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Fanny Cartellier

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Thèse de doctorat



Strategic interactions in the ecological transition: Quantitative approaches in climate finance

Thèse de doctorat de l'Institut Polytechnique de Paris
préparée à l'Ecole Nationale de la Statistique et de l'Administration Economique

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Spécialité de doctorat : Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau, le 26 juin 2024, par

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Long summary

This thesis tackles the issue of the interaction between the financial system and the low-carbon transition in four chapters, bringing mainly theoretical contributions. In particular, it focuses on modeling the strategic behavior of key players in the transition, with a specific focus on the role of the financial sector.

Chapter 1 reviews the climate stress test methodologies used by financial supervisors and proposed in academic articles. It examines the ability of the methodologies used by central banks to assess the full extent of the financial risks posed by climate change and the low-carbon transition, with particular reference to academic proposals. The results show that these methodologies give unequal consideration to the different channels of financial risk. In addition, they downplay the role that the radical uncertainty characterizing climate and transition risks can play in amplifying them. Finally, they do not take sufficient account of the role of the financial sector in the propagation of these risks.

Chapter 2 looks at the effect of divestment by banks from polluting assets whose operation is incompatible with the transition to a low-carbon economy, so-called stranded assets. In particular, it models the impact of this disinvestment on the credit risk of the companies that hold them. Under certain conditions, such divestment by one bank triggers a chain reaction, prompting other banks to divest in turn. This scenario not only raises the question of the levers available to achieve the low-carbon transition, but also the risks of economic and financial instability that the latter may provoke.

Chapter 3 models how the practice of greenwashing is influenced by investors' green preferences in a stochastic dynamic equilibrium model. It shows that investors' preference for low-carbon companies encourages them to engage in greenwashing. Nevertheless, investors can discourage this practice by penalizing it when it is revealed by environmental controversies. These results are robust to the introduction of an interaction between firms, resolved by a mean field game. Dynamically, a company's optimal communication effort appears to be countercyclical with respect to its environmental rating, which we validate empirically.

Chapter 4 develops a methodology to identify coarse correlated equilibria in linear quadratic mean field games, an emerging tool in the field of financial mathematics. Under certain conditions, this type of equilibrium achieves higher levels of social welfare than the commonly used Nash equilibrium, while being non-cooperative. As illustrated in this chapter, it is therefore particularly well suited to common-good games. By applying this method to a stylized game of climate change mitigation between countries, it shows how coarse correlated equilibria propose a non-cooperative response, at least theoretically, to the tragedy of the commons.

Résumé long

Cette thèse aborde la question de l'interaction entre le système financier et la transition bas-carbone en quatre chapitres aux contributions principalement théoriques. Elle s'attelle notamment à la modélisation du comportement stratégique d'acteurs clés dans la transition, avec une attention particulière portée au rôle du secteur financier.

Le Chapitre 1 effectue une revue des méthodologies de stress tests climatiques pratiqués par les superviseurs financiers et proposés dans des articles académiques. Il interroge la capacité des méthodologies mobilisées par les banques centrales à évaluer l'ampleur des risques financiers provoqués par le changement climatique et la transition bas-carbone, en regard des propositions académiques notamment. Il en ressort qu'elles présentent une considération inégale des différents canaux de risques financiers. En outre, elles minimisent le rôle que l'incertitude radicale caractérisant les risques climatiques et de transition peut jouer dans leur amplification. Enfin, elles ne prennent pas suffisamment en compte le rôle du secteur financier dans la propagation de ces risques.

Le Chapitre 2 s'intéresse à l'effet d'un désinvestissement par les banques d'actifs polluants dont l'exploitation est incompatible avec la transition vers une économie bas-carbone, nommés actifs échoués. En particulier, il modélise l'impact de ce désinvestissement sur le risque de crédit des entreprises qui les détiennent. Sous certaines conditions, un tel désinvestissement de la part d'une banque peut provoquer une réaction en chaîne et inciter d'autres banques à désinvestir à leur tour. Ce scénario pose à la fois la question des leviers disponibles pour la réalisation de la transition bas-carbone, mais aussi celle des risques d'instabilité économique et financière que cette dernière peut provoquer.

Le Chapitre 3 modélise la façon dont la pratique de l'écoblanchiment est influencée par les préférences vertes des investisseurs dans un modèle d'équilibre dynamique stochastique. Il montre que la préférence de ces derniers pour des entreprises faiblement carbonées les incite à faire de l'écoblanchiment. Néanmoins, les investisseurs ont la possibilité de décourager cette pratique en la pénalisant lorsqu'elle est révélée par des controverses environnementales. Ces résultats sont robustes à l'introduction d'une interaction entre les entreprises, résolue par un jeu à champ moyen. Dynamiquement, l'effort optimal de communication d'une entreprise apparaît contracyclique en regard de sa note environnementale, ce que nous validons empiriquement.

Le Chapitre 4 développe une méthodologie pour identifier des équilibres corrélés généralisés dans des jeux à champ moyen linéaires quadratiques, un outil émergent dans le champ des mathématiques financières. Sous certaines conditions, ce type d'équilibre permet d'atteindre des niveaux de bien-être social plus élevés que l'équilibre de Nash couramment utilisé, tout en étant non-coopératif. Comme illustré dans ce chapitre, il est donc particulièrement adapté à des jeux de bien commun. En appliquant cette méthode à un jeu stylisé d'atténuation du changement climatique entre pays, il montre comment l'équilibre corrélé généralisé propose une réponse non coopérative, théorique au moins, à la tragédie des communs.

Introduction

In 2015, two key events highlighted the need to consider the financial system in the fight against climate change and its consequences. The first was the Paris Agreement, in which 195 parties¹ committed to limiting the rise in global temperature to +2°C by the end of the century (+1.5°C if possible). Article 2, its main article, affirms the importance of managing financial flows in the face of climate change:

“1. This Agreement [...] aims to strengthen the global response to the threat of climate change [...] including by:

[...]

c) Making finance flows consistent with a pathway towards low greenhouse gas emissions and climate-resilient development.”

– Paris Agreement, Article 2.

The objective of mitigating climate change requires the transition of the economy towards production systems with low greenhouse gas (GHG) emissions. This transition process, known as the low-carbon transition, involves massive financing for the deployment of sustainable energies and the reallocation of capital from high-emission technologies to producing activities with a reduced environmental footprint.

The second is a landmark speech made a few weeks before the Paris Agreement by Mark Carney, then Governor of the Bank of England, who highlighted the threat that climate change poses to global financial stability. In particular, he distinguished two sources of financial risk associated with climate change:² “physical risks,” arising directly from the physical impacts of climate change, and “transition risks,” which refer to all the financial risks caused by the process of transition to a low-carbon economy.

1. As of January 1, 2024, 194 States and the European Union are signatories to the Paris Agreement, adopted at COP21 in December 2015.

2. He also mentioned a third type of risk in this speech: “litigation risks,” which threaten those responsible for greenhouse gas emissions who could be called to account in court by stakeholders for their part in worsening climate change. However, they can be seen as a sub-category of physical and transition risks (NGFS, 2021a). Moreover, they have so far received very little direct attention in the (macro)financial literature. They will therefore not be addressed specifically in this thesis.

The financial system therefore plays a crucial role in mitigating climate change. However, it is itself exposed to the risks induced by climate change as well as by the transition process required to mitigate it. It is these two facets of the interaction between climate change and the financial system that this thesis addresses.

Understanding how the financial system can promote, rather than hinder, the low-carbon transition, and identifying the channels through which the latter, as well as climate change, can be a source of financial instability, presents a number of methodological challenges. Firstly, modeling these phenomena requires combining three poles, the intersection of which is rarely modeled: the financial system, the real economy and the environment, notably through the environmental footprint of economic and financial activities and their exposure to climate hazards. Secondly, the consequences of climate change and the course of the low-carbon transition are subject to radical uncertainty, based in particular on the uncertainty as to how the agents driving the transition (politicians, producers, investors and consumers) will behave in the face of climate change. Lastly, climate change and the low-carbon transition imply a break in a number of historical economic trends, which reduces the relevance of studies based on past data to forecast future horizons.

Outline The remainder of this introduction is organized in two parts. In the first part, I develop the methodological challenges posed by modeling the two facets of the interaction between the financial system and climate change, and then review the main types of economic and financial models that can be used to address them. In the second part, I set out the main contributions of this thesis, according to its four chapters.

1 Modeling the interaction between the financial system and climate change

Modeling the interaction between the financial system and climate change raises a number of issues, presented in the first part of this section. The second part provides an overview of the different types of modeling that address them in the macroeconomic, microeconomic and financial literatures.

1.1 Methodological challenges

Modeling the interaction between the financial system and climate change faces a number of methodological challenges, linked to the nature of climate change, its consequences for society, and the place of financial modeling in economics.

Trends disruption and structural change Climate change is disrupting a large number of socio-economic trends, both through its physical impacts, which will increase in a non-linear

fashion over the course of the century, and through the mitigation efforts it is prompting.

Climate change has an impact on the productivity of human, physical and natural capital. This impact stems from the increased frequency and intensity of extreme climate hazards (floods, droughts, cyclones, etc.), but also from chronic climate and associated physical risks such as rising global temperatures, increased risk of epidemics and sea-level rise. These hazards and climatic evolutions can contribute to the degradation of physical assets, the productive obsolescence of territories that become too dry or submerged, the disruption of means of transport and telecommunications, and a reduced work capacity of the active population (Basel Committee, 2021). These effects of climate change, while beginning to appear, are still barely perceptible for some, and will continue to amplify for others, in a way that cannot be deduced by the simple statistical projection of past trends (IPCC, 2021).

In addition, mitigating climate change requires a profound structural change in the economy (Semieniuk et al., 2021; Ciarli and Savona, 2019). The transition to a low-carbon economy implies a massive increase in the share of low-carbon activities in value added, in contrast to a sharp reduction in the share of greenhouse gas-intensive activities (Campiglio and van der Ploeg, 2022). This will require a major transfer of physical and financial capital, as well as profound changes in value chains (Cahen-Fourot et al., 2020) and consumption habits. But this transition process is not a continuation of past trends. It is the result of proactive efforts to take into account a hitherto neglected negative economic externality (Campiglio and van der Ploeg, 2022). The low-carbon transition relies on the evolution of its four main drivers: (i) climate policies, including for example the carbon tax and subsidies for investment in renewable energies, (ii) technological innovations, which make renewable energies more affordable, improve the energy efficiency of production and transport systems, and capture and store carbon in the atmosphere, (iii) market sentiment, which reflects investors' expectations and preferences regarding the course of the low-carbon transition, and which influences their investment decisions, and (iv) consumer preferences, which determine the relative demand for carbon-intensive products (Task Force on Climate-Related Financial Disclosures, 2017).

The low-carbon transition relies on agents' behavior A crucial difficulty in studying the socio-economic impacts of climate change and the efforts mobilized for its mitigation is the radical uncertainty that characterizes them (Weitzman, 2011; Bolton et al., 2020). This uncertainty can be broken down into two main dimensions. The first is scientific uncertainty, which relates to both climate models (Brock and Hansen, 2018; Heal and Millner, 2013) and economic models that look at the impact of mitigation efforts on greenhouse gas emissions and the economic impacts of climate change (Barnett et al., 2021; Kriegler et al., 2014).³ This scientific uncertainty takes the form of ambiguity and misspecification in climate and economic models, according to the terminology of Barnett et al. (2021).

3. See also the number of climate-economy integrated assessment models used to simulate transition scenarios in the Sixth Assessment Report of the Intergovernmental Panel on Climate Change (IPCC, 2022, Chapter 3).

The second dimension of this radical uncertainty concerns the modalities and temporality of the low-carbon transition, which is based primarily on the behavior of agents with regards to climate change, as potential drivers of the transition: politicians, producers, investors and consumers.⁴ Thus, the assumptions underlying the “disorderly” transition scenarios used in central banks’ climate scenario analyses,⁵ established as potential sources of financial instability, place the emphasis on climate policies and the agents’ expectations regarding them, and reflect a lack of coordination between these two types of actors (NGFS, 2020; Semieniuk et al., 2021). Indeed, in a context of uncertainty, agents’ anticipation of climate policies can have a decisive influence on the very impact of these policies (Biais and Landier, 2022; Campiglio et al., 2023b). Moreover, agents heterogeneity (Cahen-Fourot et al., 2023), their behavioral biases and their limited rationality also have an impact on the effectiveness of climate policies, given their uncertainty or low credibility (Diluiso et al., 2020; Ceccarelli and Ramelli, 2024). Faced with uncertain gains from research and development in low environmental footprint technologies, corporate investment decisions may be sub-optimal without a policy adapted to the low-carbon transition objective (Bustamante and Zucchi, 2024). In particular, given the decisive role of the financial sector in the low-carbon transition, the beliefs and expectations of investors themselves play a major role in driving or slowing it down (Comerford and Spiganti, 2023; De Angelis et al., 2023).

Uncertainty as to how the low-carbon transition will unfold is an obstacle to both forecasts of the transition and forecasts of climate change, since the extent of climate change depends on efforts to mitigate it. Understanding the behavior of agents in the face of climate change therefore makes it possible to identify an important part of the levers and obstacles to the low-carbon transition, and thus to shed light on the way climate policies should be oriented to favor it. By facilitating understanding of the mitigation efforts that can be expected, it also sheds light on the intensity of future climate change and helps to prepare for it. Microeconomics and game theory provide a wide range of theoretical tools for understanding the behavior of key players in the low-carbon transition in this unprecedented context, and in particular financial system players, so as to understand their role in mitigating climate change.

4. Among the four drivers of the low-carbon transition evoked in Task Force on Climate-Related Financial Disclosures (2017) and recalled in the previous paragraph, only technological progress does not include any explicit mention of agents; however, it can be argued that this progress depends heavily on investment efforts in research and development, determined by economic, political and financial actors. More broadly, among the uncertainties concerning political actors, we can also include geopolitical issues that impact international coordination in the fight against climate change, but also, for example, wars that emit large quantities of greenhouse gases and disrupt mitigation efforts, such as the war in Ukraine which started in February 2022 in reaction to Russian invasion.

5. This terminology for a type of low-carbon transition scenario is employed by the Network of Central Banks and Supervisors for Greening the Financial System (NGFS) to describe some of its benchmark transition scenarios. It is used to describe transition scenarios in which climate policies are abrupt or delayed, implying rapid and unexpected increases in carbon taxes, for example. Because agents are poorly prepared for such increases, the costs to society of these forms of transition are high, in contrast to an “ordered” transition, which would describe an “ideal” transition scenario in terms of economic and financial stability, for the same objective of carbon neutrality in 2050.

The difficulties of modeling the role of the financial sector The financial sector has a key role to play in the low-carbon transition. It must help finance the massive deployment of renewable technologies, and participate in the effort to reallocate capital from sectors with high greenhouse gas emissions to sectors whose activities are more sustainable. Its spontaneous participation in this structural evolution is not obvious, however, due to the presence of financial frictions (Campiglio, 2016). Conversely, it risks being impacted by the massive upheaval that such a structural change in the economy implies, as well as by the incoming amplification of the physical impacts of climate change. This phenomenon is called the “double materiality”: the financial system has an impact on climate change through the orientation of its investments, and is impacted by it and by mitigation efforts.

Given that the double materiality comes into play via the real economy, understanding its two facets requires in particular to model the interaction between the financial system and the real economy, along with the environment (Battiston et al., 2021; Campiglio et al., 2023a; Svartzman et al., 2021a). Neoclassical and neo-Keynesian macroeconomics respond to this problem by introducing financial frictions into their models (Carattini et al., 2023; Pan et al., 2021). In parallel, non-equilibrium, network (Battiston et al., 2017; Roncoroni et al., 2021), agent-based (Lamperti et al., 2021) or stock-flow consistent (Bovari et al., 2018; Dunz et al., 2021; Daumas, 2023a) models often give a greater role to the financial system, notably through its destabilizing potential (Farmer et al., 2015). It is noteworthy that this second class of model, which has long remained on the sidelines, is gaining institutional legitimacy, notably by moving closer to central banks in the wake of the 2008 crisis and the emergence of the climate issue in finance (Bolton et al., 2020; Svartzman et al., 2021a; Battiston et al., 2021).

Modeling how the financial system can support, or even drive, the low-carbon transition runs up against the following obstacles. Firstly, such an objective requires the ability to model how the real economy undergoes structural changes in the context of the low-carbon transition, which is not enabled in the same way by all families of models (Ciarli and Savona, 2019). Furthermore, the literature on these issues generally separates the modeling of structural change in the economy from that of its impact on the financial system (Semieniuk et al., 2021). Measuring the financial system’s exposure to physical risks involves modeling the interaction between the real economy and climate change, translating the material damage associated with climate change into economic and then financial losses (defaults, asset devaluations, impacts on dividends). This is a highly complex task, which explains the recurrent controversies surrounding integrated climate-economy models, also named integrated assessment models (Stern et al., 2010; Nordhaus, 2007; Heal and Millner, 2014; Pindyck, 2013; Espagne et al., 2012). Added to this is the difficulty of incorporating the financial system into these models, which would allow to capture the two facets of the double materiality.

1.2 An emerging literature in response to these challenges

The disruption of socio-economic trends caused by climate change, and the potential changes in agents' behavior in response to these developments, call for a forward-looking approach based on macroeconomic, microeconomic and financial modeling (Svartzman et al., 2021a; Campiglio and van der Ploeg, 2022).⁶ Indeed, by parameterizing structural characteristics of the economy, such as agents' preferences, these modeling tools make it possible to anticipate the effects of their evolution in the context of climate change. In addition, these modeling frameworks make it possible to take into account the uncertainty faced by agents in this context.

The place of the financial system in macroeconomic modeling A very recent literature is developing the use of general equilibrium models (GEs, very often dynamic stochastic GEs, DSGEs) involving a financial intermediary (bank) and two productive sectors, one whose production technology is intensive in GHG emissions (meeting the terminologies “pollutant”, “brown”, “dirty”), the other being a low emitter (“neutral”, “green”, “clean”). Modelling a central bank allows to assess the impact of a wide range of monetary climate policies, such as subsidies for loans to the green sector or penalties on loans to the brown sector, as well as quantitative easing (QE), sometimes in conjunction with the introduction of a carbon tax and other climate policies. Thus, Benkhodja et al. (2022), Benkhodja et al. (2023) and Ferrari and Landi (2024) assess the effectiveness of these policies in terms of their ability to promote the alignment of the economy with a 2°C or 1.5°C transition trajectory, and conclude with a welfare analysis. This is also done by Comerford and Spiganti (2023), for the assessment of a very specific transition policy scenario, although no financial intermediary is explicitly modeled there. Depending on these assumptions, these articles conclude that climate policies aimed at financial intermediaries are more or less relevant in the context of the low-carbon transition, hence attributing them sometimes a rather weak role in the transition. Böser and Colesanti Senni (2021) instead raise the question of financial stability in conjunction with the achievement of environmental objectives, considering that the “brown” sector is risky due to its exposure to transition risk. Other papers also raise the issue of financial stability in this context, but answer it by adding financial frictions to their environmental DSGE (Diluiso et al., 2021; Benmir and Roman, 2020; Carattini et al., 2023; Lessmann and Kalkuhl, 2024). The introduction of financial frictions highlights how the lack of spontaneous financing of the green sector by banks can be a brake on the low-carbon transition.

In a slightly different framework, focused on the impact of physical risks, Van der Straten (2023) constructs a redistributive growth model with a single productive sector, whose physical assets are damaged by the physical consequences of climate change. By taking into account

6. As emphasized by Campiglio and van der Ploeg (2022), the call for these approaches does not invalidate the use of empirical methods, which also have their role to play (Giglio et al., 2021). However, these methodological challenges make the forward-looking approaches particularly necessary.

financial constraints for households, the author shows that the physical consequences of climate change can contribute to increasing inequalities. It should be noted that Benmir and Roman (2020) and Comerford and Spiganti (2023) also take into account, through a damage function, the physical consequences of climate change.⁷ In addition, there is a body of literature developing production-based asset pricing in the context of the low-carbon transition (Hambel et al., 2023a; Donadelli et al., 2019), even taking into account the radical uncertainty characterizing this context (Barnett et al., 2021), but without modeling any sector or financial player (Giglio et al., 2021). These sophisticated valuation models could form part of a financial stability analysis associated with the transition. The absence of a financial module, however, prevents us from understanding the role played by financial players themselves in driving the transition.

These models can be used to represent the structural evolution of the economy in the context of the low-carbon transition, or in relation to the physical consequences of climate change, sometimes in interaction with certain financial constraints. With a very simple model of the financial system, they have already succeeded in showing how the lack of financing can be a brake on the low-carbon transition, or on adaptation to climate change (Van der Straten, 2023). They thus underline the importance of focusing on the role of financial players in the transition and adaptation to climate change. However, the financial module is generally very crude, which makes it impossible to explore all the brakes and levers for transition and adaptation that can be found in the financial system, or even to explore the consequences in terms of financial stability of the low-carbon transition and climate change. Faced with this pitfall, there are two possibilities for analyzing financial stability in the context of the low-carbon transition: (i) combining economic and financial models (Semieniuk et al., 2021; ACPR, 2020) to assess financial stability, although this poses problems of modeling consistency (Hourcade et al., 2021; Jacquetin, 2021). (ii) Complete the analysis with out of equilibrium models, which can afford to be richer and incorporate network effects (Svartzman et al., 2021a). Stock-flow consistent models, for example, give greater prominence to the financial system (Bovari et al., 2018; Dunz et al., 2021; Dafermos et al., 2018; Gourdel et al., 2024, Daumas, 2023a, Chapter 2), as do agent-based models (Lamperti et al., 2021). For example, (Daumas, 2023a, Chapter 2) models two types of financial agents, a banking institution and a non-banking financial institution. Non-banking institutions are shown to be less resistant to transition risk than banks. In addition, financial institutions are found to experience both short-term and long-term financial risks associated with transi-

7. These two papers can thus be associated with the category of integrated assessment models. These models combine economic and climate modeling, by representing the cycle that links production, GHG emissions, climate change and economic damage. In particular, they allow to assess climate policies in terms of their ability to mitigate climate change. Nordhaus (1992) was the first to propose such an integrated model, in a highly stylized format. The family of integrated assessment models has since grown considerably, drawing on various macroeconomic modeling families. Although the economic part of these models has certainly been enriched, they rarely include a financial module. The constraints imposed by the climate-economy coupling are such that the economic part of integrated assessment models is often very simplified. This is why I will not mention them here. Nevertheless, in the first chapter of this thesis, the way in which these models can nonetheless be used to assess the financial risks arising from the low-carbon transition is addressed through the presentation of climate stress tests conducted by central banks.

tion, something that few modeling frameworks are able to establish. The importance of a rich modeling of the financial system in the context of low-carbon transition is also underlined by network models (Battiston et al., 2017; Roncoroni et al., 2021; Sydow et al., 2021). In addition, whether by taking into account a large number of financial frictions (Lamperti et al., 2021), or by modeling banking players' anticipations (Dunz et al., 2021), these models also provide a deeper understanding of the role of financial players in the low-carbon transition.

The contributions of microeconomic modeling Micro- or meso-economic modeling complements the macroeconomic approach in understanding the role and behavior of the financial system in the context of the low-carbon transition. It is complex for economic and financial agents to evolve in this new context, faced with these new risks, and their behavior may depend on a large number of parameters. These modeling tools enable us to “zoom in” on their motivations, interests and practices with greater creativity, as they are less constrained by the ambition of a macroeconomic framework. In particular, financial mathematics tools are particularly well-suited to modeling the behavior of agents in an uncertain world, and more specifically that of financial players. These tools can thus complement the macroeconomic approach with an exploratory focus on agent behavior, offering the possibility of deducing complex dynamic behaviors.

The theoretical literature on sustainable asset pricing, for example, makes it possible to understand the impact of preferences (Pástor et al., 2021; Pedersen et al., 2021a; Zerbib, 2022), investment strategies (Zerbib, 2022) and heterogeneous investor beliefs on equilibrium asset prices as well as on corporate emission reduction decisions (De Angelis et al., 2023) in dynamic contexts and with information uncertainty about firms' environmental quality, for example (Avramov et al., 2022a). Investor preferences can be combined with consumer preference modeling to understand their aggregate impact on the low-carbon transition (Sauzet and Zerbib, 2022b). In addition, the microeconomic approach also makes it possible to finely model how agents' expectations or beliefs adapt in a context of uncertainty as to the transition scenario, and to deduce the impact of these expectations on investment decisions (Flora and Tankov, 2023) and bond prices (Le Guenedal and Tankov, 2022a).

Although some works highlight intersectoral (Cahen-Fourot et al., 2021) and intrasectoral (Battiston et al., 2017; Roncoroni et al., 2021) propagation effects of transition risks, and although macroeconomic literature emphasizes the interdependence across sectors and their agents, few works focus on the intrasectoral interactions taking place between the key players in the low-carbon transition. Most of the contributions in this thesis address this gap in the literature, drawing on tools provided by microeconomics, game theory and financial mathematics, which are suited to this purpose.

2 Main contributions

In this thesis, I focus on how macroeconomic, but also and especially microeconomic modelling can address the issues raised in the first part of this introduction, through their forward-looking approaches and their abilities to shed light on the behaviour of key agents, notably financial, in the context of the low-carbon transition, with a particular attention paid to agents' strategic interactions. In Chapter 1, I review the climate stress test methodologies developed in recent years by financial supervisors and academic papers, which are largely based on macroeconomic and integrated assessment modeling.

The next three chapters are devoted to the development of microeconomic and game theory models to shed light on the behavior of agents and their strategic interactions in the context of the low-carbon transition. In Chapter 2, I develop a one-period financing model in which several banks finance a company whose assets are destined to be stranded in the context of the low-carbon transition, and deduce the impact of disinvestment by one of them using game-theoretic tools. In Chapter 3, I build a stochastic dynamic equilibrium model with asymmetric information about the environmental value of companies and draw conclusions about the incentives and barriers that investors create to the practice of greenwashing. Finally, in Chapter 4, I propose a methodology for identifying coarse correlated equilibria, which represent an interesting alternative to Nash equilibria for common-good games, in linear quadratic mean field games, and illustrate their relevance through an application to an emission abatement game.

It can be worth highlighting that the chapters order reflects a progression in thinking and technicality in answering the research issues raised by this thesis. The first chapter lays the foundations for the questions to which the following three chapters respond. The main idea of the second chapter, in particular, arises directly from the findings of the review on climate stress tests. This review also confirms the importance of contributing to the climate finance literature by a theoretical approach, placing the focus on the behavior of agents and their strategic interactions. In addition, it motivates the use of dynamic stochastic modeling, which echoes the long-term scenarios of the low-carbon transition and the need to better understand balance sheets dynamics. Hence, a dynamic stochastic framework, based on tools from mathematical finance, is employed in the last two chapters of the thesis. The third chapter, on greenwashing, can be considered the most comprehensive and representative of the ambitions of this thesis, as it combines sophisticated mathematical finance and game theory tools to answer a landmark question in sustainable finance, and includes an empirical validation. The fourth chapter complements this thesis by developing a conceptual and architectonic perspective, also very relevant when exploring the levers of the low-carbon transition.⁸

8. The valorization of the four chapters of the thesis is as follows: the first chapter, single-authored, has been uploaded on SSRN in August 2022 (https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4179311), and submitted without success to two top journals. Since then, as more complete reviews have been published, it has not been resubmitted. However it will be the basis to a handbook chapter. The second chapter, also single-authored, has not been released online yet, and can be considered a preliminary working paper. The paper related

2.1 Climate stress tests: a methodological review

Since Mark Carney’s speech (Carney, 2015), awareness that climate change and the low-carbon transition may be the sources of significant financial risks has emerged among central banks and a growing number of financial players (NGFS, 2018, 2019; Bolton and Kacperczyk, 2021; Krüger et al., 2020). Given the radical (knightian) uncertainty that characterizes them, their measurement is as much a crucial issue as a substantial challenge for financial supervisors who, as part of their financial stability mandate, seek to prevent them.

The methodological challenges include the following (Bolton et al., 2020). The two types of financial risk associated with climate change - physical and transition risks - affect the economy through different prisms, geographical and sectoral exposure respectively, requiring very granular data to assess firms’ climate-related risks exposure. In addition, the absence of situations comparable to climate change and to the low-carbon transition in the past diminishes the relevance of the statistical use of observed data to assess these risks, calling for an approach that is fundamentally forward-looking to measure them. Furthermore, while traditional risk channels (credit risk, market risk, liquidity risk) remain relevant for understanding the impact of climate-related risks on financial institutions, the vectors of these risks (carbon tax or increased frequency and intensity of extreme weather events, for example) have received little consideration to date, and can be transmitted in complex ways throughout the economy (value chains, redistributive effects and interdependence of the two types of risk).

The difficulty of measuring and forecasting the extent of these risks leads to information opacity about firms’ and financial institutions’ exposure to climate-related risks, and casts doubt on the ability of current prices to integrate them perfectly (Campiglio et al., 2023a; Daumas, 2023b; Acharya et al., 2023). The possibility of these risks being misjudged by financial players rightly raises the danger that they could be a source of financial instability (Semieniuk et al., 2021; Litterman et al., 2020; FSB, 2020; Bolton et al., 2020).

Climate stress tests provide a number of answers to these challenges, both as tools for assessing and exploring these risks, and as supervisory tools for central banks. A climate stress test consists in testing the resilience of financial institutions or of the financial system as a whole to climate-related risks, based on stress scenarios describing their materialization. Each scenario is based on a set of assumptions regarding the materialization of these risks, constituting its “narrative.” The scenario is then developed using economic and financial modeling tools, to establish the consequences of this narrative on the counterparties to which the financial institutions are exposed. Typically, a transition risk scenario takes as its narrative a certain rise in the price of carbon, and provides a set of economic and financial projections describing the

to the third chapter, co-authored with Peter Tankov and Olivier David Zerbib, has been released on SSRN in December, 2023 (https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4644741), and submitted to a top finance journal in March, 2023. The paper related to the last chapter, co-authored with Luciano Campi and Federico Cannerozzi, has been released on arxiv in November, 2023 (<https://arxiv.org/abs/2311.04162>), and immediately submitted to a journal of mathematical finance.

consequences of this rise on the counterparties of the financial institutions participating in the exercise. These projections are then used to assess the effect of each stress scenario on their balance sheets.

The first climate stress test I could identify, Battiston et al. (2017), comes from the academic world. It follows on from the studies on systemic risk within financial networks that were stimulated by the 2008 financial crisis (Battiston et al., 2012; Battiston and Caldarelli, 2013; Acemoglu et al., 2015; Hurd et al., 2016). A little later, researchers at the Dutch National Bank (DNB) try to assess the exposure of a large number of Dutch financial institutions to transition risks using a rich in-house dataset (Vermeulen et al., 2021),⁹ while the Banque de France (BdF), a pioneer in the field, implements a pilot exercise involving a large number of financial institutions in a “bottom-up” scheme (ACPR, 2020). The Bank of England (BoE, 2021) and the European Central Bank (ECB, 2021) followed suit, just before the practice became widespread: at the end of 2022, FSB and NGFS (2022) counted a total of 67 exercises conducted or planned by 53 institutions, central banks or financial supervisors. In particular, most of these exercises are based on the reference scenarios constructed by the Network of Central Banks and Supervisors for the Greening of the Financial System (NGFS, 2020, 2021b, 2022). The academic world has stayed left behind, also multiplying publications on the subject (Battiston et al., 2019; Roncoroni et al., 2021; Reinders et al., 2020; Nguyen et al., 2020; Bikakis, 2020; Mandel, 2021; Mandel et al., 2021; Gourdel and Sydow, 2021; Bressan et al., 2022a; Le Guenedal et al., 2022).

The first chapter of this thesis provides an overview of stress test methodologies developed by central banks and academic articles up to April 2022.¹⁰ It questions the ability of these methodologies to assess all the risks associated with climate change and the low-carbon transition affecting the financial system. First, it presents physical and transition risks and discusses the range of channels through which they can affect financial institutions, as well as how they could be a threat to financial stability. Secondly, it sets out the methodological challenges involved in assessing these risks to the financial system. It then presents the climate stress test method and discusses how this tool can meet these challenges. Thirdly, it reviews climate stress testing methodologies developed by central banks and academic papers within the defined scope. In particular, it assesses the way in which these methodologies as a whole cover the different vectors of physical and transition risks, as well as the financial channels through which these risks can affect financial institutions (credit, market and liquidity risk in particular). Finally, it outlines avenues of research to improve the way in which all the physical and transition risks affecting the financial system are taken into account.

9. The publication is dated 2021, but the corresponding working paper was put online in 2019.

10. This temporal perimeter covers the DNB stress test, the ACPR-BdF, BoE and ECB pilot exercises, as well as the academic articles cited at the end of the previous paragraph. This scope is defined as such for several reasons: as the chapter was written at the beginning of the thesis, it covers in depth only the first stress tests; moreover, some conclusions of this chapter are still relevant; finally, the ambition of exhaustiveness for this chapter is rendered unnecessary by the publication of very comprehensive literature reviews shortly afterwards (FSB and NGFS, 2022; Acharya et al., 2023). Besides, the working paper corresponding to this chapter, posted on SSRN in 2022, is cited by (Acharya et al., 2023).

This study leads to the following observations. Firstly, the approaches adopted by supervisors and in the academic literature are very different. The former seem to converge on the use of sophisticated, long-term scenarios, for what mostly looks like scenario analyses, in the frame organized by the Network of Central Banks and Supervisors for the Greening of the Financial System (NGFS).¹¹ The latter develop much more heterogeneous methodologies, especially short term, except when it comes to physical risks (Mandel et al., 2021), and often focus on a specific stage of the stress testing methodology. One consequence of these differences is that the scenarios developed in the academic literature are often more adverse. Indeed, the long-term transition scenarios on which the ACPR-BdF exercise is based, and which will form the basis of the reference scenarios subsequently constructed by the NGFS (NGFS, 2020, 2021b, 2022), appear to be more informative than adverse. This lack of adversity stems in particular from a minimization of the uncertainty faced by agents in the context of transition, from the use of economic modelling frameworks that do not allow for economic depressions, and also from the failure to taking into account the role of financial players in the propagation of transition risks and their impact on the course of the transition itself.

Secondly, climate stress test and scenario analysis methodologies unevenly cover transition risk vectors and the financial channels through which they can impact financial institutions. In particular, the vector of market sentiment is rarely considered, as is the transmission of climate risks via the liquidity risk channel.

The chapter concludes by the proposal to develop two distinct tools in response to these observations. The first one would consist in the integration of more short-term adverse scenarios into supervisors' climate exercises, to bring them into line with more standard stress testing exercises and better meet the objective of measuring risks to financial instability. In addition, the chapter proposes the development of another type of exercise, the "reallocation strategy test," that would account for the role of the financial sector in the low-carbon transition, and which would also allow to provide modeling tools suitable for representing banks' dynamic balance sheets.

2.2 Banks divesting and asset stranding in the low-carbon transition

Meeting the Paris agreement temperature target requires a significant amount of asset stranding (McGlade and Ekins, 2015). Asset stranding can be understood as the premature write-off or conversion to liabilities of assets whose values highly rely on carbon emissions, and that can not be converted at low-cost to any low-emission profitable activity (Caldecott et al., 2013; Van der Ploeg and Rezai, 2020). This phenomenon affects fossil fuel companies (McGlade and Ekins, 2015; Semieniuk et al., 2022), but can also affect other sectors due to a strong dependence on fossil fuel activities in the economy (Cahen-Fourot et al., 2020, 2021). Now, theoretical and

11. Since then, short-term and potentially more severe shock scenarios have been developed to overcome this pitfall (ECB, 2022; NGFS, 2023a).

empirical studies raise the concern that the asset stranding risk may be a threat to economic and financial stability (Carattini and Sen, 2019; Van der Ploeg and Rezai, 2020; Cahen-Fourot et al., 2021; Mercure et al., 2021; Semieniuk et al., 2022; Comerford and Spiganti, 2023).

Among the drivers of the asset stranding risk lay the market sentiment, and more broadly financing decisions of financial actors (Battiston et al., 2023; Comerford and Spiganti, 2023). Now, more and more banks have committed to carbon neutrality, for example through the Net-Zero Banking Alliance initiative, which implies in particular fossil fuel phase-out. This raises the question of whether the fulfilment of banks' net-zero pledges could provoke asset stranding. Indeed, even though the credibility of these commitments is questionable with regards to their little precision (Maio et al., 2023), banks face more and more stakeholders' pressure, and have already started to adjust their credit supply in coherence with their pledges (Kacperczyk and Peydró, 2022).

This chapter contributes to the literature in climate finance and asset stranding in the low-carbon transition by enhancing a specific interaction mechanism that could come into play between banks' financing due to the asset stranding risk. Based on a one-period heuristic toy model representing n banks financing a company that is faced with the asset stranding risk, it shows that divestment by one bank from this company could incentivize other banks to divest from this company as well.

More specifically, it contributes to the literature on the drivers of asset stranding in the low-carbon transition (Campiglio et al., 2022; Rozenberg et al., 2020; Hambel and van der Ploeg, 2024; Comerford and Spiganti, 2023; Carattini et al., 2021; van der Ploeg, 2020; Kalkuhl et al., 2020) by identifying a specific lever that banks have in hand to provoke asset stranding. It also contributes to the theoretical literature integrating transition risks into credit risk assessment (Monnin, 2018; Battiston et al., 2019; Bouchet and Le Guenedal, 2020; Reinders et al., 2020; Agliardi and Agliardi, 2021; Le Guenedal and Tankov, 2022b; Battiston et al., 2023), by estimating the specific impact of asset stranding on a company's probability of default, and by showing that the probability of default also depends on the company's financing conditions when it is faced with asset stranding risk. Finally, it contributes to the literature on banks' credit supply adjustments to transition risks (Chava, 2014; Delis et al., 2019; Degryse et al., 2022; Benincasa et al., 2022; Laeven and Popov, 2023; Ivanov et al., 2023; Kacperczyk and Peydró, 2022) by showing that this credit supply adjustment may also be the result of an interaction with one another when accounting for the asset stranding risk.

The demonstration is as follows. A one-period model represents n banks financing a single company that is faced with the asset stranding risk. Each bank decides the amount of credit to supply to the company at the initial date by optimizing the expected profit generated by the loan. This expected profit depends on the interest rate of the loan, assumed to be exogenous, on the probability of default of the company, and on a quadratic lending cost. This quadratic lending cost, parameterized by a single coefficient that is heterogeneous across banks, represents

their credit line availability to this company, which can be related to their size, their portfolio diversification needs, and their “green” preferences, such as net-zero commitment. The probability of default of the company is defined as the probability that it can not pay back its debt at the end of the period, which depends on its revenues over the period. Now, the company’s revenues depend in particular on a capital-adjustment cost, that it incurs if it can not contract enough debt at the beginning of the period, as it forces it to strand a portion of its assets. Due to this capital-adjustment (or asset stranding) cost, assumed to be convex in line with the literature (Campiglio et al., 2022), banks’ profits interact through the probability of default of the company.

Under a further specification of the model, the probability of default of the company as well as the best response functions of each bank are computed analytically. Moreover, Nash equilibria can be identified graphically when there are two banks in the banking universe. A divestment decision by a bank is assumed to be implemented for reasons that exogenous to the profit maximisation of the loan: it can be due to the implementation of a net-zero commitment, or to new regulatory constraints. A divesting move is defined as a departure from a Nash equilibrium by a reduction in credit supply for these previously cited exogenous reasons.

The results are the following. First, the best response functions illustrate the interaction between banks which arises due to the asset stranding cost. When a bank has a quadratic lending cost that is low enough to allow it to provide the company with the debt amount which maximises its expected profit, it acts as a simple substitute to the others banks’ credit supplies: its best response is to complement the other banks’ total credit supply until the optimal debt amount of the company is met. However, when a bank can not afford, on its own, to fully finance the company, intervenes an interaction effect: this bank is not willing to lend to the company if the other banks’ total credit supply is too small, because in this case the company would incur high capital-adjustment costs, which might make it insolvent.

This strong interaction effect can be further illustrated when identifying Nash equilibria between two banks of this last kind. In typical illustrative examples, two Nash equilibria can be identified: one where none of the two banks lend to the company, and one where they both lend strictly positive amounts. Assuming the prevailing equilibrium is the latter, the full or partial divestment decision by one of the two banks can have a strong multiplicative effect, by incentivizing the other bank to partially or fully divest as well. This strong multiplicative effect is robust to changes in the financing need of the company.

2.3 Can investors curb the practice of greenwashing?

A very important issue in financing the low-carbon transition, on a large scale, is that of assessing the environmental footprint of each financial asset, or each company issuing these assets. A study by the European Commission suggests that greenwashing, which consists in making stakeholders believe that the environmental footprint of a company, project or product

is lower than its true footprint in order to improve its image, is a widespread practice.¹² The recent literature on sustainable finance, namely sustainable asset pricing literature and literature on environmental disclosure, sheds light on some of the motivations that may lead companies to engage in greenwashing, through two phenomena. Firstly, companies whose environmental quality is perceived positively benefit from a lower cost of capital at equilibrium, due to the significant presence on the markets of investors who have pro-environmental preferences (Pástor et al., 2021; Pedersen et al., 2021a; Zerbib, 2022) or who integrate transition risks into their investment choices (Krueger et al., 2021). Secondly, companies can benefit from (i) the difficulty to measure their environmental footprint, (ii) the low standardization of sustainability measurement methods (Berg et al., 2022) (often based on their own, unaudited declarations) and (iii) their ability to communicate ambiguously (Fabrizio and Kim, 2019) to exaggerate, through their communication, their environmental quality (which can be understood as the inverse of their environmental footprint). Companies therefore have both the incentive and the opportunity to greenwash.

However, this practice is an obstacle to investors' impact on the low-carbon transition. Indeed, it complicates the assessment of the transition risks to which companies are exposed. It also prevents investors from favoring companies with a low environmental footprint. Lastly, it can contribute to the risks of a "green bubble," which is a threat to financial stability fostered by the low-carbon transition. In this chapter, which is the fruit of a joint work with Olivier David Zerbib and Peter Tankov, we address the question of the incentives for greenwashing caused by investors' pro-environmental preferences, and the ability of investors to limit this practice.

In doing so, we contribute to several areas of the literature. First, we contribute to the emerging financial literature on greenwashing. Gourier and Mathurin (2024); Wu et al. (2020); Duflo et al. (2013); Duchin et al. (2023) empirically demonstrate the existence of various forms of greenwashing, while Kim and Yoon (2022) and Gourier and Mathurin (2024) show, also empirically, that investors and asset managers are sensitive to greenwashing practices and can help to limit them (Bingler et al., 2022). At this date, we are the first theoretical paper on greenwashing in the financial literature, with Chen (2023a) as a competing paper, which also proposes a theoretical model to answer a question similar to ours. However, we distinguish ourselves through the following aspects: (i) we explicitly characterize equilibrium asset returns as well as the optimal greenwashing effort of firms, (ii) in a dynamic stochastic framework, (iii) allowing direct interaction between firms (in the form of competition) and (iv) we empirically validate part of our results.

Slightly more indirectly, our results echo the literature on corporate disclosure (Flammer et al., 2021; Ilhan et al., 2023) as well as that on the limited reliability of environmental ratings (Berg et al., 2022, 2021). We also contribute to the literature on sustainable asset pricing. In particular, we show that equilibrium returns can carry a risk premium associated with greenwashing

12. https://ec.europa.eu/commission/presscorner/detail/en/ip_21_269.

when investors penalize the latter, in addition to the green premium enjoyed by environmentally well-rated companies (Pástor et al., 2021; Pedersen et al., 2021b; Zerbib, 2022; De Angelis et al., 2023). Finally, we contribute to understanding the impact of sustainable investment (Green and Roth, 2021a; Oehmke and Opp, 2023; Landier and Lovo, 2020a; Hartzmark and Shue, 2023; De Angelis et al., 2023) by showing that the environmental impact of investors favoring environmentally-rated assets depends on their ability to limit the practice of greenwashing. If they succeed in doing so, their investment can have a dual impact: favoring emissions reductions and increasing the quality of environmental ratings by discouraging greenwashing.

We construct a dynamic equilibrium model in which a representative investor allocates her portfolio to assets issued by n companies. This investor, who represents an aggregated version of the demand for these assets, has pro-environmental preferences. However, there is an information asymmetry about the exact environmental footprint of each company, which is not observed by the investor. She is therefore constrained to use their environmental rating, which is an imperfect but public measure of their footprint, to guide her investments according to her preferences. These environmental ratings can be influenced, potentially misleadingly, by companies' environmental communications. However, random events that are observable to investors, namely the advent of environmental controversies, can act as revealing indicators, leading to the abrupt correction of a company's environmental rating. These controversies enable investors to penalize, a posteriori, companies that have significantly manipulated their environmental rating through greenwashing. Due to the investor's pro-environmental preferences and this penalty, the equilibrium cost of capital of the n companies is found to depend on their environmental rating and on the revelations associated with past controversies.

Companies, for their part, seek to minimize their equilibrium cost of capital, by deploying two types of effort. They can (i) communicate about their environmental footprint (in a potentially misleading way), and (ii) invest in reducing their environmental footprint, equivalently called emission abatement. Each type of effort represents a quadratic cost. The gap between a company's communication and its emission abatement effort, if positive, corresponds to its greenwashing practice at that moment. The investor's pro-environmental preferences and the penalty she applies to controversy revelations introduce a trade-off for each company, between: (a) increasing its environmental rating by any means, including greenwashing, and (b) suffering a backlash when a controversy arises, the timing of which it cannot control, as it depends on external media or societal attention. We analytically derive the optimal communication and abatement efforts of each company at each moment, from which we deduce their greenwashing efforts.

From a dynamic point of view, we find that each company's optimal environmental communication is counter-cyclical to its environmental rating: it increases when the latter decreases, and vice-versa. Furthermore, when the marginal unit cost of communication is sufficiently lower than the marginal unit cost of abatement, or when the degree of information asymmetry is

sufficiently high (i.e., the environmental rating is sufficiently imperfect), this communication dynamic mainly reflects a practice of greenwashing: the company seeks, through this counter-cyclical communication, to maintain its environmental rating at a certain level of exaggeration in relation to its actual environmental footprint. This level depends positively on the environmental sensitivity of the representative investor, and negatively on her penalty on revealed misrating.

Thus, investors' pro-environmental preferences encourage companies to engage in greenwashing, as it enables them to lower their equilibrium costs of capital. Nevertheless, by penalizing the misrating revelations provided by controversies, investors have the opportunity to limit these practices. As a result, investors can increase the environmental impact of their investment by encouraging the company to choose an effective reduction of its environmental footprint over greenwashing. However, this power to dissuade greenwashing relies on the advent of controversies, which in turn depends on the distribution of media attention and the ability of journalists and non-governmental organizations to uncover hidden practices. Thus, the dissuasive power of investors is limited by parameters that do not depend on them: our results suggest the importance of complementary political action to foster a culture of transparency.

Next, we extend the initial model by making investment strategies depend on companies' relative environmental ratings. This extension allows us to model two distinct situations, both plausible. Either investors adopt a "best-in-class" strategy, whereby they prefer to invest in the highest-rated companies in a sector by comparing companies with each other rather than by considering the absolute value of their environmental rating.¹³ This standardization introduces interaction between companies, which now need to be rated better than their competitors if they want to reduce their cost of capital. To solve the n -player game that this new program defines, we approach it with a mean field game, in which the number of firms tends to infinity.¹⁴ This passage to the limit of the number of companies, under a few additional assumptions, makes it possible to render negligible the impact of a company's strategy on the average environmental score, which represents its normalization factor. Thus, for a given average environmental score, the optimal strategy of a representative company can be expressed analytically as a function of this average. This allows us to show that the mean field version of this game admits a

13. For example, the ESG (environmental, social and governance) scores produced by the rating agency MSCI are adjusted at sector level using a benchmark revised annually (<https://www.msci.com/documents/1296102/34424357/MSCI+ESG+Ratings+Methodology.pdf>). In addition, Refinitiv LSEG ESG scores are calculated directly from percentiles, which are used to rank companies according to each metric making up the final score (https://www.lseg.com/content/dam/data-analytics/en_us/documents/methodology/lseg-esg-scores-methodology.pdf).

14. Mean field games were introduced simultaneously by Huang et al. (2006) and Lasry and Lions (2007) in the financial mathematics literature. Knowledge of these games then developed rapidly in this literature. Today, we find more and more applications in electricity markets (Aid et al., 2020; Leutscher de las Nieves, 2022; Alasseur et al., 2023), in macroeconomics (Ahn et al., 2018; Achdou et al., 2022; Bahn et al., 2017) and in finance (Carmona et al., 2017; Bertucci et al., 2020; Benazzoli et al., 2020; Lavigne and Tankov, 2023) for example. Reference can be made to Chapter 4 of this thesis and to the thesis introduction of Leutscher de las Nieves (2022) for a more detailed exposition of this tool.

unique Nash equilibrium, which we characterize by a fixed point function and approximate numerically. The results of this extension confirm the structure of the optimal communication, emission reduction and greenwashing efforts obtained in the initial model. Nevertheless, we show numerically that the normalization of environmental scores leads to a decrease in each of these efforts, and in particular in the emission reduction effort. Indeed, for the same ratio between individual and average scores, higher environmental scores yield the same reward on the cost of capital, but require more effort and therefore incur an additional cost.

Finally, using monthly data from Covalence, we show (i) that companies' monthly communication effort is almost systematically positive over the period studied (2016-2022) and (ii) that a company's communication effort decreases significantly when its environmental rating increases and vice versa at a monthly frequency. Thus, we empirically validate the countercyclical aspect of the optimal communication effort as characterized in our model. Now, given that it is reasonable to assume (a) that the communication effort is more volatile than the emissions reduction effort, (b) that it is less costly to communicate than to reduce emissions for the same result in terms of environmental rating, and (c) knowing that companies can benefit from information asymmetries about true environmental footprint (Barbalau and Zeni, 2023), these results suggest with regard to our model that this structurally positive communication effort represents, at least partially, a greenwashing effort.

In addition to its contribution to the understanding of greenwashing practices, this article makes a set of technical contributions. Firstly, it is rare to find a stochastic dynamic model in which expected returns are determined in equilibrium while being controlled by corporate strategy, as such a model is, technically, difficult to solve.¹⁵ Moreover, the application of mean field game theory allows to obtain an elegant solution to the introduction of an interaction between firms, at the cost of reasonable assumptions.¹⁶ This work therefore paints an optimistic picture of the relevance of mean field game theory to shed light on a number of strategic behaviours in economics and finance.

2.4 Coarse correlated equilibria in mean field games: application to an emission abatement game

The final chapter mainly makes a methodological contribution to the issues addressed in this thesis. Mean field games represent a family of games with a large number of players, in which

15. For example, while De Angelis et al. (2023) has a similar framework with equilibrium pricing and corporate optimization, solving it is done at the cost of assuming that investors know the dividends each company will pay in the future. Another example is Lavigne and Tankov (2023), which has a similar, less constrained framework, but does not obtain explicit solutions for the equilibrium price nor for the firm's optimal strategy.

16. The assumptions made for the passage to the limit on the number of firms are their homogeneity and the idiosyncraticity of the hazards that impact the dynamics of their environmental rating. In other words, companies are assumed to be identical and the controversies that affect them to be independent. Intuitively, the results that interest us in this mean field game should not depend on these simplifying assumptions, whose role is technical in facilitating its resolution.

each player interacts with the mass of the population rather than with each other individually.¹⁷ In particular, they can be used to approximate the solutions of dynamic stochastic n -player games. Developed in the field of financial mathematics, they have enjoyed great success since their simultaneous introduction by Huang et al. (2006) and Lasry and Lions (2007). Apart from a few very recent works (Bonesini et al., 2022; Campi and Fischer, 2022a; Campi et al., 2023), the notion of equilibrium considered to solve these games is generally the limit equivalent of the Nash equilibrium concept. However, in the field of game theory, Nash equilibrium is far from being the only equilibrium considered, and its limitations are well known. In particular, the conditions under which a population actually plays according to the Nash equilibrium are highly restrictive and require strong assumptions as to the players rationality (Aumann, 1987).

In contrast, the coarse correlated equilibrium, introduced implicitly in Hannan (1957) and then explicitly in Moulin and Vial (1978), is a notion of equilibrium that emerges more naturally from repeated games for example (Hart and Mas-Colell, 2003; Roughgarden, 2016). This type of equilibrium is a generalization of the Nash equilibrium. A Nash equilibrium is such that, if the other players play the equilibrium strategy, no player has any interest in deviating from its own equilibrium strategy. It therefore represents a fairly minimal concept of stability. A correlated equilibrium is a Nash equilibrium in which an external correlation mechanism is introduced to define the strategies that players can employ. In addition to being more natural, coarse correlated equilibria also have the advantage of being able to achieve higher levels of social utility than Nash equilibria (Dokka et al., 2022; Moulin et al., 2014). They are therefore particularly relevant in a game where the Nash equilibrium is socially inoptimal.

More precisely, a coarse correlated equilibrium is defined as follows. An external mechanism, which can be interpreted as a mediator, defines a distribution (probability law) over the space of strategy profiles of the players. This distribution is public. Based on a random draw from this distribution, the mediator privately recommends a strategy to each player. This distribution represents a coarse correlated equilibrium if each player prefers to commit to following the recommended strategy *before* receiving it,¹⁸ rather than deviating unilaterally, assuming that every other player will follow the recommendation received. The only information that the player has to decide whether to commit is the probability distribution of the recommendations. If this distribution induces independent recommendations from one player to another, the definition of a coarse correlated equilibrium reduces to that of a mixed Nash equilibrium. It is therefore the possibility of correlation between recommended strategies that makes it possible to establish a greater number of equilibria than the set of Nash equilibria (which it contains).

This concept of equilibrium in the framework of stochastic dynamic continuous-time mean

17. For a detailed introduction to this method, see for example Marcos Leutscher's thesis introduction (Leutscher de las Nieves, 2022).

18. The notion of correlated equilibrium (not "coarse") designates such a distribution when each player prefers to follow the recommended strategy *after* receiving it, rather than deviating unilaterally. It is therefore more restrictive than the notion of coarse correlated equilibrium, even if it also contains the set of Nash equilibria.

field games was defined in Campi et al. (2023), where its existence is established in a broad theoretical framework, though in a non-constructive way. In other words, the existence arguments in this paper do not allow to build nor identify precise coarse correlated equilibria. In the final chapter of this thesis, which is the fruit of a joint work with Luciano Campi and Federico Cannerozzi, we seek to identify analytically coarse correlated equilibria in linear quadratic mean field games. We explicitly characterize a subset of coarse correlated equilibria in a generic linear quadratic mean field game. The reason we restrict ourselves to such a subset is that the full set of coarse correlated equilibria is very large and complex to explore analytically. We establish a condition under which the coarse correlated equilibria in this subset exceed the Nash equilibrium in terms of utility. Note that, in a mean field game, as all players are assumed to be identical, individual utility is representative of the social utility. We therefore speak equivalently of social and individual utility in this framework. We also show that no generalized coarse correlated equilibria can exceed the optimal utility defined as the solution of the mean field control equivalent (which can be interpreted as that of a central planner). Furthermore, if the mean field control solution is not a Nash equilibrium, we show that its utility level cannot be achieved by any generalized coarse correlated equilibrium, as the latter is a non-cooperative equilibrium. We then propose a simple mechanism to explore this already rich subset and identify coarse correlated equilibria with interesting properties.

We illustrate this mechanism by applying it to a stylized game of emission abatement between countries.¹⁹ This game corresponds to a reformulation with a large number of players of a game developed by Dokka et al. (2022) out of the founding model of Barrett (1994). In the static version studied by Dokka et al. (2022) with n symmetric players, each country seeks to define an optimal emission abatement effort by maximizing its net benefits. Its gross benefits increase concavely with the total effort made by all countries, while it incurs private quadratic costs for its own abatement effort. In the mean field version of this game (i.e., with a number of players such that no player has an individual impact on the aggregation of efforts and therefore on the level of climate change; we then speak of “atomic” players), the Nash equilibrium corresponds to zero effort for each player. Indeed, any effort has a negligible impact on profits, while it has a cost. This is also the case for coarse correlated equilibria, which are reduced in this case to the single Nash equilibrium for the same reasons.

We slightly adapt this game by formulating it in a dynamic stochastic version and by adding a reputational cost for each player. For a given player, this cost is calculated as the quadratic distance between the integral of its past abatement effort and the integral of the average past abatement effort at each instant. In this game, the unique Nash equilibrium still leads to zero effort over the entire period for each player. On the other hand, the introduction of a reputational cost significantly enlarges the set of coarse correlated equilibria in this game. By

19. This application echoes that of Bahn et al. (2017) which develop another example of mean field game to study international climate negotiations, but with a completely different model.

applying in a very simple way the mechanism for exploring coarse correlated equilibria developed in our article, we identify and illustrate a subset of these equilibria that achieves high levels of emission abatement effort, as well as a level of collective net benefit well above that of the Nash equilibrium. We find that the higher the reputational cost relative to the private cost of abatement effort, the closer the collective net benefit achieved by our subset of coarse correlated equilibria is to the optimal net benefit level (calculated as the solution to the mean field control equivalent).

Thus, this work invites the rich literature on mean field games in mathematical finance to distance from the concept of Nash equilibrium, by illustrating the importance of its inefficiency in a common-good game, and by proposing a technically affordable alternative. Moving on, the application presented in this article raises two main questions. The first is the question of how to put a coarse correlated equilibrium into practice. Once an equilibrium has been identified as interesting on the basis of its social utility properties and climate ambition, how do we define the correlation mechanism and how do we force players to respect their commitment once the draw has been made? The second major question is that of modeling an international negotiation between countries using a mean field game, in which the players are typically atomic. Although this approach has already been used in the literature (Bahn et al., 2017), it is legitimate to think that it may miss a number of fundamental issues in these negotiations: the existence of dominant countries, bilateral issues, as well as the concentration of emissions in a fairly small subset of countries.

One way of answering the second question would be to apply our model to a game between a large number of small players seeking to preserve a common good. For example, we can think of the choice to take a plane in view of the environmental cost that this means of transport represents. While many individuals would benefit from climate change mitigation (ruling out the few individuals related to the flying industry in this case), the choice not to fly incurs an individual cost, as it reduces holidays or work opportunities for example. Moreover, individually, the choice not to fly has a negligible impact on global greenhouse gas emissions. In addition, it seems reasonable to assume that the reputational cost of flying is non-zero: if no one around you flies for ecological reasons, it becomes shameful to fly at all. We thus find the three key terms of the objective function defined in our application, this time with atomic players. According to the Nash equilibrium, in this case, no one prevents oneself from flying. Applied to this case, our results allow to identify coarse correlated equilibria which outperform significantly the Nash equilibrium in terms of collective (and individual) utility, with a level of significance varying on the relative cost of reputation with respect to the private cost of not taking the plane.

Then, a possible answer to the question of how to apply a coarse correlated equilibrium in this context could be as follows. If a non-governmental organization (or a media collective with a presence on social networks, for example) manages to send a sufficiently credible signal, thus playing the role of mediator, it could perhaps get the population to reach a coarse correlated

equilibrium that would be more optimal than the Nash equilibrium from an environmental point of view at least. While the exact implementation of this correlation mechanism would merit further study, the application we propose in this chapter already allows us to reflect on the conditions (comparison between individual cost and reputational cost, for example) under which such a correlation mechanism could favor the preservation of a common good while generating greater social utility.

Introduction

En 2015, deux événements clés ont souligné la nécessité de s'intéresser au système financier dans la lutte contre le changement climatique et ses conséquences. Le premier est l'accord de Paris, à l'occasion duquel 195 parties¹ s'engagent à limiter la hausse de la température globale à +2°C à la fin du siècle (+1.5°C si possible). L'Article 2, qui en est l'article principal, affirme l'importance de la gestion des flux financiers face au changement climatique :

« 1. Le présent Accord [...] vise à renforcer la riposte mondiale à la menace des changements climatiques [...] notamment en :

[...]

c) Rendant les flux financiers compatibles avec un profil d'évolution vers un développement à faible émission de gaz à effet de serre et résilient aux changements climatiques. »

– Accord de Paris, Article 2.

L'objectif d'atténuation du changement climatique nécessite la transition de l'économie vers des systèmes de production faiblement émetteurs en gaz à effet de serre (GES). Ce processus de transition, nommé transition bas-carbone, implique de financer massivement le déploiement d'énergies durables et de réallouer le capital de technologies fortement émettrices vers des moyens de production dont l'empreinte environnementale est réduite.

Par ailleurs, dans un discours qui a fait date quelques semaines avant la signature de l'accord de Paris, Mark Carney, alors Gouverneur de la Banque d'Angleterre, a mis en lumière la menace que le changement climatique représente pour la stabilité financière mondiale (Carney, 2015). Il y distingue notamment deux sources de risques financiers associés au changement climatique² : les « risques physiques », découlant directement des impacts physiques du changement climatique et

1. En date du 1^{er} janvier 2024, 194 Etats et l'Union Européenne sont signataires de l'accord de Paris, adopté lors de la COP21 en décembre 2015.

2. Il en évoque également une troisième dans ce discours : les « risques juridiques », qui menacent les responsables des émissions de gaz à effet de serre à qui des parties prenantes pourraient demander des comptes en justice pour leur participation à l'aggravation du changement climatique. Toutefois, ils peuvent être vus comme une sous-catégorie des risques physiques et de transition (NGFS, 2021a). En outre, ils ont été très peu abordés directement dans la littérature (macro)financière pour l'instant. Ils ne seront donc pas abordés spécifiquement dans cette thèse.

les « risques de transition » qui désignent l'ensemble des risques financiers causés par le processus de transition vers une économie bas-carbone.

Le système financier joue donc un rôle crucial dans l'atténuation du changement climatique. Mais il est lui-même exposé aux risques induits, non seulement par ce dernier, mais aussi par le processus de transition nécessaire à son atténuation. Ce sont les deux facettes de l'interaction entre changement climatique et système financier que cette thèse aborde.

Comprendre la façon dont le système financier peut favoriser, sans freiner, la transition bas-carbone et identifier les canaux par lesquels cette dernière, ainsi que le changement climatique, peuvent être une source d'instabilité financière présentent un certain nombre d'enjeux méthodologiques. En premier lieu, la modélisation de ces phénomènes nécessite de conjuguer trois pôles, dont l'intersection est rarement modélisée : le système financier, l'économie réelle et l'environnement, à travers notamment l'empreinte environnementale des activités économiques et financières ou leur exposition aux aléas climatiques. En second lieu, les conséquences du changement climatique et le déroulement de la transition bas-carbone font l'objet d'une incertitude radicale, qui repose notamment sur l'incertitude quant au comportement des agents vecteurs de la transition (politiques, producteurs, investisseurs et consommateurs) face au changement climatique. Enfin, le changement climatique et la transition bas-carbone induisent une rupture d'un certain nombre de tendances économiques historiques, ce qui réduit la pertinence des études sur données passées pour prévoir les horizons à venir.

Plan. La suite de cette introduction est organisée en deux parties. Dans la première partie, je développe les enjeux méthodologiques posés par la modélisation des deux facettes de l'interaction entre système financier et changement climatique, puis passe en revue les principaux types de modèles économiques et financiers qui permettent d'y répondre aujourd'hui. Dans la deuxième partie, j'énonce les principales contributions de cette thèse, selon ses quatre chapitres.

1 Modéliser l'interaction entre système financier et changement climatique

La modélisation de l'interaction entre système financier et changement climatique soulève un certain nombre d'enjeux, présentés en première partie de cette section. La deuxième partie propose un tableau des différents types de modélisation qui y répondent dans les littératures macroéconomique, microéconomique et financière.

1.1 Défis méthodologiques

La modélisation de l'interaction entre système financier et changement climatique se heurte à plusieurs difficultés méthodologiques, qui sont liées à la nature du changement climatique, à ses conséquences sur la société et à la place de la modélisation financière en économie.

Rupture de tendances et changement structurel Le changement climatique induit la rupture d'un grand nombre de tendances socio-économiques, que ce soit par ses impacts physiques qui vont s'amplifier d'une manière non linéaire au cours du siècle, mais également par les efforts d'atténuation qu'il suscite.

Le changement climatique a un impact sur la productivité des capitaux humains, physiques et naturels. Cet impact provient de la hausse de la fréquence et de l'intensité des aléas climatiques extrêmes (inondations, sécheresses, cyclones...), mais aussi d'évolutions climatiques continues comme la hausse de la température globale, la hausse du risque d'épidémies ou encore la hausse du niveau de la mer (IPCC, 2021). Ces aléas et évolutions climatiques peuvent contribuer à la dégradation d'actifs physiques, à l'obsolescence productive de territoires qui deviendraient trop secs ou submergés, à la perturbation de moyens de transport et de télécommunication et à une moindre capacité de travail de la population active (Basel Committee, 2021). Ces effets du changement climatique, s'ils commencent à apparaître, sont encore peu perceptibles pour certains, et vont continuer à s'amplifier pour d'autres, d'une manière qui ne peut être déduite par la simple projection statistique de tendances passées (IPCC, 2021).

En outre, l'atténuation du changement climatique nécessite un changement structurel profond de l'économie (Semieniuk et al., 2021; Ciarli and Savona, 2019). La transition vers une économie bas-carbone implique que la part des activités peu émettrices en carbone dans la valeur ajoutée augmente massivement, contrairement à celle des activités intensives en émissions de gaz à effet de serre qui doit fortement diminuer (Campiglio and van der Ploeg, 2022). Un important transfert de capitaux physiques et financiers doit s'effectuer pour cela, impliquant également une modification profonde des chaînes de valeur (Cahen-Fourot et al., 2020) et des habitudes de consommation. Or, ce processus de transition ne correspond pas au prolongement de tendances passées. Il découle d'efforts volontaristes pour la prise en compte d'une externalité économique négative, jusque-là négligée (Campiglio and van der Ploeg, 2022). La transition bas-carbone repose sur l'évolution de ses quatre principaux vecteurs : (i) les politiques climatiques, comprenant par exemple la taxe carbone et les subventions à l'investissement dans les énergies renouvelables, (ii) les innovations technologiques, qui permettent aux énergies renouvelables d'être plus abordables, d'améliorer l'efficacité énergétique des systèmes de production et de transport, ainsi que de capturer et stocker le carbone dans l'atmosphère, (iii) le « sentiment de marché », qui reflète les attentes et les préférences des investisseurs quant au déroulé de la transition bas-carbone et qui influence leurs décisions d'investissement et (iv) les préférences des consommateurs, qui déterminent la demande relative adressée aux produits « carbonés » ou polluants (Task Force on Climate-Related Financial Disclosures, 2017).

La transition bas-carbone repose sur le comportement des agents Une difficulté cruciale dans l'étude des impacts socioéconomiques du changement climatique et des efforts mobilisés pour son atténuation est l'incertitude radicale qui les caractérise (Weitzman, 2011; Bolton

et al., 2020). Cette incertitude peut être décomposée en deux dimensions principales. La première est une incertitude scientifique, qui porte aussi bien sur les modèles climatiques (Brock and Hansen, 2018; Heal and Millner, 2013) que sur les modèles économiques qui s'intéressent à l'impact des efforts d'atténuation sur les émissions de gaz à effet de serre et aux impacts économiques du changement climatique (Barnett et al., 2021; Kriegler et al., 2014)³. Cette incertitude scientifique se décline entre ambiguïté et erreur des modèles climatiques et économiques selon la terminologie de Barnett et al. (2021).

La deuxième dimension de cette incertitude radicale concerne les modalités et la temporalité de la transition bas-carbone, qui reposent principalement sur le comportement des agents vis-à-vis du changement climatique, en tant que potentiels vecteurs de la transition : politiques, producteurs, investisseurs et consommateurs⁴. Ainsi, les hypothèses sur lesquelles se fondent les scénarios de transition « désordonnée »⁵ utilisés par les banques centrales dans leurs analyses de scénario climatiques, établis comme sources potentielles d'instabilité financière, donnent la part belle aussi bien aux politiques climatiques qu'aux anticipations des agents les concernant et traduisent un manque de coordination entre ces deux types d'acteurs (NGFS, 2020; Semieniuk et al., 2021). En effet, dans un contexte d'incertitude, l'anticipation des agents quant aux politiques climatiques peut en effet avoir une influence déterminante sur l'impact même de ces politiques (Biais and Landier, 2022; Campiglio et al., 2023b). En outre, l'hétérogénéité des agents, leurs biais comportementaux et leur rationalité limitée ont également un impact sur l'efficacité des politiques climatiques étant donnée leur incertitude ou faible crédibilité (Diluiso et al., 2020; Ceccarelli and Ramelli, 2024). Face au gain incertain de la recherche et développement dans les technologies à faible empreinte environnementale, les décisions d'investissement des entreprises peuvent être sous-optimales sans politique adaptée par rapport à l'objectif de transition bas-carbone selon l'accord de Paris (Bustamante and Zucchi, 2024). En particulier, étant donné le rôle déterminant du secteur financier dans le déroulement de la transition bas-carbone, les croyances et anticipations des investisseurs eux-mêmes contribuent fortement à l'orienter ou la

3. Voir aussi le nombre de modèles économie-climat utilisés pour simuler les scénarios de transition dans le sixième rapport d'évaluation du Groupe d'experts intergouvernemental sur l'évolution du climat (IPCC, 2022, Chapter 3).

4. Parmi les quatre vecteurs de la transition bas-carbone évoqués dans Task Force on Climate-Related Financial Disclosures (2017) et rappelés dans le paragraphe précédent, seul le progrès technologique ne comporte pas de mention explicite d'agents ; toutefois, on peut arguer que ce progrès dépend fortement des efforts d'investissement dans la recherche et développement, déterminée par les acteurs économiques, politiques et financiers. On peut aussi inclure plus largement, parmi les incertitudes concernant les acteurs politiques, les enjeux géopolitiques qui impactent la coordination internationale dans la lutte contre le changement climatique, mais aussi par exemple les guerres qui sont fortement émettrices en gaz à effet de serre et perturbent les efforts d'atténuation, comme la guerre en Ukraine.

5. Cette terminologie désignant un type de scénarios de transition bas-carbone est employée par le Réseau des banques centrales et superviseurs pour le Verdissement du Système Financier (Network of central banks and supervisors for Greening the Financial System, NGFS) pour qualifier certains de ses scénarios de transition de référence. Elle désigne des scénarios de transition « indésirables », dans lesquels les politiques climatiques sont brusques ou tardives, ce qui implique des hausses rapides et inattendues de taxe carbone par exemple. Par une faible préparation des agents à de telles hausses, les coûts de ces formes de transition sont élevés pour la société, à l'opposé d'une transition « ordonnée », qui décrirait un scénario « idéal » de transition en termes de stabilité économique et financière, pour un même objectif de neutralité carbone en 2050.

freiner (Comerford and Spiganti, 2023; De Angelis et al., 2023).

L'incertitude quant au déroulement de la transition bas-carbone est un obstacle aussi bien aux projections la concernant qu'aux projections portant sur le changement climatique, puisque l'ampleur de ce dernier dépend des efforts faits pour son atténuation. rend difficile aussi bien les projections qui dépendent de la transition bas-carbone que sur les projections concernant le changement climatique et ses impacts, puisque leur ampleur dépend des efforts d'atténuation. Comprendre le comportement des agents face au changement climatique permet donc d'identifier une partie importante des leviers et des freins à la transition bas-carbone et ainsi d'éclairer la façon dont les politiques climatiques doivent être orientées pour la favoriser. La microéconomie et la théorie des jeux fournissent un grand nombre d'outils théoriques pour comprendre le comportement des acteurs clés de la transition bas-carbone dans ce contexte inédit, et notamment des acteurs du système financier, et ainsi comprendre leur place dans l'atténuation du changement climatique.

Les difficultés liées à la modélisation du rôle du secteur financier Le secteur financier a un rôle clé à jouer dans la transition bas-carbone. Il doit permettre de financer le déploiement massif de technologies renouvelables, et participer à l'effort de réallocation de capitaux des secteurs fortement émetteurs en gaz à effet de serre vers des secteurs dont les activités sont plus durables. Sa participation spontanée à cette évolution structurelle n'est pourtant pas évidente, du fait de la présence de frictions financières (Campiglio, 2016). A l'inverse, il risque d'être impacté par le bouleversement massif qu'un tel changement structurel de l'économie implique, ainsi que par l'amplification à venir des impacts physiques du changement climatique. C'est la double matérialité : le système financier a un impact sur le changement climatique du fait de l'orientation de ses financements, et il est impacté par ce dernier et par les efforts d'atténuation (Oman and Svartzman, 2021).

Étant donné que la double matérialité intervient à travers l'économie réelle, la compréhension de ses deux facettes nécessite en particulier de modéliser l'interaction entre le système financier et l'économie réelle, avec l'environnement (Battiston et al., 2021; Campiglio et al., 2023a; Svartzman et al., 2021a). La macroéconomie néoclassique et néokeynésienne répond à ce problème en introduisant des frictions financières dans ses modèles (Carattini et al., 2023; Pan et al., 2021). En parallèle, des modèles hors équilibre, de réseau (Battiston et al., 2017; Roncoroni et al., 2021), agent-basés (Lamperti et al., 2021) ou stock-flux cohérents (Bovari et al., 2018; Dunz et al., 2021; Daumas, 2023a) donnent souvent un plus grand rôle au système financier, notamment à travers son potentiel déstabilisateur (Farmer et al., 2015). Il est notable que cette deuxième classe de modèle, longtemps restée en marge, gagne en légitimité institutionnelle en se rapprochant notamment des banques centrales à la suite de la crise de 2008 et de l'émergence de la question climatique en finance (Bolton et al., 2020; Svartzman et al., 2021a; Battiston et al., 2021).

Modéliser la façon dont le système financier peut accompagner, voire être moteur pour la transition bas-carbone se heurte aux obstacles suivants. D’abord, un tel objectif nécessite de pouvoir modéliser la façon dont l’économie réelle subit des changements structurels dans le contexte de la transition bas-carbone, ce qui n’est pas permis de la même façon par toutes les familles de modèles (Ciarli and Savona, 2019). En outre, la littérature qui s’intéresse à ces questions sépare en général la modélisation du changement structurel de l’économie de celle de ses impacts sur le système financier (Semieniuk et al., 2021). En ce qui concerne la mesure de l’exposition aux risques physiques du système financier, elle implique de modéliser l’interaction entre économie réelle et changement climatique, en traduisant notamment les dommages matériels associés au changement climatique en termes de pertes financières (défauts, dévaluations d’actifs, impacts sur les dividendes). Or, c’est une tâche très complexe, ce qui explique les controverses récurrentes touchant les modèles intégrés économie-climat (Stern et al., 2010; Nordhaus, 2007; Heal and Millner, 2014; Pindyck, 2013; Espagne et al., 2012). A cette difficulté s’ajoute celle d’intégrer le système financier dans ces modèles, ce qui permettrait de saisir les deux facettes de la double matérialité.

1.2 Une littérature émergente en réponse à ces défis

La rupture de tendances socioéconomiques induite par le changement climatique et les potentiels changements dans le comportement des agents en réponse à ces évolutions appellent à une approche prospective fondée sur la modélisation macroéconomique, microéconomique et financière⁶ (Svartzman et al., 2021a; Campiglio and van der Ploeg, 2022). En effet, en paramétrisant certaines caractéristiques structurelles de l’économie comme les préférences des agents, ces outils de modélisation permettent d’anticiper certains effets de leur évolution dans le contexte du changement climatique. En outre, ces cadres de modélisation permettent de prendre en compte l’incertitude à laquelle font face les agents dans ce contexte.

La place du système financier dans la modélisation macroéconomique Une littérature très récente développe l’usage de modèles d’équilibre général (très souvent dynamique stochastique, EGDS) comportant un intermédiaire financier (bancaire) et deux secteurs productifs, l’un dont la technologie de production est intensive en émissions de GES (répondant aux terminologies « polluant », « marron », « sale »), l’autre considéré faiblement émetteur (« neutre », « vert », « propre »). La modélisation d’une banque centrale permet d’évaluer l’impact d’un grand nombre de politiques monétaires climatiques comme les subventions aux prêts adressés au secteur vert ou les pénalités sur les prêts au secteur marron, mais aussi un assouplissement quantitatif « vert », parfois conjointement à l’introduction d’une taxe carbone et d’autres politiques climatiques. Ainsi, Benkhodja et al. (2022, 2023); Ferrari and Landi (2024) évaluent

6. Comme le soulignent Campiglio and van der Ploeg (2022), l’appel à ces approches n’invalide pas l’utilisation de méthodes empiriques, qui ont également leur rôle à jouer (Giglio et al., 2021). Simplement, ces défis méthodologiques rendent particulièrement nécessaires les approches prospectives permises par ces types de modélisation.

l'efficacité de ces politiques à l'aune de leur capacité à favoriser l'alignement de l'économie à une trajectoire de transition à 2°C ou 1.5°C et concluent avec une analyse de welfare. C'est également ce que font Comerford and Spiganti (2023), pour l'évaluation d'un scénario de politique de transition très particulier, bien que l'intermédiaire financier n'y soit pas modélisé explicitement. En fonction de ces hypothèses, ces articles concluent à une plus ou moins grande pertinence des mesures adressées aux intermédiaires financiers dans le contexte de la transition bas-carbone, leur donnant ainsi parfois un rôle assez faible dans la transition. Böser and Colesanti Senni (2021) se pose plutôt la question de la stabilité financière conjointement à l'atteinte des objectifs environnementaux, en considérant que le secteur « marron » est risqué car exposé au risque de transition. Certains se posent des questions similaires mais en ajoutant des frictions financières à leurs EGDS environnementaux (Diluiso et al., 2021; Benmir and Roman, 2020; Carattini et al., 2023; Lessmann and Kalkuhl, 2024). L'introduction de frictions financières souligne la façon dont le manque de financement spontané du secteur vert par les banques peut être un frein à la transition bas-carbone.

Dans un cadre un peu différent, car destiné à l'évaluation de l'impact des risques physiques du changement climatique, Van der Straten (2023) construit un modèle de croissance redistributive avec un seul secteur productif, dont les actifs physiques sont menacés par les conséquences physiques du changement climatique. La prise en compte de contraintes financières pour les ménages permet de montrer que les conséquences physiques du changement climatique peuvent contribuer à augmenter les inégalités. Il est à noter que Benmir and Roman (2020), Comerford and Spiganti (2023) prennent également en compte, à travers une fonction de dommages, les conséquences physiques du changement climatique⁷. Par ailleurs, il existe une littérature proposant l'évaluation d'actifs dans le contexte de la transition bas-carbone (production-based asset pricing) (Hambel et al., 2023a; Donadelli et al., 2019), y compris en prenant en compte l'incertitude radicale caractérisant ce contexte (Barnett et al., 2021), mais sans modéliser de secteur ou d'acteurs financiers (Giglio et al., 2021). Ces modèles sophistiqués d'évaluation pourraient s'insérer dans une analyse de stabilité financière associée à la transition. L'absence de module financier empêche, néanmoins, de comprendre le rôle joué par les acteurs financiers eux-mêmes dans la conduite de la transition.

Ces modèles permettent de représenter l'évolution structurelle de l'économie dans le cadre

7. Ces derniers peuvent ainsi s'associer à la catégorie des modèles intégrés économie-climat. Ces modèles opèrent un couplage entre une modélisation économique et une modélisation climatique, en représentant le cycle qui relie production, émissions de GES, changement climatique et dommages économiques. Ils permettent, notamment, d'évaluer les politiques climatiques dans leur capacité à atténuer le changement climatique. Nordhaus (1992) est le premier à avoir proposé un tel modèle intégré, sous un format très stylisé. La famille des modèles intégrés économie-climat s'est ensuite beaucoup agrandie, puisant dans différentes familles de modélisation macroéconomiques (Hourcade et al., 2021). Malgré un enrichissement certain de la partie économique de ces modèles, ils intègrent rarement un module financier. Les contraintes imposées par le couplage économie-climat sont telles que la partie économique dans les modèles intégrés économie-climat est souvent très simplifiée. C'est pourquoi je ne les mentionne pas plus ici. Néanmoins, dans le premier chapitre de cette thèse, la façon dont ces modèles peuvent tout de même être utilisés pour évaluer les risques financiers provenant de la transition bas-carbone est abordée à travers la présentation des stress tests climatiques menés par les banques centrales.

de la transition bas-carbone, ou en relation avec les conséquences physiques du changement climatique, parfois en interaction avec certaines contraintes financières. Avec une modélisation pourtant très simple du système financier, ils parviennent déjà à montrer comment le manque de financements peut être un frein à la transition bas-carbone, ou à l'adaptation au changement climatique (Van der Straten, 2023). Ils soulignent ainsi l'importance de s'intéresser au rôle des acteurs financiers dans la transition et l'adaptation au changement climatique. Néanmoins, le module financier y est en général très fruste, ce qui ne permet pas d'explorer l'ensemble des freins et leviers à la transition et l'adaptation qu'on peut trouver dans le système financier, ni même d'explorer les conséquences en termes de stabilité financière de la transition bas-carbone et du changement climatique. Face à cet écueil, il y a deux possibilités pour l'analyse de la stabilité financière dans le contexte de la transition bas-carbone : (i) combiner des modèles économiques et financiers (Semieniuk et al., 2021; ACPR, 2020) pour évaluer la stabilité financière, bien que cela pose des problèmes de cohérence dans la modélisation (Hourcade et al., 2021; Jacquetin, 2021). (ii) Compléter l'analyse avec des modèles hors équilibre, qui peuvent se permettre d'être plus riches et intégrer des effets de réseaux (Svartzman et al., 2021a). Ainsi, les modèles stock-flux cohérents permettent de donner plus d'importance au système financier (Bovari et al., 2018; Dunz et al., 2021; Dafermos et al., 2018; Gourdel et al., 2024, Daumas, 2023a, Chapitre 2), tout comme les modèles agents-basés (Lamperti et al., 2021). Par exemple, (Daumas, 2023a, Chapitre 2) modélise deux types d'agents financiers, une institution bancaire et une institution financière non bancaire. Il trouve que les institutions non bancaires résistent moins bien que les banques au risque de transition. En outre, les institutions financières subissent aussi bien des risques financiers de court terme que de long terme associés à la transition, ce que peu de cadres de modélisation sont capables d'établir. L'importance d'une modélisation riche du système financier dans le contexte de la transition bas-carbone est également soulignée par les modèles de réseau (Battiston et al., 2017; Roncoroni et al., 2021; Sydow et al., 2021). En outre, que ce soit par la prise en compte d'un grand nombre de frictions financières (Lamperti et al., 2021), ou par la modélisation d'anticipations des acteurs bancaires (Dunz et al., 2021), ces modèles permettent également d'approfondir la compréhension du rôle des acteurs financiers dans la transition bas-carbone.

Les apports de la modélisation microéconomique La modélisation micro- ou méso-économique permet de compléter l'approche macroéconomique dans la compréhension du rôle et du comportement du système financier dans le contexte de la transition bas-carbone. Il est complexe pour les agents économiques et financiers d'évoluer dans ce nouveau contexte, face à ces risques nouveaux, et leur comportement peut dépendre d'un grand nombre de paramètres. Ces outils de modélisation permettent de « zoomer » sur leurs motivations, leurs intérêts et leurs pratiques avec plus de créativité, car ils sont moins contraints par l'ambition d'un cadre macroéconomique. Notamment, les outils de mathématiques financières sont particulièrement adaptés

à la modélisation du comportement des agents dans un monde incertain et plus spécifiquement celui des acteurs financiers. Ces outils permettent donc de compléter l’approche macroéconomique avec une visée explorative du comportement des agents, offrant la possibilité de déduire des comportements dynamiques complexes.

La littérature théorique en finance durable s’intéressant à l’évaluation d’actifs permet par exemple de comprendre l’impact des préférences (Pástor et al., 2021; Pedersen et al., 2021a; Zerbib, 2022), des stratégies d’investissement (Zerbib, 2022) et des croyances hétérogènes des investisseurs sur les prix des actifs à l’équilibre ainsi que leur impact sur les décisions de réduction d’émission des entreprises (De Angelis et al., 2023) et ce dans des contextes dynamiques et d’opacité de l’information par exemple (Avramov et al., 2022a). Les préférences des investisseurs peuvent être combinées à la modélisation des préférences des consommateurs pour comprendre leur impact agrégé sur la transition de l’économie (Sauzet and Zerbib, 2022b). En outre, l’approche microéconomique permet également de modéliser finement la façon dont les anticipations ou croyances des agents s’adaptent dans un contexte d’incertitude radicale quant au scénario de transition et d’en déduire l’impact de ces anticipations sur les décisions d’investissement (Flora and Tankov, 2023) et le prix des obligations (Le Guenedal and Tankov, 2022a).

Bien que certains travaux mettent en évidence des effets de propagation intersectorielle (Cahen-Fourot et al., 2021) et intrasectorielle (Battiston et al., 2017; Roncoroni et al., 2021) des risques de transition et bien que la littérature macroéconomique souligne l’interdépendance entre les secteurs et leurs agents, peu de travaux s’intéressent aux interactions intrasectorielles intervenant entre les acteurs clés de la transition bas-carbone. La plupart des contributions de cette thèse répondent à cette lacune de la littérature, en se fondant sur les outils fournis par la microéconomie, la théorie des jeux et les mathématiques financières, qui sont adaptés à cet objectif.

2 Contributions principales

Dans cette thèse, je m’intéresse à la façon dont les modélisations macroéconomique et surtout microéconomique et permettent de répondre aux enjeux soulevés en première partie de cette introduction, par leur approche prospective et leur capacité à éclairer le comportement d’agents clés, notamment financiers, dans le contexte de la transition bas-carbone, en donnant une attention particulière à leurs interactions stratégiques. Dans le Chapitre 1, je passe en revue les méthodologies de stress tests climatiques développées ces dernières années par des superviseurs financiers et articles académiques et qui s’appuient largement sur la modélisation macroéconomique intégrée économie-climat.

