, such as the particular case of constant states, for which this kind of direct argument is extremely powerful.

The method developed by several authors to be able to use direct energy estimates to obtain the result was to change the unknown and to consider instead a primitive $U(t,\cdot) := \int_{-\infty}^{\cdot} u(t,x) - \underline{u}(x) dx$, which forces to consider perturbations in L^1 such that the L^2 norm of the primitive is small enough. For scalar one-dimensional viscous conservation laws, in [MN94], it was proved that such a technique is enough to obtain asymptotic stability (it can be noted that, in this case, assuming that the perturbation is in L^1 implies that, up to choosing a translation of the wave, its integral starting at $-\infty$ vanishes at $+\infty$).

At last, we stress that the method does not seem adaptable to the non conservative

case considered in the present manuscript.

Finally, let us mention some use of energy estimates in hyperbolic, as well as hyperbolic-parabolic, equations to conclude nonlinear arguments : the so-called high-frequency damping estimates. They will be an important part of Chapter 1. They amount to control high order derivatives by low order ones.

Just to give a bit of context, the idea is, under certain assumptions such that $\underline{U}(x+\psi(t,x))+V(t,x+\psi(t,x))$ solves We illustrate this with a non trivial example from [RZ16]. Consider a periodic wave profile $\underline{U}=(\underline{\tau},\underline{u})$, that is a stationary solution to

$$\begin{cases} \partial_t\tau - c\partial_x\tau - \partial_x u = 0 \\ \partial_t u - c\partial_x u + \partial_x((2F^2)^{-1}\tau^{-2}) = 1-\tau u^2 + \nu\partial_x(\tau^{-2}\partial_x u), \end{cases}$$

where F and ν are some physical positive constants, and c is the wave speed. Let us look for a solution in the form $(\tau,u)(t,x)=\underline{U}(x+\psi(t,x))+V(t,x+\psi(t,x))$. Then

Proposition 3. *For any positive s, there exist positive C_s, θ_s and ε_s such that any solution (V,ψ) defined on $[0,T]$ for some T and such that $\sup_{t\in[0,T]}\|(V,\psi_t,\psi_x)\|_{H^s}<\varepsilon_s$ satisfies*

$$\|V(t)\|_{H^s}^2 \leqslant Ce^{-\theta t}\|V(0)\|_{H^s}^2 + C\int_0^t e^{-\theta(t-s)}(\|V(s)\|_{L^2}(s)+\|(\partial_t\psi(s),\partial_x\psi(s))\|_{H^s}^2)ds.$$

This is a high-frequency damping estimate on V. The result does not say whether the wave is stable or not. But it does imply that any possible instability is low-frequency and that the decay of high regularity norms is slaved to the decay of low regularity ones. The same kind of energy estimates that prove the proposition also yields a control on resolvents, useful to apply the Gearhart-Prüss-Greiner theorem discussed below.

The high-frequency damping estimates prevent blow-up of wavebreaking type. In proofs of asymptotic stability, they are typically combined with estimates on the linearized dynamics used in Duhamel formulations to control low-frequency norms (with a regularity loss compensated by the high-frequency damping).

Linear propagator and inverse Laplace transform

To prepare the next survey of techniques, and, more generally, the rest of this thesis, some classical results on linear dynamics need to be stated. The presentation here follows mainly results from [Paz83].

Recall that in the setting of semigroups, one can first rely on the inverse of the Laplace transform to derive informations on the semigroup $(e^{tA})_{t\in\mathbf{R}_+}$ generated by a

linear operator A from informations on the resolvent $R(\lambda, A) := (\lambda - A)^{-1}$, that is by a representation of the form

$$e^{tA} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\mu t + ist} R(\mu + is\lambda, A)(u_0) ds$$

for any $u_0 \in D(A^2)$, and where $\mu \in \mathbf{R}$ is a constant big enough. Of course, such a representation relies on the fact that A generates a strongly continuous semigroup.

In the context of semigroups of linear operators, a subclass that allows to study various parabolic problems are those of analytic type. To give a bit of context, they are given by a generator A such that there exist $\alpha \in \mathbf{R}$ and $M > 0$ such that $\sigma(A) \subset \{\lambda \in \mathbf{C} \,|\, \Re(\lambda) \leqslant \alpha\}$ and $\|R(\lambda, A)\| \leqslant \frac{M}{|\lambda - \alpha|}$. Moreover, in this case, the spectrum is actually contained in a strict cone and the resolvent estimate extends to the exterior of such a cone. For analytic semigroups, we have a better representation formula

$$S(t) = \frac{1}{2\pi i} \int_{\Lambda} e^{t\lambda} R(\lambda, A) d\lambda$$

where Λ is a continuous and piecewise smooth function from $\mathbf{R}$ to $\rho(A)$, such that Λ passes at the right of $\sigma(A)$ and such that $\lim_{\xi \to \pm\infty} \Re(\Lambda(\xi)) = -\infty$ and $\lim_{\xi \to \pm\infty} \Im(\Lambda(\xi)) = \pm\infty$.

The analytic representation formula readily implies that if the spectrum of A lies strictly on the left of the imaginary axis then the semigroup exhibits exponential time decay. Yet this situation can not happen naturally for our problems since 0 belongs to the spectrum because of translational invariance. One way to fix this is to work in exponentially weighted spaces requiring such a strong localization that the derivative of the profile does not belong to the functional space under consideration.

To match more closely the case of fronts or solitary waves in parabolic systems, let us assume that A is the generator of an analytic semigroup whose spectrum is contained in $\{\lambda \in \mathbf{C} \,|\, \Re(\lambda) > -2\delta\} \backslash \{0\}$ for some $\delta > 0$. Also, and that 0 is a simple eigenvalue of A. By the latter we mean that $R(\lambda, A)$ is meromorphic near 0 with a pole of order 1 at 0. Then deforming the above contour through 0 provides the following decomposition

$$\pi_0 = \frac{1}{2\pi i} \int_{\mathbb{S}(0,\delta)} R(\lambda, A) d\lambda,$$

and decay exponentially fast in time.

Historically this has been the most traditional approach [Sat]. This includes the case of parabolic conservation laws when exponential weights are used to create a spectral gap.

When adapting the above strategy to systems that are not parabolic, the only part

that needs to be significantly modified is the one converting a spectral gap into a linear decay. Indeed, in general, the presence of a spectral gap is not sufficient and some extra control on resolvents is needed. For instance, on Hilbert spaces, one may deal with general operators through the Gearhart-Prüss-Greiner Theorem as presented in [EN00, HS21] :

Theorem 8. *If A is a densely defined generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$ on a Hilbert space H, such that $\sup_{\lambda\{\Re(\lambda)>0\}} \|R(\lambda, A)\| < +\infty$ then there exist C, ω positive constants such that $\|T(t)\|_{H\to H} \leqslant Ce^{-t\omega}$.*

The conditions given are also necessary (but not sufficient) in a Banach spaces.

Examples of application of this theorem in the context of the study of the orbital stability of travelling waves are given in [KP13, EN00].

Let us recall that one of our main goals is to the stability of viscous shock waves uniformly with respect to the viscosity parameter $\epsilon > 0$. A first strategy would then be to consider the decomposition of the semigroup as described before, and to try to track done the dependence of the bounds in ε to ensure uniformity of the result in ε. However, computations do not go smoothly, and a huge issue in trying to do so is that the perturbation is singular. Indeed, the kernel of the limiting linear problem (obtained for the hyperbolic problem) involves Dirac functions. By changing variables, taking $(\tau, \xi) = (\varepsilon t, \varepsilon x)$, we obtain a new problem, not of singular perturbation this time, but for which the singularity is linked to the fact that the distance from 0 to the essential spectrum, shrinks to 0 as ε goes to 0. Because of that, one cannot obtain adequate bounds in this way. To gain uniformity, we will need to design a phase separation dramatically different from the one presented here and involving contributions from the essential spectrum.

Structure of the spectrum and weights

In scaled variables, the linear operators are of the form

$$\mathcal{L} : v \mapsto \partial_x^2 v + \partial_x(\alpha v) + \beta v$$

for some smooth functions α and β which will converge (as well as their derivative) exponentially fast at $\pm\infty$.

Here we consider as an operator on $L^2(\mathbf{R})$ with domain $H^2(\mathbf{R})$. We recall now how spectral stability may be decided. To explain how to obtain such results, one needs to first remind the decomposition of the spectrum in a point spectrum part and an essential spectrum part for such operators.

Remind that the spectrum of a linear operator $A : D(A) \to X$ densely defined on

a Banach space X is $\sigma(A) := \{\lambda \in \mathbf{C} \,|\, \lambda I - A$ is not an invertible operator from $D(A)$ to X or it is invertible and its inverse is not continuous as an operator from X to $X\}$. Furthermore, $\rho(A) := \mathbf{C}\backslash\sigma(A)$, and, for every $\lambda \in \rho(A)$, $R(\lambda, A) := (\lambda I - A)^{-1}$.

For an operator such as $\mathcal{L}$, one can actually obtain a decomposition that allows to compute rather easily one part of the spectrum , and to reduce the rest of the spectrum of the operator as the zeros of a holomorphic function.

To define them, first remind, as in [KP13], that, for an unbounded linear operator A with domain $D(A)$, its Fredholm index is defined if $Ker(A)$ is finite dimensional and $\mathrm{Ran}(A)$ is of finite codimension, and, then, $ind(A) := dim(Ker(A)) - codim(\mathrm{Ran}(A))$.

Then the splitting is

$$\sigma(\mathcal{L}) = \sigma_{ess}(\mathcal{L}) \cup \sigma_{pt}(\mathcal{L})$$

where $\sigma_{ess} = \{\lambda \in \mathbf{C} \,|\, \lambda I - \mathcal{L} - K$ does not have a Fredholm index equal to $0\}$, and $\sigma_{pt}(\mathcal{L})$, its complementary in $\sigma(\mathcal{L})$. Equivalently, when $\sigma(A) \neq \mathbf{C}$ picking $\mu \in \rho(A)$, $\mathbf{C}\backslash\sigma_{ess}(A)$ may be characterized as the set of λ such that there exists an operator K such that $D(A) \subset D(K)$, $\lambda I - (A + K)$ is invertible and $K \circ R(\mu, A)$ is compact from X to X. We refer to [EE87] for a proof of the latter. A concrete characterization of $\sigma_{ess}(\mathcal{L})$ is given below.

The full description of $\sigma_{ess}(\mathcal{L})$ in the case presented here, as well as the definition of the holomorphic function described earlier will soon be obtained.

The characterization given above enables us to check that the essential spectrum of $\mathcal{L}$ agrees with the one of $\mathcal{L}_\infty$ defined by

$$\mathcal{L}_\infty(v)(x) = \begin{cases} \partial_x^2 v(x) + a_{+\infty}\partial_x v(x) + b_{+\infty} v(x), & \text{if } x > 0 \\ \partial_x^2 v(x) + a_{-\infty}\partial_x v(x) + b_{-\infty} v(x), & \text{if } x \leqslant 0 \end{cases}$$

where $a_\pm$ and $b_\pm$ denote the respective limits of α and β at $\pm\infty$. The point is that the essential spectrum of $\mathcal{L}_\infty$ is easily computed. Indeed, let us set

$$A_\pm(\lambda) = \begin{pmatrix} 0 & 1 \\ \lambda - b_\pm & -a_\pm \end{pmatrix}$$

so that $(\lambda I - \mathcal{L}_\infty)v = 0$ implies that $V = (v, v')$ satisfies $V' = A_\pm(\lambda)V$ on $\mathbf{R}_\pm^*$. Then $\lambda I - \mathcal{L}_\infty$ is Fredholm of index 0 if and only if neither $A_+(\lambda)$ nor $A_-(\lambda)$ has purely imaginary eigenvalues and they have the same number of eigenvalues with positive real part. We refer to [KP13, Hen81] for details.

Since in the present case, the needed spatial eigenvalues may be explicitly computed this gives a concrete description of $\sigma_{ess}(L)$. In contrast, $\sigma_{pt}(\mathcal{L})$ can not be computed

explicitly but it can be characterized as the zeros of a holomorphic function. This is a more tedious task, detailed jointly with our closely related discussion of the spectral Green functions.

Let us now describe how one can use the description of the essential spectra as well as the Sturm-Liouville theory to derive orbital asymptotic stability of waves. First, the classical following result, the Sturm-Liouville theorem on the real line, needs to be stated (the form is simplified to take only into account the consequences needed here ; a more detailed statement can be found in [KP13])

Theorem 9. *With a linear operator of the form $L := \partial_x^2 + \partial_x(c\cdot) + d\cdot$ with c and d smooth and real valued, converging exponentially fast at $\pm\infty$. Then, $\sigma_{pt}(\mathcal{L})$ is finite. If it is non empty, then every eigenvalue of $\sigma_{pt}(\mathcal{L})$ is simple and real. Finally, the biggest element of $\sigma_{pt}(\mathcal{L})$ is the only one to be associated with a non vanishing eigenvector.*

In particular, with the problem under study, if $\max(b_+, b_-)$ is negative, then, as $\underline{u}'$ is an eigenfunction which has no zero associated to 0 and that $\sigma_{ess}(\mathcal{L}) \subset \{\lambda \in \mathbf{C} \,|\, \Re(\lambda) \leqslant \max(b_+, b_-)\}$, we have that $0 \in \sigma_{pt}(\mathcal{L})$.

Furthermore, as $\underline{u}'$ has no zero, $\sigma_{pt}(\mathcal{L})\backslash\{0\}$ is included in $\{\lambda \in \mathbf{C} \,|\, \Re(\lambda) \leqslant -\eta\}$ for some $\eta > 0$, and 0 is a simple eigenvalue. As a consequence, one can apply standard linear estimates. In particular,

$$e^{t\mathcal{L}} \circ (I - \pi_0) = \int_\Lambda e^{t\lambda} R(\lambda, L) \circ (I - \pi_0) d\lambda,$$

and so,

$$\|e^{t\mathcal{L}}(I - \pi_0)\|_{L^2 \to H^1} \leqslant \int_\Lambda |e^{t\lambda}| \|R(\lambda, \mathcal{L})\|_{L^2 \to H^1} |d\lambda|.$$

With Λ valued in $\rho(\mathcal{L}) \cup \{0\}$, enclosing the complementary of this set by the right, and only taking values in the set of complex numbers which have a negative real part.

And, given a semigroup of analytic operators, with a generator which has a spectrum A contained in $\{\lambda \in \mathbf{C} \,|\, \Re(\lambda) \leqslant \eta\}$, one has that for all $\delta > 0$ there exist $\nu_\delta \in \left]\frac{\pi}{2}, \pi\right[$ and $C_\delta > 0$ such that for all $\lambda \in \Omega_{\nu+\delta,\theta}$

$$\|R(\lambda, A)\| \leqslant \frac{C_\delta}{|\lambda - \nu - \delta|},$$

and so,

$$\|AR(\lambda, A)\| \leqslant 1 + \frac{C_\delta |\lambda|}{|\lambda - \eta - \delta|}.$$

Thus, in the case considered here, one has that $\partial_x^2 \circ R(\lambda, \mathcal{L}) = -I + \lambda R(\lambda, \mathcal{L}) - (\alpha' + \beta) R(\lambda, \mathcal{L}) - \alpha \partial_x \circ R(\lambda, \mathcal{L})$. Hence, as the interpolation inequality $\|\partial_x v\|_{L^2} \leqslant \sqrt{\|v\|_{L^2} \|\partial_x^2 v\|_{L^2}}$

holds, for $\lambda \in \Omega_{\nu+2\delta,\theta}$,

$$\|\partial_x \circ R(\lambda, \mathcal{L})\|_{L^2} \leqslant \frac{\tilde{C}_\delta}{\sqrt{|\lambda - \nu - \delta|}}.$$

Thus, for convection reaction diffusion equations, one can directly obtain linear bounds from some spectral gap.

To conclude on the structure of the spectrum, we explain how introducing weights may create a spectral gap. To do so we define $L^2_w(\mathbf{R})$ as the set of functions u such that $e^w u$ belongs to $L^2(\mathbf{R})$ and $H^k_w(\mathbf{R})$ as the set of functions such that their derivatives up to order k belong to $L^2_w(\mathbf{R})$. Let us now consider $\mathcal{L}$ as an operator on $L^2_w(\mathbf{R})$ with domain $H^2_w(\mathbf{R})$ for some $\mathcal{C}^2$ function w.

Equivalently one may consider $\mathcal{L}_w := e^w(\mathcal{L}(e^{-w}\cdot))$ acting on $L^2(\mathbf{R})$ with domain $H^2(\mathbf{R})$. Explicitly

$$\mathcal{L}_w(v) = (w'^2 - w'')v + \alpha' v - \alpha w' v + \alpha v' - 2w'v' + v'' + \beta v$$

In particular, by choosing w asymptotically linear at $\pm\infty$ one may improve the stability of the essential spectrum provided that α has nonzero limits at infinities.

Motivated by nonlinear analysis we shall restrict to the case when $\limsup_{\pm\infty}(w) < \infty$. Under minor extra assumptions this implies that H^1_w embeds in $L^\infty(\mathbf{R})$ and is an algebra. With this restriction, we may improve the stability of the part of essential spectrum due to one endstate with an asymptotically linear w only if the limit of α has the right sign. Roughly speaking one needs the instability to travel from infinity towards the core. For this reason, the stability obtained with such weights is often called convective stability.

1.2.3 Pointwise bounds of Green functions

When there is no spectral gap, the time decay cannot be exponential and both the Hille-Yosida theory and the Gearhart-Prüss-Greiner Theorem become useless in large-time. Still, algebraic decay rates may sometimes be obtained by other means provided that the initial data and the solution are measured in different functional spaces. We explain here how to do so through pointwise bounds on Green functions.

As a preliminary motivation, let us recall that in our analysis of the vanishing viscosity limit we are actually dealing with a spectral gap of size ϵ. It is therefore no surprise that in order to prove a uniform result we need to use techniques adapted to cases with no spectral gap.

This being said, let us point out that a good toy example is given by the heat equation, when one trades the localization of the initial data against the time decay of the solution.

Recall that

$$e^{t\partial_x^2}(u_0)(x) = \int_{\mathbf{R}} \frac{e^{-\frac{(x-y)^2}{4t}}}{2\sqrt{\pi t}} u_0(y)dy.$$

From this stems

$$\begin{aligned}\|\partial_x(e^{t\Lambda}(u_0))\|_{L_q} &\lesssim \|u_0\|_{L^p} \left\| \frac{(\cdot)e^{-\frac{.^2}{4t}}}{4t\sqrt{\pi t}} \right\|_{L^r} \\ &\lesssim \frac{\|u_0\|_{L^p}}{t^{1-\frac{1}{2r}}}.\end{aligned}$$

when $1 \leqslant p, q, r \leqslant \infty$, $q \leqslant p$, $1 + 1/q = 1/p + 1/r$.

Our present goal is to explain how to derive such bounds in a robust spectral way.

Spectral Green function

To begin with, we explain how to build a Green function representation for the resolvent operators

$$R(\lambda, \mathcal{L})(v) = \int_{\mathbf{R}} G_\lambda(x, y)v(y)dy$$

Since this is the region involved in the inverse Laplace formula for the semigroup, we focus first on the connected component of $\rho(\mathcal{L})$ containing the region where $\Re(\lambda)$ is sufficiently large.

At least formally, for any $y \in \mathbf{R}$, we expect $v = G_\lambda(\cdot, y)$ to be a $H^2(\mathbf{R})$ solution to

$$(\lambda - \mathcal{L})(v) = \delta_y,$$

which implies that v is smooth on $\mathbf{R}\backslash\{y\}$ and solves $\lambda - \beta v - (\alpha v)' - v'' = 0$.

The region we are considering is contained in the connected component of $\mathbf{C}\backslash\sigma_{ess}(\mathcal{L})$ containing λs with large real part, a region sometimes called the region of consistent splitting. As a consequence the structure of solutions to the eigenvalue equation that are L^2 near $+\infty$, (resp. $-\infty$) is fixed. In the present case, on each side the dimension of the set of solutions is 1 and one may perform, at least locally in λ, a smooth choice of such solution $w_r(\lambda, \cdot)$ (resp. $w_\ell(\lambda, \cdot)$). The construction of these elements is standard ; see for instance [BG14, Zum01]. Some high-frequency analysis is however necessary to uniformize the construction near infinity. For λ in the region of consistent splitting,

the above discussion motivates to look for $G_\lambda(x,y)$ as

$$G_\lambda(x,y) = \begin{cases} w_r(\lambda,x)c_r(\lambda,y) \text{if} x > y, \\ w_\ell(\lambda,x)c_l(\lambda,y) \text{if} x < y. \end{cases}$$

With this ansatz, the problem is reduced to ensuring that for any y

$$B(\lambda,y)\begin{pmatrix} c_r(\lambda,y) \\ -c_\ell(\lambda,y)\end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where

$$B(\lambda,y) := \begin{pmatrix} w_r(\lambda,y) & w_\ell(\lambda,y) \\ \partial_y w_r(\lambda,y) & \partial_y w_\ell(\lambda,y)\end{pmatrix}.$$

One readily checks that if for some y, $B(\lambda,y)$ is not invertible then λ possesses a nonzero eigenvector. Thus if $\lambda \in \rho(\mathcal{L})$, then $B(\lambda,y)$ is invertible for any y. This may also be deduced from a Wronskian computation

$$\det(B(\lambda,y)) = e^{\int_0^y (f'(\underline{u})-\sigma)} \det(B(\lambda,0)).$$

When $\lambda \notin \sigma(\mathcal{L})$, we may therefore conclude the construction with

$$\begin{pmatrix} c_r(\lambda,y) \\ -c_\ell(\lambda,y)\end{pmatrix} = \frac{e^{\int_0^y(\sigma - f'(\underline{u}))}}{D(\lambda)} \begin{pmatrix} -w_\ell(\lambda,y) \\ w_r(\lambda,y)\end{pmatrix}$$

where

$$D(\lambda) := \det(B(\lambda,0)),$$

is usually called an Evans' function.

Incidentally we point out that the construction shows that in the region of consistent splitting the location and order of the poles of the resolvent agree with the location and order of the roots of the Evans' function. This provides the announced characterization of the point spectrum.

Though we shall not use it now, we mention that, since $(x,y) \mapsto \overline{G_\lambda(y,x)}$ provides a Green function for $\bar{\lambda} - \mathcal{L}^*$, the coefficients $c_r(\lambda,\cdot)$ and $c_\ell(\lambda,\cdot)$ solve the ODE corresponding to the dual eigenvalue problem. On a related note we mention that the dual problem has the same region of consistent splitting and one may introduce similarly $\tilde{w}_r(\lambda,\cdot)$ (resp. $\tilde{w}_\ell(\lambda,\cdot)$).

For later use, we need to understand how far beyond the essential spectrum can this construction be extended by analytic continuation. We warn the reader that a priori the

extension will not provide a resolvent operator even in regions where such a resolvent does exist. The decisive step is the extension of the construction of $w_r(\lambda, \cdot)$ and $w_\ell(\lambda, \cdot)$. A simple inspection of the construction of w_r (resp. w_ℓ) shows that it can be continued as long as the the eigenvalues of $A_+(\lambda)$ (resp. $A_-(\lambda)$) have distinct real parts. The set of λ where this fails is called absolute spectrum and the connected component of its complement containing λs with large real part is called the set of extended consistent splitting. The conclusion of the present discussion is that an analytic continuation is possible on any simply connected subset of the region of extended consistent splitting. Furthermore, in the simple second-order model discussed here, the absolute spectrum is a half-line included in the real line, making the set of extended consistent splitting itself a simply connected set, thus making the discussion in that case simpler.

Temporal Green function

We now come to temporal Green functions. Here we mostly follow [ZH98]. We mention however that those were preceded by the closely related [SX93, LZ95, Zen94, LZ97, HZ95, Liu97]. More remotely, they also share some similarities with the much older use of temporal Green functions in short-time studies ; see for instance [Fri64]. The method pioneered in [ZH98] has been subsequently extended in [HZ00, HZ02, MZ03, MZ04a, MZ04b] so as to cover the nonlinear stability analysis of most of smooth fronts of systems with a high-frequency damping mechanism.

We want to bound $(t, x, y) \mapsto G(t, x, y)$ the temporal Green function of $\mathcal{L}$, that is, the function which satisfies, for every $\phi \in L^2(\mathbf{R})$, $(t, x) \in \mathbf{R}^+ \times \mathbf{R}$,

$$e^{t\mathcal{L}}(\phi)(x) = \int_{\mathbf{R}} G(t, x, y)\phi(y)dy.$$

The starting point is that

$$G(t, x, y) = \frac{1}{2\pi i} \int_\Lambda e^{t\lambda} G_\lambda(x, y)d\lambda.$$

where the contour is as in the inverse Laplace transform formula for $e^{t\,\mathcal{L}}$.

A crucial observation that motivates the introduction of Green functions is that one may now move contours in a way that depends on (t, x, y) (and not only t) and in particular we can make use of analytic continuations that go beyond the essential spectrum. One of the main contributions of [ZH98] is to provide a strategy to decide how to move the contour that is less sharp but much easier to apply than the saddle-point method.

We now illustrate this strategy on the heat equation. The absolute spectrum of ∂_x^2 is

$\mathbf{R}_-$ and for $\lambda \notin \mathbf{R}_-$ the analytic continuation of the spectral Green function is given by $G_\lambda(x,y) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x-y|}$. Our goal is thus, by a suitable choice of the path Λ to prove that for some $\theta > 0$

$$\left| \int_\Lambda \tfrac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}|x-y|} d\lambda \right| \lesssim \frac{1}{\sqrt{t}}\, e^{-\theta \frac{|x-y|^2}{t}}\,.$$

The path Λ is defined through

$$\Re(\sqrt{\Lambda(\xi)})\,|x-y| = (\Re(\sqrt{\Lambda_0}) + i\xi)\,|x-y|$$

with $\Lambda_0 \in \mathbf{R}_+$ still to be determined, so that the contribution of ξ to the complicated $e^{\sqrt{\lambda}|x-y|}$ can be discarded but its contribution to $e^{\lambda t}$ provides a $e^{-\xi^2 t}$ that ensures the integrability of the integrand and is responsible for the $1/\sqrt{t}$ factor. The choice of Λ_0 is done by minimization over $\Lambda_0 \in \mathbf{R}_+$ of $\Lambda_0\, t + \sqrt{\Lambda_0}\,|x-y|$. This leads to the choice $\Lambda_0 = |x-y|^2/4t$, and concludes the proof of the claim.

Following this strategy in more general contexts is obviously much more technical but leads to similar results.

1.2.4 Discontinuous waves in the hyperbolic setting

We now discuss stability results concerning discontinuous waves.

As for the local well-posedness theory, there is a clear separation between analyses carried out in topologies that can accommodate discontinuities in a soft way (such as L^1, L^∞, BV,...) and those that are very sensitive to those ($W^{1,\infty}$, piecewise smooth topologies,...). There is actually a gap between the two kinds of the result: in dimension one, the upper limit of the former being BV regularity whereas the lower limit of the latter is (piecewise) $W^{1,\infty}$.

Our analysis is clearly of the second kind and we mostly refer to [Mas99, Liu21] for some examples of the first kind. Moreover we focus on cases when some of the main features of the large-time analysis are already present at the linearized level. However we mention that this overlooks a well-known hyperbolic dissipation effect, the entropy dissipation at discontinuities of genuinely nonlinear fields. This mechanism is for instance used to prove the stability of constant solutions to one-dimensional systems with all characteristics being genuinely nonlinear. In such kind of stability result, the initial data is bounded and compactly supported but the solution is measured in L^1, with decay rate $1/\sqrt{t}$ for the scalar case, $t^{-1/4}$ in the general case.

The results of main interest for the present contribution are [DR20, DR22, GR]. In [DR20] the stability of Riemann shocks of balance laws is studied. In [DR22] a full clas-

sification of traveling waves of balance laws, including those with discontinuities and characteristic points (thus loss of ellipticity), is obtained. At last, in [GR] the stability analysis is extended to the study of the stabilization with exponential weights. In the present thesis, we discard wave profiles with characteristic points (whose analysis is expected to be dramatically different) so that only constant solutions and Riemann shocks are the only stable possibilities in unweighted topologies and the consideration of convective stability only adds more waves of the same type and some smooth fronts.

We also mention two more results in the same spirit. In [FR22], a general result has been obtained for Riemann shocks of general strictly hyperbolic systems with a spectral gap. In [JYZ21, YZ20], some discontinuous fronts of the St. Venant equations, a 2×2 system, have been proved to be stable, with algebraic decay rates. A common difficulty of all system cases, not present in the scalar case, is that the linearized problem does not fit in the semigroup framework and one needs to develop large-time tools for initial boundary value problems of hyperbolic type.

1.3 Main results

We conclude this introduction with a brief description of the main results of the following chapters.

1.3.1 Combustion models and weights

The first chapter contains the results of a collaboration with Aric Wheeler (Indiana), PhD candidate under the supervision of Kevin Zumbrun. The main part of the analysis was carried out during my two-months stay there.

In our common work the formation of singularities in two hyperbolic systems of balance laws providing combustion models was studied.

Such systems admit discontinuous traveling waves representing detonation waves. For both models, the pioneering [Joh74] was extended to show that in weightless topologies arbitrarily small perturbations of detonation waves may form further singularities in finite-time despite the fact that the perturbation is done around some traveling wave solution and not some constant.

For the smallest model, the Majda model, however, this may be cured with exponential weights. Indeed, for the latter, it is proved on one hand weighted high-frequency estimates near any detonation wave, thus discarding the formation of singularities of wavebreaking type, and on the other hand, it is proved that small-amplitude detonation waves are convectively asymptotically orbitally stable. In contrast, for the largest model,

the ZND model, it is proved that no weighted high-frequency damping estimate may hold.

Our stability analysis is carried out through energy estimates.

One aspect that affects the whole analysis, notably the convective stability part, is that the systems exhibit some characteristics that are outgoing and others that are incoming, both kinds being coupled by source terms that are not fully dissipative.

1.3.2 Riemann shocks in the inviscid limit

The second chapter is based on [BR23] and contains a proof of uniform stability of viscous approximations of Riemann shocks, that may be thought as an extension of [DR20].

In that context is proved the existence of a family of traveling-wave solutions $(\underline{u}_\epsilon, \sigma_\epsilon)$ approaching a given stable Riemann shock $(\underline{u}, \sigma)$ and their uniform asymptotic orbital stability. The stability result involves multiscale norms, accounting for the presence of shock layers of length ϵ.

These norms are of the form

$$\|v\|_{W^{1,\infty}_{\varepsilon,\theta}} := \|v\|_\infty + \left\| \frac{\partial_x v}{1 + \varepsilon^{-1} e^{-\theta\varepsilon^{-1}|\cdot|}} \right\|_\infty$$

(where $\theta > 0$ is a parameter small enough and independant of $\varepsilon > 0$.) Thus, in particular, they are equivalent for any fixed $\varepsilon > 0$ to the usual $W^{1,\infty}(\mathbf{R})$ norm, and, for functions supported away from some fixed neighborhood of 0, the equivalence is uniform in ϵ.

This is consistent at the limit with the space $W^{1,\infty}(\mathbf{R}^*)$, encoding piecewise smoothness and used in the hyperbolic case.

There are three main obstacles to overcome.

Firstly, at the linear level we need to identify a phase shift $\phi(t)$ (to encode the orbital character of stability) suitable in both limits $t \to \infty$ and $\epsilon \to 0$. This requires to revisit the whole machinery in [ZH98] and derive a completely new kind of ϵ-dependent phase splitting. In comparison, for the problem at hand, non uniform stability may be obtained by simply projecting out the elements of the kernel of the propagator as reminded earlier.

Secondly, terms of size $1/\epsilon$, such as $\partial_x v(t,x)\, \phi'(t)\, \underline{u}'_\epsilon(x)$, appear in the Duhamel formula used to bound v. Here we use that the nonlinear reminders that are not of size ε are actually spatially exponentially localized at scale ϵ, which we may transfer into a time decay at scale ϵ through a spectral gap argument. The above multiscale norm encodes that $\partial_x v$ is not worse than this.

At last, to guarantee uniformity in ϵ, we cannot close the nonlinear iteration through parabolic regularization. Instead, we rely on a maximal principle argument with a very

carefully designed weight.

It is linked to the norm described above.

1.3.3 Convective stability of Riemann shocks and smooth travelling waves in the inviscid limit

In the first part of the third chapter is also proved uniform stability results. This time, the cases of unstable Riemann shocks in weightless topologies that may be stabilized by exponential weights are studied. This provides a vanishing viscosity counterpart to the Riemann shock part of [GR].

The analysis first treats cases associated to a weight allowing the obtention of a spectral gap (up to the 0 mode associated to the derivative of the front). Then we deal with the case when the weight is barely sufficient (that is, when the weighted essential spectrum touches the imaginary axis, thus when there is not any spectral gap)

$$\frac{e^{-\theta_\epsilon |\cdot|}}{(1+|\cdot|^2)^{\frac{\alpha}{2}}}$$

with θ_ϵ critical, and α sufficiently large to close the nonlinear argument. In the critical case, we obtain algebraic time decay rates by losing spatial moments, thus spatial localization. Consistently we need to lose a sufficiently high power of moments so as to enforce that the algebraic decay is sufficiently fast to close a nonlinear stability argument. By many respects the analysis differs dramatically from the unweighted one.

In a second part, we study the same question for smooth fronts instead of Riemann shocks. Thus, this time, the problem is about the singular perturbation of a smooth traveling wave associated with the hyperbolic problem. It turns out that the corresponding stability problem may be completely analyzed with arguments of maximum principle type. Moreover, in the critical case one may also obtain a stability result without losing polynomial weights. Obviously, however, the latter stability result comes with no time decay rates, or, in other words, is not asymptotic.

Chapter 2

Majda and ZND models for detonation: nonlinear stability vs. formation of singularities

2.1 Introduction

This work is about expanding a new area [DR20, DR22, YZ20, BR23, FR22]: the inviscid global time-asymptotic stability of piecewise smooth solutions of hyperbolic balance laws: specifically, asking if given a smooth and small enough initial perturbation, that the solution to (2.1.1) will remain piecewise smooth, without any other discontinuities appearing.

$$u_t + f(u)_x = g(u). \tag{2.1.1}$$

This is quite different from the behavior of conservation laws, for which we have that, generically, other shocks are expected to form. Important physical examples include the Saint-Venant equations [Liu87, JNR$^+$19] and detonation [DF79].

The most comprehensive contributions concerning the local existence theory for such discontinuous solutions of balance laws were first obtained by Majda in [Maj83a, Maj83b] and then by Métivier in [Met]. See also the book by Benzoni-Gavage-Serre for a more in-depth exposition. Since then, a number of 1-D results have also been obtained; see [Bre00] and references therein. Conditions for finite-time blowup have also been explored, with discussions of other cases, such as other types of shocks [BGS, Bre00].

In such a framework, it is possible to ask, given a solution with a discontinuity at a single point (or, in higher spatial dimensions, on a hypersurface) to the equation (2.1.1), do smooth perturbations of the wave remain smooth outside of the given jump? Here, the goal is to understand the behavior as $t \to +\infty$ of perturbations of such waves for two systems, the Majda and ZND models, which are both detonation models that will be presented below.

A crucial part of the analysis of the Majda model is a high-frequency damping estimate. Damping estimates were initiated in [Zum04, Zum07] and later expanded in [Zum10, JNRZ14, JZN11, RZ16]. The existence of a damping estimate is a property of some systems that allows the control of higher order derivatives of the solution by knowledge of bounds on the lower order ones.

In certain contexts such as hyperbolic-parabolic systems like in [Zum07] and in the fifth chapter of [FS04], this type of result is obtained through arguments similar to those of Kawashima in [Kaw84, KS88].

Damping estimates typically take the form

$$\mathcal{E}(v)_t \leqslant -\theta \mathcal{E}(v) + C\|v\|^2_{L^2_\alpha}, \tag{2.1.2}$$

where $\mathcal{E}(u)$ is an energy equivalent to $\|u\|^2_{H^s_\alpha}$ and L^2_α and H^s_α are weighted L^2 and H^s norms with a weight parameter α. Thus, such an estimate effectively controls the H^s_α norm of the perturbation by the L^2_α norm of the perturbation.

Links with the high-frequency estimates of the resolvent are discussed in [Zum07, RZ16].

An important point is that they prevent singularity formation. In the case of conservation laws, it is often expected that singularities will form (see [Joh74, Liu79]), which can't happen in presence of such damping estimates. In this case, the first order derivative of the perturbation blows up (more precisely, its L^∞ norm blows up), but the L^p norms $(1 \leqslant p \leqslant \infty)$ of the perturbation itself does not blow up.

High frequency damping estimates (2.1.2) are tools to close nonlinear iteration. When proving stability results, they essentially reduce the problem to prove low-regularity estimates that may typically be obtained through a Duhamel formulation. See for instance [YZ20].

Again, this can be contrasted with the classical results on singularity formation [Lax73, Joh74, Liu79, Ali95, Spe16] for the case $g \equiv 0$ of hyperbolic conservation laws. Hence damping estimates depend importantly on properties of g. For further discussion, see [DR20, DR22].

A systematic treatment of damping estimates has been shown in the context of relaxation systems [MZ05a, YZ20]. Here our purpose is to explore limitations of this approach in the physically interesting context of detonation models, which are very similar in structure to and can be viewed as a degenerate type of relaxation models. We begin by showing that, for this general type of problem, the singularity formation shown by John in [Joh74] extends to this case, that is for perturbations around a wave instead of around some constant. An extensive literature on extensions of this result to more general and geometrically involved situations can be found in the books [Ali95, Chr07, Spe16]. For the Majda model, we will show an asymptotic orbital stability result directly thanks to energy estimates in some weighted space for waves of small amplitude, and damping estimates for the general case. For the ZND model, we show that blow-up will occur for some arbitrarily small initial perturbation and how this prevents the kind of damping estimates that we described above. In particular, unlike the Majda model, one cannot stabilize the shock in the ZND model by using exponentially growing weights.

2.1.1 ZND and inviscid Majda models

We focus on two closely related classical models of combustion, an inviscid variation of the model from [Maj81] (see also [Fic79]), which we will call the Majda model, and the Zeldovich-von Neumann-Doering (ZND) model. The ZND combustion model is sometimes tagged as reactive Euler systems or reactive Navier-Stokes systems depending on whether or not one includes viscous effect. Here, we focus on the inviscid case. For simplicity, we will focus on one-step reactions. The origins of this model are discussed in [Lai19]. Both of these models can be written abstractly as

$$\begin{aligned} U_t + f(U)_x &= k\vec{q}\phi(U)z, \\ z_t &= -k\phi(U)z, \end{aligned} \tag{2.1.3}$$

where $U \in \mathbb{R}^n$ is comprised of various gas-dynamical properties such as velocity, specific volume, and internal energy, $z \in \mathbb{R}$ is the mass fraction of unburned gas, $\phi(U)$ is an "ignition function", which we will take here to be a rough cutoff depending on the temperature of the gas, $\vec{q}$ corresponds to quantities produced by the reaction, in particular the amount of heat released by the reaction, and $k > 0$ corresponds to the reaction rate. Note that in the scalar case q is permitted to have either sign, with $q > 0$ corresponding to an exothermic reaction and $q < 0$ corresponding to an endothermic reaction. In the Majda model, we take $U, q \in \mathbb{R}$ to be scalar quantities, with U being a "lumped variable" representing features of the density, velocity and temperature of the gas. For the sake of

concreteness, we write the Majda model as

$$\begin{aligned} U_t + f(U)_x &= kq\phi(U)z, \\ z_t &= -k\phi(U)z. \end{aligned} \tag{2.1.4}$$

We will write the ZND model in Lagrangian coordinates, where U is now taken to be in $\mathbb{R}^3$ with $U = (v, u, E)$ for v the specific volume of the gas, u the velocity of the gas, and E the specific gas-dynamical energy (that is, $E = e + \frac{1}{2}u^2$ where e is the specific internal energy). The general system (2.1.3) for the ZND model now takes the form given in [Zum12, Zum11]

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p_x &= 0, \\ E_t + (pu)_x &= qk\phi(T)z, \\ z_t &= -k\phi(T)z. \end{aligned} \tag{2.1.5}$$

To complete the system (2.1.5), one needs to relate the temperature T and pressure p to the variables (v, u, E), or equivalently (v, u, e) for specific internal energy. One common choice to complete the system is to use the ideal gas law to define the pressure and temperature as

$$\begin{aligned} p(v, u, e) &= \frac{\Gamma e}{v}, \\ T(v, u, e) &= \frac{e}{c}, \end{aligned} \tag{2.1.6}$$

where $\Gamma > 0$ is the Gruneisen constant and c is the specific heat constant. For our purposes though, the specific forms of p and T are not so important. For other possible choices we refer to [Erp].

A right going detonation wave is a traveling (shock) solution (U, z) of (2.1.4) or (2.1.5) with speed $\sigma > 0$ satisfying

$$\lim_{x \to \pm\infty} (U, z)(x, t) = (U_\pm, z_\pm), \tag{2.1.7}$$

with $z_+ = 1$ and $z_- = 0$. For the existence of such waves in the Majda model see [Lai19] and [Maj81]. For the ZND model, a proof of the existence of such waves is given in [GS93, Wil10] for some particular choices of p and T. Physically, the shock is moving from the totally burned region to the totally unburned region. A long standing question, initiated by Erpenbeck in [Erp] for the ZND model, concerns the stability of these detonation waves. For the ZND model, there are partial stability results such as [Zum12, Zum11] and works cited therein. In [Zum12], it is shown that ZND detonations are spectrally stable in

the weak heat release and high overdrive limits by using techniques from asymptotic ODE theory. The weak heat release limit is $q \to 0$ in (2.1.5). The high overdrive limit concerns a different a choice of $\phi(T)$ in (2.1.5) than the one we've made here. It is important to note that our blowup theorem for ZND is specific to the inviscid case.
Let us emphasize that, unless stated explicitly, σ will be a positive constant here.
Finally, a closely related model that has been studied in the past is the viscous variation of the ZND model, known as the reactive Navier-Stokes, which may be written abstractly as

$$\begin{aligned} U_t + f(U)_x &= k\vec{q}\phi(T)z + \varepsilon(B(U)U_x)_x, \\ z_t &= -k\phi(T) + \varepsilon(C(U,z)z_x)_x. \end{aligned} \tag{2.1.8}$$

For the reactive Navier-Stokes equations, it is known that spectral stability implies nonlinear (orbital) stability [TZ11, Zum11].

It is also known that spectral stability of the detonation waves of the reactive Navier-Stokes for all $\varepsilon > 0$ sufficiently small implies spectral stability of the corresponding ZND detonation wave [Zum11], however, it is still open that nonlinear stability of the detonation wave for all small viscosities implies nonlinear stability of the corresponding ZND detonation wave. Finally, study of the spectral stability in the inviscid limit has been partially done in [LZ04].

Furthermore, there have also been studies of the stability of the wave in the viscous Majda model. In [LRTZ07, Sze99], it shown that nonlinear stability follows from spectral stability with the help of Green functions methods (the second one being focused on waves of small amplitudes), and the study of the spectral stability has been studied in [JYZ21, JY12, LY99] as well as [JLW05] for the low-frequency multi-dimensional variation.

Turning to the inviscid Majda model, an early result on the stability of the detonation wave is [Lev92]. In that paper, Levy shows that the weak entropy solution to some Riemann problem converges to the detonation wave as $t \to \infty$. In fact, it is proven by first showing the existence of what the author calls a normal solution to the Riemann problem with initial data $(u_{-\infty}, 0)$ (where $u_{-\infty} = \lim_{x\to-\infty} \underline{u}(x)$) on $\mathbf{R}^-$ and $(0,1)$ on $\mathbf{R}^+$. In that case, the solution may be discontinuous at $(t,0)$ for some $t > 0$ and hence does not follow from our analysis, as we note that the Riemann data is not in general a small perturbation of our wave, and that our work is centered on the preservation of the smoothness on both sides of the shock. We mention that Levy's result does not require q to be small at the expense of only working with very special initial data. As a final note,

it can be noted that a large part of the analysis relies on comparing certain solutions and obtaining monotonicity of Θ' where $\Theta(t)$ is the position of the shock, while we will rely on energy estimates in our case. We will not, in general, have monotonicity of the derivative of the phase and we will not require the initial perturbation of $\underline{u}$ to have a special sign. There are also results available on the spectral stability of the Majda model. For example, [JYZ21] proves spectral stability of the detonation wave for the inviscid Majda model for piecewise constant ignition functions using Evans function techniques. One can also study the spectral stability of the Majda via the vanishing viscosity limit as in [RV98].

2.1.2 Local existence theory

In order to study the asymptotic behavior of solutions to hyperbolic equations in the presence of a shock, we first need to recall the following result on the Cauchy problem. It is an adaptation of the results of chapter 4 from [Met](specifically Theorems 4.1.5 and 4.1.6). We state a theorem that can be applied to both models studied here. Furthermore, from now on, we will consider solutions in the Lax sense, that is, the one developed by A. Majda and G. Métivier in the references cited before [Maj83a, Maj83b, Met].

We fix an integer $n \geqslant 1$, an integer $s \geqslant 2$, two elements u_+ and u_- of $\mathbf{R}^n$. With $b, h : \mathbf{R}^n \to \mathbf{R}^n$ two smooth functions, we study the equation:

$$u_t + (b(u))_x = h(u)\,.$$

Remind some terminology associated to hyperbolic equations. We restrict ourselves to the framework needed here : hyperbolic systems of balance laws in one space variable. Consider the equation

$$u_t(t,x) + A(u)u_x = 0,$$

defined on $U \subset \mathbf{R}^n$ open where, for every $u \in \mathbf{R}^n$, where $u \mapsto A(u)$ is smooth, and valued in the set of $n \times n$ matrix. The equation is said to be constantly hyperbolic on U if, for every $u \in U$, $A(u)$ is diagonalizable with real eigenvalues of constant multiplicities.
Now, we assume that there exists b, smooth, : Assume that we have a function $b : U \to \mathbf{R}^n$ such that $db(u) = A(u)$ for every $u \in U$. Define

$$u(t,x) = \begin{cases} u_- \text{ if } x < t\sigma, \\ u_+ \text{ if } x > t\sigma \end{cases}$$

where $u_- \in \mathbf{R}^n$, $u_+ \in \mathbf{R}^n$ and $\sigma \in \mathbf{R}$. Assume that $b(u_+) - b(u_-) = \sigma(u_+ - u_-)$ and that

$v_t + A(u)v_x = 0$ is constantly hyperbolic on a neighborhood of u_- and on a neighborhood of u_+, and note $(\lambda_k(u))_{1\leqslant k\leqslant n} \in \mathbf{R}^n$, the nondecreasing sequence of eigenvalues of $A(u)$. We say that u is a one-dimensional shock solution to the equation $u_t + (b(u))_x = 0$ satisfying the stability condition if there exists $\tilde{k} \in \{0, ..., n\}$ such that $\lambda_{\tilde{k}-1}(u_-) < \sigma < \lambda_{\tilde{k}}(u)$, $\lambda_{\tilde{k}}(u_+) < \sigma < \lambda_{\tilde{k}+1}(u)$ and, with $b = u_+ - u_-$,

$$\mathbf{C}^n = \mathbf{C}b \oplus \bigoplus_{\lambda\leqslant\lambda_{\tilde{k}}(u_-)} \mathrm{Ker}(A(u_-) - \lambda_k(u_-)) \oplus \bigoplus_{\lambda\geqslant\lambda_{\tilde{k}+1}(u_+)} \mathrm{Ker}(A(u_+) - \lambda_k(u_+))\,.$$

We are looking for solutions close to a wave (with a discontinuity) satisfying the equation. More precisely, given $\sigma \in \mathbf{R}$ and a smooth function $\underline{U} : \mathbf{R}_- \to \mathbf{R}^n$ solving the equation

$$(db(\underline{U}(x)) - \sigma Id_{\mathbf{R}^n})(\underline{U}'(x)) = h(\underline{U}(x))\,,$$

with $\underline{U}(0) =: u_- \in \mathbf{R}^n$ and for a given $u_+ \in \mathbf{R}^n$ such that $h(u_+) = 0$, and also such that $b(u_+) - b(u_-) = \sigma(u_+ - u_-)$, we consider the wave is defined by the extension of $\underline{U}$ to $\mathbf{R}^+$, defined by $\underline{U}(x) = u_+$ for all $x > 0$. We also assume that $\underline{U}$ decays exponentially fast to its limit state at $-\infty$, as well as all of its derivatives decay exponentially fast to 0, and that the shock associated to u_- and u_+ is stable. We further assume there exists a neighborhood $\mathcal{U}_+$ of u_+ and a neighborhood $\mathcal{U}_-$ of u_- such that, on $\mathcal{U}_+$ $\partial_t + db(w)\partial_x$ is constantly hyperbolic and on $\mathcal{U}_-$ $\partial_t + db(w)\partial_x$ is constantly hyperbolic. We will be looking for solutions that can be written in the form $t \mapsto (\underline{U} + v(t, \cdot))(\cdot - \phi(t))$ with v in $C^0([0, T), H^s(\mathbf{R}^*))$ with $T \in \mathbf{R}^+ \cup \{+\infty\}$ and s big enough. At this stage, we remind that for any s, nonnegative integer, the Sobolev space $H^s(\mathbf{R}^*)$ is defined as $H^(\mathbf{R}^*) := \{v \in L^2(\mathbf{R}^*) \mid \text{ for any } j \in \{1, ..., k\}\, v^{(j)} \in L^2(\mathbf{R}^*)\}$, where $v^{(j)}$ is the j-th order distributional derivative of v on $\mathbf{R}^*$. In particular, on may have $v \in H^1(\mathbf{R}^*)$ but v not continuous at 0 ; its distributional derivative on $\mathbf{R}$ may contain a Dirac mass δ_0.

Notation: As in [Met], we use the following notation: $CH^s((0, T) \times \mathbf{R}^*)$ where $\mathbf{R}^* = \mathbf{R}\backslash\{0\}$, s is a nonnegative integer, T a positive number or $+\infty$ is the subset of $C^0(\overline{[0, T)}, H^s(\mathbf{R}^*))$ such that for every j a nonnegative integer with $j \leqslant s$, we have $u_t \in C^{s-j}(\overline{[0, T)}, H^j(\mathbf{R}^*))$ where $\overline{[0, T)}$ is $[0, T]$ if $T < +\infty$ and $\mathbf{R}_+$ otherwise.

Theorem 10. *In the above framework, there exists $\rho > 0$ such that for every $v_0 \in H^{s+1/2}(\mathbf{R})^n$ supported away from the shock such that $\|v_0\|_{L^\infty} \leqslant \rho$, there exists $T \in (0, +\infty]$ a unique maximum solution to the equation $u_t + (b(u))_x = h(u)$ in $CH^s((0, T) \times \mathbf{R}^*)$ with phase ϕ in $C^{s+1}([0, T))$ with initial data $\underline{U} + v_0$ for $x < 0$ and $u_+ + v_0$ for $x > 0$. Furthermore, either $T = +\infty$ or $\limsup_{t\to T} \|v(t)\|_{L^\infty}(\mathbf{R}^*) \geqslant \rho$ or $\limsup_{t\to T} \|v_x(t)\|_{L^\infty(\mathbf{R}^*)} = +\infty$. Finally, if we have $T \in \mathbf{R}^+$, then a sequence of smooth initial data $(v_n)_n$ that converges*

(strongly) in H^s to $v \in H^s(\mathbf{R})$ localized outside of a neighborhood of the point 0, with v_n giving rise to a solution u_n and u for v with all of them defined on $[0,T]$, we have that $(u_n(t))_n$ converges in L^2 to $u(t)$ for every $t \in [0,T]$.

We will not prove Theorem 10 here. We note the two adaptations needed to go from the proof presented in [Met] and the one needed here are: first, we need to add a source term, which barely changes the estimates, and, also to transform the constant state on $\mathbf{R}^-$ to a wave, which can be done by using the finite-speed of propagation of the equation. Furthermore, in ([Met]) the author proves more precise results that are not needed here.

2.1.3 Main results

In the positive direction, we have a result of asymptotic orbital stability for the Majda model with weighted norms. In fact, we have the following result, with $H^k_\varepsilon(\mathbf{R}^*)$ (where $k \in \mathbf{N}$ and $\varepsilon > 0$) being the Sobolev space defined as:

$$H^k_\varepsilon(\mathbf{R}^*) := \{v \in H^k(\mathbf{R}^*) \,|\, \forall 0 \leqslant l \leqslant k\,, (\partial_x^l v)^2 \exp(\varepsilon|\cdot|) \in L^1\}$$

with norms

$$\|v\|^2_{H^k_\varepsilon(\mathbf{R}^*)} := \sum_{j=0}^{k} \left\| \partial_x^k v \exp\left(\tfrac{\varepsilon|\cdot|}{2}\right)\right\|^2_{L^2(\mathbf{R}^*)} \tag{2.1.9}$$

Theorem 11. *Fix $k > 0$, $f : \mathbf{R} \to \mathbf{R}$, smooth and $u_0 \in \mathbf{R}^+$, such that $f'(u_0) > \sigma > f'(0)$. For $q_1 > 0$ small enough, for every $q \in [-q_1, q_1]$, there exists some wave $(\underline{u}, \underline{z}) : \mathbf{R} \to \mathbf{R}^2$ smooth on $\mathbf{R}_-$, constant on $\mathbf{R}^+$, such that $\underline{u}(0) = u_0$, $\underline{z}(0) = 1$, $\underline{u}(x) = 0 = \underline{z}(x)$ for any $x > 0$,*

$$\underline{u}'(x) = \frac{kq\underline{z}(x)}{f'(\underline{u}(x)) - \sigma},$$
$$\underline{z}'(x) = \frac{k\underline{z}(x)}{\sigma},$$

as well as $\inf_{x\in\mathbf{R}_-} f'(\underline{u}(x)) > \sigma$.
Then there exists a $\delta_0 > 0$, $q_0 > 0$, $\vartheta > 0$, $C > 0$ and $\varepsilon > 0$ such that for every $q \in [-q_0, q_0]$ and every $(v_0, \zeta_0) \in (H^{5/2}(\mathbf{R}) \cap H^2_\varepsilon(\mathbf{R}))^2$ supported away from 0 with $\|v_0\|_{H^2_\epsilon} + \|\zeta_0\|_{H^2_\epsilon} < \delta_0$, the solution (u, z) to (2.1.4) with initial data $(\underline{u} + v_0, \underline{z} + \zeta_0)$ is defined for all $t \in \mathbf{R}_+$. The position of the shock at time t, $\psi(t)$, is C^1 and for all $t \geqslant 0$

$$\|u(t, \cdot + \psi(t)) - \underline{u}\|_{H^2_\epsilon(\mathbf{R}^*)} + \|z(t, \cdot + \psi(t)) - \underline{z}\|_{H^2_\varepsilon(\mathbf{R}^*)} + |\psi'(t) - \sigma| \leqslant C(\|v_0\|_{H^2_\epsilon} + \|\zeta_0\|_{H^2_\epsilon})e^{-\vartheta t}.$$

Remark that, in Proposition 4, the existence of the q_1 mentioned in the theorem above is given for any such f, $u_0 > 0$, $k > 0$, and gives a condition on $q \in \mathbf{R}$ for such problems

ensuring the existence of the wave.
In the negative direction, we have a generalization of the blow up theorem in [Joh74] for perturbations of rapidly decaying shocks. See also [LXY22] and references cited therein for related results. Before we state Theorem 12, we recall some terminology. We recall that a smooth matrix-valued function $A(u)$ is strictly hyperbolic if $A(u)$ has all distinct real eigenvalues for all values of u in some range. By [Kat76], strict hyperbolicity implies that $A(u)$ has a smooth set of left and right eigenvectors $\eta_i(u)$ and $\xi^i(u)$ respectively. We then say that $A(u)$ has a genuinely nonlinear field if

$$\lim_{t\to 0} \frac{d}{dt}\Big(\eta_i(u)A(u+t\xi^i(u))\xi^i(u)\Big) \neq 0, \tag{2.1.10}$$

for all values of u and some index i.

Theorem 12. *Let $A(u) := DF(U)$, where $F : \mathbf{R}^n \to \mathbf{R}^n$ is smooth, be strictly hyperbolic with at least one genuinely nonlinear field. Consider a stationary shock solution $\underline{U}$ of*

$$U_t + A(U)U_x = 0. \tag{2.1.11}$$

Assume that $\underline{U}$ is smooth for $x \neq 0$ and the bounds

$$|\partial_x^k \underline{U}(x)| \leqslant c_k e^{-c|x|}, \tag{2.1.12}$$

hold for some $c > 0$, all nonzero x, and for all $k = 0, 1, 2, \ldots$. Then, for all $\theta > 0$ small enough, there exists an initial perturbation $\hat{U}(x, 0)$ satisfying

1. *$\hat{U}(x, 0)$ is compactly supported on an interval I of width one and such that the distance from I to 0 is comparable to θ^{-1}.*

2. *The C^2 norm of $\hat{U}(x, 0)$ is $\mathcal{O}(\theta)$.*

3. *There exists a $T_0 \sim \theta^{-1}$ and $T_* \leqslant T_0$ such that U given by $U = \underline{U}+\hat{U}$ is a solution of (2.1.11) on the time interval $[0, T_*)$, and the perturbation $\hat{U}(x, t)$ remains bounded in L^∞ for $t \leqslant T_*$ but*

$$\lim_{t\to T_*^-} ||\partial_x \hat{U}(\cdot, t)||_\infty = \infty. \tag{2.1.13}$$

We note here that the perturbation constructed in the proof of Theorem 12 has the property that its support remains bounded away from the underlying discontinuity on the interval $[0, T_*]$. In particular, the singularity that forms is not caused by a phase shift

of the underlying shock profile. With that in mind, our analysis is not precise enough to determine the nature of the singularity that forms.

As a corollary, we obtain a blowup result for the ZND model.

Corollary 1. *Let $\underline{U} = (\underline{v}, \underline{u}, \underline{E}, \underline{z})$ be a right going Neumann shock of the ZND model (2.1.5). Then, for all $\theta > 0$ small enough, there exists a perturbation $\hat{U}(x,0) = (\hat{v}, \hat{u}, \hat{E}, \hat{z})(x,0)$ satisfying*

1. *$\hat{U}(x,0)$ is supported on an interval I of width one with the distance of I from 0 comparable to θ^{-1} and $I \subset (-\infty, 0)$.*

2. *$\hat{z}(x,0) = 0$.*
3. *The C^2 norm of $\hat{U}(x,0)$ is $\mathcal{O}(\theta)$.*

4. *There exists a $T_0 \sim \theta^{-1}$ and $T_* \leqslant T_0$ such that there is a solution U of (2.1.5) of the form $U = \underline{U} + \hat{U}$, on the time interval $[0, T_*)$. Moreover, the perturbation $\hat{U}(x,t)$ remains bounded in L^∞ for $t \leqslant T_*$ but*

$$\lim_{t \to T_*^-} ||\partial_x \hat{U}(\cdot, t)||_\infty = \infty. \tag{2.1.14}$$

Moreover, one can arrange $\partial_x \hat{U}(x,0)$ to be "maximal" in an outgoing genuinely nonlinear direction.

This result also applies to Majda with unweighted norms.
During the proof of Theorem 12, we will show that $\hat{U}$ satisfies an equation of the form

$$\hat{U}_t + (A(\hat{U}) + B(x, \hat{U}))\hat{U}_x = G(x, \hat{U})\hat{U}, \tag{2.1.15}$$

for known matrix functions B and G which are small on the support of $\hat{U}$. What we mean by "maximal in an outgoing genuinely nonlinear direction" is that the $\hat{U}$ in the statement of Corollary 1 is such that the quantity $\max_i \sup_x |\eta_i(\hat{U}(x,0))\partial_x \hat{U}(x,0)|$ is achieved for some i with η_i associated to an eigenvalue of A which is strictly negative and genuinely nonlinear.

We note that Corollary 1 is not an immediate application of Theorem 12 due the presence of the reaction terms. The key idea of the proof of Corollary 1 from Theorem 12 is to note that $\hat{z} = 0$ effectively allows one to take $z = 0$ in the ZND model, reducing the ZND model to gas dynamics. We also note that Corollary 1 also prevents any stability

result of a similar form to Theorem 11 due to the presence of the outgoing undamped mode.

2.1.4 Discussion and open problems

One of the main questions that remains unanswered is the stability of waves for which $\underline{u}$ is not necessarily of small amplitudes. As we have obtained high-frequency estimates and that spectral stability results have been obtained, it is of interest to study the non-linear stability by using the high-frequency damping estimates and the linear stability.

In the recent work [LXY22], the authors perform an in depth study of the blow up of initially small in L^∞ but large in $W^{1,\infty}$ initial data to systems of conservation laws, as well as the details of such a blow up.

Another main question that is not answered here is whether or not the $W^{1,\infty}$-norm blows up for the ZND model for initial perturbations which are small in the weighted space H^2_α as the data v constructed in the proof of Theorem 12 has H^2_α norm of size

$$||v||_{H^2_\alpha} \sim \theta e^{\frac{\alpha}{\theta}} \gg 1.$$

Here, we only show the instability of the wave or the lack of high-frequency damping as long as the solution remains small. It does not give us finite time blow-up for initial data which is small in the weighted norm. The main issue is that our adaption of the John arguments requires the characteristics to not interact with the shock on a sufficiently large timescale determined by $\|v_x(0)\|_\infty$ for v the perturbation of the shock. Letting $V = e^{-\alpha\theta^{-1}}v$ so that $||V||_{H^2_\alpha} \sim \theta$, then gives us an expected blowup time for the perturbation V of size

$$T_* \sim \frac{e^{\frac{\alpha}{\theta}}}{\theta},$$

which is more than long enough for the characteristics emanating from an interval of distance θ^{-1} to interact with the shock.

We note that Corollary 1 implies there can be no H^s damping for $s \geqslant 2$ by Sobolev embedding. For the case of conservation laws, through a separate argument one can show that there is no H^1-damping either. To see this, we recall from the book [Bre00] and the articles [BCP00, BLY99], and references cited therein, that initial data with small BV-norm have unique solutions in BV. Since H^1 functions with compact support are BV functions, the H^1 solution agrees with the BV solution on the time of existence.

On the other hand, it is shown in [Liu77, Liu20] that (suitably rescaled) solutions of conservation laws converge to linear combinations of N-waves in L^1 as time increases. These two results can be combined to show that there is $W^{1,p}$-blowup for all $p > 1$, in particular the H^1 norm blows up as well. Briefly, the observation is the variation in the solution $V(u, I)$, for an interval I, is bounded from above by

$$V(u, I) \leqslant |I|^{1-\frac{1}{p}} ||u_x||_{L^p(I)} \leqslant |I|^{1-\frac{1}{p}} ||u_x||_{L^p(\mathbb{R})}, \tag{2.1.16}$$

for $|I|$ the width of the interval. However, because the N-wave has a discontinuity, eventually there is a point (x_*, t_*) where the solution has variation bounded *away* from zero for any interval I containing x_*. This then forces the L^p norm of u_x to be infinite at t_* for $p > 1$. Interestingly, this argument does not imply that the $W^{1,1}$-norm blows up, and indeed the equality

$$||u_x||_{L^1} = ||u||_{BV}, \tag{2.1.17}$$

for smooth solutions u seems to imply that the $W^{1,1}$-norm remains finite up to the formation of the shock.

It would be interesting to adapt these results to the case with exponentially small perturbations in the coefficients. Another interesting question is how much longer does the $W^{1,p}$ norm take to blowup? It is easy to check, and part of the construction, that for compactly supported initial data, as long as the C^1 norm of the solution remains finite, the $W^{1,p}$ norm also remains finite. On the other hand, the C^1 norm blowing up does not a priori imply that any other $W^{1,p}$ norm blows up. For scalar conservation laws, motivated by [GX92, Yu99] [CG23] provide an asymptotic expansion of the solution in the vanishing viscosity limit in the time period just before shock formation.

The key difference in our results between the Majda and ZND models is that the ZND model has an undamped outgoing genuinely nonlinear mode, whereas the outgoing mode in the Majda model is damped. This suggests that genuinely nonlinear outgoing undamped characteristics play a key role in the formation of singularities in spaces with weighted norms, as incoming characteristics can be handled by having exponential weights which "trap" the perturbations near the shock which would then prevent blowup since incoming signals would interact with the shock before they have a chance to blowup. It would be interesting to see if this mechanism is present in systems of hyperbolic balance laws where one assumes that damping is absent at the equilibrium endstate only for some characteristic direction which is outgoing.

As in [BR23], one can ask if there is a uniform stability result in the inviscid limit of the viscous Majda model. More precisely, for the viscous Majda model

$$\begin{aligned} U_t + f(U)_x &= qk\phi(U)z + \nu U_{xx}, \\ z_t &= -k\phi(U)z + \nu z_{xx}, \end{aligned} \tag{2.1.18}$$

can one show that there is a one parameter family of shocks $(\underline{U}^\nu, \underline{z}^\nu)$ such that $(\underline{U}^\nu, \underline{z}^\nu)$ converges to the detonation wave of the inviscid Majda model as $\nu \to 0$, there is a ϑ^ν such that the decay estimate in Theorem 11 holds for the viscous shock, and such that ϑ^ν can be chosen independently of ν provided ν is sufficiently small?

The final question we ask here is what does stability, in say the class BV of solutions, look like in the cases where the damping estimates fail? Could one have algebraic decay of small perturbations in time? We note that, at the linear level, the Hille-Yosida theorem prevents exponential decay in any exponentially weighted norm, and, as in [Rod15] the theorem of Datko-Pazy prevents any decay at the linear level without a change of topology between the solution and the initial data.

Outline: We establish Theorem 11 in Section 2.2 via energy estimates in weighted Sobolev spaces. We construct the background wave in Proposition 4. We then derive the evolution equations for perturbations of the constructed background wave and prove the desired bounds on the weighted energy of the perturbation and on the traces of the perturbation at the location of the shock. Finally, we conclude this section with a discussion of left going shocks in the Majda model in Proposition 7.

We establish Theorem 12 and Corollary 1 in Section 2.3. We prove Theorem 12 by adapting the argument in [Joh74] to certain variable coefficient systems. We then establish Corollary 1 and show that the shock is unstable in the weighted norm.

Acknowledgement We thank our thesis advisors L. Miguel Rodrigues and Kevin Zumbrun for suggesting this problem, and for their guidance and discussions.

2.2 Stability for Majda's model

In this section we establish the positive result Theorem 11.

2.2.1 Existence of the wave

Let $u_i > 0$, and ϕ be defined as

$$\begin{cases} \phi(u) = 1 \text{ if } u > u_i, \\ \phi(u) = 0 \text{ if } u \leqslant u_i, \end{cases}$$

and $f : \mathbf{R} \to \mathbf{R}$, a smooth function.
Let $k > 0$, $q > 0$.
We consider the following system

$$\begin{cases} u_t = kq\phi(u)z - (f(u))_x, \\ z_t = -kz\phi(u). \end{cases}$$

First, we are interested in traveling waves solutions $(t, x) \mapsto (\underline{u}_0(x - \sigma t), \underline{z}_0(x - \sigma t))$, with a jump (chosen initially at 0), smooth on $\mathbf{R}_-$, $z_0(0^-) = 1$ furthermore, with $\underline{u}_0$ smooth on $\mathbf{R}_+$ with $\inf_{\mathbf{R}^-} \underline{u}_0 > u_i > \sup_{\mathbf{R}_+} \underline{u}_0$, $\underline{u}_0(0^+) = 0$, and, here, a travelling speed in $\mathbf{R}^+$.
σ is chosen nonzero as, otherwise, the problem does not enter the framework of Lax shocks, and is thus avoided here.
As said before, the existence of such waves has already been proven in the literature, but we will recall the proof as it gives sharp bounds on the decay rate of the wave on $\mathbf{R}^-$.

Proposition 4. — *If a solution $\underline{u}_0$ to this problem exists, then $\sigma = \dfrac{f(\underline{u}_0(0^-)) - f(0)}{u_0}$, and $z_0(x) = \exp\left(\dfrac{kx}{\sigma}\right)$ if $x < 0$, $z_0(x) = 1$ (this choice can be changed up to changing the constant q by some multiplicative constant) if $x > 0$. We also have that $\underline{u}_0$ solves $(f(\underline{u}_0) - \sigma\underline{u}_0)'(x) = z_0(x)$ on $\mathbf{R}_-$ and $\underline{u}_0(x) = 0$ if $x > 0$.*

— *Furthermore, let $u_0 \in (u_i, +\infty)$ and $\sigma := \dfrac{f(u_0) - f(0)}{u_0}$.*
We denote by $\underline{u}_-$ the maximal (smooth) solution on $\mathbf{R}_-$ to

$$\begin{aligned} \underline{u}_-'(x) &= \frac{kq \exp\left(\dfrac{kx}{\sigma}\right)}{f'(\underline{u}_-) - \sigma}, \\ \underline{u}_-(0) &= u_0. \end{aligned}$$

Then, if $q > 0$ (respectively if $q < 0$), $(\underline{u}, \underline{z})$ defines a traveling wave solution to the previous problem such that $\inf_{\mathbf{R}_-} f'(\underline{u}_-(x)) > \sigma$ if and only if the equation $f(u) - \sigma u = f(u_0) - \sigma u_0 - q\sigma$ has a solution in $u \in (u_i, u_0)$ (respectively $u \in (u_0, +\infty)$) and the biggest (respectively the smallest) such u, denoted $u_{-\infty}$, satisfies

$\inf f'([u_{-\infty}, u_0]) > \sigma$ *and* $\sigma > f'(0)$. *If* $q = 0$, *then,*

$$\underline{u}(x) = u_0 \ \text{ if } x < 0\,,$$
$$\underline{u}(x) = 0 \ \text{ if } x > 0\,.$$

— *For any* $u_0 > u_i$ *and* $\sigma := \frac{f(u_0)-f(0)}{u_0}$ *such that*

$$f'(u_0) > \sigma > f'(0),$$

there exists $q_0 > 0$ *such that for all* $q \in [-q_0, q_0]$ *there exists a wave* $(\underline{u}, \underline{z})$ *solution to system, with speed* σ *and* $\inf_{x<0} f'(\underline{u}(x)) > \sigma$.