Les trois chapitres suivants sont consacrés au développement de modèles microéconomiques pour éclairer le comportement des agents dans le contexte de la transition bas-carbone. Dans le Chapitre 2, je développe un modèle de financement à une période dans lequel plusieurs banques

financent une entreprise dont les actifs ont vocation à être échoués dans le contexte de la transition et j'en déduis l'impact du désinvestissement de l'une d'elles grâce à des outils de théorie des jeux. Dans le Chapitre 3, je construis un modèle d'équilibre dynamique stochastique avec asymétrie d'information à propos de la valeur environnementale des entreprises et en tire des conclusions sur les incitations et les barrières que les investisseurs créent à la pratique de l'éco-blanchiment. Enfin, dans le Chapitre 4, je propose une méthodologie pour identifier des équilibres corrélés généralisés, qui représentent une alternative aux équilibres de Nash intéressante pour les jeux de bien commun, dans des jeux à champ moyen linéaires quadratiques et illustre leur pertinence grâce à une application à un jeu de réduction d'émissions.

Il convient de souligner que l'ordre des chapitres reflète une progression de la réflexion et de la technicité dans la réponse aux questions de recherche soulevées par cette thèse. Le premier chapitre pose les bases des questions auxquelles répondent les trois chapitres suivants. L'idée principale du deuxième chapitre, en particulier, découle directement des conclusions de la revue de littérature sur les stress tests climatiques. Cette revue de littérature confirme également l'importance de contribuer à la littérature en finance climat par une approche théorique, en mettant l'accent sur le comportement des agents et leurs interactions stratégiques. En outre, elle motive l'utilisation d'une modélisation dynamique stochastique, en écho aux scénarios de transition de long terme et à la nécessité de mieux comprendre la dynamique des bilans des institutions financières. Par conséquent, un cadre dynamique stochastique, basé sur des outils de la finance mathématique, est utilisé dans les deux derniers chapitres de la thèse. Le troisième chapitre, sur l'éco-blanchiment, peut être considéré comme le plus complet et le plus représentatif des ambitions de cette thèse, car il combine des outils sophistiqués de la finance mathématique et de la théorie des jeux pour répondre à une question clé de la finance durable, et inclut une validation empirique. Le quatrième chapitre complète cette thèse en développant une perspective conceptuelle et architectonique, également très pertinente pour explorer les leviers de la transition bas-carbone.⁸

2.1 Stress tests climatiques : revue méthodologique

Depuis le discours de Mark Carney (Carney, 2015), la conscience que le changement climatique et la transition bas-carbone peuvent être les sources d'importants risques financiers a émergé parmi les banques centrales et un nombre grandissant d'acteurs financiers (NGFS,

8. La valorisation des quatre chapitres de la thèse est la suivante : le premier chapitre, à auteur unique, a été mis en ligne sur SSRN en août 2022 (https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4179311) et soumis sans succès à deux journaux académiques de premier plan. Depuis lors, des études plus complètes ayant été publiées, il n'a pas été soumis à nouveau. Il servira toutefois de base à un chapitre de manuel. Le deuxième chapitre, également à auteur unique, n'a pas encore été publié en ligne et peut être considéré comme un document de travail préliminaire. L'article relatif au troisième chapitre, coécrit avec Peter Tankov et Olivier David Zerbib, a été publié sur SSRN en décembre 2023 (https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4644741) et soumis à une revue du top finance en mars 2023. L'article relatif au dernier chapitre, co-écrit avec Luciano Campi et Federico Cannerozzi, a été publié sur arxiv en novembre 2023 (<https://arxiv.org/abs/2311.04162>) et immédiatement soumis à un journal de finance mathématique.

2018, 2019; Bolton and Kacperczyk, 2021; Krüger et al., 2020). Face à l'incertitude radicale (knightienne) qui les caractérise, leur mesure est tout autant un enjeu crucial qu'un défi substantiel pour les superviseurs financiers qui, dans le cadre de leur mandat de stabilité financière, cherchent à les prévenir.

Les défis méthodologiques sont notamment les suivants (Bolton et al., 2020). Les deux types de risques financiers associés au changement climatique, risques physiques et de transition, affectent l'économie par des prismes différents, l'exposition géographique et sectorielle respectivement, ce qui nécessite des données très granulaires pour évaluer l'exposition des entreprises à ces risques. En outre, l'absence d'une situation comparable au changement climatique et à la transition bas-carbone dans le passé diminue la pertinence de l'usage statistique de données observées pour évaluer ces risques, appelant à une approche fondamentalement prospective pour les mesurer. En outre, si les canaux de risque classiques (risque de crédit, de marché, de liquidité) restent pertinents pour comprendre leur impact sur les institutions financières (Basel Committee, 2021), les vecteurs de ces risques (taxe carbone ou hausse de la fréquence et de l'intensité d'événements climatiques extrêmes à titre d'exemples) ont été peu considérés jusqu'alors et peuvent se transmettre de façon complexe dans l'économie (chaînes de valeur, effets redistributifs et interdépendance des deux types de risques).

La difficulté de mesurer et prévoir l'étendue de ces risques conduit à une forme d'opacité de l'information concernant l'exposition des entreprises et institutions financières à ces risques et amène à douter de la capacité des prix actuels à les intégrer parfaitement (Campiglio et al., 2023a; Daumas, 2023b; Acharya et al., 2023). La possibilité que ces risques soient mal évalués par les acteurs financiers soulève à juste titre le danger qu'ils puissent être source d'instabilité financière (Semieniuk et al., 2021; Litterman et al., 2020; FSB, 2020; Bolton et al., 2020).

Les stress tests climatiques apportent un certain nombre de réponses à ces défis, en se présentant à la fois comme outils d'évaluation et d'exploration de ces risques, mais aussi et par conséquent comme outils de supervision pour les banques centrales. Un stress test climatique consiste à tester la résilience d'institutions financières ou du système financier dans son ensemble face aux risques climatiques, à partir de scénarios de stress décrivant leur matérialisation. Chaque scénario se fonde sur un ensemble d'hypothèses quant à la matérialisation de ces risques, constituant son « narratif ». L'élaboration du scénario s'appuie alors sur des outils de modélisation économique et financière, permettant d'établir les conséquences de ce narratif sur les contreparties auxquelles sont exposées les institutions financières. Typiquement, un scénario de risque de transition a pour narratif une certaine hausse du prix du carbone et fournit un ensemble de projections économiques et financières décrivant les conséquences de cette hausse sur les contreparties des institutions financières participant à l'exercice. Ces projections permettent alors d'évaluer l'effet de chaque scénario de stress sur leur bilan.

Le premier stress test climatique qu'on peut recenser, Battiston et al. (2017), provient du monde académique. Il s'inscrit dans la lignée des études sur le risque systémique au sein des

réseaux financiers qui ont été stimulées par la crise financière de 2008 (Battiston et al., 2012; Battiston and Caldarelli, 2013; Acemoglu et al., 2015; Hurd et al., 2016). Un peu plus tard, des chercheurs de la Banque Nationale Néerlandaise (BNN) essaient d'évaluer l'exposition d'un grand nombre d'institutions financières néerlandaises aux risques de transition grâce aux riches données dont ils disposent en interne (Vermeulen et al., 2021⁹), tandis que la Banque de France (BdF), pionnière dans le genre, met en oeuvre un exercice pilote impliquant un grand nombre d'institutions financières dans un schéma « bottom-up » (ACPR, 2020). La Banque d'Angleterre (BA; BoE, 2021) et la Banque Centrale Européenne (BCE; ECB, 2021) ont pris le pas dans la foulée, juste avant que cette pratique ne se déploie largement : à la fin de l'année 2022, FSB and NGFS (2022) recense un total de 67 exercices menés ou prévus par 53 institutions, banques centrales ou superviseurs financiers. En particulier, la plupart de ces exercices s'appuient sur les scénarios de référence construits par le Réseau des Banques Centrales et Superviseurs pour le Verdissement du Système Financier (NGFS, 2020, 2021b, 2022). Le monde académique n'a pas été en reste, multipliant également les publications sur le sujet (Battiston et al., 2019; Roncoroni et al., 2021; Reinders et al., 2020; Nguyen et al., 2020; Bikakis, 2020; Mandel, 2021; Mandel et al., 2021; Gourdel and Sydow, 2021; Bressan et al., 2022a; Le Guenedal et al., 2022).

Le premier chapitre de cette thèse propose un état des lieux des méthodologies de stress test élaborées par les banques centrales et articles académiques jusqu'en avril 2022¹⁰. Il interroge la capacité de ces méthodologies à évaluer l'ensemble des risques liés au changement climatique et à la transition bas-carbone touchant le système financier. Dans un premier temps, il présente les risques physiques et de transition et discute l'ensemble des canaux par lesquels ces derniers peuvent affecter les institutions financières ainsi que la façon dont ils pourraient constituer une menace pour la stabilité financière. Dans un deuxième temps, il établit les défis méthodologiques posés par l'évaluation de ces risques pour le système financier. Il présente alors la méthode du stress test climatique en discutant la façon dont elle répond à ces défis. Dans un troisième temps, il passe en revue les méthodologies de stress tests climatiques élaborées par les banques centrales et articles académiques du périmètre défini. Notamment, il évalue la façon dont ces méthodologies dans leur ensemble couvrent les différents vecteurs des risques physiques et de transition ainsi que les canaux financiers par lesquels ces risques peuvent affecter les institutions financières (risque de crédit, de marché, de liquidité notamment). Enfin, il dresse des pistes de recherche pour améliorer la prise en compte de l'ensemble des risques physiques et de transition touchant le système financier.

9. La publication en revue est datée en 2021, mais le document de travail correspondant a été mis en ligne en 2019.

10. Ce périmètre temporel couvre le stress test de la BNN, les exercices pilotes de la BdF, de la BA et de la BCE ainsi que les articles académiques cités à la fin du paragraphe précédent. Ce périmètre est défini ainsi pour plusieurs raisons : le chapitre ayant été écrit au début de la thèse, il ne couvre en profondeur que les premiers stress tests ; en outre, certaines conclusions de ce chapitre sont toujours d'actualité ; enfin, l'ambition d'exhaustivité pour ce chapitre est rendue inutile par la publication de revues de littérature très complètes peu de temps après (FSB and NGFS, 2022; Acharya et al., 2023). Ce chapitre est d'ailleurs inscrit dans cette temporalité puisque l'article correspondant, mis sur SSRN en 2022, est cité par Acharya et al. (2023).

Cette approche permet d'établir les constats suivants. En premier lieu, les approches adoptées par les superviseurs et dans la littérature académique sont très différentes. Les premiers semblent converger vers l'emploi de scénarios de transition sophistiqués, de long-terme, dans ce qui ressemble surtout à des analyses de scénario, dans le cadre organisé par le Réseau des banques centrales et superviseurs pour le Verdissement du Système Financier (NGFS)¹¹. Les seconds développent des méthodologies beaucoup plus hétérogènes, surtout de court terme, sauf pour les risques physiques (Mandel et al., 2021), souvent centrées sur une étape spécifique de la méthodologie de stress test. Une conséquence de ces différences est que les scénarios développés dans la littérature académique sont souvent plus défavorables. En effet, les scénarios de transition de long-terme sur lesquels s'appuie l'exercice de l'ACPR-BdF, et qui seront à la fondation des scénarios de référence construits par la suite par le NGFS (NGFS, 2020, 2021b, 2022), paraissent plus informatifs que défavorables. Leur aspect peu défavorable provient notamment d'une minimisation de l'incertitude à laquelle les agents font face de le contexte de la transition, d'une modélisation économique qui ne permet pas ou peu de modéliser des dépressions économiques, mais aussi de l'absence de prise en compte du rôle des acteurs financiers dans la propagation des risques de transition et de leur impact sur le déroulement de la transition elle-même.

En second lieu, le chapitre souligne que les méthodologies de stress tests climatiques et d'analyses de scénario couvrent inégalement les vecteurs des risques de transition et les canaux financiers par lesquels ils peuvent impacter les institutions financières. Notamment, le vecteur de sentiment de marché est rarement considéré, de même que la transmission des risques climatiques par le canal du risque de liquidité.

Le chapitre se conclut par la proposition de développer deux outils distincts en réponse à ces constats. D'abord, l'intégration de scénarios adverses de court terme dans les exercices climatiques des superviseurs pour les rapprocher des exercices de stress tests plus standards et mieux répondre à l'objectif de mesurer les risques d'instabilité financière. En outre, le chapitre propose de développer un autre type d'exercice, le « test de réallocation stratégique », qui tiendrait compte du rôle du secteur financier dans la transition vers une économie bas-carbone, et qui permettrait de fournir des outils de modélisation adaptés à la représentation des bilans dynamiques des banques.

2.2 Désinvestissement des banques et actifs échoués dans la transition bas-carbone

Atteindre l'objectif de température de l'accord de Paris implique une quantité significative d'actifs échoués (McGlade and Ekins, 2015). L'échouage d'actifs peut être compris comme la radiation prématurée ou la conversion au passif d'actifs dont la valeur dépend fortement des émissions de carbone et qui ne peuvent être convertis à faible coût en une activité rentable et à

11. Depuis, des scénarios de choc de court-terme et potentiellement plus sévères ont été développés pour pallier cet écueil (ECB, 2022; NGFS, 2023a).

faibles émissions (Caldecott et al., 2013; Van der Ploeg and Rezai, 2020). Ce phénomène touche les entreprises de combustibles fossiles (McGlade and Ekins, 2015; Semieniuk et al., 2022), mais peut également affecter d'autres secteurs en raison de la forte dépendance de l'économie à l'égard des activités liées aux combustibles fossiles (Cahen-Fourot et al., 2020, 2021). Aujourd'hui, des études théoriques et empiriques font craindre que le risque d'échouage d'actifs ne constitue une menace pour la stabilité économique et financière (Carattini and Sen, 2019; Van der Ploeg and Rezai, 2020; Cahen-Fourot et al., 2021; Mercure et al., 2021; Semieniuk et al., 2022; Comerford and Spiganti, 2023).

Le sentiment de marché et, plus largement, les décisions de financement des acteurs financiers comptent parmi les moteurs du risque d'échouage d'actifs (Battiston et al., 2023; Comerford and Spiganti, 2023). Aujourd'hui, de plus en plus de banques se sont engagées à atteindre la neutralité carbone, par exemple dans le cadre de l'initiative Net-Zero Banking Alliance, ce qui implique en particulier la sortie progressive du financement des combustibles fossiles. Cela soulève la question de savoir si le respect des engagements des banques en matière de neutralité carbone pourrait provoquer l'échouage d'actifs. En effet, même si la crédibilité de ces engagements est discutable en raison de leur manque de précision (Maio et al., 2023), les banques subissent de plus en plus de pressions de la part des parties prenantes et ont déjà commencé à ajuster leur offre de crédit en fonction de leurs engagements (Kacperczyk and Peydró, 2022).

Ce chapitre contribue à la littérature en finance climat et sur les actifs échoués dans la transition bas-carbone en mettant en avant un mécanisme d'interaction spécifique qui pourrait intervenir entre les financements des banques du fait du risque d'actifs échoués. Sur la base d'un modèle heuristique à une période représentant n banques finançant une entreprise confrontée au risque d'actifs échoués, il montre que le désinvestissement d'une banque de cette entreprise pourrait inciter d'autres banques à désinvestir également de cette entreprise.

Plus précisément, il contribue à la littérature sur les vecteurs de l'échouage d'actifs dans la transition bas-carbone (Campiglio et al., 2022; Rozenberg et al., 2020; Hambel and van der Ploeg, 2024; Comerford and Spiganti, 2023; Carattini et al., 2021; van der Ploeg, 2020; Kalkuhl et al., 2020) en identifiant un levier spécifique que les banques ont en main pour provoquer ce phénomène. Il contribue également à la littérature théorique intégrant les risques de transition dans l'évaluation du risque de crédit (Monnin, 2018; Battiston et al., 2019; Bouchet and Le Guenedal, 2020; Reinders et al., 2020; Agliardi and Agliardi, 2021; Le Guenedal and Tankov, 2022b; Battiston et al., 2023), en estimant l'impact spécifique de l'échouage d'actifs sur la probabilité de défaut d'une entreprise, et en montrant que la probabilité de défaut dépend également des conditions de financement de l'entreprise lorsqu'elle est confrontée au risque d'échouage d'actif. Enfin, il contribue à la littérature sur les ajustements de l'offre de crédit des banques aux risques de transition (Chava, 2014; Delis et al., 2019; Degryse et al., 2022; Benincasa et al., 2022; Laeven and Popov, 2023; Ivanov et al., 2023; Kacperczyk and Peydró, 2022) en montrant que cet ajustement de l'offre de crédit peut aussi être le résultat d'une interaction entre les banques par

la prise en compte du risque d'échouage d'actifs.

La démonstration se déroule comme suit. Un modèle à une période représente n banques finançant une seule entreprise confrontée au risque d'échouage de ses actifs. Chaque banque décide du montant de crédit à accorder à cette entreprise à la date initiale en optimisant le profit espéré généré par le prêt. Ce profit espéré dépend du taux d'intérêt du prêt, supposé exogène, de la probabilité de défaut de l'entreprise et d'un coût de prêt quadratique. Ce coût de prêt quadratique, paramétré par un coefficient unique hétérogène entre les banques, représente la disponibilité de leur ligne de crédit pour cette entreprise, qui peut être liée à leur taille, à leur besoin de diversification de portefeuille et à leurs préférences « vertes », telles qu'un engagement de neutralité carbone. La probabilité de défaut de l'entreprise est définie comme la probabilité qu'elle ne puisse pas rembourser sa dette à la fin de la période, qui dépend de ses revenus sur la période. Or, les revenus de l'entreprise dépendent notamment d'un coût d'ajustement du capital, qu'elle subit si elle ne peut pas contracter suffisamment de dette en début de période, car cela l'oblige à échouer une partie de ses actifs. En raison de ce coût d'ajustement du capital (ou d'échouage d'actifs), supposé convexe conformément à la littérature (Campiglio et al., 2022), les profits des banques interagissent à travers la probabilité de défaut de l'entreprise.

Selon une certaine spécification du modèle, la probabilité de défaut de l'entreprise ainsi que les fonctions de réaction optimale (« best response ») de chaque banque sont calculées analytiquement. De plus, les équilibres de Nash peuvent être identifiés graphiquement lorsqu'il y a deux banques dans l'univers bancaire. La décision de désinvestissement d'une banque est supposée être mise en œuvre pour des raisons exogènes à la maximisation du profit du prêt : elle peut être due à la mise en œuvre d'un engagement de neutralité carbone ou à de nouvelles contraintes réglementaires. Le désinvestissement est défini comme un éloignement de l'équilibre de Nash par une réduction de l'offre de crédit pour les raisons exogènes précédemment citées.

Les résultats sont les suivants. Premièrement, les fonctions de réaction optimale illustrent l'interaction entre les banques qui découle du coût d'échouage des actifs. Lorsqu'une banque a un coût de prêt quadratique suffisamment bas pour lui permettre de fournir à l'entreprise le montant de dette qui maximise son profit espéré, elle agit comme un simple substitut à l'offre de crédit des autres banques : sa réaction optimale est de compléter l'offre de crédit totale des autres banques jusqu'à ce que le montant optimal de la dette de l'entreprise soit atteint. Toutefois, lorsqu'une banque n'a pas les moyens ou l'envie, à elle seule, de financer entièrement l'entreprise, un effet d'interaction intervient : cette banque n'est pas disposée à prêter à l'entreprise si l'offre totale de crédit des autres banques est trop faible, car, dans ce cas, l'entreprise supporterait des coûts élevés d'ajustement du capital, ce qui pourrait la rendre insolvable.

Ce fort effet d'interaction peut être illustré plus en détail en identifiant les équilibres de Nash entre deux banques de ce dernier type. Dans des exemples typiques, deux équilibres de Nash peuvent être identifiés : l'un où aucune des deux banques ne prête à l'entreprise, et l'autre où elles prêtent toutes deux des montants strictement positifs. Si l'on suppose que l'équilibre

qui prévaut est ce dernier, la décision de désinvestissement total ou partiel prise par l'une des deux banques peut avoir un effet multiplicateur important, en incitant l'autre banque à désinvestir partiellement ou totalement elle aussi. Cet effet multiplicatif important est robuste aux changements dans le besoin de financement de l'entreprise.

2.3 Les investisseurs peuvent-ils fléchir la pratique de l'écoblanchiment ?

Un enjeu très important du financement de la transition bas-carbone, à grande échelle, est celui de l'évaluation de l'empreinte environnementale de chaque actif financier, ou de chaque entreprise qui émet ces actifs. Or, plusieurs études aujourd'hui suggèrent que l'écoblanchiment, qui consiste à faire croire que l'empreinte environnementale d'une entreprise, d'un projet ou d'un produit est en-dessous de sa véritable empreinte, de manière à en améliorer l'image, est une pratique répandue¹². La récente littérature en finance durable qui s'intéresse à l'évaluation des actifs et de la performance environnementale des entreprises permet d'éclairer une partie des motivations qui peuvent conduire ces dernières à pratiquer l'écoblanchiment, à travers deux phénomènes. D'abord, les entreprises dont la qualité environnementale est perçue positivement bénéficient d'un coût du capital plus faible à l'équilibre, du fait de la présence significative, sur les marchés, d'investisseurs qui ont des préférences pro-environnementales (Pástor et al., 2021; Pedersen et al., 2021a; Zerbib, 2022) ou qui intègrent les risques de transition dans leurs choix d'investissement (Krueger et al., 2021). Ensuite, les entreprises peuvent bénéficier (i) de la difficulté à mesurer leur empreinte environnementale, (ii) de la faible standardisation des méthodes de mesure (Berg et al., 2022) (souvent basées sur leurs propres déclarations, non auditées) et (iii) de leur capacité à communiquer de manière ambiguë (Fabrizio and Kim, 2019) pour exagérer, par leur communication, leur qualité environnementale (qu'on peut comprendre comme l'inverse de leur empreinte environnementale). Les entreprises ont donc une incitation à et la possibilité de pratiquer l'écoblanchiment.

Or, cette pratique constitue un obstacle à l'impact des investisseurs sur la transition écologique. En effet, elle complique l'évaluation des risques de transition auxquels sont exposées les entreprises. En outre, elle empêche les investisseurs qui le souhaiteraient de favoriser les entreprises à faible empreinte environnementale. Enfin, elle peut contribuer à alimenter les risques d'une « bulle verte », qui représente une menace pour la stabilité financière favorisée par le contexte de la transition bas-carbone (Semieniuk et al., 2021). Dans ce chapitre, qui est le fruit d'un travail avec Olivier David Zerbib et Peter Tankov, nous nous posons la question des incitations à l'écoblanchiment provoquées par les préférences pro-environnementales des investisseurs, et de la capacité de ces derniers à limiter cette pratique.

Ce faisant, nous contribuons à plusieurs champs de la littérature. En premier lieu, nous contribuons à la littérature financière émergente sur l'écoblanchiment. Gourier and Mathurin (2024); Wu et al. (2020); Duflo et al. (2013); Duchin et al. (2023) démontrent empiriquement

12. https://ec.europa.eu/commission/presscorner/detail/en/ip_21_269

l'existence de formes variées d'écoblanchiment, tandis que Kim and Yoon (2022); Gourier and Mathurin (2024) montrent, également empiriquement, que les investisseurs et gestionnaires d'actifs sont sensibles aux pratiques d'écoblanchiment et peuvent contribuer à les limiter (Bingler et al., 2022). A cette date, nous sommes le premier article théorique sur l'écoblanchiment dans la littérature financière, avec Chen (2023a) comme papier concurrent, qui propose également un modèle théorique pour répondre à une question similaire à la nôtre. Toutefois, nous nous distinguons à travers les aspects suivants : (i) nous caractérisons explicitement les rendements d'actifs à l'équilibre ainsi que l'effort optimal d'écoblanchiment des entreprises, (ii) dans un cadre dynamique stochastique, (iii) permettant une interaction directe entre les entreprises (sous la forme d'une concurrence) et (iv) nous validons empiriquement une partie de nos résultats.

Un peu plus indirectement, nos résultats font écho à la littérature sur la divulgation d'information par les entreprises (Flammer et al., 2021; Ilhan et al., 2023) ainsi qu'à celle sur la fiabilité limitée des notations environnementales (Berg et al., 2022, 2021). Nous contribuons également à la littérature sur l'évaluation d'actifs en finance durable. En particulier, nous montrons que les rendements à l'équilibre peuvent comporter une prime de risque associée à la pratique d'écoblanchiment lorsque les investisseurs pénalisent cette dernière, en plus de la prime verte dont bénéficient les entreprises bien notées environnementalement (Pástor et al., 2021; Pedersen et al., 2021b; Zerbib, 2022; De Angelis et al., 2023). Enfin, nous contribuons à la compréhension de l'impact de l'investissement durable (Green and Roth, 2021a; Oehmke and Opp, 2023; Landier and Lovo, 2020a; Hartzmark and Shue, 2023; De Angelis et al., 2023) en montrant que l'impact environnemental des investisseurs favorisant les actifs bien notés environnementalement dépend de leur capacité à limiter la pratique de l'écoblanchiment. S'ils y parviennent, leur investissement peut avoir un double impact : favoriser la réduction des émissions et augmenter la qualité des notes environnementales en décourageant l'écoblanchiment.

Nous construisons un modèle d'équilibre dynamique dans lequel un investisseur représentatif alloue son portefeuille dans des actifs émis par n entreprises. Cet investisseur, qui représente une version agrégée de la demande pour ces actifs, a des préférences pro-environnementales. Toutefois, il fait face à une asymétrie d'information, qui l'empêche de connaître l'empreinte environnementale exacte de chaque entreprise. Il est donc contraint d'utiliser leur note environnementale, qui est une mesure imparfaite mais publique de leur empreinte, pour orienter ses investissements selon ses préférences. Ces notes environnementales peuvent être influencées, potentiellement de manière trompeuse, par la communication environnementale des entreprises. Toutefois, des événements aléatoires et observables par les investisseurs, représentant l'avènement de controverses environnementales, peuvent agir comme révélateurs et corriger brusquement la note environnementale de l'entreprise concernée. Ces controverses permettent à l'investisseur de pénaliser, a posteriori, les entreprises qui auraient manipulé leur note environnementale par de l'écoblanchiment. Les préférences pro-environnementales de l'investisseur ainsi que cette pénalité permettent de déduire un coût du capital d'équilibre pour les n entreprises, qui dépend de leur

note environnementale et des révélations associées aux controverses passées.

Les entreprises, elles, cherchent à minimiser leur coût du capital à l'équilibre, en déployant deux types d'efforts. Elles peuvent (i) communiquer sur leur empreinte environnementale (d'une façon potentiellement trompeuse), (ii) investir dans la réduction de leur empreinte environnementale. Chaque type d'effort d'effort représente un coût quadratique. L'écart entre la communication d'une entreprise et sa pratique effective de réduction de son empreinte environnementale, s'il est positif, correspond à sa pratique d'écoblanchiment à cet instant. Les préférences pro-environnementales de l'investisseur et la pénalité qu'il applique aux révélations des controverses introduisent un compromis pour chaque entreprise, entre : (a) augmenter par n'importe quel moyen sa note environnementale, y compris en faisant appel à de l'écoblanchiment, et (b) subir un contrecoup lors de l'avènement d'une controverse, dont elle ne maîtrise pas la temporalité car elle dépend d'une attention médiatique ou sociétale qui lui est extérieure. Nous dérivons analytiquement les comportements optimaux de communication et de réduction d'émissions de chaque entreprise à chaque instant, desquels nous déduisons son effort d'écoblanchiment.

D'un point de vue dynamique, nous montrons que la communication environnementale optimale de chaque entreprise s'articule d'une manière contracyclique à sa note environnementale : elle augmente lorsque cette dernière diminue, et vice-versa. En outre, lorsque le coût de communication est suffisamment plus faible que le coût de réduction d'émissions, ou lorsque le degré d'asymétrie d'information est suffisamment élevé (la note environnementale suffisamment imparfaite), cette dynamique de communication traduit, principalement, une pratique d'écoblanchiment : l'entreprise cherche, à travers cette communication contracyclique, à maintenir sa note environnementale à un certain niveau d'exagération par rapport à son empreinte environnementale effective. Or, ce niveau dépend positivement de la sensibilité environnementale de l'investisseur représentatif, et négativement de l'intensité de sa pénalité sur les corrections de notation au moment de controverses.

Ainsi, les préférences pro-environnementales des investisseurs encouragent les entreprises à faire de l'écoblanchiment, car cela leur permet de diminuer leur coût du capital à l'équilibre. Néanmoins, en pénalisant les révélations permises par les controverses, les investisseurs ont la possibilité de limiter ces pratiques. De ce fait, ils peuvent augmenter l'impact environnemental de leur investissement en favorisant le choix, par l'entreprise, d'une réduction effective de son empreinte environnementale au détriment de l'écoblanchiment. Toutefois, ce pouvoir de dissuasion de l'écoblanchiment dépend de l'avènement de controverses, qui dépend lui-même de la distribution de l'attention médiatique et de la capacité des journalistes et organisations non gouvernementales à mettre au jour des pratiques dissimulées. Ainsi, le pouvoir dissuasif des investisseurs est limité par des paramètres qui ne dépendent pas de lui : nos résultats suggèrent l'importance d'une action politique complémentaire favorisant une culture de la transparence.

Ensuite, nous étendons le modèle initial en faisant dépendre les stratégies d'investissement des notes environnementales *relatives* des entreprises. Cette extension permet de modéliser deux

situations distinctes, toutes deux plausibles. Soit ce sont les investisseurs qui adoptent une stratégie « best-in-class », selon laquelle ils préfèrent investir dans les entreprises les mieux notées d'un secteur en comparant les entreprises entre elles plutôt qu'en considérant la valeur absolue de leur note environnementale. Soit ce sont les agences de notation elles-mêmes qui normalisent les notes environnementales, comme elles le pratiquent régulièrement¹³. Cette normalisation introduit une interaction entre les entreprises, qui ont désormais besoin d'être mieux notées que leurs concurrentes si elles veulent obtenir une réduction de leur coût du capital. Pour résoudre le jeu à n joueurs que ce nouveau programme définit, nous l'approchons par un jeu à champ moyen, dans lequel le nombre d'entreprises tend vers l'infini¹⁴. Ce passage à la limite du nombre d'entreprises, sous quelques hypothèses supplémentaires, permet de rendre négligeable l'impact de la stratégie d'une entreprise sur la moyenne des notes environnementales, qui représente son facteur de normalisation. Ainsi, pour une moyenne de notes environnementales donnée, la stratégie optimale d'une entreprise représentative peut être exprimée analytiquement en fonction de cette moyenne. Cela nous permet de montrer que la version à champ moyen de ce jeu admet un unique équilibre de Nash, que nous caractérisons par une fonction de point fixe et approchons numériquement. Les résultats de cette extension confirment la structure des comportements de communication, de réduction d'émissions et d'écoblanchiment optimaux obtenus dans le modèle initial. Néanmoins, nous montrons numériquement que la normalisation des notes environnementales conduit à une diminution de chacun de ces efforts, et notamment de l'effort de réduction d'émission. En effet, pour le même rapport entre note individuelle et moyenne, des notes environnementales plus élevées rapportent la même récompense en termes de baisse du coût du capital, mais demandent plus d'efforts et donc un coût supplémentaire.

Enfin, à partir de données mensuelles du fournisseur de données Covalence, nous montrons (i) que l'effort mensuel de communication des entreprises est presque structurellement positif sur la période étudiée (2016-2022) tant il l'est fréquemment et (ii) que l'effort de communication d'une entreprise diminue significativement lorsque sa note environnementale augmente et inversement à une fréquence mensuelle. Ainsi, nous validons empiriquement l'aspect contracyclique de l'effort de communication optimal tel qu'on le caractérise dans notre modèle. Or, étant donné qu'il est raisonnable de supposer (a) que l'effort de communication est plus volatile que

13. Par exemple, les notes ESG (environnementales, sociales et de gouvernance) produites par l'agence de notation MSCI sont ajustées au niveau sectoriel à partir d'un référentiel révisé annuellement (<https://www.msci.com/documents/1296102/34424357/MSCI+ESG+Ratings+Methodology.pdf>). En outre, les scores ESG de Refinitiv LSEG sont calculés directement à partir de centiles qui servent à classer les entreprises selon chaque métrique composant la note finale (https://www.lseg.com/content/dam/data-analytics/en_us/documents/methodology/lseg-esg-scores-methodology.pdf).

14. Les jeux à champ moyen ont été introduits simultanément par Huang et al. (2006) et Lasry and Lions (2007) dans la littérature en mathématiques financières. La connaissance de ces jeux s'est ensuite développée à vive allure dans cette littérature. Aujourd'hui, on trouve de plus en plus d'applications aux marchés de l'électricité (Aïd et al., 2020; Leutscher de las Nieves, 2022; Alasseur et al., 2023), en macroéconomie (Ahn et al., 2018; Achdou et al., 2022; Bahn et al., 2017) et en finance (Carmona et al., 2017; Bertucci et al., 2020; Benazzoli et al., 2020; Lavigne and Tankov, 2023) par exemple. On peut se référer au Chapitre 4 de cette thèse et à l'introduction de thèse de Leutscher de las Nieves (2022) pour une exposition plus détaillée de cet outil.

l'effort de réduction d'émissions et (b) qu'il est moins coûteux de communiquer que de réduire ses émissions pour un même résultat en termes de note environnementale et (c) sachant que les entreprises peuvent bénéficier d'asymétries d'information quant à leur véritable empreinte environnementale (Barbalau and Zeni, 2023), ces résultats suggèrent en regard de notre modèle que cet effort de communication structurellement positif représente, au moins partiellement, un effort d'écoblanchiment.

En plus de sa contribution à la compréhension des pratiques d'écoblanchiment, cet article effectue un ensemble de contributions techniques. D'abord, il est rare de trouver un modèle dynamique stochastique dans lequel les rendements espérés des actifs sont déterminés à l'équilibre tout en étant influencés par la stratégie des entreprises, car un tel modèle est, techniquement, difficile à résoudre¹⁵. En outre, l'application de la théorie des jeux à champ moyen nous permet d'obtenir une solution élégante à l'introduction d'une interaction entre les entreprises, au prix d'hypothèses raisonnables¹⁶. Ce travail dresse donc un portrait optimiste de la pertinence de la théorie des jeux à champ moyen pour éclairer un certain nombre de comportements stratégiques en économie et en finance.

2.4 Equilibres corrélés généralisés dans les jeux à champ moyen : application à un jeu de réduction d'émissions

Le dernier chapitre apporte une contribution surtout méthodologique aux questions abordées dans cette thèse. Les jeux à champ moyen représentent une famille de jeux avec un grand nombre de joueurs, dans lesquels chaque joueur interagit avec la masse de la population plutôt qu'avec chaque autre joueur individuellement¹⁷. Ils permettent notamment d'approcher les solutions de jeux à n joueurs dynamiques stochastiques. Développés dans le champ des mathématiques financières, ils y connaissent un grand succès depuis leur introduction simultanée par Huang et al. (2006) et Lasry and Lions (2007). Aujourd'hui, on trouve de plus en plus d'applications aux marchés de l'électricité (Aïd et al., 2020; Leutscher de las Nieves, 2022; Alasseur et al., 2023), en macroéconomie (Ahn et al., 2018; Achdou et al., 2022; Bahn et al., 2017) et en finance (Carmona et al., 2017; Bertucci et al., 2020; Benazzoli et al., 2020; Lavigne and Tankov, 2023) par exemple¹⁸.

15. Par exemple, si De Angelis et al. (2023) a un cadre similaire avec prix d'équilibre et optimisation des entreprises, la résolution est faite au prix de l'hypothèse que les investisseurs connaissent les dividendes que chaque entreprise paiera dans le futur. On peut également citer Lavigne and Tankov (2023) qui a un cadre similaire et moins contraint, mais qui n'obtient pas de solutions explicites pour le prix d'équilibre ni pour le contrôle optimal des entreprises.

16. Les hypothèses effectuées pour le passage à la limite du nombre d'entreprises sont leur homogénéité et l'idiosyncratie des aléas qui impactent la dynamique de leur note environnementale. En d'autres termes, les entreprises sont supposées identiques et les controverses qui les frappent indépendantes. Intuitivement, les résultats qui nous intéressent dans ce jeu à champ moyen ne devraient pas dépendre de ces hypothèses simplificatrices, dont le rôle est technique en facilitant sa résolution.

17. Pour une introduction détaillée à cette méthode, on peut par exemple se référer à l'introduction de la thèse de Marcos Leutscher (Leutscher de las Nieves, 2022).

18. On peut se référer à l'introduction de thèse de Leutscher de las Nieves (2022) pour une exposition plus détaillée de cet outil.

Mis à part quelques travaux très récents (Bonesini et al., 2022; Campi and Fischer, 2022a; Campi et al., 2023), la notion d'équilibre considérée pour résoudre ces jeux est généralement l'équivalent limite du concept d'équilibre de Nash. Pourtant, dans le champ de la théorie des jeux, l'équilibre de Nash est loin d'être le seul considéré, et ses limites sont bien connues. Notamment, les conditions d'après lesquelles une population se retrouve à effectivement jouer selon l'équilibre de Nash sont très restrictives et nécessitent des hypothèses fortes quant à la rationalité des joueurs (Aumann, 1987).

Au contraire, les équilibres corrélés généralisés¹⁹ (« coarse correlated equilibria »), introduits implicitement dans Hannan (1957) puis explicitement dans Moulin and Vial (1978), sont une notion d'équilibre qui émerge plus naturellement de jeux répétés par exemple (Hart and Mas-Colell, 2003; Roughgarden, 2016). Ce type d'équilibre consiste en une généralisation de l'équilibre de Nash. Un équilibre de Nash est tel que, si les autres joueurs jouent la stratégie d'équilibre, aucun joueur n'a intérêt à dévier de sa propre stratégie d'équilibre. Il représente donc un concept assez minimal de stabilité. Un équilibre corrélé est un équilibre de Nash dans lequel on introduit un mécanisme de corrélation extérieur pour définir les stratégies que peuvent employer les joueurs. Les équilibres corrélés généralisés, en plus d'être plus naturels, ont aussi l'avantage de pouvoir atteindre des niveaux d'utilité sociale plus élevés que les équilibres de Nash (Dokka et al., 2022; Moulin et al., 2014). Ils s'avèrent donc particulièrement pertinents dans un jeu où l'équilibre de Nash est inoptimal socialement.

Plus précisément, un équilibre corrélé généralisé est défini comme suit. Un mécanisme extérieur, pouvant être interprété comme un médiateur, définit une distribution (loi de probabilité) sur l'espace de stratégie des joueurs. Cette distribution est publique. A partir d'un tirage aléatoire de cette distribution, le médiateur recommande, à chaque joueur, une stratégie de manière privée. Cette distribution représente un équilibre corrélé généralisé si chaque joueur préfère s'engager à suivre la stratégie qui lui est recommandée *avant* de l'avoir reçue²⁰, plutôt que d'en dévier unilatéralement, en supposant que chaque autre joueur va suivre la recommandation reçue. La seule information dont le joueur dispose pour décider de s'engager est la loi de probabilité des recommandations. Si cette distribution induit des recommandations indépendantes d'un joueur à l'autre, la définition d'équilibre corrélé généralisé est équivalente à celle d'un équilibre de Nash mixte. C'est donc la possibilité de corrélation entre les stratégies recommandées qui permet d'établir un plus grand nombre d'équilibres que l'ensemble des équilibres de Nash (qu'il contient).

Ce concept d'équilibre dans le cadre des jeux à champ moyen dynamiques stochastiques en temps continu a été défini dans Campi et al. (2023), où son existence est établie dans un cadre

19. Je choisis d'employer le terme « généralisé », qui permet de distinguer ce type d'équilibre de la notion plus restrictive d'équilibre corrélé (Aumann, 1974).

20. La notion d'équilibre corrélé (Aumann, 1974) désigne, elle, une telle distribution lorsque chaque joueur préfère suivre la stratégie recommandée *après* l'avoir reçue, plutôt que dévier unilatéralement. Elle est donc plus restrictive que la notion d'équilibre corrélé généralisé, même si elle contient également l'ensemble des équilibres de Nash.

théorique large, mais d'une manière non constructive. Les arguments d'existence ne permettent pas d'identifier des équilibres corrélés généralisés précis. Dans le dernier chapitre de cette thèse, qui est le fruit d'un travail conjoint avec Luciano Campi et Federico Cannerozzi, nous cherchons à identifier analytiquement des équilibres corrélés généralisés dans des jeux à champ moyen linéaires quadratiques. Nous caractérisons explicitement un sous-ensemble d'équilibres corrélés généralisés dans un jeu à champ moyen linéaire quadratique générique. Si nous nous restreignons à un tel sous-ensemble, c'est que l'ensemble complet d'équilibres corrélés généralisés est très vaste et complexe à explorer analytiquement. Nous établissons une condition selon laquelle les équilibres corrélés généralisés de ce sous-ensemble dépassent l'équilibre de Nash en termes d'utilité. Il convient de noter que, dans un jeu à champ moyen, tous les joueurs étant supposés identiques, l'utilité individuelle est représentative de l'utilité sociale. Nous parlons donc d'une manière équivalente d'utilité sociale et individuelle dans ce cadre. Nous démontrons qu'aucun équilibre corrélé généralisé ne peut dépasser l'utilité optimale définie comme la solution du contrôle à champ moyen équivalent (qui peut être interprétée comme celle d'un planificateur central). En outre, si la solution du contrôle à champ moyen n'est pas un équilibre de Nash, nous montrons que son niveau d'utilité ne peut être atteint par aucun équilibre corrélé généralisé, car ce dernier est un équilibre non-coopératif. Nous proposons ensuite un mécanisme simple pour explorer ce sous-ensemble déjà riche et identifier des équilibres corrélés généralisés aux propriétés intéressantes.

Nous illustrons ce mécanisme en l'appliquant à un jeu stylisé de réduction d'émissions entre pays²¹. Ce jeu correspond à une reformulation avec un grand nombre de joueurs d'un jeu développé par Dokka et al. (2022) à partir du modèle fondateur de Barrett (1994). Dans la version statique avec n joueurs symétriques de Dokka et al. (2022), chaque pays cherche à définir un effort optimal de réduction d'émission en maximisant ses bénéfices nets. Ses bénéfices bruts augmentent de manière concave en l'effort total effectué par l'ensemble des pays, tandis qu'il assume des coûts quadratiques individuels pour chacun de ses efforts d'abatement. Dans la version à champ moyen de ce jeu (c'est-à-dire, avec un nombre de joueurs tel qu'aucun joueur n'a d'impact individuellement sur l'agrégation des efforts et donc sur le niveau du changement climatique ; on parle alors de joueurs « atomiques »), l'équilibre de Nash correspond à un effort nul de chaque joueur. En effet, tout effort a un impact négligeable sur ses bénéfices, tandis qu'il a un coût. C'est également le cas des équilibres corrélés généralisés, qui sont réduits dans ce cas à l'unique équilibre de Nash pour les mêmes raisons.

Nous adaptons légèrement ce jeu en le formulant dans une version dynamique stochastique et en ajoutant un coût réputationnel pour chaque joueur. Pour un joueur donné, ce coût est calculé comme la distance quadratique entre l'intégrale de son effort de réduction d'émissions passé et l'intégrale de cet effort passé moyen à chaque instant. Dans ce jeu, l'unique équilibre

21. Cette application fait écho à celle de Bahn et al. (2017) qui développe un autre exemple de jeu à champ moyen pour étudier les négociations climatiques internationales, mais avec un modèle complètement différent.

de Nash conduit toujours à un effort nul sur toute la période de la part de chacun des joueurs. En revanche, l'introduction d'un coût de réputation élargit considérablement l'ensemble des équilibres corrélés généralisés dans ce jeu. En appliquant d'une façon très simple le mécanisme d'exploration d'équilibres corrélés généralisés développé dans notre article, nous identifions et illustrons un sous-ensemble de ces équilibres qui permet d'atteindre des niveaux d'effort de réduction d'émission élevés, ainsi qu'un niveau de bénéfice net collectif largement supérieur à celui de l'équilibre de Nash. Plus le coût de réputation est élevé par rapport au coût privé de l'effort de réduction d'émission, plus le bénéfice net collectif atteint par notre sous-ensemble d'équilibres corrélés généralisés se rapproche du niveau de bénéfice net optimal (calculé comme la solution du contrôle à champ moyen équivalent).

Ainsi, ce travail invite la riche littérature sur les jeux à champ moyen en mathématiques financières à prendre de la distance avec le concept d'équilibre de Nash, en illustrant l'importance de son inefficience dans un jeu avec bien commun et en proposant une alternative techniquement abordable. Pour aller plus loin, l'application que nous présentons dans cet article pose deux questions principales. La première est la question de la mise en pratique d'un équilibre corrélé généralisé. Une fois un équilibre identifié comme intéressant d'après ses propriétés d'utilité sociale et son ambition climatique, comment définir le mécanisme de corrélation et comment contraindre les joueurs à respecter leur engagement une fois le tirage effectué ? La deuxième grande question est celle de la modélisation d'une négociation internationale entre pays par un jeu à champ moyen, dans lequel les joueurs sont typiquement atomiques. Bien que cette approche ait déjà été employée dans la littérature (Bahn et al., 2017), il est légitime de penser qu'elle risque de manquer un certain nombre d'enjeux fondamentaux de ces négociations : l'existence de pays dominants, d'enjeux bilatéraux, ainsi que la concentration des émissions dans un sous-ensemble assez restreint de pays.

Une piste pour répondre à la deuxième question serait d'appliquer notre modèle à un jeu entre un grand nombre de petits joueurs qui cherchent à préserver un bien commun. Par exemple, on peut penser au choix de prendre l'avion en regard du coût environnemental que ce moyen de transport représente. Alors que de nombreuses personnes bénéficieraient de l'atténuation du changement climatique (en excluant les quelques personnes liées à l'industrie du transport aérien dans ce cas), le choix de ne pas prendre l'avion entraîne un coût individuel, car il réduit les opportunités de vacances ou de travail, par exemple. De plus, individuellement, le choix de ne pas prendre l'avion a un impact négligeable sur les émissions mondiales de gaz à effet de serre. En outre, il semble raisonnable de supposer que le coût réputationnel de prendre l'avion n'est pas nul : si personne autour de vous ne prend l'avion pour des raisons écologiques, il devient honteux de prendre l'avion. Nous retrouvons ainsi les trois termes clés de la fonction objectif définie dans notre application, cette fois avec des joueurs atomiques. Selon l'équilibre de Nash, dans ce cas, personne ne s'empêche de prendre l'avion. Appliqués à ce cas, nos résultats permettent d'identifier des équilibres corrélés généralisés qui dépassent significativement l'équilibre de Nash

en termes d'utilité collective (et individuelle), avec un niveau de dépassement qui varie selon le coût réputationnel relativement au coût privé de ne pas prendre l'avion.

Une piste de réponse à la question de la mise en pratique d'un équilibre corrélé généralisé dans ce contexte pourrait ainsi être la suivante. Si une organisation non gouvernementale (ou un collectif médiatique et présent sur les réseaux sociaux par exemple) parvient à envoyer un signal suffisamment crédible, jouant ainsi un rôle de médiateur, elle pourrait peut-être faire atteindre un équilibre corrélé généralisé à la population qui serait plus optimal que l'équilibre de Nash d'un point de vue environnemental au moins. Si la mise en oeuvre exacte de ce mécanisme de corrélation mériterait d'être approfondie, l'application que nous proposons dans ce chapitre permet déjà de réfléchir aux conditions (comparaison entre coût individuel et coût réputationnel par exemple) selon lesquelles un tel mécanisme de corrélation pourrait favoriser la préservation d'un bien commun tout en générant une plus grande utilité sociale.

Chapter 1

Climate stress testing

Abstract

Climate stress testing is a recent tool developed by academic researchers and central banks to address the challenge of climate-related risks measurement. This chapter reviews climate stress testing methodologies used by central banks in light of academic contributions to this field, by questioning their ability to assess the extent to which climate-related risks could affect financial stability. It shows that central banks' but also scholars' methodologies give unequal consideration to the different channels of financial risk. In addition, central banks' approaches tend to downplay the role that the radical uncertainty characterizing climate and transition risks can play in amplifying their impact. Finally, they do not take sufficient account of the role of the financial sector in the propagation of these risks. I propose complementary methodologies that would allow to assess the resilience of financial institutions to adverse climate-related risks scenarios and to guide them in financing the low-carbon transition.

1 Introduction

Central banks and financial supervisors have become increasingly aware during the last decade that climate change is a source of financial risks, and that it may have an impact on financial stability (Carney, 2015; NGFS, 2018, 2019; Litterman et al., 2020; FSB and NGFS, 2022). Climate change is a source of two main types of risks to the financial system. Physical risks relate to the physical consequences of climate change, such as extreme weather events or sea level rise. Transition risks come from the shift to a low-carbon economy caused by climate change mitigation efforts.

Modelling and assessing these climate-related risks to the financial system is a challenge. Both physical and transition risks are (i) radically uncertain, (ii) without any past equivalent, (iii) they affect the economy in complex ways, and (iv) some of these risks will only materialize in the very long term (Bolton et al., 2020). These difficulties call for the use of forward-looking approaches accounting for both the structural shift of the economy accompanying transition risks and the interaction between the climate and the economy to understand physical risks.

Climate stress testing and scenario analysis are flexible tools that can address at least some of these issues. They are scenario-based exercises designed to assess the resilience of financial institutions or the financial system to climate-related risks. A rapidly increasing number of central banks have conducted climate stress tests or scenario analyses (FSB and NGFS, 2022) following the pioneering pilot exercise of the Banque de France (ACPR, 2020). In parallel, researchers have developed various climate stress testing methodologies since the breaking ground work by Battiston et al. (2017).

This chapter reviews the climate stress testing methodologies used by central banks and developed in the academic literature until April 2022.¹ It discusses the main approaches adopted by central banks, in light with the findings of the academic literature. In particular, it examines the extent to which central banks' methodologies allow to provide a comprehensive overview of climate-related risks to the financial system, and draws directions for future research.

This work contributes to the literature assessing climate-related risks and to reviews about climate stress testing and scenario analysis. Bolton et al. (2020) lists the challenges that financial supervisors, regulators and central banks face in dealing with climate-related risks. Monnin (2018) discusses the challenges of assessing climate credit risk using scenario analysis. Bank for International Settlements (2021) reviews the three first climate stress testing methodologies developed by central banks: the Dutch National Bank (DNB, Vermeulen et al., 2021), the Banque de France (ACPR, 2020, referred to as the ACPR-BdF exercise) and the Bank of England (BoE, 2021). It raises the main conceptual issues of these climate stress testing exercises and provides a comparison of their practices. EIOPA (2020) provides theoretical insights on challenges and approaches to conduct a climate stress test on insurance companies. Jacquetin (2021) provides

1. This chapter covers climate stress tests carried out and articles published before April 2022. However, later works may be mobilized to put the discussion into perspective when relevant.

a detailed discussion of the challenges raised by the scenario design in climate stress testing and especially by the macroeconomic modelling. He also reviews practices of financial supervisors and central banks, and makes recommendations for future exercises.

This chapter complements these papers by providing a comprehensive overview of climate stress tests conducted on banks, insurance companies, and other financial institutions, with the perspective of assessing risks to financial stability as a whole. Moreover, it confronts central banks' methodologies with climate stress tests developed in the academic literature. To the best of my knowledge, it is the first academic review on climate stress testing to have been released online.²

This review highlights the existence of strong differences between the types of scenarios employed by central banks and developed in the academic literature. Financial supervisors tend to opt for comprehensive and comparable scenario analyses, centered on (very) long term scenarios under the guidance of the Network of Central Banks and Supervisors for Greening the Financial System (NGFS).³ As for climate stress tests designed by academic researchers, methodologies are heterogeneous and focus on one or two specific methodological contributions in general. For example, Battiston et al. (2017) highlight the importance of second-round effects of climate-related risks within the financial system due to the interconnectedness of financial institutions, while Reinders et al. (2020) propose a detailed assessment tool of carbon tax shocks to the value financial assets. Moreover, climate stress tests involving transition risks are mostly based on short term scenarios, while long-term scenarios can be designed for stress tests focused on physical risks (Mandel et al., 2021). The consequence of this diversity of approaches is the design of climate stress tests focused on a single type of risk, rarely presenting the impact of the combination of transition and physical risks.

While these differences between academic and supervisory climate stress testing can be partly explained by a difference in means, since the design of long-term comprehensive transition and physical risk scenarios requires the combination of a large number of complex modeling frameworks, it cannot be reduced to this. The philosophy used to design the scenarios is also at stake. This review shows how the long-term framework of transition risk scenarios provided by the NGFS and adopted in the ACPR-BdF exercise leads to an underestimation of the adversity of worst-case transition scenarios. Indeed, they do not allow to assess the adverse impacts of short-term shocks or of an economic depression. These exercises must therefore be clearly

2. The single-authored paper corresponding to this chapter was released on SSRN in August 2022 (Cartellier, 2022). Since then, Acharya et al. (2023), providing an in-depth review of climate stress testing methodologies with regards to academic findings, but with a focus on banks, has been published in the *Annual Review of Financial Economics*, and cites my working paper. Moreover, Brunetti et al. (2022) provide a thorough discussion of methods to assess climate-related risks to financial stability, including climate stress testing.

3. A notable exception is the pioneering work of researchers of the Dutch National Bank (Vermeulen et al., 2021), first released on 2019, which employs short term shock scenarios centered on transition risks. However, since the development of NGFS high-reference scenarios, which focus on the design of long-term scenarios, the latter are mostly used (FSB and NGFS, 2022). Aware of issues related to long-term scenarios, the ECB and NGFS have started to implement complementary short-term stress scenarios after the period covered by this chapter (ECB, 2022; NGFS, 2023a).

identified as scenario analyses rather than climate stress tests. Scenario analysis allows to raise awareness about climate-related risks among financial institutions, encourages them to develop risk assessment tools adapted to these specific risks, to collect the relevant data, and to anticipate balance sheet reallocation to accompany the low-carbon transition and protect against those risks. However, it does not allow to fully assess the resilience of financial institutions to adverse scenarios.

This review also shows that climate stress testing and scenario analysis methodologies cover unequally the different transition risks drivers and financial risks transmission channels. In particular, transition risks driven by market sentiment, and the role of the liquidity risk channel in transmitting climate-related risks to financial institutions, are overlooked. I call for the complementary use of short-term stress tests for transition risks and for the development of “reallocation strategy tests” by supervisory authorities. The former would allow to test for the adversity of different risk drivers and for the role of different risks transmission channels with more flexibility than the long-term framework. The latter would allow to help financial institutions plan their financing reallocation in the low-carbon transition and assess the impact of such reallocation strategies on the economy, financial stability and the transition itself.

Outline. The rest of the chapter is organized as follows. Section 2 exposes how climate-related risks can be a threat to financial stability, and discusses how climate stress testing addresses some of the challenges raised by their assessment. Section 3 describes the special characteristics of climate stress tests compared to other financial stress tests. Section 4 presents and compares climate stress testing methodologies. It discusses in detail the methodology of the ACPR-BdF pilot exercise for transition risks as a case study. It also assesses the collective ability of climate stress tests to fully cover the different types of climate-related risks drivers and channels to the financial system. As an answer to the limits raised in this section, Section 5 proposes two complementary approaches to the long-term deterministic framework based on the NGFS scenarios. Finally, Section 6 concludes.

2 Why climate stress testing

2.1 Climate-related risks to the financial system and financial stability

Two types of climate-related risks can be distinguished (Task Force on Climate-Related Financial Disclosures, 2017). Physical risks, which arise from the material consequences of climate change, affect a large number of economic activities across the grid of geographical and climatic exposure. Acute physical risks, such as extreme climatic events (droughts, floods, wildfires) whose intensity and frequency are increasing with climate change, can cause sudden shocks to the productivity of human, physical and natural capital in affected areas. This can affect the solvency of debtors (companies, individuals) based in these areas, as well as devaluing their as-

sets (housing value and companies' financial assets), and triggering liquidity crises through the sudden withdrawal of their deposits (Brei et al., 2019). These are all risks for financial players (banks, investors) holding securities or financial assets exposed to these counterparties. Chronic physical risks, such as rising sea levels and ocean acidification, can have slower-onset effects, such as the gradual devaluation of seaside dwellings or a decline in fishing activities. They do, however, increase the risk of abrupt changes in expectations, leading to sudden revaluations of certain assets, for example when a majority of financial players (and insurance companies in particular) become better informed about the future exposure of a given territory.

Transition risks, caused by the process of moving towards a low-carbon economy, impact companies through a sectoral prism and insertion in value chains. A significant proportion of economic activities, even those that do not directly emit greenhouse gases (GHGs), rely on GHG-intensive production sectors (Cahen-Fourot et al., 2020). The credit quality, assets and financing capacity of companies in these sectors are likely to be affected by various vectors of the low-carbon transition, be they transition policies, technological innovations or changes in investor and consumer preferences and expectations. A sharp rise in the price of CO₂ emissions, for example, could significantly worsen their revenues and employment conditions, also affecting the financial resources of their employees.

Climate-related risks can have an impact on financial institutions through roughly all standard Basel financial risk categories (Basel Committee, 2021): credit, market, liquidity, operational and underwriting risks. The way these risk categories can be affected by physical and transition risks is summarised in Tables 1.1 and 1.2 respectively.

Physical and transition risks may be a threat to financial stability, through the destabilisation of single over-exposed financial institutions at the micro-level, but also through macro-financial channels. There exists no single definition of financial stability. In this work, the notion of financial stability which is used is defined in FED (2020) as follows: "A stable financial system, when hit by adverse events, or 'shocks,' continues to meet the demands of households and businesses for financial services, such as credit provision and payment services. In contrast, in an unstable system, these same shocks are likely to have much larger effects, disrupting the flow of credit and leading to declines in employment and economic activity". This can also be understood at a microeconomic scale of individual financial institutions: they should be able to maintain their regular activities despite the shocks they face.

In the context of climate change, FSB (2020) and Litterman et al. (2020) point at the risk of widespread mispricing in financial markets due to the radical uncertainty of climate-related risks. The threat of abrupt repricing events related to changes in expectations of investors should therefore be taken seriously. Such events could be triggered by election outcomes, technological breakthroughs, new research findings about the speed and nature of physical climate impacts,

3. The operational risk is reported here to raise awareness about its existence. However, it is not considered by any climate stress test mentioned in this paper, as it deserves an assessment framework very different from the other types of risk channels. Therefore, I will not consider it further in this paper.

	Microeconomic channels and direct impacts	Macroeconomic channels and indirect impacts
Credit risk	Reduction in debtors' wealth, revenue or loan collateral (households, corporates, public institutions)	Decline in labour productivity, output reduction; increase in sovereign spreads or defaults; downgrade of (local) public institutions in vulnerable areas
Market risk	Uncertainty about physical risk materialisation: mispricing, increased volatility, increased difficulty of hedging	Same as microeconomic channels, which can spread through value chains and macroeconomic variables
Liquidity risk	Increase in funding cost due to exposure to physical risk; increased demand for liquidity following a natural disaster	Rush on liquidity following important natural disasters
Operational risk	Disruption of transportation facilities, telecommunications or market infrastructures; damages to tangible fixed assets	Same as microeconomic channels, which can spread through value chains and macroeconomic variables
Underwriting risk (for insurers)	Higher than expected increase in claims on damaged insured assets (for non-life insurers), higher than expected mortality or morbidity rates	No specific macroeconomic channel identified
Underwriting risk (for financial institutions)	Decrease in insurance coverage which would increase financial institutions' exposure to physical risks	No specific macroeconomic channel identified

Table 1.1 – Transmission channels of physical risks to financial institutions.

Source: Basel Committee (2021) and additions by the author.

	Microeconomic channels	Macroeconomic channels
Credit risk	Decrease in creditworthiness of corporates vulnerable to transition risk; higher litigation costs for corporates and governments	Changes in production value chains structure can lead to output reduction (decrease in creditworthiness of households, corporates and governments)
Market risk	Uncertainty about corporates exposure to transition risk: mispricing, risk of abrupt repricing events (ex: stranded assets), increased volatility	The spread of microeconomic market risk channels through value chains and macroeconomic variables
Liquidity risk (asset liquidity)	Assets becoming illiquid (ex: stranded assets)	Spillover effects between companies of same sectors or through value chains
Liquidity risk (funding cost)	Increase in funding cost due to exposure to transition risk or uncertainty related to this exposure	Cascading increases in funding costs due to dependence on carbon intensive activities upstream value chains

Table 1.2 – Transmission channels of transition risks to financial institutions.
Source: Basel Committee (2021) and additions by the author.

or the occurrence of major natural catastrophes (Litterman et al., 2020).

These phenomena could have an impact on the economy through feedback loops between the financial system and real economy (FSB, 2020). For example, if there was a widespread increase in risk premia due to the opacity of exposure to climate-related risks (FSB, 2020), or if banks had to restore their solvency after a huge climate physical shock, it would lead to a reduction in lending, which would decrease banks’ support to real economy. Moreover, the increasing uncertainty about the occurrence and the intensity of weather-related natural catastrophes could lead to a reduction in insurance coverage of these events, transferring this risk to economic agents. Climate-related risks could also lead to a self-reinforcing deterioration of prospects for sovereigns and their national financial institutions through the sovereign-bank nexus.

2.2 Challenges raised by the assessment of climate-related risks

Climate-related risks have specific characteristics which are somewhat new compared to other financial risks. Following the analysis of Bolton et al. (2020), their radical uncertainty and “fat tail” properties make them similar to black swans (Taleb, 2007). However, they have specific characteristics that lead Bolton et al. (2020) to qualify them as “Green Swans”. Their horizon of materiality is very long, they will lead to structural changes that cannot be observed in the

past and they are irreversible. Moreover, compared to black swans, although climate change impacts are highly uncertain, it is certain that some combination of physical and transition risks will materialize in the future. Climate catastrophes are even more serious than most systemic financial crises and could pose an existential threat to humanity. Finally, complexity related to climate change is of a higher order than for black swans owing to complex chain reactions and cascade effects.

These features make some current risk measurement methods unsuitable for assessing climate-related financial risks. Many of standard financial risk metrics and assessment methods are backward-looking, based on probabilistic estimations and historical data (I4CE, 2019). However, when it comes to assessing climate-related risks, the computation of the value at risk of a portfolio, or the measurement of the probability of default of a borrower can not solely rely on historical data.

Stress testing is a well-known tractable tool that allows to assess the impact of tail risks on the financial system with a forward-looking approach. The scenario design allows to take into account the unquantifiable uncertainty surrounding climate-related risks, without assigning a probability to their outcomes. The main challenge of climate stress testing is therefore to adapt stress testing methodologies to the specific characteristics of climate-related risks.

3 Climate vs. “standard” stress testing

Climate stress testing adapts financial stress testing principles to the specific features of climate-related risks. However, the definition of climate stress testing is not completely clear, as it has been used in very different contexts. This section therefore aims at clarifying what is meant by the notions of stress testing and climate stress testing and what are similarities and differences between the two.

3.1 Defining (climate) stress testing

Regulatory financial stress tests themselves can differ a lot from one another (Bank for International Settlements, 2017). Borio et al. (2014) consider this methodology as a “toolbox” rather than a single tool. This is probably why it is flexible enough to integrate the modelling of climate-related risks. To frame the analysis provided in this paper, I will stick to the definitions stated in this section.

Definition 1 (Stress test). A stress test is a scenario-based exercise which consists in estimating the impact adverse scenarios would have on a given system in order to test for its resilience capacity, compared to the impact of a baseline scenario.

In particular, supervisory financial stress tests aim at assessing the resilience of financial institutions to adverse financial or economic scenarios. Most of the time they focus on banks and insurance companies, as these are the most heavily supervised financial institutions.

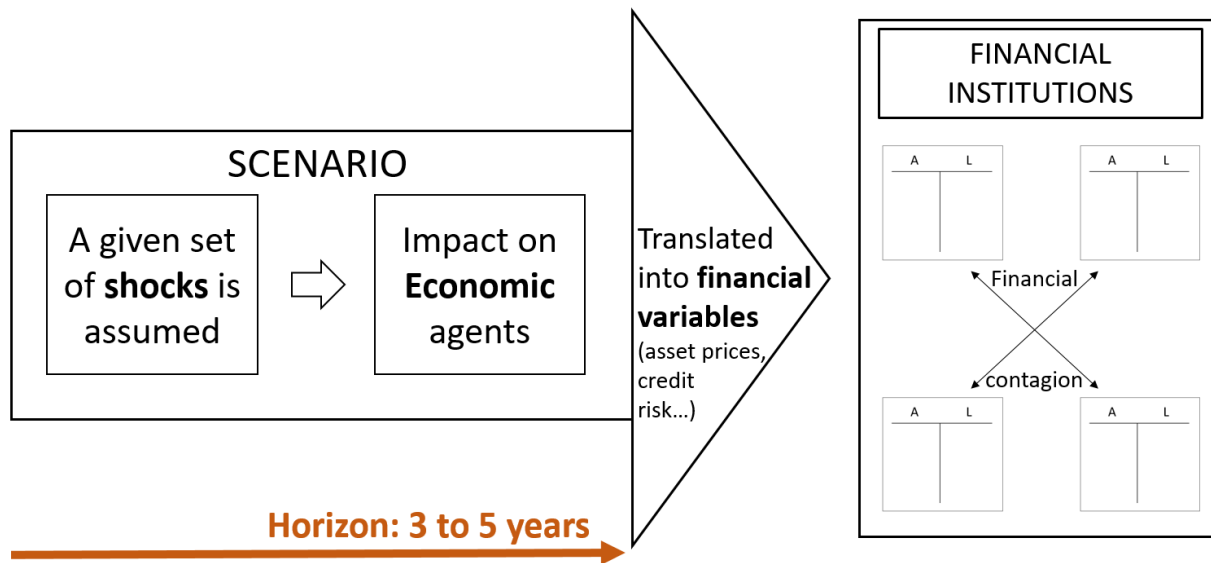


Figure 1.1 – Scheme of a standard financial stress test.

Note that the financial contagion between financial institutions is taken into account only in macroprudential stress tests.

A stress test consists of two main steps (Figure 1.2):

1. The scenario design, which provides output economic and financial variables out of a narrative that describes the materialisation of some economic and financial risks.
2. The assessment of the financial impact of each scenario on balance sheets of financial institutions. In supervisory stress tests, the impact is often measured in terms of capital losses, to ensure that capital buffers of a bank or an insurer are sufficient for the firm to resist such adverse events.

If the aim is to assess the resilience of single institutions, usually banks or insurance companies, it is called a microprudential stress test. If it aims at evaluating the resilience of the financial system as a whole, or of a set of financial institutions, it is called a macroprudential stress test. In such a case it can include an assessment of financial contagion between financial institutions as they are financial counterparties, creditors and borrowers to one another.

From this definition, I deduce a possible definition of a climate stress test.

Definition 2 (Climate stress test). A climate stress test is a stress test carried out to evaluate the resilience of financial institutions to climate-related risks. Scenario narratives describe possible combinations of climate-related risks' materialisation.

Figure 1.2 illustrates how the structure of a standard financial stress test is transformed in a climate stress test. Main differences between a non-climate financial stress test and a climate stress test lie in four aspects:

- The scenario horizon, as the materialisation of climate-related risks can happen in very long-term. In particular, even if physical risks are already starting to materialise, their

intensity will increase dramatically during the second half of the century unless drastic mitigation measures are taken in the following decades. As for transition risks, they are relevant for short, middle and long term horizons as they can materialise from the present day and during the whole century since meeting the Paris agreement target requires immediate stringent mitigation efforts to reach net-zero emissions around 2050 (NGFS, 2020).

- The scenario design, as modelling physical and transition shocks to the economy requires very specific modelling frameworks. For physical risks, one needs to use climate modelling to get an idea of the amplitude of these risks in the future and to use economic and financial modelling to understand how they can impact the economy and the price of financial assets. As for transition risks, Integrated Assessment Models (IAMs) can be useful to estimate the intensity of transition risks entailed by the Paris agreement temperature target of “well below” 2°C. Given their overall complexity, IAMs cannot afford a high-level sectoral granularity and macroeconomic complexity. Therefore, they are complemented with macroeconomic models and input-output sectoral models to understand for example the detailed impact of the implementation of a carbon tax in the various sectors of the economy and through value chains (NGFS, 2021b; Allen et al., 2020). One also needs to adapt the financial modelling to assess the impact of transition risks such as a carbon tax on financial assets, credit risk and liquidity risk.
- The assessment of the exposure of financial institutions to climate-related risks, as it relies on complex combinations of factors (Monnin, 2018). Exposure to physical risks mostly depends on geographical exposure, physical assets location and the structure of value chains of companies to which the financial institution is exposed. As for transition risk, it can vary a lot even for companies within the same industry, based primarily on their carbon emissions of scopes 1, 2 and 3,⁴ but also on their financial health, size, on the diversity of their activities and on their forward-looking strategy of carbon emissions abatement.
- The prudential status, as no climate stress test has been carried out by a central bank with consequences on capital requirement so far. Methodologies are very new, central banks and financial supervisors are still at a learning stage. However, as urgent action is needed, climate stress tests may soon be used to impose prudential constraints.⁵

4. Scope 1 emissions refer to the direct carbon emissions of a company, arising from its production process. Scope 2 emissions represent emissions indirectly related to its production process, through energy consumption for example. Scope 3 emissions include all other indirect emissions that are necessary for the company’s production upward and downward its value chain, including the life cycle of the company’s output.

5. In ECB (2020), the ECB already starts to demand banks to conduct internal climate stress tests as an internal risk management exercise to test for their capital adequacy to climate-related risks.

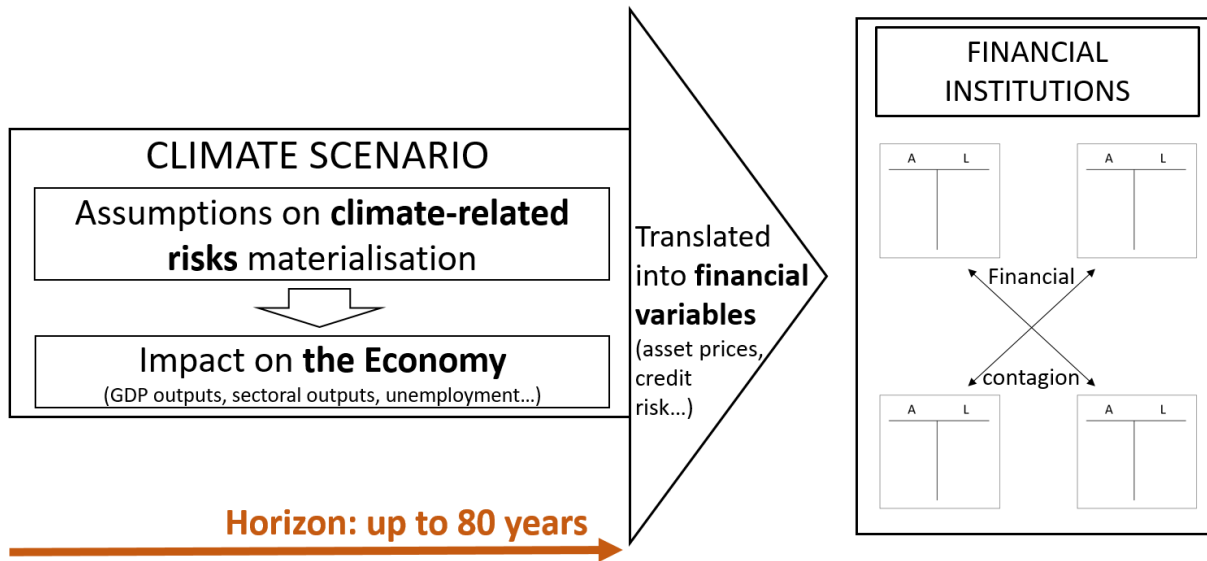


Figure 1.2 – Scheme of a climate stress test.

Note that the financial contagion between financial institutions is not always taken into account in climate stress tests.

3.2 Detailed similarities and differences between the climate and financial stress tests

Both non-climate and climate stress tests fit into what could be called the “skeleton” of a stress test, as this toolbox is very flexible. But some of the “bones” of these two types of exercises are made of different elements, due to the particularity of climate-related risks.

Figure 3 details the “skeleton” of financial stress tests and reports what each “bone” usually represents in non-climate and climate stress tests exercises. This figure sums up the main features of climate stress tests that have been observed through the review of climate stress testing exercises cited in the introduction of this chapter. It is also inspired from Bank for International Settlements (2021); Monnin (2018). As for non-climate stress tests, sources mainly come from Bank for International Settlements (2017); Adrian et al. (2020); EIOPA (2019). This table uses especially the scenario typology from Jacquetin (2021) that is reported in Box 3.2.

Scenario typology from Jacquetin (2021)

Jacquetin (2021) proposes a typology of four scenario types relevant to classify both climate and non-climate scenarios, which is used in Figure 3.

- Prospective: the scenario trajectory is pre-defined by a given target (for example a temperature or GHG emissions target). Example: a low-carbon transition scenario compatible with the Paris agreement target (rise in average global temperature of 2°C maximum). Scenarios of orderly and disorderly transition of the NGFS fall into this category (NGFS, 2021b).
- Hypothetical: the scenario describes the occurrence of one or several shocks, without any preconception on their probability of occurrence nor on their consequences, in an exploratory approach. Example: what would happen if the cost of renewables were divided by two thanks to a technological breakthrough in the following decade ?
- Historical: the scenario is built using historical observations. Due to the forward-looking aspect of climate-related risks such scenarios are of little relevance for climate stress testing.
- Enumerative: the scenario consists in the detailed implementation of all current and announced climate policies. It is characterized by its exhaustivity and realism with regards to planned policies. Narratives are usually too precise to be modelled in a climate scenario, as climate policies are varied and may impact the financial system through very different channels. However in climate stress tests there exist scenarios approximating current policy pledges. The Hot House World scenario “NDCs” (Nationally Determined Contributions) of the NGFS is one of them (NGFS, 2021b).

Structure	Definition	Standard stress tests	Climate stress tests of transition risks	Climate stress tests of physical risks
Objectives	<p>- Explore financial risks under stressed economic and financial conditions</p> <p>- Test the resistance of financial institutions to stressed conditions in terms of their solvency and liquidity provisions (adjustments of regulatory capital requirements and liquidity ratios)</p> <p>- Get more insights on financial institutions' risk management practices</p> <p>- Review and challenge their business plans</p> <p>A stress test is a toolbox that can be used for many different objectives. The specification of the objectives of a specific exercise is important to understand its structure.</p>	<p>- Explore financial risks caused by the materialisation of climate-related risks</p> <p>- Assess the losses of financial institutions due to climate-related risks & prepare them to potential regulation related to the management of these risks</p> <p>- Make them develop processes to assess and manage these new risks, raise their awareness about the materiality of climate-related risks</p> <p>- Get information on their climate strategy and make them develop a long-term strategy</p>		
Framework	<p>Microprudential exercises focus on the resistance of individual financial institutions. Macroprudential exercises focus on the financial system as a whole (usually represented by a set of supervised financial institutions, such as the banking system)</p> <p>Top-down exercises are conducted by its designers, with a harmonized methodology and the data it can get. Bottom-up exercises are conducted by financial institutions themselves, with their own methodology, to the request of a supervisor.</p>	<p>Microprudential stress tests are the most common, but macroprudential approach is developing a lot since 2008 financial crisis</p> <p>Both approaches coexist</p>	<p>Both approaches coexist</p> <p>Both approaches coexist</p>	<p>Both approaches coexist</p> <p>Both approaches coexist</p>
Choice of a set of scenarios	<p>4 types of scenarios based on Jacquetin (2021)</p> <p>Risk drivers are the sources of risk (floods, droughts, climate policy, shift in market sentiment...)</p> <p>Scenario in which it is assumed that there is no particular shock, against which the output of the stressed scenario is compared.</p>	<p>Mostly historical and hypothetical scenarios</p> <p>Various economic and financial sources of risk (oil supply shock, rise in households debt, liquidity crisis...)</p> <p>The baseline scenario is usually the "business as usual" (BAU) scenario</p>	<p>Prospective or hypothetical scenarios</p> <p>Mostly climate policy and technology changes</p>	<p>Mostly prospective scenarios</p> <p>Mostly extreme weather events and changes in weather patterns such as precipitation and heatwaves</p>
Scenario design	<p>Short description of the risks and shocks represented in the scenario</p> <p>Duration of the period within which assessed risks are assumed to materialise and have an impact on the financial system</p> <p>Modelling framework which allows to propagate the impact of risks described in the narrative in the economy</p> <p>Relevant economic and financial variables which allow to translate the economic impact of assessed risks into an impact on balance sheets of financial institutions</p> <p>Financial risk transmission channels which are considered</p> <p>Scenario outputs are more or less granular, at the portfolio-level or security-level for example</p>	<p>Mostly shocks taken from historical stress events, or shocks deemed plausible given economic and financial conditions</p> <p>Maximum 5 years, but mostly 2-3 years: business cycle temporality</p> <p>Not always used</p> <p>Varies depending on the type of risk transmission channel that is assessed (see below)</p> <p>Mostly credit and market risk, but also liquidity risk. Some take into account interaction between the three.</p> <p>Depends on the needs, but mostly sector-level or asset-level granularity</p> <p>Mostly banks and insurers, as they are the most regulated, but macroprudential supervisory stress tests focused on systemic risk can also include investment funds and pension funds</p> <p>Can include changes in the credit portfolio mix, deleveraging, as well as changes in the composition of the trading book</p>	<p>Mostly about carbon tax and technology changes, and a temperature target for prospective scenarios</p> <p>From overnight horizon to 30 years</p> <p>Often used. Usually Integrated Assessment Models, macro-financial modelling and/or sectoral equilibrium model</p> <p>Depends on the economic and financial modelling that is used + on the type of risk transmission channel that is assessed (see below)</p> <p>Credit and market risk</p> <p>Sector-level granularity</p>	<p>Mostly about the type of driver considered and the temperature rise at the end of the 21st century</p> <p>From 30 to 80 years</p> <p>Combination of climate models and long-term socio-economic projections of IPCC shared socio-economic pathways (SSPs)</p> <p>Varies depending on the type of risk transmission channel that is assessed (see below)</p> <p>Mostly underwriting risk and credit risk</p> <p>Geographical granularity from country-level to squares of 50 km</p>
Impact on the financial system	<p>Financial institutions on which the impact is assessed</p> <p>Balance sheets can be assumed to be dynamic, i.e. to vary during the scenario period, or static, to remain constant until the time horizon</p> <p>Feedback loop effects can be added, within the financial system or from the financial system to the economy</p>	<p>Mostly banks and insurers, but also investment and pension funds especially in academic stress tests</p> <p>Sometimes a dynamic balance sheet is assumed, especially for long-term scenarios: financial institutions are assumed to revise their asset allocation and insurance premia during the scenario</p> <p>A few academic stress tests measure feedback loop effects within the financial system or from banks to the economy through a general credit crunch</p>	<p>Mostly banks and insurers, but also investment and pension funds especially in academic stress tests</p> <p>Sometimes a dynamic balance sheet is assumed, especially for long-term scenarios: financial institutions are assumed to revise their asset allocation and insurance premia during the scenario</p> <p>A few academic stress tests measure feedback loop effects within the financial system or from banks to the economy through a general credit crunch</p>	<p>Mostly banks and insurers</p> <p>Static balance sheet assumption</p>

Figure 3 : A comparison between methodologies of non-climate financial stress tests and climate stress tests

4 Existing methodologies

As illustrated in Figure 3, climate stress testing methodologies differ when assessing physical risks and transition risks in terms of time horizon, modelling frameworks and the methods used to measure financial institutions' exposure. Therefore, the analyses of methodologies for physical risks and transition risks stress tests are separated.

Tables 5 and 6 in the Appendix describe the main characteristics of each climate stress test covered in this article for respectively physical risks and transition risks.

In the following, “physical risks stress tests” and “transition risk stress tests” refer to stand-alone climate stress testing exercises focusing on one of the two types of climate-related risks, but also to the respective parts of climate stress testing exercises that cover both physical and transition risks. For example, in the ACPR-BdF exercise, there are three scenarios of transition risk and three scenarios of physical risk. This stress test can be understood as a combination of two exercises, as methodologies of scenario design and financial risk assessment for the two risk types are very different and treated separately in the methodological reports.⁶

4.1 Physical risks stress tests

Climate stress tests for physical risks vary a lot in their modelling approach and financial risk assessment. To discuss existing methodologies, we chose to describe the main patterns that can be found in these different approaches, insisting on specific aspects that seem of great interest for further methodological developments.

Modelling framework and hazards coverage The scope of hazards taken into account varies depending on the modelling approach, which in turn depends on the available data and the accessibility of models. Climate scientists are having much difficulties to downscale the consequences of climate change to the level of local climate hazards and extreme weather events (Fiedler et al., 2021). Projecting the hazards related to climate change is easier for temperature variations or precipitation levels, as these phenomena are better understood by climate scientists (Pinchot et al., 2021). Therefore climate stress tests for physical risks generally either use hypothetical scenarios that do not relate the shocks to a given level of climate change (Caloia and Jansen, 2021), or focus on a specific kind of hazard that is particularly relevant and for which climate models provide usable projections such as flood risk (Mandel, 2021; Bikakis, 2020). More ambitious projects combine heavy modelling frameworks and extensive databases to take into account as many climate hazards as possible: this is what some central banks can do (BoE, ECB). IAMs can also be used to deal with physical risks damage at a macro level (BoE and ECB use these two complementary approaches).

6. Allen et al. (2020) details the design of transition risk scenarios, while CCR (2018) details the design of physical risks scenarios of the ACPR-BdF exercise.

Time horizon Time horizon in physical risks scenarios mostly ranges from 30 to 60 years. The longer the time horizon, the higher will be physical risks. To get a full picture of their magnitude, it is important to use a time horizon at least near the end of the century. However, economic projections at such a long horizon are extremely uncertain, this is why ACPR-BdF and the ECB⁷ make the choice to limit their climate projections at 2050. This leads necessarily to a lower-bound estimation of physical risks. The BoE makes a different choice in its CBES:⁸ climate outcomes that are expected between 2050 and 2080 if there was no further mitigation actions are assumed to take place during the period from 2020 to 2050. This amounts to assessing what would happen for financial institutions if they did not change their balance sheet exposure to physical risks up to 2050. A few scenarios are very short-term: this is either to build a baseline scenario as in Mandel (2021) or to build a hypothetical shock scenario as in Caloia and Jansen (2021). The latter approach is a way to acknowledge that estimating the actual magnitude of physical risks is highly uncertain. It follows the philosophy of a reverse stress test, in which one assesses the magnitude of hazards needed to trigger a significant financial instability.

Balance sheet assumptions However, in all exercises but one balance sheets are assumed to remain static. These stress tests therefore evaluate the impact of the physical risks of the second half of the century on the current balance sheets. The notion of time horizon can therefore be misleading: these stress tests actually assess what would happen in 2030, 2050 or 2080 if balance sheets remained the same, or equivalently what would happen now if hazards corresponding to these time horizons materialised now. Developing the assumption of dynamic balance sheet would allow to understand the possible adjustments reducing the exposure to physical risks, in an exploratory perspective. In particular, the role of adaptation, that would need financing efforts from the private sector and hence from financial institutions, should be key in the design of dynamic balance sheets.

Adaptation Assumptions about adaptation or infrastructures deterioration between now and the date of the shock are usually not made, or not explicit⁹ despite the major role this parameter plays in the assessment of the impact of climate-change-related hazards. Worst-case scenarios could use the hypothesis that infrastructures deteriorate in time. Exploratory scenarios with various adaptation assumptions could be made to assess the trade-off between adaptation costs and physical risks in terms of financial stability, as the ECB has done to assess the trade-off between mitigation costs and no climate action by balancing transition risks and physical risks (Alogoskoufis et al., 2021).

Second-round effects In addition to second-round effects related to financial contagion that

7. In its economy-wide climate stress test (Alogoskoufis et al., 2021).

8. Climate Biennial Exploratory Scenario.

9. A noticeable exception is Mandel et al. (2021) which mentions assumptions about adaptation efforts: scenarios either assume that investments are made so that protection levels of infrastructures remain constant, or that such investments are not made, in which case protection levels downgrade in time.

are relevant for transition risks also, physical risks can lead to a quite specific second-round effect, that is from insurance companies to banks. Physical risks can lead insurance companies to lower their coverage or raise their premia. Banks are exposed to these underwriting risks, as they finance companies that are insured for climate-change-related hazards. As for financial contagion, Mandel et al. (2021) shows the importance of taking into account international inter-exposures of financial institutions. The financial institutions of the countries in which physical risks are the highest will be highly exposed to these physical risks; and high international interconnectedness of financial institutions may foster important financial contagion.

Accounting for financial institutions' reaction The reaction of financial institutions to climate-change-related hazards can itself have an impact on physical risks, by changing the exposure and vulnerability of economic activities to the physical impacts of climate change, and by contributing to its mitigation. Their reaction can be of three kinds: they can lower their exposure to assets vulnerable to physical risks, finance climate change mitigation and finance adaptation. Assessing the impact of these various choices on their exposure and vulnerability to physical risks would be relevant to fully capture the amplitude of financial risks stemming from physical risks.

4.2 Transition risks: classification of existing methodologies

The methodologies of climate stress tests for transition risks can be divided into long-term frameworks using NGFS scenarios that are used by some central banks, especially by the bottom-up exercises of the ACPR-BdF and the BoE and short-term frameworks developed in majority by academic contributions. Figure 1.4 proposes a classification of these methodologies.

The main characteristics of long-term and short-term frameworks are the following:

- The long-term framework is based on prospective scenarios: each scenario corresponds to a given temperature target at the end of the century. It requires a heavy economic modelling including using IAMs. All methodologies subscribing to this framework have therefore relied on NGFS scenarios of Phase I and Phase II (NGFS, 2020, 2021b). They combine IAMs with macroeconomic and sectoral input-output models, as well as financial modelling to provide asset prices and default probabilities. They can include the assumption of dynamic balance sheets.