Proof. If such a solution exists, then it satisfies the Rankine-Hugoniot conditions. Thus $\sigma = \dfrac{f(u(0^-)) - f(0)}{u_-}$, z is continuous at 0, and $-\sigma z'(x) = -kz(x)$ on $\mathbf{R}_-$, $z(x) = z(0) = 1$ on $\mathbf{R}_+$ (as $z'(x) = 0$ for all $x > 0$), $u_0(x) = \lim_{y\to 0^+} u_0(y) = 0$ for all $x > 0$ (as, also, $(\sigma - f'(\underline{u}(x))\underline{u}'(x)) = 0$ for $x > 0$) and $\dfrac{kqz_0(x)}{f'(u_0(x)) - \sigma} = u_0'(x)$ on $\mathbf{R}^-$. This concludes the proof of the first point.
The conditions after are necessary as $f(\underline{u})' = \sigma\underline{u}' + kq\underline{z}$, and so by integrating we get $f(u_-) - \lim_{x\to-\infty} f(\underline{u}(x)) = \sigma(u_- - \lim_{x\to-\infty} \underline{u}(x)) + q\sigma$.
For the second point, we procede by double implication. If it defines a solution to the traveling wave problem such that $\inf_{\mathbf{R}_-} f'(\underline{u}) > \sigma$, then,

$$f(\underline{u}(0^-)) - \sigma\underline{u}(0^-) - f(\lim_{x\to-\infty} \underline{u}(x)) + \sigma \lim_{x\to-\infty} \underline{u}(x) = \int_{\mathbf{R}_-} (f'(\underline{u}) - \sigma\underline{u})\underline{u}' = q\sigma.$$

And so, with $u_{-\infty} = \lim_{x\to-\infty} \underline{u}(x)$, we have that

$$f(u_{-\infty}) - \sigma u_{-\infty} = -q\sigma + f(u_0) - \sigma u_0.$$

Furthermore, all the desired inequalities are satisfied : $\inf f'([u_{-\infty}, u_0]) > \sigma$, and $f(u) - \sigma u = f(u_0) - \sigma u_0 - q\sigma$ has a solution on $u_i, u_0)$ (respectively $(u_0, +\infty)$).
Reciprocally, for $q > 0$ (respectively $q < 0$) if the equation $f(u) - \sigma u = f(u_0) - \sigma u_0 - q\sigma$ has a solution (u_i, u_0) (respectively $(u_0, +\infty)$) with the biggest such u (respectively the smallest such u) noted $\underline{u}_{-\infty}$, satisfies $\inf_{[u_0, u_{-\infty}]} f' > \sigma$. Note $\underline{u}$, the maximal solution valued in $[u_0, u_{-\infty}]$ to

$$\underline{u}' = \frac{kq\underline{z}}{f'(\underline{u}) - \sigma}.$$

We have that where it is defined

$$(f(\underline{u}) - \sigma \underline{u})' = kq\underline{z},$$

by integrating it on $[x_0, 0]$ (for any x_0 in the domain of definition of $\underline{u}$), we have

$$f(\underline{u}(x_0)) - \sigma \underline{u}(x_0) = -q\sigma \left(1 - e^{\frac{kx_0}{\sigma}}\right) + f(u_0) - \sigma u_0.$$

Hence, by monotonicity of $\underline{u}$, $\underline{u}(x_0) \in (u_0, u_{-\infty})$.
Hence, the solution to the ODE is globally defined and gives rise to a solution to the initial problem. This solution satisfies $\lim_{x \to -\infty} \underline{u}(x) = u_{-\infty}$, and $\inf_{[u_0, u_{-\infty}]} f'(\underline{u}(x)) > \sigma$.
Finally, for the last bullet point, one just needs to remark that with such a u_0, then, for all $q \in \mathbf{R}$ close enough to 0, the solution u to $f(u) - \sigma u = f(u_0) - \sigma u_0 - q\sigma$ remains close to u_0 (on a neighborhood of size $\mathcal{O}(q)$ of u_0), and so, $f' > \sigma$ remains true on $[u, u_0]$ if q is close enough to 0. □

Note that, when the inverse function of $x \mapsto f(x) - \sigma x$ is known, one can obtain an explicit expression for $\underline{u}$.
We will study the stability of such waves according to the sign of the propagation speed σ. The case $\sigma > 0$ will be studied first to obtain a stability result in a weighted space. The case $\sigma < 0$ will give rise to instability results in various spaces. The case of $\sigma = 0$ is not treated here. In fact, even the local in time existence is not contained in the framework of Theorem 10.
Importantly, there is always, with $\underline{u}(0^-) = u_0$ fixed, such a q as there is a sufficiently small $q_0 > 0$ such that the wave is defined for every $q \in [-q_0, q_0]$. In fact, t here exists $\delta > 0$ small snough such that $\inf_{[u_0 - \delta, u_0 + \delta]} f' > \sigma$. Hence, there exists $q_0 > 0$ small enough such that for every $q \in [-q_0, q_0]$ there exists $u_{-\infty} \in [u_0 - \delta, u_0 + \delta]$ such that $f(u_{-\infty}) - \sigma u_{-\infty} = f(u_0) - \sigma u_0 - q\sigma$, and $\inf f'([u_0, u_{-\infty}]) > \sigma$. As a consequence by Proposition 4 there exists a wave $(\underline{u}, \underline{z})$ solution to the wave problem.
We now can start to give some bounds on the wave that will prove useful later on.
Consider for the moment the case of speed σ positive. We fix k, f and $u_0 > 0$ such that $f'(u_0) > \dfrac{f(u_0)}{u_0} > 0$, and consider Q, the set of $q \in \mathbf{R}$ is such that we can apply Proposition 4 and consider the associated wave $(\underline{u}, \underline{z})$. By the characterization given by the second point of the said proposition, Q is an open interval containing 0.
We obviously have the following bounds on $\mathbf{R}^-$

$$\underline{z}^{(l)}(x) = \frac{k^l}{\sigma^l} \exp\left(\frac{kx}{\sigma}\right).$$

Notice that, given $q_0 \in Q$, there exis ts some neighborhood V of $q_0 \in Q$, and $\gamma > 0$ such that for every $q \in V$, the associated $\underline{u}$ is valued in $[u_0 - \gamma, u_0 + \gamma]$.
Furthermore, there exists some $\kappa : Q \to \mathbf{R}^+$ continuous and such that $\kappa(q) = 0$ such that for every $q \in Q$ and $x < 0$

$$|\underline{u}'(x)| \leqslant \kappa(q) \exp\left(\tfrac{kx}{\sigma}\right).$$

Furthermore, as

$$\underline{u}''(x) = \left(\frac{k}{\sigma} - f''(\underline{u}(x))\frac{\underline{u}'(x)}{(f'(\underline{u}(x)) - \sigma)}\right)\underline{u}'(x),$$

and

$$\underline{u}^{(3)}(x) = \left(\frac{k}{\sigma} - \frac{f''(\underline{u}(x))\underline{u}'(x)}{(f'(\underline{u}(x) - \sigma)}\right)^2 \underline{u}'(x) - \frac{f^{(3)}(\underline{u}(x))\underline{u}'(x)^3 + f''(\underline{u}(x))\underline{u}''(x)\underline{u}'(x)}{(f'(\underline{u}(x)) - \sigma)} + \frac{f''(\underline{u}(x))^2\underline{u}'(x)^2}{(f'(\underline{u}(x)) - \sigma)^2},$$

up to changing by some multiplying κ by some function depending on q through the L^∞ norms of $(f' - \sigma)^{-1}$, of f'' and of $f^{(3)}$ on $[u_{-\infty}, u_0]$ (as $u_{-\infty}$ is a function of q for u_0, f and k fixed). One then obtains the following bounds on the derivatives of $\underline{u}$

$$|\underline{u}'(x)| + |\underline{u}''(x)| + |\underline{u}^{(3)}(x)| \leqslant \kappa(q) \exp\left(\frac{kx}{\sigma}\right)$$

for all $q \in Q$ and $x \in \mathbf{R}^-$.
Now, we are interested in damping estimates on solutions which are small perturbations to the original traveling waves. Until section 2.6, we assume $\sigma > 0$.
Thanks to the local wellposedness result in Theorem 10, we obtain that there exists $\rho > 0$ such that, given $(v_0, \zeta_0) \in H^{5/2}(\mathbf{R}\backslash\{0\})$, supported away from the shock, with L^∞ norm smaller than ρ. There exists $T \in (0, +\infty]$ such that there exists a unique maximal solution to the system with initial data $(\underline{u} + v_0, \underline{z} + \zeta_0)$. Remember that the shock localization ψ is given by the Rankine-Hugoniot condition, $\psi(0) = 0$, and for all $t \in [0, T)$ $\psi'(t) = \dfrac{f(u(t, \psi(t)^-) - f(u(t, \psi(t)^+)}{u(t, \psi(t)^-) - u(t, \psi(t)^+)}$. Furthermore, $T < +\infty$ implies that we are at least in one of the following cases:
a) the L^∞ norm of $u(t, \cdot + \psi(t)) - \underline{u}$ or of $z(t, \cdot + \psi(t)) - \underline{z}$ becomes bigger than ρ,
b) the space derivative of one of $u(t, \cdot + \psi(t)) - \underline{u}$ or of $z(t, \cdot + \psi(t)) - \underline{z}$ blows up.
We let $\zeta(t, x) := z(t, \psi(t) + x) - \underline{z}(x)$ and $v(t, x) := u(t, x + \psi(t)) - \underline{u}(x)$ for all $(t, x) \in [0, T) \times \mathbf{R}^*$. We aim to show that finite time blow-up is prevented by choosing the initial perturbation with a sufficiently small weighted H^2 norm, and that the perturbation (v, ζ) goes to 0 as t goes to $+\infty$ in the weighted norm.
To do so, we will build an energy equivalent to the H^2_ε norm that will be non-increasing

in t. To achieve this, we start by introducing norms on $L^2_\varepsilon(\mathbf{R}^+)$ and $L^2_\varepsilon(\mathbf{R}^-)$ equivalent to the norms defined by Equation 2.1.9. They are defined as the square roots of quantities of the form

$$E_\pm(w) := \int_{\mathbf{R}^\pm} w^2 \varrho$$

with the choices of the ϱ considered depending on the half-line considered ($\mathbf{R}_+$ or $\mathbf{R}_-$) and whether we consider terms in ζ or v. More precisely, we will consider them of the following forms $\varrho_{-,1}(x) := \exp\left(-\varepsilon x - \int_0^x C e^{-\varepsilon|s|} ds\right) = \exp\left(-\varepsilon x + \frac{C - Ce^{\varepsilon x}}{\varepsilon}\right)$ where C is to be chosen later (taking C big enough), $\varrho_{-,2}(x) = \exp\left(-\varepsilon x + \int_0^x C e^{-\varepsilon|s|} ds\right) = \exp\left(-\varepsilon x + \frac{Ce^{\varepsilon x} - C}{\varepsilon}\right)$ and $\varrho_+(x) = \exp(\varepsilon x)$.
Thus, on $\mathbf{R}_-$ $\exp(-\varepsilon x - C\varepsilon^{-1}) \leqslant \rho_{-,2}(x) \leqslant \exp(-\varepsilon x)$, and $\exp(\varepsilon x) \leqslant \rho_{-,1}(x) \leqslant \exp(\varepsilon x + C\varepsilon^{-1})$. Hence, we define

$$\begin{aligned} E_+(w) &:= \int_{\mathbf{R}_+} w^2 \varrho_+ \,, \\ E_{-,1}(w) &:= \int_{\mathbf{R}_-} w^2 \varrho_{-,1} \,, \\ E_{-,2}(w) &:= \int_{\mathbf{R}_-} w^2 \varrho_{-,2} \,. \end{aligned}$$

After that, the energy will be built as a sum of the terms of the form:
$C_1(k,\pm)E_{\pm,1}(\partial_x^k v(t,\cdot))$ and $C_2(k,\pm)E_{\pm,2}(\partial_x^k \zeta(t,\cdot))$ where $k \in \{0,1,2\}$, and the $C_\cdot(k,\pm)$ are constants independent of (v_0,ζ_0).
Here are the main ingredients in our proof of the existence of such an energy function, and also how fast it goes to 0:
The choices of the constants $C_j(k,\pm)$ will be made to control the terms that, when taking the time derivatives of the $E_\pm$ of v, ζ or one of their spatial derivatives up to order 2, is non-negative. Terms of the form $\partial_x^l \zeta(t,0^-)$ that will appear due to the outgoing characteristic, will be bounded through boundary terms appearing in integration by parts done on integrals appearing in the derivatives of the $E_+(\partial_x^s \zeta(t,\cdot))$ for $s \leqslant l$, and terms that have the form $|\psi'(t) - \sigma|$ will be controlled through the boundary terms of $E_\pm(v(t,\cdot))$.
Finally, the choice of a perturbations of the norms of $L^2_\varepsilon(\mathbf{R}^+)$ and $L^2_\varepsilon(\mathbf{R}^-)$ by terms of the form $\exp\left(\pm \int_0^x e^{-\varepsilon|s|} ds)\right)$ is made to bound some terms that are not small, but are of integral form with a quadratic term in v, ζ and some of their spatial derivatives times some derivative of the underlying shock profile.
Now, we can try to obtain bounds on the size of the perturbations (v,ζ), first by writing done the equations they satisfy, as well as the equations their derivatives satisfy.
From now on, we assume the initial data to be a smooth and compactly supported function in $\mathbf{R}^*$. With T, its (possibly infinite) time of existence, for every time $t \in [0,T)$,

the solution is in $H^s(\mathbf{R}^*)$ for every $s \in \mathbf{R}_+$, and, as long as the L^∞ norm of v is small enough and the solution remains bounded in $W^{1,\infty}(\mathbf{R}^*)$, we have on $\mathbf{R}^-$

$$v_t(t,\cdot) = (\psi'(t) - f'(\underline{u} + v(t,\cdot)))v_x(t,\cdot) + (\psi'(t) - \sigma + f'(\underline{u}) - f'(\underline{u} + v(t,\cdot)))\underline{u}' + kq\zeta(t,\cdot),$$

and

$$\zeta_t = (\psi'(t) - \sigma)\underline{z}' - k\zeta(t,\cdot) + \psi'(t)\zeta_x(t,\cdot).$$

Whereas

$$\begin{aligned} v_{tx}(t,.) =&(\psi' - f'(\underline{u} + v(t,\cdot)))v_{xx}(t,\cdot) - f''(\underline{u} + v(t,\cdot))(\underline{u}' + v_x(t,\cdot))v_x(t,\cdot) \\ &+ (\psi' - \sigma + f'(\underline{u}) - f'(\underline{u} + v(t,\cdot)))\underline{u}'' + (f''(\underline{u}) - f''(\underline{u} + v(t,\cdot)))\underline{u}'^2 \\ &- f''(\underline{u} + v(t,\cdot))\underline{u}'v_x(t,\cdot) + kq\zeta_x(t,\cdot), \end{aligned}$$

$$\begin{aligned} v_{txx}(t,\cdot) =&(\psi' - f'(\underline{u} + v(t,\cdot)))v_{xxx}(t,\cdot) - 2f''(\underline{u} + v(t,\cdot))(\underline{u}' + v_x(t,\cdot))v_{xx}(t,\cdot) \\ &- f^{(3)}(\underline{u} + v(t,\cdot))(\underline{u}' + v_x(t,\cdot))^2 v_x(t,\cdot) - f''(\underline{u} + v(t,\cdot))(\underline{u}'' + v_{xx}(t,\cdot))v_x(t,\cdot) \\ &+ (\psi' - \sigma + f'(\underline{u}) - f'(\underline{u} + v(t,\cdot)))\underline{u}^{(3)} + 2(f''(\underline{u}) - f''(\underline{u} + v(t,\cdot)))\underline{u}'\underline{u}'' \\ &- 2f''(\underline{u} + v(t,\cdot))\underline{u}''v_x(t,\cdot) + (f''(\underline{u}) - f''(\underline{u} + v(t,\cdot)))\underline{u}'\underline{u}'' \\ &+ (f^{(3)}(\underline{u}) - f^{(3)}(\underline{u} + v(t,\cdot)))\underline{u}'^3 - f^{(3)}(\underline{u} + v(t,\cdot))(2\underline{u}' + v_x(t,\cdot))\underline{u}'v_x(t,\cdot) \\ &- f''(\underline{u} + v(t,\cdot))\underline{u}'v_{xx}(t,\cdot) + kq\zeta_{xx}(t,\cdot), \end{aligned}$$

$$\begin{aligned} \zeta_{tx}(t,\cdot) =&(\psi'(t) - \sigma)\underline{z}'' - k\zeta_x(t,\cdot) + \psi'(t)\zeta_{xx}(t,\cdot), \\ \zeta_{txx}(t,\cdot) =&(\psi'(t) - \sigma)\underline{z}^{(3)} - k\zeta_{xx}(t,\cdot) + \psi'(t)\zeta_{xxx}(t,\cdot). \end{aligned}$$

Similarly, on $\mathbf{R}^+$

$$v_t(t,\cdot) = (\psi'(t) - f'(v(t,\cdot)))v_x(t,\cdot),$$

and

$$\zeta_t(t,\cdot) = \psi'(t)\zeta_{tx}(t,\cdot).$$

Furthermore

$$v_{tx}(t,\cdot) =(\psi'(t) - f'(v(t,\cdot)))v_{xx}(t,\cdot) - f''(v(t,\cdot))(v_x(t,\cdot))v_x(t,\cdot),$$

$$\begin{aligned} v_{txx}(t,\cdot) =&(\psi'(t) - f'(v(t,\cdot)))v_{xxx}(t,\cdot) - 2f''(v(t,\cdot))(v_x(t,\cdot))v_{xx}(t,\cdot) \\ &- f''(v(t,\cdot))(v_{xx}(t,\cdot))v_x(t,\cdot) - f^{(3)}(v(t,\cdot))(v_x(t,\cdot))^2 v_x(t,\cdot), \end{aligned}$$

$$\zeta_{tx}(t,\cdot) = \psi'\zeta_{xx}(t,\cdot),$$

$$\zeta_{txx}(t,\cdot) = \psi'(t)\zeta_{xxx}(t,\cdot).$$

2.2.2 Boundary terms

In the energy estimates, we will need to control boundary terms that can not be neglected due to the characteristics outgoing from the shock, specifically those involving ζ or one of its derivatives at 0^-, by using other boundary terms going into the shock, in particular only those involving ζ or one of its derivatives at 0^+. We also need to control the derivatives of the phase by using the boundary terms involving v or one of its derivatives both at 0^+ and 0^-, where we note both of them are linked to characteristics going into the shock.
To obtain such bounds, we will use the Rankine-Hugoniot condition, and the fact that for smooth enough solutions we can differentiate it with respect to time, and, using the equation, replace the time derivatives of v and ζ with terms involving only spatial derivatives.

Lemma 1. *For $q_0 > 0$ small enough such that the third point Proposition 4 applies there exists $\eta > 0$ small enough, such that, there exists a constant $\tilde{C}_f$ such that, for every $q \in [-q_0, q_0]$, for every $T > 0$ and every solution (v, ζ) of Equation ?? which is initially a smooth function compactly supported away from 0, such that $(v(t,\cdot), \zeta(t,\cdot))$ is defined on $[0,T]$ with its $W^{1,\infty}$ norm smaller than η, then we have, on $[0,T]$ $|\psi''(t)| \leqslant \tilde{C}_f \|(v,\zeta)\|_{L^\infty([0,T],W^{1,\infty}(\mathbf{R}^*))}$, $|\psi'(t) - \sigma|^2 \leqslant \tilde{C}_f(|v(t,0^-)|^2 + |v(t,0^+)|^2)$.*
And such that we also have, on $[0,T]$

$$\zeta(t,0^-) = \zeta(t,0^+),$$

$$\zeta_x(t,0^-)^2 \leqslant \tilde{C}_f(\zeta_x(t,0^+))^2 + \tilde{C}_f k\zeta(t,0^-)^2 + \tilde{C}_f(v(t,0^+)^2 + v(t,0^-)^2),$$

and

$$\begin{aligned}(\zeta_{xx}(t,0^-))^2 \leqslant & \tilde{C}_f(\zeta_{xx}(t,0^+))^2 + \tilde{C}_f(\zeta_x(t,0^+))^2 + \tilde{C}_f(v(t,0^-)^2 + v(t,0^+)^2) \\ & + \tilde{C}_f\zeta(t,0^+)^2 + \tilde{C}_f(v_x(t,0^-)^2 + v_x(t,0^+)^2)\end{aligned}$$

Furthermore, for any $q \in \mathbf{R}$ such that 4 applies then there exists a constant $\tilde{C}_f$ such that, for every $q \in [-q_0, q_0]$, for every $T > 0$ and every solution (v, ζ) of Equation ?? which is initially a smooth function compactly supported away from 0, such that $(v(t,\cdot), \zeta(t,\cdot))$ is defined on $[0,T]$ with its $W^{1,\infty}$ norm smaller than η, then we have, on $[0,T]$ $|\psi''(t)| \leqslant$

$\tilde{C}_f\|(v,\zeta)\|_{L^\infty([0,T],W^{1,\infty}(\mathbf{R}^*))}$, $|\psi'(t)-\sigma|^2 \leqslant \tilde{C}_f(|v(t,0^-)|^2+|v(t,0^+)|^2)$.
And such that we also have, on $[0,T]$

$$\zeta(t,0^-)=\zeta(t,0^+),$$

$$\zeta_x(t,0^-)^2 \leqslant \tilde{C}_f(\zeta_x(t,0^+))^2+\tilde{C}_f k\zeta(t,0^-)^2+\tilde{C}_f(v(t,0^+)^2+v(t,0^-)^2),$$

and

$$\begin{aligned}(\zeta_{xx}(t,0^-))^2 \leqslant & \tilde{C}_f(\zeta_{xx}(t,0^+))^2+\tilde{C}_f(\zeta_x(t,0^+))^2+\tilde{C}_f(v(t,0^-)^2+v(t,0^+)^2)\\ &+\tilde{C}_f\zeta(t,0^+)^2+\tilde{C}_f(v_x(t,0^-)^2+v_x(t,0^+)^2)\end{aligned}$$

Proof. We begin by imposing $0<\eta\leqslant\dfrac{u_0}{4}$. Assume that, as in the statement of the lemma, the $W^{1,\infty}$ norm of $(v(t,\cdot),\zeta(t,\cdot))$ remains smaller than η on $[0,T]$. We will progressively change the value of $\tilde{C}_f$ and η along this proof and during the derivation of inequalities, to ensure more conditions, and we choose not to change the notation used to avoid using to many constants.
If q is fixed, then take η small enough such that $f'-\sigma>0$ on $[\min(u_0,u_{-\infty})-\eta,\max(u_{-\infty},u_0)+\eta]$ and $f'-\sigma<0$ on $[-\eta,\eta]$. If the case of $|q|$ small is considered, then take $\eta>0$ such that $f'-\sigma>0$ on $[u_0-2\eta,u_0+2\eta]$ and $f'-\sigma<0$ on $[-\eta,\eta]$. Then, fix $q_0>$ small enough such that for every $q\in\mathbf{R}$ such that $|q|\leqslant q_0$, the associated $\underline{u}$ is defined, and $|u_0-u_{-\infty}|\leqslant\eta$. Denote $I:=[\min(u_0,u_{-\infty})-\eta,\max(u_{-\infty},u_0)+\eta]$ in the first case, and $I=[u_0-2\eta,u_0+2\eta]$ in the second case.
We then note $C_f=\max(\|(f'-\sigma)^{-1}\|_{\infty,K},\|f'\|_{\infty,K},\|f'-\sigma\|_{\infty,K},\|f''\|_{\infty,K},\|f^{(3)}\|_{\infty,K})$.

$$\begin{aligned}|\psi'(t)-\sigma| &= \left|\frac{f(u_0+v(t,0^-))-f(v(t,0^+))}{u_0+v(t,0^-)-v(t,0^+)}-\frac{f(u_0)-f(0)}{u_0}\right|\\ &= \left|\frac{(f(u_0+v(t,0^-))-f(v(t,0^+)))u_0-(f(u_0)-f(0))(u_0+v(t,0^-)-v(t,0^-))}{(u_0+v(t,0^-)-v(t,0^+))u_0}\right|,\\ &\leqslant \frac{|f(u_0+v(t,0^-))-f(u_0)|+|f(0)-f(v(t,0^+))|u_0+|f(u_0)-f(0)||v(t,0^-)-v(t,0^+)|}{|u_0(u_0+v(t,0^-)-v(t,0^+))|},\\ &\leqslant \frac{C_f(|v(t,0^-)|+|v(t,0^+)|)u_0}{u_0(u_0-2\eta)}+\frac{\sigma(|v(t,0^-)|+|v(t,0^+)|)}{u_0-2\eta},\\ &\leqslant \frac{2(C_f+\sigma)(v(t,0^-)^2+v(t,0^+)^2)}{u_0}.\end{aligned}$$

As a consequence, we have, by shrinking η if necessary, that there exists $\mu>0$ small enough such that $\psi'(t)>\mu$ for every $x>0$, $\psi'(t)-f'(v(x))>\mu$, and for every $x<0$, $\psi'(t)-f'(\underline{u}(x)+v(x))<-\mu$, and $k\geqslant\mu$.
Furthermore, there exists $\nu>0$ such that $\psi'(t)<\nu$.

We have that by differentiating in time the relation

$$\psi'(t) = \frac{f(u_0 + v(t,0^-)) - f(v(t,0^+))}{u_0 + v(t,0^-) - v(t,0^+)},$$

the following equality

$$\begin{aligned}
\psi''(t) = &-\frac{(f(u_0 + v(t,0^-)) - f(v(t,0^+))(v_t(t,0^-) - v_t(t,0^+))}{(u_0 + v(t,0^-) - v(t,0^+))^2} \\
&+ \frac{f'(u_0 + v(t,0^-))v_t(t,0^-) - f'(v(t,0^+))v_t(t,0^+)}{u_0 + v(t,0^-) - v(t,0^+)}, \\
&= -\frac{\psi'(t)(v_t(t,0^-) - v_t(t,0^+))}{u_0 + v(t,0^-) - v(t,0^+)} + \frac{f'(u_0 + v(t,0^-))v_t(t,0^-) - f'(v(t,0^+))v_t(t,0^+)}{u_0 + v(t,0^-) - v(t,0^+)}, \\
&= \frac{(f'(u_0 + v(t,0^-)) - \psi'(t))v_t(t,0^-)}{u_0 + v(t,0^-) - v(t,0^+)} + \frac{(\psi'(t) - f'(v(t,0^+))v_t(t,0^+)}{u_0 + v(t,0^-) - v(t,0^+)} \\
&= \frac{-(f'(u_0 + v(t,0^-)) - \psi'(t))^2 v_x(t,0^-)}{u_0 + v(t,0^-) - v(t,0^+)} \\
&\quad + \frac{(f'(u_0 + v(t,0^-)) - \psi'(t))\left((\psi'(t) - \sigma + f'(u_0) - f'(u_0 + v(t,0^-))\underline{u}'(0^-) + kq\zeta(t,0^-)\right)}{u_0 + v(t,0^-) - v(t,0^+)} \\
&\quad + \frac{(\psi'(t) - f'(v(t,0^+))^2 v_x(t,0^+)}{u_0 + v(t,0^-) - v(t,0^+)}.
\end{aligned}$$

And so

$$\begin{aligned}
|\psi''(t)| \leqslant &\frac{1}{u_0 - 2\eta}\Big(\nu^2(|v_x(t,0^-)| + |v_x(t,0^+)|) + \nu\tilde{C}_f(|v(t,0^-)||\underline{u}'(0^-)| + \sqrt{v(t,0^-)^2 + v(t,0^+)^2}) \\
&\qquad +k|q||\zeta(t,0^-)|\Big), \\
\leqslant &\frac{2}{u_0}\tilde{C}_f\left(|v(t,0^-)| + |v(t,0^+)| + |v_x(t,0^-)| + |v_x(t,0^+)| + |\zeta(t,0^-)|\right) \\
\leqslant &\tilde{C}_f\|(v,\zeta)\|_{L^\infty([0,T],W^{1,\infty}(\mathbf{R}*))}.
\end{aligned}$$

For the bounds on the values of ζ and its spatial derivatives at 0^-, we will use that

$$\zeta(t,0^+) = \zeta(t,0^-)$$

for every t, and so, by smoothness of ζ (it is in $H^s(\mathbf{R}^+)$ and $H^s(\mathbf{R}^-)$ for every $s \geqslant 0$) that

$$\partial_t^l \zeta(t,0^+) = \partial_t^l \zeta(t,0^-)$$

for every t and l.
Hence, we obtain the following expressions on the spatial derivatives of ζ evaluated at

$(t, 0^{\pm})$

$$\begin{aligned}
&\zeta_t(t,0^+) = \psi'(t)\zeta_x(t,0^+),\\
&\zeta_t(t,0^-) = \psi'(t)\zeta_x(t,0^-) - k\zeta(t,0^-) + (\psi' - \sigma)\frac{k}{\sigma},\\
&\zeta_{tt}(t,0^+) = \psi''(t)\zeta_x(t,0^+) + \psi'(t)\zeta_{tx}(t,0^+),\\
&\zeta_{tt}(t,0^-) = \psi'(t)\zeta_{tx}(t,0^-) + \psi''(t)\zeta_x(t,0^-) - k\zeta_t(t,0^-) + \psi''(t)\frac{k}{\sigma}.
\end{aligned}$$

As a consequence

$$\zeta_x(t,0^-) = \zeta_x(t,0^+) + \frac{k\zeta(t,0^-)}{\psi'(t)} + \frac{(\psi'(t)-\sigma)k}{\sigma\psi'(t)}.$$

Using that, for all $(a,b,c) \in \mathbb{R}$, $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, we have

$$\begin{aligned}
\zeta_x(t,0^-)^2 \leqslant& 3(\zeta_x(t,0^+))^2 + 3\left(\frac{k\zeta(t,0^-)}{\mu}\right)^2 + 3\frac{\tilde{C}_f^2(v(t,0^+)^2 + v(t,0^-)^2)k^2}{\sigma^2\mu^2},\\
\leqslant& \tilde{C}_f\left(\zeta_x(t,0^+)^2 + v(t,0^+)^2 + v(t,0^-)^2 + \zeta(t,0^-)^2\right).
\end{aligned}$$

We have, for second order derivative that

$$\begin{aligned}
\psi'(t)\zeta_{xx}(t,0^-) =& + k\zeta_x(t,0^-) - (\psi(t)-\sigma)\frac{k^2}{\sigma^2} + \psi'(t)\zeta_{xx}(t,0^+) + k\zeta_x(t,0^+) + \frac{k^2\zeta(t,0^-)}{\psi'(t)}\\
&+ \frac{k(\psi'(t)-\sigma)}{\psi'(t)}\frac{k}{\sigma} - \frac{\psi''(t)\zeta(t,0^-)}{\psi'(t)^2} + k\frac{\psi''(t)}{\psi'(t)^2},
\end{aligned}$$

Thus

$$\begin{aligned}
(\zeta_{xx}(t,0^-))^2 \leqslant& (\zeta_{xx}(t,0^+))^2 + \tilde{C}_f(\zeta_x(t,0^+))^2 + \tilde{C}_f k\zeta(t,0^-)^2\\
&+ \tilde{C}_f(v(t,0^+)^2 + v(t,0^-)^2 + v_x(t,0^+)^2 + v_x(t,0^-)^2).
\end{aligned}$$

□

2.2.3 Energy estimates

We will now obtain estimates of the time derivatives of the $E_{\pm}$ of v and ζ and their spatial derivatives. Below, δ_1 and δ_2 will denote some positive constants to be determined later. At the end, we will choose two values, one to control terms in v or v_x or v_{xx} and the other to control terms in ζ or ζ_x or ζ_{xx}.
Furthermore, we will derive estimates for $\underline{u}(0^-)$, f, k fixed while q is a parameter. Depending on q, we will obtain either a high-frequency damping estimate, or, a direct nonlinear

stability result in the weak heat release limit.
The idea, which we will soon check directly at the linear L^2 level, is that by choosing the constant C big enough and q be smaller (in absolute value) than some $q_0 > 0$, there exists three positive constants C_2, C_3 and C_4 as well as some $\nu > 0$ (independent of v and ζ) such that if $\|v(t,\cdot)\|_{W^{1,\infty}}$ and $\|\zeta(t,\cdot)\|_{W^{1,\infty}}$ are smaller than η on $[0,T]$, then, on $[0,T]$

$$\begin{aligned}\frac{d}{dt}(E_{-,1}(v) + C_2E_{-,2}(\zeta) + C_3E_+(v) + C_4E_+(\zeta))(t,\cdot) \leqslant \\ -\nu(E_{-,1}(v(t,\cdot)) + C_2E_{-,2}(\zeta(t,\cdot)) + C_3E_+(v(t,\cdot)) + C_4E_+(\zeta(t,\cdot))).\end{aligned} \tag{2.2.1}$$

From here, the idea is to obtain similar bounds for the higher order terms, while taking into account that there may be some loss of lower order derivatives when controlling higher order terms.
To begin with, we discuss how it can be done in the linear problem, as the computations are simpler, but the idea is the same. The problem is then given by

$$\begin{aligned}
v_t =&(\psi' - \sigma)\underline{u}' + kq\zeta - ((f'(\underline{u}) - \sigma)v)_x \quad \text{on } \mathbf{R}^+ \times \mathbf{R}^-,\\
\zeta_t =&\sigma\zeta_x - k\zeta + (\psi' - \sigma)\underline{z}' \quad \text{on } \mathbf{R}^+ \times \mathbf{R}^-,\\
\zeta_t =&\sigma\zeta_x \quad \text{on } \mathbf{R}^+ \times \mathbf{R}^+,\\
v_t =&(\sigma - f'(0))v_x \quad \text{on } \mathbf{R}^+ \times \mathbf{R}^+,\\
\psi' =&\frac{(f'(u_0) - \sigma)v(\cdot,0^-)}{u_0} + \frac{(\sigma - f'(0))v(\cdot,0^+)}{u_0}, \quad \zeta(\cdot,0^+) = \zeta(\cdot,0^-) \text{ on } \mathbf{R}_+.
\end{aligned}$$

In fact, we can obtain, by fixing several positive parameters C, δ, δ', $\tilde{\delta}$ and $\varepsilon := \dfrac{k}{\nu}$ that

$$\begin{aligned}
&\frac{d}{dt}\left(E_{-,1}(v) + E_+(v) + E_+(\zeta) + E_{-,2}(\zeta)\right) \leqslant\\
&v(t,0^-)^2\left(\frac{\sigma - f'(u_0)}{2} + \frac{C_f\tilde{\delta}}{2}\int_{\mathbf{R}_-}|\underline{u}'|e^{-\varepsilon\cdot - C\int_0^\cdot} + C_f\frac{\delta'}{2}\int_{\mathbf{R}_-}\underline{z}'e^{-\varepsilon+\int_0^\cdot Ce^{\varepsilon s}ds}\right)\\
&+ v(t,0^+)^2\left(\frac{f'(0) - \sigma}{2} + \frac{C_f\tilde{\delta}}{2}\int_{\mathbf{R}_-}|\underline{u}'|e^{-\varepsilon\cdot - C\int_0^\cdot} + C_f\frac{\delta'}{2}\int_{\mathbf{R}_-}\underline{z}'e^{-\varepsilon+\int_0^\cdot Ce^{\varepsilon s}ds}\right)\\
&+ \int_{\mathbf{R}_-}\zeta^2\left(\frac{k|q|}{2\delta}e^{2C\varepsilon^{-1}} + \frac{\varepsilon\sigma - 2k}{2} + \frac{(\delta'^{-1} - C)}{2}e^{\varepsilon\cdot}\right)e^{-\varepsilon\cdot + C\int_0^\cdot e^{\varepsilon s}ds}\\
&+ \int_{\mathbf{R}_-}v^2\left(-\frac{C_f\varepsilon}{2} + \frac{k|q|\delta}{2} + \frac{(k|q|C_f\tilde{\delta}^{-1} + C_fk|q| - C)e^{\varepsilon\cdot}}{2}\right)e^{-\varepsilon\cdot - C\int_0^\cdot e^{\varepsilon s}ds}\\
&+ \int_{\mathbf{R}_+}\zeta^2\left(-\frac{\varepsilon}{\sigma}\right)e^{\varepsilon\cdot} + \int_{\mathbf{R}_+}v^2\left(-\frac{(f'(0) - \sigma)}{2}\right)e^{\varepsilon\cdot}
\end{aligned}$$

We thus have that satisfying the following constraints is enough to derive an L^2 linear estimate of the form of 2.2.1

— $\dfrac{(f'(0)-\sigma)}{2}+\dfrac{C_f\sigma\tilde{\delta}e^{C\varepsilon^{-1}}}{k}+\dfrac{C_f\delta'\sigma}{k}\leqslant 0$ (constraint associated to $v(t,0^+)^2$),
— $\dfrac{k|q|e^{2C\varepsilon^{-1}}}{2\delta}+\dfrac{\varepsilon\sigma-2k}{2}+\dfrac{(\delta'^{-1}-C)}{2}<0$ (constraint associated to ζ^2),
— $\dfrac{k|q|\delta}{2}-\dfrac{C_f\varepsilon}{2}<0$ (term in v^2 on $\mathbf{R}^-$, first part),
— $f'(u_0)-\sigma\geqslant 2C_f^2e^{C\varepsilon^{-1}}\tilde{\delta}+C_f\delta'$ (term in $v(t,0^-)^2$),
— $k|q|\tilde{\delta}^{-1}+C_fk|q|-C\leqslant 0$ (term in v^2 on $\mathbf{R}^-$, second part).

We now choose the constants as follows: $\delta'\leqslant\dfrac{(\sigma-f'(0))k}{4C_f\sigma}$ (as it needs to satisfy other conditions listed before, all independent on q crucially), $C=\delta'^{-1}$, $\tilde{\delta}=\dfrac{\sigma-f'(0)}{C_f\sigma e^{2C\varepsilon^{-1}}}k$, giving us the result we were aiming for as long as $|q|$ is small enough.

We can now go back to the nonlinear model. The computations are straightforward but written here for the sake of completeness.

Now, we start by comparing it to the linear case.

The bounds on the linear level differ in that the nonlinear reminders are not there, although the main part remains the same. In fact, the idea is that up to bounds on second order derivatives, the extra terms only involve $W^{1,\infty}(\mathbf{R}^*)$ of the perturbation, times quadratic terms.

Moreover, the boundary terms associated to spatial derivatives of ζ at 0^- need to use lower order derivatives, thus to introduce constants in front of each of the 12 energies considered.

We begin by considering each one of the energies by smooth enough initial data. To state our goal at this point, it is to isolate boundary terms coming from integration by parts in order to reduce the order of the derivatives appearing in the integrals; thus to only have derivatives of order at most k in the bound of the time derivative of $E_\pm(\partial_x^k w)$ where w is v or ζ, as well as isolating terms that cannot be controlled directly by some form of dissipation (in particular, terms related to the phase or the reaction term when computing the time derivative of $E_-(v(t,\cdot))$), and, if possible, have the uncontrolled terms multiplied either by some small constant (that will depend on the energy studied being the one of v or one of its derivatives, or of ζ or of one of its derivatives) or by some exponentially localized terms in space (inside of the integral).

We have for any $\varepsilon>0$, $C>0$, $\delta_1>0$ and as long as $\sup_{0\leqslant t\leqslant T}\|(v(t,\cdot),\zeta(t,\cdot))\|_{W^{1,\infty}(\mathbf{R}^*)}<\eta$, the following inequalities hold for all t

$$\frac{1}{2}\frac{dE_{-,1}(v)}{dt}(t)\leqslant\frac{k|q|}{2\delta_1}E_{-,2}(\zeta(t,\cdot))\exp\left(\frac{2C}{\varepsilon}\right)+\frac{k|q|\delta_1}{2}E_{-,1}(v(t,\cdot))-\mu\frac{v(t,0^-)^2}{2}$$

$$
\begin{aligned}
&- \int_{\mathbf{R}^-} \mu(\varepsilon + Ce^{\varepsilon\cdot})\frac{v(t,\cdot)^2}{2}\varrho_{-,1}dx + \int_{\mathbf{R}^-} C_f(|\underline{u}'| + \eta)\frac{v(t,\cdot)^2}{2}\varrho_{-,1}dx \\
&+ \frac{1}{2}|\psi'(t) - \sigma|^2 E_{-,1}(\sqrt{\underline{u}'}) + \frac{1}{2}E_{-,1}(\sqrt{\underline{u}'}v(t,\cdot)),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\frac{dE_{-,1}(v_x)}{dt}(t) \leqslant &\int_{\mathbf{R}_-} (v_x(t,\cdot))^2 \left(-\frac{\mu\varepsilon + C\mu e^{\varepsilon\cdot}}{2} + \frac{C_f}{2}\eta + \frac{C_f}{2}|\underline{u}'| + \frac{k|q|\delta_1}{2} + \frac{|\underline{u}'|}{2} \right. \\
&\left. + \frac{C_f(\underline{u}'^2 + |\underline{u}''|)}{2} + |\underline{u}'|C_f \right) \varrho_{-,1} + \int_{\mathbf{R}_-} k|q|\delta_1 v(t,\cdot)^2 \frac{\varrho_{-,1}}{2} \\
&+ \frac{k|q|e^{2C\varepsilon^{-1}}E_{-,2}(\zeta_x)}{2\delta_1} + \frac{|\psi' - \sigma|^2 E_{-,1}(\sqrt{|\underline{u}'|})}{2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\frac{dE_{-,1}(v_{xx}(t,\cdot))}{dt}(t) \leqslant &\int_{\mathbf{R}_-} (v_{xx}(t,\cdot))^2 \left(-\frac{\mu\varepsilon + \mu Ce^{\varepsilon\cdot}}{2} \right. \\
&+ |\underline{u}'| \left(\frac{(5 + 2\eta)C_f + C_f|\underline{u}'| + 3|\underline{u}''| + \underline{u}'^2}{2} \right) + \frac{5C_f}{2}|\underline{u}''| \\
&\left. + \frac{C_f + 1}{2}|\underline{u}^{(3)}| + \frac{7C_f\eta + \delta_1 k|q| + C_f\eta^2}{2} \right) \varrho_{-,1} \\
&+ \int_{\mathbf{R}_-} (v_x(t,\cdot))^2 \left(\frac{C_f(3|\underline{u}''| + 3\eta|\underline{u}'| + \eta^2 + \underline{u}'^2)}{2} \right) \varrho_{-,1} \\
&+ \int_{\mathbf{R}_-} v(t,\cdot)^2 \left(C_f \frac{|\underline{u}^{(3)}| + 3|\underline{u}'\underline{u}''| + |\underline{u}'|^3}{2} \right) \varrho_{-,1} \\
&+ |\psi' - \sigma|^2 \frac{E_{-,1}(\sqrt{|\underline{u}^{(3)}|})}{2} + \frac{k|q|e^{2C\varepsilon^{-1}}}{2\delta_1} E_{-,2}((\zeta_{xx}(t,\cdot))).
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\frac{dE_{-,2}(\zeta)}{dt}(t) \leqslant &\frac{1}{2}|\psi'(t) - \sigma|^2 E_-(\sqrt{\underline{z}'}) + \frac{1}{2}E_{-,2}(\sqrt{\underline{z}'}\zeta^2) + \nu\frac{\zeta(0^-)^2}{2} \\
&+ \int_{\mathbf{R}^-} \nu\varepsilon\frac{\zeta^2}{2}\varrho_{-,2}dx - \int_{\mathbf{R}_-} C\mu e^{\varepsilon\cdot}\frac{\zeta^2}{2}\varrho_{-,2} - kE_{-,2}(\zeta),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\frac{dE_{-,2}(\zeta_x)}{dt}(t) \leqslant &\nu\frac{(\zeta_x)(0^-)^2}{2} + \nu\varepsilon\int_{\mathbf{R}^-} \frac{(\zeta_x)^2}{2}\varrho_{-,2} - \int_{\mathbf{R}^-} C\mu e^{\varepsilon\cdot}\frac{(\zeta_x)^2}{2}\varrho_{-,2} - kE_{-,2}(\zeta_x) \\
&+ \frac{|\psi'(t) - \sigma|^2 E_{-,2}(\sqrt{|\underline{z}''|})}{2} + \frac{E_{-,2}(\sqrt{|\underline{z}''|}\zeta_x)}{2},
\end{aligned}
$$

$$
\frac{1}{2}\frac{dE_{-,2}(\zeta_{xx})}{dt}(t) \leqslant \frac{\nu\zeta_{xx}(0^-)^2}{2} + \int_{\mathbf{R}^-} \frac{(\zeta_x)^2}{2}(\nu\varepsilon - C\mu e^{\varepsilon\cdot})\varrho_{-,2} - kE_{-,2}(\zeta_{xx})
$$

$$+\frac{|\psi'(t)-\sigma|^2 E_{-,2}(\sqrt{|\underline{z}^{(3)}|})}{2}+\frac{E_{-,2}(\sqrt{|\underline{z}^{(3)}|}\zeta_{xx})}{2},$$

$$\frac{1}{2}\frac{dE_+(\zeta)}{dt}(t)\leqslant -\mu\frac{\zeta(t,0^+)^2}{2}-\mu\int_{\mathbf{R}^+}\varepsilon\frac{\zeta^2}{2}\varrho_+,$$

$$\frac{1}{2}\frac{dE_+(\zeta_x)}{dt}(t)\leqslant -\mu\frac{\zeta_x(t,0^+)^2}{2}-\mu\varepsilon\int_{\mathbf{R}^+}\frac{(\zeta_x)^2}{2}\varrho_+,$$

$$\frac{1}{2}\frac{dE_+(\zeta_{xx})}{dt}(t)\leqslant -\mu\frac{\zeta_{xx}(t,0^+)^2}{2}-\mu\varepsilon\int_{\mathbf{R}^+}\frac{(\zeta_{xx})^2}{2}\varrho_+,$$

$$\frac{1}{2}\frac{dE_+(v(t,\cdot))}{dt}\leqslant -\frac{\mu v(t,0^+)^2}{2}-\int_{\mathbf{R}_+}\mu\frac{v(t,\cdot)^2}{2}(\varepsilon+Ce^{\varepsilon\cdot})\varrho_+ +\frac{C_f\eta}{2}\int_{\mathbf{R}_+}v(t,\cdot)^2\varrho_+,$$

$$\frac{1}{2}\frac{dE_+(v_x)}{dt}(t)\leqslant -\mu\frac{(v_x(t,0^+))^2}{2}-\int_{\mathbf{R}^+}\mu\varepsilon(v_x(t,\cdot))^2\frac{\varrho_+}{2}+\int_{\mathbf{R}^+}C_f\frac{\eta(v_x(t,\cdot))^2}{2}\varrho_+,$$

$$\begin{aligned}\frac{1}{2}\frac{dE_+(v_{xx})}{dt}(t)\leqslant&\frac{5C_f\eta-2\mu\varepsilon}{4}\int_{\mathbf{R}^+}(v_{xx}(t,\cdot))^2+\int_{\mathbf{R}^+}\frac{-C\mu e^{\varepsilon\cdot}(v_{xx}(t,\cdot))^2}{2}\varrho_+\\&-\frac{\mu v_{xx}(t,0^+)^2}{2}+\int_{\mathbf{R}^+}\frac{C_f}{2}\eta^2((v_x(t,\cdot))^2+(v_{xx}(t,\cdot))^2)\rho_+,\end{aligned}$$

Here, μ and ν are the same parameters from the end of the proof of Proposition 4.

We discuss now how first choices made on some constants encoding smallness can be made.

We now assume that η (which we remind is the upper bound on the $W^{1,\infty}(\mathbf{R}^*)$ norm of (v,ζ) on $[0,T]$) satisfies $\eta\leqslant\min\left(\frac{u_0}{4},1,\frac{\mu\varepsilon}{32}\right)$. We will impose more conditions related to the well-posedness result Theorem 10 on η later on.

When it comes to δ_1, we choose it such that $k|q|\delta_1\leqslant\frac{\mu\varepsilon}{8}$ (more precisely, we take $\delta_1=1$ if $q=0$ and $\delta_1=\frac{\mu\varepsilon}{8k|q|}$ otherwise). The following choice is made for C

$$\begin{aligned}C=2\mu^{-1}\max\Bigg(&\kappa(q)\frac{11C_f+2\eta C_f+\kappa(q)(C_f+3+\kappa(q))}{2}+\frac{1+\eta(7+\eta)+\delta_1 k|q|}{2},\\&\kappa(q)\frac{4C_f+C_f\kappa(q)+1}{2},\kappa(q)\frac{C_f+1}{2},\frac{k+k^2+k^3}{\nu}\Bigg)\end{aligned}$$

and define $\omega := \dfrac{\mu\varepsilon}{4}$.

When it comes to choosing q_0, we first fix the other constants by first taking some $q_1 > 0$ for which the $\underline{u}$ and $\underline{z}$ are defined for every $q \in [-q_1, q_1]$ and for every such q with $|q| \leqslant q_1$ $\kappa(q) \leqslant 1$, and then replacing in the definition of constants the ones obtained with $\kappa(q)$ replaced by 1 and q by q_1. Other conditions on q_0 will be imposed in the next subsection, so it will be chosen later on.

There exists a constant $\tilde{C}$ such that, as long as the $W^{1,\infty}(\mathbf{R}^*)$ norm of the perturbation remains smaller than η, we have

$$(\zeta(t,0^-))^2 \leqslant \tilde{C}(\zeta(t,0^+))^2$$

$$(\zeta_x(t,0^-))^2 \leqslant \tilde{C}(\zeta_x(t,0^+)^2 + |v(t,0^+)|^2 + |v(t,0^-)|^2 + \zeta(t,0^+)^2)$$

$$(\zeta_{xx}(t,0^-))^2 \leqslant \tilde{C}(v(t,0^+)^2 + v(t,0^-)^2 + v_x(t,0^+)^2 + v_x(t,0^-)^2 + \zeta(t,0^+)^2 + \zeta_x(t,0^+)^2)$$

Thus, we can now adjust the value of $\tilde{C}$ in a way which only depends on q_1 (while $\tilde{C}$ depends on k, f and $\underline{u}(0^-)$, we fixed these quantities independently of q and so we are not worried by that dependence) such that, for every such $q \in [-q_0, q_0]$ and solution (v, ζ) defined on $[0, T)$ which remains strictly smaller than η on $[0, T']$ we have that, on $[0, T']$, if $q \in [-q_0, q_0]$

$$\begin{aligned}
\frac{dE_{-,1}(v)(t)}{dt} &\leqslant -\omega E_{-,1}(v(t,\cdot)) + \tilde{C}|q|E_{-,2}(\zeta(t,\cdot)) - \omega v(t,0^-)^2 + \tilde{C}(v(t,0^+))^2\sqrt{\kappa(q)},\\
\frac{dE_{-,1}(v_x)(t)}{dt} &\leqslant -\omega v_x(t,0^-)^2 + \tilde{C}E_{-,1}(v(t,\cdot)) + \tilde{C}((v(t,0^-))^2 + (v(t,0^+))^2)\sqrt{\kappa(q)}\\
&\quad - \omega E_{-,1}(v_x(t,\cdot)) + \tilde{C}|q|E_{-,2}(\zeta_x(t,\cdot)),\\
\frac{dE_{-,1}(v_{xx})(t)}{dt} &\leqslant -\omega v_{xx}(t,0^-)^2 + \tilde{C}(E_{-,1}(v(t,\cdot)) + E_{-,1}(v_x(t,\cdot))) + \tilde{C}((v(t,0^-))^2\\
&\quad + (v(t,0^+))^2)\sqrt{\kappa(q)} - \omega E_{-,2}(v_{xx}(t,\cdot)) + \tilde{C}|q|E_{-,2}(\zeta_{xx}(t,\cdot)),\\
\frac{dE_{+}(v)(t)}{dt} &\leqslant -\omega\frac{v(t,0^+)^2}{2} - \omega E_{+}(v(t,\cdot)),\\
\frac{dE_{+}(v_x)(t)}{dt} &\leqslant -\omega E_{+}(v_x(t,\cdot)) - \frac{\omega(v(t,0^+))^2}{2},\\
\frac{dE_{+}(v_{xx})(t)}{dt} &\leqslant -\omega E_{+}(v_{xx}(t,\cdot)),\\
\frac{dE_{-,2}(\zeta)(t)}{dt} &\leqslant -\omega E_{-,2}(\zeta(t,\cdot)) + \tilde{C}(v(t,0^+)^2 + v(t,0^-)^2 + \zeta(t,0^+)^2),\\
\frac{dE_{-,2}(\zeta_x(t,\cdot))}{dt} &\leqslant -\omega E_{-,2}(\zeta_x(t,\cdot)) + \tilde{C}(v(t,0^+)^2 + v(t,0^-)^2) + \zeta_x(t,0^+)^2 + \zeta(t,0^+)^2),
\end{aligned}$$

$$\begin{aligned}
\frac{dE_{-,2}(\zeta_{xx})(t)}{dt} \leqslant & -\omega E_{-,2}(\zeta_{xx}(t,\cdot)) + \\
& + \tilde{C}(v(t,0^+)^2 + v(t,0^-)^2 + \zeta_{xx}(t,0^+)^2 + \zeta_x(t,0^+)^2 + \zeta(t,0^+)^2 \\
& + v_x(t,0^+)^2 + v_x(t,0^-)^2), \\
\frac{dE_+(\zeta)}{dt}(t) \leqslant & -\omega(E_+(\zeta(t,\cdot)) + \zeta(t,0^+)^2), \\
\frac{dE_+(\zeta_x)}{dt}(t) \leqslant & -\omega(E_+(\zeta_x(t,\cdot)) + \zeta_x(t,0^+)^2), \\
\frac{dE_+(\zeta_{xx})}{dt}(t) \leqslant & -\omega(E_+(\zeta_{xx}(t,\cdot)) + \zeta_{xx}(t,0^+)^2).
\end{aligned}$$

2.2.4 Nonlinear stability and high-frequency damping estimates

We will now be able to conclude the argument in the case of small q, and then obtain a high-frequency damping estimates in the other cases. For stability, we will choose the constants in order to be able to obtain the decay in time of the energy and for the high-frequency damping estimates we choose the constants to control the rest of the equations by lower order terms.
We will first focus on the small q behavior.

Proposition 5. *There exists a 12-tuple of positive constants $(C_{0,-}, C_{1,-}, C_{2,-}, C'_{0,-}, ...)$, a positive constant q_0 small enough, $C > 0$ big enough, $\vartheta > 0$ such that, for every $q \in [-q_0, q_0]$ and every perturbation of the wave that initially satisfies the conditions of 10, namely (v_0, ζ_0) supported away from zero and smooth, there exists a unique maximal solution (v, ζ) to the problem associated with q, with initial data $(\underline{u} + v_0, \underline{z} + \zeta_0)$ defined on some time interval $[0, T)$ with $T \in (0, +\infty]$. Then, for every $T' \leqslant T$ such that $\|(v, \zeta)\|_{W^{1,\infty}} < \eta$ for every $t \in [0, T')$, we have, on $[0, T')$*

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{2}\left(C_{k,-}\frac{dE_{-,1}(\partial_x^k v)}{dt}(t) + C'_{k,-}\frac{dE_{-,1}(\partial_x^k \zeta)}{dt}(t) + C_{k,+}\frac{dE_+(\partial_x^k v)}{dt}(t) + C'_{k,+}\frac{dE_+(\partial_x^k \zeta))}{dt}(t)\right) \\
&\leqslant -\frac{\vartheta}{2}\sum_{k=0}^{2}\big(C_{k,-}E_{-,1}(\partial_x^k v(t,\cdot)) + C'_{k,-}E_{-,1}(\partial_x^k \zeta(t,\cdot)) \\
&\quad + C_{k,+}E_+(\partial_x^k v(t,\cdot)) + C'_{k,+}E_+(\partial_x^k \zeta(t,\cdot))\big))
\end{aligned}$$

Thus, for some constant $M > 0$ independent on q, v_0, ζ_0 and T, for every $t \in [0, T)$, if $\|(v(s), \zeta(s)\|_{W^{1,\infty}} < \eta$ for every $s \in [0, t]$, then

$$\sum_{k=0}^{2}(\|\partial_x^k v(t,\cdot)\|^2_{L^2_\varepsilon(\mathbb{R}^*)} + \|\partial_x^k \zeta(t,\cdot)\|_{L^2_\varepsilon(\mathbf{R}^*)}) \leqslant Me^{-(t-s)\vartheta}\sum_{k=0}^{2}(\|\partial_x^k v(s,\cdot)\|^2_{L^2_\varepsilon(\mathbb{R}^*)} + \|\partial_x^k \zeta(s,\cdot)\|_{L^2_\varepsilon(\mathbf{R}^*)})$$

Proof. To obtain the desired inequality, we just need to ensure that we can force certain coefficients to non-positive if q_0 is chosen small enough and $C > 0$ big enough. We will do this by using the following bounds
We may choose $C_{0,-}$, $C_{1,-}$, $C_{2,-}$, ... such that they satisfy the following inequalities

— (to have a nonpositive factor in front of $v(t,0^+)^2$)

$$\tilde{C}((C_{0,-}+C_{1,-}+C_{2,-})\sqrt{\kappa(q)}+(C'_{0,-}+C'_{1,-}+C'_{2,-}))-\frac{\omega C_{0,+}}{2}\leqslant 0,$$

— (to have a nonpositive factor in front of $v(t,0^-)^2$)

$$\tilde{C}(\sqrt{\kappa(q)}(C_{1,-}+C_{2,-})+C'_{0,-}+C'_{1,-}+C'_{2,-})-\frac{\omega C_{0,-}}{2}\leqslant 0,$$

— (to have a negative factor in front of $E_{-,2}(\zeta_x(t,\cdot))$)

$$\tilde{C}|q|C_{1,-}-\frac{C'_{1,-}\omega}{2}\leqslant 0,$$

— (to have a negative factor in front of $E_{-,1}(v_x(t,\cdot))$)

$$\tilde{C}C_{2,-}-\frac{C_{1,-}\omega}{2}\leqslant 0,$$

— (to have a negative factor in front of $E_{-,2}(\zeta_{xx}(t,\cdot))$)

$$\tilde{C}|q|C_{2,-}-\frac{C'_{2,-}\omega}{2}\leqslant 0,$$

— (to have a nonpositive factor in front of $\zeta(t,0^+)^2$)

$$\tilde{C}(C'_{0,-}+C'_{1,-}+C'_{2,-})-\frac{\omega C'_{0,+}}{2}\leqslant 0,$$

— (to have a nonpositive factor in front of $\zeta_x(t,0^+)^2$)

$$\tilde{C}(C'_{1,-}+C'_{2,-})-\frac{\omega C'_{1,+}}{2}\leqslant 0,$$

— (to have a nonpositive factor in front of $\zeta_{xx}(t,0^-)^2$)

$$\tilde{C}C'_{2,-}-\frac{\omega C'_{2,+}}{2}\leqslant 0,$$

— (to have a nonpositive factor in front of $v_x(t,0^-)^2$)

$$\tilde{C}C'_{2,-} - \omega C_{1,-} \leqslant 0$$

and

— (to have a nonpositive factor in front of $v_x(t,0^+)^2$)

$$\tilde{C}C_{2,-} - \omega C_{1,+} \leqslant 0.$$

Thus, choosing

$$C_{0,-} = 1 = C_{0,+} = C_{1,+} = C_{2,+} = C'_{0,+} = C'_{1,+} = C'_{2,+},$$

$$C_{1,-} = \frac{\omega}{4\tilde{C}},$$

$$C_{2,-} = \min\left(C_{1,-}, \frac{C_{1,-}\omega}{2\tilde{C}}\right),$$

$$C'_{0,-} = \frac{\omega}{8\tilde{C}},$$

$$C'_{1,-} = \min\left(\frac{\omega}{8\tilde{C}}, \frac{\omega C_{1,-}}{\tilde{C}}\right),$$

we can obtain the intermediate inequalities as wanted if we choose q_0 small enough to make the terms in $|q|$ and $\kappa(q)$ small enough. Thus, for initially smooth and compactly supported functions solutions to our (perturbative) equation, we have that

$$\sum_{k=0}^{2}(C_{k,-}\frac{dE_{-,1}(\partial_x^k v)}{dt}(t) + C'_{k,-}\frac{dE_{-,1}(\partial_x^k \zeta)}{dt}(t) + C_{k,+}\frac{dE_{+}(\partial_x^k v)}{dt}(t) + C'_{k,+}\frac{dE_{+}(\partial_x^k \zeta))}{dt}(t)$$
$$\leqslant -\frac{\omega}{2}\sum_{k=0}^{2}(E_{-,1}(\partial_x^k v(t,\cdot)) + E_{-,1}(\partial_x^k \zeta(t,\cdot)) + E_{+}(\partial_x^k v(t,\cdot)) + E_{+}(\partial_x^k \zeta(t,\cdot))).$$

The desired results are then direct consequences of this inequality. □

To conclude the argument, one needs to ensure that the energy estimates of Proposition 5 remain valid in order to prove the main theorem of this part.

Proof of Theorem 11. Take the constants obtained in Proposition 5. Given η, ε, M and ϑ, we will need to apply a continuity argument. First, we work with smooth initial data, compactly supported away from the shock. As the energy is decaying exponentially fast as long as the $W^{1,\infty}$ norm is small enough, and as the energy is a norm on H^2_ε equivalent to the one defined before, we have the result as long as $\|(v,\zeta)\|_{W^{1,\infty}}$ remains small enough.

To guarantee that (v, ζ) has a $W^{1,\infty}$ norm that remains small enough, we use the Sobolev embedding theorem and a continuity argument, shrinking δ_0 if needed. After that, we can just obtain the result for initial data that may not be smooth nor compactly supported (but supported away from the shock) through a density argument, using the continuity of the flow from Theorem 10. We note (v, ζ) the associated maximum solution to a given initial data satisfying the theorem 10, as well as $(v_n, \zeta_n)_n$ the maximum solutions associated with a sequence of initial data smooth and with compact supports in $\mathbf{R}^*$ that approximate $(v(0,\cdot), \zeta(0,\cdot))$ in H^2_ε. Given that $(v(0,\cdot), \zeta(0,\cdot))$ is small enough in H^2_ε, we have that (v_n, ζ_n) is defined on $\mathbf{R}_+$ for every n big enough, and so we obtain the convergence in L^2 of the (v_n, ζ_n) for every t in the interval of existence of (v, ζ), and, as we have obtained that $(v_n, \zeta_n)_n$ is bounded in H^2_ε for n big enough, the limit is also bounded in H^2_ε, and we obtain that the sequence goes to 0 with the rate of convergence we were aiming for. □

Now, we state a proposition valid for any q satisfying the conditions of 4. In that case, we are not able to obtain a nonlinear stability result, but only a a high-frequency damping estimate. It emphasizes the importance of the assumption made before on q, as well as the fact that, in the general case, not all the computations made by energy estimates are useless.

Proposition 6. *For any q, not necessarily in $[-q_0, q_0]$, such that the wave considered in Proposition 4, with fixed f, k and $\underline{u}(0^-)$, exists then, there exists $\vartheta > 0$, $\tilde{C} > 0$ and $\eta > 0$ such that for every perturbative solutions (v, ζ) defined on some time interval $[0, T]$ where $T > 0$ such that $\|(v(t,\cdot), \zeta(t,\cdot))\|_{W^{1,\infty}(\mathbf{R}^*)} < \eta$ on $[0, T]$, we have that, for every $t \in [0, T]$*

$$\|(v(t,\cdot), \zeta(t,\cdot)\|_{H^2_\varepsilon(\mathbf{R}^*)} \leqslant Ce^{-\vartheta t}\|(v(0), \zeta(0)\|_{H^2_\varepsilon(\mathbf{R}^*)} + \int_0^t Ce^{-\vartheta(t-s)}\|(v(s), \zeta(s))\|_{L^2_\varepsilon(\mathbf{R}^*)}ds$$

The proof follows the same idea as the result before, just this time setting aside the lower order terms, that is the terms of the form $\|v\|^2_{L^2_\varepsilon}$ or $\|\zeta\|_{L^2_\varepsilon}$. The main new ingredient in this proof compared to the proof of the stability result is the use of the Sobolev embedding theorem as follows (where δ_2 is a given positive constant that will be adjusted later, and h is an element of $H^1(\mathbf{R}^*)$)

$$|h(0^-)|^2 \leqslant \delta_2\|h_x\|^2_{L^2(\mathbf{R}^-)} + \frac{\|h\|^2_{L^2(\mathbf{R}^-)}}{4\delta_2},$$
$$|h(0^+)|^2 \leqslant \delta_2\|h_x\|^2_{L^2(\mathbf{R}^+)} + \frac{\|h\|^2_{L^2(\mathbf{R}^+)}}{4\delta_2}.$$

As for the complete proof, it will not be given here. We just focus on how to obtain bounds at the H^1_ε level in the linear case, omitting the proof of the H^2_ε nonlinear problem.

We have the following equations for the spatial derivatives of (v, ζ) (for the linear problem) on $\mathbf{R}^-$

$$
\begin{aligned}
(v_x)_t &= (\sigma - f'(\underline{u}))v_{xx} - 2f''(\underline{u})\underline{u}'v_x - (f^{(3)}(\underline{u})\underline{u}'^2 + f''(\underline{u})\underline{u}'')v + kq\zeta_x + (\psi' - \sigma)\underline{u}'' \,, \\
(\zeta_x)_t &= \sigma\zeta_{xx} + (\psi' - \sigma)\underline{u}'' - k\zeta_x \,,
\end{aligned}
$$

and on $\mathbf{R}^+$

$$
\begin{aligned}
(v_x)_t &= (\sigma - f'(0))v_{xx} \,, \\
(\zeta_x)_t &= \sigma\zeta_{xx} \,, .
\end{aligned}
$$

This gives us the following energy estimates for the functions and their derivatives for smooth solutions (on $[0,T] \times \mathbf{R}^*$) for all $t \in (0,T)$

$$
\begin{aligned}
\frac{1}{2}\frac{dE_+(v)}{dt}(t) &\leqslant 0, \\
\frac{1}{2}\frac{dE_+(\zeta)}{dt}(t) &\leqslant -\frac{\sigma}{2}(\zeta(t,0^+))^2, \\
\frac{1}{2}\frac{dE_{-,1}(v)}{dt}(t) &\leqslant \frac{k|q|E_{-,1}(v(t,\cdot))}{2} + \frac{k|q|E_{-,1}(\zeta(t,\cdot))}{2} + \frac{E_{-,1}(v(t,\cdot))}{2} \\
&\quad + \left(\delta_2\|v_x(t,\cdot)\|^2_{L^2(\mathbf{R}^+)} + \delta_2\|v_x(t,\cdot)\|^2_{L^2(\mathbf{R}^-)}\right. \\
&\quad \left. + \frac{\|v(t,\cdot)\|_{L^2(\mathbf{R}^-)}}{4\delta_2} + \frac{\|v(t,\cdot)\|_{L^2(\mathbf{R}^+)}}{4\delta_2}\right)\frac{E_{-,1}(\underline{u}')}{2} \\
&\quad + \frac{C_f\sqrt{\kappa(q)}E_{-,1}(v(t,\cdot))}{2}, \\
\frac{1}{2}\frac{dE_{-,2}(\zeta)}{dt}(t) &\leqslant \frac{C_f}{2}\left(\delta_2\|v_x(t,\cdot)\|^2_{L^2(\mathbf{R}^+)} + \delta_2\|v_x(t,\cdot)\|^2_{L^2(\mathbf{R}^-)}\right. \\
&\quad \left. + \frac{\|v(t,\cdot)\|^2_{L^2(\mathbf{R}^-)}}{4\delta_2} + \frac{\|v(t,\cdot)\|_{L^2(\mathbf{R}^+)}}{4\delta_2}\right)E_{-,2}(\underline{z}') \\
&\quad + \frac{C_f E_{-,2}(\zeta(t,\cdot))}{2}, \\
\frac{1}{2}\frac{dE_+(v_x)}{dt}(t) &\leqslant -\frac{(\sigma - f'(0))}{2}v_x(t,0^+)^2 - \frac{(\sigma - f'(0))\varepsilon E_+(v_x(t,\cdot))}{2}, \\
\frac{1}{2}\frac{dE_+(\zeta_x)}{dt}(t) &\leqslant -\frac{\sigma(\zeta_x(t,0^+))^2}{2} - \frac{\sigma\varepsilon E_+(\zeta_x)}{2}, \\
\frac{1}{2}\frac{dE_{-,2}(\zeta_x)}{dt}(t) &\leqslant +\frac{\sigma(\zeta_x(t,0^-))^2}{2} - \frac{2k - \varepsilon\sigma}{2}E_{-,2}(\zeta_x(t,\cdot)) \\
&\quad + \frac{C_f E_{-,2}(\sqrt{|\underline{z}''|})}{2}\left(\delta_2\|v_x(t,\cdot)\|^2_{L^2(\mathbf{R}^+)} + \delta_2\|v_x(t,\cdot)\|^2_{L^2(\mathbf{R}^-)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\|v(t,\cdot)\|^2_{L^2(\mathbf{R}^-)}}{4\delta_2}+\frac{\|v(t,\cdot)\|_{L^2(\mathbf{R}^+)}}{4\delta_2}\Bigg)+\frac{C_f E_{-,2}(\sqrt{|\underline{z}''|}\zeta_x)}{2}\\
&+\int_{\mathbf{R}^-}\frac{-C\sigma e^{\varepsilon\cdot}(\zeta_x)^2}{2}\varrho_{-,2},\\
\frac{1}{2}\frac{dE_{-,1}(v_x)}{dt}(t)\leqslant&\frac{(\sigma-f'(u_0))}{2}(v_x(t,0^-))^2\\
&+\frac{\varepsilon(\sigma-f'(u_0))}{2}E_{-,1}(v_x(t,0^-))^2-\int_{\mathbf{R}^-}\frac{Ce^{\varepsilon\cdot}(v_x(t,\cdot))^2(f'(u_0)-\sigma)}{2}\varrho_{-,1}\\
&+2C_f\int_{\mathbf{R}^-}|\underline{u}'|v_x(t,\cdot)^2\varrho_{1,-}+\frac{C_f}{2}\int_{\mathbf{R}^-}(\underline{u}'^2+|\underline{u}''|)v_x(t,\cdot)^2\varrho_{-,1}\\
&+\frac{C_f}{2}\int_{\mathbf{R}^-}v(t,\cdot)^2\varrho_{-,1}+\frac{k|q|E_{-,1}(\zeta_x(t,\cdot))}{2\delta_1}\\
&+\frac{k|q|\delta_1 E_{-,1}(v_x(t,\cdot))}{2}+\big(\delta_2\|v_x(t,\cdot)\|^2_{L^2(\mathbf{R}^+)}+\delta_2\|v_x(t,\cdot)\|^2_{L^2(\mathbf{R}^-)}\\
&+\frac{\|v(t,\cdot)\|^2_{L^2(\mathbf{R}^-)}}{4\delta_2}+\frac{\|v(t,\cdot)\|_{L^2(\mathbf{R}^+)}}{4\delta_2}\Bigg)\frac{E_{-,1}(\sqrt{|\underline{u}''|})}{2}\\
&+\frac{E_{-,1}(\sqrt{|\underline{u}''|}v_x(t,\cdot))}{2}.
\end{aligned}
$$

Thus, choosing C, δ_1 and δ_2 as

$$
C=\max\left(\frac{C_f k^2}{\sigma^3},\frac{\kappa(q)+2C_f(3\kappa(q)+\kappa(q)^2)}{f'(u_0)-\sigma}\right),
$$

and

$$
0<\delta_2<\min\left(\frac{\varepsilon(f'(u_0)-\sigma)}{8E_{-,1}(\sqrt{|\underline{u}''|})},\frac{\varepsilon(f'(u_0)-\sigma)(2k-\varepsilon\sigma)\varepsilon\sigma}{16C_f E_{-,2}(\sqrt{|\underline{z}''|})k^2q^2e^{2C\varepsilon^{-1}}}\right)
$$

as well as

$$
\delta_1=\frac{\varepsilon(f'(u_0)-\sigma)}{4k|q|}
$$

(it does not work for $q=0$, but this case is contained in the stability result proved earlier, so we assume that $q\neq 0$) we will be able to conclude the desired damping estimate. For any $t\in(0,T)$ we obtain

$$
\begin{aligned}
\frac{k|q|e^{2C\varepsilon^{-1}}}{(2k-\varepsilon\sigma)\delta_1}\frac{d(E_{-,2}(\zeta_x))}{dt}(t)+\frac{1}{2}\frac{d(E_{-,1}(v_x))}{dt}(t)\leqslant&\frac{-\varepsilon(f'(u_0)-\sigma)E_{-,1}(v_x(t,\cdot))}{4}\\
&-\frac{k|q|e^{2C\varepsilon^{-1}}}{4\delta_1}E_{-,2}(\zeta_x(t,\cdot))+\frac{C_f E_{-,1}(v(t,\cdot))}{2}\\
&+\frac{\|v(t,\cdot)\|^2_{L^2(\mathbf{R})}E_{-,1}(\sqrt{|\underline{u}''|})}{8\delta_2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{k|q|e^{2C\varepsilon^{-1}}\|v(t,\cdot)\|^2_{L^2(\mathbf{R})}E_{-,1}(\sqrt{|\underline{z}''|})}{4\delta_2(2k-\varepsilon\sigma)\delta_1}\\
&+\left(\|v_x(t,\cdot)\|^2_{L^2(\mathbf{R}^+)}\delta_2+\frac{\|v(t,\cdot)\|^2_{L^2(\mathbf{R})}}{4\delta_2}\right)\times\\
&\left(\frac{C_f(E_{-,1}(\sqrt{|\underline{u}''|})+E_{-,2}(\sqrt{|\underline{z}''|})}{2}\right)\\
&+\frac{k|q|e^{2C\varepsilon^{-1}}(\zeta_x(t,0^+))^2\sigma}{2\delta_1(2k-\varepsilon\sigma)}.
\end{aligned}
$$

Hence, by now taking the full energy estimates (we write

$$
\begin{aligned}
\mathcal{E}(v,\zeta):=&E_{-,1}(v)+E_{-,2}(\zeta)+E_{-,1}(v)+\frac{E_{-,2}(\zeta)k|q|e^{2C\varepsilon^{-1}}}{k-\frac{\varepsilon\sigma}{2}\delta_1}+E_+(v)+ME_+(\zeta)\\
&+E_+(v_x)+ME_+(\zeta_x),
\end{aligned}
$$

where M is a positive constant to be determined), the following bounds are obtained

$$
\begin{aligned}
\frac{1}{2}\frac{d\mathcal{E}(v,\zeta)}{dt}(t)\leqslant&-\frac{M\sigma}{2}((\zeta(t,0^+))^2+(\zeta_x(t,0^+))^2)-\frac{\varepsilon(f'(u_0)-\sigma)E_{-,1}(v_x(t,\cdot))}{4}\\
&-\frac{k|q|e^{2C\varepsilon^{-1}}}{4\delta_1}E_{-,2}(\zeta_x(t,\cdot))+\left(\|v_x(t,\cdot)\|^2_{L^2(\mathbf{R}^+)}\delta_2\right)\\
&\times\left(\frac{C_f(E_{-,1}(\sqrt{|\underline{u}''|})+E_{-,2}(\sqrt{|\underline{z}''|})}{2}\right)+\frac{k|q|e^{2C\varepsilon^{-1}}(\zeta_x(t,0^+))^2\sigma}{2\delta_1(2k-\varepsilon\sigma)}\\
&+\frac{\sigma(\zeta(t,0^+))^2}{2}-\frac{\varepsilon(\sigma-f'(0))}{2}E_+(v_x(t,\cdot))+\tilde{C}(\|v(t,\cdot)\|^2_{L^2(\mathbf{R})}+\|\zeta(t,\cdot)\|^2_{L^2(\mathbf{R})})\\
&-\frac{M\sigma\varepsilon E_+(\zeta_x(t,\cdot))}{2}.
\end{aligned}
$$

For some constant $\tilde{C}$ which depends on δ_2. Thus, for M big enough and δ_2 small enough, we obtain

$$
\frac{1}{2}\frac{d(\mathcal{E}(v,\zeta))}{dt}(t)\leqslant\frac{\gamma\mathcal{E}(v,\zeta)}{dt}(t)+\tilde{C}(\|v(t,\cdot)\|^2_{L^2(\mathbf{R})}+\|\zeta(t,\cdot)\|^2_{L^2(\mathbf{R})})
$$

for some $\gamma<0$. The Proposition 5 provides a result of prevention of blow up as long as the $W^{1,\infty}$ norm remains small, and a beginning of an argument giving stability as in

2.2.5 Negative propagation speed

We now examine the case $\sigma<0$. If one has $q\leqslant 0$ and u_0 positive such that, with $\sigma:=\frac{f(u_0)-f(0)}{u_0}$, $f'(0)<\sigma$ and $f'>\sigma$ on $[u_0,+\infty[$, then there exists a unique travelling

wave $(\underline{u}, \underline{z})$ solution to

$$(f'(\underline{u}) - \sigma)\underline{u}' = kq\underline{z}'\,\sigma\underline{z}' = k\underline{z}\,,$$

on $\mathbf{R}^-$ (and smooth on this interval), with limiting values $(u_0, 1)$ at 0^-, and constant equal to $(0, 1)$ on $\mathbf{R}^+$. It is thus unbounded on $\mathbf{R}^-$. Due to the finite speed of propagation of this system, one can ensure that an initial data given by $(\underline{u}, \underline{z})$ plus a compactly supported perturbation on $\mathbf{R}^*$ will remain compactly supported on $\mathbf{R}$ for all later time $t > 0$ (as long as it remains defined). As such, perturbation of this kind can be allowed.

Proposition 7. *Let $\varepsilon \in \mathbf{R}$. Assume that $\varepsilon \neq 0$and $\sigma < 0$. There exists $\delta_0 > 0$ small enough such that there exists $(v_{0,n}, \zeta_{0,n})_{n\in\mathbf{N}} \in C_c^\infty(\mathbf{R})^{\mathbf{N}}$ that goes to 0 in H^2_ε, supported away from 0, and such that, for every $n \in \mathbf{N}$, the induced solution remains supported away from the shock on some time interval $[0, T_{n,\varepsilon})$ and $\sup_{t\in[0,T_{n,\varepsilon})} \|(v_n(t,\cdot), \zeta_n(t,\cdot)\|_{H^2_\varepsilon(\mathbf{R})}$ is larger than δ_0.*
If $\varepsilon = 0$ and for every neighborhood V of 0 in $\mathbf{R}$, f' is not constant on V.

Proof. Case $\varepsilon \leqslant 0$:
When $v_t + (f(v))_x = 0$ has a compactly supported initial perturbation, we can proceed as follows. We consider $(\phi_n)_n \in H^2(\mathbf{R})$ fixed and all nonzero, all initial data such that the induced solutions blow up in finite time in $W^{1,\infty}$, and with the sequence of their H^2 norms that goes to 0 when n goes to $+\infty$. We also assume that they all have their support contained in $[0, 1]$.
We consider the sequence of initial perturbation $(v_n, 0)$ where $v_n : x \mapsto \phi_n(\cdot - a_n)$, where $(a_n)_n$ is such that the solution induced by ϕ_n has its support included in $[0, T_n) \times (0, +\infty)$ (where T_n is its blow up time). We are now reduced to the case of a scalar conservation law. For n big enough, we have that the solution blows up in finite time in $W^{1,\infty}$, and, thus, in H^2_ε (as the support of the perturbation at time remains in a compact set K_n independent of t, we can apply the the Sobolev embedding to obtain a bound on the unweighted $W^{1,\infty}$ norm).
Now, we assume that f' is constant near 0, and $\varepsilon < 0$. By assumption, we have that $f'(0) < \sigma$. Thus, by taking any initial data nonzero localized on the right of the shock, we have that the solutions grows in L^2_ε exponentially fast in time as long as the solution is localized on the right of the shock. In particular, by translating some bump function far enough to the right, we have the desired result.
The initial data given by $(\underline{u} + v_n, \underline{z})$ (when taking n big enough, after multiplying it by some small $\delta > 0$) either induces a solution that blows up in finite time in $W^{1,\infty}$, or a solution that remains small in L^∞, is defined globally.
Case $\varepsilon > 0$:

This time, we will not obtain explosion in finite time. However, with the sequence initial data $(0, (n+1)^{-1}\phi)_n$, for t such that the solution continues to exist on $[0,t]$ we have that: $\zeta_n(t,x) = \dfrac{\phi(x-t\sigma)}{n+1}$ for all $x > 0$. Furthermore, as $\zeta_n(t,x) = 0$ for all $x < 0$ and $v_n(t,x) = 0$ for all $x \in \mathbf{R}^*$, we obtain the desired instability result: the solution is defined globally, but its $\|\cdot\|_{L^2_\varepsilon}$ norm goes to $+\infty$ as t goes to $+\infty$. □

2.3 Singularity formation for ZND

In this section, our main goal is to prove the negative result Theorem 12 and Corollary 1, as well as a similar blowup result for *un*weighted norms for the Majda model. This will be done by suitably modifying the argument in [Joh74] to variable coefficient systems. We begin with the needed assumptions and some useful background results from spectral perturbation theory in Subsection 2.3.1. We then apply the method of characteristics in Subsection 2.3.2 in order to complete the needed setup. Next, we use the equations for the perturbation and its derivative with respect to x in order to establish the needed estimates in Subsection 2.3.3. We then complete the proof of Theorem 12 in Subsection 2.3.4. Next, we discuss how to apply Theorem 12 to the ZND model in Subsection 2.3.5 and prove Corollary 1 in Subsection 2.3.6. We will also discuss the situation with weighted norms for ZND in Subsection 2.3.7.

2.3.1 Setup

Let $A(u) : \mathbb{R}^n \to M_n(\mathbb{R})$ be a smooth matrix function with the property that there exists a $\delta > 0$ so that $A(u)$ is strictly hyperbolic for $|u| \leqslant \delta$. Further assume that the (real) eigenvalues $\Lambda_i(u)$ of $A(u)$ may be ordered as

$$\Lambda_n(u) < \Lambda_{n-1}(u) < ... < \Lambda_1(u), \tag{2.3.1}$$

and that each Λ_j is simple. By shrinking δ if necessary, we may further assume that $\Lambda_i(v) < \Lambda_j(u)$ if $j < i$ and $|u|, |v| \leqslant \delta$. Assume further that there exists a smooth function $F : \mathbb{R}^n \to \mathbb{R}^n$ with

$$A(U) = D_U F(U),$$

so that

$$U_t + F(U)_x = U_t + A(U)U_x = 0, \tag{2.3.2}$$

is a system of conservation laws.

Let $\underline{U}(x-\sigma t)$ be a shock solution to

$$U_t + A(U)U_x = 0, \tag{2.3.3}$$

for some constant σ and such that $||\underline{U}||_\infty < \delta$ and $\underline{U}$ and all of its derivatives are exponentially decaying in space. For our purposes, we need A to be strictly hyperbolic with at least one genuinely nonlinear field. As both the required conditions on A are invariant under Galilean changes of coordinates, we may without essential loss of generality assume that the underlying shock speed $\sigma = 0$. Suppose we have a solution to (2.3.3) of the form

$$U = \underline{U} + \hat{U}, \tag{2.3.4}$$

with $\hat{U}$ small. Then the evolution for the perturbed solution U is given by

$$U_t + A(U)U_x = 0. \tag{2.3.5}$$

Subtracting the equation for the shock

$$A(\underline{U})\underline{U}_x = 0, \tag{2.3.6}$$

from (2.3.5) produces

$$\hat{U}_t + (A(\underline{U}+\hat{U}) - A(\underline{U}))\underline{U}_x + A(\underline{U}+\hat{U})\hat{U}_x = 0. \tag{2.3.7}$$

By applying the fundamental theorem of calculus twice, we see that

$$\begin{aligned} A(\underline{U}+\hat{U}) - A(\underline{U}) &= \int_0^1 D_u A(\underline{U}+s\hat{U})\hat{U}ds = \tilde{G}(x,\hat{U})\hat{U}, \\ A(\underline{U}+\hat{U}) &= A(\hat{U}) + \int_0^1 D_u A(\hat{U}+s\underline{U})\underline{U}ds = A(\hat{U}) + B(x,\hat{U}), \end{aligned} \tag{2.3.8}$$

where D_u denotes the Fréchet derivative with respect to u, $\tilde{G}(x,\hat{U})$ is a bilinear form and B is a matrix. We have that $\tilde{G}$ and B are piecewise smooth, smooth for x large, and that B and all of its derivatives are exponentially decaying in x.

$$\hat{U}_t + (A(\hat{U}) + B(x,\hat{U}))\hat{U}_x + G(x,\hat{U})\hat{U} = 0, \tag{2.3.9}$$

for $G(x,\hat{U})\hat{U} = \tilde{G}(x,\hat{U})(\hat{U},\underline{U}_x)$. Note that G and all of its derivatives with respect to x and $\hat{U}$ are exponentially decaying in x due to the presence of $\underline{U}_x$. Now that we have the equation for the perturbation (2.3.9), we are now going to call the perturbation u in

order to match the notation of [Joh74] more closely.

We note that there exists an $\varepsilon > 0$ and $R > 0$ depending on $B(x,u)$ and $G(x,u)$ so that B and G are smooth for $|x| \geqslant R$ and such that for all $|x| \geqslant R$ one has that

$$|B(x,u)|, |G(x,u)| < \varepsilon, \tag{2.3.10}$$

and the matrix

$$\mathcal{A}(x,u) := A(u) + B(x,u), \tag{2.3.11}$$

is strictly hyperbolic with all simple eigenvalues $\lambda_j(x,u)$ which smoothly depend on x, u [Kat76] and satisfy $\lambda_i(x,v) < \lambda_j(x,u)$ for $|u|, |v| \leqslant \delta$, shrinking δ slightly if necessary while preserving $||\underline{u}||_\infty < \delta$, and $j < i$. This is due to the exponential decay of B, G with respect to x and recalling the observation (2.3.1) that we have separation of the eigenvalues of $\mathcal{A}(x,u)$ at $x = \pm\infty$. From [Kat76], we know that we can also find left and right eigenvectors $\eta_j(x,u)$ and $\xi^j(x,u)$ of $\mathcal{A}(x,u)$ associated to the eigenvalues $\lambda_j(x,u)$ with the biorthogonality condition

$$\eta_i(x,u)\xi^j(x,u) = \delta_{ij}. \tag{2.3.12}$$

Further assume the normalization condition on the η_i

$$\eta_i \cdot \eta_i = 1. \tag{2.3.13}$$

If a solution $u(x,t)$ to (2.3.9) remains supported in $\{x : |x| \geqslant 2R\}$ for all time t on some sufficiently long time interval $0 \leqslant t \leqslant T$, then we may smooth out B and G in a such a way as to ensure that (2.3.10) holds everywhere in x by choosing smooth approximations $\hat{B}$ and $\hat{G}$ such that $\text{supp}(B - \hat{B}), \text{supp}(G - \hat{G}) \subset \{x : |x| \leqslant \frac{3}{2}R\} \times \{u : |u| \leqslant \delta\}$. As we're only trying to show blow up of a specific solution which will be supported far from $x = 0$ on the desired time of existence, there is no real harm in assuming that B and G are smooth everywhere and that the inequality (2.3.10) holds for all x. This is done purely to avoid some technicalities arising when trying to solve for the characteristics of (2.3.9) near $x = 0$.

We introduce coefficients $b_{ij}(x,u)$ and $c_{ijk}(x,u)$ in order to describe the gradient of $\mathcal{A}(x,u)$. In particular, we define

$$b_{ij}(x,u) := \eta_i(x,u)\frac{\partial \mathcal{A}}{\partial x}(x,u)\xi^j(x,u), \tag{2.3.14}$$

and

$$c_{ijk}(x,u) := \eta_i(x,u)\Big(\frac{d}{ds}\mathcal{A}(x,u+s\xi^k)|_{s=0}\Big)\xi^j(x,u). \tag{2.3.15}$$

Remark 2. *Morally, $c_{ijk}(x,u)$ can be thought of as*

$$c_{ijk}(x,u) = \eta_i(x,u)\frac{\partial \mathcal{A}}{\partial u_k}(x,u)\xi^j(x,u), \tag{2.3.16}$$

when one writes u as

$$u = \sum_{k=1}^{n} u_k \xi^k. \tag{2.3.17}$$

From this, we get the following identities

$$\begin{aligned} d\lambda_i(x,u) &= b_{ii}(x,u)dx + \sum_{k=1}^{n} c_{iik}(x,u)(du)_k, \\ d\eta_i(x,u) &= \sum_{k\neq i} \frac{b_{ik}(x,u)}{\lambda_i(x,u)-\lambda_k(x,u)}((\eta_k\cdot\eta_i)\eta_i-\eta_k)dx+ \\ &\quad + \sum_{\substack{k,m\\ k\neq i}} \frac{1}{\lambda_k(x,u)-\lambda_i(x,u)} c_{ikm}(x,u)(du)_m((\eta_k\cdot\eta_i)\eta_i-\eta_k), \end{aligned} \tag{2.3.18}$$

where we've written du as

$$du = \sum_{k=1}^{n}(du)_k\xi^k = \sum_{k=1}^{n}(\eta_k(x,u)du)\xi^k(x,u). \tag{2.3.19}$$

We record the decay in B and G as

$$|\partial_x^\alpha\partial_u^\beta B(x,u)|, |\partial_x^\alpha\partial_u^\beta G(x,u)| \leqslant C_{\alpha\beta}e^{-c|x|}, \tag{2.3.20}$$

for $|x|$ bounded away from 0.

2.3.2 Characteristics

For each $i = 1, ..., n$ we let $X_i(x,t)$ be the solution to the ODE

$$\begin{aligned} \frac{\partial X_i}{\partial t}(x,t) &= \lambda_i(X_i(x,t), u(X_i(x,t),t)), \\ X_i(x,0) &= x. \end{aligned} \tag{2.3.21}$$

Let $\overline{\lambda}$ be defined as

$$\overline{\lambda} := \max_{1\leqslant i\leqslant n}\sup_{\substack{x\in\mathbb{R}\\ |u|\leqslant\delta}}|\lambda_i(x,u)| < \infty. \tag{2.3.22}$$

The observation that $\overline{\lambda} < \infty$ follows from $||\mathcal{A}||_\infty < \infty$. Note that $\overline{\lambda} < \infty$ holds even when B arises from linearizing about a shock as $||B||_\infty < \infty$. Define parameters ν_i, μ_i by

$$\begin{aligned} \nu_i &:= \inf_{x\in\mathbb{R},\ |u|\leqslant\delta} \lambda_i(x,u), \\ \mu_i &:= \sup_{x\in\mathbb{R},\ |u|\leqslant\delta} \lambda_i(x,u). \end{aligned} \tag{2.3.23}$$

Further define σ by

$$\sigma := \min_{k<i} \nu_k - \mu_i > 0. \tag{2.3.24}$$

Let $u(x,t)$ be a solution to

$$u_t + \mathcal{A}(x,u)u_x + G(x,u)u = 0 \tag{2.3.25}$$

such that for some $\infty > T > 0$ one has the bound

$$\sup_{\substack{0\leqslant t\leqslant T \\ x\in\mathbb{R}}} |u(x,t)| \leqslant \delta. \tag{2.3.26}$$

Thus if $u(x,0)$ is compactly supported in $[\alpha_0, \beta_0]$, we then have by the method of characteristics that the solution $u(x,t)$ is compactly supported in $[\alpha_0 - \overline{\lambda}T, \beta_0 + \overline{\lambda}T]$. As such, if $[\alpha_0, \beta_0]$ is sufficiently far from $x = 0$ then $u(x,t)$ remains supported in $|x| \geqslant 2R$ for all t and hence we may smooth out B and G.

Let $w := u_x$ and write w as

$$w = \sum_{i=1}^{n} w_i \xi^i. \tag{2.3.27}$$

From this, we can write the evolution of u along the i-th characteristic as

$$\frac{du}{dt} = u_t + \lambda_i u_x = (\lambda_i - \mathcal{A}(x,u))u_x - G(x,u)u = \sum_{k=1}^{n} (\lambda_i - \lambda_k) w_k \xi^k - G(x,u)u. \tag{2.3.28}$$

Correspondingly, we have the evolution of λ_i along the i-th characteristic given by

$$\begin{aligned} \frac{d\lambda_i}{dt} &= b_{ii}\frac{\partial X_i}{\partial t} + \sum_{k=1}^{n} c_{iik}\eta_k \frac{du}{dt}, \\ &= \lambda_i b_{ii} + \sum_{k=1}^{n} \big((\lambda_i - \lambda_k)w_k - \eta_k G(x,u)u\big)c_{iik}. \end{aligned} \tag{2.3.29}$$

Similarly, the evolution of η_i along the i-th characteristic is given by

$$\begin{aligned}\frac{d\eta_i}{dt} &= \sum_{k\neq i}\frac{b_{ik}}{\lambda_i-\lambda_k}((\eta_k\cdot\eta_i)\eta_i-\eta_k)\frac{\partial X_i}{\partial t}+\sum_{\substack{k,m\\k\neq i}}\frac{1}{\lambda_k-\lambda_i}c_{ikm}(\eta_m\frac{du}{dt})((\eta_k\cdot\eta_i)\eta_i-\eta_k),\\ &= \sum_{k\neq i}\frac{\lambda_i}{\lambda_i-\lambda_k}b_{ik}((\eta_k\cdot\eta_i)\eta_i-\eta_k)+\sum_{\substack{k,m\\k\neq i}}\left(\frac{(\lambda_i-\lambda_m)}{\lambda_k-\lambda_i}c_{ikm}w_m((\eta_k\cdot\eta_i)\eta_i-\eta_k)-\right.\\ &\left.-\frac{c_{ikm}}{\lambda_k-\lambda_i}\eta_m G(x,u)u((\eta_k\cdot\eta_i)\eta_i-\eta_k)\right).\end{aligned} \tag{2.3.30}$$

This leads us to the evolution of w_i along the i-th characteristic as

$$\begin{aligned}\frac{dw_i}{dt} &= \eta_i(u_{xt}+\lambda_i u_{xx})+\frac{d\eta_i}{dt}u_x\\ &= \eta_i((\lambda_i-\mathcal{A})w_x-\mathcal{A}_x w)+\frac{d\eta_i}{dt}w-\eta_i(G(x,u)u)_x.\end{aligned} \tag{2.3.31}$$

As η_i was assumed to be a left eigenvector of $\mathcal{A}$ with eigenvalue λ_i, we are left with

$$\frac{dw_i}{dt}=\sum_{k=1}^{n}(\frac{d\eta_i}{dt}\xi^k)w_k-\sum_{k=1}^{n}\eta_i\mathcal{A}_x\xi^k w_k-\eta_i(G_x(x,u)u+G_U(x,u)(u,w)+G(x,u)w). \tag{2.3.32}$$

Plugging in (2.3.30), we find that

$$\begin{aligned}\frac{dw_i}{dt} &= \sum_{k\neq i}\frac{b_{ik}\lambda_i}{\lambda_k-\lambda_i}w_k+\sum_{\substack{k,m\\k\neq i}}\frac{\lambda_i-\lambda_m}{\lambda_k-\lambda_i}c_{ikm}((\eta_i\cdot\eta_k)w_m w_i-w_k w_m)=\\ &-\sum_{k=1}^{n}(\eta_i\mathcal{A}_x\xi^k+\eta_i G\xi^k+\eta_i G_U(u,\xi^k))w_k-\\ &-\sum_{\substack{k,m\\k\neq i}}\frac{c_{ikm}}{\lambda_k-\lambda_i}\eta_m G(x,u)u((\eta_i\cdot\eta_k)w_i-w_k)-\eta_i G_x(x,u)u.\end{aligned} \tag{2.3.33}$$

Now plugging in the expressions from (2.3.14) and (2.3.15), we get

$$\begin{aligned}\frac{dw_i}{dt} &= \sum_{k\neq i}\frac{b_{ik}\lambda_i}{\lambda_k-\lambda_i}w_k+\sum_{\substack{k,m\\k\neq i}}\frac{\lambda_i-\lambda_m}{\lambda_k-\lambda_i}c_{ikm}((\eta_i\cdot\eta_k)w_m w_i-w_k w_m)-\\ &-\sum_{k=1}^{n}b_{ik}w_k-\sum_{k,m}c_{ikm}w_k w_m-\sum_{k}(\eta_i G\xi^k+\eta_i G_U(u,\xi^k))w_k-\\ &-\sum_{\substack{k,m\\k\neq i}}\frac{c_{ikm}}{\lambda_k-\lambda_i}\eta_m G(x,u)u((\eta_i\cdot\eta_k)w_i-w_k)-\eta_i G_x(x,u)u.\end{aligned} \tag{2.3.34}$$

Introducing coefficients γ_{ikm} as in [Joh74], and new coefficients ζ_{ik} and κ_i we get

$$\frac{dw_i}{dt} = \sum_{k=1}^{n} \zeta_{ik} w_k + \sum_{k,m} \gamma_{ikm} w_k w_m + \kappa_i u. \tag{2.3.35}$$

We note that each ζ_{ik} and κ_i satisfies

$$|\partial_x^\alpha \partial_u^\beta \zeta_{ik}(x,u)|, |\partial_x^\alpha \partial_u^\beta \kappa_i(x,u)| \leqslant C_{\alpha\beta} e^{-c|x|} \tag{2.3.36}$$

for $c > 0$ the same constant as in (2.3.20), coming from the exponential decay of B and G.

Remark 1. *Comparing with [Bär22], our characteristic equation* (2.3.35) *takes a very similar form with the key difference being that here we have a forcing term independent of w. The forcing term arises due to the spatial inhomogeneity induced by linearizing* (2.3.3) *about a fixed non-constant solution.*

For initial data which is compactly supported in an interval $[\alpha_0, \beta_0]$, we let $\mathcal{R}_i$ denote the region foliated by the characteristics, i.e.

$$\mathcal{R}_i = \{(X_i(x,t), t) : x \in [\alpha_0, \beta_0], 0 \leqslant t \leqslant T\}. \tag{2.3.37}$$

We also define extremal characteristics

$$\begin{aligned} \alpha_i(t) &= X_i(\alpha_0, t), \\ \beta_i(t) &= X_i(\beta_0, t). \end{aligned} \tag{2.3.38}$$

As in the unperturbed case, when σ defined in (2.3.24) is positive, there exists a $t_0 > 0$ so that $\mathcal{R}_i \cap \mathcal{R}_j \subset \{(x,t) : 0 \leqslant t \leqslant t_0\}$. We choose the smallest such t_0 with this property, which we note is comparable to the initial width of the support $s_0 := \beta_0 - \alpha_0$.

2.3.3 Estimates

We let $u(x,t)$ be a C^2 solution to (2.3.9), which a priori remains bounded by δ for all $0 \leqslant t \leqslant T$ and all x. Assume that the initial data is small in the sense that

$$\theta := \sup_x s_0^2 |u_{xx}(x,0)| \ll 1. \tag{2.3.39}$$

In particular, assume that $\theta \leqslant \frac{1}{2}\delta$. We may further assume that there exists an index i and a $y \in [\alpha_0, \beta_0]$ so that $w_i(y,0) = \max_i \sup_x |w_i(x,0)| = W_0$. A simple consequence of

the fundamental theorem of calculus is that

$$s_0 W_0 = \mathcal{O}(\theta). \tag{2.3.40}$$

In this section, $X = \mathcal{O}(Y)$ will mean that $X \leqslant CY$ for some constant C that only depends on the matrices $A(u)$, $B(x,u)$, $G(x,u)$, δ and ε for θ sufficiently small.

Remark 3. *The argument in John does not make the assumption that W_0 is achieved by some i and y with $w_i(y,0) > 0$ in order to show that the blowup is generic. As the initial data we will show blowup for will not be generic due to being spatially supported far from $x = 0$, we are free to assume that W_0 is achieved when w_i is positive. As another remark, the coefficient γ_{iii} that leads to Riccati-type blowup can be arranged to be positive by changing the sign of η_i, ξ^i. If one assumes that γ_{iii} is negative, then one assumes that there is an i and a y such that $-w_i(y,0) = \max_i \sup_x |w_i(x,0)|$.*

The overall plan will be to follow the argument in [Joh74] as closely as possible while keeping track of the dependence on the constants on the distance of the support of $u(x,0)$ to 0 and the time of existence T. To briefly outline the argument, we begin by proving suitable bounds on u and the w_i and the maximal width of the regions $\mathcal{R}_i$. We then use these bounds to show that a specific w_i along a suitable characteristic blows up by comparison with a Riccati equation. We are going to allow ourselves the freedom to choose the distance of the support from 0 on parameters associated to the initial data, such as W_0, s_0, and ultimately T as well. What we are not allowed to do is allow the distance to depend on information about the solution for $t > 0$. We will then construct a specific initial data, originally supported on $[-\frac{1}{2}, \frac{1}{2}]$, and then scale it by θ and translate it out far enough so that all of the necessary distance conditions are satisfied.

Before we begin, we will choose a constant $\mathcal{E}(d,T)$ which only depends on the distance d of the support of $u(x,0)$ from 0 and the desired time of existence T such that $\mathcal{E}(d,T)$ controls all quantities derived from $B(x,u)$ and a sufficient number of their derivatives uniformly on $\operatorname{supp}(u(x,0)) + [-\overline{\lambda}T, \overline{\lambda}T]$ and in L^1. In a similar fashion, we choose $\mathcal{G}(d,T)$ to be a constant which controls all quantities derived from $G(x,u)$ and a sufficient number of derivatives uniformly on $\operatorname{supp}(u(x,0)) + [-\overline{\lambda}T, \overline{\lambda}T]$ and in L^1. In particular, we may choose a sufficiently large positive integer k and a large constant $C > 0$ depending only on $\mathcal{A}(x,u)$, $\mathcal{G}(x,u)$, δ and ε so that

$$\begin{aligned}
\mathcal{E}(d,T) &= C(||B||_{W^{k,\infty}(\{x<\beta_0+\overline{\lambda}T\}\times\{|u|\leqslant\delta\})} + \sup_{|u|\leqslant\delta} ||B(\cdot,u)||_{W^{k,1}((-\infty,\beta_0+\overline{\lambda}T))}), \\
\mathcal{G}(d,T) &= C(||G||_{W^{k,\infty}(\{x<\beta_0+\overline{\lambda}T\}\times\{|u|\leqslant\delta\})} + \sup_{|u|\leqslant\delta} ||G(\cdot,u)||_{W^{k,1}(-\infty,\beta_0+\overline{\lambda}T)}),
\end{aligned} \tag{2.3.41}$$

provides the needed control on B and G when $[\alpha_0, \beta_0] \subset (-\infty, 0)$ and a similar convention if $[\alpha_0, \beta_0] \subset (0, \infty)$. We remark that due to the exponential decay in $\underline{u}$, we have the bounds

$$\mathcal{E}(d,T), \mathcal{G}(d,T) \leqslant Ce^{-cd+cT}, \tag{2.3.42}$$

for C, c both finite positive constants. In particular, we emphasize that $\mathcal{E}$ and $\mathcal{G}$ are only small when $d \gg T$. Indeed, if d is too small relative to T then it is possible to prevent blow-up as we saw in Theorem 11 for the Majda model.

Define constants

$$W = \sup_i \sup_{\substack{(x,t) \\ 0 \leqslant t \leqslant T}} |w_i(x,t)|, \tag{2.3.43a}$$

$$V = \sup_i \sup_{\substack{(x,t) \notin \mathcal{R}_i \\ 0 \leqslant t \leqslant T}} |w_i(x,t)|, \tag{2.3.43b}$$

$$U = \sup_{\substack{(x,t) \\ 0 \leqslant t \leqslant T}} |u(x,t)|, \tag{2.3.43c}$$

$$S = \sup_i \sup_{0 \leqslant t \leqslant T} (\beta_i(t) - \alpha_i(t)), \tag{2.3.43d}$$

$$J = \sup_i \sup_{0 \leqslant t \leqslant T} \int_{\alpha_i(t)}^{\beta_i(t)} |w_i(x,t)| dx. \tag{2.3.43e}$$

Lemma 2. *Let $d := dist([\alpha_0, \beta_0], 0)$ be the distance of the support of $u(x,0)$ from 0. Then if $d \gtrsim \max\{1, T, |\log(s_0)|, |\log(W_0)|\}$ is such that*

$$\mathcal{E}(d,T) \leqslant \min\{1, s_0, \tilde{c}W_0\} \tag{2.3.44}$$

for some $0 < \tilde{c} \ll 1$, then one can bound U and S by

$$\begin{aligned} U &= \mathcal{O}(s_0 V + TV + J), \\ S &= \mathcal{O}(s_0 + T\mathcal{E}(d,T) + VTS + JT). \end{aligned} \tag{2.3.45}$$

Proof. We recall the proof of the estimate on U from [Joh74]. To bound u, one starts with the observation that $u(x,t)$ is supported on $[\alpha_n(t), \beta_1(t)]$ for each t. Then applying the fundamental theorem of calculus gives

$$u(x,t) = \int_{\alpha_n(t)}^{x} \sum_{k=1}^{n} w_k(y,t) \xi^k(y,t) dy, \tag{2.3.46}$$

and then bounding $u(x,t)$ by

$$|u(x,t)| \leqslant \max_k \sup_{\substack{(x,u)\\|u|\leqslant\delta}} |\xi^k(x,u)| \sum_{i=1}^{n} \int_{\alpha_n(t)}^{\beta_1(t)} |w_i(x,t)|dx. \tag{2.3.47}$$

For each i we may write this as

$$|u(x,t)| \leqslant C \sum_{i=1}^{n} \Big(\int_{\alpha_n(t)}^{\alpha_i(t)} |w_i(x,t)|dx + \int_{\alpha_i(t)}^{\beta_i(t)} |w_i(x,t)|dx + \int_{\beta_i(t)}^{\beta_1(t)} |w_i(x,t)|dx \Big). \tag{2.3.48}$$

The middle integral may be bounded by J by definition, and the other two integrals may be bounded by $(\alpha_i(t) - \alpha_n(t))V + (\beta_1(t) - \beta_i(t))V$ in a similar manner. But

$$|(\alpha_i(t)-\alpha_n(t))+(\beta_1(t)-\beta_i(t))| = |\beta_1(t)-\alpha_n(t)-(\beta_i(t)-\alpha_i(t))| \leqslant 2|\beta_1(t)-\alpha_n(t)|, \tag{2.3.49}$$

leading to

$$|u(x,t)| \leqslant 2|\beta_1(t) - \alpha_n(t)|V + J. \tag{2.3.50}$$

To finish the estimate on U, we note that

$$\Big|\frac{d(\beta_1(t) - \alpha_n(t))}{dt}\Big| = \big|\lambda_1(\beta_1(t), u(\beta_1(t),t)) - \lambda_n(\alpha_n(t), u(\alpha_n(t),t))\big| \leqslant 2\overline{\lambda}, \tag{2.3.51}$$

which gives the estimate $|\beta_1(t) - \alpha_n(t)| \leqslant s_0 + 2\overline{\lambda}T$.

The estimate for S works slightly differently from the original argument in [Joh74]. We start in the same way by looking at

$$\frac{d(\beta_i(t) - \alpha_i(t))}{dt} = \lambda_i(\beta_i(t), u(\beta_i(t),t)) - \lambda_i(\alpha_i(t), u(\alpha_i(t),t)). \tag{2.3.52}$$

Applying the fundamental theorem of calculus gives

$$\frac{d(\beta_i(t) - \alpha_i(t))}{dt} = \int_{\alpha_i(t)}^{\beta_i(t)} \frac{\partial}{\partial x}\lambda_i(x, u(x,t))dx. \tag{2.3.53}$$

Recalling the expansion for $d\lambda_i$ from (2.3.18), we find that

$$\frac{d(\beta_i(t) - \alpha_i(t))}{dt} = \int_{\alpha_i(t)}^{\beta_i(t)} b_{ii}(x, u(x,t)) + \sum_{m=1}^{n} c_{iim}(x, u(x,t))w_m(x,t)dx. \tag{2.3.54}$$

First, we bound

$$\Big|\int_{\alpha_i(t)}^{\beta_i(t)} b_{ii}(x,u(x,t))dx\Big| \leqslant C\int_{\alpha_i(t)}^{\beta_i(t)} e^{-c|x|}dx \leqslant C\big(e^{-c|\beta_i(t)|}-e^{-c|\alpha_i(t)|}\big) \leqslant \mathcal{E}(d,T), \quad (2.3.55)$$

Choosing α_0, β_0 sufficiently far from 0, depending on T, one may ensure that

$$\int_0^t \Big|\int_{\alpha_i(s)}^{\beta_i(s)} b_{ii}(x,u(x,s))dx\Big|ds \leqslant T\mathcal{E}(d,T) \quad (2.3.56)$$

for all $0 \leqslant t \leqslant T$.

Turning to the sum in (2.3.54), we split into two cases depending on whether $t \leqslant t_0$ or $t \geqslant t_0$ holds. We start with the latter case, where we see that

$$\Big|\int_{\alpha_i(t)}^{\beta_i(t)} \sum_{m=1}^{n} c_{iim}(x,u)w_m(x,t)dx\Big| \leqslant \max_{i,m}\sup_{\substack{(x,u)\\|u|\leqslant\delta}} |c_{iim}(x,u)| \sum_{m=1}^{n}\int_{\alpha_i(t)}^{\beta_i(t)} |w_m(x,t)|dx. \quad (2.3.57)$$

To use the assumption $t \geqslant t_0$, we recall that $t \geqslant t_0$ implies that (x,t) can be in at most one $\mathcal{R}_k$. As the integral in (2.3.57) is over the set of $(x,t) \in \mathcal{R}_k$ with a fixed value of t, we can further bound (2.3.57) by

$$(n-1)\int_{\alpha_i(t)}^{\beta_i(t)} V dx + \int_{\alpha_i(t)}^{\beta_i(t)} |w_i(x,t)|dx \leqslant C(SV+J). \quad (2.3.58)$$

To handle $t \leqslant t_0$, we first show w_i remains under control for $t \leqslant t_0$. To do this, we first introduce auxiliary parameters Z and Γ as follows

$$\begin{aligned} K(d,T) &= \sup_{\substack{x\in[\alpha_0-\bar{\lambda}T,\beta_0+\bar{\lambda}T]\\|u|\leqslant\delta}} \sum_{i=1}^{n} |\kappa_i(x,u)|,\\ Z &= \sup_{\substack{(x,u)\\|u|\leqslant\delta}} \sum_{ik} |\zeta_{ik}(x,u)|,\\ \Gamma &= \sup_{\substack{(x,u)\\|u|\leqslant\delta}} \sum_{ikm} |\gamma_{ikm}(x,u)|. \end{aligned} \quad (2.3.59)$$

Applying a Gronwall type estimate to the characteristic equation for w_i, we get

$$|w_i(x,t)| \leqslant y(t), \quad (2.3.60)$$

where $y(t)$ satisfies

$$\begin{aligned} y'(t) &= Zy(t) + \Gamma y(t)^2 + K(d,T)U, \\ y(0) &= W_0. \end{aligned} \tag{2.3.61}$$

Then we claim that there exists a $\overline{W} > 0$ and $0 < C < \infty$ depending only on Z, Γ, δ, the support of $u(x,0)$ and t_0 such that

$$y(t) \leqslant CW_0 \tag{2.3.62}$$

holds for all $t \leqslant t_0$ and all $0 \leqslant W_0 \leqslant \overline{W}$. We emphasize that this is a short-time bound on the solution $y(t)$. Noting that we have $U \leqslant \delta$ a priori, we are free to bound $UK(d,T)$ by $\tilde{c}\delta W_0$ for some $\tilde{c}$ sufficiently small. In particular, we can choose $\tilde{c}$ so small that $\tilde{c}\delta W_0 \leqslant ZW_0$. Hence, at time 0 $K(d,T)U \leqslant Zy(0)$; but since the right hand side of (2.3.61) is positive, the bound $K(d,T)U \leqslant Zy(t)$ persists for longer time. This allows us to bound the solution $y(t)$ by the solution $\tilde{y}$ of

$$\begin{aligned} \tilde{y}'(t) &= 2Z\tilde{y}(t) + \Gamma\tilde{y}(t)^2, \\ \tilde{y}(0) &= W_0. \end{aligned} \tag{2.3.63}$$

The claim can be proven by writing $Y(t) = \tilde{y}(t)e^{-2Zt}$ for $t \leqslant t_0$ and noting that $Y(t)$ solves the weighted Riccati equation

$$Y'(t) = (\Gamma e^{2Zt})Y(t)^2. \tag{2.3.64}$$

Now comparing $Y(t)$ with the Riccati equation

$$\begin{aligned} \tilde{Y}(t) &= (\Gamma e^{2Zt_0})\tilde{Y}(t)^2, \\ \tilde{Y}(0) &= W_0, \end{aligned} \tag{2.3.65}$$

we see that $Y(t)$ is bounded by

$$Y(t) \leqslant \frac{W_0}{1 - \Gamma e^{2Zt_0} W_0 t}, \tag{2.3.66}$$

for W_0 so small that $0 \leqslant t \leqslant t_0 < (\Gamma e^{2Zt_0} W_0)^{-1}$ holds. In particular, since $t_0 W_0 = \mathcal{O}(s_0 W_0) = \mathcal{O}(\theta)$ and Γe^{2Zt_0} is a fixed constant depending only on s_0 and the model parameters, we see that for W_0 small enough that $Y(t) \leqslant 2W_0$ for all $0 \leqslant t \leqslant t_0$.

For $0 \leqslant t \leqslant t_0$, we recall (2.3.52)

$$\frac{d(\beta_i(t) - \alpha_i(t))}{dt} = \lambda_i(\beta_i(t), u(\beta_i(t), t)) - \lambda_i(\alpha_i(t), u(\alpha_i(t), t)). \tag{2.3.67}$$

We now add and subtract $\lambda_i(\beta_i(t), 0)$ and $\lambda_i(\alpha_i(t), 0)$ to get

$$\begin{aligned}\Big|\frac{d(\beta_i(t) - \alpha_i(t))}{dt}|\Big| \leqslant |\lambda_i(\beta_i(t), u(\beta_i(t), t)) - \lambda_i(\beta_i(t), 0))| + \\ + |\lambda_i(\beta_i(t), 0) - \lambda_i(\alpha_i(t), 0)| + |\lambda_i(\alpha_i(t), u(\alpha_i(t), t)) - \lambda_i(\alpha_i(t), 0)|.\end{aligned} \tag{2.3.68}$$

By hypothesis $u(\alpha_0, 0) = u(\beta_0, 0) = 0$, and so we can bound (2.3.68) by

$$\begin{aligned}\Big|\frac{d(\beta_i(t) - \alpha_i(t))}{dt}|\Big| \leqslant C(|u(\beta_i(t), t) - u(\beta_0, 0)| + |u(\alpha_i(t), t) - u(\alpha_0, 0)|) + \\ + |\lambda_i(\beta_i(t), 0) - \lambda_i(\alpha_i(t), 0)|.\end{aligned} \tag{2.3.69}$$

Finally, we are left with controlling $|\lambda_i(\beta_i(t), 0) - \lambda_i(\alpha_i(t), 0)|$. We do this by the fundamental theorem of calculus as follows

$$|\lambda_i(\beta_i(t), 0) - \lambda_i(\alpha_i(t), 0)| = \Big| \int_{\alpha_i(t)}^{\beta_i(t)} b_{ii}(s, 0) ds \Big| \leqslant C(e^{-c|\beta_i(t)|} - e^{-c|\alpha_i(t)|}). \tag{2.3.70}$$

We then see that $|\lambda_i(\beta_i(t), 0) - \lambda_i(\alpha_i(t), 0)| = \mathcal{O}(1)$ so that for $t \leqslant t_0$ one has by the characteristic equation for u and the Gronwall type bound on w_i in (2.3.62)

$$\Big|\frac{d(\beta_i(t) - \alpha_i(t))}{dt}\Big| = \mathcal{O}(1 + t_0 W_0), \tag{2.3.71}$$

after choosing d large enough that the reaction term in (2.3.28) satisfies $\delta \mathcal{G}(d, T) \lesssim W_0$. Hence integrating and using (2.3.58) and (2.3.71) we get

$$\begin{aligned}|\beta_i(t) - \alpha_i(t)| \leqslant & s_0 + T\mathcal{E}(d, T) + \mathcal{O}(t_0 + t_0^2 W_0) + T\mathcal{O}(SV + J) \\ = & \mathcal{O}(s_0 + T\mathcal{E}(d, T) + TSV + TJ),\end{aligned} \tag{2.3.72}$$

as $t_0^2 W_0 = \mathcal{O}(s_0^2 W_0) = \mathcal{O}(\theta s_0)$ can be absorbed into $\mathcal{O}(s_0)$. □

Remark 4. *In John's argument, in the final step one assumes that $T \sim \theta^{-1} \gg 1$, which will ensure that all the distance conditions stated at the end of Lemma 2 hold.*

Proposition 8. *If we choose the distance d so large that*

$$\mathcal{E}(d, T) \leqslant \min\{1, s_0, \tilde{c} W_0\}, \tag{2.3.73}$$

holds for $\tilde{c}$ the same small constant as before and that

$$\mathcal{G}(d,T) \leqslant W_0^2, \tag{2.3.74}$$

holds as well, then we may estimate J by

$$J = \mathcal{O}(s_0 W_0 + (V + \mathcal{E}(d,T))TJ + TV(V + \mathcal{E}(d,T))S + T\mathcal{G}(d,T)US). \tag{2.3.75}$$

Proof. Following John, we introduce two new quantities

$$\begin{aligned} \rho_i(z,t) &= \frac{\partial X_i}{\partial z}(z,t), \\ v_i(z,t) &= w_i(X_i(z,t),t)\rho_i(z,t). \end{aligned} \tag{2.3.76}$$

One should think of ρ_i as some measure of the density of the ith characteristics. By a change of variables computation, one discovers that J can be computed in terms of the v_i by

$$J = \sup_i \sup_{0 \leqslant t \leqslant T} \int_{\alpha_i(t)}^{\beta_i(t)} |w_i(x,t)| dx = \sup_i \sup_{0 \leqslant t \leqslant T} \int_{\alpha_0}^{\beta_0} |v_i(z,t)| dz. \tag{2.3.77}$$

From the characteristic equation for X_i given in (2.3.21) and the expression for $d\lambda_i$ in (2.3.18), we are lead to the evolution equation for ρ_i

$$\begin{aligned} \frac{\partial}{\partial t}\rho_i = \frac{\partial}{\partial z}\frac{\partial X_i}{\partial t} &= b_{ii}\rho_i + \sum_{m=1}^{n} c_{iim} w_m \rho_i, \\ \rho_i(z,0) &= 1. \end{aligned} \tag{2.3.78}$$

It is clear that $\tilde{\rho}_i(z,t) = 0$ for all t is a solution to the first equation of (2.3.78). This ensure that our desired solution $\rho_i(z,t) > 0$ for all t by uniqueness of solutions, as the initial data is nowhere vanishing. An important consequence of $\rho_i > 0$ is that it we may look at the time evolution for $\log(\rho_i)$ given by

$$\frac{\partial}{\partial t}\log(\rho_i) = b_{ii} + \sum_{m=1}^{n} c_{iim} w_m. \tag{2.3.79}$$

By imposing the additional constraint $\operatorname{dist}([\alpha_0, \beta_0], 0) \gtrsim \max\{1, T, |\log(W_0)|\}$, we may ensure that $|b_{ik}(z,t)| \lesssim W_0$ on $[\alpha_0 - \overline{\lambda}T, \beta_0 + \overline{\lambda}T]$ for all i, k. From the short time estimate on the w_i and th distance assumption, we find that for $0 \leqslant t \leqslant t_0$ that

$$\frac{\partial}{\partial t}\log(\rho_i) = \mathcal{O}(W_0), \tag{2.3.80}$$

which upon integrating in time leads to

$$\log(\rho_i) = \mathcal{O}(t_0 W_0) = \mathcal{O}(\theta). \tag{2.3.81}$$

Exponentiating (2.3.81) leads to the bound for ρ_i

$$\rho_i = \mathcal{O}(1), \tag{2.3.82}$$

for $0 \leqslant t \leqslant t_0$.

Turning to v_i, we first write down the evolution equation as

$$\frac{\partial v_i}{\partial t} = \big(\sum_{k=1}^{n} \zeta_{ik} w_k + \sum_{k,m} \gamma_{ikm} w_k w_m + \kappa_i u \big) \rho_i + w_i \big(b_{ii} \rho_i + \sum_{m=1}^{n} c_{iim} w_m \rho_i \big). \tag{2.3.83}$$

By using the Kronecker δ, (2.3.83) can be more succintly written as

$$\frac{\partial v_i}{\partial t} = \sum_{k=1}^{n} (\zeta_{ik} + \delta_{ik} b_{ik}) \rho_i w_k + \sum_{k,m} (\gamma_{ikm} + \delta_{ik} c_{ikm}) w_k w_m \rho_i + \kappa_i \rho_i u. \tag{2.3.84}$$

For $0 \leqslant t \leqslant t_0$, appealing to the Gronwall type estimate (2.3.62), the distance condition on the support, and the estimate on ρ_i in (2.3.82), we find that

$$\frac{\partial v_i}{\partial t} = \mathcal{O}(\rho_i W_0^2 + \rho_i \mathcal{G}(d,T) U) = \mathcal{O}(W_0^2 + \mathcal{G}(d,T) U). \tag{2.3.85}$$

By assumption, $U \leqslant \delta$ so by choosing the support far enough, $\mathcal{G}(d,T)U \leqslant \delta W_0^2$, refining the short time estimate to

$$\frac{dv_i}{dt} = \mathcal{O}(W_0^2). \tag{2.3.86}$$

For $t \geqslant t_0$, we make the crucial observation that $\frac{\partial v_i}{\partial t}$ does not include w_i^2 in the sum over k, m as $\gamma_{iii} + c_{iii} = 0$. Moreover, since $(x,t) = (X_i(z,t),t)$ can only lie in one $\mathcal{R}_i$ for $t \geqslant t_0$, we get the estimate on v_i for $t \geqslant t_0$

$$\begin{aligned} \frac{\partial v_i}{\partial t}(z,t) &= \mathcal{O}(\mathcal{E}(d,T)\rho_i(|w_i| + V) + \rho_i |w_i| V + \rho_i V^2 + \rho_i \mathcal{G}(d,T) U) + \\ &= \mathcal{O}((V + \mathcal{E}(d,T))|v_i(z,t)| + (V + \mathcal{E}(d,T)) V \rho_i + \rho_i \mathcal{G}(d,T) U). \end{aligned} \tag{2.3.87}$$

Integrating with respect to time, we can bound v_i by

$$\begin{aligned}|v_i(z,t)| \leqslant |w_i(z,0)| + C\Big((V+\mathcal{E}(d,T))\int_0^T |v_i(z,s)|ds + \\ + \big(V(V+\mathcal{E}(d,T)) + \mathcal{G}(d,T)U\big)\int_0^T \rho_i(z,s)ds + t_0 W_0^2\Big),\end{aligned} \tag{2.3.88}$$

with the constant C uniform in $z \in [\alpha_0, \beta_0]$ and $0 \leqslant t \leqslant T$. Integrating in the spatial variable z, we get after applying Fubini's theorem

$$\begin{aligned}\int_{\alpha_0}^{\beta_0} |v_i(z,t)|dz \leqslant \int_{\alpha_0}^{\beta_0} |w_i(z,0)|dz + C\Big((V+\mathcal{E}(d,T))\int_0^T\int_{\alpha_0}^{\beta_0} |v_i(z,s)|dzds + \\ + \big(V(V+\mathcal{E}(d,T)) + \mathcal{G}(d,T)U\big)\int_0^T (\beta_i(s)-\alpha_i(s))ds + s_0^2 W_0^2\Big).\end{aligned} \tag{2.3.89}$$

The trivial bound $||f||_{L^1([0,T])} \leqslant T||f||_{L^\infty([0,T])}$ produces our final estimate for J, namely

$$J = \mathcal{O}(s_0 W_0 + (V+\mathcal{E}(d,T))TJ + TV(V+\mathcal{E}(d,T))S + T\mathcal{G}(d,T)US), \tag{2.3.90}$$

as $s_0 W_0$ small allows us to absorb $s_0^2 W_0^2$ into $s_0 W_0$. □

The estimate on J that we have looks slightly different from the estimate on J in John's argument, however, we've assumed that $\mathcal{E}(d,T) \leqslant W_0$. In the final step, we will start by assuming that $s_0 V = \mathcal{O}(\theta)$, and so $\mathcal{E}(d,T)$ will at least be a priori comparable to V provided $s_0 = \mathcal{O}(1)$. Note also the presence of the lower order term $T\mathcal{G}(d,T)US$.

The final estimate needed is for the estimate for V.

Proposition 9. *Suppose that* $\operatorname{dist}([\alpha_0,\beta_0],0) \gtrsim \max\{1,T,|\log(W_0)|\}$ *holds. Then one can estimate* V *by*

$$V = \mathcal{O}(s_0 W_0^2 + (V+\mathcal{E}(d,T))(VT + (V+\mathcal{E}(d,T))TJ + TV(V+\mathcal{E}(d,T))S) + \mathcal{G}(d,T)TU). \tag{2.3.91}$$

As in Lemma 2 and Proposition 8, we assume that $d \gtrsim \max\{1,T,|\log(W_0)|\}$ *so that*

$$\mathcal{E}(d,T) \leqslant \min\{1, s_0, \tilde{c}W_0, W_0^2\}, \tag{2.3.92}$$

and

$$\mathcal{G}(d,T) \leqslant W_0^2, \tag{2.3.93}$$

hold.

Proof. Let $(x,t) \notin \mathcal{R}_i$. Then there exists a $z \notin [\alpha_0, \beta_0]$ such that $(x,t) = (X_i(z,t),t)$. As z is outside the support of $u(x,0)$, we necessarily have $w(z,0) = 0$. Moreover, we can assume that z is not too far from $[\alpha_0, \beta_0]$ in the sense that $z \in [\alpha_0 - \overline{\lambda}T, \beta_0 + \overline{\lambda}T]$, as outside this larger interval w is identically zero. Integrating w_i along the characteristic starting at z, we find

$$w_i(x,t) = \int_0^t \sum_k \zeta_{ik}(X_i(z,\tau), u(X_i(z,\tau),\tau))w_k + \sum_{k,m} \gamma_{ikm}(X_i(z,\tau), u(X_i(z,\tau),\tau))w_k w_m + \\ + \kappa_i(X_i(z,\tau), u(X_i(z,\tau)))u(X_i(z,\tau),\tau)d\tau \tag{2.3.94}$$

As before, we split into two cases depending on how t compares to t_0. If $t \leqslant t_0$, then we have that the integrand of (2.3.94) is at most W_0^2 giving us the short time estimate

$$|w_i(z,t)| = \mathcal{O}(t_0 W_0^2). \tag{2.3.95}$$

In the other case, we introduce sets ω_k defined by

$$\omega_k = \{0 \leqslant \tau \leqslant T : (X_i(z,\tau),\tau) \in \mathcal{R}_k\}. \tag{2.3.96}$$

By assumption ω_i is empty. If $\tau \geqslant t_0$, then τ is contained in at most one $\mathcal{R}_k$. For that specific k, $|w_k||w_m| \leqslant V|w_k|$ otherwise $|w_k||w_m| \leqslant V^2$. For reaction term $\kappa_i u$, one can do little better than

$$\Big| \int_0^t \kappa_i(X_i(z,\tau), u(X_i(z,\tau),\tau))u(X_i(z,\tau),\tau)d\tau \Big| \leqslant \mathcal{G}(d,T)TU. \tag{2.3.97}$$

For the linear terms, we use the distance on the support to bound them by

$$\Big| \int_0^t \sum_{k=1}^n \zeta_{ik} w_k d\tau \Big| \leqslant \mathcal{E}(d,T) \sum_k \Big(\int_{\omega_k} |w_k| d\tau + \int_{[0,T]\setminus\omega_k} |w_k| d\tau \Big). \tag{2.3.98}$$

Outside of ω_k one has $|w_k| \leqslant V$, and so (2.3.98) is controlled by

$$\Big| \int_0^t \sum_{k=1}^n \zeta_{ik} w_k d\tau \Big| \leqslant \mathcal{E}(d,T)VT + \mathcal{E}(d,T) \sum_{k=1}^n \int_{\omega_k} |w_k(X_i(z,\tau),\tau)| d\tau. \tag{2.3.99}$$

We may bound the quadratic terms of (2.3.94) in a similar manner by

$$\Big| \sum_{k,m} \gamma_{ikm}(X_i(z,\tau), u(X_i(z,\tau),\tau))w_k w_m d\tau \Big| \leqslant C(V^2 T + V \sum_{k=1}^n \int_{\omega_k} |w_k(X_i(z,\tau),\tau)| d\tau). \tag{2.3.100}$$

Thus, we have reduced the problem of estimating V to estimating

$$\int_{\omega_k} |w_k(X_i(z,\tau),\tau)|d\tau. \tag{2.3.101}$$

We would like to exchange the integral with respect to the time variable τ for an integral with respect to space, as our setup gives us pointwise control on w_k outside $\mathcal{R}_k$ and spatial integrals of w_k in $\mathcal{R}_k$. To exchange the time variable for a suitable space variable in (2.3.101) fix k, and let $\tau \in \omega_k$. Then there exists some $y = y(\tau)$ so that

$$X_i(z,\tau) = X_k(y(\tau),\tau). \tag{2.3.102}$$

Differentiating (2.3.102) with respect to τ, we find that

$$\lambda_i = \lambda_k + \rho_k \frac{dy}{d\tau}. \tag{2.3.103}$$

As $\rho_k > 0$ and $\lambda_k - \lambda_i$ has a definite sign, $y(\tau)$ is a strictly monotone function of τ and is thus invertible. Changing variables in (2.3.101), we find that

$$\int_{\omega_k} |w_k(X_i(z,\tau),\tau)|d\tau = \int_{I_k} \frac{|w_k(X_k(y,\tau),\tau)|\rho_k(y,\tau)}{|\lambda_i - \lambda_k|}dy, \tag{2.3.104}$$

for some subinterval $I_k \subset [\alpha_0, \beta_0]$. By assumption, $|\lambda_i - \lambda_k|$ is bounded from below, so (2.3.104) can be bounded by

$$\int_{\omega_k} |w_k(X_i(z,\tau),\tau)|d\tau \leqslant C\int_{I_k} |v_k(y,\tau(y))|dy. \tag{2.3.105}$$

In the proof of the estimate (2.3.75), we obtained the pointwise bound (2.3.88) on v_k, valid for all $(y,\tau(y))$. Plugging this estimate into (2.3.105) produces

$$\begin{aligned}\int_{\omega_k} |w_k(X_i(z,\tau),\tau)|d\tau \leqslant C\int_{\alpha_0}^{\beta_0} \Big(|w_i(y,0)| + \Big((V+\mathcal{E}(d,T))\int_0^T |v_i(y,s)|ds + \\ + V(V+\mathcal{E}(d,T))\int_0^T \rho_i(y,s)ds + t_0W_0^2\Big)\Big)dy = \\ = \mathcal{O}(s_0W_0 + (V+\mathcal{E}(d,T))TJ + TV(V+\mathcal{E}(d,T))S).\end{aligned} \tag{2.3.106}$$

We've also expanded I_k into $[\alpha_0, \beta_0]$ as well in (2.3.106). At this point, following the same

procedure for the J estimate furnishes our final estimate on V

$$\begin{aligned} V = &\underbrace{W_0\mathcal{O}(VT + s_0W_0 + (V + \mathcal{E}(d,T))TJ + TV(V + \mathcal{E}(d,T))S)}_{\text{Linear contribution}} + \\ &+ \underbrace{V\mathcal{O}(VT + s_0W_0 + (V + \mathcal{E}(d,T))TJ + TV(V + \mathcal{E}(d,T))S)}_{\text{Quadratic contribution}} + \\ &+ \underbrace{\mathcal{G}(d,T)TU}_{\text{Reaction term}} + \underbrace{\mathcal{O}(s_0W_0^2)}_{\text{Short time}} = \\ &= \mathcal{O}(s_0W_0^2 + (V + \mathcal{E}(d,T))(VT + (V + \mathcal{E}(d,T))TJ + \\ &+ TV(V + \mathcal{E}(d,T))S) + \mathcal{G}(d,T)TU). \end{aligned} \tag{2.3.107}$$

Where we've used (2.3.40) to absorb s_0W_0V into the left hand side of (2.3.107). □

Putting these results together, we have

Theorem 13. *Suppose $u(x,t)$ is a C^2 solution, remaining bounded by δ, to (2.3.9) with initial data $u(x,0)$ compactly supported in $[\alpha_0, \beta_0]$ with*

$$\operatorname{dist}([\alpha_0, \beta_0], 0) \gtrsim \max\{1, T, |\log(s_0)|, |\log(W_0)|\}, \tag{2.3.108}$$

and satisfying

$$\max_x s_0^2 |u_{xx}(x,0)| = \theta \ll 1, \tag{2.3.109}$$

for $s_0 = \beta_0 - \alpha_0$. Then the following estimates hold

$$\begin{aligned} V &= \mathcal{O}(s_0W_0^2 + (V + \mathcal{E}(d,T))(VT + (V + \mathcal{E}(d,T))TJ + \\ &\quad + TV(V + \mathcal{E}(d,T))S) + \mathcal{G}(d,T)TU), && \text{(2.3.110a)} \\ U &= \mathcal{O}(s_0V + TV + J), && \text{(2.3.110b)} \\ S &= \mathcal{O}(s_0 + T\mathcal{E}(d,T) + VTS + JT), && \text{(2.3.110c)} \\ J &= \mathcal{O}(s_0W_0 + (V + \mathcal{E}(d,T))TJ + TV(V + \mathcal{E}(d,T))S + T\mathcal{G}(d,T)US). && \text{(2.3.110d)} \end{aligned}$$

2.3.4 Final Steps

We are now in a position to finish John's argument in showing blowup of some solution. We begin by constructing a suitable initial condition.

Lemma 3. *For a given index i and all θ small enough, there exists a nonzero function f which is C^2 with $\sup_x |f''(x)| \leqslant \theta$, compactly supported on an interval of width 1, and*

such that there exists a y in the support of f so that

$$\eta_i(y, f(y))f'(y) = \sup_i \sup_x |\eta_i(x, f(x))f'(x)|. \tag{2.3.111}$$

Proof. Let $\phi(x)$ be a scalar function which is smooth, not identically zero, and compactly supported in $[-1/2, 1/2]$. For simplicity assume that $||\phi''||_\infty = 1$. Let $f(x) = f_{i,\theta}(x) = \theta\phi(x - x_0)\xi^i_\infty(0)$, where $\xi^i_\infty(u)$ is given by

$$\xi^i_\infty(u) = \lim_{|x|\to\infty} \xi^i(x, u), \tag{2.3.112}$$

and $x_0 \in \mathbb{R}$ to be determined. By a continuity argument

$$\lim_{|x_0|\to\infty} \eta_i(x_0 + t, u)\xi^j_\infty(u) = \delta_{ij}, \tag{2.3.113}$$

with uniform convergence on $|t| \leqslant \frac{1}{2}$ and $|u| \leqslant \delta$. If we let $\eta^\infty_i(u)$ be defined a manner analogous to $\xi^i_\infty(u)$, then we note that

$$\eta^\infty_j(f(x_0 + t))\xi^i_\infty(0) = \delta_{ij} + \mathcal{O}(\theta), \tag{2.3.114}$$

for all $|t| \leqslant \frac{1}{2}$. In particular, what we find is that

$$\eta_j(x_0 + t, f(x_0 + t))\xi^i_\infty(0) \to \eta^\infty_j(\theta\phi(t)\xi^i_\infty(0))\xi^\infty_i(0) = \delta_{ij} + \mathcal{O}(\theta), \tag{2.3.115}$$

as $|x_0| \to \infty$. In particular, this implies that

$$\eta_j(x, f(x))f'(x) = \theta\phi'(x)\delta_{ij} + \mathcal{O}(\theta^2), \tag{2.3.116}$$

and so for $\theta \ll 1$, we have that there exists y in $\mathrm{supp}(f)$ such that

$$|\eta_i(y, f(y))f'(y)| = \sup_i \sup_x |\eta_i(x, f(x))f'(x)|, \tag{2.3.117}$$

and by sending $\phi \to -\phi$ if needed, we can remove the absolute value bars on the left hand side. □

Adapting a result of Lax in [Lax73] as in [Bär22], one can show that for all small smooth initial data, there is a unique classical solution to (2.3.9) on some time interval with the prescribed initial data. Our solution $u(x, t)$ will then be the unique smooth solution to (2.3.9) with the initial data constructed in Lemma 3.

We let T increase from 0 so that the inequalities

$$TW_0 \leqslant \max_i \frac{4}{\gamma_{iii}^{\infty}(0)} = \vartheta, \tag{2.3.118a}$$

$$TV \leqslant \sqrt{\theta}, \tag{2.3.118b}$$

$$J \leqslant \sqrt{\theta}, \tag{2.3.118c}$$

$$V \leqslant \theta, \tag{2.3.118d}$$

$$U \leqslant \sqrt{\theta}, \tag{2.3.118e}$$

hold.

For $\gamma_{iii}^{\infty}(0)$ the value of γ_{iii} with $B = 0$, or equivalently,

$$\gamma_{iii}^{\infty}(0) = \lim_{|x|\to\infty} \gamma_{iii}(x, 0). \tag{2.3.119}$$

Note that these inequalities are all valid at $T = 0$ for θ small enough since there $U = \mathcal{O}(\theta)$, $T = V = 0$, $J = \mathcal{O}(W_0) = \mathcal{O}(\theta)$. For the fourth estimate in Theorem 13, we may improve the estimate on J to

$$J = \mathcal{O}(W_0 + \sqrt{\theta}J + (T\mathcal{E}(d,T))J + \sqrt{\theta}(V + \mathcal{E}(d,T))S + T\mathcal{G}(d,T)US). \tag{2.3.120}$$

As we've chosen $d \sim \theta^{-1} \gg |\log(W_0)| \sim |\log(\theta)|$, we may assume that $\mathcal{E}(d,T) \leqslant \min\{1, c\vartheta W_0, s_0\}$ for some constant $c \ll 1$ so small that the term $\mathcal{O}(\mathcal{E}(d,T)T)J$ can be absorbed onto the left hand side, that is $\mathcal{O}(T\mathcal{E}(d,T)) \leqslant \frac{1}{2}$. As θ is small, the $\sqrt{\theta}J$ on the right hand side can be absorbed as well. This improves the J estimate to

$$J = \mathcal{O}(W_0 + \sqrt{\theta}VS + T\mathcal{G}(d,T)US). \tag{2.3.121}$$

To handle the reaction term, we note that $\mathcal{G}(d,T) \leqslant cW_0^2$ for some universal constant $c > 0$ sufficiently small, hence $T\mathcal{G}(d,T)U \lesssim W_0U$. This furthers improves the J estimate to

$$J = \mathcal{O}(W_0 + \sqrt{\theta}VS + W_0US) \tag{2.3.122}$$

From this, we find that JT is of the order

$$JT = \mathcal{O}(1 + \theta S + US). \tag{2.3.123}$$

Using the bounds $U \leqslant \sqrt{\theta}$ and (2.3.123) allows us to improve the S estimate to

$$S = \mathcal{O}(1 + T\mathcal{E}(d,T) + VTS + JT) = \mathcal{O}(1 + \sqrt{\theta}S + \theta S) = \mathcal{O}(1). \tag{2.3.124}$$

Plugging (2.3.124) into (2.3.122) produces our final refinement of the J estimate

$$J = \mathcal{O}(W_0 + \sqrt{\theta}V + W_0 U) = \mathcal{O}(\theta + \theta^{\frac{3}{2}}) = \mathcal{O}(\theta). \tag{2.3.125}$$

Next, we use our refined estimates on S and JT in (2.3.124) and (2.3.123) respectively, to refine the estimate on V.

$$V = \mathcal{O}(W_0^2 + (V + \mathcal{E}(d,T))(VT + (V + \mathcal{E}(d,T)) + TV(V + \mathcal{E}(d,T))) + \mathcal{G}(d,T)TU). \tag{2.3.126}$$

Applying the bound $TV \leqslant \sqrt{\theta}$ and using $\mathcal{O}(T\mathcal{E}(d,T))) \leqslant \frac{1}{2}$ as in the J estimate, we get a further refinement of the V estimate

$$V = \mathcal{O}(W_0^2 + \sqrt{\theta}V + (V + \mathcal{E}(d,T))^2 + \sqrt{\theta}(V + \mathcal{E}(d,T))^2 + \mathcal{G}(d,T)TU) + \frac{1}{2}V. \tag{2.3.127}$$

This then gives

$$V = \mathcal{O}(W_0^2 + (V + \mathcal{E}(d,T))^2 + \mathcal{G}(d,T)TU). \tag{2.3.128}$$

A priori, we have $V, W_0 = \mathcal{O}(\theta)$ and $\mathcal{E}(d,T) \lesssim W_0$, allowing us to absorb the $\mathcal{E}(d,T)V$ and V^2 terms of (2.3.128) into the left hand side; hence we conclude

$$V = \mathcal{O}(W_0^2 + \mathcal{G}(d,T)TU) = \mathcal{O}(W_0^2 + W_0 U). \tag{2.3.129}$$

From (2.3.128), we may improve the estimate on TV to

$$TV = \mathcal{O}(TW_0^2 + \mathcal{G}(d,T)T^2U) = \mathcal{O}(TW_0^2 + T^2W_0^2U). \tag{2.3.130}$$

Using $TW_0 \leqslant \vartheta = \mathcal{O}(1)$, we get the refinement

$$TV = \mathcal{O}(TW_0^2 + T^2W_0^2U) = \mathcal{O}(W_0 + U). \tag{2.3.131}$$

The final estimate to improve is the U estimate. The improvement is obtained by using (2.3.129), (2.3.131), and (2.3.122)

$$U = \mathcal{O}(s_0 V + TV + J) = \mathcal{O}(W_0^2 + W_0 U + W_0 + cU + J) = \mathcal{O}(\theta), \tag{2.3.132}$$

by using the small constant in $\mathcal{G}(d,T) \leqslant cW_0^2$ to move the U on the right hand side to the left hand side of (2.3.132). To summarize, we've bounced the estimates in (2.3.118) off of each other in order to get the better estimates

$$TV \leqslant C\theta, \tag{2.3.133a}$$

$$J \leqslant C\theta, \tag{2.3.133b}$$

$$V \leqslant C\theta^2, \tag{2.3.133c}$$

$$U \leqslant C\theta. \tag{2.3.133d}$$

for some constant C depending only on the matrices A, B, the parameters δ and ε for θ small enough under the assumptions that d is sufficiently large and $s_0 = 1$.

Suppose $u(x,t)$ remains C^2 for all $0 \leqslant t \leqslant T$ for $T = \vartheta W_0^{-1}$ and that for some i there exists a $z \in [x_0 - \frac{1}{2}, x_0 + \frac{1}{2}]$ such that

$$w_i(z,0) = W_0. \tag{2.3.134}$$

We consider the evolution of $w_i(z,0)$ along the characteristic $X_i(z,t)$, that is we look at the function $w(t)$ defined by

$$w(t) = w_i(X_i(z,t),t). \tag{2.3.135}$$

Appealing to the characteristic equation for $w(t)$, the estimate (2.3.62), and the assumption $|x_0| \gtrsim |\log(W_0)|$ we find that

$$\left|\frac{dw}{dt}\right| \leqslant C(\mathcal{E}(d,T)W_0 + W_0^2) \leqslant CW_0^2, \tag{2.3.136}$$

for all $0 \leqslant t \leqslant t_0$. Note that the reaction term $\kappa_i u$ has been absorbed into the W_0^2 by using $U\mathcal{G}(d,T) \leqslant W_0^2$. Integrating with respect to time leads to the estimate

$$|w(t) - W_0| = |w(t) - w(0)| = \mathcal{O}(t_0 W_0^2) = \mathcal{O}(W_0^2) = \mathcal{O}(\theta W_0), \tag{2.3.137}$$

where we've used $t_0 = \mathcal{O}(s_0) = \mathcal{O}(1)$ and $W_0 = \mathcal{O}(\theta)$. In particular, provided that θ is small enough, $w(t)$ will satisfy

$$w(t) > \frac{3}{4}W_0 \qquad \text{for} \qquad 0 \leqslant t \leqslant t_0. \tag{2.3.138}$$

For each $0 \leqslant t \leqslant T$ and all x, we have by a mean value theorem type estimate that

$$|\gamma_{iii}(x, u(x,t)) - \gamma_{iii}(x,0)| = \mathcal{O}(U) = \mathcal{O}(\theta). \tag{2.3.139}$$

In addition, since the perturbation $B(x,u)$ decays exponentially in x, we see that by the

fundamental theorem of calculus

$$\gamma_{iii}(x,0) = \gamma_{iii}^{\infty}(0) + \int_{-\infty}^{x} \frac{\partial \gamma_{iii}(y,0)}{\partial y} dy = \gamma_{iii}^{\infty}(0) + \mathcal{E}(d,T). \tag{2.3.140}$$

Here, we've implicitly assumed that $[\alpha_0, \beta_0] \subset (-\infty, 0)$ for the sake of definiteness, but an entirely analogous calculation works in the other case and will thus be omitted. In either case, we have that

$$|\gamma_{iii}(x, u(x,t)) - \gamma_{iii}^{\infty}(0)| = \mathcal{O}(U + \mathcal{E}(d,T)) = \mathcal{O}(\theta). \tag{2.3.141}$$

This implies that, for θ small enough and all x in the support of u that

$$\gamma_{iii}(x, u(x,t)) > \frac{1}{2}\gamma_{iii}^{\infty}(0). \tag{2.3.142}$$

For $k \neq i$ and $t > t_0$ we have that $|w_k(X_i(z,t),t)| \leqslant V$. Using this information and (2.3.141) in the characteristic equation for w, we have

$$\frac{dw}{dt} > \frac{1}{2}\gamma_{iii}^{\infty}(0)w^2 - \mathcal{E}(d,T)|w| - n\mathcal{E}(d,T)V - \Gamma(V|w| + V^2) - K(d,T)U, \tag{2.3.143}$$

for Γ and $K(d,T)$ as in (2.3.59). At $t = t_0$, we can use $\mathcal{E}(d,T) \ll W_0 \lesssim w(t_0)$, $K(d,T) \lesssim W_0^2$, and $V = \mathcal{O}(\theta W_0)$ to conclude that for θ small enough, $\frac{dw}{dt}(t_0) > 0$. Hence $w(t)$ is increasing which implies that $V = \mathcal{O}(\theta|w|)$ persists for larger times. For θ small enough, we then have for $t > t_0$

$$\frac{dw}{dt} > \frac{3}{8}\gamma_{iii}^{\infty}(0)w^2. \tag{2.3.144}$$

However, the solution to this Riccati equations blows up at some time t satisfying

$$t \leqslant t_0 + \frac{8}{3\gamma_{iii}^{\infty}(0)w(t_0)} \leqslant t_0 + \frac{2}{3}T < T. \tag{2.3.145}$$

This final conclusion comes from the observation that

$$t_0 = \mathcal{O}(\frac{\theta}{W_0}) = \mathcal{O}(\theta T) < \frac{1}{3}T, \tag{2.3.146}$$

for θ sufficiently small. As $|w_i(x,t)| = |\eta_i(x,u(x,t))w(x,t)| \leqslant |w(x,t)|$ by Cauchy-Schwarz, we see that the L^∞ norm of w blows up in finite time. This concludes the adaptation of John's argument incorporating the effects of exponentially small perturbations. We make some final concluding remarks. The first is that exponential decay in the perturbation is not essential here, so long as the perturbation $B(x,u)$ is smooth sufficiently far away from $x = 0$ and is L^1 with L^1 derivative, then the argument can be

adapted at the price of losing the ability to estimate how far the support of $u(x,0)$ needs to be from zero. Another remark is that there only needs to be one genuinely nonlinear field in this argument provided that the initial data is largest in the genuinely nonlinear direction, in the sense that the value W_0 is achieved by some w_i for which the corresponding coefficient $\gamma_{iii} \neq 0$. This is because we only showed that one specific w_i blows up in finite time, we have little control over what the other w_k are doing. Hörmander in [Hör87] does a more refined analysis of ODE of the form

$$\frac{dw}{dt} = a_0(t)w^2 + a_1(t)w + a_2(t), \tag{2.3.147}$$

to show that the blowup time is asymptotically determined by the corresponding blowup time for the Riccati equation

$$\frac{dw}{dt} = \gamma_{iii}^{\infty}(0)w^2. \tag{2.3.148}$$

As our characteristic equation is of the same form as the ODE [Hör87] studies, one can adapt the methods to show the same result here by showing the same bounds as in [Hör87] hold for the characteristic equation obtained here. An alternative argument is to note that $V = \mathcal{O}(\theta W_0)$, $\mathcal{E}(d,T) = \mathcal{O}(W_0)$, $\mathcal{G}(d,T) = \mathcal{O}(W_0^2)$, and $U = \mathcal{O}(\theta)$ can be combined to show that

$$(\gamma_{iii}(x,u(x,t)) - \mathcal{O}(\theta))w^2 \leqslant \frac{dw}{dt} \leqslant (\gamma_{iii}(x,u(x,t)) + \mathcal{O}(\theta))w^2, \tag{2.3.149}$$

for $t \geqslant t_0$. The final step is to note that $\gamma_{iii}(x,u(x,t)) \to \gamma_{iii}^{\infty}(0)$ locally uniformly as $\theta \to 0$ by $U = \mathcal{O}(\theta)$ and the distance condition on the support of u.

The final remark we make here is that John's argument shows that a large class of initial data leads to blow up in the solution, our adaption does not lead to as general a result. It would be interesting to see if generic small data supported far from $x = 0$ leads to blow up in this perturbed setup.

2.3.5 Application to the ZND model

Recall the ZND model

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v,E)_x &= 0, \\ E_t + (pu)_x &= qk\phi(T)z, \\ z_t &= -k\phi(T)z, \end{aligned} \tag{2.3.150}$$

where $\phi(T)$ is 1 for $T \geqslant T_i$ and 0 for $T < T_i$. Consider a shock solution $\underline{U} = (\underline{v}, \underline{u}, \underline{E}, \underline{z})$ with shock speed $\sigma \neq 0$ and such that the temperature $T(\underline{U}(x))$ satisfies

$$\inf_{x<0} T(\underline{U}(x)) > T_i. \tag{2.3.151}$$

Although the ZND model is not of the form (2.3.3), we will still be able to show blowup using the prior method for suitable perturbations of the shock $\underline{U}$. Making a Galilean change of coordinates into the frame where the shock $\underline{U}$ is stationary, we find that $U = (v, u, E, z)$ satisfies

$$\begin{aligned} v_t - \sigma v_x - u_x &= 0, \\ u_t - \sigma u_x + p_x &= 0, \\ E_t - \sigma E_x + (pu)_x &= qk\phi(T)z, \\ z_t - \sigma z_x &= -k\phi(T)z. \end{aligned} \tag{2.3.152}$$

Now consider a solution of the ZND model of the form $U = \underline{U} + \hat{U}$, where the support of $\hat{U}(x, 0) = [\alpha_0, \beta_0]$ is a compact subset of $(-\infty, 0)$ with $|\beta_0|$ satisfying the distance conditions in Theorem 13. Write $\hat{U} = (\hat{v}, \hat{u}, \hat{E}, \hat{z})$ and further assume that $\hat{z}(x, 0)$ is identically 0. We will show that this assumption allows us to eliminate z from the ZND model, reducing the system to gas dynamics. The importance of this reduction is that gas dynamics is a system of the form (2.3.3). If the solution $\hat{U}$ remains sufficiently small in L^∞ so that the temperature remains above T_i for all time, then $\hat{z}$ will satisfy

$$\hat{z}_t - \sigma \hat{z}_x = -k\hat{z}, \tag{2.3.153}$$

as the distance condition on $[\alpha_0, \beta_0]$ ensures that $\text{supp}(\hat{U}(x, t)) \subset (-\infty, 0)$ for all t up to T_*. Solving (2.3.153) with the initial data $\hat{z}(x, 0) = 0$ shows that $\hat{z}(x, t) = 0$ for all time. Looking at the S equation in the ZND model, and writing $E = \underline{E} + \hat{E}$ and subtracting off the equation for $\underline{E}$ we find

$$\hat{E}_t - \sigma \hat{E}_x + (pu - \underline{pu})_x = 0. \tag{2.3.154}$$

We have eliminated z from the ZND model for perturbations $\hat{U}$ of the form $\hat{U} = (\hat{v}, \hat{u}, \hat{E}, 0)$, leaving us with a system of conservation laws known to have at least one genuinely nonlinear field.