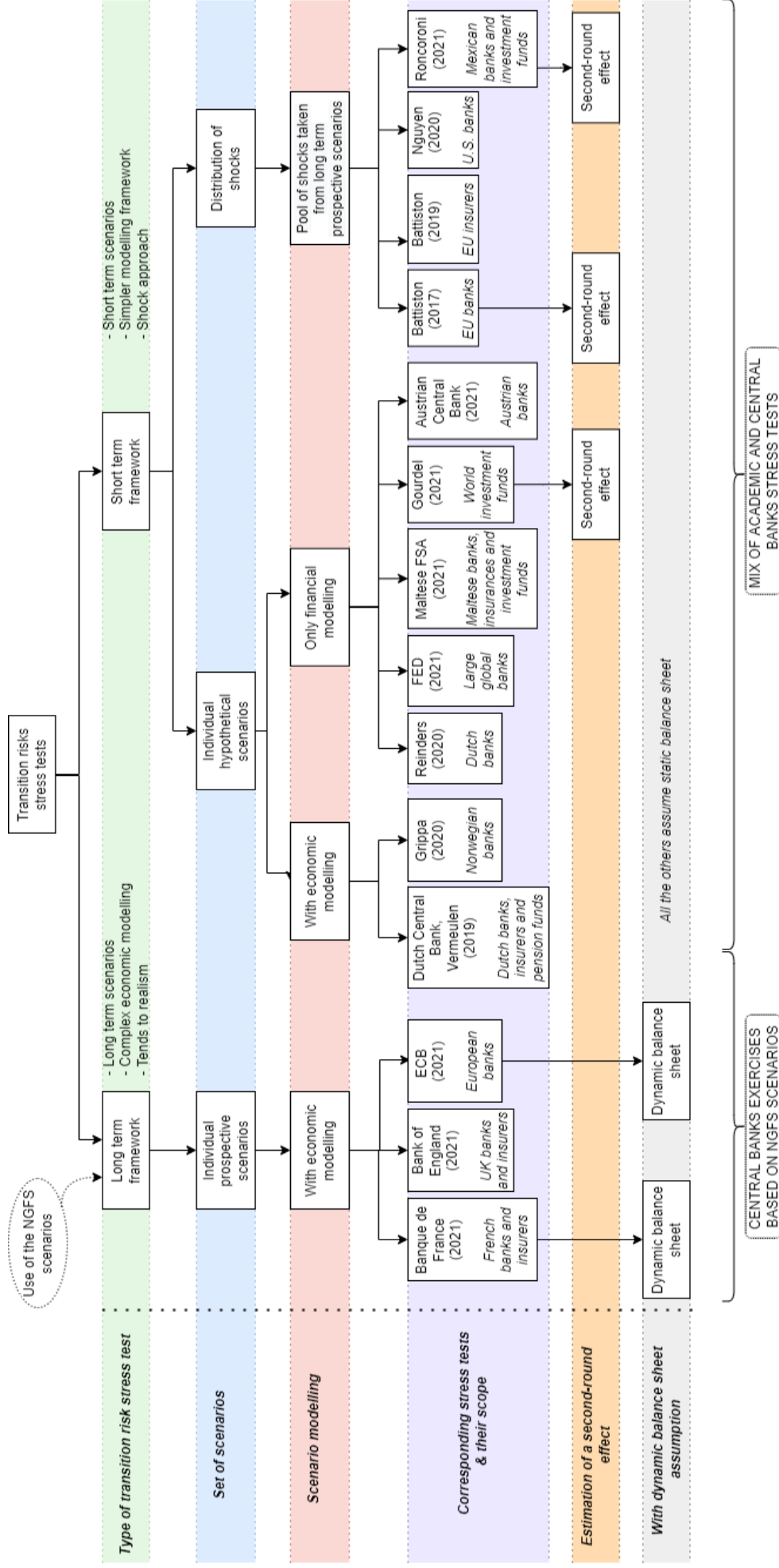


Figure 1.4 – A classification of transition risks stress tests methodologies

- The short-term framework instead is made of hypothetical scenarios with a more or less developed economic and financial modelling. It is composed of much more heterogeneous methodologies, as it is much more tractable. Each exercise of the short-term framework is often less exhaustive in terms of risk drivers or risk transmission channels, but it allows to explore a wider variety of combinations of drivers and channels.

This difference in methodological frameworks reflects both technical constraints and differences in risk assessment philosophies. Given the means that are needed for the design and use of sophisticated long-term economic scenarios, it is difficult for a single institution or research laboratory to design relevant long-term scenarios on its own. The NGFS has therefore pooled together the means of many central banks to create long-term scenarios that would be relevant for this purpose and help harmonize methodologies. On the other hand, assessing short-term impacts of transition risk through short-term stress testing framework is also a relevant methodological choice made for example by Battiston et al. (2017), Gourdel and Sydow (2021) and Jung et al. (2021). They choose to focus on short-term impacts of transition risk on financial markets and financial institutions. The two former for example use the advantages of the short-term framework to assess contagion effects between financial institutions (banks and investments funds respectively).

We further discuss the relevance of the use of long-term frameworks for transition risks stress tests through the case study of the ACPR-BdF exercise, the first complete bottom-up exercise based on NGFS scenarios that was carried out by a financial supervisor or a central bank.

The special case of a pool of shocks as plausible scenario distribution

Most climate stress tests use a small number of scenarios. However, some academic papers in the short-term framework make the choice to use a pool of scenarios (Battiston et al., 2017; Roncoroni et al., 2021; Battiston et al., 2019; Nguyen et al., 2020). In this kind of approach, adverse scenarios are modelled as shocks to economic and financial conditions and their impact is assessed on present balance sheets with a very short time horizon, less than a year. They therefore assume balance sheets to be static. The idea of this approach is to design a pool of shocks that would be representative of the distribution of transition risks. This allows to derive a “climate Value at Risk” as a measure of transition risks. To build this pool, these papers use prospective scenarios with a 2050 horizon (comparable to the NGFS scenarios) and usually define shocks as a jump from the baseline scenario to one of the adverse scenarios at any time up to 2050.¹⁰ This method therefore mixes transition shocks that might happen at different periods and assume that such a pool of shocks would be representative of transition risks at a given date.

The choice to use a probabilistic distribution of shocks for a climate stress test can be

10. Another practice used in Battiston et al. (2017) consists in considering, in each long-term scenario, any 5-year change in economic outputs as a shock.

questioned as climate-related risks are radically uncertain: it is hard to justify any estimation of the law of transition risks as of today, as they rely among other on political choices. However, if one wanted to assess the probability of scenarios, it would be interesting to discuss the way probabilities are assigned and estimated. For example, in such a case, we could expect that the magnitude of the transition risks that we may face would evolve in time. The stringency of sudden climate policies may increase with time, as a result of an increase in social acceptance of such policies; climate change awareness is indeed growing, helped among other reasons by the observation of an increasing number of extreme weather events attributed to climate change. Glette-Iversen et al. (2022) provide an interesting discussion on the concept of plausibility, showing that plausibility is a mix of a likelihood assessment derived from scientific knowledge and a judgment. Such a concept could be a useful intermediary between assigning probabilities to scenarios and not qualifying their likelihood in any way.

4.3 A case study of supervisory stress test: the Banque de France exercise

The ACPR-BdF pilot exercise is the first one to adopt a long-term framework based on NGFS scenarios for climate stress testing. The methodology that has been developed is a first attempt that is to be improved in further exercises (ACPR, 2021). We take advantage of this situation to discuss some of the methodological choices that have been made, with the hope to help for further improvements.

Table 1.3 describes the main features of the ACPR-BdF exercise that are of interest for our discussion. The exercise uses three transition scenarios designed by the NGFS: one scenario of “orderly transition” (the baseline scenario) and two scenarios of “disorderly transition” (the adverse scenarios). The orderly transition scenario describes a low-carbon transition limiting global warming to below 2°C that is implemented in an optimal way, as it minimizes both transition risks and physical risks. Climate policies are introduced early and become gradually more stringent; technological changes help a lot by being easily adopted. On the other hand, a disorderly transition means that the same temperature target is met but climate action is taken with a delay, it is therefore more stringent and may also be unexpected, sudden and abrupt. Assumptions on technological changes are also more adverse. This scenario results in higher transition risks with a similar level of physical risks.

The scenarios developed for the ACPR-BdF exercise, based on the NGFS scenarios, do not appear to be very adverse, for the following reasons.

- Scenarios are deterministic. Therefore, except for the beginning of the period in which the balance sheet is static, shocks are expected by financial institutions.¹¹ Their consequences can be smoothed out by ex ante changes in balance sheets through the dynamic balance

11. Dynamic balance sheet assumption is implemented as follows: the scenario provides changes in output variables every five-year from 2025 to 2050; balance sheet is assumed to be static between 2020 and 2025; it is the last available balance sheet in the data, from 2019. Balance sheet starts to be dynamic in 2025 up to the scenario horizon.

Type of scenarios	Prospective
Horizon	30 years (from 2020 to 2050)
Modelling	<ul style="list-style-type: none"> — IAMs used to define a carbon tax trajectory compatible with 2°C objective given assumptions about green technology efficiency and carbon capture and storage technologies — Macroeconomic and sectoral models calibrated on GDP trajectories provided by the IAM to detail the impact of mitigation efforts in the economy
Dynamic balance sheet	Financial institutions are invited to provide sectoral reallocations of their credit portfolio for each five year period from 2025 to 2050 knowing scenario outputs of the whole period
No feedback loop effect from the financial system to the economy	Balance sheet projections of financial institutions are harmonized a posteriori so that they be coherent with the pre-computed transition scenario. Therefore there is no feedback loop from banks dynamic balance sheet to the economy.

Table 1.3 – Description of the main features of the ACPR exercise

Source: ACPR (2020); Allen et al. (2020)

sheet assumption. Such an implementation of the dynamic balance sheet assumption may be suited to provide information to banks about possible future scenarios and to encourage them to develop a strategy of sectoral reallocation of financing supply for different transition pathways. However, this approach overlooks the fact that, in real time, banks will have to reallocate their financing supply given uncertainty about the transition pathway: there is no perfect information. Uncertainty and incomplete information are adverse features that should be taken into account in a stress test to better assess the magnitude of financial risks stemming from transition risks.

- The long-term modelling framework using IAMs tends to smooth out the impact of transition risks, as business cycles dynamics are not modelled. No economic recession is assumed to happen even in disorderly scenarios; they only translate into a lower growth compared to the orderly scenario (ACPR, 2021). This rules out economic crises caused by sudden and abrupt climate policies. It also overlooks the possibility of sudden repricing events caused by any beliefs updates in climate sentiment that could trigger financial losses and weaken financial institutions. Complementary short-term stress test may be more suited to assess the financial risk impact of these two kinds of adverse situations.
- The effects of the discrepancy between investors expectations and actual climate policies are not taken into account, as these expectations are not included in the modelling framework of the scenario. Yet, a misalignment of agents expectations may be an important source of adversity in disorderly transition scenarios compared to an orderly transition

(Dunz et al., 2019).

- These three characteristics contribute to reducing the difference between the baseline scenario and the two adverse disorderly scenarios (ACPR, 2021).

All these factors contribute to making the ACPR-BdF transition scenarios exploratory scenarios rather than actually adverse scenarios. They draw possible pathways for the future, without focusing on worst-case scenarios.

Moreover, the way the scenario impact is assessed on banks' balance sheets contributes to alleviating the potentially negative effects of the two disorderly transition scenarios. In particular, although interconnectedness of banks and other financial institutions may be an amplifier of the direct impact of transition risk on their balance sheets (Roncoroni et al., 2021), no contagion effect between banks has been assessed in this stress test. Moreover, as dynamic balance sheets are re-adjusted a posteriori to match the initially designed scenario trajectories, no feedback loop from the financial system to the economy is assessed, whereas we could expect that the reallocation choices of banks could have an impact on the shape of the transition itself. Battiston et al. (2021) emphasize that the financial system may hamper or accelerate the transition depending on its information and incentives to reallocate its credit supply and investments.

To design fully adverse scenarios in a transition risks stress test, the conditions under which financial institutions could have a hampering effect on the transition should be identified. Understanding the role that financial institutions could have in defining the rhythm of the transition can also be very interesting with the objective to provide exploratory rather than worst-case scenarios.

These remarks form the basis of section 5, in which alternative approaches are proposed to complement current ACPR-BdF framework.

4.4 Coverage of climate-related risk drivers and financial risk channels by current methodologies

The various climate-related risk drivers and financial risk transmission channels identified in Section 2.1 are very unequally covered by existing methodologies of climate stress tests. We report in Table 1.4 the coverage of risk driver / risk channel combinations by various stress tests.

As for climate-related risks drivers, we consider the four transition risks drivers cited in 2.1 and we group together the various physical risks drivers. It is indeed hard to distinguish between chronic and acute physical risks in many physical risks stress tests exercises given the way they are modelled. Moreover, it would only be interesting with a detailed assessment of the relative relevance of including each driver in a climate stress test given its modelling cost and uncertainty. For example, tropical cyclones are harder to model than flooding risk, see for example Le Guenedal et al. (2022) compared to Mandel et al. (2021); but they might have a

	Transition risks drivers					Physical risks	
	Climate policy	Technology changes	Market sentiment	Consumer sentiment	Unspecified	All drivers	
Financial risks transmission channels	Credit risk	ACPR (2020), Vermeulen et al. (2021), Alogoskoufis et al. (2021), Battiston et al. (2019), Grippa et al. (2020), Nguyen et al. (2020), Reinders et al. (2020), Roncoroni et al. (2021), BoE CBES (2021)	ACPR (2020), Nguyen et al. (2020), Roncoroni et al. (2021), Vermeulen et al. (2021), Alogoskoufis et al. (2021), BoE CBES (2021)	x	Alogoskoufis et al. (2021)	x	BoE CBES (2021), Alogoskoufis et al. (2021), Bikakis (2020), Mandel et al. (2021), Mandel (2021), Caloia and Jansen (2021)
	Market risk	ACPR (2020), Vermeulen et al. (2021), Battiston et al. (2017), Grippa et al. (2020), Reinders et al. (2020), Roncoroni et al. (2021), Alogoskoufis et al. (2021), Conte and Meglioli (2021), Gourdel and Sydow (2021)	ACPR (2020), Vermeulen et al. (2021), Battiston et al. (2021), Roncoroni et al. (2021), Alogoskoufis et al. (2021)	Gourdel and Sydow (2021)	Alogoskoufis et al. (2021)	Battiston et al. (2017), Jung et al. (2021)	Alogoskoufis et al. (2021), Gourdel and Sydow (2021), Bressan et al. (2022a)
	Liquidity risk (asset liquidity)	x	x	Gourdel and Sydow (2021)	x	Roncoroni et al. (2021)	Gourdel and Sydow (2021)
	Liquidity risk (funding liquidity)	x	x	Gourdel and Sydow (2021)	x	x	Gourdel and Sydow (2021)
	Underwriting risk (for insurers)	NA	NA	NA	NA	NA	ACPR (2021), BoE CBES (2021), Mandel et al. (2021)
	Underwriting risk (for other financial institutions)	NA	NA	NA	NA	NA	ACPR (2021), Mandel et al. (2021)

Table 1.4 – Crossed coverage of transition risks drivers and financial risk transmission channels
Reading note: A cross is reported when no stress testing methodology mentioned in this paper covers the couple (line,column) of (financial risk transmission channel, transition risk driver); “NA” is reported when there is no reason to assess a specific combination of channel and driver.

lower aggregate impact. We leave these tasks to further research.

As for transition risks drivers, Table 1.4 shows that market sentiment and consumer sentiment are very poorly considered by existing methodologies. Little work has been done on modelling the effect of changes in consumer sentiment and it could be interesting to further investigate this field. The way deep shifts in consumer sentiment could have an impact on financial risks remains quite uncertain and remains to be documented in the climate stress testing literature.¹² As for market sentiment, the impact of this risk driver on financial risk is more straightforward and we showed in Section 2.1 its relevance to financial stability issues. It would be important to include a more detailed analysis of this risk driver in climate stress tests. The discussion of ACPR-BdF exercise in Section 4.3 and Section 5 outline some methodological directions that would help to take into account this driver in climate stress testing.

As for financial risks transmission channels, liquidity risks, which are an important transmission channel of climate risk, are often neglected in both physical and transition risks stress tests. Banks and insurers may lose funding liquidity because of their exposure to either carbon-

12. Finance literature has already started to investigate this field. For example, Sauzet and Zerbib (2022a) provide some insights on the relationship between consumer sentiment, market sentiment and financial returns.

intensive assets or to assets highly exposed to physical risks, such as a bank or an insurer in a country very vulnerable to physical risks. Both transition and physical risks could also translate into a drop in the liquidity of asset holdings and in the value of collaterals used as credit guarantee, such as mortgage or carbon-intensive plants. This risk transmission channel should therefore be better investigated in climate stress tests.

5 A proposition of complementary stress testing frameworks for transition risks

This analysis of current methodologies leads us to emphasize the need for two approaches that would complement the long term framework of climate stress tests for transition risks that is used by central banks.

The first one would be to develop the use of short-term hypothetical scenarios. This is not a new practice, but it should be included in supervisory stress tests to complement the long-term framework. Such scenarios allow easily to test for the resilience to various short-term disruptive events. In particular, they could assess the impact of tail market sentiment risk on financial institutions, as well as the role of the liquidity channel. This complementary short-term approach is already being adopted in a second ECB stress test exercise (ECB, 2021) and should be further developed.¹³ Its flexibility is a great advantage to assess the resilience to risks that are not accounted for in the long-term framework due to modelling complexity.

The second one, more ambitious theoretically, would be to design a long-term modelling framework that accounts for the stochasticity of climate policies and technology changes and that would take into account the feedback loop from the financial system to the economy with regards to its sectoral transition. Its objective would be to allow for a more realistic dynamic balance sheet assumption, that is in which financial institutions face uncertainty about the future of the scenario to which they are asked to react. To close the loop, their reallocation decision should have impacts on the structural changes of the economy and therefore on the transition itself.

This approach requires a model in which changes in sectoral distribution of the economy would be driven by financing supply and not only by financing demand from the real economy. Such feedback loop from the financial system to the real economy is already starting to be developed, especially through stock-flow consistent models (Gourdel et al., 2022; Dafermos et al., 2018). But this is only a first step. We need in particular to better understand how financial institutions will reallocate their financing flows in the context of the low-carbon transition (Battiston et al., 2021) and how the uncertainty surrounding the timing and intensity of the transition will affect their decisions and hence the rhythm of the transition itself. De Angelis

13. It is noteworthy that it has, indeed, been further developed, as taken on by the NGFS: see NGFS (2023a) for example.

et al. (2023) show for example that uncertainty about transition shocks leads to a scarcer financing of green activities, compared to financing decisions in a deterministic transition path. Flora and Tankov (2023) model the optimal decision to exercise a real option in a context of uncertainty about the transition pathway: they show that increasing uncertainty contributes to longer waiting time before exercising the option, be it for green investment or divestment for a brown company.

We therefore call for the development of “reallocation strategy tests”. Their objective would be to assess the asset reallocation strategy of banks in the context of transition to low-carbon economy. They would consider both the risks for financial institutions and the impact of the strategies on the low-carbon transition according to the principle of double materiality (Gourdel et al., 2022). Their top-down framework would include the following characteristics:

- a) Long-term hypothetical scenarios with stochastic processes and shocks modelling the advent of climate policies and technology changes
- b) Dynamic balance sheets with a rule for reallocating financing supply in the context of uncertainty about the future
- c) Feedback loop from financing supply reallocation to the sectoral and production structure of the economy

We don’t call it a stress test as it loses a main aspect of stress tests. As illustrated in Figures 1.1 and 1.2, a stress test is based on the design of an adverse exogenous scenario and assesses the impact of this exogenous scenario on the financial system. It takes the financial system as a passive receptacle of shocks. In a reallocation strategy test, the scenario becomes endogenous to the financial system, as financial institutions reallocation choices contribute to defining the scenario itself. In this framework, the scenario cannot be designed independently of financial institutions dynamic balance sheet as it is done in the ACPR-BdF exercise. So this rationale goes further than the NGFS long-term framework, as balance sheets dynamics now have a reverse impact on real economy.

Such a reallocation strategy test would help assessing transition risks amplification through the feedback loop economy-financial system and hence be useful to design worst-case scenarios. It could also answer other objectives such as guiding financial institutions in financing the low-carbon transition and identifying financing needs. One of the stated objectives of the second ECB stress test is indeed to assess “banks’ business models and their adaptability in different long-term transition scenarios” (ECB, 2021).

6 Conclusion

Climate stress testing and scenario analysis has been used by both central banks and the academic field to assess climate-related risks to the financial system. Significant work has been done to develop various methodologies. In particular, the NGFS has made considerable efforts

to develop long-term deterministic scenarios. These scenarios have been used by several central banks for their pilot exercises and are intended to be used for future exercises and by many other central banks (ECB, 2021).

The objectives and the methodologies of existing climate stress tests vary a lot. Still, the NGFS high-level reference scenarios create a convergence of approaches among the climate stress testing exercises employed by supervisors. The influence and weight of the NGFS scenarios hence contributes to improve the necessary cross-border and cross-institutions comparability of climate stress test results and to facilitate the realisation of otherwise very complex and costly exercises. But it is also a big responsibility and such costly exercises entail path dependency costs. This paper assesses whether these methodologies are enough to assess the financial stability issues caused by climate-related risks.

It appears that using only long-term deterministic scenarios leads to overlooking some important risks drivers and financial risks channels that may be relevant in terms of financial stability issues, such as market sentiment. Market sentiment represents the risk of more or less abrupt redirections of financial flows from high-carbon to low-carbon activities (or the reverse, if a green bubble bursts), often triggered by beliefs updates and changes in expectations towards climate-related risks by investors and financial institutions. This is due to the uncertainty of climate-related risks and opacity of exposure data. It could create major repricing events and lead to considerable financial losses.

To better account for these risks, I propose to complement the long-term deterministic framework based on NGFS scenarios with short-term climate stress tests specialised in some specific risks drivers and transmission channels. This would allow especially to better assess risks related to market sentiment and other disruptive transition shocks through the credit, market and liquidity risk channels. Such short-term climate scenarios would be very flexible, easily designed and used. They would make it easier to assess the resilience of financial institutions in terms of capital buffers, like more traditional prudential stress tests. The second ECB climate stress test of 2022 goes in this direction by including a three-year horizon scenario involving a sharp rise in carbon tax (ECB, 2021).

To better account for the adversity of long-term transition scenarios and allow for a more realistic dynamic balance sheet assumption with uncertainty about the transition pathways, we also propose the development of “reallocation strategy tests”. They would be based on long-term stochastic transition pathways with feedback loops between real economy and financial institutions. They would allow to assess the macroprudential and macroeconomic relevance of financial institutions’ balance sheet adaptations to the transition. They would therefore help to pilot the low-carbon transition from the financial viewpoint by identifying financing needs and controlling banks’ strategies. However, such tests require further theoretical research to develop adapted models and tools and they require to move on from a strict stress testing scheme, as climate scenarios would become endogenous in this framework.

Appendix

Climate stress test identifier	Institution	Year	References
ACPR-BdF	Autorité de Contrôle Prudentiel et de Résolution and Banque de France	2020	ACPR (2020); Allen et al. (2020); ACPR (2021); CCR (2018)
BoE CBES (Climate Biennial Exploratory scenario)	Bank of England	2021	BoE (2021, 2022)
ECB economy-wide climate stress test	European Central Bank	2021	Alogoskoufis et al. (2021)
DNB First central bank climate stress test	Dutch National Bank	2019	Vermeulen et al. (2021)
DNB 2 nd climate stress test on flood risk	Dutch National Bank	2021	Caloia and Jansen (2021)
FED	Federal Reserve Bank	2021	Jung et al. (2021)
Maltese FSA	Maltese Financial Services Authority	2021	Conte and Meglioli (2021)
Austrian National Bank	Austrian National Bank	2021	Guth et al. (2021)

Table 1.5 – Climate stress tests of central banks and supervisors covered in this paper: list and full references

Battiston et al. (2017)	Roncoroni et al. (2021)
Battiston et al. (2019)	Mandel et al. (2021)
Bikakis (2020)	Mandel (2021)
Reinders et al. (2020)	Gourdel and Sydow (2021)
Nguyen et al. (2020)	Bressan et al. (2022a)
Grippa et al. (2020)	

Table 1.6 – List of academic climate stress tests covered in this paper

	ACPR (2020, 2021)	Bank of England (2022)	ECB (2021) (Alogoskoufis 2021)	Bikakis et al. (2020)	Mandel et al. (2021a) Financial stability	Mandel et al. (2021b) Mortgage	Gourdel et al. (2021)	Caloia et al. (2021)	Bressan et al. (2022)
SCENARIO MODELLING									
Horizon	2050	2050	2050	Immediate and 2080	2030,2050 and 2080	Immediate and 2050	A few months	1 year	2035, 2040, 2045, 2050
Number of scenarios	3	1	1	3 (1 present day scenario)	4	3 (2 present-day scenarios)	1	6	3 main scenarios
Type of scenario narrative	Prospective	Prospective	Prospective	Prospective	Prospective	Prospective	Hypothetical	Hypothetical	Prospective
Baseline scenario	Orderly transition	Scenario without any climate-related risk	Orderly transition	Not mentioned, but probably Present Day	Not reported	Present Day	Initial situation	Business as usual	Scenario without climate change
Risk drivers	Natural disasters + Health	Flooding, sea level rise, drought, heatwaves, tropical cyclones	Extreme weather events and chronic hazards	Flood risk	Flood risk	Heavy rains and tropical cyclones	Floods, heat stress, hurricanes, typhoons, water stress, and wildfires	Flood risk	Tropical cyclones, chronic hazards
Hypotheses about adaptation	None	Yes, mostly about UK flood defences	None	None	Yes	None	None	None	None
Scenario modelling	Climate model	Climate and economic modelling	Economic, climate and financial modelling	Climate and financial modelling	Climate and financial modelling	Climate and financial	Climate modelling	Macroeconomic and financial modelling	Climate, macroeconomic and financial modelling
Geographical granularity	French administrative departments	Country-level	Address level	12 UK regions	Country-level	ZIP-code 5-digit level granularity	Company-level	postal codes at the 4-digit level	Plant-level in Mexico
IMPACT ON THE FINANCIAL SYSTEM									
Top-down/bottom-up	Bottom-up	Bottom-up	Top-down	Top-down	Top-down	Top-down	Top-down	Top-down	Top-down
Balance sheet assumptions	Dynamic	Static + questionnaire to learn about adaptation of business models	Static	Static	Static	Static	Static	None	Static
Area of financial institutions	France	UK	Euro area	UK	109 countries (including OECD countries)	Florida (U.S.A)	World (mostly OECD countries)	Netherlands	EU
Type of financial institutions	Insurers, banks	Banks, insurers	Banks	Banks	Country financial sectors (excluding insurers)	Banks	Investment funds	Banks	All types of investors
Balance sheet entries	Insurers: Non-life liabilities Banks: loans	Bonds and loans for banks, liabilities for insurers	Loans, corporate bonds	Mortgage loans	Not precised	Mortgage loans	Equity, bonds	Banking and trading books exposed to households and non-financial companies	Equity holdings
Second round effect channel	Underwriting risk for banks	None	None	None	Direct cross-holdings	None	Direct cross-holdings + overlapping portfolios	None	None

Figure 5 : Climate stress tests for physical risks : comparison of existing methodologies

ACPR (Allen et al., 2020; ACPR, 2020 & 2021)	Bank of England (2021)	ECB (2021) (Alogoskoufis 2021)	DNB (Vermeulen et al., 2019)	Austrian National Bank (Guth et al., 2021)	Maltese FSA (Conte et al., 2021)	FED (Jung et al., 2021)	Battiston et al. (2017)	Battiston et al. (2019)	Roncoroni et al. (2021)	Nguyen et al. (2020)	Reinders et al. (2020)	Grippa et al. (2020)	Gourdel et al. (2021)
SCENARIO DESIGN													
Horizon	30 years	30 years	5 years	5 years	Immediate	Immediate	Immediate	Immediate	Immediate	Immediate	Immediate or 10 years	Immediate	Immediate
Number of scenarios	3 (among them one for physical risk)	2	4	3	6	1	4 + a pool of 5241 shock scenarios	A great number of shock scenarios, as Battiston et al. (2017)	A great number of shock scenarios, as Battiston et al. (2017)	A great number of shock scenarios, as Battiston et al. (2017)	12	7 (4+2+1)	4 (one is for physical risk)
Type of scenario narrative	Prospective	Prospective	Hypothetical	Hypothetical	Hypothetical	Historical	Hypothetical shock scenarios based on prospective scenarios	Hypothetical shock scenarios based on prospective scenarios	Hypothetical shock scenarios based on prospective scenarios	Hypothetical shock scenarios based on prospective scenarios	Hypothetical	Hypothetical	Hypothetical
Baseline scenario	Scenario without any climate-related risk	Orderly transition	Business as usual	Business as usual	Orderly transition	Business as usual	Business as usual	Business as usual	Business as usual	Business as usual	Not reported	One of the adverse scenarios	Initial situation
Risk drivers	Policy + technology	Policy + technology + consumer sentiment	Policy + technology	Policy	Policy	Unspecified	LIMITS database**	LIMITS database**	LIMITS database**	Policy	Policy	Policy	Policy + market sentiment
Scenario economic modelling	IAM + NIGEM	IAM+NIGEM	NIGEM + sectoral vulnerability factors	Macroeconomic + input-output models	None - Only financial valuation models	None	IAMs	IAMs	IAMs	IAMs	None - Only financial valuation models	VAR + partial equilibrium model	None
Granularity level (and sector classification if relevant)	59 sectors	22 industries at the NACE two-digit level	56 industries (NACE)	17 NACE 1-digit sectors	Company-level or sector-level when data is unavailable	NA	7 climate-policy relevant sectors*	7 climate-policy relevant sectors*	7 climate-policy relevant sectors*	Loan-level	56 industries (NACE)	85 industries (NACE 2)	Company-level
IMPACT ON THE FINANCIAL SYSTEM													
Top-down/bottom-up	Bottom-up	Top-down	Top-down	Top-down	Top-down	Top-down	Top-down	Top-down	Top-down	Top-down	Top-down	Top-down	Top-down
Exposure metric to transition risk	Direct sectoral GHG emissions + from intermediate consumption of refined oil and coke	GHG intensity (scopes 1, 2 and 3)	Embodied GHG emissions (emitted by upstream and direct producers)	GHG emissions from Eurostat data	Carbon intensity	Climate beta of banks' stock value estimated on a stranded assets portfolio	LIMITS database**	LIMITS database**	LIMITS database**	Carbon emissions scopes 1 and 2	GHG intensity (scope 1 carbon equivalent)	GHG emissions per unit of output (scope 1) + scope 3 carbon emissions	Carbon intensity (Svopes 1 and 2)
Balance sheet assumptions	Static + questionnaire to learn about adaptation of business models	Static	Static	Static	Static	Static	Static - Assumptions about sectoral distribution	Static	Static	Static - Assumptions about sectoral distribution	Static	Static	Static
Area of financial institutions	United Kingdom	Euro area	Netherlands	Austria	Malta	U.S.A.	European Union	European Union	Mexico	U.S.A.	Netherlands	Norway	World, mostly OECD countries
Type of financial institutions	Banks, insurers	Banks	Banks, insurers, pension funds	Banks	Banks, insurers, funds	Banks	Banks	Insurers	Banks and investment funds	Banks	Banks	Banks	Investment funds
Type of financial assets	Bonds and loans for banks, liabilities for insurers	Corporate loans and bonds	Equity, bonds, loans	Equity, bonds, loans	Equity, bonds, CIS (investment portfolios)	Banks' stock value	Equity	Bonds	Equity, bonds, loans	Loans	Loans, corporate debt, equity	Loans	Equity, bonds
Issuer/counterparty	Corporate	Corporate	Corporate	Corporate	Corporate	NA	Corporate	Sovereign	Corporate	Corporate	Corporate, households	Corporate	Corporate
Second round effect channel	None	None	None	None	None	None	Financial contagion	None	Direct cross-holdings + overlapping portfolios	None	None	None	Direct cross-holdings + overlapping portfolios

*Climate-policy relevant sectors is a sectoral classification first introduced in Battiston et al. (2017) to identify sectors exposed to transition risk.

**The LIMITS database is a database providing detailed information on various IPCC mitigation scenarios estimated with 7 different energy-economy and IAM models of long term horizon, targeting either 1.5°C or 2°C temperature increase at the end of the century. For each scenario, data on sectoral output is provided at each 5-year period. See Kriegler (2013) for more details.

Figure 6 : Climate stress tests for transition risks : comparison of existing methodologies

Chapter 2

Banks divesting and asset stranding in the low-carbon transition

Abstract

As part of the transition to a low-carbon economy, some companies are faced with the risk of asset stranding, that is, the risk that a portion of their assets undergo premature decommissioning. In addition, by integrating transition risks in their financing decision, and under stakeholders' pressure, more and more banks commit to carbon neutrality through divestment pledges. I build a one-period toy model in which banks interact through the financing of a company facing the asset stranding risk. I show that a bank's divestment decision due to a change in its "green" preferences or to new regulatory constraints, by creating a credit constraint for the firm, can increase its probability of default through the asset stranding risk, prompting other banks to divest from this firm as well. As a result, the asset stranding risk can create a snowball divestment effect. These results are significant for two reasons: they highlight a tool the banks can use to accelerate the low-carbon transition, and they raise awareness about potential economic and financial instability issues that this snowball effect might create.

1 Introduction

To limit global warming to 2°C, which is the Paris agreement target, McGlade and Ekins (2015) suggest that a third of oil reserves, half of gas reserves and over 80 per cent of coal reserves should remain unused from 2010 to 2050. This means that a significant portion of current fossil fuel assets must be stranded, that is to say, undergo premature decommissioning (Caldecott et al., 2013; Semieniuk et al., 2021; Daumas, 2023b). In addition, evidence suggests that fossil fuel investment plans of last years have remained strongly misaligned with the Paris agreement target (Coffin and Grant, 2020; Edwards et al., 2022). Moreover, asset stranding may also concern other sectors' assets due to downstream value chain multipliers (Cahen-Fourot et al., 2020, 2021). Hence, concerns have been raised about the impact of this phenomenon on economic and financial stability (Carattini and Sen, 2019; Van der Ploeg and Rezai, 2020; Cahen-Fourot et al., 2021; Mercure et al., 2021; Semieniuk et al., 2022; Comerford and Spiganti, 2023).

In parallel, there has been an inflation in banks' commitments to net-zero alignment these last years: the Net-Zero Banking Alliance¹ counts 144 members in April 2024, representing 41% of global banking assets. Notably, members of this alliance are asked to set sectoral intermediary targets in 2030, which mostly translate into either emissions reduction or phase-out sector-specific targets, especially for fossil fuel industries (United Nations Environment Programme, 2022). Moreover, Maio et al. (2023) mention that almost all Global Systemic Investment Banks have committed to net-zero alignment. Even though commitments are heterogeneous and can be vague or incomplete (United Nations Environment Programme, 2022; Maio et al., 2023), they are unprecedented and seem to respond to stakeholders' pressure to net-zero initiatives. What if these pledges were implemented? Contrasting with ongoing new investments in fossil fuel reserves, could their fulfillment trigger significant asset stranding, and hence potentially create risks for economic and financial stability?

Though banks' asset reallocation from carbon-intensive to environmentally innovative companies is hampered by the "asset overhang" phenomenon (Degryse et al., 2022), there is already some empirical evidence suggesting that banks are adjusting their credit supply to integrate transition risks. Delis et al. (2019) find that banks, and especially the signatories of the Principles for Responsible Banking,² charge higher loan rates to fossil fuel firms, increasingly with longer maturities. Moreover, the results of Kacperczyk and Peydró (2022) suggest that banks operate quantity adjustments by reallocating their credit supply from brown to green companies.

In this chapter, I develop a theoretical model to assess the consequences of banks divesting from companies with assets to be stranded in the low-carbon transition. I build a one-period

1. The Net-Zero Banking Alliance is convened by the United Nations Environment Programme Finance Initiative (UNEPFI). More information can be found on the UNEPFI website: <https://www.unepfi.org/net-zero-banking/>.

2. The Principles for Responsible Banking are also an initiative organized by the UNEPFI, and cover wider issues than the NZBA which focuses on climate action.

toy model wherein several banks finance a given company of which a portion of assets should be stranded to reach a given temperature target. I show that, under certain conditions, a bank’s decision to divest from this company can lead to a reduction in the supply of credit to that company by other banks, which in turn can lead to its insolvency. These results are important under two perspectives: (i) understanding the levers of the low-carbon transition, and the role that financial institutions can play in boosting it; (ii) raising awareness about potential chain reactions in a perspective of maintaining financial and macroeconomic stability.

More precisely, I build an n -player game in which n banks lend to a single company. This company holds a significant portion of assets which should be stranded for the Paris agreement target to be met. The company’s activity is modelled in a stylised one-period model. At the beginning of the period, the company contracts debt. It then produces thanks to its assets backed by its equity and contracted debt. It earns random revenues out of this production at the end of the period, with a distribution which depends on its total assets and debt. The default probability of a debt contract is defined as the probability that the company can not pay back its debt at the end of the period. In this game, the company is passive. The only agents with decision-making power are the banks. Moreover, the period should be interpreted as mid-term: the company can not consider raising equity as an alternative to raising debt.

Each bank wants to maximize the profit generated by its loan by deciding the optimal quantity to lend to the company at date 0. The profit of a loan depends on the probability of default of the company, on its interest rate, and on the bank’s quadratic lending cost. Quadratic lending costs are heterogeneous across banks, reflecting their sizes, their portfolio allocation diversity needs, but also the regulatory constraints that affect them, as well as their degrees of “greenness”.³ Banks are assumed to be price-takers. This simplifying assumption is made to focus on the impact of the credit supply quantity-adjustment rather than price-adjustment, in a first order analysis.

The main specificity of the model is the way the asset-stranding risk translates into the probability of default of the company, creating an interaction between banks’ profits. The company has assets that are made illiquid due to the risk that they be stranded. Indeed, they are priced higher in the balance sheet of the company than outside the company, due to an accounting inertia and path dependency to past investments.⁴ This inertia is reflected in the company’s profit function as follows: it has an “optimal” asset size for the period, in the sense that it minimizes the cost of adjusting the company’s asset quantity relative to its previous asset quantity. This “optimal” asset size can be interpreted as the result of a trade-off between the

3. For example, Delis et al. (2019) use participation to the Principles for Responsible Banking to proxy the “greenness” of a bank.

4. For example, the reserve structure of oil companies is very inert over time, which explains the vulnerability of certain major oil companies when the price of oil crosses certain specific thresholds (Andrade de Sá et al., 2024). This difference in valuation can also be justified by different expectations of actors related to the realisation of the low-carbon transition: the company expects them to have a higher price than what it would get from selling them off.

natural depreciation of previous capital and the optimal investment expenditure to ensure its durability. When the company's asset size at date 0 shifts away from this "optimal" asset size, the company incurs additional costs, assumed to be convex, which can be understood as follows: when exceeding this "optimal" asset size, the company must undertake costly investments; when going below, it must strand some of its assets (Campiglio et al., 2022). Therefore, for a given level of equity, the company has an optimal amount of debt to maximise its profit that depends on this capital-adjustment cost. By collectively determining the quantity of debt that the company can contract at the beginning of the period, banks determine its total asset size at date 0, and hence the capital adjustment-cost that the company has to pay. As it is convex in the amount of capital adjustment, banks' profits interact through this convex cost which plays on the probability of default of the company.

A simultaneous game is modelled between the n banks, which are assumed to be at a Nash equilibrium in the baseline case. However, some of the banks can decide to divest from the company for reasons which are exogenous to the maximisation of its profit related to this loan: it can be due to new regulatory constraints, or to the decision to meet its intermediary phase-out targets as a member of the Net-Zero Banking Alliance. This divestment decision makes it depart from the Nash equilibrium, by lowering its credit supply to the company with respect to the baseline case. The impact of such a divestment on the credit supply of the other banks is measured by summing the new total credit supply addressed to the company, defined as the Nash equilibrium between the non-divesting banks, taking as given the remaining credit supply of the divesting banks.

The model is further specified to illustrate its properties and derive the main results of the paper. The company's profit is assumed to follow a uniform distribution, parameterized by its expectation and the distance between the expectation and the two boundaries of the distribution. The expected profit is a concave quadratic polynomial in the company's contracted debt, as the sum of a linear production function and a concave quadratic polynomial representing the capital-adjustment cost. It is assumed that the total credit supply of banks is equal to the contracted debt, ruling out any price-adjustment. However, additional technical assumptions ensure that banks do not have any interest in exceeding the amount of debt which maximises the company's expected profit, which can be interpreted as its debt demand. These assumptions allow to derive explicit formula to the probability of default of the company as a function of contracted debt, and to the best response function of each bank, which can be expressed as a function of the total credit supply of the other banks. These explicit best response functions allow to identify Nash equilibria graphically when there are two banks, and to derive the impact of the divestment decision of one of the two banks on the other bank's credit supply.

The results are the following. First, under this specification, the probability of default of the company is a U-shaped function of its debt, with top branches becoming horizontal when they reach one. It is minimized in the amount of debt which maximises the expected profit.

The curvature of the U depends on the quadratic coefficient of the capital-adjustment cost: the higher the latter, the steeper the U . Hence, when the capital-adjustment cost is high enough, or when the company needs enough refinancing, the company can become insolvent due to high credit constraints. Through the asset stranding risk, banks' credit supply can therefore have a strong impact on the activity of the company.

Second, the best response function of a bank strongly depends on its quadratic lending cost, which is parameterized by a single quadratic coefficient, and which represents its a priori willingness to lend to the company, based on the bank's size, its degree of "greenness", its own regulatory constraints, as well as its portfolio diversification needs. To simplify the presentation of the results, let us call a "big" bank a bank whose quadratic lending cost is low enough so that it can, on its own, fully finance the optimal debt of the company. In contrast, we call a "small" bank one that does not meet this characteristic; that is to say, its credit line availability for this company is not enough to allow, on its own, to lower the capital-adjustment cost of the company significantly enough to make the loan profitable. In addition, "small" banks can then be ranked across their "sizes" (or a priori willingness to lend to the company for the reasons cited at the beginning of the paragraph) by ranking their quadratic lending costs: the higher the latter, the "smaller" the bank.⁵

As a result, the best response function of a "big" bank is continuous and decreasing with slope close to minus one, meaning that its credit supply purely complements the other banks' credit supplies until the total credit supply maximizes the expected profit of the company. Conversely, the best response function of a "small" bank, according to this terminology, exhibits a point of discontinuity: when the other banks' total credit supply is too low, this bank's best response is zero. When the other banks' total credit supply increases, there is a threshold at which the "small" bank jumps in with positive credit supply, at a level close to the "big" bank's best response function, and stays close to it in the whole interval above the threshold. Hence, the asset stranding risk creates an interdependence between banks' credit supplies, and the company may be credit constrained when the banking universe is sufficiently scarce.

Third, two Nash equilibria can be identified under the illustrative calibration between an "almost big"⁶ and a "small" bank: an equilibrium where none of the two banks supply credit, and one in which they both supply positive quantities. Assuming they are initially on the second Nash equilibrium, even partial divestment by one of them can have a strong multiplicative effect by incentivizing the other bank to divest partially or even fully, sometimes reducing the total credit supply to zero only by a unilateral divestment decision. This result is robust to changes in the financing need of the company, that is to say to changes in its initial equity.

5. This terminology is exclusively used to make easier the presentation of the results. As precised at the beginning of the paragraph, the quadratic lending cost provides more information than the size of the bank. In particular, it can encompass its "green" preferences.

6. An "almost big" bank here must be understood as a "small" bank whose threshold to supply positive credit is very low, close to zero, when referring to the terminology defined in the previous paragraph.

Related literature. This paper contributes to several strands of literature. First, it contributes to the literature on asset stranding in the low-carbon transition. Several macroeconomic studies identify optimal climate policy when accounting for the economic and financial risk of asset stranding in models with an emission-intensive and a low-emission sector (Campiglio et al., 2022; Rozenberg et al., 2020; Hambel and van der Ploeg, 2024), in interaction with investors (Comerford and Spiganti, 2023) and banking (Carattini et al., 2021). Other studies look at optimal climate policy in partial equilibrium (van der Ploeg, 2020) or game theory with a political economy approach (Kalkuhl et al., 2020) by focusing on the fossil fuel companies, their interests, and derive the consequences of climate policies on asset stranding. Daumas (2023a, Chapter 2) integrates the asset stranding cost in a stock-flow consistent model designed to assess financial instability along transition scenario pathways. Empirically, Semieniuk et al. (2022) trace the equity ownership of fossil fuel assets back to their ultimate owners, while the results of Carattini and Sen (2019) suggest that the asset stranding risk is material in stock markets. This paper contributes to this trend of literature by zooming in on the impact that banks could have on asset stranding, and by pointing at a potential snowball effect, in a spirit similar to the question asked by Comerford and Spiganti (2023) but with a focus on banks.

Second, it contributes to the theoretical literature on assessing credit risk in the low-carbon transition. Monnin (2018) discusses the challenges raised by credit risk assessment integrating physical and transition risks and reviews methodologies developed by practitioners though being at their infancy. Battiston et al. (2019) assess the impact of a disorderly transition on sovereign credit risk. Bouchet and Le Guenedal (2020) and Reinders et al. (2020) provide practical methodologies based on Merton model to assess the impact of increases in carbon price on probabilities of default and the value of debt instruments. Agliardi and Agliardi (2021) develop a structural bond pricing model under uncertainty about corporate earnings and climate policy. Le Guenedal and Tankov (2022b) solve a structural bond pricing model under transition scenario uncertainty, with prices being updated under a bayesian approach through continuous time revelations about the underlying transition scenario. This paper contributes to these works by assessing the specific impact on companies' probability of default of the asset stranding risk. A close paper in this regard is Battiston et al. (2023), which accounts for the economic structure of the company, and in particular integrates capital-adjustment costs, to assess the impact of changes in investors' expectations about transition risk on the valuation of the firms' debt and equity in a multiperiod model. My approach differs by solely focusing on the probability of default in a one-period framework, allowing to highlight an interaction mechanism between credit suppliers.

Finally, this paper talks to the literature on banks' financing adjustments in the face of the low-carbon transition. Empirical literature suggests that banks practice price adjustments to integrate transition risk (Chava, 2014; Delis et al., 2019). As for quantity adjustment, Degryse et al. (2022) highlight, based on a theoretical modelling and empirical validation, that banks

do not have an interest in reallocating their capital towards green innovative companies due to a path-dependency effect, as most of their current portfolio rely on carbon-intensive activities. However, empirical evidence suggest that banks' integration of transition risks affects their lending quantities and their credit supplies. Benincasa et al. (2022) and Laeven and Popov (2023) show that climate policy in a country leads to cross-border loan reallocation for high carbon risk companies, decreasing loan supply to domestic carbon-intensive companies. Ivanov et al. (2023) show that the introduction of a carbon tax leads banks to revise their credit supply in particular through shorter loan maturities and lower access to permanent forms of bank financing to high emission firms, leading to a higher share of the shadow banking in the financing of those companies. Kacperczyk and Peydró (2022) identify that after committing to carbon neutrality, banks supply less credit to carbon-intensive companies. The reduction in credit supply to those companies triggers a reduction in their total debt, leverage, total assets, and real investments. This paper contributes to this strand of literature by raising awareness about the potential impact of credit supply quantity adjustments on companies facing the asset stranding risk, and by shedding light on potential interaction effects between banks' credit supplies through another channel than the one studied by Degryse et al. (2022).

Outline. The rest of the paper is organized as follows. In section 2, the model of the company and banks' objectives are presented. In section 3, the impact of divesting decisions is assessed under a given specification of the model. Section 4 concludes.

2 Debt financing of a company with assets to be stranded

I introduce a one-period model in which, at date 0, a given universe of n banks decides how much to lend to a company. The activity of this company relies on a significant portion of assets which are to be stranded for the Paris agreement objectives to be met. Imagine, to fix ideas, an oil company.

2.1 The company

At date 0, the company holds tangible assets which are to be stranded in the low-carbon transition. It possesses some equity of amount E_0 , exogenously given. This equity can be understood as actual equity as well as long term secured debt, which does not mature during the period. The period starts when the company needs to refinance. To do so, it contracts debt with principal x at date 0, which it will have to pay back with interest rate, r , at date 1. Between the two dates, the company produces thanks to its total assets, which are equal to its equity and contracted debt, $E_0 + x$. The period is assumed to be mid-term, so that raising equity is not an alternative financing source for the company, which is assumed not to be able to raise additional equity before the end of the period.

A particularity of the company is to have a production activity which strongly relies on assets that are to be stranded for the Paris agreement temperature target to be met. Hence, these assets, already possessed by the company, are strongly illiquid. This is translated into the profit function of the company by the following: the company faces a capital-adjustment cost, which is minimized in a certain asset size, A^* , called its “optimal” asset size for this reason. This “optimal” asset size can be understood as the result of a trade-off between natural capital depreciation and operational investments to optimize its durability. It is exogenously given as it depends on the company’s tangible assets acquired in the past and on its production structure.

Profit function The profit of the company is a random function Π of its debt contracted at date 0, with principal x , as follows:

$$\Pi(x) = Q(E_0 + x) - \Gamma(A^* - (E_0 + x)) - rx + \epsilon,$$

where Q represents the EBITDA of the company as a function of its total assets; $\Gamma(A^* - (E_0 + x))$ represents the capital-adjustment cost faced by the company if its total asset size is different from its “optimal” asset size, A^* ; rx are the interests paid on debt contracted at date 0; ϵ is a centered random variable representing exogenous sources of uncertainty about its future profit, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The EBITDA function Q is assumed to be concave. The capital-adjustment cost function, Γ , is assumed to be convex, and to reach its minimum value in zero.

Several comments can be made on this specification. Note that the existence of taxes on companies’ profit is ruled out as it only adds a parameter without changing anything to the analysis.⁷ The convexity of the capital-adjustment cost function, coupled with its minimization in zero, represents the cost that the company incurs when its asset size, $E_0 + x$, goes away from its “optimal” asset size, A^* . For negative values, this cost represents the financial loss that the company incurs when it has to strand some part of its tangible assets by selling them off. For positive values, this cost represents the investment costs of capital accumulation. Both these costs are assumed to be convex, in line with Campiglio et al. (2022). Finally, the assumption of a centered distribution for the noise, ϵ , provides a tractable quantity for the expected profit. Note that the latter is concave as the sum of concave functions.

Company’s default probability. The default probability of the company is defined out of the comparison between its profit and its debt.

Definition 3. The company’s default is defined as the situation where it is unable to repay its debt and interest rates at the end of the period. Formally, the default event is defined as

7. It matters when the company has to choose between debt and equity (Leland and Toft, 1996). Here, equity is assumed to be already given, and only the possibility of debt financing is considered.

$d(x) := \{E_0 + x + \Pi(x) \leq x\}$,⁸ for a debt contracted with principal x , and where the randomness comes from the noise ϵ in the random profit $\Pi(x)$.

Hence, it is assumed that no debt renewal at date 1 can prevent the default. It makes sense from the point of view of the lending bank, which can not count on the possibility that some bank will, in the future, help the company to be solvable.

The default probability, which is the probability of the default event, $d(x)$, can then be expressed as an explicit function of contracted debt, x , as done in the next Proposition.

Proposition 1. *The default probability is a function of debt contracted at date 0, x , as follows:*

$$p(x) = F(rx + \Gamma(A^* - (E_0 + x)) - E_0 - Q(E_0 + x)),$$

with F the cumulative distribution function of ϵ .

With this definition, the probability of default fully relies on the profit function of the company. Hence, it increases with the interest rate and with the gap between the optimal and the actual asset size, $|A^* - (E_0 + x)|$, and decreases with the company's EBITDA, $Q(E_0 + x)$.

2.2 Creditors

Banks' objectives. Each bank supplies the amount of credit, x_i , which maximises the expected net revenues from its loan, at an interest rate $r \in \mathbb{R}_+$. If the company does not default, the bank's net earnings are the loan interest rates, rx_i . If the company defaults, the bank only recovers a portion $\rho \in [0, 1]$ of the credit supply plus interest rates, $(1 + r)x_i$, while it loses the loan principal, x_i . The default event, $d(\bar{x})$, depends on total credit supply, $\bar{x} := \sum_{j=1}^N x_j$. In addition, to provide a credit supply of amount x_i , bank i pays a quadratic cost with coefficient $c_i > 0$. The objective functional of bank i , for each $i \in \{1, \dots, N\}$, is hence defined as follows:

$$\forall \mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}_+^N, \quad J_i(x_i; \mathbf{x}^{-i}) = \mathbb{E} \left[\left(\mathbf{1}_{d(\bar{x})^c} r + \mathbf{1}_{d(\bar{x})} (\rho(1 + r) - 1) \right) x_i - c_i x_i^2 \right],$$

with \mathbf{x}^{-i} the vector in \mathbb{R}^{N-1} of credit supply of all banks except bank i , $d(\bar{x})^c$ the event of non default (the complementary event of default) for a total credit supply equal to \bar{x} . Hence, the interaction between banks' objective functionals lies in the default event of the company: its default depends on total credit supply of all banks. The quadratic lending cost coefficient, c_i , allows to bound the quantity of credit that bank i is prone to supply to the company for exogenous reasons, that it be the size of the bank, its diversification needs, its lending operational costs, or its willingness to finance brown companies.

Banks' objectives can be more explicitly written in function of the probability of default of the company, as done in the next Proposition. Due to the nature of the interaction between the

8. The interest rates, rx , are already subtracted in the profit $\Pi(x)$, and hence do not appear explicitly in this expression.

banks' objectives, which interact through the sum of their credit supplies, \bar{x} , bank i 's objective functional can be rewritten as an explicit function of bank i 's own credit supply, x_i , and the sum of other banks' credit supplies, $\bar{x}^{-i} := \sum_{j=1, j \neq i}^N x_j$, using the probability of default.

Proposition 2. *The objective functional of bank i can be rewritten as follows, with a convenient abuse of notation: for all $\mathbf{x} \in \mathbb{R}_+^N$,*

$$J_i(x_i; \mathbf{x}^{-i}) = J_i(x_i; \bar{x}^{-i}) = (1 - p(\bar{x}))(rx_i - c_i x_i^2) + p(\bar{x})((\rho(1 + r) - 1)x_i - c_i x_i^2), \quad (2.1)$$

with $\bar{x} = \sum_{j=1}^N x_j = x_i + \bar{x}^{-i}$.

Hence, the interaction between banks' objectives intervenes through the probability of default of the company, which weights each bank's earnings when the company defaults in their objective.

Supply function Each bank supplies the amount of credit which maximises its expected revenues from this loan, which amounts to maximizing its objective functional. Hence, the credit supply of a generic bank i is a function of total credit supply of the other banks, $\bar{x}^{-i} = \sum_{j=1, j \neq i}^N x_j$, as follows:

$$x_i(\bar{x}^{-i}) = \arg \max_{s_i \geq 0} J_i(s_i; \bar{x}^{-i}).$$

These best response functions set the framework for an n -player game between the banks financing the company.

3 Banks' best response with exogenous interest rates

In this section, I specify the company's profit function so that the best response of each bank can be computed explicitly. This toy model, despite, or thanks to, its simplicity, allows to provide some insightful intuitions on the way banks interact when lending to the same company which has illiquid assets, through syndicated loans for example. In this section, in particular, I assume that the interest rate is exogenous: banks interact only through the debt amount of the company, not its price. In the next section, a discussion on the possibility and relevance of a relaxation of this assumption is provided.

This section is organized as follows. First, I specify the profit function of the company to derive an explicit formula for the best response function. Then, I illustrate numerically and analyse the best response function that I obtain. Finally, I define the notion of equilibrium to the game, and derive graphically the impact of divesting.

3.1 Explicit formula for the best response

Under some additional assumptions about the profit function of the company, the debt market and the loan characteristics, an explicit formula for the best response of each bank can

be derived.

The profit function of the company is specified according to the following assumptions.

Assumption 1. *The EBITDA function Q represents constant returns to scale, with a rate $q > 0$:*

$$Q(x) = qx.$$

Assumption 2. *The capital-adjustment cost function Γ is quadratic, as follows, with coefficient $\gamma \geq 0$:*

$$\Gamma(y) = \gamma y^2.$$

Assumption 3. *The law of ϵ is a centered uniform distribution on $[-\frac{u}{2}, \frac{u}{2}]$, with u defined as follows, for a constant $\eta > 0$:*

$$u := 2(E_0 + qA^* + \frac{q^2}{4\gamma} + \eta).$$

Assumption 1 allows for a tractable and easily interpretable profit function while staying agnostic about the company's returns to scale. Assumption 2 is in line with quadratic costs of asset stranding and investment assumed in Campiglio et al. (2022). Assumption 3 allows to provide a tractable analytical formula for the probability of default. The specification of the range of the uniform distribution based on u solely insures that the minimal default probability is superior or equal to some positive constant for any value of the interest rate, r , and of contracted debt, x . It allows to rule out unrealistic cases in which the company would have a zero chance to default for some amounts of contracted debt. The constant η allows to adjust the level of the minimal probability of default, by a simple translation of the whole profit distribution.

Moreover, the recovery rate is assumed to be so that, when the company defaults, the loss given default is strictly superior to the debt interests. This assumption allows to make sure that, when the company defaults, each bank's payoff is negative, which helps for deriving an explicit formula to the best response function.

Assumption 4. *The recovery rate, ρ , is so that $\rho(1 + r) < 1$.*

Finally, the following simplifying assumption guarantees that the debt of the company is solely determined by credit supply. Price adjustment is considered to have a second-order impact with respect to quantity adjustment. Hence, it is ruled out in this example in order to provide a tractable analytical solution to the best response functions.

Assumption 5. *Under the following subset of assumptions, the contracted debt equals total credit supply:*

- (a) *The N banks represent the whole universe of credit suppliers available to the company.*
- (b) *The interest rate is exogenous.*

(c) *The credit demand of the company is illimited.*

Under these conditions, the company's debt is equal to the sum of the optimal credit supplies of all banks. Assumption 5 can be interpreted as follows. As banks incur quadratic lending costs, their optimal credit supplies are finite. Moreover, their willingness to supply credit is also limited by the quadratic cost on the company's profit when its asset size exceeds its "optimal" asset size A^* . Hence, Assumption 5(c) only plays a technical role by simplifying the market demand-supply encounter. In practice, the model ensures that banks have collectively no interest in significantly exceeding the quantity of debt which maximises the company's profit, which could be interpreted as its optimal credit demand.⁹

The assumptions above allow to provide a simple analytical formula for the default probability as a function of debt.

Proposition 3. *Under Assumptions 1, 2, 3, 4 and 5, the minimal default probability is always strictly positive, and reached in $g(r)$, with*

$$g : r \mapsto \arg \min_{x \in \mathbb{R}} \mathbb{E}[\Pi(x)] = \frac{q - r}{2\gamma} + A^* - E_0.$$

Moreover, the default probability is as follows:

$$\forall x \geq 0, \quad p(x) = \frac{1}{u} \left(\gamma [x - g(0)]^2 + rx + \eta \right) 1_{x \in \mathcal{D}} + 1_{x \in \mathcal{D}^c \cap \mathbb{R}_+},$$

with the domain \mathcal{D} defined as follows:

$$\begin{aligned} \mathcal{D} &= [0 \vee d_1, d_2], \quad d_1 = g(r) - \frac{\sqrt{\Delta}}{2\gamma}, \quad d_2 = g(r) + \frac{\sqrt{\Delta}}{2\gamma}, \\ \Delta &= 4\gamma \left(\gamma g(r)^2 + 2E_0 + q(E_0 + A^*) + \frac{q^2}{4\gamma} + \eta - \gamma(E_0 - A^*)^2 \right). \end{aligned}$$

This simple formula allows, then, to compute an explicit formula for the best response function of each bank. As already shown in the previous section, the best response function of bank i to credit supply of the other banks, $\mathbf{x}^{-i} = (x_j)_{j \neq i}$, can actually be written as a best response function to the aggregate credit supply of the other banks, $\bar{x}^{-i} = \sum_{j=1, j \neq i}^N x_j$. It is therefore defined as follows, with a slight abuse of notations:

$$\forall \mathbf{x}^{-i} \in \mathbb{R}_+^{N-1}, \quad \beta_i(\mathbf{x}^{-i}) = \beta_i(\bar{x}^{-i}) = \arg \max_{x_i \geq 0} J_i(x_i, \bar{x}^{-i}).$$

9. Another option would have been to interpret the expected profit-maximising debt quantity of the company as its credit demand, and use it as an upper bound to banks' total credit supply. However, as the sum of banks' optimal credit supplies could be above this boundary, it would then require an allocation rule of the company's debt demand between the banks, for which the literature does not give any specific clue. It would also complicate the best response function of each bank. As the main interest of this paper is the case in which the company's credit supply is not met, both options lead to similar results. Hence, Assumption 5(c) was retained as it is more tractable.

It can be computed explicitly, as done in the next Proposition.

Proposition 4. *The best response function of bank i can be computed analytically, as follows:*

$$\beta_i(\bar{x}^{-i}) = \begin{cases} h_i(\bar{x}^{-i}) & \text{if } \Delta > 0, J_i(h_i(\bar{x}^{-i}), \bar{x}^{-i}) > 0 \text{ and } h_i(\bar{x}^{-i}) \in \mathcal{D}(\bar{x}^{-i}), \\ 0 & \text{otherwise.} \end{cases}$$

With

$$h_i(\bar{x}^{-i}) = \frac{4\gamma K(g(0) - \bar{x}^{-i}) - 2c_i - 2Kr + \sqrt{\Delta}}{6\gamma K}, \quad K = \frac{(1-\rho)(1+r)}{u},$$

$$\Delta = \left[4\gamma K(\bar{x}^{-i} - g(0)) + 2Kr + 2c_i\right]^2 + 12\gamma K \left[r - \eta K - \gamma K(\bar{x}^{-i} - g(0))^2 - Kr\bar{x}^{-i}\right],$$

$$\mathcal{D}(\bar{x}^{-i}) = [0 \vee (d_1 - \bar{x}^{-i}), 0 \vee (d_2 - \bar{x}^{-i})].$$

The best response function is either the biggest root of a degree two polynomial, or zero. The underlying rationale, developed in the proof of this proposition, is the following. The objective of bank i is an average between two quadratic functions which represent (i) the payoff of the bank when the company does not default, $rx_i - c_ix_i^2$, which is positive when bank i 's credit supply is not too high, at a level depending on its quadratic lending cost parameter, c_i , and the interest rate, r , and (ii) the payoff of the bank when the company defaults, $(\rho(1+r) - 1)x_i - c_ix_i^2$, which is always negative according to Assumption 4. The weight used to compute the average between these two quadratic payoffs is the probability of default, $p(\bar{x})$, which notably depends on bank i 's credit supply. Depending on the total credit supply of the other banks, \bar{x}^{-i} , the credit supply of bank i needs to be more or less high to reach the minimal probability of default.

Hence, the optimal credit supply of bank i can be equal to zero in two main situations: (a) when the other banks are already supplying too much credit, such that bank i would get no profit from any additional credit supply because of a too high probability of default, and (b) when it is too costly for bank i to provide a credit supply high enough so that the probability of default is low enough to make its loan profitable. In the intermediary situation, when total credit supply of the other banks allows bank i to face a probability of default low enough to make its loan profitable, it will optimally supply a strictly positive quantity of credit. In this case, its optimal credit supply maximises the average between the two quadratic payoffs, $rx_i - c_ix_i^2$ and $(\rho(1+r) - 1)x_i - c_ix_i^2$, weighted by the probability of default, $p(\bar{x}^{-i} + x_i)$, which is shown to be the biggest root of a degree two polynomial in the proof.

3.2 Numerical illustration

The numerical application is made with calibration described in Table 2.1. This calibration is purely illustrative. Parameters are to be interpreted according to their relative values.

The default probability of the company is a U -shaped function of the company's debt (Figure

Parameter	Value
A^*	10
E_0	5
q	0.2
γ	0.8
η	0.5
ρ	0
r	0.05

Table 2.1 – Parameter values used for numerical illustration.

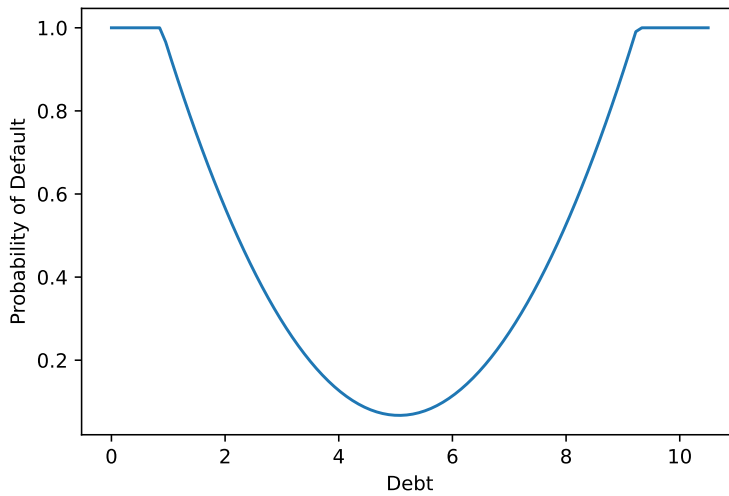


Figure 2.1 – Probability of default of the company as a function of debt

2.1). The quantity of debt which minimizes the probability of default is the quantity of debt which maximises the company’s expected profit, equal to $g(r)$.¹⁰ It can be interpreted as its optimal debt demand, $g(r)$. The U -shape is due to the convex cost of getting away from the “optimal” asset size, A^* , because of a debt which does not meet the financing need, that is to say the amount of credit that would minimize the capital-adjustment cost, $A^* - E_0$, equal to 5 under this calibration. The left arm of the U represents the cost of asset stranding, while its right arm represents the investment costs. The convexity of the curve depends on the quadratic cost parameter, γ . With this calibration (Table 2.1), it is assumed that the company can not afford to strand half of its total assets without defaulting with probability one.

Using Proposition 4, the best response function of a bank can be easily illustrated numerically (Figure 2.2). It appears to be very different across banks with different quadratic lending costs.¹¹ The best response of a “big” bank ($c_i = 0.002$) is approximately equal to the curve

10. It can be easily noted from Proposition 3 that the value of η intervenes as follows: when the interest rate, r , is zero, the minimal probability of default is equal to $\frac{\eta}{u}$. A quick computation can show that this quantity remains a good approximation of the probability of default when r is small.

11. In this framework, the values of the quadratic lending costs must be understood relatively to the interest rate, r . Indeed, the credit supply of bank i that maximises its net revenues when the probability of default is null

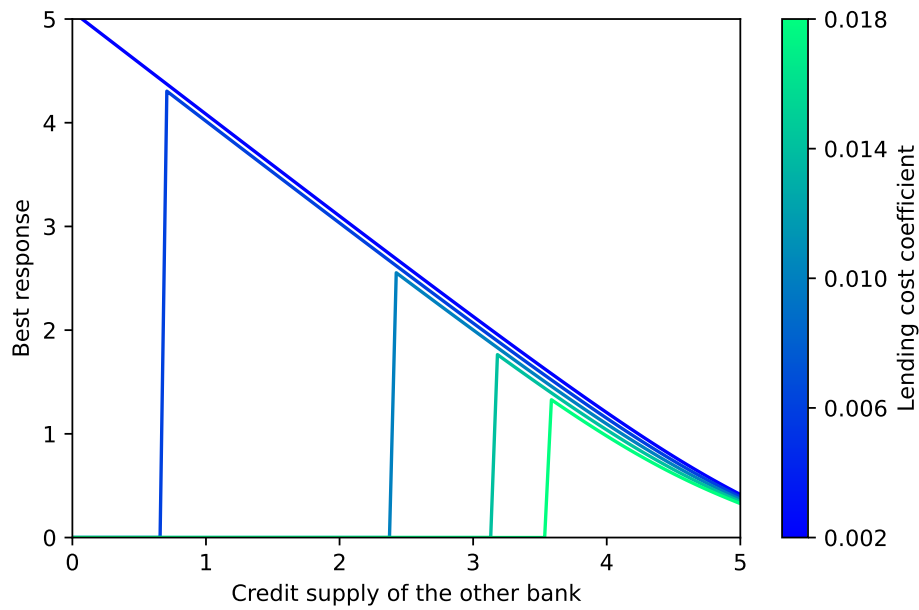


Figure 2.2 – Best response function of banks with various sizes

$x \mapsto E_0 - x = 5 - x$. Hence, this bank's credit supply purely adds to the other banks' total credit supply until the credit demand of the company is met. Conversely, the best response functions of “smaller” banks exhibit a complementarity with other banks' total credit supply. Indeed, when the latter is too small, these banks do not supply any credit. When it increases, there is a threshold above which these banks' credit supplies become positive. Above this threshold, they also adopt a “substitutive” role with respect to the other banks' total credit supply. For example, a “small” bank ($c_i = 0.18$) only accepts to lend to the company when the total credit supply of the other banks is above 3.6. Moreover, above this threshold, its best response function is well approximated by the curve $x \mapsto E_0 - x = 5 - x$.

These best response functions show that there exists a very strong interaction effect between the banks lending to the same company. The mechanism at play is the following. The threshold above which the best response function of a small bank becomes strictly positive represents the minimal level of other banks' credit supply so that the probability of default of the company is small enough for the small bank's objective to reach strictly positive values.

3.3 Banks' divesting impact on asset stranding

The strong interaction between banks' optimal credit supplies, as illustrated by their best response functions, can lead to important chain reactions when a bank decides to divest. In this framework, divesting is defined as a unilateral decision by a bank, or a group of banks, to deviate from an equilibrium for reasons that are exogenous to the maximisation of its loan-profit, J_i .

is $r/(2c_i)$. Hence, in relation to the interest rate, the quadratic lending cost scales the willingness of the bank to lend to the company.

Definition 4 (Divesting). Divesting is defined as a unilateral deviation from an equilibrium by a bank or a group of banks to a lower credit supply than the one which prevailed at equilibrium. This unilateral deviation is motivated by reasons which are exogenous to the consideration of their financial objective associated to the loan, J_i .

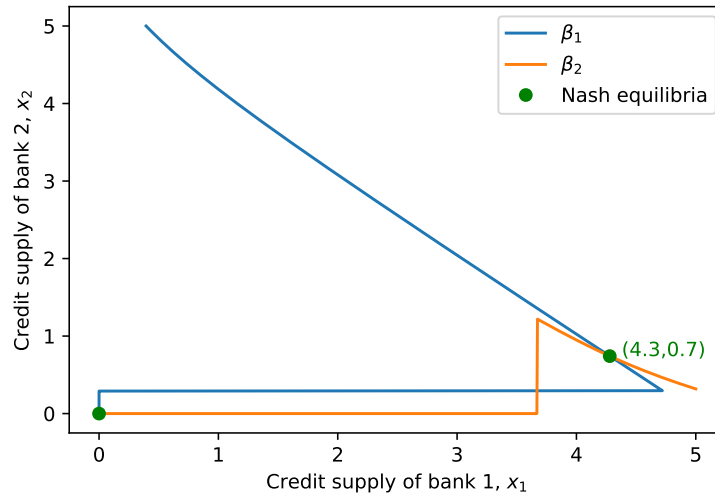
This unilateral deviation can be motivated by exogenous ethical or reputational motives, or as the consequence of regulatory constraints which concern only this bank or this group of banks among the universe of banks (e.g., due to geographical diversity), or as the consequence of a change in these banks' expectations towards the timing and impacts of the low-carbon transition, which affect their perceptions of the company's solvability, or their willingness to finance carbon-intensive companies. For example, if a bank decides to join the Net-Zero Banking Alliance, it can change its preference to invest in carbon-intensive firms.

The equilibrium prevailing before any divesting decision is taken can be represented by a Nash equilibrium, which is a conservative option with respect to the assumption of collusion for example. Indeed, total credit supply would be smaller at a Nash equilibrium, minimising the impact of divestment. Moreover, the identification of Nash equilibria allows to illustrate directly the properties of best response functions. The set of Nash equilibria can be easily identified graphically when the banking universe is made of two banks ($n = 2$). Indeed, in this case, Nash equilibria are characterized by the fixed points of the map $x \mapsto \beta_1(\beta_2(x))$ defined on \mathbb{R}_+ , where the two banks are indexed by $i \in \{1, 2\}$ (Figure 2.3).

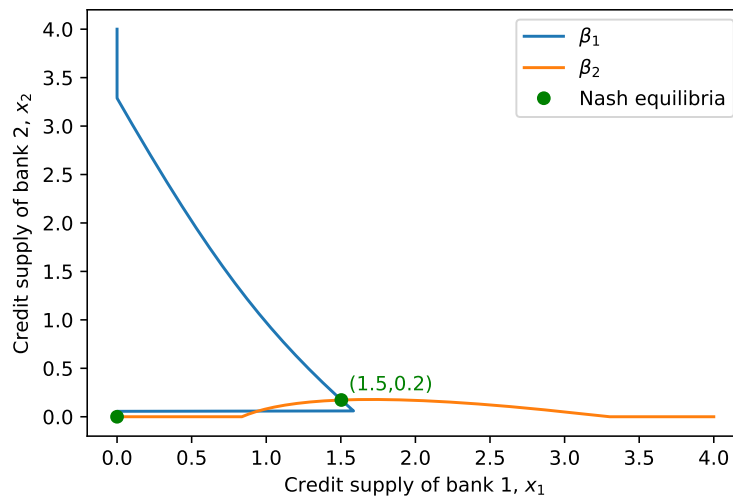
Figure 2.3 illustrates the Nash equilibria between a “big” bank (Bank 1) and a “small” bank (Bank 2), (a) for the baseline calibration (Table 2.1), and (b) when the assumption on the company's equity, E_0 , is more conservative, equal to 8.¹² In both cases, the 2-player game admits two Nash equilibria, in which either both banks supply zero credit, or both banks supply a strictly positive amount of credit. This result, in itself, is a first meaningful illustration of the impact of the strong interaction between the two banks' credit supplies.

Moreover, in both cases, if banks are at the Nash equilibrium where they both supply a strictly positive amount of credit, then the divesting decision of the “small” bank to change its supply to zero incurs a strong multiplicative effect, as it leads the “big” bank to reduce its own credit supply to zero, for pure financial considerations. Hence, in both cases, a full divesting decision by the “small” bank entails a switch between a situation where the company gets (almost) the amount of credit needed to maximise its expected profit, to a situation where it can not contract any debt. For example, in the baseline calibration (Figure 2.3a), the company needs to contract a debt of amount 5 to maximise its expected profit. At the Nash equilibrium where Bank 1 and Bank 2 supply respectively 4.3 and 0.7, the company gets its optimal debt amount. But then, if Bank 2 decides to fully divest, that is to say to reduce its credit supply from

12. Note that the quadratic lending costs of the two banks are changed in the two subfigures. Indeed, the notion of “size” of a bank as it has been defined is relative to the financing need of the company, and hence needs to be adjusted to its equity amount in order to provide enlightening examples.



(a) Equity $E_0 = 5$.



(b) Equity $E_0 = 8$

Figure 2.3 – **Nash equilibria between two banks.** These figures display the Nash equilibria between two banks when (a) the company has equity of amount 5 and banks have lending cost coefficients $c_1 = 0.0055$, $c_2 = 0.02$; (b) the company has equity of amount 8 and banks have lending cost coefficients $c_1 = 0.035$, $c_2 = 0.18$. In both Figures, the best response function of Bank 1, β_1 , is represented with a switch between the x - and the y -axes.

0.7 to zero, the optimal credit supply of Bank 1 would become zero. In other words, a reduction in total credit supply by 14%, caused by the “small” bank, actually triggers a reduction in total credit supply of 100% in this case. A similar story can be told out of Figure 2.3b, with a more conservative assumption about the refinancing needs of the company.

In Figure 2.3, the strong multiplicative effect of divesting is possible because the “big” bank would not be willing to supply, on its own, the optimal debt amount for the company. But,

even when the “big” bank is “big” enough to do so, a multiplicative effect, though smaller, can be identified when it decides to reduce its own credit supply by divesting. For example, in Figure 2.4, Bank 1’s best response is strictly positive in zero, and approximately fulfills the company’s debt demand on its own, by supplying credit of quantity 2. At the Nash equilibrium, the total credit supply is also equal to 2. But if Bank 1 decides to reduce its credit supply to 0.5, through divesting, hence reducing by 50% the total credit supply, it will actually reduce it by 75% because of the reaction of Bank 2, which will reduce its credit supply to zero according to its best response. The multiplicative effect is still quite significant in this case. Note that it would not hold if the best response of the bank which does not divest was strictly positive in zero with a best response approximating the map $x \mapsto g(r) - x$, i.e. if it was “big” enough to fulfill the company’s demand on its own.

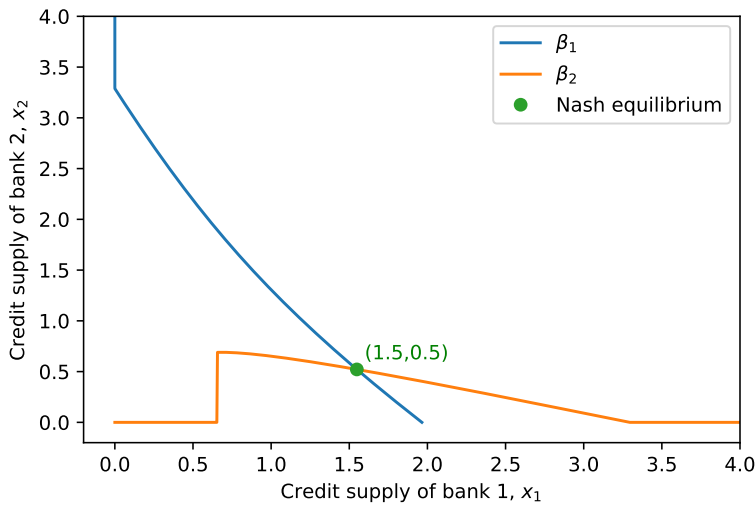


Figure 2.4 – **Nash equilibrium involving a “very big” bank.** This figure displays the Nash equilibrium between two banks when the company has equity of amount 8, and banks have lending cost coefficients $c_1 = 0.02$, $c_2 = 0.06$. The best response function of Bank 1, β_1 , is represented with a switch between the x - and the y -axes.

4 Conclusion

In conclusion, this chapter builds an original model which sheds light on an interaction mechanism between banks through the asset stranding risk faced by a company. While very simple, the model allows to point at the risk of a snowball effect leading to a quick insolvency of companies that are potentially still crucial to the economy, which could be detrimental to the economy if such a phenomenon happens too early, or in a disorderly fashion (Cahen-Fourot et al., 2021). In parallel, it highlights a tool that banks have in hand to boost the low-carbon transition. However, though the model stays quite simple, it lacks tractability. Hence the most

interesting results are derived from a numerical illustration, and highly rely on the calibration of the capital-adjustment cost which is hard to calibrate. Hence this paper should mostly be understood as a heuristic toy model.

It would be interesting to test the robustness of these results with a higher number of banks, to enrich the understanding of the interaction effect. Following the same methodology, Nash equilibria could be identified using three-dimensional graphics when there are three banks. For more than three banks, an interesting and probably tractable extension could be to consider the interaction between a major bank and a mean field of small identical banks. Furthermore, a question that is not addressed in this paper is the interaction between quantity and price adjustment. It would be interesting to extend this theoretical framework to help disentangle the price and quantity channels through which banks adjust their financing conditions to carbon-intensive companies as observed in the empirical literature.

Appendix

A Proofs

Proof of Proposition 1. Default probability, $\mathbb{P}(d(x))$, equals

$$\begin{aligned}
 \mathbb{P}(d(x)) &= \mathbb{P}(E_0 + x + \Pi(x) \leq x) \\
 &= \mathbb{P}(E_0 + \Pi(x) \leq 0) \\
 &= \mathbb{P}(\epsilon \leq -E_0 - Q(E_0 + x) + rx + \Gamma(A^* - (E_0 + x))) \\
 &= F(rx + \Gamma(A^* - (E_0 + x)) - E_0 - Q(E_0 + x)).
 \end{aligned}$$

□

Proof of Proposition 2. For $\mathbf{x} \in \mathbb{R}^N$,

$$\begin{aligned}
 J_i(x_i; \mathbf{x}^{-i}) &= \mathbb{E} \left[\left(\mathbb{1}_{d(\bar{x})^c} r + \mathbb{1}_{d(\bar{x})} (\rho(1+r) - 1) \right) x_i - c_i x_i^2 \right] \\
 &= \left(\mathbb{E} \left[\mathbb{1}_{d(\bar{x})^c} \right] r + \mathbb{E} \left[\mathbb{1}_{d(\bar{x})} \right] (\rho(1+r) - 1) \right) x_i - c_i x_i^2 \\
 &= [(1 - p(\bar{x}))r + p(\bar{x})(\rho(1+r) - 1)] x_i - c_i x_i^2 \\
 &= (1 - p(\bar{x}))(rx_i - c_i x_i^2) + p(\bar{x})((\rho(1+r) - 1)x_i - c_i x_i^2).
 \end{aligned}$$

□

Proof of Proposition 3. Under assumptions A1, A2 and A3, the default probability equals

$$p(x) = \frac{1}{u} (\gamma(E_0 + x - A^*)^2 + rs - E_0 - Q(E_0 + x) + \frac{u}{2}) \mathbb{1}_{x \in \mathcal{D}} + \mathbb{1}_{x \in \mathcal{D}^c \cap \mathbb{R}_+},$$

with $\mathcal{D} := \{x \geq 0, -\frac{u}{2} \leq \gamma(E_0 + x - A^*)^2 + rx - E_0 - Q(E_0 + x) \leq \frac{u}{2}\}$. Now,

$$\gamma(E_0 + x - A^*)^2 + rx - E_0 - Q(E_0 + x) + \frac{u}{2} = \gamma[x - g(0)]^2 + rx + \eta,$$

which is positive when $x \geq 0$. Hence, $\mathcal{D} := \{x \geq 0, \gamma(E_0 + x - A^*)^2 + rx - E_0 - Q(E_0 + x) \leq \frac{u}{2}\}$.

Then, by analysing the roots of a second degree polynomial, we get that

$$\gamma(E_0 + x - A^*)^2 + rx - E_0 - Q(E_0 + x) \leq \frac{u}{2} \iff d_1 \leq x \leq d_2,$$

with d_1, d_2 defined in the Proposition. Hence the explicit expression of the default probability. Note that, as can be seen from its explicit expression, the default probability is always strictly positive, as $x \geq 0$ and $\eta > 0$. Moreover, $p(x) = F(-\mathbb{E}[\Pi(x)] - E_0)$. As F is an increasing function, $\arg \min_{x \geq 0} p(x) = \arg \min_{x \geq 0} -\mathbb{E}[\Pi(x)]$, which can easily shown to be equal to $g(r)$. □

Proof of Proposition 4. Let be a given vector of other banks' credit supply $\mathbf{x}^{-i} \in \mathbb{R}_+^{N-1}$, and write as before the sum of its components, $\bar{x}^{-i} = \sum_{j=1, j \neq i}^N x_j$. Then,

$$\beta_i(\bar{x}^{-i}) = \arg \max_{x_i \geq 0} (1 - p(x_i + \bar{x}^{-i}))(rx_i - c_i x_i^2) + p(x_i + \bar{x}^{-i})((\rho(1+r) - 1)x_i - c_i x_i^2).$$

As the default probability function, p , takes its values in $]0, 1]$, for any $x_i \in \mathbb{R}_+$, the functional $x_i \mapsto J(x_i, \bar{x}^{-i})$ is bounded by two quadratic functions in x_i as follows:

$$(\rho(1+r) - 1)x_i - c_i x_i^2 \leq J(x_i, \bar{x}^{-i}) < rx_i - c_i x_i^2.$$

The upper bound quadratic function admits a maximum value in $\frac{r}{2c_i} > 0$, while the lower bound quadratic function is strictly decreasing on \mathbb{R}_+ as $\rho(1+r) - 1 < 0$. The two quadratic bounds equal zero in zero. Moreover, when the default probability, $p(x_i + \bar{x}^{-i})$, is equal to 1, the objective functional, $x_i \mapsto J(x_i, \bar{x}^{-i})$, is equal to its lower bound, $x_i \mapsto (\rho(1+r) - 1)x_i - c_i x_i^2$, which is negative on \mathbb{R}_+ . Therefore, out of the domain \mathcal{D} , the maximum value is 0, reached in $x_i = 0$.

Let me define the following functional:

$$f : (x_i, \bar{x}^{-i}) \mapsto -c_i x_i^2 + x_i \left(r - (1 - \rho)(1 + r) \frac{1}{u} \left(\gamma [x_i + \bar{x}^{-i} - g(0)]^2 + r(x_i + \bar{x}^{-i}) + \eta \right) \right).$$

As it is equal to the objective functional when $x_i \in \mathcal{D}(\bar{x}^{-i})$, the following holds:

$$\forall \bar{x}^{-i} \geq 0, \max_{x_i \geq 0} J(x_i, \bar{x}^{-i}) = \max(0, \max_{x_i \in \mathcal{D}(\bar{x}^{-i})} J(x_i, \bar{x}^{-i})) = \max(0, \max_{x_i \in \mathcal{D}(\bar{x}^{-i})} f(x_i, \bar{x}^{-i})).$$

Hence, I can restrict to the study of a cubic polynomial in x_i , $x_i \mapsto f(x_i, \bar{x}^{-i})$. In particular, this polynomial equals zero in zero, and is negative in $d_2 - \bar{x}^{-i}$ if the latter is positive, or negative on \mathbb{R}_+ if the latter is negative. Therefore, $\max(0, \max_{x_i \in \mathcal{D}(\bar{x}^{-i})} f(x_i, \bar{x}^{-i}))$ equals either zero or a positive local maximum of $x_i \mapsto f(x_i, \bar{x}^{-i})$ on $\mathcal{D}(\bar{x}^{-i})$.