Remark 5. *Generic shocks in the ZND model do not exponentially converge to zero as $x \to -\infty$ in every component, but that is easily remedied by writing $\underline{U} = \underline{U}(-\infty) + \tilde{\underline{U}}$, adjusting the pressure function p appropriately, with $\tilde{\underline{U}}$ and all of its derivatives decaying exponentially as $x \to -\infty$.*

There are shock solutions in the sonic case, called Chapman-Jouget waves, of the ZND model which do not decay exponentially in x; but instead exhibit power law decay with $\underline{U} - \underline{U}(-\infty)$ decaying like x^{-1}.

We also note that this argument works equally for the Majda model, showing the necessity of the weight in the stability result Theorem 11.

2.3.6 Proof of the corollary

Corollary 2. *For every $C > 0$, $\theta > 0$, $\delta > 0$, $s > \frac{3}{2}$ and $\epsilon > 0$ there exists ϕ smooth and compactly supported away from the shock such that $\|\phi\|_{L^2} \leqslant \delta$ and the solution obtained is given with a perturbation Φ from the initial wave it gives raise to is defined on some time interval $[0, T]$ and it satisfies $\|U(t)\|_{L^2} \leqslant \frac{C\epsilon}{\theta}$ on $[0, T]$, and $\|U(T)\|_{H^s} > \epsilon$.*

In particular, there can not be any damping estimates of the form: there exists $C, \theta, \delta, \epsilon$ all positive and $s > \frac{3}{2}$ such that, for every solution Φ that can be written as $\Phi(t, \cdot) = U(\cdot - \psi(t)) + v(t, \cdot)$ with $\|v(t)\|_{H^s} \leqslant \epsilon$ on $[0, T_0]$ then, on $[0, T_0]$

$$\|v(t)\|_{H^s} \leqslant Ce^{-\theta t}\|v(0)\|_{H^s} + \int_0^t Ce^{-\theta(t-s)}\|v(s)\|_{L^2}ds.$$

Furthermore, there is no orbital stability in $H^s(\mathbf{R}^*)$.

Proof. We constructed before positive constants $C' > 0$, δ and $\mu > 0$ and a sequence of smooth compactly supported away from the shock initial perturbation $(v_n)_n$ such that for every $s > \frac{3}{2}$ $\|v_n\|_{H^s(\mathbf{R}^*)} = \mathcal{O}\left(\frac{1}{n+1}\right)$ and the same is true for the L^∞ norm of v_n and $(v_n)_x$, with a support which Lebesgue measure is at most 1 and the associated solutions (the sequence $(U_n)_n$) to the original equation, with initial data the profile of the wave perturbed by v_n can be written as $U_n(t, \cdot - \sigma t) = \underline{U}(\cdot) + w_n(t, \cdot)$, w_n being in $H^s(\mathbf{R}^*)$ satisfying, on the interval of existence $\|w_n\|_{L^\infty} \leqslant C'\|v_n\|_{L^\infty}$, $\mathrm{supp}(w_n(t)) \subset \mathrm{supp}(v_n) + [-C't, C't]$ and $\|(w_n)_x(t)\|_{L^\infty} \geqslant \frac{\delta}{n\mu - t}$.

Thus, given $\epsilon > 0$, for some $t_n \leqslant n\mu$, we have $\|w_n(t_n)\|_{H^s} > \epsilon$ as a lower bound, and, also, the L^2 norm of $w_n(t)$ (for $0 \leqslant t \leqslant t_n$) is bounded by above by $\|w_n(t)\|_{L^\infty}\lambda_1(\mathrm{supp}(w_n(t)))$, and thus $\|w_n(t)\|_{L^2} \leqslant \frac{C\sqrt{C}(\sqrt{t_n} + \sqrt{\lambda_1(\mathrm{supp}(v_n))})}{n+1}$, thus, up to some multiplicative constant that does not depend on n, it is of size at most $\sqrt{n+1}^{-1}$.

Hence, given $C > 0$ and $\theta > 0$, for n big enough, we have, $\|v_n\|_{H^s} \leqslant \frac{\epsilon}{2}$, and, for t_n as before, that on $[0, t_n]$, U_n remains a Lax solution, and $\|w_n\|_{L^2} \leqslant \frac{\epsilon\theta}{2C}$ and $\|w_n(t_n)\|_{H^s} > \epsilon$ thus making the damping estimate described before impossible with these constants C, ϵ and θ. Any damping estimate of this form is, thus, impossible.

Furthermore, $(v_n)_n$ goes to 0 in H^s (for every fixed s) as n goes to $+\infty$, but $(U_n)_n$ is not even globally defined. Thus, it precludes orbital stability results in H^s. □

2.3.7 The case of weighted norms

In the following, we prove an instability result for weighted norms as used in the Majda model.

Lemma 4. *We consider the framework considered before of*

$$u_t + (A(u) + B(x,u))u_x + G(x,u)u = 0$$

satisfying the same assumptions, and the added constraint that, furthermore, $\Lambda_n(0) < 0$. Let $\alpha > 0$. There exists $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$, for all $\delta > 0$ there exists some initial data $u_0 \in C_c^\infty(\mathbf{R})$ such that $\|u_0\|_{H^s_\alpha} \leqslant \delta$ and the solution u has its H^2_α norm that gets bigger than ϵ on the interval of existence of the solution.

Proof. Under the assumptions of the lemma, with $\alpha > 0$ and $s > \frac{3}{2}$, let ξ_n be a right eigenvector of $A(0)$ associated with the eigenvalue $\Lambda_n(0)$. We also recall that $\|B(x,u)u_x\|_{L^2_\alpha} \lesssim \|u\|_{L^\infty}\|u_x\|_{L^2_\alpha}$ and $\|G(x,u)u\|_{L^2_\alpha} \lesssim (\sup_{x\leqslant x_0}\|\underline{u}_x\|)\|u\|_{L^2_\alpha}$ if $\mathrm{supp}(u) \subset (-\infty, x_0]$. We define for a given solution u, $N(u) : (t,x) \mapsto -G(x,u)u - B(x,u)u_x$. With ϕ a smooth function with support in $[-1,0]$ that is not the zero function, we have that $v_p : x \mapsto c_p e^{-\alpha p}\phi(\cdot + p)\xi_n$ goes to 0 in H^2_α with c_p a sequence of positive numbers going to 0 that will be specified later, and, furthermore, it induces a solution u_p defined on $[0, T)$ such that $w_p := \eta_n \cdot u_p$ satisfies $(w_p)_t + \lambda_n(w_p)_x = \eta_n \cdot (A(0) - A(u) - B(x,u))(u_p)_x - \eta_n \cdot G(x, u_p)u_p$.

Assume, by contradiction, that for all ϵ, there exists $p_\epsilon \in \mathbf{N}$ such that, for every $p \geqslant p_\epsilon$ u_p stays in the ball of radius ϵ and center 0 in H^1 for every $t \geqslant 0$ (and, thus, that the solution is defined on $\mathbf{R}^+$). In particular, given $T \geqslant 0$, we thus have

$$w_p(t,x) = w_p(t, x - \lambda_n(t-T)) + \int_T^t \eta_n \cdot N(u_p)(s, x - \lambda_n(t-s))ds$$

for every $t \geqslant T$.

Let

$$\tau : (-\infty, 0] \to \mathbf{R}_+,$$

such that for $x \leqslant 0$ we have $\tau(x) = \inf_{y \leqslant x} \|\underline{u}_x\|$ (with the norm being the usual sup norm on $\mathbf{R}^n$).

We want to ensure that the Duhamel term stays smaller than $\frac{e^{\alpha T} c_p \|\phi\|_{L^2_\alpha}}{2}$ with T such that $\frac{e^{\alpha T} c_p \|\phi\|_{L^2_\alpha}}{2} > \epsilon$. We will use that $\mathrm{supp}(w_p(t,\cdot)) \subset (-\infty, 1 - p + \mu_1 t]$.

Thus, as $T_p = 2\frac{\ln(\epsilon) - \ln(c_p) + \ln(2) - \ln\left(\|\phi\|_{L^2_\alpha}\right)}{\alpha}$ is big enough for our needs, we only need to choose a sequence $(c_p)_p$ that converges slowly enough to 0 to ensure that $\tau(-p + 1 + \mu_1 T_p)$ goes to 0 fast enough in comparison. Assuming that $c_p = o(p^{-1})$, we have that $T_p = o(p)$, and thus $\tau(-p+1+\mu_1 T_p) \leqslant \tau(-0.5p)$ for large p. As a consequence, we obtain the bounds

$$|\eta_n \cdot ((A(u_p) - A(0))(u_p)_x)| \lesssim |u_p||(u_p)_x|$$

$$|\eta_n \cdot (B(x, u_p)(u_p)_x)| \lesssim \tau(-0.5p)|(u_p)_x|$$

$$|\eta_n \cdot (G(x, u_p)u_p)| \lesssim \tau(-0.5p)|u_p|$$

Thus, as long as $\tau(-0.5p) = o(c_p)$, for example $c_p = \max(p^{-1}, \tau(-0.5p))^2$, we obtain that there is a contradiction for p big enough, as $\|u_p(T_p,\cdot)\|_{L^\infty} \lesssim \epsilon e^{-\frac{\alpha p}{2}}$.

$$|w_p(T_p, x)| \geqslant \phi(x - \lambda_n T_p + p) c_p e^{-\alpha p} - \int_0^{T_p} |N(t, x - \lambda_n(T_p - t))| dt$$

Hence

$$\|w_p(T_p,\cdot)\|_{L^2_\alpha} \geqslant \|\phi\|_{L^2_\alpha} e^{(p - \lambda_n T_p)\alpha} e^{-p\alpha} - \int_0^{T_p} \|N(t, x - (T_p - t)\lambda_n)\|_{L^2_\alpha} dt$$

And so

$$\|w_p(T_p,\cdot)\|_{L^2_\alpha} \geqslant \frac{\|\phi\|_{L^2_\alpha}}{2} e^{-\lambda_n T_p \alpha}$$

for p big enough. □

We now need to check that the lemma applies to ZND. We will first study the eigen-

values of $A(V)$ for a given V, for the reduced problem obtained in subsection 3.5.

$$\begin{pmatrix} 0 & -1 & 0 \\ p_v & p_u & p_E \\ up_v & up_u + p & up_E \end{pmatrix}$$

which, as its kernel is nontrivial and under the assumption that $v > 0$, $u > 0$, $E > 0$, $(p_v(v,u,E), p_E(v,u,E)) \neq (0,0)$ and $p(v,u,E) > 0$ we have that the matrix is diagonalizable with real eigenvalues if $(p_u + up_E)^2 + 4(pp_E - p_v) \geqslant 0$. As σ is positive, we have $\sigma > 0 \geqslant \Lambda_3(V)$, which allows us to apply the previous lemma to this problem, and we will do it under the assumption that the solution studied is close to a shock as described before and that we assume to be admissible, in the sense that it satisfies the assumptions of Theorem 10.

Chapter 3

Uniform stability of shocks

3.1 Introduction

In the present contribution, we prove for the very first time an asymptotic stability result, uniform with respect to the viscosity parameter, for a viscous regularization of a discontinuous traveling-wave of an hyperbolic equation.

3.1.1 The original hyperbolic result

The purely inviscid result [DR20], that we extend to the slightly viscous regimes, is itself quite recent. More generally, despite the fact that hyperbolic models are largely used for practical purposes and that for such models singularities such as shocks and characteristic points are ubiquitous, the analysis of nonlinear asymptotic stability of singular traveling-waves of hyperbolic systems is still in its infancy. The state of the art is essentially reduced to a full classification of waves of scalar equations in any dimension [DR20, DR22] (obtained using some significant insights about characteristic points from [JNR$^+$19]) and the case study of a discontinuous wave without characteristic point for a system of two equations in dimension 1 [JYZ21, YZ20].

Let us stress that, in the foregoing, stability is understood in the sense of Lyapunov, that is, globally in time, and for a topology encoding piecewise smoothness. This is consistent with the fact that concerning stability in the sense of Hadamard, that is, short-time well-posedness, for piecewise-smooth topologies, a quite comprehensive (but not complete) theory is already available even for multidimensional systems; see [Maj83a, Maj83b, Met, BGS]. At this level of regularity, being a weak solution is characterized by a free-interface initial boundary value problem, composed of equations taken in the classical sense in zones of smoothness, and the Rankine-Hugoniot transmission conditions along the free interfaces of discontinuity.

As is well-known, for hyperbolic equations, weak solutions are not unique and one needs to make an extra choice. The one we are interested in is the most classical one when the extra condition is to be obtained as a vanishing viscosity limit. For scalar equations, in any dimension, since the pioneering work of Kružkov [Kru70] (see also [Bre00, Chapters 4

and 6]), this is known to be sufficient to ensure uniqueness and to be characterized by the so-called entropy conditions, which at our level of smoothness are reduced to inequalities at the free interfaces of discontinuity. For systems, even in dimension 1, despite decisive breakthroughs achieved in [BB05], such questions are still the object of intensive research; see for instance [KV21b]. The present contribution lies at the crossroad of these questions related to the basic definitions of the notion of solution for hyperbolic equations and the ongoing development of a robust general theory for the stability of traveling waves, for which we refer the reader to [Sat73, Hen81, Zum01, San02, KP13, JNRZ14]. From the former point of view, the present contribution may be thought as a global-in-time scalar version of [GX92, GR01, Rou02]. From the latter point of view, though of a very different technical nature, by many respects, it shares similar goals with other vanishing viscosity stability programs — see for instance [BGM17, HR18] — and the present contribution is thought as being to [DR20] what [BMV16] is to [BM15].

We focus on the most basic shock stability result of [DR20]. Consider a scalar balance law in dimension 1,

$$\partial_t u + \partial_x(f(u)) = g(u) \tag{3.1.1}$$

with traveling wave solutions $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $(t,x) \mapsto \underline{u}(x - (\psi_0 + \sigma_0 t))$ with initial shock position $\psi_0 \in \mathbf{R}$, speed $\sigma_0 \in \mathbf{R}$ and wave profile $\underline{u}$ of Riemann shock type, that is,

$$\underline{u}(x) = \begin{cases} \underline{u}_{-\infty} & \text{if } x < 0 \\ \underline{u}_{+\infty} & \text{if } x > 0 \end{cases}$$

where $(\underline{u}_{-\infty}, \underline{u}_{+\infty}) \in \mathbf{R}^2$, $\underline{u}_{+\infty} \neq \underline{u}_{-\infty}$. The fact that this does define a weak solution is equivalent to

$$g(\underline{u}_{+\infty}) = 0\,, \qquad g(\underline{u}_{-\infty}) = 0\,, \qquad f(\underline{u}_{+\infty}) - f(\underline{u}_{-\infty}) = \sigma_0(\underline{u}_{+\infty} - \underline{u}_{-\infty})\,, \tag{3.1.2}$$

whereas a strict version of entropy conditions may be enforced in Oleinik's form

$$\begin{cases} \sigma_0 \ > \ f'(\underline{u}_{+\infty})\,, \\ \frac{f(\tau\,\underline{u}_{-\infty}+(1-\tau)\,\underline{u}_{+\infty})-f(\underline{u}_{-\infty})}{\tau\,\underline{u}_{-\infty}+(1-\tau)\,\underline{u}_{+\infty}-\underline{u}_{-\infty}} > \frac{f(\tau\,\underline{u}_{-\infty}+(1-\tau)\,\underline{u}_{+\infty})-f(\underline{u}_{+\infty})}{\tau\,\underline{u}_{-\infty}+(1-\tau)\,\underline{u}_{+\infty}-\underline{u}_{+\infty}} & \text{for any } \ \tau \in (0,1)\,, \\ f'(\underline{u}_{-\infty}) \ > \ \sigma_0\,. \end{cases} \tag{3.1.3}$$

Requiring a strict version of entropy conditions ensures that they still hold for nearby functions and in particular they disappear at the linearized level. In the foregoing, and throughout the text, for the sake of simplicity, we assume that $f, g \in \mathcal{C}^\infty(\mathbf{R})$ though each result only requires a small amount of regularity.

The following statement is one of the alternative versions of [DR20, Theorem 2.2] described in [DR20, Remark 2.3].

Theorem 14 ([DR20])**.** *Let* $(\sigma_0, \underline{u}_{-\infty}, \underline{u}_{+\infty}) \in \mathbf{R}^3$ *define a strictly-entropic Riemann shock of* (3.1.1) *in the above sense. Assume that it is spectrally stable in the sense that*

$$g'(\underline{u}_{+\infty}) < 0 \quad \text{and} \quad g'(\underline{u}_{-\infty}) < 0\,.$$

There exist $\delta > 0$ *and* $C > 0$ *such that for any* $\psi_0 \in \mathbf{R}$ *and* $v_0 \in BUC^1(\mathbf{R}^*)$ *satisfying*

$$\|v_0\|_{W^{1,\infty}(\mathbf{R}^*)} \leqslant \delta\,,$$

there exists $\psi \in \mathcal{C}^2(\mathbf{R}^+)$ *with initial data* $\psi(0) = \psi_0$ *such that the entropy solution to* (3.1.1)*,* u*, generated by the initial data* $u(0,\cdot) = (\underline{u}+v_0)(\cdot+\psi_0)$*, belongs to* $BUC^1(\mathbf{R}_+ \times \mathbf{R}\backslash\{\,(t,\psi(t))\,;\,t \geqslant 0\,\})$ *and satisfies for any* $t \geqslant 0$

$$\|u(t,\cdot-\psi(t)) - \underline{u}\|_{W^{1,\infty}(\mathbf{R}^*)} + |\psi'(t)-\sigma_0| \leqslant \|v_0\|_{W^{1,\infty}(\mathbf{R}^*)}\, C\, \mathrm{e}^{\max(\{g'(\underline{u}_{+\infty}),g'(\underline{u}_{-\infty})\})\,t}\,,$$

and moreover there exists ψ_∞ *such that*

$$|\psi_\infty - \psi_0| \leqslant \|v_0\|_{L^\infty(\mathbf{R}^*)} C\,,$$

and for any $t \geqslant 0$

$$|\psi(t) - \psi_\infty - t\,\sigma_0| \leqslant \|v_0\|_{L^\infty(\mathbf{R}^*)} C\, \mathrm{e}^{\max(\{g'(\underline{u}_{+\infty}),g'(\underline{u}_{-\infty})\})\,t}\,.$$

In the foregoing, we have used notation $BUC^k(\Omega)$ to denote the set of $\mathcal{C}^k$ functions over Ω whose derivatives up to order k are bounded, and uniformly continuous on every connected component of Ω. In other words, $BUC^k(\Omega)$ is the closure of $W^{\infty,\infty}(\Omega)$ for the $W^{k,\infty}(\Omega)$ topology. Working with BUC^k instead of $W^{k,\infty}$ allows to use approximation by smooth functions, an argument ubiquitous in local well-posedness theories, without imposing vanishing at ∞.

Note that expressed in classical stability terminology the previous theorem provides asymptotic orbital stability with asymptotic phase. We stress however that the role of phase shifts is here deeper than in the classical stability analysis of smooth waves since it is not only required to provide decay of suitable norms in large-time but also to ensure that these norms are finite locally in time. In particular here there is no freedom, even in finite time, in the definition of phase shifts that need to synchronize discontinuities to allow for comparisons in piecewise smooth topologies.

It is also instructive to consider the corresponding spectral problem. In a moving

frame, linearizing from $u(t,x) = \underline{u}(x-(\psi_0+\sigma_0 t)-\psi(t)) + v(t, x-(\psi_0+\sigma_0 t)-\psi(t))$ gives a linear IBVP in (v,ψ)

$$\begin{aligned}
(\partial_t + (f'(\underline{u}_{+\infty}) - \sigma_0)\partial_x - g'(\underline{u}_{+\infty}))\, v(t,\cdot) \;&=\; 0 \qquad \text{on } \mathbb{R}_+^*\,,\\
(\partial_t + (f'(\underline{u}_{-\infty}) - \sigma_0)\partial_x - g'(\underline{u}_{-\infty}))\, v(t,\cdot) \;&=\; 0 \qquad \text{on } \mathbb{R}_-^*\,,\\
\psi'(t) - \left(\frac{f'(\underline{u}_{+\infty}) - \sigma_0}{\underline{u}_{+\infty} - \underline{u}_{-\infty}} v(t,0^+) - \frac{f'(\underline{u}_{-\infty}) - \sigma_0}{\underline{u}_{+\infty} - \underline{u}_{-\infty}} v(t,0^-) \right) \;&=\; 0\,.
\end{aligned}$$

The corresponding spectrum on $BUC^1(\mathbf{R}^*) \times \mathbf{R}$ is

$$\{\ \lambda\ ;\ \Re(\lambda) \leqslant \max(\{g'(\underline{u}_{-\infty}); g'(\underline{u}_{+\infty})\})\ \} \cup \{0\}$$

and when $\max(\{g'(\underline{u}_{-\infty}); g'(\underline{u}_{+\infty})\}) < 0$, 0 has multiplicity 1 (in the sense provided by resolvent singularities) with eigenvector $(0,1)$. This shows that Theorem 14 sharply reproduces linear behavior.

3.1.2 The vanishing viscosity problem

Since even the local-in-time notion of solution involves vanishing viscosity approximations, it is natural to wonder whether Theorem 14 may have a small-viscosity extension or whether the local-in-time vanishing viscosity limits may be globalized in time about the stable Riemann shocks of Theorem 14. We answer such a question for the following parabolic approximation

$$\partial_t u + \partial_x(f(u)) = \varepsilon\, \partial_x^2 u + g(u)\,. \tag{3.1.4}$$

Note that solutions to (3.1.4) are smooth (not uniformly in ε) so that techniques based on free-interfaces IBVP formulations for (3.1.1) cannot easily be extended to the study of (3.1.4). In the reverse direction, to gain a better control on smoothness of solutions to (3.1.4), it is expedient to introduce fast variables

$$u(t,x) = \underbrace{\widetilde{u}\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right)}_{(\widetilde{t},\widetilde{x})}$$

that turn (3.1.4) into

$$\partial_{\widetilde{t}}\widetilde{u} + \partial_{\widetilde{x}}(f(\widetilde{u})) = \partial_{\widetilde{x}}^2\widetilde{u} + \varepsilon\, g(\widetilde{u})\,. \tag{3.1.5}$$

We stress however that this is indeed in original variables (t,x) that we aim at proving a uniform result. In particular, a large part of the analysis is focused on distinctions

between norms that get large and norms that get small when going from slow to fast variables. For a closely related discussion we refer the reader to [KV21a, KV21b].

In order to carry out the extension, the first step is to elucidate the existence of traveling waves to (3.1.4) near $\underline{u}$. A preliminary observation in this direction is that the formal $\varepsilon \to 0$ limit of (3.1.5) does possess a smooth traveling-wave solution $(\widetilde{t}, \widetilde{x}) \mapsto \underline{U}_0(\widetilde{x} - \sigma_0 \widetilde{t})$ of speed σ_0 and profile $\underline{U}_0$ such that

$$\lim_{-\infty} \underline{U}_0 = \underline{u}_{-\infty} \,, \qquad\qquad \lim_{+\infty} \underline{U}_0 = \underline{u}_{+\infty} \,,$$

simply obtained by solving

$$\underline{U}_0(0) = \frac{\underline{u}_{-\infty} + \underline{u}_{+\infty}}{2} \,, \qquad\qquad \underline{U}_0' \,=\, f(\underline{U}_0) - f(\underline{u}_{+\infty}) - \sigma_0 \, (\underline{U}_0 - \underline{u}_{+\infty}) \,.$$

We recall that σ_0 is tuned to ensure $f(\underline{u}_{-\infty}) - f(\underline{u}_{+\infty}) - \sigma_0 \, (\underline{u}_{-\infty} - \underline{u}_{+\infty}) = 0$ and observe that the Oleinik's entropy conditions imply that $\underline{U}_0$ is strictly monotonous. This $\varepsilon = 0$ viscous profile is often called viscous shock layer and plays the role of a short-time free-interface boundary layer. This simple limiting fast profile may be perturbed to yield profiles for (3.1.5) hence for (3.1.4). To state such a perturbation result with optimal spatial decay rates, we introduce, for $\varepsilon \geqslant 0$,

$$\begin{aligned}
\theta_\varepsilon^r &:= \frac{1}{2} |f'(\underline{u}_{+\infty}) - \sigma_\varepsilon| + \frac{1}{2} \sqrt{(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2 + 4 \, \varepsilon \, |g'(\underline{u}_{+\infty})|} \,, \\
\theta_\varepsilon^\ell &:= \frac{1}{2} \, (f'(\underline{u}_{-\infty}) - \sigma_\varepsilon) + \frac{1}{2} \sqrt{(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2 + 4 \, \varepsilon \, |g'(\underline{u}_{-\infty})|} \,.
\end{aligned}$$

Proposition 10. *Under the assumptions of Theorem 14, for any $0 < \alpha^\ell < \theta_0^\ell$, $0 < \alpha^r < \theta_0^r$ and $k_0 \in \mathbf{N}^*$, there exist $\varepsilon_0 > 0$ and $C_0 > 0$ such that there exist a unique $(\sigma_\varepsilon, \underline{U}_\varepsilon)$, with $\underline{U}_\varepsilon \in \mathcal{C}^2(\mathbf{R})$,*

$$\underline{U}_\varepsilon(0) = \frac{\underline{u}_{-\infty} + \underline{u}_{+\infty}}{2} \,, \qquad\qquad (f(\underline{U}_\varepsilon) - \sigma_\varepsilon \, \underline{U}_\varepsilon)' = \underline{U}_\varepsilon'' + \varepsilon \, g(\underline{U}_\varepsilon) \,,$$

and

$$|\sigma_\varepsilon - \sigma_0| + \| \, \mathrm{e}^{\alpha^\ell |\cdot|} (\underline{U}_\varepsilon - \underline{U}_0) \|_{W^{1,\infty}(\mathbf{R}_-)} + \| \, \mathrm{e}^{\alpha^r \cdot} (\underline{U}_\varepsilon - \underline{U}_0) \|_{W^{1,\infty}(\mathbf{R}_+)} \leqslant C_0 \, \varepsilon \,,$$

and, moreover, there also holds

$$\begin{aligned}
\| \, \mathrm{e}^{\theta_\varepsilon^\ell |\cdot|} (\underline{U}_\varepsilon - \underline{u}_{-\infty}) - \mathrm{e}^{\theta_0^\ell |\cdot|} (\underline{U}_0 - \underline{u}_{-\infty}) \|_{L^\infty(\mathbf{R}_-)} &\leqslant C_0 \, \varepsilon \,, \\
\| \, \mathrm{e}^{\theta_\varepsilon^r \cdot} (\underline{U}_\varepsilon - \underline{u}_{+\infty}) - \mathrm{e}^{\theta_0^r \cdot} (\underline{U}_0 - \underline{u}_{+\infty}) \|_{L^\infty(\mathbf{R}_+)} &\leqslant C_0 \, \varepsilon \,, \\
\| \, \mathrm{e}^{\theta_\varepsilon^\ell |\cdot|} \, \underline{U}_\varepsilon^{(k)} - \mathrm{e}^{\theta_0^\ell |\cdot|} \, \underline{U}_0^{(k)} \|_{L^\infty(\mathbf{R}_-)} &\leqslant C_0 \, \varepsilon \,, \qquad\qquad 1 \leqslant k \leqslant k_0 \,,
\end{aligned}$$

$$\| \mathrm{e}^{\theta_\varepsilon^r \cdot} \underline{U}_\varepsilon^{(k)} - \mathrm{e}^{\theta_0^r \cdot} \underline{U}_0^{(k)} \|_{L^\infty(\mathbf{R}_+)} \leqslant C_0 \, \varepsilon \, , \qquad 1 \leqslant k \leqslant k_0 \, .$$

Note that a traveling-wave $(t,x) \mapsto \underline{u}_\varepsilon(x - (\psi_0 + \sigma_\varepsilon t))$ with $\psi_0 \in \mathbf{R}$ arbitrary, is obtained from $\underline{U}_\varepsilon$ through

$$\underline{u}_\varepsilon(x) := \underline{U}_\varepsilon \left(\frac{x}{\varepsilon} \right)$$

and that, uniformly in ε,

$$\begin{aligned}
|\underline{u}_\varepsilon(x) - \underline{u}(x)| &\lesssim \mathrm{e}^{-\theta_\varepsilon^\ell \frac{|x|}{\varepsilon}} \, , && x < 0 \, , \\
|\underline{u}_\varepsilon(x) - \underline{u}(x)| &\lesssim \mathrm{e}^{-\theta_\varepsilon^r \frac{x}{\varepsilon}} \, , && x > 0 \, , \\
|\underline{u}_\varepsilon^{(k)}(x)| &\lesssim \frac{1}{\varepsilon^k} \, \mathrm{e}^{-\theta_\varepsilon^\ell \frac{|x|}{\varepsilon}} \, , && x < 0 \, , \, k \geqslant 1 \, , \\
|\underline{u}_\varepsilon^{(k)}(x)| &\lesssim \frac{1}{\varepsilon^k} \, \mathrm{e}^{-\theta_\varepsilon^r \frac{x}{\varepsilon}} \, , && x > 0 \, , \, k \geqslant 1 \, .
\end{aligned}$$

We prove Proposition 10 in Appendix 3.6. The existence and uniqueness part with suboptimal spatial rates follows from a rather standard Lyapunov-Schmidt argument. We stress however that it is crucial for our linear and nonlinear stability analyses to gain control on $\underline{U}_\varepsilon'$ with sharp spatial decay rates. We obtain the claimed upgrade from suboptimal to optimal rates essentially as a corollary to the refined spectral analysis needed to carry out the nonlinear study. We point out that, despite the fact that the literature on the subject is quite extensive — see for instance [Här00, Här03, CM07, Cro10, Gil10] and references therein —, we have not found there an existence result with the level of generality needed here, that is, including non-convex fluxes and yielding optimal spatial decay rates.

With the existence of ε-versions of traveling waves in hands, the next natural question is whether these are spectrally stable. It is settled by standard arguments, as expounded in [KP13], combining direct computations of the essential spectrum with Sturm-Liouville theory. The latter uses crucially that $\underline{U}_\varepsilon'$ is monotonous, a consequence of the Oleinik's entropy conditions. The upshot is that, in slow original variables, the spectrum of the linearization about $\underline{u}_\varepsilon$ in a co-moving frame, acting on $BUC^1(\mathbf{R})$, is stable and exhibits a spectral gap between the simple eigenvalue 0 and the rest of the spectrum of size $\min(\{|g'(\underline{u}_{+\infty})|, |g'(\underline{u}_{-\infty})|\}) + \mathcal{O}(\varepsilon)$. Note that in fast variables the spectral gap is of size $\varepsilon \times \min(\{|g'(\underline{u}_{+\infty})|, |g'(\underline{u}_{-\infty})|\}) + \mathcal{O}(\varepsilon^2)$ Details of the latter are given in Section 3.2.

The real challenge is *uniform* nonlinear asymptotic stability. Indeed, if one removes the uniformity requirement, nonlinear stability follows from spectral stability by now well-known classical arguments as expounded in [Sat73, Hen81, San02, KP13], and initially developed in, among others, [Sat, Sat77, Hen81, Kap94, WX05, Xin05]. Since the limit

is singular, it is worth spelling out what we mean by uniform stability. There are two closely related parts in the requirement. Explicitly, on initial data,

1. the most obvious one is that the restriction on the sizes of allowed initial perturbations (encoded by the smallness of δ in Theorem 14) should be uniform with respect to ε, so that the lower bound on the size of the basin of attraction provided by the analysis is nontrivial in the limit $\varepsilon \to 0$;
2. the second one is more intricate [1], it states that the ε-dependent norms, say $\|\cdot\|_{(\varepsilon)}$, used to measure this smallness (in slow original variables) should be controlled by an ε-independent norm for functions supported away from the shock, so that in particular for any $v \in \mathcal{C}_c^\infty(\mathbf{R})$ supported in $\mathbf{R}^*$, $\limsup_{\varepsilon\to 0} \|v\|_{(\varepsilon)} < +\infty$.

On the control of solutions arising from perturbations, we impose similar constraints but with upper bounds replacing lower bounds in the requirements. Constraints on the control of solutions ensure that the bounds provide a nontrivial control whereas constraints on the control of initial data ensure that nontrivial perturbations are allowed.

It may be intuitive that the stringer the norm is the larger the size of the basin of attraction is since a qualitatively better control is offered by the topology. In the present case, the discussion is on the amount of localization encoded by the norm since, though this is somewhat hidden, time decay is controlled by initial spatial localization (as opposed to cases where regularity drives decay as for instance in [BM15, BMV16, BGM17]). To offer a quantitative insight, let us use as in [HR18] a simple ODE as a toy model to predict the size constraints. Consider the stability of $y \equiv 0$ for $y' = -\tau\, y + \rho\, y^2$ where $\tau > 0$ encodes the size of the spectral gap and $\rho > 0$ measures the size of nonlinear forcing. For such an equation, a ball of radius r_0 and center 0 is uniformly attracted to 0 provided that $r_0 < \tau/\rho$. Now, if one considers (3.1.4) directly in $BUC^1(\mathbf{R})$ (or any reasonable unweighted topology) and forgets about issues related to phase definitions and possible regularity losses, the spectral gap offered by a linearization about $\underline{u}_\varepsilon$ is of order 1 whereas the forcing by nonlinear terms is of order ε^{-1} (since this is the size of $\underline{u}'_\varepsilon$) hence the rough prediction of a basin of size $\mathcal{O}(\varepsilon)$. Yet, working with weights such as $\mathrm{e}^{-\theta\,\varepsilon^{-\alpha}\,|x|}$, for some sufficiently small $\theta > 0$ and some $0 < \alpha \leqslant 1$, moves the spectrum to increase the size of the gap to the order $\varepsilon^{-\alpha}$ yielding the expectation of an $\mathcal{O}(\varepsilon^{1-\alpha})$ basin. Note that the choice $\alpha = 1$ would provide a uniform size and is consistent with the size of viscous shock layers but it would force initial perturbations to be located in an $\mathcal{O}(\varepsilon)$ spatial neighborhood of the shock location.

The foregoing simple discussion predicts quite accurately [2] what could be obtained by applying the most classical parabolic strategy to the problem at hand. The failure of

1. But our result satisfies a much simpler and stronger version of the requirement.
2. Actually it is even a bit optimistic for the unweighted and $\alpha < 1$ cases.

the classical strategy may also be read on the deeply related, but not equivalent, fact that it uses the phase only to pull out the contribution of nonlinear terms through the spectrally non-decaying 0-mode. This is inconsistent with the stronger role of the phase for the hyperbolic problem, all the spectrum contributing to the phase in the latter case. A completely different approach is needed.

Additional strong signs of the very challenging nature of the uniform stability problem may also be gathered from the examination of the viscous layer stability problem, that is, the stability of $(\widetilde{t},\widetilde{x}) \mapsto \underline{U}_0(\widetilde{x}-\sigma_0\,\widetilde{t})$ as a solution to (3.1.5) with $\varepsilon = 0$. The problem has been extensively studied, see for instance [Liu85, Goo86, Goo89, JGK93, KK98, How99a, How99b] for a few key contributions and [Zum01] for a thorough account. The spectrum of the linearization includes essential spectrum touching the imaginary axis at 0, which is still an eigenvalue, so that the decay is not exponential but algebraic and requires a trade-off, localization against decay, as for the heat equation. This is a consequence of a conservative nature of the equation, but the conservative structure may also be used to tame some of the apparent difficulties. To give one concrete example: one may remove the embedded eigenvalue 0 from the essential spectrum by using the classical antiderivative trick, dating back at least to [MN85], either directly at the nonlinear level under the restriction of zero-mean perturbations (as in [Goo89] or in [MN85] for a system case) or only to facilitate the linear analysis as in [How99a, How99b]. In fast variables, turning on $\varepsilon > 0$ moves the essential to the left, creating an $\mathcal{O}(\varepsilon)$ spectral gap but breaks the conservative structure thus rendering almost impossible, and at least quite inconvenient, the use of classical conservative tools. Our stability analysis requires a description as detailed as the one of [How99a, How99b] and, without the antiderivative trick at hand, this involves the full machinery of [ZH98, ZH02]. Roughly speaking, one of the main outcomes of our detailed spectral analysis, expressed in fast variables, is that the ε-proximity of essential spectrum and 0-eigenvalue induces that the essential spectrum has an impact of size $1/\varepsilon$ on the linear time-evolution, but that at leading-order the algebraic structure of the essential-spectrum contribution is such that it may be absorbed in a suitably designed phase modulation. Note that this is consistent with the fact that, in fast variables, variations in shock positions are expected to be of size $1/\varepsilon$ and with the fact that, in slow variables, the phase is involved in the resolution of all the hyperbolic spectral problems, not only the 0-mode.

To summarize and extend the discussion so far, we may hope

1. to overcome the discrepancy between the Rankine-Hugoniot prescription of the phase and the pure 0-mode modulation, and to phase out the hidden singularity caused by the proximity of essential spectrum and 0 eigenvalue, by carefully identifying the most singular contribution of the essential spectrum as phase variations

and including this in a carefully designed phase;

2. to guarantee uniform nonlinear decay estimates provided that we can ensure that, in sow variables, nonlinear terms of size $1/\varepsilon$ also come with a spectral-gap enhancing factor $\mathrm{e}^{-\theta\,|x|/\varepsilon}$ (for some $\theta > 0$).

The latter expectation is motivated by the fact that it is indeed the case for terms forced by $\underline{u}'_\varepsilon$ but we need to prove that it is so also for stiff terms caused by the derivatives of the perturbation itself. Concerning the latter, we stress that even if one starts with a very gentle perturbation supported away from the shock the nonlinear coupling instantaneously creates stiff parts of shock-layer type in the perturbation thus making it effectively multi-scale.

There remains a somewhat hidden issue, that we have not discussed so far. Along the foregoing discussion, we have done as if we could use Duhamel principle based on a straight-forward linearization, as in classical semilinear parabolic problems. Yet, here, closing nonlinear estimates in regularity by using parabolic regularization either explicitly through gains of derivatives or indirectly through $L^q \to L^p$, $q < p$, mapping properties, effectively induces losses in power of ε in an already ε-critical problem thus is completely forbidden. Instead, we estimate

— the variation in shock position ψ, the shape variation v and the restriction of its derivative $\partial_x v$ to an $\mathcal{O}(\varepsilon)$ neighborhood of the shock location through Duhamel formula and linear decay estimates;

— the remaining part of $\partial_x v$ by a suitably modified Goodman-type hyperbolic energy estimate.

The latter energy estimate is similar in spirit to those in [Goo89, RZ16, YZ20] but the hard part of its design is precisely in going from a classical hyperbolic estimate that would work in the complement of an $\mathcal{O}(1)$ neighborhood of the shock location to a finely tuned estimate covering the complement of an $\mathcal{O}(\varepsilon)$ neighborhood, since this is required for the combination with a lossless parabolic regularization argument. Moreover, there are two more twists in the argument: on one hand we need the estimate to include weights encoding the multi-scale nature of $\partial_x v$; on the other hand, for the sake of sharpness, to remain at the $\mathcal{C}^1$ level of regularity, we actually apply the energy estimates on a suitable nonlinear version of $\partial_x v$ so that they yield L^∞ bounds for $\partial_x v$.

The arguments sketched above, appropriately worked out, provide the main result of the present paper. To state such results, we introduce multi-scale weights and corresponding norms: for $k \in \mathbf{N}$, $\varepsilon > 0$, and $\theta \geqslant 0$,

$$\omega_{k,\varepsilon,\theta}(x) := \frac{1}{1 + \frac{1}{\varepsilon^k}\,\mathrm{e}^{-\theta\,\frac{|x|}{\varepsilon}}}\,, \qquad\qquad \|v\|_{W^{k,\infty}_{\varepsilon,\theta}(\mathbf{R})} = \sum_{j=0}^{k} \|\omega_{j,\varepsilon,\theta}\,\partial_x^j v\|_{L^\infty(\mathbf{R})}$$

Note that

1. Each norm $\|\cdot\|_{W^{k,\infty}_{\varepsilon,\theta}(\mathbf{R})}$ is equivalent to any standard norm on $\|\cdot\|_{W^{k,\infty}(\mathbf{R})}$ but non uniformly in ε and that the uniformity is restored if one restricts it to functions supported in the complement of a fixed neighborhood of the origin.
2. The norm $\|\cdot\|_{W^{0,\infty}_{\varepsilon,\theta}(\mathbf{R})}$ is uniformly equivalent to $\|\cdot\|_{L^\infty(\mathbf{R})}$.
3. If $\theta < \min(\{\theta_0^\ell, \theta_0^r\})$ then $\|\underline{u}_\varepsilon - \underline{u}\|_{W^{k,\infty}_{\varepsilon,\theta}(\mathbf{R})}$ is bounded uniformly with respect to ε.

Theorem 15. *Enforce the assumptions and notation of Theorem 14 and Proposition 10. There exists $\theta_0 > 0$ such that for any $0 < \theta \leqslant \theta_0$, there exist $\varepsilon_0 > 0$, $\delta > 0$ and $C > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$, any $\psi_0 \in \mathbf{R}$ and any $v_0 \in BUC^1(\mathbf{R})$ satisfying*

$$\|v_0\|_{W^{1,\infty}_{\varepsilon,\theta}(\mathbf{R})} \leqslant \delta\,,$$

there exists $\psi \in \mathcal{C}^1(\mathbf{R}^+)$ with initial data $\psi(0) = \psi_0$ such that the strong[3] solution to (3.1.4), u, generated by the initial data $u(0,\cdot) = (\underline{u}_\varepsilon + v_0)(\cdot + \psi_0)$, is global in time and satisfies for any $t \geqslant 0$

$$\|u(t,\cdot - \psi(t)) - \underline{u}_\varepsilon\|_{W^{1,\infty}_{\varepsilon,\theta}(\mathbf{R})} + |\psi'(t) - \sigma_\varepsilon| \leqslant \|v_0\|_{W^{1,\infty}_{\varepsilon,\theta}(\mathbf{R})}\, C\, \mathrm{e}^{\max(\{g'(\underline{u}_{+\infty}), g'(\underline{u}_{-\infty})\})\, t}\,,$$

and moreover there exists ψ_∞ such that

$$|\psi_\infty - \psi_0| \leqslant \|v_0\|_{W^{1,\infty}_{\varepsilon,\theta}(\mathbf{R})} C\,,$$

and for any $t \geqslant 0$

$$|\psi(t) - \psi_\infty - t\,\sigma_\varepsilon| \leqslant \|v_0\|_{W^{1,\infty}_{\varepsilon,\theta}(\mathbf{R})} C\, \mathrm{e}^{\max(\{g'(\underline{u}_{+\infty}), g'(\underline{u}_{-\infty})\})\, t}\,.$$

Among the many variations and extensions of Theorem 14 provided in [DR20], the simplest one to extend to a uniform small viscosity result is [DR20, Proposition 2.5] that proves that the exponential time decay also holds for higher order derivatives without further restriction on sizes of perturbations. It does not require any new insight besides the ones used to prove Theorem 15 and we leave it aside only to cut unnecessary technicalities.

Likewise, one may obtain in an even more direct way, that is, up to immaterial changes, exponential damping of norms encoding further slow spatial localization. To give an explicit example, let us extend notation $W^{k,\infty}_{\varepsilon,\theta}$, L^∞, into $W^{k,\infty}_{\varepsilon,\theta,\theta'}$, $L^\infty_{\theta'}$, accordingly to weights

$$\omega_{k,\varepsilon,\theta,\theta'}(x) := \frac{1}{\mathrm{e}^{-\theta'|x|} + \frac{1}{\varepsilon^k}\,\mathrm{e}^{-\theta\frac{|x|}{\varepsilon}}}\,, \qquad\qquad \omega_{\theta'}(x) := \mathrm{e}^{\theta'|x|}\,,$$

3. We ensure $u \in BUC^0(\mathbf{R}_+; BUC^1(\mathbf{R})) \cap \mathcal{C}^\infty(\mathbf{R}_+^*; BUC^\infty(\mathbf{R}))$.

with $\theta' \geqslant 0$ arbitrary. One may prove for instance that for any $\theta' \geqslant 0$ there exist $C_{\theta'}$ and $\varepsilon_{\theta'} > 0$ such that, under the sole further restrictions $0 < \varepsilon \leqslant \varepsilon_{\theta'}$ and $\mathrm{e}^{\theta' |\cdot|}\, v_0 \in L^\infty(\mathbf{R})$, there holds

$$\|u(t, \cdot - \psi(t)) - \underline{u}_\varepsilon\|_{L^\infty_{\theta'}(\mathbf{R})} \leqslant \|v_0\|_{L^\infty_{\theta'}(\mathbf{R})}\, C_\theta\, \mathrm{e}^{\max(\{g'(\underline{u}_{+\infty}), g'(\underline{u}_{-\infty})\})\, t}\ .$$

One point in considering these weighted topologies is that, when $\theta' > 0$, $L^\infty_{\theta'}$ is continuously embedded in $L^1 \cap L^\infty$, so that an estimate on $\|u(t, \cdot - \psi(t)) - \underline{u}\|_{L^p}$ is provided by the combination of the foregoing bound with the already known bound

$$\|\underline{u}_\varepsilon - \underline{u}\|_{L^p(\mathbf{R})} \lesssim \varepsilon^{\frac{1}{p}}\ .$$

3.1.3 Outline and perspectives

The most natural nontrivial extensions of Theorems 14/15 that we have chosen to leave for future work concern on one hand the parabolic regularization by quasilinear terms and on the other hand planar Riemann shocks in higher spatial dimensions (see [DR20, Theorem 3.4] for the hyperbolic case). We expect many parts of the present analysis to be directly relevant in quasilinear or multiD cases but we also believe that their treatments would also require sufficiently many new arguments to deserve a separate treatment.

In the multidimensional case, even the outcome is expected to be significantly different. In this direction, let us point out that the hyperbolic spectral problem is critical in the stronger sense that the spectrum includes the whole imaginary axis, instead of having an intersection with the imaginary axis reduced to $\{0\}$. This may be tracked back to the fact that the linearized Rankine-Hugoniot equation takes the form of a transport equation in transverse variables for the phase. Consistently, as proved in [DR20, Theorem 3.4], for the hyperbolic problem, perturbing a planar shock may lead asymptotically in large time to another non-planar Riemann shock sharing the same constant-states. This may still be interpreted as a space-modulated asymptotic stability result, in the sense coined in [JNRZ14] and thoroughly discussed in [Rod13, Rod15, Rod18, DR22]. A similar phenomenon is analyzed for scalar *conservation* laws in [Ser21].

Concerning the quasilinear case, the main new difficulty is expected to arise from the fact that, to close the argument, one needs to prove that the L^∞ decay of $\varepsilon\, \partial_x^2 v$, where v still denotes the shape variation, is at least as good as the one of $\partial_x v$. A priori, outside the shock layer this leaves the freedom to pick some initial typical size $\varepsilon^{-\eta_0}$, $\eta_0 \in [0, 1]$, for $\partial_x^2 v$ and to try to propagate it. Indeed, roughly speaking, in the complement of an $\mathcal{O}(\varepsilon)$ neighborhood of the shock location, this L^∞ propagation stems from arguments similar

to the ones sketched above for $\partial_x v$. The key difference is that now one cannot complete it with a bound obtained through Duhamel formula since this would involve an L^∞ bound on $\partial_x^3 v$. Thus the quasilinear study seems to require to be able to close an estimate for $\partial_x^2 v$ entirely with energy-type arguments, a highly non-trivial task.

In another direction, we expect that the study of waves with characteristic points, as arising in the full classification obtained in [DR22] for scalar balance laws, should not only involve some new patches here and there but follow very different routes and thus will require significantly new insights even at a general abstract level. As a strong token of this expectation, we point out that regularity is expected to play a paramount role there since, at the hyperbolic level, the regularity class chosen deeply modifies the spectrum when a characteristic point is present in the wave profile ; see [JNR$^+$19, DR22].

The rest of the paper is organized as follows. We have decided to shift the derivation of wave profile asymptotics, proving Proposition 10, to Appendix 3.6, because we believe that the backbone of the paper is stability and provide it mostly for completeness' sake. Next section contains a detailed examination of the required spectral preliminaries. The following one explains how to use these to obtain a practical representation of the linearized time-evolution. Though we mostly follow there the arguments in [ZH98], with some twists here and there, we provide a detailed exposition for two distinct reasons. The first one is that we need to track in constructions which parts are ε-uniform and which parts are not, a crucial point in our analysis. The second one is that most of the papers of the field requiring a detailed analysis, as we do, are either extremely long [ZH98] or cut in a few long pieces [MZ03, MZ04b] and we want to save the reader from back-and-forth consultations of the literature. This makes our analysis essentially self-contained (up to basic knowledge of spectral analysis) and we believe that it could serve as a gentle introduction to the latter massive literature. Note however, that, to keep the paper within a reasonable size, we only expound the bare minimum required by our analysis. After these two preliminary sections, we enter into the technical core of the paper, with first a section devoted to detailed linear estimates, including the identification of most-singular parts of the time-evolution as phase variations, and then a section devoted to nonlinear analysis, including adapted nonlinear maximum principles proved through energy estimates and the proof of Theorem 15.

3.2 Spectral analysis

We investigate stability for traveling waves introduced in Proposition 10. We have chosen to carry out all our proofs within co-moving fast variables. Explicitly, we introduce

new unknowns and variables through [4]

$$u(t,x) = \widetilde{u}\underbrace{\left(\frac{t}{\varepsilon}, \frac{x-\sigma_\varepsilon\, t}{\varepsilon}\right)}_{(\widetilde{t},\widetilde{x})}.$$

However, since we never go back to the original slow variables, we drop tildes on fast quantities from now on. One reason to opt for the fast variables is that it provides a simpler reading of size dependencies on ε.

Therefore our starting point is

$$\partial_t u + \partial_x(f(u) - \sigma_\varepsilon\, u) = \partial_x^2 u + \varepsilon\, g(u)\,, \tag{3.2.1}$$

about the stationary solution $\underline{U}_\varepsilon$. Accordingly we consider the operator

$$\mathcal{L}_\varepsilon := -\partial_x((f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)\,\cdot\,) + \partial_x^2 + \varepsilon g'(\underline{U}_\varepsilon) \tag{3.2.2}$$

on $BUC^0(\mathbf{R})$ with domain $BUC^2(\mathbf{R})$.

Though the elements we provide are sufficient to reconstruct the classical theory, the reader may benefit from consulting [KP13] for background on spectral analysis specialized to nonlinear wave stability. In particular, we shall make extensive implicit use of the characterizations of essential spectrum in terms of endstates of wave profiles and of the spectrum at the right-hand side [5] of the essential spectrum [6] in terms of zeroes of Evans' functions. The reader is referred to [Kat76, Dav07] for less specialized, basic background on spectral theory.

The backbone of the theory is the interpretation of spectral properties of one-dimensional differential operators in terms of spatial dynamics and a key-part of the corresponding studies is the investigation of exponential dichotomies. It starts with the identification between the eigenvalue equation

$$(\lambda - \mathcal{L}_\varepsilon)\, v \;=\; 0$$

and the system of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}\, x}\mathbf{V}(x) \;=\; \mathbf{A}_\varepsilon(\lambda, x)\, \mathbf{V}(x)$$

4. Note the slight co-moving inconsistency with the introduction.

5. We picture the complex plane with the real axis pointing to the right and the imaginary axis pointing to the top.

6. There are (at least) two reasonable definitions of essential spectrum, either through failure of satisfying Fredholm property or through failure of satisfying Fredholm property with zero index. In the context of semigroup generators both definitions provide the same right-hand boundary thus the conventional choice is immaterial to stability issues.

for the vector [7] $\mathbf{V} = (v, \partial_x v - (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)\, v)$ where

$$\mathbf{A}_\varepsilon(\lambda, x) := \begin{pmatrix} f'(\underline{U}_\varepsilon) - \sigma_\varepsilon & 1 \\ \lambda - \varepsilon\, g'(\underline{U}_\varepsilon) & 0 \end{pmatrix}. \tag{3.2.3}$$

For later use, we shall denote $\mathbf{\Phi}_\varepsilon^\lambda(x, y)$ the corresponding solution operators, mapping datum at point y to value at point x.

The essential spectrum is characterized in terms of matrices $\mathbf{A}_\varepsilon^r(\lambda) := \mathbf{A}^\varepsilon(\lambda; \underline{u}_{+\infty})$ and $\mathbf{A}_\varepsilon^\ell(\lambda) := \mathbf{A}^\varepsilon(\lambda; \underline{u}_{-\infty})$ with

$$\mathbf{A}^\varepsilon(\lambda; u) := \begin{pmatrix} f'(u) - \sigma_\varepsilon & 1 \\ \lambda - \varepsilon\, g'(u) & 0 \end{pmatrix}. \tag{3.2.4}$$

Eigenvalues of $\mathbf{A}^\varepsilon(\lambda; u)$ are given by

$$\mu_\pm^\varepsilon(\lambda; u) := \frac{f'(u) - \sigma_\varepsilon}{2} \pm \sqrt{\frac{(f'(u) - \sigma_\varepsilon)^2}{4} + \lambda - \varepsilon\, g'(u)} \tag{3.2.5}$$

and are distinct when $\lambda \neq \varepsilon\, g'(u) - \frac{1}{4}\,(f'(u) - \sigma_\varepsilon)^2$. In this case, the matrix may be diagonalized as

$$\mathbf{A}^\varepsilon(\lambda; u) \;=\; \begin{pmatrix} \mathbf{R}_+^\varepsilon(\lambda; u) & \mathbf{R}_-^\varepsilon(\lambda; u) \end{pmatrix} \begin{pmatrix} \mu_+^\varepsilon(\lambda; u) & 0 \\ 0 & \mu_-^\varepsilon(\lambda; u) \end{pmatrix} \begin{pmatrix} \mathbf{L}_+^\varepsilon(\lambda; u) \\ \mathbf{L}_-^\varepsilon(\lambda; u) \end{pmatrix}$$

with

$$\mathbf{R}_\pm^\varepsilon(\lambda; u) := \begin{pmatrix} 1 \\ -\mu_\mp^\varepsilon(\lambda; u) \end{pmatrix}, \qquad \mathbf{L}_\pm^\varepsilon(\lambda; u) := \frac{\begin{pmatrix} \pm\mu_\pm^\varepsilon(\lambda; u) & \pm 1 \end{pmatrix}}{\mu_+^\varepsilon(\lambda; u) - \mu_-^\varepsilon(\lambda; u)}. \tag{3.2.6}$$

The eigenvalues $\mu_\pm^\varepsilon(\lambda; u)$ have distinct real parts when λ does not belong to

$$\mathcal{D}_\varepsilon(u) := \varepsilon\, g'(u) - \tfrac{1}{4}\,(f'(u) - \sigma_\varepsilon)^2 + \mathbf{R}^-.$$

All our spectral studies will take place far from the half-lines $\mathcal{D}_\varepsilon(\underline{u}_{+\infty}) \cup \mathcal{D}_\varepsilon(\underline{u}_{-\infty})$, that correspond to the set termed *absolute spectrum* in [KP13].

From now on, throughout the text, we shall use $\sqrt{\cdot}$ to denote the determination of the square root on $\mathbf{C}\backslash\mathbf{R}^-$ with positive real part.

7. The use of flux variables is not necessary but it simplifies a few computations here and there.

3.2.1 Conjugation to constant coefficients

Our starting point is a conjugation of spectral problems to a piecewise constant coefficient spectral problem. This is mostly relevant in compact zones of the spectral plane and in the literature by Kevin Zumbrun and his collaborators this is known as a *gap lemma* — since a gap or in other words an exponential dichotomy is the key assumption —; see for instance [MZ05b, Lemma 2.6] for a version relevant for the present analysis. Since we need to ensure uniformity in ε for the case at hand we provide both a statement and a proof.

Proposition 11. *Let K be a compact subset of $\mathbf{C}\backslash\mathcal{D}_0(\underline{u}_{+\infty})$. There exist positive constants $(\varepsilon_0, C, \theta)$ such that there exists*[8] *a smooth map*

$$P^r \,:\, [0,\varepsilon_0]\times K\times \mathbf{R}\mapsto GL_2(\mathbf{C})\,,\qquad (\varepsilon,\lambda,x)\mapsto P^r_\varepsilon(\lambda,x)$$

locally uniformly analytic in λ on a neighborhood of K and such that, for any $(\varepsilon,\lambda,x)\in [0,\varepsilon_0]\times K\times[0,+\infty)$,

$$\|P^r_\varepsilon(\lambda,x)-I_2\|\leqslant C\ \mathrm{e}^{-\theta\,|x|}\,,\qquad\qquad \|(P^r_\varepsilon(\lambda,x))^{-1}-I_2\|\leqslant C\ \mathrm{e}^{-\theta\,|x|}\,,$$

and, for any $(\varepsilon,\lambda,x,y)\in[0,\varepsilon_0]\times K\times(\mathbf{R}_+)^2$,

$$\mathbf{\Phi}^\lambda_\varepsilon(x,y)\ =\ P^r_\varepsilon(\lambda,x)\ \mathrm{e}^{(x-y)\,\mathbf{A}^r_\varepsilon(\lambda)}\,(P^r_\varepsilon(\lambda,y))^{-1}\,.$$

The same argument applies to the conjugation on $(-\infty,0]$ with the flow of $\mathbf{A}^\ell_\varepsilon(\lambda)$ and defines a conjugation map denoted P^ℓ from now on.

Proof. The proof is essentially a quantitative "cheap" gap lemma — conjugating only one trajectory instead of solution operators — but applied in $\mathcal{M}_2(\mathbf{C})$ instead of $\mathbf{C}^2$.

Let us first observe that it is sufficient to define P^r on $[0,\varepsilon_0]\times K\times[x_0,+\infty)$ for some suitably large x_0. Indeed then one may extend P^r by

$$P^r_\varepsilon(\lambda,x)\ :=\ \mathbf{\Phi}^\lambda_\varepsilon(x,x_0)\,P^r_\varepsilon(\lambda,x_0)\ \mathrm{e}^{(x_0-x)\,\mathbf{A}^r_\varepsilon(\lambda)}$$

and bounds are extended by a continuity-compactness argument. Likewise the uniformity in ε is simply derived from a continuity-compactness argument since the construction below is continuous at the limit $\varepsilon=0$. Note moreover that in the large-x regime the bound on $(P^r_\varepsilon(\lambda,x))^{-1}$ may be derived from the bound on $P^r_\varepsilon(\lambda,x)$ by using properties of the inverse map.

8. As follows from the proof, $P^r_\varepsilon(\lambda,\cdot)$ is defined as soon as $\lambda\notin\mathcal{D}_\varepsilon(\underline{u}_{+\infty})\cup\mathcal{D}_\varepsilon(\underline{u}_{-\infty})$.

The requirements on $P_\varepsilon^r(\lambda,\cdot)$ are equivalent to the fact that it converges exponentially fast to I_2 at $+\infty$ (uniformly in ε) and that it satisfies for any x

$$\frac{\mathrm{d}}{\mathrm{d}\,x}P_\varepsilon^r(\lambda,x) \;=\; \mathcal{A}_\varepsilon^r(\lambda)(P_\varepsilon^r(\lambda,x)) + (\mathbf{A}_\varepsilon(\lambda,x) - \mathbf{A}_\varepsilon^r(\lambda))\,P_\varepsilon^r(\lambda,x)$$

where $\mathcal{A}_\varepsilon^r(\lambda) := \mathcal{A}^\varepsilon(\lambda;\underline{u}_{+\infty})$ is a linear operator on $\mathcal{M}_2(\mathbf{C})$ defined through

$$\mathcal{A}^\varepsilon(\lambda;u)(P) := [\mathbf{A}^\varepsilon(\lambda;u),P] = \mathbf{A}^\varepsilon(\lambda;u)P - P\mathbf{A}^\varepsilon(\lambda;u)\,.$$

When $\lambda \notin \mathcal{D}_\varepsilon(u)$, $\mathcal{A}^\varepsilon(\lambda;u)$ admits

$$(\mathbf{R}_+^\varepsilon(\lambda;u)\mathbf{L}_-^\varepsilon(\lambda;u),\quad \mathbf{R}_-^\varepsilon(\lambda;u)\mathbf{L}_+^\varepsilon(\lambda;u),\quad \mathbf{R}_+^\varepsilon(\lambda;u)\mathbf{L}_+^\varepsilon(\lambda;u),\quad \mathbf{R}_-^\varepsilon(\lambda;u)\mathbf{L}_-^\varepsilon(\lambda;u))$$

as a basis of eigenvectors corresponding to eigenvalues

$$(\mu_+^\varepsilon(\lambda;u) - \mu_-^\varepsilon(\lambda;u),\quad -(\mu_+^\varepsilon(\lambda;u) - \mu_-^\varepsilon(\lambda;u)),\quad 0,\quad 0)\,.$$

Note that I_2 always lies in the kernel of $\mathcal{A}^\varepsilon(\lambda;u)$. We denote by $\Pi_u^\varepsilon(\lambda;u)$, $\Pi_s^\varepsilon(\lambda;u)$, $\Pi_0^\varepsilon(\lambda;u)$ the corresponding spectral projections respectively on the unstable space, the stable space and the kernel of $\mathcal{A}^\varepsilon(\lambda;u)$ and for further later study we point out that they are given as

$$\begin{aligned}
\Pi_u^\varepsilon(\lambda;u)(P) &= \mathbf{L}_+^\varepsilon(\lambda;u)\,P\,\mathbf{R}_-^\varepsilon(\lambda;u) \quad \mathbf{R}_+^\varepsilon(\lambda;u)\mathbf{L}_-^\varepsilon(\lambda;u)\,,\\
\Pi_s^\varepsilon(\lambda;u)(P) &= \mathbf{L}_-^\varepsilon(\lambda;u)\,P\,\mathbf{R}_+^\varepsilon(\lambda;u) \quad \mathbf{R}_-^\varepsilon(\lambda;u)\mathbf{L}_+^\varepsilon(\lambda;u)\,,\\
\Pi_0^\varepsilon(\lambda;u)(P) &= \mathbf{L}_+^\varepsilon(\lambda;u)\,P\,\mathbf{R}_+^\varepsilon(\lambda;u) \quad \mathbf{R}_+^\varepsilon(\lambda;u)\mathbf{L}_+^\varepsilon(\lambda;u)\\
&\quad + \mathbf{L}_-^\varepsilon(\lambda;u)\,P\,\mathbf{R}_-^\varepsilon(\lambda;u) \quad \mathbf{R}_-^\varepsilon(\lambda;u)\mathbf{L}_-^\varepsilon(\lambda;u)\,.
\end{aligned}$$

Then the result follows when θ is sufficiently small and x_0 is sufficiently large from a use of the implicit function theorem on

$$\begin{aligned}
P_\varepsilon^r(\lambda,x) \;=\; \mathrm{I}_2 \;&-\; \int_x^{+\infty} \Pi_0^\varepsilon(\lambda;\underline{u}_{+\infty})\,((\mathbf{A}_\varepsilon(\lambda,y) - \mathbf{A}_\varepsilon^r(\lambda))\,P_\varepsilon^r(\lambda,y))\;\mathrm{d}\,y\\
&- \int_x^{+\infty} \mathrm{e}^{-(y-x)\,(\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty}))}\;\Pi_u^\varepsilon(\lambda;\underline{u}_{+\infty})\,((\mathbf{A}_\varepsilon(\lambda,y) - \mathbf{A}_\varepsilon^r(\lambda))\,P_\varepsilon^r(\lambda,y))\;\mathrm{d}\,y\\
&+ \int_{x_0}^{x} \mathrm{e}^{-(x-y)\,(\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty}))}\;\Pi_s^\varepsilon(\lambda;\underline{u}_{+\infty})\,((\mathbf{A}_\varepsilon(\lambda,y) - \mathbf{A}_\varepsilon^r(\lambda))\,P_\varepsilon^r(\lambda,y))\;\mathrm{d}\,y
\end{aligned}$$

with norm control on matrix-valued maps through

$$\sup_{x\geqslant x_0} \mathrm{e}^{\theta\,|x|}\,\|P(x) - \mathrm{I}_2\|\,.$$

□

Remark 6. *The properties of the foregoing proposition do not determine P^r uniquely. The normalizing choice made in the proof is $\Pi_s^\varepsilon(\lambda;\underline{u}_{+\infty})\,(P_\varepsilon^r(\lambda,x_0)) = 0_2$ but we could have replaced 0_2 with any analytic choice of an element of the stable space of $\mathcal{A}^\varepsilon(\lambda;\underline{u}_{+\infty})$.*

Remark 7. *The proposition is sufficient to prove classical results about the determination of the essential spectrum from endstates spectra.*

We now investigate possible failure of uniformity in the regime of large spectral parameters. In the literature by Kevin Zumbrun and his collaborators, similar purposes are achieved through comparison of unstable manifolds with their frozen-coefficients approximations by a type of lemma termed there *tracking lemma*; see for instance [HLZ09, BJRZ11]. The rationale is that to large-frequencies smooth coefficients seem almost constant and thus may be treated in some adiabatic way, a fact ubiquitous in high-frequency/semiclassical analysis.

We follow here a different path and rather effectively build a conjugation as in the foregoing gap lemma. The first step is a suitable scaling to ensure some form of uniformity in the large-x contraction argument of the proof of Proposition 11. The second-step is a high-frequency approximate diagonalization combined with an explicit solving of the leading-order part of the system ensuring that in the large-frequency regime the latter construction could actually be carried out with $x_0 = 0$.

Proposition 12. *There exist positive constants $(\varepsilon_0, C, \theta, \delta)$ such that setting*

$$\Omega_\delta := \left\{ \lambda\,;\ \Re\left(\sqrt{\lambda}\right) \geqslant \frac{1}{\delta} \right\}$$

there exists a smooth map

$$P^{r,HF}\ :\ [0,\varepsilon_0]\times\Omega_\delta\times\mathbf{R}\mapsto GL_2(\mathbf{C})\,,\qquad (\varepsilon,\lambda,x)\mapsto P_\varepsilon^{r,HF}(\lambda,x)$$

locally uniformly analytic in λ on a neighborhood of Ω_δ and such that, for any $(\varepsilon,\lambda,x)\in[0,\varepsilon_0]\times\Omega_\delta\times[0,+\infty)$,

$$\left\| \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} P_\varepsilon^{r,HF}(\lambda,x) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} - \mathrm{e}^{-\frac{1}{2}\int_x^{+\infty}(f'(\underline{U}_\varepsilon(y))-f'(\underline{u}_{+\infty}))\,\mathrm{d}\,y}\ I_2 \right\| \leqslant C\,\frac{\mathrm{e}^{-\theta\,|x|}}{\Re\left(\sqrt{\lambda}\right)}\,,$$

$$\left\| \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} (P_\varepsilon^{r,HF}(\lambda,x))^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} - \mathrm{e}^{\frac{1}{2}\int_x^{+\infty}(f'(\underline{U}_\varepsilon(y))-f'(\underline{u}_{+\infty}))\,\mathrm{d}\,y}\ I_2 \right\| \leqslant C\,\frac{\mathrm{e}^{-\theta\,|x|}}{\Re\left(\sqrt{\lambda}\right)}\,,$$

and, for any $(\varepsilon, \lambda, x, y) \in [0, \varepsilon_0] \times \Omega_\delta \times \mathbf{R}^2$,

$$\mathbf{\Phi}_\varepsilon^\lambda(x,y) \;=\; P_\varepsilon^{r,HF}(\lambda, x)\; \mathrm{e}^{(x-y)\,\mathbf{A}_\varepsilon^r(\lambda)}\; (P_\varepsilon^{r,HF}(\lambda, y))^{-1}\,.$$

We point out that for our main purposes we do not need to identify explicitly the leading order part of the conjugation.

As for Proposition 11 the same argument applies to the conjugation on $(-\infty, 0]$ with the flow of $\mathbf{A}_\varepsilon^\ell(\lambda)$ and defines a conjugation map denoted $P^{\ell,HF}$ from now on.

Proof. As a preliminary remark, we observe that the condition

$$\Re\left(\sqrt{\lambda}\right) \gg 1,$$

and

$$\Re(\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})) \gg 1$$

are equivalent, with uniform control of one by the other and vice versa.

Scaling P_ε^r to Q_ε^r defined as

$$Q_\varepsilon^r(\lambda, \cdot) := \begin{pmatrix} 1 & 0 \\ 0 & \mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty}) \end{pmatrix}^{-1} P_\varepsilon^r(\lambda, \cdot) \begin{pmatrix} 1 & 0 \\ 0 & \mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty}) \end{pmatrix}$$

removes high-frequency singularities by replacing $\mathbf{R}_\pm^\varepsilon(\lambda; \underline{u}_{+\infty})$ and $\mathbf{L}_\pm^\varepsilon(\lambda; \underline{u}_{+\infty})$ with

$$\begin{pmatrix} 1 \\ -\frac{\mu_\pm^\varepsilon(\lambda; \underline{u}_{+\infty})}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} \end{pmatrix}, \qquad \begin{pmatrix} \frac{\pm\mu_\pm^\varepsilon(\lambda; \underline{u}_{+\infty})}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} & \pm 1 \end{pmatrix},$$

whereas the only other effect is the replacement of $\mathbf{A}_\varepsilon(\lambda, y) - \mathbf{A}_\varepsilon^r(\lambda)$ with

$$\begin{pmatrix} f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}) & 0 \\ -\varepsilon\, \frac{g'(\underline{U}_\varepsilon) - g'(\underline{u}_{+\infty})}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} & 0 \end{pmatrix}.$$

At this stage, let us choose coordinates to identify $\mathcal{M}_2(\mathbf{C})$ with $\mathbf{C}^4$ in such a way that $\mathbf{C}^2 \times \{0\}^2$, $\{0\} \times \mathbf{C} \times \{0\}$, $\{0\}^3 \times \mathbf{C}$, $(1,0,0,0)$ and $(0,1,0,0)$ correspond respectively — after scaling and choice of coordinates — to the kernel of $\mathcal{A}_\varepsilon^r(\lambda)$, its unstable space, its stable space, I_2 and $\dfrac{1}{\mu_+^\varepsilon(\lambda; \underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda; \underline{u}_{+\infty})} A_{\varepsilon,HF}^r$. Then the problem to be solved takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\boldsymbol{\alpha}(x) = \mathcal{B}_0(x)\,\boldsymbol{\alpha}(x) + \mathcal{O}(\mathrm{e}^{-\theta\,|x|}) \begin{pmatrix} \beta \\ \gamma \end{pmatrix}(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\beta(x) = (\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty}) + \omega_+(x))\,\beta(x) + \mathcal{O}(\mathrm{e}^{-\theta\,|x|}) \begin{pmatrix} \boldsymbol{\alpha} \\ \gamma \end{pmatrix}(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}\,x}\gamma(x) = -(\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty}) + \omega_-(x))\,\gamma(x) + \mathcal{O}(\mathrm{e}^{-\theta\,|x|}) \begin{pmatrix} \boldsymbol{\alpha} \\ \beta \end{pmatrix}(x)$$

with $(1,0,0,0)$ as limiting value at $+\infty$, for some[9] $\theta > 0$, where $\mathcal{B}_0(x)$, $\omega_+(x)$, $\omega_-(x)$ are also of the form $\mathcal{O}(\mathrm{e}^{-\theta\,|x|})$.

It follows that when $\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})$ is sufficiently large, by a further change of variables differing from I_4 by a block off-diagonal term

$$\begin{pmatrix} \boldsymbol{\alpha}_{bis} \\ \beta_{bis} \\ \gamma_{bis} \end{pmatrix}(x) \;=\; \left(\mathrm{I}_4 + \mathcal{O}\left(\frac{\mathrm{e}^{-\theta\,|x|}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\right)\right) \begin{pmatrix} \boldsymbol{\alpha} \\ \beta \\ \gamma \end{pmatrix}(x)$$

one may transform the problem to

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\boldsymbol{\alpha}_{bis}(x) = \widetilde{\mathcal{B}}_0(x)\,\boldsymbol{\alpha}_{bis}(x) + \mathcal{O}\left(\frac{\mathrm{e}^{-\theta\,|x|}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\right) \begin{pmatrix} \beta_{bis} \\ \gamma_{bis} \end{pmatrix}(x)$$
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,x}\beta_{bis}(x) = {} & (\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty}) + \widetilde{\omega}_+(x))\,\beta_{bis}(x) \\ & + \mathcal{O}\left(\frac{\mathrm{e}^{-\theta\,|x|}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\right) \begin{pmatrix} \boldsymbol{\alpha}_{bis} \\ \gamma_{bis} \end{pmatrix}(x)\end{aligned}$$
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,x}\gamma_{bis}(x) = {} & -(\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty}) + \widetilde{\omega}_-(x))\,\gamma_{bis}(x) \\ & + \mathcal{O}\left(\frac{\mathrm{e}^{-\theta\,|x|}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\right) \begin{pmatrix} \boldsymbol{\alpha}_{bis} \\ \beta_{bis} \end{pmatrix}(x)\end{aligned}$$

with $(1,0,0,0)$ as limiting value at $+\infty$, where $\widetilde{\mathcal{B}}_0 - \mathcal{B}_0$, $\widetilde{\omega}_- - \omega_-$ and $\widetilde{\omega}_+ - \omega_+$ are all of the form

$$\mathcal{O}\left(\frac{\mathrm{e}^{-\theta\,|x|}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\right).$$

Now, we point out that there is a single solution to the leading-order part

$$\frac{\mathrm{d}}{\mathrm{d}\,x}\boldsymbol{\alpha}_{main}(x) = \widetilde{\mathcal{B}}_0(x)\,\boldsymbol{\alpha}_{main}(x)$$

with $(1,0)$ as limiting value at $+\infty$. This follows from a fixed point argument on $x \geqslant x_0$ for x_0 large followed by a continuation argument. We may be even more explicit. Indeed

9. Along the proof we allow ourselves to change the precise value of θ from line to line.

an explicit computation yields

$$\widetilde{\mathcal{B}}_0(x) \;=\; \mathcal{B}_{main}(x) + \mathcal{O}\left(\frac{\mathrm{e}^{-\theta\,|x|}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\right)$$

with

$$\mathcal{B}_{main}(x)\begin{pmatrix}1\\0\end{pmatrix} \;=\; \frac{1}{2}(f'(\underline{U}_\varepsilon(x)) - f'(\underline{u}_{+\infty}))\begin{pmatrix}1\\0\end{pmatrix}$$

so that

$$\boldsymbol{\alpha}_{main}(x) \;=\; \mathrm{e}^{-\frac{1}{2}\int_x^{+\infty}(f'(\underline{U}_\varepsilon(y))-f'(\underline{u}_{+\infty}))\,\mathrm{d}\,y}\begin{pmatrix}1\\0\end{pmatrix} + \mathcal{O}\left(\frac{\mathrm{e}^{-\theta\,|x|}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\right).$$

The proof is thus achieved by a fixed point argument on a problem of type

$$\begin{aligned}
\boldsymbol{\alpha}_{bis}(x) \;=\; \boldsymbol{\alpha}_{main}(x) \;-& \int_x^{+\infty} \Phi_0(x,y) \\
&\times \mathcal{O}\left(\frac{\mathrm{e}^{-\theta\,|y|}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\right)\begin{pmatrix}\beta_{bis}\\ \gamma_{bis}\end{pmatrix}(y)\;\mathrm{d}\,y \\
\beta_{bis}(x) = -& \int_x^{+\infty} \mathrm{e}^{-(y-x)\,(\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty})-\mu_-^\varepsilon(\lambda;\underline{u}_{+\infty}))+\int_y^x \widetilde{\omega}_+} \\
&\times \mathcal{O}\left(\frac{\mathrm{e}^{-\theta\,|y|}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\right)\begin{pmatrix}\boldsymbol{\alpha}_{bis}\\ \gamma_{bis}\end{pmatrix}(y)\;\mathrm{d}\,y \\
\gamma_{bis}(x) = \int_0^x & \mathrm{e}^{-(x-y)\,(\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty})-\mu_-^\varepsilon(\lambda;\underline{u}_{+\infty}))-\int_y^x \widetilde{\omega}_-} \\
&\times \mathcal{O}\left(\frac{\mathrm{e}^{-\theta\,|y|}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty}) - \mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\right)\begin{pmatrix}\boldsymbol{\alpha}_{bis}\\ \beta_{bis}\end{pmatrix}(y)\;\mathrm{d}\,y
\end{aligned}$$

where Φ_0 denotes the solution operator associated with $\widetilde{\mathcal{B}}_0$. □

In the following we shall complete Proposition 12 that provides $P^{r,HF}$ on Ω_δ with an application of Proposition 11 on

$$K_\delta := \left\{\,\lambda\,;\;\; d(\lambda, \mathcal{D}_0(\underline{u}_{+\infty})) \;\geqslant\; \delta \quad \text{and} \quad |\lambda| \leqslant \frac{1}{\delta}\right\}.$$

When δ is sufficiently small Ω_δ and K_δ overlaps. Yet a priori P^r and $P^{r,HF}$ differ from each other even in regions where both exist. Fortunately the implied possible mismatch disappears at the level of Green functions.