As the cubic coefficient of f , $-(1 - \rho)(1 + r) \frac{1}{u} \gamma$, is negative, and as $f(0) = 0$, f is a cubic polynomial that is either:

1. decreasing, when it has at most one real root. In this case, $\max_{x_i \geq 0} J(x_i, \bar{x}^{-i}) = 0$, and $\beta_i(\bar{x}^{-i}) = 0$.
2. admitting both a local minimum and a local maximum, when it has two real roots. In this case, if $\max_{x_i \in \mathcal{D}(\bar{x}^{-i})} f(x_i, \bar{x}^{-i}) \geq 0$, the objective function would be equal to $\arg \max_{x_i \in \mathcal{D}(\bar{x}^{-i})} f(x_i, \bar{x}^{-i})$.

Note that there is a possibility that $J(x_i, \bar{x}^{-i})$ admits two local maxima equal to zero. In this case, the best response is undetermined as it could be equal to zero or a positive value. However, this undetermination would occur only for a few combinations of parameters, and for each of them, for only one value of \bar{x}^{-i} . I therefore rule out this case as a negligible technical point.

Let me study the roots of f by solving $f'(x_i) = 0$.

$$\begin{aligned}
 f'(x_i) &= r - 2c_i s_i - K \left[3\gamma x_i^2 + \left[4\gamma(\bar{x}^{-i} - g(0)) + 2r \right] x_i + \gamma(\bar{x}^{-i} - g(0))^2 + r\bar{x}^{-i} + \eta \right] \\
 &= -3\gamma K x_i^2 - \left[4\gamma K(\bar{x}^{-i} - g(0)) + 2Kr + 2c_i \right] x_i + r - \eta K - \gamma K(\bar{x}^{-i} - g(0))^2 - Kr\bar{x}^{-i}, \\
 \Delta &= \left[4\gamma K(\bar{x}^{-i} - g(0)) + 2Kr + 2c_i \right]^2 + 12\gamma K \left[r - \eta K - \gamma K(\bar{x}^{-i} - g(0))^2 - Kr\bar{x}^{-i} \right] \\
 K &:= \frac{(1 - \rho)(1 + r)}{u}
 \end{aligned}$$

If $\Delta > 0$, the roots of f are the following:

$$\begin{aligned}
 y_1 &= \frac{4\gamma K(g(0) - \bar{x}^{-i}) - 2c_i - 2Kr + \sqrt{\Delta}}{6\gamma K} \\
 y_2 &= \frac{4\gamma K(g(0) - \bar{x}^{-i}) - 2c_i - 2Kr - \sqrt{\Delta}}{6\gamma K}
 \end{aligned}$$

As the cubic coefficient of f is negative, and as $y_1 \geq y_2$, the function f reaches its local maximum in y_1 . Therefore, if $\Delta > 0$, $f(y_1) > 0$ and $y_1 \in \mathcal{D}(\bar{x}^{-i})$, then y_1 is the best response. In all other cases, the best response is zero. \square

Chapter 3

Can investors curb greenwashing?

Abstract

We show how investors with pro-environmental preferences and who penalize revelations of past environmental controversies impact corporate greenwashing practices. Through a dynamic equilibrium model with information asymmetry, we characterize firms' optimal environmental communication, emissions reduction, and greenwashing policies, and we explain the forces driving them. Notably, under a condition that we explicitly characterize, companies greenwash to inflate their environmental score above their fundamental environmental value, with an effort and impact increasing with investors' pro-environmental preferences. However, investment decisions that penalize greenwashing, policies increasing transparency, and environment-related technological innovation contribute to mitigating corporate greenwashing. We provide empirical support for our results. This is a joint work with Peter Tankov and Olivier David Zerbib.

1 Introduction

As part of its annual screening of company websites, the European Commission focused on greenwashing practices in 2021. In 42% of cases, the authorities “had reason to believe that the [company’s] claim may be false or deceptive.”¹ This figure suggests that greenwashing, “the practice by which companies claim they are doing more for the environment than they actually are,” is extremely widespread, especially since it can be implemented in a multitude of ways and to varying degrees,² and because it is still largely unregulated.³

The latest developments in the sustainable finance literature help to understand the prevalence of greenwashing. Indeed, because part of the investors have pro-environmental preferences (Riedl and Smeets, 2017) and internalize environment-related financial risks in their investment decisions (Krüger et al., 2020), green companies benefit from a lower cost of capital in equilibrium (Pástor et al., 2021; Pedersen et al., 2021b; Zerbib, 2022). In addition, companies’ environmental footprints are challenging to measure accurately,⁴ measurement methods are not standardized (Berg et al., 2022), and companies may communicate about their environmental footprint in an ambiguous manner (Fabrizio and Kim, 2019). Thus, companies have the ability and the incentive to overstate their environmental value with the aim of increasing their environmental score.

By misinforming stakeholders about the environmental impact of companies, greenwashing creates a major obstacle to the ecological transition. Specifically, greenwashing has a negative impact on sustainable investment for two primary reasons: (i) it complicates the evaluation of the environment-related financial risks, and (ii) it reduces sustainable investors’ positive impact on the environmental practices of companies by making more challenging the evaluation of their environmental footprints. In this paper, we notably show how sustainable investors may indirectly incentivize companies to practice greenwashing, and how they can directly discourage them from doing so.

We build a dynamic equilibrium model with asymmetric information populated by n heterogeneous firms and a representative investor. Each firm has a fundamental environmental value (also referred to as “environmental value”) which it can adjust continuously by investing in green projects (or, equivalently, by “abating” its environmental footprint) at a quadratic cost. However, the investor does not observe the company’s environmental value and relies on the environmental score estimated by a third party such as a rating agency. The company can influence

1. https://ec.europa.eu/commission/presscorner/detail/en/ip_21_269

2. https://futerra-assets.s3.amazonaws.com/documents/The_Greenwash_Guide.pdf

3. With the notable exception of the European Union, where a draft European directive aimed at banning “Generic environmental claims and other misleading marketing tricks” is in preparation and could come into force in 2026 if passed by member states: <https://www.europarl.europa.eu/news/en/press-room/20230918IPR05412/eu-to-ban-greenwashing-and-improve-consumer-information-on-product-durability>.

4. For the basic example of climate change, there are several issues to contend with, such as the accuracy of disclosure for scopes 1 and 2, and the availability of information on scope 3. For other environmental topics, the challenge is often even greater; for example, the calculation of a biodiversity footprint is rudimentary and approximate, given the number of assumptions that rating agencies have to make (Garel et al., 2024).

this environmental score directly through environmental communication, which can be positive (e.g., commitment to a net-zero emissions trajectory in 2050 or the launching of green projects; referred to as “green communication”) or negative (e.g., leaving a climate coalition, adjusting emission targets upward; referred to as “brown communication”), but also true or deceptive; the environmental score increases with the company’s positive environmental communication, which also has a quadratic cost. Deceptive communication can increase the environmental score without increasing the environmental value, which gives rise to a spread between the two. However, the environmental score reverts back towards the environmental value over time through the action of two forces: (i) continuously, through the analysis of the rating agency, and (ii) discontinuously, through events, to which we also refer as “controversies,” which instantly and publicly reveal a share of the spread between the score and the environmental value. The occurrence of these events is modeled through a Poisson process. The average revelation rate of the environmental value through these two forces, which we define as the “revelation intensity,” characterizes the degree of information asymmetry affecting the environmental value of each company, with lower revelation intensity corresponding to greater information asymmetry.

The representative investor has two main features: she can have pro-environmental preferences (e.g., Pástor et al., 2021; Zerbib, 2022) and can penalize the spread between a company’s score and its environmental value, when it is revealed by controversies. We also refer to this penalty as a “penalty on revealed misrating” or “*misrating* penalty.” This penalization can be interpreted in two ways: either as a readjustment of the environmental score, which the investor considers insufficiently credible, or as an additional penalty linked to poor corporate governance. This penalty echoes other forms of misconduct penalties in related literature (e.g., Egan et al., 2022).

The investor allocates her capital among n firms with dynamic mean-variance preferences (e.g., Buffa et al., 2022) adjusted to reflect pro-environmental preferences and the penalty associated with revealed score inaccuracies. The firms dynamically choose their (i) emission abatement efforts and (ii) communication efforts to minimize the sum of their costs of capital in equilibrium and their costs of abatement and environmental communication. We show that minimizing the cost of capital in this program is equivalent to maximizing the current market value relative to the future market values. From the optimal communication and abatement efforts, we derive the greenwashing strategy of a company, which we define as positive communication in excess of a contemporaneous abatement effort when the environmental score of the company is already above (or equal to) its environmental value. Therefore, greenwashing is defined as a communication effort that aims at creating or increasing a positive gap between the environmental score and the environmental value.

Through our baseline model, we document four main results relating to (1) equilibrium expected returns, (2) companies’ optimal environmental communication and abatement strategies, (3) companies’ optimal greenwashing strategy and how investors can curb it, and (4) complemen-

tary tools available to policymakers to limit greenwashing and favor abatement. All the results we obtain are closed-form formulas, thereby allowing us to analyze the underlying effects.

First, we show that the investor's penalization of revealed misrating commands a premium on expected returns, which scales with the strength of the penalty. In addition to the green premium documented by Pástor et al. (2021), Pedersen et al. (2021b), and Zerbib (2022), the investor requires higher returns to hold stocks whose environmental score credibility is questionable in light of past controversies. As a result, she underweights these stocks in her asset allocation.

Second, optimal environmental communication and abatement efforts are structured around two forces: (i) an "incentive force," driven by the investor's pro-environmental preferences, which pulls both efforts upwards, and (ii) a "corrective force," associated with the investor's penalization of revealed misrating, which pushes both efforts in opposite directions to reduce the spread between the environmental score and the environmental value. For example, if the company is overrated compared to its fundamental environmental value, the corrective force will play negatively on environmental communication and positively on abatement. Both optimal efforts correspond to the combined effect of these two forces. In terms of sensitivity, four points are worth highlighting regarding communication and abatement efforts. (i) An increase in the investor's pro-environmental preferences leads to a linear increase in both environmental communication and abatement effort. Moreover, an increase in the penalty on revealed misrating leads companies to reduce their environmental communication and increase their abatement when the environmental score overestimates the company's environmental value. Thus, greater penalization of misrating revealed by controversies increases the investor's positive impact on companies' environmental practices. In addition, (ii) the "revelation intensity" of a company's environmental value, corresponding to the cumulative effort of the rating agencies' work and the disclosure intensity of controversies, reduces the incentive to communicate and increases the incentive to abate. Indeed, the faster the effect of communication on the score fades, the less this strategy is used by companies. Furthermore, (iii) communication and abatement efforts decrease with their marginal costs per unit of effort (also referred to as "marginal unit cost"). The more expensive it is to abate (communicate), the less the company abates (communicates). Finally, (iv) a subtle interaction effect comes into play: an increase (resp. decrease) in one of the two marginal unit costs (e.g., the marginal unit cost of abatement) leads to a joint downward (resp. upward) move of both efforts (i.e., abatement and communication), because of the investor's penalty on the spread between a company's score and its environmental value.

Third, from the optimal environmental communication and emission abatement of a company, we derive its optimal greenwashing strategy. The incentive force, driven by the investor's pro-environmental preferences, pushes the company to greenwash under an "ON-OFF" condition that guarantees the benefit of adopting such a strategy: the company greenwashes as long as it is sufficiently cheap to engage in environmental communication relative to abatement, the asymmetry of information is sufficiently strong, or the company's rate of time preference is high

enough. The condition for the existence of greenwashing does not depend on the penalty imposed by investors when a misrating is revealed by a controversy, but this penalty does impact the degree of greenwashing. Indeed, the company steers its environmental communication relative to its abatement until the difference in the environmental score and the fundamental value reaches a certain threshold, which decreases with the misrating penalty. This penalty is, therefore, a useful tool in the hands of sustainable investors to counterbalance the indirect greenwashing incentive they transmit to companies through their pro-environmental preferences. This penalty increases the company's incentive to abate its emissions, thereby enabling investors to increase the positive impact they have on companies' environmental practices.

Fourth, we examine two complementary policy instruments for reducing greenwashing: (i) regulations to increase transparency on corporate environmental practices, and (ii) support for environmental technological innovation. Whether through the investigative power of rating agencies or the ability of stakeholders to identify potential controversies, any increase in transparency encourages companies to reduce their greenwashing practices. However, the different vectors of transparency have distinct impacts on companies' greenwashing strategies. While the impact of the rating agency's work on reducing greenwashing is strong when investors do not penalize misrating as controversies arise, it becomes more marginal when investors heavily penalize misrating. Conversely, the advent of controversies is complementary to the penalization of misrating by investors: their combined effect is an effective vector for reducing greenwashing. Finally, environmental technological innovation can only reduce greenwashing when it significantly lowers the marginal unit costs of abatement compared with those of communication. Thus, maintaining a sustained and pronounced research and development effort to bring down the marginal costs of new green technologies would, in addition to increasing abatement, simultaneously help curb corporate greenwashing practices.

What if investors only care about *relative* environmental scores of companies, either because they practice best-in-class investment strategies or because rating agencies standardize scores? This practice introduces interaction between companies, which choose their optimal environmental strategies based on those of the others. We formulate an extension to the model, in which the investor normalizes each company's environmental score by the average environmental score in the investment universe. Through a mean field approximation described in the Internet Appendix, we solve this game and prove that it admits a unique Nash equilibrium. Analytically, we find that the optimal environmental strategy of a representative company follows the same pattern as in the baseline case. Hence, the qualitative conclusions stated above are robust to the introduction of such an interaction between companies. However, we show numerically that this interaction leads to lower abatement and communication efforts than in the baseline case. Indeed, since the company's objective is now to outperform its peers, the incentive for having a high absolute environmental score is weaker. These results suggest that the commonly used cross-sectional normalization of companies' environmental scores by rating

agencies and the best-in-class approaches to portfolio selection may have a detrimental impact on the improvement of firms' environmental performances.

We provide empirical evidence supporting the results of our model. Specifically, because greenwashing practices are unobservable, we focus on global companies' environmental communication from December 2015 to December 2022, and we document two main results: (i) we show that companies almost structurally engage in *green* (i.e., positive environmental) communication, and (ii) we validate the dynamics of the environmental communication found in our model.

In practice, we propose a two-step empirical method for analyzing companies' environmental communication policies and testing their dynamics in cross-section. To do so, we use monthly data from the data provider Covalence, which constructs an environmental reputation score, an environmental controversy score, and an environmental performance score from published news. The environmental reputation score includes both the environmental communications of companies and the environmental controversies that affect them. Therefore, in the first step, we construct a proxy for the environmental communication score as the orthogonal component of the environmental reputation score to the environmental controversy score, through a Within regression, wherein the environmental controversy score is instrumented by its lagged value to address simultaneity bias. We find that the monthly average flow of environmental communication is positive 98.8% of the time, that is, companies almost structurally engage in green communication (result [i]). In the second step, we provide empirical support for the environmental communication dynamics highlighted by the model (result [ii]). The fundamental environmental value is unknown, but it is legitimate to assume that it is very inert at the monthly frequency. We, therefore, perform a Within regression of the monthly change in environmental communication on the monthly change in environmental score instrumented by the past environmental score, given the simultaneity issue. Through a number of complementary estimations (different sub-samples, different starting dates, and different environmental sub-scores), we find strong evidence that companies steer their environmental communication in a counter-cyclical way according to the evolution of their environmental score, consistent with the effect of the corrective force highlighted above.

Our results show that companies have implemented, on average, a quasi-structural green (i.e., positive environmental) communication policy. There are three possible explanations for this: (i) either companies are structurally underrated by the rating agencies and communicate to raise their environmental score to the level of their fundamental environmental value, (ii) they use green communication to support their continuous abatement effort, or (iii) they are engaging in greenwashing through misleading communication, at least part of the time. Yet, (a) the academic literature has not documented any structural underestimation of the environmental scores. In addition, (b) green communication is more volatile than abatement policies, (c) the marginal unit costs of environmental communication are substantially lower than those of abatement

(Bank for International Settlement, 2017), and (d) companies can benefit from information asymmetry about their fundamental environmental values (Barbalau and Zeni, 2023). Therefore, our findings suggest that companies may engage in greenwashing, at least part of the time.

Related literature. Our findings extend prior research on greenwashing, asset pricing, and impact investing. First, our paper contributes to the nascent financial literature on greenwashing.⁵ Corporate greenwashing has increased significantly over the past five years (Gourier and Mathurin, 2024) and is particularly prevalent in cases where companies benefit from information asymmetry (Wu et al., 2020). For example, forms of greenwashing have been documented through conflicts of interest between companies and the firms auditing them (Duffo et al., 2013), as well as, indirectly, when companies sell polluting plants to companies facing weaker environmental pressures without inducing a reduction in overall pollution (Duchin et al., 2023). However, empirical evidence suggests that investors can contribute to reducing corporate greenwashing: by participating in climate initiatives using the shareholder engagement channel, investors reduce corporate cheap talk on climate issues (Bingler et al., 2023). Yet, asset managers are not exempt from suspicions of greenwashing (Kim and Yoon, 2022), and instances of greenwashing in the news lead to capital outflows from funds marketed as sustainable (Gourier and Mathurin, 2024). These results echo the literature on disclosure, which highlights investors' increased demand for transparency (Flammer, 2021; Ilhan et al., 2023), as well as the literature documenting the divergence between ESG rating providers (Berg et al., 2022), and the opacity of data construction (Berg et al., 2021), emphasizing the complex nature of investment decisions based on ESG criteria. We contribute to this literature by developing, to the best of our knowledge, the first theoretical model linking corporate greenwashing to investor pressure, along with a contemporary working paper by Chen (2023b). Specifically, we characterize the mechanisms that induce and reduce corporate greenwashing from an asset pricing perspective, and we provide empirical evidence for them.

Chen (2023b) addresses a question similar to ours through a theoretical model. However, we differ from this paper as (i) we explicitly characterize optimal asset returns and greenwashing strategies, (ii) in a dynamic setup, (iii) allowing for interaction among companies to choose their optimal policies, and (iv) providing empirical evidence for our results. In addition, from the model assumptions standpoint, we remain agnostic on the difference in NPV of green and brown projects and we allow investors to selectively penalize companies that greenwash thanks to the advent of controversies. Thus, we reach different conclusions: in Chen (2023b), investors' environmental impact decreases with pro-environmental preferences because all companies are penalized by greenwashing, while in our paper, investors' impact increases with these preferences as greenwashing is penalized at the firm level.

5. Besides the financial literature, which we review below, studies in adjacent research fields have addressed the issue of greenwashing from the business ethics standpoint, see for example, Laufer (2003), Walker and Wan (2012), Lyon and Montgomery (2015), Marquis et al. (2016).

We also contribute to the asset pricing literature. First, we add to the literature on asset pricing under asymmetric information, which has been built upon two main frameworks: the one wherein investors pay to acquire information (Grossman and Stiglitz, 1980; Admati and Pfleiderer, 1986; Hughes, 1986) and the one wherein two groups of investors—informed and uninformed investors—coexist (Easley and O’hara, 2004; Lambert et al., 2012). We contribute to this field by building an asset pricing model with random revelation times, over which investors have no control, and which allow them to constrain the companies to reduce information asymmetry. This paper is also part of the literature on sustainable asset pricing. Whether for climate (Engle et al., 2020; Choi et al., 2020; Sautner et al., 2023) or biodiversity (Giglio et al., 2023a; Garel et al., 2024; Coqueret et al., 2024) issues, the pro-environmental preferences of investors (Riedl and Smeets, 2017; Humphrey et al., 2023) and their expectations of future environmental risks (Krüger et al., 2020; Stroebel and Wurgler, 2021; Hambel et al., 2023b) command a green premium that increases the cost of capital of the most polluting companies (Pástor et al., 2021; Pedersen et al., 2021b; Zerbib, 2022; Bolton and Kacperczyk, 2021; De Angelis et al., 2023; Hsu et al., 2023), by an order of magnitude of 1.5% to 2% compared to the least polluting companies in the US (Pástor et al., 2022; Zerbib, 2022). The effect of the green premium increases with the inelasticity of the demand function of passive sustainable investors (Cheng et al., 2023), but is attenuated in the presence of uncertainty (De Angelis et al., 2023; Avramov et al., 2022b) as well as when green investors also have green consumption preferences (Sauzet and Zerbib, 2022c); it can even be almost zero when the investors’ demand function is elastic (Berk and van Binsbergen, 2021). In addition, a green premium driven by non-pecuniary motives may alter equilibrium prices in a suboptimal manner from a climate risk perspective (Goldstein et al., 2022). It is noteworthy that the rise in the cost of capital of brown companies has been associated in recent years with an increase in the financial performance of the greenest assets due to an unexpected increase in pro-environmental preferences (Pástor et al., 2022; Ardia et al., 2023), and hence, in the price impact of these flows towards green assets (Van der Beck, 2023). We contribute to the sustainable asset pricing literature by showing that investors’ penalties for misratings revealed during environmental controversies command a risk premium that increases the cost of capital of the companies whose reputations have been tarnished.

Finally, we contribute to the growing literature on impact investing. Even if the green premium induced by sustainable investment increases the cost of capital of the most polluting companies, the incentive to go green for these companies remains limited (De Angelis et al., 2023), and may even have a counter-productive effect by increasing the environmental footprint of polluting companies, which turn to brown projects that generate short-term cash flows (Hartzmark and Shue, 2023). Yet, Favilukis et al. (2023) show that constrained mandates on green investment can significantly influence the allocation of capital across sectors with a negligible impact on the cost of capital. In any case, a number of papers highlight conditions under which investors can increase their impact on the greening of corporate practices: basing investment de-

cisions on aggregate welfare by internalizing the externalities of all firms in the economy (Green and Roth, 2021b; Oehmke and Opp, 2023), funding companies that would not have been funded by regular investors otherwise (Green and Roth, 2021b), prioritizing investments where search friction is acute (Landier and Lovo, 2020b), and holding a brown stock if it has taken corrective action (Edmans et al., 2023). Nevertheless, the limits of impact investing are reflected in investors’ willingness to pay for impact (Barber et al., 2021), which is limited compared to the willingness to pay to invest in green assets (Bonnefon et al., 2022) and, when it exists, does not scale with the level of impact (Heeb et al., 2023). We contribute to this literature by showing that green investors can have a double impact on corporate practices: indirectly, by encouraging companies to greenwash through their pro-environmental preferences, and directly, by limiting corporate greenwashing and spurring emission abatement through the penalties they apply when an environmental controversy is revealed.

Outline. The remainder of this paper is structured as follows. Section 2 introduces an economy populated by firms able to greenwash but exposed to the investor penalty. Section 3 describes the equilibrium pricing equation as well as companies’ optimal abatement, communication, and greenwashing strategies. Section 4 presents an extension of the investor’s program with firm interaction and summarizes the main findings in this new setting. Section 5 provides empirical evidence supporting the findings of the model, and Section 6 concludes the paper.

2 An equilibrium model with corporate greenwashing

2.1 Empirical motivation

We motivate our study by documenting the extent and dynamics of corporate environmental communication. To do so, we use data from Covalence, a data provider which constructs an environmental reputation index and an environmental performance score, both between 0 and 100, for companies worldwide at a monthly frequency.⁶ Specifically, based on data from published news, the environmental reputation index reflects companies’ forward-looking environmental communication, which can be positive (e.g., environmental commitments) or negative (e.g., environmental performance below expectations, exit from climate action groups), as well as the occurrence of controversies.

To ensure that we cover a sufficiently reasonable number of companies per month, and given the date of signature of the Paris Agreement, which enshrined the pivotal role of investors

6. Covalence is a Switzerland-based data provider, founded in 2001, which produces ESG reputation data using media monitoring, artificial intelligence, and human analysis (<https://www.covalence.ch/>). Its services are used by asset managers, asset owners, international organizations and institutions (e.g., the EU, the WWF), and academic institutions. Its datasets have been used by several influential papers (e.g., Daubanes and Rochet (2019)). The construction methods of the indices are available in the White Paper available at this URL: https://www.covalence.ch/docs/Covalence_GreenwashingRiskIndicator_WhitePaper.pdf.

in the ecological transition,⁷ we take December 2015 as the starting month for our study. From December 2015 to December 2022, our study covers 3,769 companies.⁸ Between the end of 2015 and the end of 2019, the average environmental reputation index was fairly stable around 80, then rose rapidly between the end of 2019 and the end of 2022 to reach almost 86 (Figure 3.1a). Monthly variations in the environmental reputation index reflect companies' environmental communication flows and, when they occur, controversies. Figure 3.1b shows the average monthly changes in environmental reputation, which we also refer to as environmental reputation flows.⁹ Although they fluctuate substantially, over 96% of these flows are positive, reflecting the intensity and regularity of companies' *positive* environmental communication.

What are the drivers of environmental communication? Without pretending to answer this question at this stage, an analysis of the correlation between monthly variations in environmental reputation and environmental score reveals a surprising dynamic: the proportion of companies showing a negative correlation between variations in their environmental reputation and their previous month's environmental score varies between 63% and 78% over the years (Figure 3.2). This correlation suggests that companies could use environmental communication as a counter-cyclical instrument to adjust and correct the level of their environmental score. Does this empirical observation shed any light on companies' environmental communication practices? When do companies use environmental communication to greenwash? What role can investors play in influencing greenwashing practices? To answer these questions, we construct a model motivated by these empirical observations.

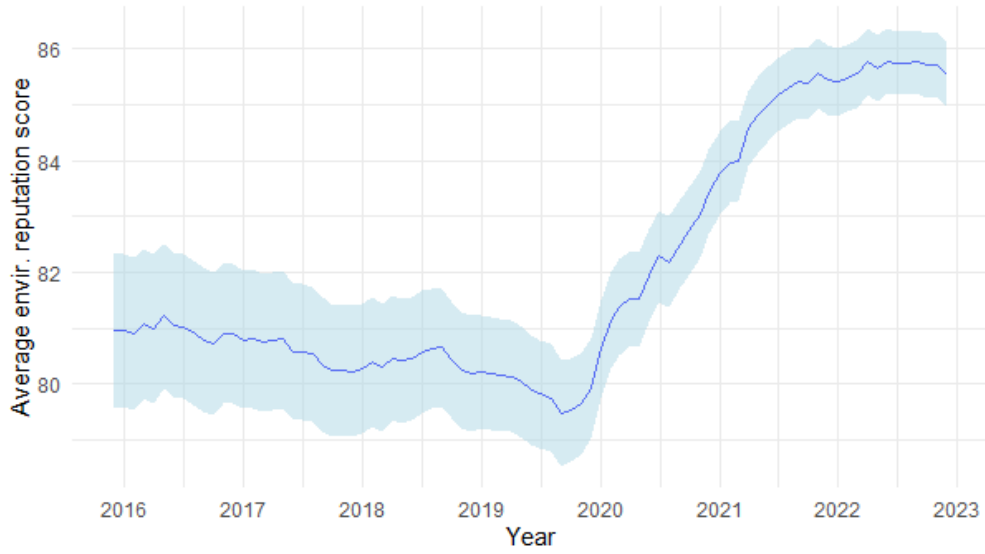
2.2 The model

Market setting. Our model is inspired by the dynamic asset pricing model of Bouchard et al. (2018), where the volatility matrix of asset prices is exogenous, the expected return vector is determined in equilibrium, and the representative investor maximizes a mean-variance objective. Unlike the above reference, we do not allow for transaction costs, but we introduce additional terms in the investor's objective function to account for non-pecuniary preferences and misrating penalty. On a filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ with infinite time horizon, we consider a market with n firms, indexed by $i \in \{1, \dots, n\}$, issuing stocks at date 0, and a representative

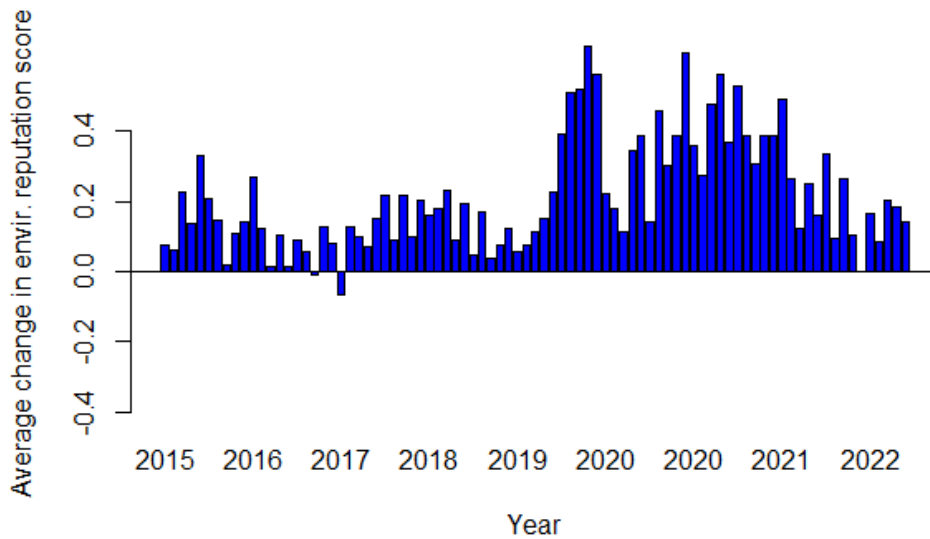
7. Article 2.1(c) of the Paris Agreement calls for "making financial flows compatible with a pathway to low greenhouse gas (GHG) emissions and climate-resilient development."

8. Until a company's reputation is monitored, Covalence assigns it a score (supposed to be neutral) of 50. To avoid introducing this bias, we exclude all the rows for which the companies have a reputation index of 50 since the launch of the database by Covalence in 2009 until the moment when their score changes value for the first time.

9. It should be noted that the two figures are not completely comparable, as Figure (b) does not represent the differentiated version of Figure (a). This is because a significant number of companies enter the sample after the initial date. Hence, the evolution of the average reputational score in Figure (a) is driven by (i) the entry of new companies, which may have lower reputation scores, and (ii) the change in reputation scores of companies already in the sample, which is mostly positive as illustrated in Figure (b).



(a) Average environmental reputation index



(b) Average monthly changes in environmental reputation index

Figure 3.1 – **Reputation index.** This figure shows the trajectory of the environmental reputation index averaged over all companies and the one-standard-deviation confidence interval (Figure 1a) as well as the average monthly changes of this index (Figure 1b).

investor.¹⁰ The price process $S \in \mathbb{R}^n$ is assumed to follow the dynamics

$$dS_t = \mu_t dt + \sigma dB_t, \quad (3.1)$$

10. We consider a representative investor to avoid unnecessary model complexity. The main conclusions remain unchanged in a model with several investors with heterogeneous preferences.

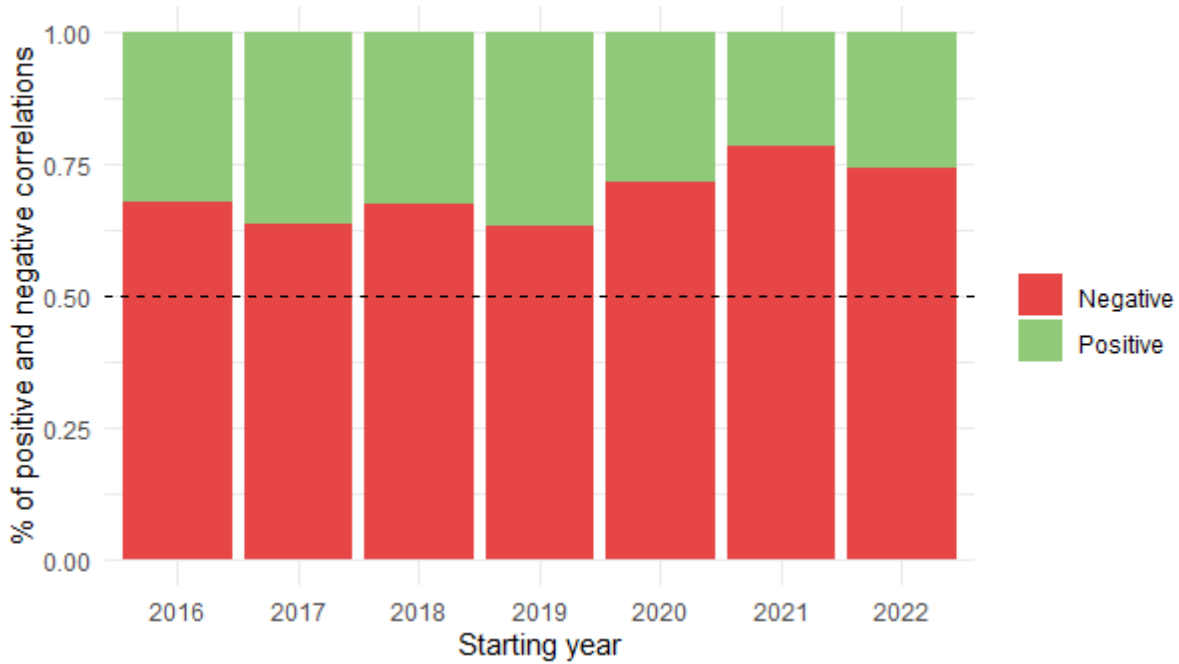


Figure 3.2 – **Correlation between changes in environmental reputation and environmental score.** This figure depicts the average correlation between the monthly changes in environmental reputation and the previous month’s environmental score between 2016 and 2022 for all companies in the universe.

where $\mu_t \in \mathbb{R}^n$ is the vector of expected returns of the assets, which is determined in equilibrium, $\sigma \in \mathbb{R}^{n \times n}$ is the exogenously specified volatility matrix, which is assumed to be constant and nonsingular,¹¹ and (B_t) is an n -dimensional Brownian motion. The quantity of stocks of each company is normalized to one. In addition to risky assets, the investor can also invest in a risk-free asset, which is assumed to have a zero rate, without loss of generality. In the remainder of this paper, the i -th component of a vector $h \in \mathbb{R}^n$ is denoted by h^i .

Environmental score. To address the question of greenwashing, we add information asymmetry to this dynamic asset pricing model. The fundamental environmental value of each company i , denoted by V^i , is not observed by the investor. Instead, she observes an environmental score, namely, a public rating provided by a rating agency, which aims to estimate the fundamental environmental value but does not perfectly reflect it due to information asymmetry. The environmental score of company i , E^i , depends on the company’s fundamental environmental value,

11. The assumption of a constant exogenous volatility matrix is consistent with what most of the sustainable asset pricing literature has assumed to date (e.g., Pástor et al., 2021; Pedersen et al., 2021b; Zerbib, 2022). The exploration of models with endogenous volatility matrix involves significant complexities, which prevent the obtention of closed-form solutions, as we allow for stochastic adapted strategies for companies, as opposed to, for example, De Angelis et al. (2023). We leave this interesting avenue for future research.

V^i , and its environmental communication effort, c^i , as follows:

$$dE_t^i = \underbrace{a(V_t^i - E_t^i)dt}_{\text{Rating agency effect}} + \underbrace{b(V_t^i - E_t^i)dN_t^i}_{\text{Controversy effect}} + \underbrace{c_t^i dt}_{\text{Communication effect}} + \underbrace{z dW_t^i}_{\text{Measurement error}}, \quad E_0^i = q^i, \quad (3.2a)$$

$$dV_t^i = \underbrace{r_t^i dt}_{\text{Abatement effect}}, \quad V_0^i = p^i, \quad (3.2b)$$

where $a, b, z, q^i, p^i \in \mathbb{R}_+$ are constant deterministic parameters, (N_t^i) is a one-dimensional Poisson process of intensity $\lambda^i \in \mathbb{R}_+^*$, and (W_t^i) a one-dimensional Brownian motion.

The fundamental environmental value of company i , V^i , is determined by its investments in green projects, or in other words, its environmental footprint reduction or “abatement” effort, r^i . However, since the rating agency does not directly observe abatement efforts, the score, E^i , can be influenced by environmental communication, c^i , and measurement noise or error, zW_t^i .¹² Environmental communication and measurement error can both contribute to creating a discrepancy between the environmental score and fundamental environmental value, but these effects are counterbalanced by two mechanisms revealing the true environmental performance. First, continuous efforts of the rating agency create a force pushing the environmental score towards the fundamental environmental value with speed a . Second, controversies related to the environmental quality of the company arise at random times and contribute to revealing its fundamental environmental value. A controversy at time t reveals a portion $b \in [0, 1]$ of the ongoing *misrating* $|E_t^i - V_t^i|$.¹³ Controversies are assumed to arise independently from the measurement error, that is, for each company i , N^i is independent from W^i .

The environmental communication, which can be seen as trying to convince the investors that an abatement effort of the same magnitude has been made, has a direct impact on the environmental score. It can be used for the purposes of green or brown communication.

Definition 1 (Green communication). We refer to *green communication* when a company engages into environmental communication with the aim of raising its environmental score, that is, when $c_t^i > 0$.

Whether truthful or deceptive, green communication refers to positive environmental communication made by a company to convince that its environmental value is higher than its current environmental score: it can be a pledge on abatement targets, environmental reporting, or attractive ways to present its environmental policy when answering rating agencies’ questionnaires.

12. Berg et al. (2022) estimate that measurement differences explain 56% of ESG scores divergence across ESG rating agencies.

13. We refer to controversies as events that reveal a discrepancy between a company’s environmental score and its environmental value. These discrepancies can be positive or negative, in line with the definition of a controversy as “a disagreement or strong debate” (Cambridge Dictionary).

Definition 2 (Brown communication). We refer to *brown communication* when a company engages into environmental communication that has a negative impact on its environmental score, that is, when $c_t^i < 0$.

Brown communication refers to any communication made by a company that adversely affects its public environmental image. The company might opt to backtrack on a previous environmental commitment, announce the abandonment of an emission reduction target, or disclose information regarding its unexpectedly substantial environmental footprint.

Now, we can define the practice of greenwashing, which is a green communication strategy whereby a company oversells its environmental image. To understand the following definition, recall that the environmental score, E^i , which is controlled by the communication effort, c^i , aims to estimate the fundamental environmental value, V^i , which is controlled by the abatement effort, r^i ; the two efforts act on their respective variables in the same way and are measured in the same units. The situation wherein the two efforts are equal may be seen as “neutral” with respect to greenwashing, as both the fundamental environmental value and the environmental score increase or decrease by the same amount.

Definition 3 (Greenwashing). Company i is *greenwashing* at time t if (i) it is overrated, that is, $E_t^i \geq V_t^i$, (ii) its environmental communication is positive, $c_t^i > 0$, and (iii) it communicates more than it abates, $c_t^i > r_t^i$. When the company is greenwashing, its *greenwashing effort* is defined as $c_t^i - r_t^i$.

The first two criteria reflect the fact that a company engages in green communication when it is already overrated in terms of its fundamental environmental value. The third criterion allows us to exclude from the scope of greenwashing cases where a company is genuinely communicating about the launch of a new green project ($c_t^i \leq r_t^i$), even though it is already overrated. Greenwashing is, therefore, any communication effort that aims at creating or increasing a positive gap between the environmental score and the fundamental environmental value, when the company is accurately rated or already overrated.

Investor’s score for environmental misrating. The investor has a preference for informative environmental scores, as she wishes to allocate her capital to green companies based on accurate information. Therefore, in her asset allocation program, she penalizes companies whose environmental scores have proven inaccurate in the past. The investor builds a score M_t^i for company i at time t , based on the environmental score inaccuracies she has observed through past controversies as follows:

$$dM_t^i = -\rho M_t^i dt + (E_t^i - E_{t-}^i)^2 dN_t^i, \quad M_0^i = u^i, \quad (3.3)$$

with $\rho, u^i \in \mathbb{R}_+$. At each controversy, that is, when $dN_t^i = 1$, the score M^i jumps upwards, according to the square of the revealed misrating, $|E_t^i - E_{t-}^i|$. This score for misrating is quadratic

in the environmental score adjustment because the effect of controversies usually induces dramatic and non-linear repricing (see, for example, the impacts of the 2010 British Petroleum, 2015 Volkswagen, and 2015 ExxonMobil controversies on asset prices). When there is no controversy, the score M^i is continuous and decreases at rate $\rho > 0$, as the investor gives more importance to recent controversies than older ones. Note that the misrating score, M^i , is positive.

It should be noted that this specification assumes that the investor penalizes all types of inaccuracies, be they positive or negative. In theory, this assumption is justified by the investor's need for transparency on the fundamental environmental value of the company to improve her capital asset allocation. In practice, as we will show below, the companies' scores are pulled upward by the investor's pro-environmental preferences, and controversies generally drive the scores down toward the companies' environmental values.

Program of the investor. The program of the representative investor combines two components: a standard mean-variance portfolio criterion (Bouchard et al., 2018) and a penalty related to non-pecuniary environmental preferences. This penalty is broken down into two parts. As in Pástor et al. (2021) and Zerbib (2022), it includes a preference term for companies with good environmental quality, measured by their public environmental score, E_t . However, the investor is aware of and averse to the low quality of ESG ratings (Berg et al., 2022), which can be biased by environmental communication. Therefore, she also penalizes companies for which past controversies have publicly revealed score inaccuracies using the misrating score, M_t . The investor determines her optimal asset allocation according to the following mean-variance-adjusted program:

$$\sup_{\omega \in \mathbb{A}^\omega} \mathbb{E} \left[\int_0^\infty e^{-\delta^I t} \left\{ \omega'_t dS_t - \frac{\gamma}{2} \langle \omega'_t dS \rangle_t + \omega'_t (\beta E_t - \alpha M_t) dt \right\} \right] \quad (3.4)$$

where $\omega \in \mathbb{A}^\omega$ denotes the vector of quantities invested in each risky asset at time t , \mathbb{A}^ω being the set of admissible strategies for the investor, which we define formally in the proofs, $S_t \in \mathbb{R}^n$ is the asset price vector at time t , and $\gamma \in \mathbb{R}_+^*$ is the risk aversion of the investor. $\beta \in \mathbb{R}_+$ is the investor's preference sensitivity for holding green assets (also referred to as investor's *pro-environmental preferences*), $E_t \in \mathbb{R}^n$ denotes the vector of environmental scores of companies at time t , observed by the investor, $\alpha \in \mathbb{R}_+$ is the sensitivity parameter to misrating revealed by past environmental controversies, and $M_t \in \mathbb{R}^n$ is the vector of misrating proxies at time t . Finally, $\delta^I \in \mathbb{R}_+^*$ is the investor's rate of time preference. The equilibrium expected returns are determined so that the investor invests optimally and the market clears.

Program of the companies. Company i dynamically determines its optimal abatement effort, r_t^i , and environmental communication effort, c_t^i , by minimizing the sum of its costs of

capital, abatement, and communication, as follows:

$$\inf_{(r^i, c^i) \in \mathbb{A}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\mu_t^i + \frac{\kappa_r^i}{2} (r_t^i)^2 + \frac{\kappa_c^i}{2} (c_t^i)^2 \right) dt \right], \quad (3.5)$$

where \mathbb{A} is the set of admissible strategies for the companies, which we define formally in the proofs. The companies face quadratic abatement and communication costs (Battaglini and Harstad, 2016), and κ_r^i and κ_c^i denote the marginal unit costs of abatement and communication, respectively.

We specify the company's program through the minimization of its cost of capital rather than the maximization of its price for four reasons: (i) the cost of capital is a critical financial variable for companies' solvency and profitability, and it is affected by their investments in sustainable projects (e.g., Pástor et al., 2021; De Angelis et al., 2023); (ii) consistent with McConnell and Sandberg (1975) and Nantell and Carlson (1975), the minimization of the cost of capital is a notion almost equivalent to the maximization of the initial price of the company: more precisely, we show that the program with minimization of the cost of capital is equivalent to a program of maximization of the price at the initial time with respect to future prices;¹⁴ (iii) the use of a program with maximization of the price in the current framework with information asymmetry would not make it possible to obtain results interpretable in closed-form formulas; (iv) expected returns, which are expressed in monetary terms,¹⁵ are homogeneous to the financial costs of environmental efforts.

It is worth noting that the companies are heterogeneous along several dimensions: their initial characteristics (fundamental environmental value, V_0^i , environmental score, E_0^i , misrating score, M_0^i), the frequency of controversies that impact them (represented by the Poisson process with intensity, λ^i), and the costs of their abatement and communication strategies (κ_r^i and κ_c^i).

3 Optimal greenwashing and investor impact

The program of the investor can be solved explicitly, allowing us to derive expected returns and allocations in equilibrium. For the sake of readability, all proofs are reported in Appendix B.

14. The program of company i is equivalent to the following one, written in terms of asset price:

$$\sup_{(r^i, c^i) \in \mathbb{A}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\delta(S_0^i - S_t^i) - \frac{\kappa_r^i}{2} (r_t^i)^2 - \frac{\kappa_c^i}{2} (c_t^i)^2 \right) dt \right].$$

It is equivalent to maximizing the discounted difference between its current and future asset price, paying quadratic costs for abatement and environmental communication.

15. As the asset prices follow a Gaussian dynamics (Equation (3.1)), the expected returns are *price returns* expressed in dollars.

Proposition 1. *The optimal asset allocation of the investor is the pointwise solution*

$$\omega_t^* = \frac{1}{\gamma} \Sigma^{-1} (\mu_t + \beta E_t - \alpha M_t),$$

and the equilibrium expected return is

$$\mu_t = \gamma \Sigma \mathbf{1}_n - \beta E_t + \alpha M_t.$$

The investor's optimal allocation breaks down into three parts: the part associated with the standard mean-variance program, $\frac{1}{\gamma} \Sigma^{-1} \mu_t$; the effect of pro-environmental preferences, $\frac{\beta}{\gamma} \Sigma^{-1} E_t$ (Pástor et al., 2021; Zerbib, 2022), which increases (decreases) the allocation in the assets with high (low) environmental scores; the new effect associated with past environmental controversies, which decreases the allocation in the assets of companies that have experienced environmental controversies revealing environmental score inaccuracies, $-\frac{\alpha}{\gamma} \Sigma^{-1} M_t$.

Similarly, expected returns are also composed of the standard mean-variance component, $\gamma \Sigma \mathbf{1}_n$, adjusted for the green premium (Pástor et al., 2021; Zerbib, 2022), $-\beta E_t$, and the premium induced by misrating revealed in the past, αM_t . The greater the inaccuracies in companies' environmental scores revealed by past controversies, the higher the return investors require to hold their assets. The effect of past revealed misrating is scaled by the sensitivity parameter α .

In view of the explicit solution for equilibrium expected returns given in Proposition 1, the optimization problem for company i takes the following form:

$$\inf_{(r^i, c^i) \in \mathbb{A}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\gamma \Sigma \mathbf{1}_n - \beta E_t^i + \alpha M_t^i + \frac{\kappa_r^i}{2} (r_t^i)^2 + \frac{\kappa_c^i}{2} (c_t^i)^2 \right) dt \right].$$

The following proposition provides a solution to this problem, which corresponds to the Stackelberg equilibrium in the game between companies and the investor, wherein the companies, choosing their abatement and communication policies, play the role of the "leader," and the investor, fixing her portfolio allocation, is the "follower."

Proposition 2 (Optimal strategies). *The optimal environmental communication effort, $c^{i,*}$, and abatement effort, $r^{i,*}$, of company i are represented in feedback form as follows:*

$$c_t^{i,*} = \frac{1}{\kappa_c^i} \left(B^i - A^i (E_t^{i,*} - V_t^{i,*}) \right), \quad (3.6a)$$

$$r_t^{i,*} = \frac{1}{\kappa_r^i} \left(\frac{\beta}{\delta} - B^i + A^i (E_t^{i,*} - V_t^{i,*}) \right), \quad (3.6b)$$

where

$$\begin{aligned} B^i &= \frac{P^i}{Q^i}, \quad P^i = \beta\left(1 + \frac{A^i}{\delta\kappa_r^i}\right), \quad Q^i = \delta + a + b\lambda^i + \frac{2A^i}{\bar{\kappa}^i}, \quad \bar{\kappa}^i = \frac{2}{\frac{1}{\kappa_r^i} + \frac{1}{\kappa_c^i}}, \\ A^i &= \frac{\bar{\kappa}^i}{4}R^i \left(\sqrt{1 + \frac{16}{\bar{\kappa}^i} \frac{T^i}{(R^i)^2}} - 1 \right), \quad R^i = \delta + 2a + \lambda^i(1 - (1 - b)^2), \quad T^i = \frac{\lambda^i b^2 \alpha}{\delta + \rho}, \end{aligned} \quad (3.7)$$

with $E^{i,*}, V^{i,*}$ solutions of (3.2) when the optimal strategies $c^{i,*}, r^{i,*}$ are employed, $A^i, B^i \geq 0$ and $\frac{\beta}{\delta} - B^i \geq 0$.

Before interpreting in detail the optimal strategies, it is noteworthy that optimal communication and abatement strategies are so that their marginal benefits equal their marginal costs, as detailed in the following proposition. The marginal benefit of communication or abatement is defined as the impact on the integrated discounted cost of capital of increasing one of these strategies over an infinitesimal time interval. This notion is formally defined in Appendix (Definition 9).

Proposition 3 (Marginal benefits of communication and abatement). *Let c^i and r^i be two admissible strategies of communication and abatement, respectively, and let E^i be the corresponding environmental score and V^i the environmental value, solutions of equation (3.2) driven by these strategies. The marginal benefit of increasing communication at time t for company i when its environmental strategy is (c^i, r^i) is as follows:*

$$\Pi_t^{c^i, i} = \frac{\beta}{\delta + a + b\lambda^i} - 2T^i \mathbb{E} \left[\int_t^\infty e^{-(\delta+a)(s-t)} (1-b)^{N_s^i - N_t^i} (E_s^i - V_s^i) ds \middle| \mathcal{F}_t \right].$$

The marginal benefit of increasing abatement at time t for company i when its environmental strategy is (c^i, r^i) is as follows:

$$\Pi_t^{r^i, i} = \frac{\beta}{\delta} - \frac{\beta}{\delta + a + b\lambda^i} + 2T^i \mathbb{E} \left[\int_t^\infty e^{-(\delta+a)(s-t)} (1-b)^{N_s^i - N_t^i} (E_s^i - V_s^i) ds \middle| \mathcal{F}_t \right].$$

At optimum, communication and abatement strategies, $c^{i,}$ and $r^{i,*}$, are so that their marginal benefits equal their marginal costs:*

$$\Pi_t^{c^{i,*}, i} = \kappa_c^i c_t^{i,*}, \quad \Pi_t^{r^{i,*}, i} = \kappa_r^i r_t^{i,*}.$$

The marginal benefits of increasing communication and abatement at time t can be understood as follows. Both are equal to the sum of (i) a constant component that does not depend on the strategy of the company, $\frac{\beta}{\delta+a+b\lambda^i}$ and $\frac{\beta}{\delta} - \frac{\beta}{\delta+a+b\lambda^i}$, respectively, which represents the impact of a rise in communication and abatement, respectively, on the integrated discounted cost of capital through an increase in the environmental score, E^i ; (ii) a stochastic term that depends on the strategy of the company through the discounted integral of its future expected

overrating, $(E_s^i - V_s^i)_{s \geq t}$. This second term represents the impact of a rise in communication and abatement, respectively, on the integrated discounted cost of capital through the channel of the misrating penalty. The marginal benefits will be interpreted in more detail in the rest of this section. Whenever we mention the notion of marginal benefits, we will refer to Proposition 3.

As a result, at the optimum, the marginal cost of the overall environmental effort, $\kappa_c^i c_t^{i,*} + \kappa_r^i r_t^{i,*}$, is equal to the marginal benefit of raising both strategies by the same amount.

Corollary 3.1 (Overall environmental effort at optimum). *At the optimum, the environmental strategy $(c^{i,*}, r^{i,*})$ of company i verifies the following equality:*

$$\kappa_r^i r_t^{i,*} + \kappa_c^i c_t^{i,*} = \frac{\beta}{\delta}. \quad (3.8)$$

This optimum equality shows that the marginal benefit of the overall environmental effort of the company, including both abatement and environmental communication, is the expected discounted impact on the cost of capital of increasing the environmental score, E^i : due to the investor's pro-environmental sensitivity β , the discounted integral of the cost of capital decreases by $\int_t^\infty e^{-\delta t} \beta dt = \beta/\delta$. Hence, the investor's penalty on revealed misrating does not influence the overall environmental effort, but only determines the distribution of efforts between abatement and communication.

To facilitate the understanding and interpretation of the optimal strategy (Proposition 2), we start by studying two limiting cases: the case wherein the investor has pro-environmental preferences but does not penalize misrating ($\beta > 0, \alpha = 0$), and the one wherein she penalizes misrating but does not have pro-environmental preferences ($\alpha > 0, \beta = 0$).

3.1 Two limiting cases

3.1.1 Pro-environmental preferences, no misrating penalty ($\beta > 0, \alpha = 0$)

In this subsection, we assume that $\beta > 0$ and $\alpha = 0$. In this limiting case, the abatement and environmental communication serve the sole purpose of optimally increasing the company's environmental score, by balancing the benefit of the reduction in cost of capital enabled by these strategies against their respective financial costs. The optimal distribution of spending between these two types of strategies depends on the degree of asymmetry of information about the fundamental environmental value of the company, which can be characterized by the following notion of "revelation intensity."

Definition 4 (Revelation intensity). We refer to $a + b\lambda^i$ as the "revelation intensity" of the environmental score.

The revelation intensity combines the effort of the rating agency, which pushes the environmental rating towards the fundamental environmental value with speed a , with the discontinuous

effect of controversies, which act as revealing events, where a portion b of misrating is revealed with intensity λ^i . This quantity represents the average speed at which the fundamental environmental value, V^i , translates into the environmental score, E^i (Equation (3.2a)), which is also the speed at which the influence of misleading green communication vanishes from the environmental score, E^i . Its inverse can be interpreted as the degree of asymmetry of information about the company's fundamental environmental value: the lower the revelation intensity, the higher the information asymmetry. Therefore, we will also use the notion of "degree of information asymmetry" to refer to the inverse of the revelation intensity.

In the rest of the paper, we assume that $a + b\lambda^i$ is strictly positive, which means that at least a minimum amount of information about the environmental value is revealed, on average, at each point in time and for each company. The revelation intensity is involved in the optimal distribution between the two types of efforts as follows.

Proposition 4 (Optimal strategies). *When the investor has pro-environmental preferences only, optimal efforts of abatement and environmental communication are constant, and have the following values:*

$$r_t^{i,*} = \frac{1}{\kappa_r^i} \left(\frac{\beta}{\delta} - \frac{\beta}{\delta + a + b\lambda^i} \right), \quad c_t^{i,*} = \frac{1}{\kappa_c^i} \frac{\beta}{\delta + a + b\lambda^i}. \quad (3.9)$$

In the absence of penalty on misrating, the marginal benefits of communication and abatement represent the benefit of increasing the environmental score through a raise in communication or abatement, respectively. This benefit is constant and positive for both strategies, as it does not depend on the stochastic overrating of the company, $E_t^{i,*} - V_t^{i,*}$. The marginal benefits of the two environmental strategies depend directly, and in opposite ways, on the degree of information asymmetry, through the revelation intensity, $a + b\lambda^i$. Indeed, information asymmetry makes environmental communication (abatement) more (less) efficient at raising the environmental score, because, on average, its impact lasts longer (is delayed).

As the following corollary shows, the two greening strategies are perfect substitutes.

Corollary 4.1 (Marginal rate of substitution). *When the investor only has pro-environmental preferences, the marginal rate of substitution between abatement and environmental communication is constant, as follows:*

$$MRS_t^{r \rightarrow c, i} = \frac{a + b\lambda^i}{\delta}.$$

On each date, the company must replace one unity of abatement effort by $\frac{a+b\lambda^i}{\delta}$ unities of environmental communication to keep the same benefit on its expected discounted cost of capital. As this ratio does not depend on the two strategies, c_t^i and r_t^i , both strategies are perfect substitutes: the company chooses indifferently between one or the other strategy, depending on their relative costs and discounted efficiencies at raising its environmental score.

The optimal greenwashing strategy of company i , in this context, is given in the following proposition.

Proposition 5 (Greenwashing effort). *When condition*

$$\frac{\kappa_r^i}{\kappa_c^i} > \frac{a + b\lambda^i}{\delta} \quad (3.10)$$

is satisfied, company i engages in positive communication effort $c_t^{i,} > 0$, which is higher than its abatement effort: $c_t^{i,*} > r_t^{i,*}$. Therefore, except when the company is underrated ($E_t^{i,*} < V_t^{i,*}$) due to measurement error, it always greenwashes. Moreover, its greenwashing effort, $c_t^{i,*} - r_t^{i,*}$, is constant and equal to the positive quantity $G_i^\beta > 0$, with $G_i^\beta = \frac{2}{\kappa^i} \frac{\beta}{\delta + a + b\lambda^i} - \frac{\beta}{\delta \kappa_r^i}$.*

When condition (3.10) is not satisfied, company i never greenwashes.

Greenwashing practices of company i depend on the “ON-OFF” condition (3.10), which compares the ratio of marginal benefits of the two strategies, $(a + b\lambda^i)/\delta$, with their relative marginal unit costs, κ_r^i/κ_c^i : when it is sufficiently cheap to engage in environmental communication relative to abatement, when the asymmetry of information is sufficiently strong, or when the company’s rate of time preference is high enough, the company greenwashes. Otherwise, it never engages in greenwashing. When condition (3.10) is satisfied, the amount of greenwashing effort, G_i^β , is constant, and it (i) increases linearly in the investor’s green sensitivity, β , (ii) increases with the degree of information asymmetry (decreases with the revelation intensity) and the marginal unit cost of abatement κ_r^i , and (iii) decreases with the marginal unit cost of communication κ_c^i .

We are now able to determine the impact of company i ’s greenwashing effort, which we define as follows.

Definition 5 (Greenwashing impact). The impact of greenwashing is the asymptotic value of the expected spread between the environmental score, E^i , and the environmental value, V^i . Formally, it writes as follows:

$$\lim_{t \rightarrow \infty} \mathbb{E}[E_t^i - V_t^i].$$

The expectation is taken to average out the measurement error. In addition, we consider the asymptotic value because this expected spread tends very quickly towards a limit value with a simple and informative expression, as illustrated in the next Proposition.

Proposition 6 (Greenwashing impact). *When condition (3.10) is satisfied, the impact of company i ’s optimal greenwashing strategy is equal to*

$$\lim_{t \rightarrow \infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}] = \frac{1}{a + b\lambda^i} G_i^\beta > 0,$$

where the convergence takes place with an exponential rate.

Company i ’s greenwashing strategy, therefore, induces a positive bias in its environmental score, which becomes, on average, higher than its environmental value. This bias increases

linearly in the greenwashing effort, G_i^β , and hence, increases linearly in the investor's green sensitivity, β .

Therefore, when condition (3.10) is satisfied, a higher marginal unit cost of abatement, degree of information asymmetry, or rate of time preference leads to an increase in greenwashing effort and impact, while abatement decreases. As for the investor's pro-environmental sensitivity, β , its increase leads to an increase in both greenwashing and abatement efforts at the same rate: their ratio remains constant (Propositions 4 and 5).

3.1.2 Misrating penalty, no pro-environmental preferences ($\alpha > 0, \beta = 0$)

In this subsection, we assume that $\beta = 0$ and $\alpha > 0$. In this limiting case, abatement and environmental communication of company i are solely directed towards increasing the accuracy of its environmental score, that is, to bring it closer to its environmental value.

Proposition 7 (Optimal strategies). *When the representative investor does not have pro-environmental preferences ($\beta = 0$), but penalizes misrating ($\alpha > 0$), the optimal abatement and communication efforts are as follows:*

$$r_t^{i,*} = \frac{A^i}{\kappa_r^i} (E_t^{i,*} - V_t^{i,*}), \quad c_t^{i,*} = -\frac{A^i}{\kappa_c^i} (E_t^{i,*} - V_t^{i,*}), \quad (3.11)$$

with $A^i > 0$ given in Proposition 2. In this context,

- (i) When the company is overrated ($E_t^{i,*} - V_t^{i,*} > 0$), it engages in brown communication ($c_t^{i,*} < 0$) and abates ($r_t^{i,*} > 0$).
- (ii) When the company is underrated ($E_t^{i,*} - V_t^{i,*} < 0$), it engages in green communication ($c_t^{i,*} > 0$) and makes brown investment ($r_t^{i,*} < 0$).

The marginal benefits of abatement and environmental communication, $A^i(E_t^{i,*} - V_t^{i,*})$ and $-A^i(E_t^{i,*} - V_t^{i,*})$, respectively, are now stochastic and depend on the company's overrating, $(E_t^{i,*} - V_t^{i,*})$, with the same coefficient but opposite signs. Thus, these strategies work in opposite directions and symmetrically at reducing the discrepancy between the environmental score and the fundamental environmental value of the company. For example, when the environmental score is higher than the environmental value, the company spends on abatement, $r_t^{i,*} > 0$, and brown communication, $-c_t^{i,*} > 0$, until their marginal costs, $\kappa_r^i r_t^{i,*}$ and $-\kappa_c^i c_t^{i,*}$, respectively, equal their marginal benefit, $A^i(E_t^{i,*} - V_t^{i,*})$. Therefore, the coefficient A^i drives a "corrective force" and represents the expected marginal discounted penalty on the company's cost of capital when the environmental score is one unit above the environmental value.

The next corollary shows that, in this limiting case also, the two types of strategies are perfect substitutes.

Corollary 7.1 (Marginal rate of substitution). *When there is the investor penalty only ($\beta = 0$), the marginal rate of substitution between abatement and environmental communication is*

constant, as follows:

$$MRS_t^{r \rightarrow c, i} = -1.$$

On each date, the company must replace one unity of abatement effort by diminishing its environmental communication by one unity to keep the same benefit. Indeed, both strategies, while playing in opposite directions, have the same efficiency at increasing the accuracy of the environmental score: environmental communication drives the environmental score in one direction, and abatement drives the company's environmental value in the other, with the same impact on the reduction of the spread between the environmental score and the environmental value. As the marginal rate of substitution between the two strategies is constant, both strategies are, again, perfect substitutes. Hence, their relative use by the company depends only on their relative marginal costs.

The next proposition characterizes the optimal greenwashing policy in this second limiting case.

Proposition 8 (Greenwashing). *When the investor does not have pro-environmental preferences ($\beta = 0$), the companies never engage in greenwashing. Therefore, their ratings are, on average, accurate: for every company i , $\lim_{t \rightarrow \infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}] = 0$, where the convergence takes place with an exponential rate.*

Since the investor does not have pro-environmental preferences, the companies have no benefit in increasing their environmental scores beyond their fundamental environmental values.¹⁶ Thus, greenwashing is suboptimal in such a case. As the measurement error in the environmental score is centered, and because there is no greenwashing effort, the expected environmental score quickly converges toward the expected environmental value.

3.2 General case: pro-environmental preferences and misrating penalty

Let us now explain the mechanisms at play in the general case (Proposition 2). From now on, we assume that $\beta > 0$ and $\alpha > 0$. In particular, this means that $B^i > 0$ and $A^i > 0$.¹⁷

3.2.1 Optimal communication and abatement

When the representative investor has pro-environmental preferences and penalizes revealed misrating, emissions abatement and environmental communication of company i jointly serve the purpose of increasing its environmental score without decoupling it too much from its fundamental environmental value. Indeed, on the one hand, increasing its environmental score allows the company to benefit from a lower cost of capital due to the investor's pro-environmental preferences. However, on the other hand, the company strives to keep its environmental score

16. See Equation (3.8), wherein the right-hand side is zero.

17. In Equations (3.7), P^i, T^i are strictly positive when $\beta, \alpha > 0$. Thus, B^i, A^i are strictly positive.

relatively close to its fundamental environmental value to minimize its exposure to the investor penalty.

Specifically, referring to Proposition 2, the abatement and communication efforts are driven by two forces: (i) an “incentive force,” which is positive and increases with the investor’s pro-environmental sensitivity, β , as in the first limiting case, and (ii) a “corrective force,” which aims at limiting the level of misrating in response to the investor’s penalty on misrating with intensity α , as in the second limiting case. More precisely, both abatement and communication efforts (i) have positive constant parts, $\frac{1}{\kappa_r^i}(\frac{\beta}{\delta} - B^i)$ and $\frac{B^i}{\kappa_c^i}$, respectively, and (ii) depend linearly on overrating, $E_t^{i,*} - V_t^{i,*}$, in opposite directions, with coefficient A^i normalized by their marginal unit cost: abatement increases with overrating, and communication decreases with this quantity, which makes them negatively correlated, as in the second limiting case. However, the optimal strategy in the general case is not a simple addition of the strategies described in the two limiting cases. While the misrating adjustment parameter A^i is unchanged, the constant B^i is different from the constant in the first limiting case ($\beta / (\delta + a + b\lambda^i)$): it now takes into account the corrective force through additional terms depending on A^i .

The combination of the investor’s pro-environmental preferences and misrating penalty induces a complementarity between the two environmental strategies, as illustrated in Figure 3.3 and based on the calibration detailed in Appendix C: the average environmental communication now decreases with the marginal unit cost of abatement, whereas in the two limiting cases it increased in this parameter due to the substitutability of the two strategies (Corollaries 4.1 and 7.1). Indeed, the drop in abatement, due to the increase in the marginal unit cost of abatement, is coupled with a drop in environmental communication, as the company is exposed to a penalty on the difference between its environmental score and its environmental value.

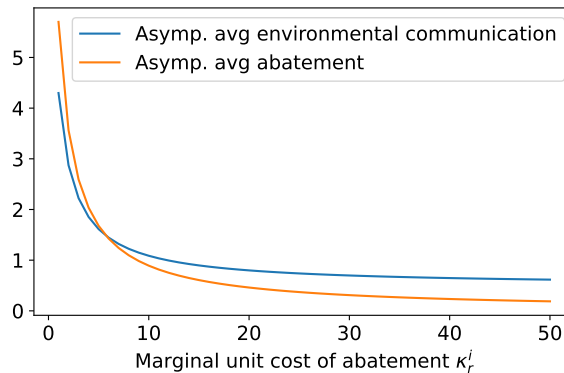


Figure 3.3 – **Average environmental communication and abatement as a function of κ_r^i .** This figure illustrates the asymptote of the expected optimal environmental communication, $\lim_{t \rightarrow \infty} \mathbb{E}[c_t^{i,*}]$, and abatement, $\lim_{t \rightarrow \infty} \mathbb{E}[r_t^{i,*}]$, as a function of the marginal unit cost of abatement, κ_r^i . The calibration is given in Appendix C.

3.2.2 Optimal greenwashing strategy and investor impact

The previous analysis of optimal environmental communication allows us to understand how and when the company practices greenwashing. In this subsection, we characterize the condition under which companies greenwash, the optimal effort of greenwashing, and the impact of greenwashing on the environmental score. This allows us to identify how investors can curb greenwashing.

Proposition 9 (Greenwashing effort). *Let us restate condition (3.10):*

$$\frac{\kappa_r^i}{\kappa_c^i} > \frac{a + b\lambda^i}{\delta}.$$

When this condition is satisfied, company i greenwashes as long as its overrating, $E_t^{i,*} - V_t^{i,*}$, is not too high: specifically, it greenwashes if, and only if, $0 \leq E_t^{i,*} - V_t^{i,*} < \frac{1}{\frac{2}{\bar{\kappa}^i} A^i} G_{max}^i$, where $G_{max}^i = \frac{2}{\bar{\kappa}^i} B^i - \frac{\beta}{\delta \bar{\kappa}_r^i}$. Its greenwashing effort, $c_t^{i,*} - r_t^{i,*}$, is maximal, equal to the positive quantity G_{max}^i , when its score is equal to the fundamental environmental value, $E_t^{i,*} = V_t^{i,*}$, and decreases linearly in the overrating, $E_t^{i,*} - V_t^{i,*}$, with slope $-\frac{2}{\bar{\kappa}^i} A^i$, reaching 0 when the overrating equals $\frac{1}{\frac{2}{\bar{\kappa}^i} A^i} G_{max}^i$.

When condition (3.10) is not satisfied, company i never greenwashes.

This proposition can be interpreted in several steps. First, when the representative investor both has pro-environmental preferences and penalizes revealed environmental misrating, company i greenwashes under the same ‘‘ON-OFF’’ condition (3.10) as when no penalty on misrating is applied (Proposition 5). Indeed, the decision to greenwash does not depend on the investor’s penalty, but solely on a condition guaranteeing that it is more beneficial to communicate than to abate to raise the environmental score, even when the company is overrated and misrating is penalized. However, the amount of greenwashing effort depends on the investor’s misrating penalty.

The greenwashing effort decreases linearly with the company’s overrating, $E_t^{i,*} - V_t^{i,*}$, through the parameter A^i , which represents the ‘‘corrective force’’ due to the penalty. Therefore, the occurrence of an environmental controversy revealing a portion of the company’s overrating triggers both a drop in its environmental score and an increase in its greenwashing effort. This effect echoes the empirical findings of Duchin et al. (2023), providing evidence for greenwashing following an ‘‘environmental risk incident.’’ A related consequence is that company i greenwashes the most when its environmental score correctly reflects the environmental value, that is, when $E_t^{i,*} = V_t^{i,*}$.

Company i no longer greenwashes once the level of overrating, $E_t^{i,*} - V_t^{i,*}$, exceeds the greenwashing threshold, $\frac{1}{\frac{2}{\bar{\kappa}^i} A^i} G_{max}^i$. When company i ’s overrating is above this threshold, the company allows its overrating to decrease (i) through the action of the rating agency and (ii) by communicating less than abating.

As a result of this greenwashing strategy, the overrating of company i quickly converges, on average, towards a positive quantity that is related to its greenwashing threshold.

Proposition 10 (Greenwashing impact). *When condition (3.10) is satisfied, the impact of company i 's greenwashing strategy can be measured as:*

$$\lim_{t \rightarrow \infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}] = \frac{1}{\frac{2}{\kappa^i} A^i + a + b\lambda^i} G_{max}^i,$$

where the convergence takes place with an exponential rate.

Consistent with the first limiting case, when condition (3.10) is satisfied, the optimal greenwashing strategy induces a positive bias on the environmental score of company i .

As shown in the proposition below, the investor can have an impact on corporate greenwashing.

Proposition 11 (Investor's impact on greenwashing). *When condition (3.10) is satisfied, the maximal greenwashing effort, G_{max}^i , increases linearly in β and exhibits a convex decrease in α .*

The effort and impact of greenwashing both increase in the pro-environmental preferences of the investor through G_{max}^i , as these preferences spur companies to display a higher environmental score. However, the investor has the ability to curb greenwashing effort and impact: by increasing her sensitivity to misrating, α , the investor lowers companies' maximal greenwashing efforts, their greenwashing thresholds, and the impact of their greenwashing strategies, which all depend on $(G_{max}^i, 1 \leq i \leq n)$. This translates into a lower average greenwashing effort, as illustrated in Figure 3.4a.¹⁸

In the proposition below, we show how the misrating penalty also affects the optimal abatement effort of companies.

Proposition 12 (Investor's impact on abatement). *The abatement effort that is not driven by the correction of the misrating, $\frac{1}{\kappa_+^i} \left(\frac{\beta}{\delta} - B^i \right)$, increases linearly in β , and, when condition (3.10) is satisfied, exhibits a concave increase in α .*

This proposition highlights the positive impact of investors' penalties for environmental misrating on companies' abatement strategies. In addition to the investor's pro-environmental preferences, which increase abatement efforts, penalizing environmental misrating not only reduces greenwashing but also further increases abatement. In particular, even a small misrating penalty appears to have a significant effect on the abatement effort (Figure 3.4b). This result adds to the emerging literature on impact investing (Landier and Lovo, 2020b; Green and Roth, 2021b; Pástor et al., 2022; De Angelis et al., 2023; Oehmke and Opp, 2023) by identifying an effective vector available to investors to encourage companies to mitigate their environmental footprints.

18. As the greenwashing and abatement efforts are linear deterministic functions of the overrating, $E_t^{i,*} - V_t^{i,*}$,

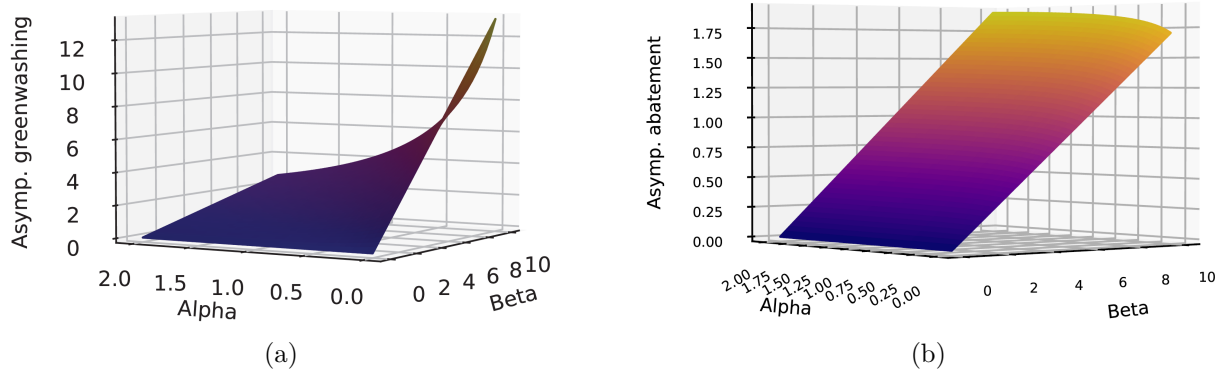


Figure 3.4 – **Average greenwashing and abatement as a function of α and β .** This figure displays the asymptotic expected optimal greenwashing ($\lim_{t \rightarrow \infty} \mathbb{E}[c_t^* - r_t^*]$; figure a) and abatement ($\lim_{t \rightarrow \infty} \mathbb{E}[r_t^*]$; figure b) efforts as a function of the pro-environmental sensitivity, β , and the misrating penalty, α . The calibration is given in Appendix C.

3.3 Complementary tools to curb greenwashing

Our model allows us to identify policy tools that could, as a complement to investor action, contribute to curbing greenwashing, namely (i) increasing transparency and (ii) fostering technological innovation in emission reduction technologies. As it is not possible to carry out an analytical analysis of these tools, their effects are illustrated through numerical sensitivity analyses.

3.3.1 Regulations increasing transparency

In this subsection, we investigate to what extent policies playing on the transparency parameters can be alternative or complementary tools to the penalization of misrating by investors.

When the investor does not penalize the observed misrating, increasing the revelation intensity can strongly deter companies from engaging in greenwashing. Analytically, this effect can be seen in the first limiting case (Proposition 4, equation (3.9)). Figure 3.5 illustrates the effect of moving each of the transparency parameters (a , b , λ^i) separately on greenwashing efforts and impacts, using the baseline calibration (Appendix C). Increasing the power of the rating agency in recovering the true environmental information through parameter a decreases substantially the greenwashing effort. In addition, increasing any transparency parameter amplifies the mitigation of the greenwashing impact; indeed, a higher revelation intensity does not only deter greenwashing practices, but also makes its effect on the environmental score less durable.

However, when the investor sufficiently penalizes the observed misrating, the action of the rating agency is not an efficient complementary tool: increasing a does not significantly reduce greenwashing efforts and impacts (Figures 3.6a and 3.6d). Conversely, the revelation of controversies is a strong complementary tool to the investor penalty of misrating: a minimum level of

their expectations also converge at an exponential rate toward their asymptotic values (see Proposition 10). This justifies the use of these asymptotic values to analyse average greenwashing and abatement efforts.

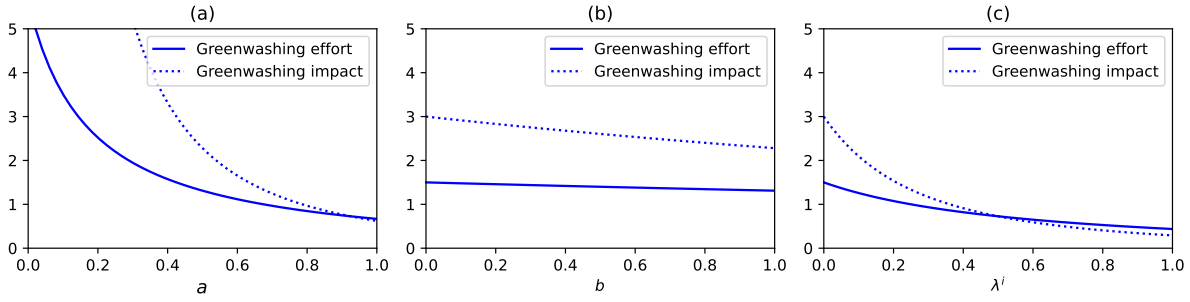


Figure 3.5 – **Greenwashing effort and impact and transparency parameters** when $\alpha = 0$. This figure displays the greenwashing effort, G_t^β , (solid lines), and greenwashing impact, $\lim_{t \rightarrow \infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}]$, (dotted lines), as a function of transparency parameters a, b, λ^i , when the investor’s penalty, α , is null. The reference calibration is given in Appendix C.

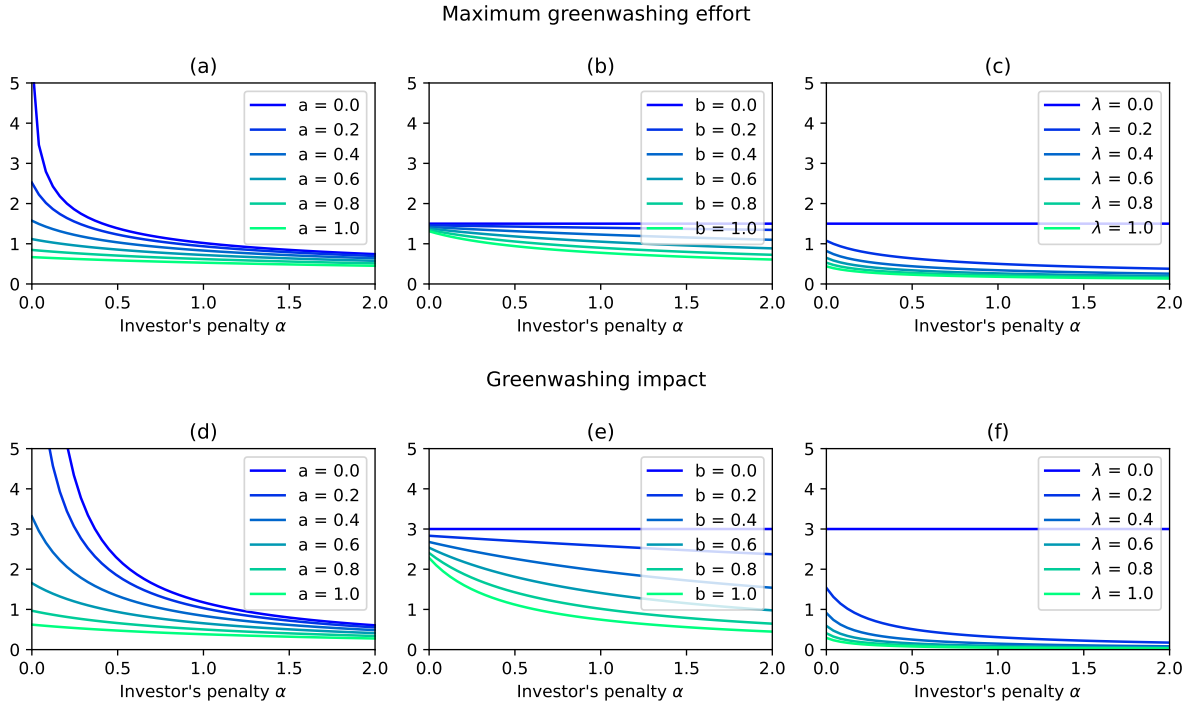


Figure 3.6 – **Greenwashing effort and impact, penalty α and transparency parameters**. This figure displays the maximum greenwashing effort, G_{max}^i , (first row), and greenwashing impact, $\lim_{t \rightarrow \infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}]$, (second row), as a function of the investor’s penalty, α , for different values of transparency parameters a, b, λ^i . The reference calibration is given in Appendix C.

intensity (λ^i) and amplitude (b) of revelation is necessary to channel the effect of the investor penalty; the mitigating effect of the penalty on greenwashing efforts and impacts significantly increases with λ^i and b (Figures 3.6b, 3.6c, 3.6e, 3.6f). Therefore, increasing the investigating power of stakeholders (hence, contributing to increasing λ^i), or triggering an in-depth re-assessment of the company’s environmental footprint once some overrating is suspected (hence, contributing to increasing b) would complement and increase the impact of investor action by

raising the pressure on companies to reduce their greenwashing practices.

3.3.2 Green technological change

Can green technological change help curb greenwashing? Figure 3.7 suggests that the marginal unit cost of abatement needs to decrease substantially before its impact on greenwashing practices becomes significant. Indeed, companies no longer practice greenwashing when the relative marginal unit cost of abatement versus communication is sufficiently low, that is when the inequality (3.10) is no longer satisfied (“ON-OFF” greenwashing condition): in the central calibration, when this ratio drops to 5.7. This result shows that maintaining a sustained and pronounced research and development effort to bring down the marginal costs of new green technologies (Popp et al., 2013) would, in addition to increasing abatement (Figure 3.3), simultaneously help curb corporate greenwashing practices.

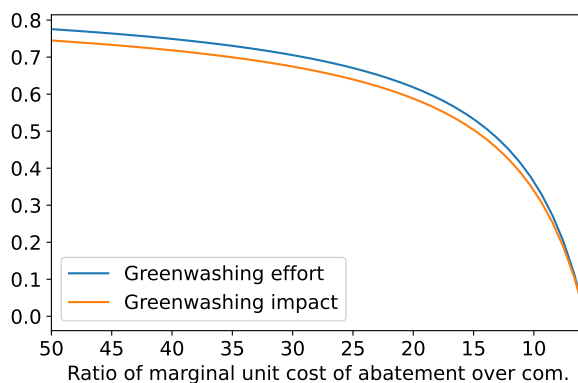


Figure 3.7 – **Greenwashing and technological change.** Maximum greenwashing effort, G_{max}^i , and impact, $\lim_{t \rightarrow \infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}]$, in function of the ratio of marginal unit costs of abatement and communication κ_r^i / κ_c^i . Consistently with Proposition 9, greenwashing is zero when the threshold represented by equation (3.10) is hit.

4 Introducing interaction between companies

Instead of caring about the absolute environmental value of each company, the investor may prefer to tilt her portfolio, at each time, towards the greenest companies in the investment universe. In this section, we present an extension of the investor’s program presented in Section 2, in which the environmental score of each company is scaled by the average environmental score of companies in the investor’s program. Through an adjustment of equilibrium expected returns, this change introduces an interaction between the companies’ objectives leading to an n -player game.

The n -player game The investor’s extended program is set as follows:

$$\sup_{\omega \in \mathbb{A}^\omega} \mathbb{E} \left[\int_0^\infty e^{-rt} \left\{ \omega'_t dS_t - \frac{\gamma}{2} \langle \omega' dS \rangle_t + \omega'_t \left(\beta \frac{E_t}{h(\frac{1}{n} \sum_j E_t^j)} - \alpha M_t \right) dt \right\} \right],$$

with h a regular function bounded from below by a strictly positive constant and approximating the identity function on \mathbb{R}_+ . This new specification is realistic as (i) rating agencies regularly rescale the environmental scores¹⁹ and (ii) ESG investors often follow a “best-in-class” investment strategy, usually at the sector level. Notice that, when h is a constant function equal to one, this program boils down to the one in Section 2.

Similarly to the initial problem, equilibrium expected returns are easily deduced from this new program. They are expressed as follows:²⁰

$$\mu_t = \gamma \Sigma \mathbf{1}_n - \beta \frac{E_t}{h(\frac{1}{n} \sum_j E_t^j)} + \alpha M_t. \quad (3.12)$$

Plugging these new equilibrium expected returns in company i ’s program gives the following:

$$\inf_{(r^i, c^i) \in \mathbb{A}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\gamma \Sigma \mathbf{1}_n - \beta \frac{E_t^i}{h(\frac{1}{n} \sum_j E_t^j)} + \alpha M_t^i + \frac{\kappa_r}{2} (r_t^i)^2 + \frac{\kappa_c}{2} (c_t^i)^2 \right) dt \right].$$

Companies’ programs are now interacting through the average environmental score of companies. Moreover, they are no longer linear quadratic: each company controls both the numerator and the denominator in the term involving its environmental score, $E_t^i / h(\frac{1}{n} \sum_j E_t^j)$. Therefore, to approximate the Nash equilibrium of this n -player game with interpretable quantities, we formulate and solve the mean field limit of this game, in other words, the limit obtained by making the number of companies n tend to infinity. Indeed, at the mean field limit, a generic company has a negligible impact on the average environmental score in the investment universe. Hence, its objective becomes a linear quadratic program, in which the average environmental score is a time-dependent deterministic parameter.

A Greenwashing mean field game In order to define a mean field game (MFG) which approximates the Greenwashing n -player game, we need to make two additional assumptions. (i) Companies are homogeneous: all parameters are the same for each company. (ii) Their environmental scores are driven by idiosyncratic noises: $(W^i, N^i)_i$ are assumed to be independent and identically distributed. Hence, at the mean field limit ($n \rightarrow \infty$), we work with a representative

19. For example, MSCI ESG ratings are industry-adjusted according to an industry benchmark, which is revised at least once a year (<https://www.msci.com/documents/1296102/34424357/MSCI+ESG+Ratings+Methodology.pdf>). Moreover, Refinitiv LSEG ESG scores are the direct result of a cross-sectional comparison between companies’ raw metrics, which are ranked to calculate percentile scores (https://www.lseg.com/content/dam/data-analytics/en_us/documents/methodology/lseg-esg-scores-methodology.pdf).

20. Being very similar to Proposition 1, the Proposition providing the equilibrium expected returns in the extension is reported, with its proof, in Appendix B.4.

company which admits $(E, V, M) \in \mathbb{R}^3$ as a state variable, solution to the following dynamics:

$$\begin{cases} dE_t = a(V_t - E_t)dt + b(V_{t-} - E_{t-})dN_t + c_t dt + z dW_t, & E_0 = \tilde{q}, \\ dV_t = r_t dt, & V_0 = \tilde{p}, \\ dM_t = -\rho M_t dt + b^2(V_{t-} - E_{t-})^2 dN_t, & M_0 = \tilde{u}, \end{cases} \quad (3.13)$$

with W a one-dimensional brownian motion, N a Poisson process with intensity $\lambda \in \mathbb{R}_+^*$ independent from W , and where $(\tilde{q}, \tilde{p}, \tilde{u})$ is a square integrable random variable valued in $\mathbb{R}^2 \times \mathbb{R}_+$ and independent from the couple (W, N) . From now on, for the n -player game, we keep the exponent i to index companies, while the state variables of the representative company considered at the mean field limit is distinguished by the absence of exponent.²¹ Note that, under the assumptions (i) and (ii), the environmental score, the environmental value, and the misrating score of the representative company and of the n companies in the n -player game follow the same distribution.

At the mean field limit ($n \rightarrow \infty$), by the law of large numbers, we expect the average environmental score of companies, $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_i E_t^i$, to be a deterministic function, $m \in \mathcal{C}^1([0, T], \mathbb{R})$. Hence, the program of the representative company at the mean field limit is equivalent to the following, up to a constant:

$$\sup_{(r,c) \in \mathbb{A}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\beta \frac{E_t}{h(m_t)} - \alpha M_t - \frac{\kappa_r}{2} (r_t)^2 - \frac{\kappa_c}{2} (c_t)^2 \right) dt \right],$$

with (E, V, M) solution of equation (3.13).

This program has an infinite horizon, while it has now a time-dependent parameter, $1/h(m_t)$, which makes it not very well suited for infinite horizon resolution. Hence, we approximate its solution by a finite horizon equivalent, with a horizon $T \in \mathbb{R}_+$ big enough:

$$\sup_{(r,c) \in \mathbb{A}_T} \mathbb{E} \left[\int_0^T e^{-\delta t} \left(\beta \frac{E_t}{h(m_t)} - \alpha M_t^i - \frac{\kappa_r}{2} (r_t)^2 - \frac{\kappa_c}{2} (c_t)^2 \right) dt \right], \quad (3.14)$$

with a new set of admissible strategies, \mathbb{A}_T , which is the set of \mathbb{F} -progressively measurable \mathbb{R}^2 -valued processes which verify $\mathbb{E} \left[\int_0^T |r_t|^2 + |c_t|^2 dt \right] < \infty$. This program, associated with the state variable dynamics of the representative company described in equation (3.13), characterizes the Greenwashing mean field game (MFG) that we solve in this extension. Before solving it, we need to define the notion of solution to a mean field game, which we call a mean field equilibrium (MFE), and which is the equivalent of a Nash equilibrium at the mean field limit.

21. Hence, the absence of exponent no longer identifies an n -dimensional vector, but a one-dimensional variable characterizing the representative company in the mean field version of the Greenwashing game.

Definition 6 (Mean field equilibrium of the Greenwashing MFG). Consider the functional

$$J(r, c, m) := \mathbb{E} \left[\int_0^T e^{-\delta t} \left(\beta \frac{E_t}{h(m_t)} - \alpha M_t - \frac{\kappa_r}{2} (r_t)^2 - \frac{\kappa_c}{2} (c_t)^2 \right) dt \right], \quad (3.15)$$

defined for any admissible strategy $(r, c) \in \mathbb{A}_T$ and deterministic function of time $m \in \mathcal{C}^1([0, T], \mathbb{R})$, with (E, V, M) solution to equation (3.13) when the environmental strategy (r, c) is employed. Then, the triplet $(r^*, c^*, m^*) \in \mathbb{A}_T \times \mathcal{C}^1([0, T], \mathbb{R})$ is a mean field equilibrium of the Greenwashing MFG if, and only if,

- (i) $\forall (r, c) \in \mathbb{A}_T, \quad J(r^*, c^*, m^*) \geq J(r, c, m^*),$
- (ii) $\forall t \in [0, T], \quad m_t^* = \mathbb{E}[E_t^*],$

with (E^*, V^*, M^*) solution to equation (3.13) when the strategy (r^*, c^*) is employed.

A mean field equilibrium is so that the representative company adopts an optimal strategy for a given environmental score average, and that this environmental score average represents the average environmental score of companies acting optimally. By definition of the representative company, its expected environmental score represents the environmental score average. Hence, to identify a mean field equilibrium, one first needs to identify the “best response” of the representative company to a given environmental score average, and then to identify the fixed point(s) of a functional which maps to a given trajectory of the environmental score average the expected environmental average of the representative company acting optimally in response to this trajectory.

For a given environmental score average, written $m \in \mathcal{C}^1([0, T], \mathbb{R})$, the optimal communication and abatement strategy of the representative company can be computed as a feedback function of its environmental score and value. This feedback function is parameterized by the environmental score average to which it is responding optimally.

Proposition 13 (Optimal strategy in the Greenwashing MFG). *For a given environmental score average, $m \in \mathcal{C}^1([0, T], \mathbb{R})$, the optimal environmental communication effort, \hat{c} , and abatement effort, \hat{r} , of the representative company are as follows:*

$$\hat{c}_t = \frac{1}{\kappa_c} \left(B(t) + A(t)(\hat{E}_t - \hat{V}_t) \right), \quad \hat{r}_t = \frac{1}{\kappa_r} \left(\beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds - B(t) - A(t)(\hat{E}_t - \hat{V}_t) \right),$$

where

$$B(t) = \beta \int_t^T e^{\int_t^s (\frac{2}{\kappa} A(u) - \delta - a - \lambda b) du} \left(\frac{1}{h(m_s)} - \frac{A(s)}{\kappa_r} \int_s^T \frac{e^{-\delta(u-s)}}{h(m_u)} du \right) ds,$$

and A is the unique solution, negative, to the Riccati equation

$$\begin{aligned} \dot{A}(t) + \frac{2}{\kappa} A(t)^2 - \left(\delta + 2a + \lambda(1 - (1 - b)^2) \right) A(t) + 2\lambda b^2 \left(\frac{\alpha}{\delta + \rho} e^{-(\delta + \rho)(T-t)} - \frac{\alpha}{\delta + \rho} \right) &= 0, \\ A(T) &= 0, \end{aligned} \quad (3.16)$$

and where \hat{E}, \hat{V} are solution to the dynamics (3.13) when the optimal strategy (\hat{r}, \hat{c}) is employed.

In the next proposition, we show that there exists a unique mean field equilibrium to the Greenwashing MFG, when the function h is increasing and admits a strictly positive lower bound.

Proposition 14 (Existence and uniqueness of the MFE). *Assume that the function h is increasing, and that there exists $\eta > 0$ so that for all $x \in \mathbb{R}$, $h(x) \geq \frac{1}{\eta}$. Then, there exists a unique mean field equilibrium to the Greenwashing mean field game.*

Proof of Proposition 14. This proof is conducted in three steps. In (i), we specify a functional, $\Psi : \mathcal{C}^1([0, T], \mathbb{R}) \mapsto \mathcal{C}^1([0, T], \mathbb{R})$, of which the fixed point(s) characterize the MFE of the Greenwashing MFG. In (ii), we show that this functional admits at least one fixed point, which means that this MFG admits at least one MFE. In (iii), we show that, if the greenwashing MFG admits a MFE, it must be unique, using the Lasry-Lions monotonicity condition. Together, (ii) and (iii) prove the result stated in this Proposition.

(i) Let us define the following map:

$$\Psi : \mathcal{C}^1([0, T], \mathbb{R}) \ni m \mapsto (\Psi_t(m))_{0 \leq t \leq T} \in \mathcal{C}^1([0, T], \mathbb{R}),$$

with, for all $t \in [0, T]$,

$$\Psi_t(m) := g_t(m) + p + \frac{1}{\kappa_r} \int_0^t [\beta f_s(m) - B_s(m) - A_s g_s(m)] ds, \quad (3.17)$$

and for the functions $f_t, B_t, g_t : \mathcal{C}^1([0, T], \mathbb{R}) \rightarrow \mathcal{C}^1([0, T], \mathbb{R})$ defined as follows:

$$f_t(m) := \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds, \quad B_t(m) := \beta \int_t^T e^{-\int_t^s (\zeta_u + \delta) du} \left(\frac{1}{h(m_s)} - \frac{A(s)}{\kappa_r} f_s(m) \right) ds,$$

$$g_t(m) = e^{-\int_0^t \zeta_s ds} x + \int_0^t e^{-\int_s^t \zeta_r dr} \left(\frac{2}{\kappa} B_s(m) - \frac{\beta}{\kappa_r} f_s(m) \right) ds,$$

writing $\zeta_u := -\frac{2}{\kappa} A(u) + a + \lambda b$

Then, let us show that the set of fixed points of Ψ characterize the set of MFE of the Greenwashing mean field game. Assume that there exists $m^* \in \mathcal{C}^1([0, T], \mathbb{R})$ so that $\Psi(m^*) = m^*$. According to Proposition 13, the optimal strategy in response to m^* , written (r^*, c^*) , verifies:

$$c_t^* = \frac{1}{\kappa_c} (B(t) + A(t)(E_t^* - V_t^*)), \quad r_t^* = \frac{1}{\kappa_r} \left(\beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s^*)} ds - B(t) - A(t)(E_t^* - V_t^*) \right), \quad (3.18)$$

where E^*, V^* are solution to the dynamics (3.13) when the optimal strategy (r^*, c^*) is employed. More generally, let us write any state variable with an index $*$ whenever it is driven by the strategy (r^*, c^*) . Let us show that (r^*, c^*, m^*) is a mean field equilibrium. By Proposition 13,

the condition (i) of the definition of a MFE is verified. Writing $X^* := E^* - V^*$ in a similar fashion as in the proof of Proposition 13, we get that for any $t \in [0, T]$, $\mathbb{E}[E_t^*] = \mathbb{E}[X_t^*] + \mathbb{E}[V_t^*]$. According to the proof of Proposition 13, equation (3.51), the explicit solution of X^* verifies the following, for $t \in [0, T]$:

$$X_t^* = \hat{\mathcal{E}}_t x + \hat{\mathcal{E}}_t \int_0^t \hat{\mathcal{E}}_s^{-1} \left\{ \left(\frac{2}{\bar{\kappa}} B(s) - \frac{\beta}{\kappa_r} \int_s^T \frac{e^{-\delta(u-s)}}{h(m_u^*)} du \right) ds + z dW_s \right\},$$

with $\hat{\mathcal{E}}_t = e^{\int_0^t (\frac{2}{\bar{\kappa}} A(s) - a) ds} (1 - b)^{Nt}$, writing $0^0 = 1$. Hence,

$$\mathbb{E}[X_s^*] = e^{-\int_0^s \zeta_u du} x + \int_0^s e^{-\int_u^s \zeta_r dr} \left(\frac{2}{\bar{\kappa}} B(u) - \frac{\beta}{\kappa_r} \int_u^T \frac{e^{-\delta(r-u)}}{h(m_r^*)} dr \right) du.$$

Moreover, the explicit expression of V^* is the following:

$$V_t^* = p + \int_0^t r_t^* dt = p + \frac{1}{\kappa_r} \int_0^t \left(\beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s^*)} ds - B(t) - A(t) X_t^* \right) dt.$$

Hence,

$$\mathbb{E}[V_t^*] = p + \frac{1}{\kappa_r} \int_0^t \left(\beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s^*)} ds - B(t) - A(t) \mathbb{E}[X_t^*] \right) dt.$$

As a result, we have, by assumption,

$$\mathbb{E}[E_t^*] = g_t(m^*) + p + \frac{1}{\kappa_r} \int_0^t [\beta f_s(m^*) - B_s(m^*) - A_s g_s(m^*)] ds = \Psi_t(m^*) = m_t^*.$$

Hence, condition (ii) of Definition 6 is verified as well. This means that (r^*, c^*, m^*) is a mean field equilibrium. Moreover, by construction of Ψ , for any mean field equilibrium (r^*, c^*, m^*) , m^* belongs to its set of fixed points.

(ii) To show that Ψ admits at least one fixed point, we apply Schauder fixed point theorem, restated in Appendix D for the sake of completeness. Let $K := \mathcal{C}^1([0, T], \mathbb{R})$, normed by $\|\cdot\| : m \in K \mapsto \int_0^T |m_t| dt$. K is a nonempty convex closed subset of a Hausdorff topological vector space, from the properties of real valued continuous functions defined on a compact set. Moreover, Ψ is continuous as it is a linear combination of continuous functions. To show that $\Psi(K)$ is included in a compact subset of K , we use Arzelà-Ascoli theorem, also restated in Appendix D. To be able to apply it to our setting, let us show that the set $\Psi(K)$ is (a) uniformly bounded, (b) uniformly equicontinuous.

(a) Let $m \in K$, $t \in [0, T]$. Then, for all $t \in [0, T]$,

$$|\Psi_t(m)| \leq |g_t(m)| + |p| + \frac{1}{\kappa_r} \int_0^t (\beta |f_s(m)| + |B_s(m)| + |A_s| |g_s(m)|) ds.$$

Now, as $\frac{1}{h(x)} \leq \eta$, $\forall x \in \mathbb{R}$, we have $|f_t(m)| = \int_t^T \frac{1}{h(m_u)} du \leq T\eta$.

Using this inequality and similar arguments, we get

$$\begin{aligned} |B_t(m)| &\leq \beta \int_t^T e^{-\int_t^s \zeta_u du} \left(\frac{1}{h(m_s)} + \frac{|A(s)|}{\kappa_r} |f_s(m)| \right) ds \leq \beta T \eta \left(1 + \frac{1}{\kappa_r} \int_0^T |A(s)| ds \right), \\ |g_t(m)| &\leq e^{-\int_0^t \zeta_u du} |x| + \int_0^t e^{-\int_u^t \zeta_r dr} \left(\frac{2}{\bar{\kappa}} |B_u(m)| + \frac{\beta}{\kappa_r} |f_u(m)| \right) du \\ &\leq |x| + \frac{2}{\bar{\kappa}} \beta T^2 \eta \left(1 + \frac{1}{\kappa_r} \int_0^T |A(s)| ds \right) + \frac{\beta}{\kappa_r} T^2 \eta. \end{aligned}$$

Hence,

$$\int_0^t |A_s| |g_s(m)| ds \leq \beta T^2 \eta \int_0^T |A_s| ds \left(|x| + \frac{2}{\bar{\kappa}} \left(1 + \frac{1}{\kappa_r} \int_0^T |A(s)| ds \right) + \frac{1}{\kappa_r} \right).$$

Summing all these upper boundaries which do not depend on t nor on m , we get an upper boundary for $|\Psi_t(m)|$ which does not depend on t nor on m . Therefore, $\Psi(K)$ is uniformly bounded.

(b) Let us show that $\Psi(K)$ is equicontinuous, i.e. that

$$\forall \epsilon > 0, \exists \delta > 0 : \forall m \in K, \forall (t_1, t_2) \in [0, T]^2, (|t_1 - t_2| \leq \delta \Rightarrow |\Psi_{t_1}(m) - \Psi_{t_2}(m)| \leq \epsilon).$$

Let $m \in K, (t_1, t_2) \in [0, T]^2$. We have

$$|\Psi_{t_1}(m) - \Psi_{t_2}(m)| \leq |g_{t_1}(m) - g_{t_2}(m)| + \frac{1}{\kappa_r} \int_{t_2}^{t_1} [\beta |f_s(m)| + |B_s(m)| + |A_s| |g_s(m)|] ds,$$

with

$$|g_{t_1}(m) - g_{t_2}(m)| \leq (1 - e^{-\int_0^T \zeta_u du}) |t_1 - t_2| |x| + \int_{t_2}^{t_1} e^{-\int_u^s \zeta_r dr} \left(\frac{2}{\bar{\kappa}} B_u(m) - \frac{\beta}{\kappa_r} f_u(m) \right) du.$$

Hence, using the boundaries established in (a), if we define the constant C as follows,

$$\begin{aligned} C := \max &\left((1 - e^{-\int_0^T \zeta_u du}) |x|, \frac{2}{\bar{\kappa}} \beta T \eta \left(1 + \frac{1}{\kappa_r} \int_0^T |A(s)| ds \right) + \frac{\beta}{\kappa_r} T \eta, \right. \\ &\left. \beta T \eta \left(1 + \frac{1}{\kappa_r} \int_0^T |A(s)| ds \right), \sup_{0 \leq s \leq T} |A_s| \left(|x| + \frac{2}{\bar{\kappa}} \beta T^2 \eta \left(1 + \frac{1}{\kappa_r} \int_0^T |A(s)| ds \right) + \frac{\beta}{\kappa_r} T^2 \eta \right) \right), \end{aligned}$$

we have

$$|\Psi_{t_1}(m) - \Psi_{t_2}(m)| \leq C |t_1 - t_2|.$$

As C does not depend on m nor on t_1, t_2 , $\Psi(K)$ is uniformly equicontinuous.

Hence, by Arzela-Ascoli theorem, we can conclude that $\Psi(K)$ is a compact subset of K . Thus, Shauder fixed point theorem can be applied, and proves that the set of fixed points of the mapping Ψ is non-empty. This means that the Greenwashing MFG admits at least one mean

field equilibrium according to (i).

(iii) To show that this MFE is unique, we only need the objective functional, thanks to Lasry-Lions monotonicity condition. Suppose (r^1, c^1, m^1) and (r^2, c^2, m^2) are two mean field equilibria, and suppose they are distinct. Solutions of equation (3.13) are noted (E^1, V^1, M^1) and (E^2, V^2, M^2) when the strategies (r^1, c^1) and (r^2, c^2) are employed respectively. Note that the two associated optimal controls, (r^1, c^1) and (r^2, c^2) , must be distinct, as otherwise we would have $m_t^1 = m_t^2$ for each t , according to condition (ii) of Definition 6. Now, because (r^1, c^1) is optimal when m^1 describes the population flow, it certainly outperforms (r^2, c^2) , and we have

$$\mathbb{E} \left[\int_0^T \left(\beta \frac{E_t^1}{h(m_t^1)} - \alpha M_t^1 - \frac{\kappa_r}{2} (r_t^1)^2 - \frac{\kappa_c}{2} (c_t^1)^2 \right) dt \right] > \mathbb{E} \left[\int_0^T \left(\beta \frac{E_t^2}{h(m_t^1)} - \alpha M_t^2 - \frac{\kappa_r}{2} (r_t^2)^2 - \frac{\kappa_c}{2} (c_t^2)^2 \right) dt \right].$$

Similarly,

$$\mathbb{E} \left[\int_0^T \left(\beta \frac{E_t^2}{h(m_t^2)} - \alpha M_t^2 - \frac{\kappa_r}{2} (r_t^2)^2 - \frac{\kappa_c}{2} (c_t^2)^2 \right) dt \right] > \mathbb{E} \left[\int_0^T \left(\beta \frac{E_t^1}{h(m_t^2)} - \alpha M_t^1 - \frac{\kappa_r}{2} (r_t^1)^2 - \frac{\kappa_c}{2} (c_t^1)^2 \right) dt \right].$$

Adding these two inequalities, and using that $\mathbb{E}[E_t^1] = m_t^1$, $\mathbb{E}[E_t^2] = m_t^2$, we get

$$\beta \int_0^T \left(\frac{m_t^1}{h(m_t^1)} + \frac{m_t^2}{h(m_t^2)} - \frac{m_t^2}{h(m_t^1)} - \frac{m_t^1}{h(m_t^2)} \right) dt > 0.$$

Now, the term inside the integral is equal to

$$\frac{(m_t^1 - m_t^2)(h(m_t^2) - h(m_t^1))}{h(m_t^1)h(m_t^2)},$$

which is non-positive as h is increasing and positive, and negative at least for some Lebesgue-non-negligible set of times t as h is monotone and m^1, m^2 are distinct from one another. Hence, a contradiction is exhibited. This proves uniqueness of the mean field equilibrium. \square

Numerical simulation of the Greenwashing MFE Finding an analytical expression to the mean field equilibrium seems inaccessible, as the representative company's program is non linear-quadratic. However, thanks to Proposition 13, we are able to express an explicit map, Ψ , from which the unique fixed point is equal to the average environmental rating, m^* , in the mean field equilibrium, (r^*, c^*, m^*) . From m^* , the optimal strategy (r^*, c^*) at the MFE can be recovered thanks to Proposition 13. The fixed point of Ψ can be approximated numerically. For this numerical approximation, we use the Fictitious Play algorithm.

Let the best response function, $\hat{\beta} : \mathcal{C}^1([0, T], \mathbb{R}) \rightarrow \mathbb{A}_T$, map the optimal strategy (\hat{r}, \hat{c}) to a given average environmental rating, m , as given in Proposition 13. Moreover, note

$$(E^{(r,c)}, M^{(r,c)}, V^{(r,c)})$$

the solution to equation (3.13) when the strategy $(r, c) \in \mathbb{A}_T$ is employed. Then, the map $\Psi : \mathcal{C}^1([0, T], \mathbb{R}) \rightarrow \mathcal{C}^1([0, T], \mathbb{R})$ is as follows: $\Psi(m) = (\mathbb{E}[E_t^{\hat{\beta}(m)}])_{0 \leq t \leq T}$. Its explicit expression is given in the proof of Proposition 14, equation (3.17).

To approximate the fixed point of Ψ , the Fictitious Play algorithm respects the following iteration rule, for $k \in \mathbb{N}^*$:

$$m_k = \frac{1}{k} \Psi(m_k) + \frac{k-1}{k} m_{k-1}.$$

Perrin et al. (2020) and Dumitrescu et al. (2023) prove the convergence of this algorithm in similar frameworks. In our framework, we can use the notion of “exploitability” to control for the convergence of our algorithm.

Definition 7 (Exploitability). The exploitability ε_k of the Fictitious Play algorithm at iteration $k \in \mathbb{N}^*$ is equal to

$$\varepsilon_k = J(\hat{\beta}(m_{k-1}), m_k) - J(\hat{\beta}(m_k), m_k),$$

with J the objective functional of the Greenwashing MFG to be minimized (3.15).

The exploitability measures potential improvement for the representative agent from the current iteration. Its interest is related to the notion of an ε -Mean Field Equilibrium, which formalizes the notion of approximate MFE.

Definition 8 (ε -Mean Field Equilibrium). An ε -Mean Field Equilibrium, for an $\varepsilon > 0$, is a triplet $(\hat{r}^\varepsilon, \hat{c}^\varepsilon, \hat{m}^\varepsilon) \in \mathbb{A}_T \times \mathcal{C}^1([0, T], \mathbb{R})$ so that for all $(r, c) \in \mathbb{A}_T$,

$$J(r, c, \hat{m}^\varepsilon) \leq J(\hat{r}^\varepsilon, \hat{c}^\varepsilon, \hat{m}^\varepsilon) + \varepsilon.$$

Note that, by definition, a 0-MFE is a MFE. The exploitability allows to characterize approximate MFE, as shown in the next Proposition.

Proposition 15. *Let ε_k be the exploitability at iteration k of the Fictitious play algorithm. Then, $(\hat{\beta}(m_{k-1}), m_k)$ is an ε_k -mean field equilibrium.*

Proof. We have $\varepsilon_k = J(\hat{\beta}(m_{k-1}), m_k) - J(\hat{\beta}(m_k), m_k)$. Hence, for all $(r, c) \in \mathbb{A}_T$, by definition of the best response function $\hat{\beta}$, $\varepsilon_k \geq J(\hat{\beta}(m_{k-1}), m_k) - J(r, c, m_k)$. This means that $(\hat{\beta}(m_{k-1}), m_k)$ is an ε_k -MFE. \square

Simulations The algorithm is implemented on the baseline calibration, except for one parameter. Indeed, to allow comparability with the case without interaction, we change β to 50, so

that companies have the same incentive to increase their environmental score at the initial date, whether or not their scores are normalized. Time horizon is set to 100, as it is enough to reach some stationary pattern between the initial and terminal conditions. The function h is set as follows: for $x \in \mathbb{R}$, $h(x) = \max(1, x)$. Hence, for $x \geq 1$, h is equal to the identity function. The initial values of the environmental score and the environmental value are set high enough so that the probability that the environmental score fall below 1 is negligible: they are set to 50 each, with a measurement error volatility $z = 0.2$. The initial value of the misrating proxy is set at 5. For the simulations, time is discretized with a time step equal to 10^{-3} . The Fictitious Play algorithm is initialized with a constant vector, m_{init} , equal to 50.

In our Fictitious Play algorithm, for which we run 500 iterations, the exploitability converges to zero very quickly (Figure 3.8). Moreover, graphically, after a few iterations, the curves representing m_k, m_{k+j} , $j \geq 0$, merge perfectly. This suggests that we are approaching very efficiently, and very precisely the MFE.

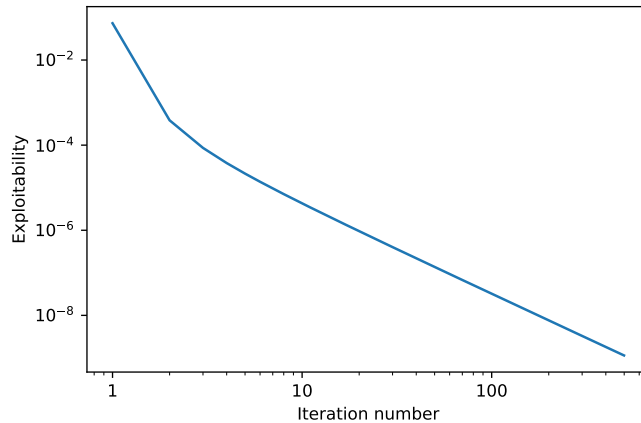


Figure 3.8 – Convergence of the exploitability (log-scaled axes).

To interpret the results, we compare the main quantities at the MFE with the ones in the “benchmark” case where there is no interaction between companies. This benchmark case corresponds to the resolution of the Greenwashing program (3.15) for the representative company when the function h is constant equal to 1 and the pro-environmental sensitivity of the investor, β , is equal to 1. This represents the optimum as in Section 3 but with finite horizon, for these results to be comparable with the MFE.

Results We derive two types of results. First, analytically, we express the optimal strategy of communication and abatement of a generic company. We find that, at the mean field equilibrium, the optimal environmental strategy follows a similar pattern as that in the baseline case (see the Internet Appendix, Proposition 15). In particular, optimal efforts follow the structure of those in Proposition 2 with similar linear coefficients B, A playing the same roles, with the

same signs, but being now time-dependent: under the baseline calibration (Appendix C),²² the normalization of companies' environmental scores leads to positive abatement, communication, and greenwashing efforts, as in the baseline case without normalization (Figure 3.9).²³ These results confirm the robustness of our main qualitative conclusions.

Second, however, the numerical analysis shows that all these efforts are lower and lead to a smaller increase in the environmental scores of companies over time compared to the baseline case (Figure 3.10). Indeed, as the investor only values relative scores, companies have less incentive to push their environmental scores as high as possible: the decrease in their costs of capital only stems from a comparison of their environmental scores to those of their peers. These results suggest that the cross-sectional normalization of firms' environmental scores by rating agencies and the best-in-class approaches to portfolio selection have a detrimental effect on the greening efforts of companies.

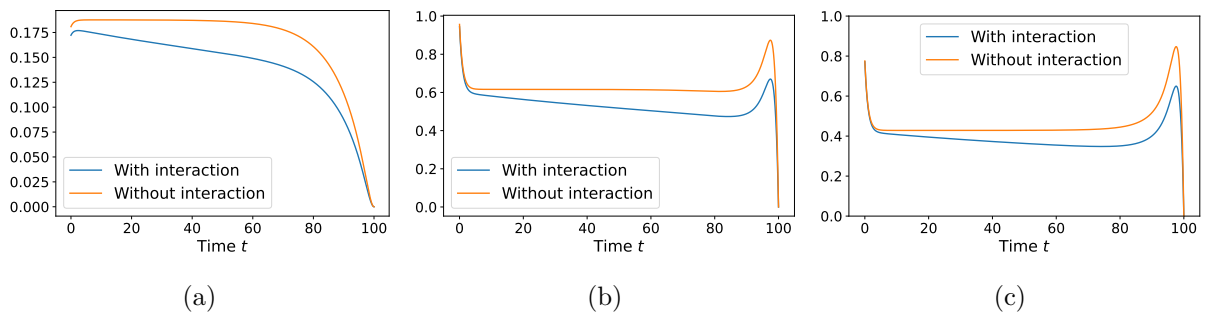


Figure 3.9 – Average abatement (a), communication (b) and greenwashing (c) efforts with and without interaction (*blue* and *orange* curves respectively).

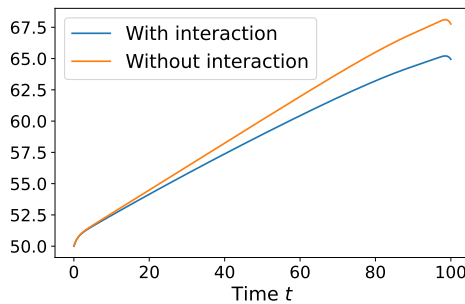


Figure 3.10 – Average environmental score with and without interaction between the companies.

22. We add to the baseline calibration described in Appendix C the initial value of the environmental score and the environmental value of the company, both set to 50. In addition, to allow comparability with the case without interaction, we modify one parameter in the baseline calibration described in Appendix C: β is changed to 50, so that companies have the same incentive to increase their environmental score at the initial date, whether or not their scores are normalized. Finally, time horizon is set to 100, as it is enough to reach some stationary pattern between the initial and terminal conditions.

23. Due to the finite time horizon, the model is not stationary anymore. Thus, peaks arise at the beginning and the end of the time period on each graph, which are due to the impact of the initial and terminal conditions, and are not of interest in the present study.

5 Empirical evidence

In this section, we carry out an empirical analysis of companies' environmental communication flow, c , at a global level to support the results of our model. We focus our empirical study on environmental communication and not on greenwashing directly, as we do not have access to sufficiently robust data on the dynamics of the abatement policies of all the companies studied. We document two main results. First, average environmental communication is positive in almost all the months studied. Second, the empirical dynamics of environmental communication are consistent with those highlighted in the model. We conclude this section by suggesting interpretations regarding corporate greenwashing practices.

5.1 Identification strategy

We develop a two-stage estimation method to analyze the cross-section of companies' environmental communications and provide support for the results of our model.

First step. In the first step, we construct a proxy for the monthly environmental communication flow.

Based on published news, the data provider Covalence constructs a monthly forward-looking environmental reputation score and a monthly environmental controversy score, both between 0 and 100, denoted by Rep_t^i and Con_t^i , respectively, for company $i \in \{1, \dots, n\}$, available at the end of month $t \in \{1, \dots, T\}$. Consistent with the rise in environmental awareness among investors after the Paris Agreement and the empirical facts documented in Section 2.1, our main analysis covers a scope of 3,769 global companies covered by Covalence at a monthly frequency between December 2015 and December 2022, representing 145,508 firm \times month observations. The description of and statistics on all the variables used in the empirical analysis are available in Appendix E (Table 3.11).

Building a proxy for environmental communication involves two challenges. First, as the environmental reputation score is driven by both companies' environmental communication and the controversies that affect them,²⁴ we construct an environmental communication *score* purged of the effect of environmental controversies, which is defined as the orthogonal component of the environmental reputation score to the environmental controversy score through a Within regression, that is, for company i at the end of month t , $\alpha_1^i + \varepsilon_{1,t}^i$ in Equation (3.19) below. Second, given the simultaneity of the reputation and controversy scores, we instrument company i 's environmental controversy score at the end of month t by company i 's environmental controversy

24. More details on the construction method of the forward-looking reputation indicator are available on page 4 of the White Paper by Covalence: https://www.covalence.ch/docs/Covalence_GreenwashingRiskIndicator_WhitePaper.pdf

score at the end of month $t - 1$. More precisely, we estimate the following specification:

$$Rep_t^i = \alpha_1^i + \beta_1 Con_t^{i,*} + \varepsilon_{1,t}^i, \quad (3.19)$$

where $Con_t^{i,*}$ is the prediction of the following regression:

$$Con_t^i = \alpha_2^i + \beta_2 Con_{t-1}^i + \varepsilon_{2,t}^i. \quad (3.20)$$

The instrument Con_t^i verifies the relevance condition: the R^2 of the regression of Con_t^i on Con_{t-1}^i is 76.4%, and the correlation between both variables is 81.3%. In addition, the weak exogeneity condition is satisfied, $\forall i \in \{1, \dots, n\}, \forall (t', t) \in \{1, \dots, T\}^2, t' \geq t, \mathbb{E}(\varepsilon_{1,t'}^i Con_t^{i,*}) = 0$, because $\forall i \in \{1, \dots, n\}, \forall t \in \{1, \dots, T\}, \forall j \in \{1, \dots, t - 1\}, \mathbb{E}(\varepsilon_{1,t}^i Con_{t-j}^i) = 0$. Indeed, the shocks to environmental reputation scores at the end of month t , $\varepsilon_{1,t}^i$, are uncorrelated with controversies that took place during month $t - j$, with $j \in \{1, \dots, t - 1\}$.

The Within estimation under weak exogeneity carries a bias that tends toward zero as the number of periods, T , increases, as shown in the following Lemma. Here, we perform the estimation on a monthly basis and work with $T = 84$ months in the baseline case. In a robustness test, we start the estimation from December 2012, that is, we use $T = 120$ months.

Lemma 5.1. *The bias of the Within estimate under weak exogeneity tends towards zero at a rate faster than or equal to $1/T$.*

By construction, the environmental communication score in month t embeds information on environmental communication from the past months. Since c_t^i is the flow of firm i 's environmental communication during month t , which can be positive, for example through good environmental news and commitments, or negative, due to poor environmental news, we approximate it as the difference in the environmental communication score between the end of month t and the end of month $t - 1$:

$$\hat{c}_t^i \equiv (\hat{\alpha}_1^i + \hat{\varepsilon}_{1,t}^i) - (\hat{\alpha}_1^i + \hat{\varepsilon}_{1,t-1}^i) = \hat{\varepsilon}_{1,t}^i - \hat{\varepsilon}_{1,t-1}^i \quad (3.21)$$

Second step. In the second step, we provide empirical support for the environmental communication dynamics highlighted by the model, focusing on the time derivative of the equilibrium equation (Equation (3.6a)).

Indeed, the fundamental environmental values of companies are unknown and probably correlated with companies' environmental scores. However, it is reasonable to assume that these values are highly inert from one month to the next. Thus, we set

$$\frac{1}{\kappa_c} A^i \Delta V_t^i = \eta_1^i + \eta_{2,t}^i, \quad (3.22)$$

with η_1^i a constant likely to be close to zero and $\eta_{2,t}^i$ an error term, and we focus on the following

general specification based on the first differences of the variables:

$$\Delta \hat{c}_t^i = \alpha_3^i + \tau_{3,t} + \beta_3 \Delta E_t^i + \varepsilon_{3,t}^i, \quad (3.23)$$

where $\Delta \hat{c}_t^i$ is the change in communication flow between month t and month $t + 1$, and ΔE_t^i is the change in environmental score between month t (set at the end of month $t - 1$) and month $t + 1$ (set at the end of month t). To the first difference of the equilibrium equation, we add time fixed effects, $\tau_{3,t}$, to control for unobserved time heterogeneity.

Given the simultaneity between the change in communication flow, $\Delta \hat{c}_t^i$, and the change in environmental score, ΔE_t^i , as the communication flow at date $t - 1$ could influence the environmental score at date t , we instrument the change in environmental score with the environmental score available throughout month $t - 1$ and calculated at the end of month $t - 2$, E_{t-2}^i . Therefore, we estimate a Within regression with robust standard errors based on the following specification:

$$\Delta \hat{c}_t^i = \alpha_3^i + \tau_{3,t} + \beta_3 \Delta E_t^{i,*} + \varepsilon_{3,t}^i, \quad (3.24)$$

where $\Delta E_t^{i,*}$ is the prediction of the following regression:

$$\Delta E_t^i = \alpha_4^i + \tau_{4,t} + \beta_4 E_{t-2}^i + \varepsilon_{4,t}^i. \quad (3.25)$$

So as to draw robust conclusions from the empirical analysis, we carry out the estimations on several samples: the entire universe of companies, as well as the 10%, 20%, ..., 90% of companies with the lowest environmental score within each sector for each month, and the 10%, 20%, ..., 90% of companies with the highest environmental score within each sector for each month. For all these samples, the instrument is relevant and strong (see Tables 3.5 and 3.6 in Appendix E). In addition, the weak exogeneity condition is satisfied, $\forall i \in \{1, \dots, n\}, \forall (t', t) \in \{1, \dots, T\}^2, t' \geq t, \mathbb{E}(\varepsilon_{3,t'}^i \Delta E_t^{i,*}) = 0$, because $\forall i \in \{1, \dots, n\}, \forall t \in \{1, \dots, T\}, \forall j \in \{2, \dots, t - 1\}, \mathbb{E}(\varepsilon_{3,t}^i E_{t-j}^i) = 0$. Indeed, we can reasonably assume that the shocks to the change in communication flow between month t and month $t + 1$, $\varepsilon_{3,t}^i$, are uncorrelated with the environmental scores set at the end of month $t - j$, with $j \in \{2, \dots, t - 1\}$. As the estimation is performed at a monthly frequency over 84 months, with a robustness test over 120 months, the bias of the Within estimate under weak exogeneity is likely to be low (Lemma 5.1).

We perform a battery of complementary estimations, including the addition of monthly systematic risk and return controls, $\beta_{t-1}^{CAPM,i}$ and R_{t-1}^i , respectively, estimated at the end of month $t - 1$ and available throughout month t , in Specification (3.24). More precisely, as a proxy for systematic risk, we use a 12-month rolling CAPM beta, $\beta_t^{CAPM,i} = Var^{-1}(R_t^m) Cov(R_t^i, R_t^m)$, where R^i and R^m denote firm i 's return and the market return, respectively. We also repeat the estimation by starting the analysis period at different dates as well as performing the estimation on several environmental sub-scores.

5.2 Estimations

The regression of the environmental reputation score on the instrumented environmental controversy score (first step, Specification (3.19)) yields a highly significant $\hat{\beta}_1 = 0.04$ (Table 3.4 in Appendix E). This estimation allows us to retrieve the fixed effects, $\hat{\alpha}_1^i$, and the residuals, $\hat{\varepsilon}_{1,t}^i$, the sum of which constitutes a proxy for the environmental communication score.

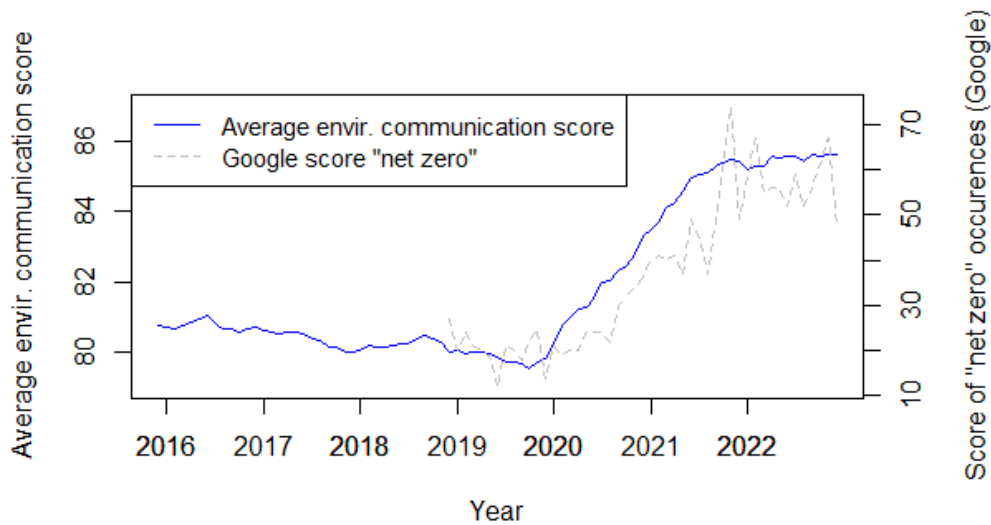
Although we are not directly interested in the absolute level of the environmental communication score proxy—whose average mean and standard deviation for all companies over the period are 81.2 and 4.9, respectively—it is worthwhile validating the reliability of its average trajectory. To do so, we compare the mean of the environmental communication score proxy with the dynamics of Google searches for the term “net zero,” which is associated with institutional commitments to adopt a strategy aligned with the emissions targets defined by the Paris Agreement. As this term was popularized in 2018 by the Intergovernmental Panel on Climate Change Special Report on Global Warming of 1.5 °C (SR15),²⁵ we focus on the Google search dynamics of the term “net zero” from December 2018, in the “Finance” category, on a global scale (Figure 3.11 [a]). The two trajectories are similar: after a period of stagnation until the end of 2019, they both substantially increased from the end of 2019 to the end of 2021. It coincides with the concomitant introduction of several key environmental regulations worldwide, the dynamics of which are well illustrated in the European Union (EU Green Deal, regulations on binding annual emission reductions, circular economy, sustainable finance, as well as EU biodiversity strategy).²⁶ Both trends then slowed down from the beginning of 2022, which coincided with the outbreak of the war in Ukraine when positive environmental communication from companies was a lower priority, especially due to the energy crisis.

The empirical analysis allows us to document two main results. First, computing the proxy for the monthly environmental communication flow, c , shows that 98.8% of the average environmental communication over the period is positive. On average, companies engage almost structurally in *green communication* as defined in the theoretical section (Figure 3.11 [b]).

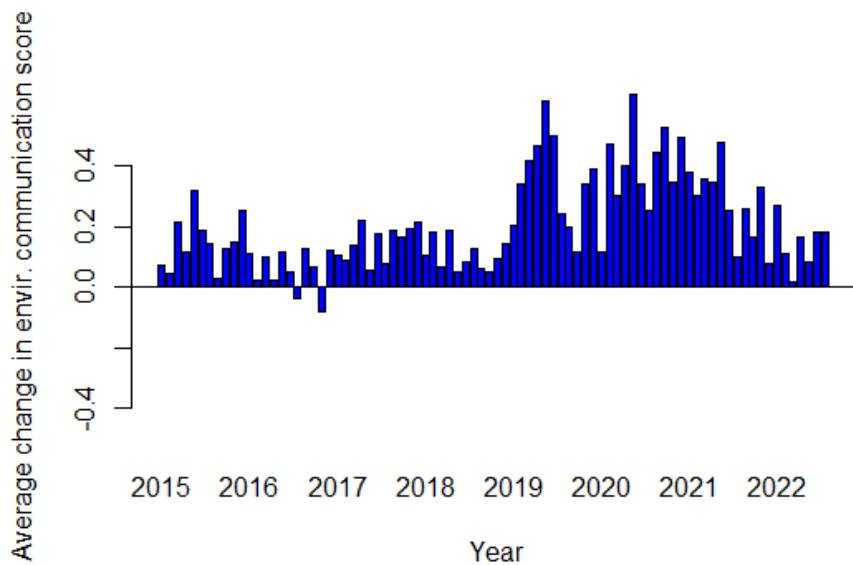
Second, we find significant empirical evidence supporting the dynamics of environmental communication derived from the theoretical section. We carry out the second-step estimation,

25. <https://www.ipcc.ch/sr15/>

26. The period from the end of 2019 to the end of 2021 is characterized by the introduction of several fundamental environmental regulations in the European Union: the presentation of the European Green Deal by the European Commission (December 2019) and, within this framework, the presentation of the “farm to fork strategy” (May 2020), aiming to make food systems fair, healthy and environmentally-friendly, declined in an action plan in May 2021, and the EU Chemicals strategy for sustainability published on 14 October 2020. The year 2019 also marks the advent of a Europe-wide EU Strategy for Plastics in the Circular Economy, translated into an action plan in 2020. Moreover, this period corresponds to the start of the application of the EU Regulation on binding annual emission reductions by Member States from 2021 to 2030 (Effort Sharing Regulation) adopted in 2018 as part of the Energy Union strategy and the EU’s implementation of the Paris Agreement. The year 2020 also sees the adoption by the European Commission of the new EU Biodiversity Strategy for 2030 and an associated Action Plan, which is an ambitious plan for protecting nature and reversing the degradation of ecosystems. Finally, this period is shaped by the launch in 2019 and adoption in 2020 and 2021 of a series of legal measures to facilitate sustainable investment (taxonomy, framework, etc.). A detailed review is available at <https://wecoop.eu/regional-knowledge-centre/eu-policies-regulations/>.



(a) Average environmental communication score, $\overline{\hat{\alpha}_1 + \hat{\varepsilon}_1}$.



(b) Average monthly flow of environmental communication, $\bar{\hat{c}}$.

Figure 3.11 – **Environmental communication.** Figure (a) depicts the average proxy for the environmental communication score, $\overline{\hat{\alpha}_1 + \hat{\varepsilon}_1}$, (left y-axis) and the score of “net zero” occurrences in Google searches (right y-axis). Figure (b) shows the average monthly flow of environmental communication, $\bar{\hat{c}}$.

whose results are presented in Tables 3.1 and 3.2. As expected, $\hat{\beta}_3$ is negative and highly significant for almost all the samples studied. (i) In the entire sample, the beta is -0.119 and the t-stat is -3.6. (ii) In addition, the 10%, 20%, ..., and 90% brownest companies in the universe have a beta ranging from -0.07 to -0.24, with t-statistics below -2.5 for all samples except the top 10% brownest companies, even reaching -4.5 for the 50% brownest companies. (iii) Finally,

the 10%, 20%, ..., and 90% greenest companies in the universe have a beta ranging from -0.24 to -0.45 and t-statistics below -3, even reaching -7.7 for the 40% greenest companies. Therefore, the empirical findings suggest that companies, especially the greenest ones, use environmental communication in a counter-cyclical way with respect to the evolution of their environmental score, in line with the results of the model.

Table 3.1 – **Main estimation (top brownest companies and entire universe)**. This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\tilde{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$. The estimations are performed for different samples: the top 10%, 20%,..., 90% brownest companies, and the entire universe. The standard deviations are shown in brackets below the estimates.

Dependent variable: $\Delta\tilde{c}_t^i$					
Top brownest companies:					
	10%	20%	30%	40%	50%
$\Delta E_t^{i,*}$	-0.071 (0.051)	-0.164** (0.065)	-0.244*** (0.073)	-0.221*** (0.067)	-0.271*** (0.060)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	18,760	30,711	44,116	56,785	68,276
R ²	0.005	0.006	0.008	0.010	0.013
Adjusted R ²	-0.061	-0.049	-0.041	-0.035	-0.029
F Statistic	0.985	3.525*	5.460**	3.608*	4.949**
Dependent variable: $\Delta\tilde{c}_t^i$					
Top brownest companies:					
	60%	70%	80%	90%	Whole sample
$\Delta E_t^{i,*}$	-0.237*** (0.053)	-0.176*** (0.049)	-0.188*** (0.046)	-0.158*** (0.040)	-0.119*** (0.033)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	83,309	97,324	110,206	123,864	145,508
R ²	0.015	0.016	0.017	0.017	0.017
Adjusted R ²	-0.023	-0.019	-0.015	-0.012	-0.008
F Statistic	3.476*	1.756	1.875	1.195	0.661

Note:

*p<0.1; **p<0.05; ***p<0.01

The results are robust to the introduction of controls for systematic risks and returns (Table 3.7 and 3.8, except for the 10% and 20% brownest companies as well as the 10% greenest companies). We carry out other robustness tests by repeating the estimation starting at different

Table 3.2 – **Main estimation (top greenest companies and entire universe)**. This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$. The estimations are performed for different samples: the top 10%, 20%,..., 90% greenest companies, and the entire universe. The standard deviations are shown in brackets below the estimates.

Dependent variable: $\Delta\hat{c}_t^i$					
Top greenest companies:					
	10%	20%	30%	40%	50%
$\Delta E_t^{i,*}$	-0.255*** (0.079)	-0.342*** (0.069)	-0.446*** (0.072)	-0.405*** (0.061)	-0.415*** (0.057)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	21,644	35,302	48,184	62,199	77,232
R ²	0.018	0.019	0.021	0.020	0.020
Adjusted R ²	-0.018	-0.013	-0.010	-0.010	-0.009
F Statistic	4.284**	8.542***	14.584***	11.377***	10.606***
Dependent variable: $\Delta\hat{c}_t^i$					
Top greenest companies:					
	60%	70%	80%	90%	Whole sample
$\Delta E_t^{i,*}$	-0.404*** (0.052)	-0.380*** (0.054)	-0.294*** (0.052)	-0.237*** (0.044)	-0.119*** (0.033)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	88,723	101,392	114,797	126,748	145,508
R ²	0.022	0.022	0.022	0.021	0.017
Adjusted R ²	-0.007	-0.006	-0.006	-0.006	-0.008
F Statistic	8.727***	6.709***	3.513*	2.169	0.661

Note:

*p<0.1; **p<0.05; ***p<0.01

dates: December 2012, December 2017, December 2019, and December 2021. The estimate $\hat{\beta}_3$ remains strongly significant (see Tables 3.9 showing the example of the 50% brownest and 50% greenest companies of the universe). Finally, we repeat the estimation applied to the three environmental subscores calculated by Covalence, which are related to (i) the environmental impacts of the products sold, (ii) the resources used, and (iii) the emissions, effluents, and waste. For all three subscores, the results are robust (see Table 3.10 showing the example of the 50% brownest and 50% greenest companies of the universe).

While the second result supports the counter-cyclical dynamic of the environmental communication highlighted by the model, the first one shows that, at least since December 2015, companies have implemented, on average, a quasi-structural green (i.e., positive environmental) communication policy. There are three possible explanations for this: (i) either companies are structurally underrated by the rating agencies and communicate to raise their environmental score to the level of their fundamental environmental value, (ii) they use green communication to support their continuous abatement effort, or (iii) they are engaging in greenwashing through misleading communication, at least part of the time. Although the scores calculated by rating agencies may be imprecise (Bar-Isaac and Shapiro, 2011) and the environmental scores tend to diverge between rating agencies (Gibson et al., 2020), the academic literature has not documented any structural underestimation of the environmental scores. In addition, it is reasonable to assume that companies' green communications are more volatile than their abatement policies.²⁷ Therefore, the greenwashing option, at least part of the time, is the most likely, especially as the marginal unit costs of communication are much lower than those of abatement,²⁸ and because companies can benefit from information asymmetry about their true environmental values (Barbalau and Zeni, 2023).

6 Conclusion

In this paper, we show why and how companies have an incentive to greenwash when investors have pro-environmental preferences. Companies greenwash, provided that the marginal unit cost of environmental communication is sufficiently low compared to the marginal unit cost of abatement, or that information asymmetry is sufficiently high. When these conditions are satisfied, companies greenwash continuously until their environmental score reaches a certain threshold above their fundamental environmental value. This threshold increases with the pro-environmental preferences of the investor, and decreases with the investor's penalty on revealed misrating.

Hence, investors can incentivize companies both to reduce the magnitude of their greenwash-

27. To recall, the standard deviation of the proxy for the environmental communication score is 5.3, which can be compared to an average of 56.7.

28. An insightful example is the comparison of the very small certification cost of a green bond compared to the issued amount (Bank for International Settlement, 2017).

ing effort and to increase the abatement of their emissions by penalizing misrating revealed by controversies. This penalty, therefore, contributes to reducing the gap between environmental scores and fundamental environmental values. In addition, policymakers have complementary tools at their disposal to curb greenwashing through (i) regulations strengthening transparency on the effective environmental practices of companies, especially the most recent ones, and (ii) pronounced and sustained support for environment-related technological innovation to substantially reduce the marginal costs of abatement. These results are robust to the introduction of interaction between companies, by assuming that investors only care about or deal with relative environmental scores. Moreover, our empirical results support the counter-cyclical dynamics of companies' optimal environmental communication.

Several avenues for future research naturally flow from this work. It would be interesting to develop a framework with an endogenous volatility matrix to understand the interaction between greenwashing and corporate financial risk, which would involve additional complexities. It would also be worthwhile to study greenwashing in a general equilibrium model, considering not only investors but also consumers who are affected by greenwashing and respond by boycotting. Finally, from an empirical viewpoint, it would be valuable to be able to estimate the dynamics of greenwashing by approximating the unknown fundamental environmental value of the companies and their abatement policies.

Appendix

A Formal definition of the marginal benefit of a strategy

To interpret the shapes of the optimal strategies in the general case and in the two limiting cases, we define the notion of “marginal benefit” of each strategy. A meaningful notion of “marginal benefit” at time t in this continuous time setting can be defined as the impact on the integrated discounted cost of capital of increasing communication or abatement over an infinitesimal time interval. This notion is formally defined below. In this section we fix a given company i and drop the superscript i to save space.

Definition 9 (Marginal benefit of communication and abatement). Let $\epsilon > 0$. For a pair of communication and abatement strategies $c, r \in \mathbb{A}$ and a pair of test functions $\delta c, \delta r \in \mathbb{A}$, let us define the associated pair of modified strategies:

$$c_s^\epsilon := c_s + \epsilon \delta c_s, \quad r_s^\epsilon := r_s + \epsilon \delta r_s.$$

The functional $J(c, r)$ is defined as the expected discounted integral of the cost of capital when the pair of strategies c, r is employed,

$$J(c, r) := \mathbb{E} \left[\int_0^\infty e^{-\delta t} \{-\gamma \Sigma \mathbf{1}_n + \beta E_t^{c,r} - \alpha M_t^{c,r}\} dt \right],$$

Then, the expected marginal benefits of communication and abatement along directions δc and δr are defined respectively as the directional (Gateaux) derivatives of J in these two directions:

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (J(c + \epsilon \delta c, r) - J(c, r)), \quad \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (J(c, r + \epsilon \delta r) - J(c, r)).$$

As we shall see below (see Section B.2), these Gateaux derivatives are linear, and can be expressed through Frechet derivatives D_t^c and D_t^r :

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (J(c + \epsilon \delta c, r) - J(c, r)) &= \mathbb{E} \left[\int_0^\infty e^{-\delta t} D_t^c J(c, r) \delta c_t dt \right], \\ \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (J(c, r + \epsilon \delta r) - J(c, r)) &= \mathbb{E} \left[\int_0^\infty e^{-\delta t} D_t^r J(c, r) \delta r_t dt \right]. \end{aligned}$$

The derivatives $D_t^c J(c, r)$ and $D_t^r J(c, r)$ shall be called marginal benefits of increasing communication or abatement at a given time t , starting from a given pair of strategies (c, r) .

B Proofs

We will use the symbol $\mathbb{H}_k^2(h)$ to denote the set of all \mathbb{F} -progressively measurable \mathbb{R}^k -valued processes $\eta = (\eta_t)_{t \in [0, T]}$ such that $\mathbb{E}[\int_0^\infty e^{-ht} \|\eta_t\|^2 dt] < \infty$ for any parameter $h \in \mathbb{R}_+^*$.

B.1 Equilibrium expected returns and optimal strategy

We restate Proposition 1 with its full set of assumptions in Proposition 16, before proving it.

Proposition 16. *Let us assume that E, M , solutions of dynamics (3.2a) and (3.3), verify $E, M \in \mathbb{H}_N^2(\delta^I)$. Moreover, let us define S as a solution to (3.1) and the set of admissible strategies \mathbb{A}^ω for the program of the investor (3.4) as $\mathbb{A}^\omega := \mathbb{H}_N^2(\delta^I)$.*

Then, the optimal portfolio choice of the investor is the pointwise solution

$$\omega_t^* = \frac{1}{\gamma} \Sigma^{-1} (\mu_t + \beta E_t - \alpha M_t),$$

and equilibrium expected returns are

$$\mu_t = \gamma \Sigma \mathbf{1}_n - \beta E_t + \alpha M_t.$$

Proof of Proposition 1. Under the assumptions of the proposition, the investor's program can be rewritten as

$$\begin{aligned} & \sup_{\omega \in \mathbb{A}^\omega} \mathbb{E} \left[\int_0^\infty e^{-\delta^I t} \omega_t' \left(\mu_t + \beta E_t - \alpha M_t - \frac{\gamma}{2} \Sigma \omega_t \right) dt \right] \\ &= \sup_{\omega \in \mathbb{A}^\omega} \mathbb{E} \left[\int_0^\infty e^{-\delta^I t} \left\{ -\frac{\gamma}{2} (\omega_t - \omega_t^*)' \Sigma (\omega_t - \omega_t^*) + \frac{\gamma}{2} \omega_t^{*'} \Sigma \omega_t^* \right\} dt \right]. \end{aligned}$$

The optimal portfolio choice of the investor is thus the pointwise solution ω_t^* . In addition, as the quantity of each asset is assumed to be normalised to one in the market, writing $\mathbf{1}_n$ a vector of ones of size n , market clearing condition writes:

$$\forall t, \omega_t^* = \mathbf{1}_n.$$

Equilibrium expected returns are therefore

$$\mu_t = \gamma \Sigma \mathbf{1}_n - \beta E_t + \alpha M_t.$$

□

Proof of Proposition 2. As the problem is symmetric for each company and depends solely on its own variables and parameters, we drop the exponent i to lighten notations throughout the proof.

The value function of the company's program is as follows:

$$\hat{v}(q, p, u) = \inf_{(r, c) \in \mathbb{A}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\gamma (\Sigma \mathbf{1}_n) - \beta E_t^q + \alpha M_t^p + \frac{\kappa_r}{2} (r_t)^2 + \frac{\kappa_c}{2} (c_t)^2 \right) dt \right],$$

with the following constraints. The state variables of the company's program are the tridimensional process (E^q, V^p, M^u) which is the unique strong solution (Protter, 2005, Chapter 5, Theorem 52) to the following SDEs:

$$\begin{cases} dE_t^q = a(V_t^p - E_t^q)dt + b(V_{t-}^p - E_{t-}^q)dN_t + c_t dt + z dW_t, & E_0^q = q, \\ dV_t^p = r_t dt, & V_0^p = p, \\ dM_t^u = -\rho M_t^u dt + b^2(V_{t-}^p - E_{t-}^q)^2 dN_t, & M_0^u = u, \end{cases} \quad (3.26)$$

for $(q, p, u) \in \mathcal{Y}$, $\mathcal{Y} := \mathbb{R}^2 \times \mathbb{R}_+$ and $(c, r) \in \mathbb{A}$. The set of admissible strategies is

$$\mathbb{A} := \left\{ (c, r) \in \mathbb{H}_2^2(\delta^I \wedge \delta) \right\}.$$

Remark that, as admissible strategies $(c, r) \in \mathbb{A}$ are in $\mathbb{H}_2^2(\delta^I \wedge \delta)$, they are both in $\mathbb{H}_2^2(\delta^I)$ and $\mathbb{H}_2^2(\delta)$.

Equivalence with an auxiliary program First, remark that \hat{v} is equivalent to the following program (which differs only through a constant):

$$\sup_{(r,c) \in \mathbb{A}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\beta E_t^q - \alpha M_t^u - \frac{\kappa_r}{2} (r_t)^2 - \frac{\kappa_c}{2} (c_t)^2 \right) dt \right]$$

Modulo adding a constant term, and using that $e^{-\delta t} \mathbb{E}[V_t^p] \xrightarrow[t \rightarrow \infty]{} 0$ according to Lemma B.1 and B.2 for any admissible control, this can be rewritten as:

$$\sup_{(r,c) \in \mathbb{A}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\beta (E_t^q - V_t^p) - \alpha M_t^u - \frac{\kappa_r}{2} \left(r_t - \frac{\beta}{\delta \kappa_r} \right)^2 - \frac{\kappa_c}{2} (c_t)^2 \right) dt \right].$$

Then, remark that for all $t \geq 0$,

$$\frac{\kappa_r}{2} \left(r_t - \frac{\beta}{\delta \kappa_r} \right)^2 + \frac{\kappa_c}{2} (c_t)^2 = \frac{\bar{\kappa}}{4} \left(c_t - r_t + \frac{\beta}{\delta \kappa_r} \right)^2 + \frac{1}{2(\kappa_r + \kappa_c)} (\kappa_c c_t + \kappa_r r_t - \frac{\beta}{\delta})^2.$$

Let $\xi_t = c_t - r_t$ with $(r, c) \in \mathbb{A}$ and introduce the new state process $X_t = E_t^q - V_t^p$, so that

$$dX_t^x = -aX_t^x dt - bX_{t-}^x dN_t + \xi_t dt + z dW_t, \quad X_0 = x = q - p,$$

$$dM_t^u = -\rho M_t^u dt + (-bX_{t-}^x)^2 dN_t, \quad M_0 = u.$$

We have $\hat{v}(q, p, u) = \tilde{v}(x, u)$, with

$$\tilde{v}(x, u) = \sup_{\substack{\xi=c-r, \\ (r,c) \in \mathbb{A}}} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\beta X_t^x - \alpha M_t^u - \frac{\bar{\kappa}}{4} \left(\xi_t + \frac{\beta}{\delta \kappa_r} \right)^2 - \frac{1}{2(\kappa_r + \kappa_c)} (\kappa_c c_t + \kappa_r r_t - \frac{\beta}{\delta})^2 \right) dt \right].$$

It is then clear that at optimum, the controls satisfy

$$\kappa_c c_t + \kappa_r r_t - \frac{\beta}{\delta} = 0. \quad (3.27)$$

We can then parameterize the two controls with a single process $\xi_t = c_t - r_t$.

This allows us to rewrite the program as a bidimensional problem, i.e. with only two state variables. Consider the auxiliary optimization problem v on $\mathcal{X} := \mathbb{R} \times \mathbb{R}_+$ as follows:

$$v(x, u) = \sup_{\xi \in \mathbb{A}^\xi} \mathbb{E} \left[\int_0^\infty e^{-\delta t} f(X_t^x, M_t^u, \xi_t) dt \right], \quad (3.28)$$

with $f(x, u, \xi) := \beta x - \alpha u - \frac{\bar{\kappa}}{4} \left(\xi + \frac{\beta}{\delta \kappa_r} \right)^2$, and the auxiliary bidimensional state variables process (X^x, M^u) as the unique strong solution to the following SDEs (Protter, 2005, Chapter 5, Theorem 52):

$$\begin{cases} dX_s^x = -aX_s ds - bX_{s-} dN_s + \xi_s ds + z dW_s, & X_0 = x, \\ dM_s^u = -\rho M_s ds + (bX_{s-})^2 dN_s, & M_0 = u, \end{cases} \quad (3.29)$$

for $(x, u) \in \mathcal{X}$ and $\xi \in \mathbb{A}^\xi$ the set of admissible strategies, verifying

$$\mathbb{A}^\xi := \left\{ \xi \in \mathbb{H}_1^2(\delta^I \wedge \delta) \right\}.$$

Note that, by construction, any control (c, r) which verifies equation (3.27) verifies that

$$(c, r) \in \mathbb{A} \iff c - r \in \mathbb{A}^\xi. \quad (3.30)$$

In particular, this is true for optimal controls.

Moreover, note that for any $\xi \in \mathbb{A}^\xi$, the bidimensional auxiliary state variable (3.29) admits the following explicit solutions:

$$\begin{cases} X_t^x = \mathcal{E}_t x + \mathcal{E}_t \int_0^t \mathcal{E}_s^{-1} \{ \xi_s ds + z dW_s \} & \text{if } 0 \leq b < 1, \\ X_t^x = \mathbf{1}_{t < \theta_1} \left(x e^{-at} + \int_0^t e^{-a(t-s)} \{ \xi_s ds + z dW_s \} \right) + \mathbf{1}_{t \geq \theta_1} \int_{\theta(t)}^t e^{-a(t-s)} \{ \xi_s ds + z dW_s \} & \text{if } b = 1, \end{cases} \quad (3.31)$$

$$M_t^u = e^{-\rho t} u + \int_0^t e^{-\rho(t-s)} (bX_{s-}^x)^2 dN_s, \quad (3.32)$$

with

$$\mathcal{E}_t = e^{-at} (1 - b)^{N_t}, \quad \theta(t) = \sup\{s \leq t : dN_s \neq 0\}, \quad \theta_1 = \inf\{s \geq 0 : dN_s \neq 0\}.$$

Solving the HJB equation of the auxiliary program We first show how the HJB equation satisfied by the value function v of the auxiliary problem may be solved explicitly, and then, in the next paragraph, prove a verification theorem which shows that the explicit solution found in this paragraph indeed coincides with the value function. Consider the following HJB equation.

$$\max_{\xi \in \mathbb{R}} \left\{ \beta x - \alpha u - \frac{\bar{\kappa}}{4} \left(\xi + \frac{\beta}{\delta \kappa_r} \right)^2 - \delta v + \frac{\partial v}{\partial x} (-ax + \xi) - \frac{\partial v}{\partial u} \rho u + \frac{z^2}{2} \frac{\partial^2 v}{\partial x^2} + \lambda [v(x(1-b), u + b^2 x^2) - v(x, u)] \right\} = 0, \quad (3.33)$$

or in other words, replacing ξ by the optimizing function $\xi^*(x, u) := \frac{2}{\bar{\kappa}} \frac{\partial v}{\partial x} - \frac{\beta}{\delta \kappa_r}$,

$$\beta x - \alpha u + \frac{1}{\bar{\kappa}} \left(\frac{\partial v}{\partial x} \right)^2 - \delta v - \frac{\partial v}{\partial x} \left(ax + \frac{\beta}{\delta \kappa_r} \right) - \frac{\partial v}{\partial u} \rho u + \frac{z^2}{2} \frac{\partial^2 v}{\partial x^2} + \lambda [v(x(1-b), u + b^2 x^2) - v(x, u)] = 0.$$

Let us use the ansatz

$$w(x, u) = \frac{1}{2} Ax^2 + Bx + Cu + w_0.$$

Substituting this function and its derivatives $\frac{\partial w}{\partial x} = Ax + B$, $\frac{\partial w}{\partial u} = C$, $\frac{\partial^2 w}{\partial x^2} = A$ into the HJB equation, we get:

$$\begin{aligned} & \beta x - \alpha u + \frac{1}{\bar{\kappa}} (Ax + B)^2 - \delta \left(\frac{1}{2} Ax^2 + Bx + Cu + w_0 \right) \\ & - (Ax + B) \left(ax + \frac{\beta}{\delta \kappa_r} \right) - C \rho u + \frac{z^2}{2} A + \lambda \left[\frac{1}{2} Ax^2 ((1-b)^2 - 1) - bBx + Cb^2 x^2 \right] = 0, \end{aligned}$$

and collecting terms with the same powers of u and x , we get that A, B, C are characterized by the following equations:

$$-\alpha - \delta C - \rho C = 0 \quad (3.34)$$

$$\frac{2}{\bar{\kappa}} A^2 - \left(\lambda(1 - (1-b)^2) + \delta + 2a \right) A + 2\lambda b^2 C = 0 \quad (3.35)$$

$$\left(\frac{2}{\bar{\kappa}} A - \delta - a - \lambda b \right) B + \beta - A \frac{\beta}{\delta \kappa_r} = 0 \quad (3.36)$$

and the candidate optimal control is

$$\hat{\xi}_t = \xi^*(\hat{X}_t^x, \hat{M}_t^u) = \frac{2}{\bar{\kappa}} \left(A \hat{X}_t^x + B \right) - \frac{\beta}{\delta \kappa_r}$$

with $(\hat{X}_t^x, \hat{M}_t^u)$ the unique strong solutions of (3.29) when the control $\hat{\xi}$ is employed (Protter, 2005, Chapter 5, Theorem 52). According to equations (3.34) and (3.36), C and B are given as follows:

$$C = -\frac{\alpha}{\rho + \delta}, \quad B = \frac{\beta \left(\frac{A}{\delta \kappa_r} - 1 \right)}{\left(\frac{2}{\bar{\kappa}} A - \delta - (a + \lambda b) \right)}.$$

The polynomial of degree 2 in A in equation (3.35) has two roots. One is strictly positive (> 0) (let us call it A^+), and the other one is negative ($A^- \leq 0$) (strictly negative if $\alpha > 0$), as follows:

$$A^- = \frac{\bar{\kappa}}{4} \left(\delta + 2a + \lambda(1 - (1 - b)^2) - \sqrt{(\delta + 2a + \lambda(1 - (1 - b)^2))^2 + 8\frac{2}{\bar{\kappa}}\lambda b^2 \frac{\alpha}{\delta + \rho}} \right), \quad (3.37)$$

$$A^+ = \frac{\bar{\kappa}}{4} \left(\delta + 2a + \lambda(1 - (1 - b)^2) + \sqrt{(\delta + 2a + \lambda(1 - (1 - b)^2))^2 + 8\frac{2}{\bar{\kappa}}\lambda b^2 \frac{\alpha}{\delta + \rho}} \right). \quad (3.38)$$

Lemma B.7 shows that the candidate optimal control associated to A^+ ,

$$\xi_t^+ = \frac{2}{\bar{\kappa}} \left(A^+ X_t^+ + B \right) - \frac{\beta}{\delta \kappa_r},$$

with X^+ the strong solution of the first SDE in (3.29) controlled by ξ^+ , is not admissible. Thus, in what follows, we will write $A := -A^-$, and show that the value function of the auxiliary problem is indeed given by the solution of the HJB equation we have just found.

Verification argument for the auxiliary program Let us define on \mathcal{X} the function

$$w(x, u) = -\frac{1}{2}Ax^2 + Bx + Cu + w_0.$$

Let us show that $v = w$.

(i) Let $\xi \in \mathbb{A}^\xi$. By Itô's formula applied to $e^{-\delta t}w(X_t^x, M_t^u)$ between 0 and the stopping time τ_n defined below, we have:

$$\begin{aligned} e^{-\delta(t \wedge \tau_n)}w(X_{t \wedge \tau_n}^x, M_{t \wedge \tau_n}^u) &= w(x, u) + \int_0^{t \wedge \tau_n} e^{-\delta s} \left[-\delta w(X_s^x, M_s^u) + \mathcal{L}^{\xi_s} w(X_s^x, M_s^u) \right] ds \\ &\quad + \int_0^{t \wedge \tau_n} e^{-\delta s} \frac{\partial w}{\partial x}(X_s^x, M_s^u) z dW_s, \end{aligned}$$

with the stopping time

$$\tau_n := \inf \{ t \geq 0 : \int_0^t e^{-\delta s} \left| \frac{\partial w}{\partial x}(X_s^x, M_s^u) \right|^2 ds \geq n \}, \quad \forall n \in \mathbb{N},$$

using the convention that $\inf \{ \emptyset \} = \infty$, and the operator $\mathcal{L}^\xi w$ defined as follows:

$$\forall (x, u) \in \mathcal{X}, \quad \mathcal{L}^\xi w(x, u) := \frac{\partial w}{\partial x}(-ax + \xi) - \frac{\partial w}{\partial u} \rho u + \frac{z^2}{2} \frac{\partial^2 w}{\partial x^2} + \lambda \left[w(x(1 - b), u + b^2 x^2) - w(x, u) \right].$$

The stopped stochastic integral is a martingale, and by taking the expectation we get

$$\begin{aligned} \mathbb{E}[e^{-\delta(t \wedge \tau_n)} w(X_{t \wedge \tau_n}^x, M_{t \wedge \tau_n}^u)] &= w(x, u) + \mathbb{E}\left[\int_0^{t \wedge \tau_n} e^{-\delta s} \left\{ -\delta w(X_s^x, M_s^u) + \mathcal{L}^{\xi_s} w(X_s^x, M_s^u) \right\} ds\right] \\ &\leq w(x, u) - \mathbb{E}\left[\int_0^{t \wedge \tau_n} e^{-\delta s} f(X_s^x, M_s^u, \xi_s) ds\right], \end{aligned}$$

from (3.33), as ξ is any admissible control. By Lemmas B.4 and B.5, we may apply the dominated convergence theorem and send n to infinity:

$$\mathbb{E}[e^{-\delta t} w(X_t^x, M_t^u)] \leq w(x, u) - \mathbb{E}\left[\int_0^t e^{-\delta s} f(X_s^x, M_s^u, \xi_s) ds\right]. \quad (3.39)$$

By sending now t to infinity, using again Lemmas B.4 and B.5, we then deduce

$$w(x, u) \geq \mathbb{E}\left[\int_0^\infty e^{-\delta s} f(X_s^x, M_s^u, \xi_s) ds\right], \quad \forall \xi \in \mathbb{A}^\xi,$$

and so $w \geq v$ on \mathcal{X} .

(ii) By repeating the above arguments and observing that the optimal control $\hat{\xi}$ achieves equality in (3.39) by construction, we have

$$\mathbb{E}[e^{-\delta t} w(\hat{X}_t^x, \hat{M}_t^u)] = w(x, u) - \mathbb{E}\left[\int_0^t e^{-\delta s} f(\hat{X}_s^x, \hat{M}_s^u, \hat{\xi}_s) ds\right].$$

From Lemma B.6, $\hat{\xi} \in \mathbb{A}^\xi$, and hence Lemma B.5 can be applied. By sending t to infinity, we then deduce

$$w(x, u) \leq \mathbb{E}\left[\int_0^\infty e^{-\delta t} |f(\hat{X}_s^x, \hat{M}_s^u, \hat{\xi}_s)| ds\right] \leq v(x, u).$$

Combining with the conclusion to (i), this shows that $w = v$ on \mathcal{X} , and that the process $\{\hat{\xi}_t = \xi^*(\hat{X}_t^x, \hat{M}_t^u), t \geq 0\}$ is an optimal control.

Now, from Lemma B.8, we get that that if ξ^1 and ξ^2 are both optimal controls, then

$$\int_0^\infty e^{-\delta t} |\xi_t^1 - \xi_t^2|^2 dt = 0,$$

hence the optimal control is unique, up to t -almost everywhere and almost sure equivalence.

Conclusion for the initial optimization program By (3.27) and (3.30), we can deduce the unique optimal control (c^*, r^*) to the equivalent program \hat{v} from the following system:

$$\begin{cases} \kappa_c c_t^* + \kappa_r r_t^* - \frac{\beta}{\delta} = 0, \\ \xi_t^* = c_t^* - r_t^*. \end{cases}$$

Hence, optimal strategies of the company are as follows:

$$r_t^* = \frac{1}{\kappa_r} \left(A(E_t^* - V_t^*) + \frac{\beta}{\delta} - B \right), \quad c_t^* = \frac{1}{\kappa_c} (-A(E_t^* - V_t^*) + B),$$

with (E^*, V^*, M^*) solutions of (3.26) controlled by (c^*, r^*) and

$$A = \frac{\bar{\kappa}}{4} \left(\delta + 2a + \lambda(1 - (1 - b)^2) \right) \left(\sqrt{1 + 8 \frac{2}{\bar{\kappa}} \lambda b^2 \frac{\alpha}{\delta + \rho}} - 1 \right), \quad B = \frac{\beta(1 + \frac{A}{\delta \kappa_r})}{\frac{2}{\bar{\kappa}} A + \delta + a + \lambda b}$$

positive coefficients, as all parameters are positive. Moreover,

$$\frac{\beta}{\delta} - B \geq 0 \iff \frac{1}{\delta} > \frac{1 + \frac{A}{\delta \kappa_r}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}} A} \iff 1 > \frac{\delta + \frac{A}{\kappa_r}}{\delta + a + \lambda b + \left(\frac{1}{\kappa_r} + \frac{1}{\kappa_c} \right) A},$$

which is always true as all parameters are non negative. \square

Lemma B.1. *If $\eta \in \mathbb{H}_1^2(\delta)$, then $\mathbb{E} \left[\int_0^\infty e^{-\delta t} |\eta_t| dt \right] < \infty$. Hence, in particular,*

$$\int_0^\infty e^{-\delta t} |\eta_t| dt < \infty \text{ a.s..}$$

Proof. We have

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty e^{-\delta t} |\eta_t| dt \right] &= \mathbb{E} \left[\int_0^\infty e^{-\delta t} |\eta_t| \mathbf{1}_{|\eta_t| \geq 1} dt + \int_0^\infty e^{-\delta t} |\eta_t| \mathbf{1}_{|\eta_t| < 1} dt \right] \\ &\leq \mathbb{E} \left[\int_0^\infty e^{-\delta t} |\eta_t|^2 dt \right] + \mathbb{E} \left[\int_0^\infty e^{-\delta t} dt \right] < \infty \end{aligned}$$

as $\eta \in \mathbb{H}_1^2(\delta)$. \square

Lemma B.2. (i) *Let η a progressively measurable process verifying $\int_0^\infty e^{-\delta t} |\eta_t| dt < \infty$ a.s. Then,*

$$\lim_{t \rightarrow \infty} e^{-\delta t} \int_0^t |\eta_s| ds = 0 \text{ a.s..}$$

(ii) *If moreover $\mathbb{E} \left[\int_0^\infty e^{-\delta t} |\eta_t| dt \right] < \infty$, then $\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E} \left[\int_0^t |\eta_s| ds \right] = 0$.*

Proof. (i) Assume $\lim_{t \rightarrow \infty} e^{-\delta t} \int_0^t |\eta_s| ds$ does not exist or is not zero for a non null probability. Therefore, there exists a measurable set $N \subset \Omega$, $\mathbb{P}(N) > 0$, so that $\lim_{t \rightarrow \infty} e^{-\delta t} \int_0^t |\eta_s| ds$ does not exist or is nonzero for every $\omega \in N$. Let us reason for a fixed $\omega \in N$. Then, there exists $c > 0$ and an increasing sequence $(t_n) \in \mathbb{R}_+^{\mathbb{N}}$ which tends to ∞ so that $e^{-\delta t_n} \int_0^{t_n} |\eta_s| ds > c$ for every n . Take two natural numbers k, l with $k \leq l$. Define $c_1 := e^{-\delta t_k} \int_0^{t_k} |\eta_s| ds$. Then $e^{-\delta t_l} \int_0^{t_k} |\eta_s| ds = c_1 e^{-\delta(t_l - t_k)}$ and hence $e^{-\delta t_l} \int_{t_k}^{t_l} |\eta_s| ds > c - c_1 e^{-\delta(t_l - t_k)}$. When t_l is big enough, there exists $\gamma > 0$ so that $c - c_1 e^{-\delta(t_l - t_k)} > \gamma$. Moreover,

$$e^{-\delta t_l} \int_{t_k}^{t_l} |\eta_s| ds \leq \int_{t_k}^{t_l} e^{-\delta s} |\eta_s| ds.$$

So for t_l big enough, we get $\int_{t_k}^{t_l} e^{-\delta s} |\eta_s| ds \geq \gamma > 0$ with a non-null probability. Now, take t_l, t_k to ∞ . As $\int_0^\infty e^{-\delta t} |\eta_t| dt$ converges almost surely, $\int_{t_k}^{t_l} e^{-\delta s} |\eta_s| ds$ tends to zero almost surely. There is a contradiction. Hence, $\lim_{t \rightarrow \infty} e^{-\delta t} \int_0^t |\eta_s| ds = 0$ a.s..

(ii) By Fubini, $\mathbb{E} \left[\int_0^\infty e^{-\delta t} |\eta_t| dt \right] < \infty$ implies that $\int_0^\infty e^{-\delta t} \mathbb{E} [|\eta_t|] dt < \infty$. By the same argument as in part (i), $\lim_{t \rightarrow \infty} e^{-\delta t} \int_0^t \mathbb{E} [|\eta_s|] ds = 0$. Applying Fubini again, conclude that $\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E} \left[\int_0^t |\eta_s| ds \right] = 0$. \square

Lemma B.3. *If $\xi \in \mathbb{H}_1^2(\delta)$, then*

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\int_0^t |X_s^x|^2 ds \right) dt \right] < \infty.$$

Moreover, $\forall t \geq 0$, $\mathbb{E} [|M_t^u|] < \infty$.

Proof. (i) By integration by parts,

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\int_0^t |X_s^x|^2 ds \right) dt \right] \\ &= \mathbb{E} \left[\int_0^\infty |X_t^x|^2 \left(\int_t^\infty e^{-\delta s} ds \right) dt - \lim_{t \rightarrow \infty} \left(\int_t^\infty e^{-\delta s} ds \right) \left(\int_0^t |X_s^x|^2 ds \right) \right] \\ &= \frac{1}{\delta} \mathbb{E} \left[\int_0^\infty e^{-\delta t} |X_t^x|^2 dt - \lim_{t \rightarrow \infty} e^{-\delta t} \left(\int_0^t |X_s^x|^2 ds \right) \right] \end{aligned}$$

Now, referring to the explicit expression of X^x in (3.31), we have, for $b < 1$, using Jensen inequality,

$$\begin{aligned} |X_t^x|^2 &\leq 3 \left(\mathcal{E}_t^2 |x|^2 + \int_0^t e^{-a(t-s)} ds \int_0^t e^{-a(t-s)} (1-b)^{2(N_t-N_s)} |\xi_s|^2 ds \right. \\ &\quad \left. + z^2 \left(\int_0^t e^{-a(t-s)} (1-b)^{N_t-N_s} dW_s \right)^2 \right) \tag{3.40} \\ &\leq 3 \left(|x|^2 + \frac{1}{a} \int_0^t |\xi_s|^2 ds + z^2 \left(\int_0^t e^{-a(t-s)} (1-b)^{N_t-N_s} dW_s \right)^2 \right). \end{aligned}$$

Noting that

$$\mathbb{E} \left[\left(\int_0^t e^{-a(t-s)} (1-b)^{N_t-N_s} dW_s \right)^2 \right] = \mathbb{E} \left[\int_0^t e^{-2a(t-s)} (1-b)^{2(N_t-N_s)} ds \right] \leq t, \tag{3.41}$$

we get

$$\mathbb{E} [|X_t^x|^2] \leq 3 \left(|x|^2 + \frac{1}{a} \mathbb{E} \left[\int_0^t |\xi_s|^2 ds \right] + z^2 t \right). \tag{3.42}$$

Hence, applying Fubini,

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} |X_t^x|^2 dt \right] \leq \tilde{C} \mathbb{E} \left[1 + \int_0^\infty e^{-\delta t} \left(\int_0^t \xi_s^2 ds \right) dt \right]$$

with a constant $\tilde{C} > 0$. By integration by parts, we have

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\int_0^t \xi_s^2 ds \right) dt \right] = \frac{1}{\delta} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \xi_t^2 dt - \lim_{t \rightarrow \infty} e^{-\delta t} \left(\int_0^t \xi_s^2 ds \right) \right].$$

As $\xi \in \mathbb{A}^\xi$, by Lemma B.2, we have $\mathbb{E} \left[\lim_{t \rightarrow \infty} e^{-\delta t} \left(\int_0^t \xi_s^2 ds \right) \right] = 0$. Therefore,

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\int_0^t \xi_s^2 ds \right) dt \right] = \frac{1}{\delta} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \xi_t^2 dt \right] < \infty.$$

Hence, we obtain that

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} |X_t^x|^2 dt \right] < \infty.$$

Using Lemma B.2 again, this implies in particular that

$$\mathbb{E} \left[\lim_{t \rightarrow \infty} e^{-\delta t} \left(\int_0^t |X_s^x|^2 ds \right) \right] = 0.$$

As a consequence,

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\int_0^t |X_s^x|^2 ds \right) dt \right] = \frac{1}{\delta} \mathbb{E} \left[\int_0^\infty e^{-\delta t} |X_t^x|^2 dt \right] < \infty$$

The same arguments can be used for $b = 1$. This concludes the first part of the proof.

(ii) Using the explicit expression of M in (3.32), we have

$$\begin{aligned} \mathbb{E} [|M_t^u|] &\leq e^{-\rho t} u + \mathbb{E} \left[\int_0^t e^{-\rho(t-s)} (bX_s^x)^2 dN_s \right] \\ &\leq u + \lambda b^2 \int_0^t \mathbb{E} [(X_s^x)^2] ds. \end{aligned}$$

Now, by Fubini, $\int_0^\infty e^{-\delta t} \left(\int_0^t \mathbb{E} [(X_s^x)^2] ds \right) dt = \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\int_0^t X_s^2 ds \right) dt \right]$, which is finite for $\xi \in \mathbb{A}^\xi$ according to (i). Thus, $\int_0^\infty e^{-\delta t} \left(\int_0^t \mathbb{E} [(X_s^x)^2] ds \right) dt$ is finite. By the property of the Lebesgue integral, it implies that $\int_0^t \mathbb{E} [(X_s^x)^2] ds$ is finite for t almost everywhere. Since $t \mapsto \int_0^t \mathbb{E} [(X_s^x)^2] ds$ is increasing, $\int_0^t \mathbb{E} [(X_s^x)^2] ds$ is actually finite for all $t \geq 0$, otherwise a contradiction can be easily exhibited. Hence, for all $t \geq 0$, $\mathbb{E} [|M_t^u|] < \infty$. This concludes the proof. □

Lemma B.4. For any admissible control $\xi \in \mathbb{A}^\xi$, for all $(x, u) \in \mathcal{X}$, we have

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} |f(X_t^x, M_t^u, \xi_t)| dt \right] < \infty.$$

Proof.