3.2.2 Evans' function and its asymptotics

Now wherever it makes sense we set

$$\begin{aligned}
\mathbf{V}^{r,s}_\varepsilon(\lambda,x) &:= \mathrm{e}^{x\,\mu^\varepsilon_-(\lambda;\underline{u}_{+\infty})}\; P^r_\varepsilon(\lambda,x)\,\mathbf{R}^\varepsilon_-(\lambda;\underline{u}_{+\infty})\,,\\
\mathbf{V}^{r,u}_\varepsilon(\lambda,x) &:= \mathrm{e}^{x\,\mu^\varepsilon_+(\lambda;\underline{u}_{+\infty})}\; P^r_\varepsilon(\lambda,x)\,\mathbf{R}^\varepsilon_+(\lambda;\underline{u}_{+\infty})\,,\\
\mathbf{V}^{\ell,s}_\varepsilon(\lambda,x) &:= \mathrm{e}^{x\,\mu^\varepsilon_-(\lambda;\underline{u}_{-\infty})}\; P^\ell_\varepsilon(\lambda,x)\,\mathbf{R}^\varepsilon_-(\lambda;\underline{u}_{-\infty})\,,\\
\mathbf{V}^{\ell,u}_\varepsilon(\lambda,x) &:= \mathrm{e}^{x\,\mu^\varepsilon_+(\lambda;\underline{u}_{-\infty})}\; P^\ell_\varepsilon(\lambda,x)\,\mathbf{R}^\varepsilon_+(\lambda;\underline{u}_{-\infty})\,,
\end{aligned}$$

and similarly for $\mathbf{V}^{r,s,HF}_\varepsilon$, $\mathbf{V}^{r,u,HF}_\varepsilon$, $\mathbf{V}^{\ell,s,HF}_\varepsilon$ and $\mathbf{V}^{\ell,u,HF}_\varepsilon$. Note that notation is used here to recall stable and unstable spaces, we also use them in areas of the spectral plane where they do not match with stable and unstable spaces. Instead, this fits analytic continuation of generators of stable/unstable spaces.

Correspondingly we define the Evans' function

$$D_\varepsilon(\lambda) := \det\begin{pmatrix}\mathbf{V}^{r,s}_\varepsilon(\lambda,0) & \mathbf{V}^{\ell,u}_\varepsilon(\lambda,0)\end{pmatrix} \tag{3.2.7}$$

and its high-frequency counterpart D^{HF}_ε. Note that we define the Evans function at point 0 but on one hand, we do not make use of any particular property due to normalization so that the point 0 could be replaced with any other point and on the other hand relations between Evans functions at different points are simply derived from Liouville's formula for Wronskians. For instance,

$$\det\begin{pmatrix}\mathbf{V}^{r,s}_\varepsilon(\lambda,x) & \mathbf{V}^{\ell,u}_\varepsilon(\lambda,x)\end{pmatrix} \;=\; D_\varepsilon(\lambda)\;\mathrm{e}^{\int_0^x \mathrm{Tr}(\mathbf{A}_\varepsilon(\lambda,\cdot))} \;=\; D_\varepsilon(\lambda)\;\mathrm{e}^{\int_0^x (f'(\underline{U}_\varepsilon)-\sigma_\varepsilon)}\;. \tag{3.2.8}$$

A simple corollary to Proposition 12 is

Corollary 3. *Uniformly in ε (sufficiently small)*

$$\lim_{\Re(\sqrt{\lambda})\to\infty}\frac{D^{HF}_\varepsilon(\lambda)}{\sqrt{\lambda}} \;=\; 2\;\mathrm{e}^{-\frac{1}{2}\int_0^{+\infty}(f'(\underline{U}_\varepsilon(y))-f'(\underline{u}_{+\infty}))\,\mathrm{d}\,y+\frac{1}{2}\int_{-\infty}^0(f'(\underline{U}_\varepsilon(y))-f'(\underline{u}_{-\infty}))\,\mathrm{d}\,y}\;.$$

To complete Corollary 3, we derive information on compacts sets of λ in the limit $\varepsilon\to 0$.

Proposition 13. *There exists $\eta_0>0$ such that for any $\delta>0$ there exist positive (ε_0,c_0) such that for any $\varepsilon\in[0,\varepsilon_0]$, $D_\varepsilon(\cdot)$ is well-defined on*

$$K_{\eta_0,\delta} := \left\{\lambda\,;\;\; d(\lambda,(-\infty,-\eta_0]) \;\geqslant\; \min\left(\left\{\delta,\frac{\eta_0}{2}\right\}\right) \quad \textit{and} \quad |\lambda|\leqslant\frac{1}{\delta}\right\}$$

and for any $\lambda \in K_{\eta_0,\delta}$,

$$|D_\varepsilon(\lambda)| \geqslant c_0 \min(\{1, |\lambda|\}) .$$

Proof. We derive the result from Sturm-Liouville theory and regularity in ε. To apply Sturm-Liouville theory, we introduce the weight

$$\omega_\varepsilon(x) := \mathrm{e}^{\frac{1}{2}\int_0^x (f'(\underline{U}_\varepsilon(y)) - \sigma_\varepsilon)\,\mathrm{d}\,y} .$$

We observe that considered as an operator on $L^2(\mathbf{R})$ with domain $H^2(\mathbf{R})$ the operator

$$L_\varepsilon := \frac{1}{\omega_\varepsilon} \mathcal{L}_\varepsilon \, (\omega_\varepsilon \, \cdot \,)$$

is self-adjoint and in the region of interest it possesses no essential spectrum and its eigenvalues agree in location and algebraic multiplicity with the roots of D_ε. As a consequence the zeroes of D_ε are real and since $\underline{U}'_\varepsilon/\omega_\varepsilon$ is a nowhere-vanishing eigenvector for the eigenvalue 0, 0 is a simple root of D_ε and D_ε does not vanish on $(0, +\infty)$. From here the corresponding bound is deduced through a continuity-compactness argument in ε. □

3.3 Green functions

Now we use the introduced spectral objects to obtain representation formulas for linearized solution operators.

3.3.1 Duality

To begin with, to provide explicit formulas for spectral Green functions to be introduced below, we extend to dual problems the conclusions of Section 3.2. Note that the duality we are referring to is not related to any particular choice of a specific Banach space but rather distributional/algebraic.

To begin with, we introduce the formal adjoint

$$\mathcal{L}_\varepsilon^{adj} := (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)\partial_x + \partial_x^2 + \varepsilon g'(\underline{U}_\varepsilon) \tag{3.3.1}$$

and note that for any sufficiently smooth v, w, and any points (x_0, x_1)

$$\mathbf{V} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w \\ \partial_x w \end{pmatrix} (x_1) - \mathbf{V} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w \\ \partial_x w \end{pmatrix} (x_0) \; = \; \int_{x_0}^{x_1} (w \, \mathcal{L}_\varepsilon v - v \, \mathcal{L}_\varepsilon^{adj} w) \tag{3.3.2}$$

with $\mathbf{V} = (v, \partial_x v - (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)\, v)$. As a first simple consequence of (3.3.2) note that if $(\lambda, y, v_y^r, v_y^\ell)$ are such that $(\lambda - \mathcal{L}_\varepsilon) v_y^r = 0$ and $(\lambda - \mathcal{L}_\varepsilon) v_y^\ell = 0$, then the function

$$\varphi_y \,:\, \mathbf{R} \to \mathbf{C}\,, \qquad x \mapsto \begin{cases} v_y^r(x) & \text{if } x > y \\ v_y^\ell(x) & \text{if } x < y \end{cases}$$

solves $(\lambda - \mathcal{L}_\varepsilon)\varphi_y \,=\, \delta_y$ if and only if

$$\begin{aligned} v_y^r(y) \,&=\, v_y^\ell(y)\,, \\ \partial_x v_y^r(y) - (f'(\underline{U}_\varepsilon)(y) - \sigma_\varepsilon)\, v_y^r(y) \,&=\, \partial_x v_y^\ell(y) - (f'(\underline{U}_\varepsilon)(y) - \sigma_\varepsilon)\, v_y^\ell(y) - 1\,. \end{aligned}$$

Specializing to the tensorized case where $v_y^r(x) \,=\, v^r(x)\,\alpha(y)$, $v_y^\ell(x) \,=\, v^\ell(x)\,\beta(y)$, note that the foregoing conditions are equivalent to

$$\begin{pmatrix} \mathbf{V}^r(y) & \mathbf{V}^\ell(y) \end{pmatrix} \begin{pmatrix} \alpha(y) \\ -\beta(y) \end{pmatrix} \,=\, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

where $\mathbf{V}^\sharp = (v^\sharp, \partial_x v^\sharp - (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)\, v^\sharp)$, $\sharp \in \{r, \ell\}$. Hence, we need to find vectors satisfying some orthogonality property to identify the inverse of the matrix :

$$\begin{pmatrix} \mathbf{V}^r(y) & \mathbf{V}^\ell(y) \end{pmatrix}$$

To go further, we identify

$$(\lambda - \mathcal{L}_\varepsilon^{adj})\, w \,=\, 0$$

and the system of ODEs

$$\frac{\mathrm{d}}{\mathrm{d}\, x} \mathbf{W}(x) \,=\, \widetilde{\mathbf{A}}_\varepsilon(\lambda, x)\, \mathbf{W}(x)$$

for the vector $\mathbf{W} = (w, \partial_x w)$ where

$$\widetilde{\mathbf{A}}_\varepsilon(\lambda, x) \,:=\, \begin{pmatrix} 0 & 1 \\ \lambda - \varepsilon\, g'(\underline{U}_\varepsilon) & -(f'(\underline{U}_\varepsilon) - \sigma_\varepsilon) \end{pmatrix}\,. \tag{3.3.3}$$

Note that

$$\widetilde{\mathbf{A}}_\varepsilon(\lambda, x) \,=\, \mathbf{A}_\varepsilon(\lambda, x) \,-\, (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)\, \mathrm{I}_2$$

so that all the proofs of Section 3.2 purely based on limiting-matrices spectral gaps arguments apply equally well to the corresponding dual problems under the exact same assumptions. Alternatively one may derive results on dual problems by using directly the

relation between solution operators

$$\widetilde{\mathbf{\Phi}}^{\lambda}_{\varepsilon}(x,y) \;=\; \mathbf{\Phi}^{\lambda}_{\varepsilon}(x,y)\; \mathrm{e}^{-\int_y^x (f'(\underline{U}_\varepsilon)-\sigma_\varepsilon)}\;.$$

Here and elsewhere throughout the text from now on we denote with a $\widetilde{\ }$ all quantities arising from dual problems. Let us point out that our choices lead to

$$\widetilde{\mu}^{\varepsilon}_{\pm}(\lambda;u) \;=\; \mu^{\varepsilon}_{\pm}(\lambda;u) - \,(f'(u)-\sigma_\varepsilon)\,,\quad \widetilde{\mathbf{R}}^{\varepsilon}_{\pm}(\lambda;u) \;=\; \mathbf{R}^{\varepsilon}_{\pm}(\lambda;u)\,,\quad \widetilde{\mathbf{L}}^{\varepsilon}_{\pm}(\lambda;u) \;=\; \mathbf{L}^{\varepsilon}_{\pm}(\lambda;u)\,,$$

and

$$\widetilde{P}^{r}_{\varepsilon}(\lambda,x) \;=\; P^{r}_{\varepsilon}(\lambda,x)\; \mathrm{e}^{\int_x^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}\,,\quad \widetilde{P}^{\ell}_{\varepsilon}(\lambda,x) \;=\; P^{\ell}_{\varepsilon}(\lambda,x)\; \mathrm{e}^{-\int_{-\infty}^x (f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}\,,$$

(and likewise for high-frequency versions).

Proposition 14. *Let K be a compact subset of $\mathbf{C}\backslash(\mathcal{D}_0(\underline{u}_{+\infty})\cup\mathcal{D}_0(\underline{u}_{+\infty}))$. There exists $\varepsilon_0>0$ such that there exist smooth maps*

$$\begin{aligned}
&\tau^r \,:\, [0,\varepsilon_0]\times K\mapsto \mathbf{C}\,,\ (\varepsilon,\lambda)\mapsto \tau^r_\varepsilon(\lambda)\,, && \rho^r \,:\, [0,\varepsilon_0]\times K\mapsto \mathbf{C}\,,\ (\varepsilon,\lambda)\mapsto \rho^r_\varepsilon(\lambda)\,,\\
&\tau^\ell \,:\, [0,\varepsilon_0]\times K\mapsto \mathbf{C}\,,\ (\varepsilon,\lambda)\mapsto \tau^\ell_\varepsilon(\lambda)\,, && \rho^\ell \,:\, [0,\varepsilon_0]\times K\mapsto \mathbf{C}\,,\ (\varepsilon,\lambda)\mapsto \rho^\ell_\varepsilon(\lambda)\,,\\
&\widetilde{\tau}^r \,:\, [0,\varepsilon_0]\times K\mapsto \mathbf{C}\,,\ (\varepsilon,\lambda)\mapsto \widetilde{\tau}^r_\varepsilon(\lambda)\,, && \widetilde{\rho}^r \,:\, [0,\varepsilon_0]\times K\mapsto \mathbf{C}\,,\ (\varepsilon,\lambda)\mapsto \widetilde{\rho}^r_\varepsilon(\lambda)\,,\\
&\widetilde{\tau}^\ell \,:\, [0,\varepsilon_0]\times K\mapsto \mathbf{C}\,,\ (\varepsilon,\lambda)\mapsto \widetilde{\tau}^\ell_\varepsilon(\lambda)\,, && \widetilde{\rho}^\ell \,:\, [0,\varepsilon_0]\times K\mapsto \mathbf{C}\,,\ (\varepsilon,\lambda)\mapsto \widetilde{\rho}^\ell_\varepsilon(\lambda)\,,
\end{aligned}$$

locally uniformly analytic in λ on a neighborhood of K and such that, for any $(\varepsilon,\lambda)\in[0,\varepsilon_0]\times K$, for any $x\in\mathbf{R}$

$$\begin{aligned}
\mathbf{V}^{r,s}_\varepsilon(\lambda,x) &= \rho^r_\varepsilon(\lambda)\,\mathbf{V}^{\ell,s}_\varepsilon(\lambda,x) + \tau^r_\varepsilon(\lambda)\,\mathbf{V}^{\ell,u}_\varepsilon(\lambda,x)\,,\\
\mathbf{V}^{\ell,u}_\varepsilon(\lambda,x) &= \rho^\ell_\varepsilon(\lambda)\,\mathbf{V}^{r,u}_\varepsilon(\lambda,x) + \tau^\ell_\varepsilon(\lambda)\,\mathbf{V}^{r,s}_\varepsilon(\lambda,x)\,,\\
\widetilde{\mathbf{V}}^{r,s}_\varepsilon(\lambda,x) &= \widetilde{\rho}^r_\varepsilon(\lambda)\,\widetilde{\mathbf{V}}^{\ell,s}_\varepsilon(\lambda,x) + \widetilde{\tau}^r_\varepsilon(\lambda)\,\widetilde{\mathbf{V}}^{\ell,u}_\varepsilon(\lambda,x)\,,\\
\widetilde{\mathbf{V}}^{\ell,u}_\varepsilon(\lambda,x) &= \widetilde{\rho}^\ell_\varepsilon(\lambda)\,\widetilde{\mathbf{V}}^{r,u}_\varepsilon(\lambda,x) + \widetilde{\tau}^\ell_\varepsilon(\lambda)\,\widetilde{\mathbf{V}}^{r,s}_\varepsilon(\lambda,x)\,,
\end{aligned}$$

and

$$\begin{aligned}
\rho^r_\varepsilon(\lambda) &= D_\varepsilon(\lambda)\,\frac{\mathrm{e}^{-\int_{-\infty}^0 (f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}}{\mu^\varepsilon_+(\lambda;\underline{u}_{-\infty})-\mu^\varepsilon_-(\lambda;\underline{u}_{-\infty})}\,,\\
\rho^\ell_\varepsilon(\lambda) &= D_\varepsilon(\lambda)\,\frac{\mathrm{e}^{\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}}{\mu^\varepsilon_+(\lambda;\underline{u}_{+\infty})-\mu^\varepsilon_-(\lambda;\underline{u}_{+\infty})}\,,
\end{aligned}$$

$$\widetilde{\rho}_\varepsilon^r(\lambda) \;=\; D_\varepsilon(\lambda)\,\frac{\mathrm{e}^{\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}}{\mu_+^\varepsilon(\lambda;\underline{u}_{-\infty})-\mu_-^\varepsilon(\lambda;\underline{u}_{-\infty})}\,,$$
$$\widetilde{\rho}_\varepsilon^\ell(\lambda) \;=\; D_\varepsilon(\lambda)\,\frac{\mathrm{e}^{-\int_{-\infty}^0(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}}{\mu_+^\varepsilon(\lambda;\underline{u}_{+\infty})-\mu_-^\varepsilon(\lambda;\underline{u}_{+\infty})}\,.$$

As a consequence for such a (ε,λ) *and any* $x\in\mathbf{R}$

$$\mathbf{V}_\varepsilon^{\ell,u}(\lambda,x)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,x)=\,0\,,\qquad \mathbf{V}_\varepsilon^{r,s}(\lambda,x)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,x)=\,0\,,$$

$$\mathbf{V}_\varepsilon^{r,s}(\lambda,x)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,x)=\,-D_\varepsilon(\lambda)\;\mathrm{e}^{-\int_{-\infty}^0(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}\,,$$
$$\mathbf{V}_\varepsilon^{\ell,u}(\lambda,x)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,x)=\,D_\varepsilon(\lambda)\;\mathrm{e}^{\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}\;.$$

Notation τ, ρ is used here to echo transmission/reflection coefficients of the classical scattering framework.

A corresponding proposition holds for the high-frequency regime.

Proof. All the properties are readily obtained by combining the fact that both $(\mathbf{V}_\varepsilon^{r,s}(\lambda,\cdot),\mathbf{V}_\varepsilon^{r,u}(\lambda,\cdot))$ and $(\mathbf{V}_\varepsilon^{\ell,s}(\lambda,\cdot),\mathbf{V}_\varepsilon^{\ell,u}(\lambda,\cdot))$ form a basis of solutions of the spectral system of ODEs, the Liouville formula for Wronskians and duality relation (3.3.2). □

We thus have that :

$$\begin{pmatrix}\mathbf{V}_\varepsilon^{r,s}(y) & \mathbf{V}_\varepsilon^{\ell,u}(y)\end{pmatrix}^{-1}=\begin{pmatrix}\dfrac{e_2\cdot\widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y)}{D_\varepsilon(\lambda)\;\mathrm{e}^{-\int_{-\infty}^0(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}} & \dfrac{e_1\cdot\widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y)}{-D_\varepsilon(\lambda)\;\mathrm{e}^{-\int_{-\infty}^0(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}}\\ -\dfrac{e_2\cdot\widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)}{D_\varepsilon(\lambda)\;\mathrm{e}^{\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}} & \dfrac{e_1\cdot\widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)}{D_\varepsilon(\lambda)\;\mathrm{e}^{\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}}\end{pmatrix}.$$

Proposition 15. *There exist positive constants* (ε_0,C,δ) *such that setting with* Ω_δ *as in Proposition 12*

$$\Omega_\delta:=\left\{\lambda\,;\;\Re\left(\sqrt{\lambda}\right)\geqslant\frac{1}{\delta}\right\}$$

there exist on $[0,\varepsilon_0]\times\Omega_\delta$ *maps* $\tau^{r,HF}$, $\rho^{r,HF}$, $\tau^{\ell,HF}$, $\rho^{\ell,HF}$, $\widetilde{\tau}^{r,HF}$, $\widetilde{\rho}^{r,HF}$, $\widetilde{\tau}^{\ell,HF}$, $\widetilde{\rho}^{\ell,HF}$, *satisfying high-frequency versions of the conclusions of Proposition 14 and moreover all these functions are uniformly bounded on* $[0,\varepsilon_0]\times\Omega_\delta$.

Proof. Most of the proof is contained in the proof of Proposition 14. The remaining part is directly derived from the observation that Proposition 12 provides asymptotics for

$(P^{\ell}_{\varepsilon}(\lambda,0))^{-1}\,P^{r}_{\varepsilon}(\lambda,0)$ thus also for the coefficients under consideration. □

This leads to the following definition (wherever it makes sense)

$$G_\varepsilon(\lambda;x,y) := \begin{cases} \dfrac{\mathrm{e}^{\int_{-\infty}^{0}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}}{D_\varepsilon(\lambda)}\ \mathbf{e}_1\cdot\mathbf{V}^{r,s}_\varepsilon(\lambda,x)\ \mathbf{e}_1\cdot\widetilde{\mathbf{V}}^{\ell,u}_\varepsilon(\lambda,y) & \text{if } x>y\,, \\ \dfrac{\mathrm{e}^{-\int_{0}^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}}{D_\varepsilon(\lambda)}\ \mathbf{e}_1\cdot\mathbf{V}^{\ell,u}_\varepsilon(\lambda,x)\ \mathbf{e}_1\cdot\widetilde{\mathbf{V}}^{r,s}_\varepsilon(\lambda,y) & \text{if } x<y\,, \end{cases} \tag{3.3.4}$$

where $\mathbf{e}_1 := (1,0)$. Note that $(\lambda-\mathcal{L}_\varepsilon)G_\varepsilon(\lambda;\cdot,y)=\delta_y$ and, for $\Re(\lambda)$ sufficiently large, $G_\varepsilon(\lambda;x,y)$ is exponentially decaying as $\|x-y\|\to\infty$. To bound $G_\varepsilon(\lambda;x,y)$, we shall refine the alternative $x<y$ *vs.* $x>y$. For instance, when $x>y$, more convenient equivalent representations of $G_\varepsilon(\lambda;x,y)$ are

$$\frac{\mathrm{e}^{\int_{-\infty}^{0}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}}{D_\varepsilon(\lambda)}\ \mathbf{e}_1\cdot\mathbf{V}^{r,s}_\varepsilon(\lambda,x)\ \mathbf{e}_1\cdot\widetilde{\mathbf{V}}^{\ell,u}_\varepsilon(\lambda,y)$$

when $x>0>y$,

$$\mathrm{e}^{\int_{-\infty}^{0}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}\ \mathbf{e}_1\cdot\mathbf{V}^{r,s}_\varepsilon(\lambda,x)\ \mathbf{e}_1\cdot\left(\frac{\widetilde{\rho}^{\ell}_\varepsilon(\lambda)}{D_\varepsilon(\lambda)}\,\widetilde{\mathbf{V}}^{r,u}_\varepsilon(\lambda,y)+\frac{\widetilde{\tau}^{\ell}_\varepsilon(\lambda)}{D_\varepsilon(\lambda)}\,\widetilde{\mathbf{V}}^{r,s}_\varepsilon(\lambda,y)\right)$$

when $x>y>0$,

$$\mathrm{e}^{\int_{-\infty}^{0}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}\ \mathbf{e}_1\cdot\left(\frac{\rho^{r}_\varepsilon(\lambda)}{D_\varepsilon(\lambda)}\,\mathbf{V}^{\ell,s}_\varepsilon(\lambda,x)+\frac{\tau^{r}_\varepsilon(\lambda)}{D_\varepsilon(\lambda)}\,\mathbf{V}^{\ell,u}_\varepsilon(\lambda,x)\right)\ \mathbf{e}_1\cdot\widetilde{\mathbf{V}}^{\ell,u}_\varepsilon(\lambda,y)$$

when $0>x>y$.

Remark 8. *The representation of spectral Green functions, thus of resolvent operators, with Evans' functions is sufficient to prove classical results about the identification of spectrum — including algebraic multiplicity — at the right-hand side of the essential spectrum with zeros of Evans' functions.*

We use similar formulas in the high-frequency regime. Yet the Green functions of the high-frequency regime and the compact-frequency regime agree where they co-exist (by uniqueness of the spectral problem (in a suitably weighted space) in some overlapping regions and uniqueness of analytic continuation elsewhere) so that we do not need to introduce a specific piece of notation for the high-frequency regime.

We also point out that it follows from Proposition 12 that in the zone of interest

$$|\partial_x G_\varepsilon(\lambda;x,y)|\leqslant C\,\max(\{1,\sqrt{|\lambda|}\})\,|G_\varepsilon(\lambda;x,y)|$$

for some uniform constant C.

3.3.2 Time-evolution

It follows from standard semigroup theory that the representation

$$S_\varepsilon(t) \;=\; \frac{1}{2\,\mathrm{i}\,\pi}\int_{\mathbf{R}} \mathrm{e}^{\Lambda(\xi)\,t}\,\Lambda'(\xi)\,(\Lambda(\xi)\mathrm{I}-\mathcal{L}_\varepsilon)^{-1}\,\mathrm{d}\,\xi \tag{3.3.5}$$

holds in $\mathcal{L}(BUC^0(\mathbf{R}))$ when $\Lambda : \mathbf{R} \to \mathbf{C}$ is a continuous, piecewise $\mathcal{C}^1$ simple curve such that

1. Λ is valued in the right-hand connected component of [10]

$$\left\{\,\lambda\,;\ \text{for } u \in \{\underline{u}_{-\infty},\underline{u}_{+\infty}\}\,,\ \Re\left(\mu^\varepsilon_+(\lambda,u)\right) > 0 > \Re\left(\mu^\varepsilon_-(\lambda,u)\right)\right\}\,;$$

2. there hold

$$\lim_{\xi\to\pm\infty} \Im(\Lambda(\xi)) \;=\; \pm\infty\,, \qquad\qquad \int_{\mathbf{R}} \mathrm{e}^{\Re(\Lambda(\xi))\,t}\,\frac{|\Lambda'(\xi)|}{1+|\Lambda(\xi)|}\,\mathrm{d}\,\xi < +\infty$$

   and there exist positive (R,c) such that for $|\xi| \geqslant R$

$$\Re(\Lambda(\xi)) \geqslant -c\,|\Im(\Lambda(\xi))|\,;$$

3. there is no root of D_ε on the right [11] of $\Lambda(\mathbf{R})$.

Failure of the third condition could be restored by adding positively-oriented small circles to the contour Λ. This is the first condition that we want to relax by going to Green functions.

For curves as above, applying the above formula to functions in $W^{\infty,\infty}(\mathbf{R})$ and testing it against functions in $\mathcal{C}^\infty_c(\mathbf{R})$ leads to a similar representation for Green functions

$$G^\varepsilon_t(x,y) \;=\; \frac{1}{2\,\mathrm{i}\,\pi}\int_{\mathbf{R}} \mathrm{e}^{\Lambda(\xi)\,t}\,\Lambda'(\xi)\,G_\varepsilon(\Lambda(\xi);x,y)\;\mathrm{d}\,\xi\,. \tag{3.3.6}$$

The point is that at fixed (t,x,y), the constraints on Λ ensuring the representation formula are significantly less stringent and one may use this freedom to optimize bounds. In particular depending on the specific regime for the triplet (t,x,y) or the kind of data one has in mind, one may trade spatial localization for time-decay and *vice versa* by adjusting contours to the right so as to gain spatial decay or to the left in order to improve time decay.

When doing so, we essentially follow the strategy of [ZH98]. The critical decay is

10. This set contains $\{\lambda;\Re(\lambda)\geqslant\omega\}$ when ω is sufficiently large.
11. The second condition implies that this makes sense.

essentially encoded in limiting-endstates spectral spatial decay and Evans' function root location. Therefore, roughly speaking, leaving aside questions related to the presence of a root of the Evans' function at zero, contours are chosen here to approximately [12] optimize bounds on

$$\int_{\mathbf{R}} \mathrm{e}^{\Re(\Lambda(\xi))\,t+\Re(\mu_\sharp^\varepsilon(\Lambda(\xi),\underline{u}_{\mathrm{sgn}(x)\infty}))\,x+\Re(\widetilde{\mu}_\flat^\varepsilon(\Lambda(\xi),\underline{u}_{\mathrm{sgn}(y)\infty}))\,y}\,\frac{|\Lambda'(\xi)|}{|D_\varepsilon(\Lambda(\xi))|}\,\mathrm{d}\,\xi$$

with $(\sharp,\flat)\in\{+,-\}^2$. More precisely, at fixed (t,x,y), one picks Λ_0 real in $[-\frac{1}{2}\eta_0,+\infty]$ (with η_0 as in Proposition 13), approximately minimizing

$$\Re(\lambda)\,t+\Re(\mu_\sharp^\varepsilon(\lambda,\underline{u}_{\mathrm{sgn}(x)\infty}))\,x+\Re(\widetilde{\mu}_\flat^\varepsilon(\lambda,\underline{u}_{\mathrm{sgn}(y)\infty}))\,y$$

among such real λ in $[-\frac{1}{2}\eta_0,+\infty]$ and then depending on cases one defines Λ through one of the equations

$$\begin{aligned}
&\Re(\mu_\sharp^\varepsilon(\Lambda(\xi),\underline{u}_{\mathrm{sgn}(x)\infty}))\,x+\Re(\widetilde{\mu}_\flat^\varepsilon(\Lambda(\xi),\underline{u}_{\mathrm{sgn}(y)\infty}))\,y\\
&\qquad=\Re(\mu_\sharp^\varepsilon(\Lambda_0,\underline{u}_{\mathrm{sgn}(x)\infty}))\,x+\Re(\widetilde{\mu}_\flat^\varepsilon(\Lambda_0,\underline{u}_{\mathrm{sgn}(y)\infty}))\,y+\mathrm{i}\,\xi\,\zeta_{\mathrm{sgn}(\xi)}\,(\sharp\,x+\flat\,y)\\
&\Re(\widetilde{\mu}_\flat^\varepsilon(\Lambda(\xi),\underline{u}_{\mathrm{sgn}(y)\infty}))\,y=\Re(\widetilde{\mu}_\flat^\varepsilon(\Lambda_0,\underline{u}_{\mathrm{sgn}(y)\infty}))\,y+\mathrm{i}\,\xi\,\zeta_{\mathrm{sgn}(\xi)}\ \times(\flat\,y)\\
&\Re(\mu_\sharp^\varepsilon(\Lambda(\xi),\underline{u}_{\mathrm{sgn}(x)\infty}))\,x=\Re(\mu_\sharp^\varepsilon(\Lambda_0,\underline{u}_{\mathrm{sgn}(x)\infty}))\,x+\mathrm{i}\,\xi\,\zeta_{\mathrm{sgn}(\xi)}\ \times(\sharp\,x)
\end{aligned}$$

with $\zeta_\pm$ conveniently chosen to ensure a condition analogous to the second condition of the semigroup representation and including

$$\lim_{|\xi|\to\infty}\Re(\sqrt{\Lambda(\xi)})=+\infty\,,\qquad\qquad\lim_{\xi\to\pm\infty}\Im(\Lambda(\xi))=\pm\infty\,.$$

This should be thought as an approximate/simplified version of the saddlepoint method in the sense that $\Lambda_0=\Lambda(0)$ is an approximate maximizer of the exponential decay rate among real numbers, but a minimizer along the curve $\Lambda(\cdot)$.

Computational details — carried out in next section — are cumbersome but the process is rather systematic.

3.4 Linear stability

We now make the most of our spectral preparation to derive linear stability estimates.

12. In some cases a genuine optimization — as in direct applications of the Riemann saddle point method — would be impractical.

To motivate the analysis, let us anticipate that our achievement is the splitting of $(S_\varepsilon(t))_{t\geqslant 0}$ as

$$S_\varepsilon(t)(w)(x) \;=\; \underline{U}'_\varepsilon(x)\, s^{\mathrm{p}}_\varepsilon(t)(w) \;+\; \widetilde{S}_\varepsilon(t)(w)(x)\,. \tag{3.4.1}$$

for some $(s^{\mathrm{p}}_\varepsilon(t))_{t\geqslant 0}$, $(\widetilde{S}_\varepsilon(t))_{t\geqslant 0}$ with $\widetilde{S}_\varepsilon(0) = \mathrm{Id}$, so that the following proposition holds.

Proposition 16. *There exists $\varepsilon_0 > 0$ such that*

1. *there exists $C > 0$ such that for any $t \geqslant 0$, any $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and any $w \in BUC^0(\mathbf{R})$*

$$\begin{aligned}\|\widetilde{S}_\varepsilon(t)(w)\|_{L^\infty(\mathbf{R})} + \min\big(\{1,\sqrt{t}\}\big)\,\|\partial_x\widetilde{S}_\varepsilon(t)(w)\|_{L^\infty(\mathbf{R})} + |\partial_t s^p_\varepsilon(t)(w)| \\ \leqslant\; C\;\mathrm{e}^{-\min\left(\left\{|g'(\underline{u}_{-\infty})|,|g'(\underline{u}_{+\infty})|\right\}\right)\varepsilon\, t}\;\|w\|_{L^\infty(\mathbf{R})}\,,\end{aligned}$$

   and when moreover $w \in BUC^1(\mathbf{R})$

$$\|\widetilde{S}_\varepsilon(t)(w)\|_{W^{1,\infty}(\mathbf{R})} \leqslant\; C\;\mathrm{e}^{-\min\left(\left\{|g'(\underline{u}_{-\infty})|,|g'(\underline{u}_{+\infty})|\right\}\right)\varepsilon\, t}\;\|w\|_{W^{1,\infty}(\mathbf{R})}\,,$$

2. *for any $\theta > 0$ there exist positive $(C_\theta, \omega_\theta)$ such that for any $t \geqslant 0$, any $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and any $w \in BUC^0(\mathbf{R})$*

$$\begin{aligned}\|\widetilde{S}_\varepsilon(t)(w)\|_{L^\infty(\mathbf{R})} + \min\big(\{1,\sqrt{t}\}\big)\,\|\partial_x\widetilde{S}_\varepsilon(t)(w)\|_{L^\infty(\mathbf{R})} + |\partial_t s^p_\varepsilon(t)(w)| \\ \leqslant\; C_\theta\;\mathrm{e}^{-\omega_\theta\, t}\;\|\,\mathrm{e}^{\theta\,|\,\cdot\,|}\;w\|_{L^\infty(\mathbf{R})}\,.\end{aligned}$$

Estimates on operators are derived through pointwise bounds on Green kernels from the trivial fact that if $\mathbf{T}$ is defined through

$$\mathbf{T}(w)(x) \;=\; \int_{\mathbf{R}} \mathbf{K}(x,y)\, w(y)\;\mathrm{d}\,y$$

then

$$\|\mathbf{T}(w)\|_{L^\infty(\mathbf{R})} \;\leqslant\; \|\mathbf{K}\|_{L^\infty_x(L^1_y)}\,\|w\|_{L^\infty(\mathbf{R})}\,.$$

3.4.1 Auxiliary lemmas

To begin with, to gain a practical grasp on the way the placement of spectral curves impacts decay rates, we provide two lemmas, that will be of ubiquitous use when establishing pointwise bounds on Green functions.

Both lemmas are motivated by the fact that when $\beta \geqslant 0$ and $t > 0$ the minimization of

$$\Lambda_0\, t + \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \,b\, + \Lambda_0}\right)\,\beta$$

over $\Lambda_0 \in (-\frac{\alpha^2}{4} + b\,, +\infty)$ is equivalent to

$$2\sqrt{\frac{\alpha^2}{4} - b + \Lambda_0} = \frac{\beta}{t}\,. \tag{3.4.2}$$

The first lemma directly elucidates the consequences of this choice of Λ_0 in the approximate saddlepoint method sketched above.

Lemma 5. *Let $t > 0$, $\alpha \in \mathbf{R}$, $\beta \geqslant 0$, $\beta_0 \geqslant 0$ $b \in \mathbf{R}$, and $(\zeta_-, \zeta_+) \in \mathbf{C}^2$ such that*

$$\Re(\zeta_\pm) > |\Im(\zeta_\pm)|\,, \qquad\qquad \mp\Im(\zeta_\pm) > 0\,.$$

Then the curve $\Lambda : \mathbf{R} \to \mathbf{C}$ defined through[13]

$$2\sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)} = \frac{\beta_0}{t} + \mathrm{i}\,\xi\,\zeta_{\mathrm{sgn}(\xi)}\,,$$

satisfies for any $\xi \in \mathbf{R}$, when either $\beta = \beta_0$ or ($\beta \geqslant \beta_0$ and $\alpha \leqslant 0$)

$$\begin{aligned}\Re\left(\Lambda(\xi)\,t + \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right)\beta\right) &\leqslant -\left(\frac{\alpha^2}{4} - b\right)t + \frac{\alpha}{2}\,\beta_0 - \frac{\beta_0^2}{4\,t} - \frac{\xi^2}{4}\,\Re(\zeta^2_{\mathrm{sgn}(\xi)})\,t \\ &= b\,t - \frac{(\beta_0 - \alpha\,t)^2}{4\,t} - \frac{\xi^2}{4}\,\Re(\zeta^2_{\mathrm{sgn}(\xi)})\,t\end{aligned}$$

and for any $\xi \in \mathbf{R}^*$

$$|\Lambda'(\xi)| \leqslant |\zeta_{\mathrm{sgn}(\xi)}|\left(1 + \frac{\Re(\zeta_{\mathrm{sgn}(\xi)})}{|\Im(\zeta_{\mathrm{sgn}(\xi)})|}\right)\Re\left(\sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right).$$

We omit the proof of Lemma 5 as straightforward and elementary.

The second lemma is designed to deal with cases when the natural choice (3.4.2) is not available because of extra constraints arising from Evans' function possible annulation in $(-\frac{\alpha^2}{4} + b\,, 0)$. Explicitly, it focuses on the case when $\beta/t \leqslant \omega_0$ when ω_0 is typically picked as either $\omega_r^{\eta_0}$ or $\omega_r^{\eta_0}$ with

$$\omega_r^{\eta_0} := 2\sqrt{\frac{(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2}{4} - \frac{\eta_0}{2}}\,, \qquad \omega_\ell^{\eta_0} := 2\sqrt{\frac{(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2}{4} - \frac{\eta_0}{2}}\,, \tag{3.4.3}$$

where η_0 is as in Proposition 13. Since $\beta/t \leqslant \omega_0$ should be thought as a bounded-domain restriction, it is useful to let the second lemma also encode the possible trade-off between spatial localization and time decay.

13. Sign conditions on $\Im(\zeta_\pm)$ ensure that this is a licit definition.

Lemma 6. *Let $t > 0$, $\alpha \in \mathbf{R}$, $\beta \geqslant 0$, $b < 0$, $(\zeta_-, \zeta_+) \in \mathbf{C}^2$ such that*

$$\Re(\zeta_\pm) > |\Im(\zeta_\pm)|\,, \qquad\qquad \mp\Im(\zeta_\pm) > 0\,,$$

and $\omega_0 \geqslant 0$ such that

$$\beta \leqslant \omega_0\, t\,.$$

Then the curve $\Lambda : \mathbf{R} \to \mathbf{C}$ defined through

$$2\sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)} = \omega_0 + \mathrm{i}\,\xi\,\zeta_{\mathrm{sgn}(\xi)}\,,$$

satisfies for any $\xi \in \mathbf{R}^*$

$$|\Lambda'(\xi)| \leqslant |\zeta_{\mathrm{sgn}(\xi)}| \left(1 + \frac{\Re(\zeta_{\mathrm{sgn}(\xi)})}{|\Im(\zeta_{\mathrm{sgn}(\xi)})|}\right) \Re\left(\sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right)\,.$$

and for any $\xi \in \mathbf{R}$ and $\eta > 0$,

$$\begin{aligned}
&\Re\left(\Lambda(\xi)\, t + \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right)\beta\right) \\
&\leqslant -\left(\frac{\alpha^2 - (1+\eta)\,\omega_0^2}{4} - b\right) t - \frac{\omega_0 - \alpha}{2}\,\beta - \frac{\xi^2}{4}\left(\Re(\zeta_{\mathrm{sgn}(\xi)}^2) - \frac{1}{\eta}\,|\Im(\zeta_{\mathrm{sgn}(\xi)})|^2\right) t
\end{aligned}$$

and, when moreover $\omega_0 < |\alpha|$,

$$\begin{aligned}
&\Re\left(\Lambda(\xi)\, t + \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right)\beta\right) \\
&\leqslant b\, t - \frac{(|\alpha|\, t - \beta)^2}{4\, t}\left(1 - (1+\eta)\,\frac{\omega_0^2}{\alpha^2}\right) - \frac{\xi^2}{4}\left(\Re(\zeta_{\mathrm{sgn}(\xi)}^2) - \frac{1}{\eta}\,|\Im(\zeta_{\mathrm{sgn}(\xi)})|^2\right) t\,.
\end{aligned}$$

Note that to guarantee for some $\eta > 0$ both

$$\Re(\zeta_{\mathrm{sgn}(\xi)}^2) > \frac{1}{\eta}\,|\Im(\zeta_{\mathrm{sgn}(\xi)})|^2\,, \qquad\qquad 1 > (1+\eta)\,\frac{\omega_0^2}{\alpha^2}\,,$$

one needs to enforce

$$\Re(\zeta_{\mathrm{sgn}(\xi)}) > \frac{|\alpha|}{\sqrt{\alpha^2 - \omega_0^2}}\,|\Im(\zeta_{\mathrm{sgn}(\xi)})|\,. \tag{3.4.4}$$

Likewise when $\omega_0 > |\alpha|$, one may extract large-time decay for $|\xi| \geqslant \xi_0 > 0$ provided that $\Re(\zeta_{\mathrm{sgn}(\xi)})$ is sufficiently large.

Proof. The starting point is that for any $\eta > 0$,

$$\begin{aligned}
&\Re\left(\Lambda(\xi)\,t + \left(\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - b + \Lambda(\xi)}\right)\beta\right)\\
&\leqslant -\left(\frac{\alpha^2 - \omega_0^2 - \eta\,\left(\omega_0 - \frac{\beta}{t}\right)^2}{4} + |b|\right)t - \frac{\omega_0 - \alpha}{2}\,\beta - \frac{\xi^2}{4}\left(\Re(\zeta^2_{\mathrm{sgn}(\xi)}) - \frac{1}{\eta}\,|\Im(\zeta_{\mathrm{sgn}(\xi)})|^2\right)t\\
&= -|b|\,t - \frac{(\alpha\,t - \beta)^2}{4\,t} - \frac{\xi^2}{4}\left(\Re(\zeta^2_{\mathrm{sgn}(\xi)}) - \frac{1}{\eta}\,|\Im(\zeta_{\mathrm{sgn}(\xi)})|^2\right)t + \frac{(\omega_0\,t - \beta)^2}{4\,t}\,(1+\eta)\,.
\end{aligned}$$

The first bound on the real part is then obtained by using the first formulation of the foregoing bound jointly with

$$\left(\omega_0 - \frac{\beta}{t}\right)^2 \leqslant \omega_0^2$$

whereas the second bound, specialized to the case $|\alpha| > \omega_0$, stems from the second formulation and

$$(|\alpha|\,t - \beta)^2 \leqslant (\alpha\,t - \beta)^2\,, \qquad\qquad (\omega_0\,t - \beta)^2 \leqslant \frac{\omega_0^2}{\alpha^2}\,(|\alpha|\,t - \beta)^2\,.$$

□

3.4.2 First separations

We would like to split $G_t^\varepsilon(x,y)$ into pieces corresponding to different behaviors. Yet we must take into account that our description of $G_\varepsilon(\lambda;x,y)$ is different in high-frequency and compact regimes. To do so, we pick some curves and break them into pieces.

Explicitly, motivated by (3.4.4) with ω_0 either $\omega_r^{\eta_0}$ or $\omega_r^{\eta_0}$ — defined in (3.4.3) with η_0 as in Proposition 13 —, we first choose $\zeta_\pm^{HF}$ such that

$$\Re(\zeta_\pm^{HF}) \geqslant 2\,\frac{\max\left(\{|f'(\underline{u}_{+\infty}) - \sigma_\varepsilon|, f'(\underline{u}_{-\infty}) - \sigma_\varepsilon\}\right)}{\sqrt{2\,\eta_0}}\,|\Im(\zeta_\pm^{HF})|\,, \qquad \mp\Im(\zeta_\pm^{HF}) > 0\,.$$

Then we define curves Λ_ε^r, Λ_ε^ℓ through

$$\begin{aligned}
2\sqrt{\frac{(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2}{4} - \varepsilon\,g'(\underline{u}_{+\infty}) + \Lambda_\varepsilon^r(\xi)} &= \omega_r^{HF} + \mathrm{i}\,\xi\,\zeta^{HF}_{\mathrm{sgn}(\xi)}\,,\\
2\sqrt{\frac{(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2}{4} - \varepsilon\,g'(\underline{u}_{-\infty}) + \Lambda_\varepsilon^\ell(\xi)} &= \omega_\ell^{HF} + \mathrm{i}\,\xi\,\zeta^{HF}_{\mathrm{sgn}(\xi)}\,,
\end{aligned}$$

where ω_r^{HF} and ω_ℓ^{HF} are fixed such that

$$\omega_r^{HF} > |f'(\underline{u}_{+\infty}) - \sigma_0|\,, \qquad\qquad \omega_\ell^{HF} > |f'(\underline{u}_{-\infty}) - \sigma_0|\,.$$

Note that this is sufficient to guarantee that both curves satisfy requirements ensuring (3.3.5) thus also (3.3.6).

We shall do a particular treatment of the parts of the curves corresponding to $|\xi| \leqslant \xi^{HF}$ where we choose ξ^{HF} as

$$\xi^{HF} := \frac{2 \max\left(\left\{\omega_r^{HF} - \omega_r^{\eta_0}, \omega_\ell^{HF} - \omega_\ell^{\eta_0}\right\}\right)}{\min\left(\left\{|\Im(\zeta_+^{HF})|, |\Im(\zeta_-^{HF})|\right\}\right)} .$$

Once again the motivation for the definition of ξ^{HF} stems from Lemma 6. Indeed the definition ensures that for $\omega \in \mathbf{R}$, a curve Λ, defined through

$$2\sqrt{\frac{(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2}{4} - \varepsilon\, g'(\underline{u}_{+\infty}) + \Lambda(\xi)} = \omega + \mathrm{i}\,\xi\,\zeta^{\omega}_{r,\mathrm{sgn}(\xi)} ,$$

respectively through

$$2\sqrt{\frac{(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2}{4} - \varepsilon\, g'(\underline{u}_{-\infty}) + \Lambda(\xi)} = \omega + \mathrm{i}\,\xi\,\zeta^{\omega}_{\ell,\mathrm{sgn}(\xi)} ,$$

with

$$\zeta^{\omega}_{\sharp,\pm} := \Re(\zeta_\pm^{HF}) + \mathrm{i}\left(\Im\left(\zeta_\pm^{HF}\right) \mp \frac{\omega_\sharp^{HF} - \omega}{\xi^{HF}}\right) , \quad \sharp \in \{r, \ell\} ,$$

satisfies

$$\Lambda(\pm\xi^{HF}) = \Lambda^r_\varepsilon(\pm\xi^{HF}) , \qquad \text{respectively} \quad \Lambda(\pm\xi^{HF}) = \Lambda^\ell_\varepsilon(\pm\xi^{HF}) ,$$

whereas, for $\sharp \in \{r, \ell\}$, $\omega \in [\omega_\sharp^{\eta_0}, \omega_\sharp^{HF}]$,

$$\Re(\zeta^{\omega}_{\sharp,\pm}) = \Re(\zeta_\pm^{HF}) , \qquad \mp\Im(\zeta^{\omega}_{\sharp,\pm}) > 0 , \qquad |\Im(\zeta^{\omega}_{\sharp,\pm})| \leqslant \frac{3}{2}|\Im(\zeta_\pm^{HF})| .$$

In the following, for $\sharp \in \{r, \ell\}$, we use notation $\Lambda^{\sharp,LF}_\varepsilon := (\Lambda^\sharp_\varepsilon)_{|[-\xi^{HF},\xi^{HF}]}$ and $\Lambda^{\sharp,HF}_\varepsilon := (\Lambda^\sharp_\varepsilon)_{|\mathbf{R}\setminus[-\xi^{HF},\xi^{HF}]}$.

To ensure that Lemma 6 provides exponential time decay for the part of the evolution arising from $\Lambda^{\sharp,HF}_\varepsilon$, we reinforce the constraint on $\Re(\zeta_\pm^{HF})$ by adding

$$\Re(\zeta_\pm^{HF}) \geqslant \sqrt{\Im(\zeta_\pm^{HF})^2 + 2\,\frac{\max\left(\{(\omega_r^{HF})^2 - (f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2, (\omega_\ell^{HF})^2 - (f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2\}\right)}{(\xi^{HF})^2}} .$$

Anticipating our needs when analyzing small-λ expansions, we point out that by lowering η_0 and ω_ℓ^{HF}, ω_r^{HF}, we may enforce that for $\sharp \in \{r, \ell\}$, when $\omega = \omega_\sharp^{\eta_0}$, and Λ is

defined as above, there exists $\omega' > 0$ and some $\delta > 0$, such that for any $\xi \in [-\xi^{HF}, \xi^{HF}]$

$$\Re(\Lambda(\xi)) \leqslant -\omega' \,, \quad \Re\left(\sqrt{\frac{(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2}{4} - \varepsilon\, g'(\underline{u}_{+\infty}) + \Lambda(\xi)}\right) \leqslant \frac{(-f'(\underline{u}_{+\infty}) + \sigma_\varepsilon)}{2} - \delta \,,$$

respectively

$$\Re(\Lambda(\xi)) \leqslant -\omega' \,, \quad \Re\left(\sqrt{\frac{(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2}{4} - \varepsilon\, g'(\underline{u}_{-\infty}) + \Lambda(\xi)}\right) \leqslant \frac{(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)}{2} - \delta \,.$$

After these preliminaries, to account for different behaviors, when $t > 0$ we break G_t^ε as

$$G_t^\varepsilon \;=\; G_t^{\varepsilon,\mathrm{pt}} + G_t^{\varepsilon,\mathrm{ess}} \tag{3.4.5}$$

with $G_t^{\varepsilon,\mathrm{pt}}$ and $G_t^{\varepsilon,\mathrm{ess}}$ defined as follows. First

$$G_t^{\varepsilon,\mathrm{pt}}(x,y) = 0 \,, \qquad \text{if } xy > 0 \text{ and } \big(y \geqslant \omega_r^{HF}\, t \text{ or } y \leqslant -\omega_\ell^{HF}\, t \,\big)$$

$$G_t^{\varepsilon,\mathrm{pt}}(x,y) \;=\; \frac{1}{2\,\mathrm{i}\,\pi} \int_\Lambda \mathrm{e}^{\lambda\, t}\; G_\varepsilon(\lambda; x, y)\; \mathrm{d}\,\lambda \qquad \text{if } xy < 0 \,,$$

and when $xy > 0$ and $-\omega_\ell^{HF}\, t < y < \omega_r^{HF}\, t$

$$G_t^{\varepsilon,\mathrm{pt}}(x,y)$$

$$= \begin{cases} \frac{1}{2\,\mathrm{i}\,\pi} \int_{\Lambda_\varepsilon^{r,LF}} \mathrm{e}^{\lambda\, t}\; \mathrm{e}^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))}\; \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(\lambda, x)\, \dfrac{\widetilde{\tau}_\varepsilon^\ell(\lambda)}{D_\varepsilon(\lambda)}\; \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda, y)\; \mathrm{d}\,\lambda \\ \quad \text{if } x > y > 0 \\ \frac{1}{2\,\mathrm{i}\,\pi} \int_{\Lambda_\varepsilon^{\ell,LF}} \mathrm{e}^{\lambda\, t}\; \mathrm{e}^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{-\infty}))}\; \dfrac{\tau_\varepsilon^r(\lambda)}{D_\varepsilon(\lambda)}\; \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,u}(\lambda, x)\; \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda, y)\; \mathrm{d}\,\lambda \\ \quad \text{if } 0 > x > y \\ \frac{1}{2\,\mathrm{i}\,\pi} \int_{\Lambda_\varepsilon^{\ell,LF}} \mathrm{e}^{\lambda\, t}\; \mathrm{e}^{-\int_0^{+\infty} (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))}\; \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,u}(\lambda, x)\, \dfrac{\widetilde{\tau}_\varepsilon^r(\lambda)}{D_\varepsilon(\lambda)}\; \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda, y)\; \mathrm{d}\,\lambda \\ \quad \text{if } 0 > y > x \\ \frac{1}{2\,\mathrm{i}\,\pi} \int_{\Lambda_\varepsilon^{r,LF}} \mathrm{e}^{\lambda\, t}\; \mathrm{e}^{-\int_0^{+\infty} (f'(\underline{U}_\varepsilon) - f'(\underline{u}_{+\infty}))}\; \dfrac{\tau_\varepsilon^\ell(\lambda)}{D_\varepsilon(\lambda)}\; \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(\lambda, x)\; \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda, y)\; \mathrm{d}\,\lambda \\ \quad \text{if } y > x > 0 \,, \end{cases}$$

where, here and in the definition of $G_t^{\varepsilon,\mathrm{ess}}$, Λ is either $\Lambda = \Lambda_\varepsilon^r$ or $\Lambda = \Lambda_\varepsilon^\ell$, and we use compact notation for integrals over curves instead of explicitly parametrized versions.

Second,

$$G_t^{\varepsilon,\mathrm{ess}}(x,y) = 0\,, \qquad\qquad \text{if } xy<0$$

$$G_t^{\varepsilon,\mathrm{ess}}(x,y) = \frac{1}{2\,\mathrm{i}\,\pi}\int_\Lambda \mathrm{e}^{\lambda\,t}\; G_\varepsilon(\lambda;x,y)\;\mathrm{d}\,\lambda\,, \quad \text{if } xy>0 \text{ and } \left(\, y \geqslant \omega_r^{HF}\,t \text{ or } y \leqslant -\omega_\ell^{HF}\,t\,\right)\,,$$

and when $xy>0$ and $-\omega_\ell^{HF}\,t < y < \omega_r^{HF}\,t$

$$G_t^{\varepsilon,\mathrm{ess}}(x,y)$$
$$= \begin{cases} \frac{1}{2\,\mathrm{i}\,\pi}\int_{\Lambda_\varepsilon^{r,LF}} \mathrm{e}^{\lambda\,t}\; \mathrm{e}^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}\; \mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{r,s}(\lambda,x)\,\dfrac{\widetilde{\rho}_\varepsilon^{\ell}(\lambda)}{D_\varepsilon(\lambda)}\;\mathbf{e}_1\cdot\widetilde{\mathbf{V}}_\varepsilon^{r,u}(\lambda,y)\;\mathrm{d}\,\lambda \\ \quad +\frac{1}{2\,\mathrm{i}\,\pi}\int_{\Lambda_\varepsilon^{r,HF}} \mathrm{e}^{\lambda\,t}\; G_\varepsilon(\lambda;x,y)\;\mathrm{d}\,\lambda \qquad\qquad \text{if } x>y>0 \\ \frac{1}{2\,\mathrm{i}\,\pi}\int_{\Lambda_\varepsilon^{\ell,LF}} \mathrm{e}^{\lambda\,t}\; \mathrm{e}^{\int_{-\infty}^0 (f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}\; \dfrac{\rho_\varepsilon^{r}(\lambda)}{D_\varepsilon(\lambda)}\;\mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{\ell,s}(\lambda,x)\;\mathbf{e}_1\cdot\widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y)\;\mathrm{d}\,\lambda \\ \quad +\frac{1}{2\,\mathrm{i}\,\pi}\int_{\Lambda_\varepsilon^{\ell,HF}} \mathrm{e}^{\lambda\,t}\; G_\varepsilon(\lambda;x,y)\;\mathrm{d}\,\lambda \qquad\qquad \text{if } 0>x>y \\ \frac{1}{2\,\mathrm{i}\,\pi}\int_{\Lambda_\varepsilon^{\ell,LF}} \mathrm{e}^{\lambda\,t}\; \mathrm{e}^{-\int_0^{+\infty} (f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}\; \mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{\ell,u}(\lambda,x)\,\dfrac{\widetilde{\rho}_\varepsilon^{r}(\lambda)}{D_\varepsilon(\lambda)}\;\mathbf{e}_1\cdot\widetilde{\mathbf{V}}_\varepsilon^{\ell,s}(\lambda,y)\;\mathrm{d}\,\lambda \\ \quad +\frac{1}{2\,\mathrm{i}\,\pi}\int_{\Lambda_\varepsilon^{\ell,HF}} \mathrm{e}^{\lambda\,t}\; G_\varepsilon(\lambda;x,y)\;\mathrm{d}\,\lambda \qquad\qquad \text{if } 0>y>x \\ \frac{1}{2\,\mathrm{i}\,\pi}\int_{\Lambda_\varepsilon^{r,LF}} \mathrm{e}^{\lambda\,t}\; \mathrm{e}^{-\int_0^{+\infty} (f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}\; \dfrac{\rho_\varepsilon^{\ell}(\lambda)}{D_\varepsilon(\lambda)}\;\mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{r,u}(\lambda,x)\;\mathbf{e}_1\cdot\widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)\;\mathrm{d}\,\lambda \\ \quad +\frac{1}{2\,\mathrm{i}\,\pi}\int_{\Lambda_\varepsilon^{r,HF}} \mathrm{e}^{\lambda\,t}\; G_\varepsilon(\lambda;x,y)\;\mathrm{d}\,\lambda \qquad\qquad \text{if } y>x>0\,. \end{cases}$$

Note that the above splitting implies

$$\partial_x G_t^\varepsilon \;=\; \partial_x G_t^{\varepsilon,\mathrm{pt}} + \partial_x G_t^{\varepsilon,\mathrm{ess}}$$

where here and elsewhere throughout the text, ∂_x acting on either $G_t^{\varepsilon,\mathrm{pt}}$ or $G_t^{\varepsilon,\mathrm{ess}}$ is understood as a pointwise derivative wherever these functions are continuous.

The rationale behind the splitting is that the large-time decay of $G_t^{\varepsilon,\mathrm{ess}}$ is essentially limited by spatial decay hence may be thought as purely explained by essential spectrum considerations whereas the large-time asymptotics of $G_t^{\varepsilon,\mathrm{pt}}$ is driven by the presence near the spectral curves of a root of D_ε at $\lambda=0$, hence is due to the interaction of essential and point spectra.

Some extra complications in the splitting are due to the fact that we need to prepare the identification of the most singular part as a phase modulation, which comes into a tensorized form. This explains why we define zones in terms of the size of $|y|$, instead of the otherwise more natural $|x-y|$.

3.4.3 First pointwise bounds

We begin our use of Lemmas 5 and 6 with short-time bounds.

Lemma 7. *There exist positive $(\varepsilon_0, C, \omega, \theta)$ such that for any $t > 0$, any $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and any $(x, y) \in \mathbf{R}^2$*

$$|G_t^{\varepsilon,pt}(x,y)| + \min\left(\{1, \sqrt{t}\}\right) |\partial_x G_t^{\varepsilon,pt}(x,y)| \leqslant C\, \mathrm{e}^{\omega\, t}\, \mathrm{e}^{-\theta\, |x|}\, \frac{1}{\sqrt{t}}\, \mathrm{e}^{-\theta\, \frac{y^2}{t}}\,,$$

The foregoing lemma does not contain estimates on $G_t^{\varepsilon,\mathrm{ess}}$ because those would be redundant with the corresponding large-time estimates. The point of Lemma 7 is to show that for short-time estimates the singularity at $\lambda = 0$ may be avoided whereas this singularity is not present in $G_t^{\varepsilon,\mathrm{ess}}$.

Proof. To bound $G_t^{\varepsilon,\mathrm{pt}}(x,y)$ when $xy < 0$, we separate between $x > 0 > y$ and $x < 0 < y$. The analyses being completely similar, we only discuss here the former case. To treat it, we move curves as in Lemmas 5 and 6 with $\beta_0 = \beta = |y|$, $\alpha = f'(\underline{u}_{-\infty}) - \sigma_\varepsilon$, $b = \varepsilon\, g'(\underline{u}_{-\infty})$ and note that

$$\Re(\mu_-^\varepsilon(\lambda, \underline{u}_{+\infty})) \leqslant \frac{1}{2}\left(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon\right) < 0\,.$$

More explicitly, we use Lemma 5 to bound the regime $|y| \geqslant \omega_\ell^{HF}\, t$ which leads to the claimed heat-like bound since

$$\mathrm{e}^{-\frac{(|y| - \alpha\, t)^2}{4\, t}} \leqslant \mathrm{e}^{-\left(1 - \frac{|\alpha|}{\omega_\ell^{HF}}\right)\frac{y^2}{4\, t}}\,, \qquad\qquad |y| \geqslant \omega_\ell^{HF}\, t\,.$$

In the remaining zone where $|y| \leqslant \omega_\ell^{HF}\, t$ we use instead Lemma 6 to derive a bound that may be converted into a heat-like bound through

$$\mathrm{e}^{-\frac{\omega_\ell^{HF} - \alpha}{2}|y|} \leqslant \mathrm{e}^{-\left(1 - \frac{|\alpha|}{\omega_\ell^{HF}}\right)\frac{y^2}{2\, t}}\,, \qquad\qquad |y| \leqslant \omega_\ell^{HF}\, t\,.$$

The estimates on $G_t^{\varepsilon,\mathrm{pt}}(x,y)$ when $xy > 0$ are obtained in exactly the same way. □

We proceed with bounds on $G_t^{\varepsilon,\mathrm{ess}}$.

Lemma 8. *There exist positive $(\varepsilon_0, C, \omega, \theta)$ such that for any $t > 0$, any $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and any $(x, y) \in \mathbf{R}^2$*

$$\begin{aligned}
&|G_t^{\varepsilon,ess}(x,y)| + \min\left(\{1, \sqrt{t}\}\right) |\partial_x G_t^{\varepsilon,ess}(x,y)| \\
&\quad \leqslant C\, \mathbf{1}_{|x-y| \leqslant |y|}\, \mathrm{e}^{-\min\left(\left\{|g'(\underline{u}_{-\infty})|, |g'(\underline{u}_{+\infty})|\right\}\right)\varepsilon\, t}\, \frac{1}{\sqrt{t}} \left(\mathrm{e}^{-\theta\, \frac{|x-y-(f'(\underline{u}_{+\infty})-\sigma_\varepsilon)\, t|^2}{t}}\right.
\end{aligned}$$

$$+ \mathrm{e}^{-\theta\,\frac{|x-y-(f'(\underline{u}_{-\infty})-\sigma_\varepsilon)\,t|^2}{t}}\Big)$$

$$+\ C\ \mathrm{e}^{-\omega\,t}\ \frac{1}{\sqrt{t}}\,\mathrm{e}^{-\theta\,\frac{y^2}{t}}\ +C\,\mathrm{e}^{-\omega t}\,\mathrm{e}^{-\theta|x-y|}\ .$$

This also implies that there exist positive $(\varepsilon_0, C, \theta)$ *such that for any* $\theta' > 0$ *there exists* $\omega' > 0$ *such that for any* $t > 0$*, any* $0 \leqslant \varepsilon \leqslant \varepsilon_0$ *and any* $(x, y) \in \mathbf{R}^2$

$$\mathrm{e}^{-\theta'|y|}\ \Big(|G_t^{\varepsilon,ess}(x,y)| + \min\big(\{1,\sqrt{t}\}\big)\,|\partial_x G_t^{\varepsilon,ess}(x,y)|\Big)$$
$$\leqslant\ C\ \mathrm{e}^{-\omega'\,t}\ \frac{1}{\sqrt{t}}\Big(\mathrm{e}^{-\theta\,\frac{|x-y-(f'(\underline{u}_{+\infty})-\sigma_\varepsilon)\,t|^2}{t}} + \mathrm{e}^{-\theta\,\frac{|x-y-(f'(\underline{u}_{-\infty})-\sigma_\varepsilon)\,t|^2}{t}}$$
$$+\,\mathrm{e}^{-\theta\,\frac{y^2}{t}}\Big)\ + Ce^{-t\omega'}e^{-\theta|x-y|}.$$

In the foregoing statement and throughout the text we use $\mathbf{1}_A$ to denote a characteristic function for the condition A.

Proof. To deduce the second bound from the first we observe that for any α

$$\frac{\theta}{2\,t}\,|x-y-\alpha\,t|^2 + \theta'|x-y| \geqslant \begin{cases} \frac{1}{2}\,\theta'\,|\alpha|\,t & \text{if } |x-y-\alpha\,t| \leqslant \frac{1}{2}|\alpha|\,t \\ \frac{\theta}{4}\,|\alpha|^2\,t & \text{if } |x-y-\alpha\,t| \geqslant \frac{1}{2}|\alpha|\,t \end{cases}.$$

To prove the first bound we should distinguish between regimes defined by $0 < y < x$, $y < x < 0$, $0 > y > x$ and $y > x > 0$. Regimes $0 < y < x$ and $0 > y > x$ on one hand and $y < x < 0$ and $y > x > 0$ on the other hand may be treated similarly and we give details only for the cases $y < x < 0$ and $0 < y < x$.

Note that when $y < x < 0$, we have $|x-y| \leqslant |y|$. When $y < x < 0$ and $|y| \geqslant \omega_\ell^{HF}\,t$, we choose the curve according to Lemma 5 with $\beta_0 = \beta = |x-y|$, $\alpha = f'(\underline{u}_{-\infty}) - \sigma_\varepsilon$, $b = \varepsilon\,g'(\underline{u}_{-\infty})$ if $|x-y| \geqslant t\omega_\ell^{\eta_0}$, and, otherwise, according to Lemma 6 with $\omega_0 = \omega_\ell^{\eta_0}$ and $\beta = |x-y|$, as well as $\alpha = f'(\underline{u}_{-\infty}) - \sigma_\varepsilon$. To analyze the regime when $y < x < 0$ and $|y| < \omega_\ell^{HF}\,t$, we never move the curve $\Lambda_\varepsilon^{\ell,HF}$ (but bound its contribution according to Lemma 6) whereas we move $\Lambda_\varepsilon^{\ell,LF}$ as in Lemma 5 with $\zeta_\pm = \zeta_{\ell,\pm}^{|x-y|/t}$ when $|x-y| \geqslant \omega_\ell^{\eta_0}\,t$ or as in Lemma 6 with $\zeta_\pm = \zeta_{\ell,\pm}^{\omega_\ell^{\eta_0}}$ when $|x-y| \leqslant \omega_\ell^{\eta_0}\,t$.

To bound the contribution of the regime $0 < y < x$, we may proceed as when $y < x < 0$ provided that $|x-y| \leqslant |y|$ or $-\omega_\ell^{HF}\,t \leqslant y \leqslant \omega_r^{HF}\,t$. The remaining case is dealt with by applying Lemma 5 with $\beta_0 = |y|$ and $\beta = |x-y|$ using the fact that $f'(\underline{u}_{+\infty}) - \sigma_\varepsilon < 0$. □

3.4.4 Linear phase separation

The large-time estimates for $G_t^{\varepsilon,\mathrm{pt}}$ require a phase separation. To carry it out we first recall that there exist $(a_\varepsilon^r, a_\varepsilon^\ell) \in \mathbf{R}^2$, each uniformly bounded from below and above, such that

$$\mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{r,s}(0,\cdot) = a_\varepsilon^r \, \underline{U}_\varepsilon' \,, \qquad\qquad \mathbf{e}_1 \cdot \mathbf{V}_\varepsilon^{\ell,u}(0,\cdot) = a_\varepsilon^\ell \underline{U}_\varepsilon' \,.$$

Then we split $G_t^{\varepsilon,\mathrm{pt}}$ as

$$G_t^{\varepsilon,\mathrm{pt}}(x,y) \;=\; \underline{U}_\varepsilon'(x)\, G_t^{\varepsilon,\mathrm{p}}(y) \;+\; \widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x,y)$$

with

$$G_t^{\varepsilon,\mathrm{p}}(y) = 0 \,, \qquad\qquad \text{if } \left(y \geqslant \omega_r^{HF}\, t \text{ or } y \leqslant -\omega_\ell^{HF}\, t \right) ,$$

whereas when $-\omega_\ell^{HF}\, t < y < \omega_r^{HF}\, t$

$$G_t^{\varepsilon,\mathrm{p}}(y) = \begin{cases} \dfrac{1}{2\,\mathrm{i}\,\pi} \displaystyle\int_{\Lambda_\varepsilon^{r,LF}} \mathrm{e}^{\lambda t}\; \mathrm{e}^{-\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}\; a_\varepsilon^\ell\; \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)\, \dfrac{\mathrm{d}\,\lambda}{D_\varepsilon(\lambda)} & \text{if } y > 0 \\ \dfrac{1}{2\,\mathrm{i}\,\pi} \displaystyle\int_{\Lambda_\varepsilon^{\ell,LF}} \mathrm{e}^{\lambda t}\; \mathrm{e}^{\int_{-\infty}^{0}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}\; a_\varepsilon^r\; \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y) \dfrac{\mathrm{d}\,\lambda}{D_\varepsilon(\lambda)} & \text{if } y < 0 \,. \end{cases}$$

As a result, when $xy < 0$ and $-\omega_\ell^{HF}\, t < y < \omega_r^{HF}\, t$

$$\begin{aligned} &\widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x,y) \\ &= \begin{cases} \frac{1}{2\,\mathrm{i}\,\pi} \int_{\Lambda_\varepsilon^{r,LF}} \mathrm{e}^{\lambda t}\; \mathrm{e}^{\int_{-\infty}^{0}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))}\; \mathbf{e}_1 \cdot (\mathbf{V}_\varepsilon^{r,s}(\lambda,x) - \mathbf{V}_\varepsilon^{r,s}(0,x))\; \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y)\, \dfrac{\mathrm{d}\,\lambda}{D_\varepsilon(\lambda)} \\ \quad + \frac{1}{2\,\mathrm{i}\,\pi} \int_{\Lambda_\varepsilon^{r,HF}} \mathrm{e}^{\lambda t}\; G_\varepsilon(\lambda;x,y)\; \mathrm{d}\,\lambda \qquad\qquad \text{if } x > 0 > y \\ \frac{1}{2\,\mathrm{i}\,\pi} \int_{\Lambda_\varepsilon^{\ell,LF}} \mathrm{e}^{\lambda t}\; \mathrm{e}^{-\int_0^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))}\; \mathbf{e}_1 \cdot \left(\mathbf{V}_\varepsilon^{\ell,u}(\lambda,x) - \mathbf{V}_\varepsilon^{\ell,u}(0,x) \right) \mathbf{e}_1 \cdot \widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)\; \dfrac{\mathrm{d}\,\lambda}{D_\varepsilon(\lambda)} \\ \quad + \frac{1}{2\,\mathrm{i}\,\pi} \int_{\Lambda_\varepsilon^{\ell,HF}} \mathrm{e}^{\lambda t}\; G_\varepsilon(\lambda;x,y)\; \mathrm{d}\,\lambda \qquad\qquad \text{if } y > 0 > x \,, \end{cases} \end{aligned}$$

whereas when $(y \geqslant \omega_r^{HF}\, t$ or $y \leqslant -\omega_\ell^{HF}\, t)$,

$$\widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x,y) = \begin{cases} 0 \,, & \text{if } xy > 0 \\ \frac{1}{2\,\mathrm{i}\,\pi} \int_\Lambda \mathrm{e}^{\lambda t}\; G_\varepsilon(\lambda;x,y)\; \mathrm{d}\,\lambda & \text{if } xy < 0 \,, \end{cases}$$

and when $xy > 0$ and $-\omega_\ell^{HF}\, t < y < \omega_r^{HF}\, t$, $\widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x,y)$ equals

$$\begin{cases} \frac{1}{2\,\mathrm{i}\,\pi}\ \int_{\Lambda_\varepsilon^{r,LF}} \mathrm{e}^{\lambda\, t}\ \mathrm{e}^{\int_{-\infty}^{0}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))} & \big(\widetilde{\tau}_\varepsilon^{\ell}(\lambda)\, \mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{r,s}(\lambda,x) - \widetilde{\tau}_\varepsilon^{\ell}(0)\, \mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{r,s}(0,x)\big) \\ & \times \dfrac{\mathbf{e}_1\cdot\widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)}{D_\varepsilon(\lambda)}\ \mathrm{d}\,\lambda \\ \text{if } x>y>0\,, & \\ \frac{1}{2\,\mathrm{i}\,\pi}\ \int_{\Lambda_\varepsilon^{\ell,LF}} \mathrm{e}^{\lambda\, t}\ \mathrm{e}^{\int_{-\infty}^{0}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{-\infty}))} & \big(\tau_\varepsilon^{r}(\lambda)\ \mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{\ell,u}(\lambda,x) - \tau_\varepsilon^{r}(0)\ \mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{\ell,u}(0,x)\big) \\ & \times \dfrac{\mathbf{e}_1\cdot\widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y)}{D_\varepsilon(\lambda)}\ \mathrm{d}\,\lambda \\ \text{if } 0>x>y\,, & \\ \frac{1}{2\,\mathrm{i}\,\pi}\ \int_{\Lambda_\varepsilon^{\ell,LF}} \mathrm{e}^{\lambda\, t}\ \mathrm{e}^{-\int_{0}^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))} & \big(\widetilde{\tau}_\varepsilon^{r}(\lambda)\ \mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{\ell,u}(\lambda,x) - \widetilde{\tau}_\varepsilon^{r}(0)\ \mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{\ell,u}(0,x)\big) \\ & \times \dfrac{\mathbf{e}_1\cdot\widetilde{\mathbf{V}}_\varepsilon^{\ell,u}(\lambda,y)}{D_\varepsilon(\lambda)}\ \mathrm{d}\,\lambda \\ \text{if } 0>y>x\,, & \\ \frac{1}{2\,\mathrm{i}\,\pi}\ \int_{\Lambda_\varepsilon^{r,LF}} \mathrm{e}^{\lambda\, t}\ \mathrm{e}^{-\int_{0}^{+\infty}(f'(\underline{U}_\varepsilon)-f'(\underline{u}_{+\infty}))} & \big(\tau_\varepsilon^{\ell}(\lambda)\ \mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{r,s}(\lambda,x) - \tau_\varepsilon^{\ell}(0)\ \mathbf{e}_1\cdot\mathbf{V}_\varepsilon^{r,s}(0,x)\big) \\ & \times \dfrac{\mathbf{e}_1\cdot\widetilde{\mathbf{V}}_\varepsilon^{r,s}(\lambda,y)}{D_\varepsilon(\lambda)}\ \mathrm{d}\,\lambda \\ \text{if } y>x>0\,, & \end{cases}$$

Proceeding as above for the rest of bounds, we obtain the following lemmas.

Lemma 9. *There exist positive $(\varepsilon_0, C, \omega, \theta)$ such that for any $t>0$, any $0\leqslant\varepsilon\leqslant\varepsilon_0$ and any $(x,y)\in\mathbf{R}^2$*

$$\begin{aligned} &|\widetilde{G}_t^{\varepsilon,pt}(x,y)| + \min\big(\{1,\sqrt{t}\}\big)\,|\partial_x\widetilde{G}_t^{\varepsilon,pt}(x,y)| \\ &\quad\leqslant C\ \mathrm{e}^{-\min\left(\left\{|g'(\underline{u}_{-\infty})|,|g'(\underline{u}_{+\infty})|\right\}\right)\varepsilon\, t}\ \mathrm{e}^{-\theta\,|x|}\ \frac{1}{\sqrt{t}}\left(\mathrm{e}^{-\theta\,\frac{|y-(f'(\underline{u}_{+\infty})-\sigma_\varepsilon)\,t|^2}{t}} + \mathrm{e}^{-\theta\,\frac{|y+(f'(\underline{u}_{-\infty})-\sigma_\varepsilon)\,t|^2}{t}}\right) \\ &\qquad + C\ \mathrm{e}^{-\omega\, t}\ \mathrm{e}^{-\theta\,|x|}\ \mathrm{e}^{-\theta\,|x-y|}\ . \end{aligned}$$

This also implies that there exist positive $(\varepsilon_0, C, \theta)$ such that for any $\theta'>0$ there exists $\omega'>0$ such that for any $t>0$, any $0\leqslant\varepsilon\leqslant\varepsilon_0$ and any $(x,y)\in\mathbf{R}^2$

$$\begin{aligned} &\mathrm{e}^{-\theta'|y|}\left(|\widetilde{G}_t^{\varepsilon,pt}(x,y)| + \min\big(\{1,\sqrt{t}\}\big)\,|\partial_x\widetilde{G}_t^{\varepsilon,pt}(x,y)|\right) \\ &\qquad\leqslant C\ \mathrm{e}^{-\omega' t}\ \left(\frac{1}{\sqrt{t}}\left(\mathrm{e}^{-\theta\,\frac{|y-(f'(\underline{u}_{+\infty})-\sigma_\varepsilon)\,t|^2}{t}} + \mathrm{e}^{-\theta\,\frac{|y+(f'(\underline{u}_{-\infty})-\sigma_\varepsilon)\,t|^2}{t}}\right) + \mathrm{e}^{-\theta\,|x|}\ \mathrm{e}^{-\theta\,|x-y|}\ \mathrm{e}^{-\theta'|y|}\right)\ . \end{aligned}$$

Lemma 10. *There exist positive $(\varepsilon_0, C, \omega, \theta)$ such that for any $t>0$, any $0\leqslant\varepsilon\leqslant\varepsilon_0$*

and any $y \in \mathbf{R}$

$$|G_t^{\varepsilon,p}(y)| \leqslant C\, \mathrm{e}^{\omega\, t}\, \frac{1}{\sqrt{t}}\, \mathrm{e}^{-\theta\, \frac{y^2}{t}}\ .$$

Moreover, there exist positive $(\varepsilon_0, C, \omega, \theta)$ *such that for any* $t > 0$*, any* $0 \leqslant \varepsilon \leqslant \varepsilon_0$ *and any* $y \in \mathbf{R}$

$$|\partial_t G_t^{\varepsilon,p}(y)| \ \leqslant C\, \mathrm{e}^{-\min\left(\left\{|g'(\underline{u}_{-\infty})|, |g'(\underline{u}_{+\infty})|\right\}\right)\varepsilon\, t}\, \frac{1}{\sqrt{t}} \left(\mathrm{e}^{-\theta\, \frac{|y-(f'(\underline{u}_{+\infty})-\sigma_\varepsilon)\, t|^2}{t}} + \mathrm{e}^{-\theta\, \frac{|y+(f'(\underline{u}_{-\infty})-\sigma_\varepsilon)\, t|^2}{t}} \right).$$

This also implies that there exist positive $(\varepsilon_0, C, \theta)$ *such that for any* $\theta' > 0$ *there exists* $\omega' > 0$ *such that for any* $t > 0$*, any* $0 \leqslant \varepsilon \leqslant \varepsilon_0$ *and any* $y \in \mathbf{R}$

$$\mathrm{e}^{-\theta'|y|}\ |\partial_t G_t^{\varepsilon,p}(y)| \leqslant C\, \mathrm{e}^{-\omega'\, t}\, \frac{1}{\sqrt{t}} \left(\mathrm{e}^{-\theta\, \frac{|y-(f'(\underline{u}_{+\infty})-\sigma_\varepsilon)\, t|^2}{t}} + \mathrm{e}^{-\theta\, \frac{|y+(f'(\underline{u}_{-\infty})-\sigma_\varepsilon)\, t|^2}{t}} \right).$$

To conclude and prove Proposition 16, we pick a smooth cut-off function χ on $[0, +\infty)$ such that $\chi \equiv 1$ on $[2, +\infty)$ and $\chi \equiv 0$ on $[0, 1]$ and define $s_\varepsilon^{\mathrm{p}}$, $\widetilde{S}_\varepsilon$ by

$$\begin{aligned} s_\varepsilon^{\mathrm{p}}(t)(w) \ &:= \int_{\mathbf{R}} \chi(t)\, G_t^{\varepsilon,\mathrm{p}}(y)\, w(y)\ \mathrm{d}\, y\,, \\ \widetilde{S}_\varepsilon(t)(w)(x) \ &:= \int_{\mathbf{R}} \Big(\chi(t)\, (G_t^{\varepsilon,\mathrm{ess}}(x,y) + \widetilde{G}_t^{\varepsilon,\mathrm{pt}}(x,y)) + (1-\chi(t))\, G_t^{\varepsilon}(x,y) \Big)\ w(y)\ \mathrm{d}\, y\,. \end{aligned}$$

The definitions are extended to $t = 0$ by $s_\varepsilon^{\mathrm{p}}(0)(w) = 0$ and $\widetilde{S}_\varepsilon(0)(w) = w$.

As explained near its statement, Proposition 16 follows then from $L_x^\infty L_y^1$ bounds on Green kernels, which themselves are derived from pointwise bounds proved above.

3.5 Nonlinear stability

In the present section we conclude the proof of Theorem 15.

To do so, we seek for u solving (3.2.1) under the form

$$u(t,x) \ = \underline{U}_\varepsilon(x + \psi(t)) + v(t, x + \psi(t)) \tag{3.5.1}$$

with (v, ψ') exponentially decaying in time. In these terms the equation becomes

$$\begin{aligned} &\partial_t v + (f'(\underline{U}_\varepsilon + v) - \sigma_\varepsilon + \psi')\partial_x v - \partial_x^2 v \\ &\quad = \varepsilon\, (g(\underline{U}_\varepsilon + v) - g(\underline{U}_\varepsilon)) - (f'(\underline{U}_\varepsilon + v) - f'(\underline{U}_\varepsilon) + \psi')\underline{U}'_\varepsilon\,. \end{aligned} \tag{3.5.2}$$

Equation (3.5.2) may be solved through

$$v(t,\cdot) \;=\; \widetilde{S}_\varepsilon(t)\,(v_0) \;+\; \int_0^t \widetilde{S}_\varepsilon(t-s)\,\mathcal{N}_\varepsilon[v(s,\cdot)),\psi'(s)]\;\mathrm{d}\,s\,, \tag{3.5.3}$$

$$\psi'(t) \;=\; \partial_t s^{\mathrm{p}}_\varepsilon(t)\,(v_0) \;+\; \int_0^t \partial_t s^{\mathrm{p}}_\varepsilon(t-s)\,\mathcal{N}_\varepsilon[v(s,\cdot)),\psi'(s)]\;\mathrm{d}\,s\,, \tag{3.5.4}$$

with

$$\begin{aligned}\mathcal{N}_\varepsilon[w,\varphi] := &-(f'(\underline{U}_\varepsilon+w)-f'(\underline{U}_\varepsilon)+\varphi)\partial_x w+\varepsilon\,(g(\underline{U}_\varepsilon+w)-g(\underline{U}_\varepsilon)-g'(\underline{U}_\varepsilon)\,w)\\ &-(f'(\underline{U}_\varepsilon+w)-f'(\underline{U}_\varepsilon)-f''(\underline{U}_\varepsilon)\,w)\underline{U}'_\varepsilon\,.\end{aligned}$$

In the present section, for notational concision's sake, we denote

$$\omega_\infty = \min(|g'(\underline{u}_{-\infty})|,|g'(\underline{u}_{+\infty})|)\,.$$

To begin with, we observe that estimates of the foregoing section are almost sufficient to run a continuity argument on (3.5.3)-(3.5.4). Indeed they provide the following proposition.

Proposition 17. *There exist $\theta_0>0$ and $\varepsilon_0>0$ such that for any $0<\theta\leqslant\theta_0$ and $\delta>0$, there exist $C>0$ such that for any $0<\varepsilon\leqslant\varepsilon_0$ and $T>0$, if (v,ψ') solves* (3.5.3)-(3.5.4) *on $[0,T]$, with*

$$\|v(t,\cdot)\|_{L^\infty(\mathbf{R})}\leqslant\delta\,,\qquad\qquad t\in[0,T]\,,$$

then, for any $t\in[0,T]$,

$$\begin{aligned}&|\psi'(t)|+\|v(t,\cdot)\|_{W^{1,\infty}(\mathbf{R})}\\ &\quad\leqslant C\,\|v(0,\cdot)\|_{W^{1,\infty}(\mathbf{R})}\;\mathrm{e}^{-\varepsilon\,\omega_\infty t}\\ &\qquad\times\exp\left(C\sup_{0\leqslant s\leqslant T}\mathrm{e}^{\varepsilon\,\omega_\infty s}(|\psi'(s)|+\|v(s,\cdot)\|_{L^\infty(\mathbf{R})}+\|(\varepsilon+\mathrm{e}^{-\theta|\cdot|})^{-1}\partial_x v(s,\cdot)\|_{L^\infty(\mathbf{R})})\right).\end{aligned}$$

The estimate fails to close by the fact that $\|\partial_x w\|_{L^\infty(\mathbf{R})}$ provides a weaker ε-uniform control on w than $\|(\varepsilon+\mathrm{e}^{-\theta|\cdot|})^{-1}\partial_x w\|_{L^\infty(\mathbf{R})}$. Note however that for any $x_*>0$,

$$\|(\varepsilon+\mathrm{e}^{-\theta|\cdot|})^{-1}\partial_x w\|_{L^\infty([-x_*,x_*])}\leqslant\mathrm{e}^{\theta x_*}\;\|\partial_x w\|_{L^\infty(\mathbf{R})}$$

so that we only need to improve the estimates on $\partial_x v(t,\cdot)$ on the complement of some compact neighborhood of 0.

3.5.1 Maximum principle and propagation of regularity

To close our nonlinear estimates without using neither localization nor parabolic smoothing — which would cause loss in powers of ε—, we shall use a maximum principle argument.

To begin with, we state and prove a convenient classical abstract maximum principle. We provide a proof mostly to highlight that it may be thought as an energy estimate on a suitable nonlinear function.

Lemma 11. *Let $T > 0$, $x_* \in \mathbf{R}$, $\delta > 0$, $a \in L^1([0,T]; W^{1,\infty}([x_*, +\infty)))$ bounded from above away from zero and $h \in \mathcal{C}^0([0,T] \times [x_*, +\infty) \times \mathbf{R})$. If $w \in \mathcal{C}^2((0,T) \times [x_*, +\infty)) \cap \mathcal{C}^0([0,T] \times [x_*, +\infty))$ is a bounded function such that*

$$\partial_t w + a(\cdot,\cdot)\, \partial_x w \leqslant \partial_x^2 w + h(\cdot,\cdot,w)\,, \qquad \text{on } [0,T] \times [x_*, +\infty)$$

and M is a positive constant such that

$$\begin{aligned} M &\geqslant w(\cdot, x_*)\,, && \text{on } [0,T]\,,\\ M &\geqslant w(0,\cdot,)\,, && \text{on } [x_*, +\infty)\,,\\ -\delta\cdot &\geqslant \mathbf{1}_{\cdot > M}\, h(t,x,\cdot)\,, \end{aligned}$$

then

$$w \leqslant M\,, \qquad \text{on } [0,T] \times [x_*, +\infty)\,.$$

Proof. When moreover

$$M > \limsup_{x \to \infty} w(\cdot, x)\,, \qquad \text{on } [0,T]$$

the claim is proved by a Grönwall argument on

$$t \mapsto \int_{x_*}^{+\infty} (w(t,x) - M)_+ \,\mathrm{d}\, x\,.$$

The general case is recovered by applying this special case to $(t,x) \mapsto \mathrm{e}^{-\theta\,(x - x_*)}\, w(t,x)$ with $\theta > 0$ sufficiently small and taking the limit $\theta \to 0$. □

We now use the foregoing lemma to derive a weighted bound on $\partial_x v$ outside a sufficiently large compact neighborhood of 0. We shall insert such a bound in a continuity argument so that we only need to prove that as long as $\partial_x v$ does not become too large it remains small. This is the content of the following proposition.

Proposition 18. *There exists $\theta_0 > 0$ such that for any $0 < \theta \leqslant \theta_0$, there exist $x_* > 0$, $\varepsilon_0 > 0$, $\delta > 0$ and $C > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$ and $T > 0$, if (v, ψ') solves (3.5.2) on $[0,T] \times \mathbf{R}$, with*

$$
\begin{aligned}
|\psi'(t)| + \|v(t,\cdot)\|_{L^\infty(\mathbf{R})} &\leqslant \delta\, \mathrm{e}^{-\varepsilon\, \omega_\infty t}\,, && t \in [0,T]\,,\\
\frac{|\partial_x v(t,x)|}{\varepsilon + \mathrm{e}^{-\theta|x|}} &\leqslant \delta\, \mathrm{e}^{-\varepsilon\, \omega_\infty t}\,, && (t,x) \in [0,T] \times \mathbf{R}\,,
\end{aligned}
$$

then for any $(t,x) \in [0,T] \times (\mathbf{R} \backslash [-x_*, x_*])$

$$
\begin{aligned}
\frac{|\partial_x v(t,x)|}{\varepsilon + \mathrm{e}^{-\theta|x|}} \leqslant C\ \mathrm{e}^{-\varepsilon\, \omega_\infty t} \times \Big(& \sup_{0 \leqslant s \leqslant T} \mathrm{e}^{\varepsilon\, \omega_\infty s} (|\psi'(s)| + \|v(s,\cdot)\|_{L^\infty(\mathbf{R})} + \|\partial_x v(s,\cdot)\|_{L^\infty([-x_*, x_*])}) \\
& + \|(\varepsilon + \mathrm{e}^{-\theta|\cdot|})^{-1} \partial_x v(0,\cdot)\|_{L^\infty(\mathbf{R})} \Big)\,.
\end{aligned}
$$

Proof. We may argue separately to deal with bounds on $x \geqslant x_*$ on one hand and on $x \leqslant -x_*$ on the other hand, and provide details only for the former. From now on we focus on $x \geqslant x_*$.

We would like to apply Lemma 11 to both $A_\varepsilon\, \partial_x v$ and $-A_\varepsilon\, \partial_x v$ for a suitable weight A_ε equivalent to $(t,x) \mapsto \mathrm{e}^{\varepsilon\, \omega_\infty t}\, (\varepsilon + \mathrm{e}^{-\theta|x|})^{-1}$. Our choice is

$$
A_\varepsilon(t,x) := \frac{\mathrm{e}^{\omega_\infty\, \varepsilon\, t}\, \mathrm{e}^{\int_t^{+\infty} \varepsilon\, \mathrm{e}^{-\varepsilon\, \omega_\infty s}\, \mathrm{d}\, s}}{\varepsilon + \mathrm{e}^{-\theta\, |x|}}\,.
$$

Note that one has

$$
\begin{aligned}
\partial_t(A_\varepsilon \partial_x v) + \left((f'(\underline{U}_\varepsilon + v) - \sigma_\varepsilon + \psi') + 2 \frac{\theta\, \mathrm{e}^{-\theta|x|}}{\varepsilon + \mathrm{e}^{-\theta|x|}} \right) \partial_x(A_\varepsilon \partial_x v) - \partial_x^2(A_\varepsilon \partial_x v) \\
= (f'(\underline{U}_\varepsilon) - f'(\underline{U}_\varepsilon + v) - \psi') A_\varepsilon \underline{U}_\varepsilon'' + (f''(\underline{U}_\varepsilon) - f''(\underline{U}_\varepsilon + v)) A_\varepsilon \underline{U}_\varepsilon'^2 \\
- A_\varepsilon \partial_x v \Big(\varepsilon\, \mathrm{e}^{-\varepsilon\, \omega_\infty t} - (f'(\underline{U}_\varepsilon + v) - \sigma_\varepsilon + \psi') \frac{\theta\, \mathrm{e}^{-\theta|x|}}{\varepsilon + \mathrm{e}^{-\theta|x|}} \\
- \varepsilon\, (\omega_\infty + g'(\underline{U}_\varepsilon + v)) - \frac{\theta^2\, \mathrm{e}^{-\theta x}\, (2\, \varepsilon\, + \mathrm{e}^{-\theta x})}{(\varepsilon + \mathrm{e}^{-\theta x})^2} + f''(\underline{U}_\varepsilon + v)(2 \underline{U}_\varepsilon' + \partial_x v) \Big)\,.
\end{aligned}
$$

Fixing first $\theta > 0$ sufficiently small, then x_* sufficiently large and δ and ε sufficiently small, one enforce that the term in front of $\partial_x(A_\varepsilon \partial_x v)$ is bounded from above away from zero and the term in front of $A_\varepsilon \partial_x v$ is bounded from below by a multiple of $\varepsilon\, \mathrm{e}^{-\varepsilon\, \omega_\infty t} + \theta\, \mathrm{e}^{-\theta|x|}$. This is sufficient to apply Lemma 11 and derive the claimed upper bound on $x \geqslant x_*$. □

3.5.2 Proof of Theorem 15

Our very first task when proving Theorem 15 is to convert classical local well-posedness yielding maximal solutions u to (3.2.1) into convenient local existence results for (v, ψ').

This follows from the following simple observation. By design, $s_\varepsilon^{\rm p}(t) \equiv 0$ when $0 \leqslant t \leqslant 1$. Thus if u solves (3.2.1) on $[0,T] \times \mathbf{R}$ then (v, ψ') satisfying (3.5.1)-(3.5.3)-(3.5.4) may be obtained recursively through

$$\psi(t) = \psi_0\,, \qquad v(t,\cdot) = u(t, \cdot - \psi(t)) - \underline{U}_\varepsilon\,, \qquad\qquad \text{when } 0 \leqslant t \leqslant \min(\{1, T\})\,,$$

and, for any $n \in \mathbf{N}$,

$$\begin{aligned}
\psi'(t) \;&=\; \partial_t s_\varepsilon^{\rm p}(t)\,(v_0) \;+\; \int_0^{t-1} \partial_t s_\varepsilon^{\rm p}(t-s)\,\mathcal{N}_\varepsilon[v(s,\cdot)), \psi'(s)]\;\mathrm{d}\,s\,,\\
\psi(t) &= \psi_0 + \int_0^t \psi'(s)\;\mathrm{d}\,s\,,\\
v(t,\cdot) &= u(t, \cdot - \psi(t)) - \underline{U}_\varepsilon\,, \qquad \text{when } \min(\{n, T\}) \leqslant t \leqslant \min(\{n+1, T\})\,.
\end{aligned}$$

Now, combining together Propositions 17 and 18, one obtains that for any $\theta > 0$ sufficiently small, there exist $\varepsilon_0 > 0$, $\delta > 0$ and $C \geqslant 1$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$, and (v_0, ψ_0) with

$$\|v_0\|_{L^\infty(\mathbf{R})} + \|(\varepsilon + \mathrm{e}^{-\theta|\cdot|})^{-1}\partial_x v_0\|_{L^\infty(\mathbf{R})} \;\leqslant\; \delta\,,$$

the corresponding solution u to (3.2.1), in the form (3.5.1), satisfies that if for some $T > 0$ and any $0 \leqslant t \leqslant T$

$$\begin{aligned}
|\psi'(t)| + \|v(t,\cdot)\|_{L^\infty(\mathbf{R})} &+ \|(\varepsilon + \mathrm{e}^{-\theta|\cdot|})^{-1}\partial_x v(t,\cdot)\|_{L^\infty(\mathbf{R})}\\
&\leqslant\; 2\,C\;\mathrm{e}^{-\varepsilon\,\omega_\infty t}\;\big(\|v_0\|_{L^\infty(\mathbf{R})} + \|(\varepsilon + \mathrm{e}^{-\theta|\cdot|})^{-1}\partial_x v_0\|_{L^\infty(\mathbf{R})}\big)
\end{aligned}$$

then for any $0 \leqslant t \leqslant T$

$$\begin{aligned}
|\psi'(t)| + \|v(t,\cdot)\|_{L^\infty(\mathbf{R})} &+ \|(\varepsilon + \mathrm{e}^{-\theta|\cdot|})^{-1}\partial_x v(t,\cdot)\|_{L^\infty(\mathbf{R})}\\
&\leqslant\; C\;\mathrm{e}^{-\varepsilon\,\omega_\infty t}\;\big(\|v_0\|_{L^\infty(\mathbf{R})} + \|(\varepsilon + \mathrm{e}^{-\theta|\cdot|})^{-1}\partial_x v_0\|_{L^\infty(\mathbf{R})}\big)\;.
\end{aligned}$$

From this and a continuity argument stem that u is global and that the latter estimate holds globally in time. One achieves the proof of Theorem 15 by deriving bounds on ψ by integration of those on ψ' and going back to original variables.

3.6 Wave profiles

In the present Appendix, we prove Proposition 10. Let us first reformulate the wave profile equation in terms of

$$\widetilde{\underline{U}}_\varepsilon := \frac{\underline{U}_\varepsilon - \underline{U}_0}{\varepsilon}\,, \qquad\qquad \widetilde{\sigma}_\varepsilon := \frac{\sigma_\varepsilon - \sigma_0}{\varepsilon}\,.$$

The equation to consider is

$$\begin{aligned}\widetilde{\underline{U}}_\varepsilon'' - \Big((f'(\underline{U}_0) - \sigma_0)\,\widetilde{\underline{U}}_\varepsilon\Big)' \;=\; & -g(\underline{U}_0 + \varepsilon\,\widetilde{\underline{U}}_\varepsilon) - \Big(\widetilde{\sigma}_\varepsilon\,(\underline{U}_0 + \varepsilon\,\widetilde{\underline{U}}_\varepsilon)\Big)' \\ & + \left(\frac{f(\underline{U}_0 + \varepsilon\,\widetilde{\underline{U}}_\varepsilon) - f(\underline{U}_0) - \varepsilon\,f'(\underline{U}_0)\,\widetilde{\underline{U}}_\varepsilon}{\varepsilon}\right)'\,,\end{aligned}$$

with $\widetilde{\underline{U}}_\varepsilon(0) = 0$, $(\widetilde{\sigma}_\varepsilon, \mathrm{e}^{\theta\,|\cdot|}\,\widetilde{\underline{U}}_\varepsilon, \mathrm{e}^{\theta\,|\cdot|}\,\widetilde{\underline{U}}_\varepsilon')$ uniformly bounded, for some sufficiently small $\theta > 0$. As announced in the introduction the framework we first consider is suboptimal from the point of view of spatial localization but we shall refine it in a second step. To carry out the first step we introduce spaces $W^{k,\infty}_\theta$ and their subspaces BUC^k_θ, corresponding to norms

$$\|v\|_{W^{k,\infty}_\theta(\mathbf{R})} = \sum_{j=0}^{k} \|\,\mathrm{e}^{\theta\,|\cdot|}\;\partial_x^j v\|_{L^\infty(\mathbf{R})}\,.$$

In this first step, we just pick some $0 < \theta < \min(\{\theta_0^\ell, \theta_0^r\})$ and let all the constants depend on this particular choice.

We begin with two preliminary remarks. Firstly note that a simple integration yields that a necessary constraint is

$$\widetilde{\sigma}_\varepsilon \;=\; -\frac{1}{\underline{u}_{+\infty} - \underline{u}_{-\infty}} \int_{\mathbf{R}} g(\underline{U}_0 + \varepsilon\,\widetilde{\underline{U}}_\varepsilon) \;=:\; \widetilde{\Sigma}_\varepsilon[\widetilde{\underline{U}}_\varepsilon]$$

and that

$$\begin{aligned}\widetilde{\mathcal{N}}_\varepsilon[\widetilde{\underline{U}}_\varepsilon] := & -g(\underline{U}_0 + \varepsilon\,\widetilde{\underline{U}}_\varepsilon) - \Big(\widetilde{\Sigma}_\varepsilon[\widetilde{\underline{U}}_\varepsilon]\,(\underline{U}_0 + \varepsilon\,\widetilde{\underline{U}}_\varepsilon)\Big)' \\ & + \left(\frac{f(\underline{U}_0 + \varepsilon\,\widetilde{\underline{U}}_\varepsilon) - f(\underline{U}_0) - \varepsilon\,f'(\underline{U}_0)\,\widetilde{\underline{U}}_\varepsilon}{\varepsilon}\right)'\end{aligned}$$

defines a continuous map from BUC^1_θ to the closed subspace of BUC^0_θ whose range is contained in the set of functions with zero integral and that, on any ball of BUC^1_θ, has an $\mathcal{O}(\varepsilon)$-Lipschitz constant.

Secondly, denoting L_0 the operator defined by

$$L_0(v) \,:=\, v'' - (f'(\underline{U}_0) - \sigma_0)v)'$$

on BUC^0_θ, with domain BUC^2_θ, we observe that L_0 is Fredholm of index 0 (as a continuous operator from BUC^2_θ to BUC^0_θ), its kernel is spanned by $\underline{U}'_\varepsilon$ and the kernel of its adjoint is reduced to constant functions. The foregoing claims are easily proved by direct inspection but may also be obtained with the arguments of Sections 3.2 and 3.3, combining spatial dynamics point of view with a Sturm-Liouville argument. Since evaluation at 0 acts continuously on BUC^2_θ and $\underline{U}'_0(0) \neq 0$, this implies that the restriction of L_0 from the closed subspace of BUC^2_θ consisting of functions with value 0 at 0 to the closed subspace of BUC^0_θ consisting of functions with zero integral is boundedly invertible. Indeed, the inverse of this restriction is readily seen to be given by

$$L_0^\dagger(h)(x) := -\int_0^x \int_z^{+\infty} \frac{\underline{U}'_0(x)}{\underline{U}'_0(z)}\, h(y)\, \mathrm{d}\, y\, \mathrm{d}\, z \,=\, \int_0^x \int_{-\infty}^z \frac{\underline{U}'_0(x)}{\underline{U}'_0(z)}\, h(y)\, \mathrm{d}\, y\, \mathrm{d}\, z\,.$$

Note that from the profile equation itself stems that if $\widetilde{\underline{U}}_\varepsilon$ is a BUC^1_θ-solution it is also a BUC^2_θ-solution so that the problem reduces to

$$\widetilde{\sigma}_\varepsilon = \widetilde{\Sigma}[\widetilde{\underline{U}}_\varepsilon]\,, \qquad\qquad \widetilde{\underline{U}}_\varepsilon = L_0^\dagger(\widetilde{\mathcal{N}}_\varepsilon[\widetilde{\underline{U}}_\varepsilon])\,.$$

If C_0 is chosen such that $C_0 > \|L_0^\dagger \widetilde{\mathcal{N}}_\varepsilon[0_{\mathbf{R}}]\|_{W^{1,\infty}_\theta(\mathbf{R})}$, it follows that, when ε is sufficiently small, the map $L_0^\dagger \circ \widetilde{\mathcal{N}}_\varepsilon$ sends the complete space

$$\left\{ v \in BUC^1_\theta(\mathbf{R})\,;\, v(0) = 0 \text{ and } \|v\|_{W^{1,\infty}_\theta(\mathbf{R})} \leqslant C_0 \right\}$$

into itself and is strictly contracting with an $\mathcal{O}(\varepsilon)$-Lipschitz constant. Thus resorting to the Banach fixed-point theorem achieves the first step of the proof of Proposition 10.

Note that, in order to conclude the proof, it is sufficient to provide asymptotic descriptions of $\mathrm{e}^{\theta^\ell_\varepsilon|\cdot|}(\underline{U}_\varepsilon - \underline{u}_{-\infty})$, $\mathrm{e}^{\theta^r_\varepsilon\cdot}(\underline{U}_\varepsilon - \underline{u}_{+\infty})$, $\mathrm{e}^{\theta^\ell_\varepsilon|\cdot|}\,\underline{U}'_\varepsilon$ and $\mathrm{e}^{\theta^r_\varepsilon\cdot}\,\underline{U}'_\varepsilon$. Indeed, on one hand, the asymptotic comparisons for $\mathrm{e}^{\theta^\ell_\varepsilon|\cdot|}\,\underline{U}^{(k)}_\varepsilon$ and $\mathrm{e}^{\theta^r_\varepsilon\cdot}\,\underline{U}^{(k)}_\varepsilon$, $k \geqslant 2$, are then deduced recursively by using the profile equation (differentiated $(k-2)$ times). On the other hand, since, for $\# \in \{\ell, r\}$, $\theta^\#_\varepsilon = \theta^\#_0 + \mathcal{O}(\varepsilon)$, the asymptotic descriptions are sufficient to upgrade the existence part of the first step arbitrarily close to optimal spatial decay rates, $\alpha^\# \to \theta^\#_0$.

As a further reduction, we observe that the asymptotics for $\mathrm{e}^{\theta^\ell_\varepsilon|\cdot|}(\underline{U}_\varepsilon - \underline{u}_{-\infty})$, $\mathrm{e}^{\theta^r_\varepsilon\cdot}(\underline{U}_\varepsilon - \underline{u}_{+\infty})$, may be deduced from the ones for $\mathrm{e}^{\theta^\ell_\varepsilon|\cdot|}\,\underline{U}'_\varepsilon$ and $\mathrm{e}^{\theta^r_\varepsilon\cdot}\,\underline{U}'_\varepsilon$ by integration since

$$\mathrm{e}^{-\theta^\ell_\varepsilon\, x}\,(\underline{U}_\varepsilon(x) - \underline{u}_{-\infty}) - \mathrm{e}^{-\theta^\ell_0\, x}\,(\underline{U}_0(x) - \underline{u}_{-\infty})$$

$$= \int_{-\infty}^{x} \mathrm{e}^{\theta_\varepsilon^\ell\,(y-x)} \left(\mathrm{e}^{-\theta_\varepsilon^\ell\,y}\ \underline{U}'_\varepsilon(y) - \mathrm{e}^{-\theta_0^\ell\,y}\ \underline{U}'_0(y) \right) \mathrm{d}\,y$$
$$+ \int_{-\infty}^{x} \mathrm{e}^{\theta_0^\ell\,(y-x)} \left(\mathrm{e}^{(\theta_\varepsilon^\ell-\theta_0^\ell)\,(y-x)}\ -1 \right) \mathrm{e}^{-\theta_0^\ell\,y}\ \underline{U}'_0(y)\ \mathrm{d}\,y$$

and likewise near $+\infty$.

To conclude, we derive the study of $\mathrm{e}^{\theta_\varepsilon^\ell|\,\cdot\,|}\,\underline{U}'_\varepsilon$ and $\mathrm{e}^{\theta_\varepsilon^r\,\cdot}\,\underline{U}'_\varepsilon$ from the analysis of Proposition 11 (with $K = \{0\}$). Indeed,

$$\theta_\varepsilon^\ell = \mu_+^\varepsilon(0; \underline{u}_{-\infty})\,, \qquad\qquad \theta_\varepsilon^\ell = -\mu_-^\varepsilon(0; \underline{u}_{+\infty})\,,$$

and

$$\underline{U}'_\varepsilon(x) = \frac{(\underline{u}_{+\infty} - \underline{u}_{-\infty})}{2}\ \frac{\mathrm{e}^{\theta_\varepsilon^\ell\,x}\ \ \mathbf{e}_1 \cdot P_\varepsilon^\ell(0,x)\,\mathbf{R}_+^\varepsilon(0; \underline{u}_{-\infty})}{\int_{-\infty}^{0} \mathrm{e}^{\theta_\varepsilon^\ell\,y}\ \ \mathbf{e}_1 \cdot P_\varepsilon^\ell(0,y)\,\mathbf{R}_+^\varepsilon(0; \underline{u}_{-\infty})\ \mathrm{d}\,y}\,,$$
$$= \frac{(\underline{u}_{+\infty} - \underline{u}_{-\infty})}{2}\ \frac{-\,\mathrm{e}^{\theta_\varepsilon^r\,x}\ \ \mathbf{e}_1 \cdot P_\varepsilon^r(0,x)\,\mathbf{R}_-^\varepsilon(0; \underline{u}_{+\infty})}{\int_{0}^{+\infty} \mathrm{e}^{-\theta_\varepsilon^r\,y}\ \ \mathbf{e}_1 \cdot P_\varepsilon^r(0,y)\,\mathbf{R}_-^\varepsilon(0; \underline{u}_{+\infty})\ \mathrm{d}\,y}\,.$$

Thus the claimed expansion stems from the smoothness in ε afforded by Proposition 11.

Chapter 4

Convective uniform stability

4.1 Introduction

In this chapter, we continue the study initiated in the third chapter. We are again studying the stability of particular waves solving partial differential equations of the form

$$\partial_t u + \partial_x(f(u)) = g(u) + \varepsilon\partial_x^2 u. \tag{4.1.1}$$

We are interested in particular classes of waves, that are singular perturbations of waves solutions to the hyperbolic problem obtained as $\varepsilon \to 0$.

However, in contrast with Chapter 3, the goal is to study stability of waves in weighted spaces, which are spaces of the form

$$BUC^0_\omega(\mathbf{R}) := \{ve^{-\omega} \,|\, v \in BUC^0(\mathbf{R})\},$$

and higher-order versions thereof, where $\omega : \mathbf{R} \to \mathbf{R}$ is typically a locally bounded function.

We begin by discussing the choice of the functional space and the introducing of such weights. This choice is an important factor in the determination of the spectrum of the linearized operator, and thus can crucially affect the stability properties of such waves. In fact, in various problems, some parts of the spectrum may be moved to the left half of the complex plane or even entirely deleted by an other functional space.

A seminal work developping this approach of choosing a particular functional space to obtain nonlinear stability in a similar framework is [Sat76], by Sattinger. It discusses the stability of waves solutions to equations of the form

$$\partial_t u = \partial_x^2 u + f(u, \partial_x u),$$

and the use of weight allows him to move the essential spectrum to the left of the imaginary axis in various problems. In fact, as this part of the spectrum is a consequence of properties of the asymptotic matrices at $\pm\infty$ associated to the linearized operator, and the introduction of weights mostly changes the prescription of the behavior of functions

near $\pm\infty$. Let us point out that, actually, one may even get rid of the eigenvalue at $\lambda = 0$ by doing so, thus obtaining potentially a classical asymptotic stability result, and not just a classical orbital one.

In particular, in [Sat76] stability in some reaction-diffusion equations and the viscous Burgers equation are studied in such spaces. The analysis of [Sat76] relies mostly on abstract semigroup arguments and parabolic regularization estimates, those being sufficient to carry out the nonlinear analysis. In this chapter, we will need to use sharper methods to study the uniform stability of parabolic regularizations Riemann shocks, and in particular, as in the third chapter, Green function bounds.

We will, here, mostly restrict the discussion to weights of the form e^{ω} where $\omega : \mathbf{R} \to \mathbf{R}$ is equal to $c_+ x$ for all $x \geqslant 0$, and to $c_- x$ for $x \leqslant 0$, where $c_+ \leqslant 0$ and $c_- \geqslant 0$ are constants. This should be contrasted with works introducing weights of Gaussian types, or, in the part of [DR22] related to smooth waves with a characteristic point, in which choosing spaces associated with norms controlling higher order derivatives allows to obtain a bigger spectral gap, and, hence, better time decay.

Note that we restrict a priori to weights strengthening the localization because we want the corresponding L^∞ based spaces to form algebras. This is almost mandatory in order to close a nonlinear argument.

Finally, the importance of such studies, and more generally of unstables fronts in the study of the long-time behavior of solutions to PDEs, already clearly apparent in the pioneering [AW78]. In particular, let us mention that, when one studies a reaction-diffusion scalar equation, an initial data which is initially compactly supported will have its dynamics strongly influenced by particular fronts that are unstable in the unweighted topology. Finally, in the hyperbolic case of balance laws, works have also been done to study such behavior, as in [MS97, Sin96], where it is proved that at least in the strictly convex case and for particular topologies, one should expect that an initial data constant at $+\infty$ and $-\infty$ will be asymptotically described by piecing waves together.

The present chapter is a vanishing viscosity counterpart to [GR] and we describe now its main results. It extends some parts of [DR20, DR22] to other functional spaces. More precisely, in [DR20, DR22], stability and instability properties of fronts are studied in the framework of L^∞ based unweighted spaces. In such unweighted topologies, the authors obtain a complete description of non-degenerate stable and unstable traveling waves. The authors of [GR] analyze the effects of weights described above on the classification.

An important part of [GR] consists in obtaining stability in the critical case. This refers to the case of an exponential weight with the biggest possible $c_+ < 0$ for which the spectrum is confined in the left half of the complex plane. In the case of smooth traveling waves, the zero mode is not in the weighted space, and the authors obtain a classical

stability result. Furthermore, when adding an extra polynomial weight, one can get back some form of decay on the perturbation. More specifically, with a weight of the form $\|\cdot\|_{c,\beta} := \|(1+(\cdot)_+)^\beta e^{c_+(\cdot)_+} v\|_{L^\infty}$ with $e^{c\cdot}$ the critical weight, and for some $\beta > 0$, a decay of speed at least $\frac{1}{1+t^\beta}$ is obtained. In the case of a Riemann shock, however, an asymptotic orbital result is obtained and it requires that the set of initial data considered have slightly more localization, since an extra polynomial localization is needed with $\beta > 1$. The latter extra constraint originates in the necessity to control the orbital shift.

Let us now describe briefly the content of the present chapter, beginning with proofs of existence of wave solutions to the viscous equations, approximating hyperbolic waves. The case of Riemann shocks is similar to those studied before. In the case of smooth waves, however, it relies on singular perturbations techniques. In fact, the spatial eigenvalues of the profile ODE at $\pm\infty$ are asymptotically equivalent to $g'(\underline{u}_{\pm\infty})/(f'(\underline{u}_{\pm\infty}) - \sigma)$ and $(\sigma - f'(\underline{u}_{\pm\infty}))/\varepsilon$. The latter is stiff and requires a specific treatment.

It is worth noting that, in the smooth case, there exists a wave profile for each speed greater than a given critical value, both in the hyperbolic and parabolic problems. In contrast, when it comes to the Riemann shock, the speed is uniquely determined by the endstates.

Moreover, we do not study here the problems of other types of instabilities. First of all, nullstable front in the case of Riemann shocks (that is, with $g'(\underline{u}_{-\infty}) > 0$ and $g'(\underline{u}_{+\infty}) > 0$) is expected to be similarly solvable. Likewise we expect that the case when the wave is marginally stable in unweighted topologies, because for instance $g'(\underline{u}_{+\infty}) = 0$, could be treated similarly but we do not treat it here. In contrast, the analysis of the degenerate case of waves with critical speed $\sigma_* := \sup_{[\underline{u}_{-\infty}, \underline{u}_{+\infty}]} f'$, also not carried out here, is likely to require new insights.

We are studying here fronts that are monostable. Here, instabilities at the spectral level are restricted to those coming from constant states. Monostability means that one endstate is unstable while the other is stable. In such a context, as we are considering heteroclinic with noncharacteristic convective parts, we can ensure that, adding weights on that side of the real line, the linearized operator is actually spectrally stable, but some sign information on the convection is needed to ensure that the weight is increasing localization. For viscous shocks and other traveling waves considered here, this sign condition is met and one may hope to close a nonlinear argument, as we do.

We conclude this short introduction by gathering the weighted topologies used throughout the chapter. For $\theta, \beta \geqslant 0$, we define the following spaces

$$
\begin{aligned}
L^\infty_\theta(\mathbf{R}) &:= \{ u e^{-\theta(\cdot)_+} \,|\, u \in L^\infty(\mathbf{R}) \}, \\
W^{1,\infty}_\theta &:= \{ u \in L^\infty_\theta(\mathbf{R}) \,|\, u' \in L^\infty_\theta(\mathbf{R}) \},
\end{aligned}
$$

$$
\begin{aligned}
&BUC^0_\theta(\mathbf{R}) := \{ue^{-\theta(\cdot)_+} \,|\, u \in BUC^0(\mathbf{R})\},\\
&BUC^1_\theta := \{u \in BUC^0_\theta(\mathbf{R}) \,|\, u' \in BUC^0_\theta(\mathbf{R})\},\\
&L^\infty_{\theta,\beta}(\mathbf{R}) := \{(1+(\cdot))^{-\beta} ue^{-\theta\chi} \,|\, u \in L^\infty(\mathbf{R})\},\\
&W^{1,\infty}_\theta := \{u \in L^\infty_{\theta,\beta}(\mathbf{R}) \,|\, u' \in L^\infty_\theta(\mathbf{R})\},\\
&BUC^0_\theta(\mathbf{R}) := \{(1+(\cdot)_+)^{-\beta} ue^{-\theta(\cdot)_+} \,|\, u \in BUC^0(\mathbf{R})\},\\
&BUC^1_\theta := \{u \in BUC^0_\theta(\mathbf{R}) \,|\, u' \in BUC^0_{\theta,\beta}(\mathbf{R})\}.
\end{aligned}
$$

The associated norms, defined, for any v in, respectively, $L^\infty_\theta(\mathbf{R})$, $W^{1,\infty}_\theta(\mathbf{R})$, $L^\infty_{\theta,\beta}(\mathbf{R})$, $W^{1,\infty}_{\theta,\beta}(\mathbf{R})$ are given by

$$
\begin{aligned}
&\|v\|_{L^\infty_\theta} := \|e^{\theta(\cdot)_+} v\|_{L^\infty},\\
&\|v\|_{W^{1,\infty}_\theta} := \|e^{\theta(\cdot)_+} v\|_{L^\infty_\theta} + \|e^{\theta(\cdot)_+} v'\|_{L^\infty_\theta}\\
&\|v\|_{L^\infty_{\theta,\beta}} := \|e^{\theta(\cdot)_+}(1+(\cdot)_+)^\beta v\|_{L^\infty},\\
&\|v\|_{W^{1,\infty}_{\theta,\beta}} := \|e^{\theta(\cdot)_+} v\|_{L^\infty_{\theta,\beta}} + \|e^{\theta(\cdot)_+} v'\|_{L^\infty_{\theta,\beta}}.
\end{aligned}
$$

4.2 Riemann shock

Consider $\underline{u}_{\pm\infty}$, distinct real such that $\sigma_0 = \dfrac{f(\underline{u}_{+\infty}) - f(\underline{u}_{-\infty})}{\underline{u}_{+\infty} - \underline{u}_{-\infty}}$ with $f'(\underline{u}_{+\infty}) < \sigma_0 < f'(\underline{u}_{-\infty})$ and $g(\underline{u}_{+\infty}) = 0 = g(\underline{u}_{-\infty})$, as well as $g'(\underline{u}_{-\infty}) < 0 < g'(\underline{u}_{+\infty})$.

Define θ_ε such that, for every $\varepsilon > 0$, it is the smallest positive root of the polynom $\varepsilon X^2 + (f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)X + g'(\underline{u}_{+\infty})$.

Under these conditions, viscous waves satsifying

$$\varepsilon \underline{U}_\varepsilon'' + (\sigma_\varepsilon - f'(\underline{U}_\varepsilon))\underline{U}_\varepsilon' + g(\underline{U}_\varepsilon) = 0,$$

on $\mathbf{R}$, with $\underline{U}_\varepsilon(x) \to \underline{u}_{\pm\infty}$ as $x \to \pm\infty$ and $\underline{U}_\varepsilon'(x) \to 0$ as $x \to \pm\infty$, and $\underline{U}_\varepsilon(x)$ converges as $\varepsilon \to 0^+$ towards $\underline{u}_{+\infty}$ if $x > 0$ and towards $\underline{u}_{-\infty}$ if $x < 0$. Their existence can be proven as in the previous Chapter.

Furthermore, define $\tau(\varepsilon, \theta) = \varepsilon\theta^2 + (f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)\theta + g'(\underline{u}_{+\infty})$. This will be the decay rate in the space weighted like $\exp(\theta(\cdot)_+)$ (when bigger in absolute value than $-g'(\underline{u}_{-\infty})$).

The following theorem describes the orbital asymptotic stability of the waves $(\underline{u}_\varepsilon)_{\varepsilon>0}$ which will be constructed later on. More precisely, the uniformity of the stability with respect to the viscous parameter will be an important point of focus here. We are interested in the case of weighted spaces, and especially of those allowing to create a spectral gap, and, also, of those for which the weight is critical. In that case, with the help of some extra polynomial weight, we obtain asymptotic orbital stability.

Theorem 16. *Given $\eta > 0$, there exist positive ε_0, α, δ and C such that, for every $\varepsilon \in (0, \varepsilon_0]$, $v_0 \in BUC^1_{\theta_0+\eta}(\mathbf{R})$ satisfying*

$$\|v_0\|_{L^\infty_{\theta_0+\eta}} \leqslant \delta, \qquad \left\| \frac{\varepsilon \partial_x v_0}{\varepsilon + e^{-\frac{\alpha|x|}{\varepsilon}}} \right\|_{L^\infty_{\theta_0+\eta}} \leqslant \delta,$$

the solution u to Equation 4.1.1 with initial data $\underline{U}_\varepsilon + v_0$ is defined on $\mathbf{R}_+$, and there exists $\psi \in C^1(\mathbf{R}_+, \mathbf{R})$ such that for every $t \in \mathbf{R}^+$, $(u(t, \cdot - \sigma_\varepsilon t + \psi(t)) - \underline{u}_\varepsilon) \in BUC^1_{\theta+\eta}(\mathbf{R})$, and

$$\begin{aligned} \| \left(u(t, \cdot - \sigma_\varepsilon t + \psi(t)) - \underline{u}_\varepsilon\right) \|_{L^\infty_{\theta_0+\eta}} + \left\| \frac{\varepsilon \partial_x \left(u(t, \cdot - \sigma_\varepsilon t + \psi(t)) - \underline{u}_\varepsilon\right)}{\varepsilon + e^{-\frac{\alpha|\cdot|}{\varepsilon}}} \right\|_{L^\infty_{\theta_0+\eta}} + |\psi'(t)| \\ \leqslant C e^{-t\tau(\varepsilon, \theta_0+\eta)} \left(\|v_0\|_{L^\infty_{\theta_0+\eta}} + \left\| \frac{\varepsilon \partial_x v_0}{e^{-\frac{\alpha|\cdot|}{\varepsilon}} + \varepsilon} \right\|_{L^\infty_{\theta_0+\eta}} \right) \end{aligned}$$

Furthermore, for every $\beta > 0$, there exist positive ε_0, δ and C such that for every $\varepsilon \in$

$(0, \varepsilon_0]$, *for every* $v_0 \in BUC^1_{\theta_\varepsilon,\beta}(\mathbf{R})$ *such that*

$$\|v_0\|_{L^\infty_{\theta_\varepsilon,\beta}} \leqslant \delta, \qquad \left\| \frac{\varepsilon \partial_x v_0}{\varepsilon + e^{-\alpha|x|}} \right\|_{L^\infty_{\theta_\varepsilon,\beta}} \leqslant \delta,$$

the solution u *to Equation 4.1.1 with initial data* $\underline{U}_\varepsilon + v_0$ *is defined on* $\mathbf{R}_+$*, and there exists* $\psi \in C^1(\mathbf{R}_+, \mathbf{R})$ *such that for every* $t \in \mathbf{R}^+$*,* $(u(t, \cdot - \sigma_\varepsilon t + \psi(t)) - \underline{u}_\varepsilon) \in BUC^1_{\theta_\varepsilon,\beta}(\mathbf{R})$*, and*

$$\begin{aligned} \| \left(u(t, \cdot - \sigma_\varepsilon t + \psi(t)) - \underline{u}_\varepsilon\right) \|_{L^\infty_{\theta_\varepsilon}} + \left\| \frac{\varepsilon \partial_x \left(u(t, \cdot - \sigma_\varepsilon t + \psi(t)) - \underline{u}_\varepsilon\right)}{\varepsilon + e^{-\frac{\alpha|\cdot|}{\varepsilon}}} \right\|_{L^\infty_{\theta_\varepsilon}} + |\psi'(t)| \\ \leqslant \frac{C}{(1+t)^\beta} \left(\|v_0\|_{L^\infty_{\theta_\varepsilon,\beta}} + \left\| \frac{\varepsilon \partial_x v_0}{\varepsilon + e^{-\frac{\alpha|\cdot|}{\varepsilon}}} \right\|_{L^\infty_{\theta_\varepsilon,\beta}} \right) \end{aligned}$$

In the critical case, that is, for $\theta = \theta_\varepsilon$, as in the hyperbolic counterpart to this result, a result is only obtained by choosing initial data having an extra polynomial localization. This is actually needed to control, in the Duhamel formulation, terms involving the phase. In fact, for terms of the form $\psi' \partial_x v$, no extra spatial localization is obtained, which can be opposed to terms such as v^2 or $v \partial_x v$, which are localized like $e^{-2\theta_\varepsilon |\cdot|}$.

Following [BR23], we will prove the result by working in fast variables. Explicitly we perform the following substitutions

$$\tilde{u}(\tilde{t}, \tilde{x}) := u(\varepsilon\, \tilde{t}, \varepsilon\, \tilde{x})\,, \qquad \underline{U}_\varepsilon(\tilde{x}) := \underline{u}_\varepsilon(\varepsilon\, \tilde{x})\,.$$

For the sake of notational simplicity, we drop tildes in the following and simply consider t the fast dynamical equation

$$\partial_t u + \partial_x(f(u)) = \partial_x^2 u + \varepsilon g(u)\,, \tag{4.2.1}$$

and the fast profile equation

$$\underline{U}''_\varepsilon = (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)\underline{U}'_\varepsilon - \varepsilon g(\underline{U}_\varepsilon). \tag{4.2.2}$$

The proof of the existence of wave profiles is essentially identical the the one in [BR23] and is thus omitted here.

To a large extent, the next two subsections follow the analysis of the corresponding ones in [BR23]. Let us thus stress what are the departures. Two of the main changes are simply due to different expository choices: in the present chapter, our normalization

of dual functions is slightly different (in order to simplify some expressions) and our high-frequency analysis follows the same ideas, but the bounds obtained are voluntarily sub-optimal to emphasize how little information is needed on that part of the complex plane. However some other differences have a much deeper origin and motivation. In particular, two things that could be roughly bounded in [BR23] must be very carefully tracked in the present analysis. Explicitly

1. For obvious reasons, here one cannot mix contributions arising to $+\infty$ and to $-\infty$.
2. In heat-like type bounds, one must obtain sharp expressions for standard deviations.

The latter point is necessary to allow for the sharp range of spatial localization rates including the critical $\theta = \theta_\varepsilon$ and to derive the corresponding sharp time decay rates $\tau(\varepsilon, \theta)$.

4.2.1 Conjugation of the flows

We start the work on the spectral problem by obtaining some result relating the solutions to some ODE with asymptotically constant coefficients (with convergence rate fast enough) and the ODE associated with the limiting coefficients.
It is written in a general form.

Proposition 19. *Consider Ω, an open subset of the complex plane, and $\varepsilon_0 > 0$. Let $(\lambda, \varepsilon, x) \mapsto A(\lambda, \varepsilon, x)$, be a smooth function defined on the set $\Omega \times [0, \varepsilon_0] \times \mathbf{R}$, with value in $\mathbb{M}_2(\mathbf{C})$. Assume that $A(\lambda, \varepsilon, x)$ has a limit for fixed (λ, ε) at $x \to +\infty$ noted $A_\infty(\lambda, \varepsilon)$, which is also smooth in (λ, ε) and holomorphic in λ. Furthermore, we assume that for every $(\varepsilon, x) \in [0, \varepsilon_0] \times \mathbf{R}$, the function $\lambda \mapsto A(\lambda, \varepsilon, x)$ is holomorphic.*
Assume also that there exist $\varepsilon_0 > 0$ such that for every $K \subset \Omega$ compact there exist some positive constants $C(K)$ and $\theta(K)$ and a smooth map (holomorphic in λ) $A^r : \Omega \times [0, \varepsilon_0] \to \mathbb{M}_2(\mathbb{C})$ such that for every $\varepsilon \in [0, \varepsilon_0]$, every $x \in \mathbf{R}$, every $\lambda \in \Omega$, we have that $\|A(\lambda, \varepsilon, x) - A_\infty(\lambda, \varepsilon)\| \leqslant C(K) e^{-\theta(K)|x|}$. Also, for every $\lambda \in \Omega$ and $\varepsilon \in [0, \varepsilon_0]$, we assume that the eigenvalues of $A_\infty(\lambda, \varepsilon)$ have distinct real parts. Then, noting $(\lambda, \varepsilon, x, y) \mapsto S(\lambda, \varepsilon, x, y)$, the solution operator associated to $(\lambda, \varepsilon, x) \mapsto A(\lambda, \varepsilon, x)$ (that is, $S(\lambda, \varepsilon, \cdot, y)$ is the solution to $z' = A(\lambda, \varepsilon, \cdot) z$ with initial data $z(y) = I_2$) we have that there exists a smooth map (holomorphic in λ) $(\lambda, \varepsilon, x) \mapsto P^r(\lambda, \varepsilon, x)$ such that, for every $(x, y) \in \mathbf{R}^2$ we have that $S(\lambda, \varepsilon, x, y) P^r(\lambda, \varepsilon, y) = P^r(\lambda, \varepsilon, x) e^{(x-y) A_\infty(\lambda, \varepsilon)}$ and, furthermore, for every compact set K included in Ω, there exist $C > 0$ and $\theta > 0$ such that for every $(\lambda, \varepsilon, x) \in K \times [0, \varepsilon_0] \times \mathbf{R}$, $\|P^r(\lambda, \varepsilon, x) - I_2\| \leqslant C e^{-\theta|x|}$.

Proof. Fix such a Ω, ε_0 and A, smooth, defined on $\Omega \times [0, \varepsilon_0] \times \mathbf{R}$ and holomorphic in $\lambda \in \Omega$. We start by noticing that such a $P_r(\lambda, \varepsilon, x)$, for a fixed $(\lambda, \varepsilon) \in \Omega \times [0, \varepsilon_0]$, satisfies

the desired conditions if and only if it is a solution of

$$\partial_x Y(x) = -Y(x)A_\infty(\lambda,\varepsilon) + A(\lambda,\varepsilon,x)Y(x)$$

that is of

$$Y' = [Y, A_\infty(\lambda,\varepsilon)] + (A(\lambda,\varepsilon,x) - A_\infty(\lambda,\varepsilon))Y(x)$$

and that it satisfies a condition of the form $\|P^r(\lambda,\varepsilon,x) - I_2\| \leqslant Ce^{-\theta|x|}$ with some positive constants C, θ.

Noting $\mu_+(\varepsilon,\lambda)$ and $\mu_-(\varepsilon,\lambda)$ the eigenvalues of $A_\infty(\lambda,\varepsilon)$ with the convention

$$\Re(\mu_-(\lambda,\varepsilon)) < \Re(\mu_+(\lambda,\varepsilon))$$

They are well defined and as smooth as $(\lambda,\varepsilon) \mapsto A(\lambda,\varepsilon)$ due to the implicit function theorem applied to $(\lambda,\varepsilon,\mu) \mapsto \det(\mu I_2 - A(\lambda,\varepsilon))$ around $(\lambda_0,\varepsilon_0,\mu_+(\lambda_0,\varepsilon_0))$ in one case, and around $(\lambda_0,\varepsilon_0,\mu_-(\lambda_0,\varepsilon_0))$ in the other case.

We note $R_+(\lambda,\varepsilon)$ and $R_-(\lambda,\varepsilon)$, right eigenvectors associated to the eigenvalues of $A_\infty(\lambda,\varepsilon)$ (and L_+ as well as L_- are left eigenvectors) such that the relations of normalization $L_+R_+ = 1$ and $L_-R_- = 1$ are satisfied.

Remind that we can construct locally the projections on the stable subspace, the unstable subspace and the kernel of the linear operator $B \mapsto [A_\infty(\lambda,\varepsilon), B]$ by considering the operators given by

$$\Pi_u(\lambda,\varepsilon)(B) = L_+(\lambda,\varepsilon)BR_-(\lambda,\varepsilon)\ R_+(\lambda,\varepsilon)L_-(\lambda,\varepsilon),$$

(where $L_+(\lambda,\varepsilon)BR_-(\lambda,\varepsilon)$ is identified to the scalar contained in the 1×1 matrix) and

$$\Pi_s(\lambda,\varepsilon)(B) = L_-(\lambda,\varepsilon)BR_+(\lambda,\varepsilon)R_-(\lambda,\varepsilon)L_+(\lambda,\varepsilon).$$

The said eigenvectors can be constructed locally such that they are as smooth as $A_\infty(\lambda,\varepsilon)$ by considering the application $X \mapsto (A_\infty(\lambda,\varepsilon) - \mu_\pm(\lambda,\varepsilon))X$ defined on the affine space $\{R_+(\lambda_0,\varepsilon_0) + \nu R_-(\lambda_0,\varepsilon_0) \,|\, \nu \in \mathbf{R}\}$ for R_+, and similarly for R_-, and then $L_\pm$, and we then normalize $L_\pm$ by dividing it by $L_\pm R_\pm$. As a consequence, as we have that $\Pi_u(\lambda,\varepsilon)$ and $\Pi_s(\lambda,\varepsilon)$ are uniquely defined, they are locally smooth due to the previous equality, the smoothness of $R_\pm$ and $L_\pm$, and, hence, they are smooth on $\Omega\times[0,\varepsilon_0]$ (with holomorphy with respect to $\lambda\in\Omega$).

Now, consider K, a compact subset of Ω, and fix some constant $\vartheta > 0$ such that $2\vartheta \leqslant \min_{(\lambda,\varepsilon)\in K\times[0,\varepsilon_0]} \Re(\mu_+(\lambda,\varepsilon) - \mu_-(\lambda,\varepsilon))$ and $2\vartheta \leqslant \theta$.

We can now define the operator $T_{x_0,\lambda,\varepsilon} : E_{x_0} \to E_{x_0}$ where E_{x_0} is defined as $E_{x_0} =$

$\{I_2+u|$ for $u \in C^0([x_0,+\infty),\mathcal{M}_2)$ such that $\|e^{\vartheta\cdot}(u(\cdot)-I_2)\|_{L^\infty} < +\infty$ (on which we consider the natural distance defined for $(u,v) \in E^2_{x_0}$ by $d(u,v) = \|e^{\vartheta\cdot}(u-v)(\cdot)\|_{L^\infty}$) such that

$$\begin{aligned} T_{x_0,\lambda,\varepsilon}(u)(x) = I_2 &+ \int_{x_0}^{x} e^{(x-\tau)[A_\infty(\lambda,\varepsilon),\cdot]}\Pi_s((A(\lambda,\varepsilon,\tau)-A_\infty(\lambda,\varepsilon))u(\tau))d\tau \\ &+ \int_{+\infty}^{x} e^{(x-\tau)[A_\infty(\lambda,\varepsilon),\cdot]}(Id_{\mathcal{M}_2(\mathbf{C})}-\Pi_s)((A(\lambda,\varepsilon,\tau)-A_\infty(\lambda,\varepsilon))u(\tau))d\tau \end{aligned}$$

We can now show that for x_0 big enough (depending on K) such that $T_{x_0,\lambda,\varepsilon}$ is a contraction map on some closed neighborhood of the map $x \mapsto I_2$. We have that for every $u \in E_{x_0}$ (we use here that there exists some constant $\tilde{C} > 0$ uniform on $K \times [0,\varepsilon_0]$

$$\begin{aligned} d(T_{x_0,\lambda,\varepsilon}(u), I_2) \leqslant & C(K)\tilde{C}\left(\int_{x_0}^{x} e^{-2\vartheta(x-\tau)}e^{-2\vartheta\tau}d\tau + \int_{+\infty}^{x} -e^{-2\vartheta\tau}d\tau\right)(1+\|u-I_2\|_{L^\infty}) \\ \leqslant & \tilde{C}C(K)\left((x-x_0)e^{-2\vartheta x} + \frac{e^{-2\vartheta x}}{2\vartheta}\right)(1+\|u-I_2\|_{L^\infty}) \\ \leqslant & \tilde{C}C(K)\left(\frac{e^{-\vartheta x_0}e^{-\vartheta x}}{\vartheta} + \frac{e^{-2\theta x}}{2\vartheta}\right)(1+\|u-I_2\|_{L^\infty}) \end{aligned}$$

and that for every $(u,v) \in E_{x_0}$

$$d(T_{x_0,\lambda,\varepsilon}(u), T_{x_0,\lambda,\varepsilon}(u)) \leqslant \tilde{C}C(K)\left(\int_{x_0}^{x} e^{-2\vartheta(x-\tau)}e^{-2\vartheta\tau}d\tau + \int_{+\infty}^{x} -e^{-2\vartheta\tau}d\tau\right)\|u-v\|_{L^\infty([x_0,+\infty))}.$$

Hence, by taking x_0 big enough, we have that $T_{x_0,\lambda,\varepsilon}$ is contracting for every $(\lambda,\varepsilon) \in K \times [0,\varepsilon_0]$. Furthermore, the regularity in (λ,ε) is obtained by using the implicit function theorem to $(\lambda,\varepsilon,u) \mapsto u - T_{x_0,\lambda,\varepsilon}(u)$. Finally, $P^r(\lambda,\varepsilon,\cdot)$ is extended for a fixed $(\lambda,\varepsilon) \in \Omega \times [0,\varepsilon_0]$ through the relation for every $x \in (-\infty,x_0)$

$$P^r(\lambda,\varepsilon,x) := S(\lambda,\varepsilon,x,x_0)P^r(\lambda,\varepsilon,x_0)e^{(x_0-x)A_\infty(\lambda,\varepsilon)}.$$

The regularity in x is obtained by using the fact that $x \mapsto S\ (\lambda,\varepsilon,x,x_0)$ is smooth, and so $x \mapsto P^r(\lambda,\varepsilon,x)$ is also smooth. □

Here, consider $(x,y) \mapsto \mathbf{\Phi}^\lambda_\varepsilon(x,y)$ the solution operator to the ordinary differential equation

$$Y' = \begin{pmatrix} f'(\underline{U}_\varepsilon - \sigma_\varepsilon & 1 \\ \lambda - \varepsilon g'(\underline{U}_\varepsilon) & 0 \end{pmatrix} Y$$

that is $\mathbf{\Phi}^\lambda_\varepsilon(x,y)Z_0$ is the only solution that satisfies this equation and which that the

value Z_0 at y. Furthermore, note

$$\mathbf{A}_\varepsilon^r(\lambda) := \begin{pmatrix} f'(\underline{u}_{+\infty}) - \sigma_\varepsilon & 1 \\ \lambda - \varepsilon g'(\underline{u}_{+\infty}) & 0 \end{pmatrix}.$$

Proposition 20. *There exist positive constants $(\varepsilon_0, C, \theta, \delta)$ such that setting*

$$\Omega_\delta := \left\{ \lambda\,;\ \Re\left(\sqrt{\lambda}\right) \geqslant \frac{1}{\delta} \right\}$$

there exists a smooth map

$$P^{r,HF} \,:\, [0,\varepsilon_0] \times \Omega_\delta \times \mathbf{R} \mapsto GL_2(\mathbf{C})\,, \qquad (\varepsilon,\lambda,x) \mapsto P_\varepsilon^{r,HF}(\lambda,x)$$

locally uniformly analytic in λ on a neighborhood of Ω_δ and such that, for any $(\varepsilon,\lambda,x) \in [0,\varepsilon_0] \times \Omega_\delta \times [0,+\infty)$,

$$\left\| \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} P_\varepsilon^{r,HF}(\lambda,x) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} - \mathrm{e}^{-\frac{1}{2}\int_x^{+\infty}(f'(\underline{U}_\varepsilon(y))-f'(\underline{u}_{+\infty}))\,\mathrm{d}\,y}\ I_2 \right\| \leqslant C\,\frac{\mathrm{e}^{-\theta\,|x|}}{\Re\left(\sqrt{\lambda}\right)}\,,$$

$$\left\| \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} (P_\varepsilon^{r,HF}(\lambda,x))^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix} - \mathrm{e}^{\frac{1}{2}\int_x^{+\infty}(f'(\underline{U}_\varepsilon(y))-f'(\underline{u}_{+\infty}))\,\mathrm{d}\,y}\ I_2 \right\| \leqslant C\,\frac{\mathrm{e}^{-\theta\,|x|}}{\Re\left(\sqrt{\lambda}\right)}\,,$$

and, for any $(\varepsilon,\lambda,x,y) \in [0,\varepsilon_0] \times \Omega_\delta \times \mathbf{R}^2$,

$$\mathbf{\Phi}_\varepsilon^\lambda(x,y) \;=\; P_\varepsilon^{r,HF}(\lambda,x)\ \mathrm{e}^{(x-y)\,\mathbf{A}_\varepsilon^r(\lambda)}\,(P_\varepsilon^{r,HF}(\lambda,y))^{-1}\,.$$

The proof is identical to the one in the previous part, that is Chapter 3 and is thus omitted.

4.2.2 Spectral stability and Evans function

To obtain an expression for the Green function associated to the operator $\lambda - \mathcal{L}_\varepsilon$ where

$$\mathcal{L}_\varepsilon : v \mapsto \left(\partial_x^2 + \partial_x((\sigma_\varepsilon - f'(\underline{U}_\varepsilon))v) + \varepsilon g'(\underline{U}_\varepsilon)v\right),$$

we first try to obtain some particular basis of the ODE. Parametrized by $(\lambda,\varepsilon,x) \in \Omega \times [0,\varepsilon_0] \times \mathbf{R}$ where Ω is defined by

$$\Omega := \mathbf{C}\backslash\left(-\infty, \frac{\alpha_0}{2}\right],$$

with

$$\alpha_0 = \max\left(-\frac{(\sigma_0 - f'(\underline{u}_{-\infty}))^2}{4}, -\frac{(\sigma_0 - f'(\underline{u}_{+\infty}))^2}{4}\right),$$

and $\varepsilon_0 > 0$ small enough such that $\underline{U}_\varepsilon$ is defined for every $\varepsilon \in [0, \varepsilon_0]$ and also such that $\alpha_\varepsilon \leqslant \frac{\alpha_0}{2}$,here, α_ε is

$$\alpha_\varepsilon = \max\left(\varepsilon g'(\underline{u}_{-\infty}) - \frac{(\sigma_\varepsilon - f'(\underline{u}_{-\infty}))^2}{4}, \varepsilon g'(\underline{u}_{+\infty}) - \frac{(\sigma_\varepsilon - f'(\underline{u}_{+\infty}))^2}{4}\right),$$

and consider the matrices

$$A(\lambda, \varepsilon, x) := \begin{pmatrix} (f'(\underline{U}_\varepsilon(x)) - \sigma_\varepsilon) & 1 \\ \lambda - \varepsilon g'(\underline{U}_\varepsilon(x)) & 0 \end{pmatrix}.$$

We know due to Proposition 19 that, as the family of matrices satisfy its assumptions, we can conjugate the solutions to

$$Y' = A(\lambda, \varepsilon, \cdot)Y,$$

to the asymptotic matrices $\lim_{x\to+\infty} A(\lambda, \varepsilon, x)$ and $\lim_{x\to-\infty} A(\lambda, \varepsilon, x)$.

We, thus, obtain smooth conjugation matrices $P^r : \Omega \times [0, \varepsilon_0] \times \mathbf{R}$ and $P^\ell : \Omega \times [0, \varepsilon_0] \times \mathbf{R} \to \mathcal{M}_2(\mathbf{C})$ such that we have the conjugation associated to the constant coefficient equations $Y' = A(\lambda, \varepsilon, +\infty)Y$ and $Y' = A(\lambda, \varepsilon, -\infty)Y$.

Considering

$$R^r_-(\lambda, \varepsilon) = \begin{pmatrix} 1 \\ -\mu^r_+(\lambda, \varepsilon) \end{pmatrix}, \qquad R^r_+(\lambda, \varepsilon) = \begin{pmatrix} 1 \\ -\mu^r_-(\lambda, \varepsilon) \end{pmatrix},$$
$$R^\ell_+(\lambda, \varepsilon) = \begin{pmatrix} 1 \\ -\mu^\ell_-(\lambda, \varepsilon) \end{pmatrix}, \qquad R^\ell_-(\lambda, \varepsilon) = \begin{pmatrix} 1 \\ -\mu^\ell_+(\lambda, \varepsilon) \end{pmatrix}.$$

and

$$V^{r,s}(\lambda, \varepsilon) : x \mapsto P^r(\lambda, \varepsilon, x)e^{xA_\infty(\lambda,\varepsilon)}R^r_-(\lambda, \varepsilon),$$
$$V^{r,u}(\lambda, \varepsilon) : x \mapsto P^r(\lambda, \varepsilon)e^{xA(\lambda,\varepsilon,+\infty)}R^r_+(\lambda, \varepsilon),$$
$$V^{\ell,u}(\lambda, \varepsilon) : x \mapsto P^\ell(\lambda, \varepsilon, x)e^{xA(\lambda,\varepsilon,-\infty)}R^\ell_+(\lambda, \varepsilon),$$
$$V^{\ell,s}(\lambda, \varepsilon) : x \mapsto P^\ell(\lambda, \varepsilon)e^{xA(\lambda,\varepsilon,-\infty}R^\ell_-(\lambda, \varepsilon)$$

we can define

$$D_\varepsilon(\lambda) := \det(V^{r,s}(\lambda, \varepsilon, 0), V^{\ell,u}(\lambda, \varepsilon, 0)).$$

We are now able to state a result which corresponds to the bounds on the eigenvalues of the linearized operator $\mathcal{L}_\varepsilon$ in weighted spaces. First remind the theorem of Sturm-Liouville from the introduction 9.

Proposition 21. *Given $\delta > 0$, there exist $\eta_0 > 0$ and $\varepsilon_0 > 0$, as well as $C > 0$ such that for every $\lambda \in \mathbf{C}$ such that $d(\lambda, (-\infty, -\eta_0]) \geqslant \delta$ and $|\lambda| \leqslant \dfrac{1}{\delta}$ we have that $|D_\varepsilon(\lambda)| \geqslant C|\lambda|$.*

Proof. Fix $\delta > 0$.

We consider, for $\varepsilon \in \mathbf{R}_+$ such that $\underline{U}_\varepsilon$ is defined, the weight defined on $\mathbf{R}$ by

$$\begin{aligned}
\omega_\varepsilon(x) :=& \exp\left(\frac{1}{2}\int_0^x (f'(\underline{U}_\varepsilon(\tau)) - \sigma_\varepsilon)d\tau\right),\\
\mathcal{L}_\varepsilon(\omega_\varepsilon\cdot)\omega_\varepsilon^{-1} =& \partial_x^2\cdot + \frac{\omega_\varepsilon'' + \omega_\varepsilon'^2 + (\sigma_\varepsilon - f'(\underline{U}_\varepsilon))\omega_\varepsilon'}{\omega_\varepsilon}\cdot + (\varepsilon g'(\underline{U}_\varepsilon) - f''(\underline{U}_\varepsilon)\underline{U}_\varepsilon')\cdot,\\
=& \partial_x^2\cdot + \left(\varepsilon g'(\underline{U}_\varepsilon) - \frac{f''(\underline{U}_\varepsilon)\underline{U}_\varepsilon'}{2} + \frac{-(\sigma_\varepsilon - f'(\underline{U}_\varepsilon))^2}{2}\right)\cdot.
\end{aligned}$$

And so, $L_\varepsilon := \omega_\varepsilon^{-1}\mathcal{L}_\varepsilon(\omega_\varepsilon\cdot)$, defined as an unbounded operator on $L^2(\mathbf{R})$ with domain $H^2(\mathbf{R})$ is a self-adjoint operator, and, thanks to Sturm-Liouville theory, the point spectrum of L_ε is a finite set of real values . Furthermore, these eigenvalues are all simple and, if this set is nonempty, then its maximum corresponds to the eigenvalue which has an eigenfunction which has no zero. Thus, in our case, to $\lambda = 0$. Furthermore, other eigenvalues correspond with vanishing of the Evans function and, still in this part, the vanishing of the Evans function always gives raise to a simple zero of D_ε.