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} |f(X_t^x, M_t^u, \xi_t)| dt \right] \leq \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\beta |X_t^x| + \alpha M_t^u + \frac{\bar{\kappa}}{4} \left(\xi_t + \frac{\beta}{\delta \kappa_r} \right)^2 \right) dt \right]$$

We have, for $b < 1$,

$$\begin{aligned} |X_t^x| &\leq \mathcal{E}_t |x| + \mathcal{E}_t \left| \int_0^t \mathcal{E}_s^{-1} \{ \xi_s ds + z dW_s \} \right| \\ &\leq |x| + \int_0^t |\xi_s| ds + z \left(1 + \left(\int_0^t e^{-a(t-s)} (1-b)^{N_t - N_s} dW_s \right)^2 \right). \end{aligned} \quad (3.43)$$

By (3.41), we deduce

$$\mathbb{E} [|X_t^x|] \leq \mathbb{E} \left[|x| + \int_0^t |\xi_s| ds + z(1+t) \right] \quad (3.44)$$

Moreover we have, by integration by parts:

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\int_0^t |\xi_s| ds \right) dt \right] = \mathbb{E} \left[\frac{1}{\delta} \left(\int_0^\infty e^{-\delta t} |\xi_t| dt + \lim_{t \rightarrow \infty} e^{-\delta t} \int_0^t |\xi_s| ds \right) \right].$$

As $\xi \in \mathbb{A}^\xi$, the expectation of the left term of the sum is finite by Lemma B.1. Moreover, by Lemma B.2, the expectation of the “lim” term is null. Applying Fubini, we finally get that

$$\mathbb{E} \left[\int_0^\infty e^{-\delta t} \beta |X_t^x| dt \right] < \infty.$$

The method with $b = 1$ follows the same argument.

As for $\mathbb{E} \left[\int_0^\infty e^{-\delta t} |M_t^u| dt \right]$,

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty e^{-\delta t} |M_t^u| dt \right] &\leq \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(e^{-\rho t} u + \int_0^t e^{-\rho(t-s)} (bX_s^x)^2 dN_s \right) dt \right] \\ &\leq \int_0^\infty e^{-\delta t} \left(e^{-\rho t} u + \lambda \int_0^t \mathbb{E} \left[(bX_s^x)^2 \right] ds \right) dt, \end{aligned}$$

which is finite for $\xi \in \mathbb{A}^\xi$ according to Lemma B.3.

Finally, $\mathbb{E} \left[\int_0^\infty e^{-\delta t} \frac{\bar{\kappa}}{4} \left(\xi_t + \frac{\beta}{\delta \kappa_r} \right)^2 dt \right] \leq \mathbb{E} \left[\int_0^\infty e^{-\delta t} \frac{\bar{\kappa}}{2} \left(\xi_t^2 + \left(\frac{\beta}{\delta \kappa_r} \right)^2 \right) dt \right]$ which is finite as $\xi \in \mathbb{H}_1^2(\delta)$. \square

Lemma B.5. *For every $\xi \in \mathbb{A}^\xi$ and every $t > 0$,*

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} |w(X_s^x, M_s^u)| \right] < \infty.$$

Moreover,

$$\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E} [w(X_t^x, M_t^u)] = 0.$$

Proof. (i) Let us show that $\mathbb{E}[\sup_{0 \leq s \leq t} |w(X_s^x, M_s^u)|] < \infty$. We have

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq s \leq t} |w(X_s^x, M_s^u)| \right] &\leq \frac{1}{2} A \mathbb{E} \left[\sup_{0 \leq s \leq t} (X_s^x)^2 \right] + B \mathbb{E} \left[\sup_{0 \leq s \leq t} |X_s^x| \right] \\ &\quad + |C| \left(u + \mathbb{E} \left[\sup_{0 \leq s \leq t} \int_0^s e^{-\rho(s-y)} (bX_y^x)^2 dN_y \right] \right). \end{aligned}$$

If $b < 1$, referring to (3.40) and using Burkholder-Davis-Gundy inequality, there exists a positive constant \tilde{C} so that for every $t \geq 0$,

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} |X_s^x|^2 \right] \leq \tilde{C} \mathbb{E} \left[|x|^2 + \int_0^t |\xi_u|^2 du + z^2 t \right].$$

This upper boundary is finite as $\mathbb{E}[\int_0^t |\xi_u|^2 du] \leq \mathbb{E}[e^{\delta t} \int_0^t e^{-\delta u} |\xi_u|^2 du]$, which is finite as $\xi \in \mathbb{H}_1^2(\delta)$. Thus, $\mathbb{E} \left[\sup_{0 \leq s \leq t} |X_s^x|^2 \right]$ is finite.

Moreover, recalling (3.43), and applying again Burkholder-Davis-Gundy inequality, we similarly get that $\mathbb{E} \left[\sup_{0 \leq s \leq t} |X_s^x| \right] < \infty$ for $\xi \in \mathbb{A}^\xi$, using this time Lemma B.1 to say that $\mathbb{E}[\int_0^\infty e^{-\delta u} |\xi_u| du] < \infty$.

Finally, as $s \mapsto \int_0^s e^{\rho y} (bX_y^x)^2 dN_y$ is increasing for each trajectory, we have

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} \int_0^s e^{-\rho(s-y)} (bX_y^x)^2 dN_y \right] \leq \mathbb{E} \left[\sup_{0 \leq s \leq t} \int_0^s e^{\rho y} (bX_y^x)^2 dN_y \right] \leq \mathbb{E} \left[\int_0^t e^{\rho y} (bX_y^x)^2 dy \right],$$

which is finite since M is integrable for admissible strategies by Lemma B.3.

The same reasoning can be applied when $b = 1$. Therefore, we can conclude by a finite sum of finite terms that, for $0 \leq b \leq 1$,

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} |w(X_s^x, M_s^u)| \right] < \infty.$$

(ii) Now, let us show that $\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E} [w(X_t^x, M_t^u)] = 0$. We have

$$\begin{aligned} \lim_{t \rightarrow \infty} |e^{-\delta t} \mathbb{E} [w(X_t^x, M_t^u)]| &\leq \lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E} \left[\frac{1}{2} A (X_t^x)^2 + B |X_t^x| + |C| \int_0^t e^{-\rho(t-s)} (bX_s^x)^2 dN_s \right] \\ &= \lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E} \left[\frac{1}{2} A (X_t^x)^2 + B |X_t^x| + |C| b^2 \lambda \int_0^t (X_s^x)^2 ds \right] \end{aligned}$$

using the explicit expression of M^u in (3.32), and as M^u is integrable for admissible strategies by Lemma B.3. Again, assume $b < 1$. Since, by (3.42),

$$e^{-\delta t} \mathbb{E} [|X_t^x|^2] \leq 3e^{-\delta t} \left(|x|^2 + \frac{1}{a} \mathbb{E} \left[\int_0^t |\xi_s|^2 ds \right] + z^2 t \right),$$

and by Lemma B.2, $e^{-\delta t} \mathbb{E} \left[\int_0^t |\xi_s|^2 ds \right] \rightarrow 0$, (as $\xi \in \mathbb{A}^\xi$), we conclude that

$$\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E} \left[\frac{1}{2} A(X_t^x)^2 \right] = 0.$$

Now, let us deal with $e^{-\delta t} \mathbb{E}[|X_t|]$. Similarly, by (3.44),

$$e^{-\delta t} \mathbb{E} [|X_t^x|] \leq e^{-\delta t} \left(|x| + \mathbb{E} \left[\int_0^t |\xi_s| ds \right] + z(1+t) \right).$$

Moreover, $\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E} \left[\int_0^t |\xi_s| ds \right] = 0$ by applying successively Lemma B.1 and B.2, as $\xi \in \mathbb{A}^\xi$. Therefore, $\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E} [B|X_t^x|] = 0$.

Finally, as $\mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\int_0^t (X_s^x)^2 ds \right) dt \right] < \infty$ for admissible strategies (belonging to \mathbb{A}^ξ) according to Lemma B.3, it implies in particular, applying Fubini and due to the property of an infinite integral with positive integrand,

$$\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E} \left[\left(\int_0^t (X_s^x)^2 ds \right) \right] = 0.$$

The method is the same for $b = 1$. This concludes the proof. □

Lemma B.6. *The optimal control is admissible, i.e. $\hat{\xi} \in \mathbb{A}^\xi$.*

Proof. As $\hat{\xi}_t = \frac{2}{\kappa} \left(-A\hat{X}_t^x + B \right) - \frac{\beta}{\delta \kappa_r}$, \hat{X}^x, \hat{M}^u are solutions to the following SDEs:

$$\begin{cases} dX_s = (-\zeta X_s + \nu) ds - bX_{s-} dN_s + z dW_s, & X_0 = x, \\ dM_s = -\rho M_s ds + (bX_{s-})^2 dN_s, & M_0 = u, \end{cases}$$

with $\zeta := a + \frac{2}{\kappa} A$, $\nu := \frac{2}{\kappa} B - \frac{\beta}{\delta \kappa_r}$.

The explicit solutions for \hat{X}^x, \hat{M}^u are therefore as follows:

$$\begin{cases} \hat{X}_t^x = \hat{\mathcal{E}}_t x + \hat{\mathcal{E}}_t \int_0^t \hat{\mathcal{E}}_s^{-1} \{ \nu ds + z dW_s \} & \text{if } 0 \leq b < 1, \\ \hat{X}_t^x = \mathbf{1}_{t < \theta_1} \left(e^{-\zeta t} x + \int_0^t e^{-\zeta(t-s)} \{ \nu ds + z dW_s \} \right) + \mathbf{1}_{t \geq \theta_1} \int_{\theta(t)}^t e^{-\zeta(t-s)} \{ \nu ds + z dW_s \} & \text{if } b = 1, \end{cases} \quad (3.45)$$

$$\hat{M}_t^u = e^{-\rho t} u + \int_0^t e^{-\rho(t-s)} (b\hat{X}_{s-}^x)^2 dN_s, \quad (3.46)$$

with $\theta(t), \theta_1$ defined in (3.31) and

$$\hat{\mathcal{E}}_t = e^{-\zeta t}(1-b)^{N_t}.$$

Let us show that $\hat{\xi} \in \mathbb{H}_1^2(\delta^I \wedge \delta)$. We show it for $b < 1$. The method is the same for $b = 1$. For $b < 1$, using the explicit expression of \hat{X}^x in (3.45), we have

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty e^{-(\delta^I \wedge \delta)t} |\hat{\xi}_t|^2 dt \right] &\leq \tilde{C} \left(1 + \mathbb{E} \left[\int_0^\infty e^{-(\delta^I \wedge \delta)t} |\hat{X}_t^x|^2 dt \right] \right) \\ &\leq \tilde{C} \left(1 + \mathbb{E} \left[\int_0^\infty e^{-(\delta^I \wedge \delta)t} (|x|^2 + \int_0^t \nu^2 ds \right. \right. \\ &\quad \left. \left. + z^2 \left(\int_0^t e^{-\zeta(t-s)} (1-b)^{N_t - N_s} dW_s \right)^2 \right) dt \right] \right) \\ &\leq \tilde{C} \left(1 + \mathbb{E} \left[\int_0^\infty e^{-(\delta^I \wedge \delta)t} (|x|^2 + t\nu^2 + z^2 t) dt \right] \right) < \infty, \end{aligned}$$

with a positive constant \tilde{C} , and using (3.40) and (3.41) along with Fubini. This concludes the proof. \square

Lemma B.7. *Let ξ^+ be the strategy defined by $\xi^+ := \frac{2}{\kappa} (A^+ X_t^+ + B) - \frac{\beta}{\delta \kappa r}$, with A^+ given in (3.38) and X^+ the strong solution of the first SDE in (3.29) controlled by ξ^+ . ξ^+ is not an admissible strategy, i.e. $\xi^+ \notin \mathbb{A}^\xi$.*

Proof. In part (ii) of the proof of Lemma B.5, we show that, if $\xi \in \mathbb{A}^\xi$, then $\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E}[B|X_t^x]| = 0$, for any $x \in \mathbb{R}$. As $|\mathbb{E}[X_t^x]| \leq \mathbb{E}[|X_t^x|]$, this implies that $\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E}[X_t^x] = 0$, for any $x \in \mathbb{R}$.

Let us show that $\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E}[X_t^+] = 0$ is not true for every initial condition $x \in \mathbb{R}$ (we do not write explicitly the dependence of X^+ on its initial condition in its exponent to lighten notations). X^+ is solution to the following SDE:

$$dX_s^+ = \left(\left(\frac{2}{\kappa} A^+ - a \right) X_s^+ + \nu \right) ds - b X_{s-}^+ dN_s + z dW_s, \quad X_0^+ = x,$$

with $\nu = \frac{2}{\kappa} B - \frac{\beta}{\delta \kappa r}$. Hence, its expectation verifies the following ODE:

$$d\mathbb{E}[X_t^+] = \left(\left(\frac{2}{\kappa} A^+ - a - b\lambda \right) \mathbb{E}[X_t^+] + \nu \right) dt.$$

This ODE has a unique solution which is, writing $\zeta^+ := \frac{2}{\kappa} A^+ - a - b\lambda$,

$$\mathbb{E}[X_t^+] = e^{\zeta^+ t} \left(x + \frac{\nu}{\zeta^+} \right) - \frac{\nu}{\zeta^+}.$$

Now, remark that

$$\begin{aligned}\zeta^+ - \delta &= \frac{1}{2} \left(\sqrt{(\delta + 2a + \lambda(1 - (1 - b)^2))^2 + 8\frac{2}{\bar{\kappa}}\lambda b^2 \frac{\alpha}{\delta + \rho} - \delta - \lambda b^2} \right) \\ &\geq \frac{1}{2} \left(\delta + 2a + \lambda(1 - (1 - b)^2) - \delta - \lambda b^2 \right) = a + \lambda(b - b^2) > 0,\end{aligned}$$

as $a + b\lambda > 0$ and $0 \leq b \leq 1$. Hence,

$$\lim_{t \rightarrow \infty} e^{-\delta t} \mathbb{E}[X_t^+] = \lim_{t \rightarrow \infty} e^{(\zeta^+ - \delta)t} \left(x + \frac{\nu}{\zeta^+} \right) - e^{-\delta t} \frac{\nu}{\zeta^+} = \pm\infty \quad \text{if } x \neq -\frac{\nu}{\zeta^+}.$$

Therefore, ξ^+ can not be an admissible strategy. □

Lemma B.8. *For every $\xi \in \mathbb{A}^\xi$, $\forall(x, u) \in \mathcal{X}$, the functional*

$$\mathcal{J} : (x, u, \xi) \mapsto \mathbb{E} \left[\int_0^\infty e^{-\delta t} (-f(X_t^x, M_t^u, \xi_t)) dt \right]$$

is strictly convex in ξ and for $\theta \in [0, 1]$, and for $\xi^1, \xi^2 \in \mathbb{A}^\xi$,

$$\theta \mathcal{J}(x, u, \xi_1) + (1 - \theta) \mathcal{J}(x, u, \xi_2) - \mathcal{J}(x, u, \theta \xi_1 + (1 - \theta) \xi_2) \geq \frac{\bar{\kappa}}{4} \int_0^\infty e^{-\delta t} |\xi_t^1 - \xi_t^2|^2 dt.$$

Proof. Let us show that $\forall t \geq 0$, $\mathbb{E}[-f(X_t^x, M_t^u, \xi_t)]$ is convex in ξ . By linearity of integrals and applying Fubini thanks to Lemma B.4, it will be so for \mathcal{J} . We first deal with the case $b < 1$. We have

$$\mathbb{E}[-f(X_t^x, M_t^u, \xi_t)] = -\beta \mathbb{E}[X_t^x] + \alpha \mathbb{E}[M_t^u] + \mathbb{E} \left[\frac{\bar{\kappa}}{4} \left(\xi_t + \frac{\beta}{\delta \kappa_r} \right)^2 \right].$$

X_t^x is linear in ξ according to its explicit expression (3.31). The last term is obviously strictly convex in ξ_t . As for M_t^u , using its explicit expression (3.32) and the properties of admissible strategies ($\in \mathbb{A}^\xi$),

$$\mathbb{E}[M_t^u] = e^{-\rho t} u + \lambda b^2 \mathbb{E} \left[\int_0^t e^{-\rho(t-s)} (X_s^x)^2 ds \right].$$

Now, $(X_s^x)^2$ is strictly convex in ξ by Jensen inequality. Therefore, by addition of linear and strictly convex terms in ξ , $\mathbb{E}[-f(X_t^x, M_t^u, \xi_t)]$ is strictly convex in ξ , and so is \mathcal{J} . More precisely, by focusing only on the third part, it is easy to show that for $\theta \in [0, 1]$, and for $\xi^1, \xi^2 \in \mathbb{A}^\xi$,

$$\theta \mathcal{J}(x, u, \xi_1) + (1 - \theta) \mathcal{J}(x, u, \xi_2) - \mathcal{J}(x, u, \theta \xi_1 + (1 - \theta) \xi_2) \geq \frac{\bar{\kappa}}{4} \int_0^\infty e^{-\delta t} |\xi_t^1 - \xi_t^2|^2 dt.$$

□

B.2 Marginal benefit of a strategy

Proof of Proposition 3. (i) Let us start with the marginal benefit of communication

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (J(c + \epsilon \delta c, r) - J(c, r)) \\ &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left\{ \beta \frac{1}{\epsilon} (E_t^{c+\epsilon \delta c, r} - E_t^{c, r}) - \alpha \frac{1}{\epsilon} (M_t^{c+\epsilon \delta c, r} - M_t^{c, r}) \right\} dt \right] \\ &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left\{ \beta \frac{1}{\epsilon} (X_t^{c+\epsilon \delta c, r} - X_t^{c, r}) - \alpha \frac{1}{\epsilon} (M_t^{c+\epsilon \delta c, r} - M_t^{c, r}) \right\} dt \right], \end{aligned}$$

as $V_t^{c+\epsilon \delta c, r} - V_t^{c, r} = 0$.

Using the explicit expression of X (3.31), as $\xi = c - r$, we have, when $b < 1$, and for any $\epsilon > 0$,

$$\frac{1}{\epsilon} (X_t^{c+\epsilon \delta c, r} - X_t^{c, r}) = \mathcal{E}_t \int_0^t \mathcal{E}_s^{-1} \delta c_s ds.$$

Therefore, by integration by parts,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \beta \frac{1}{\epsilon} (X_t^{c+\epsilon \delta c, r} - X_t^{c, r}) dt \right] \\ &= \beta \mathbb{E} \left[\int_0^\infty e^{-\delta t} \mathcal{E}_t \int_0^t \mathcal{E}_s^{-1} \delta c_s ds dt \right] \\ &= \beta \mathbb{E} \left[\int_0^\infty \mathcal{E}_t^{-1} \delta c_t \int_t^\infty e^{-\delta s} \mathcal{E}_s ds dt - \lim_{t \rightarrow \infty} \left(\int_0^t \mathcal{E}_s^{-1} \delta c_s ds \right) \left(\int_t^\infty e^{-\delta s} \mathcal{E}_s ds \right) \right] \end{aligned}$$

Now, using first that \mathcal{E}_s is decreasing, and then Lemma B.2 which can be applied as δc is a test function (assumed to be admissible),

$$\begin{aligned} & \left| \left(\int_0^t \mathcal{E}_s^{-1} \delta c_s ds \right) \left(\int_t^\infty e^{-\delta s} \mathcal{E}_s ds \right) \right| \leq \left(\int_0^t \mathcal{E}_s^{-1} |\delta c_s| ds \right) \left(\int_t^\infty e^{-\delta s} \mathcal{E}_s ds \right) \\ & \leq \mathcal{E}_t \left(\int_0^t |\delta c_s| ds \right) \left(\int_t^\infty e^{-\delta s} ds \right) \mathcal{E}_t^{-1} \\ & = \frac{1}{\delta} e^{-\delta t} \left(\int_0^t |\delta c_s| ds \right) \xrightarrow{t \rightarrow \infty} 0 \text{ a.s.} \end{aligned}$$

and therefore

$$\left| \mathbb{E} \left[\lim_{t \rightarrow \infty} \left(\int_0^t \mathcal{E}_s^{-1} \delta c_s ds \right) \left(\int_t^\infty e^{-\delta s} \mathcal{E}_s ds \right) \right] \right| \leq \mathbb{E} \left[\lim_{t \rightarrow \infty} \frac{1}{\delta} e^{-\delta t} \left(\int_0^t |\delta c_s| ds \right) \right],$$

which equals 0 by Lemma B.2, as δc is a test function (assumed to be admissible). Hence,

$$\lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \beta \frac{1}{\epsilon} (X_t^{c+\epsilon \delta c, r} - X_t^{c, r}) dt \right] = \beta \mathbb{E} \left[\int_0^\infty e^{-\delta t} \delta c_t \mathbb{E} \left[\int_t^\infty e^{-\delta(s-t)} \mathcal{E}_t^{-1} \mathcal{E}_s ds \middle| \mathcal{F}_t \right] dt \right].$$

As a consequence, the part of the Frechet derivative that is due to the X term is given by

$$\beta \mathbb{E} \left[\int_t^\infty e^{-\delta(s-t)} \mathcal{E}_t^{-1} \mathcal{E}_s ds \middle| \mathcal{F}_t \right] = \frac{\beta}{a + \delta + b\lambda}.$$

Similar computations can be made when $b = 1$, leading to the same result.

As for the terms in M , using its explicit expression (3.32), we get, using the fact that δc is admissible,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(M_t^{c+\epsilon\delta c, r} - M_t^{c, r} \right) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t e^{-\rho(t-s)} b^2 \left\{ (X_{s-}^{c+\epsilon\delta c, r})^2 - (X_{s-}^{c, r})^2 \right\} dN_s \\ &= 2b^2 \int_0^t e^{-\rho(t-s)} X_{s-}^{c, r} \mathcal{E}_{s-} \left(\int_0^s \mathcal{E}_y^{-1} \delta c_y dy \right) dN_s \end{aligned}$$

Therefore, by integration by parts, using that $\mathbb{E} \left[\int_0^\infty e^{-\delta t} M_t ds \right] < \infty$ for admissible strategies as proved in Lemma B.4,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \alpha \frac{1}{\epsilon} \left(M_t^{c+\epsilon\delta c, r} - M_t^{c, r} \right) dt \right] \\ &= 2b^2 \alpha \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\int_0^t e^{-\rho(t-s)} X_{s-}^{c, r} \mathcal{E}_{s-} \left(\int_0^s \mathcal{E}_y^{-1} \delta c_y dy \right) dN_s \right) dt \right] \\ &= 2 \frac{b^2 \alpha}{\delta + \rho} \mathbb{E} \left[\int_0^\infty e^{-\delta s} X_{s-}^{c, r} \mathcal{E}_{s-} \left(\int_0^s \mathcal{E}_y^{-1} \delta c_y dy \right) dN_s \right] \\ &= 2 \frac{b^2 \alpha \lambda}{\delta + \rho} \mathbb{E} \left[\int_0^\infty e^{-\delta s} X_s^{c, r} \mathcal{E}_s \left(\int_0^s \mathcal{E}_y^{-1} \delta c_y dy \right) ds \right] \\ &= 2 \frac{b^2 \alpha \lambda}{\delta + \rho} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \delta c_t \mathbb{E} \left[\int_t^\infty e^{-\delta(s-t)} X_s^{c, r} \mathcal{E}_s \mathcal{E}_t^{-1} ds \middle| \mathcal{F}_t \right] dt \right]. \end{aligned}$$

Indeed, in a similar fashion as for the X term, it can be shown that

$$\mathbb{E} \left[\lim_{t \rightarrow \infty} \left(\int_t^\infty e^{-\delta s} ds \right) \left(\int_0^t e^{-\rho(t-s)} X_{s-}^{c, r} \mathcal{E}_{s-} \left(\int_0^s \mathcal{E}_y^{-1} \delta c_y dy \right) dN_s \right) \right] = 0.$$

Hence, the part of the Frechet derivative that is due to the M term is given by

$$-\frac{2b^2 \alpha \lambda}{\delta + \rho} \mathbb{E} \left[\int_t^\infty e^{-\delta(s-t)} X_s^{c, r} \mathcal{E}_s \mathcal{E}_t^{-1} ds \middle| \mathcal{F}_t \right]$$

Joining together the X term and the M term, we finally obtain:

$$D_t^c J(c, r) = \frac{\beta}{a + \delta + b\lambda} - \frac{2b^2 \alpha \lambda}{\delta + \rho} \mathbb{E} \left[\int_t^\infty e^{-\delta(s-t)} X_s^{c, r} \mathcal{E}_s \mathcal{E}_t^{-1} ds \middle| \mathcal{F}_t \right].$$

(ii) As for D_t^r ,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (J(c, r + \epsilon \delta r) - J(c, r)) \\ &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left\{ \beta \frac{1}{\epsilon} (E_t^{c, r + \epsilon \delta r} - E_t^{c, r}) - \alpha \frac{1}{\epsilon} (M_t^{c, r + \epsilon \delta r} - M_t^{c, r}) \right\} dt \right] \\ &= \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left\{ \beta \frac{1}{\epsilon} (X_t^{c, r + \epsilon \delta r} - X_t^{c, r}) + \beta \frac{1}{\epsilon} (V_t^{c, r + \epsilon \delta r} - V_t^{c, r}) - \alpha \frac{1}{\epsilon} (M_t^{c, r + \epsilon \delta r} - M_t^{c, r}) \right\} dt \right]. \end{aligned}$$

Using the explicit expression of X (3.31), as $\xi = c - r$, we have, when $b < 1$, and for any $\epsilon > 0$,

$$\frac{1}{\epsilon} (X_t^{c, r + \epsilon \delta r} - X_t^{c, r}) = -\mathcal{E}_t \int_0^t \mathcal{E}_s^{-1} \delta r_s ds.$$

Similarly to the Gateaux derivative of c , the part of the Frechet derivative that is due to the X term is given by

$$-\beta \mathbb{E} \left[\int_t^\infty e^{-\delta(s-t)} \mathcal{E}_t^{-1} \mathcal{E}_s ds \middle| \mathcal{F}_t \right] = -\frac{\beta}{a + \delta + b\lambda}$$

The term in V is immediate, using the explicit expression of V , $V_t = p + \int_0^t r_s ds$:

$$\frac{1}{\epsilon} (V_t^{c, r + \epsilon \delta r} - V_t^{c, r}) = \int_0^t \delta r_s ds.$$

Therefore, by integration by parts, similarly to the treatment of the X term in (i),

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \beta \frac{1}{\epsilon} (V_t^{c, r + \epsilon \delta r} - V_t^{c, r}) dt \right] &= \beta \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left(\int_0^t \delta r_s ds \right) dt \right] \\ &= \frac{\beta}{\delta} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \delta r_t dt \right], \end{aligned}$$

so that the part of the Frechet derivative that is due to the V term is given by $\frac{\beta}{\delta}$.

Finally, using the explicit expression of M in (3.32), we get, using the admissibility of δr ,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (M_t^{c + \epsilon \delta c, r} - M_t^{c, r}) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t e^{-\rho(t-s)} b^2 \left\{ (X_{s-}^{c + \epsilon \delta c, r})^2 - (X_{s-}^{c, r})^2 \right\} dN_s \\ &= -2b^2 \int_0^t e^{-\rho(t-s)} X_{s-}^{c, r} \mathcal{E}_{s-} \left(\int_0^s \mathcal{E}_y^{-1} \delta c_y dy \right) dN_s. \end{aligned}$$

By computations similar to the case of δc , the part of the Frechet derivative that is due to the M term is given by

$$2 \frac{b^2 \alpha \lambda}{\delta + \rho} \mathbb{E} \left[\int_t^\infty e^{-\delta(s-t)} X_s^{c, r} \mathcal{E}_s \mathcal{E}_t^{-1} ds \middle| \mathcal{F}_t \right]$$

Joining together the X term, the V term and the M term, we finally obtain:

$$D_t^r J(c, r) = \frac{\beta}{\delta} - \frac{\beta}{a + \delta + b\lambda} + \frac{2b^2 \alpha \lambda}{\delta + \rho} \mathbb{E} \left[\int_t^\infty e^{-\delta(s-t)} X_s^{c, r} \mathcal{E}_s \mathcal{E}_t^{-1} ds \middle| \mathcal{F}_t \right].$$

(iii) In view of the form of the company's optimization functional (3.5), the optimal communication and abatement strategies c^* and r^* equalize marginal benefits and marginal costs:

$$D_t^c J(c^*, r^*) = \kappa_c c_t^*, \quad D_t^r J(c^*, r^*) = \kappa_r r_t^*.$$

□

B.3 Interpretation of the optimal strategy

Proof of Corollary 3.1. From (3.6),

$$\kappa_c^i c_t^{i,*} = B^i - A^i(E_t^{i,*} - V_t^{i,*}), \quad \kappa_r^i r_t^{i,*} = \frac{\beta}{\delta} - B^i + A^i(E_t^{i,*} - V_t^{i,*}).$$

Hence, $\kappa_c^i c_t^{i,*} + \kappa_r^i r_t^{i,*} = \frac{\beta}{\delta}$.

□

Proof of Proposition 4. Referring to Proposition 2 describing optimal controls in the general case, notice that $A^i = 0$ when $\alpha = 0$, and hence $B^i = \frac{\beta}{\delta + a + b\lambda^i}$. Therefore, optimal controls are given by (3.9).

□

Proof of Corollary 4.1. The marginal rate of substitution from abatement to environmental communication, $MRS^{r \rightarrow c, i}$, can be computed as follows:

$$MRS_t^{r \rightarrow c, i} = \frac{\Pi_t^{r, i}}{\Pi_t^{c, i}},$$

with $\Pi_t^{r, i}, \Pi_t^{c, i}$ given in Proposition 3. When $\alpha = 0$, we have $T^i = 0$, and hence

$$\Pi_t^{r, i} = \frac{\beta}{\delta} - \frac{\beta}{\delta + a + b\lambda^i}, \quad \Pi_t^{c, i} = \frac{\beta}{\delta + a + b\lambda^i}.$$

A quick computation gives the result.

□

Proof of Proposition 5. We drop the exponent i in the proof for simplicity. According to Proposition 4, $c_t^* = \frac{1}{\kappa_c} \frac{\beta}{\delta + a + b\lambda}$. As we have assumed that $\beta > 0$ in this subsection, we always have $c_t^* > 0$. Moreover, using the expressions of c_t^*, r_t^* given in Proposition 4, we have

$$c_t^* > r_t^* \iff \frac{r_t^*}{c_t^*} < 1 \iff \frac{\kappa_r^i}{\kappa_c^i} > \frac{a + b\lambda^i}{\delta}. \quad (3.47)$$

(i) Let us assume that condition (3.10) is satisfied. Referring to Definition 3, it implies that the company greenwashes if, and only if, $E_t^* \geq V_t^*$. Now, if the company is overrated at time t , i.e. if $E_t^* \geq V_t^*$, as $0 \leq a, b \leq 1$ and $c_t^* > 0, c_t^* > r_t^*$, the only possibility to get $E_s^* < V_s^*$ is through the measurement noise $z dW_t$ in (3.2a). Indeed, referring to the explicit expression of

$X_t^* = E_t^* - V_t^*$ in (3.45), all terms are positive except the Itô integral which can be negative, and which represents the measurement error.

Then, greenwashing effort can be computed as follows, using Proposition 4:

$$G^\beta := c_t^* - r_t^* = \frac{1}{\kappa_c} \frac{\beta}{\delta + a + b\lambda} - \frac{1}{\kappa_r} \left(\frac{\beta}{\delta} - \frac{\beta}{\delta + a + b\lambda} \right) = \frac{2}{\bar{\kappa}} \frac{\beta}{\delta + a + b\lambda} - \frac{\beta}{\delta \kappa_r}.$$

Moreover, $G^\beta > 0$ as $c_t^* - r_t^* > 0$ under condition (3.10).

(ii) If condition (3.10) is not verified, we have $c_t^* \leq r_t^*$ for all t , and hence the company never greenwashes. \square

Proof of Proposition 6. We drop the exponent i for simplicity. Inserting optimal strategies of equation (3.9) into the dynamics of the environmental score (3.2a), one can deduce that $\mathbb{E}[E_t^* - V_t^*]$ verifies the following ODE:

$$\begin{aligned} d\mathbb{E}[E_t^* - V_t^*] &= \left(-(a + \lambda b)\mathbb{E}[E_t^* - V_t^*] + \frac{2}{\bar{\kappa}} \frac{\beta}{\delta + a + b\lambda^i} - \frac{\beta}{\delta \kappa_r} \right) dt, \\ \mathbb{E}[E_0^* - V_0^*] &= q - p. \end{aligned}$$

The solution to this ODE exists, is unique and given by

$$\mathbb{E}[E_t^* - V_t^*] = e^{-(a+b\lambda)t} (q - p) + \left(\frac{2}{\bar{\kappa}} \frac{\beta}{\delta + a + b\lambda^i} - \frac{\beta}{\delta \kappa_r} \right) \frac{1}{a + b\lambda} \left(1 - e^{-(a+b\lambda)t} \right).$$

Therefore,

$$\mathbb{E}[E_t^* - V_t^*] \xrightarrow{t \rightarrow \infty} \frac{1}{a + b\lambda} \left(\frac{2}{\bar{\kappa}} \frac{\beta}{\delta + a + b\lambda} - \frac{\beta}{\delta \kappa_r} \right) =: L_\beta.$$

(Reminding that $a + b\lambda > 0$ as assumed after Definition 4.) Now,

$$L_\beta > 0 \iff \frac{2}{\bar{\kappa}} \frac{\beta}{\delta + a + b\lambda} - \frac{\beta}{\delta \kappa_r} > 0 \iff \frac{\kappa_r}{\kappa_c} > \frac{a + b\lambda}{\delta}.$$

This concludes the proof. \square

Proof of Proposition 7. Referring to Proposition 2, when $\beta = 0$, we have that $B^i = 0$, and A^i is unchanged. This gives optimal controls as in (3.11).

(i) and (ii) can be deduced from the shapes of the optimal controls in equation (3.11), using that $A^i > 0$ if $\alpha > 0$, as it can be seen in equation (3.7), and recording the definitions of the two types of environmental communication (Definitions 1 and 2). \square

Proof of Corollary 7.1. The proof is similar to the proof of Corollary 4.1. This time, as $\beta = 0$, we get

$$\Pi_t^{r^i, i} = 2T^i \mathbb{E} \left[\int_t^\infty e^{-(\delta+a)(s-t)} (1-b)^{N_s^i - N_t^i} \left(E_s^i - V_s^i \right) ds \middle| \mathcal{F}_t \right],$$

$$\Pi_t^{c^i, i} = -2T^i \mathbb{E} \left[\int_t^\infty e^{-(\delta+a)(s-t)} (1-b)^{N_s^i - N_t^i} (E_s^i - V_s^i) ds \middle| \mathcal{F}_t \right]$$

from Proposition 3. Hence the result. \square

Proof of Proposition 8. According to equation (3.11), as $A^i \geq 0$ (refer to Proposition 2), when $E_t^{i,*} \geq V_t^{i,*}$, $c_t^* \leq 0$. And, when $E_t^{i,*} < V_t^{i,*}$, $c_t^{i,*} > 0$. Therefore, the cases $E_t^* \geq V_t^*$ and $c_t^* > 0$ never happen at the same instants. Referring to the definition of greenwashing (Definition 3), one can conclude that the company never practices greenwashing in this limiting case.

Now, let us show that $\lim_{t \rightarrow \infty} \mathbb{E}[E_t^{i,*} - V_t^{i,*}] = 0$. Inserting optimal strategies of equation (3.11) into the dynamics of the environmental score (3.2a), one can deduce that $\mathbb{E}[E_t^* - V_t^*]$ verifies the following ODE:

$$\begin{aligned} d\mathbb{E}[E_t^* - V_t^*] &= -(a + \lambda b + \frac{2}{\bar{\kappa}}A)\mathbb{E}[E_t^* - V_t^*]dt, \\ \mathbb{E}[E_0^* - V_0^*] &= q - p. \end{aligned}$$

The solution to this ODE exists, is unique and given by

$$\mathbb{E}[E_t^* - V_t^*] = e^{-(a+b\lambda+\frac{2}{\bar{\kappa}}A)t} (q - p).$$

Therefore, $\mathbb{E}[E_t^* - V_t^*] \xrightarrow{t \rightarrow \infty} 0$, where the convergence takes place with an exponential rate. \square

Proof of Proposition 9. We drop the i indices in the proof for simplicity. We have

$$c_t^* - r_t^* = \frac{2}{\bar{\kappa}} (-A(E_t^* - V_t^*) + B) - \frac{\beta}{\delta \kappa_r}. \quad (3.48)$$

Hence, when $E_t^* \geq V_t^*$, the maximum value of $c_t^* - r_t^*$ is equal to $\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta \kappa_r}$. Now,

$$\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta \kappa_r} > 0 \iff \frac{\kappa_r}{\kappa_c} > \frac{a + b\lambda}{\delta}.$$

Therefore, referring to Definition 3, if $\frac{\kappa_r}{\kappa_c} \leq \frac{a+b\lambda}{\delta}$, the company never greenwashes.

Then, we have $c_t^* > 0 \iff E_t^* - V_t^* < \frac{B}{A}$ according to the optimal communication strategy given in Proposition 2. Moreover, one can deduce out of equation (3.48) that

$$c_t^* - r_t^* > 0 \iff E_t^* - V_t^* < \frac{1}{\frac{2}{\bar{\kappa}}A} \left(\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta \kappa_r} \right) = \frac{B}{A} - \frac{1}{\frac{2}{\bar{\kappa}}} \frac{\beta}{\delta \kappa_r} < \frac{B}{A}.$$

Combining the two conditions and referring to Definition 3, the company greenwashes if, and only if, $0 \leq E_t^* - V_t^* < \frac{1}{\frac{2}{\bar{\kappa}}A} \left(\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta \kappa_r} \right)$, which is a non-empty event only under condition (3.10) which guarantees that $\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta \kappa_r} > 0$, as stated above.

Moreover, as stated above, when $\frac{\kappa_r}{\kappa_c} > \frac{a+b\lambda}{\delta}$, $\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta \kappa_r} > 0$ and it is the maximal value of $c_t^* - r_t^*$ when $E_t^* \geq V_t^*$, hence the maximal value of greenwashing effort. Moreover, greenwashing

effort $c_t^* - r_t^*$ decreases linearly in $E_t^* - V_t^*$ according to equation (3.48). Finally, when $E_t^* - V_t^* = \frac{1}{\frac{2}{\bar{\kappa}}A} \left(\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_r} \right)$, greenwashing effort is null, as can be derived from the same equation. \square

Proof of Proposition 10. Let us compute $\lim_{t \rightarrow \infty} \mathbb{E}[E_t^* - V_t^*]$. We have

$$c^* - r_t^* = \frac{2}{\bar{\kappa}} \left(-A(E_t^* - V_t^*) + B \right) - \frac{\beta}{\delta\kappa_r}$$

and hence

$$d\mathbb{E}[E_t^* - V_t^*] = \left(-(a + b\lambda + \frac{2}{\bar{\kappa}}A)\mathbb{E}[E_t^* - V_t^*] + \frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_r} \right) dt.$$

This ODE has a unique solution which is

$$\mathbb{E}[E_t^* - V_t^*] = e^{-(a+b\lambda+\frac{2}{\bar{\kappa}}A)t} \left(q - p - \frac{\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_r}}{a + b\lambda + \frac{2}{\bar{\kappa}}A} \right) + \frac{\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_r}}{a + b\lambda + \frac{2}{\bar{\kappa}}A}.$$

Therefore, we have

$$\lim_{t \rightarrow \infty} \mathbb{E}[E_t^* - V_t^*] = \frac{\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_r}}{a + b\lambda + \frac{2}{\bar{\kappa}}A} = \frac{1}{a + b\lambda + \frac{2}{\bar{\kappa}}A} G_{max},$$

where the convergence takes place with an exponential rate. \square

Proof of Proposition 11. As usual, we drop the index i for the sake of simplicity.

(i) According to Propositions 9 and 2, we have

$$G_{max} = \frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_r} = \beta \left(\frac{2}{\bar{\kappa}} \frac{1 + \frac{A}{\delta\kappa_r}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}}A} - \frac{1}{\delta\kappa_r} \right).$$

Noticing that $\frac{2}{\bar{\kappa}}B - \frac{\beta}{\delta\kappa_r} > 0$ when condition (3.10) is satisfied, as $\beta > 0$, we deduce that

$$\frac{2}{\bar{\kappa}} \frac{1 + \frac{A}{\delta\kappa_r}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}}A} - \frac{1}{\delta\kappa_r} > 0$$

under this condition. It means that G_{max} increases linearly in β when condition (3.10) is satisfied.

(ii) Using the expression of A in Proposition 2, we get that

$$\frac{\partial G_{max}}{\partial \alpha} = \beta \frac{\frac{1}{\delta\kappa_r} (\delta + a + \lambda b) - \frac{2}{\bar{\kappa}}}{\left(\delta + a + \lambda b + \frac{2}{\bar{\kappa}}A \right)^2} \frac{1}{\sqrt{1 + \frac{16}{\bar{\kappa}} \frac{T}{R^2}}} \frac{4}{\bar{\kappa}} \frac{1}{R} \frac{\lambda^i b^2}{\delta + \rho}.$$

Now,

$$\frac{1}{\delta\kappa_r} (\delta + a + \lambda b) - \frac{2}{\bar{\kappa}} < 0 \iff \frac{\kappa_r}{\kappa_c} > \frac{a + b\lambda}{\delta}.$$

Hence, as all the other terms are positive, G_{max} decreases with α when condition (3.10) is verified. Moreover, as T increases with α and A increases with α through T , we can see that $\frac{\partial G_{max}}{\partial \alpha}$ increases with α when condition (3.10) is verified, as it is negative under this condition. Hence, G_{max} is convex in α under condition (3.10). \square

Proof of Proposition 12. As usual, we drop the index i for the sake of simplicity.

(i) According to Proposition 2, the constant in the optimal abatement strategy is equal to

$$\frac{1}{\kappa_r} \left(\frac{\beta}{\delta} - B \right) = \beta \frac{1}{\kappa_r} \left(\frac{1}{\delta} - \frac{1 + \frac{A}{\delta \kappa_r}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}} A} \right)$$

Now, using that $\frac{2}{\bar{\kappa}} = \frac{1}{\kappa_r} + \frac{1}{\kappa_c}$, we have

$$\frac{1}{\delta} > \frac{1 + \frac{A}{\delta \kappa_r}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}} A} \iff 1 > \frac{\delta + \frac{A}{\kappa_r}}{\delta + a + \lambda b + \left(\frac{1}{\kappa_r} + \frac{1}{\kappa_c} \right) A}$$

which is always true as all parameters are positive. Hence, $\frac{1}{\delta} - \frac{1 + \frac{A}{\delta \kappa_r}}{\delta + a + \lambda b + \frac{2}{\bar{\kappa}} A} > 0$, which means that $\frac{1}{\kappa_r} \left(\frac{\beta}{\delta} - B \right)$ increases linearly in β .

(ii) Using the expression of A in Proposition 2, we get that

$$\frac{\partial \left(\frac{1}{\kappa_r} \left(\frac{\beta}{\delta} - B \right) \right)}{\partial \alpha} = - \frac{1}{\kappa_r} \beta \frac{\frac{1}{\delta \kappa_r} (\delta + a + \lambda b) - \frac{2}{\bar{\kappa}}}{\left(\delta + a + \lambda b + \frac{2}{\bar{\kappa}} A \right)^2} \frac{1}{\sqrt{1 + \frac{16}{\bar{\kappa}} \frac{T}{R^2}}} \frac{8}{\bar{\kappa}^2} \frac{1}{R} \frac{\lambda^i b^2}{\delta + \rho}.$$

which is positive under condition (3.10), according to the proof of Proposition 11 (ii). Moreover, using similar arguments as in that proof, we get that, under condition (3.10), $\frac{1}{\kappa_r} \left(\frac{\beta}{\delta} - B \right)$ is concave in α . \square

B.4 A Greenwashing mean field game

Proposition 17 (Equilibrium expected returns in the n -player game). *Let us assume that E, M , solutions of dynamics (3.13), verify $E, M \in \mathbb{H}_n^2(\delta^I)$. Moreover, let us define S as a solution to (1) and the set of admissible strategies \mathbb{A}^ω for the program of the investor as $\mathbb{A}^\omega := \mathbb{H}_n^2(\delta^I)$.*

Then, the optimal portfolio choice of the investor is the pointwise solution

$$\omega_t^* = \frac{1}{\gamma} \Sigma^{-1} \left(\mu_t + \beta \frac{E_t}{h\left(\frac{1}{n} \sum_i E_t^i\right)} - \alpha M_t \right),$$

and equilibrium expected returns are

$$\mu_t = \gamma \Sigma \mathbf{1}_n - \beta \frac{E_t}{h\left(\frac{1}{n} \sum_i E_t^i\right)} + \alpha M_t.$$

Proof of Proposition 17. Under the assumptions of the proposition, the investor's program can be rewritten as

$$\begin{aligned} & \sup_{\omega \in \mathbb{A}^\omega} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \omega'_t \left(\mu_t + \beta \frac{E_t}{h(\frac{1}{n} \sum_i E_t^i)} - \alpha M_t - \frac{\gamma}{2} \Sigma \omega_t \right) dt \right] \\ &= \sup_{\omega \in \mathbb{A}^\omega} \mathbb{E} \left[\int_0^\infty e^{-\delta t} \left\{ -\frac{\gamma}{2} (\omega_t - \omega_t^*)' \Sigma (\omega_t - \omega_t^*) + \frac{\gamma}{2} \omega_t^{*'} \Sigma \omega_t^* \right\} dt \right]. \end{aligned}$$

The optimal portfolio choice of the investor is thus the pointwise solution ω_t^* . In addition, as the quantity of each asset is assumed to be normalised to one in the market, writing $\mathbf{1}_n$ a vector of ones of size n , market clearing condition writes:

$$\forall t, \omega_t^* = \mathbf{1}_n.$$

Equilibrium expected returns are therefore

$$\mu_t = \gamma \Sigma \mathbf{1}_n - \beta \frac{E_t}{h(\frac{1}{n} \sum_i E_t^i)} + \alpha M_t.$$

□

Proof of Proposition 13. Let us define the value function, \hat{v} , of the representative company:

$$\hat{v}(q, p, u) := \sup_{(r^i, c^i) \in \mathbb{A}_T} \mathbb{E} \left[\int_0^T e^{-\delta t} \left(\beta \frac{E_t^q}{h(m_t)} - \alpha M_t^u + \frac{\kappa_r}{2} (r_t)^2 - \frac{\kappa_c}{2} (c_t)^2 \right) dt \right]$$

with the following constraints. The state variables of the representative company's program are the tridimensional process (E^q, V^p, M^u) which is the unique strong solution (Protter, 2005, Chapter 5, Theorem 52) to the following SDEs:

$$\begin{cases} dE_t^q = a(V_t^p - E_t^q)dt + b(V_{t-}^p - E_{t-}^q)dN_t + c_t dt + z dW_t, & E_0^q = q, \\ dV_t^p = r_t dt, & V_0^p = p, \\ dM_t^u = -\rho M_t^u dt + b^2(V_{t-}^p - E_{t-}^q)^2 dN_t, & M_0^u = u, \end{cases} \quad (3.49)$$

for $(q, p, u) \in \mathcal{Y}$, $\mathcal{Y} := \mathbb{R}^2 \times \mathbb{R}_+$ and $(r, c) \in \mathbb{A}_T$.

The program can be rewritten, up to a constant (depending on m), as follows:

$$\sup_{(r, c) \in \mathbb{A}_T} \mathbb{E} \left[\int_0^T e^{-\delta t} \left(\frac{\beta}{h(m_t)} (E_t - V_t) - \alpha M_t - \frac{\kappa_r}{2} \left(r_t - \frac{\beta}{\kappa_r} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right)^2 - \frac{\kappa_c}{2} (c_t)^2 \right) dt \right].$$

Then, remark that

$$\begin{aligned} & \frac{\kappa_r}{2} \left(r_t - \frac{\beta}{\kappa_r} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right)^2 + \frac{\kappa_c}{2} (c_t)^2 \\ &= \frac{\bar{\kappa}}{4} \left(c_t - r_t + \frac{\beta}{\kappa_r} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right)^2 + \frac{1}{2(\kappa_r + \kappa_c)} \left(\kappa_c c_t + \kappa_r r_t - \beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right)^2. \end{aligned}$$

Let $\xi_t = c_t - r_t$ with $(r, c) \in \mathbb{A}_T$ and introduce the new state process $X_t = E_t^q - V_t^p$, so that

$$dX_t^x = -aX_t^x dt - bX_{t-}^x dN_t + \xi_t dt + z dW_t, \quad X_0 = x = q - p,$$

$$dM_t^u = -\rho M_t^u dt + (-bX_{t-}^x)^2 dN_t, \quad M_0 = u.$$

We have $\hat{v}(q, p, u) = \tilde{v}(x, u)$, with

$$\begin{aligned} \tilde{v}(x, u) = \sup_{\substack{\xi=c-r, \\ (r,c) \in \mathbb{A}_T}} \mathbb{E} \left[\int_0^T e^{-\delta t} \left(\frac{\beta}{h(m_t)} X_t^x - \alpha M_t^u - \frac{\bar{\kappa}}{4} \left(c_t - r_t + \frac{\beta}{\kappa_r} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right)^2 \right. \right. \\ \left. \left. - \frac{1}{2(\kappa_r + \kappa_c)} \left(\kappa_c c_t + \kappa_r r_t - \beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right)^2 \right) dt \right]. \end{aligned}$$

It is then clear that at optimum, the controls satisfy

$$\kappa_c c_t + \kappa_r r_t - \beta \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds = 0,$$

and that we can concentrate on the following auxiliary two dimensional problem:

$$\tilde{v}(x, u) = \sup_{\substack{\xi=c-r, \\ (r,c) \in \mathbb{A}_T}} \mathbb{E} \left[\int_0^T e^{-\delta t} \left(\frac{\beta}{h(m_t)} X_t^x - \alpha M_t^u - \frac{\bar{\kappa}}{4} \left(\xi_t + \frac{\beta}{\kappa_r} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right)^2 \right) dt \right].$$

Let $\mathcal{X}_T := [0, T] \times \mathbb{R} \times \mathbb{R}_+$, and $(t, x, u) \in \mathcal{X}_T$. We define, on \mathcal{X}_T , the value function in time as follows:

$$v(t, x, u) = \sup_{\xi \in \mathbb{A}_T^\xi} \mathbb{E} \left[\int_t^T e^{-\delta(s-t)} f_T(s, X_s^{t,x}, M_s^{t,u}) ds \right],$$

with $f_T(s, x, u, \xi) := \frac{\beta}{h(m_s)} x - \alpha u - \frac{\bar{\kappa}}{4} \left(\xi + \frac{\beta}{\kappa_r} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr \right)^2$, \mathbb{A}_T^ξ the set of \mathbb{F} -progressively measurable \mathbb{R} -valued processes ξ which verify $\mathbb{E} \left[\int_0^T |\xi_t|^2 dt \right] < \infty$, and the auxiliary bidimensional state variables process $(X^{t,x}, M^{t,u})$ as the unique strong solution to the following SDEs (Protter,

2005, Chapter 5, Theorem 52):

$$\begin{cases} dX_s^{t,x} = -aX_s t, x ds - bX_{s-t}, x dN_s + \xi_s ds + z dW_s, & X_t = x, \\ dM_s^{t,u} = -\rho M_s^{t,u} ds + (bX_{s-}^{t,u})^2 dN_s, & M_t = u. \end{cases} \quad (3.50)$$

Moreover, note that for any $\xi \in \mathbb{A}_T^\xi$, the bidimensional auxiliary state variable (3.50) admits the following explicit solution:

$$X_s^{t,x} = \mathcal{E}_s^t x + \mathcal{E}_s^t \int_t^s (\mathcal{E}_r^t)^{-1} \{ \xi_r dr + z dW_r \}, \quad (3.51)$$

$$M_s^{t,u} = e^{-\rho(s-t)} u + \int_t^s e^{-\rho(s-r)} (bX_{r-}^{t,x})^2 dN_r, \quad (3.52)$$

with $\mathcal{E}_s^t = e^{-a(s-t)}(1-b)^{N_s-N_t}$, and writing $0^0 = 1$.

HJB equation The value function satisfies the following HJB equation, for all $(t, x, u) \in \mathcal{X}_T$, omitting the argument (t, x, u) of the function v and its partial derivatives when it is clear:

$$\begin{aligned} & \max_{\xi \in \mathbb{R}} \left\{ \frac{\beta}{h(m_t)} x - \alpha u - \frac{\bar{\kappa}}{4} \left(\xi + \frac{\beta}{\kappa_r} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right)^2 \right. \\ & \left. - \delta v + \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} (-ax + \xi) - \frac{\partial v}{\partial u} \rho u + \frac{z^2}{2} \frac{\partial^2 v}{\partial x^2} + \lambda \left[v(t, x(1-b), u + b^2 x^2) - v(t, x, u) \right] \right\} = 0, \end{aligned} \quad (3.53)$$

or in other words, replacing ξ by the optimizing function $\xi^*(t, x, u) := \frac{2}{\bar{\kappa}} \frac{\partial v}{\partial x} - \frac{\beta}{\kappa_r} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds$,

$$\begin{aligned} & \frac{\beta}{h(m_t)} x - \alpha u + \frac{1}{\bar{\kappa}} \left(\frac{\partial v}{\partial x} \right)^2 - \delta v + \frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} \left(ax + \frac{\beta}{\kappa_r} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds \right) - \frac{\partial v}{\partial u} \rho u \\ & + \frac{z^2}{2} \frac{\partial^2 v}{\partial x^2} + \lambda \left[v(t, x(1-b), u + b^2 x^2) - v(t, x, u) \right] = 0. \end{aligned}$$

Let us use the ansatz

$$w(t, x, u) = \frac{1}{2} A(t) x^2 + B(t) x + C(t) u + w_0(t).$$

Substituting this into the equation and collecting terms with the same powers of u and x , we get:

$$\begin{aligned} & -\alpha + \dot{C}(t) - (\rho + \delta)C(t) = 0, \quad C(T) = 0 \\ & \dot{A}(t) + \frac{2}{\bar{\kappa}} A(t)^2 - \left(\delta + 2a + \lambda(1 - (1-b)^2) \right) A(t) + 2\lambda b^2 C(t) = 0, \quad A(T) = 0 \\ & \dot{B}(t) + \left(\frac{2}{\bar{\kappa}} A(t) - \delta - a - \lambda b \right) B(t) + \frac{\beta}{h(m_t)} - A(t) \frac{\beta}{\kappa_r} \int_t^T \frac{e^{-\delta(s-t)}}{h(m_s)} ds = 0, \quad B(T) = 0, \end{aligned}$$

and the optimal control is, for all $s \in [t, T]$,

$$\hat{\xi}_s = \frac{2}{\kappa} \left(A(s) \hat{X}_s^{t,x} + B(s) \right) - \frac{\beta}{\kappa_r} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr,$$

with $\hat{X}^{t,x}$ solution of the dynamics (3.50) when the strategy $\hat{\xi}$ is employed.

Solutions to the above solvable equations are :

$$C(t) = \frac{\alpha}{\delta + \rho} e^{-(\delta+\rho)(T-t)} - \frac{\alpha}{\delta + \rho},$$

$$B(t) = \beta \int_t^T e^{\int_t^s (\frac{2}{\kappa} A(u) - \delta - a - \lambda b) du} \left(\frac{1}{h(m_s)} - \frac{A(s)}{\kappa_r} \int_s^T \frac{e^{-\delta(u-s)}}{h(m_u)} du \right) ds.$$

Existence and negativity of the Riccati solution Before stating the verification argument, let us show that the Riccati equation (3.16) admits a unique solution, which is negative. Let the Riccati equation (3.16) be rewritten, with a suitable function f , as $\dot{A}(t) = f(t, A(t))$. Fix a large constant M . The equation $\dot{A}_M(t) = -M \vee f(t, A_M(t)) \wedge M$ has a C^1 solution on $[0, T]$ by Cauchy-Lipschitz theorem. Let us define A^-, A^+ solutions of the two following Riccati equations:

$$\dot{A}^-(t) = -\frac{2}{\kappa} A^-(t)^2 + (\delta + 2a + \lambda(1 - (1-b)^2))A^-(t) + 2\lambda b^2 \frac{\alpha}{\delta + \rho}, \quad A^-(T) = 0, \quad (3.54)$$

$$\dot{A}^+(t) = -\frac{2}{\kappa} A^+(t)^2 + (\delta + 2a + \lambda(1 - (1-b)^2))A^+(t), \quad A^+(T) = 0. \quad (3.55)$$

They admit the following explicit solutions: for all $t \in [0, T]$, $A^+(t) = 0$, and

$$A^-(t) = \frac{2\lambda b^2 \alpha}{\delta + \rho} \frac{e^{2\sqrt{R}(T-t)} - 1}{-\left(\sqrt{R} + \left(a + \frac{\delta + \lambda(1 - (1-b)^2)}{2}\right)\right) e^{2\sqrt{R}(T-t)} + a + \frac{\delta + \lambda(1 - (1-b)^2)}{2} - \sqrt{R}},$$

$$R = \left(a + \frac{\delta + \lambda(1 - (1-b)^2)}{2} \right)^2 + 2\lambda b^2 \frac{2}{\kappa} \frac{\alpha}{\delta + \rho}.$$

Let us rewrite them $\dot{A}^-(t) = g(t, A^-(t))$, $\dot{A}^+(t) = h(t, A^+(t))$ by defining g, h accordingly. We have that, $\forall t \in [0, T]$, $\forall x \in \mathbb{R}$, $h(t, x) \leq f(t, x) \leq g(t, x)$, and $A^+(T) = A(T) = A^-(T) = 0$. Moreover, note that A^- and A^+ are bounded according to their explicit solutions. Then, by the comparison theorem we have that for M sufficiently large, $\forall t \in [0, T]$, $A^+(t) \geq A_M(t) \geq A^-(t)$. By the boundedness of A^- and A^+ , we have that, for M sufficiently large, $f(t, A_M(t)) \in [-M, M]$, which means that A_M solves the original Riccati equation on $[0, T]$. Finally, Cauchy-Lipschitz theorem guarantees that $A_M = A$ is unique. In particular, as $A \leq A^+ = 0$, A is negative.

Verification argument for the auxiliary program Let us define on \mathcal{X}_T the function

$$w(t, x, u) = \frac{1}{2}A(t)x^2 + B(t)x + C(t)u + w_0(t).$$

Let us show that $v = w$.

(i) Let $\xi \in \mathbb{A}_T^\xi$ and $(t, x, u) \in \mathcal{X}_T$. Let $s \in [t, T]$. By Itô's formula, for the stopping time τ_n defined below, we have:

$$\begin{aligned} e^{-\delta(s \wedge \tau_n)} w(s \wedge \tau_n, X_{s \wedge \tau_n}^{t,x}, M_{s \wedge \tau_n}^{t,u}) \\ = e^{-\delta t} w(t, x, u) + \int_t^{s \wedge \tau_n} e^{-\delta r} \left(-\delta w + \frac{\partial w}{\partial t} + \mathcal{L}^{\xi_r} w \right) (r, X_r^{t,x}, M_r^{t,u}) dr \\ + \int_t^{s \wedge \tau_n} e^{-\delta r} \frac{\partial w}{\partial x} (r, X_r^{t,x}, M_r^{t,u}) z dW_r, \end{aligned}$$

with the stopping time

$$\tau_n := \inf\{s \geq t : \int_t^s e^{-\delta r} \left| \frac{\partial w}{\partial x} (r, X_r^{t,x}, M_r^{t,u}) \right|^2 dr \geq n\}, \quad \forall n \in \mathbb{N},$$

using the convention that $\inf\{\emptyset\} = \infty$, and the operator $\mathcal{L}^\xi w$ defined as follows, omitting the argument (t, x, u) of the function w and its partial derivatives when it is clear:

$$\forall (t, x, u) \in \mathcal{X}_T,$$

$$\mathcal{L}^\xi w(t, x, u) := \frac{\partial w}{\partial x}(-ax + \xi) - \frac{\partial w}{\partial u} \rho u + \frac{z^2}{2} \frac{\partial^2 w}{\partial x^2} + \lambda \left[w(t, x(1-b), u + b^2 x^2) - w(t, x, u) \right].$$

The stopped stochastic integral is a martingale, and by taking the expectation we get

$$\begin{aligned} \mathbb{E} \left[e^{-\delta(s \wedge \tau_n)} w(s \wedge \tau_n, X_{s \wedge \tau_n}^{t,x}, M_{s \wedge \tau_n}^{t,u}) \right] \\ = \mathbb{E} \left[e^{-\delta t} w(t, x, u) + \int_t^{s \wedge \tau_n} e^{-\delta r} \left(-\delta w + \frac{\partial w}{\partial t} + \mathcal{L}^{\xi_r} w \right) (r, X_r^{t,x}, M_r^{t,u}) dr \right] \\ \leq e^{-\delta t} w(t, x, u) - \mathbb{E} \left[\int_t^{s \wedge \tau_n} e^{-\delta r} f_T(r, X_r^{t,x}, M_r^{t,u}, \xi_r) dr \right], \end{aligned}$$

using equation (3.53), as ξ is any admissible control. By Lemmas B.10 and B.11, we may apply the dominated convergence theorem and send n to infinity:

$$\mathbb{E}[e^{-\delta s} w(s, X_s^{t,x}, M_s^{t,u})] \leq e^{-\delta t} w(t, x, u) - \mathbb{E} \left[\int_t^s e^{-\delta r} f_T(r, X_r^{t,x}, M_r^{t,u}, \xi_r) dr \right]. \quad (3.56)$$

By sending now s to T , as w is continuous and $w(T, X_T^{t,x}, M_T^{t,u}) = 0$, using again Lemmas B.10

and B.11, we then deduce

$$w(t, x, u) \geq \mathbb{E} \left[\int_t^T e^{-\delta(r-t)} f_T(r, X_r^{t,x}, M_r^{t,u}, \xi_r) dr \right], \quad \forall \xi \in \mathbb{A}_T^\xi,$$

and so $w \geq v$ on \mathcal{X}_T .

(ii) By repeating the above arguments and observing that the optimal control $\hat{\xi}$ achieves equality in (3.56) by construction, we have

$$\mathbb{E}[e^{-\delta s} w(s, X_s^{t,x}, M_s^{t,u})] = e^{-\delta t} w(t, x, u) - \mathbb{E} \left[\int_t^s e^{-\delta r} f_T(r, \hat{X}_r^{t,x}, \hat{M}_r^{t,u}, \hat{\xi}_r) dr \right].$$

From Lemma B.12, $\hat{\xi} \in \mathbb{A}^\xi$, and hence Lemma B.11 can be applied. By sending s to T , we then deduce

$$w(t, x, u) \leq \mathbb{E} \left[\int_t^T e^{-\delta(r-t)} f_T(r, \hat{X}_r^{t,x}, \hat{M}_r^{t,u}, \hat{\xi}_r) dr \right] \leq v(x, u).$$

Combining with the conclusion to (i), this shows that $w = v$ on \mathcal{X}_T , and that the process

$$\{\hat{\xi}_s = \xi^*(s, \hat{X}_s^{t,x}, \hat{M}_s^{t,u}), s \in [t, T]\}$$

is an optimal control.

Now, from Lemma B.13, we get that that if ξ^1 and ξ^2 are both optimal controls, then

$$\int_0^T e^{-\delta t} |\xi_t^1 - \xi_t^2|^2 dt = 0,$$

hence the optimal control is unique, up to t -almost everywhere and almost sure equivalence.

Conclusion for the initial optimization program We can deduce the unique optimal control (\hat{c}, \hat{r}) to the equivalent program \hat{v} from the following system:

$$\begin{cases} \kappa_c \hat{c}_s + \kappa_r \hat{r}_s - \beta \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr = 0, \\ \hat{\xi}_s = \frac{2}{\kappa} \left(A(s) \hat{X}_s^{t,x} + B(s) \right) - \frac{\beta}{\kappa_r} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr, \end{cases}$$

so that, for all $s \in [t, T]$, $(q, p, u) \in \mathcal{Y}$,

$$\hat{r}_s = \frac{1}{\kappa_r} \left(\beta \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr - B(s) - A(s) (\hat{E}_s^{t,q} - \hat{V}_s^{t,p}) \right), \quad \hat{c}_s = \frac{1}{\kappa_c} \left(B(s) + A(s) (\hat{E}_s^{t,q} - \hat{V}_s^{t,p}) \right),$$

with $(\hat{E}^q, \hat{V}^p, \hat{M}^u)$ solutions of (3.49) when the strategy (\hat{c}, \hat{r}) is employed. \square

Lemma B.9. *If $\xi \in \mathbb{A}_T^\xi$, then for all $(t, x, u) \in \mathcal{X}_T$,*

$$\mathbb{E} \left[\int_t^T e^{-\delta s} \left(\int_t^s |X_u^x|^2 du \right) ds \right] < \infty.$$

Moreover, $\forall s \in [t, T]$, $\mathbb{E}[|M_s^{t,u}|] < \infty$.

Proof. (i) By integration by parts,

$$\begin{aligned} \mathbb{E} \left[\int_t^T e^{-\delta s} \left(\int_t^s |X_r^{t,x}|^2 dr \right) ds \right] &= \mathbb{E} \left[\int_t^T \left(\int_s^T e^{-\delta r} dr \right) |X_s^{t,x}|^2 ds \right] \\ &\leq \frac{1}{\delta} \mathbb{E} \left[\int_t^T e^{-\delta s} |X_s^{t,x}|^2 ds \right]. \end{aligned}$$

Now, referring to the explicit expression of $X^{t,x}$ in (3.51), we have, using Jensen inequality,

$$\begin{aligned} |X_s^{t,x}|^2 &\leq 3 \left((\mathcal{E}_s^t)^2 |x|^2 + \int_t^s e^{-a(s-r)} dr \int_t^s e^{-a(s-r)} (1-b)^{2(N_s-N_r)} |\xi_r|^2 dr \right. \\ &\quad \left. + z^2 \left(\int_t^s e^{-a(s-r)} (1-b)^{N_s-N_r} dW_r \right)^2 \right) \quad (3.57) \\ &\leq 3 \left(|x|^2 + \frac{1}{a} \int_t^s |\xi_r|^2 dr + z^2 \left(\int_t^s e^{-a(s-r)} (1-b)^{N_s-N_r} dW_r \right)^2 \right). \end{aligned}$$

Noting that

$$\mathbb{E} \left[\left(\int_t^s e^{-a(s-r)} (1-b)^{N_s-N_r} dW_r \right)^2 \right] = \mathbb{E} \left[\int_t^s e^{-2a(s-r)} (1-b)^{2(N_s-N_r)} dr \right] \leq s-t, \quad (3.58)$$

we get

$$\mathbb{E} \left[|X_s^{t,x}|^2 \right] \leq 3 \left(|x|^2 + \frac{1}{a} \int_t^s |\xi_r|^2 dr + z^2 (s-t) \right). \quad (3.59)$$

Hence, applying Fubini,

$$\mathbb{E} \left[\int_t^T e^{-\delta s} |X_s^{t,x}|^2 ds \right] \leq \tilde{C} \mathbb{E} \left[1 + \int_t^T e^{-\delta s} \left(\int_t^s \xi_r^2 dr \right) dt \right]$$

with a constant $\tilde{C} > 0$. By integration by parts, we have

$$\mathbb{E} \left[\int_t^T e^{-\delta s} \left(\int_t^s \xi_r^2 dr \right) dt \right] \leq \frac{1}{\delta} \mathbb{E} \left[\int_t^T e^{-\delta s} \xi_s^2 ds \right],$$

which is finite as $\xi \in \mathbb{A}_T^\xi$. This allows to conclude the first part of the proof.

(ii) Let $s \in [t, T]$. Using the explicit expression of M in (3.52), we have

$$\begin{aligned} \mathbb{E} \left[|M_s^{t,u}| \right] &\leq e^{-\rho(s-t)} u + \mathbb{E} \left[\int_t^s e^{-\rho(s-r)} (bX_{r-}^{t,x})^2 dN_r \right] \\ &\leq u + \lambda b^2 \int_t^s \mathbb{E} \left[(X_{r-}^{t,x})^2 \right] dr. \end{aligned}$$

Now, by Fubini, $\int_t^T e^{-\delta s} (\int_t^s \mathbb{E} [|X_r^{t,x}|^2] dr) ds = \mathbb{E} \left[\int_t^T e^{-\delta s} (\int_t^s |X_r^{t,x}|^2 dr) ds \right]$, which is finite for $\xi \in \mathbb{A}_T^\xi$ according to (i). Thus, $\int_t^T e^{-\delta s} (\int_t^s \mathbb{E} [|X_r^{t,x}|^2] dr) ds$ is finite. By the property of the Lebesgue integral, it implies that $\int_t^s \mathbb{E} [|X_r^{t,x}|^2] dr$ is finite for $s \in [t, T]$ almost everywhere. Since $s \mapsto \int_t^s \mathbb{E} [|X_r^{t,x}|^2] dr$ is increasing, $\int_t^s \mathbb{E} [|X_r^{t,x}|^2] dr$ is actually finite for all $s \in [t, T]$, otherwise a contradiction can be easily exhibited. Hence, for all $s \in [t, T]$, $\mathbb{E} [|M_s^{t,u}|] < \infty$. This concludes the proof. \square

Lemma B.10. *For any admissible control $\xi \in \mathbb{A}_T^\xi$, for all $(t, x, u) \in \mathcal{X}_T$, we have*

$$\mathbb{E} \left[\int_t^T e^{-\delta s} |f_T(s, X_s^{t,x}, M_s^{t,u}, \xi_s)| ds \right] < \infty.$$

Proof. We have, using that h is inferiorly bounded by a strictly positive term that we write $1/\eta$ with the constant $\eta \in \mathbb{R}_+^*$,

$$\begin{aligned} & \mathbb{E} \left[\int_t^T e^{-\delta s} |f_T(s, X_s^{t,x}, M_s^{t,u}, \xi_s)| ds \right] \\ & \leq \mathbb{E} \left[\int_t^T e^{-\delta s} \left(\eta \beta |X_s^{t,x}| + \alpha M_s^{t,u} + \frac{\bar{\kappa}}{4} \left(\xi_s + \frac{\beta}{\kappa_r} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr \right)^2 \right) ds \right] \end{aligned}$$

Now, from the explicit expression of $X^{t,x}$ in (3.51), it holds:

$$\begin{aligned} |X_s^{t,x}| & \leq \mathcal{E}_s^t |x| + \mathcal{E}_s^t \left| \int_t^s (\mathcal{E}_r^t)^{-1} \{ \xi_r dr + z dW_r \} \right| \\ & \leq |x| + \int_t^s |\xi_r| dr + z \left(1 + \left(\int_t^s e^{-a(s-r)} (1-b)^{N_s - N_r} dW_r \right)^2 \right). \end{aligned} \quad (3.60)$$

By (3.58), we get

$$\mathbb{E} [|X_s^{t,x}|] \leq \mathbb{E} \left[|x| + \int_t^s |\xi_r| dr + z(1 + s - t) \right]. \quad (3.61)$$

Moreover, by integration by parts,

$$\mathbb{E} \left[\int_t^T e^{-\delta s} \left(\int_t^s |\xi_r| dr \right) ds \right] \leq \mathbb{E} \left[\frac{1}{\delta} \int_t^T |\xi_s| ds \right].$$

As $\xi \in \mathbb{A}_T^\xi$, the expectation in the right-hand side is finite. Hence,

$$\mathbb{E} \left[\int_t^T e^{-\delta s} |X_s^{t,x}| ds \right] < \infty.$$

As for $\mathbb{E} \left[\int_t^T e^{-\delta s} |M_s^{t,u}| ds \right]$, we have, by the explicit expression of $M^{t,u}$ in equation (3.52):

$$\begin{aligned} \mathbb{E} \left[\int_t^T e^{-\delta s} |M_s^{t,u}| ds \right] &\leq \mathbb{E} \left[\int_t^T e^{-\delta s} \left(e^{-\rho(s-t)} u + \int_0^s e^{-\rho(s-r)} (bX_{r-}^{t,x})^2 dN_r \right) ds \right] \\ &\leq \int_t^T e^{-\delta s} \left(u + \lambda \int_t^s \mathbb{E}[(bX_{r-}^{t,x})^2] dr \right) ds, \end{aligned}$$

which is finite for $\xi \in \mathbb{A}^\xi$ according to Lemma B.9.

Finally, $\mathbb{E} \left[\int_t^T e^{-\delta s} \left(\xi_s + \frac{\beta}{\kappa_r} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr \right)^2 ds \right] \leq \mathbb{E} \left[\int_t^T e^{-\delta s} \left(\xi_s^2 + \left(\frac{\beta}{\kappa_r} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr \right)^2 \right) ds \right]$, which is finite as $\xi \in \mathbb{A}_T^\xi$. \square

Lemma B.11. *For every $\xi \in \mathbb{A}^\xi$ and every $(t, x, u) \in \mathcal{X}_T$,*

$$\mathbb{E} \left[\sup_{t \leq r \leq s} |w(X_r^{t,x}, M_r^{t,u})| \right] < \infty.$$

Proof. We have

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq r \leq s} |w(X_r^{t,x}, M_r^{t,u})| \right] &\leq \frac{1}{2} A \mathbb{E} \left[\sup_{t \leq r \leq s} (X_r^{t,x})^2 \right] + B \mathbb{E} \left[\sup_{t \leq r \leq s} |X_r^{t,x}| \right] \\ &\quad + |C| \left(u + \mathbb{E} \left[\sup_{t \leq r \leq s} \int_t^r e^{-\rho(r-y)} (bX_{y-}^{t,x})^2 dN_y \right] \right). \end{aligned}$$

Referring to (3.57) and using Burkholder-Davis-Gundy inequality, there exists a positive constant \tilde{C} so that for every $t \geq 0$,

$$\mathbb{E} \left[\sup_{t \leq r \leq s} (X_r^{t,x})^2 \right] \leq \tilde{C} \mathbb{E} \left[|x|^2 + \int_t^s |\xi_r|^2 dr + z^2(s-t) \right].$$

This upper boundary is finite as $\xi \in \mathbb{A}_T^\xi$. Thus, $\mathbb{E} \left[\sup_{t \leq r \leq s} (X_r^{t,x})^2 \right]$ is finite.

Moreover, recalling (3.60), and applying again Burkholder-Davis-Gundy inequality, we similarly get that $\mathbb{E} \left[\sup_{t \leq r \leq s} |X_r^{t,x}| \right] < \infty$ for $\xi \in \mathbb{A}_T^\xi$.

Finally, as $r \mapsto \int_t^r e^{\rho y} (bX_{y-}^{t,x})^2 dN_y$ is increasing for each trajectory, we have

$$\mathbb{E} \left[\sup_{t \leq r \leq s} \int_t^r e^{-\rho(r-y)} (bX_{y-}^{t,x})^2 dN_y \right] \leq \mathbb{E} \left[\sup_{t \leq r \leq s} \int_t^r e^{\rho y} (bX_{y-}^{t,x})^2 dN_y \right] \leq \mathbb{E} \left[\int_t^s e^{\rho y} (bX_{y-}^{t,x})^2 dy \right],$$

which is finite since M is integrable for admissible strategies by Lemma B.9.

Therefore, we can conclude by a finite sum of finite terms that $\mathbb{E} \left[\sup_{0 \leq s \leq t} |w(X_s^x, M_s^u)| \right]$ is finite. \square

Lemma B.12. *The optimal control is admissible, i.e. $\hat{\xi} \in \mathbb{A}_T^\xi$.*

Proof. As $\forall s \in [t, T]$, $\hat{\xi}_s = \frac{2}{\kappa} \left(A(s) \hat{X}_s^{t,x} + B(s) \right) - \frac{\beta}{\kappa_r} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr$, the explicit solutions for

$\hat{X}^{t,x}, \hat{M}^{t,u}$ are as follows, for $s \in [t, T]$:

$$\hat{X}_s^{t,x} = \hat{\mathcal{E}}_s^t x + \hat{\mathcal{E}}_s^t \int_t^s (\hat{\mathcal{E}}_r^t)^{-1} \{ \nu_r dr + z dW_r \}, \quad (3.62)$$

$$\hat{M}_s^{t,u} = e^{-\rho(s-t)} u + \int_t^s e^{-\rho(s-r)} (b \hat{X}_{r-}^{t,x})^2 dN_r, \quad (3.63)$$

with $\nu_s := \frac{2}{\bar{\kappa}} B(s) - \frac{\beta}{\kappa_r} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr$, and $\hat{\mathcal{E}}_s^t = e^{-\int_t^s (a - \frac{2}{\bar{\kappa}} A(r)) dr} (1-b)^{N_s - N_t}$, and writing $0^0 = 1$. Note that $\forall s \in [t, T]$, $a - \frac{2}{\bar{\kappa}} A(s) \geq 0$, as A is negative.

Now, using the explicit expression of $\hat{X}^{t,x}$ above, equations (3.57) and (3.58), and Fubini, we have

$$\begin{aligned} \mathbb{E} \left[\int_t^T |\hat{\xi}_s|^2 ds \right] &\leq \tilde{C} \left(1 + \mathbb{E} \left[\int_t^T |\hat{X}_s^{t,x}|^2 ds \right] \right) \\ &\leq \tilde{C} \left(1 + \mathbb{E} \left[\int_t^T \left(|x|^2 + \int_t^s \nu_r^2 dr + z^2 (s-t) \right) ds \right] \right) < \infty, \end{aligned}$$

with a positive constant \tilde{C} .

Moreover, $\hat{\xi}$ is \mathbb{F} -progressively measurable as $\forall s \in [t, T]$, $\hat{\xi}_s = g((\hat{X}_r^{t,x})_{t \leq r \leq s})$, with g a continuous function. Hence, $\hat{\xi}$ is an admissible strategy, i.e. is $\in \mathbb{A}_T^\xi$. \square

Lemma B.13. For every $\xi \in \mathbb{A}_T^\xi$, $\forall (t, x, u) \in \mathcal{X}_T$, the functional

$$\mathcal{J} : (t, x, u, \xi) \mapsto \mathbb{E} \left[\int_t^T e^{-\delta s} \left(-f_T(s, X_s^{t,x}, M_s^{t,u}, \xi_s) \right) ds \right]$$

is strictly convex in ξ and for $\theta \in [0, 1]$, and for $\xi^1, \xi^2 \in \mathbb{A}_T^\xi$,

$$\theta \mathcal{J}(t, x, u, \xi_1) + (1 - \theta) \mathcal{J}(t, x, u, \xi_2) - \mathcal{J}(t, x, u, \theta \xi_1 + (1 - \theta) \xi_2) \geq \frac{\bar{\kappa}}{4} \int_t^T e^{-\delta s} |\xi_s^1 - \xi_s^2|^2 ds.$$

Proof. Let us show that $\forall s \in [t, T]$, $\mathbb{E}[-f_T(s, X_s^{t,x}, M_s^{t,u}, \xi_s)]$ is convex in ξ . By linearity of integrals and applying Fubini thanks to Lemma B.10, it will be so for \mathcal{J} . We have

$$\mathbb{E} \left[-f_T(s, X_s^{t,x}, M_s^{t,u}, \xi_s) \right] = -\frac{\beta}{h(m_s)} \mathbb{E} \left[X_s^{t,x} \right] + \alpha \mathbb{E} \left[M_s^{t,u} \right] + \mathbb{E} \left[\frac{\bar{\kappa}}{4} \left(\xi_s + \frac{\beta}{\kappa_r} \int_s^T \frac{e^{-\delta(r-s)}}{h(m_r)} dr \right)^2 \right].$$

Now, $X_s^{t,x}$ is linear in ξ according to its explicit expression (3.51). The last term is obviously strictly convex in ξ_s . As for $M_s^{t,u}$, using its explicit expression (3.52) and the properties of admissible strategies ($\in \mathbb{A}_T^\xi$),

$$\mathbb{E} \left[M_s^{t,u} \right] = e^{-\rho(s-t)} u + \lambda b^2 \mathbb{E} \left[\int_t^s e^{-\rho(s-r)} (X_r^{t,x})^2 dr \right].$$

Now, $(X_r^{t,x})^2$ is strictly convex in ξ by Jensen inequality. Therefore, by addition of linear and

strictly convex terms in ξ , $\mathbb{E}[-f(s, X_s^{t,x}, M_s^{t,u}, \xi_s)]$ is strictly convex in ξ , and so is \mathcal{J} . More precisely, by focusing only on the third part, it is easy to show that for $\theta \in [0, 1]$, and for $\xi^1, \xi^2 \in \mathbb{A}_T^\xi$,

$$\theta \mathcal{J}(t, x, u, \xi_1) + (1 - \theta) \mathcal{J}(t, x, u, \xi_2) - \mathcal{J}(t, x, u, \theta \xi_1 + (1 - \theta) \xi_2) \geq \frac{\bar{\kappa}}{4} \int_t^T e^{-\delta s} |\xi_s^1 - \xi_s^2|^2 ds.$$

□

B.5 Empirics

Proof of Lemma 5.1. Let us prove this lemma by considering the following generic specification, for all $i \in \{1, \dots, n\}$ and $t \in \{1, \dots, T\}$:

$$Y_{i,t} = \alpha_i + X_{i,t}\beta + \varepsilon_{i,t},$$

where for each t , $(X_{i,t}, \varepsilon_{i,t})_i$ are integrable i.i.d. variables, for each i , $(X_{i,t}, \varepsilon_{i,t})$ is stationary, and $\forall t' \geq t$, $\mathbb{E}(X_{i,t}\varepsilon_{i,t'}) = 0$ (weak exogeneity assumption).

Let us set $\bar{X}_i = \frac{1}{T} \sum_{t=1}^T X_{i,t}$ and $\tilde{X}_{i,t} = X_{i,t} - \bar{X}_i$, and define $\bar{Y}_i, \tilde{Y}_{i,t}, \bar{\varepsilon}_i, \tilde{\varepsilon}_{i,t}$ similarly. The Within estimator, $\hat{\beta}$, verifies

$$\hat{\beta} = \left(\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T (\tilde{X}_{i,t})^2 \right)^{-1} \left(\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \tilde{X}_{i,t} \tilde{Y}_{i,t} \right),$$

that is,

$$\hat{\beta} = \beta + \left(\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T (\tilde{X}_{i,t})^2 \right)^{-1} \left(\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \tilde{X}_{i,t} \varepsilon_{i,t} \right),$$

where $\tilde{\varepsilon}_{i,t}$ is replaced by $\varepsilon_{i,t}$ because $\frac{1}{T} \sum_{t=1}^T \tilde{X}_{i,t} \bar{\varepsilon}_i = \bar{\varepsilon}_i \frac{1}{T} \sum_{t=1}^T \tilde{X}_{i,t} = 0$. Therefore, we can write the bias of the Within estimation as

$$\hat{\beta} - \beta = \left(\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T (\tilde{X}_{i,t})^2 \right)^{-1} \left(\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \tilde{X}_{i,t} \varepsilon_{i,t} \right).$$

By the law of large numbers, writing (X_t, ε_t) with the same distribution as $(X_{i,t}, \varepsilon_{i,t})$ for any i ,

$$\text{plim}_{N \rightarrow \infty} (\hat{\beta} - \beta) = \left(\frac{1}{T} \sum_{t=1}^T \mathbb{E} [\tilde{X}_t^2] \right)^{-1} \left(\frac{1}{T} \sum_{t=1}^T \mathbb{E} [\tilde{X}_t \varepsilon_t] \right),$$

writing plim for the convergence in probability.

Now, for each t , $\mathbb{E}(\tilde{X}_t \varepsilon_t) = \mathbb{E}((X_t - \bar{X})\varepsilon_t) = -\mathbb{E}(\bar{X}\varepsilon_t)$. Therefore, because $(X_t, \varepsilon_t)_t$ is stationary and $\mathbb{E}[X_t \varepsilon_t] = 0$ (from the weak exogeneity assumption), one can rewrite this bias as

$$\text{plim}_{N \rightarrow \infty} (\hat{\beta} - \beta) = \left(\mathbb{E} [\tilde{X}_t^2] \right)^{-1} \left(-\mathbb{E} (\bar{X} \bar{\varepsilon}) \right), \quad t \in \{1, \dots, T\}.$$

However, from Cauchy-Schwartz,

$$|\mathbb{E} (\bar{X} \bar{\varepsilon})| \leq \left(\text{Var}(\bar{X}) \text{Var}(\bar{\varepsilon}) \right)^{1/2} = O(1/T)$$

if X_t and ε_t are weakly dependent.

Therefore, the bias tends to zero, and its limit in probability is upper-bounded by a variable that tends to zero at a rate $1/T$, which proves Lemma 5.1. □

C Calibration

The reference calibration used in Section 3 is made to illustrate the properties of the model for a generic company.

We calibrate the frequency of controversies, λ , using the Environmental Controversy score provided by Covalence: λ is the average frequency for which this score is above 25 (over 100) across the 13,298 companies in the whole Covalence database from January 2009 to December 2022. We assume that when a controversy occurs, the fundamental environmental value of the company is fully revealed ($b = 1$). We also assume that the rating agency progressively recovers the fundamental environmental value of the company over two years on average ($a = 0.5$). We choose the marginal unit cost of environmental communication relative to the marginal unit cost of abatement ($\kappa_r/\kappa_c = 50$) in line with the ratio of a EUR 3,000,000 green bond emission to its certification costs (of the order of EUR 60,000).

We set the pro-environmental sensitivity of the investor, β , equal to the generic value of 1. Since the green premium and the misrating penalty premium are homogeneous metrics, we also assume that $\alpha = 1$. It is worth noting that α and β do not impact the “ON-OFF” greenwashing condition (equation (9)), but only contribute to scaling abatement, communication, and greenwashing efforts. We consider a rate of time preference of 10% for both the company and the investor.

As such, the calibration verifies the following two realistic conditions:

1. It is much more costly to abate than to do environmental communication ($\kappa_r \gg \kappa_c$).
2. The relative marginal unit costs κ_r/κ_c , asymmetry of information $a + b\lambda$, and rate of time preference δ , are so that condition (3.10) is satisfied.

In short, the calibration is reported in Table 3.3.

Table 3.3 – Calibration.