Finally, as D_ε converges to D_0 uniformly on every compact of Ω, it is also the case of D_ε' and of D_ε'' (towards D_0' and D_0'' respectively). Thus, we have that there exists some $\eta_0 > 0$ (we allow ourselves to reduce it such that the inequality $\dfrac{|D_0'(0)|}{2}\eta_0 - 3|D_0''(0)|\eta_0^2 \geqslant 0$ is also satisfied) and $\varepsilon_0 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$

$$|D_\varepsilon'| \geqslant \frac{|D_0'(0)|}{2}, \qquad |D_\varepsilon''(\lambda)| \leqslant \frac{3|D_0''(0)|}{2},$$

on $\mathbb{D}(0, \eta_0)$ and, for every $\lambda \in \mathbf{C}\backslash(-\infty, -\eta_0]$ such that $|\lambda| \leqslant \frac{1}{\delta}$ and $d(\lambda, (-\infty, -\eta_0]) \geqslant \delta$, as well as, with $\Omega_{\delta,\eta_0} := \{\mu \in \mathbf{C}\backslash(-\infty, -\eta_0] \,|\, |\lambda| \leqslant \dfrac{1}{\delta}\,,\, d(\lambda, (s - \infty, -\eta_0] \geqslant \delta$, that it satisfies

$$|D_\varepsilon(\lambda) - D_0(\lambda)| \leqslant \frac{\inf_{\{\mu\in\Omega_{\delta,\eta_0}, |\mu|\geqslant\eta_0\}} |D_0(\mu)|}{2}.$$

Thus, for every $\varepsilon \in [0, \varepsilon_0]$, we have that

$$|D_\varepsilon(\lambda)| \geqslant \frac{|D_0(\lambda)|}{2},$$

if $\lambda \in \Omega_{\delta,\eta_0}$ and $|\lambda| \geqslant \eta_0$, as well as

$$|D_\varepsilon(\lambda)| \geqslant \frac{|D_0'(0)||\lambda|}{4},$$

for $|\lambda| \leqslant \eta_0$. Thus, for every $\lambda \in \Omega_{\delta,\eta_0}$ and $\varepsilon \in [0, \varepsilon_0]$, we have that

$$|D_\varepsilon(\lambda)| \geqslant \min\left(\frac{\delta \inf_{\{\mu \in \Omega_{\delta,\eta_0} \,|\, |\mu| \geqslant \delta\}} |D_0(\mu)|}{2}, \frac{|D_0'(0)|}{4}\right) |\lambda|.$$

□

We are now really almost ready to be able to give an expression of the spectral Green function on a given compact set of the form given in 21 (that is for any fixed $\delta > 0$).

To do so, we define the dual spectral problem, which is associated to the following operator (defined on $L^2(\mathbf{R})$ with domain $H^2(\mathbf{R})$ and associated to an ODE in the 2D space that differs from the one associated with the original one by a smooth function in x and ε times I_2)

$$v \mapsto v'' + (f'(\underline{U}_\varepsilon) - \sigma_\varepsilon)v' + \varepsilon g'(\underline{U}_\varepsilon)v.$$

$$\tilde{A}(\lambda, \varepsilon, x) := \begin{pmatrix} 0 & 1 \\ \lambda - \varepsilon g'(\underline{U}_\varepsilon) & \sigma_\varepsilon - f'(\underline{U}_\varepsilon) \end{pmatrix}$$

Thus, the solutions on $\mathbf{R}$ of the system $Y'(x) = \tilde{A}(\lambda, \varepsilon, x)Y(x)$ are the elements of

$$\{x \mapsto \exp\left(\int_0^x (\sigma_\varepsilon - f'(\underline{U}_\varepsilon(\tau)))d\tau\right) Z(x) \,|\, Z \text{ solves the equation } Z'(x) = A(\lambda, \varepsilon, x)Z(x) \text{ on } \mathbf{R}\}.$$

We can define

$$\begin{aligned}
\tilde{V}^{r,s}(\lambda, \varepsilon, \cdot) &= e^{\int_0^\cdot (\sigma - f'(\underline{U}_\varepsilon(\tau))d\tau} V^{r,s}(\lambda, \varepsilon, \cdot), \\
\tilde{V}^{r,u}(\lambda, \varepsilon, \cdot) &= e^{\int_0^\cdot (\sigma - f'(\underline{U}_\varepsilon(\tau))d\tau} V^{r,u}(\lambda, \varepsilon, \cdot), \\
\tilde{V}^{\ell,u}(\lambda, \varepsilon, \cdot) &= e^{\int_0^\cdot (\sigma - f'(\underline{U}_\varepsilon(\tau))d\tau} V^{\ell,u}(\lambda, \varepsilon, \cdot), \\
\tilde{V}^{\ell,s}(\lambda, \varepsilon, \cdot) &= e^{\int_0^\cdot (\sigma - f'(\underline{U}_\varepsilon(\tau))d\tau} V^{\ell,s}(\lambda, \varepsilon, \cdot).
\end{aligned}$$

Remind that, for every V solving $V' = A(\lambda, \varepsilon, \cdot)V$ on $\mathbf{R}$ and every W solving $W' =$

$\tilde{A}(\lambda,\varepsilon,\cdot)W$ on $\mathbf{R}$ we have that

$$V\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}W$$

is constant on $\mathbf{R}$, we can thus compute the scalar products of the form below by taking the limits at $\pm\infty$.

Thus, we have that for every $x\in\mathbf{R}$

$$\begin{aligned}
V^{\ell,u}(\lambda,\varepsilon,x)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\tilde{V}^{\ell,s}(\lambda,\varepsilon,x) &= \lim_{y\to-\infty}V^{\ell,u}(\lambda,\varepsilon,y)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\tilde{V}^{\ell,s}(\lambda,\varepsilon,y)\\
&= e^{\int_0^{-\infty}(f'(\underline{u}_{-\infty})-f'(\underline{U}_\varepsilon))}(\mu_-^\ell(\lambda,\varepsilon)-\mu_+^\ell(\lambda,\varepsilon)),\\
V^{\ell,u}(\lambda,\varepsilon,x)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\tilde{V}^{\ell,u}(\lambda,\varepsilon,x) &= \lim_{y\to-\infty}V^{\ell,u}(\lambda,\varepsilon,y)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\tilde{V}^{\ell,u}(\lambda,\varepsilon,y)=0,\\
V^{r,s}(\lambda,\varepsilon,x)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\tilde{V}^{r,u}(\lambda,\varepsilon,x) &= \lim_{y\to+\infty}V^{r,s}(\lambda,\varepsilon,y)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\tilde{V}^{r,u}(\lambda,\varepsilon,y)\\
&= e^{\int_0^{+\infty}(f'(\underline{u}_{+\infty})-f'(\underline{U}_\varepsilon))}(\mu_+^r(\lambda,\varepsilon)-\mu_-^r(\lambda,\varepsilon)),\\
V^{r,s}(\lambda,\varepsilon,x)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\tilde{V}^{r,s}(\lambda,\varepsilon,x) &= 0,\\
V^{r,s}(\lambda,\varepsilon,\cdot) &= \tau_r(\lambda,\varepsilon)V^{\ell,u}(\lambda,\varepsilon,\cdot)+\rho_r(\lambda,\varepsilon)V^{\ell,s}(\lambda,\varepsilon,\cdot),\\
V^{\ell,u}(\lambda,\varepsilon,\cdot) &= \tau_\ell(\lambda,\varepsilon)V^{r,s}(\lambda,\varepsilon,\cdot)+\rho_\ell(\lambda,\varepsilon)V^{r,u}(\lambda,\varepsilon,\cdot),\\
\tilde{V}^{r,s}(\lambda,\varepsilon,\cdot) &= \tilde{\tau}_r(\lambda,\varepsilon)\tilde{V}^{\ell,u}(\lambda,\varepsilon,\cdot)+\tilde{\rho}_r(\lambda,\varepsilon)\tilde{V}^{\ell,s}(\lambda,\varepsilon,\cdot),\\
\tilde{V}^{\ell,u}(\lambda,\varepsilon,\cdot) &= \tilde{\tau}_\ell(\lambda,\varepsilon)\tilde{V}^{r,s}(\lambda,\varepsilon,\cdot)+\tilde{\rho}_\ell(\lambda,\varepsilon)\tilde{V}^{r,u}(\lambda,\varepsilon,\cdot).
\end{aligned}$$

Hence, for every $x\in\mathbf{R}$

$$\begin{aligned}
\rho_r(\lambda,\varepsilon)\det(V^{\ell,u}(\lambda,\varepsilon,x),V^{\ell,s}(\lambda,\varepsilon,x)) &= \det(V^{\ell,u}(\lambda,\varepsilon,x),V^{r,s}(\lambda,\varepsilon,x)),\\
&= -D_\varepsilon(\lambda)e^{\int_0^x(f'(\underline{U}_\varepsilon(\tau))-\sigma_\varepsilon)d\tau},
\end{aligned}$$

and

$$\begin{aligned}
\rho_\ell(\lambda,\varepsilon)\det(V^{r,s}(\lambda,\varepsilon,x),V^{r,u}(\lambda,\varepsilon,x) &= \det(V^{r,s}(\lambda,\varepsilon,x),V^{\ell,u}(\lambda,\varepsilon,x))\\
&= D_\varepsilon(\lambda)e^{\int_0^x(f'(\underline{U}_\varepsilon(\tau))-\sigma_\varepsilon)d\tau}.
\end{aligned}$$

From the last two equalities, we get the following expressions for ρ_r and ρ_ℓ

$$\rho_r(\lambda,\varepsilon)=-\frac{e^{\int_0^x(f'(\underline{U}_\varepsilon(\tau))-\sigma_\varepsilon)d\tau}D_\varepsilon(\lambda)}{\det(V^{\ell,u}(\lambda,\varepsilon,x),V^{\ell,s}(\lambda,\varepsilon,x))},$$

$$= - \frac{e^{\int_0^x (f'(\underline{U}_\varepsilon(s)) - f'(\underline{u}_{-\infty}))ds} D_\varepsilon(\lambda)}{\det\left(P^\ell(\lambda,\varepsilon,x) \begin{pmatrix} 1 \\ -\mu_-^\ell(\lambda,\varepsilon) \end{pmatrix}, P^\ell(\lambda,\varepsilon,x) \begin{pmatrix} 1 \\ -\mu_+^\ell(\lambda,\varepsilon) \end{pmatrix} \right)},$$

$$\rho_\ell(\lambda,\varepsilon) = \frac{D_\varepsilon(\lambda) e^{\int_0^x (f'(\underline{U}_\varepsilon(\tau)) - \sigma_\varepsilon) d\tau}}{\det(V^{r,s}(\lambda,\varepsilon,x), V^{r,u}(\lambda,\varepsilon,x))},$$

$$= \frac{D_\varepsilon(\lambda) e^{\int_0^x (f'(\underline{U}_\varepsilon(s)) - f'(\underline{u}_{+\infty}))ds}}{\det\left(P^r(\lambda,\varepsilon,x) \begin{pmatrix} 1 \\ -\mu_+^r(\lambda,\varepsilon) \end{pmatrix}, P^r(\lambda,\varepsilon,x) \begin{pmatrix} 1 \\ -\mu_-^r(\lambda,\varepsilon) \end{pmatrix} \right)},$$

and, for $\tilde{\rho}_r$ and $\tilde{\rho}_\ell$

$$\tilde{\rho}_r(\lambda,\varepsilon) = \rho_r(\lambda,\varepsilon),$$

$$\tilde{\rho}_l(\lambda,\varepsilon) = \rho_\ell(\lambda,\varepsilon).$$

Thus, by taking x going to $+\infty$ (or $-\infty$, depending on the case)

$$\begin{aligned}
\rho_r(\lambda,\varepsilon) =& \frac{e^{-\int_{-\infty}^0 (f'(\underline{U}_\varepsilon(\tau)) - f'(\underline{u}_{-\infty}))d\tau} D_\varepsilon(\lambda)}{\mu_+^\ell(\lambda,\varepsilon) - \mu_-^\ell(\lambda,\varepsilon)}, \\
\rho_\ell(\lambda,\varepsilon) =& \frac{D_\varepsilon(\lambda) e^{\int_0^{+\infty} (f'(\underline{U}_\varepsilon(\tau)) - f'(\underline{u}_{+\infty}))d\tau}}{\mu_+^r(\lambda,\varepsilon) - \mu_-^r(\lambda,\varepsilon)}, \\
\tilde{\rho}_r(\lambda,\varepsilon) =& \rho_r(\lambda,\varepsilon), \\
\tilde{\rho}_\ell(\lambda,\varepsilon) =& \rho_\ell(\lambda,\varepsilon),
\end{aligned}$$

as well as, for every $x \in \mathbf{R}$

$$\begin{aligned}
V^{r,s}(\lambda,\varepsilon,x) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{V}^{\ell,u}(\lambda,\varepsilon,x) =& \tilde{\rho}_\ell(\lambda,\varepsilon) \lim_{y \to +\infty} V^{r,s}(\lambda,\varepsilon,y) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{V}^{r,u}(\lambda,\varepsilon,y), \\
=& e^{\int_0^{+\infty} (f'(\underline{u}_{+\infty}) - f'(\underline{U}_\varepsilon))} \tilde{\rho}_\ell(\lambda,\varepsilon)(\mu_-^r(\lambda,\varepsilon) - \mu_+^r(\lambda,\varepsilon)), \\
=& - D_\varepsilon(\lambda) e^{\int_0^{+\infty} (f'(\underline{u}_{+\infty}) - f'(\underline{U}_\varepsilon))} e^{\int_0^{+\infty} (f'(\underline{U}_\varepsilon(\tau)) - f'(\underline{u}_{+\infty}))d\tau} \\
=& - D_\varepsilon(\lambda), \\
V^{\ell,u}(\lambda,\varepsilon,x) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{V}^{r,s}(\lambda,\varepsilon,x), =& e^{-\int_{-\infty}^0 (f'(\underline{u}_{-\infty}) - f'(\underline{U}_\varepsilon(\tau)))d\tau} \tilde{\rho}_r(\lambda,\varepsilon)(\mu_-^\ell(\lambda,\varepsilon) - \mu_+^\ell(\lambda,\varepsilon)) \\
=& - D_\varepsilon(\lambda).
\end{aligned}$$

Thus, for all $y \in \mathbf{R}$,

$$\begin{pmatrix} V^{\ell,u}(\lambda,\varepsilon,y) & V^{r,s}(\lambda,\varepsilon,y) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\tilde{V}^{r,s}(\lambda,\varepsilon,x)}{-D_\varepsilon(\lambda)} & \dfrac{\tilde{V}^{\ell,u}(\lambda,\varepsilon,x)}{-D_\varepsilon(\lambda)} \end{pmatrix}^{\mathsf{T}} = I_2.$$

And so, for every $y \in \mathbf{R}$

$$\begin{pmatrix} V^{\ell,u}(\lambda,\varepsilon,y) & V^{r,s}(\lambda,\varepsilon,y) \end{pmatrix} \begin{pmatrix} \alpha(\lambda,\varepsilon,y) \\ -\beta(\lambda,\varepsilon,y) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

where α and β are given by the formula $\alpha(\lambda,\varepsilon,y) = \mathbf{e}_1 \cdot \tilde{V}^{r,s}(\lambda,\varepsilon,y)\, D_\varepsilon(\lambda)^{-1}$ and $\beta(\lambda,\varepsilon,y) = \mathbf{e}_1 \cdot \tilde{V}^{\ell,u}(\lambda,\varepsilon,y)\, D_\varepsilon(\lambda)^{-1}$.

Thus, we obtain the following representation of the spectral Green function. If $x > y$, then

$$G(\lambda,\varepsilon,x,y) = \frac{1}{D_\varepsilon(\lambda)} \mathbf{e}_1 \cdot V^{r,s}(\lambda,\varepsilon,x)\mathbf{e}_1 \cdot \tilde{V}^{\ell,u}(\lambda,\varepsilon,y)$$

and, if $x < y$

$$G(\lambda,\varepsilon,x,y) = \frac{1}{D_\varepsilon(\lambda)} \mathbf{e}_1 \cdot V^{\ell,u}(\lambda,\varepsilon,x)\mathbf{e}_1 \cdot \tilde{V}^{r,s}(\lambda,\varepsilon,y).$$

Obviously, using decompositions of $V^{r,u}$, $V^{r,s}$, $V^{\ell,u}$ and $V^{\ell,s}$, we can obtain the following important decompositions of G

For $x > y > 0$

$$G(\lambda,\varepsilon,x,y) = \left(\frac{\tilde{\rho}_\ell(\lambda,\varepsilon)\mathbf{e}_1 \cdot \tilde{V}^{r,u}(\lambda,\varepsilon,y)\mathbf{e}_1 \cdot V^{r,s}(\lambda,\varepsilon,x)}{D_\varepsilon(\lambda)} + \frac{\tilde{\tau}_\ell(\lambda,\varepsilon)\mathbf{e}_1 \cdot V^{r,s}(\lambda,\varepsilon,x)\mathbf{e}_1 \cdot \tilde{V}^{r,s}(\lambda,\varepsilon,y)}{D_\varepsilon(\lambda)} \right),$$

for $y > x > 0$

$$G(\lambda,\varepsilon,x,y) = \left(\frac{\rho_\ell(\lambda,\varepsilon)\mathbf{e}_1 \cdot \tilde{V}^{r,s}(\lambda,\varepsilon,y)\mathbf{e}_1 \cdot V^{r,u}(\lambda,\varepsilon,x)}{D_\varepsilon(\lambda)} + \frac{\tau_\ell(\lambda,\varepsilon)\mathbf{e}_1 \cdot V^{r,s}(\lambda,\varepsilon,x)\mathbf{e}_1 \cdot \tilde{V}^{r,s}(\lambda,\varepsilon,y)}{D_\varepsilon(\lambda)} \right),$$

for $y < x < 0$

$$G(\lambda,\varepsilon,x,y) = \left(\frac{\rho_r(\lambda,\varepsilon)\mathbf{e}_1 \cdot \tilde{V}^{\ell,u}(\lambda,\varepsilon,y)\mathbf{e}_1 \cdot V^{\ell,s}(\lambda,\varepsilon,x)}{D_\varepsilon(\lambda)} + \frac{\tau_r(\lambda,\varepsilon)\mathbf{e}_1 \cdot V^{\ell,u}(\lambda,\varepsilon,x)\mathbf{e}_1 \cdot \tilde{V}^{\ell,u}(\lambda,\varepsilon,y)}{D_\varepsilon(\lambda)} \right),$$

for $x < y < 0$ it is possible to write

$$G(\lambda,\varepsilon,x,y) = \left(\frac{\tilde{\rho}_r(\lambda,\varepsilon)\mathbf{e}_1 \cdot \tilde{V}^{\ell,s}(\lambda,\varepsilon,y)\mathbf{e}_1 \cdot V^{\ell,u}(\lambda,\varepsilon,x)}{D_\varepsilon(\lambda)} + \frac{\tilde{\tau}_r(\lambda,\varepsilon)\mathbf{e}_1 \cdot V^{\ell,u}(\lambda,\varepsilon,x)\mathbf{e}_1 \cdot \tilde{V}^{\ell,u}(\lambda,\varepsilon,y)}{D_\varepsilon(\lambda)} \right).$$

Remind that, here, $\tilde{\rho}_\sharp = \rho_\sharp$ due to the choices made, and, importantly, we have that $\frac{\rho_\sharp(.,\varepsilon)}{D_\varepsilon(\lambda)}$ is defined and holomorphic on the open set

$$\mathbf{C}\backslash\left(-\infty, \frac{\alpha_0}{2}\right],$$

(for $\sharp = \ell$ or $\sharp = r$), and both are bounded uniformly with respect to ε, locally in λ. Finally, we can derive bounds in the high-frequency regime. They are rough bounds, to emphasize how little information is needed in that regime.

Lemma 12. *For every $\nu > 0$, there exists $R > 0$, as well as $C, \theta > 0$ such that, for every $\varepsilon \in [0, \varepsilon_0]$, for every $\lambda \in \mathbf{C}$, if $|\lambda| > R$ and $|arg(\lambda)| \leqslant \pi - \nu$, we have that $D_\varepsilon(\lambda) \neq 0$. Hence, the Green function $G(\lambda, \varepsilon, \cdot, \cdot)$ is well defined, and it satisfies, for every $(x, y) \in \mathbf{R}^2$,*

$$|G(\lambda,\varepsilon,x,y)| + \frac{|\partial_x G(\lambda,\varepsilon,x,y)|}{\sqrt{|\lambda|}} \leqslant \frac{Ce^{-\Re(\sqrt{\lambda})\theta|x-y|}}{\sqrt{|\lambda|}}.$$

Proof. As before, using Proposition 20, one obtain that for $\nu > 0$ fixed, there exists some $R > 0$ big enough such that $P^{\sharp,hHF}$ is defined and satisfies the bounds as in the aforementioned lemma for $\sharp \in \{\ell, r\}$ as long as $|\lambda| \geqslant R$ and $|\arg(\lambda)| \leqslant \pi - \nu$. Moreover, $\Re(\sqrt{|\lambda|}) \leqslant \sqrt{|\lambda|} \leqslant \frac{\Re(\sqrt{\lambda})}{\sin(\nu)}$. Furthermore, we can define $V^{r,s,HF}(\lambda,\cdot)$, $V^{r,u,HF}(\lambda,\varepsilon,\cdot)$, $V^{l,u,HF}(\lambda,\varepsilon,\cdot)$ and $V^{l,s,HF}(\lambda,\varepsilon,\cdot)$ by

$$V^{r,s,HF}(\lambda,\varepsilon,x) := e^{x\mu^r_-(\lambda,\varepsilon)} P^{r,HF}(\lambda,\varepsilon,x) \begin{pmatrix} 1, \\ -\mu^r_+(\lambda,\varepsilon) \end{pmatrix},$$

$$V^{r,u,HF}(\lambda,\varepsilon,x) := e^{x\mu^r_+(\lambda,\varepsilon)} P^{r,HF}(\lambda,\varepsilon,x) \begin{pmatrix} 1 \\ -\mu^r_-(\lambda,\varepsilon) \end{pmatrix},$$

$$V^{\ell,u,HF}(\lambda,\varepsilon,x) := e^{x\mu^\ell_+(\lambda,\varepsilon)} P^{\ell,HF}(\lambda,\varepsilon,x) \begin{pmatrix} 1 \\ -\mu^\ell_-(\lambda,\varepsilon) \end{pmatrix},$$

$$V^{\ell,s,HF}(\lambda,\varepsilon,x) := e^{x\mu^\ell_-(\lambda,\varepsilon)} P^{\ell,HF}(\lambda,\varepsilon,x) \begin{pmatrix} 1 \\ -\mu^\ell_+(\lambda,\varepsilon) \end{pmatrix}.$$

As before, we define the dual basis of eigenfuctions : for every (λ, ε) as above

$$\begin{aligned}
\tilde{V}^{l,u,HF}(\lambda,\varepsilon,x) &:= e^{\int_0^x(\sigma_\varepsilon - f'(\underline{U}_\varepsilon)} V^{l,u,HF}(\lambda,\varepsilon,x)\,,\\
\tilde{V}^{l,s,HF}(\lambda,\varepsilon,x) &:= e^{\int_0^x(\sigma_\varepsilon - f'(\underline{U}_\varepsilon)} V^{l,s,HF}(\lambda,\varepsilon,x)\,,\\
\tilde{V}^{r,u,HF}(\lambda,\varepsilon,x) &:= e^{\int_0^x(\sigma_\varepsilon - f'(\underline{U}_\varepsilon)} V^{r,u,HF}(\lambda,\varepsilon,x)\,,\\
\tilde{V}^{r,s,HF}(\lambda,\varepsilon,x) &:= e^{\int_0^x(\sigma_\varepsilon - f'(\underline{U}_\varepsilon)} V^{r,s,HF}(\lambda,\varepsilon,x)\,.
\end{aligned}$$

As well as $D_\varepsilon^{HF}(\lambda)$ by

$$D_\varepsilon^{HF} := \left(V^{r,s,HF}(\lambda,\varepsilon,0), V^{l,u,HF}(\lambda,\varepsilon,0)\right),$$

and then, we also $\rho_{r,HF}$, $\rho_{\ell,HF}$, $\tau_{r,HF}$ and $\tau_{\ell,HF}$ such that, for such λ and ε

$$\begin{aligned}
V^{l,u,HF}(\lambda,\varepsilon,\cdot) &= \rho_{\ell,HF}(\lambda,\varepsilon)V^{r,u,HF}(\lambda,\varepsilon,\cdot) + \tau_{\ell,HF}(\lambda,\varepsilon)V^{r,s,HF}(\lambda,\varepsilon,\cdot)\,,\\
V^{r,s,HF}(\lambda,\varepsilon,\cdot) &= \rho_{r,HF}(\lambda,\varepsilon)V^{l,s,HF}(\lambda,\varepsilon,\cdot) + \tau_{r,HF}(\lambda,\varepsilon)V^{l,u,HF}(\lambda,\varepsilon,\cdot)\,.
\end{aligned}$$

We have that

$$\sqrt{|\lambda|} \leqslant |D_\varepsilon^{HF}(\lambda)| \leqslant 4\sqrt{|\lambda|},$$

for every $|\lambda| \geqslant R$ and $|\arg(\lambda)| \leqslant \pi - \nu$. Furthermore, as ρ^{HF} and τ^{HF} are bounded, we have that for some constant $M > 0$, for any λ and $\varepsilon \geqslant 0$ as above

$$\left|\frac{\rho^{HF}(\lambda,\varepsilon)}{D_\varepsilon^{HF}(\lambda)}\right| + \left|\frac{\tau^{HF}(\lambda,\varepsilon)}{D_\varepsilon^{HF}(\lambda)}\right| \leqslant \frac{M}{\sqrt{|\lambda|}}\,.$$

Furthermore, one can obtain, as for the low-frequency regime, the expression

$$\begin{aligned}
G(\lambda,\varepsilon,x,y) &= \frac{\mathbf{e}_1\cdot V^{r,s,HF}(\lambda,\varepsilon,x)\mathbf{e}_1\cdot\tilde{V}^{\ell,u,HF}(\lambda,\varepsilon,y)}{D_\varepsilon^{HF}(\lambda)}\,\text{if}\,x>y\,,\\
G(\lambda,\varepsilon,x,y) &= \frac{\mathbf{e}_1\cdot \tilde{V}^{r,s,HF}(\lambda,\varepsilon,x)\mathbf{e}_1\cdot V^{\ell,u,HF}(\lambda,\varepsilon,y)}{D_\varepsilon^{HF}(\lambda)}\,\text{if}\,x<y\,.
\end{aligned}$$

Hence, for $x > y > 0$

$$\begin{aligned}
G(\lambda,\varepsilon,x,y) = \Bigg(&\frac{\rho_{\ell,HF}(\lambda,\varepsilon)}{D_\varepsilon^{HF}(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{r,u,HF}(\lambda,\varepsilon,y)\\
&+\frac{\tau_{\ell,HF}(\lambda,\varepsilon)}{D_\varepsilon^{HF}(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{r,s}(\lambda,\varepsilon,y)\Bigg)\,\mathbf{e}_1\cdot V^{r,s}(\lambda,\varepsilon,x),
\end{aligned}$$

and similarly for $x < y < 0$, $y < x < 0$ and $0 < x < y$.

Notice that if $x > y > 0$, then we have that

$$\begin{aligned}|\partial_x G(\lambda,\varepsilon,x,y)| \leqslant & \frac{M}{\sqrt{|\lambda|}} e^{x\Re(\mu_-^r(\lambda,\varepsilon))} \left\| P^{r,HF}(\lambda,\varepsilon,x) \begin{pmatrix} 1 \\ -\mu_-^r(\lambda,\varepsilon) \end{pmatrix} \right\| \times \\ & e^{-y\Re(\mu_-^r(\lambda,\varepsilon))} \left(\left| \mathbf{e}_1 \cdot P^{r,HF}(\lambda,\varepsilon,y) \begin{pmatrix} 1 \\ -\mu_-^r(\lambda,\varepsilon) \end{pmatrix} \right| \right. \\ & \left. + \left| \mathbf{e}_1 \cdot P^{r,HF}(\lambda,\varepsilon,y) \begin{pmatrix} 1 \\ -\mu_+^r(\lambda,\varepsilon) \end{pmatrix} \right| \right), \\ \leqslant & 2CMe^{\Re(\mu_-^r(\lambda,\varepsilon))(x-y)}.\end{aligned}$$

for some $C \geqslant 1$ independent of λ and ε.

For $x > 0 > y$

$$\begin{aligned}|\partial_x G(\lambda,\varepsilon,x,y)| \leqslant & e^{x\Re(\mu_-^r(\lambda,\varepsilon))} \left\| P^{r,HF}(\lambda,\varepsilon,x) \begin{pmatrix} 1 \\ -\mu_-^r(\lambda,\varepsilon) \end{pmatrix} \right\| e^{y\Re(\mu_+^\ell(\lambda,\varepsilon))} \\ & \left| \mathbf{e}_1 \cdot P^{\ell,HF}(\lambda,\varepsilon,y) \begin{pmatrix} 1 \\ -\mu_+^\ell(\lambda,\varepsilon) \end{pmatrix} \right|, \\ \leqslant & Ce^{(\Re(\mu_+^\ell(\lambda,\varepsilon))+f'(\underline{u}_{+\infty})-\sigma_\varepsilon)y} e^{\Re(\mu_-^r(\lambda,\varepsilon))x},\end{aligned}$$

for some big enough constant C independent of λ and ε.

Hence, we have that for constants $C, \theta > 0$, respectively big enough and small enough, for any λ, ε such that $|\lambda| \geqslant R$, $|\arg(\lambda)| \leqslant \pi - \nu$ and $0 \leqslant \varepsilon \leqslant \varepsilon_0$,

$$|\partial_x G(\lambda,\varepsilon,x,y)| \leqslant Ce^{-\theta\Re(\sqrt{|\lambda|})|x-y|}.$$

(One just needs to use here that $|x-y| \leqslant 2\max(|x|,|y|)$ for some cases.)

The bounds on G are then obtained by integrating in x the one on $\partial_x G$

$$|G(\lambda,\varepsilon,x,y)| \leqslant \frac{Ce^{-\theta\Re(\sqrt{\lambda})|x-y|}}{\sqrt{|\lambda|}}.$$

Hence, for (λ,ε) such that $|\lambda| \geqslant R$, $|\arg(\lambda)| \leqslant \pi - \nu$ and $\varepsilon \in [0,\varepsilon_0]$, we have that $G(\lambda,\varepsilon,\cdot,\cdot)$ is defined and satisfy the desired bounds. □

4.2.3 Pointwise bounds

We consider, with $\eta_0 > 0$ such as in Proposition 21. Define

$$\omega^r_{\eta_0} := \sqrt{(f'(\underline{u}_{+\infty}) - \sigma_0)^2 - \frac{\eta_0}{2}}, \qquad \omega^\ell_{\eta_0} := \sqrt{(f'(\underline{u}_{-\infty}) - \sigma_0)^2 - \frac{\eta_0}{2}}.$$

Thus, for $\varepsilon > 0$ small enough, for any ω_0 such that $\omega_0 \geqslant \omega^r_{\eta_0}$, the solution λ_0 to

$$2\sqrt{(\sigma_\varepsilon - f'(\underline{u}_{+\infty})^2 - \varepsilon g'(\underline{u}_{+\infty}) + \lambda} = \omega_0$$

will satisfy $\lambda \geqslant -\eta_0$.

Similarly, one obtains this type of statement for ℓ in place of r.

First, for any $\theta \in \left(\frac{\pi}{2}, \pi\right)$, fix a path, divided in two parts, of the form $\rho \mapsto \tilde{\delta} - \rho e^{-i\theta}$ on $\mathbf{R}_-$ and $\rho \mapsto \tilde{\delta} + \rho e^{i\theta}$ on $\mathbf{R}_+$ for a given constant $\tilde{\delta} > 0$ big enough, such that this path is valued in $\bigcap_{\varepsilon\in[0,\varepsilon_0]} \rho(\mathcal{L}_\varepsilon)$, which exists and can be chosen independently on ε. Note that one can choose θ as close to π as needed, up to making $\tilde{\delta}$ bigger (which will be needed later on).

Furthermore, for any ρ_0 (which will be chosen just after) noting $\lambda_0 = \tilde{\delta} - \rho_0 \cos(\theta)$ and $\lambda_1 = \rho_0 \sin(\theta)$, as well as $\zeta^\sharp : [\omega^\sharp_{\eta_0}, +\infty)$ for $\sharp \in \{\ell, r\}$ such that

$$\zeta^r(\omega) = -i\left(\sqrt{\lambda_0 + i\lambda_1 - (f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2 + 4\varepsilon g'(\underline{u}_{+\infty})} - \omega\right),$$

for every $\omega \geqslant \omega^r_{\eta_0}$ and, $\zeta^\ell : [\omega^\ell_{\eta_0}, +\infty)$such that

$$\zeta^\ell(\omega) = -i\left(\sqrt{\lambda_0 + i\lambda_1 - (f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2 + 4\varepsilon g'(\underline{u}_{-\infty})} - \omega\right),$$

for every $\omega \geqslant \omega^\ell_{\eta_0}$. The dependence of $\zeta^\sharp$ on ε is implicit, to simplify a bit the notations. We are now ready to impose the first conditions on ρ_0 and θ. For $\sharp \in \{\ell, r\}$, if θ is close enough to π, and ρ_0 big enough, $\zeta^\sharp(\omega^\sharp_{\eta_0})$ satisfies $\inf_{\varepsilon\in[0,\varepsilon_0]} \Re\left(\zeta^\sharp(\omega^\sharp_{\eta_0})\right) > \sup_{\varepsilon\in[0,\varepsilon_0]} \frac{\max(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon, \sigma_\varepsilon - f'(\underline{u}_{-\infty}))}{\sqrt{\min((f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)^2 - (\omega^\ell_{\eta_0})^2, (\sigma_\varepsilon - f'(\underline{u}_{-\infty}))^2 - (\omega^r_{\eta_0})^2)}} |\Im(\zeta^\sharp(\omega)|$. Then, with ρ_0 big enough, we have $\rho_0 \cos(\theta) + \tilde{\delta} \leqslant -1$. The first condition will allow us to use Lemma 6.

Λ_{LF} is related to the part of the curve associated to $[-\rho_0, 0] \cup [0, \rho_0]$. Λ_{HF} is the related to the rest of the integral, for which decay uniform in ε is available, as we will see later on. In particular, sharp pointwise bounds techniques will only be needed for the low frequency (Λ_{LF}) part, as well as the decomposition in a decaying part and a phase part will only be needed for that part. The last condition needed on ρ_0 are that, for every

$\rho \geqslant \rho_0$, $|\tilde{\delta} + \rho e^{i\theta}| \geqslant R$, with $R > 0$ as in 12 (associated to $\nu = \theta$), which ensures that one can obtain pointwise bounds on that part of the integral.

Now, we define the low frequency paths. They will be characterized (except for the original path and the small time one) by a point $\omega \in \mathbf{R}$, $\varepsilon \in [0, \varepsilon_0]$ and a side (ℓ or r), and, given $\sharp \in \{\ell, r\}$ (say r), as well as an associated admissible $\omega \geqslant \omega^r_{\eta_0}$. The path Λ is then defined by, for every $\xi \in \mathbf{R}_+$,

$$\begin{aligned}\Lambda(\xi) &= \frac{(\omega + i\xi\zeta^r(\omega))^2 - (f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2 + 4\varepsilon g'(\underline{u}_{+\infty})}{4},\\ \Lambda(-\xi) &= \overline{\Lambda(\xi)}.\end{aligned}$$

Where the dependence of Λ on ω and ε is omitted to simplify the notations.

Finally, such choices imply that for any ω_{max}, the curves associated to any admissible ω smaller than ω_{max} for any $\varepsilon \in [0, \varepsilon_0]$ have their derivatives Λ' uniformly bounded on $[-1, 1]$ (uniformly in ξ, ε and ω).

Notice that there exist $\tilde{c}_\ell(\varepsilon)$ and $\tilde{c}_r(\varepsilon)$ such that

$$\begin{aligned}\mathbf{e}_1 \cdot V^{l,u}(0, \varepsilon, \cdot) &= \tilde{c}_\ell(\varepsilon)\underline{U}'_\varepsilon,\\ \mathbf{e}_1 \cdot V^{r,s}(0, \varepsilon, \cdot) &= \tilde{c}_r(\varepsilon)\underline{U}'_\varepsilon,\end{aligned}$$

as both are solutions to $\partial_x^2 v + \partial_x((\sigma_\varepsilon - f'(\underline{U}_\varepsilon)v) + \varepsilon g'(\underline{U}_\varepsilon)v = 0$ decaying exponentially fast at $-\infty$ and $+\infty$ and there is only one nonzero solution up to a multiplication by a scalar in $L^2(\mathbf{R})$.

In particular,

$$V^{r,s}(0, \varepsilon, \cdot) = \frac{\tilde{c}_r(\varepsilon)}{\tilde{c}_l(\varepsilon)} V^{\ell,u}(0, \varepsilon, \cdot),$$

and

$$\tilde{c}_\ell(\varepsilon) = \tau_\ell(0, \varepsilon)\tilde{c}_r(\varepsilon),$$

as well as

$$\tilde{c}_r(\varepsilon) = \tau_r(0, \varepsilon)\tilde{c}_\ell(\varepsilon).$$

We now consider the decomposition

$$G^\varepsilon_t(x, y) = \tilde{G}^\varepsilon_t(x, y) + G^{\varepsilon,p}_t(y)\underline{U}'_\varepsilon(x)$$

where we define $G^{\varepsilon,p}_t(y)$ as, for $0 < y \leqslant t\omega^{HF}_r$,

$$G^{\varepsilon,p}_t(y) := \frac{\tilde{c}_\ell(\varepsilon)}{2\pi i} \int_{\Lambda_{LF}} \frac{e^{t\lambda}}{D_\varepsilon(\lambda)} \mathbf{e}_1 \cdot \tilde{V}^{r,s}(\lambda, \varepsilon, y) d\lambda,$$

and, for $-t\omega_\ell^{HF} \leqslant y < 0$

$$G_t^{\varepsilon,p}(y) := \frac{\tilde{c}_r(\varepsilon)}{2\pi i}\int_{\Lambda_{LF}} \frac{e^{t\lambda}}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{\ell,u}(\lambda,\varepsilon,y)d\lambda.$$

In particular, we have that, for $x<0<y<t\omega_{HF}^r$

$$\begin{aligned}\tilde{G}_t^\varepsilon(x,y) =&\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\mathbf{e}_1\cdot(V^{\ell,u}(\lambda,\varepsilon,x)-V^{\ell,u}(0,\varepsilon,x))}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{r,s}(\lambda,\varepsilon,y)d\lambda\\ &+\frac{1}{2\pi i}\int_{\Lambda_{HF}} e^{t\lambda}G(\lambda,\varepsilon,x,y)d\lambda.\end{aligned}$$

for $0<y<\min(t\omega_{HF}^r,x)$

$$\begin{aligned}\tilde{G}_t^\varepsilon(x,y) =&\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\rho_\ell(\lambda,\varepsilon)\mathbf{e}_1\cdot\tilde{V}^{r,u}(\lambda,\varepsilon,y)\mathbf{e}_1\cdot V^{r,s}(\lambda,\varepsilon,x)}{D_\varepsilon(\lambda)}d\lambda\\ &+\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\tau_\ell(\lambda,\varepsilon)\mathbf{e}_1\cdot V^{r,s}(\lambda,\varepsilon,x)-\tau_\ell(0,\varepsilon)\mathbf{e}_1\cdot V^{r,s}(0,\varepsilon,x)}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{r,s}(\lambda,\varepsilon,y)d\lambda\\ &+\frac{1}{2\pi i}\int_{\Lambda_{HF}} e^{t\lambda}G(\lambda,\varepsilon,x,y)d\lambda,\end{aligned}$$

for $0<x<y<t\omega_{HF}^r$

$$\begin{aligned}\tilde{G}_t^\varepsilon(x,y) =&\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\mathbf{e}_1\cdot(V^{\ell,u}(\lambda,\varepsilon,x)-V^{\ell,u}(0,\varepsilon,x))}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{r,s}(\lambda,\varepsilon,y)d\lambda\\ &+\frac{1}{2\pi i}\int_{\Lambda_{HF}} e^{t\lambda}G(\lambda,\varepsilon,x,y)d\lambda\\ =&\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\mathbf{e}_1\cdot(\tau_\ell(\lambda,\varepsilon)V^{r,s}(\lambda,\varepsilon,x)-\tau_\ell(0,\varepsilon)V^{r,s}(0,\varepsilon,x))}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{r,s}(\lambda,\varepsilon,y)d\lambda\\ &\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\rho_\ell(\lambda,\varepsilon)\mathbf{e}_1\cdot V^{r,u}(\lambda,\varepsilon,x)}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{r,s}(\lambda,\varepsilon,y)d\lambda\\ &+\frac{1}{2\pi i}\int_{\Lambda_{HF}} e^{t\lambda}G(\lambda,\varepsilon,x,y)d\lambda,\end{aligned}$$

for $-t\omega_{HF}^\ell < y < 0 < x$

$$\begin{aligned}\tilde{G}_t^\varepsilon(x,y) =&\frac{1}{2\pi i}\int_{\Lambda LF} e^{t\lambda}\frac{\mathbf{e}_1\cdot(V^{r,s}(\lambda,\varepsilon,x)-V^{r,s}(0,\varepsilon,x)))}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{\ell,u}(\lambda,\varepsilon,y)\\ &+\frac{1}{2\pi i}\int_{\Lambda_{HF}} e^{t\lambda}G(\lambda,\varepsilon,x,y)d\lambda,\end{aligned}$$

for $\max(x, -t\omega^\ell_{HF}) < y < 0$.

$$\begin{aligned}\tilde{G}^\varepsilon_t(x,y) =&\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\mathbf{e}_1\cdot(V^{r,s}(\lambda,\varepsilon,x)-V^{r,s}(0,\varepsilon,x))}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{\ell,u}(\lambda,\varepsilon,y)d\lambda\\ =&\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\mathbf{e}_1\cdot V^{\ell,s}(\lambda,\varepsilon,x)}{D_\varepsilon(\lambda)}\rho_r(\lambda,\varepsilon)\mathbf{e}_1\cdot\tilde{V}^{\ell,u}(\lambda,\varepsilon,y)d\lambda\\ &+\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\tau_r(\lambda,\varepsilon)\mathbf{e}_1\cdot V^{\ell,u}(\lambda,\varepsilon,x)-\tau_r(0,\varepsilon)\mathbf{e}_1\cdot V^{\ell,u}(0,\varepsilon,x)}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{\ell,u}(\lambda,\varepsilon,y)d\lambda\\ &+\frac{1}{2\pi i}\int_{\Lambda_{HF}} e^{t\lambda}G(\lambda,\varepsilon,x,y)d\lambda,\end{aligned}$$

for $-t\omega^\ell_{HF} < y < x < 0$

$$\begin{aligned}\tilde{G}^\varepsilon_t(x,y) =&\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\mathbf{e}_1\cdot(V^{r,s}(\lambda,\varepsilon,x)-V^{r,s}(0,\lambda,x))}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{\ell,u}(\lambda,\varepsilon,y)d\lambda\\ =&\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\mathbf{e}_1\cdot(\tau_r(\lambda,\varepsilon)V^{\ell,u}(\lambda,\varepsilon,x)-\tau_r(0,\varepsilon,x)V^{\ell,u}(0,\varepsilon,x))}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{\ell,u}(\lambda,\varepsilon,y)d\lambda\\ &+\frac{1}{2\pi i}\int_{\Lambda_{LF}} e^{t\lambda}\frac{\rho_r(\lambda,\varepsilon)\mathbf{e}_1\cdot V^{\ell,s}(\lambda,\varepsilon,x)}{D_\varepsilon(\lambda)}\mathbf{e}_1\cdot\tilde{V}^{\ell,s}(\lambda,\varepsilon,y)d\lambda\\ &+\frac{1}{2\pi i}\int_{\Lambda_{HF}} e^{t\lambda}G(\lambda,\varepsilon,x,y)d\lambda.\end{aligned}$$

Finally, for every $x\in\mathbf{R}$, for $y > t\omega^r_{HF}$ and for $y < -t\omega^\ell_{HF}$, we have that

$$\tilde{G}^\varepsilon_t(x,y) = \frac{1}{2\pi i}\int_\Lambda e^{t\lambda}G(\lambda,\varepsilon,x,y)d\lambda.$$

When it comes to obtaining bounds on the spatial derivatives, we remind that despite the lack of smoothness (and even of continuity) of $\partial_x G(\lambda,\varepsilon,\cdot,y)$ at $x=y$, the temporal Green function of a parabolic operator with smooth coefficients is actually smooth. As a consequence, it is enough to obtain bounds on $\partial_x G^\varepsilon_t(x,y)$ (or equivalently on $\tilde{G}^\varepsilon_t(x,y)$ as $G^{\varepsilon,p}_t(y)\underline{U}'_\varepsilon$ is anyway smooth in x) for x in a dense subset of $\mathbf{R}$ for $t>0$ and $y\in\mathbf{R}$ (y avoiding a finite number of values depending on t) being both fixed. In fact, the bounds on the linear operators here will be obtained by using a classical convolution estimate (for any $(\phi,\psi)\in L^\infty\times L^1$, for any $x\in\mathbf{R}$, $|\phi*\psi|(x)\leqslant\|\phi\|_{L^\infty}\|\psi\|_{L^1}$) and direct bounds (such as the fact that for any $\phi\in L^\infty_x(\mathbf{R},L^1_y(\mathbf{R}))$ and any $\psi\in L^\infty(\mathbf{R})$, $\left|\int_{\mathbf{R}}\phi(x,y)\psi(y)dy\right|\leqslant\|\phi\|_{L^\infty_x(\mathbf{R},L^1_y(\mathbf{R}))}\|\psi\|_{L^\infty}$). Thus, for $t>0$ fixed, we will try to bound for all but a finite set of y the values of $|G^{\varepsilon,p}_t(y)|$, and, for such (t,y), to bound $\tilde{G}^\varepsilon_t(x,y)$ for almost all values of x. We stress that, for hyperbolic problems, due to the presence of Dirac masses, the analysis is quite different.

Proposition 22. *We have the following bounds on $\tilde{G}^\varepsilon_t(x,y)$, $\partial_x G^\varepsilon_t(x,y)$ and $G^{\varepsilon,p}_t(y)$.*

There exist $C, \theta, \omega', \omega''$, positive constants, such that, for every $(x,y,t) \in \mathbf{R}^2 \times]0,2]$

$$|G_t^\varepsilon(x,y)| + \sqrt{t}|\partial_x G_t^\varepsilon(x,y)| \leqslant \frac{Ce^{-\theta\frac{y^2}{t}}}{\sqrt{t}},$$

and for every $(x,y,t) \in \mathbf{R}^2 \times [1,+\infty[$

$$\begin{aligned}
|\tilde{G}_t^\varepsilon(x,y)| + |\partial_x \tilde{G}_t^\varepsilon(x,y)| \leqslant & \frac{Ce^{\varepsilon g'(\underline{u}_{-\infty})t}}{\sqrt{t}} \left(\mathbb{1}_{y<0} e^{-\frac{(y-(\sigma_\varepsilon - f'(\underline{u}_{-\infty}))^2}{4t}} e^{-\theta|x|} \right. \\
& \left. + \mathbb{1}_{y,x<0} e^{-\frac{(y-x-t(\sigma_\varepsilon - f'(\underline{u}_{-\infty})))^2}{4t}} \right) \\
& + \frac{Ce^{\varepsilon g'(\underline{u}_{+\infty})t}}{\sqrt{t}} \left(\mathbb{1}_{y>0} e^{-\frac{(y-(\sigma_\varepsilon - f'(\underline{u}_{+\infty}))^2}{4t}} e^{-\theta|x|} \right. \\
& \left. + \mathbb{1}_{y,x>0} e^{-\frac{(y-x-t(\sigma_\varepsilon - f'(\underline{u}_{+\infty})))^2}{4t}} \right) \\
& + Ce^{-\omega t} e^{-\theta|x-y|}, \\
|G_t^{\varepsilon,p}(y)| \leqslant & \frac{C}{\sqrt{t}} e^{t\omega''} e^{-\frac{\theta y^2}{t}}, \\
|\partial_t G_t^{\varepsilon,p}(y)| \leqslant & \frac{C}{\sqrt{t}} e^{\varepsilon g'(\underline{u}_{+\infty})t} e^{-\frac{(y-t(\sigma_\varepsilon - f'(\underline{u}_{+\infty})))^2}{4t}} \mathbb{1}_{y>0} \\
& + \frac{C}{\sqrt{t}} e^{\varepsilon g'(\underline{u}_{-\infty})t} e^{-\frac{(y-t(\sigma_\varepsilon - f'(\underline{u}_{-\infty})))^2}{4t}} \mathbb{1}_{y<0} + \frac{Ce^{-\omega' t} e^{-\frac{\theta y^2}{t}}}{\sqrt{t}}.
\end{aligned}$$

Before starting the proof, we explain here the main idea. One can notice that, from the proof and the statement of the proposition stem that, actually, the main contribution comes from the cases given for x and y being of the same sign, and $x-y$ is, at the right, when $y-x$ is close to $t(\sigma_\varepsilon - f'(\underline{u}_{+\infty})$, and, at the left, when $y-x$ is close to $t(\sigma_\varepsilon - f'(\underline{u}_{-\infty}))$. It makes sense considering the way the hyperbolic problem actually behaves. This observation will lead us, in the proof, to analyze carefully the points (t,x,y) such that $y-x$ is close to $t(\sigma_\varepsilon - f'(\underline{u}_\pm))$, and also the point such that y is close to $t(\sigma_\varepsilon - f'(\underline{u}_\pm))$. These ones will actually be associated with an extreme localization (of the solution in the original variable) of the form $x \mapsto e^{-\theta\frac{|x|}{\varepsilon}}$.

Finally, let us emphasize the main differences with [BR23]. First, we obviously need here to separate the rate at $+\infty$ and at $-\infty$. In particular, it requires to keep track, in the Gaussian, of the terms $\frac{(y-t(\sigma_\varepsilon - f'(\underline{u}_{+\infty})))^2}{4t}$ or $\frac{(y-x-t(\sigma_\varepsilon - f'(\underline{u}_{+\infty})))^2}{4t}$, and not transform the $\frac{1}{4}$ into some $\alpha > 0$ small, as the critical exponents would, otherwise, be unattainable.

Also notice that the Gaussians of the form $\frac{(x-y-t(\sigma_\varepsilon-f'(\underline{u}_{+\infty})))^2}{4t}$ need to be avoided, as, otherwise, one cannot conclude either.

Proof. We first bound G_ε^t for $0 < t \leqslant 2$. In fact, classical parabolic theory (see [Fri64] for the general case) implies this result. We will still prove it by hand here. In order to do so, with $\nu > 0$ defined by $\pi - 2\arctan(3) = \nu$, we consider constants R, C and θ' associated to the high-frequency regime Lemma 12, that is such that for every $\lambda \in \mathbf{C}$ such that $|\lambda| \geqslant R$ and $|\arg(\lambda)| \leqslant \pi - \nu$, we have $|G(\lambda, \varepsilon, x, y)| \leqslant \dfrac{Ce^{-\theta'\Re(\sqrt{\lambda})|x-y|}}{\sqrt{|\lambda|}}$ for every $(x, y) \in \mathbf{R}^2$. for every $(x, y) \in \mathbf{R}^2$. Here, by choosing the path according to $\Lambda(\xi) = \left(\dfrac{\theta'|x-y|}{2t} + i\xi(3 - \mathrm{sgn}(\xi)i)\right)^2$ if $\frac{\theta'|x-y|}{2t} \geqslant 2R$ and $\Lambda(\xi) = (R + i\xi(3 - \mathrm{sgn}(\xi)i))^2$ if $\dfrac{\theta'|x-y|}{2t} < 2R$, the existence of positive constants C and θ such that, for every $\varepsilon > 0$ small enough, for every x, y reals, for every $0 < t \leqslant 2$,

$$|G_t^\varepsilon(x, y)| + |\partial_x G_t^\varepsilon(x, y)|\sqrt{t} \leqslant \frac{Ce^{-\frac{\theta y^2}{t}}}{\sqrt{t}}.$$

through direct computations (by doing similarly with ∂_x).

Now, we go back to the general case, in which we will bound $G_t^{\varepsilon,p}(y)$ and $\tilde{G}_t^\varepsilon(x, y)$ for $t > 1$.

The bounds on the high-frequency regime for $t > 1$ are obtained directly. One can obtain, in that regime, a bound of the form

$$|G_t^\varepsilon(x, y)| \leqslant \frac{C}{t}e^{-\omega' t}e^{-\theta|x-y|},$$
$$|\partial_x G_t^\varepsilon(x, y)| \leqslant \frac{C}{\sqrt{t}^3}e^{-\omega' t}e^{-\theta|x-y|}.$$

by just keeping the curve of the form $\xi \mapsto \xi e^{i\mathrm{sgn}(\xi)\theta} + \tilde{\delta}$ for $|\xi| \geqslant \rho_0$.

As we are in the regime $t > 1$, we thus obtain the desired inequalities.

Now, we will focus on the low-frequency part.

In that context, bounds on $\partial_x \tilde{G}_t^\varepsilon(x, y)$ are obtained similarly to those on $\tilde{G}_t^\varepsilon(x, y)$ as we restrict ourselves to a low-frequency analysis here, and, furthermore, here, the curves can be chosen identically on each part, as the bounds needed on the integrand are true both in the derivated case and in the case without the derivative.

The derivation of the bounds will be decomposed in parts involving convolution (that is, when the term in the exponential is of the form $\Re(\mu_-^r(\lambda, \varepsilon))|x - y|$) for example), and those in a product form (for example when we have as an argument of the exponential

$\Re(\tilde{\mu}^r_-(\lambda,\varepsilon))|y| + \Re(\mu^r_-(\lambda,\varepsilon))|x|)$. The curves will then be chosen, with $\sharp = l$ or $\sharp = r$ and with $\omega_\sharp(t,z)$ ($z = \dfrac{|y|}{t}$ or $z = \dfrac{|x-y|}{t}$) and noting any $\alpha \in \mathbf{R}$ and $\beta \in \mathbf{R}$ and Λ as described before, associated with $\sharp$, α and β such that in the exponential the term can be written as $\Re(t\Lambda(\xi)) + \Re\left(\dfrac{\alpha - \sqrt{\alpha^2 - 4\varepsilon\beta + 4\Lambda(\xi)}}{2} tz\right) \leqslant +\varepsilon\beta t - \dfrac{(z-\alpha t)^2}{4t} - \dfrac{\xi^2 \Re(\zeta^2_\pm)t}{4}$.

Note $\tilde{\alpha} = \sup_{\varepsilon\in[0,\varepsilon_0]} \max(\sigma_\varepsilon - f'(\underline{u}_{+\infty}), f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)$, $\tilde{\beta} = \varepsilon_0 \max(g'(\underline{u}_{+\infty}), g'(\underline{u}_{-\infty}))$. Taking $\tilde{\omega} > 0$ such that $\tilde{\omega} \geqslant 4\tilde{\alpha}$ and $\tilde{\omega}^2 \geqslant 32\tilde{\beta}$, one obtains that with $\Lambda(\xi) := \dfrac{\left(\tilde{\omega} + i\xi\zeta_{\mathrm{sgn}(\xi)}\right)^2}{4} - \dfrac{\alpha^2}{4} + b$, one has, for any α of the form $\pm(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)$ or $\pm(f'(\underline{u}_{-\infty}) - \sigma_\varepsilon)$ as well as b of the form $\varepsilon g'(\underline{u}_{\pm\infty})$, we have, for any $z \geqslant t\tilde{\omega}$,

$$t\Re(\Lambda(\xi)) + \Re\left(\frac{\alpha - \sqrt{\alpha^2 - b + \Lambda(\xi)}}{2}\right) z \leqslant \frac{|\xi||\Im(\zeta_{\mathrm{sgn}(xi)}))}{2}(\tilde{\omega}t - z) - \frac{\xi^2\Re(\zeta_{\mathrm{sgn}(\xi)})t}{4} - \frac{\tilde{\omega}}{16}(t\tilde{\omega} + tz) + bt,$$

which can be bounded by above by

$$-\frac{t\tilde{\omega}z}{32} - \frac{t\tilde{\omega}^2}{32} - \frac{\xi^2\Re(\zeta_{\mathrm{sgn}(\xi)})t}{4}.$$

Define

$$\omega^\ell(t,z) := \max\left(\omega^\ell_{\eta_0}, \min\left(\tilde{\omega}, \frac{|z|}{t}\right)\right), \qquad \omega^r(t,z) := \max\left(\omega^r_{\eta_0}, \min\left(\tilde{\omega}, \frac{|z|}{t}\right)\right).$$

For the rest of this proof, we will apply Proposition 19 with a compact as in Lemma 21, this time choosing $\delta > 0$ small enough such that for every $\sharp \in \{\ell, r\}$, for every $\varepsilon \in [0, \varepsilon_0]$, for every $z \in [\omega^\sharp_{\eta_0}, \tilde{\omega}]$, the path $\Lambda^\sharp$ associated to z is valued in that compact, and also such that the image of the initial path for $\rho \in [-\rho_0, \rho_0]$ is contained in the compact. This is possible as we chose the bounds of the intervals $(\omega^\sharp_{\eta_0}, \tilde{\omega})$ to ensure that the paths remain bounded, and bounded away from the absolute spectrum, as this is the union of the image of two compact sets by continuous mappings. The mappings are $(\xi, \varepsilon, z) \in [-1,1] \times [0,\varepsilon_0] \times [\omega^\sharp_{\eta_0}, \tilde{\omega}] \mapsto \Lambda^\sharp(\xi)$ where $\Lambda^\sharp$ is associated to ε (through $\zeta^\sharp(\omega)$ constructed before) and such that $\Lambda^\sharp(0) = z$, and $\sharp \in \{\ell, r\}$. It is possible to ensure the desired conditions as long as ε_0 is small enough.

If $t\omega^\sharp_{\eta_0} \leqslant z \leqslant t\tilde{\omega}$, by Lemma 5, one can bound the term in the exponential as follows

$$t\Re(\Lambda(\xi)) + \Re\left(\frac{\alpha - \sqrt{\alpha^2 - 4b + 4\Lambda(\xi)}}{2}\right) z \leqslant bt - \frac{(z - t\alpha)^2}{4t} - \frac{\xi^2\Re(\zeta^2)t}{4}.$$

Finally, for some constants C and ω' positive, for every $\varepsilon \in [0, \varepsilon_0]$, for every $(t,z) \in$

$(\mathbf{R}^+)^2$,if $z \leqslant t\omega^\sharp_{\eta_0}$ (say $\sharp = r$), then the conditions on $\zeta^\sharp$ ensures that, for Λ associated to $\omega^r_{\eta_0}$ satisfies that for every $\xi \in [-1,1]$,

$$\begin{aligned}
&t\Re(\Lambda(\xi)) + \Re\left(\frac{\pm(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon) - \sqrt{(f'(\underline{u}_{+\infty}) - \sigma_\varepsilon)^2 - 4\varepsilon g'(\underline{u}_{+\infty}) + 4\Lambda(\xi)}}{2}\right) tz \\
\leqslant &-\omega' t \leqslant -\frac{\omega' t}{2} - \frac{\omega' z}{2\omega^r_{\eta_{\eta_0}}}.
\end{aligned}$$

Now, we are ready to obtain the bounds on the low-frequency part of the integrals.

We will focus here on the part associated to $y > 0$.

If $x < 0$, then a direct bound is applied using that for every $\lambda \in K$ $|G(\lambda,\varepsilon,x,y)| \leqslant Ce^{-\theta|x|}e^{Re\left(\tilde{\mu}^r_-(\lambda,\varepsilon)y\right)}$. In fact, one then obtains, with $\omega_l\left(t,\frac{y}{t}\right)$

$$\begin{aligned}
&\left|\int_\Lambda e^{t\lambda}G(\lambda,\varepsilon,x,y)d\lambda\right| \\
\lesssim &\int_{-1}^1 e^{t\Re(\lambda)-\theta|x|+\Re\left(\frac{\sigma_\varepsilon - f'(\underline{u}_{-\infty}) + \sqrt{(\sigma_\varepsilon - f'(\underline{u}_{-\infty}))^2 - 4\varepsilon g'(\underline{u}_{-\infty}) + 4\Lambda(\xi)}}{2}\right)y} \frac{|\Lambda'(\xi)|}{|D_\varepsilon(\Lambda(\xi))|}d\xi, \\
\leqslant &C\frac{e^{\varepsilon g'(\underline{u}_{+\infty})t}e^{-\theta|x|}e^{-\frac{(y-(\sigma_\varepsilon - f'(\underline{u}_{+\infty}))t)^2}{4t}}}{\sqrt{t}}.
\end{aligned}$$

If $x > 0$ and $y > \max(t\omega_{HF}, x)$, then, we decompose the integral in two part, one containing the ρ_ℓ, and which can bounded with the choice of the curve Λ_1 associated to $\omega_r\left(t,\frac{y-x}{t}\right)$, and the τ_ℓ part with is bounded by choosing the path Λ_2 associated to $\omega_r\left(t,\frac{y}{t}\right)$. For example, if $t\tilde{\omega} > y - x > t\omega^r_{\eta_0}$, then we have the bounds

$$\begin{aligned}
\left|\int_\Lambda e^{t\lambda}G(\lambda,\varepsilon,x,y)d\lambda\right| \lesssim &\left|\int_{\Lambda_1} e^{t\Re(\lambda)}\frac{\rho_\ell(\Lambda(\xi),\varepsilon)}{D_\varepsilon(\lambda)}\Lambda'(\xi)e^{\Re(\mu_-(\lambda,\varepsilon))(y-x)}d\xi\right| \\
&+ \left|\int_{\Lambda_2} e^{t\lambda}\frac{\tau_\ell(\Lambda(\xi),\varepsilon)\Lambda'(\xi)}{D_\varepsilon(\Lambda(\xi))}d\xi\right| \\
\lesssim &\int_{-1}^1 e^{t\Re(\Lambda)}\frac{|\rho_\ell(\Lambda(\xi)|}{|D_\varepsilon(\Lambda(\xi)|}|\Lambda'(\xi)|e^{\Re\left(\frac{\sigma_\varepsilon - f'(\underline{u}_{+\infty}) - \sqrt{(\sigma_\varepsilon - f'(\underline{u}_{+\infty}))^2 - 4\varepsilon g'(\underline{u}_{+\infty}) + 4\Lambda(\xi)}}{2}\right)(y-x)}d\xi \\
&+ \int_{-1}^1 e^{-\theta x}e^{t\Re(\Lambda(\xi))}e^{\Re\left(\frac{\sigma_\varepsilon - f'(\underline{u}_{+\infty}) - \sqrt{(\sigma_\varepsilon - f'(\underline{u}_{+\infty}))^2 - 4\varepsilon g'(\underline{u}_{+\infty}) + 4\Lambda(\xi)}}{2}\right)y}|\Lambda_2'(\xi)|\frac{|\tau_\ell(\Lambda_2(\xi),\varepsilon)|}{|D_\varepsilon(\Lambda(\xi))|}d\xi \\
\leqslant &Ce^{\varepsilon t g'(\underline{u}_{+\infty})}\left(\frac{e^{-\theta x}e^{-\frac{(y-t(\sigma_\varepsilon - f'(\underline{u}_{+\infty})))^2}{4t}} + e^{-\frac{(y-x-t(\sigma_\varepsilon - f'(\underline{u}_{+\infty})))^2}{4t}}}{\sqrt{t}}\right)
\end{aligned}$$

If $x > y > t\omega^r_{HF}$, then the $\tilde{\rho}_\ell (= \rho_\ell)$ part is bounded with the curve passing through $\omega_r(t, x-y)$, and the τ_ℓ one with the curve passing through $\omega_l(t,y)$. If $y < t\omega^r_{HF}$, then we will need a new ingredient. To bound terms of the form

$$\frac{\mathbf{e}_1 \cdot (V^{r,s}(\lambda,\varepsilon,x)\tau_\ell(\lambda,\varepsilon) - \tau_\ell(0,\varepsilon)V^{r,s}(0,x)}{D_\varepsilon(\lambda)},$$

uniformly with respect to λ, by some term exponentially localized in x, we decompose K in two parts. Outside of any given small enough neighborhood of 0 in K, it is obtained directly, and, using the Cauchy formula, it can be extended in the neighborhood itself. In this regime, assuming thus $y < t\omega^r_{HF}$ and $x < 0$, then $\omega_r\left(t, \frac{y}{t}\right)$. If $x < y$, then we use for the τ_ℓ part the path associated to $\omega_r\left(t, \frac{y}{t}\right)$, and, for the ρ_ℓ part, the path associated to $\omega_r\left(t, \frac{y-x}{t}\right)$. If $x > y$ then, use the path associated to $\omega_r\left(t, \frac{x-y}{t}\right)$ for the ρ_ℓ part, and the path associated to $\omega_r\left(t, \frac{y}{t}\right)$ for the τ_ℓ part.

We decompose in four parts those associated to $\partial_t G^{\varepsilon,p}_t(y)$. If $y \notin [-t\omega^\ell_{HF}, t\omega^r_{HF}]$ then we have that $\partial_t G^{\varepsilon,p}_t(y) = 0$.

For $y \in (0, t\omega^r_{\eta_0})$

$$\begin{aligned}|\partial_t G^{\varepsilon,p}_t(y)| \lesssim& \left|\int_\Lambda \lambda e^{t\lambda}\mathbf{e}_1 \cdot \tilde{V}^{r,s}(\lambda,\varepsilon,y)\frac{d\lambda}{D_\varepsilon(\lambda)}\right|,\\ \leqslant& \frac{C}{\sqrt{t}}e^{-\omega' t}e^{-\frac{\theta y^2}{t}}.\end{aligned}$$

For $y \in (t\omega^r_{\eta_0}, t\omega^r_{HF})$

$$\begin{aligned}|\partial_t G^{\varepsilon,p}_t(y)| \lesssim& \left|\int_\Lambda \lambda e^{t\lambda}\mathbf{e}_1 \cdot \tilde{V}^{r,s}(\lambda,\varepsilon,y)\frac{d\lambda}{D_\varepsilon(\lambda)}\right|,\\ \leqslant& \frac{C}{\sqrt{t}}e^{\varepsilon g'(\underline{u}_{+\infty})t}e^{-\frac{(y-t(\sigma_\varepsilon - f'(\underline{u}_{+\infty})))^2}{4t}}.\end{aligned}$$

Similarly, for $y \in (-t\omega^l_{\eta_0}, 0)$,

$$\begin{aligned}|\partial_t G^{\varepsilon,p}_t(y)| \lesssim& \left|\int_\Lambda \lambda e^{t\lambda}\mathbf{e}_1 \cdot \tilde{V}^{l,u}(\lambda,\varepsilon,y)\frac{d\lambda}{D_\varepsilon(\lambda)}\right|,\\ \leqslant& \frac{C}{\sqrt{t}}e^{\varepsilon g'(\underline{u}_{+\infty})t}e^{-\frac{(y-t(\sigma_\varepsilon - f'(\underline{u}_{+\infty})))^2}{4t}}.\end{aligned}$$

For $y \in (-t\omega^l_{HF}, -t\omega^l_{\eta_0})$,

$$|\partial_t G^{\varepsilon,p}_t(y)| \lesssim \left|\int_\Lambda \lambda e^{t\lambda}\mathbf{e}_1 \cdot \tilde{V}^{l,u}(\lambda,\varepsilon,y)\frac{d\lambda}{D_\varepsilon(\lambda)}\right|,$$

$$\leqslant \frac{C}{\sqrt{t}} e^{\varepsilon g'(\underline{u}_{-\infty})t} e^{-\frac{(y-t(\sigma_\varepsilon - f'(\underline{u}_{-\infty})))^2}{4t}}.$$

For the bounds on $G_t^{\varepsilon,p}(y)$, one can keep the original path and derive crudely a Gaussian type estimate, with a time-increasing bound, as it is enough to obtain some $\theta > 0$ and some ω'' independant of $\varepsilon \in [0, \varepsilon_0]$. □

4.2.4 Linear stability

In this section, we will obtain, by using the pointwise bounds on Green functions derived in the previous subsection, results on the linear propagators, and this will, in particular, imply boundedness of the linear propagators in weighted spaces.

Fix $\chi : \mathbf{R}_+ \to \mathbf{R}_+$, smooth and nondecreasing such that $\chi(t) = 0$ for every $t \leqslant 1$ and $\chi(t) = 1$ for every $t \geqslant 2$.

The linear form $s_{\varepsilon,p}(t)$ is defined, for $t > 0$, by, for any $v_0 \in BUC^0(\mathbf{R})$

$$s_{\varepsilon,p}(t)(v_0) = \chi(t) \int_{\mathbf{R}} G^{\varepsilon,p}(t,y) v_0(y) dy,$$

and $\tilde{S}_\varepsilon(t)$ is defined, for $t > 0$, by, for every $v_0 \in BUC^0(\mathbf{R})$

$$\tilde{S}_\varepsilon(t)(v_0) = \chi(t) \int_{\mathbf{R}} \tilde{G}_t^\varepsilon(x,y) v_0(y) dy + (1 - \chi(t)) \int_{\mathbf{R}} G_t^\varepsilon(x,y) v_0(y) dy.$$

For $s_{\varepsilon,p}(t)$ one can easily obtain bounds of the form (for every $v_0 \in BUC^0(\mathbf{R})$)

$$|s_{\varepsilon,p}(t)(v_0)| \leqslant C e^{t\omega''} \|v_0\|_{L^\infty},$$

for some positive constants C, ω''. In particular, show that $t \mapsto s_{\varepsilon,p}(t)$ is well defined. However, on $t \mapsto \partial_t s_{\varepsilon,p}(t)$, sharper bounds are needed.

First, fix $\eta > 0$. We can derive the following bounds on the liner propagators $\mathcal{L}_\varepsilon$ from the pointwise Green functions derived earlier.

Proposition 23. *There exists $C > 0$ big enough such that the following holds*

For every ω', as well as α, and ε_0, all positive and small enough, for every $\varepsilon \in [0, \varepsilon_0]$, $v_0 \in BUC^0_{\varepsilon(\theta_0+\eta)}$

$$\begin{aligned}
&\|\tilde{S}_\varepsilon(t)(v_0)\|_{L^\infty_{\varepsilon(\theta_0+\eta)}} + \min(1, \sqrt{t}) \|\partial_x \tilde{S}_\varepsilon(t)(v_0)\|_{L^\infty_{\varepsilon(\theta_0+\eta)}} + |\partial_t s_{\varepsilon,p}(t)(v_0)| \\
&\leqslant C e^{-\varepsilon\tau(\varepsilon,\theta_0+\eta)t} \|v_0\|_{L^\infty_{\varepsilon(\theta_0+\eta)}}, \\
&\|\tilde{S}_\varepsilon(t)(v_0)\|_{L^\infty_{\varepsilon(\theta_0+\eta)}} + \min(1, \sqrt{t}) \|\partial_x \tilde{S}_\varepsilon(t)(v_0)\|_{L^\infty_{\varepsilon(\theta_0+\eta)}} + |\partial_t s_{\varepsilon,p}(t)(v_0)|
\end{aligned}$$

$$\leqslant Ce^{-\omega' t}\|v_0 e^{\alpha|\cdot|}\|_{L^\infty}.$$

Proof. The only point needed to be explained is how to bound

$$\int_{\mathbf{R}_+} \frac{e^{-\frac{((y-x)-(t\sigma_\varepsilon - tf'(\underline{u}_{+\infty})))^2}{4t}}}{\sqrt{t}} e^{-\varepsilon(\theta_0+\eta)y} dy,$$

given $\eta > 0$ and $x > 0$. Since

$$\begin{aligned}(y - x - t(\sigma_\varepsilon - f'(\underline{u}_{+\infty})))^2 + 4\varepsilon(\theta_0+\eta)ty =& (y - x - t(\sigma_\varepsilon - f'(\underline{u}_{+\infty}) - 2\varepsilon(\theta_0+\eta)))^2 \\ &+ t^2(2\varepsilon(\theta_0+\eta)(2\sigma_\varepsilon - 2f'(\underline{u}_{+\infty}) - 2\varepsilon(\theta_0+\eta))) \\ &+ 4\varepsilon(\theta_0+\eta)tx.\end{aligned}$$

Which gives

$$\begin{aligned}\int_{\mathbf{R}_+} \frac{e^{-\frac{((y-x)-(t\sigma_\varepsilon - tf'(\underline{u}_{+\infty})))^2}{4t}}}{\sqrt{t}} e^{-\varepsilon(\theta_0+\eta)y} dy &\leqslant \int_{\mathbf{R}_+} e^{-\frac{y^2}{4t}} \frac{e^{\varepsilon(\theta_0+\eta)(\sigma_\varepsilon - f'(\underline{u}_{+\infty}) - \varepsilon(\theta_0+\eta))t} e^{-\varepsilon(\theta_0+\eta)x}}{\sqrt{t}} dy, \\ &\leqslant Ce^{-\varepsilon(\tau(\varepsilon,\theta_0+\eta)+g'(\underline{u}_{+\infty}))t} e^{-\varepsilon(\theta_0+\eta)x}\end{aligned}$$

The rest of the bounds are similar are trivial. □

Now, we consider the critical case, that is, when $\theta = \theta_\varepsilon$ but some algebraic spatial localization is added to the exponential localization. Fix $\beta \geqslant 0$ to encode the latter.