Parameter	Value
a	0.5
b	1
λ	7.5%
κ_c	1
κ_r	50
β	1
α	1
ρ	0.1
δ	0.1

D Standard theorems

Theorem 18 (Schauder fixed point theorem). *If K is a nonempty convex closed subset of a Hausdorff topological vector space V and f is a continuous mapping of K into itself such that $f(K)$ is contained in a compact subset of K , then the set of fixed points of f is non-empty.*

Theorem 19 (Arzela-Ascoli). *Consider a sequence of real-valued continuous functions $(f_n)_{n \in \mathbb{N}}$ defined on a closed and bounded interval $[a, b]$ of the real line. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ that converges uniformly.*

E Variables and Tables

Table 3.4 – **First-step estimation.** This Table gives the results of the first-step estimation, which is a 2SLS Within panel regression with robust standard errors of the environmental reputation index at the end of the month t , Rep_t , on the environmental controversy index at the end of the month t that is instrumented by the environmental controversy index at the end of the month $t - 1$, Con_t^* . The standard deviation is shown in brackets below the estimate.

Dependent variable: Rep_t	
Con_t^*	0.040*** (0.013)
Firm FE	Yes
Observations	152,821
R ²	0.002
Adjusted R ²	-0.023
F Statistic	240.292***
<i>Note:</i>	*p<0.1; **p<0.05; ***p<0.01

Table 3.5 – **Relevance of the instrument used in the second-step estimation (top brownest companies and entire universe)**. This table shows the results of the Within regression with robust standard errors of the change in environmental score, ΔE_t^i , on the lagged environmental score, E_{t-1}^i . Both variables are used in the step-2 regression: the former is the independent variable and the latter is the instrument. The estimations are performed for different samples: the top 10%, 20%,..., 90% brownest companies, and the entire universe. The standard deviation is shown in brackets.

<i>Dependent variable: ΔE_t^i</i>					
Top brownest companies:					
	10%	20%	30%	40%	50%
E_{t-2}^i	-0.259*** (0.028)	-0.162*** (0.015)	-0.108*** (0.009)	-0.084*** (0.006)	-0.071*** (0.004)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	19,942	32,667	46,884	60,320	72,470
R ²	0.218	0.123	0.074	0.054	0.044
Adjusted R ²	0.167	0.074	0.028	0.012	0.004
F Statistic	5,224.462***	4,325.124***	3,572.930***	3,310.524***	3,215.831***
<i>Dependent variable: ΔE_t^i</i>					
Top brownest companies:					
	60%	70%	80%	90%	Whole sample
E_{t-2}^i	-0.059*** (0.003)	-0.048*** (0.002)	-0.039*** (0.002)	-0.030*** (0.001)	-0.023*** (0.001)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	88,223	102,884	116,290	130,457	152,821
R ²	0.034	0.026	0.021	0.015	0.011
Adjusted R ²	-0.002	-0.007	-0.010	-0.013	-0.014
F Statistic	2,981.010***	2,644.557***	2,366.927***	1,951.302***	1,673.251***

Note:

*p<0.1; **p<0.05; ***p<0.01

Table 3.6 – **Relevance of the instrument used in the second-step estimation (top greenest companies and entire universe)**. This table shows the results of the Within regression with robust standard errors of the change in environmental score, ΔE_t^i , on the lagged environmental score, E_{t-1}^i . Both variables are used in the step-2 regression: the former is the independent variable and the latter is the instrument. The estimations are performed for different samples: the top 10%, 20%,..., 90% greenest companies, and the entire universe. The standard deviation is shown in brackets.

<i>Dependent variable: ΔE_t^i</i>					
Top greenest companies:					
	10%	20%	30%	40%	50%
E_{t-2}^i	−0.053*** (0.005)	−0.046*** (0.003)	−0.042*** (0.002)	−0.039*** (0.002)	−0.034*** (0.002)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	22,364	36,531	49,937	64,598	80,351
R ²	0.041	0.034	0.030	0.027	0.022
Adjusted R ²	0.006	0.002	−0.0001	−0.003	−0.006
F Statistic	917.932***	1,234.863***	1,507.587***	1,723.228***	1,764.086***
<i>Dependent variable: ΔE_t^i</i>					
Top greenest companies:					
	60%	70%	80%	90%	Whole sample
E_{t-2}^i	−0.031*** (0.002)	−0.027*** (0.001)	−0.025*** (0.001)	−0.023*** (0.001)	−0.023*** (0.001)
Firm FE	Yes	Yes	Yes	Yes	Yes
Month FE	Yes	Yes	Yes	Yes	Yes
Observations	92,501	105,937	120,154	132,879	152,821
R ²	0.019	0.016	0.013	0.012	0.011
Adjusted R ²	−0.010	−0.012	−0.014	−0.015	−0.014
F Statistic	1,708.114***	1,643.758***	1,541.759***	1,551.436***	1,673.251***

Note:

*p<0.1; **p<0.05; ***p<0.01

Table 3.7 – **Main estimation with time fixed effects and controls (top brownest companies and entire universe)**. This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$, including time fixed effects as well as controls for systematic risk and return. The estimations are performed for different samples: the top 10%, 20%,..., 90% brownest companies, and the entire universe. The standard deviations are shown in brackets below the estimates.

Dependent variable: $\Delta\hat{c}_t^i$					
Top brownest companies:					
	10%	20%	30%	40%	50%
$\Delta E_t^{i,*}$	0.168 (0.161)	-0.150 (0.136)	-0.253** (0.128)	-0.241*** (0.087)	-0.459*** (0.159)
R_{t-1}^i	0.216 (0.260)	0.135 (0.182)	0.324* (0.197)	0.180 (0.140)	0.139 (0.145)
$\beta_{t-1}^{CAPM,i}$	0.012 (0.023)	0.038** (0.016)	-0.010 (0.021)	0.009 (0.012)	0.021* (0.013)
Firm FE	Yes	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Observations	6,044	9,190	12,473	15,507	18,033
R ²	0.006	0.008	0.013	0.014	0.019
Adjusted R ²	-0.073	-0.056	-0.044	-0.037	-0.028
F Statistic	1.020	1.575	2.704	1.828	4.724
Dependent variable: $\Delta\hat{c}_t^i$					
Top brownest companies:					
	60%	70%	80%	90%	Whole sample
$\Delta E_t^{i,*}$	-0.281** (0.130)	-0.195* (0.105)	-0.164* (0.091)	-0.166** (0.072)	-0.083* (0.050)
R_{t-1}^i	0.188 (0.147)	0.366** (0.163)	0.374** (0.154)	0.322** (0.137)	0.252** (0.124)
$\beta_{t-1}^{CAPM,i}$	0.012 (0.011)	0.021 (0.014)	0.009 (0.010)	0.012 (0.008)	0.010 (0.007)
Firm FE	Yes	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Observations	21,749	25,249	28,980	33,168	41,252
R ²	0.018	0.017	0.017	0.019	0.016
Adjusted R ²	-0.025	-0.023	-0.019	-0.013	-0.012
F Statistic	2.420	4.896	4.795	4.225	3.014

Note:

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*p<0.1; **p<0.05; ***p<0.01

Table 3.8 – **Main estimation with time fixed effects and controls (top greenest companies and entire universe)**. This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$, including time fixed effects as well as controls for systematic risk and return. The estimations are performed for different samples: the top 10%, 20%,..., 90% greenest companies, and the entire universe. The standard deviations are shown in brackets below the estimates.

Dependent variable: $\Delta\hat{c}_t^i$					
Top greenest companies:					
	10%	20%	30%	40%	50%
$\Delta E_t^{i,*}$	-0.205 (0.182)	-0.380** (0.178)	-0.261* (0.142)	-0.243** (0.096)	-0.280*** (0.093)
R_{t-1}^i	-0.335 (0.287)	-0.222 (0.245)	-0.002 (0.217)	0.348 (0.241)	0.480** (0.232)
$\beta_{t-1}^{CAPM,i}$	0.005 (0.015)	0.008 (0.014)	-0.013 (0.027)	0.008 (0.013)	-0.009 (0.014)
Firm FE	Yes	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Observations	8,084	12,272	16,003	19,503	23,219
R ²	0.016	0.021	0.023	0.022	0.020
Adjusted R ²	-0.023	-0.012	-0.008	-0.009	-0.009
F Statistic	1.504	3.582	1.748	3.120	5.449
Dependent variable: $\Delta\hat{c}_t^i$					
Top greenest companies:					
	60%	70%	80%	90%	Whole sample
$\Delta E_t^{i,*}$	-0.385*** (0.093)	-0.284*** (0.086)	-0.251*** (0.093)	-0.193*** (0.067)	-0.083* (0.050)
R_{t-1}^i	0.375* (0.220)	0.185 (0.170)	0.316* (0.171)	0.255* (0.153)	0.252** (0.124)
$\beta_{t-1}^{CAPM,i}$	0.005 (0.011)	0.008 (0.011)	-0.011 (0.012)	-0.0002 (0.010)	0.010 (0.007)
Firm FE	Yes	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes	Yes
Observations	25,745	28,779	32,062	35,208	41,252
R ²	0.023	0.022	0.023	0.022	0.016
Adjusted R ²	-0.007	-0.007	-0.006	-0.006	-0.012
F Statistic	5.711	2.722	4.029	2.754	3.014

Note:

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*p<0.1; **p<0.05; ***p<0.01

Table 3.9 – **Main estimation with different starting dates (focus on the 50% brownest and 50% greenest companies)**. This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$, from different starting dates. This table focuses on the sample of the 50% brownest companies (upper panel) and 50% greenest companies (lower panel) in each sector on each date. The standard deviations are shown in brackets below the estimates.

Dependent variable: $\Delta\hat{c}_t^i$				
50% brownest companies				
	Since 2012	Since 2017	Since 2019	Since 2021
$\Delta E_t^{i,*}$	-0.271*** (0.060)	-0.226*** (0.057)	-0.220*** (0.072)	-0.237*** (0.087)
Firm FE	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes
Observations	68,276	57,626	43,107	19,098
R ²	0.013	0.014	0.019	0.022
Adjusted R ²	-0.029	-0.034	-0.042	-0.093
F Statistic	4.949**	3.497*	3.420*	4.817**
Dependent variable: $\Delta\hat{c}_t^i$				
50% greenest companies				
	Since 2012	Since 2017	Since 2019	Since 2021
$\Delta E_t^{i,*}$	-0.415*** (0.057)	-0.457*** (0.061)	-0.449*** (0.065)	-0.353*** (0.069)
Firm FE	Yes	Yes	Yes	Yes
Time FE	Yes	Yes	Yes	Yes
Observations	77,232	64,719	48,000	20,768
R ²	0.020	0.022	0.026	0.029
Adjusted R ²	-0.009	-0.012	-0.020	-0.075
F Statistic	10.606***	13.629***	18.549***	9.557***

Note:

* p<0.1; ** p<0.05; *** p<0.01

Table 3.10 – **Main estimation applied to different environmental subscores (focus on the 50% brownest and 50% greenest companies)**. This Table gives the results of the step-2 estimation, which is a Within panel regression with robust standard errors of the change in the proxy for the environmental communication flow, $\Delta\hat{c}_t^i$, on the change in environmental score instrumented by the lagged environmental score, $\Delta E_t^{i,*}$, applied to different environmental subscores, which are related to (i) the environmental impacts of the products sold ($E_t^{Imp,i,*}$), (ii) the resources used ($E_t^{Res,i,*}$), and (iii) the emissions, effluents, and waste ($E_t^{Emi,i,*}$). This table focuses on the sample of the 50% brownest companies in each sector on each date. The standard deviations are shown in brackets below the estimates.

Dependent variable: $\Delta\hat{c}_t^i$			
50% brownest companies			
	(1)	(2)	(3)
$\Delta E_t^{Imp,i,*}$	-0.142*** (0.046)		
$\Delta E_t^{Res,i,*}$		-0.180*** (0.047)	
$\Delta E_t^{Emi,i,*}$			-0.204*** (0.051)
Firm FE	Yes	Yes	Yes
Time FE	Yes	Yes	Yes
Observations	68,276	68,276	68,276
R ²	0.006	0.005	0.015
Adjusted R ²	-0.036	-0.037	-0.027
F Statistic	2.087	3.580*	3.978**
Dependent variable: $\Delta\hat{c}_t^i$			
50% greenest companies			
	(1)	(2)	(3)
$\Delta E_t^{Imp,i,*}$	-0.269*** (0.042)		
$\Delta E_t^{Res,i,*}$		-0.252*** (0.038)	
$\Delta E_t^{Emi,i,*}$			-0.225*** (0.036)
Firm FE	Yes	Yes	Yes
Time FE	Yes	Yes	Yes
Observations	77,232	77,232	77,232
R ²	0.013	0.009	0.014
Adjusted R ²	-0.016	-0.020	-0.016
F Statistic	5.953**	8.354***	8.135***

Note: *p<0.1; **p<0.05; ***p<0.01

Table 3.11 – **Variables.** Description of and statistics on the variables used in the empirical analysis.

Variable	Mean	Std. Dev.	Min.	Max.	# values	Description
<i>Rep</i>	83.0	18.7	0	100	155,852	<i>Rep</i> is the monthly environmental reputation score based on forward-looking news data, provided by Covalence, “reflecting companies’ sustainability commitments, targets, and ambitions” regarding the environment. “The basic metrics used are quantities of news items gathered on the web that can be coded as having a positive or negative polarity towards named companies. Positive news articles are called “endorsements,” while articles with negative polarity are “controversies.” A historical erosion factor is applied to the quantities of positive and negative news with recent articles weighting more than older ones. The sentiment, or reputation score, is given by the share of positive news over the total of positive and negative news.”
<i>Con</i>	8.7	19.6	0	100	155,707	<i>Con</i> is the monthly environmental controversy score provided by Covalence. It is constructed in a similar way to the environmental reputation score using news-based data focused on environmental controversies.
<i>E</i>	54.9	10.3	17.9	94.0	155,707	<i>E</i> is the monthly environmental score provided by Covalence. It is constructed based on two types of data: quantitative indicators (data disclosed annually by companies) and news-based data (published by the media and other stakeholders).
ΔE	0.10	0.7	-22.0	21.0	152,821	ΔE is the change in environmental score over two consecutive months.
<i>c</i>	0.2	2.9	-67.9	73.0	149,136	<i>c</i> is the monthly environmental communication variable, which is constructed through the first step of the empirical analysis in Section 4.1.
Δc	-0.004	4.0	-79.0	101.3	145,508	Δc is the change in environmental communication over two consecutive months.
<i>R</i>	0.01	0.1	-0.9	8.8	44,825	<i>R</i> is the monthly realized return on the stocks issued by the companies.
β^{CAPM}	1.1	1.1	-16.7	26.1	44,067	β^{CAPM} is the CAPM beta of each stock defined as $\beta_t^{CAPM,i} = Var^{-1}(r_t^n)Cov(r_t^i, r_t^n)$, where r^i and r^n denote firm <i>i</i> ’s return and the market return, respectively.

Chapter 4

Coarse correlated equilibria in mean field games: application to an emission abatement game

Abstract

Coarse correlated equilibria (CCE) are a good alternative to Nash equilibria (NE), as they arise more naturally as outcomes of learning algorithms and may exhibit higher payoffs. CCEs include a device which allows players' strategies to be correlated without any cooperation, only through information sent by a mediator. We develop a methodology to concretely compute mean field CCEs in a linear-quadratic mean field game (MFG) framework. We compare their performance to mean field control solutions and mean field NE (usually named MFG solutions). Our approach is implemented in the mean field version of an emission abatement game between greenhouse gas emitters. In particular, we exhibit a simple and tractable class of mean field CCEs which allows to outperform very significantly the mean field NE payoff and abatement levels, bridging the gap between the mean field NE and the social optimum obtained by mean field control. This is a joint work with Luciano Campi and Federico Cannerozzi.

1 Introduction

Mean field games (MFGs) have been introduced in mid 2000s in Lasry and Lions (2007) and independently in Huang et al. (2006). They arise as limit systems of large dynamic symmetric games with interactions of mean field type. In the limit, the concept of Nash equilibrium translates into a fixed point problem in the space of flows of measures. This equilibrium concept is commonly defined as an MFG solution, for two main reasons. On the one hand, approximate Nash equilibria with vanishing approximation error can be constructed starting from such an MFG solution (see, e.g., Campi and Fischer, 2018; Carmona and Delarue, 2013; Lacker and Le Flem, 2023). On the other hand, Nash equilibria (NEs) for the N -player game can be shown to converge to such MFG solutions (see, e.g., Lacker, 2020; Lacker and Le Flem, 2023). In this sense, MFG solutions can be considered as the infinitely many players analogue of Nash equilibria, so that one can (and we will) refer to commonly called MFG solutions also as mean field Nash equilibria (mean field NE, for short).

Despite their popularity, Nash equilibria present some flaws. First, they raise numerical complexity issues, see for instance Gilboa and Zemel (1989). Second, it is well-known in game theory that agents are proved to actually behave according to a Nash equilibrium only under strong rationality assumptions. Finally, they can be highly inefficient compared to social optimum. As an alternative to Nash equilibria, correlated equilibria (CEs) and coarse correlated equilibria (CCEs) have been introduced in game theory literature. They can be understood as a generalization of the notion of Nash equilibrium by the introduction of a correlation device, which allows agents to adopt correlated strategies without any cooperation. While CEs were introduced by Aumann (1974), CCEs were introduced in Hannan (1957) and explicitly by Moulin and Vial (1978) as a generalization of CEs. CCEs have been shown in game theory and computational literature to arise naturally from no-regret adaptive learning procedures (Hart and Mas-Colell, 2003; Roughgarden, 2016, Section 17.4). Moreover, they are computationally “easier” as shown by Gilboa and Zemel (1989). Finally, they are shown to be able to outperform NE payoffs in standard game theory (Dokka et al., 2022; Moulin et al., 2014) even in situations where correlated equilibria cannot, for instance in potential games (Neyman, 1997). For these reasons in this paper we focus on CCEs.

CCEs can be interpreted as follows in an N -player setting. A moderator, or correlation device, picks a strategy profile for the N players randomly according to some publicly known distribution; then, she recommends it privately to the players. Before the lottery is run, each player has to decide whether to commit to the moderator recommendation (whatever it will be), assuming that all other players commit, only knowing the lottery distribution. If a player commits, then she is communicated in private her (and only her) selected strategy, and must follow it. Instead, if a player deviates, she will do so without any information on the outcome of the lottery, assuming that all other players follow the private suggestion they receive. A

lottery is a CCE if every player prefers to commit rather than unilaterally deviate, assuming that all others do commit. CCEs are a generalization of Aumann’s notion of correlated equilibria (Aumann, 1974, 1987), since in the latter each player is asked to commit to moderator’s lottery after having seen her suggested strategy.

Lately, correlated and coarse correlated equilibria have made their appearance in MFG literature. Bonesini (2023); Bonesini et al. (2022); Campi and Fischer (2022b) establish existence and convergence results for correlated equilibria in mean field games with discrete time and finite state and action spaces. A second group of papers by Muller et al. (Muller et al., 2022, 2021) considers both CEs and CCEs in a similar setting. In addition, they provide an extensive discussion of learning algorithms for both types of equilibria in MFG. Lastly, in Campi et al. (2023), CCEs have been introduced in both continuous time stochastic differential games and mean field games. The notion of coarse correlated solution to the MFG is justified by proving an approximation result. An existence result is also proved, by means of a minimax theorem. Although its generality, this result is not constructive, and the question of how to construct coarse correlated solutions to MFG is left open.

This paper’s goal is to develop a methodology for computing mean field CCEs, and to effectively compare them to mean field NEs and mean field control (MFC) solutions (see Carmona et al. (2013, 2019) for an insightful discussion on the differences between such two notions and a quantitative comparison). For this reason, we do not consider the N -player game, but we limit our analysis to the mean field game. Since we search for explicit solutions, we restrict our analysis to linear-quadratic stochastic MFGs, working in a setting closely related to Graber (2016). Applying our methodology to a toy model, we show that mean field CCEs indeed allow to significantly outperform the mean field NE in terms of payoffs under identified conditions.

We propose a notion of mean field CCE which is strongly inspired by the notion of coarse correlated solution to the MFG of Campi et al. (2023). As for a mean field NE, our notion of mean field CCEs is any suitable pair made of a strategy and a flow of moments, with the following important differences. The flow of moments can be stochastic, and the strategy can be correlated to the flow of moments even without the presence of a common noise, as it is the case in this paper. The way they are correlated is chosen by the moderator at the beginning of the game as part of the equilibrium. We call such pairs *correlated flows*. In few words, any of such correlated flows is a mean field CCE if the representative player has no incentive to deviate before knowing the flow realisation, and if the flow is consistent, i.e., at any time t the flow of moments equals the conditional expectation of the representative player’s state given the whole flow of moments up to terminal time.

Our main contributions can be summarised as follows:

- After focusing on a suitable class of suggested strategies and flows of moments verifying the consistency condition, we reduce the search of a mean field CCE to an inequality involving

only the law of stochastic flow of moments at the equilibrium.

- We compare the payoffs of mean field CCEs with those of mean field NEs and MFC solutions. We show that the MFC optimal payoff is the unattainable upper bound for all mean field CCEs and provide a condition on the law of the stochastic flow of moments so that mean field CCEs in a specific class yield a higher payoff than mean field NE.
- Finally, we apply our results to an emission abatement game between countries, inspired by environmental economics literature on international environmental agreements Barrett (1994); Dokka et al. (2022). We show that it is possible to build simple mean field CCEs that both yield much higher payoffs than the mean field NE and guarantee higher average abatement levels.

The application also shows an additional interest of CCEs, which is to help a regulator not only to lead the population to a more optimal payoff than the free-riding NE, but also or otherwise to lead it to match other and potentially payoff-conflicting targets, such as the abatement level of players in this application. To the best of our knowledge, no attempt has been made so far to identify CCEs analytically in a mean field game, nor to explore and illustrate their potential in outperforming the payoffs of mean field Nash equilibria.

The rest of the paper is organised as follows: in Section 2 we state the assumptions, which will be in force throughout the whole paper, and give the definition of mean field CCE. In Section 3, we develop the methodology for computing mean field CCEs, while in Section 4 the comparison between mean field CCEs, MFC solutions and mean field NEs is carried out. In Section 5, we apply the results of the previous sections to the abatement game, and we analyse and explore the resulting characterization of the set of mean field CCEs which outperform the payoff of the unique mean field NE. Finally, we collect in the Appendix the most standard proofs, which we choose to include for the sake of completeness.

2 Setting

Let $T > 0$ be a fixed time horizon. Let $d, k \in \mathbb{N}$. For $n \in \mathbb{N}$, denote by \mathcal{S}^n the set of $n \times n$ symmetric matrices and by I_n the identity matrix in \mathcal{S}^n . We are going to work under the following set of assumptions.

Assumptions A. Consider the following vector valued or matrix valued functions:

- (1) $A, \sigma \in L^\infty([0, T]; \mathbb{R}^{d \times d})$;
- (2) $B \in L^\infty([0, T]; \mathbb{R}^{d \times k})$;
- (3) $Q, \bar{Q}, \tilde{Q} \in L^\infty([0, T]; \mathcal{S}^d)$, $R \in \mathcal{C}([0, T]; \mathcal{S}^k)$, $H, \bar{H}, \tilde{H} \in \mathcal{S}^d$;
- (4) $H, \bar{H}, \tilde{H} \geq 0$, $Q_t \geq d_1 I_d$ for every $t \in [0, T]$, $d_1 \geq 0$, $R_t \geq d_2 I_k$ for every $t \in [0, T]$, $d_2 > 0$;

(5) $S \in L^\infty([0, T]; \mathbb{R}^{k \times d})$, $\sup_{t \in [0, T]} |S_t|^2 < d_1 d_2$ if $d_1 > 0$, $S_t = 0$ for every $t \in [0, T]$ otherwise;

(6) $L, q \in L^\infty([0, T]; \mathbb{R}^d)$, $r \in L^\infty([0, T]; \mathbb{R}^k)$.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space satisfying usual assumptions, let W be a d dimensional \mathbb{F} -Brownian motion and let ξ be an \mathbb{R}^d -valued \mathcal{F}_0 -measurable random variable with law ν . Denote by ν_1 and ν_2 the first and second moments of ν respectively. Suppose that ξ and W are independent. Throughout the paper, we assume the following assumption:

Assumption U. The σ -algebra \mathcal{F}_0 is large enough to support a \mathcal{F}_0 -measurable uniform random variable independent of ξ and W .

In the following, we denote by $\mathbb{F}^1 = (\mathcal{F}_t^1)_{t \in [0, T]}$ the filtration generated by ξ and W , which we assume without loss of generality to satisfy the usual conditions.

Given an arbitrary filtration \mathbb{G} , we will use the standard notation $\mathbb{H}^2(\mathbb{G})$ for the set of all \mathbb{G} -progressively measurable \mathbb{R}^k -valued processes $\alpha = (\alpha_t)_{t \in [0, T]}$ such $\mathbb{E}[\int_0^T |\alpha_t|^2 dt] < \infty$.

We introduce the notion of correlated flow and mean field coarse correlated equilibrium.

Definition 1 (Correlated flow). A correlated flow is a pair (λ, μ) satisfying the following properties:

- i) $\lambda = (\lambda_t)_{t \in [0, T]}$ is a process in $\mathbb{H}^2(\mathbb{F})$.
- ii) $\mu = (\mu_t)_{t \in [0, T]}$ is an \mathcal{F}_0 -measurable $\mathcal{C}([0, T]; \mathbb{R}^d)$ -random variable.
- iii) μ is independent of both ξ and W .

We refer to λ as the *recommended strategy* and to μ as the *random flow of moments*.

We can interpret a correlated flow (λ, μ) as follows: moderator's lottery is run before the game starts and independently of the idiosyncratic shocks that determine the random evolution of representative player's state. This is made possible by Assumption **U**, which allows for some independent extra randomness. We stress that, while the recommended strategy λ is correlated both to ξ and W and to μ , μ is independent of the initial datum and the noise. We will sometimes use the equivalent expressions “correlated strategy” or “suggested strategy” to refer to λ .

Let us consider a correlated flow (λ, μ) . We now assign dynamics and payoff functional. We consider a state variable with linear dynamics given by

$$dX_t = (A_t X_t + B_t \lambda_t) dt + \sigma_t dW_t, \quad X_0 = \xi, \quad (4.1)$$

and a linear-quadratic payoff functional

$$\begin{aligned} \mathfrak{J}(\lambda, \mu) = & \mathbb{E} \left[\int_0^T \left(\langle L_t, \mu_t \rangle - \frac{1}{2} \langle \bar{Q}_t \mu_t, \mu_t \rangle - \left(\frac{1}{2} \langle Q_t X_t, X_t \rangle + \langle \tilde{Q}_t X_t, \mu_t \rangle + \langle q_t, X_t \rangle + \frac{1}{2} \langle R_t \lambda_t, \lambda_t \rangle \right. \right. \right. \\ & \left. \left. + \langle S_t X_t, \lambda_t \rangle + \langle r_t, \lambda_t \rangle \right) dt - \frac{1}{2} \langle \bar{H} \mu_T, \mu_T \rangle - \left(\frac{1}{2} \langle H X_T, X_T \rangle + \langle \tilde{H} X_T, \mu_T \rangle \right) \right]. \end{aligned} \quad (4.2)$$

When needed, we will stress the dependence of the process X on the control λ by using the notation X^λ .

Now, in order to move to the definition of mean field CCE, two cases must be distinguished. If the representative player decides to trust the mediator and therefore accepts to follow her recommendation λ before knowing it, the dynamics is given by equation (4.1), and the player gets the reward $\mathfrak{J}(\lambda, \mu)$. If instead she decides to deviate, she uses a strategy $\beta \in \mathbb{H}^2(\mathbb{F}^1)$, her state dynamics is given by equation (4.1) with β instead of λ , and her reward is $\mathfrak{J}(\beta, \mu)$. Observe that when she deviates, her strategy β is measurable only with respect to the initial datum and the idiosyncratic noise, since she has no information on the outcome of the moderator's lottery. The deviating player can only use her knowledge of the law of the correlated flow (λ, μ) , which is assumed to be publicly known. As a consequence, when deviating, the state process X of the representative player is independent of the random flow of moments μ , which, however, still appears in her payoff.

Definition 2 (Mean field coarse correlated equilibrium). A correlated flow (λ, μ) is a mean field CCE if the following holds:

- (i) Optimality: for every deviation $\beta \in \mathbb{H}^2(\mathbb{F}^1)$, it holds

$$\mathfrak{J}(\lambda, \mu) \geq \mathfrak{J}(\beta, \mu). \quad (4.3)$$

- (ii) Consistency: let $X = (X_t)_{t \in [0, T]}$ be the solution to equation (4.1) with the control process λ . For every time $t \in [0, T]$, μ_t is a version of the conditional expectation of X_t given μ , that is,

$$\mu_t = \mathbb{E}[X_t | \mu] \quad \mathbb{P}\text{-a.s.} \quad \forall t \in [0, T]. \quad (4.4)$$

The definition of mean field CCE has two fundamental differences with the usual definition of mean field NE. First of all, as already mentioned, the optimality condition features an asymmetry between the suggested strategy, which belongs to $\mathbb{H}^2(\mathbb{F})$, and deviating player's strategies, which belong to the smaller class $\mathbb{H}^2(\mathbb{F}^1)$, since the former depends also on the information used by the moderator to run her lottery while the latter does not. As for the consistency condition, we notice that, coherently with μ being stochastic, it is formulated in terms of conditional expectations, although no common noise is present. It should be interpreted in the following way: if all players commit to the mediator's lottery outcomes before knowing them, then the flow of measures should arise from aggregation of the individual behaviors. In the mean field limit, the influence of the idiosyncratic noise on the flow of moments vanishes, while the influence of moderator's lottery does not. Therefore, μ stays stochastic and its stochasticity should derive from moderator's lottery only. We refer to Campi et al. (2023) for more considerations and a deeper analysis of the connection with the N -player game.

Remark 1. The reader might have noticed that μ does not appear in the state dynamics (4.1).

While computing mean field NEs and MFC solutions in the linear-quadratic case with the flow of moments in the dynamics is standard, computing mean field CCEs can be more delicate when μ appears in the state dynamics. We refer to Section 3.1 and to Remark 4 therein for more explanations.

Remark 2. In Campi et al. (2023), moderator’s lottery was modeled in the following way: an auxiliary probability space was chosen by the moderator to support the extra randomness for her lottery. As a consequence, the recommended strategy, dynamics and payoff were naturally defined on a suitable product space supporting ξ, W and such extra randomness. Here, thanks to Assumption **U**, the given filtration \mathbb{F} is already big enough to allow for any extra randomization the moderator might want to use. In both formulations moderator’s lottery is run independently of ξ and W and deviations are measurable with respect to ξ and W only. Moreover, while Campi et al. (2023) considers a stochastic flow of measures, and the consistency condition is given in terms of conditional probabilities, here it is enough to consider a flow of moments and conditional expectations, due to the linear-quadratic structure of the MFG.

3 Computing mean field coarse correlated equilibria

The set of coarse correlated equilibria is typically very wide and it is difficult to characterize in a continuous time setting. We therefore focus on a tractable class of correlated flows for which we are able to characterize a sufficient condition for being a mean field CCE. To do so, we adopt the following procedure:

- We fix a correlated flow (λ, μ) . We suppose that the representative player does not commit to the moderator’s lottery and we compute her best deviating strategy $\hat{\beta}$, i.e.

$$\hat{\beta} = \arg \max_{\beta \in \mathbb{H}^2(\mathbb{F}^1)} \mathfrak{J}(\beta, \mu).$$

This is the content of Proposition 1. Observe that $\hat{\beta}$ will depend upon the law of (λ, μ) itself, but not on its actual realization.

- We define a parameterised class of correlated flows (λ, μ) of similar shape as the best deviating strategy $\hat{\beta}$ so that the consistency condition (4.4) is fulfilled. The correlation is due to a suitable random parameter δ . This is accomplished in subsection 3.2.
- Finally, for (λ, μ) in such a class, with corresponding parameter δ , we express the optimality condition

$$\mathfrak{J}(\lambda, \mu) \geq \mathfrak{J}(\hat{\beta}, \mu)$$

as an inequality involving the law of μ and δ only. Such an inequality is established in Theorem 2.

As a result, we reduce the search for a mean field CCE to finding a law for μ and δ that verifies an optimality inequality. The choice of focusing on a class of correlated flows with shape similar to the best deviation allows for explicit analytical comparison between the two payoffs in the optimality condition (4.3).

Remark 3. Interestingly, the outlined procedure does not involve the usual two steps procedure used to compute mean field NEs: first, optimize with a fixed flow of moments and, second, perform a fixed point argument to determine the flow. Indeed, we first impose the consistency condition and then we verify the optimality condition, more in line with an MFC fashion. This sheds light on one important feature of mean field CCEs: they can be regarded as a middle ground between mean field NE and MFC solutions. The comparison will be carried out in Section 4, and in Section 5 through the study of a simple yet important example.

3.1 Deviating player's optimization problem

Suppose that the representative player does not commit to the lottery. Therefore, as anticipated in Section 2, she chooses a strategy on her own before the moderator sends his recommendation, hence in particular without any information on the realisation of the correlated flow. The only information she has about (λ, μ) is the joint law of the pair itself, which is assumed to be publicly known. Due to the linear-quadratic structure of the MFG and the fact that any admissible deviation β is independent of μ , it turns out that knowing the expectation of μ_t for all $t \in [0, T]$ is enough.

Since the term $\int_0^T (\langle L_t, \mu_t \rangle - \frac{1}{2} \langle \bar{Q}_t \mu_t, \mu_t \rangle) dt - \langle \bar{H} \mu_T, \mu_T \rangle$ in (4.2) can be viewed as an uncontrolled constant for the deviating player's optimization problem, we can focus on the equivalent optimization problem

$$\min_{\beta \in \mathbb{H}^2(\mathbb{F}^1)} \mathfrak{J}'(\beta, \mu),$$

where

$$\begin{aligned} \mathfrak{J}'(\beta, \mu) = \mathbb{E} \left[\int_0^T \left(\frac{1}{2} \langle Q_t X_t, X_t \rangle + \langle \tilde{Q}_t X_t, \mu_t \rangle + \langle q_t, X_t \rangle + \frac{1}{2} \langle R_t \beta_t, \beta_t \rangle + \langle S_t X_t, \beta_t \rangle + \langle r_t, \beta_t \rangle \right) dt \right. \\ \left. + \frac{1}{2} \langle H X_T, X_T \rangle + \langle \tilde{H} X_T, \mu_T \rangle \right] \end{aligned} \quad (4.5)$$

under the constraint

$$dX_t = (A_t X_t + B_t \beta_t) dt + \sigma_t dW_t, \quad X_0 = \xi. \quad (4.6)$$

Since $\beta \in \mathbb{H}^2(\mathbb{F}^1)$, it follows that X is \mathbb{F}^1 -adapted, and therefore is independent of the flow of

moments μ , which implies that deviating player's payoff can be written as:

$$\begin{aligned}
 \mathfrak{J}'(\beta, \mu) &= \int_0^T \mathbb{E} \left[\mathbb{E} \left[\frac{1}{2} \langle Q_t X_t, X_t \rangle + \langle \tilde{Q}_t X_t, \mu_t \rangle + \langle q_t, X_t \rangle + \frac{1}{2} \langle R_t \beta_t, \beta_t \rangle + \langle S_t X_t, \beta_t \rangle \right. \right. \\
 &\quad \left. \left. + \langle r_t, \beta_t \rangle \mid \mathcal{F}_t^1 \right] \right] dt + \mathbb{E} \left[\mathbb{E} \left[\frac{1}{2} \langle H X_T, X_T \rangle + \langle \tilde{H} X_T, \mu_T \rangle \mid \mathcal{F}_T^1 \right] \right] \\
 &= \mathbb{E} \left[\int_0^T \left(\frac{1}{2} \langle Q_t X_t, X_t \rangle + \langle \tilde{Q}_t \mathbb{E}[\mu_t] + q_t, X_t \rangle + \frac{1}{2} \langle R_t \beta_t, \beta_t \rangle + \langle S_t X_t, \beta_t \rangle + \langle r_t, \beta_t \rangle \right) dt \right. \\
 &\quad \left. + \frac{1}{2} \langle H X_T, X_T \rangle + \langle \tilde{H} \mathbb{E}[\mu_T], X_T \rangle \right].
 \end{aligned} \tag{4.7}$$

This is now a standard linear quadratic control problem, which can be solved by the stochastic maximum principle.

Proposition 1 (Optimal strategy for the deviating player). *Let ϕ , ψ and θ be the solutions of the following ODEs:*

$$\begin{cases} \dot{\phi}_t + \phi_t A_t + A_t^\top \phi_t + Q_t - (\phi_t B_t + S_t^\top) R_t^{-1} (B_t^\top \phi_t + S_t) = 0, & \phi_T = H, \\ \dot{\psi}_t + A_t^\top \psi_t + \tilde{Q}_t - (\phi_t B_t + S_t^\top) R_t^{-1} B_t^\top \psi_t = 0, & \psi_T = \tilde{H}, \\ \dot{\theta}_t + \psi_t \frac{d\mathbb{E}[\mu_t]}{dt} + A_t^\top \theta_t + q_t - (\phi_t B_t + S_t^\top) R_t^{-1} (B_t^\top \theta_t + r_t) = 0, & \theta_T = 0. \end{cases} \tag{4.8}$$

There exists a unique optimal strategy for the deviating player, which is given by

$$\hat{\beta}_t = -R_t^{-1} ((B_t^\top \phi_t + S_t) X_t + B_t^\top \psi_t \mathbb{E}[\mu_t] + B_t^\top \theta_t + r_t). \tag{4.9}$$

We postpone the proof to the [Appendix](#). We observe only that the optimal control is actually feedback in the state X_t and in the expectation $\mathbb{E}[\mu_t]$. Moreover, while the functions ϕ and ψ do not depend upon μ or its expectation, the flow of expectations $\mathbb{E}[\mu_t]$ appears in the equation for θ , through its time derivative $\frac{d\mathbb{E}[\mu_t]}{dt}$.

Remark 4. This first step towards calculating mean field CCEs requires a filtering procedure, since the deviating player does not observe the actual realisation of μ . If the dynamics of the deviating player were dependent on μ , this step would require a much more involved analysis. Indeed, the state process X and μ would not be independent, even if $\beta \in \mathbb{H}^2(\mathbb{F}^1)$, which would lead to considering the projections on \mathbb{F}^1 of the processes X^i , $X^i X^j$ and $X^j \mu^i$, $1 \leq i, j \leq d$. This is why we have opted for a flow-free state dynamics, and postponed the analysis of the more general case to future research.

3.2 Correlated flow

We now consider a class of correlated flows (λ, μ) with a similar structure as the deviating player's best strategy $\hat{\beta}$ in (4.9). Our goal is to easily compare the payoff functionals $\mathfrak{J}'(\lambda, \mu)$ and $\mathfrak{J}'(\hat{\beta}, \mu)$. Hence we use the same functions ϕ and ψ , whereas we replace $\mathbb{E}[\mu_t]$ with μ_t itself

and the term $R_t^{-1}(B_t^\top \theta_t + r_t)$ with a free parameter $\delta = (\delta_t)_{t \in [0, T]}$. Given any such δ , we define μ so that the consistency condition (4.4) is satisfied, so that we will be left with taking care of the optimality condition only.

More precisely, let \mathcal{G} be the set of all correlated flows (λ, μ) defined as

$$\begin{aligned} \lambda_t &= -R_t^{-1}((B_t^\top \phi_t + S_t)X_t + B_t^\top \psi_t \mu_t + \delta_t), \\ \dot{\mu}_t &= (A_t - B_t R_t^{-1}(B_t^\top \phi_t + S_t + B_t^\top \psi_t))\mu_t - B_t R_t^{-1} \delta_t, \quad \mu_0 = \nu_1, \end{aligned} \quad (4.10)$$

where $\delta = (\delta_t)_{t \in [0, T]}$ is any process in $\mathbb{H}^2(\mathcal{F}_0)$ independent of ξ and W , and ϕ and ψ are as in (4.8). The parameter δ represents the extra source of randomness in the correlated flow with respect to ξ and W .

Lemma 3.1. *Any correlated flow $(\lambda, \mu) \in \mathcal{G}$ satisfies the consistency condition (4.4).*

Proof. Let $(\lambda, \mu) \in \mathcal{G}$ corresponding to some $\delta \in \mathbb{H}^2(\mathcal{F}_0)$ independent of ξ and W . To ease the notation, set

$$\Phi_t = R_t^{-1}(B_t^\top \phi_t + S_t), \quad \Psi_t = R_t^{-1} B_t^\top \psi_t, \quad \Theta_t = R_t^{-1}(B_t^\top \theta_t + r_t). \quad (4.11)$$

Notice that μ satisfies the measurability requests of Definition 1. The dynamics of the representative player state is given by

$$\begin{aligned} dX_t &= \left((A_t - B_t \Phi_t)X_t - B_t(\Psi_t \mu_t + R_t^{-1} \delta_t) \right) dt + \sigma_t dW_t, \\ X_0 &= \xi, \end{aligned} \quad (4.12)$$

which implies that the process $(\mu_t - X_t)_{t \in [0, T]}$ satisfies the stochastic differential equation

$$d(\mu_t - X_t) = (A_t - B_t \Phi_t)(\mu_t - X_t)dt - \sigma_t dW_t, \quad (4.13)$$

Since $\delta \in \mathbb{H}^2(\mathcal{F}_0)$, equation (4.12) admits a unique continuous adapted solution X satisfying $\mathbb{E}[\sup_{t \in [0, T]} |X_t|^2] < \infty$. Since ξ , W and δ are independent by assumption, by taking the conditional expectation with respect to μ in (4.13), we get

$$d\mathbb{E}[\mu_t - X_t | \mu] = (A_t - B_t \Phi_t)\mathbb{E}[\mu_t - X_t | \mu]dt, \quad \mathbb{E}[\mu_0 - X_0 | \mu] = \mu_0 - \mathbb{E}[\xi] = 0, \quad \mathbb{P}\text{-a.s.},$$

which implies $\mathbb{E}[\mu_t - X_t | \mu] = 0$ \mathbb{P} -a.s. for every t , i.e. (4.4). \square

Remark 5. Although the structure of the class \mathcal{G} is simple and quite specific, we will see later in the application section (Section 5) that it is rich enough to contain a large set of mean field CCEs with some desirable properties, such as significantly outperforming the mean field NE.

3.3 Optimality condition

Let $(\lambda, \mu) \in \mathcal{G}$. Since consistency has already been verified in Lemma 3.1, the goal is now to restate the optimality condition (4.3) in terms of quantities dependent upon the law of μ and δ only.

Theorem 2. *Let $(\lambda, \mu) \in \mathcal{G}$ corresponding to some $\delta \in \mathbb{H}^2(\mathcal{F}_0)$ independent of ξ and W . Let Φ , Ψ and Θ be given by in (4.11). Set*

$$M_t = Q_t + \Phi_t^\top R_t \Phi_t - 2\Phi_t^\top S_t, \quad N_t = \tilde{Q}_t + \Psi_t^\top R_t \Phi_t - \Psi_t^\top S_t, \quad G_t = \Psi_t^\top R_t \Psi_t. \quad (4.14)$$

Let $f(\mu) = (f_t(\mu))_{t \in [0, T]}$ be given by

$$\begin{cases} \dot{f}_t(\mu) = (A_t - B_t \Phi_t) f_t(\mu) + B_t \left(\Psi_t (\mu_t - \mathbb{E}[\mu_t]) + R_t^{-1} \delta_t - \Theta_t \right), & 0 \leq t \leq T, \\ f_0(\mu) = 0. \end{cases} \quad (4.15)$$

Then, (λ, μ) is a mean field CCE if and only if the following condition is satisfied:

$$\begin{aligned} & \int_0^T \left(\mathbb{E}[\langle N_t(\mu_t - \mathbb{E}[\mu_t]), \mu_t - \mathbb{E}[\mu_t] \rangle] + \frac{1}{2} (\mathbb{E}[\langle G_t \mu_t, \mu_t \rangle] - \langle G_t \mathbb{E}[\mu_t], \mathbb{E}[\mu_t] \rangle) \right. \\ & \quad \left. + \frac{1}{2} \mathbb{E}[\langle R_t^{-1} \delta_t, \delta_t \rangle] - \frac{1}{2} \langle R_t \Theta_t, \Theta_t \rangle \right) dt + \mathbb{E}[\langle \tilde{H}(\mu_T - \mathbb{E}[\mu_T]), \mu_T - \mathbb{E}[\mu_T] \rangle] \\ & \leq \int_0^T \left(\frac{1}{2} (\mathbb{E}[\langle M_t(\mu_t + f_t(\mu)), \mu_t + f_t(\mu) \rangle] - \mathbb{E}[\langle M_t \mu_t, \mu_t \rangle]) + \mathbb{E}[\langle N_t f_t(\mu), \mathbb{E}[\mu_t] \rangle] \right. \\ & \quad \left. + \langle q_t - \Phi_t^\top r_t, \mathbb{E}[f_t(\mu)] \rangle + \langle B_t^\top (\phi_t + \psi_t) \mathbb{E}[\mu_t], \Theta_t \rangle - \mathbb{E}[\langle B_t^\top (\phi_t + \psi_t) \mu_t, R_t^{-1} \delta_t \rangle] \right. \\ & \quad \left. + \mathbb{E}[\langle B_t^\top \phi_t f_t(\mu), \Theta_t \rangle] - \mathbb{E}[\langle r_t, \Theta_t - R_t^{-1} \delta_t \rangle] \right) dt \\ & \quad + \frac{1}{2} (\mathbb{E}[\langle H(\mu_T + f_T(\mu)), \mu_T + f_T(\mu) \rangle] - \mathbb{E}[\langle H \mu_T, \mu_T \rangle]) + \langle \tilde{H} \mathbb{E}[f_T(\mu)], \mathbb{E}[\mu_T] \rangle. \end{aligned} \quad (4.16)$$

Proof. Since $(\lambda, \mu) \in \mathcal{G}$ satisfies the consistency condition (4.4) by Lemma 3.1, we focus on the optimality condition (4.3). This is equivalent to verifying that

$$\mathfrak{J}'(\lambda, \mu) \leq \mathfrak{J}'(\hat{\beta}, \mu), \quad (4.17)$$

with $\hat{\beta}$ given by (4.9) and $\mathfrak{J}'(\lambda, \mu)$ and $\mathfrak{J}'(\hat{\beta}, \mu)$ are defined by (4.7). Denote by $\hat{X} = (\hat{X}_t)_{t \in [0, T]}$ the state of the deviating player when she uses the strategy $\hat{\beta}$ defined in (4.9), i.e.

$$d\hat{X}_t = ((A_t - B_t \Phi_t) \hat{X}_t - B_t (\Psi_t \mathbb{E}[\mu_t] + \Theta_t)) dt + \sigma_t dW_t, \quad \hat{X}_0 = \xi,$$

and by $X = (X_t)_{t \in [0, T]}$ the state of the representative player corresponding to the correlated flow (4.10), i.e.

$$dX_t = \left((A_t - B_t \Phi_t) X_t - B_t (\Psi_t \mu_t + R_t^{-1} \delta_t) \right) dt + \sigma_t dW_t, \quad X_0 = \xi.$$

We rewrite the cost functionals $\mathfrak{J}'(\lambda, \mu)$ and $\mathfrak{J}'(\hat{\beta}, \mu)$ by taking advantage of the explicit form of λ and $\hat{\beta}$ and functions (4.14):

$$\begin{aligned} \mathfrak{J}'(\lambda, \mu) = & \mathbb{E} \left[\int_0^T \left(\frac{1}{2} \langle M_t X_t, X_t \rangle + \langle N_t X_t, \mu_t \rangle + \frac{1}{2} \langle G_t \mu_t, \mu_t \rangle + \langle q_t - \Phi_t^\top r_t, X_t \rangle \right. \right. \\ & + \langle (R_t \Phi_t - S_t) X_t, R_t^{-1} \delta_t \rangle + \langle R_t \Psi_t \mu_t, R_t^{-1} \delta_t \rangle - \langle \Psi_t^\top r_t, \mu_t \rangle \\ & \left. \left. + \frac{1}{2} \langle R_t^{-1} \delta_t, \delta_t \rangle - \langle r_t, R_t^{-1} \delta_t \rangle \right) dt + \frac{1}{2} \langle H X_T, X_T \rangle + \langle \tilde{H} X_T, \mu_T \rangle \right], \end{aligned}$$

and

$$\begin{aligned} \mathfrak{J}'(\hat{\beta}, \mu) = & \mathbb{E} \left[\int_0^T \left(\frac{1}{2} \langle M_t \hat{X}_t, \hat{X}_t \rangle + \langle N_t \hat{X}_t, \mathbb{E}[\mu_t] \rangle + \frac{1}{2} \langle G_t \mathbb{E}[\mu_t], \mathbb{E}[\mu_t] \rangle + \langle q_t - \Phi_t^\top r_t, \hat{X}_t \rangle \right. \right. \\ & + \langle (R_t \Phi_t - S_t) \hat{X}_t, \Theta_t \rangle + \langle R_t \Psi_t \mathbb{E}[\mu_t], \Theta_t \rangle - \langle \Psi_t^\top r_t, \mathbb{E}[\mu_t] \rangle \\ & \left. \left. + \frac{1}{2} \langle R_t \Theta_t, \Theta_t \rangle - \langle r_t, \Theta_t \rangle \right) dt + \frac{1}{2} \langle H \hat{X}_T, \hat{X}_T \rangle + \langle \tilde{H} \hat{X}_T, \mathbb{E}[\mu_T] \rangle \right] \end{aligned}$$

By Itô's formula, we get

$$d(\hat{X}_t - X_t) = (A_t - B_t \Phi_t)(\hat{X}_t - X_t) dt + B_t \left(\Psi_t(\mu_t - \mathbb{E}[\mu_t]) + R_t^{-1} \delta_t - \Theta_t \right) dt, \quad \hat{X}_0 - X_0 = 0,$$

so that it holds

$$\hat{X}_t = X_t + f_t(\mu), \quad 0 \leq t \leq T, \quad \mathbb{P}\text{-a.s.} \quad (4.18)$$

In particular, we note that $f(\mu)$ is $\sigma(\mu)$ -measurable. Then, we have

$$\begin{aligned} \mathbb{E}[\langle M_t \hat{X}_t, \hat{X}_t \rangle] &= \mathbb{E}[\langle M_t(X_t + f_t(\mu)), X_t + f_t(\mu) \rangle] \\ &= \mathbb{E}[\langle M_t X_t, X_t \rangle] + \mathbb{E}[\langle M_t(\mu_t + f_t(\mu)), \mu_t + f_t(\mu) \rangle] - \mathbb{E}[\langle M_t \mu_t, \mu_t \rangle], \end{aligned}$$

where we have used the fact that X satisfies the consistency condition (4.4). Therefore, we have

$$\begin{aligned} \mathfrak{J}'(\hat{\beta}, \mu) = & \mathbb{E} \left[\int_0^T \left(\frac{1}{2} \langle M_t X_t, X_t \rangle + \frac{1}{2} \langle M_t(\mu_t + f_t(\mu)), \mu_t + f_t(\mu) \rangle - \frac{1}{2} \langle M_t \mu_t, \mu_t \rangle + \langle N_t X_t, \mathbb{E}[\mu_t] \rangle \right. \right. \\ & + \langle N_t f_t(\mu), \mathbb{E}[\mu_t] \rangle + \frac{1}{2} \langle G_t \mathbb{E}[\mu_t], \mathbb{E}[\mu_t] \rangle + \langle q_t - \Phi_t^\top r_t, X_t \rangle + \langle q_t - \Phi_t^\top r_t, f_t(\mu) \rangle \\ & + \langle (R_t \Phi_t - S_t) X_t, \Theta_t \rangle + \langle (R_t \Phi_t - S_t) f_t(\mu), \Theta_t \rangle + \langle R_t \Psi_t \mathbb{E}[\mu_t], \Theta_t \rangle - \langle \Psi_t^\top r_t, \mathbb{E}[\mu_t] \rangle \\ & + \frac{1}{2} \langle R_t \Theta_t, \Theta_t \rangle - \langle r_t, \Theta_t \rangle \left. \right) dt + \frac{1}{2} \langle H X_T, X_T \rangle + \frac{1}{2} \langle H(\mu_T + f_T(\mu)), \mu_T + f_T(\mu) \rangle - \frac{1}{2} \langle H \mu_T, \mu_T \rangle \\ & \left. + \langle \tilde{H} X_T, \mathbb{E}[\mu_T] \rangle + \langle \tilde{H} f_T(\mu), \mathbb{E}[\mu_T] \rangle \right]. \end{aligned}$$

Since the correlated flow (λ, μ) satisfies the consistency condition (4.4), and noticing that

$$\begin{aligned}\mathbb{E}[\langle N_t \mu_t, \mathbb{E}[\mu_t] - \mu_t \rangle] &= -\mathbb{E}[\langle N_t(\mathbb{E}[\mu_t] - \mu_t), \mathbb{E}[\mu_t] - \mu_t \rangle], \\ \mathbb{E}[\langle (R_t \Phi_t - S_t) \mu_t, \Theta_t \rangle] &= \langle (R_t \Phi_t - S_t) \mathbb{E}[\mu_t], \Theta_t \rangle,\end{aligned}$$

we obtain

$$\begin{aligned}\mathfrak{J}'(\hat{\beta}, \mu) - \mathfrak{J}'(\lambda, \mu) &= \mathbb{E} \left[\int_0^T \left(\frac{1}{2} \langle M_t(\mu_t + f_t(\mu)), \mu_t + f_t(\mu) \rangle - \frac{1}{2} \langle M_t \mu_t, \mu_t \rangle \right. \right. \\ &\quad - \langle N_t(\mathbb{E}[\mu_t] - \mu_t), \mathbb{E}[\mu_t] - \mu_t \rangle + \langle N_t f_t(\mu), \mathbb{E}[\mu_t] \rangle + \frac{1}{2} \langle G_t \mathbb{E}[\mu_t], \mathbb{E}[\mu_t] \rangle - \frac{1}{2} \langle G_t \mu_t, \mu_t \rangle \\ &\quad + \langle q_t - \Phi_t^\top r_t, f_t(\mu) \rangle + \langle (R_t(\Phi_t + \Psi_t) - S_t) \mathbb{E}[\mu_t], \Theta_t \rangle - \langle (R_t(\Phi_t + \Psi_t) - S_t) \mu_t, R_t^{-1} \delta_t \rangle \\ &\quad + \langle (R_t \Phi_t - S_t) f_t(\mu), \Theta_t \rangle + \frac{1}{2} \langle R_t \Theta_t, \Theta_t \rangle - \frac{1}{2} \langle R_t^{-1} \delta_t, \delta_t \rangle - \langle r_t, \Theta_t - R_t^{-1} \delta_t \rangle \Big) dt \\ &\quad + \frac{1}{2} \langle H(\mu_T + f_T(\mu)), \mu_T + f_T(\mu) \rangle - \frac{1}{2} \langle H \mu_T, \mu_T \rangle \\ &\quad \left. - \langle \tilde{H}(\mathbb{E}[\mu_T] - \mu_T), \mathbb{E}[\mu_T] - \mu_T \rangle + \langle \tilde{H} f_T(\mu), \mathbb{E}[\mu_T] \rangle \right].\end{aligned}$$

Therefore, the correlated flow (λ, μ) defined by (4.10) is a correlated flow if and only if the RHS above is non-negative. By rearranging the terms and using the equalities

$$R_t \Phi_t - S_t = B_t^\top \phi_t, \quad R_t \Psi_t = B_t^\top \psi_t, \quad (4.19)$$

we get condition (4.16). \square

The condition for a correlated flow of class \mathcal{G} to be a mean field CCE is now reduced to an optimality condition which only depends on the population average state and the correlating device of the mediator, i.e. on the joint law of (δ, μ) . Even though the inequality looks quite long, it can become very tractable and easy to interpret when one specifies some class of dynamics for μ as done in Section 5.

4 Comparison with MFC solution and mean field NE

In this section, we analyze the relationship between mean field CCEs, mean field NEs and MFC solutions. In more detail, we prove the following results:

- We compute the MFC solution $\hat{\alpha}^{MFC}$ and we show that it is unique and optimal in the broader class of controls $\mathbb{H}^2(\mathbb{F})$. This is accomplished in Proposition 3 and Lemma 4.1. Then, we show that no mean field CCE can outperform the payoff of the MFC solution. Moreover, if the MFC solution is not a mean field NE, we establish that the MFC payoff is unattainable by a mean field CCE. This is accomplished in Theorem 4.
- As for mean field NE, we show in Proposition 5 that there exists a unique mean field NE

in our setting. Then, we show that for a correlated flow (λ, μ) to be a mean field CCE different from the mean field NE, it is necessary that the flow of moments μ is stochastic, which is the content of Theorem 6.

- Finally, Theorem 7 gives a condition so that a mean field CCE $(\lambda, \mu) \in \mathcal{G}$ yields a higher payoff than the mean field NE.

We remark that the results in the first two points above are fully general, in the sense that they do not restrict to correlated flows in the class \mathcal{G} defined by (4.10), while the condition on a mean field CCE (λ, μ) to outperform the payoff of the mean field NE is provided only for correlated flows in \mathcal{G} .

We recall here for reader's convenience the definitions of both mean field NE and MFC solution.

Definition 3. We say that a pair $(\hat{\alpha}, \hat{m}) \in \mathbb{H}^2(\mathbb{F}^1) \times \mathcal{C}([0, T]; \mathbb{R}^d)$ is a mean field Nash equilibrium if the following properties hold:

- (i) Optimality: $\hat{\alpha}$ maximizes $\mathfrak{J}(\cdot, \hat{m})$ over $\mathbb{H}^2(\mathbb{F}^1)$, i.e.,

$$\mathfrak{J}(\hat{\alpha}, \hat{m}) = \max_{\beta \in \mathbb{H}^2(\mathbb{F}^1)} \mathfrak{J}(\beta, \hat{m}). \quad (4.20)$$

- (ii) Consistency: let $X^{NE} = (X_t^{NE})_{t \in [0, T]}$ be the solution to equation (4.1) with the control process $\hat{\alpha}$. For every time $t \in [0, T]$, \hat{m}_t equals the expectation of X_t^{NE} , i.e.,

$$\hat{m}_t = \mathbb{E}[X_t^{NE}], \quad \forall t \in [0, T]. \quad (4.21)$$

Definition 4. For any $\beta \in \mathbb{H}^2(\mathbb{F}^1)$, let X^β be the solution of equation (4.1) with β instead of λ . Denote by $\mathbb{E}[X^\beta] = (\mathbb{E}[X_t^\beta])_{t \in [0, T]}$ the corresponding flow of first order moments. We say that a strategy $\hat{\alpha}^{MFC}$ is a MFC solution, if

$$\mathfrak{J}(\hat{\alpha}^{MFC}, \mathbb{E}[X^{\hat{\alpha}^{MFC}}]) = \max_{\beta \in \mathbb{H}^2(\mathbb{F}^1)} \mathfrak{J}(\beta, \mathbb{E}[X^\beta]). \quad (4.22)$$

4.1 Comparison with MFC solution

In this subsection we compare the expected payoffs of mean field CCEs and the MFC solution. In our setting, there exists a unique MFC solution. Since computations are very standard, we postpone them to the [Appendix](#).

Proposition 3. Let ϕ^{MFC} and θ^{MFC} be the solutions of the following equations:

$$\begin{cases} \dot{\phi}_t^{MFC} + \phi_t^{MFC} A_t + A_t^\top \phi_t^{MFC} + (Q_t + 2\tilde{Q}_t + \bar{Q}_t) - (\phi_t^{MFC} B_t + S_t^\top) R_t^{-1} (B_t^\top \phi_t^{MFC} + S_t) = 0, \\ \phi_T^{MFC} = H + 2\tilde{H} + \bar{H}, \\ \dot{\theta}_t^{MFC} + A_t^\top \theta_t^{MFC} + q_t - L_t - (\phi_t^{MFC} B_t + S_t^\top) R_t^{-1} (B_t^\top \theta_t^{MFC} + r_t) = 0, \\ \theta_T^{MFC} = 0. \end{cases} \quad (4.23)$$

Define \bar{A} and \bar{B} as

$$\bar{A}_t = A_t - B_t R_t^{-1} B_t \phi_t^{MFC} - B_t R_t^{-1} S_t, \quad \bar{B}_t = B_t R_t^{-1} (B_t^\top \theta_t^{MFC} + r_t). \quad (4.24)$$

Let $\bar{\psi}$ and $\bar{\theta}$ be the solutions of the following equations:

$$\begin{cases} \dot{\bar{\psi}}_t + \bar{A}_t^\top \bar{\psi}_t + A_t^\top \bar{\psi}_t + (\bar{Q}_t + 2\tilde{Q}_t) - (\phi_t B_t + S_t^\top) R_t^{-1} B_t^\top \bar{\psi}_t = 0, & \bar{\psi}_T = \bar{H} + 2\tilde{H}, \\ \dot{\bar{\theta}}_t - \bar{\psi}_t \bar{B}_t + A_t^\top \bar{\theta}_t + q_t - (\phi_t B_t + S_t^\top) R_t^{-1} (B_t^\top \bar{\theta}_t + r_t) = 0, & \bar{\theta}_T = 0. \end{cases} \quad (4.25)$$

Let ϕ be the solution of the matrix Riccati equation in (4.8). There exists a unique MFC solution $\hat{\alpha}^{MFC}$, which is given by

$$\hat{\alpha}_t^{MFC} = -R_t^{-1} ((B_t^\top \phi_t + S_t) X_t^{MFC} + B_t^\top \bar{\psi}_t \bar{x}_t^{MFC} + (B_t^\top \bar{\theta}_t + r_t)), \quad (4.26a)$$

$$\dot{\bar{x}}_t^{MFC} = (A_t - B_t R_t^{-1} B_t \phi_t^{MFC} - B_t R_t^{-1} S_t) \bar{x}_t^{MFC} - B_t R_t^{-1} (B_t^\top \theta_t^{MFC} + r_t), \quad \bar{x}_0^{MFC} = \nu_1, \quad (4.26b)$$

where X^{MFC} is the solution of

$$\begin{cases} dX_t^{MFC} = ((A_t - R_t^{-1} (B_t^\top \phi_t + S_t)) X_t^{MFC} - R_t^{-1} B_t^\top \bar{\psi}_t \bar{x}_t^{MFC} - R_t^{-1} (B_t^\top \bar{\theta}_t + r_t)) dt + \sigma_t dW_t, \\ X_0 = \xi. \end{cases} \quad (4.27)$$

In particular, it holds $\bar{x}_t^{MFC} = \mathbb{E}[X_t^{MFC}]$ for every $t \in [0, T]$.

Showing that no mean field CCE can outperform the payoff of the MFC solution requires first to show that the MFC solution is actually optimal over the larger control set $\mathbb{H}^2(\mathbb{F})$, as it is done in the following preliminary lemma:

Lemma 4.1. Let $\hat{\alpha}^{MFC}$ be the solution of the MFC problem. Then, for any β in $\mathbb{H}^2(\mathbb{F})$, $\beta \neq \hat{\alpha}$, it holds

$$\mathfrak{J}(\hat{\alpha}^{MFC}, \bar{x}^{MFC}) > \mathfrak{J}(\beta, \mathbb{E}[X^\beta]). \quad (4.28)$$

Proof. To ease the notation, we set

$$\mathfrak{J}^{MFC}(\alpha) := \mathfrak{J}(\alpha, \mathbb{E}[X^\alpha]), \quad (4.29)$$

where the process X has dynamics given by (4.1), for any $\alpha \in \mathbb{H}^2(\mathbb{F})$. We observe that

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{2} \langle \bar{Q}_t \mathbb{E}[X_t], \mathbb{E}[X_t] \rangle + \langle \tilde{Q}_t X_t, \mathbb{E}[X_t] \rangle + \frac{1}{2} \langle Q_t X_t, X_t \rangle + \frac{1}{2} \langle R_t \alpha_t, \alpha_t \rangle + \langle S_t X_t, \alpha_t \rangle \right] \\ &= \mathbb{E} \left[\frac{1}{2} \langle (\bar{Q}_t + 2\tilde{Q}_t) \mathbb{E}[X_t], \mathbb{E}[X_t] \rangle + \frac{1}{2} \langle Q_t X_t, X_t \rangle + \frac{1}{2} \langle R_t \alpha_t, \alpha_t \rangle + \langle S_t X_t, \alpha_t \rangle \right] \\ &= \mathbb{E} \left[\frac{1}{2} \langle (\bar{Q}_t + 2\tilde{Q}_t + Q_t) \mathbb{E}[X_t], \mathbb{E}[X_t] \rangle + \frac{1}{2} \langle Q_t (X_t - \mathbb{E}[X_t]), (X_t - \mathbb{E}[X_t]) \rangle \right. \\ & \quad \left. + \frac{1}{2} \langle R_t \alpha_t, \alpha_t \rangle + \langle S_t (X_t - \mathbb{E}[X_t]), \alpha_t \rangle + \langle S_t \mathbb{E}[X_t], \alpha_t \rangle \right]. \end{aligned}$$

By Assumptions **A**, this equality implies that the running payoff in the functional \mathfrak{J}^{MFC} is strictly concave jointly in $\mathbb{E}[X_t]$, $X_t - \mathbb{E}[X_t]$ and α_t , for every $t \in [0, T]$. Since \mathfrak{J}^{MFC} is also upper semi-continuous, this implies that the maximum exists and that it is unique over the broader class $\mathbb{H}^2(\mathbb{F})$.

We are left to show that the maximum point is indeed $\hat{\alpha}^{MFC}$. For the sake of clarity, we set

$$\begin{aligned} f(t, x, m, a) &= \langle L_t, m \rangle - \frac{1}{2} \langle \bar{Q}_t m, m \rangle - \frac{1}{2} \langle Q_t x, x \rangle - \langle \tilde{Q}_t x, m \rangle - \langle q_t, x \rangle \\ & \quad - \frac{1}{2} \langle R_t a, a \rangle - \langle S_t x, a \rangle - \langle r_t, a \rangle, \\ g(x, m) &= - \left(\frac{1}{2} \langle \bar{H} m, m \rangle + \frac{1}{2} \langle H x, x \rangle + \langle \tilde{H} x, m \rangle \right). \end{aligned} \tag{4.30}$$

Let β in $\mathbb{H}^2(\mathbb{F})$. We define the following process:

$$\tilde{\beta}_t = \mathbb{E}[\beta_t | \mathcal{F}_t^1], \quad t \in [0, T]. \tag{4.31}$$

Since \mathbb{F}^1 satisfies the usual assumptions, $\tilde{\beta}$ can be taken \mathbb{F}^1 -progressively measurable (see, e.g., Brémaud and Yor, 1978, Section 2). Let $\tilde{X} = (\tilde{X}_t)_{t \in [0, T]}$ be the solution of

$$d\tilde{X}_t = (A_t \tilde{X}_t + B_t \tilde{\beta}_t) dt + \sigma_t dW_t, \quad \tilde{X}_0 = \xi.$$

Then, using the explicit expression for the solution \tilde{X} of the SDE above, it can be shown by direct computation that

$$\tilde{X}_t = \mathbb{E}[X_t^\beta | \mathcal{F}_t^1] \quad \mathbb{P}\text{-a.s.}, \quad t \in [0, T].$$

Due to the concave linear quadratic structure of f , we have the following:

$$\begin{aligned}
 \mathfrak{J}^{MFC}(\beta) &= \mathbb{E} \left[\int_0^T f(t, X_t^\beta, \mathbb{E}[X_t^\beta], \beta_t) dt + g(X_T^\beta, \mathbb{E}[X_T^\beta]) \right] \\
 &= \int_0^T \mathbb{E} \left[\mathbb{E} \left[f(t, X_t^\beta, \mathbb{E}[X_t^\beta], \beta_t) \mid \mathcal{F}_t^1 \right] \right] dt + \mathbb{E} \left[\mathbb{E} \left[g(X_T^\beta, \mathbb{E}[X_T^\beta]) \mid \mathcal{F}_T^1 \right] \right] \\
 &\leq \mathbb{E} \left[\int_0^T f(t, \mathbb{E}[X_t^\beta | \mathcal{F}_t^1], \mathbb{E}[X_t^\beta], \mathbb{E}[\beta_t | \mathcal{F}_t^1]) dt + g(\mathbb{E}[X_T^\beta | \mathcal{F}_T^1], \mathbb{E}[X_T^\beta]) \right] \\
 &= \mathbb{E} \left[\int_0^T f(t, \tilde{X}_t, \mathbb{E}[\tilde{X}_t], \tilde{\beta}_t) dt + g(\tilde{X}_T, \mathbb{E}[\tilde{X}_T]) \right] = \mathfrak{J}^{MFC}(\tilde{\beta}),
 \end{aligned}$$

where we have used the fact that $\mathbb{E}[X_t^\beta] = \mathbb{E}[\tilde{X}_t]$ for every time t . Since $\tilde{\beta}$ belongs to $\mathbb{H}^2(\mathbb{F}^1)$, Proposition 3 implies

$$\mathfrak{J}^{MFC}(\beta) \leq \mathfrak{J}^{MFC}(\tilde{\beta}) \leq \mathfrak{J}^{MFC}(\hat{\alpha}^{MFC}).$$

By strict concavity, we deduce that the inequality is strict for any $\beta \neq \hat{\alpha}$. \square

In the next theorem we prove that the MFC solution provides an upper bound to the payoffs of any mean field CCEs. Moreover, this upper bound can not be attained unless the MFC solution is a mean field NE.

Theorem 4 (No outperformance over the MFC solution). *Let (λ, μ) a mean field CCE. Then, the following holds:*

(i) *If $\mathfrak{J}(\lambda, \mu) \geq \mathfrak{J}(\hat{\alpha}^{MFC}, \bar{x}^{MFC})$, then $(\lambda, \mu) = (\hat{\alpha}^{MFC}, \bar{x}^{MFC})$, so $\mathfrak{J}(\lambda, \mu) = \mathfrak{J}(\hat{\alpha}^{MFC}, \bar{x}^{MFC})$;*

(ii) *If the MFC solution is not a mean field NE, then $\mathfrak{J}(\lambda, \mu) < \mathfrak{J}(\hat{\alpha}^{MFC}, \bar{x}^{MFC})$. In particular, the MFC solution is not a mean field CCE either.*

Proof of (i). By using the payoff functional \mathfrak{J}^{MFC} defined by (4.29), the payoffs' inequality reads as

$$\mathfrak{J}(\lambda, \mu) \geq \mathfrak{J}(\hat{\alpha}^{MFC}, \bar{x}^{MFC}) = \mathfrak{J}^{MFC}(\hat{\alpha}^{MFC}). \quad (4.32)$$

We reformulate the MFC problem weakly, by taking advantages of the results of (Carmona and Delarue, 2018, Paragraph 6.6). We define the set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R}^d \times \mathcal{C}([0, T]; \mathbb{R}^d) \times L^2([0, T]; \mathbb{R}^k))$ of admissible probability measures in the following way: take any filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual assumptions, equipped with a d -dimensional \mathbb{F} -Brownian motion W and an \mathcal{F}_0 -measurable random variable ξ independent of W . Let $\alpha \in \mathbb{H}^2(\mathbb{F})$, which we regard as random variable taking values in $L^2([0, T]; \mathbb{R}^k)$. Let $X = X^\alpha$ be the solution of

$$dX_t^\alpha = (A_t X_t^\alpha + B_t \alpha_t) dt + dW_t, \quad X_0^\alpha = \xi. \quad (4.33)$$

Then, a probability measure P belongs to \mathcal{A} if $P = \mathbb{P} \circ (\xi, X^\alpha, \alpha)^{-1}$. For any $P \in \mathcal{A}$, set $\bar{x}_t = \int_{\mathbb{R}^d} y(P \circ x_t^{-1})(dy)$. By recalling the definitions of f and g in (4.30), define the payoff

functional

$$\begin{aligned} \mathcal{J}(P) &= \int_{\mathbb{R}^d \times \mathcal{C}([0,T], \mathbb{R}^d) \times L^2([0,T]; \mathbb{R}^k)} \left(\int_0^T f(t, x_t, \bar{x}_t, a) dt + g(x_T, \bar{x}_T) \right) P(dz, dx, da) \\ &= \mathbb{E}^{\mathbb{P}} \left[\int_0^T \left(f(t, X_t, \mathbb{E}^{\mathbb{P}}[X_t], \alpha_t) \right) dt + g(X_T, \mathbb{E}^{\mathbb{P}}[X_T]) \right]. \end{aligned} \quad (4.34)$$

By (Carmona and Delarue, 2018, Theorem 6.37), there exists a probability measure P^* in \mathcal{A} so that

$$\mathcal{J}(P^*) \geq \mathcal{J}(P) \quad \forall P \in \mathcal{A}. \quad (4.35)$$

Let $(\xi, X^{MFC}, \hat{\alpha}^{MFC})$ be the MFC solution given by Proposition 3 and let \hat{P} be its law. Let (λ, μ) be a mean field CCE and $(\xi, X^\lambda, \lambda)$ be the corresponding initial state, state process and correlated strategy. We show the following properties:

1. The maximum point P^* is unique and it is equal to \hat{P} .
2. For every m in the support of μ , there exists a version of the regular conditional probability of $(\xi, \lambda, X^\lambda)$ given $\mu = m$; if we set $P^m = \mathbb{P}((\xi, \lambda, X^\lambda) \in \cdot \mid \mu = m)$, then P^m belongs to \mathcal{A} , and it holds

$$\mathfrak{J}(\lambda, \mu) = \int_{\mathcal{C}([0,T]; \mathbb{R}^d)} \mathcal{J}(P^m) \rho(dm),$$

where ρ denotes the law of μ .

3. We use the above equality to show that $\mu = \bar{x}^{MFC}$ \mathbb{P} -a.s. and deduce $\lambda = \hat{\alpha}^{MFC} d\mathbb{P} \otimes dt$ -a.e..

As for point 1, let P^* be the admissible probability measure that maximizes \mathcal{J} . Let $(\Omega^*, \mathcal{F}^*, \mathbb{F}^*, \mathbb{P}^*)$, W^* , ξ^* , α^* and X^* be so that $P^* = \mathbb{P}^* \circ (\xi^*, \alpha^*, X^*)^{-1}$. By applying Proposition 3 in this probability space, there exists an optimal control $\hat{\beta}$ which maximizes \mathfrak{J} over $\mathbb{H}^2(\mathbb{F}^*)$. Since the flow of moments of $X^{\hat{\beta}}$ is still given by (4.26b) and (4.27) admits a strong solution, we have $\mathbb{P}^* \circ (\xi^*, X^{\hat{\beta}}, \hat{\beta})^{-1} = \mathbb{P} \circ (\xi, X^{MFC}, \hat{\alpha}^{MFC})^{-1} = \hat{P}$. Therefore, we can conclude that

$$\begin{aligned} \mathcal{J}(P^*) &= \mathbb{E}^{\mathbb{P}^*} \left[\int_0^T \left(f(t, X_t^*, \mathbb{E}^{\mathbb{P}^*}[X_t^*], \alpha_t^*) \right) dt + g(X_T^*, \mathbb{E}^{\mathbb{P}^*}[X_T^*]) \right] \\ &\geq \mathbb{E}^{\mathbb{P}^*} \left[\int_0^T \left(f(t, X_t^{\hat{\beta}}, \mathbb{E}^{\mathbb{P}^*}[X_t^{\hat{\beta}}], \hat{\beta}_t) \right) dt + g(X_T^{\hat{\beta}}, \mathbb{E}^{\mathbb{P}^*}[X_T^{\hat{\beta}}]) \right] = \mathcal{J}(\hat{P}), \end{aligned}$$

with the inequality being strict if $\beta^* \neq \hat{\beta}$. This shows point 1.

As for point 2, we can suppose without loss of generality that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Polish probability space. We note that the state process X^λ is adapted to the filtration generated by ξ , W and λ , which is countably generated. This implies that there exists a version of the regular conditional probability of \mathbb{P} given $\mu = m$, that we denote by \mathbb{P}^m . Since ξ and W are independent of μ , it is straightforward to see that W is a Brownian motion under \mathbb{P}^m as well, the law of ξ under \mathbb{P}^m is

ν and that X^λ still satisfies equation (4.1). Let $P^m = \mathbb{P}^m \circ (\xi, X^\lambda, \lambda)^{-1}$ and observe that P^m belongs to \mathcal{A} for ρ -a.e m in $\mathcal{C}([0, T], \mathbb{R}^d)$. The consistency condition implies that $\mathbb{E}^{\mathbb{P}^m}[X_t^\lambda] = m_t$ for ρ -a.e. m , which in turn implies that

$$\begin{aligned} \mathfrak{J}(\lambda, \mu) &= \mathbb{E}^{\mathbb{P}} \left[\int_0^T \left(f(t, X_t^\lambda, \mu_t, \lambda_t) \right) dt + g(X_T, \mu_T) \right] \\ &= \mathbb{E}^{\mathbb{P}} \left[\mathbb{E}^{\mathbb{P}} \left[\int_0^T \left(f(t, X_t^\lambda, \mu_t, \lambda_t) \right) dt + g(X_T, \mu_T) \middle| \mu \right] \right] \\ &= \int_{\mathcal{C}([0, T], \mathbb{R}^d)} \mathbb{E}^{\mathbb{P}^m} \left[\int_0^T \left(f(t, X_t^\lambda, \mathbb{E}^{\mathbb{P}^m}[X_t^\lambda], \lambda_t) \right) dt + g(X_T, \mathbb{E}^{\mathbb{P}^m}[X_T^\lambda]) \right] \rho(dm) \\ &= \int_{\mathcal{C}([0, T], \mathbb{R}^d)} \mathcal{J}(P^m) \rho(dm). \end{aligned}$$

By (4.32) and (4.35), we have

$$\int_{\mathcal{C}([0, T], \mathbb{R}^d)} (\mathcal{J}(\hat{P}) - \mathcal{J}(P^m)) \rho(dm) \leq 0, \quad \mathcal{J}(\hat{P}) \geq \mathcal{J}(P^m),$$

which implies $\mathcal{J}(\hat{P}) = \mathcal{J}(P^m)$ for ρ -a.e. m . Since \hat{P} is the unique maximizer of \mathcal{J} by point 1, we get $P^m = \hat{P}$ ρ -a.e.. In particular, this implies

$$m_t = \mathbb{E}^{\mathbb{P}^m}[X_t^\lambda] = \int_{\mathbb{R}^d} y(P^m \circ x_t)^{-1}(dy) = \int_{\mathbb{R}^d} y(\hat{P} \circ x_t)^{-1}(dy) = \mathbb{E}[X_t^{MFC}] = \bar{x}_t^{MFC} \text{ for } \rho\text{-a.e. } m.$$

Thus, μ is a.s. equal to \bar{x}^{MFC} , so that the consistency condition (4.4) for the mean field CCE (λ, μ) rewrites as $\bar{x}_t^{MFC} = \mathbb{E}[X_t^\lambda]$. Therefore, we have

$$\mathfrak{J}(\lambda, \mu) = \mathbb{E}^{\mathbb{P}} \left[\int_0^T \left(f(t, X_t^\lambda, \mathbb{E}[X_t^\lambda], \lambda_t) \right) dt + g(X_T, \mathbb{E}[X_T^\lambda]) \right] = \mathfrak{J}^{MFC}(\lambda).$$

Since, by Lemma 4.1, $\hat{\alpha}^{MFC}$ is unique, the previous equality implies that λ is equal to $\hat{\alpha}^{MFC}$ $d\mathbb{P} \otimes dt$ -a.e., which concludes the proof. \square

Proof of (ii). Let us assume that the MFC solution $(\hat{\alpha}^{MFC}, \bar{x}^{MFC})$ is not a mean field NE (see upcoming Definition 3). By item (i) of Theorem 4, every mean field CCE yields a lower payoff than the MFC solution; moreover, if there was a mean field CCE yielding the same payoff as the MFC solution, it would be the MFC solution itself. Therefore, we just need to prove that the MFC solution is not a mean field CCE.

The pair $(\hat{\alpha}^{MFC}, \bar{x}^{MFC})$ is a correlated flow which satisfies the consistency condition in the definition of mean field CCE. Moreover, since $\hat{\alpha}^{MFC} \in \mathbb{H}^2(\mathbb{F}^1)$ and \bar{x}^{MFC} is deterministic, it satisfies the consistency condition of the definition of the mean field NE as well. Since by assumption the MFC solution is not a mean field NE, it is the optimality condition (4.20) in the upcoming definition of mean field NE which is not satisfied. Therefore, there exists $\beta \in \mathbb{H}^2(\mathbb{F}^1)$

so that $\mathfrak{J}(\beta, \bar{x}^{MFC}) > \mathfrak{J}(\hat{\alpha}^{MFC}, \bar{x}^{MFC})$. Since such β is an admissible deviation to the correlated flow $(\hat{\alpha}^{MFC}, \bar{x}^{MFC})$, the optimality condition (4.3) in definition of mean field CCE is not satisfied either. This means that the MFC solution is not a mean field CCE. \square

4.2 Comparison with mean field Nash equilibria

As for mean field NE, we first show that the only mean field CCE with deterministic flow of moments is the mean field NE itself. In particular, this implies that randomization of the flow of moments is needed for mean field CCEs to reach higher payoffs than the mean field NE. Differently from the MFC case, no general outperformance result can be established for mean field CCEs. Instead, one can derive an outperformance condition for correlated flows in the class \mathcal{G} , in a similar approach as for the optimality condition in subsection 3.3.

As shown by next proposition, there exists a unique mean field NE. The proof is a standard application of the Pontryagin maximum principle approach together with the fixed point argument of (Carmona and Delarue, 2018, Chapter 4). We include it in the [Appendix](#) for the sake of completeness.

Proposition 5. *Let ϕ^{NE} and θ^{NE} the solutions of the following system:*

$$\begin{cases} \dot{\phi}_t^{NE} + \phi_t^{NE} A + A^\top \phi_t^{NE} + (Q + \tilde{Q}) - (\phi_t^{NE} B + S^\top) R^{-1} (B^\top \phi_t^{NE} + S) = 0, \\ \phi_T^{NE} = H + \tilde{H}, \\ \dot{\theta}_t^{NE} + A^\top \theta_t^{NE} + q - (\phi_t^{NE} B + S^\top) R^{-1} (B^\top \theta_t^{NE} + r_t) = 0, \\ \theta_T^{NE} = 0. \end{cases} \quad (4.36)$$

Define \hat{A} and \hat{B} as

$$\hat{A}_t = A_t - B_t R_t^{-1} B_t \phi_t^{NE} - B_t R_t^{-1} S, \quad \hat{B}_t = B_t R_t^{-1} (B_t^\top \theta_t^{NE} + r_t). \quad (4.37)$$

Let $\theta^{\hat{m}}$ be the solution of the following equation:

$$\dot{\theta}_t^{\hat{m}} - \psi_t \frac{d\hat{m}_t}{dt} + A_t^\top \theta_t^{\hat{m}} + q_t - (\phi_t B_t + S^\top) R_t^{-1} (B_t^\top \theta_t^{\hat{m}} + r_t) = 0, \quad \theta_T^{\hat{m}} = 0. \quad (4.38)$$

Let ϕ be the solution of the matrix Riccati equation in (4.8). There exists a unique mean field NE $(\hat{\alpha}, \hat{m})$, which is given by

$$\dot{\hat{m}}_t = (A_t - B_t R_t^{-1} B_t \phi_t^{NE} - B_t R_t^{-1} S) \hat{m}_t - B_t R_t^{-1} (B_t^\top \theta_t^{NE} + r_t), \quad \hat{m}_0 = \nu_1, \quad (4.39a)$$

$$\hat{\alpha}_t = -R_t^{-1} ((B_t^\top \phi_t + S) X_t^{NE} + B_t^\top \psi_t \hat{m}_t + B_t^\top \theta_t^{\hat{m}} + r_t), \quad (4.39b)$$

where X^{NE} is the solution of equation (4.1) with the control $\hat{\alpha}$.

We observe that, by definition, a mean field NE is a mean field CCE with deterministic flow

of measures \hat{m} . The converse is true as well, as shown by the following Theorem:

Theorem 6. *Let (λ, μ) be a mean field CCE with deterministic μ . Then, (λ, μ) is the mean field NE.*

Proof. We start by observing that, by using the same concavity and projections arguments as in the proof of Lemma 4.1, we have

$$\mathfrak{J}(\hat{\alpha}, \hat{m}) > \mathfrak{J}(\beta, \hat{m}) \quad \forall \beta \in \mathbb{H}^2(\mathbb{F}), \beta \neq \hat{\alpha}.$$

Let (λ, μ) be a coarse correlated equilibrium with deterministic flow of moments μ . Then, the consistency condition (4.4) becomes $\mu_t = \mathbb{E}[X_t^\lambda]$ for every time t . By optimality, it holds $\mathfrak{J}(\lambda, \mu) \geq \mathfrak{J}(\beta, \mu)$ for every $\beta \in \mathbb{H}^2(\mathbb{F}^1)$. By reasoning as in the proof of Lemma 4.1, there exists a strategy $\tilde{\lambda} \in \mathbb{H}^2(\mathbb{F}^1)$ so that

$$\tilde{\lambda}_t = \mathbb{E}[\lambda_t | \mathcal{F}_t^1], \quad X_t^{\tilde{\lambda}} = \mathbb{E}[X_t^\lambda | \mathcal{F}_t^1], \quad \mathbb{P}\text{-a.s.}, \forall t \in [0, T],$$

where $X^{\tilde{\lambda}}$ is the solution of equation (4.1) corresponding to the strategy $\tilde{\lambda}$. Since μ is deterministic, by exploiting the convex linear quadratic structure of the payoff functional \mathfrak{J}' , we have

$$\mathfrak{J}(\beta, \mu) \leq \mathfrak{J}(\lambda, \mu) \leq \mathfrak{J}(\tilde{\lambda}, \mu), \quad \forall \beta \in \mathbb{H}^2(\mathbb{F}^1). \quad (4.40)$$

Since μ is deterministic by assumption, the consistency condition holds true for the correlated flow $(\tilde{\lambda}, \mu)$ as well, so that (4.40) implies that $(\tilde{\lambda}, \mu)$ is itself a mean field NE. By uniqueness of the mean field NE, we deduce $(\tilde{\lambda}, \mu) = (\hat{\alpha}, \hat{m})$, so that in particular $\mu = \hat{m}$ \mathbb{P} -a.s.. Since $\hat{\alpha}$ is the unique maximizer of $\mathfrak{J}(\cdot, \hat{m})$ over $\mathbb{H}^2(\mathbb{F})$, we deduce $\mathfrak{J}(\hat{\alpha}, \hat{m}) \geq \mathfrak{J}(\lambda, \hat{m})$. Since (λ, μ) is a mean field CCE by assumption, it holds $\mathfrak{J}(\hat{\alpha}, \hat{m}) \leq \mathfrak{J}(\lambda, \hat{m})$, which, by uniqueness, implies that $\lambda = \hat{\alpha}$ $dt \otimes \mathbb{P}$ -a.e. as well. \square

Finally, consider again correlated flows (λ, μ) in the class \mathcal{G} . By using their specific structure as described in (4.10), we are able to provide a condition under which they yield a higher payoff than the mean field NE.