Proposition 24. *There exists $C > 0$ big enough such that the following holds*

For every ε_0, ω' and α, all positive and small enough, for every $\varepsilon \in [0, \varepsilon_0]$, for every $v_0 \in BUC^0_{\theta_\varepsilon,\beta}$ and for every $t > 0$

$$\begin{aligned}&\|\tilde{S}_\varepsilon(t)(v_0)\|_{L^\infty_{\theta_\varepsilon}} + \min(1,\sqrt{t})\|\partial_x \tilde{S}_\varepsilon(t)(v_0)\|_{L^\infty_{\theta_\varepsilon}} + |\partial_t s_{\varepsilon,p}(t)(v_0)| \\ &\leqslant \frac{C}{(1+(\varepsilon t))^\beta}\|(1+(\varepsilon\cdot)_+)^\beta v_0\|_{L^\infty_{\theta_\varepsilon}}, \\ &\|\tilde{S}_\varepsilon(t)(v_0)\|_{L^\infty_{\theta_\varepsilon}} + \min(1,\sqrt{t})\|\partial_x \tilde{S}_\varepsilon(t)(v_0)\|_{L^\infty_{\theta_\varepsilon}} + |\partial_t s_{\varepsilon,p}(t)(v_0)| \\ &\leqslant Ce^{-\omega' t}\|v_0 e^{\alpha|\cdot|}\|_{L^\infty}.\end{aligned}$$

Proof. The only missing point is to obtain the polynomial decay in time from the algebraic extra localization when considering contributions from integrals over $y \in \mathbf{R}_+$ when $x > 0$. To do so, one may use that, for every decay $t > 0$ and $x > 0$, for any $a > 0$, there exist

$C > 0$ and $\omega' > 0$ such that for any $\varepsilon > 0$ small enough, for any $\alpha \geqslant a$,

$$\begin{aligned}\int_{\mathbf{R}_+} \frac{e^{-\frac{(y-x-t\alpha)^2}{4t}}}{\sqrt{t}(1+\varepsilon y)^\beta} dy \leqslant & Ce^{-\frac{t\alpha^2}{16}} + \int_{x+\frac{\alpha t}{2}}^{x+\frac{3\alpha t}{2}} \frac{e^{-\frac{(y-x-t\alpha)^2}{4t}}}{\sqrt{t}(1+\varepsilon y)^\beta} dy, \\ \leqslant & Ce^{-\omega' t} + \frac{C}{(1+\varepsilon t)^\beta}.\end{aligned}$$

□

4.2.5 Nonlinear stability

Now, to obtain the nonlinear stability, we are just missing a last key ingredient, a way to obtain sharper bounds on the spatial derivative of the solution. In order to do so, we will use a maximum principle. We start by discussing the case of an algebraic decay. Fix $\beta > 1$. We need to fix some $\chi : \mathbf{R} \to \mathbf{R}$ such that $\chi(x) = 0$ for every $x < 0$ small enough, and $\chi(x) = x$ for every $x > 0$ big enough, and χ is smooth and nondecreasing. Also fix $0 < \alpha \leqslant \frac{\min(\sigma_0 - f'(\underline{u}_{+\infty}), f'(\underline{u}_{-\infty}) - \sigma)}{2}$. In order to adapt the strategy from [BR23] to that case, we consider the following weight (defined for $t, x \geqslant 0$), where $\mu > 0$ is small enough. Define the weight $A_{\varepsilon,\beta}$ on $\mathbf{R}_+ \times \mathbf{R}^*$ as

$$A_{\varepsilon,\beta}(t,x) := \frac{\exp\left(\int_t^{+\infty} \varepsilon(1+(\varepsilon s))^{-\beta} ds\right) (1+(\varepsilon(\chi(x)+\mu t)))^\beta e^{\varepsilon\theta_\varepsilon \chi(x)}}{\varepsilon + e^{-\alpha|x|}}.$$

We will focus on this case here. The case of the exponential decay, that is with $\varepsilon(\theta_0+\eta)$, is similar to the work in [BR23].

We have the following on $\mathbf{R}_+ \times \mathbf{R}^*$

$$\begin{aligned}
&\frac{\partial_t A_{\varepsilon,\beta}}{A_{\varepsilon,\beta}}(t,x) = \left(-\frac{\varepsilon}{(1+\varepsilon t)^\beta} + \frac{\varepsilon\mu\beta}{(1+\varepsilon(\chi(x)+\mu t))}\right), \\
&\frac{\partial_x A_{\varepsilon,\beta}}{A_{\varepsilon,\beta}}(t,x) = \frac{\varepsilon\beta}{1+\varepsilon(\chi(x)+\mu t)} + \varepsilon\theta_\varepsilon\chi'(x) + \frac{\mathrm{sgn}(x)\alpha e^{-\alpha|x|}}{\varepsilon + e^{-\alpha|x|}}, \\
&\frac{\partial_x^2 A_{\varepsilon,\beta}}{A_{\varepsilon,\beta}}(t,x) = \left(\frac{\varepsilon\beta}{1+\varepsilon(\chi(x)+\mu t)} + \varepsilon\theta_\varepsilon\chi'(x) + \frac{\mathrm{sgn}(x)\alpha e^{-\alpha|x|}}{\varepsilon + e^{-\alpha|x|}}\right)^2 - \frac{\varepsilon^2\beta}{(1+\varepsilon(\chi(x)+\mu t))^2} \\
&- \frac{\alpha^2 e^{-2\alpha|x|}\varepsilon}{(\varepsilon + e^{-\alpha|x|})^2}.
\end{aligned}$$

Hence, if the solution is defined on $[0,T]$, then, on $[0,T] \times \mathbf{R}$

$$\partial_t(A_{\varepsilon,\beta}\partial_x v) - \partial_x^2(A_{\varepsilon,\beta}\partial_x v) - \left(\sigma_\varepsilon - \psi'(t) - f'(\underline{U}_\varepsilon + v) - 2\frac{\partial_x A_{\varepsilon,\beta}}{A_{\varepsilon,\beta}}\right)(\partial_x(A_{\varepsilon,\beta}\partial_x v))$$

$$= \left(\frac{\partial_t A_{\varepsilon,\beta}}{A_{\varepsilon,\beta}} - (\sigma_\varepsilon - \psi'(t) - f'(\underline{U}_\varepsilon + v)) \frac{\partial_x A_{\varepsilon,\beta}}{A_{\varepsilon,\beta}} - f''(\underline{U}_\varepsilon + v)(\underline{U}'_\varepsilon + \partial_x v) \right.$$
$$+ \varepsilon g'(\underline{U}_\varepsilon + v) - f''(\underline{U}_\varepsilon + v)\partial_x v + 2 \left(\frac{\partial_x A_{\varepsilon,\beta}}{A_{\varepsilon,\beta}} \right)^2$$
$$\left. - \frac{\partial_x^2 A_{\varepsilon,\beta}}{A_{\varepsilon,\beta}} \right) A_{\varepsilon,\beta} \partial_x v + \varepsilon (g'(\underline{U}_\varepsilon + v) - g'(\underline{U}_\varepsilon)) \underline{U}'_\varepsilon A_{\varepsilon,\beta}$$
$$+ (f''(\underline{U}_\varepsilon) - f''(\underline{U}_\varepsilon + v)) \underline{U}'^2_\varepsilon A_{\varepsilon,\beta} + (f'(\underline{U}_\varepsilon) - f'(\underline{U}_\varepsilon + v) - \psi') \underline{U}''_\varepsilon A_{\varepsilon,\beta}$$

In order to apply the maximum principle from Chapter 3, we observe that for any $K > 0$, when ε_0, α, δ and $\tilde{\delta}$ are sufficiently small and x_* is sufficiently large, there exists $\gamma > 0$ such that the right-hand side of the above equation multiplied by $\operatorname{sgn}(A_{\varepsilon,\beta}\partial_x v)$ is bounded above by

$$-\gamma \tilde{\delta} \left(\frac{\varepsilon}{(1+\varepsilon t)^\beta} + \alpha\, e^{-\alpha\, x} \right)$$

at any (t,x) such that $|x| \geqslant x_*$, and

$$|\psi'(t)| + c|v(t,x)| + e^{\varepsilon \theta_\varepsilon\, x_+} |\partial_x v(t,x)| \leqslant \frac{K\,\delta}{(1+\varepsilon\, t)^\beta}\,, \qquad |A_{\varepsilon,\beta}(t,x)\partial_x v(t,x)| \geqslant \tilde{\delta}\,.$$

With this in hands the proof is then achieved essentially as the one of the main result in Chapter 3. See the end of the next section for some omitted details about time integrals.

In the exponentially decaying case (when the weight is associated to $\theta_0 + \eta$, and not the critical θ_ε), we consider a weight of the form

$$\tilde{A}_{\varepsilon,\eta}(t,x) := \frac{\exp\left(\int_t^{+\infty} \varepsilon e^{-\varepsilon\tau(\varepsilon,\theta_0+\eta)s} ds \right) e^{\varepsilon(\theta_0+\eta)\chi(x)} e^{\varepsilon\tau(\varepsilon,\theta_0+\eta)t}}{\varepsilon + e^{-\alpha|x|}}.$$

The strategy is similar.

Note that for the case of a weight of the $e^{\varepsilon(\theta_0+\eta)x_+}$, the bounds obtained are of the following form. For some $\gamma > 0$ small enough, for every $K > 0$, when ε_0, α, δ and $\tilde{\delta}$ are sufficiently small and x_* is sufficiently large, the right-hand side of the above equation multiplied by $\operatorname{sgn}(\tilde{A}_{\varepsilon,\eta}\partial_x v)$ is bounded by above by

$$-\gamma \tilde{\delta} \left(\varepsilon e^{-\varepsilon\tau(\varepsilon,\theta_0+\eta)t} + \alpha e^{-\alpha|x|} \right)$$

at any (t,x) such that $|x| \geqslant x_*$, and

$$|\psi'(t)| + e^{\varepsilon(\theta_0+\eta)x_+} |v(t,x)| + e^{\varepsilon(\theta_0+\eta)x_+} |\partial_x v(t,x)| \leqslant K\,\delta e^{\varepsilon\tau(\varepsilon,\theta_0+\eta)t}\,, \quad |\tilde{A}_{\varepsilon,}(t,x)\partial_x v(t,x)| \geqslant \tilde{\delta}\,.$$

Which also allows to conclude in that case.

4.3 Some smooth traveling waves

Fix real numbers $\underline{u}_{-\infty} < \underline{u}_{+\infty}$.

The profile of the traveling wave solution to the hyperbolic limiting problem solves

$$(f'(\underline{U}_0) - \sigma)\underline{U}_0' = g(\underline{U}_0),$$

with the assumptions that $\underline{U}_0$ is valued in $(\underline{u}_{-\infty}, \underline{u}_{+\infty})$, converges towards $\underline{u}_{\pm\infty}$ at $\pm\infty$, where $\underline{u}_{\pm\infty}$ are zeros of g, and such that $f' - \sigma$ and g do not vanish on $(\underline{u}_{-\infty}, \underline{u}_{+\infty})$. Furthermore, we assume that $\underline{u}_{\pm\infty}$ are not zeros of g'. These assumptions imply that g and $f' - \sigma$ are, if $\underline{u}_{-\infty} < \underline{u}_{+\infty}$ (respectively $\underline{u}_{-\infty} > \underline{u}_{+\infty}$), of the same (fixed) sign on $(\underline{u}_{-\infty}, \underline{u}_{+\infty})$ (respectively of opposite signs).

From now on, we will assume that $\underline{u}_{-\infty} < \underline{u}_{+\infty}$. Applying the transformation $u \mapsto -u$ allows one to go back to the other case, and it does not change the stability properties of the wave.

Also, $g'(\underline{u}_{+\infty})$ (respectively $g'(\underline{u}_{-\infty})$) is of the same sign as $-g$ (respectively of the same sign as g) on $(\underline{u}_{-\infty}, \underline{u}_{+\infty})$. We assume from now on that $g'(\underline{u}_{-\infty}) < 0 < g'(\underline{u}_{+\infty})$. The application transforming a function u of the variables (t, x) to $\tilde{u} : (t, x) \mapsto -u(t, -x)$ enables us to go from one case to the other. Hence, g and $f' - \sigma$ take only negative values on $(\underline{u}_{-\infty}, \underline{u}_{+\infty})$.

Under such assumptions, there exist a wave profile solution to $(f'(\underline{U}_0) - \sigma)\underline{U}_0' = g(\underline{U}_0)$, converging towards $\underline{u}_{\pm}$ at $\mathbf{R}_{\pm}$. First, we will show that, in that context, and for $\varepsilon > 0$ small enough, there exists a wave profile $\underline{U}_\varepsilon$ solution to $\varepsilon\underline{U}_\varepsilon'' = (f'(\underline{U}_\varepsilon) - \sigma)\underline{U}_\varepsilon' - g(\underline{U}_\varepsilon)$, converging towards $\underline{u}_{\pm}$ at $\pm\infty$. Furthermore, we will obtain, in particular, that, up to some spatial translation, $\underline{U}_\varepsilon$ converges pointwise towards $\underline{U}_0$.

We consider $(\theta_\varepsilon)_{0 \leqslant \varepsilon \leqslant \varepsilon_0}$ defined by $\theta_0 := \frac{g'(\underline{u}_{+\infty})}{\sigma - f'(\underline{u}_{+\infty})} > 0$, and

$\theta_\varepsilon := \frac{\sigma - f'(\underline{u}_{+\infty}) - \sqrt{(\sigma - f'(\underline{u}_{+\infty}))^2 - 4\varepsilon g'(\underline{u}_{+\infty})}}{2\varepsilon}$, and so, we have that $\theta_\varepsilon = \theta_0 + \mathcal{O}_\varepsilon(\varepsilon)$. As in the previous section, we note $\tau(\varepsilon, \theta) := \varepsilon\theta^2 + (f'(\underline{u}_{+\infty}) - \sigma)\theta + g'(\underline{u}_{+\infty})$.

We fix, for a given $\eta > 0$ some function $\zeta : \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}_+$ such that if $\tau(\varepsilon, \theta) \leqslant g'(\underline{u}_{-\infty}) - \eta$, then $\zeta(t, \varepsilon, \theta) = 1$, and $\zeta(t, \varepsilon, \theta) = t$ otherwise. It is there to take into account some issues to bounds an integral of the form

$$\int_0^t e^{-\alpha(t-s)} e^{\alpha s} ds,$$

compared to

$$\int_0^t e^{-(\alpha+\eta)(t-s)} e^{\alpha s} ds.$$

Theorem 17. *Let f and g, smooth functions from $\mathbf{R}$ to $\mathbf{R}$, $\sigma \in \mathbf{R}$, $\theta_{max} >$, $\beta_{max} > 0$*

and enforce the previous assumptions on f, g and $(\underline{u}_{-\infty}, \underline{u}_{+\infty}) \in \mathbf{R}^2$.

There exist $\varepsilon_0 > 0$, $(\underline{U}_\varepsilon)_{0 \leqslant \varepsilon \leqslant \varepsilon_0}$, $C > 0$, $\delta > 0$, such that, for every $\varepsilon \in (0, \varepsilon_0]$, for every $\theta_{max} \geqslant \theta \geqslant \theta_\varepsilon$, and for every initial data $v_0 \in BUC^1_\theta(\mathbf{R})$ such that $\|v_0\|_{W^{1,\infty}_\theta} \leqslant \delta$, the solution w to 4.1.1 generated by $\underline{U}_\varepsilon + v_0$ satisfies

$$\|w(t,\cdot) - \underline{U}_\varepsilon\|_{L^\infty_\theta} \leqslant C e^{t \max(g'(\underline{u}-\infty), \tau(\varepsilon,\theta))} \|v_0\|_{L^\infty_\theta},$$
$$\|\partial_x w(t,\cdot) - \underline{U}'_\varepsilon\|_{L^\infty_\theta} \leqslant C\zeta(t,\theta,\varepsilon) e^{t \max(g'(\underline{u}-\infty), \tau(\varepsilon,\theta))} \|v_0\|_{W^{1,\infty}_\theta}.$$

Furthermore, if the initial data v_0 satisfies also $v_0 \in BUC^1_{\theta_\varepsilon,\beta}(\mathbf{R})$, and $\|v_0\|_{W^{1,\infty}_{\theta_\varepsilon,\beta}} \leqslant \delta$, we have that w satisfies

$$\|w(t,\cdot) - \underline{U}_\varepsilon\|_{W^{1,\infty}_{\theta_\varepsilon}} \leqslant \frac{C}{1+t^\beta} \|v_0\|_{W^{1,\infty}_{\theta_\varepsilon,\beta}}.$$

Let us emphasize here that the weights considered are actually strong enough to get rid of the zero mode. As a consequence, we obtain an asymptotic nonlinear stability result, not an orbital one.

4.3.1 Existence of the parabolic approximations

We are looking for wave profiles solutions to

$$(f'(\underline{U}_\varepsilon) - \sigma)\underline{U}'_\varepsilon = g(\underline{U}_\varepsilon) + \varepsilon \underline{U}''_\varepsilon$$

which approximate the wave profile $\underline{U}_0$ above in the inviscid limit.

In fact, we will ensure that by building $(\underline{U}_\varepsilon)_\varepsilon$ as a family of solution to order one ODE associated with a family of nonlinearities of the form $(F_\varepsilon)_\varepsilon$, such that F_ε converges, in some sense to be precised after, to $\dfrac{g}{f'-\sigma}$ (which will be noted F, the dependance on the given speed σ being implicit) when ε goes to 0. As the limiting front only exist if $\mathrm{sgn}(F_{(\underline{u}_{-\infty},\underline{u}_{+\infty})}) = \mathrm{sgn}(\underline{u}_{+\infty} - \underline{u}_{-\infty})$, we limit ourselves to that case, and, furthermore, we have that we are in that case, in our framework, if and only if $f' - \sigma$ is of the opposite sign of the one of $g'(\underline{u}_{+\infty})$ (or, equivalently, of the same sign as the one of $g'(\underline{u}_{-\infty})$).

We want to build some function defined on $[\underline{u}_{-\infty}, \underline{u}_{+\infty}]$, R_ε of size ε in L^∞ and of class C^1 on $(\underline{u}_{-\infty}, \underline{u}_{+\infty})$ such that solutions to

$$y' = F(y)(1 + R_\varepsilon(y))$$

with values in $(\underline{u}_{-\infty}, \underline{u}_{+\infty})$, are actually solutions to

$$\varepsilon y'' = (f'(y) - \sigma)y' - g(y).$$

Which is true if and only if the following holds on $(\underline{u}_{-\infty}, \underline{u}_{+\infty})$,

$$\varepsilon F(1+R_\varepsilon)(F'(1+R_\varepsilon)+FR'_\varepsilon) = (f'-\sigma)F(R_\varepsilon+1) - g.$$

This is equivalent, for ε small enough, to the following ODE

$$R'_\varepsilon - \frac{(f'-\sigma)^2}{\varepsilon g}R_\varepsilon = -\frac{(f'-\sigma)^2}{\varepsilon g(1+R_\varepsilon)}R_\varepsilon^2 - \frac{g'}{g}(R_\varepsilon+1) + \frac{f''}{f'-\sigma}(1+R_\varepsilon)$$

From which we can obtain a Duhamel formulation. From now on, we assume that $\underline{u}_{+\infty} > \underline{u}_{-\infty}$, $g'(\underline{u}_{+\infty}) > 0 > g'(\underline{u}_{-\infty})$ (and, hence, $f'-\sigma < 0$ on $[\underline{u}_{-\infty}, \underline{u}_{+\infty}]$).

To obtain the bounds necessary to apply a fixed point argument, we will need to use the fact in the exponential, one has a term bounded by above by $v \mapsto -\dfrac{\delta}{\varepsilon(\underline{u}_{+\infty}-v)(v-\underline{u}_{-\infty})}$, allowing to tame the singularity.

Proposition 25. *With the assumption made before, we have that there exist $\varepsilon_0 > 0$, $C > 0$ and $\theta > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exists a unique smooth solution of $\varepsilon \underline{U}''_\varepsilon + (\sigma - f'(\underline{U}_\varepsilon))\underline{U}'_\varepsilon + g(\underline{U}_\varepsilon) = 0$ such that $\underline{U}_\varepsilon(0) = \underline{U}_0(0)$, that converges to $\underline{u}_{\pm\infty}$ at $\pm\infty$.*

Finally, it satisfies

$$\begin{aligned}
&|\underline{U}_\varepsilon(x) - \underline{u}_{+\infty}| \leqslant Ce^{-\theta x} \text{ on } \mathbf{R}_+,\\
&|\underline{U}_\varepsilon(x) - \underline{u}_{-\infty}| \leqslant Ce^{\theta x} \text{ on } \mathbf{R}_-,\\
&|\underline{U}'_\varepsilon(x)| \leqslant Ce^{-\theta|x|} \text{ on } \mathbf{R},\\
&|\underline{U}''_\varepsilon(x)| \leqslant Ce^{-\theta|x|} \text{ on } \mathbf{R}.
\end{aligned}$$

Proof. In order to prove the existence the wave profile, we will begin by showing the existence of R_ε, as announced before. To do so, we will apply a fixed point argument to the function (which needs to be shown to be well defined)

$$\begin{aligned}
T_\varepsilon : C^0([\underline{u}_{-\infty}, \underline{u}_{+\infty}], \mathbf{R}) &\to C^0([\underline{u}_{-\infty}, \underline{u}_{+\infty}], \mathbf{R})\\
R &\mapsto \left(v \mapsto \int_{\underline{u}_{-\infty}}^v \exp\left(\int_w^v \frac{(f'(s)-\sigma)^2}{\varepsilon g(s)}ds\right)\left(\frac{f''(w)}{f'(w)-\sigma}(1+R(w))\right.\right.\\
&\qquad \left.\left. -\frac{g'(w)(1+R(w))}{g(w)} - \frac{(f'(w)-\sigma)^2}{\varepsilon g(w)(1+R(w))}R(w)^2\right) dw\right).
\end{aligned}$$

To show that T_ε is well defined, one needs some identities. For every $v \in (\underline{u}_{-\infty}, \underline{u}_{+\infty})$,

$\varepsilon > 0$ and $\alpha > 0$,

$$\int_{\underline{u}_{-\infty}}^{v} \exp\left(\int_{w}^{v} \frac{-\alpha}{\varepsilon(s-\underline{u}_{-\infty})} ds\right) \frac{1}{\varepsilon(w-\underline{u}_{-\infty})} ds = \frac{1}{\alpha},$$
$$\int_{\underline{u}_{-\infty}}^{v} \exp\left(\int_{w}^{v} \frac{-\alpha}{\varepsilon(\underline{u}_{+\infty}-s)} ds\right) \frac{1}{\varepsilon(\underline{u}_{+\infty}-w)} ds = \frac{1}{\alpha}\left(1 - \left(\frac{v-\underline{u}_{-\infty}}{\underline{u}_{+\infty}-\underline{u}_{-\infty}}\right)^{\frac{\alpha}{\varepsilon}}\right),$$

and also, for every $R \in C^0([\underline{u}_{-\infty}, \underline{u}_{+\infty}], \mathbf{R})$,

$$\lim_{v \to \underline{u}_{-\infty}} \int_{\underline{u}_{-\infty}}^{v} \exp\left(\int_{w}^{v} \frac{-\alpha}{\varepsilon(s-\underline{u}_{-\infty})} ds\right) \frac{R(w)}{\varepsilon(w-\underline{u}_{-\infty})} ds = \frac{R(\underline{u}_{-\infty})}{\alpha},$$
$$\lim_{v \to \underline{u}_{+\infty}} \int_{\underline{u}_{-\infty}}^{v} \exp\left(\int_{w}^{v} \frac{-\alpha}{\varepsilon(\underline{u}_{+\infty}-s)} ds\right) \frac{R(w)}{\varepsilon(\underline{u}_{+\infty}-w)} dw = \frac{R(\underline{u}_{+\infty})}{\alpha}.$$

They allow us to show of the form, that T_ε is well defined. It also allows us to show that for some positive constants ε_0, C_1 and C_2, for every $\varepsilon \in (0, \varepsilon_0]$, it maps the ball of center 0 and radius $C_1\varepsilon$ into itself, and more precisely for every $\varepsilon > 0$ small enough and every $R \in C^0([\underline{u}_{-\infty}, \underline{u}_{+\infty}], \mathbf{R})$ such that $\|R\|_{L^\infty} \leqslant C_1\varepsilon$

$$\|T_\varepsilon(R)\|_{L^\infty} \leqslant \frac{C_1}{2}(\varepsilon + \|R\|_{L^\infty}^2),$$

and, restricted to that ball, has a Lipschitz constant bounded by $C_2\varepsilon$, more precisely we have that there exists $\tilde{C} > 0$ such that for every $\varepsilon > 0$ small enough and every $(R_1, R_2) \in C^0([\underline{u}_{-\infty}, \underline{u}_{+\infty}], \mathbf{R})$ such that $\max(\|R_1\|_{L^\infty}, \|R_2\|_{L^\infty}) \leqslant C_1\varepsilon$

$$\|T_\varepsilon(R_1) - T_\varepsilon(R_2)\|_{L^\infty} \leqslant \tilde{C}\left(\varepsilon + \|R_1\|_{L^\infty} + \|R_2\|_{L^\infty}\right)\|R_1 - R_2\|_{L^\infty}.$$

Now, $\underline{U}_\varepsilon$, the solution to :

$$\underline{U}_\varepsilon' = F(\underline{U}_\varepsilon)(1 + R_\varepsilon(\underline{U}_\varepsilon)) \text{ on } \mathbf{R},$$
$$\underline{U}_\varepsilon(0) = \underline{U}_0(0).$$

The bounds are obtained directly that for some constants $C, \theta > 0$, for all $\varepsilon \in (0, \varepsilon_0]$, by noticing that for $x > 0$ big enough,

$$(\underline{u}_{+\infty} - \underline{U}_\varepsilon)' \leqslant -\theta\left(\underline{u}_{+\infty} - \underline{U}_\varepsilon\right),$$

Hence, for all nonnegative x,

$$|\underline{U}_\varepsilon(x) - \underline{u}_{+\infty}| \leqslant Ce^{-\theta x},$$
$$|\underline{U}_\varepsilon(-x) - \underline{u}_{-\infty}| \leqslant Ce^{-\theta x}.$$

From those, it is easy to derive from these identities that by eventually taking C bigger and γ smaller, but still positive, that for any x nonnegative

$$|\underline{U}'_\varepsilon(x)| + |\underline{U}'_\varepsilon(-x)| + |\underline{U}''_\varepsilon(x)| + |\underline{U}''_\varepsilon(-x)| \leqslant Ce^{-\theta x}.$$

In fact, it is proved for $\underline{U}'_\varepsilon$ by using that, for $\varepsilon > 0$ small enough, for every $v \in [\underline{u}_{-\infty}, \underline{u}_{+\infty}]$,

$$|F(v)(1 + R_\varepsilon(v)| \leqslant 2\|F'\|_\infty \min(\underline{u}_{+\infty} - v, v - \underline{u}_{-\infty}).$$

For $\underline{U}''_\varepsilon$, as

$$\underline{U}''_\varepsilon = F'(\underline{U}_\varepsilon)(1 + R_\varepsilon(\underline{U}_\varepsilon))\underline{U}'_\varepsilon + F(\underline{U}_\varepsilon)R'_\varepsilon(\underline{U}_\varepsilon)\underline{U}'_\varepsilon,$$

it is enough to use that $F'(1 + R_\varepsilon) + FR'_\varepsilon$ is bounded on $(\underline{u}_{-\infty}, \underline{u}_{+\infty})$.

□

Let us stress out that, despite the integral form of T_ε, due to singularity at the endpoint $\underline{u}_{-\infty}$, one may have $T_\varepsilon(R)_\varepsilon(\underline{u}_{-\infty}) \neq 0$ for some $R \in C^0([\underline{u}_{-\infty}, \underline{u}_{+\infty}], \mathbf{R})$, which is crucial as, we knew before the start that if R_ε exists, then $R_\varepsilon(\underline{u}_{-\infty}) \neq 0$.

4.3.2 Maximum principle and stability of evolution systems

To bound the linear evolution systems, we will rely on the use of maximum principles, and in particular, the weights created to do so will need to take advantage of the convective term.

Proposition 26. *Let $a \in C^0([0,T], BUC^1(\mathbf{R}))$ and $h \in C^0([0,T] \times \mathbf{R} \times \mathbf{R})$, ($h$ being lipschitzian in the third coordinate) and $\delta > 0$ such that $\mathbb{1}_{(\delta,+\infty)}(v)h(\cdot,\cdot,v) \leqslant 0$ for all $v \in \mathbf{R}$ (independently of the other variables). We have that any mild solution in $BUC^0(\mathbf{R})$ to*

$$\partial_t v(\cdot,\cdot) + a(\cdot,\cdot)\partial_x v(\cdot,\cdot) = h(\cdot,\cdot,v(\cdot,\cdot)) + \partial_x^2 v(\cdot,\cdot)$$

such that $v(0,\cdot) \leqslant \delta$ on $\mathbf{R}$, and $v(0,\cdot) \in BUC^0(\mathbf{R})$ satisfies $\|v(t,\cdot)_+\|_{L^\infty} \leqslant \delta$ on $[0,T]$.

Proof. We fix such a, h, but assume them to be smooth for the moment, and such that $\mathbb{1}_{(\delta,+\infty)}(v)h(\cdot,\cdot,v) \leqslant -\gamma \mathbb{1}_{(\delta,+\infty)}(v)$ for some fixed positive γ, independent on v, t and x. Through a perturbation argument, we will obtain the general case.

In that case, we first assume that $v(t,\cdot)$ goes to 0 at $x \to \pm\infty$ for every $t \in [0,T]$ and apply the Gronwall lemma to $t \mapsto \int_{\mathbf{R}} (v(t)-\delta)_+$, obtaining that $v \leqslant \delta$ on $[0,T]\times\mathbf{R}$.

In order to relax the vanishing condition at $\pm\infty$, we replace v by $w_\theta(t,x) = v(t,x)e^{-\theta\sqrt{1+x^2}}$. By taking any θ sufficiently small, we can apply the previous result, using that $\gamma > 0$. Letting θ go to 0 gives the result in that case.

The general case is then obtained by regularization and then taking the limit. In particular, one needs to add a small perturbation to the regularization of h. It is done in order to ensure that the $\tilde{h}$ obtained satisfies that there exists some $\gamma > 0$ positive such that, for $v \geqslant \delta$,

$$h(\cdot,\cdot,v)\mathbb{1}_{(\delta,+\infty)}(v) \leqslant -\gamma \mathbb{1}_{(\delta,+\infty)}(v).$$

The details are classical and thus omitted here. □

Now, we turn to the linear estimates, applying the maximum principle we proved earlier 26 to the linear problem.

Proposition 27. *Let m and M be two positive constants and $\phi \in L^1 \cap \mathcal{C}^1(\mathbf{R})$ such that $0 \leqslant \phi \leqslant M$ and $|\phi'| \leqslant M\,\phi$.*
There exist positive ε_0 and C such that, for any $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $b \in \mathbf{R}$, $T > 0$, $\alpha \in \mathcal{C}^1 \cap W^{1,\infty}([0,T]\times\mathbf{R})$ and $\beta \in \mathcal{C}^0([0,T]\times\mathbf{R})$ satisfying

$$m \leqslant |\alpha| \leqslant M\,, \qquad\qquad \beta \leqslant M\,\phi - b\,,$$

any solution v to

$$\partial_t v - \varepsilon\partial_x^2 v + \alpha\partial_x v = \beta v$$

satisfies

$$\|v(t,\cdot)\|_{L^\infty} \leqslant Ce^{-bt}\|v(0,\cdot)\|_{L^\infty}.$$

Proof. It is sufficient to obtain the bound, for every $t \in [0,T]$ and $x \in \mathbf{R}$

$$v(t,x) \leqslant Ce^{-bt}\|v(0,\cdot)\|_{L^\infty},$$

$w := ve^{\kappa}$ that satisfies a modified equation of the form as the linearity of the operator gives the rest of the bound.

The idea is to build a new unknown

$$\partial_t w + \tilde{\alpha}\partial_x w = \varepsilon\partial_x^2 w + \tilde{\beta} w$$

looking for $\tilde{\beta}$ nonpositive on $[0,T]\times\mathbf{R}$

$$\partial_t w - \varepsilon\partial_x^2 w + (\alpha + 2\varepsilon\partial_x\kappa)\partial_x w = (\beta + \partial_t\kappa + (\alpha+2\varepsilon\partial_x\kappa)\partial_x\kappa - \varepsilon(\partial_x^2\kappa + (\partial_x\kappa)^2)w.$$

Thus, we want to ensure that

$$\beta + \partial_t\kappa + (\alpha + 2\varepsilon\partial_x\kappa)\partial_x\kappa - \varepsilon(\partial_x^2\kappa + (\partial_x\kappa)^2) \leqslant 0$$

on $\mathbf{R}$.

It is straighforward to check that $\kappa(t,x) := bt + L\int_0^x \phi$ satisfies the above inequality provided that ε_0 is small enough, L has the sign of α and $|L|$ is large enough (independently of ε).

Since ϕ is integrable, the proof is then concluded by applying Lemma 26. □

Now, we can obtain results that apply to evolution systems, and in particular to obtain bounds in weighted spaces. We will place ourselves in weighted spaces of the form $BUC^0_\alpha(\mathbf{R})$.

Let us note that we only consider weights on $\mathbf{R}_+$.

Proposition 28. *We consider f and g as before, as well as ε_0 small enough such as in the proposition 25, and some fixed $\beta_{max} > 0$ and θ_{max}. There exist $\varepsilon_1 \in (0,\varepsilon_0]$, $\delta > 0$ and $C > 1$, for every $0 < \varepsilon < \varepsilon_1$ and every $T > 0$ and every $v \in C^0([0,T], BUC^1(\mathbf{R}))$ such that $\|v\|_{L^\infty([0,T],BUC^0_\theta(\mathbf{R}))} \leqslant \delta$ we have the solutions w to*

$$\partial_t w + (f'(\underline{U}_\varepsilon + v) - \sigma)\partial_x(w) = \varepsilon\partial_x^2 w + g'(\underline{U}_\varepsilon)w + f''(\underline{U}_\varepsilon)w$$

satisfy

$$\|w(t)\|_{L^\infty_{\theta,+}} \leqslant C\|w(0)\|_{L^\infty_{\theta,+}} \exp\left(\int_0^t \max\left(g'(\underline{u}_{-\infty}), g'(\underline{u}_{+\infty}) + \theta((f'(\underline{u}_{+\infty}) - \sigma)) + \varepsilon\theta^2\right)\right)$$

for every $\theta_{\max} \geqslant \theta \geqslant 0$.

Furthermore, we also have that if $\|w(t,.)\|_{L^\infty_\theta} \leqslant \delta$ for every $\beta_0 > \beta > 0$ and $\theta = \theta_\varepsilon$ that

$$\|(1+(x)_+^\beta)w(t,\cdot)\|_{L^\infty_\theta} + (1+t^\beta)\|w(t,\cdot)\|_{L^\infty_\theta} \leqslant C\|(1+(x)_+^\beta)w(0,\cdot)\|_{L^\infty_\theta}$$

Proof. Again we want to design a suitable space-time-dependent weight ρ so as to replace

w with $\tilde{w}_\rho$. Then $\tilde{w}_\rho$ satisfies

$$\partial_t\tilde{w}_\rho + \left(f'(\underline{U}_\varepsilon + v) - \sigma + 2\varepsilon\frac{\partial_x\rho}{\rho}\right)\partial_x\tilde{w}_\rho = \varepsilon\partial_x^2\tilde{w}_\rho + \left(g'(\underline{U}_\varepsilon) + (f'(\underline{U}_\varepsilon + v) - \sigma)\frac{\partial_x\rho}{\rho} - \varepsilon\frac{\partial_x^2\rho}{\rho} + 2\varepsilon\frac{(\partial_x\rho)^2}{(\rho)^2} + \partial_t\rho + f''(\underline{U}_\varepsilon)\underline{U}'_\varepsilon\right)\tilde{w}_\rho$$

To derive the first bound, we choose the weight of the form

$$\rho(t,x) := e^{\theta\chi(x)},$$

where χ is a smooth nondecreasing function which is constant equal to L on $(-\infty, L-1]$ and, for every $x \in [L+1, +\infty)$, $\chi(x) = x$, $\|\chi'\|_{L^\infty} \leqslant 10$ and $\|\chi''\|_{L^\infty} \leqslant 10$, where $L > 10$ is a constant. Thus

$$\partial_t\tilde{w}_\rho + (f'(\underline{U}_\varepsilon + v) - \sigma + 2\varepsilon\theta)\partial_x\tilde{w}_\rho - \varepsilon\partial_x^2\tilde{w}_\rho = (g'(\underline{U}_\varepsilon) + (f'(\underline{U}_\varepsilon + v) - \sigma)\theta\chi'(x) - \varepsilon(\theta\chi'(x))^2 - \varepsilon\theta\chi''(x) + 2\varepsilon(\theta\chi'(x))^2 + f''(\underline{U}_\varepsilon)\underline{U}_{\varepsilon'})\,\tilde{w}_\rho.$$

Hence the result in that case, as the crucial term $g'(\underline{U}_\varepsilon) + (f'(\underline{U}_\varepsilon + v) - \sigma)\theta\chi'(x) - \varepsilon(\theta\chi'(x))^2 - \varepsilon\theta\chi''(x) + 2\varepsilon(\theta\chi'(x))^2 + f''(\underline{U}_\varepsilon)\underline{U}_{\varepsilon'}$ is equal, on $\mathbf{R}_+$, to a term exponentially localized plus $g'(\underline{u}_{+\infty}) + (f'(\underline{u}_{+\infty}) - \sigma)\theta + \varepsilon\theta^2$, and, on $\mathbf{R}_-$, is equal to $g'(\underline{u}_{-\infty})$ plus an exponentially localized term, and one can thus directly apply to that case the previous lemma.

Now, we assume that we are in the case associated to $\theta = \theta_\varepsilon$.

For a weight of the form

$$\rho(t,x) := (ct + \chi(x))^\beta e^{\theta_\varepsilon\chi(x)}.$$

Thus

$$\begin{aligned}
\partial_t\rho(t,x) =& c\beta\frac{\rho(t,x)}{ct+\chi(x)},\\
\partial_x\rho(t,x) =& \beta\chi'(x)\frac{\rho(t,x)}{ct+\chi(x)} + \theta_\varepsilon\chi'(x)\rho(t,x),\\
\partial_x^2\rho(t,x) =& \theta_\varepsilon\chi''(x)\rho(t,x) + \left(\theta_\varepsilon\chi'(x) + \frac{\beta\chi'(x)}{ct+\chi(x)}\right)^2\rho(t,x)\\
&+ \frac{\beta\chi''(x)}{ct+\chi(x)}\rho(t,x) - \frac{\beta\chi'(x)^2}{(ct+\chi(x))^2}\rho(t,x).
\end{aligned}$$

Therefore

$$
\begin{aligned}
&g'(\underline{U}_\varepsilon) + (f'(\underline{U}_\varepsilon + v) - \sigma)\left(\theta_\varepsilon\chi'(x) + \frac{\beta\chi'(x)}{\chi(x)+ct}\right) + \frac{c\beta\chi'(x)}{\chi(x)+ct} \\
&+ \varepsilon\left(\theta_\varepsilon\chi'(x) + \frac{\beta\chi'(x)}{\chi(x)+ct}\right)^2 + \frac{\varepsilon\chi''(x)\beta}{\chi(x)+ct} + \varepsilon\theta_\varepsilon\chi''(x) + \varepsilon\beta\frac{\chi'(x)^2}{(\chi(x)+ct)^2} + f''(\underline{U}_\varepsilon)\underline{U}'_\varepsilon = \\
&g'(\underline{U}_\varepsilon) - g'(\underline{u}_{+\infty}) + (f'(\underline{U}_\varepsilon + v) - f'(\underline{u}_{+\infty}))\left(\theta_\varepsilon\chi'(x) + \frac{\beta\chi'(x)}{\chi(x)+ct}\right) + \varepsilon\beta\frac{\chi'(x)^2}{(\chi(x)+ct)^2} \\
&+ \varepsilon\theta_\varepsilon\chi''(x) + \varepsilon\beta\frac{\chi''(x)}{\chi(x)+ct} + \varepsilon\beta^2\frac{\chi'(x)^2}{(\chi(x)+ct)^2} + f''(\underline{U}_\varepsilon)\underline{U}'_\varepsilon \\
&+ (2\varepsilon\theta_\varepsilon + c + f'(\underline{u}_{+\infty}) - \sigma)\beta\chi'(x)\frac{\chi'(x)}{\chi(x)+ct},
\end{aligned}
$$

on $\mathbf{R}_+$.

Taking c positive sufficiently small, one concludes the proof by applying Proposition 27 with $b = 0$ and $\phi(x) = C/(1+|x|)^2$ with some $C > 0$ big enough. The equality above gives the inequality on $\mathbf{R}_+$ for C big enough, and on $\mathbf{R}_-$ as χ', χ'' is zero on some neighborhood of $-\infty$, $g'(\underline{u}_{-\infty}) < 0$ and $g'(\underline{U}_\varepsilon)$ converges exponentially fast towards $g'(\underline{U}_{-\infty})$, it is obtained directly. □

4.3.3 Nonlinear stability

Now, we can turn to the proof of the main statement of this part, Theorem 17.

Proof. We first prove the exponential bound. Let us observe that $\tau(\varepsilon,\cdot)$ is nonincreasing on $[\theta_\varepsilon, 2\theta_{max}+1]$ (provided that ε_0 is chosen sufficiently small) and bounded away from zero uniformly in ε on $[2\theta_\varepsilon, 2\theta_{max}+1]$. By considering quadratic zeroth order terms as source terms and using a Duhamel formula, this is sufficient to conclude that there exists a constant C uniform with respect to ε such that if $\|v\|_{L^\infty([0,T];BUC^0_\theta(\mathbf{R}))} \leqslant 3Ce^{-t\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta))}\|v_0\|_{BUC^0_\theta(\mathbf{R})}$, then provided that δ is sufficiently small,
$\|v\|_{L^\infty([0,T];BUC^0_\theta)} \leqslant 2Ce^{-t\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta))}\|v_0\|_{BUC^0_\theta(\mathbf{R})}$ since

$$
\int_0^t e^{-(t-s)\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,2\theta))}\, e^{-2s\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta))}ds \lesssim e^{-t\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta))}.
$$

A continuity argument then concludes the relevant bound in $L^\infty(\mathbf{R}_+; BUC^0_\theta(\mathbf{R}))$.

To bound $\partial_x v$, we use a similar argument but this time we consider as source terms in the Duhamel formula terms that are quadratic in $(v, \partial_x v)$ or linear in v with coefficients

expontially decaying to zero at $\pm\infty$. In doing so we use that for any $\nu > 0$

$$\int_0^t e^{-(t-s)\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta+\nu))}\, e^{-s\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta))} ds \lesssim e^{-t\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta))}\,,$$

if $g'(\underline{u}_{-\infty}) > \tau(\varepsilon,\theta+\nu)$, with a constant that blows up as $\tau(\varepsilon,\theta+\nu)$ gets close to $g'(\underline{u}_{-\infty})$, and that, anyway

$$\int_0^t e^{-(t-s)\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta+\nu))}\, e^{-s\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta))} ds \lesssim t e^{-t\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta))}\,.$$

We finally also need that

$$\int_0^t e^{-(t-s)\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta+\nu))}\, s^2 e^{-2s\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta))} ds \lesssim t e^{-t\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta))}\,.$$

The proof of the algebraic decay rates is completely similar, using that

$$\int_0^t e^{-(t-s)\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,2\theta_\varepsilon))}\, \frac{ds}{(1+s)^{2\beta}} \lesssim \frac{1}{(1+t)^\beta}$$

and for $\nu > 0$

$$\int_0^t e^{-(t-s)\max(g'(\underline{u}_{-\infty}),\tau(\varepsilon,\theta+\nu))}\, \frac{ds}{(1+s)^{\beta}} \lesssim \frac{1}{(1+t)^\beta}\,.$$

□

Chapter 5

Prospects

We conclude this thesis by gathering what we believe to be the main open questions related to the present work.

5.1 Combustion models

5.1.1 Majda model

In the special case of main interest when $f(U) = U^2/2$, all combustion waves of the Majda model with step-like ignition function have been proved to be spectrally stable [JY12]. This naturally suggests as an avenue to describe the large-time dynamics near waves of arbitrary amplitude the possibility to extend Lyapunov-type results that take spectral stability as assumptions and prove nonlinear stability, as in [YZ20, FR22], so as to cover the present case. Note that our high-frequency damping estimates provide a significant part of the expected analysis. The missing part is essentially linear and we expect that the insights gained in [YZ20, FR22] could be of direct use.

5.1.2 ZND model

In the ZND model, we note here two different kinds of questions. First, for exponentially localized weights, only instability was proved, but the question of the formation of singularity of wave breaking type needs to be explored.

The second question which needs to be answered about the ZND model is whether or not one can recover high-frequency damping properties for the ZND model by introducing crossed weights, that is, as ZND is a system, one can choose to write a function in a particular basis, and each component is chosen in a different weighted space.

5.2 Stability of shocks

The study, at the hyperbolic level, of the asymptotic behavior of waves contaning shocks is well developped in the scalar case, in particular with [DR20, DR22, GR]. How-

ever, in the system case, a lot of work still needs to be done. Several steps are clearly identified currently as the needed one to be realized to generalize [YZ20, FR22].

To indicate next steps in different directions, extensions should be able to cover for general systems

1. Riemann shocks without spectral gap
2. waves that are not piecewise constant but exhibit a spectral gap
3. Riemann shocks of multidimensional systems.

Obviously after such separate generalizations, one should look to treat cases mixing them or situations even worse (waves with caracteristic points, periodic waves,...).

5.3 Uniform stability in the vanishing viscosity limit

Even for scalar equations, some questions remain concerning the uniform stability of viscous approximations of stable traveling waves of hyperbolic equations. An obvious one is the consideration of waves with characteristic points whose stability has been analyzed in [DR22]. Even if one restricts to Riemann shocks a few important questions are still open.

One of those is the consideration of quasilinear regularizations, replacing $\varepsilon\partial_x^2$ with $\varepsilon\partial_x(a(v)\partial_x v)$ where a is bounded from below away from zero. Applying our strategy would require in the Duhamel part of the estimates a control of $\partial_x^2 v$. Yet, unlike what we have done for $\partial_x v$, even in the near field, one cannot gain a control of $\partial_x^2 v$ by the Duhamel formula itself. A significant modification of the strategy of proof would be needed here.

Another one is the consideration of multidimensional scalar equations. The hyperbolic counterpart is available in [DR20, GR]. In the multidimensional case, the phase shift also depends on transverse variables. Moreover, in the hyperbolic case, transverse derivatives of the phase shift do not decay, but this is compensated by some special uncoupling between the phase shift and the shape perturbation. In the parabolic case the uncoupling property is lost but one expects a non-uniform algebraic decay for the transverse derivatives. How this could be combined still needs to be worked out.

BIBLIOGRAPHY

[Ali95] Serge Alinhac. *Blowup for nonlinear hyperbolic equations*, volume 17 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, aj2, 1995.

[AW78] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.*, 30(1):33–76, 1978.

[Bär22] Johannes Bärlin. Formation of singularities in solutions to nonlinear hyperbolic systems with general source terms. *arXiv preprint arXiv:2204.12101*, 2022.

[BB00] Stefano Bianchini and Alberto Bressan. BV solutions for a class of viscous hyperbolic systems. *Indiana Univ. Math. J.*, 49(4):1673–1713, 2000.

[BB01] Stefano Bianchini and Alberto Bressan. A case study in vanishing viscosity. *Discrete Contin. Dynam. Systems*, 7(3):449–476, 2001.

[BB02a] Stefano Bianchini and Alberto Bressan. A center manifold technique for tracing viscous waves. *Commun. Pure Appl. Anal.*, 1(2):161–190, 2002.

[BB02b] Stefano Bianchini and Alberto Bressan. On a Lyapunov functional relating shortening curves and viscous conservation laws. *Nonlinear Anal.*, 51(4, Ser. A: Theory Methods):649–662, 2002.

[BB05] Stefano Bianchini and Alberto Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math. (2)*, 161(1):223–342, 2005.

[BCP00] Alberto Bressan, Graziano Crasta, and Benedetto Piccoli. Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws. *Mem. Amer. Math. Soc.*, 146(694):viii+134, 2000.

[BG14] Sylvie Benzoni-Gavage. Calcul différentiel et équations différentielles : Cours et exercices corrigés ed. 2. 2014.

[BGM17] Jacob Bedrossian, Pierre Germain, and Nader Masmoudi. On the stability threshold for the 3D Couette flow in Sobolev regularity. *Ann. of Math. (2)*, 185(2):541–608, 2017.

[BGS]

[BJRZ11] Blake Barker, Mathew A. Johnson, L. Miguel Rodrigues, and Kevin Zumbrun. Metastability of solitary roll wave solutions of the St. Venant equations with viscosity. *Phys. D*, 240(16):1289–1310, 2011.

[BLY99] Alberto Bressan, Tai-Ping Liu, and Tong Yang. L^1 stability estimates for $n \times n$ conservation laws. *Arch. Ration. Mech. Anal.*, 149(1):1–22, 1999.

[BM15] Jacob Bedrossian and Nader Masmoudi. Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations. *Publ. Math. Inst. Hautes Études Sci.*, 122:195–300, 2015.

[BMV16] Jacob Bedrossian, Nader Masmoudi, and Vlad Vicol. Enhanced dissipation and inviscid damping in the inviscid limit of the Navier-Stokes equations near the two dimensional Couette flow. *Arch. Ration. Mech. Anal.*, 219(3):1087–1159, 2016.

[BR23] Paul Blochas and L. Miguel Rodrigues. Uniform asymptotic stability for convection-reaction-diffusion in the inviscid limit towards Riemann shocks. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2023.

[Bre00] Alberto Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[BW22] Paul Blochas and Aric Wheeler. Majda and ZND models for detonation: nonlinear stability vs. formation of singularities. *arXiv preprint arXiv:2210.16039*, 2022.

[CG23] Sanchit Chaturvedi and Cole Graham. The inviscid limit of viscous Burgers at nondegenerate shock formation. *Ann. PDE*, 9(1):Paper No. 1, 80, 2023.

[CH93] M. C. Cross and P. C. Hohenberg. Pattern formation outside of equilibrium. *Rev. Mod. Phys.*, 65:851–1112, Jul 1993.

[Chr07] Demetrios Christodoulou. *The formation of shocks in 3-dimensional fluids*. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2007.

[CM07] E. C. M. Crooks and C. Mascia. Front speeds in the vanishing diffusion limit for reaction-diffusion-convection equations. *Differential Integral Equations*, 20(5):499–514, 2007.

[Cou03] Jean-François Coulombel. Stability of multidimensional undercompressive shock waves. *Interfaces Free Bound.*, 5(4):367–390, 2003.

[Cro10] E. C. M. Crooks. Front profiles in the vanishing-diffusion limit for monostable reaction-diffusion-convection equations. *Differential Integral Equations*, 23(5-6):495–512, 2010.

[Daf00] Constantine M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2000.

[Dav07] E. Brian Davies. *Linear operators and their spectra*, volume 106 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[DF79] William C. Davis and Wildon Fickett. Detonation: Theory and experiment. 1979.

[DR20] Vincent Duchêne and L. Miguel Rodrigues. Large-time asymptotic stability of Riemann shocks of scalar balance laws. *SIAM J. Math. Anal.*, 52(1):792–820 889, 2020.

[DR22] Vincent Duchêne and Luis Miguel Rodrigues. Stability and instability in scalar balance laws: fronts and periodic waves. *Anal. PDE*, 15(7):1807–1859, 2022.

[EE87] D. E. Edmunds and W. D. Evans. *Spectral theory and differential operators*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1987. Oxford Science Publications.

[EN00] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[Erp] J J Erpenbeck. Stability of steady-state equilibrium detonations. *Physics of Fluids (U.S.)*.

[Fic79] Wildon Fickett. Detonation in miniature. *American Journal of Physics*, 47:1050–1059, 1979.

[Fif79] Paul C. Fife. Long time behavior of solutions of bistable nonlinear diffusion equations. *Arch. Rational Mech. Anal.*, 70(1):31–46, 1979.

[FM77] Paul C. Fife and J. B. McLeod. The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Rational Mech. Anal.*, 65(4):335–361, 1977.

[FR22] Gregory Faye and L. Miguel Rodrigues. Exponential asymptotic stability of riemann shocks of hyperbolic systems of balance laws. *arXiv preprint arXiv:2207.12686*, 2022.

[Fri64] Avner Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[FS04] S. Friedlander and D. Serre, editors. *Handbook of mathematical fluid dynamics. Vol. III*. North-Holland, Amsterdam, 2004.

[Gil10] Brian H. Gilding. On front speeds in the vanishing diffusion limit for reaction-convection-diffusion equations. *Differential Integral Equations*, 23(5-6):445–450, 2010.

[Goo86] Jonathan Goodman. Nonlinear asymptotic stability of viscous shock profiles for conservation laws. *Arch. Rational Mech. Anal.*, 95(4):325–344, 1986.

[Goo89] Jonathan Goodman. Stability of viscous scalar shock fronts in several dimensions. *Trans. Amer. Math. Soc.*, 311(2):683–695, 1989.

[GR] Louis Garéneaux and L. Miguel Rodrigues. Convective stability in scalar balance laws. *Worki in progress.*

[GR01] Emmanuel Grenier and Frédéric Rousset. Stability of one-dimensional boundary layers by using Green's functions. *Comm. Pure Appl. Math.*, 54(11):1343–1385, 2001.

[GS93] I. Gasser and P. Szmolyan. A geometric singular perturbation analysis of detonation and deflagration waves. *SIAM J. Math. Anal.*, 24(4):968–986, 1993.

[GX92] Jonathan Goodman and Zhou Ping Xin. Viscous limits for piecewise smooth solutions to systems of conservation laws. *Arch. Rational Mech. Anal.*, 121(3):235–265, 1992.

[Ḧ97] Lars Hörmander. *Lectures on nonlinear hyperbolic differential equations*, volume 26 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1997.

[Här00] Jörg Härterich. Viscous profiles for traveling waves of scalar balance laws: the uniformly hyperbolic case. *Electron. J. Differential Equations*, pages No. 30, 22, 2000.

[Här03] J. Härterich. Viscous profiles of traveling waves in scalar balance laws: the canard case. *Methods Appl. Anal.*, 10(1):97–117, 2003.

[Hen81] Daniel Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1981.

[HLZ09] Jeffrey Humpherys, Gregory Lyng, and Kevin Zumbrun. Spectral stability of ideal-gas shock layers. *Arch. Ration. Mech. Anal.*, 194(3):1029–1079, 2009.

[Hör87] Lars Hörmander. The lifespan of classical solutions of nonlinear hyperbolic equations. In *Pseudodifferential operators (Oberwolfach, 1986)*, volume 1256 of *Lecture Notes in Math.*, pages 214–280. Springer, Berlin, 1987.

[How99a] Peter Howard. Pointwise estimates on the Green's function for a scalar linear convection-diffusion equation. *J. Differential Equations*, 155(2):327–367, 1999.

[How99b] Peter Howard. Pointwise Green's function approach to stability for scalar conservation laws. *Comm. Pure Appl. Math.*, 52(10):1295–1313, 1999.

[HR18] Maxime Herda and L. Miguel Rodrigues. Large-time behavior of solutions to Vlasov-Poisson-Fokker-Planck equations: from evanescent collisions to diffusive limit. *J. Stat. Phys.*, 170(5):895–931, 2018.

[HS21] B. Helffer and J. Sjöstrand. Improving semigroup bounds with resolvent estimates. *Integral Equations Operator Theory*, 93(3):Paper No. 36, 41, 2021.

[HZ95] David Hoff and Kevin Zumbrun. Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow. *Indiana Univ. Math. J.*, 44(2):603–676, 1995.

[HZ00] David Hoff and Kevin Zumbrun. Asymptotic behavior of multidimensional scalar viscous shock fronts. *Indiana Univ. Math. J.*, 49(2):427–474, 2000.

[HZ02] David Hoff and Kevin Zumbrun. Pointwise Green's function bounds for multidimensional scalar viscous shock fronts. *J. Differential Equations*, 183(2):368–408, 2002.

[JGK93] Christopher K. R. T. Jones, Robert Gardner, and Todd Kapitula. Stability of travelling waves for nonconvex scalar viscous conservation laws. *Comm. Pure Appl. Math.*, 46(4):505–526, 1993.

[JLW05] Helge Kristian Jenssen, Gregory Lyng, and Mark Williams. Equivalence of low-frequency stability conditions for multidimensional detonations in three models of combustion. *Indiana Univ. Math. J.*, 54(1):1–64, 2005.

[JNR+19] Mathew A. Johnson, Pascal Noble, L. Miguel Rodrigues, Zhao Yang, and Kevin Zumbrun. Spectral stability of inviscid roll waves. *Comm. Math. Phys.*, 367(1):265–316, 2019.

[JNRZ14] Mathew A. Johnson, Pascal Noble, L. Miguel Rodrigues, and Kevin Zumbrun. Behavior of periodic solutions of viscous conservation laws under localized and nonlocalized perturbations. *Invent. Math.*, 197(1):115–213, 2014.

[Joh74] Fritz John. Formation of singularities in one-dimensional nonlinear wave propagation. *Comm. Pure Appl. Math.*, 27:377–405, 1974.

[JY12] Soyeun Jung and Jinghua Yao. Stability of ZND detonations for Majda's model. *Quart. Appl. Math.*, 70(1):69–76, 2012.

[JYZ21] Soyeun Jung, Zhao Yang, and Kevin Zumbrun. Stability of strong detonation waves for Majda's model with general ignition functions. *Quart. Appl. Math.*, 79(2):357–365, 2021.

[JZN11] Mathew A. Johnson, Kevin Zumbrun, and Pascal Noble. Nonlinear stability of viscous roll waves. *SIAM J. Math. Anal.*, 43(2):577–611, 2011.

[Kap94] Todd Kapitula. On the stability of travelling waves in weighted L^∞ spaces. *J. Differential Equations*, 112(1):179–215, 1994.

[Kat76] Tosio Kato. *Perturbation theory for linear operators*. Springer-Verlag, Berlin, second edition, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132.

[Kaw84] Shuichi Kawashima. Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics. 1984.

[KK98] Gunilla Kreiss and Heinz-Otto Kreiss. Stability of systems of viscous conservation laws. *Comm. Pure Appl. Math.*, 51(11-12):1397–1424, 1998.

[KP13] Todd Kapitula and Keith Promislow. *Spectral and dynamical stability of nonlinear waves*, volume 185 of *Applied Mathematical Sciences*. Springer, New York, 2013. With a foreword by Christopher K. R. T. Jones.

[Kru70] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[KS88] Shuichi Kawashima and Yasushi Shizuta. On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws. *Tohoku Math. J. (2)*, 40(3):449–464, 1988.

[Kuz76] N.N. Kuznetsov. Accuracy of some approximate methods for computing the weak solutions of a first-order quasi-linear equation. *USSR Computational Mathematics and Mathematical Physics*, 16(6):105–119, 1976.

[KV21a] Moon-Jin Kang and Alexis Vasseur. Contraction property for large perturbations of shocks of the barotropic Navier-Stokes system. *J. Eur. Math. Soc. (JEMS)*, 23(2):585–638, 2021.

[KV21b] Moon-Jin Kang and Alexis F. Vasseur. Uniqueness and stability of entropy shocks to the isentropic Euler system in a class of inviscid limits from a large family of Navier-Stokes systems. *Invent. Math.*, 224(1):55–146, 2021.

[KV22] Moon-Jin Kang and Alexis F. Vasseur. Well-posedness of the Riemann problem with two shocks for the isentropic Euler system in a class of vanishing physical viscosity limits. *J. Differential Equations*, 338:128–226, 2022.

[Lai19] Geng Lai. Detonation wave solution to a 1D piston problem for the Zeldovich-von Neumann-Döring combustion model. *J. Differential Equations*, 267(9):4949–4974, 2019.

[Lax73] Peter D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973.

[Lev92] Arnon Levy. On Majda's model for dynamic combustion. *Comm. Partial Differential Equations*, 17(3-4):657–698, 1992.

[Liu77] Tai Ping Liu. Decay to N-waves of solutions of general systems of nonlinear hyperbolic conservation laws. *Comm. Pure Appl. Math.*, 30(5):586–611, 1977.

[Liu79] Tai Ping Liu. Development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations. *J. Differential Equations*, 33(1):92–111, 1979.

[Liu85] Tai-Ping Liu. Nonlinear stability of shock waves for viscous conservation laws. *Mem. Amer. Math. Soc.*, 56(328):v+108, 1985.

[Liu87] Tai-Ping Liu. Hyperbolic conservation laws with relaxation. *Comm. Math. Phys.*, 108(1):153–175, 1987.

[Liu97] Tai-Ping Liu. Pointwise convergence to shock waves for viscous conservation laws. *Comm. Pure Appl. Math.*, 50(11):1113–1182, 1997.

[Liu20] Tai-Ping Liu. N-waves for conservation laws with linear defeneracy. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 15(3):187–216, 2020.

[Liu21] Tai-Ping Liu. *Shock waves*, volume 215 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, [2021] ©2021.

[LRTZ07] Gregory Lyng, Mohammadreza Raoofi, Benjamin Texier, and Kevin Zumbrun. Pointwise Green function bounds and stability of combustion waves. *J. Differential Equations*, 233(2):654–698, 2007.

[LXY22] Jun Li, Gang Xu, and Huicheng Yin. On the blowup mechanism of smooth solutions to 1d quasilinear strictly hyperbolic systems with large initial data, 2022.

[LY99] Tai-Ping Liu and Shih-Hsien Yu. Nonlinear stability of weak detonation waves for a combustion model. *Comm. Math. Phys.*, 204(3):551–586, 1999.

[LZ95] Tai-Ping Liu and Kevin Zumbrun. Nonlinear stability of an undercompressive shock for complex Burgers equation. *Comm. Math. Phys.*, 168(1):163–186, 1995.

[LZ97] Tai-Ping Liu and Yanni Zeng. Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws. *Mem. Amer. Math. Soc.*, 125(599):viii+120, 1997.

[LZ04] Gregory Lyng and Kevin Zumbrun. One-dimensional stability of viscous strong detonation waves. *Arch. Ration. Mech. Anal.*, 173(2):213–277, 2004.

[MÓ4] Guy Métivier. *Small viscosity and boundary layer methods*. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Boston, Inc., Boston, MA, 2004. Theory, stability analysis, and applications.

[Maj81] Andrew Majda. A qualitative model for dynamic combustion. *SIAM J. Appl. Math.*, 41(1):70–93, 1981.

[Maj83a] Andrew Majda. The existence of multidimensional shock fronts. *Mem. Amer. Math. Soc.*, 43(281):v+93, 1983.

[Maj83b] Andrew Majda. The stability of multidimensional shock fronts. *Mem. Amer. Math. Soc.*, 41(275):iv+95, 1983.

[Maj84] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1984.

[Mas99] Corrado Mascia. Onde viaggianti ed equazioni di reazione-convezione. *Bollettino della Unione Matematica Italiana A*, 2:111–114, 01 1999.

[Mas07] Nader Masmoudi. Examples of singular limits in hydrodynamics. In *Handbook of differential equations: evolutionary equations. Vol. III*, Handb. Differ. Equ., pages 195–275. Elsevier/North-Holland, Amsterdam, 2007.

[Met]

[Mét01] Guy Métivier. Stability of multidimensional shocks. In *Advances in the theory of shock waves*, volume 47 of *Progr. Nonlinear Differential Equations Appl.*, pages 25–103. Birkhäuser Boston, Boston, MA, 2001.

[MN85] Akitaka Matsumura and Kenji Nishihara. On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas. *Japan J. Appl. Math.*, 2(1):17–25, 1985.

[MN94] Akitaka Matsumura and Kenji Nishihara. Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity. *Comm. Math. Phys.*, 165(1):83–96, 1994.

[MR12] Nader Masmoudi and Frédéric Rousset. Uniform regularity for the Navier-Stokes equation with Navier boundary condition. *Arch. Ration. Mech. Anal.*, 203(2):529–575, 2012.

[MS97] Corrado Mascia and Carlo Sinestrari. The perturbed Riemann problem for a balance law. *Adv. Differential Equations*, 2(5):779–810, 1997.

[MZ03] Corrado Mascia and Kevin Zumbrun. Pointwise Green function bounds for shock profiles of systems with real viscosity. *Arch. Ration. Mech. Anal.*, 169(3):177–263, 2003.

[MZ04a] C. Mascia and K. Zumbrun. Stability of small-amplitude shock profiles of symmetric hyperbolic-parabolic systems. *Comm. Pure Appl. Math.*, 57(7):841–876, 2004.

[MZ04b] Corrado Mascia and Kevin Zumbrun. Stability of large-amplitude viscous shock profiles of hyperbolic-parabolic systems. *Arch. Ration. Mech. Anal.*, 172(1):93–131, 2004.

[MZ05a] Corrado Mascia and Kevin Zumbrun. Stability of large-amplitude shock profiles of general relaxation systems. *SIAM J. Math. Anal.*, 37(3):889–913, 2005.

[MZ05b] Guy Métivier and Kevin Zumbrun. Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems. *Mem. Amer. Math. Soc.*, 175(826):vi+107, 2005.

[Paz83] Ammon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[PW84] Murray H. Protter and Hans F. Weinberger. *Maximum principles in differential equations*. Springer-Verlag, New York, 1984. Corrected reprint of the 1967 original.

[Rod13] L. Miguel Rodrigues. *Asymptotic stability and modulation of periodic wavetrains, general theory & applications to thin film flows*. Habilitation à diriger des recherches, Université Lyon 1, 2013.

[Rod15] Luis Miguel Rodrigues. Space-modulated stability and averaged dynamics. *Journées équations aux dérivées partielles*, 2015. talk:8.

[Rod18] L. Miguel Rodrigues. Linear asymptotic stability and modulation behavior near periodic waves of the Korteweg–de Vries equation. *J. Funct. Anal.*, 274(9):2553–2605, 2018.

[Rou02] Frédéric Rousset. Viscous limits for strong shocks of one-dimensional systems of conservation laws. In *Journées "Équations aux Dérivées Partielles" (Forges-les-Eaux, 2002)*, pages Exp. No. XVI, 12. Univ. Nantes, Nantes, 2002.

[RV98] Jean-Michel Roquejoffre and Jean-Paul Vila. Stability of ZND detonation waves in the Majda combustion model. *Asymptot. Anal.*, 18(3-4):329–348, 1998.

[RZ16] L. Miguel Rodrigues and Kevin Zumbrun. Periodic-coefficient damping estimates, and stability of large-amplitude roll waves in inclined thin film flow. *SIAM J. Math. Anal.*, 48(1):268–280, 2016.

[San02] Björn Sandstede. Stability of travelling waves. In *Handbook of dynamical systems, Vol. 2*, pages 983–1055. North-Holland, Amsterdam, 2002.

[Sat]

[Sat73] David H. Sattinger. *Topics in stability and bifurcation theory*. Lecture Notes in Mathematics, Vol. 309. Springer-Verlag, Berlin-New York, 1973.

[Sat76] D. H. Sattinger. On the stability of waves of nonlinear parabolic systems. *Advances in Math.*, 22(3):312–355, 1976.

[Sat77] D. H. Sattinger. Weighted norms for the stability of traveling waves. *J. Differential Equations*, 25(1):130–144, 1977.

[Ser21] Denis Serre. Asymptotic stability of scalar multi-D inviscid shock waves. *arXiv preprint arXiv:2103.09615*, 2021.

[Sin96] Carlo Sinestrari. Asymptotic profile of solutions of conservation laws with source. *Differential Integral Equations*, 9(3):499–525, 1996.

[Spe16] Jared Speck. *Shock formation in small-data solutions to 3D quasilinear wave equations*, volume 214 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2016.

[SX93] Anders Szepessy and Zhou Ping Xin. Nonlinear stability of viscous shock waves. *Arch. Rational Mech. Anal.*, 122(1):53–103, 1993.

[Sze99] Anders Szepessy. Dynamics and stability of a weak detonation wave. *Comm. Math. Phys.*, 202(3):547–569, 1999.

[Tao06] Terence Tao. *Nonlinear dispersive equations*, volume 106 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. Local and global analysis.

[Tar79] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, volume 39 of *Res. Notes in Math.*, pages 136–212. Pitman, Boston, Mass.-London, 1979.

[TZ11] Benjamin Texier and Kevin Zumbrun. Transition to longitudinal instability of detonation waves is generically associated with Hopf bifurcation to time-periodic galloping solutions. *Comm. Math. Phys.*, 302(1):1–51, 2011.

[VVV94] Aizik I. Volpert, Vitaly A. Volpert, and Vladimir A. Volpert. *Traveling wave solutions of parabolic systems*, volume 140 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1994. Translated from the Russian manuscript by James F. Heyda.

[VW15] Alexis Vasseur and Yi Wang. The inviscid limit to a contact discontinuity for the compressible Navier-Stokes-Fourier system using the relative entropy method. *SIAM J. Math. Anal.*, 47(6):4350–4359, 2015.

[Wil10] Mark Williams. Heteroclinic orbits with fast transitions: a new construction of detonation profiles. *Indiana Univ. Math. J.*, 59(3):1145–1209, 2010.

[WX05] Yaping Wu and Xiuxia Xing. The stability of travelling fronts for general scalar viscous balance law. *J. Math. Anal. Appl.*, 305(2):698–711, 2005.

[Xin05] Xiu-xia Xing. Existence and stability of viscous shock waves for non-convex viscous balance law. *Adv. Math. (China)*, 34(1):43–53, 2005.

[Yu99] Shih-Hsien Yu. Zero-dissipation limit of solutions with shocks for systems of hyperbolic conservation laws. *Arch. Ration. Mech. Anal.*, 146(4):275–370, 1999.

[YZ20] Zhao Yang and Kevin Zumbrun. Stability of hydraulic shock profiles. *Arch. Ration. Mech. Anal.*, 235(1):195–285, 2020.

[Zen94] Yanni Zeng. L^1 asymptotic behavior of compressible, isentropic, viscous 1-D flow. *Comm. Pure Appl. Math.*, 47(8):1053–1082, 1994.

[ZH98] Kevin Zumbrun and Peter Howard. Pointwise semigroup methods and stability of viscous shock waves. *Indiana Univ. Math. J.*, 47(3):741–871, 1998.

[ZH02] K. Zumbrun and P. Howard. Errata to: "Pointwise semigroup methods, and stability of viscous shock waves" [Indiana Univ. Math. J. **47** (1998), no. 3, 741–871; MR1665788 (99m:35157)]. *Indiana Univ. Math. J.*, 51(4):1017–1021, 2002.

[Zum01] Kevin Zumbrun. Multidimensional stability of planar viscous shock waves. In *Advances in the theory of shock waves*, volume 47 of *Progr. Nonlinear Differential Equations Appl.*, pages 307–516. Birkhäuser Boston, Boston, MA, 2001.

[Zum04] Kevin Zumbrun. Stability of large-amplitude shock waves of compressible Navier-Stokes equations. In *Handbook of mathematical fluid dynamics. Vol. III*, pages 311–533. North-Holland, Amsterdam, 2004. With an appendix by Helge Kristian Jenssen and Gregory Lyng.

[Zum07] Kevin Zumbrun. Planar stability criteria for viscous shock waves of systems with real viscosity. In *Hyperbolic systems of balance laws*, volume 1911 of *Lecture Notes in Math.*, pages 229–326. Springer, Berlin, 2007.

[Zum10] Kevin Zumbrun. Conditional stability of unstable viscous shock waves in compressible gas dynamics and MHD. *Arch. Ration. Mech. Anal.*, 198(3):1031–1056, 2010.

[Zum11] Kevin Zumbrun. Stability of detonation profiles in the ZND limit. *Arch. Ration. Mech. Anal.*, 200(1):141–182, 2011.

[Zum12] Kevin Zumbrun. High-frequency asymptotics and one-dimensional stability of Zel'dovich–von Neumann–Döring detonations in the small-heat release and high-overdrive limits. *Arch. Ration. Mech. Anal.*, 203(3):701–717, 2012.

COLLEGE MATHS, TELECOMS
DOCTORAL INFORMATIQUE, SIGNAL
BRETAGNE SYSTEMES, ELECTRONIQUE

Université de Rennes

Titre : Stabilité des chocs et de leurs approximations visqueuses.

Mot clés : Ondes Progressives, Chocs, Équations Aux Dérivées Partielles Non Linéaires

Résumé : Cette thèse concerne la stabilité d'ondes contenant une discontinuité. Plus précisément, le problème est de comprendre quels sont les chocs qui sont orbitalement stables en temps long dans des topologies encodant de la régularité par morceaux. Ce travail suit le cadre d'existence et stabilité en temps court de perturbations de chocs de référence fourni par des résultats récents. De plus, le problème de l'approximation visqueuse de telles ondes est également considérée, et est le point principal considéré ici. Dans le premier chapitre, la stabilité et l'instabilité d'ondes discontinues dans des problèmes de combustion. Les problèmes sont hyperboliques, et considérées dans des espaces à poids. Le deuxième chapitre et le troisième chapitre sont centrés sur la stabilité asymptotique oritale d'approximations visqueuses de chocs, dans un cadre scalaire. Le premier cas traité est celui d'un choc de Riemann dont le linéarisé a un spectre qui contient un trou spectral. Le second cas est associé à un choc de Riemann spectralement instable, qui est stabilisé au niveau spectral par l'introduction de poids. Un front lisse au niveau hyperbolique est également étudié de cette façon.

Title: Stability of shocks and their viscous approximations.

Keywords: Travelling waves, Shocks, Nonlinear Partial Differential Equations

Abstract: In this thesis, the stability of waves containing some discontinuity is studied. More precisely, the goal is to prove results of asymptotic orbital stability of shocks in piecewise smooth topologies. The framework follows the one given by recent results obtained to understand conditions ensuring local wellposedness of perturbations around a shock of reference. Moreover, the study of uniform asymptotic stability of viscous approximations of shocks is the main point considered in this thesis. The first chapter is related to the stability and instability properties of various waves containing a discontinuity in the framework of combustion models. This is done in the context of hyperbolic equations, using weighted spaces. The second and third chapter are about the stability of Riemann shocks in the viscous regime. The first one is the case of stable Riemann shocks, with a spectral gap at the spectral level. The last one is associated to an unstable Riemann shocks, which is stabilized at the linear level through the use of weights. A smooth traveling wave at the hyperbolic level has its viscous regularization also studied that way.