Theorem 7. *Let $\theta^{\hat{m}}$ be the solution of (4.38). Set*

$$\Theta_t^{\hat{m}} = R_t^{-1}(B^\top \theta_t^{\hat{m}} + r_t). \quad (4.41)$$

Let $(\lambda, \mu) \in \mathcal{G}$ corresponding to some $\delta \in \mathbb{H}^2(\mathcal{F}_0)$. Then, $\mathfrak{J}(\lambda, \mu)$ is higher than the payoff $\mathfrak{J}(\hat{\alpha}, \hat{m})$

given by the mean field NE if and only if the following inequality is satisfied:

$$\begin{aligned}
 & \int_0^T \left(\frac{1}{2} (\langle \bar{Q} \hat{m}_t, \hat{m}_t \rangle - \mathbb{E}[\langle \bar{Q} \mu_t, \mu_t \rangle]) + \frac{1}{2} (\langle M_t \hat{m}_t, \hat{m}_t \rangle - \mathbb{E}[\langle M_t \mu_t, \mu_t \rangle]) \right. \\
 & + \frac{1}{2} (\langle G_t \hat{m}_t, \hat{m}_t \rangle - \mathbb{E}[\langle G_t \mu_t, \mu_t \rangle]) + \langle N_t \hat{m}_t, \hat{m}_t \rangle - \mathbb{E}[\langle N_t \mu_t, \mu_t \rangle] \\
 & + \langle L_t - q_t + R_t^{-1} (B_t^\top (\phi_t + \psi_t) + S_t)^\top r_t, \mathbb{E}[\mu_t] - \hat{m}_t \rangle + \langle B_t^\top \phi_t \hat{m}_t, \Theta_t^{\hat{m}} \rangle \\
 & - \mathbb{E}[\langle B_t^\top \phi_t \mu_t, R_t^{-1} \delta_t \rangle] - \mathbb{E}[\langle B_t^\top \psi_t \mu_t, R_t^{-1} \delta_t \rangle] \\
 & + \langle B_t^\top \psi_t \hat{m}_t, \Theta_t^{\hat{m}} \rangle + \frac{1}{2} (\langle R_t \Theta_t^{\hat{m}}, \Theta_t^{\hat{m}} \rangle - \mathbb{E}[\langle R_t^{-1} \delta_t, \delta_t \rangle]) - \langle r_t, \Theta_t^{\hat{m}} - R_t^{-1} \mathbb{E}[\delta_t] \rangle \Big) dt \\
 & + \frac{1}{2} (\langle \bar{H} \hat{m}_T, \hat{m}_T \rangle - \mathbb{E}[\langle \bar{H} \mu_T, \mu_T \rangle]) + \frac{1}{2} (\langle H \hat{m}_T, \hat{m}_T \rangle - \mathbb{E}[\langle H \mu_T, \mu_T \rangle]) \\
 & + \langle \tilde{H} \hat{m}_T, \hat{m}_T \rangle - \mathbb{E}[\langle \tilde{H} \mu_T, \mu_T \rangle] \geq 0.
 \end{aligned} \tag{4.42}$$

The proof is similar to the one of Theorem 2. For the sake of completeness, we include it in the [Appendix](#). We observe that, although the inequality (4.42) is not easy to interpret, it involves only the law of μ and its associated δ . Moreover, it can be verified separately from the optimality condition (4.16), giving some room for mean field CCEs to outperform the mean field NE payoff. This will be accomplished for the abatement game in Section 5.

5 Application to an emission abatement game

In this section we consider an emission abatement game inspired by environmental economics literature on international environmental agreements, in line with the very popular model of Barrett (1994). Previous section's findings allow us to exhibit a simple class of coarse correlated equilibria which (highly) outperforms the mean field NE in this game.

The emission abatement game has the following payoff and dynamics of the representative player state:

$$\mathfrak{J}(\alpha, \mu) = \mathbb{E} \left[\int_0^T \left(a \mu_t - \frac{b}{2} \mu_t^2 - \frac{1}{2} \alpha_t^2 - \frac{\varepsilon}{2} (\mu_t - X_t)^2 \right) dt \right], \tag{4.43a}$$

$$dX_t = \alpha_t dt + dW_t, \quad X_0 = \xi, \tag{4.43b}$$

with a, b non-negative constants and $\varepsilon > 0$. The strategy α_t represents the abatement rate of the player at time t , while X_t models the cumulated abatement over the interval $[0, t]$.

We translate a slightly modified version of the abatement game of Barrett (1994) into a dynamic stochastic mean field game. We follow the N -player formulation of Dokka et al. (2022) by considering symmetric players, and a normalization of the number of players is implicitly added by replacing the sum of abatement efforts by the flow of moments μ . We also add the last term in ε , inspired by further developments of this model in the literature (Grüning and Peters, 2010), which can be interpreted as a reputational cost. It appears to be necessary when

one wants mean field CCEs outperforming the mean field NE at the mean field limit. Indeed, when $\varepsilon = 0$, there exists only a unique mean field CCE, corresponding to the mean field NE. This is straightforward by direct computations and can be also deduced from Proposition 8 (see upcoming Remark 7).

Following Barrett (1994), the other terms of the payoff can be interpreted as follows. The term $a\mu_t - \frac{b}{2}\mu_t^2$, which depends solely on the mean field component μ , is the “abatement benefit”. It represents the individual benefit of global climate change mitigation allowed by aggregate abatement efforts, with a decreasing marginal benefit. The quadratic term in the control, i.e. $-\frac{1}{2}\alpha_t^2$, is an “abatement cost” that the representative country privately pays for its abatement effort.

We do not claim that a mean field approximation of the abatement game of Barrett (1994) is a right way to approach the problem of international environmental agreements economically. We rather use this payoff functional as a toy example that allows us to illustrate very efficiently the interest of mean field CCEs in a context of common good, and to contribute to the findings of Dokka et al. (2022).

Remark 6. Going from static to dynamic games also induces some additional assumptions that were not included in reference models from Barrett (1994) and Dokka et al. (2022). We chose to represent the “abatement benefit” as a running payoff rather than a terminal one, considering that environmental objectives are not only to reach a given level of emissions at a terminal time, but also to abate as much as possible, as early as possible.

5.1 Translation and interpretation of findings in the abatement game

In this subsection we apply the theory developed in the previous section to compute mean field CCEs in the abatement game. In the next subsection, we will make a step further and exhibit a simple but interesting subclass of correlated flows (λ, μ) which verify both the optimality inequality (4.16) and the NE outperformance inequality (4.42).

We use the setting of Section 2 with $d = k = 1$. The parameters are given by

$$A_t = 0, B_t = 1, \sigma_t = 1, L_t = a, \bar{Q}_t = b + \varepsilon, Q_t = \varepsilon, \tilde{Q}_t = -\varepsilon, R_t = 1, \quad \forall t \in [0, T], \quad (4.44)$$

and remaining parameters equal 0. According to Proposition 1 with the abatement game parameters as in (4.44), for a given correlated flow (λ, μ) , the best deviating strategy and the corresponding state process are given by

$$\begin{aligned} \hat{\beta}_t &= \phi_t(\mathbb{E}[\mu_t] - \hat{X}_t) - \theta_t, \\ d\hat{X}_t &= \hat{\beta}_t dt + dW_t, \quad \hat{X}_0 = \xi, \end{aligned} \quad (4.45)$$

with ϕ and θ satisfying equations

$$\begin{cases} \dot{\phi}_t + \epsilon - \phi_t^2 = 0, & \phi_T = 0, \\ \dot{\theta}_t - \phi_t \left(\theta_t + \frac{d\mathbb{E}[\mu_t]}{dt} \right) = 0, & \theta_T = 0. \end{cases} \quad (4.46)$$

Note that ψ does not appear as in this case $\psi = -\phi$. We stress that, as only unilateral deviation is allowed, the deviating player can not act on the abatement benefit, and therefore does not consider a and b in her optimal strategy.

The family \mathcal{G} of correlated flows defined by (4.10) is composed of any correlated flow (λ, μ) so that:

$$\begin{aligned} \lambda_t &= \phi_t(\mu_t - X_t) - \delta_t, \\ \dot{\mu}_t &= -\delta_t, \quad \mu_0 = \nu_1, \end{aligned}$$

for some $\delta \in \mathbb{H}^2(\mathcal{F}_0)$ and where X solves

$$dX_t = \lambda_t dt + dW_t, \quad X_0 = \xi.$$

In particular, we note that it holds $\delta_t = -\frac{d\mu}{dt}$. Therefore, in this model, the class \mathcal{G} is composed of correlated flows (λ, μ) verifying

$$\lambda_t = \phi_t(\mu_t - X_t) + \frac{d\mu}{dt}, \quad \mathbb{E} \left[\int_0^T \left(\frac{d\mu}{dt} \right)^2 dt \right] < \infty. \quad (4.47)$$

As the correlated strategy depends on μ itself, we remark that the state variable becomes actually mean-reverting. The extra term $\frac{d\mu}{dt}$ allows the state of the representative player following the suggested strategy to satisfy the consistency condition by following the suggested variations of μ .

As shown by the next proposition, in the abatement game, the optimality condition only depends on the law of μ , the reputational cost parameter ϵ , and the final time horizon T .

Proposition 8 (Optimality condition for the abatement game). *Let (λ, μ) be a correlated flow in \mathcal{G} . Let $f(\mu) = (f_t(\mu))_{t \in [0, T]}$ be given by*

$$\begin{cases} \dot{f}_t(\mu) = - \left(\phi_t (f_t(\mu) + \mu_t - \mathbb{E}[\mu_t]) + \frac{d\mu}{dt} + \theta_t \right), & 0 \leq t \leq T, \\ f_0(\mu) = 0. \end{cases} \quad (4.48)$$

Then, (λ, μ) is a mean field CCE if and only if the following condition is satisfied:

$$\int_0^T \mathbb{E} \left[\left(\frac{d\mu}{dt} \right)^2 \right] dt \leq \int_0^T \mathbb{E} \left[(\phi_t f_t(\mu) + \theta_t)^2 + \phi_t^2 (\mu_t - \mathbb{E}[\mu_t] + f_t(\mu))^2 + (\epsilon - \phi_t^2) f_t(\mu)^2 \right] dt. \quad (4.49)$$

Proof. Referring to (4.11) and (4.14), the auxiliary functions for the abatement game are as

follows:

$$\begin{aligned}\Phi_t &= \phi_t, & \Psi_t &= -\phi_t, & \Theta_t &= \theta_t, \\ M_t &= \varepsilon + \phi_t^2, & N_t &= -\varepsilon - \phi_t^2, & G_t &= \phi_t^2.\end{aligned}\tag{4.50}$$

This implies that $f(\mu)$ given in (4.15) takes the form of equation (4.48), recalling that $\delta_t = -d\mu/dt$ by (4.47). After a few computations, we get that the optimality condition (4.16) rewrites as

$$\begin{aligned}\int_0^T \mathbb{E} [\delta_t^2] dt &\leq \int_0^T \mathbb{E} [(\phi_t f_t(\mu) + \theta_t)^2 + \phi_t^2(\mu_t - \mathbb{E}[\mu_t] + f_t(\mu))^2 + (\varepsilon - \phi_t^2) f_t^2(\mu) \\ &\quad + 2\varepsilon(\mu_t - \mathbb{E}[\mu_t])(\mu_t - \mathbb{E}[\mu_t] + f_t(\mu))] dt,\end{aligned}$$

using that

$$\mathbb{E}[\mu_t^2 - \mathbb{E}[\mu_t]^2] = \mathbb{E}[(\mu_t - \mathbb{E}[\mu_t])^2].$$

Since $f_t(\mu) = \hat{X}_t - X_t$ \mathbb{P} -a.s., for every time t by (4.18) and $f(\mu)$ is $\sigma(\mu)$ -measurable by definition, we have

$$f_t(\mu) = \mathbb{E}[f_t(\mu)|\mu] = \mathbb{E}[\hat{X}_t - X_t|\mu] = \mathbb{E}[\hat{X}_t] - \mu_t$$

where we used the consistency condition (4.4) and the fact that \hat{X} and μ are independent. This implies that

$$\mathbb{E}[(\mu_t - \mathbb{E}[\mu_t])(\mu_t - \mathbb{E}[\mu_t] + f_t(\mu))] = \mathbb{E}[\mu_t - \mathbb{E}[\mu_t]](-\mathbb{E}[\mu_t] + \mathbb{E}[\hat{X}_t]) = 0.$$

Therefore, (λ, μ) is a mean field CCE if and only if condition (4.49) is satisfied. \square

By Proposition 5, there exists a unique mean field NE $(\hat{\alpha}, \hat{m})$, which is given by

$$\hat{\alpha}_t = \phi_t(\hat{m}_t - X_t^{NE}),\tag{4.51a}$$

$$\hat{m}_t = \nu_1, \forall t \in [0, T],\tag{4.51b}$$

since we have $\phi^{NE} = \theta^{NE} = 0$ in (4.36), which implies $\theta^{\hat{m}} = 0$ in (4.38) as well.

The mean field NE consists, on average, to null abatement, as \hat{m}_t stays constant equal to its initial value. This corresponds to a free-riding equilibrium, where everybody does as little as possible, and prefers to take advantage of the others' efforts. As a result, nobody does anything.

Remark 7. One can easily see from the optimality condition in equation (4.49) that, if $\varepsilon = 0$, the only mean field CCE is the mean field NE. Indeed, in this case $\phi_t = \theta_t = 0$ and $f_t(\mu) = d\mu/dt$, for all $t \in [0, T]$. Hence the right-hand side term in (4.49) is null, forcing $d\mu/dt = 0$, $t \in [0, T]$. As $\lambda_t = d\mu/dt$, we get $\lambda_t = \hat{\alpha}_t = 0$, $\hat{m}_t = \nu_1$, which is the mean field NE when $\varepsilon = 0$. This seems consistent with the findings of Dokka et al. (2022). Indeed, in an equivalent N -player static deterministic game without the reputational cost ($\varepsilon = 0$), the authors find that, the more players, the less the payoff-maximising CCE outperforms the payoff of the NE. This probably

comes from the fact that, at the mean field limit, there is only one mean field CCE, which is the mean field Nash equilibrium itself.

By Proposition 3, there exists a unique MFC solution $(\hat{\alpha}^{MFC}, \bar{x}_t^{MFC})$ which reads:

$$\hat{\alpha}_t^{MFC} = \phi_t(\bar{x}_t^{MFC} - X_t^{MFC}) - \bar{\eta}_t \bar{x}_t^{MFC} - \bar{\chi}_t, \quad (4.52a)$$

$$\dot{\bar{x}}_t^{MFC} = -\bar{\eta}_t \bar{x}_t^{MFC} - \bar{\chi}_t, \quad \bar{x}_0^{MFC} = \nu_1, \quad (4.52b)$$

with

$$\begin{cases} \dot{\bar{\eta}}_t = \bar{\eta}_t^2 - b, & \bar{\eta}_T = 0, \\ \dot{\bar{\chi}}_t = \bar{\eta}_t \bar{\chi}_t + a, & \bar{\chi}_T = 0. \end{cases} \quad (4.53)$$

The MFC solution adds to the mean-reversion two terms which depend on a and b , i.e. on the coefficients of the abatement benefit. One can note that the MFC solution and the mean field NE coincide if and only if $a = b = 0$. To the contrary, when the ‘‘common good’’ aspect of climate is accounted for in the payoff through the abatement benefit, the central planner can reach higher payoffs by preventing any inefficient free-riding behaviour. This gives some room for mean field CCEs to bridge the gap between the free-riding mean field NE and the central planner optimum.

To find mean field CCEs outperforming the mean field NE, the following condition should be fulfilled.

Proposition 9 (Outperformance condition over mean field NE). *Let (λ, μ) be a correlated flow in \mathcal{G} . Then,*

$$\mathfrak{J}(\lambda, \mu) \geq \mathfrak{J}(\hat{\alpha}, \hat{m}) \iff \mathbb{E} \left[\int_0^T \left(a(\mu_t - \hat{m}_t) - \frac{b}{2}(\mu_t^2 - \hat{m}_t^2) \right) dt \right] \geq \frac{1}{2} \mathbb{E} \left[\int_0^T \left(\frac{d\mu}{dt} \right)^2 dt \right].$$

Proof. By recalling the identities in (4.50), inequality (4.42) takes the following form:

$$\begin{aligned} \mathfrak{J}(\lambda, \mu) - \mathfrak{J}(\hat{\alpha}, \hat{m}) &= \int_0^T \left(\frac{1}{2}(b + \varepsilon)(\hat{m}_t^2 - \mathbb{E}[\mu_t^2]) + \frac{1}{2}(\varepsilon + \phi_t^2)(\hat{m}_t^2 - \mathbb{E}[\mu_t^2]) + \frac{1}{2}\phi_t^2(\hat{m}_t^2 - \mathbb{E}[\mu_t^2]) \right. \\ &\quad \left. - (\varepsilon + \phi_t^2)(\hat{m}_t^2 - \mathbb{E}[\mu_t^2]) + a(\mathbb{E}[\mu_t] - \hat{m}) - \phi_t \mathbb{E}[\mu_t \delta_t] + \phi \mathbb{E}[\delta_t \mu_t] - \frac{1}{2}\mathbb{E}[\delta_t^2] \right) dt \geq 0 \end{aligned}$$

By rearranging the terms and recalling that $\delta_t = -d\mu/dt$, we get the desired inequality. \square

The equivalence in Proposition 9 clearly illustrates that, when $a = b = 0$, the best payoff mean field CCE is actually the mean field NE. This was also implied by the fact that, when $a = b = 0$, the MFC solution is a mean field NE as we already mention above.

5.2 A tractable class of mean field CCEs

In this subsection we show that, when $a \neq 0$ or $b \neq 0$, the optimality and outperformance conditions are not empty, and neither is their intersection. In this case, the MFC solution is

distinct from the mean field NE, which implies, according to Theorem 4, that the MFC solution is not a mean field CCE, as the required control does not resist any unilateral deviation which tends to a less costly free-riding option. However, by introducing correlation through correlated flows, one can manage to drive the population at quite high abatement levels, leading to more desirable social outcomes than the one of the mean field NE.

The optimality condition (4.49) is very convenient to use when one focuses on a specific class of dynamics for μ . In this subsection, we consider a subclass $\mathcal{G}_l \subseteq \mathcal{G}$ where the flows of moments are linear in time.

More precisely, let \mathcal{G}_l be the set of all correlated flows $(\lambda, \mu) \in \mathcal{G}$ such that

$$\mu_t = \nu_1 + tZ, \quad t \in [0, T], \quad (4.54)$$

for some $Z \in L^2(\mathcal{F}_0)$ independent of ξ and W . Then, for all correlated flows $(\lambda, \mu) \in \mathcal{G}_l$ we have

$$\lambda_t = \phi_t(\mu_t - X_t) + Z, \quad t \in [0, T].$$

In the rest of the paper, we will use the notations $z_1 := \mathbb{E}[Z]$, $z_2 := \mathbb{E}[Z^2]$, $\sigma_z^2 := \mathbb{V}[Z]$.

Proposition 10 (Optimality condition for \mathcal{G}_l). *Let $(\lambda, \mu) \in \mathcal{G}_l$. Then (λ, μ) is a mean field CCE if and only if*

$$z_1^2 c_M + \sigma_z^2 c_V \geq 0 \quad (4.55)$$

with

$$c_M = \int_0^T \left((\phi_t r_t + g_t)^2 + \varepsilon r_t^2 \right) dt - T, \quad c_V = \int_0^T \left(\phi_t^2 (v_t - t)^2 + \varepsilon v_t^2 \right) dt - T, \quad (4.56)$$

and

$$g_t = \int_t^T \phi_s e^{-\int_t^s \phi_u du} ds, \quad r_t = \int_0^t (1 - g_s) e^{-\int_s^t \phi_u du} ds, \quad v_t = \int_0^t (s\phi_s + 1) e^{-\int_s^t \phi_u du} ds. \quad (4.57)$$

Proof. For any given $(\lambda, \mu) \in \mathcal{G}_l$ we have $\frac{d\mu}{dt} = Z$ a.s. and $\mathbb{E}[\mu_t] = \nu_1 + tz_1$ so that

$$\begin{aligned} \int_0^T \mathbb{E} \left[\left(\frac{d\mu}{dt} \right)^2 \right] dt &= Tz_2, & \dot{\theta}_t &= \phi_t(\theta_t + z_1), \quad \theta_T = 0, \\ \dot{f}_t(\mu) &= -(\phi_t(f_t(\mu) + t(Z - z_1)) + Z + \theta_t), & f_0(\mu) &= 0. \end{aligned}$$

Let us set

$$p_t := \int_0^t e^{-\int_s^t \phi_u du} ds.$$

By using p_t and the auxiliary functions defined in (4.57), $f_t(\mu)$ and θ_t can be rewritten as follows:

$$\theta_t = -z_1 g_t, \quad f_t(\mu) = -Z p_t - (Z - z_1)(v_t - p_t) + z_1(p_t - r_t).$$

We compute the different terms appearing in the integral of the right-hand side of the optimality condition:

$$\begin{aligned}\mathbb{E}[(f_t(\mu))^2] &= \sigma_z^2 v_t^2 + z_1^2 r_t^2, & \mathbb{E}[(\mu_t - \mathbb{E}[\mu_t] + f_t(\mu))^2] &= \sigma_z^2 (v_t - t)^2 + z_1^2 r_t^2, \\ \mathbb{E}[(\phi_t f_t(\mu) + \theta_t)^2] &= \sigma_z^2 \phi_t^2 v_t^2 + z_1^2 (\phi_t r_t + g_t)^2.\end{aligned}$$

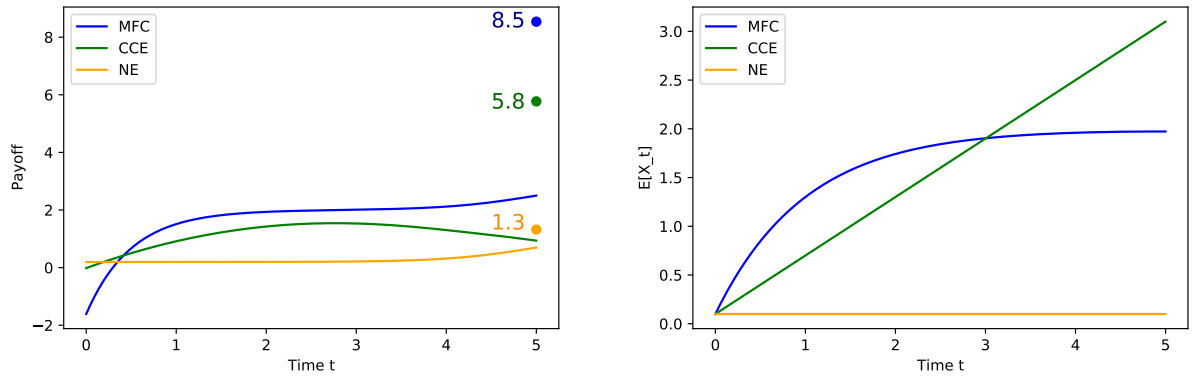
After summing, simplifying and factorising, the optimality condition becomes an inequality on the moments of Z as follows:

$$Tz_2 \leq z_1^2 \int_0^T ((\phi_t r_t + g_t)^2 + \varepsilon r_t^2) dt + \sigma_z^2 \int_0^T (\phi_t^2 (v_t - t)^2 + \varepsilon v_t^2) dt.$$

As $z_2 = \sigma_z^2 + z_1^2$, we get (4.55) and (4.56). \square

Thanks to this simple optimality condition, the set \mathcal{G}_l of mean field CCEs can be easily explored numerically and analytically. In Figure 4.1a we represent the running expected payoffs (time derivative of the payoff) of a mean field CCE, the MFC solution and the mean field NE as curves, and their total payoffs as dots. Figure 4.1b represents the average of the state variables at each time for the same equilibria. As one can see, the mean field CCE in the figure outperforms the mean field NE in terms of both payoff and abatement levels. Moreover, Figure 4.1b shows that this mean field CCE also outperforms the MFC solution in terms of average level of cumulated abatement at the end of the period, i.e. $\mathbb{E}[\mu_T] > \bar{x}_T^{MFC}$.

Implications of the optimality condition for \mathcal{G}_l can be further analysed by stating some of its analytical properties.



(a) Running expected utility and payoff of a mean field CCE, the MFC solution and the mean field NE.

(b) Average level of cumulated abatement as a function of time for this same mean field CCE, the MFC solution and the mean field NE.

Figure 4.1 – A mean field CCE in \mathcal{G}_l bridging the gap between the mean field NE and the MFC solution. Parameter values: $T = 5, a = 2, b = 1, \varepsilon = 1, \nu_1 = 0.1, z_1 = 0.6, \sigma_z^2 = 0.06$.

Proposition 11. *The coefficients c_M, c_V defined in Proposition 10 verify the following:*

(i) $c_M < 0$,

(ii) $c_V > 0$ if and only if $\varepsilon T^2 \geq 3$.

Proof. (i) We argue by contradiction. Suppose $c_M \geq 0$. By (4.55), this is equivalent to the existence of a mean field CCE in \mathcal{G}_l so that the associated random variable Z satisfies $\sigma_Z^2 = 0$ and $z_1 > 0$. Since $\sigma_Z^2 = 0$, (λ, μ) is a mean field CCE with deterministic flow of moments $\mu_t = \nu_1 + tz_1$, for any t in $[0, T]$. By Theorem 6, this implies that $(\lambda, \mu) = (\hat{\alpha}, \hat{m})$. Since $\hat{m}_t = \nu_1$ for every time t , this implies that $z_1 = 0$, leading to a contradiction.

(ii) We now show that $c_V > 0$ if and only if $\varepsilon T^2 \geq 3$. Since $c_M < 0$ by point (i), condition (4.55) implies that $c_V \geq 0$ if and only if there exists a mean field CCE in \mathcal{G}_l so that the associated random variable Z satisfies $z_1 = 0$ and $\sigma_z^2 > 0$. We now identify the conditions so that such a correlated flow is a mean field CCE. In particular, it verifies $\mathbb{E}[\mu_t] = \nu_1$, for all $t \in [0, T]$. By equation (4.45), the optimal strategy of the deviating player is given by $\hat{\beta}_t = \phi_t(\nu_1 - \hat{X}_t)$, where \hat{X} is deviating player's state process. Such a correlated flow is a mean field CCE if and only if $\mathfrak{J}(\lambda, \mu) \geq \mathfrak{J}(\hat{\beta}, \mu)$, which is in turn equivalent to $\mathfrak{J}'(\lambda, \mu) \leq \mathfrak{J}'(\hat{\beta}, \mu)$, where

$$\begin{aligned}\mathfrak{J}'(\hat{\beta}, \mu) &= \frac{1}{2} \mathbb{E} \left[\int_0^T \left((\varepsilon + \phi_t^2) \mathbb{E}[(\nu_1 - \hat{X}_t)^2] + \varepsilon t^2 \sigma_z^2 \right) dt \right], \\ \mathfrak{J}'(\lambda, \mu) &= \frac{1}{2} \mathbb{E} \left[\int_0^T \left((\varepsilon + \phi_t^2) \mathbb{E}[(\mu_t - X_t)^2] + \sigma_z^2 \right) dt \right].\end{aligned}$$

By computing and comparing their dynamics, it can be shown that

$$\mathbb{E}[(\mu_t - X_t)^2] = \mathbb{E}[(\nu_1 - \hat{X}_t)^2], \quad \forall t \in [0, T].$$

Therefore, (λ, μ) is a mean field CCE if and only if $\varepsilon \frac{T^3}{3} - T \geq 0$. This allows to conclude that $c_V \geq 0$ is equivalent to $\varepsilon T^2 \geq 3$. Since c_V is null if and only if $\varepsilon = 0$ and $\varepsilon \neq 0$ by assumption, we deduce that $c_V > 0$ is equivalent to $\varepsilon T^2 \geq 3$. \square

Proposition 11 implies that, if the reputational cost coefficient ε and time horizon T are small enough, the only mean field CCE in \mathcal{G}_l is the mean field NE. On the contrary, when T, ε are big enough, for any expectation of Z there exists a variance level so that any correlated flow with same expectation and higher variance is a mean field CCE.

5.3 Comparison with mean field NE

We have seen above that increasing the variance of Z is a way to build mean field CCEs easily. However, increasing the variance of Z comes at the cost of lowering the odds to outperform the mean field NE, as shown in the next Proposition.

Proposition 12 (Outperformance over the mean field NE in \mathcal{G}_l). *A correlated flow $(\lambda, \mu) \in \mathcal{G}_l$ outperforms the mean field NE in terms of payoff if and only if*

$$Tz_1(a - b\nu_1) - (z_1^2 + \sigma_z^2) \left(b\frac{T^2}{3} + 1 \right) \geq 0 \quad (4.58)$$

Proof. This result follows directly from Proposition 9. The inequality is assessed in the specific case of correlated flows in \mathcal{G}_l , using the following equalities:

$$\hat{m}_t = \nu_1, \quad \mathbb{E}[\mu_t] = \nu_1 + tz_1, \quad \mathbb{E}[\mu_t^2] = \nu_1^2 + 2\nu_1tz_1 + t^2(z_1^2 + \sigma_z^2), \quad \frac{d\mu}{dt} = Z, \quad t \in [0, T]. \quad (4.59)$$

□

The optimality and outperformance conditions for correlated flows in \mathcal{G}_l in, respectively, Proposition 10 and Proposition 12, are both expressed in terms of the first and second moments of associated variable Z . This allows us to characterize analytically a region of mean field CCEs outperforming the mean field NE in the plane (z_1, σ_z^2) .

Proposition 13. *Assume $\varepsilon T^2 \geq 3$. Then, a correlated flow in \mathcal{G}_l is a mean field CCE outperforming the mean field NE in terms of payoff if and only if the associated random variable Z verifies*

$$-\frac{c_M}{c_V} z_1^2 \leq \sigma_z^2 \leq z_1 \frac{3T(a - b\nu_1)}{bT^2 + 3} - z_1^2. \quad (4.60)$$

Moreover, the set of mean field CCEs outperforming the mean field NE is not reduced to the mean field NE if and only if $a - b\nu_1 > 0$.

Proof. By combining Propositions 8, 11 and 12, we can see that a correlated flow in \mathcal{G}_l with moments z_1, σ_z^2 for Z is a mean field CCE outperforming the mean field NE in terms of expected payoff if and only if equation (4.60) is verified. Let us denote by $f, g : \mathbb{R} \rightarrow \mathbb{R}$ respectively the left hand-side and the right-hand side of that equation as function of z_1 . They are both parabola intersecting at the point $(0, 0)$. The second derivative of f , f'' , is strictly increasing as c_V is positive according to Proposition 11, while g'' is strictly decreasing. Simple arguments therefore imply that the region between the two curves characterized in equation (4.60), i.e.,

$$\{(x, y) \in \mathbb{R}^2 : f(x) \leq y \leq g(x)\},$$

is not equal to the point $(0, 0)$ if and only if $g'(0) > f'(0)$. This is the case if and only if $a - b\nu_1 > 0$. If the region between the two curves was reduced to the point $(0, 0)$, the only mean field CCEs outperforming the mean field NE would verify $z_1 = \sigma_z^2 = 0$, which corresponds to the mean field NE. □

Figure 4.2 represents the region of mean field CCEs outperforming the payoff of the mean field NE in the plane (z_1, σ_z^2) for the same parameters as in Figure 4.1. The outperformance

condition parabola (“upper parabola”) is represented in red, while the optimality condition parabola (“lower parabola”) is in blue.

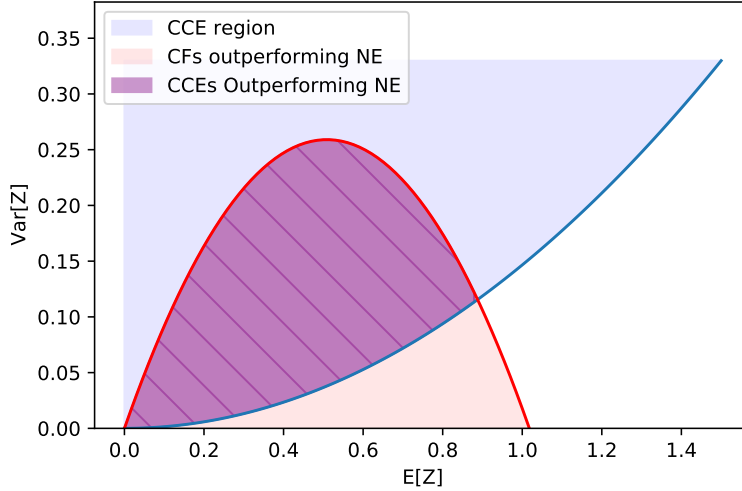


Figure 4.2 – Region of mean field CCEs in \mathcal{G}_l which outperform the Nash equilibrium in the plane (z_1, σ_z^2) .

Proposition 13 shows that each of the parameters a , b and ε of the payoff plays specific roles in identifying mean field CCEs that outperform the payoff of the mean field NE. The upper parabola comes from the outperformance condition and only depends on a, b while the lower parabola comes from the optimality condition and only depends on ε . The existence of the abatement benefit leaves space for more correlated flows to outperform the free-riding equilibrium payoff, as the upper parabola increases in a and decreases in b . Moreover, Figure 4.3 shows that the ratio $-c_M/c_V$ and hence the lower parabola is decreasing in ε . Therefore, the reputational cost helps correlated flows to be CCEs. Indeed, with a higher reputational cost, countries have more interest in staying close to one another, and therefore the correlation device is more enforcing.

Figure 4.4 represents the payoffs of the mean field CCEs belonging to \mathcal{G}_l and which outperform the mean field NE in terms of payoff, i.e., verifying equation (4.60). According to this graph, using the simple and tractable class of correlated flows \mathcal{G}_l , one is able to explore a large part of the payoffs attainable by mean field CCEs in this game. Indeed, mean field CCEs payoffs get pretty close to the unattainable bound provided by the MFC solution payoff, relatively to the payoff of the mean field NE.

The mean field CCEs in \mathcal{G}_l which are optimal in terms of expected payoffs can be identified analytically.

Proposition 14. *Assume $\varepsilon T^2 \geq 3$. Then, the expected payoff of mean field CCEs in \mathcal{G}_l is*

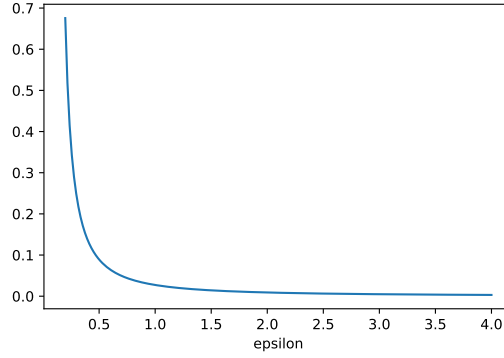


Figure 4.3 – Representation of $-c_M/c_V$ in function of ε , where ε verifies $\varepsilon T^2 \geq 3$. Parameterization: $T = 5$.

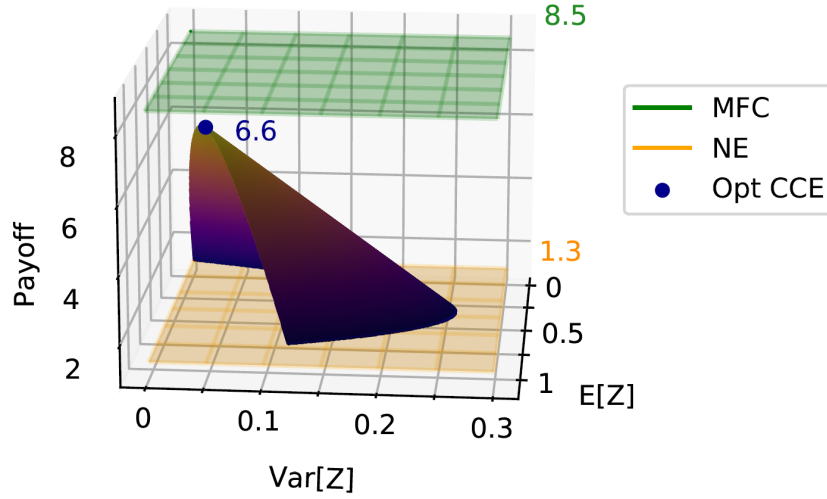


Figure 4.4 – Expected payoff of mean field CCEs in \mathcal{G}_l which outperform the mean field NE in terms of payoff (i.e. verifying equation (4.60)) in the 3D space with z_1 as the x-axis, σ_z^2 as the y-axis and expected payoff as the z-axis. Parameters: same as Figure 4.1.

maximised by a correlated flow (λ, μ) so that the associated random variable Z satisfies

$$z_1 = \frac{T(a - b\nu_1)}{2(1 - \frac{c_M}{c_V})(b\frac{T^2}{3} + 1)}, \quad \sigma_z^2 = -z_1^2 \frac{c_M}{c_V}. \quad (4.61)$$

Proof. We note that

$$\mathfrak{J}(\lambda, \mu) = \mathfrak{J}(\hat{\alpha}, m) + \frac{T^2}{2} z_1 (a - b\nu_1) - \frac{(\sigma_z^2 + z_1^2)}{2} \left(b\frac{T^3}{3} + T \right).$$

Therefore, $\mathfrak{J}(\lambda, \mu)$ is strictly decreasing in σ_z^2 . Moreover, since $c_V > 0$ according to Proposition 11, the optimality condition for correlated flows in \mathcal{G}_l of Proposition 10 implies that (λ, μ) is a mean field CCE if and only if

$$\sigma_z^2 \geq -z_1^2 \frac{c_M}{c_V}.$$

As $-c_M/c_V > 0$, for any given z_1 the mean field CCE with the highest expected payoff verifies $\sigma_z^2 = -z_1^2 c_M/c_V$. From now on, let us set σ_z^2 to this value. We get

$$\mathfrak{J}(\lambda, \mu) = \mathfrak{J}(\hat{\alpha}, m) + \frac{T^2}{2} z_1 (a - b\nu_1) - \frac{z_1^2 (1 - \frac{c_M}{c_V})}{2} \left(b \frac{T^3}{3} + T \right),$$

which is a polynomial in z_1 whose maximum point is given by z_1 as in (4.61). \square

Figure 4.5a shows the payoffs of mean field CCEs in \mathcal{G}_l with payoff-maximizing variance for Z , i.e. verifying $\sigma_z^2 = -z_1^2 c_M/c_V$. These payoffs are expressed as a function of z_1 , on the x -axis, and they are compared to the payoffs of the MFC solution and the mean field NE, with same parameter settings as in the other figures. Figure 4.5b represents the average cumulated abatement over the whole time interval for the same equilibria, in the same fashion.

One can see out of Figure 4.5 that for “little ambitious” mean field CCEs in \mathcal{G}_l , i.e., with relatively small z_1 , there is actually a significant increase in both abatement levels and payoffs with regards to the mean field NE. However, there is a critical value of z_1 , given by the payoff-maximising value of Proposition 14 and represented by the grey vertical line, after which increasing abatement comes at the cost of decreasing the payoff. This is in partial contrast with the results of Dokka et al. (2022) where a much stronger trade-off was observed. The difference is due to the presence of the reputational cost.

Characterizing a surface of mean field CCEs allows any moderator to choose the mean field CCE which corresponds to its goal, which might be to maximise expected payoff, or to consider positive externalities of abatement which are not “priced” in \mathfrak{J} , and therefore to favor high abatement over maximising payoffs.

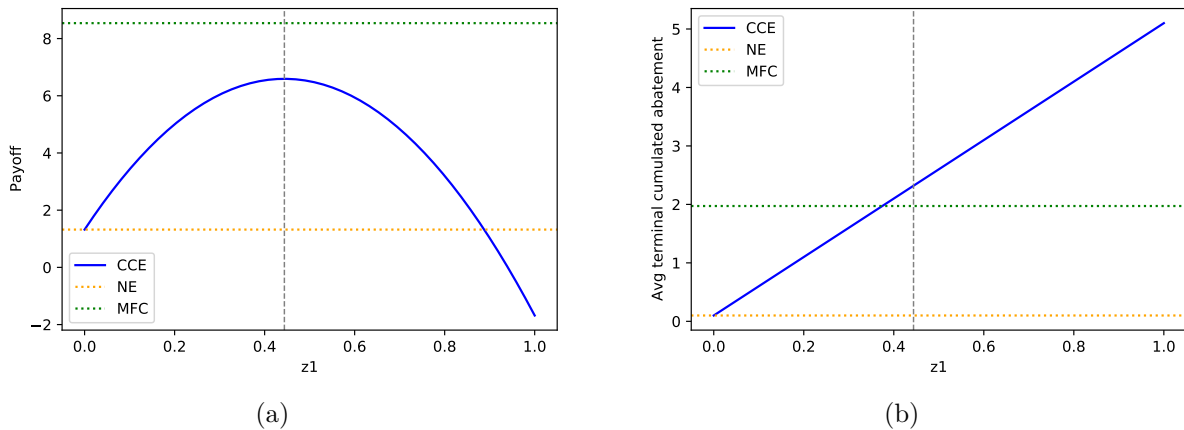


Figure 4.5 – (a) Payoff of mean field CCEs of \mathcal{G}_l with optimal variance as a function of z_1 , and (b) their average cumulated abatement $\mathbb{E}[\mu_T]$, compared to the same quantities for the MFC solution and mean field NE. Parameter values are the same as in Figure 4.1.

Conclusion

In conclusion, Campi et al. (2023) show that mean field coarse correlated equilibria are a relevant notion of mean field game solution. But proofs of existence and convergence in this paper are not constructive in the sense that the shape of actual CCEs remains very abstract. To allow for any analytical computations, we focus on linear-quadratic mean field games, in line with Carmona et al. (2019) for example. In this paper, we outline a methodology to characterize CCEs in linear-quadratic MFGs by restraining the analysis to a tractable class of correlated flows. We establish that they can not outperform the payoff of the MFC solution, and that they equalize their payoff only if they coincide with a mean field NE. We identify a condition under which the mean field NE payoff can be overtaken by mean field CCEs. Finally, we illustrate these different conditions with a mean field version of an emission abatement game, and show that mean field CCEs can indeed very significantly outperform the payoff of a mean field NE. We also show that the set of payoffs attainable through mean field CCEs can be efficiently (even though not exhaustively) explored using very simple and tractable shapes of correlated flows. Even though it looks out of reach to characterize the whole set of mean field CCEs even in a linear-quadratic mean field game, our case study shows that the simple sub-class that we consider has very desirable properties in terms of payoffs and abatement levels, establishing them as relevant alternatives to mean field NE. Allowing for state dynamics depending on the stochastic flow of moments as well as going beyond the linear-quadratic setting will be the topic of future research.

Acknowledgements

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Appendix

A Proofs

Proof of Proposition 1. We follow the approach of (Yong and Zhou, 1999, Chapter 6). We start by noticing that the equation for ϕ is a matrix Riccati equation, which admits a unique solution $\phi \in \mathcal{C}^1([0, T], \mathcal{S}^d)$ by Chapter 6, Theorem 7.2 therein. This implies the existence and uniqueness for ψ and θ as well as they satisfy linear ODEs.

First, thanks to Assumptions **A**(4), the cost functional \mathfrak{J}' is strictly convex and therefore has a unique minimizer. Indeed, by looking at (4.7), we have

$$\begin{aligned} & \frac{1}{2} \langle Q_t X_t, X_t \rangle + \langle \tilde{Q}_t \mathbb{E}[\mu_t] + q_t, X_t \rangle + \frac{1}{2} \langle R_t \beta_t, \beta_t \rangle + \langle S_t X_t, \beta_t \rangle + \langle \tilde{Q}_t \mathbb{E}[\mu_t] + q_t, X_t \rangle + \langle r_t, \beta_t \rangle \\ & \geq \frac{1}{2} d_1 |X_t|^2 + \frac{1}{2} d_2 |\beta_t|^2 - \sup_t |S_t| |X_t| |\beta_t| + \langle \tilde{Q}_t \mathbb{E}[\mu_t] + q_t, X_t \rangle + \langle r_t, \beta_t \rangle \\ & > \frac{1}{2} d_1 |X_t|^2 + \frac{1}{2} d_2 |\beta_t|^2 - \sqrt{d_1} \sqrt{d_2} |S_t| |X_t| |\beta_t| + \langle \tilde{Q}_t \mathbb{E}[\mu_t] + q_t, X_t \rangle + \langle r_t, \beta_t \rangle. \end{aligned}$$

This inequality and the assumption $H \geq 0$ imply that the cost functional is strictly convex and lower semicontinuous, which yields that the minimizer exists and it is unique. Observe that this holds for any $(\mathbb{E}[\mu_t])_{t \in [0, T]}$, since it appears only in the linear terms $\langle \tilde{Q}_t \mathbb{E}[\mu_t] + q_t, X_t \rangle$ and $\langle r_t, \beta_t \rangle$.

We apply the stochastic maximum principle, as in (Yong and Zhou, 1999, Chapter 6, Proposition 5.5). In the following, for the sake of clarity, we omit the dependence on time in all the matrices appearing in the coefficients and in the cost functions. Let $\mathcal{H}(t, x, y, \beta)$ be the reduced Hamiltonian of the system, defined as

$$\mathcal{H}(t, x, y, \beta) = \langle Ax + B\beta, y \rangle + \frac{1}{2} \langle Qx, x \rangle + \langle \tilde{Q}\mathbb{E}[\mu_t] + q, x \rangle + \langle Sx, \beta \rangle + \frac{1}{2} \langle R\beta, \beta \rangle + \langle r, \beta \rangle.$$

Then, the control $\hat{\beta} = (\hat{\beta}_t)_{t \in [0, T]}$ is optimal if and only if there exists a 4-tuple $(X, \hat{\beta}, Y, Z)$ which satisfies

$$\begin{cases} dX_t = (A_t X_t + B_t \hat{\beta}_t) dt + \sigma_t dW_t, & X_0 = \xi, \\ dY_t = - \left(A_t^\top Y_t + Q_t X_t + \tilde{Q}_t \mathbb{E}[\mu_t] + q_t + S_t^\top \hat{\beta}_t \right) dt + Z_t dW_t, & Y_T = H X_T + \tilde{H} \mathbb{E}[\mu_T], \\ B_t^\top Y_t + S_t X_t + R_t \hat{\beta}_t + r_t = 0. \end{cases} \quad (4.62)$$

We make the following ansatz on Y :

$$Y_t = \phi_t X_t + \psi_t \mathbb{E}[\mu_t] + \theta_t,$$

with ϕ , ψ and θ deterministic functions taking values in $\mathbb{R}^{d \times d}$, $\mathbb{R}^{d \times d}$ and \mathbb{R}^d respectively. Since R is invertible for every time t by assumption **A**(4), by comparing the stochastic differential

of the ansatz with (4.62), we find that $Z_t = \hat{\phi}_t \sigma$ and that ϕ , ψ and θ must satisfy equations (4.8). \square

Proof of Proposition 3. We follow the Pontryagin maximum principle approach for MFC problems of (Carmona and Delarue, 2018, Chapter 6). Let \mathcal{H} be the Hamiltonian of the system:

$$\begin{aligned} \mathcal{H}(t, x, y, m, \alpha) = & \langle Ax + B\alpha, y \rangle - \frac{1}{2} \langle \bar{Q}m, m \rangle + \langle L, m \rangle - \left(\frac{1}{2} \langle Qx, x \rangle + \langle \tilde{Q}m + q, x \rangle \right. \\ & \left. + \langle Sx, \alpha \rangle + \frac{1}{2} \langle R\alpha, \alpha \rangle + \langle r, \alpha \rangle \right), \quad (t, x, y, m, \alpha) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^k. \end{aligned}$$

Then, a control $\hat{\alpha} = (\hat{\alpha}_t)_{t \in [0, T]}$ is optimal if and only if there exists a 4-tuple $(X, \hat{\alpha}, Y, Z)$ which satisfies

$$\begin{cases} dX_t = (A_t X_t + B_t \hat{\alpha}_t) dt + \sigma dW_t, \\ X_0 = \xi, \\ dY_t = - \left(A_t^\top Y_t + Q_t X_t + q_t + S_t^\top \hat{\alpha}_t + (\bar{Q}_t + 2\tilde{Q}_t) \mathbb{E}[X_t] - L_t \right) dt + Z_t dW_t, \\ Y_T = H X_T + (\bar{H} + 2\tilde{H}) \mathbb{E}[X_T], \\ B_t^\top Y_t + S_t X_t + R_t \hat{\alpha}_t + r_t = 0. \end{cases} \quad (4.63)$$

Set $x_t = \mathbb{E}[X_t]$, $y_t = \mathbb{E}[Y_t]$ and $a_t = \mathbb{E}[\hat{\alpha}_t]$. Then, by taking expectation, we get the following system

$$\begin{cases} \dot{x}_t = A_t x_t + B_t a_t, & x_0 = \mathbb{E}[\xi], \\ \dot{y}_t = - \left(A_t^\top y_t + (Q_t + 2\tilde{Q}_t + \bar{Q}_t) x_t + q_t - L_t + S_t^\top a_t \right), & y_T = (H + 2\tilde{H} + \bar{H}) x_T, \\ B_t^\top y_t + S_t x_t + R_t a_t + r_t = 0. \end{cases} \quad (4.64)$$

To find a solution, we make the following ansatz on y :

$$y_t = \phi_t^{MFC} x_t + \theta_t^{MFC},$$

with ϕ^{MFC} and θ^{MFC} suitable deterministic functions taking values in $\mathbb{R}^{d \times d}$ and \mathbb{R}^d respectively. Since R is invertible for every time t by assumption **A**(4), by comparing the differential of the ansatz with (4.64), we get to equations (4.23). By (Yong and Zhou, 1999, Chapter 6, Theorem 7.2) there exists a unique solution for the matrix Riccati equation for $\phi^{MFC} \in \mathcal{C}^1([0, T], \mathcal{S}^d)$. We note that the flow of expectations $\bar{x}^{MFC} = (\bar{x}_t^{MFC})_{t \in [0, T]}$ satisfies the differential equation (4.26b).

To prove the existence of a solution to the forward backward system (4.63), we can make the ansatz

$$Y_t = \hat{\phi}_t X_t + \bar{\psi}_t \bar{x}_t^{MFC} + \bar{\theta}_t.$$

with $\hat{\phi}$, $\bar{\psi}$ and $\bar{\theta}$ deterministic functions taking values in $\mathbb{R}^{d \times d}$, $\mathbb{R}^{d \times d}$ and \mathbb{R}^d respectively. By differentiating the ansatz, comparing it with (4.63) and using the invertibility of R for any time t , we find that $Z_t = \hat{\phi}_t \sigma$ and that $\hat{\phi}$ satisfies the same equation as ϕ , so that $\hat{\phi} = \phi$, and equations (4.25) must be satisfied by $\bar{\psi}$ and $\bar{\theta}$. \square

Proof of Proposition 5. We follow the Pontryagin maximum principle approach together with the fixed point argument of (Carmona and Delarue, 2018, Chapter 4). Let \mathcal{H} be the Hamiltonian of the system:

$$\begin{aligned} \mathcal{H}(t, x, y, m, \alpha) = & \langle Ax + B\alpha, y \rangle - \frac{1}{2} \langle \bar{Q}m, m \rangle + \langle L, m \rangle - \left(\frac{1}{2} \langle Qx, x \rangle + \langle \tilde{Q}m + q, x \rangle \right. \\ & \left. + \langle Sx, \alpha \rangle + \frac{1}{2} \langle R\alpha, \alpha \rangle + \langle r, \alpha \rangle \right), \quad (t, x, y, m, \alpha) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^k. \end{aligned}$$

Then, a control $\hat{\alpha} = (\hat{\alpha}_t)_{t \in [0, T]}$ is optimal if and only if there exists a 4-tuple $(X, \hat{\alpha}, Y, Z)$ which satisfies

$$\begin{cases} dX_t = (AX_t + B\hat{\alpha}_t)dt + \sigma dW_t, & X_0 = \xi, \\ dY_t = - \left(A^\top Y_t + QX_t + \tilde{Q}\hat{m}_t + q + S^\top \hat{\alpha}_t \right) dt + Z_t dW_t, & Y_T = HX_T + \tilde{H}\hat{m}_T, \\ B^\top Y_t + SX_t + R\hat{\alpha}_t + r_t = 0. \end{cases} \quad (4.65)$$

Set $x_t = \mathbb{E}[X_t]$, $y_t = \mathbb{E}[Y_t]$ and $a_t = \mathbb{E}[\hat{\alpha}_t]$. Then, the consistency condition $\mathbb{E}[X_t] = \hat{m}_t$ for every $0 \leq t \leq T$ holds if and only if the following system

$$\begin{cases} \dot{x}_t = Ax_t + Ba_t, & x_0 = \mathbb{E}[\xi], \\ \dot{y}_t = - \left(A^\top y_t + (Q + \tilde{Q})x_t + q + S^\top a_t \right), & y_T = (H + \tilde{H})x_T, \\ B^\top y_t + Sx_t + Ra_t + r_t = 0, \end{cases} \quad (4.66)$$

admits a unique solution. We make the following ansatz on y :

$$y_t = \phi_t^{NE} x_t + \theta_t^{NE},$$

with ϕ^{NE} and θ^{NE} suitable deterministic functions taking values in $\mathbb{R}^{d \times d}$ and \mathbb{R}^d respectively. Since R is invertible for every time t by assumption **A**(4), by comparing the differential of the ansatz with (4.66), we get to equations (4.36). We note that the flow of moments $\hat{m} = (\hat{m}_t)_{t \in [0, T]}$ satisfies the differential equation (4.39a).

The last step it to prove the existence of a solution to the forward backward system (4.65). We make the ansatz

$$Y_t = \hat{\phi}_t X_t + \hat{\psi}_t \hat{m}_t + \theta_t^{\hat{m}}.$$

with $\hat{\phi}$, $\hat{\psi}$ and $\theta^{\hat{m}}$ deterministic functions taking values in $\mathbb{R}^{d \times d}$, $\mathbb{R}^{d \times d}$ and \mathbb{R}^d respectively. By the same reasoning of Proposition 1, we find that $Z_t = \hat{\phi}_t \sigma$, that $\hat{\phi}$ and $\hat{\psi}$ satisfy the same

equations as $\hat{\phi}$ and $\hat{\psi}$, so that $\hat{\phi} = \phi$ and $\hat{\psi} = \psi$, and that equation (4.38) must be satisfied by $\theta^{\hat{m}}$. \square

Proof of Theorem 7. By using $(\phi, \psi, \theta^{\hat{m}})$, we write the dynamics of the state process X^{NE} as

$$\begin{aligned} dX_t^{NE} &= \left((A_t - B_t R_t^{-1} (B_t^\top \phi_t + S)) X_t^{NE} - B_t R_t^{-1} (B_t^\top \psi_t \hat{m}_t + B_t^\top \theta_t^{\hat{m}} + r_t) \right) dt + \sigma_t dW_t \\ &= \left((A_t - B_t \Phi_t) X_t^{NE} - B_t (\Psi_t \hat{m}_t + \Theta_t^{\hat{m}}) \right) dt + \sigma_t dW_t, \end{aligned}$$

with Φ and Ψ defined by (4.14) and $\Theta^{\hat{m}}$ by (4.41). We remark that $\theta^{\hat{m}}$ and thus $\Theta^{\hat{m}}$ depend on \hat{m} through its time derivative $d\hat{m}/dt$. Let $f_t^{\hat{m}, \mu} = (f_t^{\hat{m}, \mu})_{t \in [0, T]}$ be the solution of

$$\dot{f}_t^{\hat{m}, \mu} = (A_t - B_t \Phi_t) f_t^{\hat{m}, \mu} + B_t (\Psi_t (\hat{m}_t - \mu_t) + \Theta_t^{\hat{m}} - \delta_t), \quad f_0^{\hat{m}, \mu} = 0.$$

By Itô's formula, we have that

$$f_t^{\hat{m}, \mu} = X_t - X_t^{NE}.$$

Since $f(\mu)$ is $\sigma(\mu)$ -measurable and X^{NE} is \mathbb{F}^1 -progressively measurable, we have both that X^{NE} and $f_t^{\hat{m}, \mu}$ are independent and that

$$f_t^{\hat{m}, \mu} = \mathbb{E}[f_t^{\hat{m}, \mu} | \mu] = \mathbb{E}[X_t - X_t^{NE} | \mu] = \mu_t - \hat{m}_t, \quad (4.67)$$

by consistency condition. Since it holds

$$\begin{aligned} \mathfrak{J}(\lambda, \mu) - \mathfrak{J}(\hat{\alpha}, \hat{m}) &= \int_0^T \left(\frac{1}{2} \langle \bar{Q} \hat{m}_t, \hat{m}_t \rangle - \frac{1}{2} \mathbb{E}[\langle \bar{Q} \mu_t, \mu_t \rangle] + \langle L_t, \mathbb{E}[\mu_t] - \hat{m}_t \rangle \right) dt \\ &\quad + \frac{1}{2} \langle \bar{H} \hat{m}_T, \hat{m}_T \rangle - \frac{1}{2} \mathbb{E}[\langle \bar{H} \mu_T, \mu_T \rangle] + \mathfrak{J}'(\hat{\alpha}, \hat{m}) - \mathfrak{J}'(\lambda, \mu), \end{aligned}$$

we focus on the difference $\mathfrak{J}'(\hat{\alpha}, \hat{m}) - \mathfrak{J}'(\lambda, \mu)$. In a very similar way as in the proof of Theorem 2 we obtain

$$\begin{aligned} \mathfrak{J}'(\hat{\alpha}, \hat{m}) - \mathfrak{J}'(\lambda, \mu) &= \int_0^T \left(\frac{1}{2} (\langle M_t \hat{m}_t, \hat{m}_t \rangle - \mathbb{E}[\langle M_t (\hat{m}_t + f_t^{\hat{m}, \mu}), \hat{m}_t + f_t^{\hat{m}, \mu} \rangle]) \right. \\ &\quad + \frac{1}{2} (\langle G_t \hat{m}_t, \hat{m}_t \rangle - \mathbb{E}[\langle G_t \mu_t, \mu_t \rangle]) + \langle N_t \hat{m}_t, \hat{m}_t - \mathbb{E}[\mu_t] \rangle - \mathbb{E}[\langle N_t f_t^{\hat{m}, \mu}, \mu_t \rangle] \\ &\quad - \langle q_t - \Phi_t^\top r_t, \mathbb{E}[f_t^{\hat{m}, \mu}] \rangle - \langle \Psi_t^\top r_t, \hat{m}_t - \mathbb{E}[\mu_t] \rangle + \langle (R_t \Phi_t - S_t) \hat{m}_t, (\Theta_t^{\hat{m}} - R_t^{-1} \mathbb{E}[\delta_t]) \rangle \\ &\quad - \mathbb{E}[\langle (R_t \Phi_t - S_t) f_t^{\hat{m}, \mu}, R_t^{-1} \delta_t \rangle] - \mathbb{E}[\langle R_t \Psi_t \mu_t, R_t^{-1} \delta_t \rangle] \\ &\quad + \langle R_t \Psi_t \hat{m}_t, \Theta_t^{\hat{m}} \rangle + \frac{1}{2} (\langle R_t \Theta_t^{\hat{m}}, \Theta_t^{\hat{m}} \rangle - \mathbb{E}[\langle R_t^{-1} \delta_t, \delta_t \rangle]) - \langle r_t, \Theta_t^{\hat{m}} - R_t^{-1} \mathbb{E}[\delta_t] \rangle \Big) dt \\ &\quad + \frac{1}{2} \langle H \hat{m}_T, \hat{m}_T \rangle - \frac{1}{2} \langle H (\hat{m}_T + \hat{f}_T(\mu)), \hat{m}_T + \hat{f}_T(\mu) \rangle + \langle \tilde{H} \hat{m}_T, \hat{m}_T - \mathbb{E}[\mu_T] \rangle - \mathbb{E}[\langle \tilde{H} \hat{f}_T(\mu), \mu_T \rangle]. \end{aligned}$$

Finally, we observe that, by using (4.67), we have

$$\begin{aligned}
 & \langle M_t \hat{m}_t, \hat{m}_t \rangle - \mathbb{E}[\langle M_t(\hat{m}_t + f_t^{\hat{m}, \mu}), \hat{m}_t + f_t^{\hat{m}, \mu} \rangle] = \langle M_t \hat{m}_t, \hat{m}_t \rangle - \mathbb{E}[\langle M_t \mu_t, \mu_t \rangle], \\
 & \langle N_t \hat{m}_t, \hat{m}_t - \mathbb{E}[\mu_t] \rangle - \mathbb{E}[\langle N_t f_t^{\hat{m}, \mu}, \mu_t \rangle] = \langle N_t \hat{m}_t, \hat{m}_t \rangle - \mathbb{E}[\langle N_t \mu_t, \mu_t \rangle], \\
 & \langle (R_t \Phi_t - S_t) \hat{m}_t, (\Theta_t^{\hat{m}} - R_t^{-1} \mathbb{E}[\delta_t]) \rangle - \mathbb{E}[\langle (R_t \Phi_t - S_t) f_t^{\hat{m}, \mu}, R_t^{-1} \delta_t \rangle] \\
 & = \langle (R_t \Phi_t - S_t) \hat{m}_t, \Theta_t^{\hat{m}} \rangle - \mathbb{E}[\langle (R_t \Phi_t - S_t) \mu_t, R_t^{-1} \delta_t \rangle], \\
 & \langle q_t - \Phi_t^\top r_t, \mathbb{E}[f_t^{\hat{m}, \mu}] \rangle + \langle \Psi_t^\top r_t, \hat{m}_t - \mathbb{E}[\mu_t] \rangle = \langle q_t - (\Phi_t + \Psi_t)^\top r_t, \mathbb{E}[\mu_t] - \hat{m}_t \rangle.
 \end{aligned}$$

By using these identities together with (4.19), we get to (4.42). □

Conclusion

In summary, the first chapter of this thesis provides a methodological review of climate stress tests and scenario analyses developed by central banks and the academic literature. The ability of these methodologies to assess the magnitude of financial risks caused by climate change and the low-carbon transition is discussed. In particular, this review shows that the impact of the uncertainty characterizing these two phenomena tends to be underestimated in the assessment of the financial risks they represent. This chapter, together with the perspective provided in the introduction to this thesis, helped to set the stage for the research issues addressed in the following chapters: the importance of agent behavior in the low-carbon transition, the key role of the financial sector and the opacity of data assessing corporate exposure to physical and transition risks.

The following three chapters present three models and modeling tools from the fields of financial microeconomics and game theory, in which key agents in the low-carbon transition interact strategically. The second chapter of this thesis answers the need to understand banks' financing decisions and their impact in the context of the low-carbon transition, by adopting a microeconomic approach focused on their interests. Although still in working paper stage, it suggests the existence of a potential snowball effect when a bank ceases to finance a company whose assets must be stranded in the low-carbon transition. Indeed, a bank's decision to disinvest from such a company could, under certain conditions, prompt the other banks financing this company to interrupt their financing as well, leading the company into a situation of insolvency. The results of this study can thus be interpreted as the identification of a lever for the low-carbon transition, but also as highlighting a potential risk of economic and financial instability associated with it.

The third, more mature chapter looks at corporates' greenwashing behavior as a strategic choice to minimize their cost of capital, in a context of asymmetric information about their environmental footprint. It shows that, if communication is sufficiently cheap relative to emission abatement efforts, or if information asymmetry regarding their environmental footprint is sufficiently high, investors' environmental preferences encourage companies to engage in greenwashing. However, investors can limit this incentive by penalizing revelations about companies' environmental footprints when they are affected by a controversy.

The fourth chapter of this thesis, although mainly addressed to the emerging literature on mean field games in the financial mathematics community, proposes an application of its technical contributions to an emission abatement game between countries. In particular, it shows that coarse correlated equilibria increase the utility of each player, which is representative of collective utility since agents are considered symmetric in a mean field game, with respect to a Nash equilibrium. This final contribution highlights the relevance of focusing on coarse correlated equilibria in a common-good game, and opens the prospect of further applications to climate change mitigation efforts.

Thus, through these four chapters of mainly theoretical contributions, this thesis fuels research on the use of prospective macroeconomic, microeconomic and financial models to better understand the interaction between the financial system and the low-carbon transition. By focusing on the strategic interactions between agents, it helps to decipher the drivers and inhibitors of climate change mitigation. In particular, it sheds light on the interest and capacity of financial players to participate in mitigation efforts by developing appropriate models. It also fuels research into the risks to financial stability posed by climate change and the low-carbon transition.

In addition, the last two chapters of this thesis make contributions at the intersection of financial mathematics, microeconomics and game theory. The choice of a continuous-time stochastic dynamic framework and the use of mean field games within this same framework, while favoring an analytical resolution, enable to propose rich models endowed with a complex system of interactions while remaining interpretable. These last two chapters thus underline the relevance of using these sophisticated modeling tools to shed light on the strategic behavior of agents in the context of the low-carbon transition.

This thesis opens up a number of avenues for further research. In particular, it calls for the further development of theoretical tools to study the role of the financial system in the low-carbon transition, an issue that is far from exhausted at present. For example, the second chapter addresses the question of banks' financing choices in relation to the low-carbon transition from just one angle, that of disinvestment. Although it would be interesting to refine the approach to divestment, in particular by seeking to link divestment more directly to its impact in terms of carbon emissions, this approach should also be complemented by theoretical work addressing the issue of financing companies making mitigation efforts. This would provide a better understanding of the levers available to banks to promote mitigation efforts. More generally, it would also inform the work of central banks in climate stress testing exercises with dynamic balance sheets, and in their handling of climate-related risks in interaction with their financial stability mandate.

Furthermore, the conclusions of the third chapter on corporate greenwashing are based exclusively on the equilibrium cost of capital, leaving aside the impact of greenwashing, asymmetric information and controversies on asset price volatility. However, investors' penalization of reve-

lations at the time of controversies should have significant effects on asset price volatility, which it would be interesting to investigate in future research. This question is not addressed in the third chapter, as it represents an additional degree of complexity, which would merit another chapter in its own right and would probably limit the readability of the results as presented in the third chapter of this thesis.

As far as the fourth chapter is concerned, while its contributions are aimed particularly at the literature on mean field games in the field of financial mathematics, it also opens up perspectives in terms of application to common-good games, and thus notably in the context of the low-carbon transition. However, the development of other convincing applications requires clarification of how these equilibria can be achieved in practice, so as to make their discussion more concrete. In particular, the conditions under which the initial commitment mechanism can realistically be binding on participants need to be questioned. In addition, since the outperformance in terms of utility of the coarse correlated equilibrium is based on the correlation of recommendations, we need to understand in concrete terms how this correlation can be implemented. One approach would be to look for the source of the randomness of recommendations within an uncertainty about the state of the world, and to identify a signal rooted in this randomness that is sufficiently credible and known by agents to serve as a reference point. While the possibility of a commitment to which agents are constrained may seem unrealistic as currently described, it should be possible to reinterpret coarse correlated equilibria without mentioning such a commitment, by focusing on a phenomenon of path dependence which, in effect, binds agents to past choices. These avenues are to be pursued in future research.

Secondly, this conclusion is the occasion to point out that certain aspects of the interaction between the financial system and climate change are not addressed in this thesis. Indeed, there are at least two blind spots in the analysis of this interaction, reflecting in part a lesser interest for these issues in the financial literature. The first is the litigation risk, already raised by Mark Carney in 2015 (Carney, 2015), which threatens economic actors considered responsible for climate change or environmental degradation with reputational damage and potentially substantial fines. This risk can affect financial institutions both directly, if they are sued for their participation in the financing of companies whose activity is incompatible with the low-carbon transition, and indirectly, through the litigation risk hanging over the companies they finance. Although this risk is starting to be addressed in the literature (NGFS, 2021a; Solana, 2020), notably in an attempt to quantify it (Setzer and Higham, 2022) and show that it is an actual financial risk (Sato et al., 2023), it is still little studied and deserves ad hoc work combining empirical study and theory to address its specificity.

The second is the financing of adaptation to climate change, which mainly consists of investment in long-term infrastructure to make economies and territories more resilient to the physical impacts of climate change (Monasterolo et al., 2022). There is a significant adaptation financing gap, particularly in developing countries (Watkiss et al., 2023). This gap cannot be

filled without mobilizing private finance; however, such investments are not very attractive to the latter, notably because of the high level of risk of such investments (Watkiss et al., 2021) and an underestimation of the financial impact of physical risks (Bressan et al., 2022b). Private finance must therefore be mobilized with the support of public finance to reduce risk levels and make such financing more attractive (Watkiss et al., 2021), but also by the conviction that it is directly exposed to high risks (Monasterolo et al., 2022; Bressan et al., 2022b). Finance research could contribute to understanding the factors behind this financing gap, and help bridge it by designing appropriate financial instruments, articulated to public funding, to make these investments attractive to private funds.

Finally, while this thesis focuses on the issue of climate change, the footprint of human activities on the environment also has other consequences that are sometimes even more worrying than climate change because of the speed at which they are evolving, such as the record-breaking degradation of biodiversity, which is essential to our economies, and to our lives (IPBES, 2019). The question of the dual materiality of biodiversity to finance has, even more recently than that of climate change, begun to be investigated in the financial literature (Flammer et al., 2023; Giglio et al., 2023b; Karolyi and Tobin-de la Puente, 2023) and to interest central banks (Svartzman et al., 2021b; NGFS, 2023b) as well as the wider financial community (Task Force on Nature-Related Financial Disclosures, 2023). Given that understanding the financial system's relationship with climate change and biodiversity involves common issues, a number of the analyses in this thesis can easily be extended to a more comprehensive consideration of the sustainability of our economic activities. Nevertheless, the specific nature of some of the issues raised by biodiversity would probably merit the development of appropriate theoretical tools.

Conclusion

En résumé, cette thèse propose dans un premier chapitre une revue méthodologique des stress tests climatiques et analyses de scénario développés par les banques centrales et dans la littérature académique. La capacité de ces méthodologies à évaluer l'ampleur des risques financiers causés par le changement climatique et la transition bas-carbone y est discutée. En particulier, cette revue montre que l'impact de l'incertitude caractérisant ces deux phénomènes tend à être sous-estimé dans l'évaluation des risques financiers qu'ils représentent. Ce chapitre, accompagné de la mise en perspective effectuée en introduction de cette thèse, a contribué à poser les jalons des problématiques de recherche auxquelles répondent les chapitres suivants : importance du comportement des agents dans la transition bas-carbone, rôle clé du secteur financier et opacité des données évaluant l'exposition des entreprises aux risques physiques et de transition.

Les trois chapitres qui suivent présentent ainsi trois modèles et outils de modélisation appartenant aux champs de la microéconomie financière et de la théorie des jeux dans lesquels des agents clés de la transition bas-carbone interagissent stratégiquement. Le deuxième chapitre de cette thèse répond au besoin de comprendre les décisions de financement des banques et leur impact dans le contexte de la transition bas-carbone, en adoptant une approche microéconomique, centrée sur leurs intérêts. Bien qu'encore sous forme de document de travail, il suggère l'existence d'un potentiel effet boule de neige lorsqu'une banque cesse de financer une entreprise dont les actifs doivent être échoués pour permettre l'atteinte de l'objectif de limitation de température de l'accord de Paris. En effet, la décision de désinvestissement d'une telle entreprise par une banque pourrait, sous certaines conditions, pousser les autres banques finançant cette entreprise à interrompre leur financement également, conduisant l'entreprise à une situation d'insolvabilité. Les résultats de cette étude peuvent ainsi être interprétés comme l'identification d'un levier de la transition bas-carbone, mais aussi comme une mise en lumière d'un potentiel risque d'instabilité économique et financière qui y est associé.

Le troisième chapitre, plus mature, s'intéresse au comportement d'écoblanchiment des entreprises comme un choix stratégique dans le but de minimiser leur coût du capital, dans un contexte d'asymétrie d'information concernant leur empreinte environnementale. Il montre ainsi que, si la communication est suffisamment peu coûteuse relativement aux efforts de réduction

d'émissions, ou si l'asymétrie d'information au regard de leur empreinte environnementale est suffisamment élevée, les préférences environnementales des investisseurs incitent les entreprises à faire de l'écoblanchiment. Toutefois, les investisseurs ont la possibilité de limiter cet effet d'incitation en pénalisant les révélations sur l'empreinte environnementale des entreprises lorsqu'elles sont touchées par une controverse.

Le quatrième chapitre de cette thèse, même s'il s'adresse principalement à la littérature émergente sur les jeux à champ moyen dans la communauté des mathématiques financières, propose une application de ses contributions techniques à un jeu de réduction d'émissions entre pays. En particulier, il montre que les équilibres corrélés généralisés permettent d'augmenter l'utilité de chaque joueur, représentative de l'utilité collective puisque les agents sont considérés symétriques dans un jeu à champ moyen, par rapport à un équilibre de Nash. Cette dernière contribution souligne la pertinence de s'intéresser aux équilibres corrélés généralisés dans un jeu de bien commun et ouvre la perspective d'autres applications aux efforts d'atténuation du changement climatique.

Ainsi, à travers ces quatre chapitres aux contributions principalement théoriques, cette thèse alimente la recherche sur l'usage de modèles prospectifs, macroéconomiques, microéconomiques et financiers, pour mieux comprendre l'interaction entre système financier et transition bas-carbone. En mettant l'accent sur les interactions stratégiques entre agents, elle aide à décrypter les moteurs et les freins à l'atténuation du changement climatique. En particulier, elle contribue à éclairer la question de l'intérêt et de la capacité des acteurs financiers à participer aux efforts d'atténuation en développant des modèles adéquats. Elle nourrit également la recherche sur les risques que le changement climatique et la transition bas-carbone font peser sur la stabilité financière.

En outre, à travers ses deux derniers chapitres, les contributions de cette thèse s'inscrivent à l'intersection des mathématiques financières, de la microéconomie et de la théorie des jeux. Le choix d'un cadre dynamique stochastique en temps continu et l'utilisation des jeux à champ moyen dans ce même cadre, en favorisant une résolution analytique, permettent de proposer des modèles riches dotés d'un système complexe d'interactions tout en restant interprétables. Ces deux derniers chapitres soulignent ainsi la pertinence d'utiliser ces outils de modélisation sophistiqués pour éclairer le comportement stratégique des agents dans le contexte de la transition bas-carbone.

Cette thèse ouvre un certain nombre de pistes de recherche. Elle invite notamment à poursuivre le développement d'outils théoriques pour étudier le rôle du système financier dans la transition bas-carbone, question qui est loin d'être épuisée pour l'instant. Par exemple, le deuxième chapitre aborde la question des choix de financement des banques vis-à-vis de la transition bas-carbone sous un seul angle, celui du désinvestissement. S'il serait intéressant d'affiner l'approche sur le désinvestissement, notamment en cherchant comment relier plus directement le désinvestissement à son impact en termes d'émissions carbone, il faudrait également compléter

cette approche par des travaux théoriques abordant la question du financement des entreprises effectuant des efforts d'atténuation. Cela permettrait de mieux comprendre quels leviers les banques ont à disposition pour favoriser les efforts d'atténuation. Plus généralement, cela permettrait également d'alimenter le travail des banques centrales dans les exercices de stress tests climatiques avec bilan dynamique et dans leur façon d'aborder le changement climatique et la transition bas-carbone en interaction avec leur mandat de stabilité financière.

En outre, les conclusions du troisième chapitre sur l'écoblanchiment des entreprises se fondent exclusivement sur le coût du capital à l'équilibre, laissant de côté l'impact de l'écoblanchiment, de l'asymétrie d'information et des controverses sur la volatilité du prix des actifs. Or, la pénalisation par les investisseurs des révélations au moment de controverses devrait avoir des effets importants sur la volatilité du prix des actifs qu'il serait intéressant d'investiguer dans une future recherche. Cette question n'est pas abordée dans le troisième chapitre car elle représente un degré supplémentaire de complexité, qui mériterait un autre chapitre à part entière et limiterait probablement la lisibilité des résultats tels qu'ils sont présentés dans le troisième chapitre de cette thèse.

En ce qui concerne le quatrième chapitre, si en l'état ses contributions s'adressent particulièrement à la littérature sur les jeux à champ moyen dans le champ des mathématiques financières, il ouvre également des perspectives en termes d'application à des jeux de bien commun, donc notamment dans le contexte de la transition bas-carbone. Toutefois, le développement d'autres applications convaincantes nécessite de préciser la façon dont ces équilibres peuvent être atteints concrètement, de manière à rendre plus concrète la discussion de la pertinence de ces équilibres. Notamment, les conditions selon lesquelles le mécanisme d'engagement initial peut être contraignant pour les participants d'une manière réaliste sont à interroger. En outre, la surperformance en termes d'utilité de l'équilibre corrélé généralisé reposant sur la corrélation des recommandations, il s'agit de comprendre concrètement comment se met en oeuvre cette corrélation. Une esquisse de réponse serait de chercher la source de l'aléa des recommandations dans une incertitude sur l'état du monde et d'identifier un signal ancré dans cet aléa qui soit suffisamment crédible et connu par les agents pour servir de point de référence. Si la possibilité d'un engagement auquel les agents sont contraints semble peu réaliste tel qu'il est décrit pour l'instant, il devrait être possible de réinterpréter l'équilibre corrélé généralisé sans parler d'un tel engagement en s'intéressant à un phénomène de dépendance au chemin qui, de fait, lie les agents à des choix passés. Ces pistes sont à poursuivre dans des recherches futures.

Ensuite, cette conclusion est l'occasion de souligner que certains aspects de l'interaction entre système financier et changement climatique ne sont pas abordés dans cette thèse. En effet, on peut noter au moins deux angles morts dans l'analyse de cette interaction, qui reflètent en partie un moindre intérêt de la littérature financière pour ces questions. Le premier est le risque de litige juridique, déjà soulevé par Mark Carney en 2015 (Carney, 2015), qui menace d'une dégradation de réputation et d'amendes potentiellement conséquentes les acteurs économiques considérés

comme responsables du changement climatique ou de la dégradation de l'environnement. Ce risque peut affecter les institutions financières à la fois directement, si elles sont poursuivies en justice pour leur participation au financement d'entreprises dont l'activité est incompatible avec la transition bas-carbone, et indirectement, à travers le risque de litige juridique qui pèse sur les entreprises qu'elles financent. S'il commence à être abordé dans la littérature (NGFS, 2021a; Solana, 2020), notamment pour essayer de le chiffrer (Setzer and Higham, 2022) et montrer qu'il s'agit d'un véritable risque financier (Sato et al., 2023), ce risque est encore peu étudié et mériterait des travaux ad hoc combinant étude empirique et théorie pour aborder sa spécificité.

Le deuxième est celui du financement de l'adaptation au changement climatique, qui consiste principalement en de l'investissement dans des infrastructures de long terme pour rendre les économies et territoires plus résilients aux impacts physiques du changement climatique (Monasterolo et al., 2022). Il existe un important déficit de financement de l'adaptation, particulièrement important dans les pays en développement (Watkiss et al., 2023). Ce déficit ne peut être comblé sans mobilisation de la finance privée ; toutefois, ces investissements sont peu attractifs pour cette dernière du fait notamment du haut niveau de risque de ces investissements (Watkiss et al., 2021) et d'une sous-estimation de l'impact financier des risques physiques (Bressan et al., 2022b). La finance privée doit donc être mobilisée avec le soutien de la finance publique pour réduire les niveaux de risques et rendre ces financements plus attractifs (Watkiss et al., 2021), mais aussi par la conviction qu'elle est directement exposée à des risques élevés (Monasterolo et al., 2022; Bressan et al., 2022b). La recherche en finance pourrait contribuer à la compréhension des facteurs de ce déficit de financement et aider à le combler par la conception d'instruments financiers adéquats, articulés à des financements publics, pour rendre ces investissements attractifs aux fonds privés.

Enfin, si cette thèse se concentre sur la question du changement climatique, l'empreinte des activités humaines sur l'environnement a également d'autres conséquences parfois plus inquiétantes encore que le changement climatique du fait de leur vitesse d'évolution, comme la dégradation à une vitesse record de la biodiversité, pourtant essentielle à notre (sur)vie (IPBES, 2019). La question de la double matérialité de la biodiversité pour la finance a, encore plus récemment que celle concernant le changement climatique, commencé à être investiguée dans la littérature financière (Flammer et al., 2023; Giglio et al., 2023b; Karolyi and Tobin-de la Puente, 2023) et à intéresser les banques centrales (Svartzman et al., 2021b; NGFS, 2023b) et plus largement la communauté financière (Task Force on Nature-Related Financial Disclosures, 2023). Etant donné que la compréhension des relations que le système financier entretient avec le changement climatique et la biodiversité comporte des problématiques communes, un certain nombre d'analyses de cette thèse peuvent facilement s'élargir à une considération plus exhaustive de la durabilité de nos activités économiques. Néanmoins, la spécificité de certains enjeux soulevés par la biodiversité mériterait probablement le développement d'outils théoriques adéquats en complément.

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Titre : Interactions stratégiques dans la transition écologique : Approches quantitatives en finance climat

Mots clés : Finance climat, Stress test climatique, Actifs échoués, Ecoblanchiment, Equilibres corrélés, Jeux à champ moyen

Résumé : Cette thèse modélise le comportement et les interactions stratégiques d'acteurs clés dans la transition bas-carbone ainsi que l'impact qu'ils ont sur son déroulement. Elle porte notamment une attention spécifique au rôle des acteurs financiers dans l'atténuation du changement climatique.

Le Chapitre 1 effectue une revue des méthodologies de stress tests climatiques pratiqués par les superviseurs financiers et proposés dans des articles académiques. Il propose des pistes de réflexion pour améliorer leur capacité à évaluer l'ensemble des risques physiques et de transition qui pourraient menacer la stabilité financière.

Le Chapitre 2 modélise l'effet d'un désinvestissement par les banques d'actifs polluants dont l'exploitation est incompatible avec la transition vers une économie bas-carbone, nommés actifs échoués. Sous certaines conditions, un tel désinvestissement de la part d'une banque peut provoquer une réaction en chaîne et inciter d'autres banques à désinvestir à leur tour. Ce scénario pose à la fois la question des leviers disponibles pour la réalisation de la transition bas-carbone, mais aussi celle des risques d'instabilité économique et financière que cette dernière peut provoquer.

Le Chapitre 3 modélise la façon dont la pratique de l'écoblanchiment est influencée par les préférences vertes des investisseurs dans un modèle d'équilibre dynamique stochastique. Il montre que la préférence de ces derniers pour des entreprises faiblement carbonées les incite à faire de l'écoblanchiment. Néanmoins, les investisseurs ont la possibilité de décourager cette pratique en la pénalisant lorsqu'elle est révélée par des controverses environnementales. Ces résultats sont robustes à l'introduction d'une interaction entre les entreprises et sont validés empiriquement.

Le Chapitre 4 développe une méthodologie pour identifier des équilibres corrélés généralisés dans des jeux à champ moyen linéaires quadratiques. Sous certaines conditions, ce type d'équilibre permet d'atteindre des niveaux de bien-être social plus élevés que l'équilibre de Nash couramment utilisé, tout en étant également non-coopératif. En appliquant cette méthode à un jeu stylisé d'atténuation du changement climatique entre pays, il montre comment l'équilibre corrélé généralisé propose une réponse non coopérative, théorique au moins, à la tragédie des communs.

Title: Strategic interactions in the ecological transition: Quantitative approaches in climate finance

Keywords: Climate finance, Climate stress testing, Stranded assets, Greenwashing, Coarse correlated equilibria, Mean field games

Abstract: This thesis models the strategic behavior and interactions of key players in the low-carbon transition, and the impact they have on its progress. It pays particular attention to the role of financial actors in climate change mitigation.

Chapter 1 reviews the climate stress test methodologies used by financial supervisors and proposed in academic articles. It suggests research avenues for improving their ability to assess the full range of physical and transition risks that could threaten financial stability.

Chapter 2 looks at the effect of divestment by banks from polluting assets whose operation is incompatible with the transition to a low-carbon economy, so-called stranded assets. Under certain conditions, such divestment by one bank triggers a chain reaction, prompting other banks to divest in turn. This scenario not only raises the question of the levers available to achieve the low-carbon transition, but also the risks of economic and financial instability that the latter may provoke.

Chapter 3 models how the practice of greenwashing is influenced by investors' green preferences in a stochastic dynamic equilibrium model. It shows that investors' preference for low-carbon companies encourages them to engage in greenwashing. Nevertheless, investors can discourage this practice by penalizing it when it is revealed by environmental controversies. These results are robust to the introduction of an interaction between firms, resolved by a mean field game, and are validated empirically.

Chapter 4 develops a methodology to identify coarse correlated equilibria in linear quadratic mean field games, an emerging tool in the field of financial mathematics. Under certain conditions, this type of equilibrium achieves higher levels of social welfare than the commonly used Nash equilibrium, while also being non-cooperative. By applying this method to a stylized game of climate change mitigation between countries, it shows how coarse correlated equilibria propose a non-cooperative response, at least theoretically, to the tragedy of the commons.